Graphs (Part 2)

TODO: FILL ME OUT

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Graphs (Part 2)

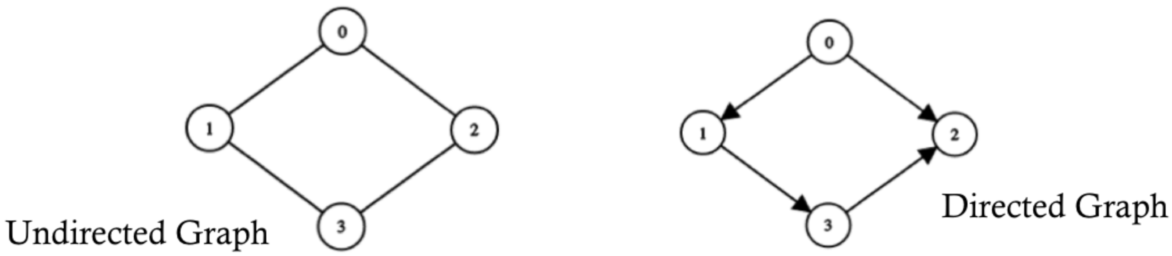
Road Map

- Graph basics
  - Definitions and terms
  - Applications
- Graph representations
  - Adjacency matrix
  - Adjacency list
- Graph algorithms
  - Graph traversal
  - Shortest path problem
  - Many more

Graph Representation

- *Graph*: a set of vertices V and edges E
- Representation: the way in which we store a graph in programs
  - Adjacency matrix: a 2D array `A[n][n]` where $n = |V|$
    * `A[v][w]` is the weight of (v, w)
    * `A[v][w]` is infinity if v, w is not in the graph
  - Adjacency list
    * A group of $|V|$ linked lists
    * List `v`: a list of all nodes w such that there is an edge (v, w)

Adjacency Matrix



	0	1	2	3
0	∞	1	1	∞
1	1	∞	∞	1
2	1	∞	∞	1
3	∞	1	1	∞

	0	1	2	3
0	∞	1	1	∞
1	∞	∞	∞	1
2	∞	∞	∞	∞
3	∞	∞	1	∞

Figure 1: A comparison of features between undirected and directed graphs

Adjacency List

Using the same image as above for reference, we have the following adjacency lists:

- Undirected graph
  - $0 \to 1 \to 2$
  - $1 \to 0 \to 3$
  - $2 \to 0 \to 3$
  - $3 \to 1 \to 2$
- Directed graph
  - $0 \to 1 \to 2$
  - $1 \to 3$
  - $2 \to$ `null` (empty list)
  - $3 \to 2$

Complexity

- Adjacency matrix

  - Space cost: $O(|V|^2)$
  - Initialization time: $O(|V|^2)$
  - Fine with dense graphs, not so great for sparse ones

- Adjacency list
  - Space cost: $O(|V| + |E|) = O(|E|)$
  - Initialization time: linear from a list of edges
    * Most cases, we skip the checking of duplicate edges (even though they might come with different edge weights)

- How do we add, remove, or find a node or edge?

Graph Representation Example

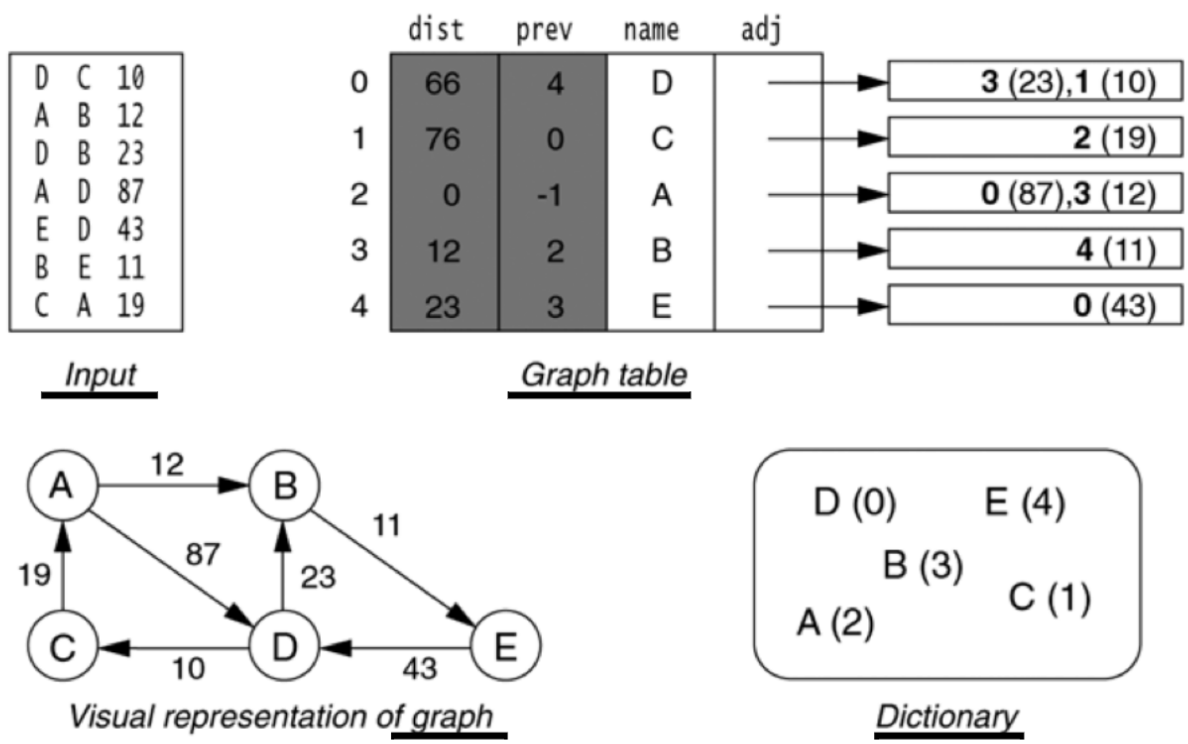


Figure 2: An abstract scenario of the data structures used in a shortest-path calculation, with an input graph taken from a file. The shortest weighted path from A to C is A to B to E to D to C (cost is 76). From Weiss, Figure 14.4

Graph Traversal and Searching

- Basis of many graph algorithms

- The general idea
  - Given a starting point, visit, check, or update every vertex in a graph by following the given edges
- Application examples
  - Connected components
  - Topological sorting
  - Minimum spanning tree (MST)
  - Shortest path
  - Many more
- Procedure
  - Given a starting point, visit, check, or update every vertex in a graph by following the given edges
  - Order: which vertex do we visit next?
- Basic approaches
  - *Breadth-first*: visit vertices in layers – those closest to the start are visited first, and those most distant are visited last
  - *Depth-first*: from the starting vertex, explore as far as possible along each path before backtracking
  - Remember tree traversals? They're baaaaaaack!

Breadth-First Traversal

- Given the starting point S
  - Visit all the nodes that are one edge away (S's direct neighbors)
  - Visit all nodes that are two edges away (neighbors of neighbors)
  - Visit all nodes that are three edges away (neighbors of neighbors of neighbors)
  - …
  - Repeat this until all nodes have been visited

Traversal Example: Breadth-First

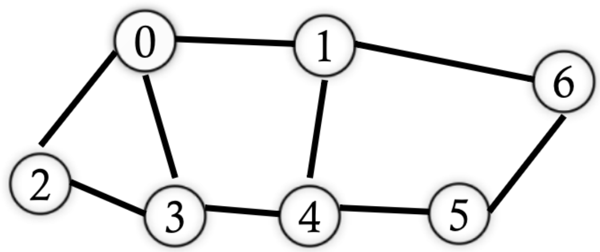


Figure 3: Example 1

- A breadth-first traversal starting with 0: $\{0\}$
  - Visit all the nodes adjacent to 0: $\{0, 1, 2, 3\}$
  - Visit all the neighbors of those nodes: $\{0, 1, 2, 3, 4, 6\}$
  - Continue the process: $\{0, 1, 2, 3, 4, 6, 5\}$
  - We've reached all the nodes, so we can stop. We've also found that every node in this graph can be reached by a path of at most length 3.

Breadth-First Traversal Implementation

- Have we done something similar to this before?
  - Yes, and depending on your progress in Project 3 you might still be doing it.
- We can use a queue
  - Initialize by enqueueing the starting vertex
  - Mark it as visited when we enqueue
  - Process the vertives in a first-in first-out (FIFO) order:
    * Dequeue the first vertex v
    * Enqueue v's neighbor that has not yet been visited or marked

Depth-First Traversal

- Given the starting point S
  - Visit the first neighbor of S
  - Visit the first neighbor of the first neighbor of S
  - Visit the first neighbor fo the first neighbor of the first neighbor of S

    - Repeat this process until there are no more nodes to go, then back, trying the second neighbor of S
    - Repeat that process until we've processed all of the neighbors of S

Traversal Example: Depth-First

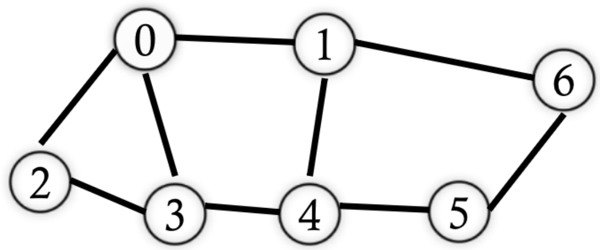


Figure 4: Example 2

A depth-first traversal starting with 0: $\{0\}$

Pick one neighbor of 0: 1. Then the set of nodes visited is $\{0, 1\}$.

- Pick one neighbor of 1: 4. Then the set of nodes visited is $\{0, 1, 4\}$.
    - Pick one neighbor of 4: 3. Then the set of nodes visited is $\{0, 1, 4, 3\}$.
        * Pick one neighbor of 3: 2. Then the set of nodes visited is $\{0, 1, 4, 3, 2\}$.
            · Pick one neighbor of 2: 2 has no neighbors that we haven't visited already, so we backtrack to 3.
        * Pick one neighbor of 3: 3 has no neighbors that we haven't visited already, so we backtrack to 4. Then the set of nodes visited is $\{0, 1, 4, 3, 2\}$.
    - Pick one neighbor of 4: 5. Then the set of nodes visited is $\{0, 1, 4, 3, 2, 5\}$.
        * Pick one neighbor of 5: 6. Then the set of nodes visited is $\{0, 1, 4, 3, 2, 5, 6\}$.
            · Pick one neighbor of 6: 6 has no neighbors that we haven't visited already, so we backtrack to 5.
        * Pick one neighbor of 5: 5 has no neighbors that we haven't visited already, so we backtrack to 4.
    - Pick one neighbor of 4: 4 has no neighbors that we haven't visited already, so we backtrack to 1.
- Pick one neighbor of 1: 1 has no neighbors that we haven't visited already, so we backtrack to 0.

Pick one neighbor of 0: 0 has no neighbors that we haven't visited already, and we cannot backtrack further, so we're done.

Therefore, our result is $\{0, 1, 4, 3, 2, 5\}$.

Depth-First Traversal Implementation

- Implementation qualms:
  - How do we implement backtracking?
    * With recursion, or equivalently, a stack
  - Is a post-order tree traversal depth-first when applied to graphs? What about a pre-order traversal? What about an in-order traversal?
- Using recursion (or a stack)
  - We push nodes with unvisited neighbors onto the stack
  - Pick an unvisited neighbor to continue
    * If there are no more unvisited neighbors, pop out the node and backtrack

Visualization

- Breadth-first traversal
  - https://www.cs.usfca.edu/~galles/visualization/BFS.html
- Depth-first traversal
  - https://www.cs.usfca.edu/~galles/visualization/DFS.html

Shortest Path Problem

- Task: given a designated vertex S, find the shortest path for every vertex v starting from S
  - For an unweighted graph, length is measured by the number of edges of the path
  - For a weighted graph, length is measured by the sum of the weights of all the edges of the path
- Algorithms
  - Unweighted graph: breadth-first traversal (Weiss 14.2)
  - Positive-weighted graph: Dijkstra's algorithm (Weiss 14.3)
  - Negative-weighted graph: Bellman-Ford algorithm (Weiss 14.4)

Unweighted Shortest Path Problem

- The breadth first traversal already gives us a solution to this version of the problem!
  1. Start with vertex S
  2. Visit all the nodes adjacent to S (shortest path length = 1)
  3. Visit all the nodes not yet visited but adjacent to nodes visited in the last step (shortest path length = 2)
  4. Repeat step 3 with an increased path length, until all the nodes have been visited

Shortest Path Example

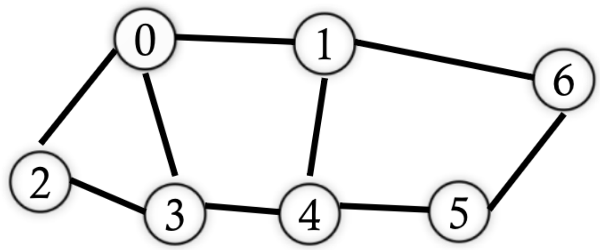


Figure 5: Example 3

- Using a breadth-first traversal, and starting with 0: $\{0\}$
  - Visit all the nodes adjacent to 0: $\{0, 1, 2, 3\}$ (one edge away)
    - Paths established: $\{0, 1\}$, $\{0, 3\}$, $\{0, 2\}$
  - Visit neighbors of neighbors: $\{0, 1, 2, 3, 4, 6\}$
    - Paths established: $\{0, 1\}$, $\{0, 3\}$, $\{0, 2\}$, $\{0, 1, 4\}$, $\{0, 1, 6\}$
  - Visit neighbors of neighbors: $\{0, 1, 2, 3, 4, 6, 5\}$
    - Paths established: $\{0, 1\}$, $\{0, 3\}$, $\{0, 2\}$, $\{0, 1, 4\}$, $\{0, 1, 6\}$, $\{0, 1, 4, 5\}$

Node	Prev	Dist	Adjacency List
0	0	0	1, 2, 3
1	0	1	0, 4, 6
2	0	1	0, 3
3	0	1	0, 2, 4
4	1	2	1, 3, 5
5	4	3	4, 6
6	1	2	1, 5