Rudin's PRINCIPLES OF MATHEMATICAL ANALYSIS, 3RD ED

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## Basic Topology – Selected Exercises

5. Construct a bounded set of real numbers with exactly three limit points.

*Proof.* The point $p$ is a limit point of $\{\frac{1}{n} : n \in \mathbb{N}\}$ if $\forall r > 0, N_r(p) \cap E$ has infinitely many points.

Claim: Zero is a limit point of $\{\frac{1}{n} : n \in \mathbb{N}\}$.

For zero, $N_r(0) = \{x : d(x, 0) < r\} = \{x : |x - 0| < r\} = \{x : -r < x < r\} = (-r, r)$. Since $(r, r) \subseteq \mathbb{R}, \forall r > 0$, there are uncountably infinitely many points and $(r, r) \cap \{\frac{1}{n} : n \in \mathbb{N}\}$ has infinitely many points.

It is important to note that we can shift the location of this limit point. Consider the set $\{\frac{1}{n} + 1 : n \in \mathbb{N}\}$. For the same reasons above we can see that the limit point here is one.

In general, we can make a bounded set of real numbers with exactly $p, p \in \mathbb{Z}^+$, limit points with the set

$$\bigcup_{k=0}^{p} \{\frac{1}{n} + k : n \in \mathbb{N}\}.$$

So, if we want three limit points, our set is as

$$\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \bigcup \left\{\frac{1}{n} + 1 : n \in \mathbb{N}\right\} \bigcup \left\{\frac{1}{n} + 2 : n \in \mathbb{N}\right\}.$$

$\square$

6. Let $E'$ be the set of all limit points of a set $E$. Prove that $E'$ is closed. Prove that $E$ and $\bar{E}$ have the same limit points. (Recall that $\bar{E} = E \cup E'$.) Do $E$ and $E'$ always have the same limit points?

*Proof.* □

7. Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.

   (a) If $B_n = \cup_{i=1}^{n} A_i$, prove that $\bar{B}_n = \cup_{i=1}^{n} \bar{A}_i$, for $n = 1, 2, 3, \ldots$.

   (b) If $B = \cup_{i=1}^{\infty} A_i$, prove that $\bar{B} \supset \cup_{i=1}^{\infty} \bar{A}_i$.

   Show, by an example, that this inclusion can be proper.

*Proof.* □

8. Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of $E$? Answer the same question for closed sets in $\mathbb{R}^2$.

*Proof.* $\square$

9. Let $E^\circ$ denote the set of all interior points of a set $E$.

   (a) Prove that $E^\circ$ is always open.
   (b) Prove that $E$ is open if and only if $E^\circ = E$.
   (c) If $G \subset E$ and $G$ is open, prove that $G \subset E^\circ$.
   (d) Prove that the complement of $E^\circ$ is the closure of the complement of $E$.
   (e) Do $E$ and $\bar{E}$ always have the same interiors?
   (f) Do $E$ and $E^\circ$ always have the same closures?

*Proof.* □