Homework 4
Connor Baker, March 2017

1. Prove that if R is a partial order on a set A, then R^{-1} (the inverse relation) is also a partial order on A.

Proof. For R to be a partial order on a set A, it must be reflexive, transitive, and anti-symmetric. R^{-1} must also have these properties to be a partial order on A. We first prove reflexivity:
Assume that $\forall x \in A, (x, x) \in R$. Then, by the reflexivity of R, it must be that case that $(x, x) \in R^{-1}$. As such, R^{-1} is reflexive. We now prove that R^{-1} is anti-symmetric.
Assume that $(x, y) \in R$. Then, since R^{-1} is the inverse of R, $(y, x) \in R^{-1}$. If it was the case that $(y, x) \in R$, then $x = y$ (by the definition of anti-symmetry). As such, if $(x, y) \in R^{-1}$, then $y = x$, and R^{-1} is anti-symmetric.
Assume that $(x, y) \in R$, and $(y, z) \in R$. Then, since R was transitive, $(x, z) \in R$. Because R^{-1} is the inverse of R, if the assumption is true, then $(y, x) \in R^{-1}$, $(z, y) \in R^{-1}$, and $(z, x) \in R^{-1}$. As such, R^{-1} is transitive.
Therefore, because R^{-1} is reflexive, anti-symmetric, and transitive, R^{-1} is a partial order on A. $\square$

2. Let R be a relation on the set A. Prove that if S is a symmetric relation on A, and $R \subseteq S$, then $R^{-1} \subseteq S$.

Proof. Since S is a symmetric relation on A, if $(x, y) \in R$ (which is a subset of S) then $(x, y) \in S$, and by symmetry, $(y, x) \in S$. Then, $(y, x) \in R^{-1}$ since it is the inverse of R. We know this to be in S, so $R \subseteq S$. $\square$

3. Let R be an antisymmetric relation on the nonempty set A. Prove that if R is symmetric and $\mathrm{dom}(R) = A$, then $R = I_A$ (the identity relation on A).

Proof. For R to be an identity relation on A, it must be the case that $R = \{(x, y) \in A \times A : x = y\}$. Since R is symmetric, and $\mathrm{dom}(R) = A$, then $\forall x, y \in A$ it must be the case that $(x, y) \in R$ and $(y, x) \in R$. Furthermore, since R is anti-symmetric, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$. As such, $\forall x \in A, (x, x) \in R$. Therefore, $R = I_A$. $\square$

4. Prove that the subset of every well-ordered set is well ordered.

Proof. Assume that the linear ordering R on A is well ordered. Then, every nonempty subset B of A has a least element in B. Let $S \subseteq B$. Then, $\forall S \in \mathcal{P}(B) - \{\emptyset\}$, there is an element $x \in S$ such that $\forall y \in S, x \leq y$. As such, every subset S of B has a least element in S. Therefore, every subset of a well-ordered set is well-ordered. $\square$

5. Prove that R is transitive on a set A if and only if $R \circ R \subseteq R$.

Proof. Assume that $R \circ R \subseteq R$. Then, if $(x, y) \in R$ and $(y, z) \in R$, $(x, z) \in R \circ R$. Since $R \circ R \subseteq R$, it must be the case that $(x, z) \in R$. As such, R is transitive. $\square$