Homework 3
Connor Baker, February 2017

1. Let R be a relation from a nonempty set A to itself. Prove that if R is symmetric, transitive, and $dom(R) = A$, then R is an equivalence relation.

Proof. Let $x \in A$. Then since the the domain of R is A, $y : xRy$. By symmetry, yRx. By transitivity, xRx, so the relaiton is also reflixive. Since the relation is symmetric, transitive, and reflexive, R must be an equivalence relation. $\square$

2. Use the Principle of Mathematical Induction to prove $3^n \geq 2^n + 1$ for all $n \in \mathbb{N}$.

Proof. Let $n = 1$. Then,

$$3^1 \geq 2^1 + 1. \tag{1}$$

Next, we must now prove that if the inequality is true for some $n \in \mathbb{N}$, the above inequality also holds for $n + 1$. Let $n = k$. Then, assume the following equation to be true:

$$3^k \geq 2^k + 1. \tag{2}$$

We now try to prove that the inequality holds for $n = k + 1$:

$$3^{k+1} \geq 2^{k+1} + 1, \tag{3}$$

which is equivalent to

$$3^k \cdot 3 \geq 2^k \cdot 2 + 1. \tag{4}$$

Multiplying Inequality (2) by two gives us:

$$3^k \cdot 2 \geq 2^k \cdot 2 + 2. \tag{5}$$

So:

$$3^{k+1} = 3^k \cdot 3 > 3^k \cdot 2 \geq 2^k \cdot 2 + 2 > 2^k \cdot 2 + 1 = 2^{k+1} + 1$$

Therefore, by the Principal of Mathematical Induction, $3^n \geq 2^n + 1$, $\forall n \in \mathbb{N}$. $\square$

3. Use the Principle of Mathematical Induction to prove

$$\sqrt{2\sqrt{2\sqrt{2\sqrt{2\ldots}}}} \leq 2.$$

(Hint: Construct a recursively defined sequence.)

Proof. We construct a recursively defined sequence to model the left side of the inequality we want to prove. Let

$$a_1 = \sqrt{2}$$

and,

$$a_n = \sqrt{2 \cdot a_{n-1}}.$$

Now, with this recursive sequence, we can prove by induction.
Let $n = 1$. Then,

$$\sqrt{2} \leq 2 \tag{1}$$

holds true.
Next, we must now prove that if the inequality is true for some $n \in \mathbb{N}$, the above inequality also holds for $n + 1$.
Let $n = k$. Then, assume the following equation to be true:

$$a_k = \sqrt{2 \cdot a_{k-1}} \leq 2. \tag{2}$$

We now try to prove that the inequality holds for $n = k + 1$:

$$a_{k+1} = \sqrt{2 \cdot a_k} \leq 2 \tag{3}$$

Since $a_k \leq 2$, it must be the case that $\sqrt{a_k} \leq \sqrt{2}$. Therefore, since

$$a_{k+1} = \sqrt{2} \cdot \sqrt{a_k} \tag{4}$$

it must be the case that

$$a_{k+1} \leq 2. \tag{5}$$

Therefore, by the Principle of Mathematical Induction, it has been proven that $\sqrt{2\sqrt{2\sqrt{2\sqrt{2\ldots}}}} \leq 2$. □

4. Let $a_1 = 2, a_2 = 4$, and $a_{n+2} = 5a_{n+1} - 6a_n$ for $n \geq 1$. Prove that $a_n = 2^n$ for all natural numbers n.

Proof. Let $n = 1$. Then:

$$a_1 = 2 = 2^1.$$

Next, we must now prove that if the inequality is true for some $n \in \mathbb{N}$, the above inequality also holds for $n+1$. Let $n = k$. Then, assume the following equation to be true $\forall n \leq k$ as well – that is, assume the equality holds for Equation (1) and Equation (2).

$$a_{k+1} = 5a_k - 6a_{k-1} = 2^{k-1}, \tag{1}$$

$$a_{k+2} = 5a_{k+1} - 6a_k = 2^k. \tag{2}$$

We now try to prove that the inequality holds for $n = k+1$:

$$a_{k+3} = 5a_{k+2} - 6a_{k+1} = 2^{k+1}. \tag{3}$$

We substitute Equation (1) into Equation (2):

$$a_{k+3} = 5(2^k) - 6(2^{k-1}) = 2^{k+1}. \tag{4}$$

This simplifies as follows:

$$\begin{gathered} a_{k+3} = 5(2 \cdot 2^{k-1}) - 6(2^{k-1}) = 2^{k+1}, \\ a_{k+3} = 10 \cdot 2^{k-1}) - 6 \cdot 2^{k-1}) = 2^{k+1}, \\ a_{k+3} = 2^{k-1}(10 - 6) = 2^{k+1}, \\ a_{k+3} = 2^{k-1}(4) = 2^{k+1}, \\ a_{k+3} = 2^{k-1}(2^2) = 2^{k+1}, \\ a_{k+3} = 2^{k+1} = 2^{k+1}. \end{gathered} \tag{5}$$

Therefore, by the Principle of Strong Mathematical Induction, $a_n = 2^n$ holds for all natural numbers n. $\square$

5. Use the Principle of Mathematical Induction to prove that

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1.$$

Proof. Let $n = 1$. Then:

$$\begin{aligned} 1 \cdot 1! &= (1+1)! - 1 \\ 1 &= 2 - 1 \\ 1 &= 1. \end{aligned}$$

Next, we must now prove that if the equality is true for some $n \in \mathbb{N}$, the above equality also holds for $n+1$. Let $n = k$. Then, assume the following equation to be true:

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + k \cdot k! = (k+1)! - 1. \tag{1}$$

We now try to prove that the equality holds for $n = k+1$:

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + (k+1) \cdot (k+1)! = ((k+1)+1)! - 1. \tag{2}$$

Equation (2) can be re-written to show more terms so that we can more clearly substitute in Equation (1):

$$1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + (k+1) \cdot (k+1)! = ((k+1)+1)! - 1,$$

which when we substitute Equation (1) into becomes:

$$(k+1)! - 1 + (k+1) \cdot (k+1)! = ((k+1)+1)! - 1.$$

Factoring out the $(k+1)!$ yeilds:

$$(k+1)! \cdot ((k+1)+1) - 1 = (k+2)! - 1.$$

This can be re-written as

$$(k+1)! \cdot (k+2) - 1 = (k+2)! - 1,$$

which is the same as

$$(k+2)! - 1 = (k+2)! - 1. \tag{3}$$

Therefore, it has been proven by the Principle of Mathematical Induction that the equality holds for all natural numbers n. ∎