Homework 3
Connor Baker, February 2017

1. Let R be a relation from a nonempty set A to itself. Prove that if R is symmetric, transitive, and $dom(R) = A$, then R is an equivalence relation.

Proof. $\square$

2. Use the Principle of Mathematical Induction to prove $3^n \geq 2^n + 1$ for all $n \in \mathbb{N}$.

Proof. Let $n = 1$. Then,

$$3^1 \geq 2^1 + 1. \tag{1}$$

Next, we must now prove that if the inequality is true for some $n \in \mathbb{N}$, the above inequality also holds for $n+1$. Let $n = k$. Then, assume the following equation to be true:

$$3^k \geq 2^k + 1. \tag{2}$$

We now try to prove that the inequality holds for $n = k + 1$:

$$3^{k+1} \geq 2^{k+1} + 1, \tag{3}$$

which is equivalent to

$$3^k * 3 \geq 2^k * 2 + 1. \tag{4}$$

Multiplying Inequality (2) by two gives us:

$$3^k * 2 \geq 2^k * 2 + 2. \tag{5}$$

So:

$$3^{k+1} = 3^k * 3 > 3^k * 2 \geq 2^k * 2 + 2 > 2^k * 2 + 1 = 2^{k+1} + 1$$

Therefore, by the Principal of Mathematical Induction, $3^n \geq 2^n + 1$, $\forall n \in \mathbb{N}$. $\square$

3. Use the Principle of Mathematical Induction to prove

$$\sqrt{2\sqrt{2\sqrt{2\sqrt{2\ldots}}}} \leq 2.$$

(Hint: Construct a recursively defined sequence.)

Proof. We construct a recursively defined sequence to model the left side of the inequality we want to prove. Let

$$a_1 = \sqrt{2}$$

and,

$$a_n = \sqrt{2 * a_{n-1}}.$$

Now, with this recursive sequence, we can prove by induction.
Let $n = 1$. Then,

$$\sqrt{2} \leq 2 \tag{1}$$

holds true.
Next, we must now prove that if the inequality is true for some $n \in \mathbb{N}$, the above inequality also holds for $n + 1$.
Let $n = k$. Then, assume the following equation to be true:

$$a_k = \sqrt{2 * a_{k-1}} \leq 2. \tag{2}$$

We now try to prove that the inequality holds for $n = k + 1$:

$$a_{k+1} = \sqrt{2 * a_k} \leq 2 \tag{3}$$

Since $a_k \leq 2$, it must be the case that $\sqrt{a_k} \leq \sqrt{2}$. Therefore, since

$$a_{k+1} = \sqrt{2} * \sqrt{a_k} \tag{4}$$

it must be the case that

$$a_{k+1} \leq 2. \tag{5}$$

Therefore, by the Principle of Mathematical Induction, it has been proven that $\sqrt{2\sqrt{2\sqrt{2\sqrt{2\ldots}}}} \leq 2$. $\square$

4. Let $a_1 = 2, a_2 = 4$, and $a_{n+2} = 5a_{n+1} - 6a_n$ for $n \geq 1$. Prove that $a_n = 2^n$ for all natural numbers n.

Proof. Let $n = 1$. Then:

$$a_1 = 2 = 2^1.$$

Next, we must now prove that if the inequality is true for some $n \in \mathbb{N}$, the above inequality also holds for $n + 1$. Let $n = k$. Then, assume the following equation to be true:

$$a_{k+2} = 5a_{k+1} - 6a_k = 2^k. \tag{1}$$

We now try to prove that the inequality holds for $n = k + 1$:

$$a_{k+3} = 5a_{k+2} - 6a_{k+1} = 2^{k+1}. \tag{2}$$

We substitute Equation (1) into Equation (2):

$$a_{k+3} = 5(2^k) - 6a_{k+1} = 2^{k+1}. \tag{3}$$

Then, we find the equation for a_{k+1} so that we can substitute that as well:

$$a_{k+1} = 5a_k - 6a_{k-1} = 2^{k-1}, \tag{4}$$

which we then substitute into Equation (3):

$$a_{k+3} = 5(2^k) - 6(2^{k-1}) = 2^{k+1}. \tag{5}$$

Expanding this equaiton yeilds:

$$2^{k-1}(5 * 2 - 6) = 2^{k-1}(10 - 6) = 2^{k-1} * 4 = 2^{k+1}. \tag{6}$$

Therefore, by the Principle of Mathematical Induction, $a_n = 2^n$ holds for all natural numbers n. $\square$

5. Use the Principle of Mathematical Induction to prove that

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + ... + n \cdot n! = (n+1)! - 1.$$

Proof. $\square$