Rudin's Principles of Mathematical Analysis, 3rd ed

Connor Baker, June 2017

Basic Topology: Selected Exercises

5. Construct a bounded set of real numbers with exactly three limit points.

Lemma 1. *Consider the set E:*

$$E = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

Claim: $|E| = |\mathbb{N}|$.

Let $f : E \to \mathbb{N}$ where $\frac{1}{n} \mapsto n$. We will show that f is one-to-one.
Assume $f(a) = f(b)$. Then $a = b$, so f is one-to-one.
Choose $n \in \mathbb{N}$. Then there exists exactly one $\frac{1}{n} \in E$ such that $f(\frac{1}{n}) = n$ (since the reciprocal of any natural number is unique relative to the reciprocal of any other number in the naturals). As such, f is onto.

Since there is a one-to-one correspondence between E and $\mathbb{N}$, $|E| = |\mathbb{N}|$.

CHECK ME. Let E be the set of numbers such that:

$$E = \bigcup_{k=0}^{2} A_k, \qquad A_k = \left\{ \frac{1}{n} + k : n \in \mathbb{N} \right\}.$$

By Lemma 1, we know that A_0 has the same cardinality as N. Since $\lim_{n\to\infty} \frac{1}{n} = 0$ the value of k allows us to change the value the limit approaches. As such, every value of A_1 is one larger than the corresponding value of A_0, as is A_2 with respect to A_1. The cardinality remains unchanged.
Since the cardinality of A_k is that of the naturals, the neighborhood $N_r(k)$ for some $r > 0$ intersect A_k is infinite. Therefore the set of limit points of E must contain at least the points 0, 1, and 2, and as such, $\{0, 1, 2\} \subseteq E'$.

We now show that E' contains only 0, 1, and 2.

Suppose that there is some limit point $x \notin \{0, 1, 2\}$. Let $\epsilon = \min(d(x, 0),\ d(x, 1),\ d(x, 2))$ (we want the smallest radius possible). Then the neighborhood

$$N_{\frac{\epsilon}{2}}(x) = \left\{ y : d(x, y) < \frac{\epsilon}{2} \right\}$$

does not contain the set $(0, \frac{\epsilon}{2}) \cup (1, 1 + \frac{\epsilon}{2}) \cup (2, 2 + \frac{\epsilon}{2})$. Each interval in that union has no least lower bound, since $\mathbb{N}$ is not bounded above, we can pick larger and larger n for $\frac{1}{n}$, which the interval is composed of. Since this portion of the interval is not included, the neighborhood is finite. Since the neighborhood is finite, the intersection with E is finite and therefore x is not a limit point.

As such, $x \notin E'$, and $E' = \{0, 1, 2\}$. □

6. Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\bar{E}$ have the same limit points. (Recall that $\bar{E} = E \cup E'$.) Do E and E' always have the same limit points?

Proof. Prove that E' is closed.

The set E' is closed if it contains its own limit points.

Let $x \in \bar{E}'$.

$\square$

Proof. Proof that E and $\bar{E}$ have the same limit points.

By the previous proof, E' is closed and therefore contains its own limit points.

Let $\bar{E}'$ be the set of all limit points of $\bar{E}$.

We begin by showing that $E' \subseteq \bar{E}'$.

Let $x \in E'$. Then $x \in$

$\square$

CHECK ME. Do E and E' always have the same limit points?

Consider the set

$$E = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

Since $\lim_{n\to\infty} \frac{1}{n} = 0$, and $|E| = |\mathbb{N}|$ (by Lemma 1), $N_r(0) \cap E$ has infinitely many points, zero must be a limit point of E, so $E' = \{0\}$.

However, the limit points of E' are the empty set, since no matter the point we pick to center the neighborhood, the intersection will contain at most one element (zero). As such, E and E' do not always have the same limit points.

$\square$

7. Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.

   (a) If $B_n = \cup_{i=1}^n A_i$, prove that $\bar{B}_n = \cup_{i=1}^n \bar{A}_i$, for $n = 1, 2, 3, \ldots$.

   (b) If $B = \cup_{i=1}^\infty A_i$, prove that $\bar{B} \supset \cup_{i=1}^\infty \bar{A}_i$.

   Show, by an example, that this inclusion can be proper.

CHECK ME. Note that:

$$\bar{B}_n = B_n \cup B_n'$$
$$\bigcup_{i=1}^n \bar{A}_i = \bigcup_{i=1}^n \{A_i \cup A_i'\} =$$

Claim: $\bar{B}_n \subseteq \cup_{i=1}^n \bar{A}_i$.
Assume $x \in \bar{B}_n$.

Case 1: $x \in B_n$.

Since $B_n \subseteq \cup_{i=1}^n A_i$, and $x \in B_n$, $x \in \cup_{i=1}^n A_i \subseteq \cup_{i=1}^n \bar{A}_i$. Therefore $x \in \cup_{i=1}^n \bar{A}_i$.

Case 2: $x \in B_n'$.

Since $B_n = \cup_{i=1}^n A_i$, and $x \in B_n'$, then x must be a limit point of some A_i in $\cup_{i=1}^n A_i$, so $x \in A_i'$, and as such is in the union $\cup_{i=1}^n A_i'$. Therefore $x \in \cup_{i=1}^n \bar{A}_i$.

As such, $\bar{B}_n \subseteq \cup_{i=1}^n \bar{A}_i$.

Claim: $\cup_{i=1}^n \bar{A}_i \subseteq \bar{B}_n$.

Assume $y \in \cup_{i=1}^n \bar{A}_i$.

Case 1: $y \in \cup_{i=1}^n A_i$.

Since $B_n = \cup_{i=1}^n A_i, y \in B_n$.
Case 2: $y \in \cup_{i=1}^n A_i'$.

Since $B_n = \cup_{i=1}^n A_i$, they must have the same limit points. As such, $y \in B_n'$.
As a result, $\cup_{i=1}^n \bar{A}_i \subseteq \bar{B}_n$.

Therefore, $\bar{B}_n = \cup_{i=1}^n \bar{A}_i$. $\square$

8. Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

CHECK ME. Since E is open, all elements of E are interior points.

Let $p \in E$. Choose $r > 0$, and let $N_r(p)$ be a neighborhood of p. Since p is an interior point, $\exists \delta > 0 : N_\delta(p) \subseteq E$.

Choose $0 < \epsilon \leq \min(r, \delta)$. Then, $N_\epsilon(p) \subseteq N_\delta(p) \subseteq E$. Since $N_\epsilon(p)$ contains infinitely many points of E, and $N_\epsilon(p) \subseteq N_r(p)$, $N_r(p)$ does as well.

Closed sets, by definition contain all of their own limit points, so any closed subset of $\mathbb{R}^2$ will contain its own limit points. However, it is not the case that any finite set of E will have limit points since the neighborhood intersect E will be finite. $\square$

9. Let E° denote the set of all interior points of a set E.

   (a) Prove that E° is always open.

   (b) Prove that E is open if and only if $E^\circ = E$.

   (c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.

   (d) Prove that the complement of E° is the closure of the complement of E.

   (e) Do E and $\bar{E}$ always have the same interiors?

   (f) Do E and E° always have the same closures?

Proof. Proof that E° is always open.

$E^\circ = \{p : N_r(p) \subseteq E \text{ for some } r\}$. □

CHECK ME. Proof that E is open if and only if $E^\circ = E$.

Assume that $E^\circ = E$. Then, E is the set of all interior points of E. Since every point of E is an interior point, E is open. □

CHECK ME. If $G \subset E$ and G is open, prove that $G \subset E^\circ$.

Since G is open, every point of G is an interior point of G. Since $G \subseteq E$, G is a set of some number of interior points of E, so $G \subseteq E^\circ$. □

Proof. Prove that the complement of E° is the closure of the complement of E.

$(E^\circ)^c = \bar{E}^c = (E^c \cup E'^c)$? □

Proof. Do E and $\bar{E}$ always have the same interiors? □

Proof. Do E and E° always have the same closures?

Since $\bar{E} = E \cup E'$, and $\bar{E}^\circ = E^\circ \cup E'^\circ$, we must show that $\bar{E} \subset \bar{E}^\circ$ and $\bar{E}^\circ \subset \bar{E}$.

$\bar{E} \subset \bar{E}^\circ$:

$\bar{E}^\circ \subset \bar{E}$: □