Rudin's Principles of Mathematical Analysis, 3rd ed

Connor Baker, May 2017

Basic Topology – Selected Exercises

5. Construct a bounded set of real numbers with exactly three limit points.

Proof. The point p is a limit point of $\{\frac{1}{n} : n \in \mathbb{N}\}$ if $\forall r > 0, N_r(p) \cap E$ has infinitely many points.

Claim: Zero is a limit point of $\{\frac{1}{n} : n \in \mathbb{N}\}$.

For zero, $N_r(0) = \{x : d(x, 0) < r\} = \{x : |x - 0| < r\} = \{x : -r < x < r\} = (-r, r)$. Since $(r, r) \subseteq \mathbb{R}, \forall r > 0$, there are uncountably infinitely many points and $(r, r) \cap \{\frac{1}{n} : n \in \mathbb{N}\}$ has infinitely many points.

It is important to note that we can shift the location of this limit point. Consider the set $\{\frac{1}{n} + 1 : n \in \mathbb{N}\}$. For the same reasons above we can see that the limit point here is one.

In general, we can make a bounded set of real numbers with exactly $p, p \in \mathbb{Z}^+$, limit points with the set

$$\bigcup_{k=0}^{p} \{\frac{1}{n} + k : n \in \mathbb{N}\}.$$

So, if we want three limit points, our set is as

$$\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \bigcup \left\{\frac{1}{n} + 1 : n \in \mathbb{N}\right\} \bigcup \left\{\frac{1}{n} + 2 : n \in \mathbb{N}\right\}.$$

$\square$

6. Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\bar{E}$ have the same limit points. (Recall that $\bar{E} = E \cup E'$.) Do E and E' always have the same limit points?

Proof. Set of all limit points of a set is closed.

The set E' is closed if it contains its own limit points.

Assume that p is a limit point of E'. Then, $\exists x \neq p, x \in E' : \forall r > 0, d(x,p) < \frac{r}{2}$.

Since $x \in E'$, x is a limit point of E, by the definition of E'. As such, $\exists y \in E : d(x,y) < \frac{r}{2}$. By the triangle inequality:

$$d(y,p) \leq d(y,x) + d(x,p).$$

Substituting using the inequalities described above:

$$d(y,p) < \frac{r}{2} + \frac{r}{2} = r,$$

so p is a limit point of E. Since p is a limit point of E, it must be in E'. Therefore, since E' contains its own limit points, it is closed. □

Proof. A set and a set which is the union of that set and the set containing its limit points have the same limit points.

By the previous proof, E' is closed and therefore contains its own limit points.

Let $\bar{E}'$ be the set of all limit points of $\bar{E}$.

We begin by showing that $E' \subseteq \bar{E}'$.

□

Proof. A set and the set that contains that set's limit points do not necessarily have the same limit points. Consider the set

$$E = \{\frac{1}{n} : n \in \mathbb{N}\}.$$

Then the limit point of that set is zero, so $E' = 0$. However, the limit points of E' are the empty set. We know this because to be a limit point, the intersection of the neighborhood and the set must have infinitely many points. This is not possible for our E', since we have at most a single point in the intersection.

In general, the limit points of a finite set are the empty set.

□

7. Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.

   (a) If $B_n = \cup_{i=1}^{n} A_i$, prove that $\bar{B}_n = \cup_{i=1}^{n} \bar{A}_i$, for $n = 1, 2, 3, \ldots$.

   (b) If $B = \cup_{i=1}^{\infty} A_i$, prove that $\bar{B} \supset \cup_{i=1}^{\infty} \bar{A}_i$.

   Show, by an example, that this inclusion can be proper.

Proof. □

8. Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

Proof. $\square$

9. Let E° denote the set of all interior points of a set E.

   (a) Prove that E° is always open.
   (b) Prove that E is open if and only if $E^\circ = E$.
   (c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.
   (d) Prove that the complement of E° is the closure of the complement of E.
   (e) Do E and $\bar{E}$ always have the same interiors?
   (f) Do E and E° always have the same closures?

Proof. $\square$