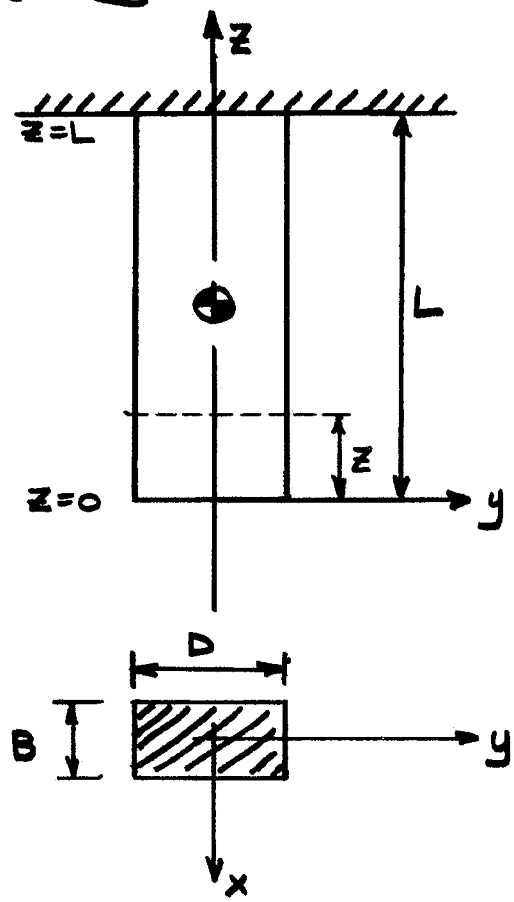
Simple Extension of a Prismatic Bar under Self-Weight

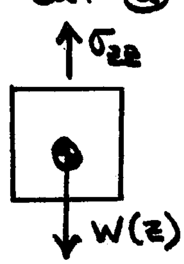
- Engineering (Strength of Materials) Solution



Assuming a 1-D stress field ($\sigma_{xx}=\sigma_{yy}=\sigma_{xy}=\sigma_{yz}=\sigma_{xz}=0$)

$$\sigma_{zz} = E\,\epsilon_{zz}$$

Taking a cut @ z and drawing the f.b.d.



$$\Sigma F_z = 0$$

$$\sigma_{zz} D B = \gamma D B z$$

$$\boxed{\sigma_{zz}(z) = \gamma z}$$

and the strain is,

$$\epsilon_{zz} = \frac{\partial u_z}{\partial z} = \frac{1}{E}\sigma_{zz} = \frac{1}{E}\gamma z$$

integrating,

$$u_z = \int \frac{1}{E}\gamma z\, dz = \frac{\gamma}{E}\frac{z^2}{2} + C$$

and now we evaluate the constant C from the displacement boundary condition,

$$u_z(L) = 0$$

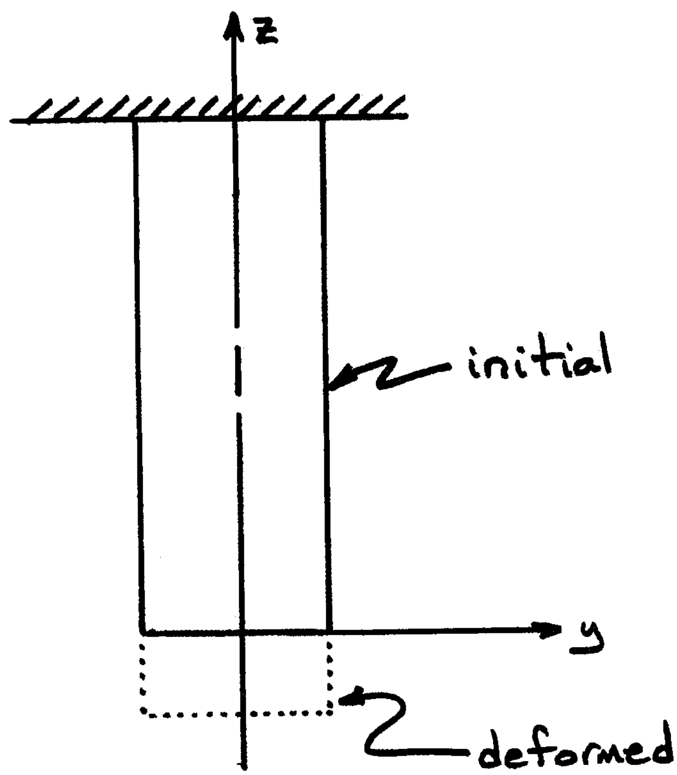
$$\text{or} \quad 0 = \frac{\gamma}{E}\frac{L^2}{2} + C \longrightarrow \boxed{C = \frac{-\gamma L^2}{2E}}$$

So that,

$$u_z(z) = \frac{\gamma}{2E}\left(z^2 - L^2\right)$$

$$u_z^{max} = u_z(0) = -\frac{\gamma L^2}{2E}$$

and the whole cross-section is assumed to move uniformly.



② Elasticity Solution

First we write the equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + f_x = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + f_y = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z = 0$$

Also, at $z = L$

$$\int_{-B/2}^{B/2} \int_{-D/2}^{D/2} \sigma_{zz}(x,y,L)\, dx\, dy = \gamma B D L$$

which says that the traction integral ~~at~~ over the area at the boundary is equal to the weight of the bar.

We know the body forces are,

$$f_x = f_y = 0$$
$$f_z = -\gamma$$

Method of solution $\rightarrow$ Semi-inverse Method with

$$\sigma_{zz} = \gamma z$$
$$\sigma_{xx} = \sigma_{yy} = \sigma_{xz} = \sigma_{yz} = \sigma_{xz} = 0$$

which satisfies equilibrium.

The strains are,

$$\begin{cases} \epsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] = \frac{-\nu}{E}\gamma z \\ \epsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right] = \frac{-\nu}{E}\gamma z \\ \epsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right] = \frac{1}{E}\gamma z \\ \epsilon_{xy} = \epsilon_{yz} = \epsilon_{xz} = 0 \end{cases}$$

Since the strains are linear, the compatibility relations are identically satisfied. Thus, we should be able to obtain the displacement field via integration.

$$\begin{cases} \epsilon_{xx} = \dfrac{\partial u_x}{\partial x} = \dfrac{-\nu}{E}\gamma z & \ldots\ldots\ldots\ldots(1) \\ \epsilon_{yy} = \dfrac{\partial u_y}{\partial y} = \dfrac{-\nu}{E}\gamma z & \ldots\ldots\ldots\ldots(2) \\ \epsilon_{zz} = \dfrac{\partial u_z}{\partial z} = \dfrac{1}{E}\gamma z & \ldots\ldots\ldots\ldots(3) \\ \epsilon_{xy} = \dfrac{1}{2}\left(\dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x}\right) = 0 & \ldots\ldots\ldots\ldots(4) \\ \epsilon_{yz} = \dfrac{1}{2}\left(\dfrac{\partial u_y}{\partial z} + \dfrac{\partial u_z}{\partial y}\right) = 0 & \ldots\ldots\ldots\ldots(5) \\ \epsilon_{xz} = \dfrac{1}{2}\left(\dfrac{\partial u_x}{\partial z} + \dfrac{\partial u_z}{\partial x}\right) = 0 & \ldots\ldots\ldots\ldots(6) \end{cases}$$

integrating (3) we get,

$$u_z = \frac{1}{E}\gamma\frac{z^2}{2} + f(x,y) \quad \ldots\ldots\ldots\ldots(7)$$

differentiating (7) and substituting into (5) and (6),

$$\frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial f(x,y)}{\partial y}\right) = 0 \longrightarrow \frac{\partial u_y}{\partial z} = \frac{-\partial f(x,y)}{\partial y} \quad \ldots\ldots(8)$$

$$\frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial f(x,y)}{\partial x}\right) = 0 \longrightarrow \frac{\partial u_x}{\partial z} = \frac{-\partial f(x,y)}{\partial x} \quad \ldots\ldots(9)$$

integrating (8) and (9) we get,

$$u_y = -z\frac{\partial f(x,y)}{\partial y} + g(x,y) \quad \ldots\ldots\ldots\ldots(10)$$

$$u_x = -z\frac{\partial f(x,y)}{\partial x} + h(x,y) \quad \ldots\ldots\ldots\ldots(11)$$

now substitute (10) and (11) into (1) and (2),

$$-z\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial h(x,y)}{\partial x} = \frac{-\nu}{E}\gamma z \quad \ldots\ldots\ldots\ldots (12)$$

$$-z\frac{\partial^2 f(x,y)}{\partial y^2} + \frac{\partial g(x,y)}{\partial y} = \frac{-\nu}{E}\gamma z \quad \ldots\ldots\ldots\ldots (13)$$

from (12) and (13) the following constraints can

be established,

$$\frac{\partial h(x,y)}{\partial x} = 0 \quad (h \text{ is not a fn. of } x) \ldots\ldots\ldots\ldots (14)$$

$$\frac{\partial^2 f(x,y)}{\partial x^2} = \frac{\nu}{E}\gamma \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

$$\frac{\partial g(x,y)}{\partial y} = 0 \quad (g \text{ is not a fn. of } y) \ldots\ldots\ldots\ldots (16)$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{\nu}{E}\gamma \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

also, substituting (10) and (11) into (4) yields,

$$-2z\frac{\partial^2 f(x,y)}{\partial x \partial y} + \frac{\partial h(x,y)}{\partial y} + \frac{\partial g(x,y)}{\partial x} = 0 \ldots\ldots (18)$$

which separates into,

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19)$$

and $$\frac{\partial h(x,y)}{\partial y} + \frac{\partial g(x,y)}{\partial x} = 0 \ldots\ldots\ldots\ldots (20)$$

equations (14), (16) and (20) require that $g(x,y)$ and $h(x,y)$ be of the form,

$$g(x,y) = c_1 r(x) + c_2 \quad \ldots\ldots\ldots\ldots (21)$$

$$h(x,y) = c_3 t(y) + c_4 \quad \ldots\ldots\ldots\ldots (22)$$

and
$$c_3 \frac{\partial t(y)}{\partial y} + c_1 \frac{\partial r(x)}{\partial x} = 0 \quad \ldots\ldots\ldots\ldots (23)$$

where $r(x)$ and $t(y)$ are arbitrary functions.

Therefore, choose the following

$$\begin{cases} t(y) = y & \ldots\ldots\ldots\ldots (24) \\ r(x) = x & \ldots\ldots\ldots\ldots (25) \end{cases}$$

So that from (23) we find that,

$$\boxed{c_3 = -c_1}$$

Thus,

$$g(x,y) = c_1 x + c_2 \quad \ldots\ldots\ldots\ldots (26)$$

$$h(x,y) = -c_1 y + c_4 \quad \ldots\ldots\ldots\ldots (27)$$

now substitute (26) and (27) into (12) and (13), integrate, and add to obtain,

$$f(x,y) = \frac{\nu\gamma}{2E}(x^2+y^2) + C_5 x + C_6 y + C_7 \quad \ldots\ldots (28)$$

Therefore, from (7) we have that,

$$u_z = \frac{\gamma}{2E}\left[z^2 + \nu(x^2+y^2)\right] + C_5 x + C_6 y + C_7 \quad \ldots\ldots (29)$$

also, from (10) and (11) we have that,

$$u_x = \frac{-\nu\gamma xz}{E} - C_1 y + C_4 - C_5 z \quad \ldots\ldots\ldots\ldots (30)$$

$$u_y = \frac{-\nu\gamma yz}{E} + C_1 x + C_2 - C_6 z \quad \ldots\ldots\ldots\ldots (31)$$

We now have 3 displacements in terms of 6 arbitrary constants ($C_1, C_2, C_4 \rightarrow C_7$).

These we obtain by imposing constraints on the displacements and average rotations at the wall,

$$\begin{cases} u_x = u_y = u_z = 0 & \text{at} \quad z = L,\ x = y = 0 \\ \omega_{xy} = \omega_{xz} = \omega_{yz} = 0 & \text{at} \quad \text{"} \qquad \text{"} \end{cases}$$

Now apply boundary conditions,

$$u_x(0,0,L) = C_4 - C_5 L = 0$$

$$u_y(0,0,L) = C_2 - C_6 L = 0$$

$$u_z(0,0,L) = \frac{\gamma L^2}{2E} + C_7 = 0$$

$$\omega_{xy}(0,0,L) = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right)$$

$$= \frac{1}{2}\left[(0 - C_1) - (0 + C_1)\right]\Big/_{x=0}$$

$$= \frac{1}{2}\left[-C_1 - C_1\right] = 0$$

$$\longrightarrow \boxed{C_1 = 0}$$

$$\omega_{xz}(0,0,L) = \frac{1}{2}\left[-C_5 - C_5\right] = 0 \longrightarrow \boxed{C_5 = 0}$$

$$\omega_{yz}(0,0,L) = \frac{1}{2}\left[-C_6 - C_6\right] = 0 \longrightarrow \boxed{C_6 = 0}$$

then we see that,

$$\boxed{\begin{aligned} C_2 &= C_4 = 0 \\ C_7 &= \frac{-\gamma L^2}{2E} \end{aligned}}$$

The final displacements are,

$$\begin{cases} u_x = \dfrac{-\nu}{E}\gamma x z \\ u_y = \dfrac{-\nu}{E}\gamma y z \\ u_z = \dfrac{\gamma}{2E}\left[z^2 + \nu(x^2+y^2) - L^2\right] \end{cases}$$

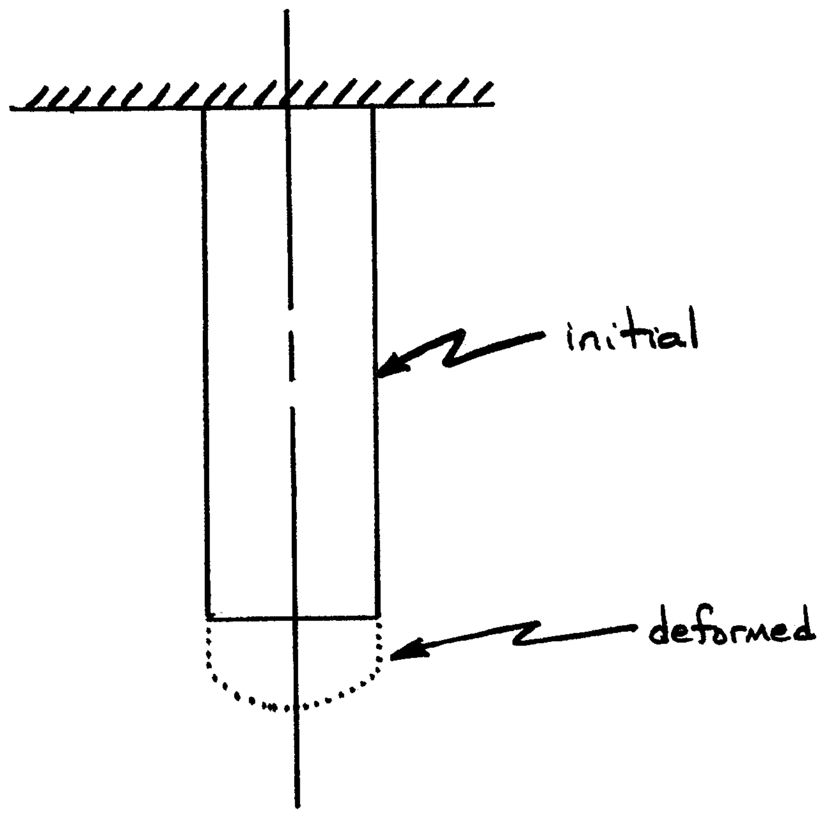
Note:

(1) all points not on the ℄ have contractions in the x-y plane — Poisson effect.

(2) the axial displacement is not uniform on the cross-section, but is a parabolic surface with a maximum along the z-axis.

(3) the engineering (SOM) solution matches the elasticity solution along the z-axis: $u_z(0,0,z)$

Clearly, we see the nearly universal use (and appeal) of the engineering or "strength of materials" solution to the problem.



Strength of Materials	Elasticity
$u_z = \frac{\gamma}{2E}\left(z^2 - L^2\right)$	$u_z = \frac{\gamma}{2E}\left[z^2 - L^2 + \nu(x^2+y^2)\right]$
$u_x = 0$	$u_x = \frac{-\nu}{E}\gamma x z$
$u_y = 0$	$u_y = \frac{-\nu}{E}\gamma y z$

$$u_z\,(\text{S.O.M.}) = u_z(0,0,z)\Big|_{\text{elasticity}}$$