Rudin's Principles of Mathematical Analysis, 3rd ed

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Numerical Sequences and Series: Selected Exercises

1. Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?

Proof. 1a)

Assume that $\{s_n\}$ is a convergent Cauchy sequence, and let $\epsilon > 0$. Since $\{s_n\}$ is a Cauchy sequence, for every $\epsilon > 0, \exists N \in \mathbb{Z}$ such that $|s_n - s_m| < \epsilon, \forall m, n \geq N$.

Claim: $||p| - |q|| \leq |p - q|$.

Case 1: $p \geq 0, q \geq 0$.
Then $||p| - |q|| = |p - q|$.

Case 2: $p \leq 0, q \leq 0$.
Then $||p| - |q|| = |(-p) - (-q)| = |q - p| = |-1(p - q)| = |-1| \cdot |p - q| = |p - q|$.

Case 3: $p \leq 0, q \geq 0$.
Then $||p| - |q|| = |(-p) - q| = |-1(p + q)| = |-1| \cdot |p + q| = |p + q| \leq |p - q|$, since subtracting a non-negative number from a non-positive number increases the magnitude of the result more so than adding a non-negative number to a non-positive number.

Case 4: $p \geq 0, q \leq 0$.
Then $||p| - |q|| = |p - (-q)| = |p + q| \leq |p - q|$, since subtracting a non-positive number from a non-negative number increases the magnitude of the result more so than adding a non-positive number to a non-negative number.

As such, $||s_n| - |s_m|| \leq |s_n - s_m| < \epsilon$ for all $m, n \geq N$.

Therefore, $\{|s_n|\}$ is a convergent Cauchy sequence. $\square$

Proof. 1b)

Since $||p| - |q|| \leq |p - q|$, the converse is not true – having $||s_n| - |s_m|| < \epsilon, \forall m, n \geq N$ tells us nothing about whether $|s_n - s_m| < \epsilon, \forall m, n \geq N$ is true. $\square$

2. Calculate $\lim_{n\to\infty}(\sqrt{n^2+n}-n)$.

Proof. We begin by multiplying by the algebraic conjugate:

$$\lim_{n\to\infty}\left[(\sqrt{n^2+n}-n)\cdot\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n}\right]=\lim_{n\to\infty}\left[\frac{n^2+n-n^2}{\sqrt{n^2+n}+n}\right]=\lim_{n\to\infty}\left[\frac{n}{\sqrt{n^2+n}+n}\right].$$

Tentatively trying to evaluate the limit yields a composition of algebraic indeterminate forms:

$$\lim_{n\to\infty}\left[\frac{n}{\sqrt{n^2+n}+n}\right]=\frac{\infty}{\sqrt{\infty}+\infty}=\frac{\infty}{\infty+\infty}.$$

We proceed by multiplying by a form of one:

$$\lim_{n\to\infty}\left[\frac{n}{\sqrt{n^2+n}+n}\cdot\frac{\frac{1}{n}}{\frac{1}{n}}\right]=\lim_{n\to\infty}\left[\frac{\frac{n}{n}}{\sqrt{\frac{n^2}{n^2}+\frac{n}{n^2}}+\frac{n}{n}}\right]=\lim_{n\to\infty}\left[\frac{1}{\sqrt{1+\frac{1}{n}}+1}\right].$$

From this point, we can now successfully evaluate the limit.

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{1+\frac{1}{n}}+1}\right]=\frac{1}{\sqrt{1+\frac{1}{\infty}}+1}=\frac{1}{\sqrt{1+0}+1}=\frac{1}{2}.$$

$\square$

3. If $s_1 = \sqrt{2}$, and

$$s_{n+1} = \sqrt{2 + s_n}, (n = 1, 2, 3, \dots),$$

prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1, 2, 3, \dots$.

Proof. Claim: $s_n < 2, \forall n \in \mathbb{N}$.

Base case: Let $n = 1$. Then $s_1 = \sqrt{2} < 2$ and the base case holds.

Inductive hypothesis: Let $n = k$. Assume that for all $n \leq k$, $s_k < 2$.

Inductive step: Let $n = k + 1$. Then $s_{k+1} = \sqrt{2 + s_k}$, which, by our inductive hypothesis, is less than $\sqrt{2+2} = \sqrt{4} = 2$.

By the Principle of Mathematical Induction, $s_n < 2, \forall n \in \mathbb{N}$. As such, s_n is bounded above.

Claim: $s_n < s_{n+1}, \forall n \in \mathbb{N}$.

Base case: Let $n = 1$. Then $s_1 = \sqrt{2} < \sqrt{2 + \sqrt{2}} = s_2$ and the base case holds.

Inductive hypothesis: Let $n = k$. Assume that for all $n \leq k$, $s_k < s_{k+1}$.

Inductive step: Let $n = k + 1$. Then $s_{k+1} = \sqrt{2 + s_k}$, which, by our inductive hypothesis, is less than $\sqrt{2 + s_{k+1}} = s_{k+2}$.

By the Principle of Mathematical Induction, $s_n < s_{n+1}, \forall n \in \mathbb{N}$. As such, s_n is monotonic increasing.

Since s_n is both bounded above and monotonic increasing, by the monotone convergence theorem, s_n converges. $\square$

4. Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_1 = 0; \quad s_{2m} = \frac{s_{2m-1}}{2}; \quad s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Proof. We begin by trying to find the closed form of the sequence $\{s_n\}$.

m	s_{2m}	s_{2m+1}
1	$s_2 = \frac{0}{2}$	$s_3 = \frac{1}{2} + 0 = \frac{1}{2}$
2	$s_4 = \frac{\frac{1}{2}}{2} = \frac{1}{2^2}$	$s_5 = \frac{1}{2} + \frac{1}{2^2} = \frac{2+1}{2^2}$
3	$s_6 = \frac{\frac{2+1}{2^2}}{2} = \frac{2+1}{2^3}$	$s_7 = \frac{1}{2} + \frac{2+1}{2^3} = \frac{2^2+2+1}{2^3}$
$\vdots$	$\vdots$	$\vdots$
m	$s_{2m} = \frac{\sum 2^k{}_{k=0}^{m-2}}{2^m}$	$s_{2m+1} = \frac{\sum 2^k{}_{k=0}^{m-1}}{2^m}$

We then try to find the closed forms of the sums that we're using – since these are geometric series, we can use the formula that follows:

$$\sum_k^n ar^k = \frac{a(1-r^n)}{1-r}.$$

For our series, $a = 1$, and $r = 2$, so:

$$\frac{a(1-2^n)}{1-2} = \frac{1-2^n}{-1} = 2^n - 1.$$

As such, we now have:

$$s_{2m} = \frac{\sum 2^k{}_{k=0}^{m-2}}{2^m} = \frac{2^{m-1}-1}{2^m} = \frac{1}{2} - \frac{1}{2^m},$$

and

$$s_{2m+1} = \frac{\sum 2^k{}_{k=0}^{m-1}}{2^m} = \frac{2^m - 1}{2^m} = 1 - \frac{1}{2^m}.$$

Claim: The closed forms of s_{2m} and s_{2m+1} we have derived hold $\forall m \in \mathbb{N}$. We proceed with induction.

Base case: Let $m = 1$. Then:

$$s_2 = \frac{s_1}{2} = \frac{0}{2} = 0,$$

and

$$\frac{2^{1-1}-1}{2^1} = \frac{0}{2} = 0 = s_2.$$

In addition,

$$s_3 = \frac{s_2}{2} = \frac{1}{2} + s_2 = \frac{1}{2} + 0 = \frac{1}{2},$$

and

$$\frac{2^1 - 1}{2^1} = \frac{1}{2} = s_3.$$

Assume that $\forall m \leq r$, our closed forms of s_{2m} and s_{2m+1} hold. Then

$$s_{2r} = \frac{1}{2} - \frac{1}{2^r}; \quad s_{2r+1} = 1 - \frac{1}{2^r}.$$

Let $m = r + 1$. Then

$$s_{2(r+1)} = s_{2r+2} = \frac{s_{2r+1}}{2} = \frac{1 - \frac{1}{2^r}}{2} = \frac{1}{2} - \frac{1}{2^{r+1}},$$

and

$$s_{2(r+1)+1} = s_{2r+3} = \frac{1}{2} + s_{2r+2} = \frac{1}{2} + \frac{1}{2} - \frac{1}{2^{r+1}} = 1 - \frac{1}{2^{r+1}}.$$

By the Principle of Mathematical Induction, $\forall m \in \mathbb{N}$,

$$s_{2m} = \frac{1}{2} - \frac{1}{2^m}; \quad s_{2m+1} = 1 - \frac{1}{2^m}$$

hold.

We now look at what the sequences s_{2m} and s_{2m+1} converge to:

$$\lim_{m\to\infty} s_{2m} = \lim_{m\to\infty} \frac{1}{2} - \frac{1}{2^m} = \frac{1}{2},$$

$$\lim_{m\to\infty} s_{2m+1} = \lim_{m\to\infty} 1 - \frac{1}{2^m} = 1.$$

Since the sequence converges, any sub-sequence must converge to the same point. As such, when we consider the limit supremum and infimum – that is, the supremum and infimum of all subsequenes, they must converge to the same point that the limit of the sequence converges to. As such,

$$\lim_{m\to\infty} \inf s_m = \frac{1}{2},$$

$$\lim_{m\to\infty} \sup s_m = 1.$$

$\square$

5. For any two real sequences $\{a_n\}, \{b_n\}$, prove that

$$\lim_{n\to\infty} \sup(a_n + b_n) \leq \lim_{n\to\infty} \sup(a_n) + \lim_{n\to\infty} \sup(b_n)$$

provided the sum on the right is not of the form $\infty - \infty$.

Proof. □

6. Investigate the behavior (convergence or divergence) of $\sum a_n$ if
   (a) $a_n = \sqrt{n+1} - \sqrt{n}$.
   (b) $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$.
   (c) $a_n = (\sqrt[n]{n} - 1)^n$.

A note on notation: An easy test to see if a series diverges is to check if $lim_{n\to\infty} \sup a_n \neq lim_{n\to\infty} \inf a_n$. If the series converges, then the two are equal, and we do not need to use the sup and inf notation. Below, we do not use sup and inf because either the series converges and it was needless, or it does not and we are done with the problem.

Proof. 6a)

We begin by finding a more useful form of a_n:

$$a_n = \sqrt{n+1} - \sqrt{n} = (\sqrt{n+1} - \sqrt{n}) \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

Since $\sqrt{n} < \sqrt{n+1}$,

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} > \frac{1}{\sqrt{n+1} + \sqrt{n+1}} = \frac{1}{2\sqrt{n+1}}.$$

Since we know that $\frac{1}{\sqrt{n+1}}$ diverges (it is a p-series where $p \leq 1$), we can use the limit comparison test.

$$\lim_{n\to\infty} \left(\frac{\frac{1}{2\sqrt{n+1}}}{\frac{1}{\sqrt{n+1}}} \right) = \frac{1}{2}.$$

Since $0 < \frac{1}{2} < \infty$, by the limit comparison test, a_n diverges. □

Proof. 6b)

We begin by finding a more useful form of a_n:

$$a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n} = (\frac{\sqrt{n+1} - \sqrt{n}}{n}) \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{n+1-n}{n(\sqrt{n+1} + \sqrt{n})} = \frac{1}{n(\sqrt{n+1} + \sqrt{n})}.$$

Since $\sqrt{n} < \sqrt{n+1}$,

$$\frac{1}{n(\sqrt{n+1} + \sqrt{n})} < \frac{1}{n(\sqrt{n} + \sqrt{n})} = \frac{1}{2n\sqrt{n})} = \frac{1}{2n^{3/2}}.$$

To more clearly see that the sum of our newly created sequence is a p-series, we can re-write it as such:

$$\sum^{\infty} \left(\frac{1}{2n^{3/2}} \right) = \frac{1}{2} \cdot \sum^{\infty} \left(\frac{1}{n^{3/2}} \right).$$

Since $p = \frac{3}{2}$, and $p > 1$, the sum must converge. Furthermore, since a_n is smaller than the sequence which when summed converges, a_n must also sum when converged. □

Proof. 6c)

Let $\alpha = \lim_{n\to\infty}(\sqrt[n]{(\sqrt[n]{n} - 1)^n})$. Then

$$\alpha = \lim_{n\to\infty} (\sqrt[n]{n} - 1) = \lim_{n\to\infty} (\sqrt[n]{n}) - \lim_{n\to\infty} (1) = \lim_{n\to\infty} (e^{\ln(\sqrt[n]{n})}) - 1 = \lim_{n\to\infty} (e^{\frac{\ln(n)}{n}}) - 1.$$

Plugging in for n, we arrive the basic indeterminate form $\frac{\infty}{\infty}$, so we can proceed with L'Hôpital's rule.

$$\alpha = \lim_{n\to\infty} (e^{\frac{1}{n}}) - 1 = e^0 - 1 = 1 - 1 = 0.$$

Since $\alpha = 0$, and $\alpha < 1$, by the root test $\sum a_n$ converges. □

7. Prove that the convergence of $\sum a_n$ implies the convergence of

$$\sum \left(\frac{\sqrt{a_n}}{n} \right),$$

if $a_n \geq 0$.

Proof. Since $\sum a_n$ converges, $\exists m \in \mathbb{N} : \forall n \geq m, a_n = 0$. Furthermore, $\exists N \in \mathbb{N} : \forall n \in \mathbb{N}, s_n \in [0, N]$. Take s_{m-1}.
We know that $\exists p \in \mathbb{R}$:

$$s_{m-1} = \sum_{n}^{m-1} \frac{\sqrt{a_n}}{n} = p.$$

Claim: $s_\infty - s_{m-1} = 0$.
It is clear that

$$s_\infty - s_{m-1} = \sum_{n=m}^{\infty} \frac{\sqrt{a_n}}{n}.$$

Since $n \geq m$, $a_n = 0$ and the sum is zero. then, $s_\infty = s_{m-1} = p$, and the sum converges to p. $\square$

11. Suppose $a_n > 0$, $s_n = a_1 + a_2 + \cdots + a_n$, and $\sum a_n$ diverges.

(a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges.

(b) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

and that $\sum \frac{a_n}{s_n}$ diverges.

(c) Prove that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and that $\sum \frac{a_n}{s_n^2}$ converges.

(d) What can be said about

$$\sum \frac{a_n}{1 + n \cdot a_n} \quad \text{and} \quad \sum \frac{a_n}{1 + n^2 \cdot a_n}?$$

Proof. 11a)

By re-writing the sum, we see that

$$\sum \left(\frac{a_n}{1 + a_n} \right) = \sum \left(\frac{1}{(\frac{1+a_n}{a_n})} \right) = \sum \left(\frac{1}{(\frac{1}{a_n} + 1)} \right).$$

Since $a_n > 0, \exists N \in \mathbb{N} : N \geq \frac{1}{a_n}$, by the Archimedian principle. As such, $\frac{1}{a_n} \in [0, N]$. Then, $(\frac{1}{a_n} + 1) \in [1, N+1]$, and

$$\left(\frac{1}{(\frac{1}{a_n} + 1)} \right) \geq \frac{1}{N+1}.$$

By the divergence theorem, since the terms of this infinite sum are positive, the sum must diverge. □

Proof. 11b)

□

Proof. 11c)

□

Proof. 11d)

□