Homework 5
Connor Baker, March 2017

1. Prove that if the real-valued function f is strictly increasing or strictly decreasing on $\mathbb{R}$, then f is one-to-one (Note: You cannot assume f is differentiable).

Proof. Case 1: f is strictly decreasing.
Let $x, a \in \text{dom}(f)$. Assume that $f(x) = f(a)$. If $x \neq a$, then by trichotomy either $x < a$ or $x > a$.
Case A: If $x < a$, then $f(x) > f(a)$, and $f(x) \neq f(a)$.
Case B: If $x > a$, then $f(x) < f(a)$, and $f(x) \neq f(a)$.
In either case, if $x \neq a$, then $f(x) \neq f(a)$, and f is one-to-one.
Case 2: f is strictly increasing.
Let $x, a \in \text{dom}(f)$. Assume that $f(x) = f(a)$. If $x \neq a$, then by trichotomy either $x < a$ or $x > a$.
Case A: If $x < a$, then $f(x) < f(a)$, and $f(x) \neq f(a)$.
Case B: If $x > a$, then $f(x) > f(a)$, and $f(x) \neq f(a)$.
In either case, if $x \neq a$, then $f(x) \neq f(a)$, and f is one-to-one.
As such, f is one-to-one if it is strictly increasing or strictly decreasing on $\mathbb{R}$. □

2. Prove the following are metrics:

   (a) $X = \mathbb{R}, d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

   (b) $X = \mathbb{R} \times \mathbb{R}, d((x, y), (z, w)) = \sqrt{(x - z)^2 + (y - w)^2}$

   Definition 1 (Metric)**.** A metric on a set X is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$,

   (a) $d(x, y) \geq 0$

   (b) $d(x, y) = 0$ if and only if $x = y$

   (c) $d(x, y) = d(y, x)$,

   (d) $d(x, y) + d(y, z) \geq d(x, z)$.

Proof. We begin by proving that the first function is a metric.

1. The $\text{rang}(d) = \{0, 1\}$ so the function is definitely greater than or equal to zero for any inputted pair of values.

2. By the definition of d, $d(x, y) = 0$ if and only if $x = y$.

3. Since the equals relationship is symmetric, $x = y \implies y = x$. As such, $d(x, y) = d(y, x)$, since the order of the inputs does not generate a unique output.

4. Not completed.

Therefore the first function is a metric.

We now prove that the second function is a metric.

1. The $\text{rang}(d) = \{0, 1\}$ so the function is definitely greater than zero.

2. By the definition of d, if and only if $x = y$ does $d(x, y) = 1$.

3. Since the equals relationship is symmetric, $x = y \implies y = x$. As such, $d(x, y) = d(y, x)$.

4. Not completed.

Therefore the function is a metric. □

3. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be given by $f(m, n) = 2^{m-1}(2n - 1)$. Prove that f is one-to-one and onto.

Proof. We begin by showing that the prime factorization is unique up to commutativity. We have to match the factors of two, and then the odd number factor which is the prodcut of all the odd prime factors.
We need to make sure we note all of the solutions are in $\mathbb{N}$. $\square$

4. Let $f : A \to B$ be a function from a nonempty set A. Prove that the set $\mathcal{C} = \{f^{-1}(b) : b \in \operatorname{rang}(f)\}$ is a partition of A. Note: $\mathcal{C}$ is a subset of $\mathcal{P}(A)$.

Definition 1 (Partition)**.**

Proof. $\square$

5. Let $f : A \to B$ be a function from a nonempty set A which is surjective. Find a new function $g : C \to B$ which is one-to-one such that $C \subseteq A$, $\text{rang}(g) = \text{rang}(f)$, and for every $x \in C, f(x) = g(x)$. Explain why g has an inverse function, g^{-1}. Then, compute $f(g^{-1}(x))$ for all $x \in B$.

Proof. $\square$