Rudin's Principles of Mathematical Analysis, 3rd ed

Connor Baker, June 2017

Basic Topology: Selected Exercises

5. Construct a bounded set of real numbers with exactly three limit points.

Theorem (Limit points of E_p). *Consider the set E_p:*

$$E_p = \left\{ \frac{1}{n} + p : n \in \mathbb{N} \right\}.$$

When $p \in \mathbb{Z}$, the limit point of E_p is the point p.

Proof. We begin by showing that $|E_p| = |\mathbb{N}|$.

Claim: $|E_p| = |\mathbb{N}|$.

Let $f : E_p \to \mathbb{N}$ where $\frac{1}{n} + p \mapsto n$. Assume that $f(a) = f(b)$. Then $\frac{1}{a} + p = \frac{1}{b} + p$. By cancellation, $\frac{1}{a} = \frac{1}{b}$, so $a = b$, and f is one-to-one.

Choose $n \in \mathbb{N}$. Since the reciprocal of any natural number is unique relative to the reciprocal of any other number in the naturals, there exists exactly one $\frac{1}{n} + p \in E_p$ such that $f(\frac{1}{n} + p) = n$. As such, f is onto.

Since there is a one-to-one correspondence between E and $\mathbb{N}$, $|E| = |\mathbb{N}|$.

We now show that the set E_p has a limit point p.

Since the cardinality of E_p is that of the naturals, for $r > 0, N_r(p) \cap E_p$ is infinite (we choose p because that is what our sequence converges to – the choice of any other point $\frac{1}{n} + p$ results in $\epsilon : \forall r > \epsilon, N_r(\frac{1}{n} + p) \cap E_p$ is finite). Therefore the set of limit points of E_p must contain at least the point p; and as such, $\{p\} \subseteq E'$.

We now show that E'_p contains only the point p.

Suppose that there is some limit point $x \neq p$. Let $\delta = d(x, p)$. Then the neighborhood

$$N_{\frac{\delta}{2}}(x) = \left\{ y : d(x, y) < \frac{\delta}{2} \right\}$$

does not contain the interval $(p, x - \frac{\delta}{2}]$ (inclusive since the neighborhood includes $(x - \frac{\delta}{2}, x + \frac{\delta}{2})$, by definition). The interval $(p, x - \frac{\delta}{2}]$ is equivalent to the set $\{\frac{1}{n} + p : \frac{1}{n} \leq x - \frac{\delta}{2}\}$. This set has no least lower bound, since $\mathbb{N}$ is not bounded above, so we are able to pick larger and larger n for $\frac{1}{n}$ which satisfy the inequality. By not including this portion of the interval, the neighborhood becomes finite. Since the neighborhood is finite, $N_{\frac{\delta}{2}} \cap E_p$ is finite and therefore x is not a limit point of E_p.

As such, $x \notin E'_p$, and $E'_p = \{p\}$. □

Corollary ($E_p \backslash N_r(p)$ is finite). *Given the set E_p, excluding the neighborhood of p makes the cardinality of the set finite. That is to say that $E_p \backslash N_r(p)$ is finite.*

The solution to this example is then trivial, as we can take the finite union of any E_p that we choose to create sets with an arbitrary finite number of limit points. For example, the set $\cup_{i=0}^{2} E_i$ contains exactly three limit points (the points 0, 1, and 2).

6. Let E' be the set of all limit points of a set E. Prove:
   (a) E' is closed.
   (b) E and $\bar{E}$ have the same limit points.
   (c) Whether or not E and E' always have the same limit points.

Proof. 6a)
Let $p \in E''$. Then, for $r > 0$,

$$\exists q \in N_{\frac{r}{2}}(p) : q \in E', q \neq p.$$

Let

$$\delta = \frac{d(p,q)}{2} < \frac{r}{2}.$$

Since $q \in E'$, for $r > 0$, $\exists s \in N_\delta(q) : s \in E$, and $s \neq q$. Due to the definition of the radius of $N_\delta(q)$, $s \neq p$ since the neighborhood's radius is not large enough to contain p.
Then:

$$d(p,q) < \frac{r}{2}$$

$$d(q,s) < \delta$$

$$d(p,s) \leq d(p,q) + d(q,s) < \frac{r}{2} + \delta < \frac{r}{2} + \frac{r}{2} = r.$$

So, $p \in E'$, and E' is closed. □

Proof. 6b)
Proof that E and $\bar{E}$ have the same limit points.

By the previous proof, E' is closed and therefore contains its own limit points.

Let $\bar{E}'$ be the set of all limit points of $\bar{E}$.

We begin by showing that $E' \subseteq \bar{E}'$.

Let $x \in E'$. Then for all $r > 0, N_r(x)$ contains some point $y \neq x$. Since $E \subseteq \bar{E}$, $y \in \bar{E}$, so $x \in \bar{E}'$, and $E \subseteq \bar{E}$.

We now show that $\bar{E}' \subseteq E'$.

Let $x \in \bar{E}'$. Then for all $r > 0, N_r(x)$ contains some point $y \neq x$. Since $\bar{E} = E \cup E'$, $y \in E$ or $y \in E'$. If $y \in E$, then $x \in E'$. If $y \in E'$, choose $\epsilon > 0$ such that $\epsilon < d(x,y)$ and $N_\epsilon(y) \subseteq N_r(x)$. Since y is a limit point of the set E, then there is some point $z \in E$ such that $z \neq y$ and $z \in N_\epsilon(y)$. Since $z \neq x$ (due to $x \notin N_\epsilon(y)$), and $N_\epsilon(y) \subseteq N_r(x)$, any neighborhood of x contains $z \neq x$ with $x \in E$, so x must be a limit point of E. □

Proof. 6c)
Consider the set

$$E = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

By the theorem Limit points of E_p, we know that $E' = \{0\}$.

However, the limit points of E' are the empty set, since no matter the point we pick to center the neighborhood, the intersection will contain at most one element (zero). As such, E and E' do not always have the same limit points.

□

7. Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.

   (a) If $B_n = \cup_{i=1}^n A_i$, prove that $\bar{B}_n = \cup_{i=1}^n \bar{A}_i$, for $n = 1, 2, 3, \ldots$.

   (b) If $B = \cup_{i=1}^\infty A_i$, prove that $\bar{B} \supset \cup_{i=1}^\infty \bar{A}_i$, and show by an example that this inclusion can be proper.

Lemma. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. 7a)
Claim: $(A \cup B)' \subseteq A' \cup B'$.
Case 1: The trivial case is that $A \cup B$ is finite. Then $(A \cup B)'$ is the empty set, since finite sets cannot have limit points, which is a subset of any set so our claim holds true.
Case 2: $A \cup B$ is not finite. Then let $x \in (A \cup B)'$. Choose any $r > 0$, and consider $N_r(x) \cap A$, and $N_r(x) \cap B$. Suppose both are finite. Then $N_r(x) \cap (A \cup B)$ is finite, and x is not a limit point. This is a contradiction, so our assumption that both are finite is incorrect. Therefore, at least one must be infinite, so $N_r(x) \cap A$ or $N_r(x) \cap B$ must be infinite, and therefore $x \in A' \cup B'$.
Claim: $A' \cup B' \subseteq (A \cup B)'$.
Let $x \in A' \cup B'$. Choose any $r > 0$. Then, $N_r(x) \cap A$ or $N_r(x) \cap B$ is infinite, and $x \in (A \cup B)'$.
Therefore, $(A \cup B)' = A' \cup B'$, and

$$\overline{A \cup B} = (A \cup B) \cup (A \cup B)' = A \cup A' \cup B \cup B' = \overline{A} \cup \overline{B}.$$

□

Proof. 7a) Cont.
We proceed with proof by induction.
Let $n = 1$. Then:

$$\overline{\bigcup_{i=1}^{1} A_i} = \overline{A_1} = \bigcup_{i=1}^{1} \overline{A_i}.$$

Let $n = k$. Assume that

$$\overline{\bigcup_{i=1}^{k} A_i} = \bigcup_{i=1}^{k} \overline{A_i},$$

for all $k \in \mathbb{N}$.
Let $n = k + 1$. Then,

$$\overline{\bigcup_{i=1}^{k+1} A_i} = \overline{\bigcup_{i=1}^{k} A_i \cup A_{k+1}},$$

which by the lemma above is equivalent to (using our assumption for $n = k$)

$$\overline{\bigcup_{i=1}^{k} A_i} \cup \overline{A_{k+1}} = \bigcup_{i=1}^{k} \overline{A_i} \cup \overline{A_{k+1}} = \bigcup_{i=1}^{k+1} \overline{A_i}.$$

Therefore

$$B_n = \bigcup_{i=1}^{n} A_i, \qquad \overline{B_n} = \overline{\bigcup_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A_i}$$

by the Principle of Mathematical Induction. □

Proof. 7b)
Since $B_i \supseteq \cup A_i$, and $B \supseteq A_i \ \forall i$, then $\overline{B} \supseteq \overline{A}_i \ \forall i$. Then, $\overline{B} \supseteq \cup_{i=1}^\infty \overline{A}_i$. However, it is not necessarily true that $\overline{B} \subseteq \cup_{i=1}^\infty \overline{A}_i$.

Let $A_i = \{r_i\}$ where the set $\{r_1, r_2, \ldots\}$ is an enumeration of $\mathbb{Q}$. Then $\overline{B} = \mathbb{R}$, and $A_i' = \emptyset \forall i$, so $\cup \overline{A}_i = B$. As such, $\overline{B} \supset \cup_{i=1}^\infty \overline{A}_i$. □

8. Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

Proof. Since E is open, all elements of E are interior points.

Let $p \in E$. Choose $r > 0$, and let $N_r(p)$ be a neighborhood of p. Since p is an interior point, $\exists \delta > 0 : N_\delta(p) \subseteq E$.

Choose $0 < \epsilon \leq \min(r, \delta)$. Then, $N_\epsilon(p) \subseteq N_\delta(p) \subseteq E$. Since $N_\epsilon(p)$ contains infinitely many points of E, and $N_\epsilon(p) \subseteq N_r(p)$, $N_r(p)$ does as well.

It is not the case that any finite set B will have limit points since the neighborhood intersect B will be finite. $\square$

9. Let E° denote the set of all interior points of a set E.

   (a) Prove that E° is always open.

   (b) Prove that E is open if and only if $E^\circ = E$.

   (c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.

   (d) Prove that the complement of E° is the closure of the complement of E.

   (e) Do E and $\bar{E}$ always have the same interiors?

   (f) Do E and E° always have the same closures?

Proof. 9a) *Didn't show that all points of E° are interior to E°.*
Since every point of E° is an interior point (by definition of E°), the set E° is open. □

Proof. 9b) *Must prove both directions.*
A set is open if every point of the set is an interior point of the set (the definition of open). If a set E is equal to the set of it's interior points, E°, then every point of E is an interior point of E, and E is open. As such, E is open if and only if $E^\circ = E$. □

Proof. 9c) *Proof is lacking.*
Since G is open, $G = G^\circ$.

Claim: $G^\circ \subseteq E^\circ$.

Assume $x \in G^\circ$. Then $x \in E$. Assume that $x \notin E^\circ$. Then x is not an interior point of E, and will not be contained in any open subset of E, like G°, which is a contradiction.

Therefore $x \in E^\circ$ and $G^\circ \subseteq E^\circ$. Since $G^\circ = G$, and $G^\circ \subseteq E^\circ, G \subseteq E^\circ$. □

Proof. 9d)
Claim: $(E^\circ)^c \subseteq \overline{(E^c)}$.
Claim: $\overline{(E^c)} \subseteq (E^\circ)^c$. □

Proof. 9e)
Pick two open intervals, say $(0, 1) \cup (1, 2)$. Then 1 is not an interior point, and the closure of the set is $[0, 1] \cup [1, 2] = [0, 2]$, where one is an interior point.
As such, the set and its closure do not necessarily have the same interiors. □

Proof. 9f)
Let $E = \{1\}$.
Then $E^\circ = \emptyset$, $\overline{E^\circ} = \overline{\emptyset} = \emptyset$.
In addition, $\overline{E} = \{1\} \neq \emptyset$.
As such, the set and the set of it's interior points do not necessarily have the same closures. □

12. Let $K \subseteq \mathbb{R}^1$ consist of zero and the numbers $1/n$ for $n = 1, 2, 3, \ldots$. Prove that K is compact directly from the definition (without using Heine-Borel theorem).

Proof. Let $\mathcal{C}$ is an open covering of K.
Since $0 \in K$, there must be some $G_{\alpha,0} \in \mathcal{C}$ that contains zero as well. Since all G are open, there exists $r > 0$ such that $N_r(0) \subseteq G_{\alpha,0}$.
Since 0 is the only limit point of K (by the theorem Limit points of E_p), $K \backslash N_r(0)$ is finite (by the corollary to the theorem Limit points of E_p), and can be can therefore be covered by finite number of subcovers, $\cup_{\alpha}^{n} G_{\alpha}$.
Therefore, K is compact. $\square$

13. Construct a compact set of real numbers whose limit points form a countable set.

Proof. Limit points are $\mathbb{R}$ in $(0,1)$, not contained in the union, so the set is not compact. Try again.
The theorem Limit points of E_p can easily be trivially extended to handle $p \in \mathbb{Q}$, so that the limit point of a set E_p is known to be p.

Let E_p, where $p \in \mathbb{Q}$, be defined as follows:

$$E_p = \left\{ \frac{1}{n} + p : n \in \mathbb{N}, n \geq 2 \right\}.$$

Let $\{r_1, r_2, \ldots\}$ be an indexed set of the rationals such that $0 \leq r_i < \frac{1}{2}$.

Then the set A such that

$$A = \bigcup_{i=1}^{\infty} (E_{r_i} \cup r_i)$$

has limit points $\{r_1, r_2, \ldots\}$, which is a set containing countably many points (since there are infinitely many rational numbers between zero and one-half).

The set A is bounded on both sides (zero and one respectively), and since it contains it's own limit points, it's closed.

Since A is closed and bounded, it is compact. $\square$

14. Give an example of an open cover of the segment $(0, 1)$ which has no finite subcover.

Proof. Well Ordering Principle when used to order the rationals does not preserve the less than or greater than operator. Proof does not work. Try again.
Let R be the indexed list of rationals $\{r_0, r_1, \dots\}$ such that $0 \leq r_i < r_{i+1}1$.

Let $E = \cup_{i=0}^{\infty}(r_i, r_{i+2})$. Then $E = (0, 1)$, since $r_0 = 0$ and $r_\infty = 1$ (*note: would it be better to write that the sequence r_i approaches 1 as $i \to \infty$?*).

By using the indices i and $i+2$, we create overlap between any two intervals. Should we exclude some interval (like taking the finite union), we will have a gap in our subcover. As such, this open cover has no finite subcover. $\square$

15. Show that Theorem 2.36 and its Corollary become false (in $\mathbb{R}^1$, for example) if the word "compact" is replaced by "closed" or "bounded."

Theorem. *If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\cap K_\alpha$ is nonempty.*

Corollary. *If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$ ($n = 1, 2, 3, \dots$), then $\cap_1^\infty K_n$ is not empty.*

Proof. Consider the closed set $A_n = [n, \infty)$. For any choice of n, this set has no limit point since it is a list of natural numbers, and we can very easily choose any $r < 1$ such that $N_r(p) \cap A_n$ is finite. As such, it contains its own non-existent limit points and is closed.

Since the intersection of any finite collection of A_ns will have a nonzero number of elements in common, then $\cap A_n$ should be nonempty. However, $\cap_{i=1}^\infty A_n = \emptyset$.

Therefore, Theorem 2.36 and its Corollary become false if the word "compact" is replaced by "closed." $\square$

Proof. Consider the bounded set $A_n = (0, \frac{1}{n})$. It is bounded since for any n, we can choose a number $M : d(p,q) < M, \forall p, q \in A_n$.

Since the intersection of any finite collection of A_n will have a nonzero number of element ins common, then $\cap A_n$ should be nonempty. However, $\cap_{i=1}^\infty A_n = \emptyset$, since as we near $n \to \infty$, $\frac{1}{n} \to 0$, yielding the interval $(0, 0)$, which is equivalent to the empty set, and the empty set intersect any set is the empty set.

Therefore, Theorem 2.36 and its Corollary become false if the word "compact" is replaced by "bounded." $\square$

16. Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space, with $d(p, q) = |p - q|$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show:

    (a) E is closed and bounded in $\mathbb{Q}$, but that E is not compact.

    (b) Whether E is open in $\mathbb{Q}$.

Proof. 16a) $\square$

Proof. 16b) $\square$

17. Let $E = \{x : x \in [0, 1]$ and x's decimal expansion contains only the digits 4 and 7$\}$. Show:

    (a) Whether E is countable.

    (b) Whether E is dense in $[0, 1]$.

    (c) Whether E is compact.

    (d) Whether E is perfect.

Proof. 17a)
E is uncountably infinite since we can create an element that is not in any list containing the elements of E (similar to the example we had done in class to prove that $\mathbb{R}$ is uncountable). $\square$

Proof. 17b)
E is not dense in $[0, 1]$ since not every point in $[0, 1]$ is a point or limit point of E (for example, 0.1). $\square$

Proof. 17c)
Claim: E is bounded.

E is bounded since we can choose a number M (like 2) such that $d(p, q) < M, \forall p, q \in E$.

Claim: E is closed.

Let $a, b \in E, a \neq b$. For two elements of E to be distinct, they must differ by at least one digit. It is important to note that no element of E has a finite decimal expansion (since such an expansion could be re-written with trailing zeros, and would then not be the decimal expansion of an element of E). Define a and b as the non-terminating decimal expansions below:

$$a = 0.a_1a_2a_3\ldots,$$

$$b = 0.b_1b_2b_3\ldots.$$

Work in progress...
Since E contains its own limit points, it is closed. Since E is closed and bounded, it is compact. $\square$

Proof. 17d)
Since E is compact, and every point of E is a limit point of E, it is perfect. $\square$

18. Is there a nonempty perfect set in $\mathbb{R}^1$ which contains no rational number?

Proof. $\square$

19. (a) If A and B are disjoint closed sets in some metric space X, prove that they are separated.
    (b) Prove the same for disjoint open sets.
    (c) Fix $p \in X, \delta > 0$, define A to be the set of all $q \in X$ for which $d(p,q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.
    (d) Prove that every connected metric space with at least two points in uncountable. *Hint*: Use (c).

Proof. 19a)
For A and B to be separated, they must be disjoint and share no limit points. Since A and B are disjoint, $A \cap B = \emptyset$. Since A and B are closed, $A = \overline{A}$ and $B = \overline{B}$, $\overline{A} \cap B = \emptyset$, and $A \cap \overline{B} = \emptyset$. Therefore, A and B are separated. □

Proof. 19b)
□

Proof. 19c)
□

Proof. 19d)
□