

## Homework 6

### Connor Baker, April 2017

1. An algebraic number is any number that is a root of a polynomial with rational coefficients. Prove that the algebraic numbers are countable. A number is transcendental if it is not algebraic. Prove there are uncountable many transcendental numbers.

*Proof.* Let  $M_n$  be the set of all monomials of degree  $n, n \in \mathbb{Z}^+$ , which have rational coefficients:

$$M_n = \{q_n x^n : q_n \in \mathbb{Q}\}.$$

Let  $P_n$  be the set of all polynomials of (at most) degree  $n$ , which have rational coefficients:

$$P_n = \{M_n + M_{n-1} + \cdots + M_1 + M_0\}.$$

Then,  $|P_n| = |M_0 \cup M_1 \cup \cdots \cup M_n| = |\mathbb{Q}|$ , since the countable union of countable sets is countable.

Let  $R_n$  be the set of all roots of  $P_n$ :

$$R_n = \{x \in \mathbb{R} : \exists f \in P_n : f(x) = 0\}.$$

For each  $i \in \mathbb{N}$ , let  $p_i^n$  be a unique polynomial from  $P_n$ . Then, let  $r_i^n \subseteq R_n$  be the roots of  $p_i^n$ . By the fundamental theorem of algebra, every polynomial of degree  $n$  has at most  $n$  real roots.

Then,

$$R_n = \bigcup_{i \in \mathbb{N}} r_i^n.$$

As the countable union of countable sets,  $R_n$  is countable, and  $|R_n| = |\mathbb{Q}|$ .

Let  $\mathbb{A} = \bigcup_{n \in \mathbb{N}} R_n$  be the set of algebraic numbers. Again, the countable union of countable sets is countable, and the algebraic numbers are countable.

We now prove that there are uncountable many transcendental numbers by contradiction.

Suppose that the set of all transcendental numbers  $\mathbb{T}$  is countable. Then, since  $\mathbb{R}$  is defined as  $\mathbb{A} \cup \mathbb{T}$ ,  $|\mathbb{R}| = |\mathbb{A} \cup \mathbb{T}|$ . However, since the union of any countable set with another countable set is countable,  $|A \cup T| = |\mathbb{Q}|$ , which by transitivity means  $|\mathbb{Q}| = |\mathbb{R}|$ , which is a contradiction since the reals are uncountable. Therefore, by contradiction,  $\mathbb{T}$  must be uncountable.  $\square$

2. Let  $A$  be the set of all functions  $f : \mathbb{N} \rightarrow \{0, 1\}$ . Find the cardinality of  $A$ .

*Proof.* To show that  $A$  has the same cardinality as  $\mathbb{R}$ , we must show that there is a bijective mapping between the two. Instead of creating a mapping to  $\mathbb{R}$ , we instead use  $(0, 1)$ , which has the same cardinality (as proved in class).

Consider the function  $g : A \rightarrow (0, 1)$ . The function  $g$  takes each function in  $f$  and creates a list of the outputs of that function as a decimal number in  $(0, 1)$ :

$$g(f_1) = 0.a_{11}a_{12}a_{13} \dots$$

$$g(f_2) = 0.a_{21}a_{22}a_{23} \dots$$

$$g(f_3) = 0.a_{31}a_{32}a_{33} \dots$$

$$\vdots$$

$$g(f_n) = 0.a_{n1}a_{n2}a_{n3} \dots$$

$$\vdots$$

where  $a_{ij}$  is either 0 or 1, the range of  $f$ . Then  $g$  is obviously a one-to-one mapping to  $(0, 1)$ .

Now, consider  $g^{-1} : (0, 1) \rightarrow A$ . Any real number in  $(0, 1)$  has a binary representation, finite or not. As such,  $g^{-1}$  maps any binary number in the interval to  $A$  and is one-to-one.

As such,  $g$  is bijective, and  $|A| = |(0, 1)| = |\mathbb{R}|$ . □

3. Let  $A$  be the set of all functions  $f : \mathbb{N} \rightarrow \{0, 1\}$  that are “eventually zero” (We say that  $f$  is eventually zero if there is a positive integer  $N$  such that  $f(n) = 0$  for all  $n \geq N$ ). Find the cardinality of  $A$ .

*Proof.* To show that  $A$  has the same cardinality as  $\mathbb{N}$ , we must show that there is a bijective mapping between the two.

Consider the function  $g : A \rightarrow (0, 1)$ . The function  $g$  takes each function in  $f$  and creates a list of the outputs of that function, excluding all trailing zeros (that is, all zeros that are not followed at some point by a nonzero number), then reverses that list (ex:  $010010100000 \mapsto 1010010$ ), creating a unique binary number in  $\mathbb{N}$ :

$$g(f_1) = a_{13}a_{12}a_{11} \dots$$

$$g(f_2) = a_{23}a_{22}a_{21} \dots$$

$$g(f_3) = a_{33}a_{32}a_{31} \dots$$

$$\vdots$$

$$g(f_n) = a_{n3}a_{n2}a_{n1} \dots$$

$$\vdots$$

where  $a_{ij}$  is either 0 or 1, the range of  $f$ . Then  $g$  is obviously a one-to-one mapping to  $\mathbb{N}$ .

Now, consider  $g^{-1} : \mathbb{N} \rightarrow A$ . All natural numbers have a unique finite binary representation. As such,  $g^{-1}$  maps any number in  $\mathbb{N}$  to  $A$  and is one-to-one.

As such,  $g$  is bijective, and  $|A| = |\mathbb{N}|$ . □

4. Use the axiom of choice to prove that if there exists  $f : A \rightarrow B$  that is onto, then there exists a function  $g : B \rightarrow A$  that is one-to-one.
5. We say that  $|A| \geq |B|$  if there exists a function  $f : A \rightarrow B$  which is onto. Prove that if  $|A| \geq |B|$ , and  $|B| \geq |A|$ , then  $|A| = |B|$ . (Hint: Use 4).