Homework 2
Connor Baker, January 2017

1. Determine whether the following expressions are true or false. Give a complete explanation for each part.
   (a) $\emptyset \subseteq \{\emptyset, \{\emptyset\}\}$
   (b) $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$
   (c) $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\}$
   (d) For every set A, $\{\emptyset\} \subseteq A$.
   (e) $\{1, 2\} \in \{\{1, 2, 3\}, \{1, 3\}, 1, 2\}$
   (f) $\{\{4\}\} \subseteq \{1, 2, 3, \{4\}\}$

Definition 1 (Subset)**.** Given two subsets A, B, A is said to be a subset of B if and only if all elements of A are also in B. That is to say:

$$X \subseteq Y \iff \forall x(x \in X \implies x \in Y)$$

Proof. (a) Let $A = \emptyset, B = \{\emptyset, \{\emptyset\}\}$. Then, by the definition of a subset,

$$\emptyset \subseteq \{\emptyset, \{\emptyset\}\} \iff \forall a(a \in \emptyset \implies a \in \{\emptyset, \{\emptyset\}\})$$

However, $a \notin \emptyset$ (the empty set contains no elements). As such, the statement is vacuously true (because for all a, of which there are none, we cannot tell whether it is in both sets or not).

Therefore, by the definition of a subset, the expression is true. □

Proof. (b) Let $A = \{\emptyset\}, B = \{\emptyset, \{\emptyset\}\}$. Then, by the definition of a subset,

$$\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\} \iff \forall a(a \in \{\emptyset\} \implies a \in \{\emptyset, \{\emptyset\}\})$$

Let $a = \emptyset$, the only element of A. Then,

$$\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\} \iff (\emptyset \in \{\emptyset\} \implies \emptyset \in \{\emptyset, \{\emptyset\}\})$$

which is true.

Therefore, by the definition of a subset, the expression is true. □

Proof. (c) Let $A = \{\{\emptyset\}\}, B = \{\emptyset, \{\emptyset\}\}$. Then, by the definition of a subset,

$$\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\} \iff \forall a(a \in \{\{\emptyset\}\} \implies a \in \{\emptyset, \{\emptyset\}\})$$

Let $a = \{\emptyset\}$, the only element of A. Then,

$$\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\} \iff (\{\emptyset\} \in \{\{\emptyset\}\} \implies \emptyset \in \{\emptyset, \{\emptyset\}\})$$

which is true.

Therefore, by the definition of a subset, the expression is true. □

Proof. (d) Let $B = \{\emptyset\}$. Then, by the definition of a subset,

$$\{\emptyset\} \subseteq C \iff \forall b(b \in \{\emptyset\} \implies b \in C)$$

Let $b = \emptyset$, the only element of B. Then,

$$\{\emptyset\} \subseteq C \iff (\emptyset \in \{\emptyset\} \implies \emptyset \in C)$$

which is contingent on the elements of C. There is no guarantee that C contains the empty set.

Therefore, by the definition of a subset, the expression is false. □

Proof. (e) This statement is false because the set does not contain the set $\{1, 2\}$. $\square$

Proof. (f) Let $A = \{\{4\}\}, B = \{1, 2, 3, \{4\}\}$. Then, by the definition of a subset,

$$\{\{4\}\} \subseteq \{1, 2, 3, \{4\}\} \iff \forall a(a \in \{\{4\}\} \implies a \in \{1, 2, 3, \{4\}\})$$

Let $a = \{4\}$, the only element of A. Then,

$$\{\{4\}\} \subseteq \{1, 2, 3, \{4\}\} \iff (\{4\} \in \{\{4\}\} \implies \{4\} \in \{1, 2, 3, \{4\}\})$$

which is true.

Therefore, by the definition of a subset, the expression is true. $\square$

2. Let $\Delta = [0, 1) = \{x \in \mathbb{R} : 0 \leq x < 1\}$ and let $A_\alpha = (-\alpha, \alpha] = \{x \in \mathbb{R} : -\alpha < x \leq \alpha\} \subseteq \mathbb{R}$, where $\alpha \in \Delta$. Prove that

$$\bigcup_{\alpha \in \Delta} A_\alpha = (-1, 1),$$

and

$$\bigcap_{\alpha \in \Delta} A_\alpha = \emptyset$$

Proof. (a) Assume □

Proof. (b) For this to be true, then for all $\alpha, \beta \in \Delta$, $A_\alpha \neq A_\beta$, which means the union of all these sets is pairwise disjoint. □

3. Let A, B, C, and D be sets with $C \subseteq A$ and $D \subseteq B$. Prove that $C \cup D \subseteq A \cup B$.

Proof. Since $C \subseteq A$, the set A must be at least as large as C and contain every element C has. The same follows for D and B, since $D \subseteq B$. Then, $A \cup B$ is the set containing at least every element in the sets C and D, and as such must contain $C \cup D$. $\square$

4. Prove that if $\mathcal{A}$ is a non-empty family of sets, then

$$\bigcap_{A \in \mathcal{A}} A \subseteq \bigcup_{A \in \mathcal{A}} A.$$

Proof. The intersection over a family of sets is defined as:

$$\bigcap_{A \in \mathcal{A}} A = \{x | (\forall A)(A \in \mathcal{A} \implies x \in A)\}$$

And the union over a family of sets is defined as:

$$\bigcup_{A \in \mathcal{A}} A = \{x | (\exists A)((A \in \mathcal{A}) \wedge (x \in A))\}$$

$\square$

5. Use the principle of mathematical induction to prove $4^{n+4} > (n+4)^4$, for all natural numbers n.

Proof. Let $n = 1$: $4^5 > 5^4$. We see that the base case is true. Then, let $n = k$: $4^{k+4} > (k+4)^4$. Let $n = k+1$: $4^{k+5} > (k+5)^4$. Then:

$$4^{k+5} = 4^k * 4^5$$

$$4^{k+4} = 4^k * 4^4$$

$$4(4^{k+4}) > 4^{k+4}$$

$$4^k * 4^5 = 4(4^{k+4})$$

$$(k+5)^4 = k^4 + 20k^3 + 150k^2 + 500k + 625$$

$$(k+4)^4 = k^4 + 16k^3 + 96k^2 + 256k + 256$$

$$4(k+4)^4 = 4k^4 + 64k^3 + 384k^2 + 1024k + 1024 > (k+5)^4$$

$$4^{k+5} = 4(4^{k+4}) > 4(k+4)^4 > (k+5)^4$$

Then, by the principle of mathematical induction, $4^{n+4} > (n+4)^4 \ \forall n \in \mathbb{N}$. $\square$