

# Numbers in Non-Integer Bases

Connor Baker

Dr. Tyler White

Northern Virginia Community College

VMATYC, Spring 2017

## 1 Whole Numbers

- Definitions
- Examples
- Language Generated by Whole Number Bases

## 2 Non-Integer Algebraic Numbers

- Definitions
- Examples
- Properties of Non-Integer Bases
- Language Generated by Non-Integer Bases

## 3 Questions and Answers

# Overview

## 1 Whole Numbers

- Definitions
- Examples
- Language Generated by Whole Number Bases

## 2 Non-Integer Algebraic Numbers

- Definitions
- Examples
- Properties of Non-Integer Bases
- Language Generated by Non-Integer Bases

## 3 Questions and Answers

## Base

A base (or radix) is a mathematical building block used to describe a number system whose digits are used to represent numbers.

# Definition

## Base

A base (or radix) is a mathematical building block used to describe a number system whose digits are used to represent numbers.

## Example (Different Whole Number Bases)

# Definition

## Base

A base (or radix) is a mathematical building block used to describe a number system whose digits are used to represent numbers.

## Example (Different Whole Number Bases)

$125_{10}$

# Definition

## Base

A base (or radix) is a mathematical building block used to describe a number system whose digits are used to represent numbers.

## Example (Different Whole Number Bases)

$125_{10}$

$888_9$

# Definition

## Base

A base (or radix) is a mathematical building block used to describe a number system whose digits are used to represent numbers.

## Example (Different Whole Number Bases)

$125_{10}$

$888_9$

$48.5_8$



# Definition

## Base

A base (or radix) is a mathematical building block used to describe a number system whose digits are used to represent numbers.

## Example (Different Whole Number Bases)

$125_{10}$

$888_9$

$48.5_8 ?$

## Maximum Allowed Numeral in Base

The radix is the number of numerals allowed in a base, and it includes zero.

## Maximum Allowed Numeral in Base

The radix is the number of numerals allowed in a base, and it includes zero. The number of numerals in a base is one less than the base.

# Definition

## Maximum Allowed Numeral in Base

The radix is the number of numerals allowed in a base, and it includes zero. The number of numerals in a base is one less than the base.

## Example (Allowed Numbers in Different Bases)

# Definition

## Maximum Allowed Numeral in Base

The radix is the number of numerals allowed in a base, and it includes zero. The number of numerals in a base is one less than the base.

## Example (Allowed Numbers in Different Bases)

$125_{10}$

# Definition

## Maximum Allowed Numeral in Base

The radix is the number of numerals allowed in a base, and it includes zero. The number of numerals in a base is one less than the base.

## Example (Allowed Numbers in Different Bases)

$125_{10}$

$888_9$

# Definition

## Maximum Allowed Numeral in Base

The radix is the number of numerals allowed in a base, and it includes zero. The number of numerals in a base is one less than the base.

## Example (Allowed Numbers in Different Bases)

$125_{10}$

$888_9$

$48.5_8$

# Definition

## Maximum Allowed Numeral in Base

The radix is the number of numerals allowed in a base, and it includes zero. The number of numerals in a base is one less than the base.

## Example (Allowed Numbers in Different Bases)

$125_{10}$

$888_9$

$48.5_9$



## Radix Point

A point used to separate the integer part of a number from the fractional part.

# Definition

## Radix Point

A point used to separate the integer part of a number from the fractional part.

## Example (Radix Point Expressions)

# Definition

## Radix Point

A point used to separate the integer part of a number from the fractional part.

## Example (Radix Point Expressions)

$10.5_{10}$

# Definition

## Radix Point

A point used to separate the integer part of a number from the fractional part.

## Example (Radix Point Expressions)

$10.5_{10}$

$A5.E_{16}$

# Definition

## Radix Point

A point used to separate the integer part of a number from the fractional part.

## Example (Radix Point Expressions)

$10.5_{10}$

$A5.E_{16}$

$1.1_2$

## Beta Expansion

A means of rewriting a number as the digits multiplied by the base which is raised to the power of position of the digit.

# Definition

## Beta Expansion

A means of rewriting a number as the digits multiplied by the base which is raised to the power of position of the digit.

## Example (Example of Beta Expansions)

# Definition

## Beta Expansion

A means of rewriting a number as the digits multiplied by the base which is raised to the power of position of the digit.

## Example (Example of Beta Expansions)

$$125_{10} = 1 \times 10^2 + 2 \times 10^1 + 5 \times 10^0$$



# Definition

## Beta Expansion

A means of rewriting a number as the digits multiplied by the base which is raised to the power of position of the digit.

## Example (Example of Beta Expansions)

$$125_{10} = 1 \times 10^2 + 2 \times 10^1 + 5 \times 10^0$$

$$A5.E_{16} = 10 \times 16^1 + 5 \times 16^0 + 14 \times 16^{-1}$$

# Definition

## Beta Expansion

A means of rewriting a number as the digits multiplied by the base which is raised to the power of position of the digit.

## Example (Example of Beta Expansions)

$$125_{10} = 1 \times 10^2 + 2 \times 10^1 + 5 \times 10^0$$

$$A5.E_{16} = 10 \times 16^1 + 5 \times 16^0 + 14 \times 16^{-1}$$

$$10_2 = 1 \times 10^1 + 0 \times 2^0$$

# Overview

## 1 Whole Numbers

- Definitions
- **Examples**
- Language Generated by Whole Number Bases

## 2 Non-Integer Algebraic Numbers

- Definitions
- Examples
- Properties of Non-Integer Bases
- Language Generated by Non-Integer Bases

## 3 Questions and Answers

# Let's Play a Game

- Suppose that you're given the interval  $[0, 1]$ , with a point somewhere on it

# Let's Play a Game

- Suppose that you're given the interval  $[0, 1]$ , with a point somewhere on it



# Let's Play a Game

- Suppose that you're given the interval  $[0, 1]$ , with a point somewhere on it



- You're allowed one operation
  - Multiplication by a whole number, followed by modulo one

# Let's Play a Game

- Suppose that you're given the interval  $[0, 1]$ , with a point somewhere on it



- You're allowed one operation
  - Multiplication by a whole number, followed by modulo one
  - Once you choose a multiplier, you must use it for the duration

# Let's Play a Game

- Suppose that you're given the interval  $[0, 1]$ , with a point somewhere on it



- You're allowed one operation
  - Multiplication by a whole number, followed by modulo one
  - Once you choose a multiplier, you must use it for the duration
- We keep track of the interval that it falls in after the operation



# Dealing with Radix Point Expressions (Geometrically)

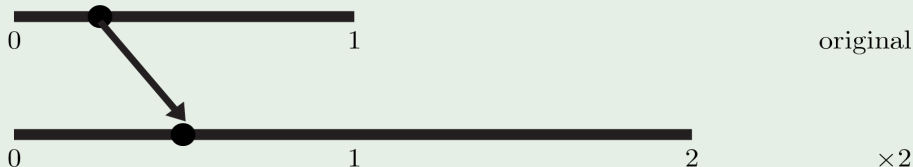
Example (Suppose you have a point...)



original

# Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)



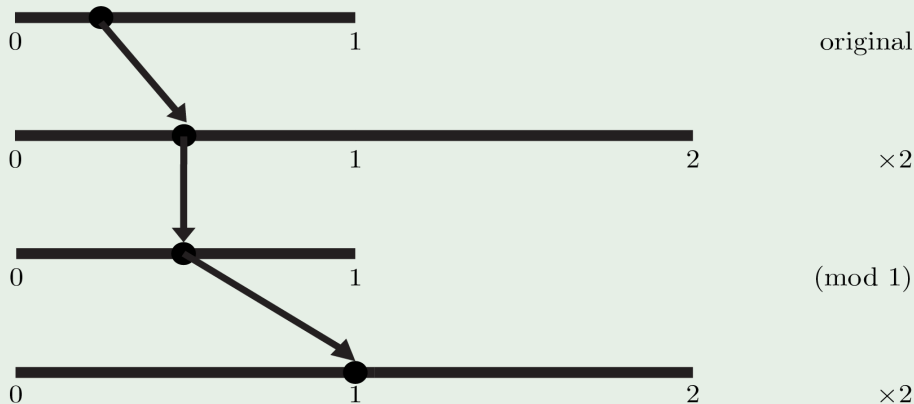
# Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)



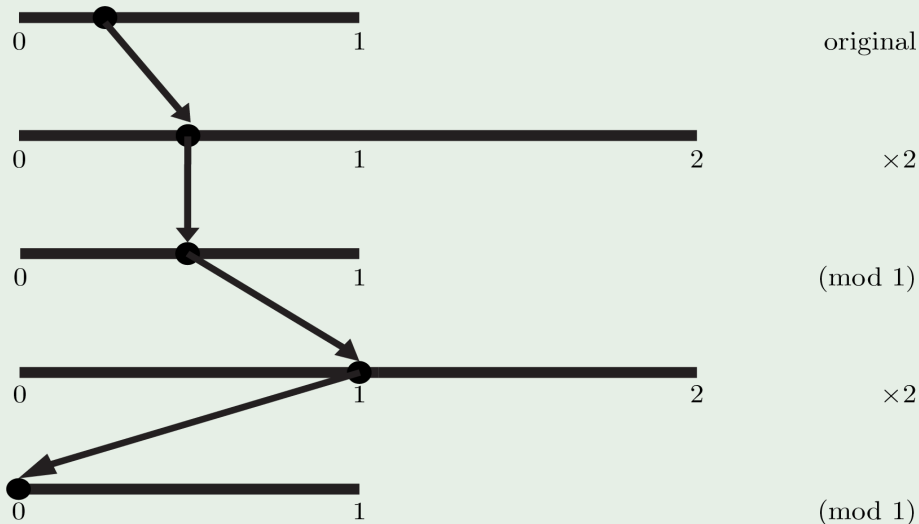
# Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)



# Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)



# How Partitioning Works

- We partition before we multiply

# How Partitioning Works

- We partition before we multiply
- We multiply so that the first partition becomes the unit interval

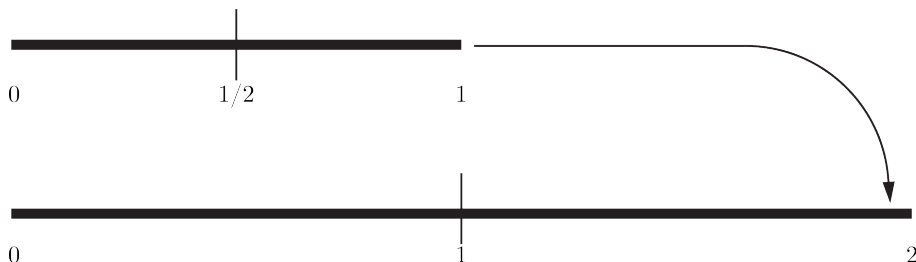
# How Partitioning Works

- We partition before we multiply
- We multiply so that the first partition becomes the unit interval
  - The number we multiply by is the base that we are converting to



# How Partitioning Works

- We partition before we multiply
- We multiply so that the first partition becomes the unit interval
  - The number we multiply by is the base that we are converting to
- So for binary, this is what we were doing:



# What this Tells Us

- Where the point was originally

# What this Tells Us

- Where the point was originally
  - The conversion of  $0.25_{10}$  to a number in binary:  $0.01_2$
  - We can check this with a beta expansion:
$$0.01_2 = 0 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2} = 1/4 = 0.25_{10}$$

# What this Tells Us

- Where the point was originally
  - The conversion of  $0.25_{10}$  to a number in binary:  $0.01_2$
  - We can check this with a beta expansion:
$$0.01_2 = 0 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2} = 1/4 = 0.25_{10}$$
- That there is a finite radix point expression in a given base

# What this Tells Us

- Where the point was originally
  - The conversion of  $0.25_{10}$  to a number in binary:  $0.01_2$
  - We can check this with a beta expansion:
$$0.01_2 = 0 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2} = 1/4 = 0.25_{10}$$
- That there is a finite radix point expression in a given base
- That our choice of partitioning and multiplication matter... this is what happens geometrically when we change bases

# What this Tells Us

- Where the point was originally
  - The conversion of  $0.25_{10}$  to a number in binary:  $0.01_2$
  - We can check this with a beta expansion:
$$0.01_2 = 0 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2} = 1/4 = 0.25_{10}$$
- That there is a finite radix point expression in a given base
- That our choice of partitioning and multiplication matter... this is what happens geometrically when we change bases
- Let's do another example

# Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)



original

# Dealing with Radix Point Expressions (Geometrically)

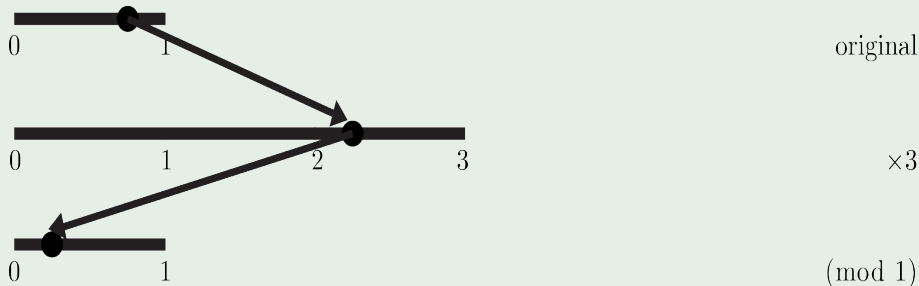
Example (Suppose you have a point...)





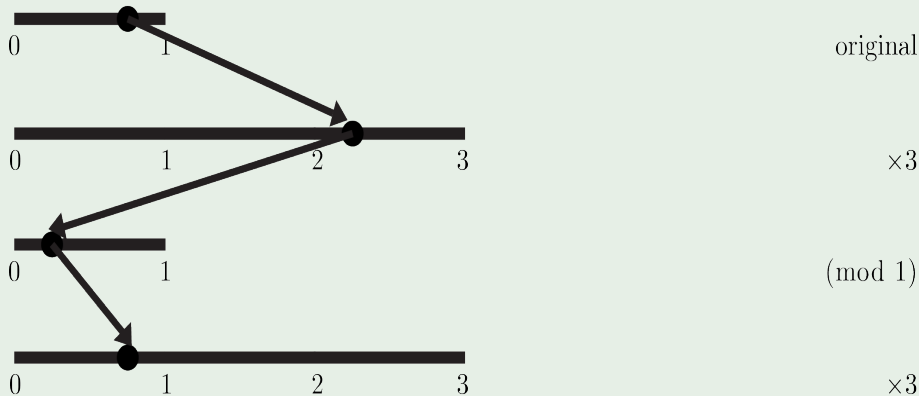
# Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)



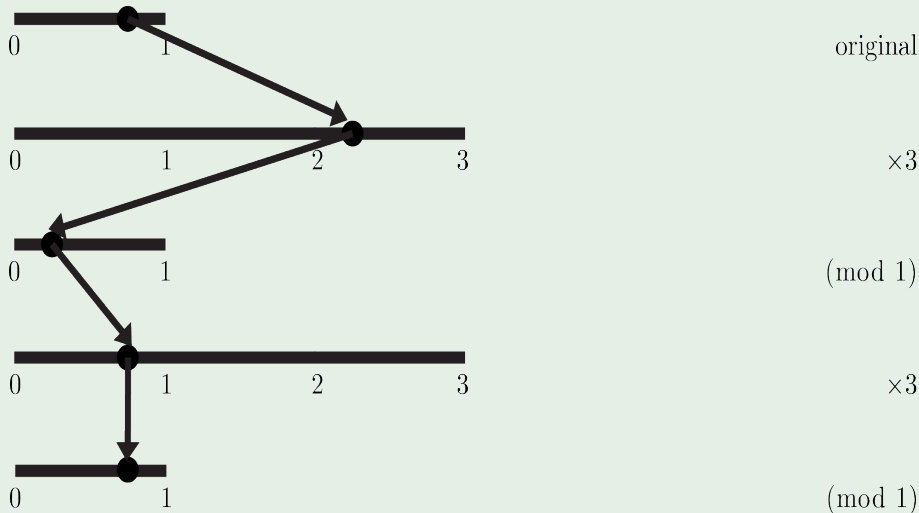
# Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)



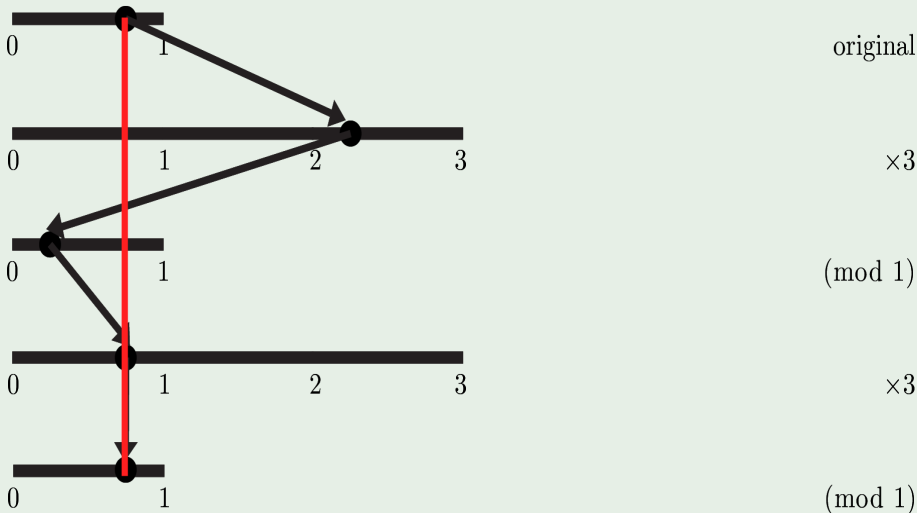
# Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)



# Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)



# What this Tells Us

- Getting stuck in this cycle means this number is not representable as a finite radix point expression in base 3

# What this Tells Us

- Getting stuck in this cycle means this number is not representable as a finite radix point expression in base 3
- In general, this can and does happen in any base

# What this Tells Us

- Getting stuck in this cycle means this number is not representable as a finite radix point expression in base 3
- In general, this can and does happen in any base
- We can explain this phenomenon arithmetically

# Theorem

## Finite Radix Point Expression



# Theorem

## Finite Radix Point Expression

A number  $p$  is said to have a finite radix point expression if

$$p = \frac{n}{m_1 m_2 \dots m_k}$$

where  $n \in \mathbb{N}$ , and  $m_1 m_2 \dots m_k$  are all the prime factors of the base raised to some power.

---

<http://mathworld.wolfram.com/DecimalExpansion.html>

# Theorem

## Finite Radix Point Expression

A number  $p$  is said to have a finite radix point expression if

$$p = \frac{n}{m_1 m_2 \dots m_k}$$

where  $n \in \mathbb{N}$ , and  $m_1 m_2 \dots m_k$  are all the prime factors of the base raised to some power.

---

<http://mathworld.wolfram.com/DecimalExpansion.html>

## Example (1/3 in Base 10)

$$\frac{1}{3} \neq \frac{n}{2^\alpha 5^\beta}$$

where  $n, \alpha, \beta \in \mathbb{N}$ .

# Dealing with Radix Point Expressions (Arithmetically)

- We can do this operation with arithmetic

# Dealing with Radix Point Expressions (Arithmetically)

- We can do this operation with arithmetic
  - We must first qualify what we were doing in our “game” previously

## Beta Map

The transformation  $T : [0, 1) \rightarrow [0, 1)$  defined by  $T = (x \cdot \beta) \pmod{1}$ .

# Dealing with Radix Point Expressions (Arithmetically)

Example (Convert  $0.1_{10}$  to binary.)

# Dealing with Radix Point Expressions (Arithmetically)

Example (Convert  $0.1_{10}$  to binary.)

$$0.1 * 2 = 0.2$$

$$0.2 \bmod (1) = 0.2$$

0.0

# Dealing with Radix Point Expressions (Arithmetically)

Example (Convert  $0.1_{10}$  to binary.)

$0.1 * 2 = 0.2$	$0.2 \bmod (1) = 0.2$	0.0
$0.2 * 2 = 0.4$	$0.4 \bmod (1) = 0.4$	0.00



# Dealing with Radix Point Expressions (Arithmetically)

Example (Convert  $0.1_{10}$  to binary.)

$0.1 * 2 = 0.2$	$0.2 \bmod (1) = 0.2$	0.0
$0.2 * 2 = 0.4$	$0.4 \bmod (1) = 0.4$	0.00
$0.4 * 2 = 0.8$	$0.8 \bmod (1) = 0.8$	0.000

# Dealing with Radix Point Expressions (Arithmetically)

Example (Convert  $0.1_{10}$  to binary.)

$0.1 * 2 = 0.2$	$0.2 \bmod (1) = 0.2$	0.0
$0.2 * 2 = 0.4$	$0.4 \bmod (1) = 0.4$	0.00
$0.4 * 2 = 0.8$	$0.8 \bmod (1) = 0.8$	0.000
$0.8 * 2 = 1.6$	$1.6 \bmod (1) = 0.6$	0.0001

# Dealing with Radix Point Expressions (Arithmetically)

## Example (Convert $0.1_{10}$ to binary.)

$0.1 * 2 = 0.2$	$0.2 \bmod (1) = 0.2$	0.0
$0.2 * 2 = 0.4$	$0.4 \bmod (1) = 0.4$	0.00
$0.4 * 2 = 0.8$	$0.8 \bmod (1) = 0.8$	0.000
$0.8 * 2 = 1.6$	$1.6 \bmod (1) = 0.6$	0.0001
$0.6 * 2 = 1.2$	$1.2 \bmod (1) = 0.2$	0.00011

# Dealing with Radix Point Expressions (Arithmetically)

## Example (Convert $0.1_{10}$ to binary.)

$0.1 * 2 = 0.2$	$0.2 \bmod (1) = 0.2$	0.0
$0.2 * 2 = 0.4$	$0.4 \bmod (1) = 0.4$	0.00
$0.4 * 2 = 0.8$	$0.8 \bmod (1) = 0.8$	0.000
$0.8 * 2 = 1.6$	$1.6 \bmod (1) = 0.6$	0.0001
$0.6 * 2 = 1.2$	$1.2 \bmod (1) = 0.2$	0.00011
$0.2 * 2 = 0.4$	$0.4 \bmod (1) = 0.4$	...

# Dealing with Radix Point Expressions (Arithmetically)

## Example (Convert $0.1_{10}$ to binary.)

$0.1 * 2 = 0.2$	$0.2 \bmod (1) = 0.2$	0.0
$0.2 * 2 = 0.4$	$0.4 \bmod (1) = 0.4$	0.00
$0.4 * 2 = 0.8$	$0.8 \bmod (1) = 0.8$	0.000
$0.8 * 2 = 1.6$	$1.6 \bmod (1) = 0.6$	0.0001
$0.6 * 2 = 1.2$	$1.2 \bmod (1) = 0.2$	0.00011
$0.2 * 2 = 0.4$	$0.4 \bmod (1) = 0.4$	...
		$0.0001\overline{1}_2$

## 1 Whole Numbers

- Definitions
- Examples
- Language Generated by Whole Number Bases

## 2 Non-Integer Algebraic Numbers

- Definitions
- Examples
- Properties of Non-Integer Bases
- Language Generated by Non-Integer Bases

## 3 Questions and Answers

# Definition

## Word

A word is a sequence of characters from an alphabet.

# Definition

## Word

A word is a sequence of characters from an alphabet.

## Example (Examples of Words)



# Definition

## Word

A word is a sequence of characters from an alphabet.

## Example (Examples of Words)

101 (Binary)

# Definition

## Word

A word is a sequence of characters from an alphabet.

## Example (Examples of Words)

101 (Binary)

99.9 (Decimal)

# Definition

## Language

A language is the set of all allowed words.

# Definition

## Language

A language is the set of all allowed words.

## Example (Examples of Languages)

# Definition

## Language

A language is the set of all allowed words.

## Example (Examples of Languages)

Binary

# Definition

## Language

A language is the set of all allowed words.

## Example (Examples of Languages)

Binary

Ternary

# Definition

## Language

A language is the set of all allowed words.

## Example (Examples of Languages)

Binary

Ternary

Numerical Bases

# Language Generated by Whole Number Bases

- A word is not allowed in a language if it contains a forbidden subword



# Language Generated by Whole Number Bases

- A word is not allowed in a language if it contains a forbidden subword
  - Acceptable words in base 10 include 9, or 5.5 – but not CAT or 5Z: they contain forbidden subwords

# Language Generated by Whole Number Bases

- A word is not allowed in a language if it contains a forbidden subword
  - Acceptable words in base 10 include 9, or 5.5 – but not CAT or 5Z: they contain forbidden subwords
  - Acceptable words in base 2 include 1, or 0.110000 – but not 3, or 4: they contain forbidden subwords

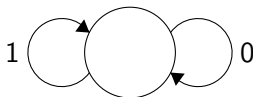
# Language Generated by Whole Number Bases

- A word is not allowed in a language if it contains a forbidden subword
  - Acceptable words in base 10 include 9, or 5.5 – but not CAT or 5Z: they contain forbidden subwords
  - Acceptable words in base 2 include 1, or 0.110000 – but not 3, or 4: they contain forbidden subwords
- How can we represent the language generated by these bases?

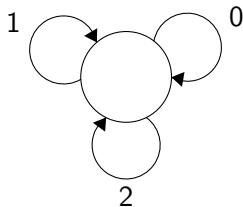
## Finite State Machine

A finite state machine (FSM) is a mathematical structure that allows us to imagine a machine with no memory.

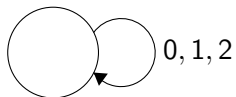
# FSM for Binary



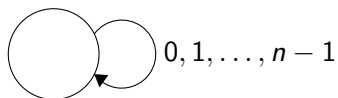
# FSM for Ternary



# Simplified FSM for Ternary



# FSM for Whole Number Bases



- $n \in \mathbb{N}, n > 1$



# Overview

## 1 Whole Numbers

- Definitions
- Examples
- Language Generated by Whole Number Bases

## 2 Non-Integer Algebraic Numbers

- Definitions
- Examples
- Properties of Non-Integer Bases
- Language Generated by Non-Integer Bases

## 3 Questions and Answers

## Maximum Allowed Numeral in Base

If the radix is a whole number, the previous definition holds.

If the radix is not a whole number, then the maximum allowed numeral is the floor function of the base.

# Definition

## Maximum Allowed Numeral in Base

If the radix is a whole number, the previous definition holds.

If the radix is not a whole number, then the maximum allowed numeral is the floor function of the base.

## Example (Allowed Numbers in Different Bases)

# Definition

## Maximum Allowed Numeral in Base

If the radix is a whole number, the previous definition holds.

If the radix is not a whole number, then the maximum allowed numeral is the floor function of the base.

## Example (Allowed Numbers in Different Bases)

$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$ , maximum numeral is 1

# Definition

## Maximum Allowed Numeral in Base

If the radix is a whole number, the previous definition holds.

If the radix is not a whole number, then the maximum allowed numeral is the floor function of the base.

## Example (Allowed Numbers in Different Bases)

$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$ , maximum numeral is 1

$\alpha = 1 + \sqrt{2} \approx 2.14$ , maximum numeral is 2

# Definition

## Algebraic Integer

A number  $x$  is an algebraic integer if it is the root of a monic polynomial  $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ , where  $a_{n-1}, \dots, a_0 \in \mathbb{Z}$ .

## Example (Examples of Algebraic Polynomials)

# Definition

## Algebraic Integer

A number  $x$  is an algebraic integer if it is the root of a monic polynomial  $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ , where  $a_{n-1}, \dots, a_0 \in \mathbb{Z}$ .

## Example (Examples of Algebraic Polynomials)

$$\varphi^2 - \varphi - 1 = 0, \quad x = \frac{1 \pm \sqrt{5}}{2}$$

# Definition

## Algebraic Integer

A number  $x$  is an algebraic integer if it is the root of a monic polynomial  $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ , where  $a_{n-1}, \dots, a_0 \in \mathbb{Z}$ .

## Example (Examples of Algebraic Polynomials)

$$\varphi^2 - \varphi - 1 = 0, \quad x = \frac{1 \pm \sqrt{5}}{2}$$

$$\alpha^2 - 2\alpha - 1 = 0, \quad x = 1 \pm \sqrt{2}$$



# Definition

## Algebraic Integer

A number  $x$  is an algebraic integer if it is the root of a monic polynomial  $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ , where  $a_{n-1}, \dots, a_0 \in \mathbb{Z}$ .

## Example (Examples of Algebraic Polynomials)

$$\varphi^2 - \varphi - 1 = 0, \quad x = \frac{1 \pm \sqrt{5}}{2}$$

$$\alpha^2 - 2\alpha - 1 = 0, \quad x = 1 \pm \sqrt{2}$$

$$x^3 - x - 1 = 0, \quad x = \frac{1}{3} \sqrt[3]{\frac{27 - 3\sqrt{69}}{2}} + \frac{\sqrt[3]{\frac{1}{2}(9 + \sqrt{69})}}{3^{2/3}}$$

# Partitioning Differently

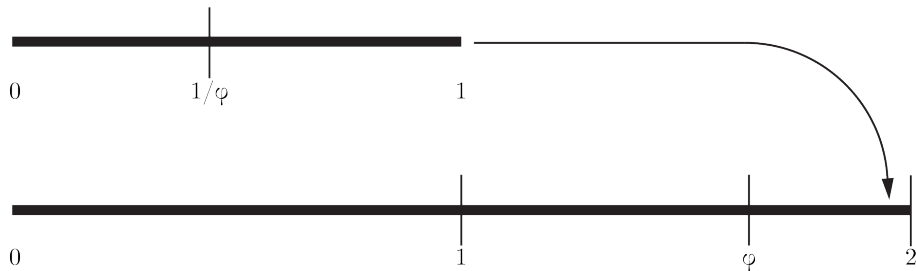
- We don't have to use a whole number

# Partitioning Differently

- We don't have to use a whole number
- The partitions won't be even, but that's not a problem

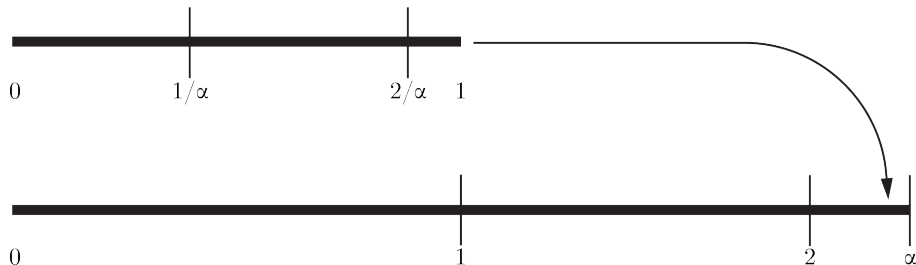
# Partitioning Differently

- We don't have to use a whole number
- The partitions won't be even, but that's not a problem



# Partitioning Differently

- We don't have to use a whole number
- The partitions won't be even, but that's not a problem



# Overview

## 1 Whole Numbers

- Definitions
- Examples
- Language Generated by Whole Number Bases

## 2 Non-Integer Algebraic Numbers

- Definitions
- **Examples**
- Properties of Non-Integer Bases
- Language Generated by Non-Integer Bases

## 3 Questions and Answers

# Examples of Non-Integer Bases

- Base Golden Ratio
- Base Silver Ratio

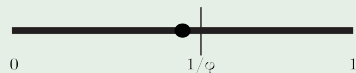
## Golden Ratio

The positive solution to the equation  $\varphi^2 = \varphi + 1$ .  
 $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$ .



# Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)



original

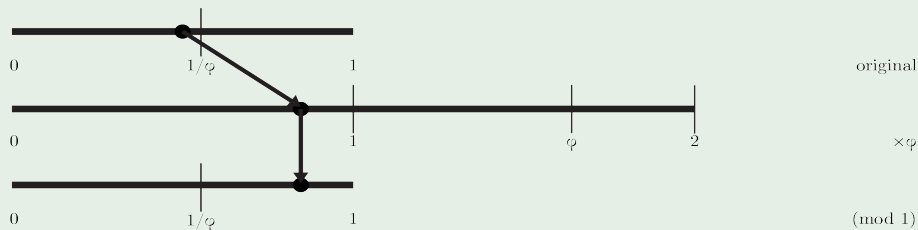
# Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)



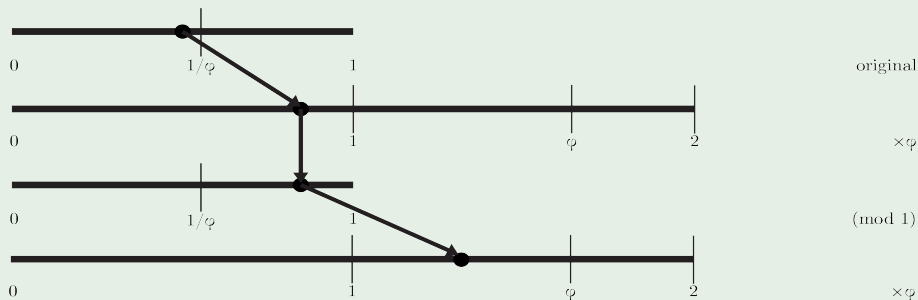
# Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)



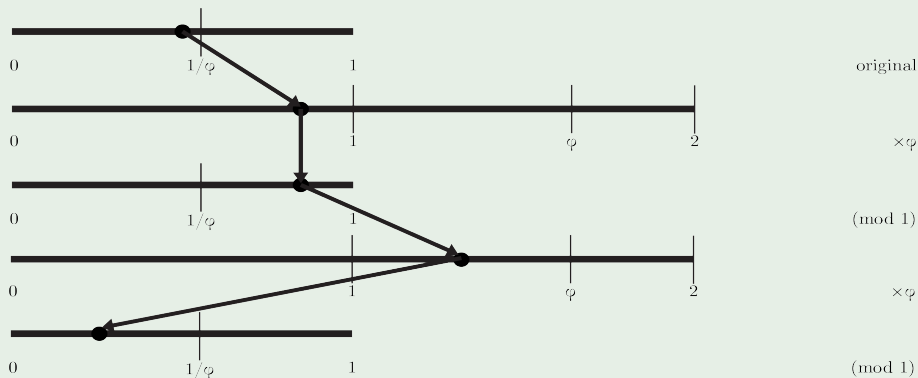
# Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)



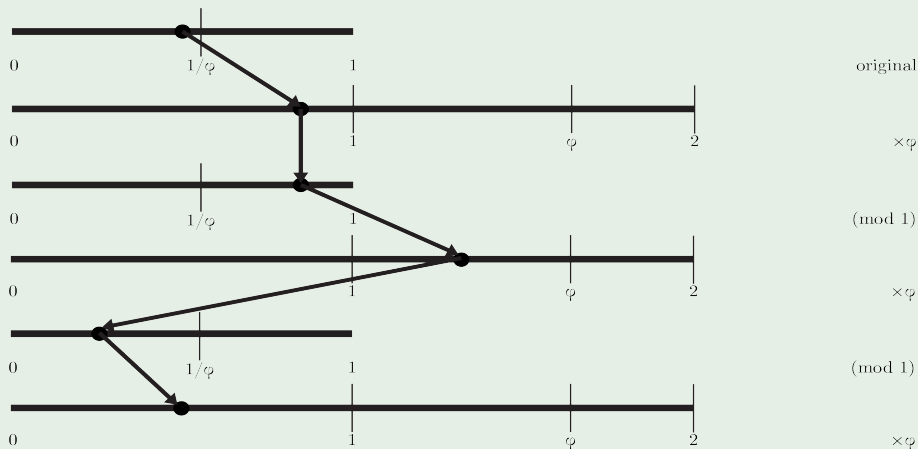
# Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)



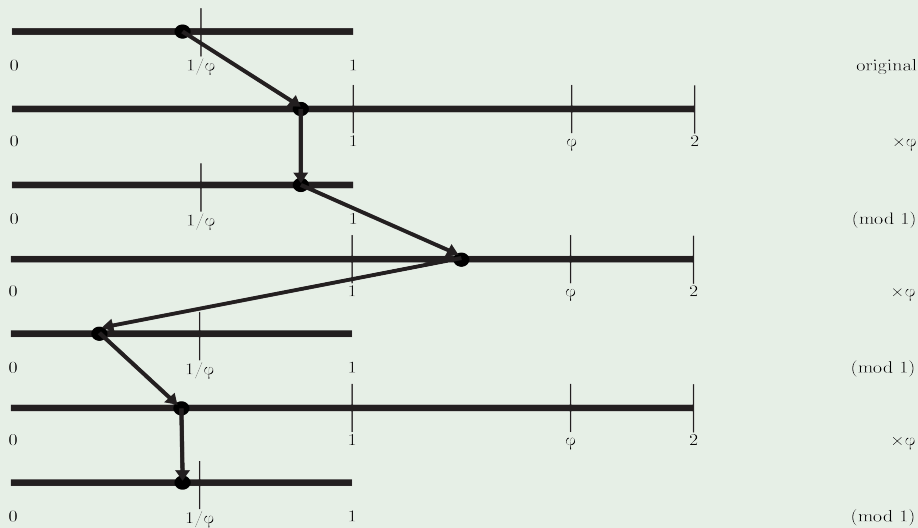
# Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)



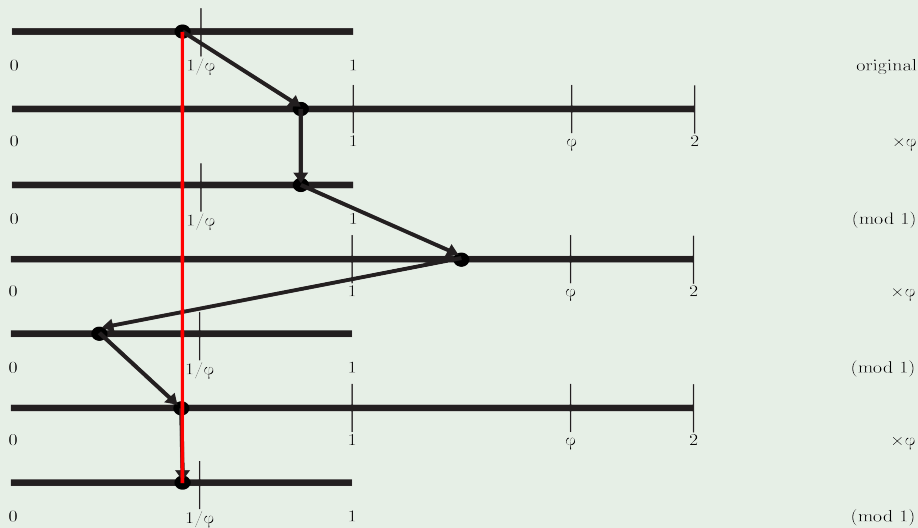
# Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)



# Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)





# What this Tells Us

- Getting stuck in this cycle means this number is not representable as a finite radix point expression in base golden ratio

# What this Tells Us

- Getting stuck in this cycle means this number is not representable as a finite radix point expression in base golden ratio
  - $0.5_{10} = 0.\overline{010}_\varphi$

# What this Tells Us

- Getting stuck in this cycle means this number is not representable as a finite radix point expression in base golden ratio
  - $0.5_{10} = 0.\overline{010}_\varphi$
- In general, whole numbers (excluding the base and its multiples) are typically the only numbers with finite representations in non-integer, algebraic bases

# What this Tells Us

- Getting stuck in this cycle means this number is not representable as a finite radix point expression in base golden ratio
  - $0.5_{10} = 0.\overline{010}_\varphi$
- In general, whole numbers (excluding the base and its multiples) are typically the only numbers with finite representations in non-integer, algebraic bases
- Now, an arithmetic example

# Dealing with Radix Point Expressions (Arithmetically)

Example (Convert  $(\varphi/2)_{10}$  to base golden ratio.)

# Dealing with Radix Point Expressions (Arithmetically)

Example (Convert  $(\varphi/2)_{10}$  to base golden ratio.)

$$\frac{\varphi}{2} \cdot \varphi = \frac{\varphi^2}{2} \quad \frac{\varphi^2}{2} \bmod (1) = \frac{\varphi^2}{2} - 1 \quad 0.1$$

# Dealing with Radix Point Expressions (Arithmetically)

Example (Convert  $(\varphi/2)_{10}$  to base golden ratio.)

$$\frac{\varphi}{2} \cdot \varphi = \frac{\varphi^2}{2} \quad \frac{\varphi^2}{2} \bmod (1) = \frac{\varphi^2}{2} - 1 \quad 0.1$$

$$\left(\frac{\varphi^2}{2} - 1\right) \cdot \varphi = \frac{\varphi^3}{2} - \varphi \quad \frac{\varphi^3}{2} - \varphi \bmod (1) = \frac{\varphi^3}{2} - \varphi \quad 0.10$$

# Dealing with Radix Point Expressions (Arithmetically)

Example (Convert  $(\varphi/2)_{10}$  to base golden ratio.)

$$\frac{\varphi}{2} \cdot \varphi = \frac{\varphi^2}{2} \quad \frac{\varphi^2}{2} \bmod (1) = \frac{\varphi^2}{2} - 1 \quad 0.1$$

$$\left(\frac{\varphi^2}{2} - 1\right) \cdot \varphi = \frac{\varphi^3}{2} - \varphi \quad \frac{\varphi^3}{2} - \varphi \bmod (1) = \frac{\varphi^3}{2} - \varphi \quad 0.10$$

$$\left(\frac{\varphi^3}{2} - \varphi\right) \cdot \varphi = \frac{\varphi^4}{2} - \varphi^2 \quad \frac{\varphi^4}{2} - \varphi^2 \bmod (1) = \frac{\varphi^4}{2} - \varphi^2 \quad 0.100$$



# Dealing with Radix Point Expressions (Arithmetically)

Example (Convert  $(\varphi/2)_{10}$  to base golden ratio.)

$$\frac{\varphi}{2} \cdot \varphi = \frac{\varphi^2}{2} \quad \frac{\varphi^2}{2} \bmod (1) = \frac{\varphi^2}{2} - 1 \quad 0.1$$

$$\left(\frac{\varphi^2}{2} - 1\right) \cdot \varphi = \frac{\varphi^3}{2} - \varphi \quad \frac{\varphi^3}{2} - \varphi \bmod (1) = \frac{\varphi^3}{2} - \varphi \quad 0.10$$

$$\left(\frac{\varphi^3}{2} - \varphi\right) \cdot \varphi = \frac{\varphi^4}{2} - \varphi^2 \quad \frac{\varphi^4}{2} - \varphi^2 \bmod (1) = \frac{\varphi^4}{2} - \varphi^2 \quad 0.100$$

$$\frac{\varphi^4}{2} - \varphi^2 = \frac{\varphi^2(\varphi^2 - 2)}{2} = \frac{(\varphi + 1)(\varphi - 1)}{2} = \frac{\varphi^2 - 1}{2}$$

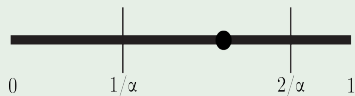
$$0.\overline{100}_{\varphi}$$

## Silver Ratio

The positive solution to the equation  $\alpha^2 = 2\alpha + 1$ .  
 $\alpha = 1 + \sqrt{2} \approx 2.414$ .

# Dealing with Radix Point Expressions (Geometrically)

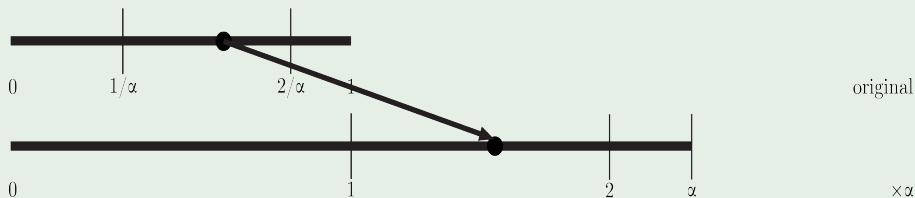
Example (Suppose you have a point...)



original

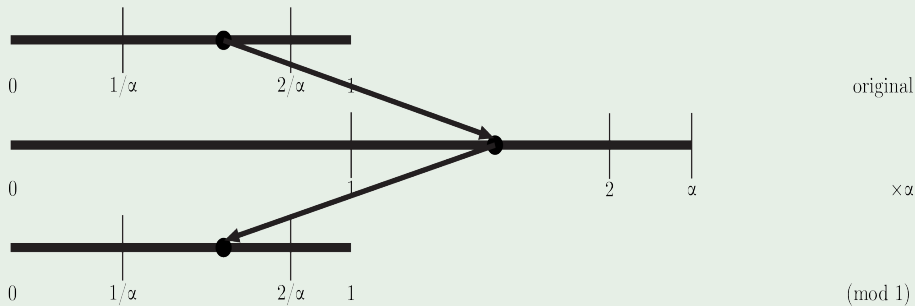
# Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)



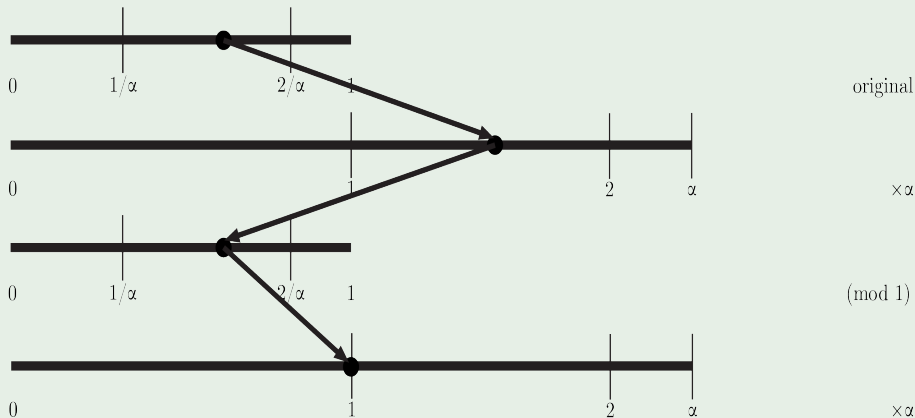
# Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)



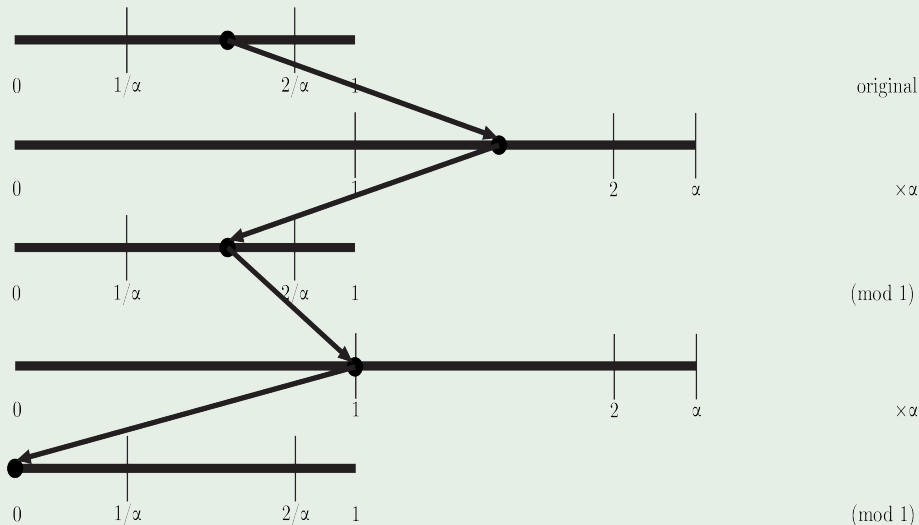
# Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)



# Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)



# Overview

## 1 Whole Numbers

- Definitions
- Examples
- Language Generated by Whole Number Bases

## 2 Non-Integer Algebraic Numbers

- Definitions
- Examples
- **Properties of Non-Integer Bases**
- Language Generated by Non-Integer Bases

## 3 Questions and Answers



# Properties of Non-Integer Bases

- There will be forbidden words

# Properties of Non-Integer Bases

- There will be forbidden words
- It is easy/interesting to represent certain numbers in these bases

# Properties of Non-Integer Bases

- There will be forbidden words
- It is easy/interesting to represent certain numbers in these bases
- Arithmetic is more of a pain than usual

## Simplest Form

A number is in simplest form if we are unable to rewrite the beta expansion such that it includes a higher power.

# Example of Simplest Form

## Simplest Form

Let

$$\varphi^2 = \varphi + 1.$$

# Example of Simplest Form

## Simplest Form

Let

$$\varphi^2 = \varphi + 1.$$

Using the positive solution of that equation,  $\varphi \approx 1.618$ , we can show that  $0.11_\varphi = 1.00_\varphi$  by using a beta expansion.

# Example of Simplest Form

## Simplest Form

Let

$$\varphi^2 = \varphi + 1.$$

Using the positive solution of that equation,  $\varphi \approx 1.618$ , we can show that  $0.11_\varphi = 1.00_\varphi$  by using a beta expansion.

$$0.11_\varphi = 1 \times \varphi^{-1} + 1 \times \varphi^{-2} = \frac{1}{\varphi} + \frac{1}{\varphi^2}.$$

# Example of Simplest Form

## Simplest Form

Let

$$\varphi^2 = \varphi + 1.$$

Using the positive solution of that equation,  $\varphi \approx 1.618$ , we can show that  $0.11_\varphi = 1.00_\varphi$  by using a beta expansion.

$$0.11_\varphi = 1 \times \varphi^{-1} + 1 \times \varphi^{-2} = \frac{1}{\varphi} + \frac{1}{\varphi^2}.$$

Then,

$$\frac{\varphi}{\varphi^2} + \frac{1}{\varphi^2} = \frac{\varphi + 1}{\varphi^2},$$



# Example of Simplest Form

## Simplest Form

Let

$$\varphi^2 = \varphi + 1.$$

Using the positive solution of that equation,  $\varphi \approx 1.618$ , we can show that  $0.11_\varphi = 1.00_\varphi$  by using a beta expansion.

$$0.11_\varphi = 1 \times \varphi^{-1} + 1 \times \varphi^{-2} = \frac{1}{\varphi} + \frac{1}{\varphi^2}.$$

Then,

$$\frac{\varphi}{\varphi^2} + \frac{1}{\varphi^2} = \frac{\varphi + 1}{\varphi^2},$$

and by the definition of the base, we can rewrite the numerator:

$$\frac{\varphi^2}{\varphi^2} = 1_\varphi$$

# Golden Ratio Addition

Example  $(1_\varphi + 1_\varphi)$

# Golden Ratio Addition

Example ( $1_\varphi + 1_\varphi$ )

$$1_\varphi = 0.11_\varphi.$$

# Golden Ratio Addition

## Example ( $1_\varphi + 1_\varphi$ )

$$1_\varphi = 0.11_\varphi.$$

$$1_\varphi + 0.11_\varphi = 1.11_\varphi = 10.01_\varphi.$$

## Example ( $1_\varphi + 1_\varphi$ )

$$1_\varphi = 0.11_\varphi.$$

$$1_\varphi + 0.11_\varphi = 1.11_\varphi = 10.01_\varphi.$$

We can double check this operation by doing a beta expansion and using the definition of the base.

# Golden Ratio Addition

## Example ( $1_\varphi + 1_\varphi$ )

$$1_\varphi = 0.11_\varphi.$$

$$1_\varphi + 0.11_\varphi = 1.11_\varphi = 10.01_\varphi.$$

We can double check this operation by doing a beta expansion and using the definition of the base.

$$10.01_\varphi = 1 \times \varphi^1 + 0 \times \varphi^0 + 0 \times \varphi^{-1} + 1 \times \varphi^{-2} = \varphi + 1/\varphi^2 = (\varphi^2 + 1)/\varphi^2 = 2_{10}.$$

## Example ( $1_\varphi + 1_\varphi$ )

$$1_\varphi = 0.11_\varphi.$$

$$1_\varphi + 0.11_\varphi = 1.11_\varphi = 10.01_\varphi.$$

We can double check this operation by doing a beta expansion and using the definition of the base.

$$10.01_\varphi = 1 \times \varphi^1 + 0 \times \varphi^0 + 0 \times \varphi^{-1} + 1 \times \varphi^{-2} = \varphi + 1/\varphi^2 = (\varphi^2 + 1)/\varphi^2 = 2_{10}.$$

We can calculate any whole number through this method.

# Overview

## 1 Whole Numbers

- Definitions
- Examples
- Language Generated by Whole Number Bases

## 2 Non-Integer Algebraic Numbers

- Definitions
- Examples
- Properties of Non-Integer Bases
- Language Generated by Non-Integer Bases

## 3 Questions and Answers



# Example of Simplest Form

## Simplest Form

Let

$$\varphi^2 = \varphi + 1.$$

Using the positive solution of that equation,  $\varphi \approx 1.618$ , we can show that  $0.11_\varphi = 1.00_\varphi$  by using a beta expansion.

$$0.11_\varphi = 1 \times \varphi^{-1} + 1 \times \varphi^{-2} = \frac{1}{\varphi} + \frac{1}{\varphi^2}.$$

Then,

$$\frac{\varphi}{\varphi^2} + \frac{1}{\varphi^2} = \frac{\varphi + 1}{\varphi^2},$$

and by the definition of the base, we can rewrite the numerator:

$$\frac{\varphi^2}{\varphi^2} = 1_\varphi$$

# What Makes a Word Forbidden in Non-Integer Bases?

- In general, a word is forbidden if it is not in simplest form

# What Makes a Word Forbidden in Non-Integer Bases?

- In general, a word is forbidden if it is not in simplest form
- In actuality, it is because it escapes the range of the beta map

# Escaping the Beta Map

Example (Golden Base Ratio:  $T(0.01101_\varphi)$ )

Using our previous example:

$$T(0.01101_\varphi) = 0.1101_\varphi = 1.0001_\varphi.$$

# Escaping the Beta Map

Example (Golden Base Ratio:  $T(0.01101_\varphi)$ )

Using our previous example:

$$T(0.01101_\varphi) = 0.1101_\varphi = 1.0001_\varphi.$$

But that can't happen! By definition,  $T : [0, 1) \rightarrow [0, 1)$ . As such, we say that this is a forbidden word.

# Escaping the Beta Map

Example (Golden Base Ratio:  $T(0.01101_\varphi)$ )

Using our previous example:

$$T(0.01101_\varphi) = 0.1101_\varphi = 1.0001_\varphi.$$

But that can't happen! By definition,  $T : [0, 1) \rightarrow [0, 1)$ . As such, we say that this is a forbidden word.

Geometrically, this can't happen. If we multiply a number in  $[0, 1)$  by any number and perform (mod 1), we will never get a number larger than one. It's the arithmetic equivalent of having  $x \cdot \beta > \beta$ ,  $x < 1$ .

## Orbit

The unique, non-terminating radix point representation of a number in any base.

# Definition

## Orbit

The unique, non-terminating radix point representation of a number in any base.

## Example (Orbit of One in Different Bases)



# Definition

## Orbit

The unique, non-terminating radix point representation of a number in any base.

## Example (Orbit of One in Different Bases)

$$1_{10} = 0.\overline{99}_{10}$$

# Definition

## Orbit

The unique, non-terminating radix point representation of a number in any base.

## Example (Orbit of One in Different Bases)

$$1_{10} = 0.\overline{99}_{10}$$

$$1_9 = 0.\overline{88}_9$$

# Definition

## Orbit

The unique, non-terminating radix point representation of a number in any base.

## Example (Orbit of One in Different Bases)

$$1_{10} = 0.\overline{99}_{10}$$

$$1_9 = 0.\overline{88}_9$$

$$1_2 = 0.\overline{11}_2$$

# Finding Forbidden Words

- Take the orbit of one in the base

# Finding Forbidden Words

- Take the orbit of one in the base
- Find the period of the orbit and take one segment

# Finding Forbidden Words

- Take the orbit of one in the base
- Find the period of the orbit and take one segment
  - Any sequence of numerals greater than the numerals of the period is not allowed

# Finding Forbidden Words

- The orbit of one is composed of the maximum allowed numeral in a language

# Finding Forbidden Words

- The orbit of one is composed of the maximum allowed numeral in a language
- All the forbidden words then will subwords of the orbit, with one added until the word is the maximum numeral possible



# Language Generated by base Golden Ratio

- $0.\overline{10}_\varphi$  is the orbit of one in base golden ratio

# Language Generated by base Golden Ratio

- $0.\overline{10}_\varphi$  is the orbit of one in base golden ratio
- Since the maximum allowed numeral in the language is 10, we add one and find that the forbidden word is 11

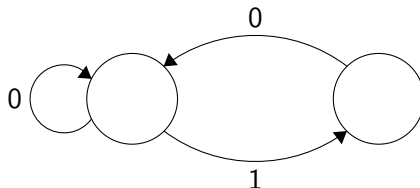
# Language Generated by base Golden Ratio

- $0.\overline{10}_\varphi$  is the orbit of one in base golden ratio
- Since the maximum allowed numeral in the language is 10, we add one and find that the forbidden word is 11
- There can be more than one forbidden word, but since the maximum allowed numeral in base golden ratio is 1, we can't add one again

# Language Generated by base Golden Ratio

- $0.\overline{10}_\varphi$  is the orbit of one in base golden ratio
- Since the maximum allowed numeral in the language is 10, we add one and find that the forbidden word is 11
- There can be more than one forbidden word, but since the maximum allowed numeral in base golden ratio is 1, we can't add one again
- With this in mind, we can now create a finite state machine

# FSM For Language Generated by Base Golden Ratio



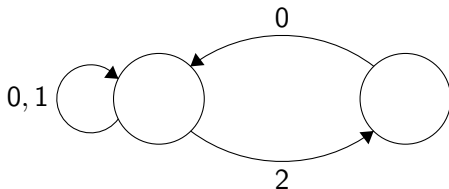
# Language Generated by base Silver Ratio

- $0.\overline{20}_\alpha$  is the orbit of one in base silver ratio

# Language Generated by base Silver Ratio

- $0.\overline{20}_\alpha$  is the orbit of one in base silver ratio
- Therefore, 21 and 22 are forbidden words in the language

# FSM For Language Generated by Base Silver Ratio





- Positive solution of the monic polynomial  $x^2 = 5x + 5$

# Language Generated by Base $\frac{5+(3\sqrt{5})}{2}$

- Positive solution of the monic polynomial  $x^2 = 5x + 5$
- Base is  $\approx 5.85$ , so it's equivalent to base 5

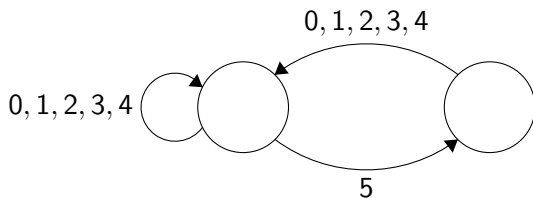
# Language Generated by Base $\frac{5+(3\sqrt{5})}{2}$

- Positive solution of the monic polynomial  $x^2 = 5x + 5$
- Base is  $\approx 5.85$ , so it's equivalent to base 5
- $0.\overline{54}_\beta$  is the orbit of one

# Language Generated by Base $\frac{5+(3\sqrt{5})}{2}$

- Positive solution of the monic polynomial  $x^2 = 5x + 5$
- Base is  $\approx 5.85$ , so it's equivalent to base 5
- $0.\overline{54}_\beta$  is the orbit of one
- Therefore, 55 is the forbidden word in the language

# FSM For Language Generated by Base $\frac{5+(3\sqrt{5})}{2}$



# Questions?