Projective toric designs, difference sets, and quantum state designs

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Abstract

Trigonometric cubature rules are sets of points on the torus over which sums reproduce integrals of degree t monomials over the full torus. They can be thought of as t-designs on the torus. Motivated by the projective structure of quantum mechanics, we develop the notion of t-designs on the *projective* torus, which, surprisingly, have a much more restricted structure than their counterparts on full tori. We provide various constructions of these projective toric designs and prove some bounds on their size and characterizations of their structure. We draw connections between projective toric designs and a diverse set of mathematical objects, including Sidon sets from the field of additive combinatorics, symmetric, informationally complete positive operator valued measures (SIC-POVMs) from quantum information theory, and complete sets of mutually unbiased bases (MUBs), which are conjectured to relate to finite projective geometry. We also use projective toric designs to construct families of quantum state designs. Finally, we discuss many open questions about the properties of these projective toric designs and how they relate to other questions in number theory, geometry, and quantum information.

Contents

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1 Introduction

Given a measure space (M, μ) and a set of polynomials on M, a *t-design* on M is a measure space $(X \subset M, \nu)$ satisfying $\int_X f \, \mathrm{d}\nu = \int_M f \, \mathrm{d}\mu$ for all polynomials f of degree $\leq t$ [1–13]. Classic examples are Gaussian quadrature rules [1] and spherical designs [6, 7], where the measure space M is the hypercube and hypersphere, respectively. Typically, one is interested in finding designs where X is a discrete measure space such that the integral over X with respect to ν reduces to a weighted sum that is often simpler to compute. However, this is not always possible; in the case of rigged designs (defined below), it is often crucial that X be a non-discrete measure space [14].

Specific forms of t-designs for particular choices of measure spaces M have found a plethora of uses in the field of quantum information theory [15–48]. In particular, complex projective space $\mathbb{CP}^{d-1}$ describes the space of d-dimensional quantum states [49], so t-designs on $M = \mathbb{CP}^{d-1}$ are called *complex-projective* or *quantum state t-designs*. These quantum state designs also relate to other mathematical objects such as symmetric, informationally complete positive operator valued measures (SIC-POVMs) and complete sets of mutually unbiased bases (MUBs), which themselves are conjectured to relate to finite projective geometry. Finite-dimensional quantum state designs can be generalized to designs on infinite-dimensional, or continuous-variable, quantum systems by defining *rigged quantum state t-designs*, which are designs on the space of tempered distributions $M = S(\mathbb{R})'$ [14]. Finally, the (projective) unitary group $\mathrm{PU}(d)$ describes the space of noiseless dynamics of quantum states, and these too admit constructions of *unitary t-designs*. Therefore a better understanding of various kinds of t-designs can also lead to deep insights about quantum information.

Consider the complex sphere Ω_d; that is, the set of unit vectors in $\mathbb{C}^d$. Any vector in Ω_d can be written (non-uniquely) as $|q, \phi\rangle \coloneqq \sum_{n=1}^{d} \sqrt{q_n} \mathrm{e}^{\mathrm{i}\phi_n} |n\rangle$, where $\{|n\rangle\}_{n=1}^{d}$ forms an orthonormal basis, $q = (q_n)_{n=1}^{d}$ is a discrete probability distribution ($\sum_n q_n = 1$), and $\phi = (\phi_n)_{n=1}^{d}$ is a set of phases. Therefore, q belongs to the $(d-1)$-simplex Δ^{d-1} and ϕ to the d-torus T^d. Via this mapping $\Delta^{d-1} \times T^d \to \Omega_d$, one can combine simplex designs and toric designs to form complex spherical designs [2]. Identifying $\mathbb{CP}^{d-1}$ with $\Omega_d/\mathrm{U}(1)$ (that is, quantum states are complex unit vectors with a global phase redundancy), we have a similar mapping $\Delta^{d-1} \times P(T^d) \to \mathbb{CP}^{d-1}$ defined as $(q, [\phi]) \mapsto [|q, \phi\rangle]$, where $P(T^d) = T^d/\mathrm{U}(1)$ is the projective torus (see Definition 4) and $[\cdot]$ denotes equivalence classes in the respective quotient spaces. In a similar way as before, via this mapping one can combine simplex designs and *projective toric designs* (see Definition 5) to form quantum state designs [3, 14].

In what follows, we flesh out and formalize this argument. Specifically, we formalize the notion of projective toric designs—both finite- and infinite-dimensional—and provide various constructions thereof. We discuss the connection between projective toric designs and difference sets [50–52], and use this correspondence to construct more projective toric designs, including some minimal ones. We illustrate the connection to quantum state designs and various other mathematical objects. Using minimal projective toric 2-designs, we construct an infinite family of almost-minimal complex-projective 2-designs. Finally, we discuss many exciting open questions regarding projective toric designs, some of which are deeply connected to long-outstanding conjectures in mathematics, such as some conjectures relating to finite affine and projective spaces.

Relation to prior work. Toric designs have been considered before. Trigonometric cubature rules are such designs on the torus [9–11]. Ref. [2] generalized the idea of trigonometric cubature to more general algebraic tori. Ref. [14] studied designs on projective tori and showed an equivalence to a specific case of Ref. [2], and further showed that such projective toric designs are related to complete sets of MUBs [53]. However we believe the presentation given in Section 2 gives new clarity and focus on the subject. Furthermore, Section 2.1 compiles, to the best of our knowledge, all previously known constructions of projective toric t-designs[1], and indeed generalizes some of these constructions. In Section 2.2, we prove a new characterization of minimal designs.

The main novel contributions of our work lie in Sections 3 and 4. To the best of our knowledge, Section 3 on difference sets and their application to constructing minimal projective toric designs is entirely new. The relationship between projective toric designs and quantum state designs described in Section 4 was first noted in Refs. [2, 14], though we believe that Section 4 greatly clarifies it. In Section 4, we also construct an

[1] Of course, many toric designs are known, and these always project to projective toric designs. Such constructions are not compiled in this manuscript.

infinite family of *almost-minimal* quantum state 2-designs—that is, quantum state 2-designs of size exactly one more than minimal. While these specific almost-minimal designs have been noted before in Ref. [54], we arrive at the construction via an entirely different route that utilizes projective toric designs. We believe that this route has a much better hope of generalizing to other infinite families and $t > 2$.

Finally, Section 5 compiles a number of new interesting open problems involving projective toric designs, highlighting their connection to a number of other open problems in mathematics.

2 Theory of toric designs

We begin with some basic definitions that we will use for the rest of the paper.

Definition 1 (Torus)**.** *Let $T := \mathbb{R}/2\pi\mathbb{Z}$. When $n \in \mathbb{N}$, let $I_n := \{1, 2, \dots n\}$; when $n = \infty$, let $I_n = I_\infty := \mathbb{N}$. For such n, let $T^n := \prod_{i \in I_n} T$ with the product topology. Define the projection maps $p_i \colon T^n \to T$ as $(\phi_j)_{j \in I_n} \mapsto \phi_i$. For all $n \in \mathbb{N} \cup \{\infty\}$, let μ_n denote T^n's unit-normalized Haar measure.*

Note that by Tychonoff's theorem, T^∞ is compact. For all n, T^n is therefore a compact abelian group and thus has a unique unit-normalized Haar measure.

By definition, the product topology on T^∞ is the coarsest topology such that the projection maps p_i are continuous. Similarly, we endow T^∞ with the smallest σ-algebra such that the projections p_i are measurable. This σ-algebra is generated by sets of the form $A = \prod_{i \in \mathbb{N}} A_i$, where each A_i is a measurable subset of T and all but finitely many A_i are equal to T. Define a measure μ' on T^∞ by $\mu'(A) = \prod_{i \in \mathbb{N}} \mu_1(A_i)$. From Ref. [55, Thm. 10.6.1] (or Ref. [56] for a shorter proof), this definition of μ' on such subsets uniquely determines μ' on the whole space. Clearly μ' is transitionally-invariant and unit-normalized, and therefore $\mu' = \mu_\infty$.

Definition 2 (Toric design)**.** *Fix an $n \in \mathbb{N} \cup \{\infty\}$ and $t_1, t_2 \in \mathbb{N}$. Let $X \subset T^n$ and (X, Σ, ν) be a measure space. (X, Σ, ν) (henceforth X for short) is called a T^n (t_1, t_2)-*design *if for all $a \in I_n^{t_1}$ and $b \in I_n^{t_2}$,*

$$\int_X \exp\left(\mathrm{i}\sum_{j=1}^{t_1} p_{a_j} - \mathrm{i}\sum_{k=1}^{t_2} p_{b_k}\right) \mathrm{d}\nu = \int_{T^n} \exp\left(\mathrm{i}\sum_{j=1}^{t_1} p_{a_j} - \mathrm{i}\sum_{k=1}^{t_2} p_{b_k}\right) \mathrm{d}\mu_n. \tag{1}$$

X is called discrete *if ν is a counting measure, and is called* finite *if it is discrete and $|X| < \infty$. If X is finite, then $|X|$ is called the* size *of X.*

The definition of toric designs closely resembles that of trigonometric cubature rules [9–11], however the two are not equivalent.

Definition 3 (Trigonometric cubature rule)**.** *A* trigonometric cubature rule *of dimension n and degree s is a measure space $(X \subset T^n, \Sigma, \nu)$ such that*

$$\int_X \exp\left(\mathrm{i}\sum_{j=1}^{n} \alpha_j p_j\right) \mathrm{d}\nu = \int_{T^n} \exp\left(\mathrm{i}\sum_{j=1}^{n} \alpha_j p_j\right) \mathrm{d}\mu_n \tag{2}$$

for all $\alpha \in \mathbb{Z}^n$ satisfying $\sum_{j=1}^n |\alpha_j| \leq s$.

Throughout this work, we will use double braces to denote multisets, whereas single braces will denote sets as usual; that is, $\{\!\{1, 2, 2\}\!\} = \{\!\{2, 1, 2\}\!\} \neq \{\!\{1, 2\}\!\}$, whereas $\{1, 2, 2\} = \{1, 2\} = \{2, 1\}$. Since the integrand in Eq. (1) contains only a finite number of projection maps, we can use Fubini's theorem to compute the integral on the right-hand side to be

$$\int_{T^n} \exp\left(\mathrm{i}\sum_{j=1}^{t_1} p_{a_j} - \mathrm{i}\sum_{k=1}^{t_2} p_{b_k}\right) \mathrm{d}\mu_n = \begin{cases} 1 & \text{if } \{\!\{a_i \mid i \in \{1, \dots, t_1\}\}\!\} = \{\!\{b_i \mid i \in \{1, \dots, t_2\}\}\!\} \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Obviously, the integral is zero whenever $t_1 \neq t_2$. By picking a subset of the indices, we clearly see that a T^n (t_1, t_2)-design is also a T^{n-1} (t_1, t_2)-design. By setting $a_{t_1} = b_{t_2}$, we get an arbitrary monomial of degree $(t_1 - 1, t_2 - 1)$. Thus, a (t_1, t_2)-design is also a $(t_1 - 1, t_2 - 1)$-design.

We now consider the projective torus, an important object in the study of quantum mechanics because it removes a global phase redundancy (see Section 4).

Definition 4 (Projective torus)**.** *Let $P(T^n)$ denote the* projective torus $P(T^n) \coloneqq T^n/T$*, where here T denotes the inclusion $T \hookrightarrow T^n$ by $T \ni \theta \mapsto (\theta, \theta, \dots) \in T^n$.*

Clearly, for any $f : T^n \to \mathbb{C}$ to descend to a well-defined function on $P(T^n)$ it must be constant on the cosets of the diagonal subgroup; in other words, it must satisfy $f(\mathrm{e}^{\mathrm{i}\phi_1+\mathrm{i}\theta}, \mathrm{e}^{\mathrm{i}\phi_2+\mathrm{i}\theta}, \dots) = f(\mathrm{e}^{\mathrm{i}\phi_1}, \mathrm{e}^{\mathrm{i}\phi_2}, \dots)$ for all $\theta \in T$. Hence, when studying designs on $P(T^n)$, we need only consider monomials on T^n where the degree and conjugate degree are equal. A degree t monomial on $P(T^n)$ therefore lifts to $\exp\Big(\mathrm{i}\sum_{k=1}^{t}(\phi_{a_k} - \phi_{b_k})\Big)$ for $a, b \in I_n^t$. We are thus now in a position to define a $P(T^n)$ t-design.

Definition 5 (Projective toric design)**.** *Fix an $n \in \mathbb{N} \cup \{\infty\}$ and $t \in \mathbb{N}$. Let $X \subset P(T^n)$ and (X, Σ, ν) be a measure space. X is called a $P(T^n)$ t-*design *if for all $a, b \in I_n^t$,*

$$\int_X \exp\left(\mathrm{i}\sum_{j=1}^{t}(p_{a_j} - p_{b_j})\right) \mathrm{d}\nu = \int_{P(T^n)} \exp\left(\mathrm{i}\sum_{j=1}^{t}(p_{a_j} - p_{b_j})\right) \mathrm{d}\mu_{n-1}. \tag{4}$$

Here we denote the unit-normalized Haar measure on $P(T^n)$ as simply μ_{n-1} since $P(T^n) \cong T^{n-1}$.

Clearly a $P(T^n)$ t-design is also a $(t-1)$-design, since we can let $a_t = b_t$ and have the integrand become an arbitrary degree $(t-1)$ monomial. Additionally, a $P(T^n)$ t-design is also a $P(T^{n-1})$ t-design, as can be seen by picking a subset of indices.

By choosing a set of representatives of $P(T^n)$ to be those phases ϕ for which $p_1(\phi) = \phi_1 = 0$, we can think of $P(T^n)$ as $\{0\} \times T^{n-1}$. In this way, we have that $p_1(\phi) = 0$ for all ϕ. It follows that $X \subset \{0\} \times T^{n-1}$ is a $P(T^n)$ t-design if

$$\int_X \exp\left(\mathrm{i}\sum_{j=1}^{t}(p_{a_j} - p_{b_j})\right) \mathrm{d}\nu = \int_{\{0\}\times T^{n-1}} \exp\left(\mathrm{i}\sum_{j=1}^{t}(p_{a_j} - p_{b_j})\right) \mathrm{d}\mu_{n-1} \tag{5a}$$

$$= \begin{cases} 1 & \text{if } \{\!\{a_i \mid i \in \{1, \dots, t\}\}\!\} = \{\!\{b_i \mid i \in \{1, \dots, t\}\}\!\} \\ 0 & \text{otherwise} \end{cases}. \tag{5b}$$

Suppose that we set each $b_j = 1$. It follows that X must match integration of polynomials on T^{n-1} of degree t and conjugate degree 0. Similarly, we can set each $a_j = 1$, and thus X must match integration of degree 0 and conjugate degree t. More generally, we see that it must match on monomials on T^{n-1} of degree (t_1, t_2) whenever $t_1 \leq t$ and $t_2 \leq t$. It follows that a trigonometric cubature rule of degree $2t$ on T^{n-1} is a $P(T^n)$ t-design, and a $P(T^n)$ t-design is a trigonometric cubature rule of degree t on T^{n-1}. The reverse implications, however do not hold in general.

Since $P(T^n) \cong T^{n-1}$, obviously a $P(T^n)$ t-design is a T^{n-1} (t,t)-design, and vice versa. Note also though that a $P(T^n)$ t-design is a T^n (t,t)-design, and vice versa. The backward direction comes from the fact that for every ϕ in a T^n (t,t)-design, we can subtract each ϕ_i by ϕ_1, resulting in a point $\phi \in \{0\} \times T^{n-1} \cong P(T^n)$. The resulting set of such ϕ will constitute a $P(T^n)$ t-design. From this and the previous paragraph, it therefore follows that a T^n (t,t)-design is a T^n (t_1, t_2)-design for all $t_1 \leq t$ and $t_2 \leq t$. We have hence seen that a $P(T^n)$ t-design is a t_1-design for all $t_1 \leq t$ and a T^n (t,t)-design is a (t_1, t_2)-design for all $t_1, t_2 \leq t$. Indeed, it is for this reason that t-designs and (t,t)-designs are of most interest as compared to (t_1, t_2)-designs for $t_1 \neq t_2$.

In fact, (t_1, t_2)-designs can be trivial when $t_1 \neq t_2$. In this case, we simply need the integral of every (t_1, t_2) monomial to be zero. This is achieved simply by the measure space $X = \{(0, 0, \dots), (\pi/\Delta, \pi/\Delta, \dots)\}$ with the counting measure. Here $\Delta \coloneqq t_1 - t_2$. With this measure space, we have

$$\int_X \exp\left(\mathrm{i}\sum_{j=1}^{t_1} p_{a_j} - \mathrm{i}\sum_{k=1}^{t_2} p_{b_k}\right) \mathrm{d}\nu = 1 + \exp\left(\frac{\mathrm{i}\pi}{\Delta} \cdot \Delta\right) = 1 - 1 = 0. \tag{6}$$

Since a (t,t)-design is a (t_1, t_2)-design for all $t_1, t_2 \leq t$ and a (t_1, t_2)-design can be trivial, we now restrict our attention to T^n (t,t)-designs. Because of the aforementioned equivalence between $P(T^n)$ t-designs

and T^n (t,t)-designs, we will henceforth refer to them interchangeably as T^n t-designs while occasionally emphasizing the projective aspect when it is useful. By linearity, a $P(T^n)$ t-design exactly integrates all polynomials on $P(T^n)$ of degree t or less. It is really the projective nature of the polynomials that we are integrating that give toric designs their interesting structure that is quite different than the structure of trigonometric cubature rules. For example, as we will see, for finite n, T^n 2-designs must be of size at least $n(n-1)+1$, and indeed this can be saturated for many n; in contrast, it is known that a trigonometric cubature rule of degree 4 requires size at least $2n^2$, of degree 3 requires at least $4n$ points (which can often be achieved), and of degree 2 requires at least $2n$ points (and $2n+1$ can often be achieved) [10]. Indeed, the difference between toric designs and projective toric designs is analogous to the difference between (complex) spherical designs and (complex) projective designs.

2.1 Constructions of toric designs

In this section, we will present a few simpler constructions in order to get a handle on toric designs. Later, in Section 3, we will construct more (and smaller) toric designs by utilizing difference sets and Sidon sets from additive combinatorics [50].

Our first example is a T^n 2-design of size n^2 whenever n is prime, and slightly larger when n is not prime. Note that this construction can be generalized to be size n^2 whenever n is a prime power, but we do not do this here.

Theorem 6 (Thm. C.9 of [14]). *Let $n \in \mathbb{N}$. Define p to be the smallest prime number strictly larger than $\max(2,n)$ (by the prime number theorem, $p \in \mathcal{O}(n+\log n)$). Let $X \subset T^n$ be the set*

$$X = \left\{ \left(0, 2\pi(q_1+q_2)/p, 2\pi(2q_1+4q_2)/p, \dots, 2\pi((n-1)q_1+(n-1)^2 q_2)/p\right) \mid q_1 \in \mathbb{Z}_p, q_2 \in \mathbb{Z}_p \right\} \tag{7}$$

and v the constant map $v(\phi) = 1/p^2$. Then X with the counting measure weighted by v is a T^n 2-design.

We can extend this construction to the case when $n = \infty$.

Theorem 7. *Let $X \subset T^\infty$ be the set*

$$X = \left\{ \left(0, \vartheta+\varphi, 2\vartheta+4\varphi, \dots, j\vartheta + j^2\varphi, \dots\right) \mid \vartheta, \varphi \in [0,2\pi] \right\} \tag{8}$$

and ν the unit normalized Lebesgue measure on $[0,2\pi]^2$ (i.e. $\mathrm{d}\nu = \mathrm{d}\vartheta\,\mathrm{d}\varphi/(2\pi)^2$). Then X is a T^∞ 2-design.

Proof. For any $a,b,c,d \in \mathbb{N}$,

$$\int_X \exp(\mathrm{i}(p_a + p_b - p_c - p_d))\,\mathrm{d}\nu = \int_{[0,2\pi]^2} \exp\left(\mathrm{i}\vartheta(a+b-c-d) + \mathrm{i}\varphi(a^2+b^2-c^2-d^2)\right) \frac{\mathrm{d}\vartheta\,\mathrm{d}\varphi}{(2\pi)^2} \tag{9a}$$

$$= \begin{cases} 1 & \text{if } a+b=c+d \ \wedge \ a^2+b^2=c^2+d^2 \\ 0 & \text{otherwise} \end{cases} \tag{9b}$$

$$= \begin{cases} 1 & \text{if } \{\!\{a,b\}\!\} = \{\!\{c,d\}\!\} \\ 0 & \text{otherwise} \end{cases}, \tag{9c}$$

where in the last line we used [14, Lem. C.10]. □

We now consider arbitrary t.

Theorem 8 (Thm. C.8 of [14]). *Let $n \in \mathbb{N}$, $t \in \mathbb{N}$, $X \subset T^n$ be the set*

$$X = \left\{ (2\pi d_1/(t+1), 2\pi d_2/(t+1), \dots, 2\pi d_n/(t+1)) \mid d \in \mathbb{Z}_{t+1}^n \right\}, \tag{10}$$

and v be the constant map $v(\phi) = (t+1)^{-n}$. Then X with the counting measure weighted by v is a T^n t-design.

We now extend this construction to $n = \infty$.

Theorem 9. *Let $t \in \mathbb{N}$ and $X_1 \subset T$ be the discrete probability space $X_1 = \{2\pi d/(t+1) \mid d \in \mathbb{Z}_{t+1}\}$. Let $X = \prod_{i\in\mathbb{N}} X_1$ and its σ-algebra be generated by sets of the form $\prod_{i\in\mathbb{N}} A_i$ where each A_i in the power set $A_i \in \mathcal{P}(X_1)$ and for all but finitely many i we have $A_i = X_1$. Define ν by its action $\nu(A) = \prod_{i\in\mathbb{N}}(|A_i| / |X_1|)$, and note that ν uniquely extends to a measure on X [55, Thm. 10.6.1]. Then X is a T^∞ t-design.*

Proof. Let $m = \max(\max_j a_j, \max_j b_j)$. Since t is finite, we are only ever dealing with a finite number of projection maps p_i in the integrand. Therefore, we can apply Fubini's theorem to separate the integral $\int_X$ into a product of an integral over X_1^m and an integral over the rest of the space. Hence,

$$\int_X \exp\left(\mathrm{i}\sum_{j=1}^{t}(p_{a_j} - p_{b_j})\right) \mathrm{d}\nu = \frac{1}{\left|\mathbb{Z}_{t+1}^m\right|} \sum_{d\in\mathbb{Z}_{t+1}^m} \exp\left(\frac{2\pi\mathrm{i}}{t+1}\sum_{j=1}^{t}(d_{a_j} - d_{b_j})\right) \tag{11a}$$

$$= \begin{cases} 1 & \text{if } \{\!\{a_j \mid j \in \{1,\dots,t\}\}\!\} = \{\!\{b_j \mid j \in \{1,\dots,t\}\}\!\} \\ 0 & \text{otherwise} \end{cases}. \tag{11b}$$

□

Finally, we note the asymptotic existence theorem proven in [2].

Theorem 10 (Thm. 3.3 and Cor. 5.4 of [2])**.** *Asymptotically in $n \to \infty$ but for finite n, a T^n t-design must have size at least $\frac{n^t(1-o(1))}{\lceil t/2\rceil!\lfloor t/2\rfloor!}$ and there exists t-designs of size $n^t(1+o(1))$.*

2.2 Minimal toric designs

A very natural question that one can ask is *what is the size of the smallest toric t-design?* We call such designs *minimal*. We now prove a lower bound on the size of minimial toric 2-designs, and in Section 3, we will show that this bound can be saturated in many dimensions.

Proposition 11 (Prop. C.11 of [14])**.** *Let $n \in \mathbb{N}$ and (X, Σ, ν) be a finite T^n 2-design. Then $|X| > n(n-1)$.*

Proof. Since X is finite and a discrete measure space, denote $\int_X (\cdot)\,\mathrm{d}\nu$ by $\sum_{\phi\in X} v(\phi)(\cdot)$. The toric 2-design condition can be expressed as follows. Let

$$\Gamma = \{(0,\dots,0), (1,-1,0,\dots,0), (1,0,-1,0,\dots,0), \dots, (-1,1,0,\dots,0), \dots, (0,\dots,0,-1,1)\}, \tag{12}$$

so that $|\Gamma| = n(n-1)+1$. Let each $\phi \in X$ label a basis element of $V \coloneqq \mathbb{C}^{|X|}$ so that $\{|\phi\rangle \mid \phi \in X\}$ is an orthonormal basis of V. Then for $k \in \Gamma$, define $|k\rangle = \sum_{\phi\in X} \sqrt{v(\phi)}\mathrm{e}^{\mathrm{i}k\cdot\phi}\,|\phi\rangle$. The 2-design condition is summed up by $\langle k|k'\rangle = \delta_{kk'}$. Hence, $\{|k\rangle \mid k \in \Gamma\}$ must be orthonormal in V, meaning that $|\Gamma| \le \dim V = |X|$. □

Furthermore, we can prove that a minimal design must be uniformly weighted.

Proposition 12. *Let $S \subset T^n$ and let $v\colon S \to (0,\infty)$ define a weighted discrete measure on S. Suppose the measure space defined by S and v is a minimal 2-design. Then $v(\theta) = 1/|S|$.*

Proof. This proof essentially follows that of Ref. [10, Thm. 2.2]. Let Γ be as in Proposition 11. The T^n 2-design condition is written as $MM^\dagger = \mathbb{I}_{|\Gamma|\times|\Gamma|}$, where $M_{k,\theta} = \sqrt{v(\theta)}\mathrm{e}^{\mathrm{i}k\cdot\theta}$. If S is minimal—that is, if $|S| = |\Gamma|$—then M is a square matrix so that $MM^\dagger = \mathbb{I}$ if and only if $M^\dagger M = \mathbb{I}$. From the latter condition, it follows that $\delta_{\theta,\theta'} = \sqrt{v(\theta)v(\theta')}\sum_{k\in\Gamma}\mathrm{e}^{\mathrm{i}k\cdot(\theta-\theta')}$. When $\theta = \theta'$, we therefore find that $v(\theta) = 1/|\Gamma| = 1/|S|$. □

In Section 3, we show how minimal 2-designs are related to difference sets. Using this connection, we construct a family of minimal 2-designs. We would also like to generalize the lower bound for $t = 2$ to arbitrary t, but the difficulty in this is determining what Γ becomes in the arbitrary t case. Indeed, this is closely related to a similar difficulty in studying difference sets, which we comment on more in Sections 3 and 5.

3 Relation to difference sets

We say that $X \subset T^n$ is a *group toric t-design* if X is a t-design and also inherits group structure from T^n. We begin by considering the case when X is a cyclic group for finite n and a circle group for $n = \infty$. Here we will find connections to Sidon sets and difference sets [50]. We will then discuss the case of more general groups.

3.1 Cyclic designs

We begin with the infinite case. Suppose that $X \subset T^\infty$ is a t-design and isomorphic to the circle group U(1). Then there is a single element $z \in \mathbb{Z}^\infty$ such that $X = \{\theta z = (\theta z_1, \theta z_2, \dots) \mid \theta \in [0, 2\pi]\}$. In order for X to be a design, it must be that

$$\int_0^{2\pi} \exp\left(\mathrm{i}\theta \sum_{j=1}^{t} (z_{a_j} - z_{b_j})\right) \frac{\mathrm{d}\theta}{2\pi} = \begin{cases} 1 & \text{if } \{\!\{a_j \mid j \in \{1,\dots,t\}\}\!\} = \{\!\{b_j \mid j \in \{1,\dots,t\}\}\!\} \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

for all $a, b \in \mathbb{N}^t$. It follows that z must satisfy

$$\left(\sum_{j=1}^{t} z_{a_j} = \sum_{j=1}^{t} z_{b_j}\right) \iff \left(\{\!\{a_j \mid j \in \{1,\dots,t\}\}\!\} = \{\!\{b_j \mid j \in \{1,\dots,t\}\}\!\}\right). \tag{14}$$

In other words, the sum of any t elements of z must be unique. If we restrict z to be in $\mathbb{Z}^\infty_{\geq 0}$, then Eq. (14) is exactly the condition for z to be a B_t *set*[2] [50, Def. 4.27]. In the special case of $t = 2$, we need to find a $z \in \mathbb{Z}^\infty_{\geq 0}$ such that $z_a + z_b = z_c + z_d$ if and only if $\{\!\{a, b\}\!\} = \{\!\{c, d\}\!\}$. Such a z is called a *Sidon set* [50].

We have therefore proven the following proposition.

Proposition 13. *Group T^∞ t-designs isomorphic to the circle group are in one-to-one correspondence with B_t sets.*

We consider the following simple B_t set defined by $z_a = t^a$. In this case, z_a written in base t is $100\dots0$, a 1 followed by a 0s. It follows easily that every sum is unique up to reordering.

We now discuss finite n. Suppose that $X \subset T^n$ is a t-design and isomorphic to the cyclic group $\mathbb{Z}_m$. It follows that X is a size m t-design and is generated by a fixed $z \in \mathbb{Z}_m^n$. In order for X to be a design, it must be that

$$\sum_{d=0}^{m-1} \exp\left(\frac{2\pi \mathrm{i} d}{m} \sum_{j=1}^{t} (z_{a_j} - z_{b_j})\right) = \begin{cases} 1 & \text{if } \{\!\{a_j \mid j \in \{1,\dots,t\}\}\!\} = \{\!\{b_j \mid j \in \{1,\dots,t\}\}\!\} \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

for all $a, b \in I_n^t$. It follows that z must satisfy

$$\left(\sum_{j=1}^{t} z_{a_j} \equiv \sum_{j=1}^{t} z_{b_j} \pmod m\right) \iff \left(\{\!\{a_j \mid j \in \{1,\dots,t\}\}\!\} = \{\!\{b_j \mid j \in \{1,\dots,t\}\}\!\}\right). \tag{16}$$

In other words, the sum of any t elements of z must be unique, or equivalently,

$$\left|\left\{\sum_{j=1}^{t} z_{a_j} \bmod m \mid a \in I_n^t\right\}\right| = \binom{n+t-1}{t}. \tag{17}$$

Eq. (16) is precisely the condition for z to be a $B_t \bmod m$ *set of size* n [50]. We have therefore shown the following proposition.

[2]Note that we are considering z to be a tuple and yet calling it a difference "set". It is understood that we are talking about the set $\{z_a \mid a \in \mathbb{N}\}$.

Proposition 14. *Group T^n t-designs isomorphic to the cyclic group $\mathbb{Z}_m$ are in one-to-one correspondence with B_t mod m sets of size n.*

In the special case where $t = 2$, such a set is called a *Sidon set of size n mod m*. Notably, by a simple counting argument, any Sidon set of size n mod m must satisfy $m \geq n(n-1)+1$.[3] Further, for many but not all n, this bound can be saturated, as we will discuss later. When the bound is saturated, we say the Sidon set is *dense*. Hence, for every n for which there is a Sidon set of size n mod $n(n-1)+1$, there is a *minimal* T^n 2-design—that is, a T^n 2-design of size $n(n-1)+1$, hence saturating the lower bound from Proposition 11.

As an example of such a Sidon set, consider $n = 6$ and $m = n(n-1)+1 = 31$. Then one can easily check that $z = (0, 1, 3, 8, 12, 18)$ is a Sidon set and thus gives rise to a T^6 2-design of size 31. A simple numerical search however reveals that there does not exist a Sidon set of size 7 mod $7(7-1)+1 = 43$. Therefore, we have the following corollary.

Corollary 15. *Either there are no T^7 2-designs of size saturating the lower bound given in Proposition 11, or such a saturating design cannot be isomorphic to a cyclic group.*

There is a general construction of dense Sidon sets—called Singer sets—whenever $n-1$ is a prime power [57]. Thus, we have constructed minimal T^n 2-designs whenever $n-1$ is a prime power, and these designs are isomorphic to the cyclic group $\mathbb{Z}_{n(n-1)+1}$. We review the Singer set construction in Appendix A.

4 Relation to quantum state designs

Toric designs are closely connected to complex-projective designs [15–18, 23, 32, 41, 44–48], continuous-variable (CV) rigged designs [14], and complete sets of mutually unbiased bases (MUBs) [53]. These connections arise by concatenating toric and simplex designs in order to generate elements in complex-projective space, which in turn satisfy the design condition. We discuss the connection here.

Let $\mathbb{CP}^{d-1}$ be complex-projective space $S^{2d-1}/\mathrm{U}(1)$, where S^{2d-1} is viewed as subset of $\mathbb{C}^d$. Pick an orthonormal basis $\{|n\rangle \mid n \in \{1, \ldots, d\}\}$ of $\mathbb{C}^d$. A polynomial f on S^{2d-1} descends to a well-defined polynomial on $\mathbb{CP}^{d-1}$ if and only if it is invariant under the action of U(1)—that is, $f(\mathrm{e}^{\mathrm{i}\theta}|\psi\rangle) = f(|\psi\rangle)$ for all θ. It follows that all monomials on $\mathbb{CP}^{d-1}$ are of the form $\prod_{i=1}^t \langle a_i|\psi\rangle\langle\psi|b_i\rangle$ for $a, b \in I_d^t$. A $\mathbb{CP}^{d-1}$ t-design is thus a measure space (X, Σ, ν) such that, for all $a, b \in I_d^t$,

$$\int_X \left(\prod_{i=1}^t \langle a_i|\psi\rangle\langle\psi|b_i\rangle\right) \mathrm{d}\nu(\psi) = \int_{\mathbb{CP}^{d-1}} \left(\prod_{i=1}^t \langle a_i|\psi\rangle\langle\psi|b_i\rangle\right) \mathrm{d}\psi = \frac{\Pi_t^{(d)}(a;b)}{\operatorname{Tr}\Pi_t^{(d)}}, \tag{18}$$

where $\Pi_t^{(d)}$ is the projector onto the symmetric subspace of $(\mathbb{C}^d)^{\otimes t}$, $\Pi_t^{(d)}(a;b) \coloneqq \left(\bigotimes_{i=1}^t \langle a_i|\right) \Pi_t^{(d)} \left(\bigotimes_{i=1}^t |b_i\rangle\right)$, and $\mathrm{d}\psi$ denotes the Fubini-Study volume measure on $\mathbb{CP}^{d-1}$. The last equality is a simple consequence of Schur's lemma and the unitary invariance of $\mathrm{d}\psi$ [16, 47][14, Ap. C3].

Let Δ^{d-1} denote the $(d-1)$-dimensional simplex. Simplex t-designs have analogous definitions to those of toric and complex-projective designs [2, 3, 8, 12, 13]. Any vector $|\psi\rangle \in S^{2d-1}$ can be represented as $|p, \phi\rangle \coloneqq \sum_{n=1}^d \sqrt{p_n}\mathrm{e}^{\mathrm{i}\phi_n}|n\rangle$ for some (not necessarily unique) $p \in \Delta^{d-1}$ and $\phi \in T^d$. Furthermore, if we let $\pi \colon \Delta^{d-1} \times P(T^d) \mapsto \mathbb{CP}^{d-1}$ be defined by the above map, then the pullback of the Fubini-Study volume form along π is precisely the Lebesgue measure on Δ^{d-1} times the Lebesgue measure on $P(T^d)$ (see Appendix B). Together, this implies that the concatenation of a Δ^{d-1} t-design and a $P(T^d)$ t-design yields a $\mathbb{CP}^{d-1}$ t-design [2, 14].

We note that the analogous result holds for the complex sphere $\Omega_n = \{z \in \mathbb{C}^n \mid \sum_i |z_i|^2 = 1\}$; namely, concatenation of a Δ^{d-1} t-design and a trigonometric $2t$-design yields a Ω_d t-design. The reason that we only need a projective toric design in the $\mathbb{CP}^{d-1}$ case, as opposed to a full trigonometric design in the Ω_d case, is because polynomials on $\mathbb{CP}^{d-1}$ are more restricted than on Ω_d. On Ω_d, $z_1 z_2 \bar{z}_3$ is a valid monomial. On the other hand, this is an invalid monomial on $\mathbb{CP}^{d-1} = \Omega_d/\mathrm{U}(1)$ since it varies under the action of U(1).

[3] The Sidon set condition can be restated as stipulating that $z_a - z_c \equiv z_d - z_b$ if and only if $\{\!\{a, b\}\!\} = \{\!\{c, d\}\!\}$. We therefore need $z_a - z_c$ to be unique for all a and c. First choose an $a \in I_n$ and then choose a $c \in I_n$ with $c \neq a$. This gives us $n(n-1)$ distinct values. Further, we have one more value—namely 0—coming from when $a = c$.

One particularly nice simplex 2-design contains the extremal points $(1,0,\ldots,0),\ldots,(0,0,\ldots,1)$ and the centroid $c = (1/d, 1/d, \ldots, 1/d)$ (see e.g. [14, Thm. C4]). When concatenating the extremal points with a toric design, we get the basis vectors $[|n\rangle] \in \mathbb{CP}^{d-1}$, where $[\cdot]$ denotes the equivalence class in $\mathbb{CP}^{d-1}$. When concatenating the centroid with a finite-sized toric design X, we get a collection of points $\big\{[|c,\phi\rangle] \in \mathbb{CP}^{d-1} \mid \phi \in X\big\}$. Hence, the total number of points in the resulting complex-projective design is $d + |X|$. Recalling Proposition 11, we have that $|X| \geq d(d-1)+1$. Furthermore, from Section 3, we found an explicit construction using Singer sets of these minimal toric designs whenever $d+1$ is a prime power. It follows that the resulting complex-projective 2-design is of size d^2+1. Interestingly, the smallest possible complex-projective 2-design—also called a SIC-POVM—has size d^2. The existence of SIC-POVM's in all dimensions d is still an open problem.

These *almost-minimal* $\mathbb{CP}^{d-1}$ 2-designs that we just constructed—$\mathbb{CP}^{d-1}$ 2-designs of size d^2+1—were first constructed in Ref. [54]. Notably, however, our utilization of toric designs indicates a possible path toward extending such constructions to higher t-designs.

In Ref. [14, Ap. F], it was shown that toric designs are closely related to complete sets of MUBs. Let $S \subset P(T^n)$ be a set of size $|S| = d^2$. By concatenating S with the simplex 2-design described above (extremal points and centroid), one finds that the set of phases S constitute a complete set of MUBs if and only if they satisfy an (1) orthonormality condition, and a (2) mutually unbiased condition. It was shown in Ref. [14, Ap. F] that the second condition (2) can be replaced with the requirement that S be a toric 2-design.

Finally, Ref. [58] introduced the notion of a continuous variable (CV) t-design. Ref. [14] proved that such designs do not exist and therefore introduced rigged CV t-designs. A simplex design can be generalized to the unnormalized infinite-dimensional simplex. It then follows that the concatenation of an infinite-dimensional simplex t-design and a $P(T^\infty)$ t-design yields a rigged CV t-design. We therefore see that designs on the infinite-dimensional projective torus $P(T^\infty)$ are closely related to designs on other infinite-dimensional spaces.

5 Conclusion and open questions

In this work, we have developed the theory of projective toric designs and their relation to various other objects in and areas of mathematics and physics. There is still much unknown and we believe there are still many exciting connections to be made. We now discuss various future research directions relating to projective toric designs.

Minimal projective toric designs In this work, we showed that if X is a $P(T^n)$ 2-design, then $|X| \geq n(n-1)+1$. Furthermore, using Sidon sets, we showed that the bound can be saturated when $n-1$ is a prime power. However, we also showed that the bound cannot always be satisfied using the Sidon set construction; for example, when $n=7$, the Sidon set construction does not yield a minimal projective toric 2-design. We thus have the following open question: do projective toric 2-designs saturating the bound exist for all n?

We showed that if the 2-design is a cyclic group, then the constructions are in one-to-one correspondence with Sidon sets. In the case of e.g. $n=7$, $n(n-1)+1 = 43$ is prime so that the only group design could be a cyclic group. Therefore, if one can prove that a minimal design must be a group, then one would prove that the bound cannot be saturated for all n. Must the minimal design be a group?

For general t, even less is known. Can one prove a tight lower bound on the size of projective toric t-designs for arbitrary t? Can one construct saturating designs? One difficulty in proving a lower bound is related to the counting problem of determining a bound on the size of dense modular difference sets. As we saw in Proposition 11, the lower bound on the size of projective toric 2-designs comes matches the lower bound on the size of dense modular Sidon sets. We believe that the analogous statement holds for more general t.

Connection to affine/projective planes A *finite projective plane* is a tuple (P, L) of a finite set of points P and lines $L \subseteq 2^P$ such that:

1. Any two points are elements of a unique common line

2. Any two lines intersect at a unique point

3. There exist four points in P such that no line contains more than two of them.

Affine planes are defined similarly. A tuple (P, L) can only be a finite projective plane if there exists some $d \in \mathbb{N}$ such that $|P| = |L| = d^2 + d + 1$. However, finite projective planes have only been constructed for d a prime power, and are known to *not* exist if d is both not the sum of two squares and $d \equiv 1$ or 2 mod 4. These numeric similarities, along with deep connections between combinatorial designs and finite geometry, hint at a deeper connection between projective toric designs and finite projective planes. In addition, projective planes appear in the construction of Sidon sets, and are conjectured to correspond to dense ones [59].

Further, a complete set of MUBs yields a finite projective plane, while a SIC-POVM in prime power dimensions yields a finite affine plane [53, 60, 61]. As mentioned above, MUBs are closely related to projective toric designs, while SIC-POVMs are minimal complex-projective designs. All of this circumstantial evidence begs the question: are there interesting direct connections one can make between projective toric designs and finite planes, either projective or affine?

Connection to other designs Recall that complex projective designs can be constructed by concatenating simplex and projective toric designs. One can ask: how much can this result be generalized? Can we use similar constructions for toric varieties and flag varieties? Indeed $\mathbb{CP}^n$ is a toric variety with moment map to the associated polytope being the simplex Δ^n. The moment map allows us to project $\mathbb{CP}^n$ designs to Δ^n designs. Projective toric designs allow us to pullback along the moment map and build $\mathbb{CP}^n$ designs from Δ^n designs. How much more general can this result be made?

Approximate designs One can consider approximate projective toric t-design, which are points on the torus that integrate monomials of degree $\leq t$ up to an error of ε. How does the size of the minimal approximate t-design depend on t and ε? If one takes an ε_1-approximate simplex t-design and ε_2-approximate projective toric design and concatenates them, what is the ε with which we get a ε-approximate complex-projective t-design?

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A Singer sets

For cyclic groups of size $(p^m)^2 + (p^m) + 1$ with p a prime, there are constructions of Sidon sets of size $p^m + 1$, called Singer sets [[57, p. 380-381]; [51, Sec. 3.5]; [52]]. In other words, there is a n-torus 2-design of size $(n-1)^2 + n = n^2 - n + 1$, i.e., a minimal one, whenever $n - 1$ is prime-power.

The construction of such Sidon sets goes as follows. Let θ be the generator of $\mathbb{F}^\times_{(n-1)^{t+1}}$, and then let

$$T_t := \{0\} \cup \{a \in [(n-1)^{t+1} - 1] \ : \ (\theta^a - \theta) \in \mathbb{F}_{n-1} \subset \mathbb{F}_{(n-1)^{t+1}}\}. \tag{A1}$$

The inclusion $\mathbb{F}_{n-1} \hookrightarrow \mathbb{F}_{(n-1)^{t+1}}$ is done by identifying the generator of $\mathbb{F}^\times_{(n-1)}$ with $\theta^{\frac{(n-1)^{t+1}-1}{n-2}}$, which makes sense as for any finite field $\mathbb{F}_q$, $|\mathbb{F}_q^\times| = q - 1$, and $\mathbb{F}_q^\times$ is cyclic.

Further, note that $\mathbb{F}_{(n-1)^{t+1}}$ is a $(t+1)$-dimensional $\mathbb{F}_{n-1}$-vector space. Thus, $\{\theta^b\}_{b=0}^t$ is a $\mathbb{F}_{n-1}$-basis of $\mathbb{F}_{(n-1)^{t+1}}$. This means that all $\theta^a = \sum_{i=0}^t k_i \theta^i$ for some unique $k_i \in \mathbb{F}_{(n-1)}$. However, if $\frac{(n-1)^{t+1}-1}{n-2} | a$, we know all $i \geq 1$ have $k_i = 0$.

Then, let

$$S_t((n-1),\theta) := \left\{ l \in \mathbb{Z}_{\frac{(n-1)^{t+1}-1}{n-2}} \;:\; l \equiv a \mod \left(\frac{(n-1)^{t+1}-1}{n-2}\right), a \in T_t \right\} \tag{A2}$$

be the residues of T_t mod $\frac{(n-1)^{t+1}-1}{n-2}$. We now recount proofs of some of $S_t((n-1),\theta)$'s properties.

Lemma A.1. $|S_t((n-1),\theta)| = n$.

Proof. First we note there are n distinct elements of $\mathbb{F}_{(n-1)^{t+1}}$ of the form $\theta + \gamma_a$, $\gamma_a \in \mathbb{F}_{n-1}$ by the $\mathbb{F}_{n-1}$-linear independence of θ and 1. As all elements of $\mathbb{F}_{(n-1)^{t+1}}$ equal θ^a for some unique $a \in [(n-1)^{t+1}-1]$, we see that $|T_t| = n$. Now, we must show that every element of T_t has a different residue modulo $\frac{(n-1)^{t+1}-1}{n-2}$.

Suppose $a, a' := a + k\frac{(n-1)^{t+1}-1}{n-2} \in T$, $k \in \mathbb{Z}_{>0}$. Then $r := \theta^{a'}/\theta^a = \theta^{k\frac{(n-1)^{t+1}-1}{n-2}} \in \mathbb{F}_{n-1}$. But by definition of T_t, $\theta^a = \theta + \gamma_a$, $\theta^{a'} = \theta + \gamma_{a'}$. But

$$\theta^{a'} = r\theta^a = r\theta + r\gamma_a. \tag{A3}$$

Thus, $r = 1$, meaning $(n-2)|k$, which means that only a can be in $[(n-1)^{t+1}-1]$, and thus that no two elements of T_t can have the same residue modulo $\frac{(n-1)^{t-1}+1}{n-2}$. □

Lemma A.2. *$S_t((n-1),\theta)$ is a B_t (mod $\frac{(n-1)^{t+1}-1}{n-2}$) set.*

Proof. Recall that $\{\theta^i\}_{i=0}^t$ is a $\mathbb{F}_{n-1}$-basis of $\mathbb{F}_{(n-1)^{t+1}}$. In other words, there exist no non-elementwise-zero tuples $(c_i)_{i=0}^t \in \mathbb{F}_{n-1}^{t+1}$ such that

$$\sum_{i=0}^{t} c_i\theta^i = 0. \tag{A4}$$

Equivalently, θ cannot be the root of any polynomial of degree $\leq t$ with $\mathbb{F}_{n-1}$-coefficients.

Now, consider two multisets A, B, $|A| = |B| \leq t$, taking entries from $S_t((n-1),\theta)$. Then, by the definition of $S_t((n-1),\theta)$ and T_t, we see that for all $a \in A \cup B$

$$\theta^a = \alpha_a(\theta + \gamma_a) \tag{A5}$$

for some $\alpha_a \in \mathbb{F}_{n-1}$. Now, consider $\Pi_A := \prod_{a\in A}\theta^a$ and $\Pi_B := \prod_{b\in B}\theta^b$. It is clear that $\Pi_B/\Pi_A \in \mathbb{F}_{n-1}$ and only if

$$\sum_{a\in A} a \equiv \sum_{b\in B} b \mod \frac{(n-1)^{t+1}-1}{n-2}. \tag{A6}$$

Thus, $\Pi_A - \beta_{A,B}\Pi_B = 0$ for some $\beta_{A,B} \in \mathbb{F}_{n-1}$ if and only if Eq. (A6) holds. However, for any $\beta \in \mathbb{F}_{n-1}$, we see that $\Pi_A - \beta\Pi_B$ is a degree-t polynomial equation in θ with $\mathbb{F}_{n-1}$ coefficients, meaning it cannot have any solutions, meaning the B_t (mod $\frac{(n-1)^{t+1}-1}{n-2}$) condition is satisfied. □

A.1 Explicit example of dense modular Sidon set

In this appendix, we work through an explicit example of the construction of the Sidon set $S_{t=2}((n-1),\theta)$ for $n = 5 = 2^2+1$. We begin by constructing T_t. Consider the field $\mathbb{F}_{(n-1)^{t+1}} = \mathbb{F}_{4^3} = \mathbb{F}_{2^6}$. With the irreducible polynomial $f(x) = 1 + x^5 + x^6 \in \mathbb{F}_2[x]$, we work in the polynomial representation $\mathbb{F}_{2^6} \cong \mathbb{F}_2[x]/(f(x))$.

One can check that the generator θ of the multiplicative group $\mathbb{F}_{2^6}^\times$ is x in this representation—in other words, $|\{x^m \bmod f(x) \mid m \in \mathbb{Z}_{63}\}| = 63$. We identify $\mathbb{F}_{n-1} = \mathbb{F}_{2^2} \subset \mathbb{F}_{2^6}$ via generating $\mathbb{F}_{2^2}^\times$ with

$$y = x^{\frac{(n-1)^{t+1}-1}{n-2}} = x^{21}, \tag{A7}$$

so that $\mathbb{F}_{2^2} = \{0\} \cup \{y^k \mid k \in \mathbb{Z}_3\}$. Then

$$T_{t=2} = \{0\} \cup \{a \in \mathbb{Z}_{4^3-1} \setminus \{0\} \mid (x^a - x) \pmod{f(x)} \in \mathbb{F}_{2^2}\}. \tag{A8}$$

Clearly, $1 \in T_{t=2}$. With that out of the way, we can rephrase this as

$$T_{t=2} = \{0,1\} \cup \left\{ a \in \mathbb{Z}_{4^3-1} \setminus \{0,1\} \mid \exists k \in \mathbb{Z}_3 \colon \; x^a - x \equiv y^k \pmod{f(x)} \right\}. \tag{A9}$$

One can straightforwardly numerically verify that $T_2 = \{0,1,14,25,58\}$. To ensure understanding of the construction, we will work through why $14 \in T_2$. We need to show that $x^{14} - x \equiv y^k \pmod{f(x)}$ for $k = 0, 1$ or 2. It turns out that $k = 2$ satisfies this equation. In particular,

$$(x^{14} - x) \pmod{f(x)} = x^3 + x^4 + x^5 = y^2 \pmod{f(x)} = x^{42} \pmod{f(x)}, \tag{A10}$$

where recall we're working with polynomials over the field $\mathbb{F}_2$. Similarly, for 25,

$$(x^{25} - x) \pmod{f(x)} = 1 + x^3 + x^4 + x^5 = y^1 \pmod{f(x)} = x^{21} \pmod{f(x)}, \tag{A11}$$

and for 58,

$$(x^{58} - x) \pmod{f(x)} = 1 = y^0 \pmod{f(x)}. \tag{A12}$$

Hence, we have found that $T_2 = \{0,1,14,25,58\}$. To get our Sidon set, we compute the residues $S_2 = T_2 \bmod \frac{(n-1)^{t+1}-1}{n-2} = T_2 \bmod 21$, giving

$$S_2 = \{0,1,14,4,16\} = \{0,1,4,14,16\}. \tag{A13}$$

One can easily confirm that this is a Sidon set mod 21. In particular, the set of all sums $a + b \bmod 21$ for $a, b \in S_2$ is $\{0,1,2,4,5,7,8,9,11,14,15,16,17,18,20\}$, which has size $15 = \binom{n+t-1}{t} = \binom{6}{2}$, which is the maximal possible size.

B Pullback of the Fubini-Study volume form

It is shown in Ref. [49, Sec. 4.5, 4.7, 7.6] that the volume measure on complex projective space is the product of the flat measure on the simplex and the flat measure on the torus. For completeness, in this appendix, we show the same result via a different method.

Let $[Z_0 : \cdots : Z_n]$ be homogeneous coordinates on $\mathbb{CP}^n$. Consider the coordinate patches $C_0, \ldots, C_n$ on $\mathbb{CP}^n$, where $C_i = \{[Z_0 : \cdots : Z_n] \mid Z_i \neq 0\}$. The volume of $\mathbb{CP}^{d-1} \setminus C_0$ is zero, and therefore for the purposes of volume integration we can restrict our attention to C_0. On C_0, we use the coordinates $z_i \coloneqq Z_i/Z_0$ for $i = 1, \ldots, n$. The (unnormalized) Fubini-Study volume form ω can then be written as

$$\omega = \frac{1}{(1 + \sum_{i=1}^n |z_i|^2)^{n+1}} \, \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \ldots \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n. \tag{B1}$$

We can write $Z_i = \sqrt{p_i} \mathrm{e}^{\mathrm{i}\phi_i}$ for $i = 0, \ldots, n$ and $\sum_{i=0}^n p_i = 1$. In other words, p is a point on the simplex $p \in \Delta^n \coloneqq \{p \in [0,1]^n \mid \sum_i p_i \leq 1\}$ (with $p_0 \coloneqq 1 - \sum_{i=1}^n p_i$) and ϕ is a point on the projective torus $\phi \in P(T^{n+1})$ (e.g. we can choose a representative with $\phi_0 = 0$). Therefore, $z_i = \sqrt{\frac{p_i}{p_0}} \mathrm{e}^{\mathrm{i}\phi_i - \mathrm{i}\phi_0}$.

Consider the map $\pi \colon \tilde{\Delta}^n \times P(T^{n+1}) \to C_0$, where $\tilde{\Delta}^n$ is all $p \in \Delta^n$ satisfying $p_0 > 0$. The map is $\pi^i(p, \phi) = \sqrt{\frac{p_i}{p_0}} \mathrm{e}^{\mathrm{i}\phi_i - \mathrm{i}\phi_0}$.

Proposition B.1. *The pullback $\pi^*\omega$ is*

$$\pi^*\omega = (-1)^{n/2} \, \mathrm{d}p_1 \wedge \ldots \mathrm{d}p_n \wedge \mathrm{d}\phi_1 \wedge \ldots \mathrm{d}\phi_n. \tag{B2}$$

It follows from this proposition that the unit-volume normalized volume measure on $\mathbb{CP}^n$ is equal to the product of the Lebesgue measure on the simplex Δ^n and the Lebesgue measure on $P(T^{n+1})$ (where recall the latter is equal to the Lebesgue measure on T^n).

Proof of the proposition. We can without loss of generality fix $\phi_0 = 0$. We can rewrite

$$\omega = p_0^{n+1} \, \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \ldots \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n. \tag{B3}$$

Therefore,

$$\pi^*\omega = p_0^{n+1} \det(J)\, \mathrm{d}p_1 \wedge \ldots \mathrm{d}p_n \wedge \mathrm{d}\phi_1 \wedge \ldots \mathrm{d}\phi_n, \tag{B4}$$

where

$$J = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{B5}$$

is the Jacobian with

$$A_{ij} = \frac{\partial \pi^i}{\partial \phi_j}, \quad B_{ij} = \frac{\partial \pi^i}{\partial p_j}, \quad C_{ij} = \frac{\partial \bar{\pi}^i}{\partial \phi_j}, \quad D_{ij} = \frac{\partial \bar{\pi}^i}{\partial p_j}. \tag{B6}$$

We can check that

$$\frac{\partial \pi^i}{\partial p_j} = \frac{1}{2}\pi^i(p,\phi)\left(\frac{\delta_{ij}}{p_i} + \frac{1}{p_0}\right), \qquad \frac{\partial \pi^i}{\partial \phi_j} = \mathrm{i}\delta_{ij}\pi^i(p,\phi). \tag{B7}$$

Therefore, A and C are diagonal and thus commute, meaning that $\det(J) = \det(AD - CB)$. The matrix elements are $(AD - CB)_{ij} = \frac{\mathrm{i}}{p_0}\left(\delta_{ij} + \frac{p_i}{p_0}\right)$.

By the matrix determinant lemma [62], $\det(M + uv^T) = (1 + v^T M^{-1} u)\det(M)$ with $M = \frac{\mathrm{i}}{p_0}\delta_{ij}$ and $u_i = \mathrm{i}/p_0$ and $v_i = p_i/p_0$, we find that

$$\det(J) = \left(\frac{\mathrm{i}}{p_0}\right)^n \left(1 + \sum_{i=1}^n \frac{p_i}{p_0}\right) = \left(\frac{i}{p_0}\right)^n \frac{1}{p_0} = \frac{(-1)^{n/2}}{p_0^{n+1}}. \tag{B8}$$

The proposition follows. □

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