

Homework 2

Connor Baker, January 2017

1. Determine whether the following expressions are true or false. Give a complete explanation for each part.

- (a) $\emptyset \subseteq \{\emptyset, \{\emptyset\}\}$
- (b) $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$
- (c) $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\}$
- (d) For every set A , $\{\emptyset\} \subseteq A$.
- (e) $\{1, 2\} \in \{\{1, 2, 3\}, \{1, 3\}, 1, 2\}$
- (f) $\{\{4\}\} \subseteq \{1, 2, 3, \{4\}\}$

Definition 1 (Subset). Given two subsets A, B , A is said to be a subset of B if and only if all elements of A are also in B . That is to say:

$$X \subseteq Y \iff \forall x(x \in X \implies x \in Y)$$

Proof. (a) Let $A = \emptyset, B = \{\emptyset, \{\emptyset\}\}$. Then, by the definition of a subset,

$$\emptyset \subseteq \{\emptyset, \{\emptyset\}\} \iff \forall a(a \in \emptyset \implies a \in \{\emptyset, \{\emptyset\}\})$$

However, $a \notin \emptyset$ (the empty set contains no elements). As such, the statement is vacuously true (because for all a , of which there are none, we cannot tell whether it is in both sets or not).

Therefore, by the definition of a subset, the expression is true. □

Proof. (b) Let $A = \{\emptyset\}, B = \{\emptyset, \{\emptyset\}\}$. Then, by the definition of a subset,

$$\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\} \iff \forall a(a \in \{\emptyset\} \implies a \in \{\emptyset, \{\emptyset\}\})$$

Let $a = \emptyset$, the only element of A . Then,

$$\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\} \iff (\emptyset \in \{\emptyset\} \implies \emptyset \in \{\emptyset, \{\emptyset\}\})$$

which is true.

Therefore, by the definition of a subset, the expression is true. □

Proof. (c) Let $A = \{\{\emptyset\}\}, B = \{\emptyset, \{\emptyset\}\}$. Then, by the definition of a subset,

$$\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\} \iff \forall a(a \in \{\{\emptyset\}\} \implies a \in \{\emptyset, \{\emptyset\}\})$$

Let $a = \{\emptyset\}$, the only element of A . Then,

$$\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\} \iff (\{\emptyset\} \in \{\{\emptyset\}\} \implies \{\emptyset\} \in \{\emptyset, \{\emptyset\}\})$$

which is true.

Therefore, by the definition of a subset, the expression is true. □

Proof. (d) Let $B = \{\emptyset\}$. Then, by the definition of a subset,

$$\{\emptyset\} \subseteq C \iff \forall b(b \in \{\emptyset\} \implies b \in C)$$

Let $b = \emptyset$, the only element of B . Then,

$$\{\emptyset\} \subseteq C \iff (\emptyset \in \{\emptyset\} \implies \emptyset \in C)$$

which is contingent on the elements of C . There is no guarantee that C contains the empty set.

Therefore, by the definition of a subset, the expression is false. □

Proof. (e) This statement is false because the set does not contain the set $\{1, 2\}$. □

Proof. (f) Let $A = \{\{4\}\}$, $B = \{1, 2, 3, \{4\}\}$. Then, by the definition of a subset,

$$\{\{4\}\} \subseteq \{1, 2, 3, \{4\}\} \iff \forall a(a \in \{\{4\}\} \implies a \in \{1, 2, 3, \{4\}\})$$

Let $a = \{4\}$, the only element of A . Then,

$$\{\{4\}\} \subseteq \{1, 2, 3, \{4\}\} \iff (\{4\} \in \{\{4\}\} \implies \{4\} \in \{1, 2, 3, \{4\}\})$$

which is true.

Therefore, by the definition of a subset, the expression is true. □

2. Let $\Delta = [0, 1) = \{x \in \mathbb{R} : 0 \leq x < 1\}$ and let $A_\alpha = (-\alpha, \alpha] = \{x \in \mathbb{R} : -\alpha < x \leq \alpha\} \subseteq \mathbb{R}$, where $\alpha \in \Delta$. Prove that

$$\bigcup_{\alpha \in \Delta} A_\alpha = (-1, 1),$$

and

$$\bigcap_{\alpha \in \Delta} A_\alpha = \emptyset$$

Proof. Assume

□

3. Let A, B, C , and D be sets with $C \subseteq A$ and $D \subseteq B$. Prove that $C \cup D \subseteq A \cup B$.

Proof. Assume

□

4. Prove that if \mathcal{A} is a non-empty family of sets, then

$$\bigcap_{A \in \mathcal{A}} A \subseteq \bigcup_{A \in \mathcal{A}} A.$$

Proof. Assume

□

5. Use the principle of mathematical induction to prove $4^{n+4} > (n+4)^4$, for all natural numbers n .

Proof. Let $n = 1$: $4^5 > 5^4$. We see that the base case is true. Then, let $n = k$: $4^{k+4} > (k+4)^4$. Let $n = k + 1$: $4^{k+5} > (k+5)^4$. Then:

$$4^{k+5} = 4^k * 4^5$$

$$4^{k+4} = 4^k * 4^4$$

$$4(4^{k+4}) > 4^{k+4}$$

$$4^k * 4^5 = 4(4^{k+4})$$

$$(k+5)^4 = k^4 + 20k^3 + 150k^2 + 500k + 625$$

$$(k+4)^4 = k^4 + 16k^3 + 96k^2 + 256k + 256$$

$$4(k+4)^4 = 4k^4 + 64k^3 + 384k^2 + 1024k + 1024 > (k+5)^4$$

$$4^{k+5} = 4(4^{k+4}) > 4(k+4)^4 > (k+5)^4$$

Then $4^{n+4} > (n+4)^4 \forall n \in \mathbb{N}$. □