Examples from Class
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Example 1 (Prove that if $(3|a) \wedge (3|b)$ then $9|(ab)$)**.** Assume that $\exists k, j \in \mathbb{N} : 3k = a, 3j = b$. Then, $ab = 9kj$. Since $k, j \in \mathbb{N}$, and $(kj) \in \mathbb{N}$, then $(9kj) \in \mathbb{N}$. By the definition of divisibility, $9|(ab)$.

Example 2 (Let $m, n \in \mathbb{N}$ and q prime. Then $q|m \iff q|m^2$.)**.** If $q|m$, then $q|m^2$. Then $\exists k \in \mathbb{N} : qk = m$. Then, $m^2 = q^2k^2$. By definition, since it has the same factor twice, $q|m^2$.
If $q|m^2$, then $q|m$. Let the unique prime factor decomposition of $m = p_1^{n_1} \cdot p_2^{n_2} \cdots p_k^{n_k}$. Then $m^2 = p_1^{2n_1} \cdot p_2^{2n_2} \cdots p_k^{2n_k}$. Since $q|m^2, q = p_i^{2n_i}$ for some $i \in \mathbb{N}, i \leq k$. Since q is prime and is in $\mathbb{N}, 2n_i \geq 2 \implies n_i \geq 1$. Furthermore, q must be in the unique prime factorization of m (which we can infer from q's being prime and a factor of m^2 – it must have a factor of at least q^2). As such, $q|m$.
Therefore, $q|m \iff q|m^2$.

Example 3 ($\sqrt{2}$ is irrational)**.** If $(x > 0) \wedge (x^2 = 2)$, then x is irrational. We will prove by contradiction that x is irrational.
Assume that x is rational and $x > 0, x^2 = 2$. Then, since x is rational, $\exists m, n \in \mathbb{N} : x = \frac{m}{n}$, and m, n have no common factors. As such, $2 = x^2 = \frac{m^2}{n^2} \implies m^2 = 2n^2$, and $2|m^2$, which by the previous example, means $2|m$.
Since $2|m, \exists k \in \mathbb{N}$ where $m = 2k$. As such, $x = \frac{2k}{n} \implies x^2 = \frac{4k^2}{n^2} = 2 \implies 2k^2 = n^2$.
Therfore, $2|n^2 \implies 2|n$.
So, m, n both have no factors in common, yet they have a factor of two, which is a contradiction. Therefore, it must the case that if $(x > 0) \wedge (x^2 = 2)$, then x is irrational.