Rudin's Principles of Mathematical Analysis, 3rd ed

Connor Baker, June 2017

Numerical Sequences and Series: Selected Exercises

1. Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?

Proof. $\square$

2. Calculate $\lim_{n\to\infty}(\sqrt{n^2+n}-n)$.

Proof. We begin by multiplying by the algebraic conjugate:

$$\lim_{n\to\infty}\left[(\sqrt{n^2+n}-n)\cdot\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n}\right]=\lim_{n\to\infty}\left[\frac{n^2+n-n^2}{\sqrt{n^2+n}+n}\right]=\lim_{n\to\infty}\left[\frac{n}{\sqrt{n^2+n}+n}\right].$$

Tentatively trying to evaluate the limit yields a composition of algebraic indeterminate forms:

$$\lim_{n\to\infty}\left[\frac{n}{\sqrt{n^2+n}+n}\right]=\frac{\infty}{\sqrt{\infty}+\infty}=\frac{\infty}{\infty+\infty}.$$

We proceed by multiplying by a form of one:

$$\lim_{n\to\infty}\left[\frac{n}{\sqrt{n^2+n}+n}\cdot\frac{\frac{1}{n}}{\frac{1}{n}}\right]=\lim_{n\to\infty}\left[\frac{\frac{n}{n}}{\sqrt{\frac{n^2}{n^2}+\frac{n}{n^2}}+\frac{n}{n}}\right]=\lim_{n\to\infty}\left[\frac{1}{\sqrt{1+\frac{1}{n}}+1}\right].$$

From this point, we can now successfully evaluate the limit.

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{1+\frac{1}{n}}+1}\right]=\frac{1}{\sqrt{1+\frac{1}{\infty}}+1}=\frac{1}{\sqrt{1+0}+1}=\frac{1}{2}.$$

$\square$

3. If $s_1 = \sqrt{2}$, and

$$s_{n+1} = \sqrt{2 + s_n}, (n = 1, 2, 3, \dots),$$

prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1, 2, 3, \dots$.

Proof. We begin by proving that $s_n < 2, \forall n \in \mathbb{N}$.

Let $n = 1$. Then $s_1 = \sqrt{2} < 2$, and the base case holds.

Inductive hypothesis: Assume that for $n \leq k$, the following is true: $s_k < 2$.

Let $n = k + 1$. Then $s_{k+1} = \sqrt{2 \cdot s_k} = \sqrt{2} \cdot \sqrt{s_k}$. By the inductive hypothesis, $\sqrt{2} \cdot \sqrt{s_k} < \sqrt{2} \cdot \sqrt{2} = 2$.

By the Principle of Mathematical Induction, $s_n < 2$ for all $n \in \mathbb{N}$. $\square$

4. Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_1 = 0; \quad s_{2m} = \frac{s_{2m-1}}{2}; \quad s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Proof. □

5. For any two real sequences $\{a_n\}, \{b_n\}$, prove that

$$\lim_{n\to\infty} \sup(a_n + b_n) \leq \lim_{n\to\infty} \sup(a_n) + \lim_{n\to\infty} \sup(b_n)$$

provided the sum on the right is not of the form $\infty - \infty$.

Proof. $\square$