Solution 2-D Elasticity Problems by Fourier Series

We have shown that many solutions can be obtained for 2-D problems by using a polynomial series,

$$\phi(x,y) = \sum_m \sum_n C_{mn} x^m y^n \qquad (1).$$

A much greater generality is possible by taking a Fourier series expansion. For example,

$$\phi(x,y) = \sin\left(\frac{m\pi x}{L}\right) f(y) \qquad (2)$$

where m is an integer and f(y) is a function of y only. Here we have used a Fourier series expansion for x only. This stress function must satisfy,

$$\nabla^4 \phi = 0 \qquad (3)$$

$$\phi_{,xxxx} + 2\phi_{,xxyy} + \phi_{,yyyy} = 0$$

Substituting the Fourier expansion in eq. (2) into equation (3) yields,

$$f'''' - 2\alpha^2 f'' + \alpha^4 f = 0 \qquad (4)$$

The general solution of equation (4) is,

$$f(y) = C_1 \cosh(\alpha y) + C_2 \sinh(\alpha y) + C_3 y \cosh(\alpha y) + C_4 y \sinh(\alpha y) \qquad (5)$$

Here we have used the substitution:

$$\alpha = \left(\frac{m\pi}{\ell}\right) \qquad (6)$$

The stress function is then,

$$\phi(x,y) = \sin\alpha x\Big(C_1\cosh(\alpha y) + C_2\sinh(\alpha y) + C_3 y\cosh(\alpha y) + C_4 y\sinh(\alpha y)\Big) \quad (7)$$

From here we can find the stresses:

$$\sigma_{xx} = \phi_{,yy} = \sin\alpha x\Big[C_1\alpha^2\cosh(\alpha y) + C_2\alpha^2\sinh(\alpha y) + C_3\alpha\big(2\sinh(\alpha y) + \alpha y\cosh(\alpha y)\big) + C_4\alpha\big(2\cosh(\alpha y) + \alpha y\sinh(\alpha y)\big)\Big] \quad (8)$$

$$\sigma_{yy} = \phi_{,xx} = -\alpha^2\sin\alpha x\Big(C_1\cosh(\alpha y) + C_2\sinh(\alpha y) + C_3 y\cosh(\alpha y) + C_4 y\sinh(\alpha y)\Big) \quad (9)$$

$$\sigma_{xy} = -\phi_{,xy} = -\alpha\cos\alpha x\Big[C_1\alpha\sinh(\alpha y) + C_2\alpha\cosh(\alpha y)$$

$$+ C_3\big(\cosh(\alpha y) + \alpha y\sinh(\alpha y)\big)$$

$$+ C_4\big(\sinh(\alpha y) + \alpha y\cosh(\alpha y)\big)\Big] \quad (10)$$

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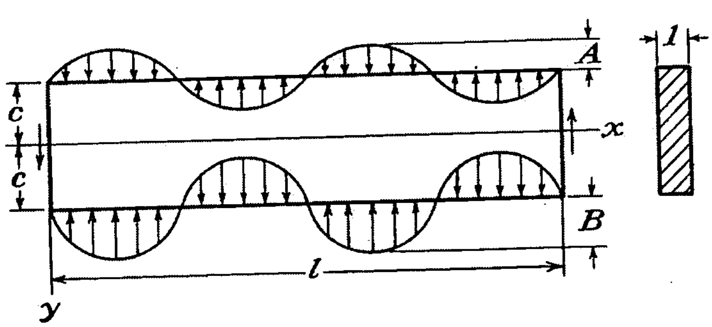
Now let's consider an example. A rectangular beam is supported at the ends and subjected along the upper and lower edges to continuously distributed vertical loadings.



where, in this case;

$$\alpha = \frac{4\pi}{l}$$

The stress distribution for this case can be obtained from eqs. (8) – (10). The constants $C_1 \rightarrow C_4$ are obtained from the boundary conditions on the upper and lower surfaces of the beam.

$$
\begin{cases}
@\ y = +c & \sigma_{xy} = 0 \\
 & \sigma_{yy} = -B \sin \alpha x \\
@\ y = -c & \sigma_{xy} = 0 \\
 & \sigma_{yy} = -A \sin \alpha x
\end{cases}
$$

From the conditions on the shear stress and equation (10) we can show that,

$$C_3 = -C_2 \frac{\alpha \cosh(\alpha c)}{\cosh(\alpha c) + \alpha c \sinh(\alpha c)} \qquad (11)$$

$$C_4 = -C_1 \frac{\alpha \sinh(\alpha c)}{\sinh(\alpha c) + \alpha c \cosh(\alpha c)} \qquad (12)$$

Then we can use the conditions on σ_{yy} and equation (9) to obtain the constants in their final form.

$$C_1 = \frac{A+B}{\alpha^2} \frac{\sinh(\alpha c) + \alpha c \cosh(\alpha c)}{\sinh(2\alpha c) + 2\alpha c} \qquad (13)$$

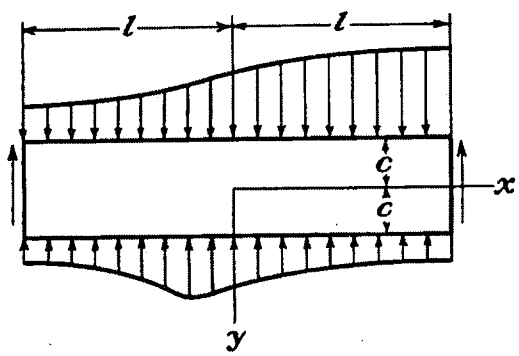
$$C_2 = \frac{-(A-B)}{\alpha^2} \frac{\cosh(\alpha c) + \alpha c \sinh(\alpha c)}{\sinh(2\alpha c) - 2\alpha c} \qquad (14)$$

$$C_3 = \frac{A-B}{\alpha^2} \frac{\alpha \cosh(\alpha c)}{\sinh(2\alpha c) - 2\alpha c} \qquad (15)$$

$$C_4 = \frac{-(A+B)}{\alpha^2} \frac{\alpha \sinh(\alpha c)}{\sinh(2\alpha c) + 2\alpha c} \qquad (16)$$

The stress can now be found * by substitution into eqs. (8)-(10).

Now let's take a more general case of loading along the upper and lower edges of a beam.



The loadings can be represented by,

$$\begin{pmatrix}\text{upper}\\ \text{surface}\end{pmatrix} \qquad q_u = A_0 + \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi x}{l}\right) + \sum_{m=1}^{\infty} A_m' \cos\left(\frac{m\pi x}{l}\right) \tag{17}$$

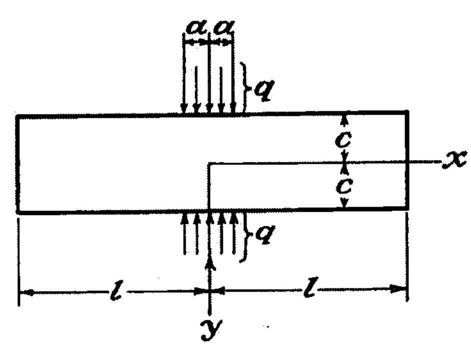
$$\begin{pmatrix}\text{lower}\\ \text{surface}\end{pmatrix} \qquad q_l = B_0 + \sum_{m=1}^{\infty} B_m \sin\left(\frac{m\pi x}{l}\right) + \sum_{m=1}^{\infty} B_m' \cos\left(\frac{m\pi x}{l}\right) \tag{18}$$

1) $A_0, B_0 \longrightarrow$ represent uniform loading of the beam.

2) Stresses produced by terms containing $\sin\left(\frac{m\pi x}{l}\right)$ are found from eqs. (8)-(10).

3) Stresses produced by terms containing $\cos\left(\frac{m\pi x}{l}\right)$ are found from eqs. (8)-(10) by exchanging $\sin(\alpha x)$ for $\cos(\alpha x)$ and vice versa, and by changing the sign of σ_{xy}.

Illustrative Example



The general Fourier series

$$f(x) = A_0 + \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi x}{\ell}\right) + \sum_{m=1}^{\infty} A_m' \cos\left(\frac{m\pi x}{\ell}\right) \quad (19)$$

has the coefficients,

$$\begin{cases} A_0 = \dfrac{1}{2\ell}\displaystyle\int_{-\ell}^{\ell} f(x)\,dx & (20) \\ A_m = \dfrac{1}{\ell}\displaystyle\int_{-\ell}^{\ell} f(x)\sin\left(\dfrac{m\pi x}{\ell}\right)dx & (21) \\ A_m' = \dfrac{1}{\ell}\displaystyle\int_{-\ell}^{\ell} f(x)\cos\left(\dfrac{m\pi x}{\ell}\right)dx & (22) \end{cases}$$

In the example considered,

$$f(x) = q \qquad -a \le x \le a \quad (23)$$

Therefore, we find that

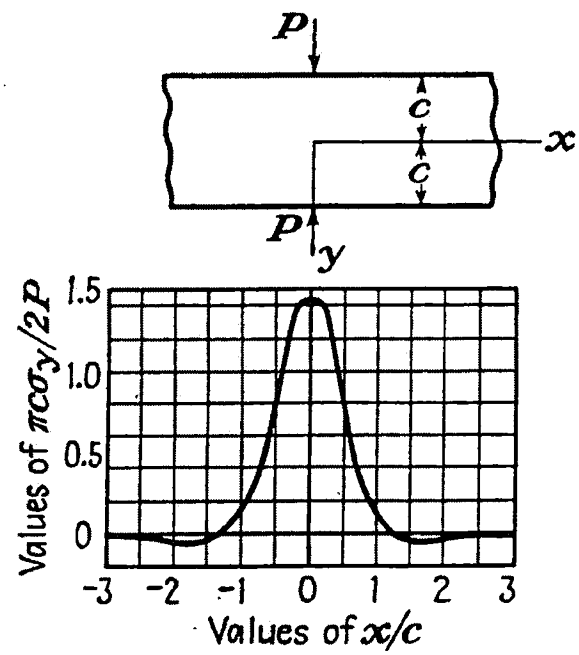
$$\begin{cases} A_0 = B_0 = \dfrac{qa}{\ell} & (24) \\ A_m' = B_m' = \dfrac{2q\sin\left(\dfrac{m\pi a}{\ell}\right)}{m\pi} & (25) \end{cases}$$

Note that since the loading is symmetric, terms with $\sin\left(\frac{m\pi x}{\ell}\right)$ vanish, i.e. $A_m = B_m = 0$

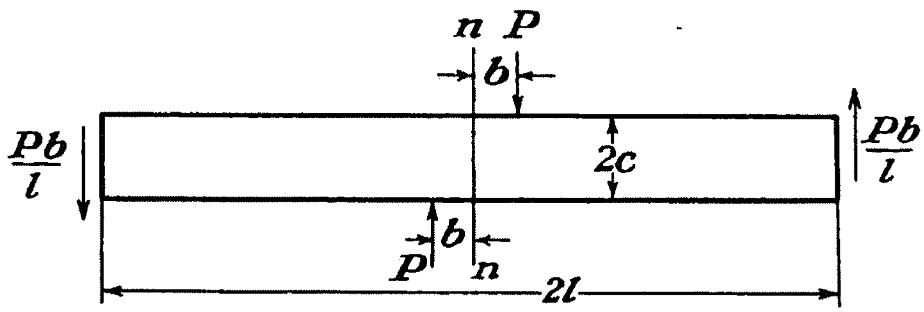
Consider the middle plane $(y=0)$ on which there is only σ_{yy} :

$$\sigma_{yy}(y=0) = \frac{-qa}{\ell} - \frac{4q}{\pi}\sum_{m=1}^{\infty}\frac{\sin\left(\frac{m\pi a}{\ell}\right)}{m} \cdot \left\{\frac{\left(\frac{m\pi c}{\ell}\right)\cosh\left(\frac{m\pi c}{\ell}\right) + \sinh\left(\frac{m\pi c}{\ell}\right)}{\sinh\left(\frac{2m\pi c}{\ell}\right) + \left(\frac{2m\pi c}{\ell}\right)}\right\}\cos\left(\frac{m\pi x}{\ell}\right) \qquad (26)$$

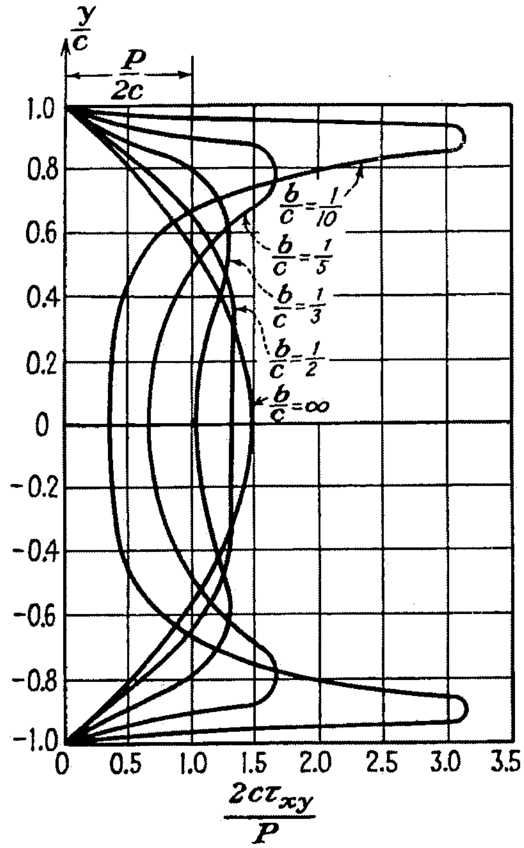
This problem was evaluated by Filon (1903) for a concentrated force $P = 2qa$.



The stress diminishes rapidly with x and at $x = 1.35c$ it is equal to zero. Filon also looked at the case where the loads are offset from eachother.



The solution procedure is the same and a matter of practical importance is the distribution of shear stress σ_{xy} across the cross-section n-n.

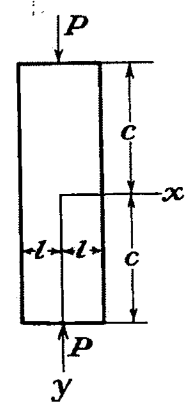


Remember that according to elementary beam theory the distribution should be parabolic. For small values of b/c the shear stress is concentrated near the surfaces and is quite high.

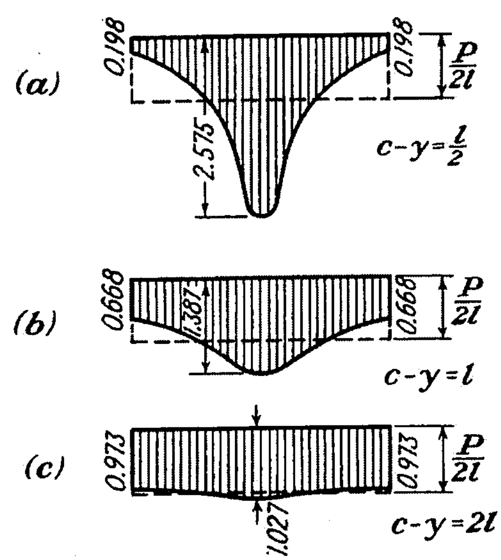
Now consider the case where the depth of the beam (2c) is large in comparison to the length (2l). We can find the transverse normal stress from equation (9) with $\cos \alpha x$ instead of $\sin \alpha x$ so that,

$$\sigma_{yy} = \frac{-q\,a}{l} - \frac{4q}{\pi} \sum_{m=1}^{\infty} \frac{\sin \alpha a}{m} \left\{ \frac{(\alpha c \cosh \alpha c + \sinh \alpha c)\cosh \alpha y}{\sinh 2\alpha c + 2\alpha c} - \frac{\alpha y \sinh \alpha y \sinh \alpha c}{\sinh 2\alpha c + 2\alpha c} \right\} \cos \alpha x \tag{27}$$

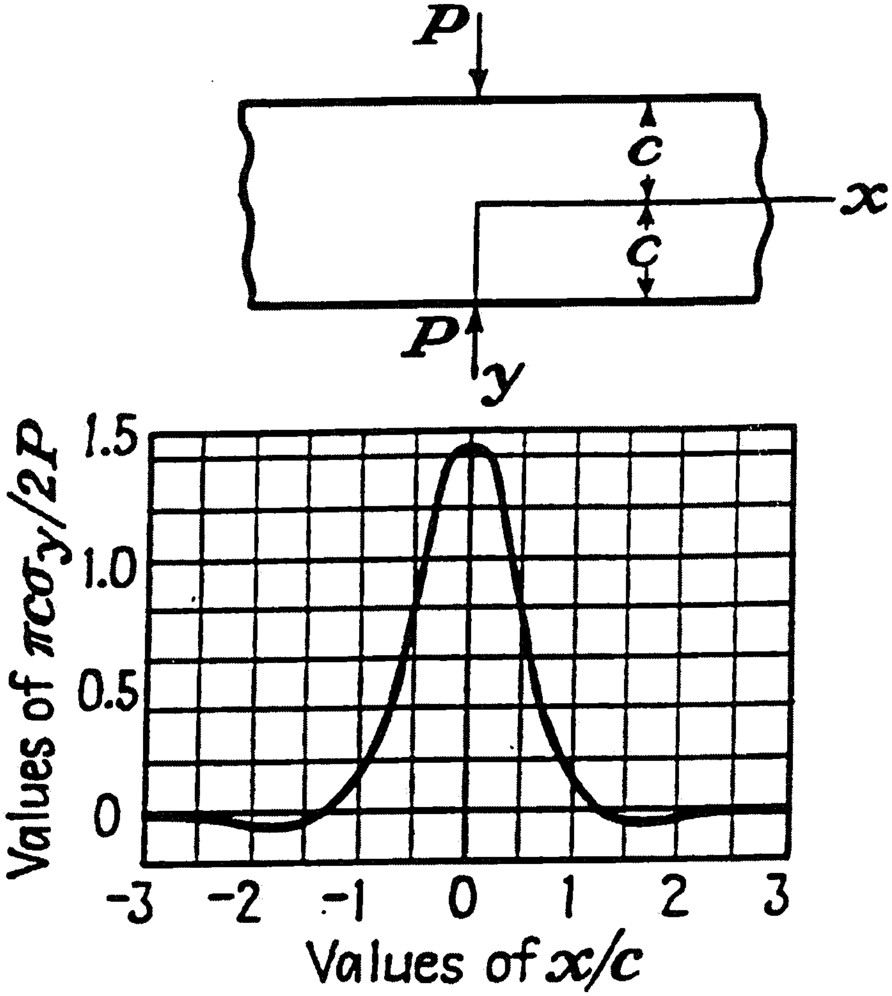
where $qa = P/2$.

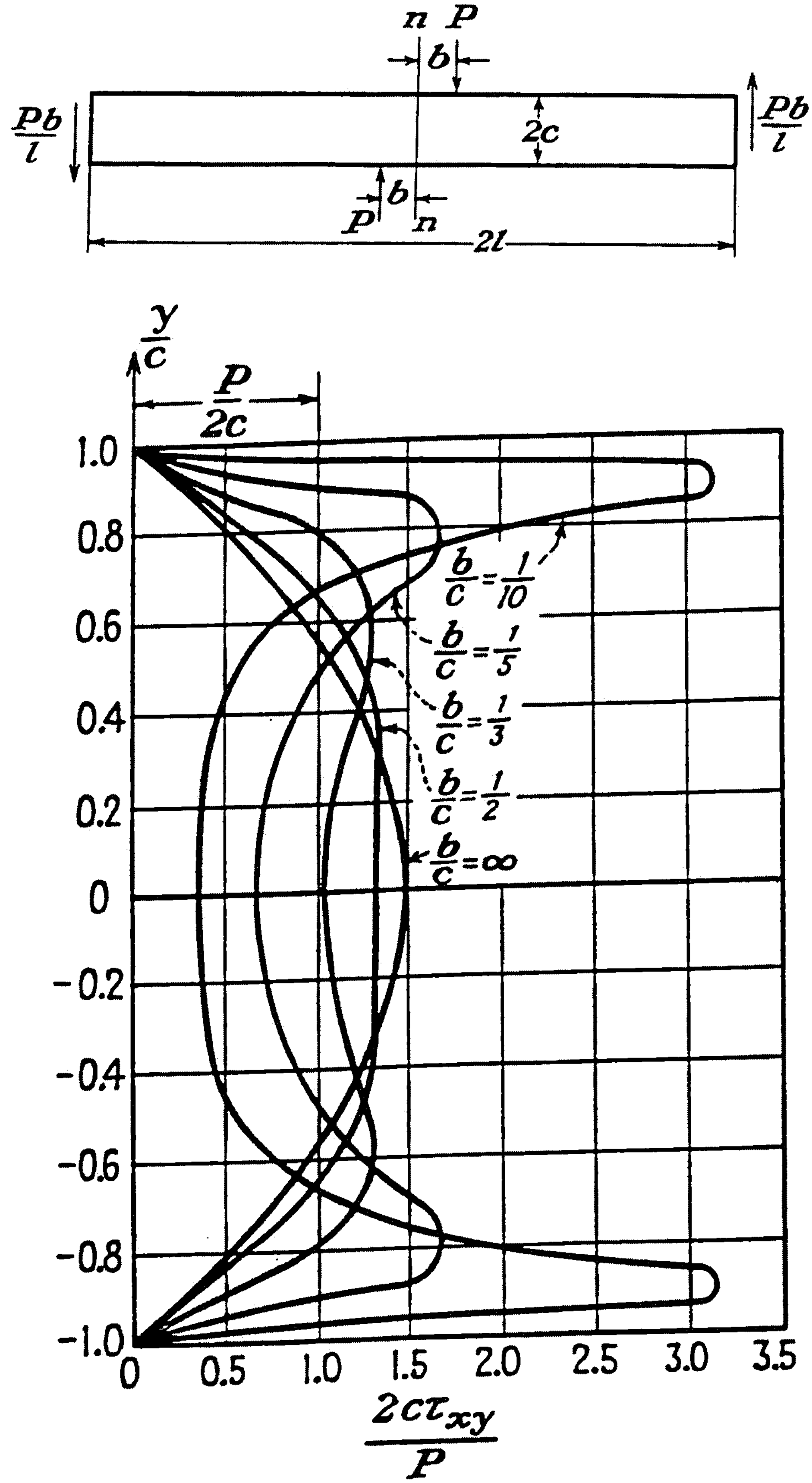


Stress distributions:

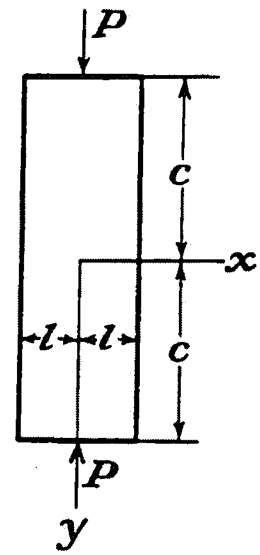


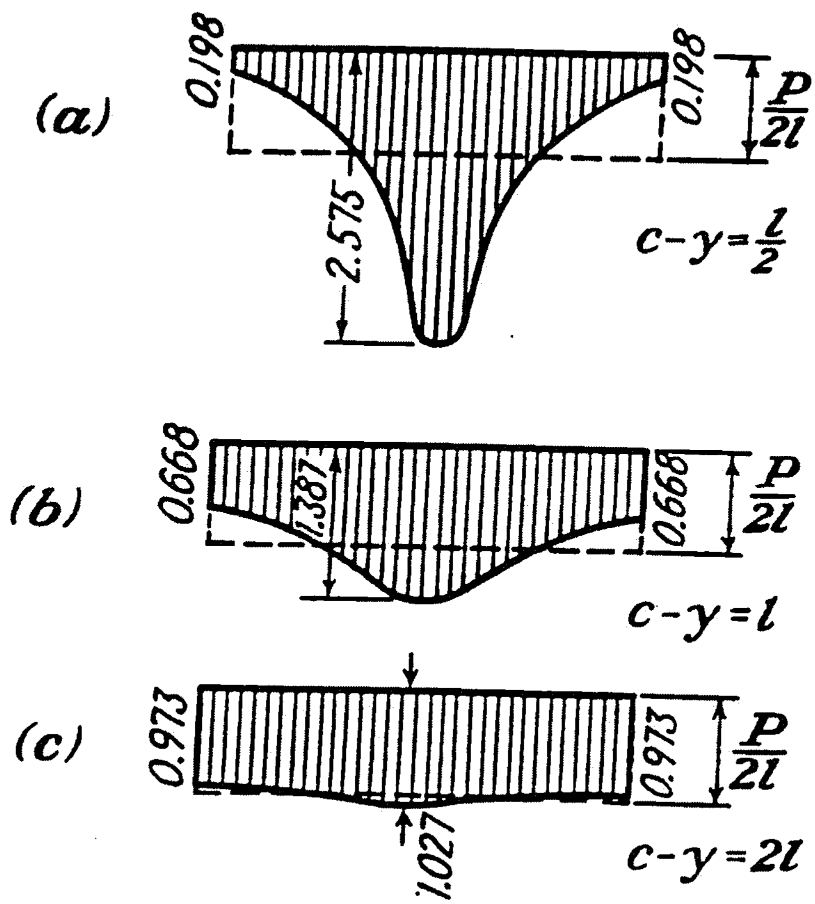
→ distance from top/bottom = length of beam


P
c
x
c
P
y
Values of $\pi c\sigma_y/2P$
1.5
1.0
0.5
0
-3
-2
-1
0
1
2
3
Values of x/c



n
P
b
Pb/l
2c
Pb/l
b
P
n
2l
y/c
P/2c
1.0
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1.0
b/c = 1/10
b/c = 1/5
b/c = 1/3
b/c = 1/2
b/c = ∞
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
2cτxy/P


P
c
x
l
l
c
P
y


(a)
0.198
0.198
P/2l
2.575
c−y=l/2
(b)
0.668
1.387
0.668
P/2l
c−y=l
(c)
0.973
0.973
P/2l
1.027
c−y=2l