**Homework 1**
**Connor Baker, January 2017**

1. Prove by contradiction that if $a - b$ is odd, then $a + b$ is odd.

   *Proof.* Assume that $a, b \in \mathbb{Z}$, $a-b$ is odd, and $a+b$ is even. If $a-b$ is odd, then by definition, $a-b = 2x+1$ for some number $x \in \mathbb{Z}$. If $a+b$ is even, then by definition, $a+b = 2y$ for some number $y \in \mathbb{Z}$.

   Combining the system of equations with addition yields $2a = 2x + 2y + 1$. This can be rewritten as $2a = 2(x+y)+1$. This implies that an even number (the product $2a$) can be equal to an odd number (the even numer resulting from the product of $2(x+y)$ plus one). As such, the equation is a false.

   This is contradiction of our original assumption. Therefore, through proof by condtradiction, if $a - b$ is odd, then $a + b$ is odd. $\square$

2. Write a proof by contrapositive to show that if $xy$ is odd, then both $x$ and $y$ are odd.

   *Proof.* We will prove that if $x$ or $y$ is even, then the product $xy$ is even. Assume that $x$ is even, and that $x, y \in \mathbb{Z}$. By definition, $x = 2k$ for all $k \in \mathbb{Z}$. Then $xy = 2ky$, which, since $k, y \in \mathbb{Z}$, is by definition, even. So, regardless of the parity of $y$, the product $xy$ will be even so long as at least one is even. If $x$ was odd, and $y$ was even, then the above would still hold, due to multiplication being commutative.

   Since either $x$ or $y$ is even, and $xy$ is even, we can infer by the contrapositive that if $xy$ is odd, then both $x$ and $y$ are odd. $\square$

3. Prove that there do not exist integers $m$ and $n$ such that $12m + 15n = 1$.

   *Proof.* The equation $12m+15n = 1$ is equivalent to $3(4m+5n) = 1$. For this statement to be true, $4m+5n$ must be the multiplicative inverse of 3, which is not in the set of natural numbers. Therefore, there do not exist integers $m$ and $n$ such that $12m + 15n = 1$. $\square$

4. Prove there is a natural number $M$ such that for every natural number $n$, $\frac{1}{n} < M$.

   *Proof.* The number $1/n$ is less than or equal to one for all $n \in \mathbb{N}$. As such, $M$ can be any number in the set of natural numbers that is greater than or equal to two

   Therefore, the first natural number $M$ larger than $1/n$ for all $n \geq 1$ is 2. $\square$

5. Prove that if $-2 < x < 1$ or $x > 3$, then $\frac{(x-1)(x+2)}{(x-3)(x+4)} > 0$.

   *Proof.* The function has two $x$-intercepts, and two vertical asymptotes, at $x = -2, 1$ and $x = -4, 3$ respectively. By the Intermediate Value Theorem, those four $x$-values are the only places that the function can change the sign of its output.

   The sign of the function on the interval $(-\infty, -2)$ is not of concern: we care only about the interval $(-2, 1)$ and $(3, \infty)$. By picking a point on the intervals $(-2, 1)$ and $(3, \infty)$ and verifying the sign, the Intermediate Value Theorem proves that the function value has the same sign on the entirety of the interval.

   Let $x = 0$. Then, on the interval $(-2, 1)$, the function is positive.

   Let $x = 4$. Then, on the interval $(3, \infty)$, the function is positive.

   Therefore, by the Intermediate Value Theorem, if $-2 < x < 1$ or $x > 3$, then $\frac{(x-1)(x+2)}{(x-3)(x+4)} > 0$. $\square$