Homework 2
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1. Determine whether the following expressions are true or false. Give a complete explanation for each part.
   (a) $\emptyset \subseteq \{\emptyset, \{\emptyset\}\}$
   (b) $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$
   (c) $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\}$
   (d) For every set A, $\{\emptyset\} \subseteq A$.
   (e) $\{1, 2\} \in \{\{1, 2, 3\}, \{1, 3\}, 1, 2\}$
   (f) $\{\{4\}\} \subseteq \{1, 2, 3, \{4\}\}$

Definition 1 (Subset)**.** Given two subsets A, B, A is said to be a subset of B if and only if all elements of A are also in B. That is to say:

$$X \subseteq Y \iff \forall x(x \in X \implies x \in Y)$$

Proof. (a) Let $A = \emptyset, B = \{\emptyset, \{\emptyset\}\}$. Then, by the definition of a subset,

$$\emptyset \subseteq \{\emptyset, \{\emptyset\}\} \iff \forall a(a \in \emptyset \implies a \in \{\emptyset, \{\emptyset\}\})$$

However, $a \notin \emptyset$ (the empty set contains no elements). As such, the statement is vacuously true (because for all a, of which there are none, we cannot tell whether it is in both sets or not).

Therefore, by the definition of a subset, the expression is true. □

Proof. (b) Let $A = \{\emptyset\}, B = \{\emptyset, \{\emptyset\}\}$. Then, by the definition of a subset,

$$\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\} \iff \forall a(a \in \{\emptyset\} \implies a \in \{\emptyset, \{\emptyset\}\})$$

Let $a = \emptyset$, the only element of A. Then,

$$\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\} \iff (\emptyset \in \{\emptyset\} \implies \emptyset \in \{\emptyset, \{\emptyset\}\})$$

which is true.

Therefore, by the definition of a subset, the expression is true. □

Proof. (c) Let $A = \{\{\emptyset\}\}, B = \{\emptyset, \{\emptyset\}\}$. Then, by the definition of a subset,

$$\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\} \iff \forall a(a \in \{\{\emptyset\}\} \implies a \in \{\emptyset, \{\emptyset\}\})$$

Let $a = \{\emptyset\}$, the only element of A. Then,

$$\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\} \iff (\{\emptyset\} \in \{\{\emptyset\}\} \implies \emptyset \in \{\emptyset, \{\emptyset\}\})$$

which is true.

Therefore, by the definition of a subset, the expression is true. □

Proof. (d) Let $B = \{\emptyset\}$. Then, by the definition of a subset,

$$\{\emptyset\} \subseteq C \iff \forall b(b \in \{\emptyset\} \implies b \in C)$$

Let $b = \emptyset$, the only element of B. Then,

$$\{\emptyset\} \subseteq C \iff (\emptyset \in \{\emptyset\} \implies \emptyset \in C)$$

which is contingent on the elements of C. There is no guarantee that C contains the empty set.

Therefore, by the definition of a subset, the expression is false. □

Proof. (e) This statement is false because the set does not contain the set $\{1, 2\}$. □

Proof. (f) Let $A = \{\{4\}\}, B = \{1, 2, 3, \{4\}\}$. Then, by the definition of a subset,

$$\{\{4\}\} \subseteq \{1, 2, 3, \{4\}\} \iff \forall a(a \in \{\{4\}\} \implies a \in \{1, 2, 3, \{4\}\})$$

Let $a = \{4\}$, the only element of A. Then,

$$\{\{4\}\} \subseteq \{1, 2, 3, \{4\}\} \iff (\{4\} \in \{\{4\}\} \implies \{4\} \in \{1, 2, 3, \{4\}\})$$

which is true.

Therefore, by the definition of a subset, the expression is true. □

2. Let $\Delta = [0, 1) = \{x \in \mathbb{R} : 0 \leq x < 1\}$ and let $A_\alpha = (-\alpha, \alpha] = \{x \in \mathbb{R} : -\alpha < x \leq \alpha\} \subseteq \mathbb{R}$, where $\alpha \in \Delta$. Prove that

$$\bigcup_{\alpha \in \Delta} A_\alpha = (-1, 1),$$

and

$$\bigcap_{\alpha \in \Delta} A_\alpha = \emptyset$$

Proof. (a) We will first prove that

$$\bigcup_{\alpha \in \Delta} A_\alpha \subseteq (-1, 1).$$

There exists some $x \in \Delta = [0, 1)$ such that $a \in \ A_x = (-x, x]$. Since $0 \leq x < 1$, letting x be the largest value it can, the set A_x which is $(-x, x]$ must be a subset of the set $(-1, 1)$.

We will now prove that

$$(-1, 1) \subseteq \bigcup_{\alpha \in \Delta} A_\alpha.$$

Picking some x in $(-1, 1)$, it is guaranteed that $|x| \in \Delta$. Then, let $n = (1 + |x|)/(2)$. We know that $x \in A_n = (-n, n]$. As such, $(-1, 1) \in A_n$.

Therefore, it has been proven that

$$\bigcup_{\alpha \in \Delta} A_\alpha = (-1, 1).$$

□

Proof. (b) We will first prove that

$$\bigcap_{\alpha \in \Delta} A_\alpha \subseteq \emptyset.$$

Let $\alpha = 0$. Then $A_\alpha = (0, 0] = \emptyset$. The intersection with the any set is the empty set (since they have no elements to share in common). Since I proved in Problem (1), Part (a), the empty set is a subset of any set,

$$\bigcap_{\alpha \in \Delta} A_\alpha \subseteq \emptyset.$$

By the same proof as mentioned above,

$$\emptyset \subseteq \bigcap_{\alpha \in \Delta} A_\alpha.$$

Therefore, it has been proven that

$$\bigcap_{\alpha \in \Delta} A_\alpha = \emptyset.$$

□

3. Let A, B, C, and D be sets with $C \subseteq A$ and $D \subseteq B$. Prove that $C \cup D \subseteq A \cup B$.

Proof. By the definition of a subset, $(\forall c)(c \in C \implies c \in A)$. Again, by the same definition, $(\forall d)(d \in D \implies d \in B)$.

Then, by the definition of the union, $(\forall c)(c \in C \implies c \in C \cup D)$ and $(\forall d)(d \in D \implies d \in C \cup D)$.

Additionally, by the same definition, it follows from the first two statements that since all $c, d \in C \cup D$, then all $c, d \in A \cup B$.

Therefore, since all $c, d \in C \cup D$ and all $c, d \in A \cup B$, it follows that $C \cup D \subseteq A \cup B$. $\square$

4. Prove that if $\mathcal{A}$ is a non-empty family of sets, then

$$\bigcap_{A \in \mathcal{A}} A \subseteq \bigcup_{A \in \mathcal{A}} A.$$

Proof. Let the set $X \in \mathcal{A}$. Then, let $x \in X$. If $x \in \cap_{A \in \mathcal{A}} A$, then x is in all sets in the family. As such, $\cap_{A \in \mathcal{A}} A \subseteq X$.

If x from the set X as described above is in $\cup_{A \in \mathcal{A}} A$, then x is in at least one set in the family. As such, $X \subseteq \cup_{A \in \mathcal{A}} A$.

Then, $\cap_{A \in \mathcal{A}} A \subseteq X \subseteq \cup_{A \in \mathcal{A}} A$. Therefore, $\cap_{A \in \mathcal{A}} A \subseteq \cup_{A \in \mathcal{A}} A$. $\square$

5. Use the principle of mathematical induction to prove $4^{n+4} > (n+4)^4$, for all natural numbers n.

Proof. Let $n = 1$:

$$4^5 > 5^4$$

so the base case is true. Then, let $n = k$:

$$4^{k+4} > (k+4)^4. \tag{1}$$

Since $k \in \mathbb{N}$, then $(k+1) \in \mathbb{N}$ as well. As such, when $n = k+1$:

$$4^{k+5} > (k+5)^4. \tag{2}$$

To prove that this inequality holds for all natural numbers, we will establish bounds using Inequality (1). Equation (3) shows the left hand side of Inequality (2) re-written in expanded form.

$$4^{k+5} = 4^k * 4^5 \tag{3}$$

With this knowledge, we can see that:

$$4^{k+5} = 4^k * 4^5 = 4(4^{k+4}). \tag{4}$$

which we can compare directly to the left hand side of Inequality (1)

$$4(4^{k+4}) > 4^{k+4}$$

and find that it is greater by four times. In terms of building a series of inequalities, we now have:

$$4^{k+5} = 4(4^{k+4}) > 4^{k+4} \tag{5}$$

Now, let us look at the right hand side of Inequalities (1) and (2):

$$(k+4)^4 = k^4 + 16k^3 + 96k^2 + 256k + 256 \tag{6}$$

$$(k+5)^4 = k^4 + 20k^3 + 150k^2 + 500k + 625. \tag{7}$$

Referring to Equation (4), where we used four times the left hand side of Inequality (2), we now use four times the right hand side of Inequality (1) to keep the relationship that the two share:

$$4(4^{k+4}) > 4(k+4)^4. \tag{8}$$

From this point, it is important to note the distributed form of $4(k+4)^4$ is greater than $(k+5)^4$ (given of course that $k \in \mathbb{N}$):

$$4(k+4)^4 = 4k^4 + 64k^3 + 384k^2 + 1024k + 1024 > k^4 + 20k^3 + 150k^2 + 500k + 625 = (k+5)^4. \tag{9}$$

Therefore

$$4(k+4)^4 > (k+5)^4. \tag{10}$$

Finally, we build the last inequality, taking parts of Equation (5), Inequality (8), and Inequality (10):

$$4^{k+5} = 4(4^{k+4}) > 4(k+4)^4 > (k+5)^4 \tag{11}$$

Then, by the principle of mathematical induction, $4^{n+4} > (n+4)^4$ $\forall n \in \mathbb{N}$. $\square$