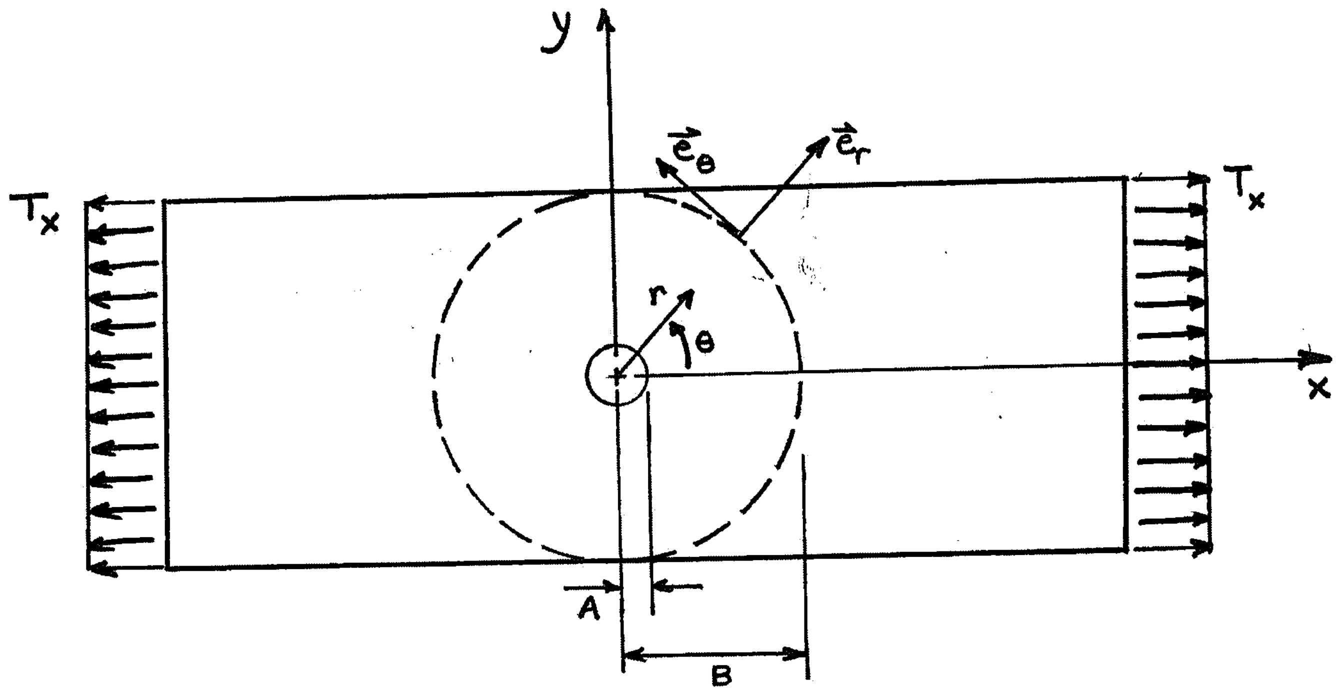
2-D Elasticity Example

SHEET WITH A SMALL CIRCULAR HOLE



Consider a <u>thin</u> sheet under uniaxial uniform tension with a small hole of radius A in the interior.

Without the hole the stresses would be,

$$\begin{cases} \sigma_{xx} = T_x \\ \sigma_{yy} = \sigma_{xy} = 0 \end{cases}$$

By St. Venant's Principle we would expect modification of the uniform stress field only near the hole, the stress field remaining uniform everywhere else.

The problem is simplified if we construct a hypothetical circle of radius B equal to half the sheet width. Then we will determine the state of stress in polar coordinates at $r = B$.

We first assume that B is large enough so that the stress state is uniform. From boundary conditions we know that,

$$[\sigma] = \begin{bmatrix} T_x & 0 \\ 0 & 0 \end{bmatrix} \qquad (1)$$

From the figure we know that,

$$\alpha_{11} = \cos\theta$$
$$\alpha_{12} = \sin\theta$$
$$\alpha_{21} = -\sin\theta$$
$$\alpha_{22} = \cos\theta$$

so that,

$$[R] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad (2)$$

Then, the transformed stress state is,

$$[\sigma'] = [R][\sigma][R]^T = \begin{bmatrix} T_x\cos^2\theta & -T_x\sin\theta\cos\theta \\ -T_x\sin\theta\cos\theta & T_x\sin^2\theta \end{bmatrix}$$

$$= \left.\begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} \\ \sigma_{r\theta} & \sigma_{\theta\theta} \end{bmatrix}\right|_{r=B} \qquad (3)$$

We can construct a solution to this problem by superimposing solutions of two separate problems of tractions applied to thin disks of radius B.

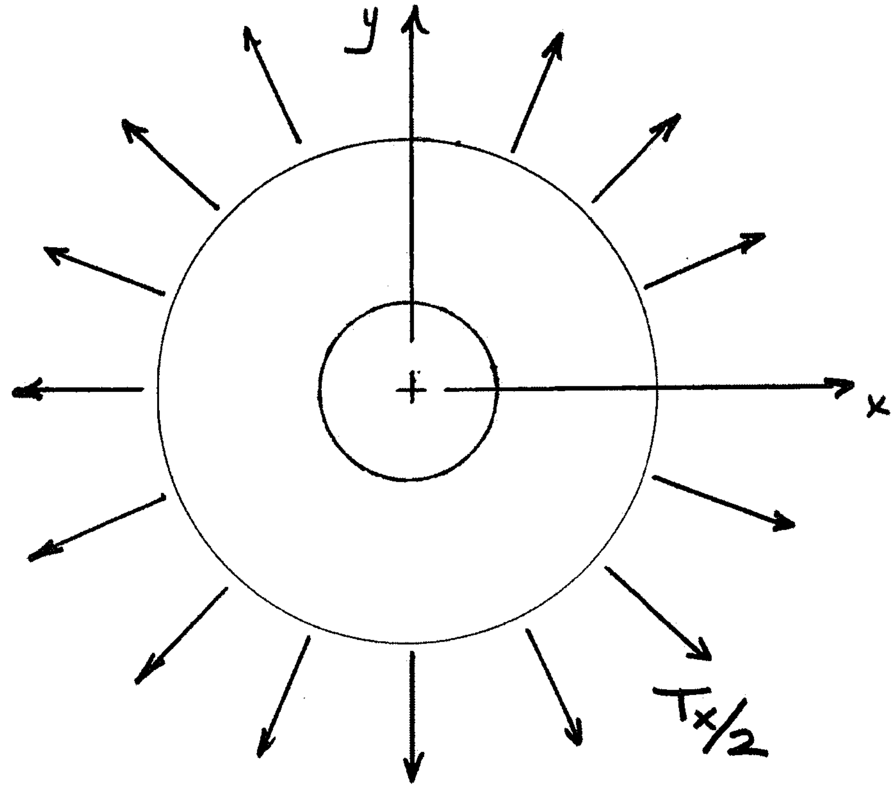
First, we note that

$$\begin{cases} \sigma_{rr} = T_x \cos^2\theta = \dfrac{T_x}{2}\left(1 + \cos 2\theta\right) & (4) \\ \sigma_{r\theta} = -T_x \sin\theta\cos\theta = -\dfrac{T_x}{2}\sin 2\theta & (5) \end{cases}$$

Therefore,

Original Problem	=	Problem #1	+	Problem #2
		(axisymmetric)		(non-axisymmetric)
		$\sigma_{rr}(r=B) = \dfrac{T_x}{2}$		$\sigma_{rr}(r=B) = \dfrac{T_x}{2}\cos 2\theta$
		$\sigma_{r\theta}(r=B) = 0$		$\sigma_{r\theta}(r=B) = -\dfrac{T_x}{2}\sin 2\theta$
		* uniform pressure		

Problem #1 has already been solved previously.



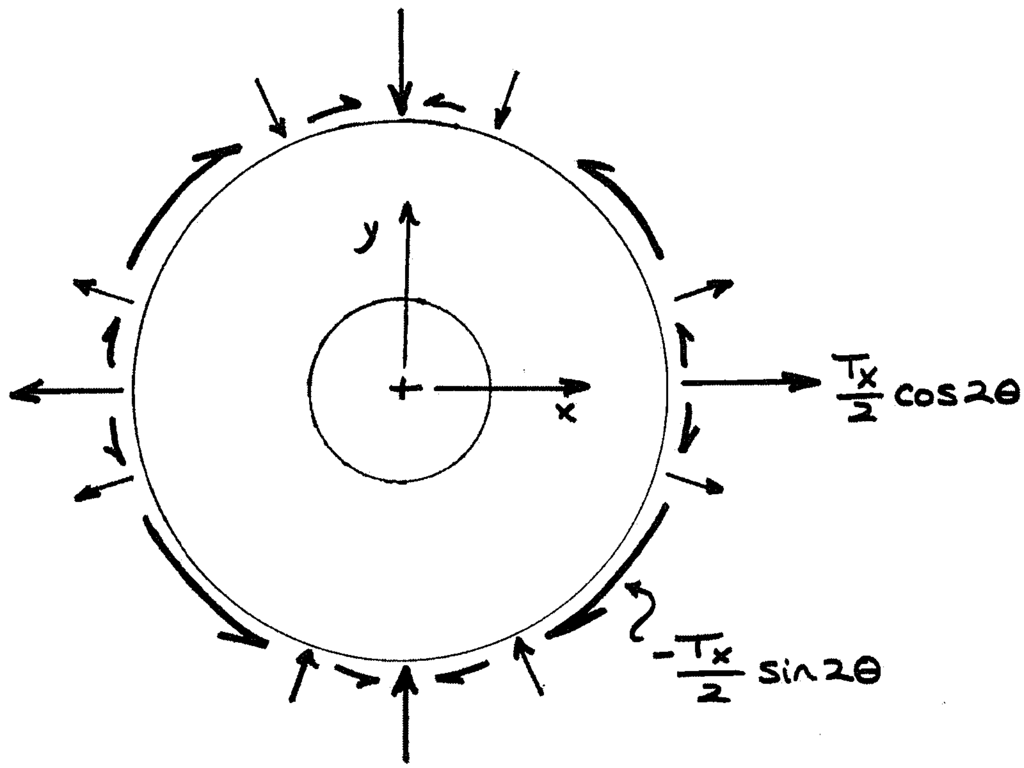
$p_i = 0 \quad , \quad p_o = -\frac{T_x}{2}$

$R_i = A \quad , \quad R_o = B$

Thus, the solution for the axisymmetric case is,

$$\begin{cases} \sigma_{rr}^{(1)}(r,\theta) = \frac{T_x}{2}\left(\frac{B^2}{B^2-A^2}\right) - \frac{T_x}{2}\left(\frac{A^2B^2}{B^2-A^2}\right)\frac{1}{r^2} & (6) \\ \sigma_{\theta\theta}^{(1)}(r,\theta) = \frac{T_x}{2}\left(\frac{B^2}{B^2-A^2}\right) + \frac{T_x}{2}\left(\frac{A^2B^2}{B^2-A^2}\right)\frac{1}{r^2} & (7) \end{cases}$$

This leaves the second (non-axisymmetric) case to solve.



$$\nabla^4\phi = \left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]\left[\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right]=0 \tag{8}$$

Recognizing that the boundary conditions are periodic, we assume a stress function of the form,

$$\phi(r,\theta) = F(r)\cos 2\theta \qquad (9)$$

Substituting into the biharmonic we can reduce the P.D.E. in r & θ to an O.D.E. in r,

$$\frac{d^4F}{dr^4} + \frac{2}{r}\frac{d^3F}{dr^3} - \frac{9}{r^2}\frac{d^2F}{dr^2} + \frac{9}{r^3}\frac{dF}{dr} = 0 \qquad (10)$$

To solve this equation we let,

$$F = r^\lambda \qquad (11)$$

Then, we substitute eq. (11) into (10) and perform the differentiations to get,

$$\lambda(\lambda-1)(\lambda-2)(\lambda-3)r^{\lambda-4} + \frac{2}{r}\lambda(\lambda-1)(\lambda-2)r^{\lambda-3} - \frac{9}{r^2}\lambda(\lambda-1)r^{\lambda-2}$$

$$+ \frac{9}{r^3}\lambda r^{\lambda-1} = 0 \qquad (12)$$

Dividing equation (12) by $\lambda r^{\lambda-4}$ leaves,

$$(\lambda-1)(\lambda-2)(\lambda-3) + 2(\lambda-1)(\lambda-2) - 9(\lambda+1) + 9 = 0 \tag{13}$$

and we've already identified one root as $\lambda = 0$.

Expanding equation (13),

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 + 2\lambda^2 - 6\lambda + 4 - 9\lambda + 9 + 9 = 0 \tag{14}$$

Factoring,

$$(\lambda^2-4)(\lambda-4) = 0$$
$$(\lambda-2)(\lambda+2)(\lambda-4) = 0 \tag{15}$$

Therefore, the roots are,

$$\begin{cases} \lambda = 2 \\ \lambda = -2 \\ \lambda = 4 \\ \lambda = 0 \end{cases}$$

Thus, the stress function is,

$$\phi(r,\theta) = \underbrace{\left(c_1 + c_2 r^2 + c_3 r^4 + c_4 \frac{1}{r^2}\right)}_{F(r)} \cos 2\theta \qquad (16)$$

According to definition, the stresses are then,

$$\left\{\begin{aligned} \sigma_{rr}^{(2)}(r,\theta) &= \frac{1}{r}\phi_{,r} + \frac{1}{r^2}\phi_{,\theta\theta} \\ &= -\left(4c_1\frac{1}{r^2} + 2C_2 + 6C_4\frac{1}{r^4}\right)\cos 2\theta \qquad (17) \\ \sigma_{\theta\theta}^{(2)}(r,\theta) &= \phi_{,rr} \\ &= \left(2C_2 + 12C_3 r^2 + 6C_4\frac{1}{r^4}\right)\cos 2\theta \qquad (18) \\ \sigma_{r\theta}^{(2)}(r,\theta) &= \frac{1}{r^2}\phi_{,\theta} - \frac{1}{r}\phi_{,r\theta} \end{aligned}\right.$$

$$= \left(-2c_1\frac{1}{r^2} + 2c_2 + 6C_3 r^2 - 6C_4\frac{1}{r^4}\right)\sin 2\theta \qquad (19)$$

The four unknown constants are found from the boundary conditions:

$$
\begin{cases}
\sigma_{rr}^{(2)}(B,\theta) = \dfrac{T_x}{2}\cos 2\theta \\[2mm]
\sigma_{r\theta}^{(2)}(B,\theta) = \dfrac{-T_x}{2}\sin 2\theta \\[2mm]
\left.\begin{array}{l}\sigma_{rr}^{(2)}(A,\theta) = 0 \\[2mm] \sigma_{r\theta}^{(2)}(A,\theta) = 0\end{array}\right\} \text{stress-free at the hole}
\end{cases}
$$

Substituting these relations into equations (17) and (19) yields a set of 4 algebraic eqs.,

$$
\begin{bmatrix}
-4/A^2 & -2 & 0 & -6/A^4 \\
-2/A^2 & 2 & 6A^2 & -6/A^4 \\
-4/B^2 & -2 & 0 & -6/B^4 \\
-2/B^2 & 2 & 6B^2 & -6/B^4
\end{bmatrix}
\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ T_x/2 \\ -T_x/2 \end{Bmatrix}
\qquad (20)
$$

If we let $\frac{A}{B} \to 0$, that is, the hole is very small compared to the plate width then,

$$\begin{cases} C_1 = \dfrac{A^2}{2} T_x \\ C_2 = \dfrac{-T_x}{4} \\ C_3 = 0 \\ C_4 = \dfrac{-A^4}{4} T_x \end{cases} \qquad (21)$$

so that the stresses are now,

$$\begin{cases} \sigma_{rr}^{(2)}(r,\theta) = -\left(\dfrac{2A^2}{r^2} - \dfrac{1}{2} - \dfrac{3A^4}{2r^4}\right) T_x \cos 2\theta & (22) \\ \sigma_{\theta\theta}^{(2)}(r,\theta) = -\left(\dfrac{1}{2} + \dfrac{3A^4}{2r^4}\right) T_x \cos 2\theta & (23) \\ \sigma_{r\theta}^{(2)}(r,\theta) = \left(-\dfrac{A^2}{r^2} - \dfrac{1}{2} + \dfrac{3A^4}{2r^4}\right) T_x \sin 2\theta & (24) \end{cases}$$

Now we can add the two stress fields by superposition (with $\frac{A}{B} \to 0$) in order to find the solution to the original problem :

$$\left\{ \begin{aligned} \sigma_{rr}(r,\theta) &= \sigma_{rr}^{(1)} + \sigma_{rr}^{(2)} \\ &= \frac{T_x}{2}\left(1 - \frac{A^2}{r^2}\right) + \frac{T_x}{2}\left(1 - 4\frac{A^2}{r^2} + 3\frac{A^4}{r^4}\right)\cos 2\theta \qquad (25) \\ \sigma_{\theta\theta}(r,\theta) &= \frac{T_x}{2}\left(1 + \frac{A^2}{r^2}\right) - \frac{T_x}{2}\left(1 + 3\frac{A^4}{r^4}\right)\cos 2\theta \qquad (26) \\ \sigma_{r\theta}(r,\theta) &= -\frac{T_x}{2}\left(1 + 2\frac{A^2}{r^2} - 3\frac{A^4}{r^4}\right)\sin 2\theta \qquad (27) \end{aligned} \right.$$

Observations :

(1) when <u>r is large</u> ,

$$\sigma_{rr} \approx \frac{T_x}{2}(1+\cos 2\theta)$$

$$\sigma_{r\theta} \approx -\frac{T_x}{2}\sin 2\theta$$

which means that the stresses σ_{rr} & $\sigma_{r\theta}$ approach the boundary condition at $r=B$

(2) when <u>$r \to A$</u> ,

$$\begin{cases} \sigma_{rr} \approx 0 \\ \sigma_{r\theta} \approx 0 \\ \sigma_{\theta\theta} \approx T_x(1-2\cos 2\theta) \end{cases}$$

(3) $\sigma_{\theta\theta}^{max}$ occurs at $\theta = 90°, 270°$

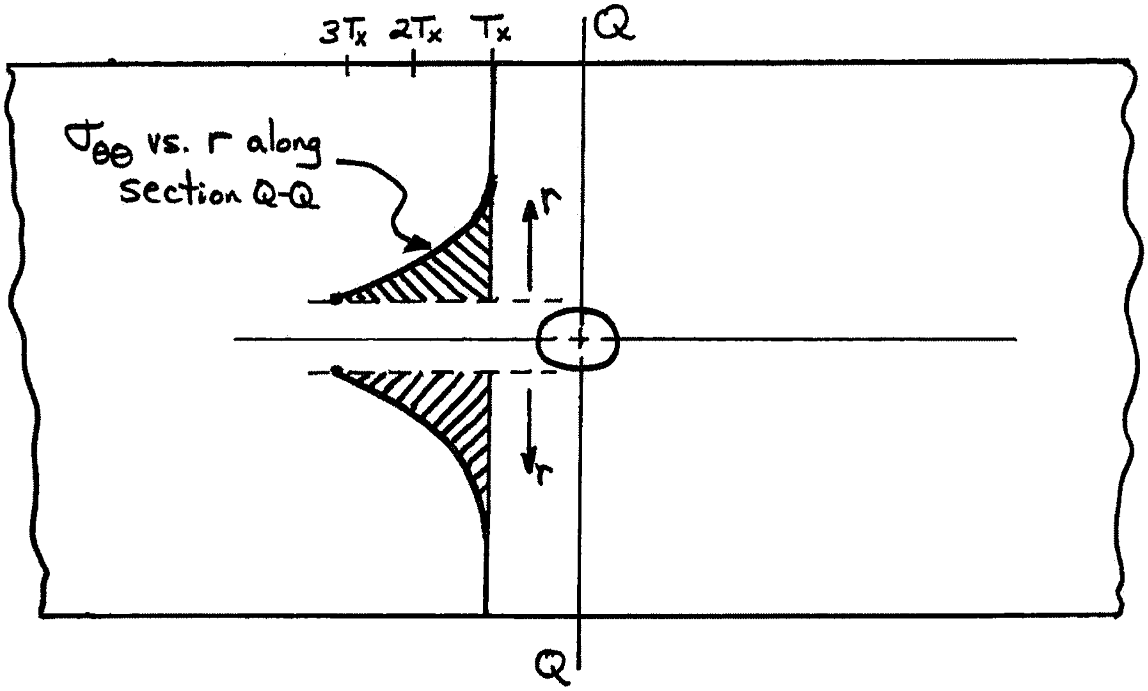
$$|\sigma_{\theta\theta}^{max}| = 3T_x$$

(4) The ratio of the maximum to "nominal" stress is,

$$\frac{|\sigma_{\theta\theta}^{max}|}{T_x} = 3 \equiv K_\sigma$$

where K_σ is called the stress concentration factor.

(5) From the $\frac{1}{r^2}$ & $\frac{1}{r^4}$ factors we deduce that the stress concentrations are localized, decaying rapidly as we move away from the hole.



3Tx
2Tx
Tx
Q
σθθ vs. r along section Q-Q
r
r
Q