Homework 1
Connor Baker, January 2017

1. Prove by contradiction that if $a - b$ is odd, then $a + b$ is odd.

   Proof. Assume that $a, b \in \mathbb{Z}$, $a - b$ is even, and $a + b$ is odd. If $a - b$ is even, then by definition, $a - b = 2x$ for some number $x \in \mathbb{Z}$. If $a + b$ is odd, then by definition, $a + b = 2y + 1$ for some number $y \in \mathbb{Z}$.

   Combining the system of equations with addition yields $2a = 2x + 2y + 1$. By the definition of an even number, the product $2a$ will be even, as will the products $2x$ and $2y$. The sum of the two even products $2x$ and $2y$ is even. By definition, an even number plus one is odd. As a result, the equation is a false: an even integer cannot equal an odd integer.

   This is contradiction of our original assumption. Therefore, through proof by condtradiction, if $a - b$ is odd, then $a + b$ must odd. ∎

2. Write a proof by contrapositive to show that if xy is odd, then both x and y are odd.

   Proof. Assume that x is even, and that $x, y \in \mathbb{Z}$. By definition, $x = 2k$ for all $k \in \mathbb{Z}$. Then $xy = 2ky$, which, since $k, y \in \mathbb{Z}$, is by definition, even. So, regardless of the parity of y, the product xy will be even so long as at least one is even. If x was odd, and y was even, then the above would still hold, due to multiplication being commutative.

   Since either x or y is even, and xy is even, we can infer by the contrapositive that if xy is odd, then both x and y are odd. ∎

3. Prove that there do not exist integers m and n such that $12m + 15n = 1$.

   Proof. The equation $12m+15n = 1$ is equivalent to $3(4m+5n) = 1$. For this statement to be true, $4m+5n$ must be the multiplicative inverse of 3, which is not in the set of natural numbers. Therefore, there do not exist integers m and n such that $12m + 15n = 1$. ∎

4. Prove there is a natural number M such that for every natural number n, $\frac{1}{n} < M$.

   Proof. Let $n = 1$. Then $1/n = 1$. As n increases, the value of the ratio decreases since the top is constant. As such, since $n \in \mathbb{N}$, for all $n > 2$, $1/n < 1$.

   Therefore, the first natural number M larger than $1/n$ for all $n \geq 1$ is 2. ∎

5. Prove that if $-2 < x < 1$ or $x > 3$, then $\frac{(x-1)(x+2)}{(x-3)(x+4)} > 0$.

   Proof. The function has two x-intercepts, and two vertical asymptotes, at $x = -2, 1$ and $x = -4, 3$ respectively. By the Intermediate Value Theorem, those four x-values are the only places that the function can change the sign of its output.

   The sign of the function on the interval $(-\infty, -2)$ is not of concern: we care only about the interval $(-2, 1)$ and $(3, \infty)$. By picking a point on the intervals $(-2, 1)$ and $(3, \infty)$ and verifying the sign, the Intermediate Value Theorem proves that the function value has the same sign on the entirety of the interval.

   Let $x = 0$. Then, on the interval $(-2, 1)$, the function is positive.

   Let $x = 4$. Then, on the interval $(3, \infty)$, the function is positive.

   Therefore, by the Intermediate Value Theorem, if $-2 < x < 1$ or $x > 3$, then $\frac{(x-1)(x+2)}{(x-3)(x+4)} > 0$. ∎