Homework 2
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1. Determine whether the following expressions are true or false. Give a complete explanation for each part.
   (a) $\emptyset \subseteq \{\emptyset, \{\emptyset\}\}$
   (b) $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$
   (c) $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\}$
   (d) For every set A, $\{\emptyset\} \subseteq A$.
   (e) $\{1, 2\} \in \{\{1, 2, 3\}, \{1, 3\}, 1, 2\}$
   (f) $\{\{4\}\} \subseteq \{1, 2, 3, \{4\}\}$

Definition 1 (Subset)**.** Given two subsets A, B, A is said to be a subset of B if and only if all elements of A are also in B. That is to say:

$$X \subseteq Y \iff \forall x(x \in X \implies x \in Y)$$

Proof. (a) Let $A = \emptyset, B = \{\emptyset, \{\emptyset\}\}$. Then, by the definition of a subset,

$$\emptyset \subseteq \{\emptyset, \{\emptyset\}\} \iff \forall a(a \in \emptyset \implies a \in \{\emptyset, \{\emptyset\}\})$$

However, $a \notin \emptyset$ (the empty set contains no elements). As such, the statement is vacuously true (because for all a, of which there are none, we cannot tell whether it is in both sets or not).

Therefore, by the definition of a subset, the expression is true. $\square$

Proof. (b) Let $A = \{\emptyset\}, B = \{\emptyset, \{\emptyset\}\}$. Then, by the definition of a subset,

$$\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\} \iff \forall a(a \in \{\emptyset\} \implies a \in \{\emptyset, \{\emptyset\}\})$$

Let $a = \emptyset$, the only element of A. Then,

$$\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\} \iff (\emptyset \in \{\emptyset\} \implies \emptyset \in \{\emptyset, \{\emptyset\}\})$$

which is true.

Therefore, by the definition of a subset, the expression is true. $\square$

Proof. (c) Let $A = \{\{\emptyset\}\}, B = \{\emptyset, \{\emptyset\}\}$. Then, by the definition of a subset,

$$\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\} \iff \forall a(a \in \{\{\emptyset\}\} \implies a \in \{\emptyset, \{\emptyset\}\})$$

Let $a = \{\emptyset\}$, the only element of A. Then,

$$\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\} \iff (\{\emptyset\} \in \{\{\emptyset\}\} \implies \emptyset \in \{\emptyset, \{\emptyset\}\})$$

which is true.

Therefore, by the definition of a subset, the expression is true. $\square$

Proof. (d) Let $B = \{\emptyset\}$. Then, by the definition of a subset,

$$\{\emptyset\} \subseteq C \iff \forall b(b \in \{\emptyset\} \implies b \in C)$$

Let $b = \emptyset$, the only element of B. Then,

$$\{\emptyset\} \subseteq C \iff (\emptyset \in \{\emptyset\} \implies \emptyset \in C)$$

which is contingent on the elements of C. There is no guarantee that C contains the empty set.

Therefore, by the definition of a subset, the expression is false. $\square$

Proof. (e) This statement is false because the set does not contain the set $\{1, 2\}$. $\square$

Proof. (f) Let $A = \{\{4\}\}, B = \{1, 2, 3, \{4\}\}$. Then, by the definition of a subset,

$$\{\{4\}\} \subseteq \{1, 2, 3, \{4\}\} \iff \forall a(a \in \{\{4\}\} \implies a \in \{1, 2, 3, \{4\}\})$$

Let $a = \{4\}$, the only element of A. Then,

$$\{\{4\}\} \subseteq \{1, 2, 3, \{4\}\} \iff (\{4\} \in \{\{4\}\} \implies \{4\} \in \{1, 2, 3, \{4\}\})$$

which is true.

Therefore, by the definition of a subset, the expression is true. $\square$

2. Let $\Delta = [0, 1) = \{x \in \mathbb{R} : 0 \leq x < 1\}$ and let $A_\alpha = (-\alpha, \alpha] = \{x \in \mathbb{R} : -\alpha < x \leq \alpha\} \subseteq \mathbb{R}$, where $\alpha \in \Delta$. Prove that

$$\bigcup_{\alpha \in \Delta} A_\alpha = (-1, 1),$$

and

$$\bigcap_{\alpha \in \Delta} A_\alpha = \emptyset$$

Proof. (a) The union of a family of sets creates a set containing all elements that exist in any set in the family (by definition).

As α increases, so too does the range of the interval. For example, $A_{0.5} = (-0.5, 0.5]$ is smaller than $A_{0.9} = (-0.9, 0.9]$. For every α, the next largest α $\forall \alpha \in \Delta$, which I will call α', as an index will is a set that will contain the previous set ($A_\alpha \subseteq A_{\alpha'}$). For example (though 0.9 is most certainly not the number directly after 0.5 in the set of all real numbers), $A_{0.5} \subseteq A_{0.9}$.

Since the next set in the index contains the set that is immediately before it, the union of all sets in the index will simply be the interval contained by the last set in the union. In this case, since the largest index possible given the constraints is one exclusive, the final set in the family will be $(-1, 1)$. □

Proof. (b) Let $\alpha = 0$. Then $A_0 = [0, 0)$, which is not possible (the set both contains and does not contain zero), so $A_0 = \emptyset$. Letting $\alpha = x$, $A_x = (-x, x]$. For all $x \in (0, 1)$, the set A_x contains at least zero. Then, the intersection of all the sets excluding the set A_0 is zero. However, the intersection of all sets including the set A_0 is the empty set: The first set is the empty set, no other sets contain the empty set, so they have no shared elements. □

3. Let A, B, C, and D be sets with $C \subseteq A$ and $D \subseteq B$. Prove that $C \cup D \subseteq A \cup B$.

Proof. Since C is a subset of A, $\forall c \in C \implies c \in A$, and since D is a subset of B, $\forall d \in D \implies d \in B$.

The union of the sets C and D contains all elements that exist in either. The same is true for the union of the sets A and B. Since A contains C, and B contains D, the union of the sets A and B contain the union of the sets C and D. $\square$

4. Prove that if $\mathcal{A}$ is a non-empty family of sets, then

$$\bigcap_{A \in \mathcal{A}} A \subseteq \bigcup_{A \in \mathcal{A}} A.$$

Proof. The intersection of a family of sets creates a set containing only elements shared between all sets in the family (by definition).

The union of a family of sets creates a set containing all elements that exist in any set in the family (by definition).

Let x be in the union of the family of sets. If x is in the union, then that means that it is in at least one of the sets in the family. If it is in every set in the family, then it is in the intersection. However, if x was to be in any number of sets (excluding being contained in every set), then x would not be in the intersection of the family of sets. Because of this, elements of the union are not necessarily all in the intersection, and therefore the set generated by the union over the family is not necessarily a subset of the intersection.

Let y be in the intersection of the family of sets. This implies that y is shared between every set in the family, and is therefore also in the union. As such, the intersection over a family of sets contains elements found in the union over the same family, and is a subset. $\square$

5. Use the principle of mathematical induction to prove $4^{n+4} > (n+4)^4$, for all natural numbers n.

Proof. Let $n = 1$:

$$4^5 > 5^4$$

so the base case is true. Then, let $n = k$:

$$4^{k+4} > (k+4)^4 \tag{1a}$$

Since $k \in \mathbb{N}$, then $(k+1) \in \mathbb{N}$ as well. As such, when $n = k+1$:

$$4^{k+5} > (k+5)^4 \tag{1b}$$

To prove that this inequality holds for all natural numbers, we will establish bounds using the inequality in Equation (1a).

$$4^{k+5} = 4^k * 4^5 \tag{2a}$$

$$4^{k+4} = 4^k * 4^4 \tag{2b}$$

Equations (2a) and (2b) show the left hand sides of Equations (1a) and (1b) re-written in expanded form. With this knowledge, we can see that:

$$4^{k+5} = 4^k * 4^5 = 4(4^{k+4}) \tag{2c}$$

which we can compare directly to the left hand side of the inequality from Equation (1a)

$$4(4^{k+4}) > 4^{k+4}$$

and find that it is greater by a factor of exactly four. In terms of building a series of inequalities, we now have:

$$4^{k+5} = 4(4^{k+4}) > 4^{k+4} \tag{3}$$

Now, let us look at the right hand side of the inequalities in Equations (1a) and (1b):

$$(k+5)^4 = k^4 + 20k^3 + 150k^2 + 500k + 625 \tag{4a}$$

$$(k+4)^4 = k^4 + 16k^3 + 96k^2 + 256k + 256 \tag{4b}$$

Referring to Equation (3), where we used four times the left hand side of the inequality from Equation (1a), we now use four times the right hand side of the inequality from Equation (1a) to keep the relationship that the two share:

$$4(k+4)^4 > 4(k+4)^4 \tag{5a}$$

From this point, it is important to note the distributed form of $4(k+4)^4$ in comparison to $(k+5)^4$:

$$4(k+4)^4 = 4k^4 + 64k^3 + 384k^2 + 1024k + 1024 > k^4 + 20k^3 + 150k^2 + 500k + 625 = (k+5)^4 \tag{5b}$$

So therefore

$$4(k+4)^4 > (k+5)^4 \tag{5c}$$

Finally, we build the last inequality, taking parts of Equation (3), Equation (5a), and Equation (5c):

$$4^{k+5} = 4(4^{k+4}) > 4(k+4)^4 > (k+5)^4 \tag{6}$$

Then, by the principle of mathematical induction, $4^{n+4} > (n+4)^4 \ \forall n \in \mathbb{N}$. $\square$