

# Homework 1

## Connor Baker, January 2017

1. Prove by contradiction that if  $a - b$  is odd, then  $a + b$  is odd.

**Assumptions** Assume that  $a, b \in \mathbb{Z}$ ,  $a - b$  is even, and  $a + b$  is odd.

**Lemma** Let  $x, y, z \in \mathbb{Z}$ ,  $x = 2z$ ,  $y = 2z + 1$ . For all  $z$ ,  $x$  is even (any number with a factor of two is even), and  $y$  is odd (any even number plus one is odd).

**Proof** If  $a - b$  is even, then by the previous lemma,  $a - b = 2z$  for some unknown number  $z$ . If  $a + b$  is odd, then by the previous lemma,  $a + b = 2z + 1$  for some unknown number  $z$ . Substituting  $2z + b$  for  $a$  (from the first equation) into  $a + b = 2z + 1$  yields  $2z + b + b = 2z + 1$ . Simplifying yields  $b = 1/2$ , which means that  $b$  is *not* in the set of integers. Therefore, the original assumption is a contradiction. As a result, if  $a - b$  is odd, then  $a + b$  must odd.

2. Write a proof by contrapositive to show that if  $xy$  is odd, then both  $x$  and  $y$  are odd.

**Assumptions** Assume that  $x, y \in \mathbb{Z}$ , either  $x$  or  $y$  is even, and  $xy$  is even.

**Lemma** Let  $a, b, c \in \mathbb{Z}$ ,  $a = 2c$ ,  $b = 2c + 1$ . For all  $c$ ,  $a$  is even (any number with a factor of two is even), and  $b$  is odd (any even number plus one is odd).

**Proof** If  $x$  is even, and  $y$  is odd, then by the previous lemma,  $xy = (2c)(2c + 1)$ , which is always even for some number  $c$ . If  $x$  is odd, and  $y$  is even, then by the previous lemma,  $xy = (2c + 1)(2c)$ , which is also always even for some number  $c$ .

Since either  $x$  or  $y$  is even, and  $xy$  is even, we can infer by the contrapositive that if  $xy$  is odd, then both  $x$  and  $y$  are odd.

3. Prove that there do not exist integers  $m$  and  $n$  such that  $12m + 15n = 1$ .

**Proof** The equation  $12m + 15n = 1$  is equivalent to  $3(4m + 5n) = 1$ , which is equivalent to  $4m + 5n = 1/3$ .

Any integer multiplied by an integer is an integer. As such, four (an integer) multiplied by any integer  $m$  is an integer, and five (an integer) multiplied by any integer  $n$  is also an integer.

Let  $a, b \in \mathbb{Z}$ , where  $a = 4m$  and  $b = 5n$ . The equation can now be expressed as  $a + b = 1/3$ . Additionally, the sum of any two integers is an integer.

Let  $c \in \mathbb{Z}$ , where  $c = a + b$ . The equation can now be expressed as  $c = 1/3$ .

However,  $1/3$  is *not* in the set of integers. Therefore, by direct proof, there do not exist integers  $m$  and  $n$  such that  $12m + 15n = 1$ .

4. Prove there is a natural number  $M$  such that for every natural number  $n$ ,  $\frac{1}{n} < M$ .

**Proof** Let  $n = 1$ . Then  $1/n = 1 < M$ . If  $n = 1$ , then the first natural number that  $M$  can be to make the statement true is  $M = 2$ .

Let  $k = n + 1$ , where  $k \in \mathbb{N}$ . Then  $1/n > 1/(n + 1) = 1/k$ . Therefore every natural number  $n$  greater than one will yield a number less than one, so the natural number  $M$  such that for every natural number  $n$ ,  $\frac{1}{n} < M$  is two.

5. Prove that if  $-2 < x < 1$  or  $x > 3$ , then  $\frac{(x-1)(x+2)}{(x-3)(x+4)} > 0$ .