Homework 1
Connor Baker, January 2017

1. Prove by contradiction that if $a - b$ is odd, then $a + b$ is odd.

Proof Assume that $a, b \in \mathbb{Z}$, $a - b$ is even, and $a + b$ is odd. If $a - b$ is even, then by definition, $a - b = 2x$ for some number $x \in \mathbb{Z}$. If $a + b$ is odd, then by definition, $a + b = 2y + 1$ for some number $y \in \mathbb{Z}$.

Combining the system of equations with addition yields $2a = 2x+2y+1$. By the definition of an even number, the product $2a$ will be positive, as will the products $2x$ and $2y$. The sum of the two even products $2x$ and $2y$ is even. By definition, an even number plus one is odd. As a result, the equation is a false: an even integer cannot equal an odd integer.

This is indicative of our original assumption that $a-b$ is even, and $a+b$ is odd being false. Therefore, through proof by condtradiction, if $a - b$ is odd, then $a + b$ must odd.

2. Write a proof by contrapositive to show that if xy is odd, then both x and y are odd.

Proof Assume that x is even, and that $x, y \in \mathbb{Z}$. By definition, $x = 2k$ for all $k \in \mathbb{Z}$. Then $xy = 2ky$, which, since $k, y \in \mathbb{Z}$, is by definition, even. So, regardless of the parity of y, the product xy will be even so long as the multiplier and or multiplicand is even. If x was odd, and y was even, then the above would still hold, due to multiplication being commutative.

Since either x or y is even, and xy is even, we can infer by the contrapositive that if xy is odd, then both x and y are odd.

3. Prove that there do not exist integers m and n such that $12m+15n = 1$.

Proof The equation $12m + 15n = 1$ is equivalent to $3(4m + 5n) = 1$. For this statement to be true, $4m + 5n$ must be the multiplicative inverse of 3, which is not in the set of natural numbers. Therefore, there do not exist integers m and n such that $12m + 15n = 1$.

4. Prove there is a natural number M such that for every natural number n, $\frac{1}{n} < M$.

Proof Let $n = 1$. Then $1/n = 1$. As n increases, the value of the ratio decreases since the top is constant. As such, since $n \in \mathbb{N}$, for all $n > 2$, $1/n < 1$.

Therefore, the first natural number M larger than $1/n$ for all $n \geq 1$ is 2.

5. Prove that if $-2 < x < 1$ or $x > 3$, then $\frac{(x-1)(x+2)}{(x-3)(x+4)} > 0$.

Proof