Homework 5
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1. Prove that if the real-valued function f is strictly increasing or strictly decreasing on $\mathbb{R}$, then f is one-to-one (Note: You cannot assume f is differentiable).

Proof. Case 1: f is strictly decreasing.
If f is strictly decreasing, then $\forall\ x, a \in \text{dom}(f),\ x \leq a,\ f(x) \geq f(a)$. If it is the case that $f(x) = f(a)$, then it must also be the case that $x = a$. Then, $f(x) = f(a)$ if and only if $x = a$, and $f(x) > f(a)\ \forall x, a \in \text{dom}(f),\ x < a$, the function f must be one-to-one.
Case 2: f is strictly increasing.
If f is strictly increasing, then $\forall\ x, a \in \text{dom}(f),\ x \leq a,\ f(x) \leq f(a)$. If it is the case that $f(x) = f(a)$, then it must also be the case that $x = a$. Then, $f(x) = f(a)$ if and only if $x = a$, and $f(x) < f(a)\ \forall x, a \in \text{dom}(f),\ x < a$, the function f must be one-to-one. $\square$

2. Prove the following are metrics:

   (a) $X = \mathbb{R}, d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

   (b) $X = \mathbb{R} \times \mathbb{R}, d((x, y), (z, w)) = \sqrt{(x - z)^2 + (y - w)^2}$

   Definition 1 (Metric)**.** A metric on a set X is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$,

   (a) $d(x, y) \geq 0$

   (b) $d(x, y) = 0$ if and only if $x = y$

   (c) $d(x, y) = d(y, x)$,

   (d) $d(x, y) + d(y, z) \geq d(x, z)$.

Proof. We begin by proving that the first function is a metric.

1. The $\text{rang}(d) = \{0, 1\}$ so the function is definitely greater than or equal to zero for any inputted pair of values.
2. By the definition of d, if and only if $x = y$ does $d(x, y) = 1$.
3. Since the equals relationship is symmetric, $x = y \implies y = x$. As such, $d(x, y) = d(y, x)$.
4. As shown below, it is the case that $d(x, y) + d(y, z) \geq d(x, z)$.

Therefore the first function is a metric.

We now prove that the second function is a metric.

1. The $\text{rang}(d) = \{0, 1\}$ so the function is definitely greater than zero.
2. By the definition of d, if and only if $x = y$ does $d(x, y) = 1$.
3. Since the equals relationship is symmetric, $x = y \implies y = x$. As such, $d(x, y) = d(y, x)$.
4. As shown below, it is the case that $d(x, y) + d(y, z) \geq d(x, z)$:

Therefore the function is a metric. $\square$

3. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be given by $f(m, n) = 2^{m-1}(2n - 1)$. Prove that f is one-to-one and onto.

Proof. Let $a, b, c \in \mathbb{N}$. Then, $f(b, c) \geq 1$, and $\forall a \in \mathbb{N}$, $\exists (b, c) \in \mathbb{N} \times \mathbb{N} :\ f(b, c) = a$, since a, b and c are all natural numbers, and the natural numbers are closed under multiplication. As such, f is surjective.
The function f is injective if $\forall x, y \in \mathbb{N}\times\mathbb{N}, x \neq y, f(x) \neq f(y)$. Then, assuming $f(x) = f(y)$ we have $2^{a-1}(2b-1) = 2^{c-1}(2d - 1)$. Assuming that $x \neq y$, $x = (a, c)$ and $y = (c, d)$, it must be that the case that $a \neq c$ or $b \neq d$.
Since the prime factor decomposition of a number is unique up to commutativity, they must have the same factor of two, which must come from the term outside the parenthesis (since $2k - 1\ \forall k \in \mathbb{N}$ is the definition of an odd number and will therefore have no factor of two), so $a = c$.
Furthermore, since the two sides of the equation have the same factor of two and the prime factor decomposition is unique, the terms in the parenthesis must be equal. Using cancellation, we find that $b = d$.
This is a contradiction of our assumption that $x \neq y$, so it must be that $f(x) = f(y)$ if and only if $x = y$, and f is injective.
Since f is both injective and surjective, it must be bijective. $\square$

4. Let $f : A \to B$ be a function from a nonempty set A. Prove that the set $\mathcal{C} = \{f^{-1}(b) : b \in \text{rang}(f)\}$ is a partition of A. Note: $\mathcal{C}$ is a subset of $\mathcal{P}(A)$.

Proof. Since $\square$

5. Let $f : A \to B$ be a function from a nonempty set A which is surjective. Find a new function $g : C \to B$ which is one-to-one such that $C \subseteq A$, $\operatorname{rang}(g) = \operatorname{rang}(f)$, and for every $x \in C, f(x) = g(x)$. Explain why g has an inverse function, g^{-1}. Then, compute $f(g^{-1}(x))$ for all $x \in B$.

Proof. Since $\square$