Homework 5
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1. Prove that if the real-valued function f is strictly increasing or strictly decreasing on $\mathbb{R}$, then f is one-to-one (Note: You cannot assume f is differentiable).

Proof. Case 1: f is strictly decreasing.
If f is strictly decreasing, then $\forall\ x, a \in \operatorname{dom}(f),\ x \leq a,\ f(x) \geq f(a)$. If it is the case that $f(x) = f(a)$, then it must also be the case that $x = a$. Then, $f(x) = f(a)$ if and only if $x = a$, and $f(x) > f(a)\ \forall x, a \in \operatorname{dom}(f),\ x < a$, the function f must be one-to-one.
Case 2: f is strictly increasing.
If f is strictly increasing, then $\forall\ x, a \in \operatorname{dom}(f),\ x \leq a,\ f(x) \leq f(a)$. If it is the case that $f(x) = f(a)$, then it must also be the case that $x = a$. Then, $f(x) = f(a)$ if and only if $x = a$, and $f(x) < f(a)\ \forall x, a \in \operatorname{dom}(f),\ x < a$, the function f must be one-to-one. $\square$

2. Prove the following are metrics:

   (a) $X = \mathbb{R}, d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

   (b) $X = \mathbb{R} \times \mathbb{R}, d((x, y), (z, w)) = \sqrt{(x - z)^2 + (y - w)^2}$

   Definition 1 (Metric)**.** A metric on a set X is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$,

   (a) $d(x, y) \geq 0$

   (b) $d(x, y) = 0$ if and only if $x = y$

   (c) $d(x, y) = d(y, x)$,

   (d) $d(x, y) + d(y, z) \geq d(x, z)$.

Proof. We begin by proving that the first function is a metric.

1. The rang$(d) = \{0, 1\}$ so the function is definitely greater than or equal to zero for any inputted pair of values.
2. By the definition of d, if and only if $x = y$ does $d(x, y) = 1$.
3. Since the equals relationship is symmetric, $x = y \implies y = x$. As such, $d(x, y) = d(y, x)$.
4. As shown below, it is the case that $d(x, y) + d(y, z) \geq d(x, z)$.

Therefore the first function is a metric.

We now prove that the second function is a metric.

1. The rang$(d) = \{0, 1\}$ so the function is definitely greater than zero.
2. By the definition of d, if and only if $x = y$ does $d(x, y) = 1$.
3. Since the equals relationship is symmetric, $x = y \implies y = x$. As such, $d(x, y) = d(y, x)$.
4. As shown below, it is the case that $d(x, y) + d(y, z) \geq d(x, z)$:

Therefore the function is a metric. □

3. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be given by $f(m, n) = 2^{m-1}(2n - 1)$. Prove that f is one-to-one and onto.

Proof. Let $a, b, c \in \mathbb{N}$. Then, $f(b, c) \geq 1$, and $\forall a \in \mathbb{N}$, $\exists (b, c) \in \mathbb{N} \times \mathbb{N}$: $f(b, c) = a$, since a, b and c are all natural numbers, and the natural numbers are closed under multiplication. As such, f is surjective.
The function f is injective if $\forall x, y \in \mathbb{N} \times \mathbb{N}, x \neq y, f(x) \neq f(y)$. □

4. Let $f : A \to B$ be a function from a nonempty set A. Prove that the set $\mathcal{C} = \{f^{-1}(b) : b \in \text{rang}(f)\}$ is a partition of A. Note: $\mathcal{C}$ is a subset of $\mathcal{P}(A)$.

Proof. Since $\square$

5. Let $f : A \to B$ be a function from a nonempty set A which is surjective. Find a new function $g : C \to B$ which is one-to-one such that $C \subseteq A$, $\text{rang}(g) = \text{rang}(f)$, and for every $x \in C, f(x) = g(x)$. Explain why g has an inverse function, g^{-1}. Then, compute $f(g^{-1}(x))$ for all $x \in B$.

Proof. Since $\square$