Homework 6
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1. An algebraic number is any number that is a root of a polynomial with rational coefficients. Prove that the algebraic numbers are countable. A number is transcendental if it is not algebraic. Prove there are uncountable many transcendental numbers.

Proof. Let P_n be the set of all polynomials that have rational coefficients, and variables raised to nonzero, whole exponents of (at most) degree n:

$$P_n = \{a_k x^n + a_{k-1}x^{n-1} + \cdots + a_1 x^1 + a_0\} : a_i \in \mathbb{Q}\}.$$

Then, since all polynomials are functions on $\mathbb{R}$, P_n is a set of functions (which all have solutions). Let R_n be the set of all roots of P_n:

$$R_n = \{z \in \mathbb{C} : \exists f \in P_n : f(z) = 0\}.$$

By the fundamental theorem of algebra, every polynomial in P_n has at most n roots (including complex roots). Additionally, $|P_n|\ \forall n = |\mathbb{Q}|$. We can consider each coefficient to be a choice of an element from $\mathbb{Q}$, and the number of possible polynomials to be $|\cup_{i\in\mathbb{N}}^{n} \mathbb{Q}|$ (the union of the sets containing the coefficients we can choose from). Since this is the countable union of countable sets, the result is countable. Then since any P_n is countable:

$$\left| P_n \bigcup P_{n-1} \bigcup \cdots \bigcup P_1 \right| = |\mathbb{Q}|.$$

Since P_n is countable, R_n contains a countable number of solutions for countably many polynomials (technically, n times countably infinite solutions – which is still countably infinite). As such:

$$\left| R_n \bigcup R_{n-1} \bigcup \cdots \bigcup R_1 \right| = |\mathbb{Q}|.$$

Since there are countably many roots in all, algebraic numbers are countable.
We now prove that there are uncountable many transcendental numbers.
Let A be the set of all algebraic numbers. We know that $|A| = |\mathbb{Q}|$. The reals are the union of the sets of algebraic numbers A, and the set of transcendental numbers T. Since the union of any countable set with another countable set is countable, and we know that $|A \cup T| = |\mathbb{R}|$, T must be uncountable. $\square$

2. Let A be the set of all functions $f : \mathbb{N} \to \{0, 1\}$. Find the cardinality of A.

Proof. Let $A = \{f : f : \mathbb{N} \to \{0, 1\}\}$. Let

$$G = \{g : \mathbb{N} \to \{0, 1\}\}, g(n) = \begin{cases} 1 & rn \geq 0 \\ 0 & rn < 0 \end{cases}$$

where $r \in \mathbb{R}$. Then, since $\mathbb{R}$ is uncountable, G contains uncountable many functions.
Since G is a set of functions that map $\mathbb{N}$ to the set $\{0, 1\}$, $G \subseteq A$. Furthermore, since $|G| = |\mathbb{R}|$, it must also be the case that $|A| = |\mathbb{R}|$. $\square$

3. Let A be the set of all functions $f : \mathbb{N} \to \{0, 1\}$ that are "eventually zero" (We say that f is eventually zero if there is a positive integer N such that $f(n) = 0$ for all $n \geq N$). Find the cardinality of A.

Proof. Let $A = \{f : f : \mathbb{N} \to \{0, 1\}, \text{"and } f \text{ eventually zero"}\}$. Let

$$G = \{g : \mathbb{N} \to \{0, 1\}\}, g(n) = \begin{cases} 1 & n \leq |r| \\ 0 & n > |r| \end{cases}$$

where $r \in \mathbb{R}$. Then, since $\mathbb{R}$ is uncountable, G contains uncountable many functions.
Since G is a set of functions that map $\mathbb{N}$ to the set $\{0, 1\}$, $G \subseteq A$. Furthermore, since $|G| = |\mathbb{R}|$, it must also be the case that $|A| = |\mathbb{R}|$. $\square$

4. Use the axiom of choice to prove that if there exists $f : A \to B$ that is onto, then there exists a function $g : B \to A$ that is one-to-one.

5. We say that $|A| \geq |B|$ if there exists a function $f : A \to B$ which is onto. Prove that if $|A| \geq |B|$, and $|B| \geq |A|$, then $|A| = |B|$. (Hint: Use 4).