

Numbers in Non-Integer Bases

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NVCC

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1 Whole Numbers

- Definitions
- Examples
- Language Generated by Whole Number Bases

2 Non-Integer Algebraic Numbers

- Definitions
- Examples
- Properties of Non-Integer Bases
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Example (Different Whole Number Bases)

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Radix Point

A point used to separate the integer part of a number from the fractional part.

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$$20_2 = 2 \times 10^1 + 0 \times 2^0$$

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Let's Play a Game

- Suppose that you're given the interval $[0, 1]$, with a point somewhere on it

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 - Once you choose a multiplier, you must use it for the duration
- We keep track of the interval that it falls in after the operation

Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)



original

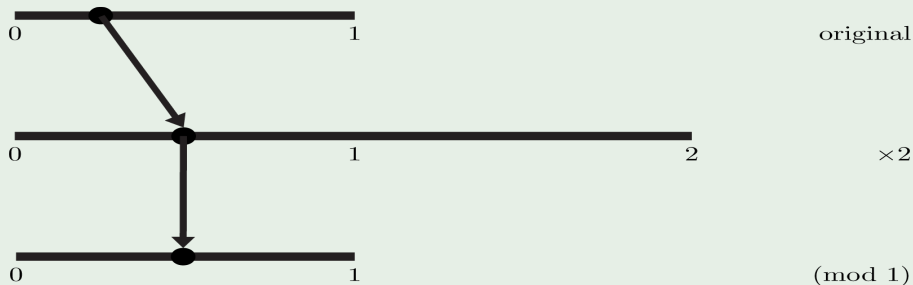
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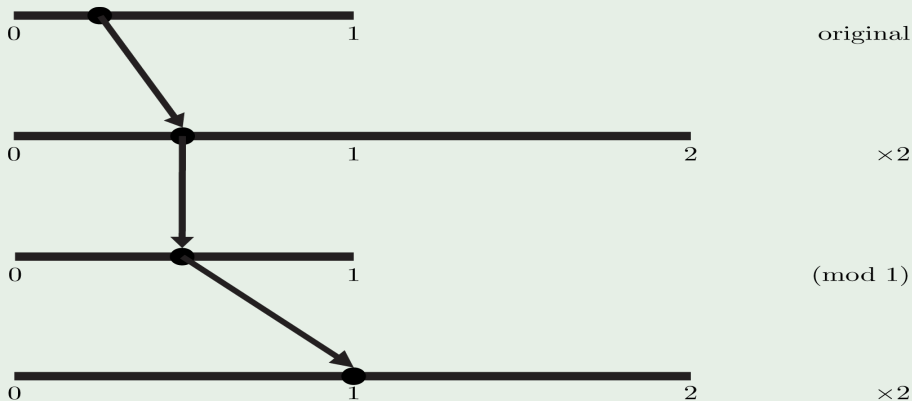
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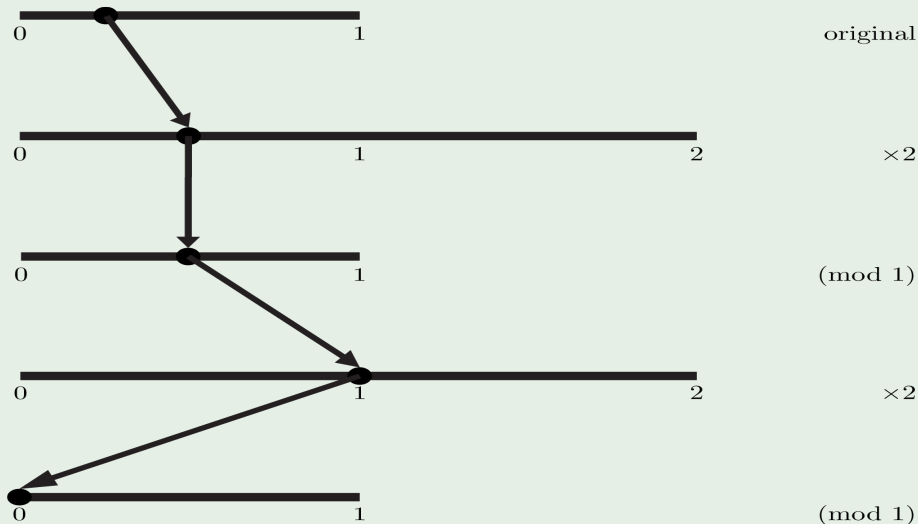
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How Partitioning Works

- We partition before we multiply

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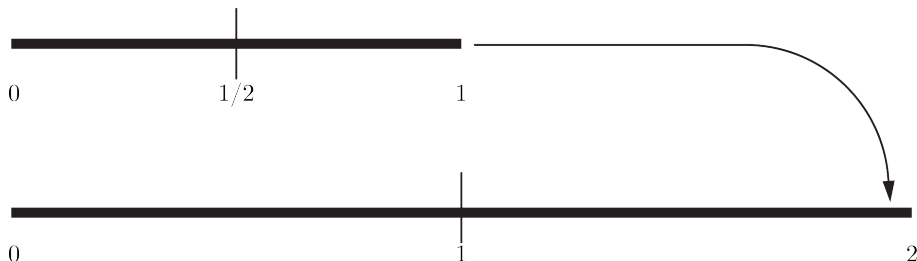
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 - The number we multiply by is the base that we are converting to
- So for binary, this is what we doing:



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 - The conversion of 0.25_{10} to a number in binary: 0.01_2
 - We can check this with a beta expansion:
$$0.01_2 = 0 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2} = 1/4 = 0.25_{10}$$

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- That there is a finite radix point expression in a given base
- That our choice of partitioning and multiplication matter... this is what happens geometrically when we change bases
- Let's do another example

Dealing with Radix Point Expressions (Geometrically)

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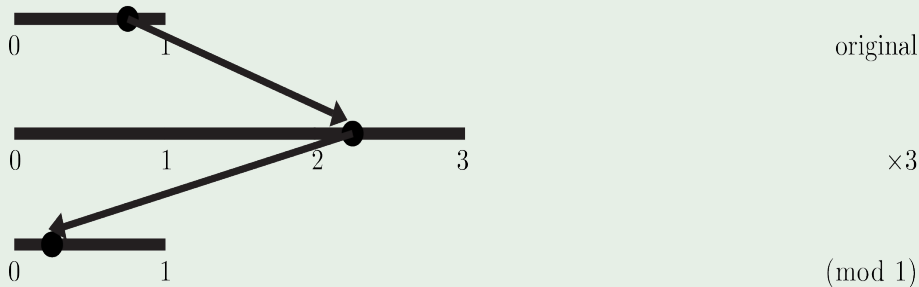
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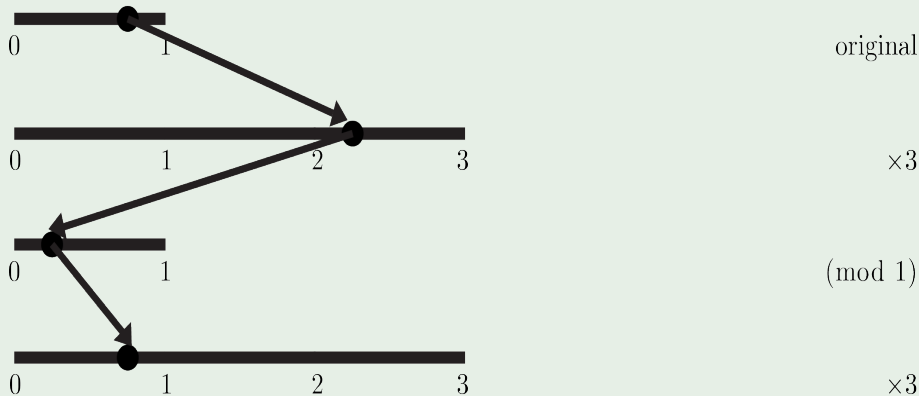
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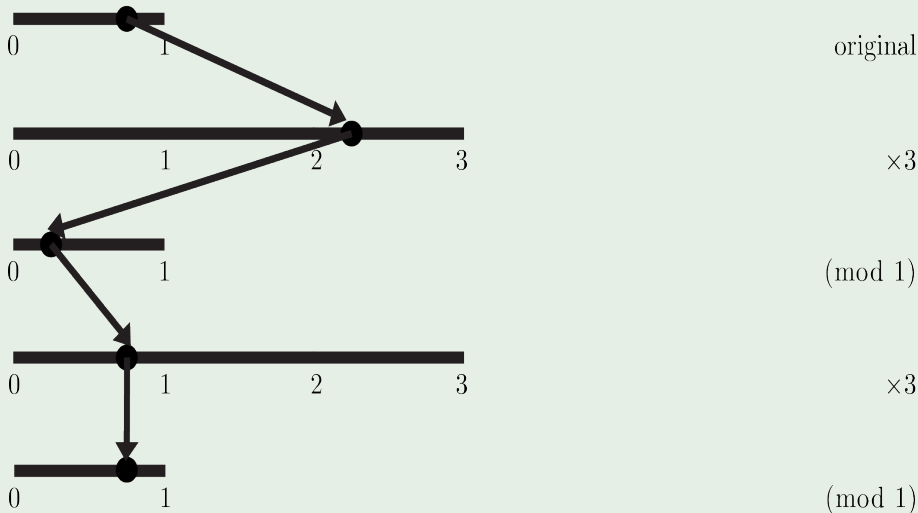
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- Getting stuck in this cycle means this number is not representable as a finite radix point expression in base 3

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- Getting stuck in this cycle means this number is not representable as a finite radix point expression in base 3
- This can happen in any base
 - We can explain this phenomenon arithmetically

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Finite Radix Point Expression

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A number p is said to have a finite radix point expression if

$$p = \frac{n}{m_1 m_2 \dots m_k}$$

where $n \in \mathbb{N}$, and $m_1 m_2 \dots m_k$ are all the prime factors of the base raised to some power.

<http://mathworld.wolfram.com/DecimalExpansion.html>

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Example (1/3 in Base 10)

$$\frac{1}{3} \neq \frac{n}{2^\alpha 5^\beta}$$

where $n, \alpha, \beta \in \mathbb{N}$.

Dealing with Radix Point Expressions (Arithmetically)

- We can do this operation with arithmetic

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 - We must first qualify what we were doing in our “game” previously

Beta Map

The transformation $T : [0, 1) \rightarrow [0, 1)$ defined by $T = (x \cdot \beta) \pmod{1}$.

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Example (Convert 0.1_{10} to binary.)

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$$0.1 * 2 = 0.2$$

$$0.2 \bmod (1) = 0.2$$

0.0

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$0.1 * 2 = 0.2$	$0.2 \bmod (1) = 0.2$	0.0
$0.2 * 2 = 0.4$	$0.4 \bmod (1) = 0.4$	0.00

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$0.4 * 2 = 0.8$	$0.8 \bmod (1) = 0.8$	0.000

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$0.4 * 2 = 0.8$	$0.8 \bmod (1) = 0.8$	0.000
$0.8 * 2 = 1.6$	$1.6 \bmod (1) = 0.6$	0.0001

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$0.6 * 2 = 1.2$	$1.2 \bmod (1) = 0.2$	0.00011
$0.2 * 2 = 0.4$	$0.4 \bmod (1) = 0.4$...
		$0.0001\overline{1}_2$

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Definition

Word

A word is a sequence of characters allowed by the definition of its language.

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99.9 (Decimal)

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Numerical Bases

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Example (Examples of Languages)

English

Numerical Bases

Sounds a bird makes while chirping

Language Generated by Whole Number Bases

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 - Acceptable words in base 2 include 1, or 0.110000 – but not 3, or 4: they contain forbidden subwords

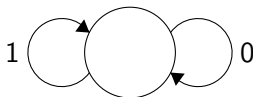
Language Generated by Whole Number Bases

- A word is not allowed in a language if it contains a forbidden subword
 - Acceptable words in base 10 include 9, or 5.5 – but not CAT or 5Z: they contain forbidden subwords
 - Acceptable words in base 2 include 1, or 0.110000 – but not 3, or 4: they contain forbidden subwords
- How can we represent the language generated by these bases?

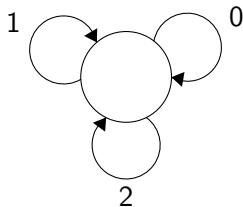
Finite State Machine

A finite state machine (FSM) is a mathematical structure that allows us to imagine a machine with no memory.

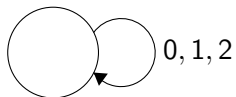
FSM for Binary



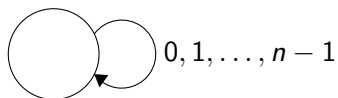
FSM for Ternary



Simplified FSM for Ternary



FSM for Whole Number Bases



- $n \in \mathbb{N}, n > 1$

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Definition

Orbit

The unique, non-terminating radix point representation of a number in any base.

Algebraic Integer

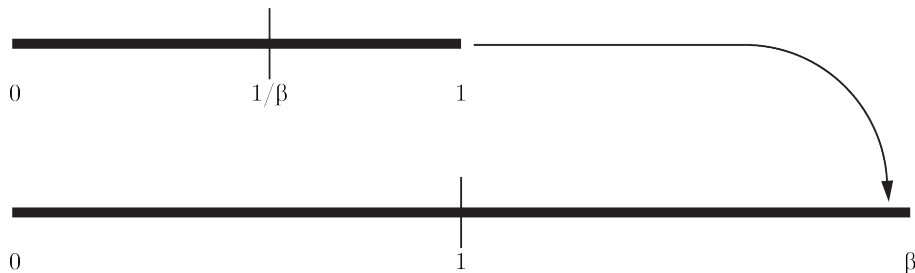
A number x is an algebraic integer if it is the root of a monic polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where $a_{n-1}, \dots, a_0 \in \mathbb{Z}$.

Simplest Form

A number is in simplest form if we are unable to rewrite the beta expansion such that it includes a higher power.

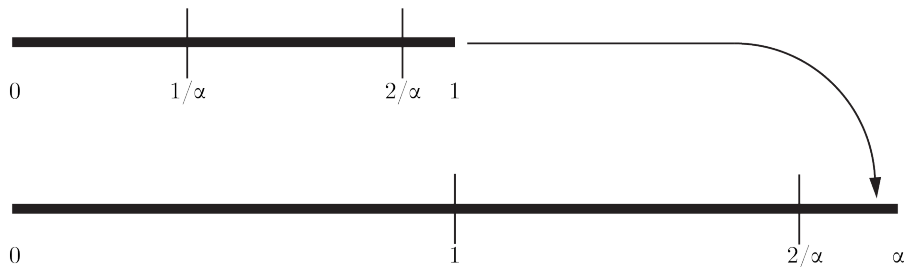
Partitioning Differently

- We don't have to use a whole number
- The partitions won't be even, but that's not a problem



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- **Properties of Non-Integer Bases**
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Examples of Non-Integer Bases

- Base Golden Ratio
- Base Silver Ratio

Properties of Non-Integer Bases

- There will be forbidden words
 - We can find these by using the orbit of any whole number in these bases (one is the easiest choice)
- It is easy/interesting to represent certain numbers in these bases
- Arithmetic is more of a pain than usual

Golden Ratio

The positive solution to the equation $\varphi^2 = \varphi + 1$.
 $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$.

Golden Ratio Addition

Example ($1_\varphi + 1_\varphi$)

$$1_\varphi = 0.11_\varphi$$

$$1_\varphi + 0.11_\varphi = 1.11_\varphi = 10.01_\varphi$$

We can double check this operation by doing a beta expansion and using the definition of the base.

Golden Ratio Addition

Example ($1_\varphi + 1_\varphi$)

$$10.01_\varphi = 1 \times \varphi^1 + 0 \times \varphi^0 + 0 \times \varphi^{-1} + 1 \times \varphi^{-2} = \varphi + 1/\varphi^2 = (\varphi^2 + 1)/\varphi^2 = 2_{10}$$

Silver Ratio

The positive solution to the equation $\alpha^2 = 2\alpha + 1$.
 $\alpha = 1 + \sqrt{2} \approx 2.414$.

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Finding Forbidden Words

- Take the orbit of one
- Find the period
- Any sequence of numerals greater than the numerals of the period is not allowed

What Makes a Word Forbidden in Non-Integer Bases?

- In general, a word is forbidden if it is not in simplest form
- In actuality, it is because it escapes the range of the beta map

Escaping the Beta Map

Example ($T(0.011_\varphi)$)

$T(0.011_\varphi) = 0.11_\varphi = 1.00_\varphi$ But that can't happen! By definition, $T : [0, 1) \rightarrow [0, 1)$. As such, we say that this is a forbidden word. How do we find forbidden words?

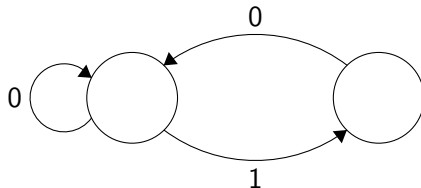
Finding Forbidden Words

- The orbit of one is composed of the maximum allowed word in a language
- All the forbidden words then will subwords of the orbit, with one added until the word is the maximum allowed in the language

Language Generated by base Golden Ratio

- $0.\overline{10}_\varphi$ is the orbit of one in base golden ratio
- Since the maximum allowed word in the language is 10, we add one and find that the forbidden word is 11.
- There can be more than one forbidden word, but since the maximum allowed numeral in base golden ratio is 1, we can't add one again without changing the length.
- With this in mind, we can now create a finite state machine

FSM For Language Generated by Base Golden Ratio



Language Generated by base Silver Ratio

- $0.\overline{20}_\alpha$ is the orbit of one in base silver ratio
- Therefore, 21 and 22 are forbidden words in the language

FSM For Language Generated by Base Silver Ratio

