Examples from Class
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Example 1 (Prove that if $(3|a) \wedge (3|b)$ then $9|(ab)$)**.** Assume that $\exists k, j \in \mathbb{N} : 3k = a, 3j = b$. Then, $ab = 9kj$. Since $k, j \in \mathbb{N}$, and $(kj) \in \mathbb{N}$, then $(9kj) \in \mathbb{N}$. By the definition of divisibility, $9|(ab)$.

Example 2 (Let $m, n \in \mathbb{N}$ and q prime. Then $q|m \iff q|m^2$.)**.** If $q|m$, then $q|m^2$. Then $\exists k \in \mathbb{N} : qk = m$. Then, $m^2 = q^2k^2$. By definition, since it has the same factor twice, $q|m^2$.
If $q|m^2$, then $q|m$. Let the unique prime factor decomposition of $m = p_1^{n_1} \cdot p_2^{n_2} \cdots p_k^{n_k}$. Then $m^2 = p_1^{2n_1} \cdot p_2^{2n_2} \cdots p_k^{2n_k}$. Since $q|m^2, q = p_i^{2n_i}$ for some $i \in \mathbb{N}, i \leq k$. Since q is prime and is in $\mathbb{N}, 2n_i \geq 2 \implies n_i \geq 1$. Furthermore, q must be in the unique prime factorization of m (which we can infer from q's being prime and a factor of m^2 – it must have a factor of at least q^2). As such, $q|m$.
Therefore, $q|m \iff q|m^2$.

Example 3 ($\sqrt{2}$ is irrational)**.** If $(x > 0) \wedge (x^2 = 2)$, then x is irrational. We will prove by contradiction that x is irrational.
Assume that x is rational and $x > 0, x^2 = 2$. Then, since x is rational, $\exists m, n \in \mathbb{N} : x = \frac{m}{n}$, and m, n have no common factors. As such, $2 = x^2 = \frac{m^2}{n^2} \implies m^2 = 2n^2$, and $2|m^2$, which by the previous example, means $2|m$.
Since $2|m, \exists k \in \mathbb{N}$ where $m = 2k$. As such, $x = \frac{2k}{n} \implies x^2 = \frac{4k^2}{n^2} = 2 \implies 2k^2 = n^2$.
Therfore, $2|n^2 \implies 2|n$.
So, m, n both have no factors in common, yet they have a factor of two, which is a contradiction. Therefore, it must the case that if $(x > 0) \wedge (x^2 = 2)$, then x is irrational.

Example 4 ($\mathcal{A} = \{(-a, a) : (a \in \mathbb{R}) \wedge (a > 0)\}$. Show that the union over the indexed family of sets is $\mathbb{R}$.)**.** We must show that given the definition of $\mathcal{A}$ above:

$$\bigcup_{A \in \mathcal{A}} A = \mathbb{R}.$$

We begin by proving that:

$$\bigcup_{A \in \mathcal{A}} A \subseteq \mathbb{R}.$$

Let $x \in \cup_{A \in \mathcal{A}} A$. Then, $\exists a \in \mathbb{R}, a > 0 : x \in (-a.a)$. Then $-a < x < a$. Since $a \in \mathbb{R}, x \in (-a, a)$, it must be that $x \in \mathbb{R}$, and $\cup_{A \in \mathcal{A}} A \subseteq \mathbb{R}$.
We now prove that:

$$\mathbb{R} \subseteq \bigcup_{A \in \mathcal{A}} A.$$

Let $x \in \mathbb{R}$. Then $0 \leq |x| \leq (|x| + 1)$ and $-|x| \leq x \leq |x|$. Using this inequality, we see that $-(|x| + 1) < x \leq |x| < (|x| + 1)$. Then $x \in (-(|x| + 1), (|x| + 1)) \in A$, so by the definition of union, $x \in \cup_{A \in \mathcal{A}} A$, which implies that all elements of $\mathbb{R}$ are in the union, so $\mathbb{R} \subseteq \cup_{A \in \mathcal{A}} A$.
Therefore,

$$\bigcup_{A \in \mathcal{A}} A = \mathbb{R}.$$

Example 5 ($\mathcal{A} = \{(-a, a) : (a \in \mathbb{R}) \wedge (a > 0)\}$. Show that the intersection over the indexed family of sets is $\{0\}$.)**.** We must show that given the definition of $\mathcal{A}$ above:

$$\bigcap_{A \in \mathcal{A}} A = \{0\}.$$

We begin by proving that:

$$\bigcap_{A \in \mathcal{A}} A \subseteq \{0\}.$$

Let $x \in \cap_{A \in \mathcal{A}} A$. Then, if $x \in \{0\}, x = 0$. However, suppose that $x \neq 0$. Then since $x \in \mathbb{R}$, $\forall a \in \mathbb{R}^+, x \in (-a, a)$. We now prove that x is in every interval, but not one specific interval. Let $(-|x|/2, |x|/2) \in \mathcal{A}$. Then, since $|x|/2 < |x|$, it follows that $-|x| < -|x|/2 < |x|/2 < |x|$. Furthermore, $x = -|x|$ or $x = |x|$, so $x \notin (-|x|/2, |x|/2)$.

This is a contradiction of our assumption that $x \in \cap_{A\in\mathcal{A}} A$ and $x \neq 0$. It must be the case that $x = 0$ and is the only element of the intersection. As such, $\cap_{A\in\mathcal{A}} A \subseteq \{0\}$.
We now prove that:

$$\{0\} \subseteq \bigcap_{A\in\mathcal{A}} A.$$

Let $I \in \mathcal{A}$. Then $\exists a \in \mathbb{R}^+$ such that when $I = (-a, a), -a < 0 < a$ (since $a > 0$).
Therefore $0 \in (-a, a)\ \forall a > 0$. So, $0 \in I, \forall I \in \mathcal{A}$. As such, it must be the case that $\{0\} \subseteq \cap_{A\in\mathcal{A}} A$.
Therefore,

$$\bigcap_{A\in\mathcal{A}} A = \{0\}.$$