Homework 1
Connor Baker, January 2017

1. Prove by contradiction that if $a - b$ is odd, then $a + b$ is odd.

Proof. Assume that $a, b \in \mathbb{Z}$, $a - b$ is odd, and $a + b$ is even. If $a - b$ is odd, then by definition, $a - b = 2x + 1$ for some number $x \in \mathbb{Z}$. If $a + b$ is even, then by definition, $a + b = 2y$ for some number $y \in \mathbb{Z}$.

Combining the system of equations with addition yields $2a = 2x+2y+1$. This can be rewritten as $2a = 2(x+y)+1$. This implies that an even number (the product $2a$) can be equal to an odd number (the even numer resulting from the product of $2(x + y)$ plus one), which is a contradicition.

Therefore, through by law of the excluded middle, if $a - b$ is odd, then $a + b$ is odd. $\square$

2. Write a proof by contrapositive to show that if xy is odd, then both x and y are odd.

Proof. We will prove that if x or y is even, then the product xy is even. Assume that x is even, and that $x, y \in \mathbb{Z}$. By definition, $x = 2k$ for all $k \in \mathbb{Z}$. Then $xy = 2ky$ is even. So, regardless of the parity of y, the product xy will be even so long as at least one is even. If x was odd, and y was even, then the above would still hold, due to multiplication being commutative.

Since either x or y is even, and xy is even, we can infer by the contrapositive that if xy is odd, then both x and y are odd. $\square$

3. Prove that there do not exist integers m and n such that $12m + 15n = 1$.

Proof. The equation $12m + 15n = 1$ is equivalent to $3(4m + 5n) = 1$. For this statement to be true, $4m + 5n$ must be the multiplicative inverse of 3, which is not in the set of natural numbers. Therefore, there do not exist integers m and n such that $12m + 15n = 1$. $\square$

4. Prove there is a natural number M such that for every natural number n, $\frac{1}{n} < M$.

Proof. Because $n \in \mathbb{N}$, $n \geq 1$ for all choices of n. As a result, $1/n \leq 1$, for all choices of n. Therefore, M can be any number such that $M \geq 2$. $\square$

5. Prove that if $-2 < x < 1$ or $x > 3$, then $\frac{(x-1)(x+2)}{(x-3)(x+4)} > 0$.

Proof. Let the function $f(x)$ be as follows:

$$f(x) = \frac{(x-1)(x+2)}{(x-3)(x+4)}$$

Then $f(x)$ has two x-intercepts at $x = -2, 1$, and two vertical asymptotes at $x = -4, 3$. By the Intermediate Value Theorem, those four x-values are the only places that the function can change the sign of its output. As such, it has been established that $f(x)$ does not change sign over the given intervals.

The sign of the function on the interval $(-\infty, -2)$ is not of concern: we care only about the interval $(-2, 1)$ and $(3, \infty)$. By picking a point on the intervals $(-2, 1)$ and $(3, \infty)$ and verifying the sign, the Intermediate Value Theorem proves that the function value has the same sign on the entirety of the interval.

Let $x = 0$. Then, on the interval $(-2, 1)$, the function is positive.

Let $x = 4$. Then, on the interval $(3, \infty)$, the function is positive.

Therefore, by the Intermediate Value Theorem, if $-2 < x < 1$ or $x > 3$, then $\frac{(x-1)(x+2)}{(x-3)(x+4)} > 0$. $\square$