Homework 5
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1. Prove that if the real-valued function f is strictly increasing or strictly decreasing on $\mathbb{R}$, then f is one-to-one (Note: You cannot assume f is differentiable).

Proof. Case 1: f is strictly decreasing.
Let $x, a \in \text{dom}(f)$. Assume that $f(x) = f(a)$. If $x \neq a$, then by trichotomy either $x < a$ or $x > a$.
Case A: If $x < a$, then $f(x) > f(a)$, and $f(x) \neq f(a)$.
Case B: If $x > a$, then $f(x) < f(a)$, and $f(x) \neq f(a)$.
In either case, if $x \neq a$, then $f(x) \neq f(a)$, and f is one-to-one.
Case 2: f is strictly increasing.
Let $x, a \in \text{dom}(f)$. Assume that $f(x) = f(a)$. If $x \neq a$, then by trichotomy either $x < a$ or $x > a$.
Case A: If $x < a$, then $f(x) < f(a)$, and $f(x) \neq f(a)$.
Case B: If $x > a$, then $f(x) > f(a)$, and $f(x) \neq f(a)$.
In either case, if $x \neq a$, then $f(x) \neq f(a)$, and f is one-to-one.
As such, f is one-to-one if it is strictly increasing or strictly decreasing on $\mathbb{R}$. $\square$

2. Prove the following are metrics:

   (a) $X = \mathbb{R}, d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

   (b) $X = \mathbb{R} \times \mathbb{R}, d((x, y), (z, w)) = \sqrt{(x - z)^2 + (y - w)^2}$

   Definition 1 (Metric)**.** A metric on a set X is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$,

   (a) $d(x, y) \geq 0$

   (b) $d(x, y) = 0$ if and only if $x = y$

   (c) $d(x, y) = d(y, x)$,

   (d) $d(x, y) + d(y, z) \geq d(x, z)$.

Proof. We begin by proving that the first function is a metric.

1. The $\text{rang}(d) = \{0, 1\}$ so the function is definitely greater than or equal to zero for any inputted pair of values.
2. By the definition of d, $d(x, y) = 0$ if and only if $x = y$.
3. Since the equals relationship is symmetric, $x = y \implies y = x$. As such, $d(x, y) = d(y, x)$, since the order of the inputs does not generate a unique output.
4. Not completed.

Therefore the first function is a metric.

We now prove that the second function is a metric.

1. The $\text{rang}(d) = \{0, 1\}$ so the function is definitely greater than zero.
2. By the definition of d, if and only if $x = y$ does $d(x, y) = 1$.
3. Since the equals relationship is symmetric, $x = y \implies y = x$. As such, $d(x, y) = d(y, x)$.
4. Not completed.

Therefore the function is a metric. $\square$

3. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be given by $f(m, n) = 2^{m-1}(2n - 1)$. Prove that f is one-to-one and onto.

Proof. If we can prove that there is a unique solution $f(m, n)$ for every $m, n \in \mathbb{N}$, then we will have proven that f is one-to-one and onto.
We begin by breaking apart the function. Let us consider two functions:

$$g(m) = 2^{m-1}$$

and

$$h(n) = 2n - 1,$$

such that

$$f(m, n) = g(m) \cdot h(n).$$

The function g is clearly one-to-one and onto for any m. Any value of m produces a power of two, all of which are in $\mathbb{N}$. Looking at f, we see that if $m > 1$, g in effect creates the even factors found in the result, $f(m, n)$.
Considering h, we see that it is also one-to-one and onto for any n. Any value of n produces an odd number (because h is the definition of an odd number) – in fact, h produces every odd number in $\mathbb{N}$.
There are two cases to consider.
Case 1: $m = 1$. In this case, $f(1, n)$ will map n to every odd number in $\mathbb{N}$. If $m = 1$, the function f is one-to-one and onto.
Let $p_1 p_2 \ldots p_q$ be the prime factorization of $f(m, n)$, where p_i, $1 \leq i \leq q$, is a prime factor of $f(m, n)$ raised to some power. By the fundamental theorem of arithmetic, the prime factorization of a number is unique up to commutativity. In this case, we find that $f(1, n) = 2n - 1$, so the prime factorization is entirely dependent on the value of n.
Case 2: $m > 1$. In this case, $f(m, n)$ has some power of two (that is not one) multiplying an odd number. Since any odd number multiplied by an even number is even, $f(m, n)$ will be even for all n.
Let $p_1 p_2 \ldots p_q$ be the prime factorization of $f(m, n)$, where p_i, $1 \leq i \leq q$, is a prime factor of $f(m, n)$ raised to some power. By the fundamental theorem of arithmetic, the prime factorization of a number is unique up to commutativity. In this case, we find that $f(m, n) = 2^{m-1}(2n - 1)$, so the prime factorization is entirely dependent on the value of both m and n.
Since different values of n will yield different odd numbers, the prime factors will be the same if and only if the value of the input is the same. The same is true for m.
As such, as long as $(m, n) \neq (p, q)$, $f(m, n)$ and $f(p, q)$ have different prime factors. Therefore, $f(m, n)$ is one-to-one.
Furthermore, there is always at least one solution for all m, n, so $f(m, n)$ is onto as well. $\square$

4. Let $f : A \to B$ be a function from a nonempty set A. Prove that the set $\mathcal{C} = \{f^{-1}(b) : b \in \operatorname{rang}(f)\}$ is a partition of A. Note: $\mathcal{C}$ is a subset of $\mathcal{P}(A)$.

Definition 1 (Partition)**.**

Proof. $\square$

5. Let $f : A \to B$ be a function from a nonempty set A which is surjective. Find a new function $g : C \to B$ which is one-to-one such that $C \subseteq A$, $\mathrm{rang}(g) = \mathrm{rang}(f)$, and for every $x \in C, f(x) = g(x)$. Explain why g has an inverse function, g^{-1}. Then, compute $f(g^{-1}(x))$ for all $x \in B$.

Proof. $\square$