Homework 5
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1. Prove that if the real-valued function f is strictly increasing or strictly decreasing on $\mathbb{R}$, then f is one-to-one (Note: You cannot assume f is differentiable).

Proof. Case 1: f is strictly decreasing.
If f is strictly decreasing, then $\forall\ x, a \in \text{dom}(f),\ x \leq a,\ f(x) \geq f(a)$. If it is the case that $f(x) = f(a)$, then it must also be the case that $x = a$. Then, $f(x) = f(a)$ if and only if $x = a$, and $f(x) > f(a)\ \forall x, a \in \text{dom}(f),\ x < a$, the function f must be one-to-one.
Case 2: f is strictly increasing.
If f is strictly increasing, then $\forall\ x, a \in \text{dom}(f),\ x \leq a,\ f(x) \leq f(a)$. If it is the case that $f(x) = f(a)$, then it must also be the case that $x = a$. Then, $f(x) = f(a)$ if and only if $x = a$, and $f(x) < f(a)\ \forall x, a \in \text{dom}(f),\ x < a$, the function f must be one-to-one. $\square$

2. Prove the following are metrics:

   (a) $X = \mathbb{R}, d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

   (b) $X = \mathbb{R} \times \mathbb{R}, d((x, y), (z, w)) = \sqrt{(x - z)^2 + (y - w)^2}$

   Definition 1 (Metric)**.** A metric on a set X is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$,

   (a) $d(x, y) \geq 0$

   (b) $d(x, y) = 0$ if and only if $x = y$

   (c) $d(x, y) = d(y, x)$,

   (d) $d(x, y) + d(y, z) \geq d(x, z)$.

Proof. We begin by proving that the first function is a metric.

1. The $\text{rang}(d) = \{0, 1\}$ so the function is definitely greater than or equal to zero for any inputted pair of values.

2. By the definition of d, $d(x, y) = 0$ if and only if $x = y$.

3. Since the equals relationship is symmetric, $x = y \implies y = x$. As such, $d(x, y) = d(y, x)$, since the order of the inputs does not generate a unique output.

4. Not completed.

Therefore the first function is a metric.

We now prove that the second function is a metric.

1. The $\text{rang}(d) = \{0, 1\}$ so the function is definitely greater than zero.

2. By the definition of d, if and only if $x = y$ does $d(x, y) = 1$.

3. Since the equals relationship is symmetric, $x = y \implies y = x$. As such, $d(x, y) = d(y, x)$.

4. Not completed.

Therefore the function is a metric. $\square$

3. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be given by $f(m,n) = 2^{m-1}(2n-1)$. Prove that f is one-to-one and onto.

Proof: f is one-to-one. Let $m, n, p, q \in \mathbb{N}$ and $f(m,n) = f(p,q)$. Then,

$$2^{m-1}(2n-1) = 2^{p-1}(2q-1).$$

Each side has two distinct factors: one that is even (a power of two), and one that is odd (two times a number less one). For these two numbers to be even, they must have the same power factor of two, so it must be the case that:

$$2^{m-1} = 2^{p-1}.$$

As such, by taking the log_2 of both sides and using cancellation on the negative one shared on both sides, we find $m = p$. Additionally, since they have the same power factor of two, it must be the case that:

$$(2n-1) = (2q-1).$$

Using cancellation on the one and then the two, we find that $n = q$. As such, when $f(m,n) = f(p,q)$, it must also be the case that $(m,n) = (p,q)$, and $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is one-to-one. $\square$

Proof: f is onto. Let $m, n, p \in \mathbb{N}$. Then, for all values m, n in the natural numbers, there is a natural number p such that $f(m,n) = p$. $\square$

Proof: f is bijective. By the above two proofs, since f is both one-to-one and onto, f is bijective. $\square$

4. Let $f : A \to B$ be a function from a nonempty set A. Prove that the set $\mathcal{C} = \{f^{-1}(b) : b \in \text{rang}(f)\}$ is a partition of A. Note: $\mathcal{C}$ is a subset of $\mathcal{P}(A)$.

Definition 1 (Partition)**.** A set B is said to be a partition of a set A if:

1. $\emptyset \notin B$
2. $\forall C, D \in B, C \cap D = \emptyset$ when $C \neq D$
3. $\cap_{C \in B} C = A$
4. $B \subseteq \mathcal{P}(A)$

Proof. The set C does not have $\emptyset$ as an element since that would mean that A contains the empty set – but it does not, since A is a nonempty set. As such, the first condition is met.
The fourth condition is met because it is given. $\square$

5. Let $f : A \to B$ be a function from a nonempty set A which is surjective. Find a new function $g : C \to B$ which is one-to-one such that $C \subseteq A$, $\text{rang}(g) = \text{rang}(f)$, and for every $x \in C, f(x) = g(x)$. Explain why g has an inverse function, g^{-1}. Then, compute $f(g^{-1}(x))$ for all $x \in B$.

Proof. Since $\text{rang}(g) = \text{rang}(f)$, and $\text{rang}(f) = B$ (because f is surjective), $\text{rang}(g) = B$, and g is surjective. Additionally, since g was given as injective, g must be bijective. The function g has an inverse because it is bijective, and that inverse is bijective. For all $x \in B$, the function $f(g^{-1}(x))$ is equivalent to $f(C)$ (since $\text{rang}(g^{-1}) = C$). Furthermore, $f(C) = B$, since $C \subseteq A$ and $\forall c \in C, f(x) = g(x)$. $\square$