Numbers in Non-Integer Bases

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NVCC

VMATYC, Spring 2017

Outline

1. Whole Numbers
   - Definitions
   - Examples
   - Language Generated by Whole Number Bases
2. Non-Integer Algebraic Numbers
   - Definitions
   - Examples
   - Properties of Non-Integer Bases
   - Language Generated by Non-Integer Bases

Overview

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- We keep track of the interval that it falls in after the operation

Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)

original

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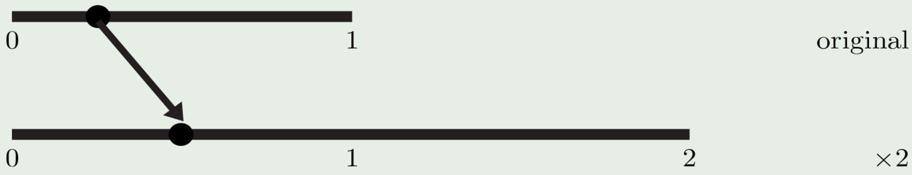

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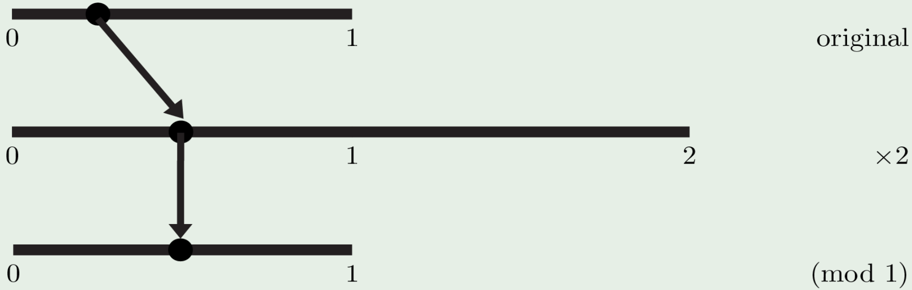

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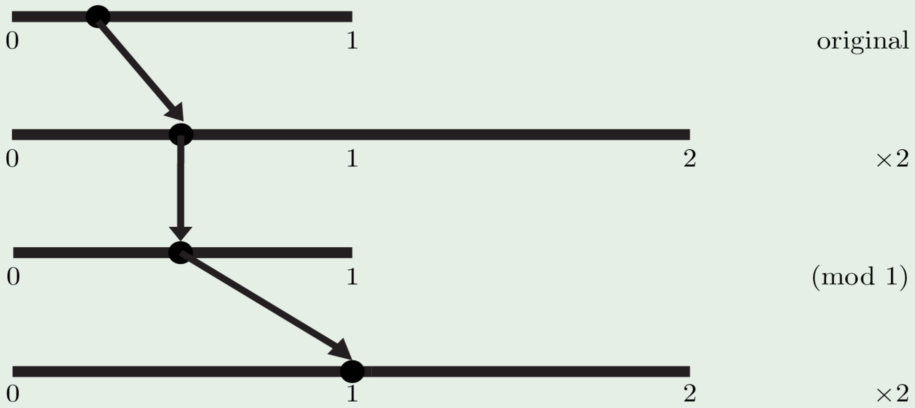

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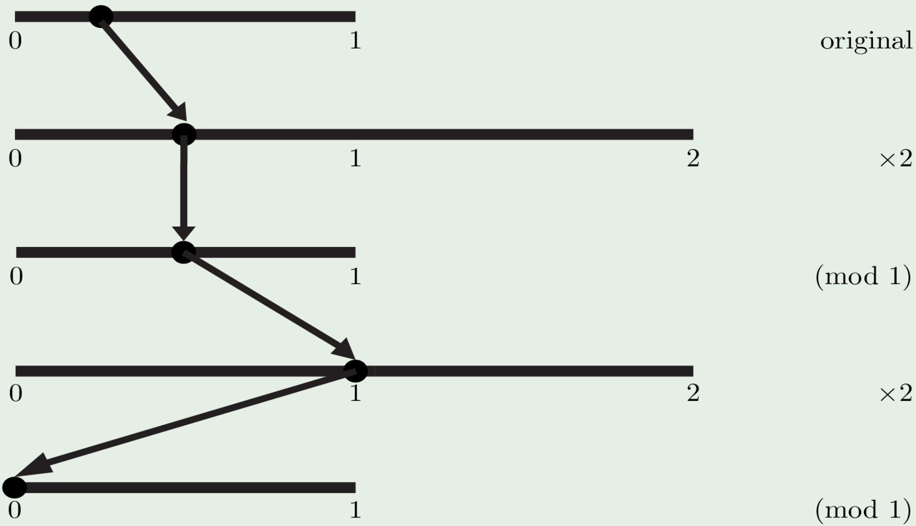

How Partitioning Works

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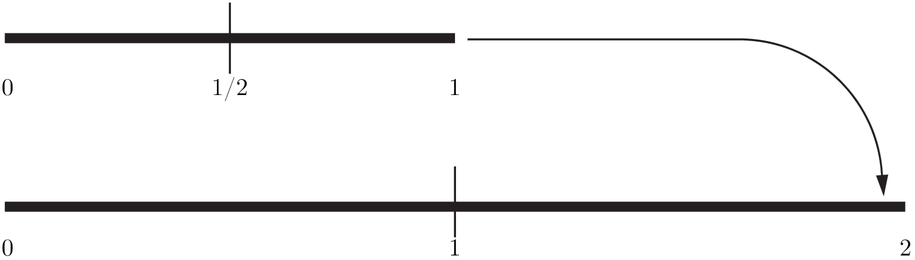

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  - The conversion of 0.25_{10} to a number in binary: 0.01_2
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- That there is a finite radix point expression in a given base
- That our choice of partitioning and multiplication matter... this is what happens geometrically when we change bases
- Let's do another example

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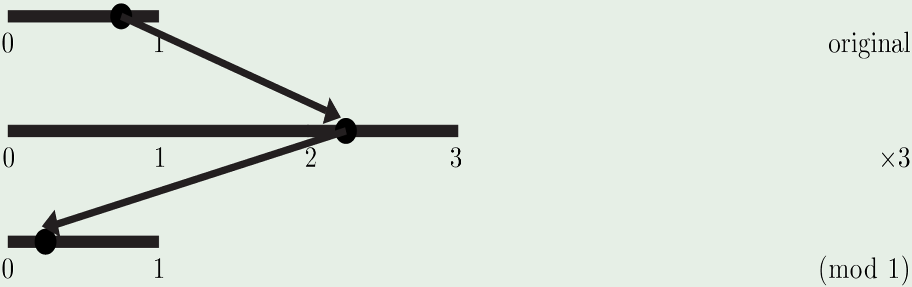

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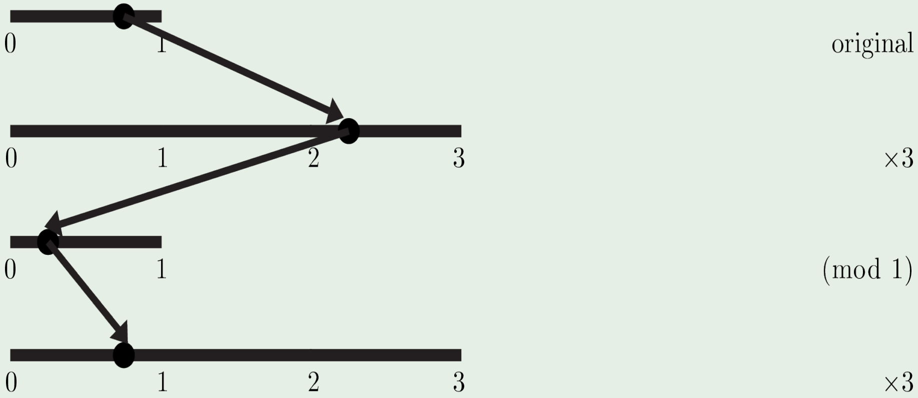

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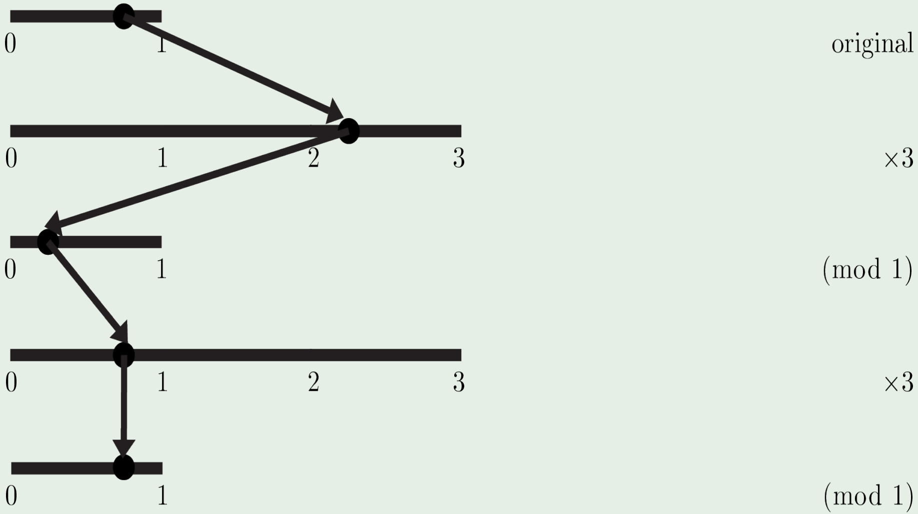

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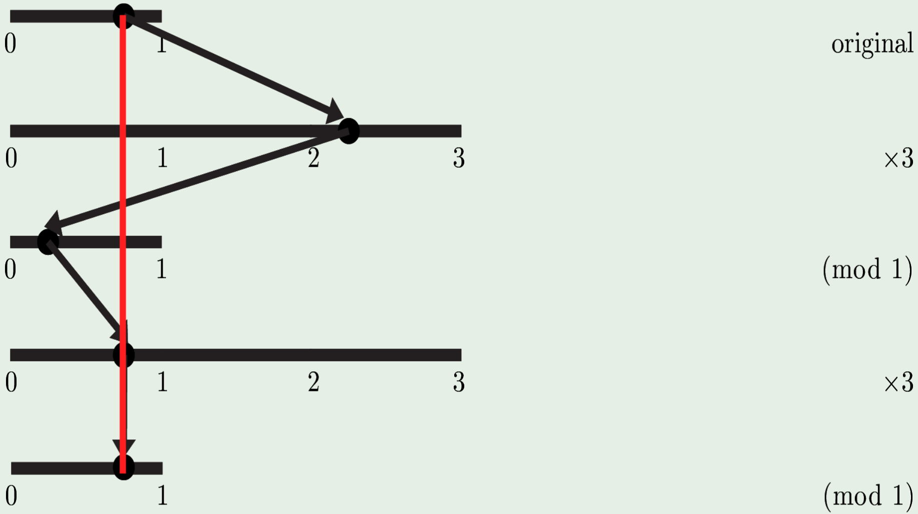

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- In general, this can and does happen in any base
- We can explain this phenomenon arithmetically

Theorem

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A number p is said to have a finite radix point expression if

$$p = \frac{n}{m_1 m_2 \ldots m_k}$$

where $n \in \mathbb{N}$, and $m_1 m_2 \ldots m_k$ are all the prime factors of the base raised to some power.

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http://mathworld.wolfram.com/DecimalExpansion.html

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Example (1/3 in Base 10)

$$\frac{1}{3} \neq \frac{n}{2^\alpha 5^\beta}$$

where $n, \alpha, \beta \in \mathbb{N}$.

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  - We must first qualify what we were doing in our "game" previously

Definition

Beta Map

The transformation $T : [0, 1) \to [0, 1)$ defined by $T = (x \cdot \beta) \pmod 1$.

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Example (Convert 0.1_{10} to binary.)

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Numerical Bases

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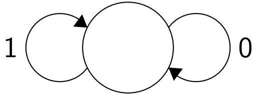
- How can we represent the language generated by these bases?

Definition

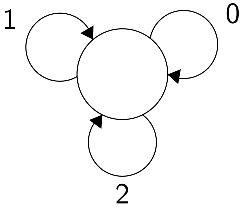
Finite State Machine

A finite state machine (FSM) is a mathematical structure that allows us to imagine a machine with no memory.

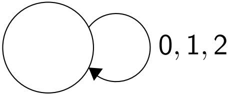
FSM for Binary



FSM for Ternary



Simplified FSM for Ternary



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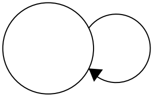

- $n \in \mathbb{N}, \;\; n > 1$

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$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$, maximum numeral is 1
$\alpha = 1 + \sqrt{2} \approx 2.14$, maximum numeral is 2

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Algebraic Integer

A number x is an algebraic integer if it is the root of a monic polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_{n-1}, \ldots, a_0 \in \mathbb{Z}$.

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$$\alpha^2 - 2\alpha - 1 = 0, \quad x = 1 \pm \sqrt{2}$$

$$x^3 - x - 1 = 0, \quad x = \frac{1}{3}\sqrt[3]{\frac{27 - 3\sqrt{69}}{2}} + \frac{\sqrt[3]{\frac{1}{2}(9 + \sqrt{69})}}{3^{2/3}}$$

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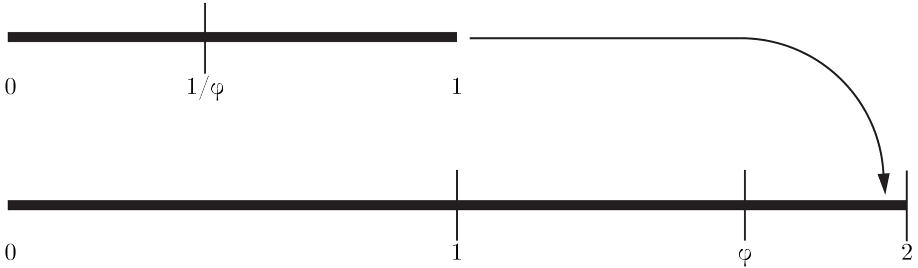

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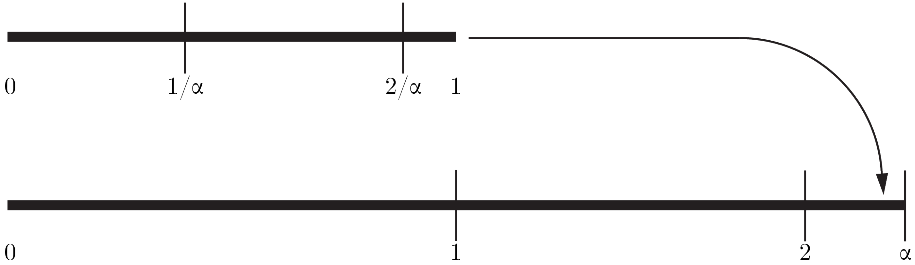

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Examples of Non-Integer Bases

- Base Golden Ratio
- Base Silver Ratio

Golden Ratio

Golden Ratio

The positive solution to the equation $\varphi^2 = \varphi + 1$.
$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$.

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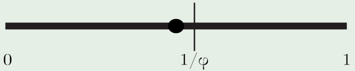

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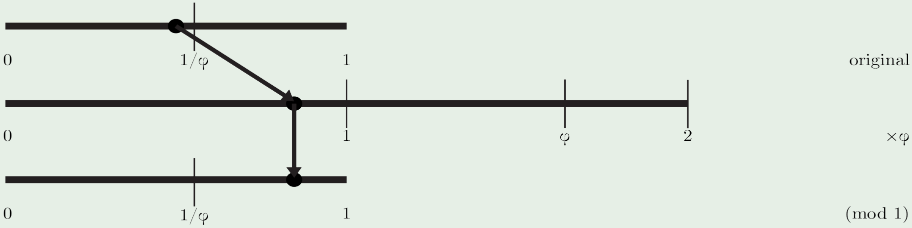

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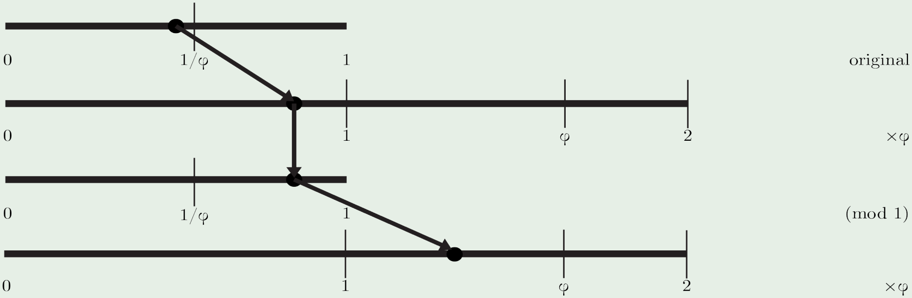

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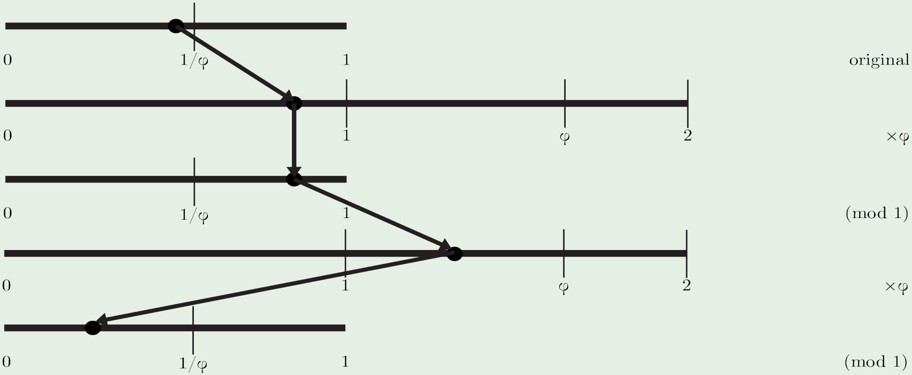

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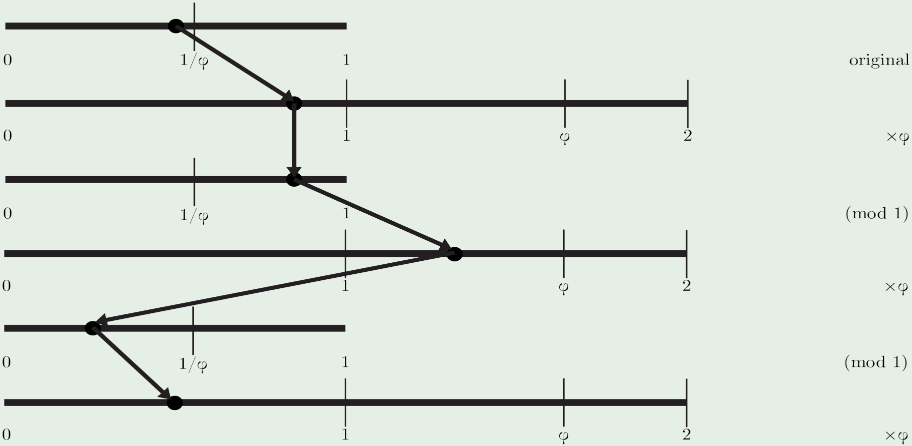

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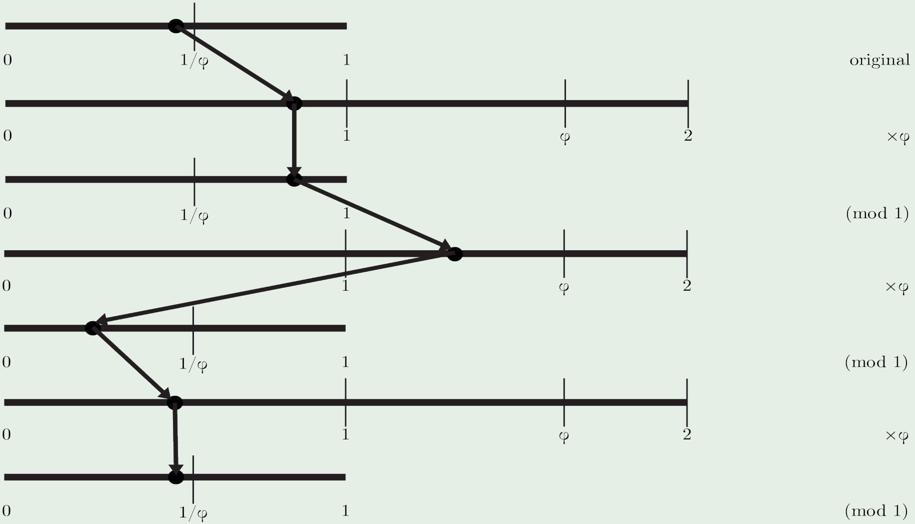

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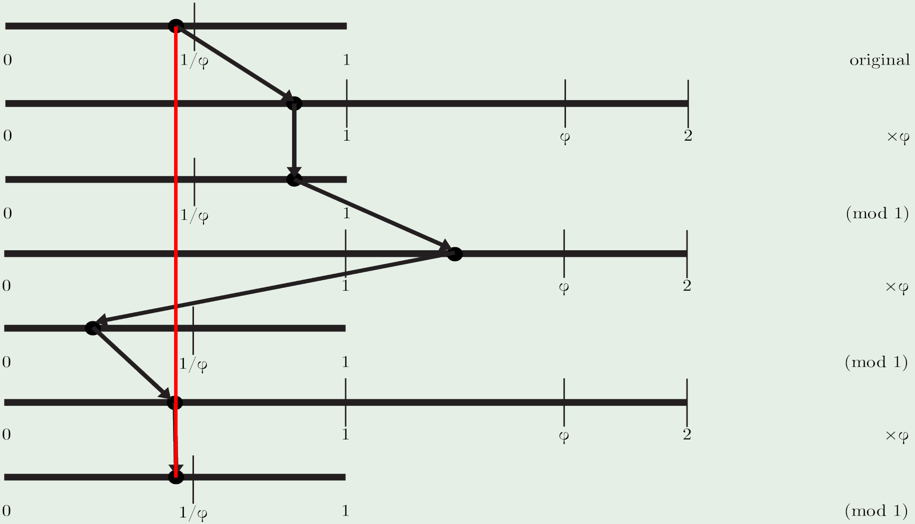

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- Getting stuck in this cycle means this number is not representable as a finite radix point expression in base golden ratio

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- In general, whole numbers (excluding the base and its multiples) are typically the only numbers with finite representations in non-integer, algebraic bases

Silver Ratio

Silver Ratio

The positive solution to the equation $\alpha^2 = 2\alpha + 1$.
$\alpha = 1 + \sqrt{2} \approx 2.414$.

Overview

1. Whole Numbers
   - Definitions
   - Examples
   - Language Generated by Whole Number Bases
2. Non-Integer Algebraic Numbers
   - Definitions
   - Examples
   - Properties of Non-Integer Bases
   - Language Generated by Non-Integer Bases

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  - We can find these by using the orbit of any whole number in these bases (one is the easiest choice)
- It is easy/interesting to represent certain numbers in these bases
- Arithmetic is more of a pain than usual

Definition

Simplest Form

A number is in simplest form if we are unable to rewrite the beta expansion such that it includes a higher power.

Example of Simplest Form

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$$\varphi^2 = \varphi + 1.$$

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and by the definition of the base, we can rewrite the numerator:

$$\frac{\varphi^2}{\varphi^2} = 1_\varphi$$

Golden Ratio Addition

Example ($1_\varphi + 1_\varphi$)

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$10.01_\varphi = 1\times\varphi^1 + 0\times\varphi^0 + 0\times\varphi^{-1} + 1\times\varphi^{-2} = \varphi + 1/\varphi^2 = (\varphi^2+1)/\varphi^2 = 2_{10}$.

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We can calculate any whole number through this method.

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What Makes a Word Forbidden in Non-Integer Bases?

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- In general, a word is forbidden if it is not in simplest form
- In actuality, it is because it escapes the range of the beta map

Escaping the Beta Map

Example (Golden Base Ratio: $T(0.011_\varphi)$)

Using our previous example:

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$$T(0.011_\varphi) = 0.11_\varphi = 1.00_\varphi.$$

But that can't happen! By definition, $T : [0,1) \to [0,1)$. As such, we say that this is a forbidden word.
Geometrically, this can't happen. If we multiply a number in [0,1) by any number and perform (mod 1), we will never get a number larger than one. It's the arithmetic equivalent of having $x \cdot \beta > \beta,\ x < 1$.

Finding Forbidden Words

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- Find the period of the orbit and take one segment
  - Any sequence of numerals greater than the numerals of the period is not allowed

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$1_2 = 0.\overline{11}_8$

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- The orbit of one is composed of the maximum allowed numeral in a language
- All the forbidden words then will subwords of the orbit, with one added until the word is the maximum numeral lexicographically possible

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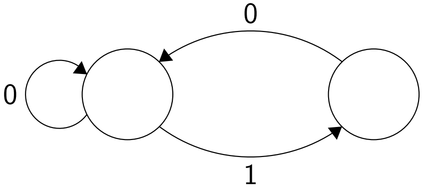
- Since the maximum allowed numeral in the language is 10, we add one and find that the forbidden word is 11
- There can be more than one forbidden word, but since the maximum lexicographically allowed numeral in base golden ratio is 1, we can't add one again
- With this in mind, we can now create a finite state machine

FSM For Language Generated by Base Golden Ratio



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- $0.\overline{20}_\alpha$ is the orbit of one in base silver ratio

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- $0.\overline{20}_\alpha$ is the orbit of one in base silver ratio
- Therefore, 21 and 22 are forbidden words in the language

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