Rudin's Principles of Mathematical Analysis, 3rd ed

Connor Baker, June 2017

Basic Topology: Selected Exercises

5. Construct a bounded set of real numbers with exactly three limit points.

Lemma 1. *Consider the set E:*

$$E = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

Claim: $|E| = |\mathbb{N}|$.

Let $f : E \to \mathbb{N}$ where $\frac{1}{n} \mapsto n$. We will show that f is one-to-one.
Assume $f(a) = f(b)$. Then $a = b$, so f is one-to-one.
Choose $n \in \mathbb{N}$. Then there exists exactly one $\frac{1}{n} \in E$ such that $f(\frac{1}{n}) = n$ (since the reciprocal of any natural number is unique relative to the reciprocal of any other number in the naturals). As such, f is onto.

Since there is a one-to-one correspondence between E and $\mathbb{N}$, $|E| = |\mathbb{N}|$.

Proof. Let E be the set of numbers such that:

$$E = \bigcup_{k=0}^{2} A_k, \qquad A_k = \left\{ \frac{1}{n} + k : n \in \mathbb{N} \right\}.$$

By Lemma 1, we know that A_0 has the same cardinality as N. Since $\lim_{n\to\infty} \frac{1}{n} = 0$ the value of k allows us to change the value the limit approaches. As such, every value of A_1 is one larger than the corresponding value of A_0, as is A_2 with respect to A_1. The cardinality remains unchanged.
Since the cardinality of A_k is that of the naturals, the neighborhood $N_r(k)$ for some $r > 0$ intersect A_k is infinite. Therefore the set of limit points of E must contain at least the points 0, 1, and 2, and as such, $\{0, 1, 2\} \subseteq E'$.

We now show that E' contains only 0, 1, and 2.

Suppose that there is some limit point $x \notin \{0, 1, 2\}$. Let $\epsilon = \min(d(x, 0),\ d(x, 1),\ d(x, 2))$ (we want the smallest radius possible). Then the neighborhood

$$N_{\frac{\epsilon}{2}}(x) = \left\{ y : d(x, y) < \frac{\epsilon}{2} \right\}$$

does not contain the set $(0, \frac{\epsilon}{2}) \cup (1, 1 + \frac{\epsilon}{2}) \cup (2, 2 + \frac{\epsilon}{2})$. Each interval in that union has no least lower bound, since $\mathbb{N}$ is not bounded above, we can pick larger and larger n for $\frac{1}{n}$, which the interval is composed of. Since this portion of the interval is not included, the neighborhood is finite. Since the neighborhood is finite, the intersection with E is finite and therefore x is not a limit point.

As such, $x \notin E'$, and $E' = \{0, 1, 2\}$. □

6. Let E' be the set of all limit points of a set E. Prove:

   (a) E' is closed.

   (b) E and $\bar{E}$ have the same limit points.

   (c) Whether or not E and E' always have the same limit points.

Proof. 6a)
Let $p \in E''$. Then, for $r > 0$,

$$\exists q \in N_{\frac{r}{2}}(p) : q \in E', q \neq p.$$

Let

$$\delta = \frac{d(p,q)}{2} < \frac{r}{2}.$$

Since $q \in E'$, for $r > 0$, $\exists s \in N_\delta(q) : s \in E$, and $s \neq q$. Due to the definition of the radius of $N_\delta(q)$, $s \neq p$ since the neighborhood's radius is not large enough to contain p.
Then:

$$d(p,q) < \frac{r}{2}$$

$$d(q,s) < \delta$$

$$d(p,s) \leq d(p,q) + d(q,s) < \frac{r}{2} + \delta < \frac{r}{2} + \frac{r}{2} = r.$$

So, $p \in E'$, and E' is closed. □

Proof. 6b)
Proof that E and $\bar{E}$ have the same limit points.

By the previous proof, E' is closed and therefore contains its own limit points.

Let $\bar{E}'$ be the set of all limit points of $\bar{E}$.

We begin by showing that $E' \subseteq \bar{E}'$.

Let $x \in E'$. Then $x \in$

□

Proof. 6c)
Consider the set

$$E = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

Since $\lim_{n \to \infty} \frac{1}{n} = 0$, and $|E| = |\mathbb{N}|$ (by Lemma 1), $N_r(0) \cap E$ has infinitely many points, zero must be a limit point of E, so $E' = \{0\}$.

However, the limit points of E' are the empty set, since no matter the point we pick to center the neighborhood, the intersection will contain at most one element (zero). As such, E and E' do not always have the same limit points.

□

7. Let $A_1, A_2, A_3, \dots$ be subsets of a metric space.

   (a) If $B_n = \cup_{i=1}^n A_i$, prove that $\bar{B}_n = \cup_{i=1}^n \bar{A}_i$, for $n = 1, 2, 3, \dots$.

   (b) If $B = \cup_{i=1}^\infty A_i$, prove that $\bar{B} \supset \cup_{i=1}^\infty \bar{A}_i$, and show by an example that this inclusion can be proper.

Lemma 2. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. 6a)
Claim: $(A \cup B)' \subseteq A' \cup B'$.
Case 1: The trivial case is that $A \cup B$ is finite. Then $(A \cup B)'$ is the empty set, since finite sets cannot have limit points, which is a subset of any set so our claim holds true.
Case 2: $A \cup B$ is not finite. Then let $x \in (A \cup B)'$. Choose any $r > 0$, and consider $N_r(x) \cap A$, and $N_r(x) \cap B$. Suppose both are finite. Then $N_r(x) \cap (A \cup B)$ is finite, and x is not a limit point. This is a contradiction, so our assumption that both are finite is incorrect. Therefore, at least one must be infinite, so $N_r(x) \cap A$ or $N_r(x) \cap B$ must be infinite, and therefore $x \in A' \cup B'$.
Claim: $A' \cup B' \subseteq (A \cup B)'$.
Let $x \in A' \cup B'$. Choose any $r > 0$. Then, $N_r(x) \cap A$ or $N_r(x) \cap B$ is infinite, and $x \in (A \cup B)'$.
Therefore, $(A \cup B)' = A' \cup B'$, and

$$\overline{A \cup B} = (A \cup B) \cup (A \cup B)' = A \cup A' \cup B \cup B' = \overline{A} \cup \overline{B}.$$

□

Proof. 6a) Cont.
We proceed with proof by induction.
Let $n = 1$. Then:

$$\overline{\bigcup_{i=1}^{1} A_i} = \overline{A_1} = \bigcup_{i=1}^{1} \overline{A_i}.$$

Let $n = k$. Assume that

$$\overline{\bigcup_{i=1}^{k} A_i} = \bigcup_{i=1}^{k} \overline{A_i},$$

for all $k \in \mathbb{N}$.
Let $n = k + 1$. Then,

$$\overline{\bigcup_{i=1}^{k+1} A_i} = \overline{\bigcup_{i=1}^{k} A_i \cup A_{k+1}},$$

which by the lemma above is equivalent to (using our assumption for $n = k$)

$$\overline{\bigcup_{i=1}^{k} A_i} \cup \overline{A_{k+1}} = \bigcup_{i=1}^{k} \overline{A_i} \cup \overline{A_{k+1}} = \bigcup_{i=1}^{k+1} \overline{A_i}.$$

Therefore

$$B_n = \bigcup_{i=1}^{n} A_i, \qquad \overline{B_n} = \overline{\bigcup_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A_i}$$

by the Principle of Mathematical Induction. □

8. Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

Proof. Since E is open, all elements of E are interior points.

Let $p \in E$. Choose $r > 0$, and let $N_r(p)$ be a neighborhood of p. Since p is an interior point, $\exists \delta > 0 : N_\delta(p) \subseteq E$.

Choose $0 < \epsilon \leq \min(r, \delta)$. Then, $N_\epsilon(p) \subseteq N_\delta(p) \subseteq E$. Since $N_\epsilon(p)$ contains infinitely many points of E, and $N_\epsilon(p) \subseteq N_r(p)$, $N_r(p)$ does as well.

Closed sets, by definition contain all of their own limit points, so any closed subset of $\mathbb{R}^2$ will contain its own limit points. However, it is not the case that any finite set of E will have limit points since the neighborhood intersect E will be finite. $\square$

9. Let E° denote the set of all interior points of a set E.

   (a) Prove that E° is always open.

   (b) Prove that E is open if and only if $E^\circ = E$.

   (c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.

   (d) Prove that the complement of E° is the closure of the complement of E.

   (e) Do E and $\bar{E}$ always have the same interiors?

   (f) Do E and E° always have the same closures?

Proof. 9a) □

Proof. 9b) □

Proof. 9c)
Let $p \in G$, where G is an open set. Since G is open, p is an interior point of G. Choose any $r > 0$, and $N_r(p) \subseteq G$.
`Next step: Show that` $N_r(p) \subseteq E$`, so that` $p \in E$`, so` $G \subseteq E^\circ$`.` □

Proof. 9d)
Claim: $(E^\circ)^c \subseteq \overline{(E^c)}$. Since $E^\circ \subseteq E$ (why exactly is this?), $(E^\circ)^c$ is closed since it is the complement of an open set, and is equal to $(E^\circ)^c$, which is a super-set of $\overline{(E^c)}$.
Claim: $\overline{(E^c)} \subseteq (E^\circ)^c$. □

Proof. 9e)
Pick two open intervals, say $(0, 1) \cup (1, 2)$. Then 1 is not an interior point, and the closure of the set is $([0, 1] \cup [1, 2] = [0, 2]$, where one is an interior point.
As such, the set and its closure do not necessarily have the same interiors. □

Proof. 9f)
Let $E = \{1\}$.
Then $E^\circ = \emptyset$, $\overline{E^\circ} = \overline{\emptyset} = \emptyset$.
In addition, $\overline{E} = \{1\} \neq \emptyset$.
As such, the set and the set of it's interior points do not necessarily have the same closures. □

12. Let $K \subseteq \mathbb{R}^1$ consist of zero and the numbers $1/n$ for $n = 1, 2, 3, \ldots$. Prove that K is compact directly from the definition (without using Heine-Borel theorem).

Proof. Suppose that $K \subseteq \mathcal{C}$, where $\mathcal{C}$ is an open covering of K. Since $\mathcal{C}$ is an open covering, it is the set of $\cup_\alpha G_\alpha$ where $K \subseteq \cup_\alpha G_\alpha$.
Since $0 \in K$, there must be some $G_{\alpha,0}$ that contains zero as well. Since all G are open, there exists $r > 0$ such that $N_r(0) \subseteq G_{\alpha,0}$.
Since 0 is the only limit point of K (by Lemma 1), $K \backslash N_r(0)$ is finite, and can be can therefore be covered by finite number of subcovers, $\cup_\alpha^n G_\alpha$.
Therefore, K is compact. $\square$

13. Construct a compact set of real numbers whose limit points form a countable set.

Proof. □

14. Give an example of an open cover of the segment $(0, 1)$ which has no finite subcover.

Proof. $\square$

15. Show that Theorem 2.36 and its Corollary become false (in $\mathbb{R}^1$, for example) if the word "compact" is replaced by "closed" or "bounded".

Theorem 3. *If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every*

Proof. $\square$

16. Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space, with $d(p, q) = |p - q|$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show:

   (a) E is closed and bounded in $\mathbb{Q}$, but that E is not compact.

   (b) Whether E is open in $\mathbb{Q}$.

Proof. 16a) $\square$

Proof. 16b) $\square$

17. Let $E = \{x : x \in [0,1]$ and x's decimal expansion contains only the digits 4 and 7$\}$. Show:

   (a) Whether E is countable.
   (b) Whether E is dense in $[0,1]$.
   (c) Whether E is compact.
   (d) Whether E is perfect.

Proof. 17a) $\square$

Proof. 17b) $\square$

Proof. 17c) $\square$

Proof. 17d) $\square$

18. Is there a nonempty perfect set in $\mathbb{R}^1$ which contains no rational number?

Proof. $\square$

19. (a) If A and B are disjoint closed sets in some metric space X, prove that they are separated.

    (b) Prove the same for disjoint open sets.

    (c) Fix $p \in X, \delta > 0$, define A to be the set of all $q \in X$ for which $d(p, q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.

    (d) Prove that every connected metric space with at least two points in uncountable. *Hint*: Use (c).

Proof. $\square$