

# Numbers in Non-Integer Bases

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Northern Virginia Community College

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## 1 Whole Numbers

- Definitions
- Examples
- Language Generated by Whole Number Bases

## 2 Non-Integer Algebraic Numbers

- Definitions
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- Properties of Non-Integer Bases
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- We keep track of the interval that it falls in after the operation



# Dealing with Radix Point Expressions (Geometrically)

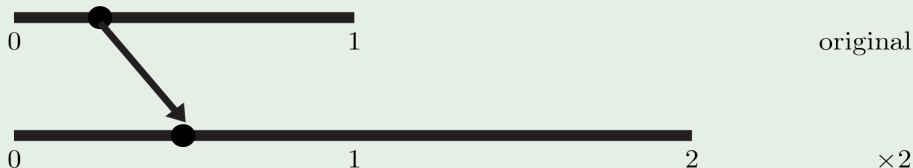
Example (Suppose you have a point...)



original

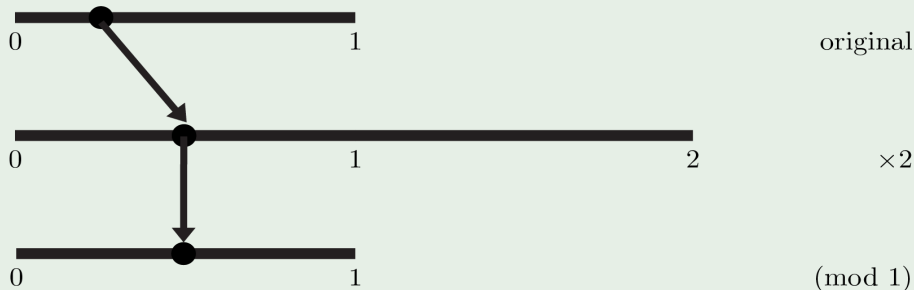
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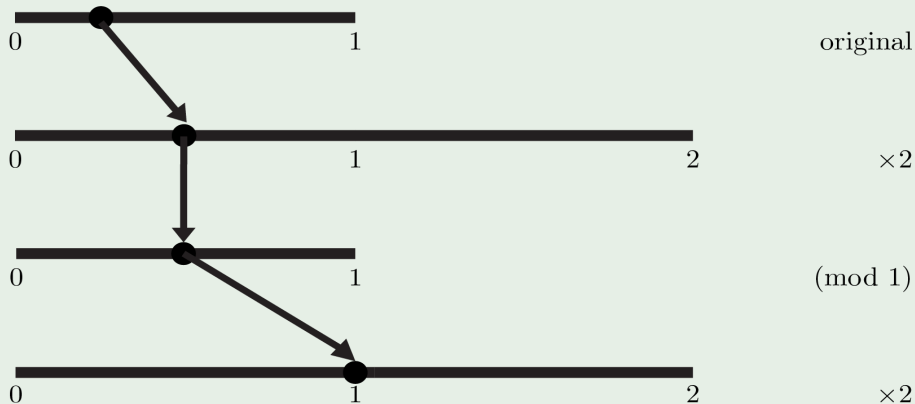
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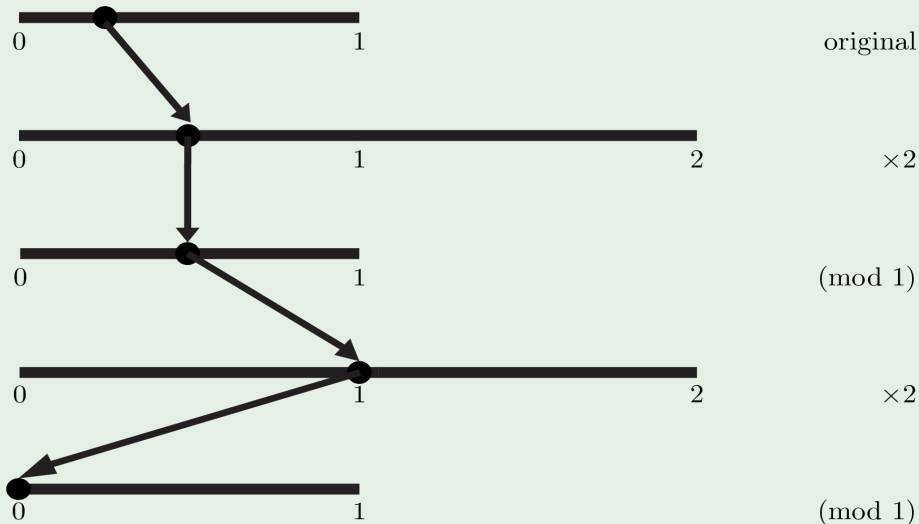
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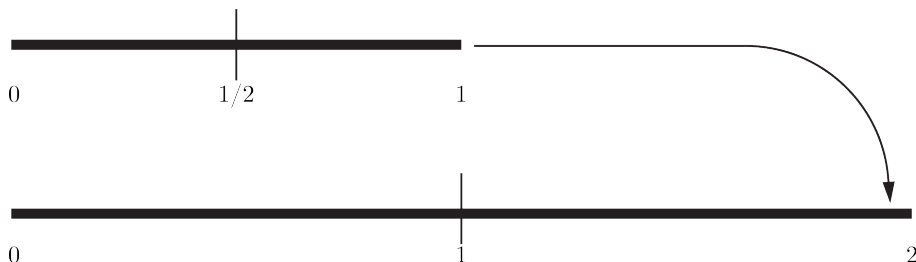
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- So for binary, this is what we were doing:



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- That there is a finite radix point expression in a given base
- That our choice of partitioning and multiplication matter... this is what happens geometrically when we change bases
- Let's do another example

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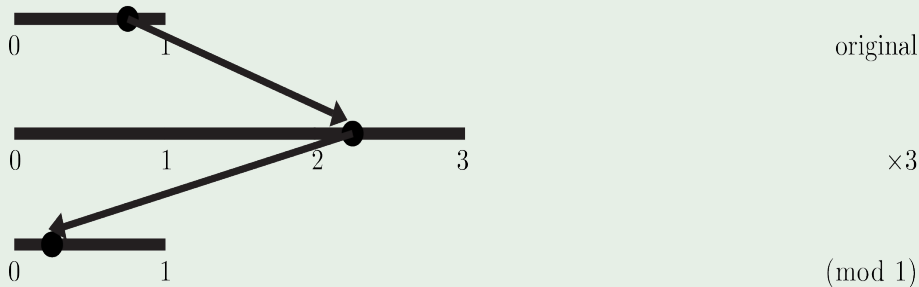
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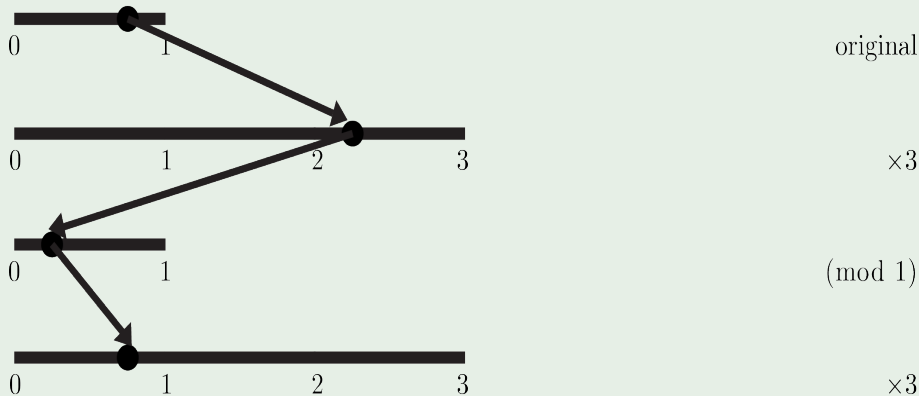
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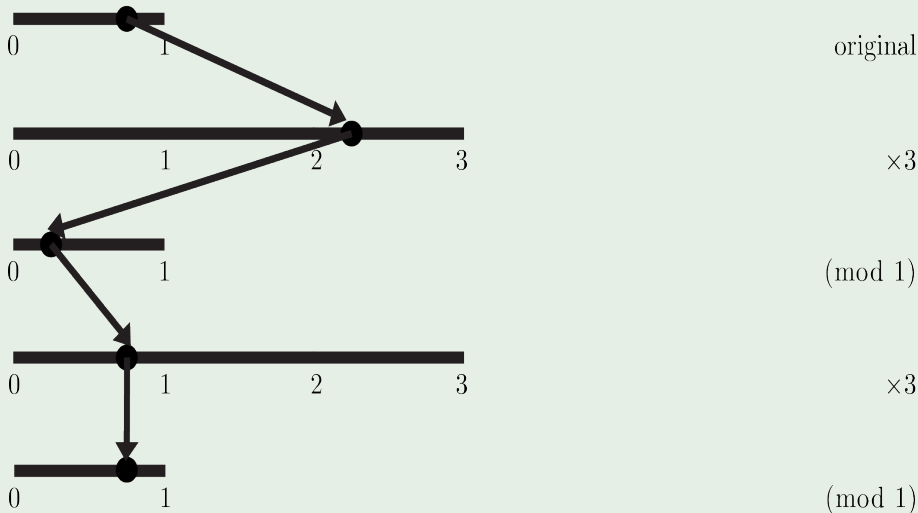
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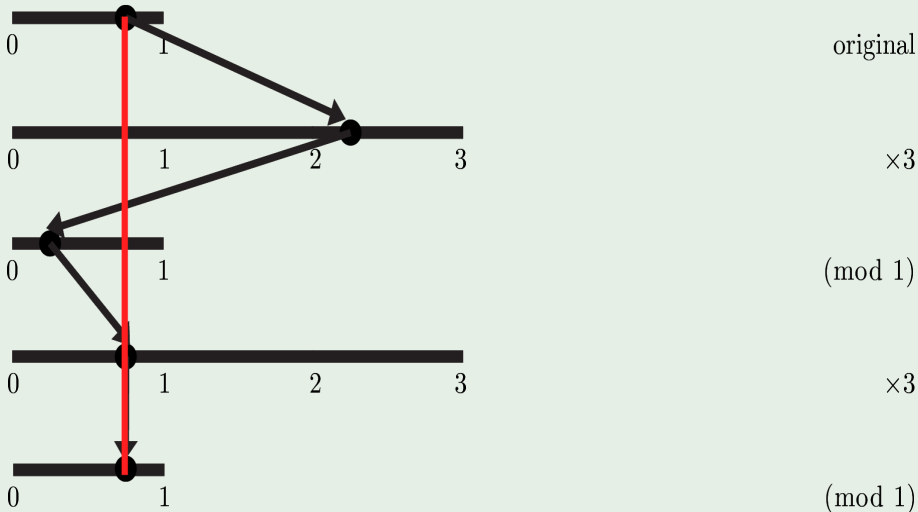
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- In general, this can and does happen in any base
- We can explain this phenomenon arithmetically

# Theorem

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A number  $p$  is said to have a finite radix point expression if

$$p = \frac{n}{m_1 m_2 \dots m_k}$$

where  $n \in \mathbb{N}$ , and  $m_1 m_2 \dots m_k$  are all the prime factors of the base raised to some power.

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<http://mathworld.wolfram.com/DecimalExpansion.html>

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## Example (1/3 in Base 10)

$$\frac{1}{3} \neq \frac{n}{2^\alpha 5^\beta}$$

where  $n, \alpha, \beta \in \mathbb{N}$ .

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  - We must first qualify what we were doing in our “game” previously

## Beta Map

The transformation  $T : [0, 1) \rightarrow [0, 1)$  defined by  $T = (x \cdot \beta) \pmod{1}$ .

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$0.2 * 2 = 0.4$	$0.4 \bmod (1) = 0.4$	0.00



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$0.8 * 2 = 1.6$	$1.6 \bmod (1) = 0.6$	0.0001

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101 (Binary)

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Numerical Bases

# Language Generated by Whole Number Bases

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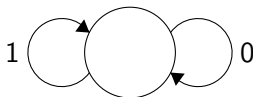
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- How can we represent the language generated by these bases?

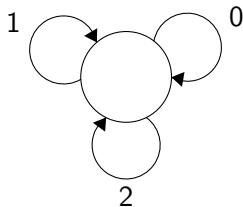
## Finite State Machine

A finite state machine (FSM) is a mathematical structure that allows us to imagine a machine with no memory.

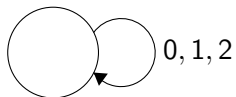
# FSM for Binary



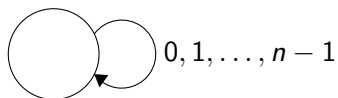
# FSM for Ternary



# Simplified FSM for Ternary



# FSM for Whole Number Bases



- $n \in \mathbb{N}, n > 1$



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## Example (Allowed Numbers in Different Bases)

$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$ , maximum numeral is 1

$\alpha = 1 + \sqrt{2} \approx 2.14$ , maximum numeral is 2

# Definition

## Algebraic Integer

A number  $x$  is an algebraic integer if it is the root of a monic polynomial  $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ , where  $a_{n-1}, \dots, a_0 \in \mathbb{Z}$ .

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$$\varphi^2 - \varphi - 1 = 0, \quad x = \frac{1 \pm \sqrt{5}}{2}$$

$$\alpha^2 - 2\alpha - 1 = 0, \quad x = 1 \pm \sqrt{2}$$

$$x^3 - x - 1 = 0, \quad x = \frac{1}{3} \sqrt[3]{\frac{27 - 3\sqrt{69}}{2}} + \frac{\sqrt[3]{\frac{1}{2}(9 + \sqrt{69})}}{3^{2/3}}$$

# Partitioning Differently

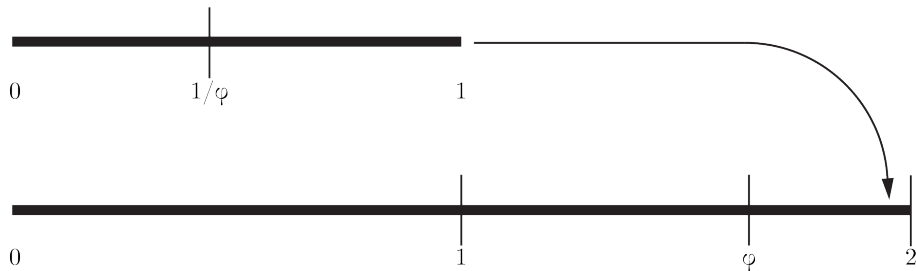
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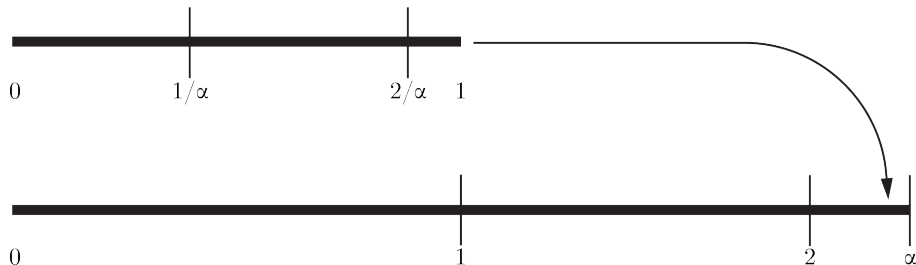
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# Examples of Non-Integer Bases

- Base Golden Ratio
- Base Silver Ratio

## Golden Ratio

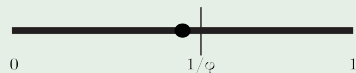
The positive solution to the equation  $\varphi^2 = \varphi + 1$ .

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618.$$



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Example (Suppose you have a point...)



original

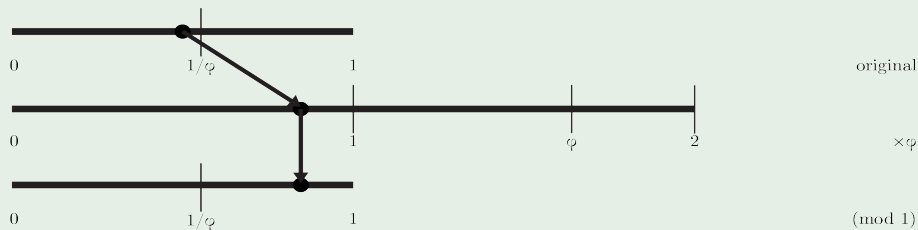
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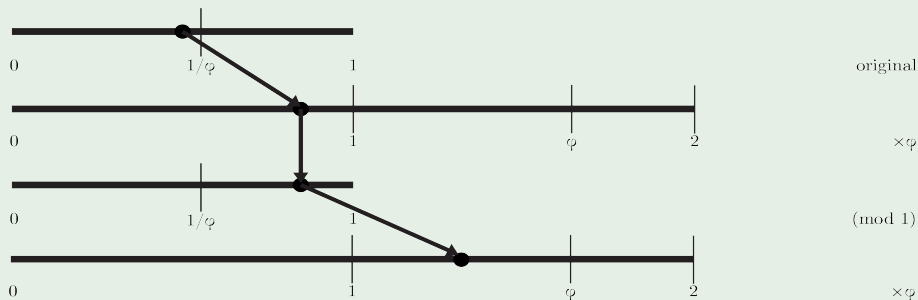
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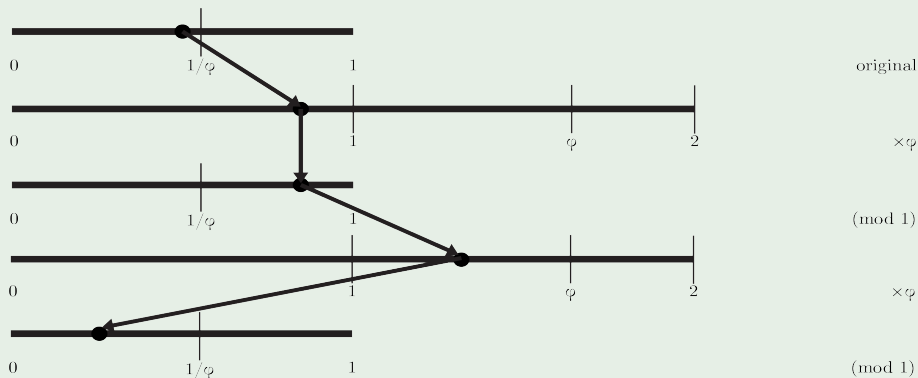
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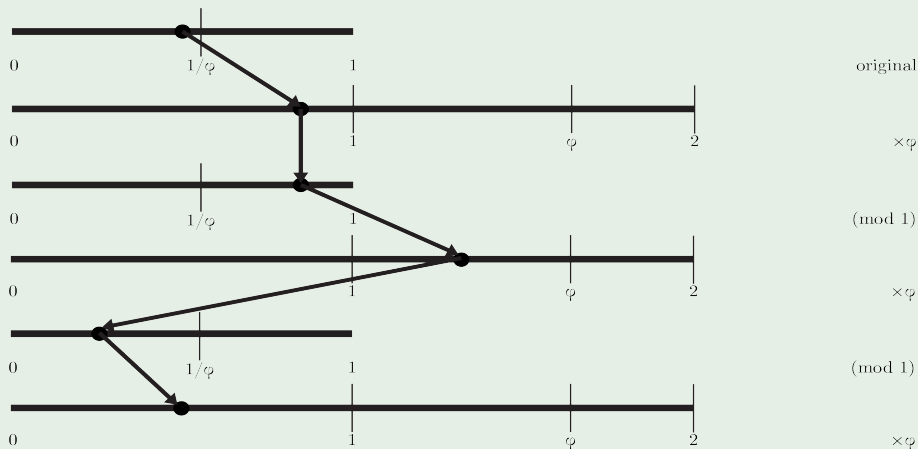
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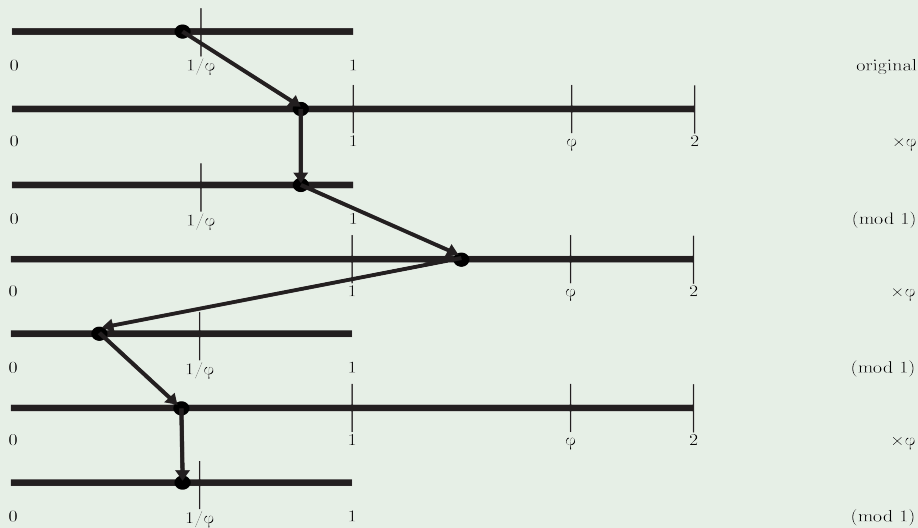
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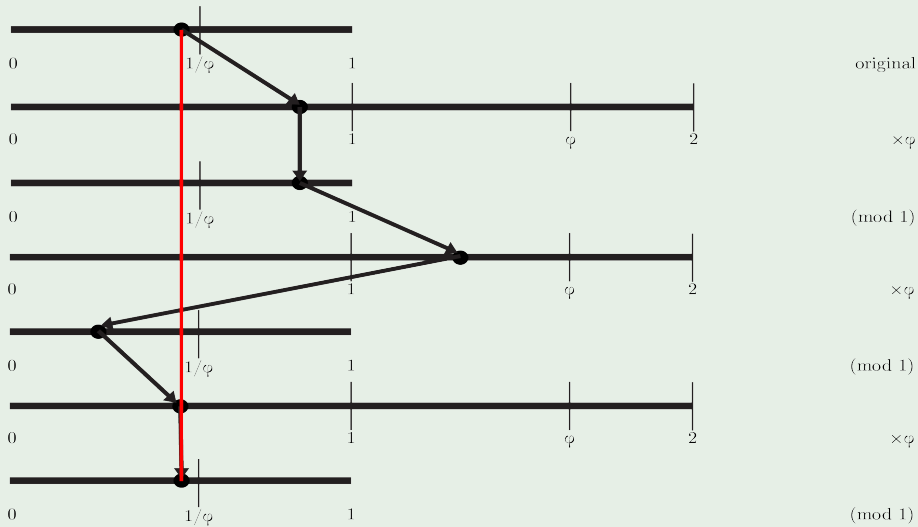
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- In general, whole numbers (excluding the base and its multiples) are typically the only numbers with finite representations in non-integer, algebraic bases

## Silver Ratio

The positive solution to the equation  $\alpha^2 = 2\alpha + 1$ .  
 $\alpha = 1 + \sqrt{2} \approx 2.414$ .

## 1 Whole Numbers

- Definitions
- Examples
- Language Generated by Whole Number Bases

## 2 Non-Integer Algebraic Numbers

- Definitions
- Examples
- **Properties of Non-Integer Bases**
- Language Generated by Non-Integer Bases

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- It is easy/interesting to represent certain numbers in these bases
- Arithmetic is more of a pain than usual

## Simplest Form

A number is in simplest form if we are unable to rewrite the beta expansion such that it includes a higher power.



# Example of Simplest Form

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and by the definition of the base, we can rewrite the numerator:

$$\frac{\varphi^2}{\varphi^2} = 1_\varphi$$

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We can calculate any whole number through this method.

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- In general, a word is forbidden if it is not in simplest form
- In actuality, it is because it escapes the range of the beta map

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Geometrically, this can't happen. If we multiply a number in  $[0, 1)$  by any number and perform (mod 1), we will never get a number larger than one. It's the arithmetic equivalent of having  $x \cdot \beta > \beta$ ,  $x < 1$ .

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- All the forbidden words then will subwords of the orbit, with one added until the word is the maximum numeral lexicographically possible

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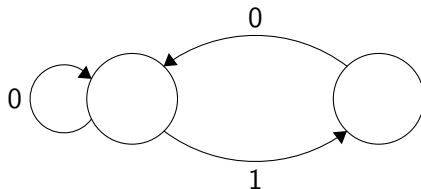
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- There can be more than one forbidden word, but since the maximum lexicographically allowed numeral in base golden ratio is 1, we can't add one again
- With this in mind, we can now create a finite state machine



# FSM For Language Generated by Base Golden Ratio



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- $0.\overline{20}_\alpha$  is the orbit of one in base silver ratio
- Therefore, 21 and 22 are forbidden words in the language

# FSM For Language Generated by Base Silver Ratio

