Homework 5
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1. Prove that if the real-valued function f is strictly increasing or strictly decreasing on $\mathbb{R}$, then f is one-to-one (Note: You cannot assume f is differentiable).

Proof. Case 1: f is strictly decreasing.
If f is strictly decreasing, then $\forall\ x, a \in \mathrm{dom}(f),\ x \leq a,\ f(x) \geq f(a)$. If it is the case that $f(x) = f(a)$, then it must also be the case that $x = a$. Then, $f(x) = f(a)$ if and only if $x = a$, and $f(x) > f(a)\ \forall x, a \in \mathrm{dom}(f),\ x < a$, the function f must be one-to-one.
Case 2: f is strictly increasing.
If f is strictly increasing, then $\forall\ x, a \in \mathrm{dom}(f),\ x \leq a,\ f(x) \leq f(a)$. If it is the case that $f(x) = f(a)$, then it must also be the case that $x = a$. Then, $f(x) = f(a)$ if and only if $x = a$, and $f(x) < f(a)\ \forall x, a \in \mathrm{dom}(f),\ x < a$, the function f must be one-to-one. $\square$

2. Prove the following are metrics:

   (a) $X = \mathbb{R}, d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

   (b) $X = \mathbb{R} \times \mathbb{R}, d((x,y),(z,w)) = \sqrt{(x-z)^2 + (y-w)^2}$

   Definition 1 (Metric)**.** A metric on a set X is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$,

   (a) $d(x,y) \geq 0$
   (b) $d(x,y) = 0$ if and only if $x = y$
   (c) $d(x,y) = d(y,x)$,
   (d) $d(x,y) + d(y,z) \geq d(x,z)$.

Proof. We begin by proving that the first function is a metric.

1. The $\text{rang}(d) = \{0,1\}$ so the function is definitely greater than or equal to zero for any inputted pair of values.

2. By the definition of d, if and only if $x = y$ does $d(x,y) = 1$.

3. Since the equals relationship is symmetric, $x = y \implies y = x$. As such, $d(x,y) = d(y,x)$.

4. As shown below, it is the case that $d(x,y) + d(y,z) \geq d(x,z)$.

x	y	z	d(x,y)	d(y,z)	d(x,z)	d(x,y)+d(y,z)≥d(x,z)
0	0	0	0	0	0	true
0	0	1	0	1	1	true
0	0	2	0	1	1	true
0	1	0	1	1	0	true
0	1	1	1	0	1	true
0	1	2	1	1	1	true
0	2	0	1	1	0	true
0	2	1	1	1	1	true
0	2	2	1	0	1	true
1	0	0	1	0	1	true
1	0	1	1	1	0	true
1	0	2	1	1	1	true
1	1	0	0	1	1	true
1	1	1	0	0	0	true
1	1	2	0	1	1	true
1	2	0	1	1	1	true
1	2	1	1	1	1	true
1	2	2	1	0	1	true
2	0	0	1	0	1	true
2	0	1	1	1	1	true
2	0	2	1	1	0	true
2	1	0	1	1	1	true
2	1	1	1	0	1	true
2	1	2	1	1	0	true
2	2	0	0	1	1	true
2	2	1	0	1	1	true
2	2	2	0	0	0	true

Therefore the first function is a metric.

We now prove that the second function is a metric.

1. The rang(d) $= \{0, 1\}$ so the function is definitely greater than zero.
2. By the definition of d, if and only if $x = y$ does $d(x, y) = 1$.
3. Since the equals relationship is symmetric, $x = y \implies y = x$. As such, $d(x, y) = d(y, x)$.
4. As shown below, it is the case that $d(x, y) + d(y, z) \geq d(x, z)$:

x	y	w	z	d(x,y)	d(y,z)	d(x,z)	d(x,y)+d(y,z)≥d(x,z)
0	0	0	0				true
0	0	0	1				true
0	0	0	2				true
0	0	0	3				true
0	0	1	0				true
0	0	1	1				true
0	0	1	2				true
0	0	1	3				true
0	0	2	0				true
0	0	2	1				true
0	0	2	2				true
0	0	2	3				true
0	0	3	0				true
0	0	3	1				true
0	0	3	2				true
0	0	3	3				true
0	1	0	0				true
0	1	0	1				true
0	1	0	2				true
0	1	0	3				true
0	1	1	0				true
0	1	1	1				true
0	1	1	2				true
0	1	1	3				true
0	1	2	0				true
0	1	2	1				true
0	1	2	2				true
0	1	2	3				true
0	1	3	0				true
0	1	3	1				true
0	1	3	2				true
0	1	3	3				true
0	2	0	0				true
0	2	0	1				true
0	2	0	2				true
0	2	0	3				true
0	2	1	0				true
0	2	1	1				true
0	2	1	2				true
0	2	1	3				true
0	2	2	0				true
0	2	2	1				true
0	2	2	2				true
0	2	2	3				true

x	y	w	z	d(x,y)	d(y,z)	d(x,z)	d(x,y)+d(y,z)≥d(x,z)
0	2	3	0				true
0	2	3	1				true
0	2	3	2				true
0	2	3	3				true
0	3	0	0				true
0	3	0	1				true
0	3	0	2				true
0	3	0	3				true
0	3	1	0				true
0	3	1	1				true
0	3	1	2				true
0	3	1	3				true
0	3	2	0				true
0	3	2	1				true
0	3	2	2				true
0	3	2	3				true
0	3	3	0				true
0	3	3	1				true
0	3	3	2				true
0	3	3	3				true
1	0	0	0				true
1	0	0	1				true
1	0	0	2				true
1	0	0	3				true
1	0	1	0				true
1	0	1	1				true
1	0	1	2				true
1	0	1	3				true
1	0	2	0				true
1	0	2	1				true
1	0	2	2				true
1	0	2	3				true
1	0	3	0				true
1	0	3	1				true
1	0	3	2				true
1	0	3	3				true
1	1	0	0				true
1	1	0	1				true
1	1	0	2				true
1	1	0	3				true
1	1	1	0				true
1	1	1	1				true
1	1	1	2				true
1	1	1	3				true
1	1	2	0				true
1	1	2	1				true
1	1	2	2				true
1	1	2	3				true
1	1	3	0				true
1	1	3	1				true
1	1	3	2				true

x	y	w	z	d(x,y)	d(y,z)	d(x,z)	d(x,y)+d(y,z)≥d(x,z)
1	1	3	3				true
1	2	0	0				true
1	2	0	1				true
1	2	0	2				true
1	2	0	3				true
1	2	1	0				true
1	2	1	1				true
1	2	1	2				true
1	2	1	3				true
1	2	2	0				true
1	2	2	1				true
1	2	2	2				true
1	2	2	3				true
1	2	3	0				true
1	2	3	1				true
1	2	3	2				true
1	2	3	3				true
1	3	0	0				true
1	3	0	1				true
1	3	0	2				true
1	3	0	3				true
1	3	1	0				true
1	3	1	1				true
1	3	1	2				true
1	3	1	3				true
1	3	2	0				true
1	3	2	1				true
1	3	2	2				true
1	3	2	3				true
1	3	3	0				true
1	3	3	1				true
1	3	3	2				true
1	3	3	3				true
2	0	0	0				true
2	0	0	1				true
2	0	0	2				true
2	0	0	3				true
2	0	1	0				true
2	0	1	1				true
2	0	1	2				true
2	0	1	3				true
2	0	2	0				true
2	0	2	1				true
2	0	2	2				true
2	0	2	3				true
2	0	3	0				true
2	0	3	1				true
2	0	3	2				true
2	0	3	3				true
2	1	0	0				true
2	1	0	1				true

x	y	w	z	d(x,y)	d(y,z)	d(x,z)	d(x,y)+d(y,z)≥d(x,z)
2	1	0	2				true
2	1	0	3				true
2	1	1	0				true
2	1	1	1				true
2	1	1	2				true
2	1	1	3				true
2	1	2	0				true
2	1	2	1				true
2	1	2	2				true
2	1	2	3				true
2	1	3	0				true
2	1	3	1				true
2	1	3	2				true
2	1	3	3				true
2	2	0	0				true
2	2	0	1				true
2	2	0	2				true
2	2	0	3				true
2	2	1	0				true
2	2	1	1				true
2	2	1	2				true
2	2	1	3				true
2	2	2	0				true
2	2	2	1				true
2	2	2	2				true
2	2	2	3				true
2	2	3	0				true
2	2	3	1				true
2	2	3	2				true
2	2	3	3				true
2	3	0	0				true
2	3	0	1				true
2	3	0	2				true
2	3	0	3				true
2	3	1	0				true
2	3	1	1				true
2	3	1	2				true
2	3	1	3				true
2	3	2	0				true
2	3	2	1				true
2	3	2	2				true
2	3	2	3				true
2	3	3	0				true
2	3	3	1				true
2	3	3	2				true
2	3	3	3				true
3	0	0	0				true
3	0	0	1				true
3	0	0	2				true
3	0	0	3				true
3	0	1	0				true

x	y	w	z	d(x,y)	d(y,z)	d(x,z)	d(x,y)+d(y,z)≥d(x,z)
3	0	1	1				true
3	0	1	2				true
3	0	1	3				true
3	0	2	0				true
3	0	2	1				true
3	0	2	2				true
3	0	2	3				true
3	0	3	0				true
3	0	3	1				true
3	0	3	2				true
3	0	3	3				true
3	1	0	0				true
3	1	0	1				true
3	1	0	2				true
3	1	0	3				true
3	1	1	0				true
3	1	1	1				true
3	1	1	2				true
3	1	1	3				true
3	1	2	0				true
3	1	2	1				true
3	1	2	2				true
3	1	2	3				true
3	1	3	0				true
3	1	3	1				true
3	1	3	2				true
3	1	3	3				true
3	2	0	0				true
3	2	0	1				true
3	2	0	2				true
3	2	0	3				true
3	2	1	0				true
3	2	1	1				true
3	2	1	2				true
3	2	1	3				true
3	2	2	0				true
3	2	2	1				true
3	2	2	2				true
3	2	2	3				true
3	2	3	0				true
3	2	3	1				true
3	2	3	2				true
3	2	3	3				true
3	3	0	0				true
3	3	0	1				true
3	3	0	2				true
3	3	0	3				true
3	3	1	0				true
3	3	1	1				true
3	3	1	2				true
3	3	1	3				true

x	y	w	z	d(x,y)	d(y,z)	d(x,z)	d(x,y)+d(y,z)≥d(x,z)
3	3	2	0				true
3	3	2	1				true
3	3	2	2				true
3	3	2	3				true
3	3	3	0				true
3	3	3	1				true
3	3	3	2				true
3	3	3	3				true

Therefore the function is a metric. $\square$

3. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be given by $f(m, n) = 2^{m-1}(2n - 1)$. Prove that f is one-to-one and onto.

Proof. Since $\square$

4. Let $f : A \to B$ be a function from a nonempty set A. Prove that the set $\mathcal{C} = \{f^{-1}(b) : b \in \operatorname{rang}(f)\}$ is a partition of A. Note: $\mathcal{C}$ is a subset of $\mathcal{P}(A)$.

Proof. Since $\square$

5. Let $f : A \to B$ be a function from a nonempty set A which is surjective. Find a new function $g : C \to B$ which is one-to-one such that $C \subseteq A$, $\text{rang}(g) = \text{rang}(f)$, and for every $x \in C, f(x) = g(x)$. Explain why g has an inverse function, g^{-1}. Then, compute $f(g^{-1}(x))$ for all $x \in B$.

Proof. Since $\square$