Definitions and Theorems
Connor Baker, March 2017

Definition 1 (Statement)**.** Any sentence which can be evaluated as either true or false.

Definition 2 (Compound Statement)**.** A statement made up of one or more component statements connected by logical connectors.

Definition 3 (Equivalence of Logical Operators)**.** Two sets of logical operators are said to be equivalent if they produce the same output.

Definition 4 (Tautology)**.** A statement that's always true.

Definition 5 (Contradiction)**.** A statement that's always false.

Definition 6 (A Set)**.** Any collection of objects.

Definition 7 (Set Builder Notation)**.** {expression: rule}

Definition 8 (Universal Set)**.** The given or implied set that contains all other sets in the problem. This set fixes Russel's Paradox.

Definition 9 (Tautology)**.** A statement that's always true.

Definition 10 (Natural Numbers)**.** The set $\mathbb{N} : \{1, 2, 3, \dots\}$.

Definition 11 (Integers)**.** The set $\mathbb{Z} : \{\dots, -1, 0, 2, \dots\}$.

Definition 12 (Rational Numbers)**.** The set $\mathbb{Q} : \{\frac{a}{b} : a \in \mathbb{Z} \text{ and } b \in \mathbb{N}\}$.

Definition 13 (Real Numbers)**.** The set $\mathbb{R} : \{a_n a_{n-1} \dots a_1 a_0 a_{-1} a_{-2} \cdots : n \in \mathbb{N} \cup \{0\} \text{ and } a_i \in \{0, \dots, 9\}\}$.

Definition 14 (Complex Numbers)**.** The set $\mathbb{C} : \{a + bi : i^2 = -1 \text{ and } a, b \in \mathbb{R}\}$.

Definition 15 (Subset)**.** Given two sets A and B, $A \subseteq B \iff \forall a \in A \implies a \in B$.

Definition 16 (Open Sentence (AKA Predicate))**.** A statement that contains a variable. The truth value depends on the variable.

Definition 17 (Truth Set)**.** The set of values that make the statement true.

Definition 18 (Quantifiers and Negations)**.**

1. Universal Quantifier: $\forall$ – Must be true for all x in the universal set such that $P(x)$ is true: $(\forall x)(P(x))$.
2. Existential Quantifier: $\exists$ – True if for at least one x in the universal set such that $P(x)$ is true: $(\exists x)(P(x))$.
3. Unique Quantifier: $\exists!$ – True if there exists only one x in the universal set such that $P(x)$ is true: $(\exists! x)(P(x))$.
4. Negation of the Universal Quantifier: $\sim (\forall x)(P(x))$ is $(\exists x)(\sim P(x))$.
5. Negation of the Existential Quantifier: $\sim (\exists x)(P(x))$ is $(\forall x)(\sim P(x))$.

Definition 19 (Direct Proof)**.** $P \implies Q$.

Definition 20 (Contrapositive Proof)**.** $(\sim Q) \implies (\sim P)$.

Definition 21 (Proof by Contradiction)**.** We start with $P \implies Q$. Assume that $\sim P \wedge Q$ is true. Then $\sim P \implies A_1 \implies A_2 \implies \cdots \implies R$. And, if $Q \implies B_1 \implies B_2 \implies \cdots \implies \sim R$. Then,$\sim R \wedge R$ must be true, which is a contradiction, so the original assumption is false, and $P \implies Q$.

Definition 22 (Axioms of the Natural Numbers)**.**

1. Successor property
   (a) One is a natural number

(b) One is not the successor of any number
(c) Every natural number has a unique successor

2. Closure under addition and multiplication
3. Associativity
4. Commutativity
5. Distribution of multiplication over addition
6. Cancellation
   (a) Real numbers have this property unless the number being cancelled is a zero
   (b) Matrix multiplication does not have this property

Definition 23 (Divisible)**.** Let $a, b \in \mathbb{N}$. Then $a|b$ if $\exists k \in \mathbb{N} : ak = b$.

Definition 24 (Prime)**.** A number p, where $p \in \mathbb{N}$, is prime if $p > 1$ and its only divisors are one and itself.

Definition 25 (Factor)**.** A number q, where $q \in \mathbb{N}$, is a factor of r if $q|r$.

Definition 26 (Prime Factor Decomposition)**.** Let $p_1, p_2, \ldots, p_k$ be all primes less than q. Then, the prime factor decomposition of q is $p_1^{n_1} p_2^{n_2}, \ldots, p_k^{n_k}$ where $n_i \in (\mathbb{N} \cup \{0\})$.

Theorem 27 (Fundamental Theorem of Arithmetic)**.** All natural numbers have a unique prime factorization up to commutativity.

Definition 28 (Union over $\mathcal{A}$)**.** Let $\mathcal{A}$ be a family of sets. The union over $\mathcal{A}$ is defined as:

$$\bigcup_{A \in \mathcal{A}} = \{x : (\exists A \in \mathcal{A})(x \in A)$$

which is equivalent to:

$$\bigcup_{A \in \mathcal{A}} = \{x : (\exists A)((A \in \mathcal{A}) \wedge (x \in A))$$

Definition 29 (Intersection over $\mathcal{A}$)**.** Let $\mathcal{A}$ be a family of sets. The intersection over $\mathcal{A}$ is defined as:

$$\bigcap_{A \in \mathcal{A}} = \{x : (\forall A \in \mathcal{A})(x \in A)$$

which is equivalent to:

$$\bigcap_{A \in \mathcal{A}} = \{x : (\forall A)((A \in \mathcal{A}) \implies (x \in A))$$

Theorem 30 (Relative Cardinality of Intersection and Union)**.** For every set $B \in \mathcal{A}$:

$$B \subseteq \bigcup_{A \in \mathcal{A}} A,$$

$$\bigcap_{A \in \mathcal{A}} A \subseteq B.$$

The intersection is no bigger than the smallest set, and the union is no smaller than the biggest set.
Assume that $\mathcal{A} \neq \emptyset$. Then:

$$\bigcap_{A \in \mathcal{A}} A \subseteq \bigcup_{A \in \mathcal{A}} A.$$

If $\mathcal{A} \neq \emptyset$, the union isn't a problem but the intersection would be the set of all sets, and as such is undefined.

Definition 31 (Family of Sets)**.** Let Δ be a nonempty set. Then, $\forall \alpha \in \Delta$, there is a corresponding set A_α. The family of sets $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$.