MTH 295: Homework 1

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1. Prove by contradiction that if $a - b$ is odd, then $a + b$ is odd.

Proof. Assume that both a and b are integers. We proceed using proof by contradiction, and assume that $a - b$ is even implies that $a + b$ is odd.

Since $a - b$ is even, it can be written as

$$a - b = 2k \tag{1}$$

for some integer k, and since $a + b$ is odd, it can be written as

$$a + b = 2j + 1 \tag{2}$$

for some integer j. Solving both equations for a yields

$$a = 2k + b \tag{3a}$$
$$a = 2j + 1 - b \tag{3b}$$

and setting them equal to the other (by transitivity) gives:

$$2k + b = 2j + 1 - b. \tag{3c}$$

Simplifying by means of collecting b on the left hand side and factoring out a two gives

$$2(k + b) = 2j + 1. \tag{3d}$$

Since an even number can never be equal to an odd number (by definition), we have arrived at a contradiction, and our original assumption that $a - b$ is even implies that $a + b$ is odd must be incorrect.

Therefore, by means of proof by contradiction, if $a - b$ is odd, then $a + b$ is odd. □

2. Write a proof by contrapositive to show that if xy is odd, then both x and y are odd.

Proof. Assume that x and y are integers. We proceed with proof by contrapositive, so we assume that if x or y are even, then xy is even.

We begin with the case in which x is even. Then, x can be rewritten as $x = 2k$ for some integer k. Then the product xy can be written as

$$\begin{align} xy &= 2k \cdot y \tag{1} \\ &= 2(ky) \tag{2} \end{align}$$

which is even, by definition. Without loss of generality, the case in which y is even is the same (simply swap the variables x and y).

Finally, we have the case in which both x and y are even. In this case, x and y can be written

$$\begin{align} x &= 2k \tag{3} \\ y &= 2j \tag{4} \end{align}$$

for some integers k and j. With these equivalencies, we can rewrite the product xy as follows:

$$\begin{align} xy &= 2k \cdot 2j \tag{5} \\ &= 2(2kj) \tag{6} \end{align}$$

which is even, by definition.

Since either x or y being even implies xy is even, we can infer by the contrapositive that if xy is odd, then both x and y are odd. □

3. Prove that there do not exist integers m and n such that $12m + 15n = 1$.

Proof. The equation $12m + 15n = 1$ is equivalent to $3(4m + 5n) = 1$. For this statement to be true, $4m + 5n$ must be the multiplicative inverse of 3, which is not in the set of natural numbers. Therefore, there do not exist integers m and n such that $12m + 15n = 1$. $\square$

4. Prove there is a natural number M such that for every natural number n, $\frac{1}{n} < M$.

Proof. Because $n \in \mathbb{N}$, $n \geq 1$ for all choices of n. As a result, $1/n \leq 1$, for all choices of n. Therefore, M can be any number such that $M \geq 2$. $\square$

5. Prove that if $-2 < x < 1$ or $x > 3$, then $\frac{(x-1)(x+2)}{(x-3)(x+4)} > 0$.

Proof. Let the function $f(x)$ be as follows:

$$f(x) = \frac{(x-1)(x+2)}{(x-3)(x+4)} \tag{1}$$

Then $f(x)$ has two x-intercepts at $x = -2$ and at $x = 1$, and two vertical asymptotes at $x = -4$ and $x = 3$. By the Intermediate Value Theorem, those four x-values are the only places that the function can change sign. As such, it has been established that $f(x)$ does not change sign over the intervals

$$(-\infty, -4), (-4, -2), (-2, 1), (1, 3), (3, \infty).$$

By picking a point on the intervals $(-2, 1)$ and $(3, \infty)$ and verifying the sign, then by the Intermediate Value Theorem proves, the function value has the same sign on the entirety of the interval.

Let $x = 0$. Then, on the interval $(-2, 1)$, the function is positive.

Let $x = 4$. Then, on the interval $(3, \infty)$, the function is positive.

Therefore, by the Intermediate Value Theorem, if $-2 < x < 1$ or $x > 3$, then $\frac{(x-1)(x+2)}{(x-3)(x+4)} > 0$. $\square$