Priority Queues and Binary Heaps (Part 2)

Heap sorting, creation of heaps, and heap creation complexity

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Priority Queues and Binary Heaps (Part 2)

Heaps for Sorting

- How would you use a priority queue or a heap to sort a collection of values?
  - Max heap: Sort in descending order
  - Min heap: Sort in ascending order
- Steps: insert and delete
  - First, insert each value into the heap
  - Then, remove each value one-by-one until none remain

Out-Of-Place Heap Sort: Issues

- Data duplication is required
  - We need to create a copy of the original data set, store it in a priority queue, and then copy it back
    - This doubles the memory requirement
- For large data sets, this duplication hurts
  - We ideally want an approach to perform in-place sorting

In-Place Heap Sort

- Task: given a non-heap array, sort it using the ideas that we've seen used to make heap-sort work
- Three main issues:
  1. The array is full with some value at index 0
     - Our heap has a dummy 0-index item
  2. Where do we store the value we delete from the heap?
  3. How do we make the non-heap array a heap?

Issue 1: Changed Root Location

- Root at 1
  - `static int root(){ return 1; }`
  - `static int left(int i){ return i * 2; }`
  - `static int right(int i){ return i * 2 + 1; }`

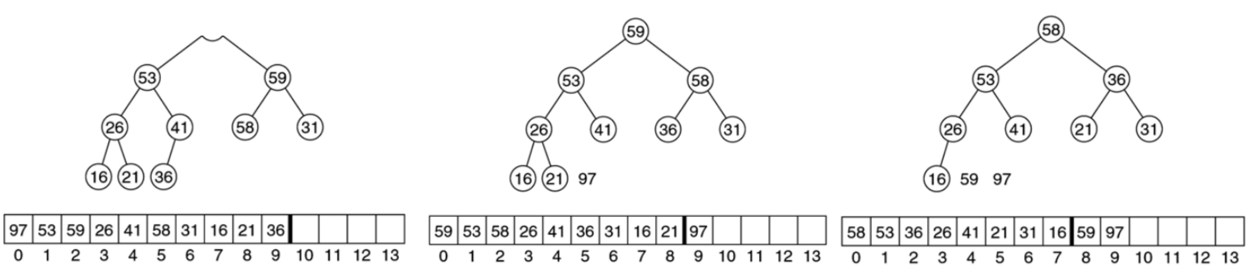
  - `static int parent(int i){ return i / 2; }`
- Root at 0
  - `static int root(){ return 0; }`
  - `static int left(int i){ return i * 2 + 1; }`
  - `static int right(int i){ return i * 2 + 2; }`
  - `static int parent(int i){ return (i - 1)/ 2; }`

Issue 2: Space Reuse

- If we have a heap already…
- Space available for values removed from the heap
  - Remove an element from a heap
  - Now there's open space at the end of the array (since the complete tree is shrinking, it must first withdraw from that portion of the array)
  - Put the removed element at the end of the array
  - Repeat this process until the array is empty

Space Reuse: Example

- Images, left to right
  - Initial heap
    * 10 unsorted values
  - After the first `deleteMax()`
    * 9 unsorted and 1 sorted
  - After the second `deleteMax()`
    * 8 unsorted and 2 sorted



Issue 3: "Heapify"

- We need to be able to convert an existing array into a heap
- We can build the heap bottom up through repeated application of `percolateDown()`
  - Start one level above the bottom
  - Work right to left, bottom up
  - Apply `percolateDown()` for each non-leaf node
    - ⋆ Compare the non-leaf node with its children
    - ⋆ Swap if the heap order is violated

Example: Min Heap

```
int[] arr = [92, 47, 21, 20, 12, 45, 63, 61, 17, 55, 37, 25, 64, 83, 73];
```

Then we perform the following steps to convert it to a heap (noting that `percolateDown()` takes as an argument the *index* of the array to percolate):

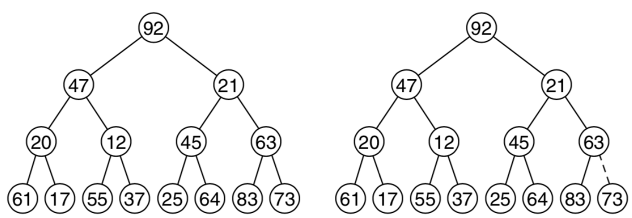


Figure 1: Left: Initial heap; Right: after `percolateDown(7)`

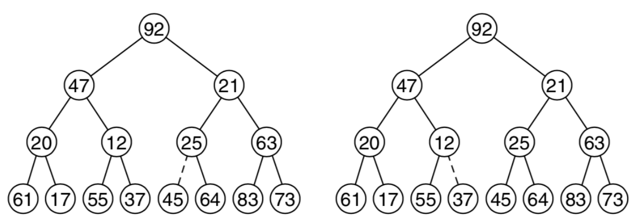


Figure 2: Left: After `percolateDown(6)`; Right: After `percolateDown(5)`

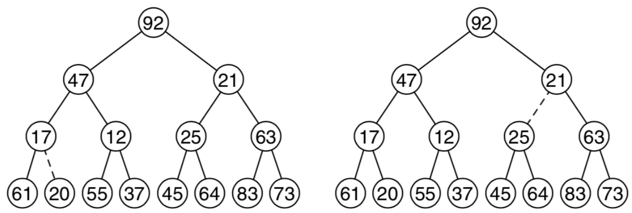


Figure 3: Left: After `percolateDown(4)`; Right: After `percolateDown(3)`

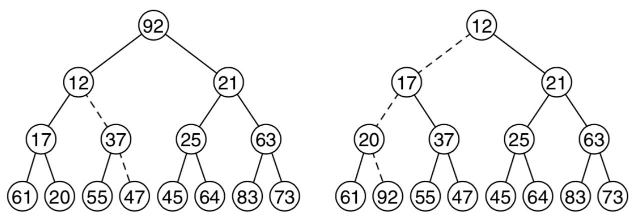


Figure 4: Left: After `percolateDown(2)`; Left: After `percolateDown(1)` and `buildHeap` terminate

Heapify Implementation

```java
public void buildHeap() {
  for (int i = parent(this.size); i >= root(); i--) {
    this.percolateDown(i);
  }
}
```

- What is the complexity?
  - **Answer**
- How many loop iterations are there?
  - **Answer**
- How many edges do we have to move down for each iteration?
  - **Answer**
- Worst case?
  - **Answer**

Heapify Complexity

- Assume the tree height is h, and count the work as the number of comparisons/swaps done at each level

  - At the bottom (level 0) there are (at most) 2^h nodes
    * We do not do anything, so the work is zero
  - Level 1 has 2^{h-1} nodes
    * Each might move down (at most) 1 level
  - Level 2 has 2^{h-2} nodes
    * Each might move down (at most) 2 levels
  - Level i is the ith from the bottom and has 2^{h-i} nodes
  - Level h is the root, has $2^{h-h} = 2^0 = 1$ node

- Each level i node can move at most i steps down, so

$$\text{moves} = \sum_{i=1}^{h} i \times 2^{h-i} = \sum_{i=1}^{\log_2(n)} i \times 2^{\log_2(n-i)} = \sum_{i=1}^{\log_2(n)} i \times \frac{2^{\log_2(n)}}{2^i} = n \sum_{i=1}^{\log_2(n)} \frac{i}{2^i}.$$

  Since

$$\sum_{i=1}^{\infty} \frac{i}{2^i} \to 2,$$

  we know that

$$n \sum_{i=1}^{\log_2(n)} \frac{i}{2^i} \leq n \times 2$$

  and

$$n \times 2 \in O(n).$$

  Therefore

$$\text{moves} \in O(n).$$

Weiss Theorem

- *Theorem 21.1:* For a perfect tree of height h with $n = 2^{h+1} - 1$ nodes, the sum of the heights of the nodes is $n - h - 1$
  - This is the upper bound for a complete binary tree
  - The sum of the heights is equivalent to the max number of swaps required, so it is $O(n)$
- We can arrive at the proof by darkening h edges for each node of height h in the tree while keeping the edges disjoint
  - We go left once, then right all the way down
- Leaf node marks nothing
- Height is one
  - Mark every height one node's left edge

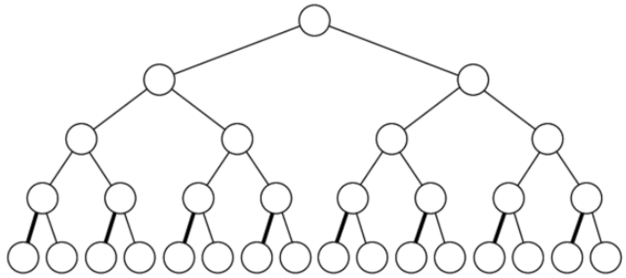


Figure 5: Marking the left edges for height 1 nodes

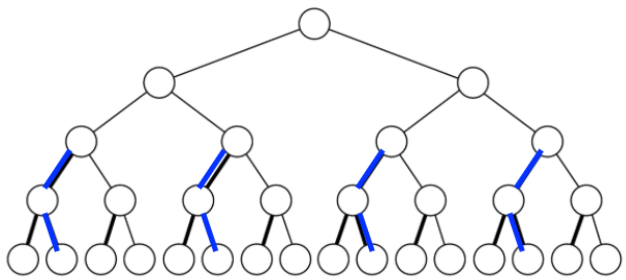


Figure 6: Marking the first left edge and the subsequent right edge for height 2 nodes

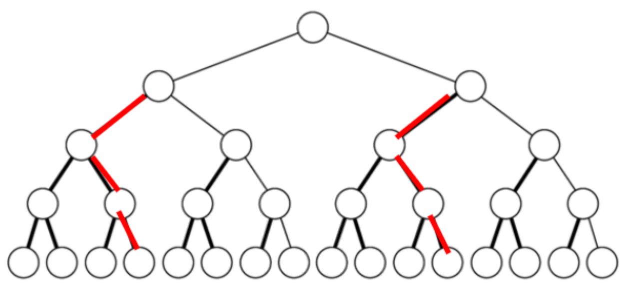


Figure 7: Marking the first left edge and the subsequent two right edges for height 3 nodes

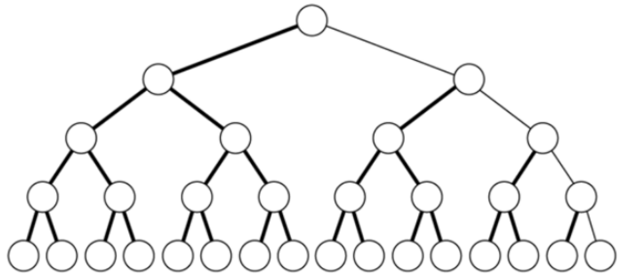


Figure 8: Completed Marking

- After we're finished darkening all the nodes, the rightmost path is still not marked
  - There are h edges in that path
  - The total number of edges is $n - 1$
  - The total number of edges marked is $n - h - 1$

Heap Sort Summary

- Time complexity
  - $O(n \log(n))$
- Space complexity
  - Delete and put it into a second array: $O(n)$ additional memory
  - Delete only at the end of the array: no extra memory involved
- Stable?
  - A *stable sort* maintains order among equal items. An example of sorting the following items: $1_a, 2, 1_b$
    - Stable sort: $1_a, 1_b, 2$
    - Unstable sort: $1_a, 1_b, 2$ or $1_b, 1_a, 2$

Next Lecture

- Topic: Balanced Binary Search Trees
  - Reading: Chapter 19