**Homework 5**
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1. Prove that if the real-valued function $f$ is strictly increasing or strictly decreasing on $\mathbb{R}$, then $f$ is one-to-one (Note: You cannot assume $f$ is differentiable).

*Proof.* Case 1: $f$ is strictly decreasing.
Let $x, a \in \mathrm{dom}(f)$. Assume that $f(x) = f(a)$. If $x \neq a$, then by trichotomy either $x < a$ or $x > a$.
Case A: If $x < a$, then $f(x) > f(a)$, and $f(x) \neq f(a)$.
Case B: If $x > a$, then $f(x) < f(a)$, and $f(x) \neq f(a)$.
In either case, if $x \neq a$, then $f(x) \neq f(a)$, and $f$ is one-to-one.
Case 2: $f$ is strictly increasing.
Let $x, a \in \mathrm{dom}(f)$. Assume that $f(x) = f(a)$. If $x \neq a$, then by trichotomy either $x < a$ or $x > a$.
Case A: If $x < a$, then $f(x) < f(a)$, and $f(x) \neq f(a)$.
Case B: If $x > a$, then $f(x) > f(a)$, and $f(x) \neq f(a)$.
In either case, if $x \neq a$, then $f(x) \neq f(a)$, and $f$ is one-to-one.
As such, $f$ is one-to-one if it is strictly increasing or strictly decreasing on $\mathbb{R}$. $\square$

2. Prove the following are metrics:

   (a) $X = \mathbb{R}, d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

   (b) $X = \mathbb{R} \times \mathbb{R}, d((x,y),(z,w)) = \sqrt{(x-z)^2 + (y-w)^2}$

   **Definition 1** (Metric)**.** A metric on a set $X$ is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$,

   (a) $d(x,y) \geq 0$

   (b) $d(x,y) = 0$ if and only if $x = y$

   (c) $d(x,y) = d(y,x)$,

   (d) $d(x,y) + d(y,z) \geq d(x,z)$.

*Proof.* We begin by proving that the first function is a metric.

1. The $\text{rang}(d) = \{0,1\}$ so the function is definitely greater than or equal to zero for any inputted pair of values.

2. By the definition of $d$, $d(x,y) = 0$ if and only if $x = y$.

3. Since the equals relationship is symmetric, $x = y \implies y = x$. As such, $d(x,y) = d(y,x)$, since the order of the inputs does not generate a unique output.

4. Not completed.

Therefore the first function is a metric.

We now prove that the second function is a metric.

1. The $\text{rang}(d) = \{0,1\}$ so the function is definitely greater than zero.

2. By the definition of $d$, if and only if $x = y$ does $d(x,y) = 1$.

3. Since the equals relationship is symmetric, $x = y \implies y = x$. As such, $d(x,y) = d(y,x)$.

4. Not completed.

Therefore the function is a metric. $\square$

3. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be given by $f(m, n) = 2^{m-1}(2n - 1)$. Prove that $f$ is one-to-one and onto.

*Proof.* If we can prove that there is a unique solution $f(m, n)$ for every $m, n \in \mathbb{N}$, then we will have proven that $f$ is one-to-one and onto.
We begin by breaking apart the function. Let us consider two functions:

$$g(m) = 2^{m-1}$$

and

$$h(n) = 2n - 1,$$

such that

$$f(m, n) = g(m) \cdot h(n).$$

The function $g$ is clearly one-to-one and onto for any $m$. Any value of $m$ produces a power of two, all of which are in $\mathbb{N}$. Looking at $f$, we see that if $m > 1$, $g$ in effect creates the even factors found in the result, $f(m, n)$.
Considering $h$, we see that it is also one-to-one and onto for any $n$. Any value of $n$ produces an odd number (because $h$ is the definition of an odd number) – in fact, $h$ produces every odd number in $\mathbb{N}$.
There are two cases to consider.
Case 1: $m = 1$. In this case, $f(1, n)$ will map $n$ to every odd number in $\mathbb{N}$. If $m = 1$, the function $f$ is one-to-one and onto.
Let $p_1 p_2 \dots p_q$ be the prime factorization of $f(m, n)$, where $p_i$, $1 \leq i \leq q$, is a prime factor of $f(m, n)$ raised to some power. By the fundamental theorem of arithmetic, the prime factorization of a number is unique up to commutativity. In this case, we find that $f(1, n) = 2n - 1$, so the prime factorization is entirely dependent on the value of $n$.
Case 2: $m > 1$. In this case, $f(m, n)$ has some power of two (that is not one) multiplying an odd number. Since any odd number multiplied by an even number is even, $f(m, n)$ will be even for all $n$.
Let $p_1 p_2 \dots p_q$ be the prime factorization of $f(m, n)$, where $p_i$, $1 \leq i \leq q$, is a prime factor of $f(m, n)$ raised to some power. By the fundamental theorem of arithmetic, the prime factorization of a number is unique up to commutativity. In this case, we find that $f(m, n) = 2^{m-1}(2n - 1)$, so the prime factorization is entirely dependent on the value of both $m$ and $n$.
Since different values of $n$ will yield different odd numbers, the prime factors will be the same if and only if the value of the input is the same. The same is true for $m$.
As such, as long as $(m, n) \neq (p, q)$, $f(m, n)$ and $f(p, q)$ have different prime factors. Therefore, $f(m, n)$ is one-to-one.
Furthermore, there is always at least one solution for all $m, n$, so $f(m, n)$ is onto as well. $\square$

4. Let $f : A \to B$ be a function from a nonempty set $A$. Prove that the set $\mathcal{C} = \{f^{-1}(b) : b \in \text{rang}(f)\}$ is a partition of $A$. Note: $\mathcal{C}$ is a subset of $\mathcal{P}(A)$.

**Definition 1** (Partition)**.** The set $\mathcal{C}$ is a partition of $A$ if:

1. $\forall D \in \mathcal{C},\ D \neq \emptyset$
2. $\forall D, E \in \mathcal{C}$, then either $D \cap E = \emptyset$ or $D = E$
3. $\cup_{D \in \mathcal{C}} D = A$

*Proof.* We begin by showing that $\mathcal{C}$ does not contain the empty set. By the definition of the inverse, we know that $f^{-1} = \{a \in A : f(a) = b\}$. Choosing any $D \in \mathcal{C} \subseteq \mathcal{P}(A)$, we can see that $\exists b \in \text{rang}(f) : f^{-1}(b) = D$. The set $D$ is nonempty, since if $b \in \text{rang}(f)$, then there exists $a \in A : f(a) = b$.
We now show that any two subsets of $\mathcal{C}$ are pairwise disjoint if they are not the same set. Since $f$ is a function, there is only one $f(a)$ for any $a \in A$. Then, the intersection of any two subsets of $\mathcal{C}$, which is the set containing the inverse of all elements of the range, will be the empty set unless they are the same: this follows as a result of there being only one $f(a)$ for any $a \in A$ – if there were more than one, then multiple elements in the range could have the same preimage, but this is not the case. Indeed, the intersect of any two sets in $\mathcal{C}$ is not the empty set if they are the same set.
Finally, we show that the union of all sets in $\mathcal{C}$ are equal to $A$. since $f$ is a function, $\text{dom}(f) = A$. That means that every point in $A$ is either mapped to $B$ by $f$, or does not have a solution. By the definition of $\mathcal{C}$, the other partition must be the set of $a \in A : f^{-1}(b) \neq a, b \in B$. The union of these two partitions is equal to $A$.
Therefore, $\mathcal{C}$ is a partition of $A$.

$\square$

5. Let $f : A \to B$ be a function from a nonempty set $A$ which is surjective. Find a new function $g : C \to B$ which is one-to-one such that $C \subseteq A$, $\text{rang}(g) = \text{rang}(f)$, and for every $x \in C, f(x) = g(x)$. Explain why $g$ has an inverse function, $g^{-1}$. Then, compute $f(g^{-1}(x))$ for all $x \in B$.

*Proof.* Since $f$ is surjective, there is at least one solution for $f(a)$ for all $a \in A$. Consider the function $g : C \to B$, where $C$ is the set of all values in $A$ such that a single element is chosen from every different partition of $A$. Then, since we can partition $A$ such that each partition's elements send all its points to the same point in $B$, by restricting the domain to a single point from each partition, we can make $g$ one-to-one – and since $f$ was onto, simply restricting the domain doesn't cause $g$ to cease being onto, which makes $g$ bijective. Furthermore, $\text{rang}(g) = \text{rang}(f)$, and for every $x \in C, f(x) = g(x)$, since $g$ is essentially $f$ with a restricted domain. Since $g$ is bijective, it has an inverse that is also bijective.
The composed function $f(g^{-1}(x))$ for all $x \in B$ is $B$. The inverse $g^{-1}$ takes every element of $B$ and maps it to a partition of $A$. Then, $f$ operates on those elements (which are essentially a single element from every partition of $A$, making $f$ look like $g$ since it's operating on what is essentially a restricted domain) and sends them to the range. Since $f$ is onto, every element that $g^{-1}$ returned in $A$ is sent back to $B$. As such, the composition yields the set $B$. $\square$