Numbers in Non-Integer Bases

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NVCC

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Outline

Overview

Definition

Base

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A point used to separate the integer part of a number from the fractional part.

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$20_2 = 2 \times 10^1 + 0 \times 2^0$

Overview

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- Suppose that you're given the interval $[0, 1]$, with a point somewhere on it

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- Suppose that you're given the interval $[0, 1]$, with a point somewhere on it

- You're allowed one operation
  - Multiplication by a whole number, followed by modulo one
  - Once you choose a multiplier, you must use it for the duration
- We keep track of the interval that it falls in after the operation

Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)

original

Dealing with Radix Point Expressions (Geometrically)

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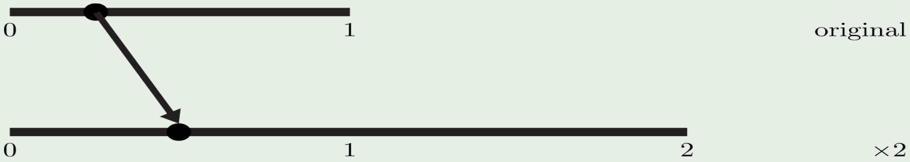

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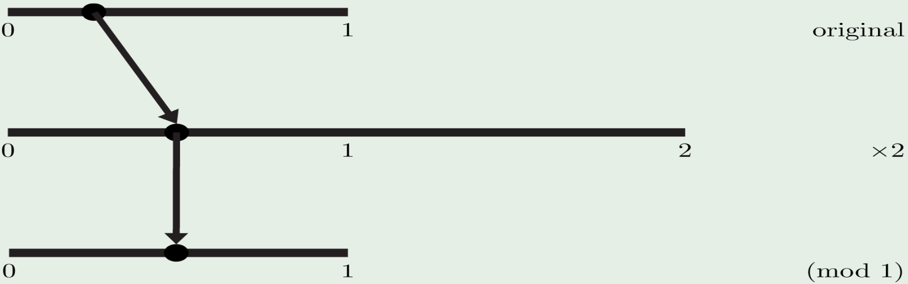

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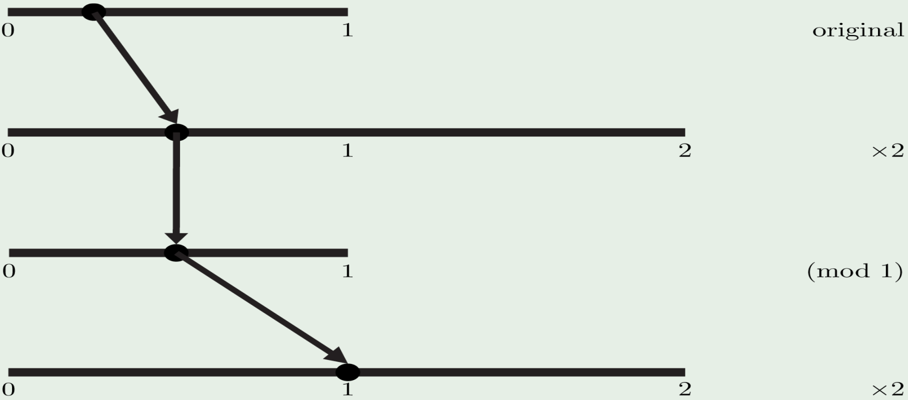

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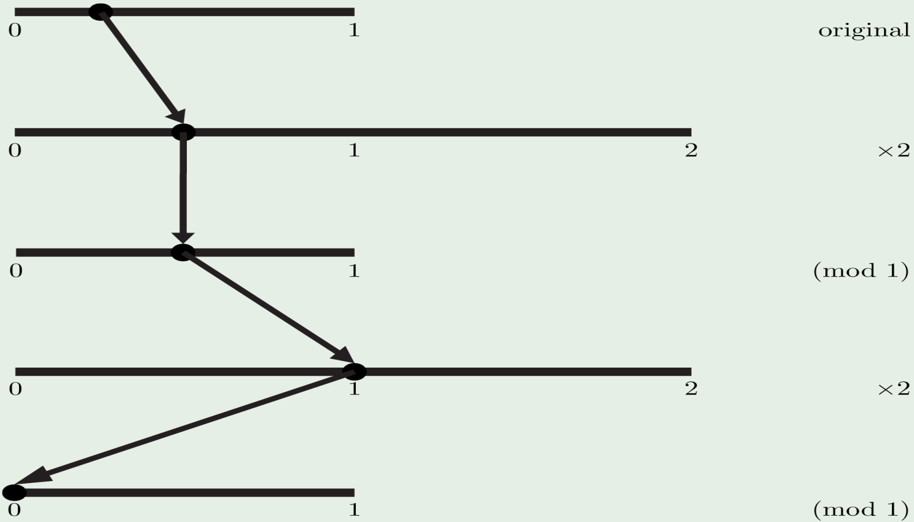

How Partitioning Works

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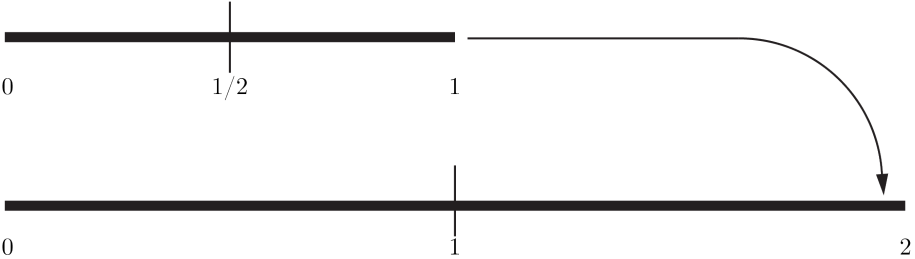
- So for binary, this is what we doing:



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  - The conversion of 0.25_{10} to a number in binary: 0.01_2
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- That there is a finite radix point expression in a given base
- That our choice of partitioning and multiplication matter... this is what happens geometrically when we change bases
- Let's do another example

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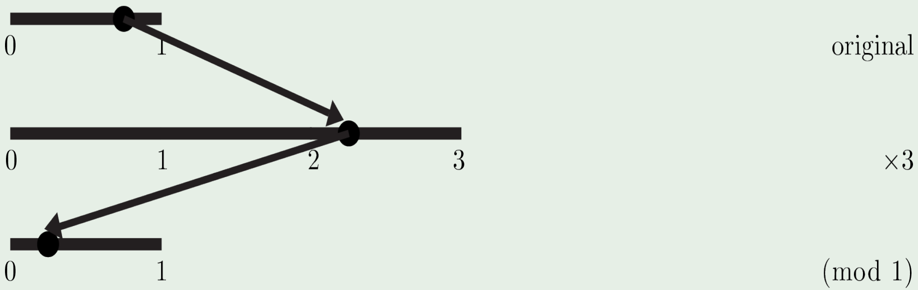

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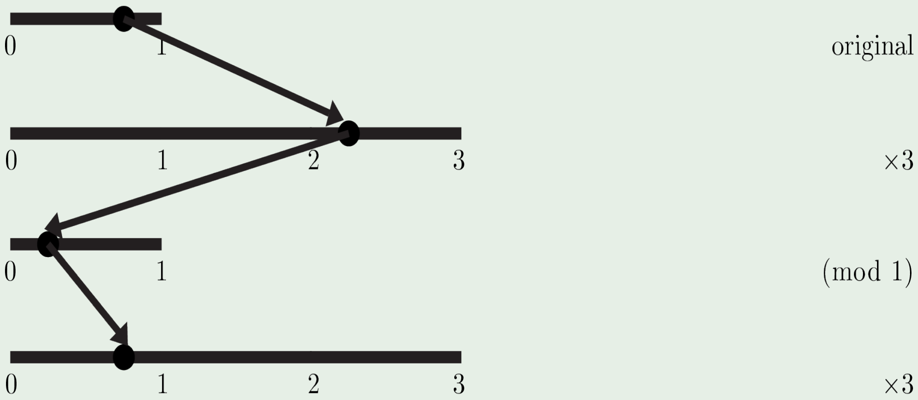

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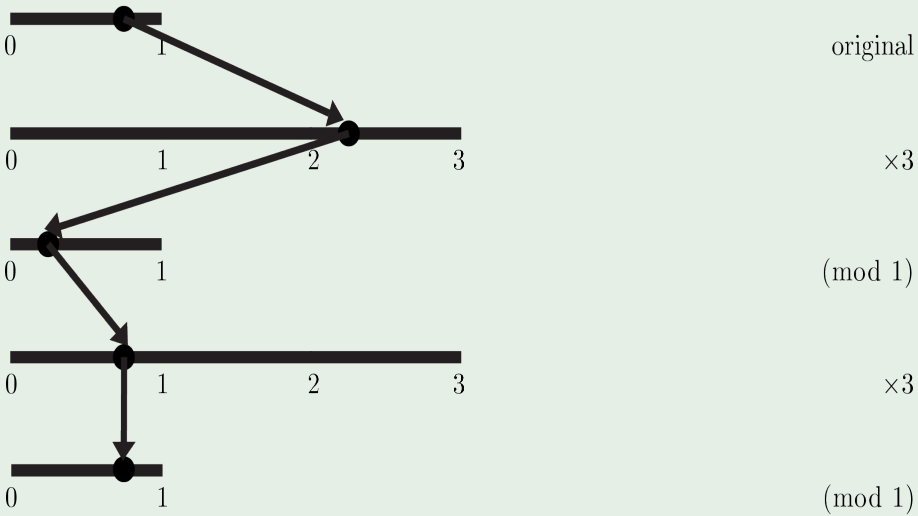

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- In general, this can and does happen in any base
- We can explain this phenomenon arithmetically

Theorem

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A number p is said to have a finite radix point expression if

$$p = \frac{n}{m_1 m_2 \ldots m_k}$$

where $n \in \mathbb{N}$, and $m_1 m_2 \ldots m_k$ are all the prime factors of the base raised to some power.

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http://mathworld.wolfram.com/DecimalExpansion.html

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Example (1/3 in Base 10)

$$\frac{1}{3} \neq \frac{n}{2^\alpha 5^\beta}$$

where $n, \alpha, \beta \in \mathbb{N}$.

Dealing with Radix Point Expressions (Arithmetically)

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  - We must first qualify what we were doing in our “game” previously

Definition

Beta Map

The transformation $T : [0, 1) \to [0, 1)$ defined by $T = (x \cdot \beta) \pmod 1$.

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Example (Convert 0.1_{10} to binary.)

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Dealing with Radix Point Expressions (Arithmetically)

Example (Convert 0.1_{10} to binary.)

$0.1 * 2 = 0.2$	$0.2 \bmod (1) = 0.2$	0.0
$0.2 * 2 = 0.4$	$0.4 \bmod (1) = 0.4$	0.00

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$0.4 * 2 = 0.8$	$0.8 \bmod (1) = 0.8$	0.000

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$0.4 * 2 = 0.8$	$0.8 \bmod (1) = 0.8$	0.000
$0.8 * 2 = 1.6$	$1.6 \bmod (1) = 0.6$	0.0001

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$0.2 * 2 = 0.4$	$0.4 \bmod (1) = 0.4$	...

Dealing with Radix Point Expressions (Arithmetically)

Example (Convert 0.1_{10} to binary.)

$$
\begin{array}{lll}
0.1 * 2 = 0.2 & 0.2 \bmod (1) = 0.2 & 0.0 \\
0.2 * 2 = 0.4 & 0.4 \bmod (1) = 0.4 & 0.00 \\
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 & & 0.0\overline{0011}_2
\end{array}
$$

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Example (Examples of Words)

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101 (Binary)
99.9 (Decimal)

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Language

A language is the set of all allowed words defined by some rule.

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Numerical Bases

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Example (Examples of Languages)

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Numerical Bases

Sounds a bird makes while chirping

Language Generated by Whole Number Bases

- A word is not allowed in a language if it contains a forbidden subword

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  - Acceptable words in base 10 include 9, or 5.5 – but not CAT or 5Z: they contain forbidden subwords

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    - Acceptable words in base 2 include 1, or 0.110000 – but not 3, or 4: they contain forbidden subwords

Language Generated by Whole Number Bases

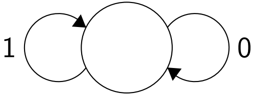
- A word is not allowed in a language if it contains a forbidden subword
  - Acceptable words in base 10 include 9, or 5.5 – but not CAT or 5Z: they contain forbidden subwords
  - Acceptable words in base 2 include 1, or 0.110000 – but not 3, or 4: they contain forbidden subwords
- How can we represent the language generated by these bases?

Definition

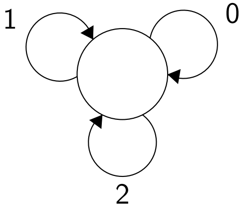
Finite State Machine

A finite state machine (FSM) is a mathematical structure that allows us to imagine a machine with no memory.

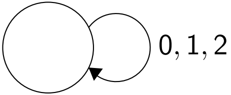
FSM for Binary



FSM for Ternary



Simplified FSM for Ternary



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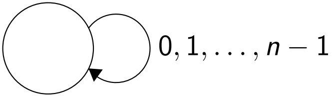

- $n \in \mathbb{N}, \;\; n > 1$

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If the radix is a whole number, the previous definition holds.
If the radix is not a whole number, then the maximum allowed characters is the floor function of the base.

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$\beta = \frac{1+\sqrt{5}}{2} \approx 1.618$, maximum numeral is 1

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Example (Allowed Numbers in Different Bases)

$\beta = \frac{1+\sqrt{5}}{2} \approx 1.618$, maximum numeral is 1
$\alpha = 1 + \sqrt{2} \approx 2.14$, maximum numeral is 2

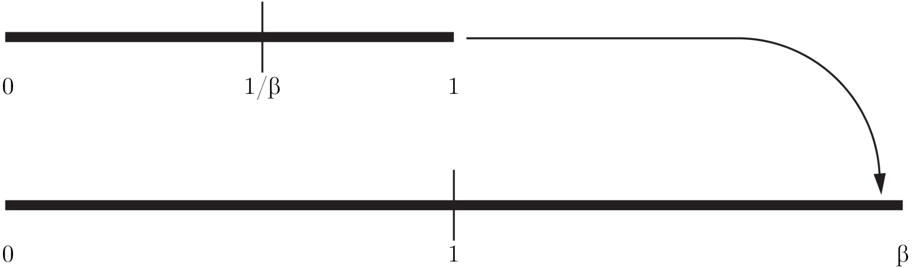
Definition

Algebraic Integer

A number x is an algebraic integer if it is the root of a monic polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_{n-1}, \ldots, a_0 \in \mathbb{Z}$.

Partitioning Differently

- We don't have to use a whole number
- The partitions won't be even, but that's not a problem



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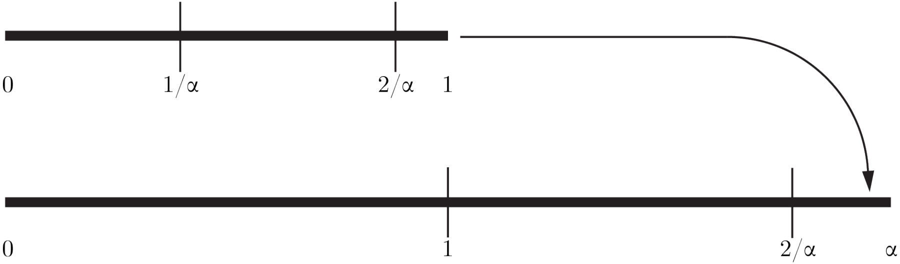

Overview

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Examples of Non-Integer Bases

- Base Golden Ratio
- Base Silver Ratio

Properties of Non-Integer Bases

- There will be forbidden words
  - We can find these by using the orbit of any whole number in these bases (one is the easiest choice)
- It is easy/interesting to represent certain numbers in these bases
- Arithmetic is more of a pain than usual

Golden Ratio

Golden Ratio

The positive solution to the equation $\varphi^2 = \varphi + 1$.
$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$.

Golden Ratio Addition

Example ($1_\varphi + 1_\varphi$)

$1_\varphi = 0.11\varphi$
$1_\varphi + 0.11\varphi = 1.11\varphi = 10.01_\varphi$
We can double check this operation by doing a beta expansion and using the definition of the base.

Golden Ratio Addition

Example ($1_\varphi + 1_\varphi$)

$10.01_\varphi = 1 \times \varphi^1 + 0 \times \varphi^0 + 0 \times \varphi^{-1} + 1 \times \varphi^{-2} = \varphi + 1/\varphi^2 = (\varphi^2 + 1)/\varphi^2 = 2_{10}$

Silver Ratio

Silver Ratio

The positive solution to the equation $\alpha^2 = 2\alpha + 1$.
$\alpha = 1 + \sqrt{2} \approx 2.414$.

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The unique, non-terminating radix point representation of a number in any base.

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Orbit of One in Different Bases $1_{10} = 0.\overline{99}_{10}$
$1_9 = 0.\overline{88}_9$
$1_2 = 0.\overline{11}_8$

Definition

Simplest Form

A number is in simplest form if we are unable to rewrite the beta expansion such that it includes a higher power.

Example of Simplest Form

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Let $\varphi^2 = \varphi + 1$.

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Let $\varphi^2 = \varphi + 1$.
Using the positive solution of that equation $\varphi \approx 1.618$, we can show that $0.11_\varphi = 1.00_\varphi$ using a beta expansion.

$$0.11_\varphi = 1 \times \varphi^{-1} + 1 \times \varphi^{-2}.$$

$$\frac{1}{\varphi} + \frac{1}{\varphi^2} = \frac{\varphi}{\varphi^2} + \frac{1}{\varphi^2}.$$

$$\frac{\varphi + 1}{\varphi^2},$$

and by the definition of the base, we can rewrite the numerator:

$$\frac{\varphi^2}{\varphi^2} = 1_\varphi$$

Finding Forbidden Words

- Take the orbit of one
- Find the period
- Any sequence of numerals greater than the numerals of the period is not allowed

What Makes a Word Forbidden in Non-Integer Bases?

- In general, a word is forbidden if it is not in simplest form
- In actuality, it is because it escapes the range of the beta map

Escaping the Beta Map

Example ($T(0.011_\varphi)$)

$T(0.011_\varphi) = 0.11_\varphi = 1.00_\varphi$ But that can't happen! By definition, $T : [0,1) \to [0,1)$. As such, we say that this is a forbidden word.
How do we find forbidden words?

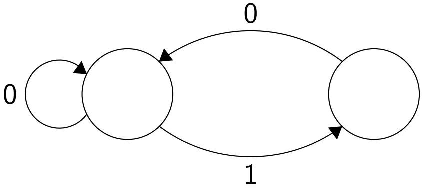
Finding Forbidden Words

- The orbit of one is composed of the maximum allowed word in a language
- All the forbidden words then will subwords of the orbit, with one added until the word is the maximum allowed in the language

Language Generated by base Golden Ratio

- $0.\overline{10}_\varphi$ is the orbit of one in base golden ratio
- Since the maximum allowed word in the language is 10, we add one and find that the forbidden word is 11.
- There can be more than one forbidden word, but since the maximum allowed numeral in base golden ratio is 1, we can't add one again without changing the length.
- With this in mind, we can now create a finite state machine

FSM For Language Generated by Base Golden Ratio



Language Generated by base Silver Ratio

- $0.\overline{20}_\alpha$ is the orbit of one in base silver ratio
- Therefore, 21 and 22 are forbidden words in the language

FSM For Language Generated by Base Silver Ratio