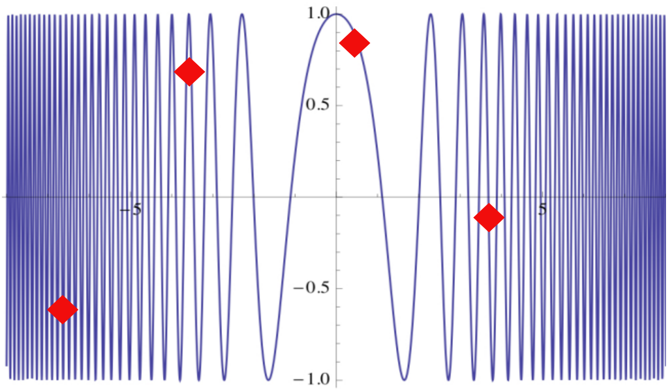
Lefschetz Thimble Quantum Monte Carlo for Spin Systems

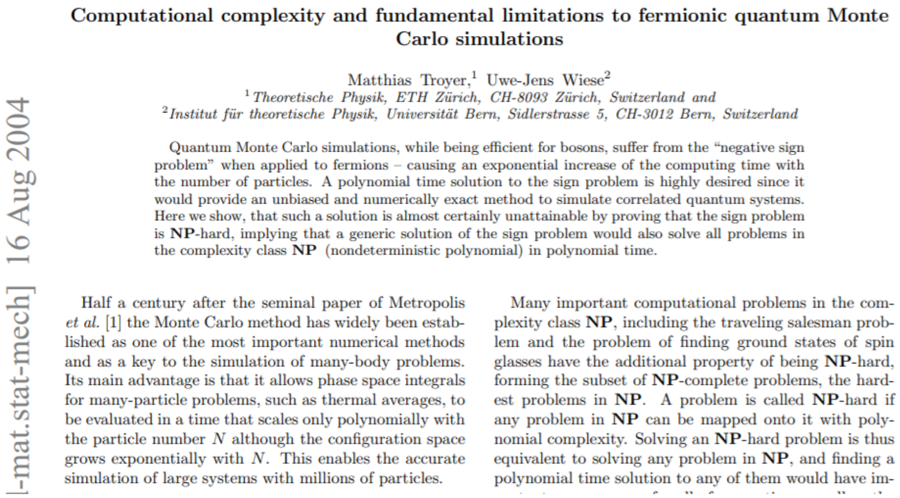
Connor Mooney, Jacob Bringewatt, Lucas Brady

Gorshkov Group meeting
August 20, 2021

Overview

The Sign Problem: Numerical instabilities due to oscillations in phases of Boltzmann weights





Computational complexity and fundamental limitations to fermionic quantum Monte Carlo simulations

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Quantum Monte Carlo simulations, while being efficient for bosons, suffer from the "negative sign problem" when applied to fermions – causing an exponential increase of the computing time with the number of particles. A polynomial time solution to the sign problem is highly desired since it would provide an unbiased and numerically exact method to simulate correlated quantum systems. Here we show, that such a solution is almost certainly unattainable by proving that the sign problem is **NP**-hard, implying that a generic solution of the sign problem would also solve all problems in the complexity class **NP** (nondeterministic polynomial) in polynomial time.

Half a century after the seminal paper of Metropolis *et al.* [1] the Monte Carlo method has widely been established as one of the most important numerical methods and as a key to the simulation of many-body problems. Its main advantage is that it allows phase space integrals for many-particle problems, such as thermal averages, to be evaluated in a time that scales only polynomially with the particle number N although the configuration space grows exponentially with N. This enables the accurate simulation of large systems with millions of particles.

Many important computational problems in the complexity class **NP**, including the traveling salesman problem and the problem of finding ground states of spin glasses have the additional property of being **NP**-hard, forming the subset of **NP**-complete problems, the hardest problems in **NP**. A problem is called **NP**-hard if any problem in **NP** can be mapped onto it with polynomial complexity. Solving an **NP**-hard problem is thus equivalent to solving any problem in **NP**, and finding a polynomial time solution to any of them would have im-

Overview

Picard-Lefschetz Theory: A part of Algebraic Geometry that studies topology via critical points of holomorphic maps

- Lefschetz Thimbles: Multidimensional stationary phase manifolds attached to critical points of a function
- Semi-classical expansion: Expanding path integrals as sums over Lefschetz thimbles attached to classical paths (critical points)

Proceedings of Symposia in Pure Mathematics
Volume 40 (1983), Part 2

VANISHING HOMOLOGIES AND THE *n* VARIABLE SADDLEPOINT METHOD[1]

FRÉDÉRIC PHAM[2]

Introduction. Our aim is to study the asymptotic behavior, as $\tau \to +\infty$, of an oscillatory integral

$$(0) \qquad F(\tau) = \int_{\mathbf{R}^n} e^{-i\tau\varphi(x_1,\ldots,x_n)} a(x_1,\ldots,x_n)\, dx_1,\ldots,dx_n$$

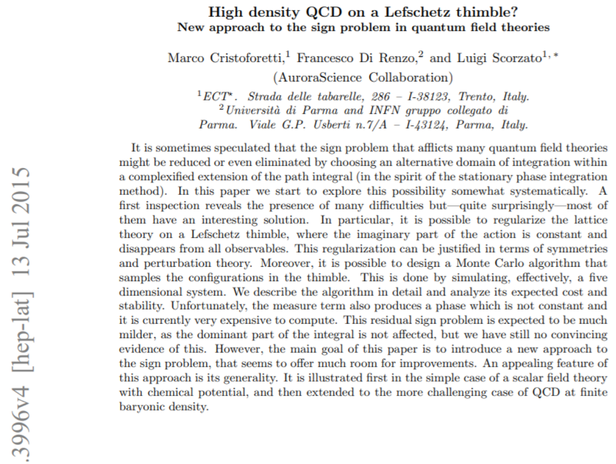
where φ is a polynomial with real coefficients (the "phase function"), whereas a is a polynomial with possibly complex coefficients. Our starting point has been Malgrange's paper [**8**], but our method will be closer to the idea of the classical (one variable) *saddlepoint method* (*méthode du col*), which consists in pushing the integration contour in $\mathbf{C}^n$ towards the directions of "steepest descent" of $\operatorname{Im}\varphi$, thus achieving two goals:

Goal 1: give a meaning to the integral (which perhaps did not converge at infinity to start with);

Goal 2: replace the "Fourier-like" integral (0) by a rapidly convergent ("Laplace-like") integral, whose asymptotic behavior will be governed by the local behavior of the integrand near the—real or complex—critical points of the phase function.

Overview

Lefschetz Thimble Monte Carlo: Monte Carlo methods mitigating the sign problem with Picard-Lefschetz Theory. Works for Hilbert spaces with continuous bases (i.e. position)



.3996v4 [hep-lat] 13 Jul 2015

High density QCD on a Lefschetz thimble?
New approach to the sign problem in quantum field theories

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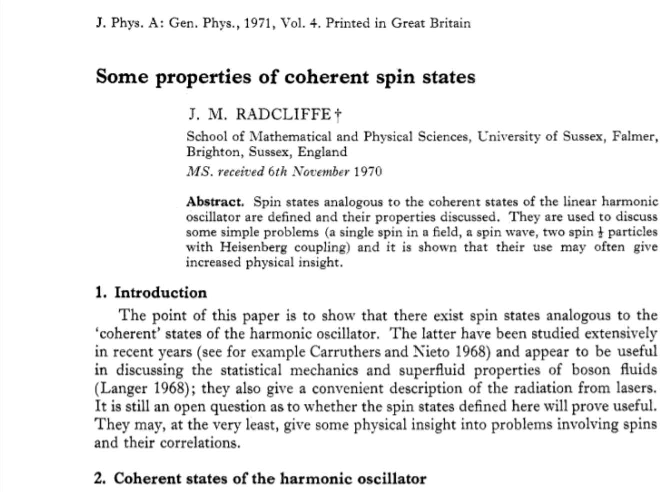
It is sometimes speculated that the sign problem that afflicts many quantum field theories might be reduced or even eliminated by choosing an alternative domain of integration within a complexified extension of the path integral (in the spirit of the stationary phase integration method). In this paper we start to explore this possibility somewhat systematically. A first inspection reveals the presence of many difficulties but—quite surprisingly—most of them have an interesting solution. In particular, it is possible to regularize the lattice theory on a Lefschetz thimble, where the imaginary part of the action is constant and disappears from all observables. This regularization can be justified in terms of symmetries and perturbation theory. Moreover, it is possible to design a Monte Carlo algorithm that samples the configurations in the thimble. This is done by simulating, effectively, a five dimensional system. We describe the algorithm in detail and analyze its expected cost and stability. Unfortunately, the measure term also produces a phase which is not constant and it is currently very expensive to compute. This residual sign problem is expected to be much milder, as the dominant part of the integral is not affected, but we have still no convincing evidence of this. However, the main goal of this paper is to introduce a new approach to the sign problem, that seems to offer much room for improvements. An appealing feature of this approach is its generality. It is illustrated first in the simple case of a scalar field theory with chemical potential, and then extended to the more challenging case of QCD at finite baryonic density.

Overview

Spin Coherent States: A continuous overcomplete family spanning the spin Hilbert space and resolving the identity



J. Phys. A: Gen. Phys., 1971, Vol. 4. Printed in Great Britain

Some properties of coherent spin states

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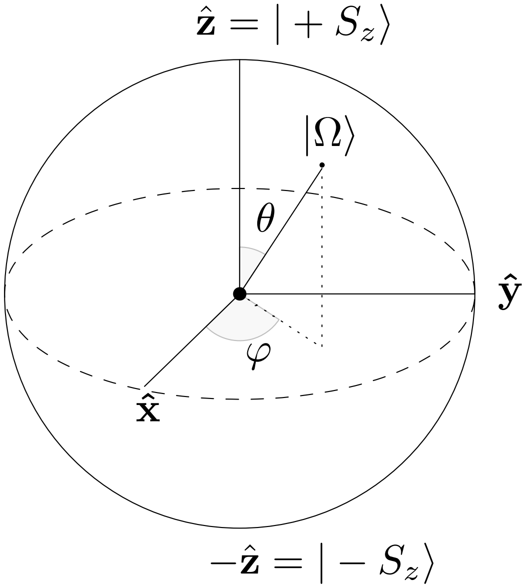
MS. received 6th November 1970

Abstract. Spin states analogous to the coherent states of the linear harmonic oscillator are defined and their properties discussed. They are used to discuss some simple problems (a single spin in a field, a spin wave, two spin $\frac{1}{2}$ particles with Heisenberg coupling) and it is shown that their use may often give increased physical insight.

1. Introduction

The point of this paper is to show that there exist spin states analogous to the 'coherent' states of the harmonic oscillator. The latter have been studied extensively in recent years (see for example Carruthers and Nieto 1968) and appear to be useful in discussing the statistical mechanics and superfluid properties of boson fluids (Langer 1968); they also give a convenient description of the radiation from lasers. It is still an open question as to whether the spin states defined here will prove useful. They may, at the very least, give some physical insight into problems involving spins and their correlations.

2. Coherent states of the harmonic oscillator



The Sign Problem

Path Integrals

“Sum over histories”

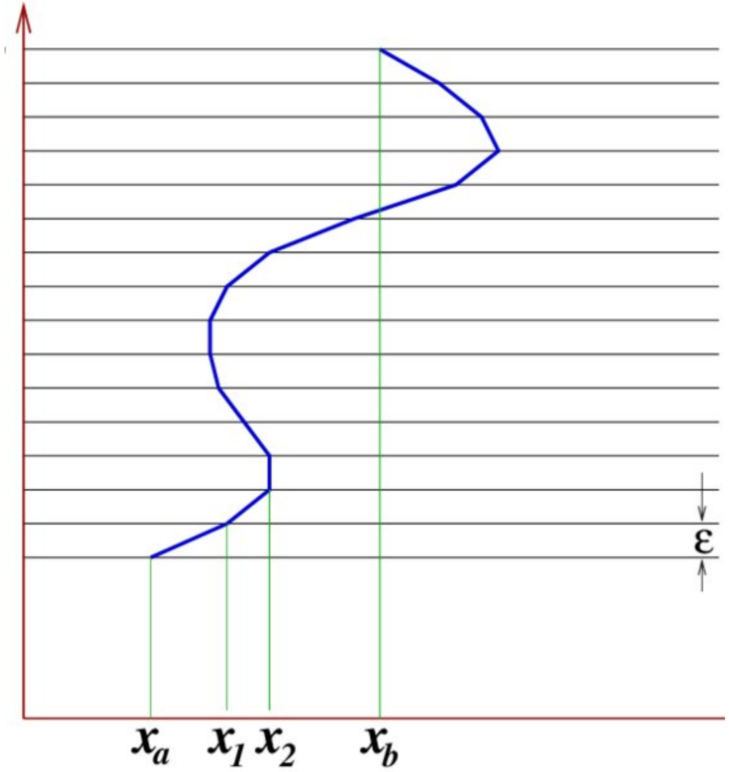
$$\mathcal{Z} = \mathrm{Tr}\{e^{-\beta H}\} = \int dx_0 \langle x_0| \left(e^{-\beta H/T}\right)^T |x_0\rangle$$

$$= \int \left(\prod_{j=0}^{T-1} dx_j\right) \prod_{k=0}^{T-1} \langle x_{k+1}|e^{-\beta H/T}|x_k\rangle$$

$$= \oint \mathcal{D}x e^{-\mathcal{S}[x]}$$

In $T \mapsto \infty$ limit, we can split up diagonal and off-diagonal parts:

$$e^{-\frac{\beta}{T}H} = e^{-\frac{\beta}{T}H_d} e^{-\frac{\beta}{T}H_o} + \mathcal{O}\left(T^{-2}\right)$$



Path Integral Quantum Monte Carlo

We can sample over trajectories with distribution $p(\{x_k\}) = \frac{1}{\mathcal{Z}} \prod_{k=0}^{T-1} \langle x_k | e^{-\beta H/T} | x_{k-1} \rangle$ $\quad p[x(t)] = \frac{e^{-\mathcal{S}[x(t)]}}{\mathcal{Z}}$

Monte Carlo simulations

Expectations

$$\langle O \rangle = \left\langle \frac{\langle x_0 | O e^{-\beta H/T} | x_{T-1} \rangle}{\langle x_0 | e^{-\beta H/T} | x_{T-1} \rangle} \right\rangle_p$$

Reweighting

What if $\mathcal{S}$ isn't always real?

The answer: **Reweighting**

- Sample w.r.t weight's magnitude
- Push the phase into the observable

$$\mathcal{Z} = \oint \mathcal{D}\varphi e^{-\mathcal{S}[\varphi]}$$

$$\begin{aligned}\langle O \rangle &= \frac{1}{\mathcal{Z}} \oint \mathcal{D}\varphi O e^{-\mathcal{S}[\varphi]} \\ &= \frac{1}{\mathcal{Z}} \oint \mathcal{D}\varphi O e^{-i\mathrm{Im}\mathcal{S}[\varphi]} e^{-\mathrm{Re}\mathcal{S}[\varphi]} \\ &= \frac{\langle O e^{-i\mathrm{Im}\mathcal{S}} \rangle_{\mathrm{Re}}}{\langle e^{-i\mathrm{Im}\mathcal{S}} \rangle_{\mathrm{Re}}}\end{aligned}$$

The Sign Problem

$$\langle e^{-i\mathrm{Im}S}\rangle_{Re} = \mathcal{Z}/\mathcal{Z}_{Re} = e^{-\beta\Delta f N}$$ where $$\mathcal{Z}_{Re} = \oint \mathcal{D}x e^{-\mathrm{Re}\mathcal{S}[x]}$$

Take M samples

$$\frac{\Delta\left(e^{-i\mathrm{Im}S}\right)}{\langle e^{-i\mathrm{Im}S}\rangle_{Re}} = \frac{\sqrt{\left(\langle\left|\left(e^{-i\mathrm{Im}S}\right)\right|^2\rangle_{Re} - \left|\langle\left(e^{-i\mathrm{Im}S}\right)\rangle_{Re}\right|^2\right)/M}}{\langle e^{-i\mathrm{Im}S}\rangle_{Re}}$$

$$\sim \frac{e^{\beta N\Delta f}}{\sqrt{M}}$$

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$$\sim \boxed{\frac{e^{\beta N\Delta f}}{\sqrt{M}}}$$

At constant relative error, the number of samples needed scales exponentially with particle number and inverse temperature

Troyer and Wiese, arXiv:cond-mat/0408370

Countermeasures

Stoquastic Hamiltonians: non-positive real off-diagonal entries

- No sign problem

Basis dependent $$\underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\text{z-basis}} \to \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\text{y-basis}}$$

Curing generally is NP hard

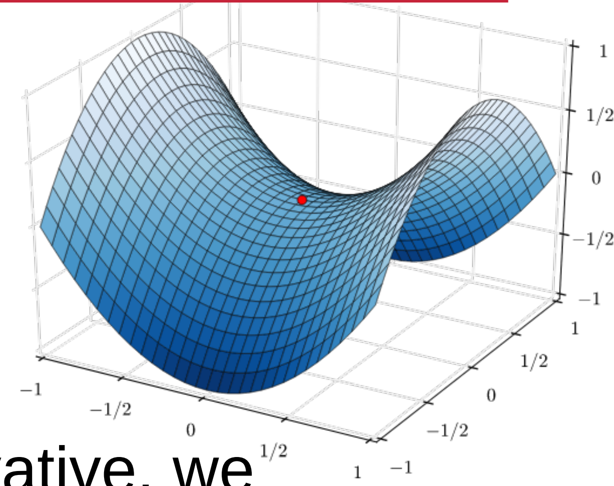
Curing the sign problem: Finding a basis in which H is stoquastic

Mitigation strategies:

- Diagrammatic Monte Carlo — arXiv:0802.2923
- Majorana Monte Carlo — arXiv:1805.08219
- Stochastic Quantization — arXiv:0810.2089
- Fixed-Node Quantum Monte Carlo — PRL.72.2442

Picard-Lefschetz Theory

Saddle-Point Method



- Given improper integral

$$\int_{-\infty}^{\infty} dx e^{-if(x)}$$

- Replace x with z
- Given a saddle point with non-zero second derivative, we deform the contour through the point, where $\mathrm{Re} f$ is constant.
  - Also the steepest descent for $\mathrm{Im} f$
- No longer oscillatory

Setup

- n-dimensional improper integrals

$$\int_{\mathbb{R}^n} d^n x e^{-\mathcal{S}(\mathbf{x})}$$

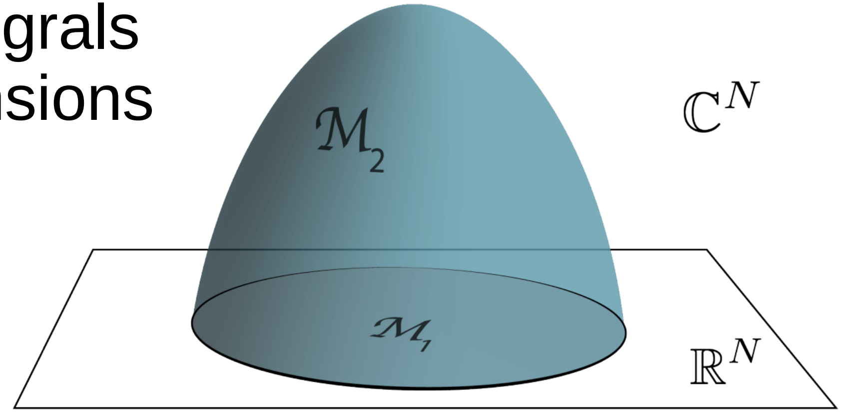
- We promote $\mathcal{S}(\mathbf{x})$ to $\mathcal{S}(\mathbf{z})$, holomorphic on $\mathbb{C}^n$ ($\partial\mathcal{S}/\partial\overline{\mathbf{z}} = 0$)
- Assume $\mathcal{S}(\mathbf{z})$ has nondegenerate critical points, i.e. its critical points have nonsingular Hessian
  - $-\mathrm{Re}\mathcal{S}$ is a Morse function
- Call higher dimensional contours integration cycles

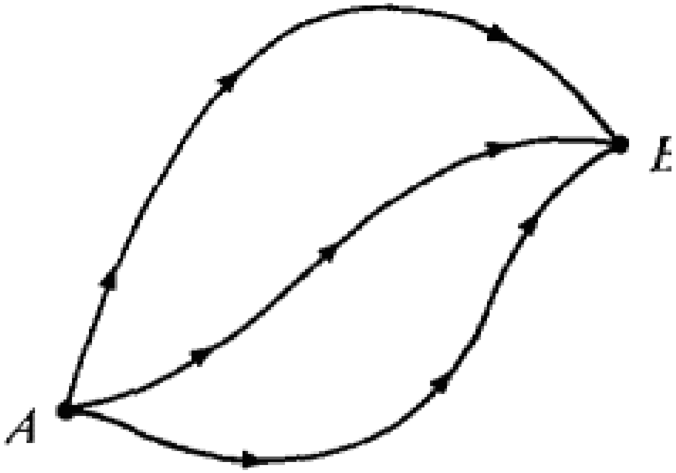
Cauchy's Theorem

"If f is analytic in a simply connected region R, then for any closed curve γ completely in the region, $\oint_\gamma f(z)dz = 0$."

Path independence

Improper integrals
Higher dimensions





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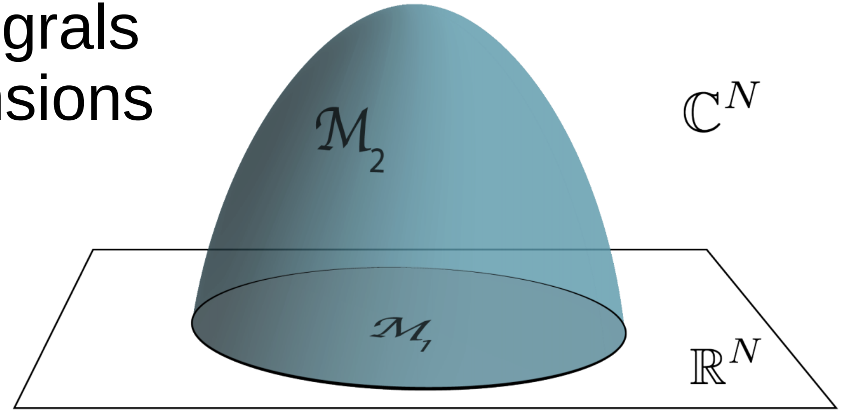
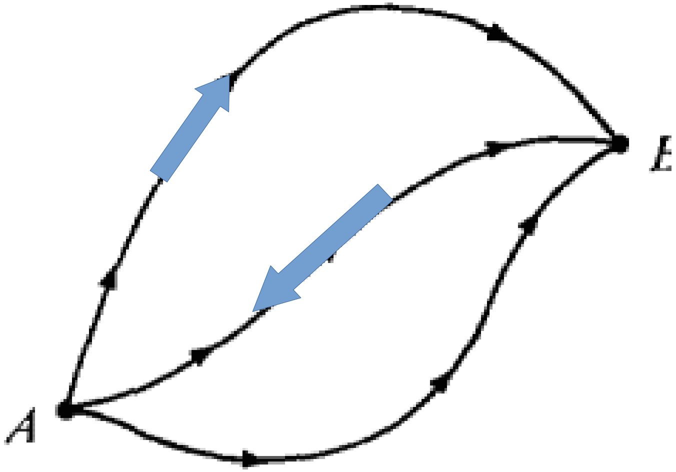
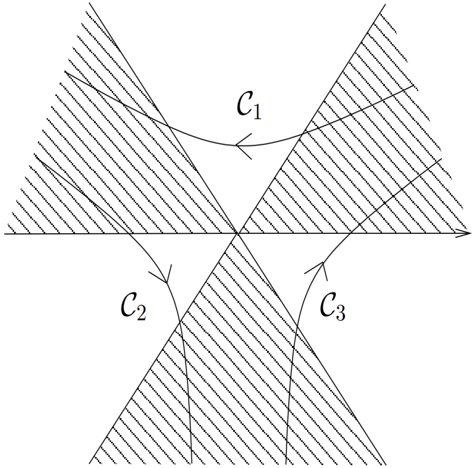




Improper Contour Integrals

- Complex functions behave differently asymptotically in different regions of complex space
- Infinite ends of improper cycles have to be in the regions where the function goes to 0
  - Contours with ends in the same region are equal
    - Not deforming through singularity at infinity
- Want to describe equivalence classes of cycles



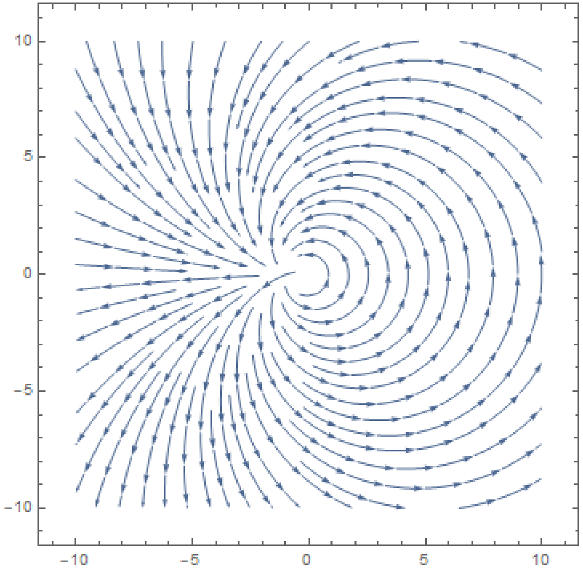
$$e^{i\left(z^3/3+z\right)}$$

The Holomorphic Flow

Let $h(z) \equiv 2\mathrm{Re}\mathcal{S}$ and let the metric be $ds^2 = \sum_i |dz^i|^2$

Gradient flow: $\frac{dz^i}{d\tau} = \overline{\frac{\partial \mathcal{S}}{\partial z^i}}$ & $\frac{d\overline{z}^i}{d\tau} = \frac{\partial \mathcal{S}}{\partial \overline{z}^i}$

$$\frac{d\mathcal{S}}{d\tau} = \frac{\partial \mathcal{S}}{\partial z^i}\frac{\partial z^i}{\partial \tau} = \sum_i \frac{\partial \mathcal{S}}{\partial z^i}\overline{\frac{\partial \mathcal{S}}{\partial z^i}} \geq 0$$



Let $\varphi^{\tau}(z)$ denote z flowed by τ

Imaginary part stays constant!

Flow preserves equivalence class

Lefschetz Thimbles

Call the critical points p_σ

Lefschetz thimble: $\mathcal{J}_\sigma = \{z | \varphi^{-\infty}(z) = p_\sigma\}$

- Unstable manifold
- Steepest ascent of $\mathrm{Re}\mathcal{S}$ from p_σ

Anti thimble: $\mathcal{K}_\sigma = \{z | \varphi^{\infty}(z) = p_\sigma\}$

- Stable manifold

Imaginary part constant

No sign problem integrating on $\mathcal{J}_\sigma$

Thimble Decomposition

(anti) thimbles have same dimension as original cycle

Morse (in)equality: There are the same number of critical points and equivalence classes of cycles

Intersection pairing: $\langle \mathcal{K}_\sigma, \mathcal{J}_{\sigma'} \rangle = \delta_{\sigma\sigma'}$

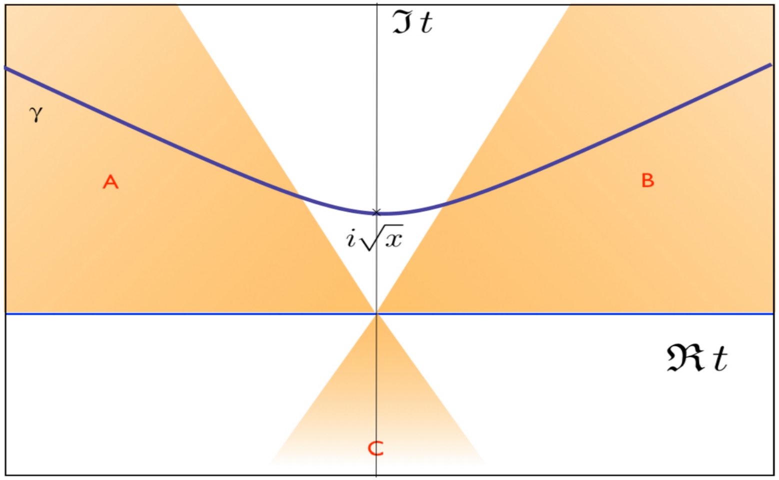
We can show the thimbles are linearly independent:

$$\sum_\sigma a_\sigma \mathcal{J}_\sigma = 0 \implies \text{for all } \sigma', \left\langle \mathcal{K}_{\sigma'}, \sum_\sigma a_\sigma \mathcal{J}_\sigma \right\rangle = 0$$

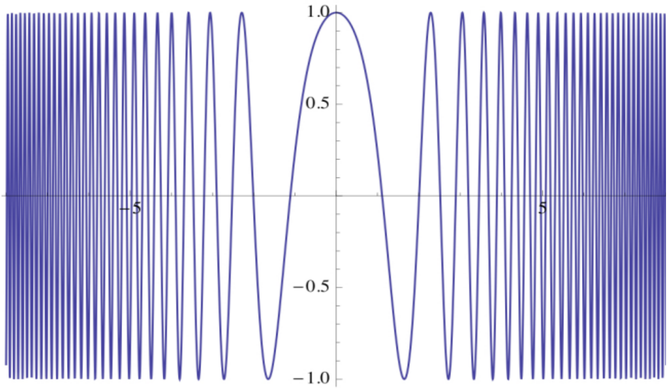
$$\implies \text{for all } \sigma' \ a_{\sigma'} = 0$$

Thus, thimbles form a “basis” for the equivalence class of convergent cycles

Example: The Airy Function



$$\mathrm{Ai}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \exp\left(i\left(t^3/3 + xt\right)\right)$$



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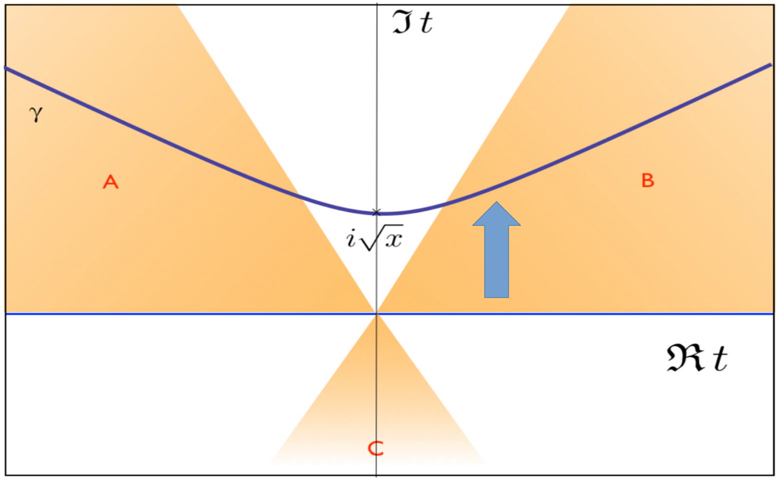

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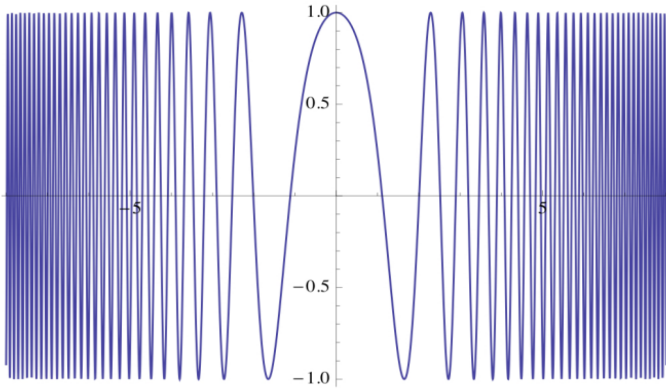

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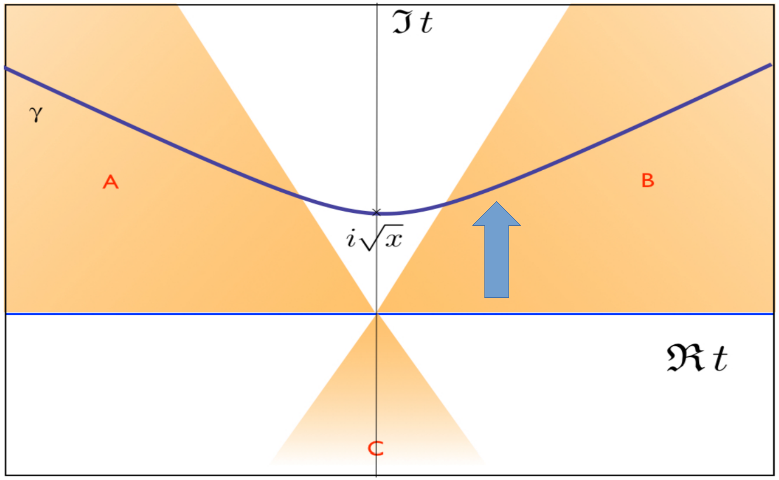
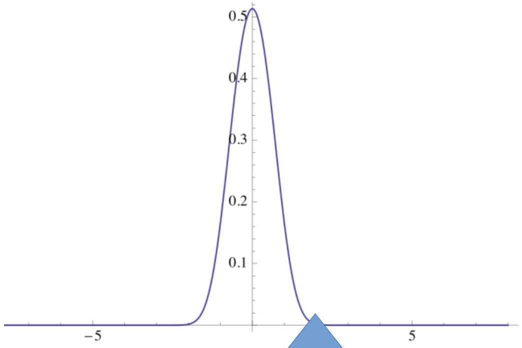
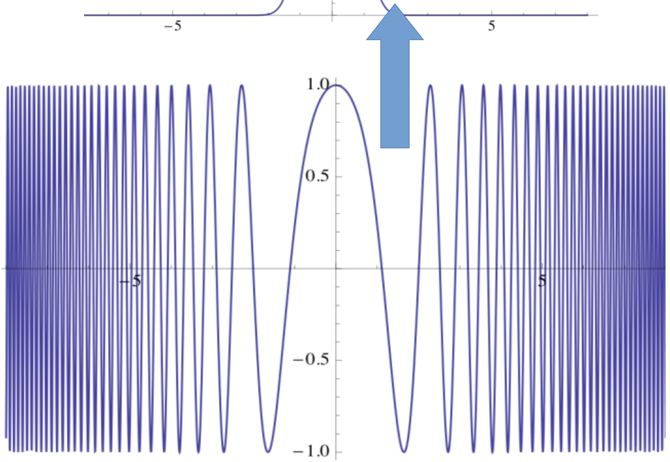






$$\mathrm{Ai}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \exp\left(i\left(t^3/3 + xt\right)\right)$$

Lefschetz Thimble Monte Carlo

The Contraction Algorithm

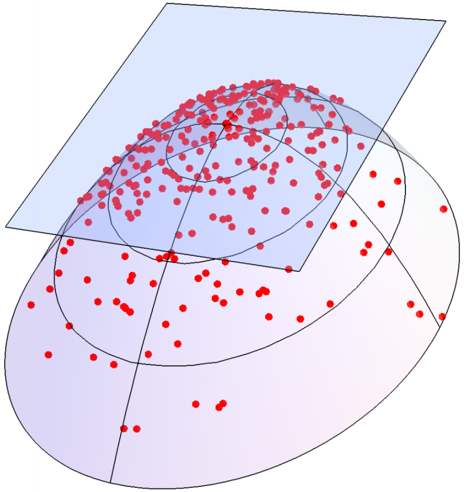
$$\varphi^{\infty}\left(T_{p_\sigma}\left(\mathcal{J}_\sigma\right)\right) = \mathcal{J}_\sigma$$

Sampling a thimble: Need a basis for tangent space

-Takagi vectors: $\rho^{(a)}$ such that $H\rho^{(a)} = \lambda^{(a)}\overline{\rho}^{(a)}$

for positive real $\lambda^{(a)}$

1) Start with $v = p_\sigma + \sum_a x_a \rho^{(a)}$
2) Flow, calculate $\mathcal{S}_{eff} = \mathcal{S}(\varphi^t(v)) - \log\det J(v;t)$
3) Propose point v' and calculate $\mathcal{S}'_{eff}$



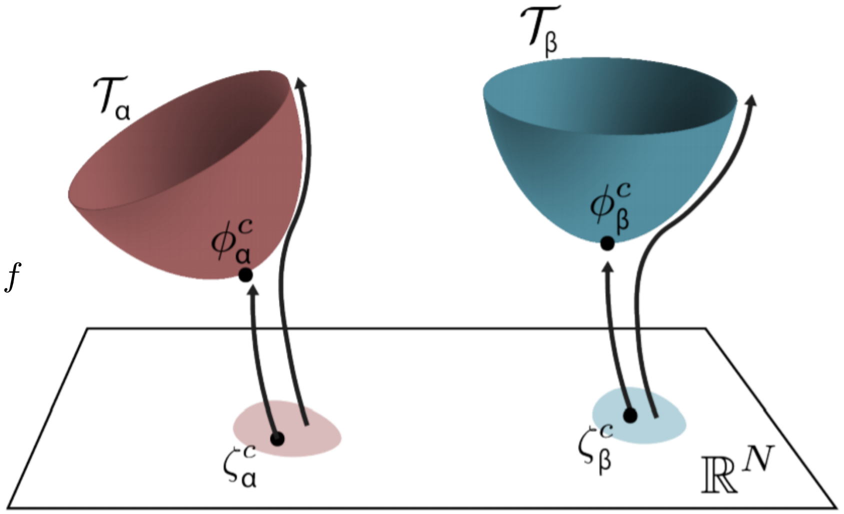
$$\frac{dJ}{d\tau} = \overline{H(\varphi^{\tau}(z))J(\tau)}$$

Generalized Thimble Algorithm

What if there are multiple thimbles?

$$\varphi^{\infty}(\mathbb{R}^n) = \sum_{\sigma} \langle \mathcal{K}_\sigma, \mathbb{R}^n \rangle \mathcal{J}_\sigma$$

1) Start with real point x
2) Flow, calculate $\mathcal{S}_{eff} = \mathcal{S}(\varphi^t(x)) - \log\det J(x;t)$
3) Propose new real point x' and calculate $\mathcal{S}'_{eff}$



Computational Concerns

- Trapping
- The Jacobian
- Inter-thimble sign problem

Application to Spin Systems

$$\langle H \rangle = \mathrm{Tr}\{He^{-\beta H}\} = \sum_{s_0=-S}^{S} \langle s_0 | He^{-\beta H} | s_0 \rangle$$

$$= \sum_{s_0, s_1, \cdots, s_{T-1}} \frac{\langle s_0 | He^{-\beta H/T} | s_{T-1} \rangle}{\langle s_0 | e^{-\beta H/T} | s_{T-1} \rangle} \prod_{k=0}^{T-1} \langle s_{k+1} | e^{-\beta H/T} | s_k \rangle$$

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Wait a second...

Application to Spin Systems

$$\langle H \rangle = \mathrm{Tr}\{ \ldots \sum^{S} \ldots$$

$$= \sum_{s_0, s_1, \ldots} \ldots |e^{-\beta H/T}|s_k\rangle$$

Wait a second...

Spin Coherent States

Spin Coherent States

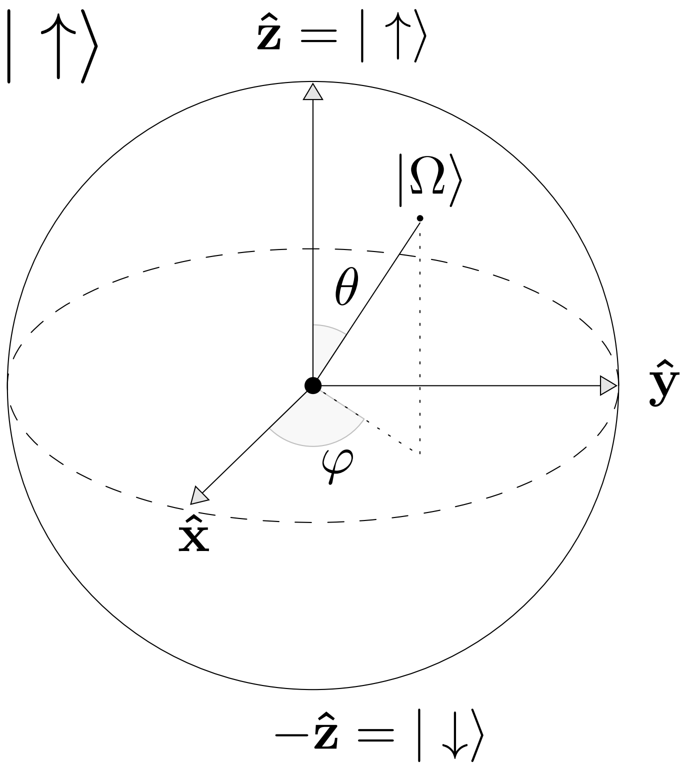
Euler angle representation: $|\Omega\rangle \equiv e^{-i\varphi S_z} e^{-i\theta S_x} |\uparrow\rangle$

Max spin eigenstate along $\hat{\mathbf{n}}(\theta, \varphi)$

Can view as $2S$ spin ½ particles

aligned along θ and φ

Saturates Uncertainty Principle



Resolution of the Identity:

$$I = \frac{2S+1}{4\pi} \int d\Omega |\Omega\rangle\langle\Omega|$$

Spin Coherent States

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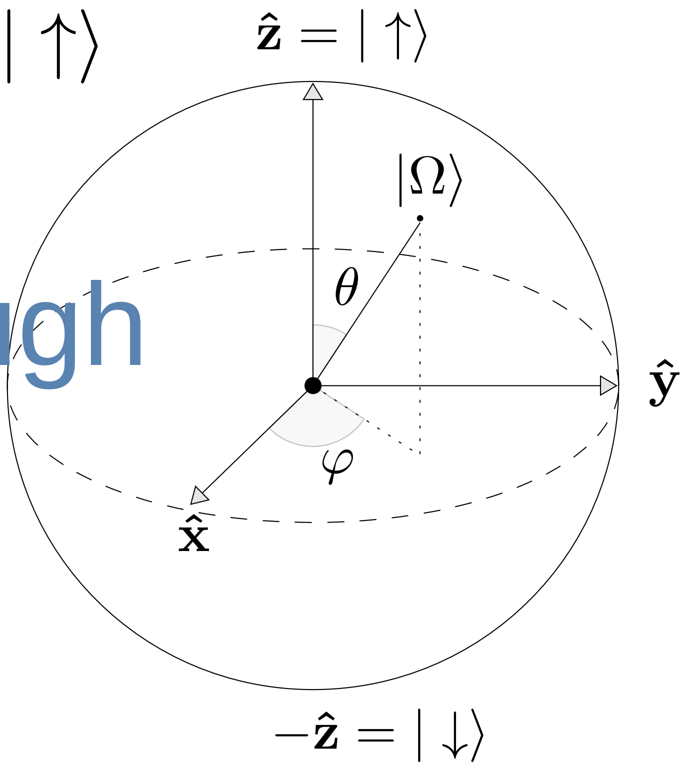
Not Good Enough

Saturates Uncertainty Principle



Resolution of the Identity:

$$I = \frac{2S+1}{4\pi} \int d\Omega |\Omega\rangle\langle\Omega|$$

Altland and Simons, *Condensed Matter Field Theory*

Spin Coherent States

Euler angle vs. Rotation around axis:

$$e^{-i\varphi S_z} e^{-i\theta S_x} |\uparrow\rangle = e^{-i\theta(-\sin\varphi S_x + \cos\varphi S_y)} |\uparrow\rangle$$

Spin Coherent States

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Expand trig and combine into ladder operators:

$$= \exp\left(\frac{\theta}{2}\left(S_- e^{i\varphi} - S_+ e^{-i\varphi}\right)\right) |\uparrow\rangle$$

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$$= \exp\left(\frac{\theta}{2}\left(S_- e^{i\varphi} - S_+ e^{-i\varphi}\right)\right) |\uparrow\rangle$$

$$= \exp\left(-e^{i\varphi} \tan\frac{\theta}{2} S_-\right) e^{-\log\left(1+\tan^2\frac{\theta}{2}\right) S_z} \exp\left(-e^{-i\varphi} \tan\frac{\theta}{2} S_+\right) |\uparrow\rangle$$

$$= \left(1 + \tan^2\frac{\theta}{2}\right)^{-S} \exp\left(-e^{i\varphi} \tan\frac{\theta}{2} S_-\right) |\uparrow\rangle$$

Arecchi et al., PRA.6.2211

Spin Coherent States

Euler angle vs. Rotation around axis:

$$e^{-i\varphi S_z} e^{-i\theta S_x} |\uparrow\rangle = e^{-i\theta(-\sin\varphi S_x + \cos\varphi S_y)} |\uparrow\rangle$$

Expand trig and combine into ladder operators:

$$= \exp\left(\frac{\theta}{2}\left(S_- e^{i\varphi} - S_+ e^{-i\varphi}\right)\right) |\uparrow\rangle$$

$$= \exp\left(-e^{i\varphi} \tan\frac{\theta}{2} S_-\right) e^{-\log\left(1+\tan^2\frac{\theta}{2}\right) S_z} \exp\left(-e^{-i\varphi} \tan\frac{\theta}{2} S_+\right) |\uparrow\rangle$$

$$= \left(1 + \tan^2\frac{\theta}{2}\right)^{-S} \exp\left(-\boxed{e^{i\varphi} \tan\frac{\theta}{2}} S_-\right) |\uparrow\rangle$$

Arbitrary complex number

Arecchi et al., PRA.6.2211

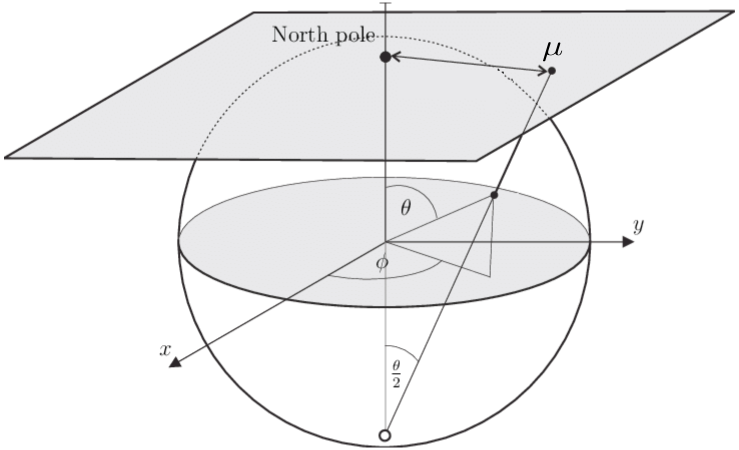
The Stereographic Parametrization

$$|\mu\rangle = \frac{1}{\left(1+|\mu|^2\right)^S} \exp\left(\mu S_-\right)|\uparrow\rangle$$

If $S \to \infty$, goes to canonical coherent states

Closest as possible to classical state

Saturates Uncertainty Principle



$$I = \frac{2S+1}{\pi} \int \frac{d\mathrm{Re}\left(\mu\right) d\mathrm{Im}\left(\mu\right)}{\left(1+|\mu|^2\right)^2} |\mu\rangle\langle\mu|$$

Radcliffe, doi: 10.1088/0305-4470/4/3/009

Spin Coherent State Path Integral

$$\mathcal{Z} = \oint \mathcal{D}\theta\mathcal{D}\phi e^{-\mathcal{S}[\theta,\phi]} \qquad \mathcal{S}[\theta,\phi] = \int_0^\beta \left(-iS\dot{\phi}\cos\theta + H^{cl}(\theta,\phi)\right)$$

$$\mathcal{Z} = \oint \mathcal{D}z\mathcal{D}\overline{z} e^{-\mathcal{S}[z,\overline{z}]} \qquad \mathcal{S}[z,\overline{z}] = \int_0^\beta d\tau \left(-S\frac{\dot{\overline{z}}z - \overline{z}\dot{z}}{1+|z|^2} + H^{cl}(z,\overline{z})\right)$$

- Not holomorphic, so expand $z = x + iy$ and $\overline{z} = x - iy$
- Push volume form into exponential too

- Continuous form not entirely well-founded
- More accurate at higher spins

arXiv: quant-ph/9807005
arXiv: cond-mat/0111139

Altland and Simons; Kochetov, doi: 10.1063/1.530913

Spin Coherent State Path Integral

- Expanded out:

$$\mathcal{S}[x,y] = \int_0^\beta d\tau \left(2iS\frac{\dot{y}x - \dot{x}y}{1+x^2+y^2} + H^{cl}(x,y) \right)$$

- First term Berry Phase
- Spin operators for H^{cl}:

$$J_x \to \frac{2Sx}{1+x^2+y^2} \quad J_y \to \frac{2Sy}{1+x^2+y^2} \quad J_z \to \frac{S(1-x^2-y^2)}{1+x^2+y^2}$$

Spin Coherent State Path Integral

- Expanded out:

$$\mathcal{S}[x,y] = \int_0^\beta d\tau \left(2iS \frac{\dot{y}x - \dot{x}y}{\boxed{1 + x^2 + y^2}} + H^{cl}(x,y) \right)$$

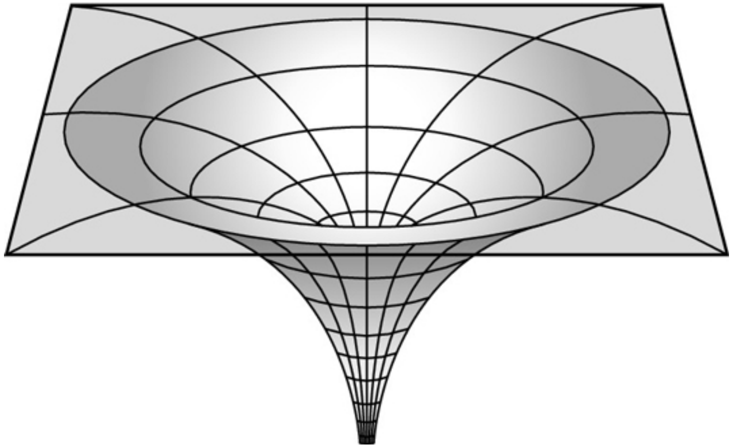
Diverge if $x^2 + y^2 = -1$

- First term Berry Phase
- Spin operators for H^{cl}:

$$J_x \to \frac{2Sx}{1 + x^2 + y^2} \quad J_y \to \frac{2Sy}{\boxed{1 + x^2 + y^2}} \quad J_z \to \frac{S(1 - x^2 - y^2)}{1 + x^2 + y^2}$$

Holomorphic Flow Blow-up

- Flows can blow up in finite time!
- For us, at $x_i^2 + y_i^2 = -1$
- Potentially also at ∞
- Numeric breakdown
- Sometimes tuning isn’t enough



Gradient Flow without Blow-ups

- Change metric so distance blows up at singularities too

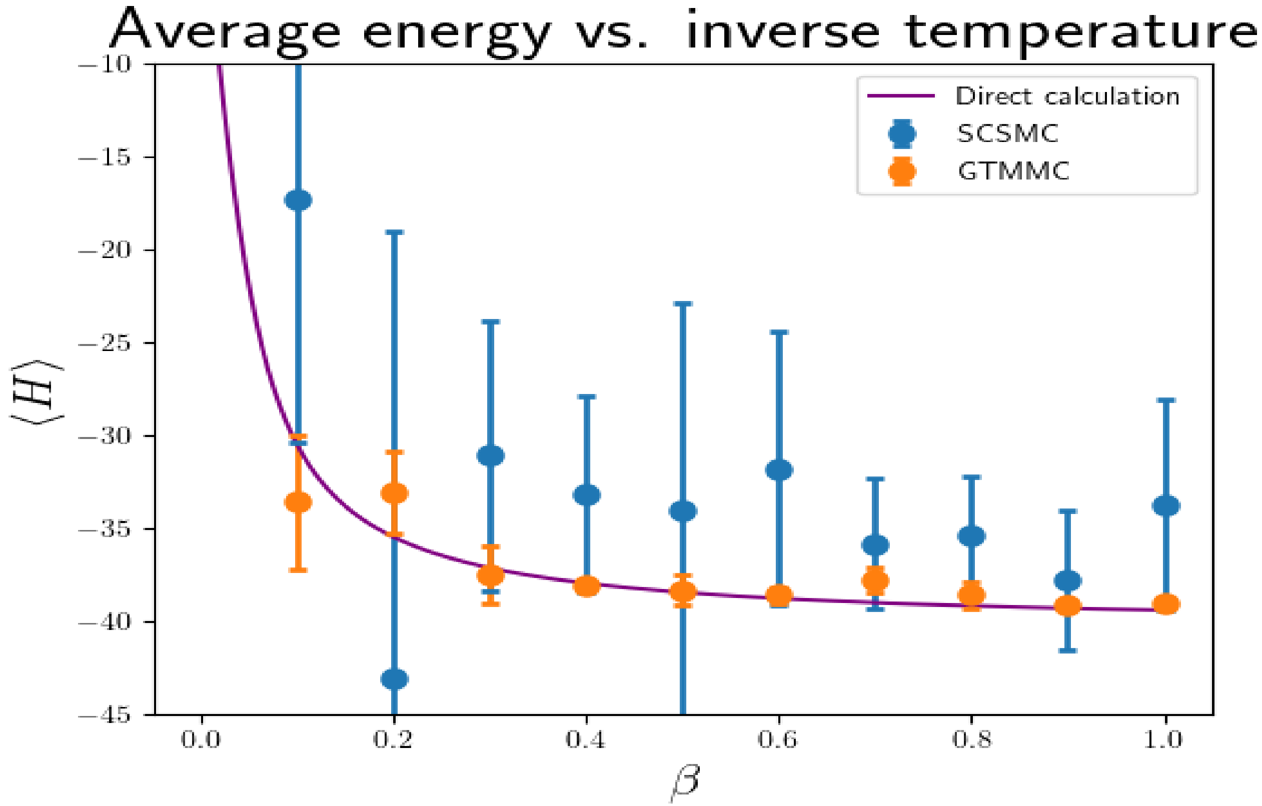
$$\sum_i |dz^i|^2 \to e^{2\mathrm{Re}\mathcal{S}/\Lambda} \left(\sum_i |dz^i|^2 \right)$$

- Gradient flow decelerates at singularities

$$\frac{dz^i}{d\tau} = e^{-2\mathrm{Re}\mathcal{S}/\Lambda} \overline{\frac{\partial \mathcal{S}}{\partial z^i}}$$

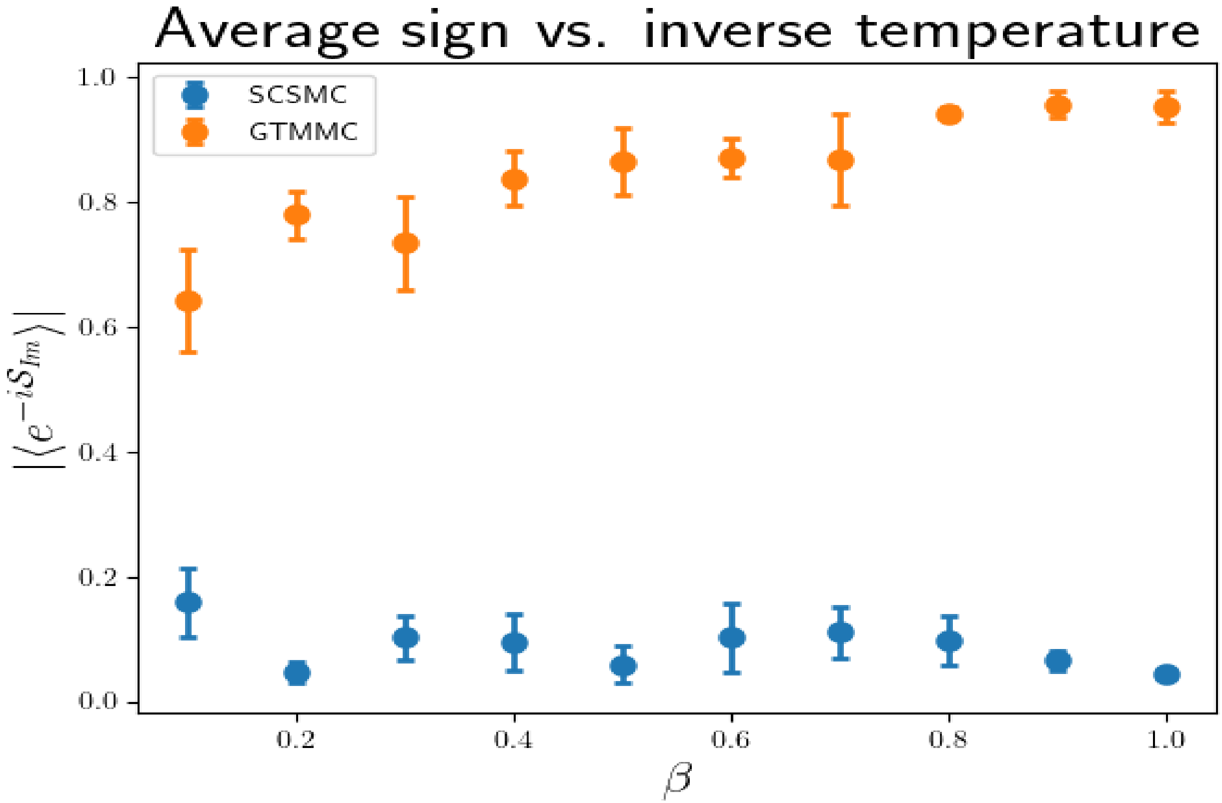
- Trade-off: sign problem vs. blow-up

Our Results



Average energy vs. inverse temperature
Direct calculation
SCSMC
GTMMC
−10
−15
−20
−25
−30
−35
−40
−45
⟨H⟩
0.0
0.2
0.4
0.6
0.8
1.0
β

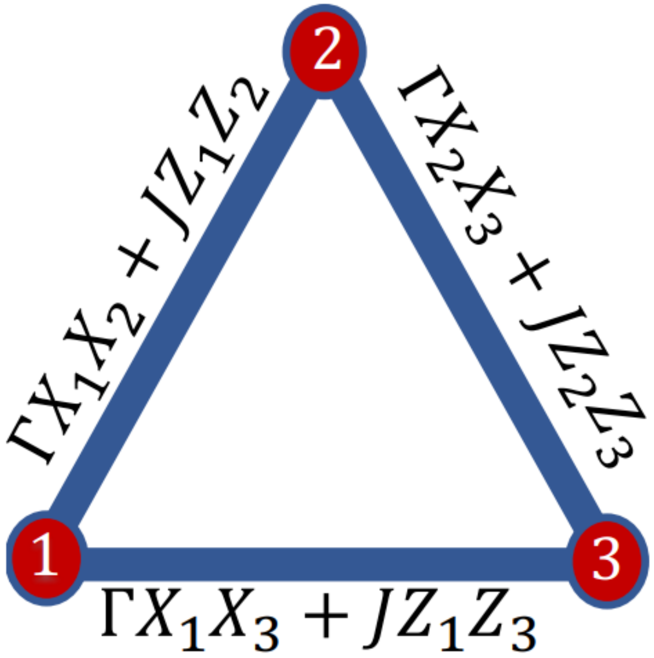
Our Results



Average sign vs. inverse temperature
SCSMC
GTMMC
1.0
0.8
0.6
0.4
0.2
0.0
0.2
0.4
0.6
0.8
1.0
$|\langle e^{-iS_{Im}} \rangle|$
β

Future Directions

- So far, just **proof of concept**
- Multi-spin systems
- Actual sign problems
  - Frustrated triplet
- *Mitigate,* not solve



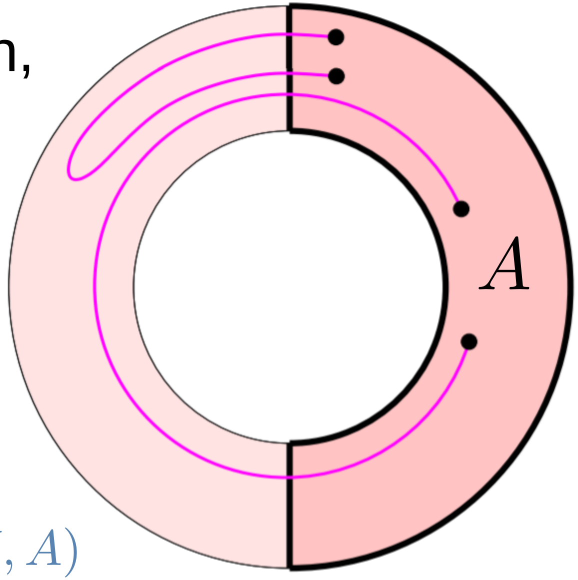
Backup Slides

Relative Homology

All relative to subspace A

- Relative n-cycles are n-dimensional hypersurfaces with either no boundary or boundary in
- Relative n-boundaries are the n-hypersurfaces which, up to n-hypersurfaces in A, are boundaries of $(n+1)$-hypersurfaces.
- Every relative boundary is a relative cycle
- Define $Z_n(X, A)$, $B_n(X, A)$ as integer-coefficient

  formal sums of relative cycles, boundaries
- n^{th} relative homology group: $H_n(X, A) \equiv Z_n(X, A)/B_n(X, A)$

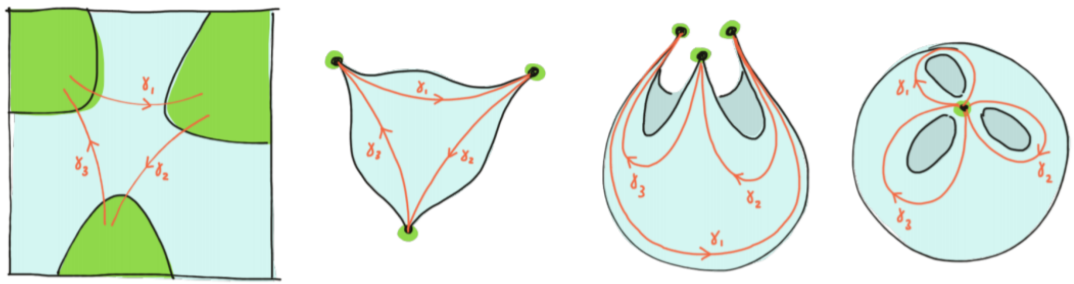


Relative Homology

Define $X_{\geq T} \equiv \{z \in X | \mathrm{Re}\mathcal{S}(z) \geq T\}$

Assume large-T limit

- $Z_n(X, X_{\geq T})$ are convergent integration cycles for $e^{-\mathcal{S}(z)}$
- $B_n(X, X_{\geq T})$ are cycles that integrate to 0 for $e^{-\mathcal{S}(z)}$
- $H_n(X, X_{\geq T})$ are equivalence classes of convergent cycles partitioned by value of their integral



Thimble Decomposition

(anti) thimbles have same dimension as original cycle - h is a perfect Morse function

Morse (in)equality: If dimension n, and Σ critical points,

$$\Sigma = \mathrm{Rank} H_n(X, X_{\geq T})$$

Intersection pairing: $\langle \mathcal{K}_\sigma, \mathcal{J}_{\sigma'} \rangle = \delta_{\sigma\sigma'}$

Thus, $[C] = \sum_\sigma \langle \mathcal{K}_\sigma, C \rangle \mathcal{J}_\sigma$

A Caveat

I've lied to you

Flows between critical points

$\langle \mathcal{J}_\sigma, \mathcal{K}_{\sigma'} \rangle \neq \delta_{\sigma\sigma'}$ if we have *Stokes Rays*

- Thimbles don't form basis
- Only possible if $\mathrm{Im}\mathcal{S}(p_\sigma) = \mathrm{Im}\mathcal{S}(p_{\sigma'})$
- Not really a problem