Homework 6
Connor Baker, April 2017

1. An algebraic number is any number that is a root of a polynomial with rational coefficients. Prove that the algebraic numbers are countable. A number is transcendental if it is not algebraic. Prove there are uncountable many transcendental numbers.

Proof. Let M_n be the set of all monomials of degree $n, n \in \mathbb{Z}^+$, which have rational coefficients:

$$M_n = \{q_n x^n : q_n \in \mathbb{Q}\}.$$

Let P_n be the set of all polynomials of (at most) degree n, which have rational coefficients:

$$P_n = \{M_n + M_{n-1} + \cdots + M_1 + M_0\}.$$

Then, $|P_n| = |M_0 \cup M_1 \cup \cdots \cup M_n| = |\mathbb{Q}|$, since the countable union of countable sets is countable.
Let R_n be the set of all roots of P_n:

$$R_n = \{x \in \mathbb{R} : \exists f \in P_n : f(x) = 0\}.$$

For each $i \in \mathbb{N}$, let p_i^n be a unique polynomial from P_n. Then, let $r_i^n \subseteq R_n$ be the roots of p_i^n. By the fundamental theorem of algebra, every polynomial of degree n has at most n real roots.
Then,

$$R_n = \bigcup_{i \in \mathbb{N}}^{\infty} r_i^n.$$

As the countable union of countable sets, R_n is countable, and $|R_n| = |\mathbb{Q}|$.
Let $\mathbb{A} = \cup_{n \in \mathbb{N}} R_n$ be the set of algebraic numbers. Again, the countable union of countable sets is countable, and the algebraic numbers are countable.
We now prove that there are uncountable many transcendental numbers by contradiction.
Suppose that the set of all transcendental numbers $\mathbb{T}$ is countable. Then, since $\mathbb{R}$ is defined as $\mathbb{A} \cup \mathbb{T}$, $|\mathbb{R}| = |\mathbb{A} \cup \mathbb{T}|$. However, since the union of any countable set with another countable set is countable, $|A \cup T| = |\mathbb{Q}|$, which by transitivity means $|\mathbb{Q}| = |\mathbb{R}|$, which is a contradiction since the reals are uncountable. Therefore, by contradiction, T must be uncountable. $\square$

2. Let A be the set of all functions $f : \mathbb{N} \to \{0, 1\}$. Find the cardinality of A.

Proof. Let $A = \{f : f : \mathbb{N} \to \{0, 1\}\}$. Let

$$G = \{g : \mathbb{N} \to \{0, 1\}\}, g(n) = \begin{cases} 1 & rn \geq 0 \\ 0 & rn < 0 \end{cases}$$

where $r \in \mathbb{R}$. Then, since $\mathbb{R}$ is uncountable, G contains uncountable many functions.
Since G is a set of functions that map $\mathbb{N}$ to the set $\{0, 1\}$, $G \subseteq A$. Furthermore, since $|G| = |\mathbb{R}|$, it must also be the case that $|A| = |\mathbb{R}|$. $\square$

3. Let A be the set of all functions $f : \mathbb{N} \to \{0, 1\}$ that are "eventually zero" (We say that f is eventually zero if there is a positive integer N such that $f(n) = 0$ for all $n \geq N$). Find the cardinality of A.

Proof. Let $A = \{f : f : \mathbb{N} \to \{0, 1\}, \text{"and } f \text{ eventually zero"}\}$. Let

$$G = \{g : \mathbb{N} \to \{0, 1\}\}, g(n) = \begin{cases} 1 & n \leq |r| \\ 0 & n > |r| \end{cases}$$

where $r \in \mathbb{R}$. Then, since $\mathbb{R}$ is uncountable, G contains uncountable many functions.
Since G is a set of functions that map $\mathbb{N}$ to the set $\{0, 1\}$, $G \subseteq A$. Furthermore, since $|G| = |\mathbb{R}|$, it must also be the case that $|A| = |\mathbb{R}|$. $\square$

4. Use the axiom of choice to prove that if there exists $f : A \to B$ that is onto, then there exists a function $g : B \to A$ that is one-to-one.

5. We say that $|A| \geq |B|$ if there exists a function $f : A \to B$ which is onto. Prove that if $|A| \geq |B|$, and $|B| \geq |A|$, then $|A| = |B|$. (Hint: Use 4).