Homework 1
Connor Baker, January 2017

1. Prove by contradiction that if $a - b$ is odd, then $a + b$ is odd.

Assumptions Assume that $a, b \in \mathbb{Z}$, $a - b$ is even, and $a + b$ is odd.

Lemma Let $x, y, z \in \mathbb{Z}$, $x = 2z$, $y = 2z + 1$. For all z, x is even (any number with a factor of two is even), and y is odd (any even number plus one is odd).

Proof If $a - b$ is even, then by the previous lemma, $a - b = 2z$ for some unknown number z. If $a + b$ is odd, then by the previous lemma, $a + b = 2z + 1$ for some unknown number z. Substituting $2z + b$ for a (from the first equation) into $a + b = 2z + 1$ yields $2z + b + b = 2z + 1$. Simplifying yields $b = 1/2$, which means that b is *not* in the set of integers. Therefore, the original assumption is a contradiction. As a result, if $a - b$ is odd, then $a + b$ must odd.

2. Write a proof by contrapositive to show that if xy is odd, then both x and y are odd.

Assumptions Assume that $x, y \in \mathbb{Z}$, either x or y is even, and xy is even.

Lemma Let $a, b, c \in \mathbb{Z}$, $a = 2c$, $b = 2c + 1$. For all c, a is even (any number with a factor of two is even), and b is odd (any even number plus one is odd).

Proof If x is even, and y is odd, then by the previous lemma, $xy = (2c)(2c+1)$, which is always even for some number c. If x is odd, and y is even, then by the previous lemma, $xy = (2c + 1)(2c)$, which is also always even for some number c.

Since either x or y is even, and xy is even, we can infer by the contrapositive that if xy is odd, then both x and y are odd.

3. Prove that there do not exist integers m and n such that $12m + 15n = 1$.

Proof The equation $12m + 15n = 1$ is equivalent to $3(4m + 5n) = 1$, which is equivalent to $4m + 5n = 1/3$.

Any integer multiplied by an integer is an integer. As such, four (an integer) multiplied by any integer m is an integer, and five (an integer) multiplied by any integer n is also an integer.

Let $a, b \in \mathbb{Z}$, where $a = 4m$ and $b = 5n$. The equation can now be expressed as $a + b = 1/3$. Additionaly, the sum of any two integers is an integer.

Let $c \in \mathbb{Z}$, where $c = a + b$. The equation can now be expressed as $c = 1/3$.

However, $1/3$ is *not* in the set of integers. Therefore, by direct proof, there do not exist integers m and n such that $12m + 15n = 1$.

4. Prove there is a natural number M such that for every natural number n, $\frac{1}{n} < M$.

Proof **Basis Step:** Let $n = 1$. Then $1/n = 1 < M$. If $n = 1$, then the first natural number that M can be to make the statement true is $M = 2$.

Induction: Let $k = n + 1$, where $k \in \mathbb{N}$. Then $1/n > 1/(n + 1) = 1/k$. Therefore, by induction, every natural number n greater than one will yield a number less than one. As such, the first natural number M such that for every natural number n, $\frac{1}{n} < M$, is two.

5. Prove that if $-2 < x < 1$ or $x > 3$, then $\frac{(x-1)(x+2)}{(x-3)(x+4)} > 0$.

Proof The function has vertical asymptotes at $x = -4$ and $x = 3$. The only two intercepts are at $x = -2$ and $x = 1$. Since both have multiplicity one, the function crosses the axis at both points. The limit as x approaches -2 from the right is positive, as is the limit as x approaches 1 from the left. Then, by the Intermediate Value Theorem, the function on $(-2, 1)$ is positive.

Since there are no other roots, if the limit approaching from the right of $x = 3$ is positive (which it is), the function is positive on $(3, \infty)$.

Therefore, if $-2 < x < 1$ or $x > 3$, then $\frac{(x-1)(x+2)}{(x-3)(x+4)} > 0$.