

Numbers in Non-Integer Bases

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1 Whole Numbers

- Definitions
- Examples
- Language Generated by Whole Number Bases

2 Non-Integer Algebraic Numbers

- Definitions
- Examples
- Properties of Non-Integer Bases
- Language Generated by Non-Integer Bases

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Radix Point

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$$20_2 = 2 \times 10^1 + 0 \times 2^0$$

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- Suppose that you're given the interval $[0, 1]$, with a point somewhere on it

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 - Once you choose a multiplier, you must use it for the duration
- We keep track of the interval that it falls in after the operation

Dealing with Radix Point Expressions (Geometrically)

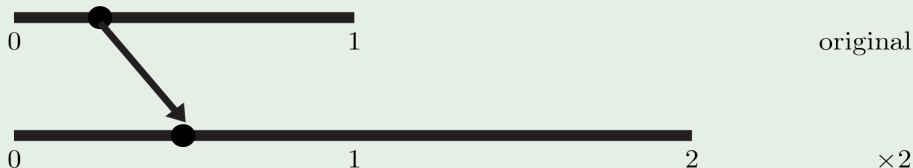
Example (Suppose you have a point...)



original

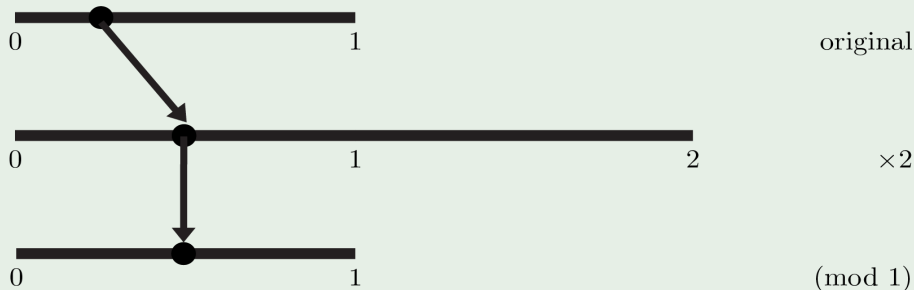
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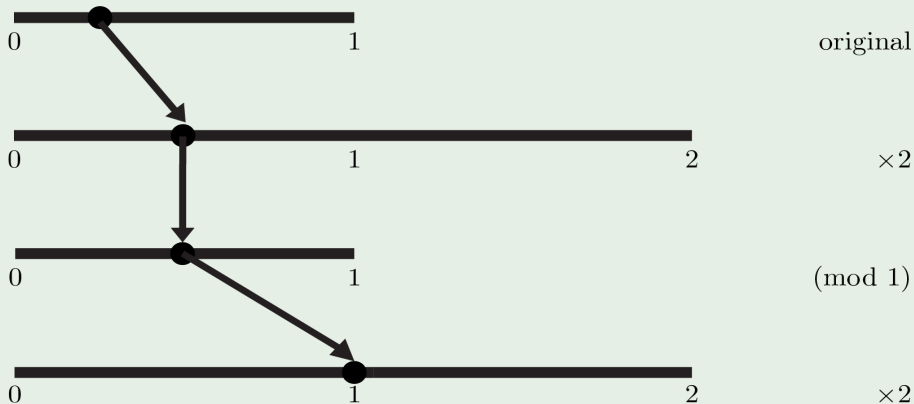
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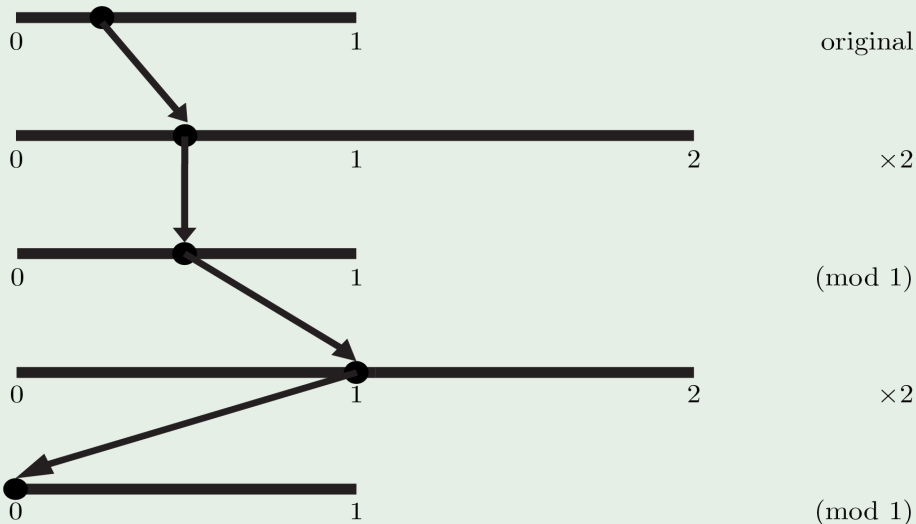
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How Partitioning Works

- We partition before we multiply

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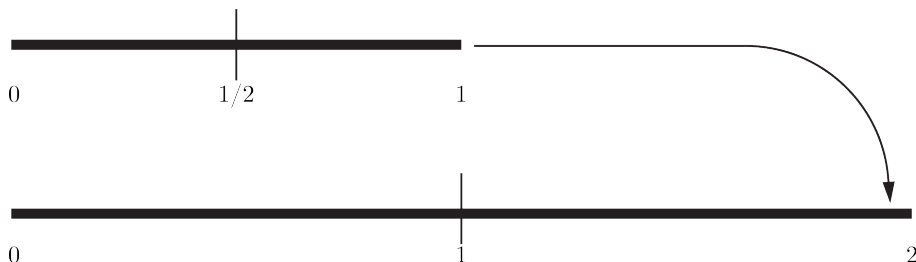
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- So for binary, this is what we were doing:



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 - The conversion of 0.25_{10} to a number in binary: 0.01_2
 - We can check this with a beta expansion:
$$0.01_2 = 0 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2} = 1/4 = 0.25_{10}$$

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- That there is a finite radix point expression in a given base
- That our choice of partitioning and multiplication matter... this is what happens geometrically when we change bases
- Let's do another example

Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)



original

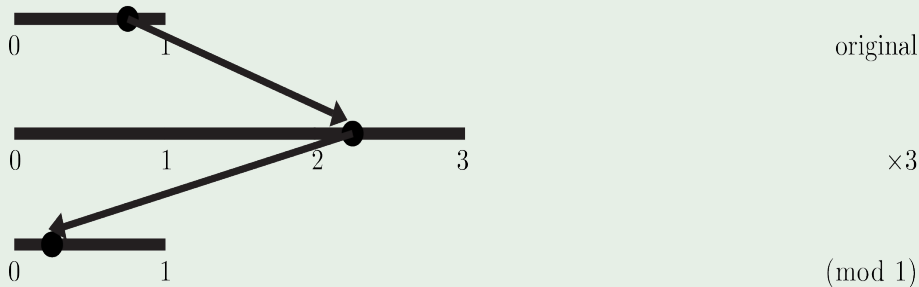
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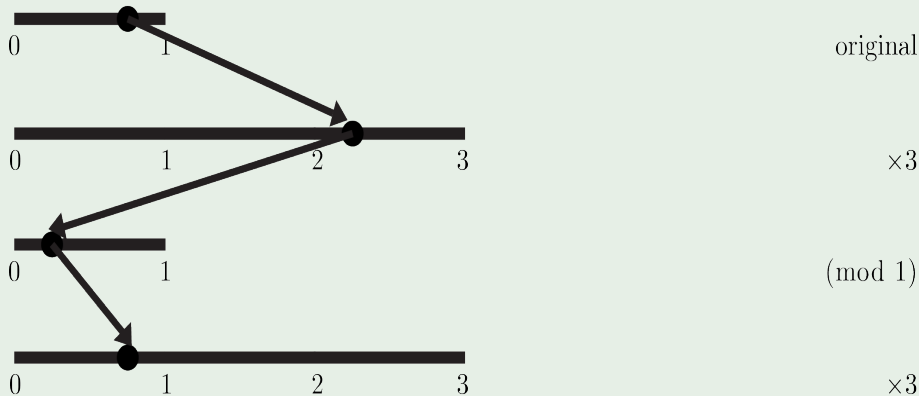
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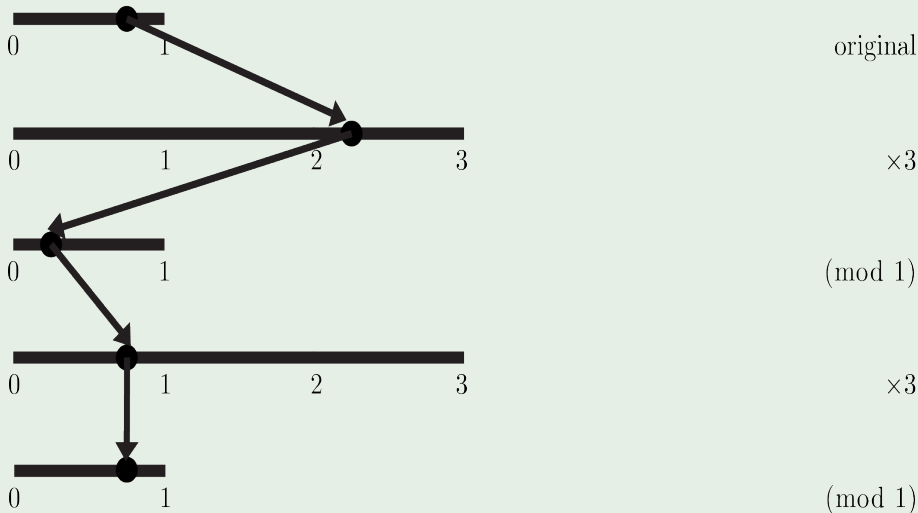
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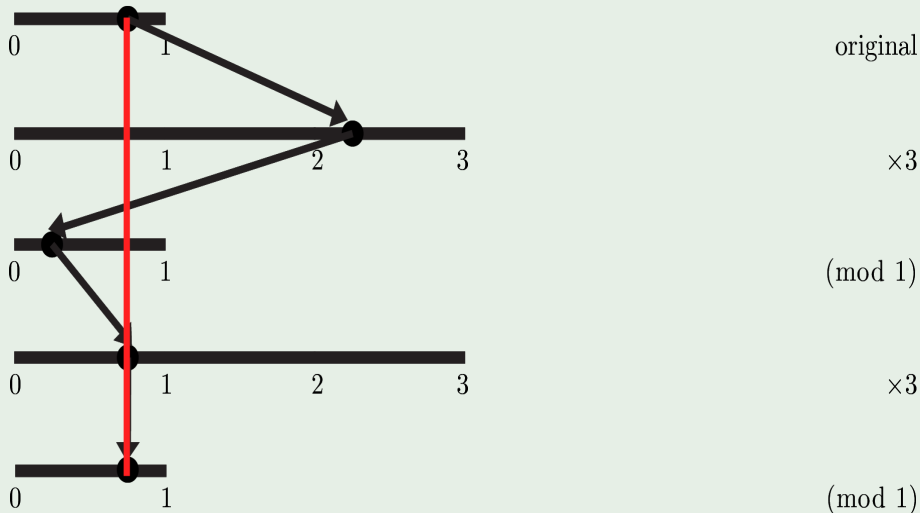
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- In general, this can and does happen in any base
- We can explain this phenomenon arithmetically

Theorem

Finite Radix Point Expression

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A number p is said to have a finite radix point expression if

$$p = \frac{n}{m_1 m_2 \dots m_k}$$

where $n \in \mathbb{N}$, and $m_1 m_2 \dots m_k$ are all the prime factors of the base raised to some power.

<http://mathworld.wolfram.com/DecimalExpansion.html>

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Example (1/3 in Base 10)

$$\frac{1}{3} \neq \frac{n}{2^\alpha 5^\beta}$$

where $n, \alpha, \beta \in \mathbb{N}$.

Dealing with Radix Point Expressions (Arithmetically)

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- We can do this operation with arithmetic
 - We must first qualify what we were doing in our “game” previously

Beta Map

The transformation $T : [0, 1) \rightarrow [0, 1)$ defined by $T = (x \cdot \beta) \pmod{1}$.

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Example (Convert 0.1_{10} to binary.)

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$$0.2 \bmod (1) = 0.2$$

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Dealing with Radix Point Expressions (Arithmetically)

Example (Convert 0.1_{10} to binary.)

$0.1 * 2 = 0.2$	$0.2 \bmod (1) = 0.2$	0.0
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$0.4 * 2 = 0.8$	$0.8 \bmod (1) = 0.8$	0.000
$0.8 * 2 = 1.6$	$1.6 \bmod (1) = 0.6$	0.0001

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$0.8 * 2 = 1.6$	$1.6 \bmod (1) = 0.6$	0.0001
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		$0.0001\overline{11}_2$

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Example (Examples of Words)

101 (Binary)

99.9 (Decimal)

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Language

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Numerical Bases

Language Generated by Whole Number Bases

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 - Acceptable words in base 2 include 1, or 0.110000 – but not 3, or 4: they contain forbidden subwords

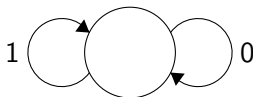
Language Generated by Whole Number Bases

- A word is not allowed in a language if it contains a forbidden subword
 - Acceptable words in base 10 include 9, or 5.5 – but not CAT or 5Z: they contain forbidden subwords
 - Acceptable words in base 2 include 1, or 0.110000 – but not 3, or 4: they contain forbidden subwords
- How can we represent the language generated by these bases?

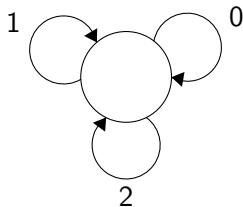
Finite State Machine

A finite state machine (FSM) is a mathematical structure that allows us to imagine a machine with no memory.

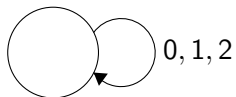
FSM for Binary



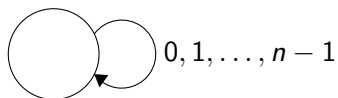
FSM for Ternary



Simplified FSM for Ternary



FSM for Whole Number Bases



- $n \in \mathbb{N}, n > 1$

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If the radix is a whole number, the previous definition holds.

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Example (Allowed Numbers in Different Bases)

$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$, maximum numeral is 1

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$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$, maximum numeral is 1

$\alpha = 1 + \sqrt{2} \approx 2.14$, maximum numeral is 2

Definition

Algebraic Integer

A number x is an algebraic integer if it is the root of a monic polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where $a_{n-1}, \dots, a_0 \in \mathbb{Z}$.

Example (Examples of Algebraic Polynomials)

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$$\varphi^2 - \varphi - 1 = 0, \quad x = \frac{1 \pm \sqrt{5}}{2}$$

$$\alpha^2 - 2\alpha - 1 = 0, \quad x = 1 \pm \sqrt{2}$$

$$x^3 - x - 1 = 0, \quad x = \frac{1}{3} \sqrt[3]{\frac{27 - 3\sqrt{69}}{2}} + \frac{\sqrt[3]{\frac{1}{2}(9 + \sqrt{69})}}{3^{2/3}}$$

Partitioning Differently

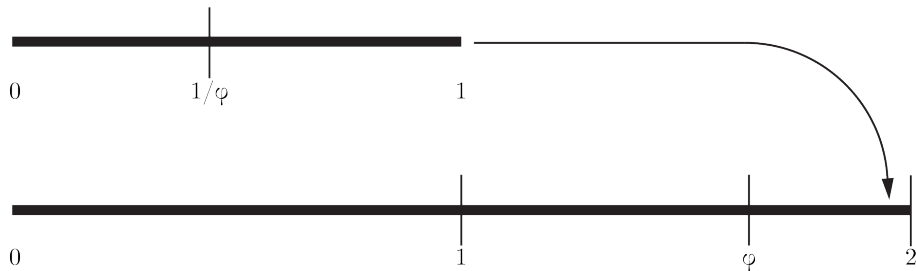
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Partitioning Differently

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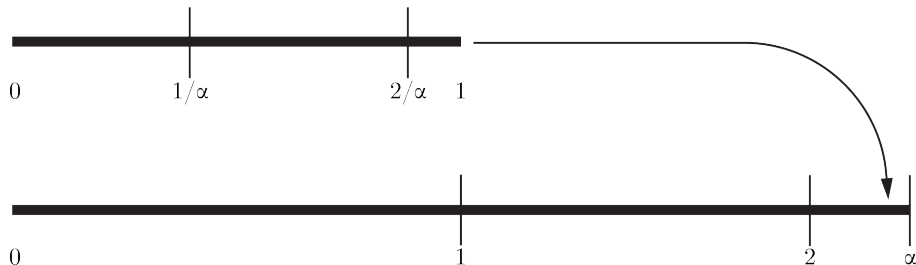
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Examples of Non-Integer Bases

- Base Golden Ratio
- Base Silver Ratio

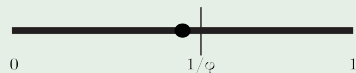
Golden Ratio

The positive solution to the equation $\varphi^2 = \varphi + 1$.

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618.$$

Dealing with Radix Point Expressions (Geometrically)

Example (Suppose you have a point...)



original

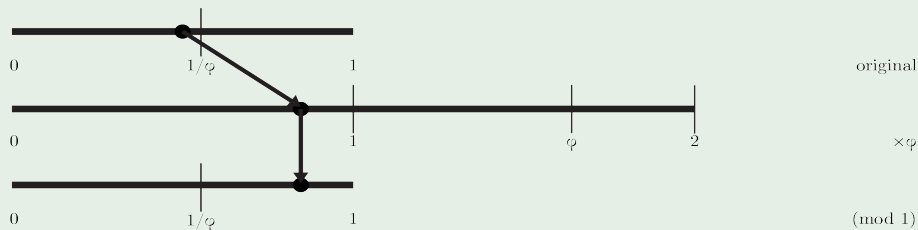
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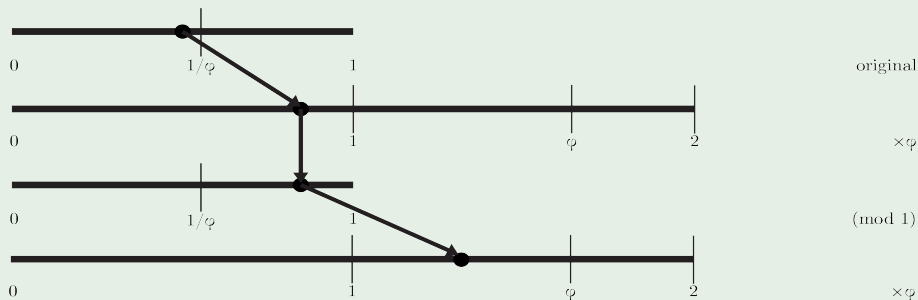
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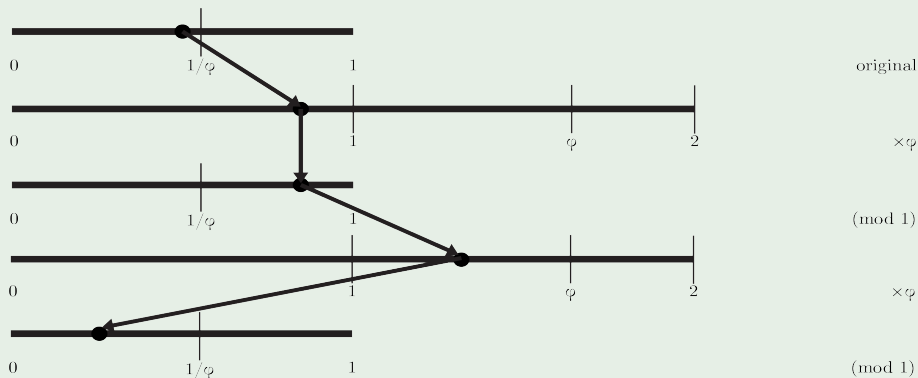
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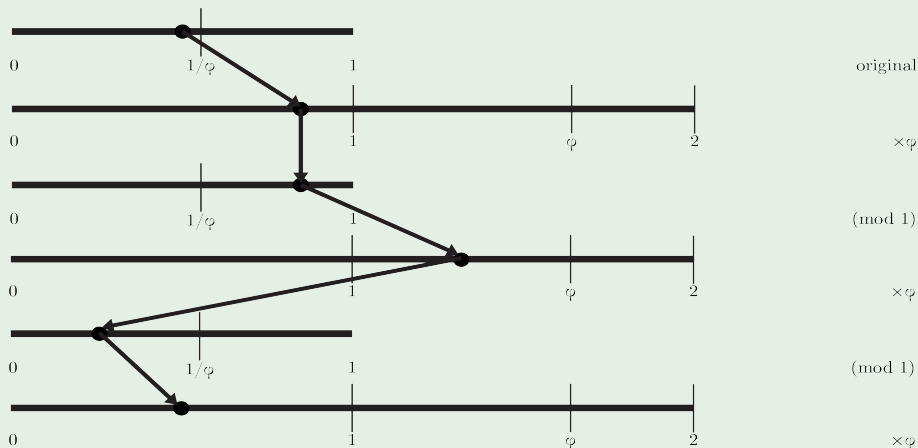
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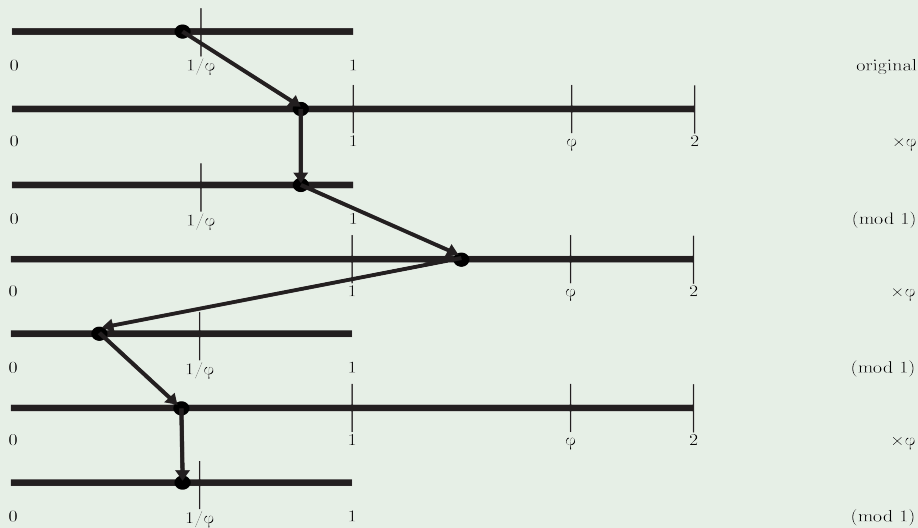
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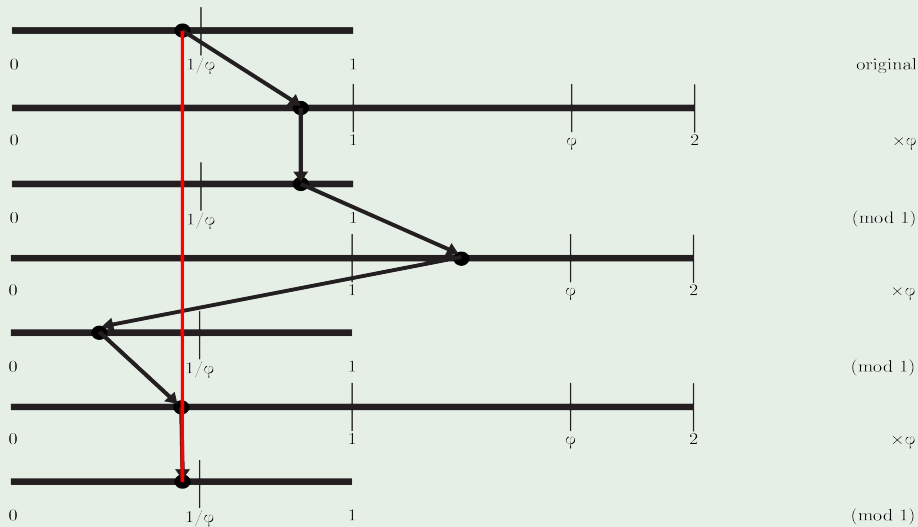
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- In general, whole numbers (excluding the base and its multiples) are typically the only numbers with finite representations in non-integer, algebraic bases

Silver Ratio

The positive solution to the equation $\alpha^2 = 2\alpha + 1$.
 $\alpha = 1 + \sqrt{2} \approx 2.414$.

1 Whole Numbers

- Definitions
- Examples
- Language Generated by Whole Number Bases

2 Non-Integer Algebraic Numbers

- Definitions
- Examples
- **Properties of Non-Integer Bases**
- Language Generated by Non-Integer Bases

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 - We can find these by using the orbit of any whole number in these bases (one is the easiest choice)
- It is easy/interesting to represent certain numbers in these bases
- Arithmetic is more of a pain than usual

Simplest Form

A number is in simplest form if we are unable to rewrite the beta expansion such that it includes a higher power.

Example of Simplest Form

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and by the definition of the base, we can rewrite the numerator:

$$\frac{\varphi^2}{\varphi^2} = 1_\varphi$$

Golden Ratio Addition

Example $(1_\varphi + 1_\varphi)$

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We can calculate any whole number through this method.

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- In actuality, it is because it escapes the range of the beta map

Escaping the Beta Map

Example (Golden Base Ratio: $T(0.011_\varphi)$)

Using our previous example:

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Geometrically, this can't happen. If we multiply a number in $[0, 1)$ by any number and perform (mod 1), we will never get a number larger than one. It's the arithmetic equivalent of having $x \cdot \beta > \beta$, $x < 1$.

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- Find the period of the orbit and take one segment
 - Any sequence of numerals greater than the numerals of the period is not allowed

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- The orbit of one is composed of the maximum allowed numeral in a language
- All the forbidden words then will subwords of the orbit, with one added until the word is the maximum numeral lexicographically possible

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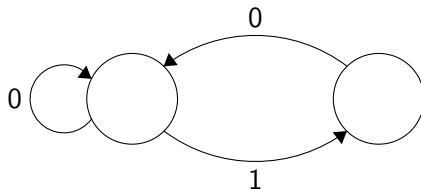
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- There can be more than one forbidden word, but since the maximum lexicographically allowed numeral in base golden ratio is 1, we can't add one again
- With this in mind, we can now create a finite state machine

FSM For Language Generated by Base Golden Ratio



Language Generated by base Silver Ratio

- $0.\overline{20}_\alpha$ is the orbit of one in base silver ratio

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- $0.\overline{20}_\alpha$ is the orbit of one in base silver ratio
- Therefore, 21 and 22 are forbidden words in the language

FSM For Language Generated by Base Silver Ratio

