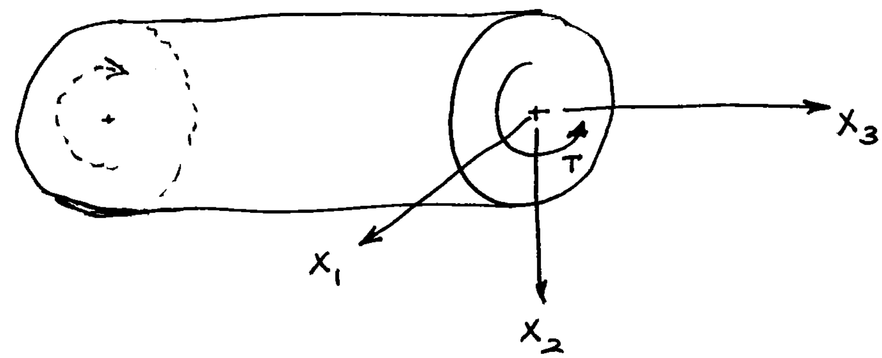
Torsion



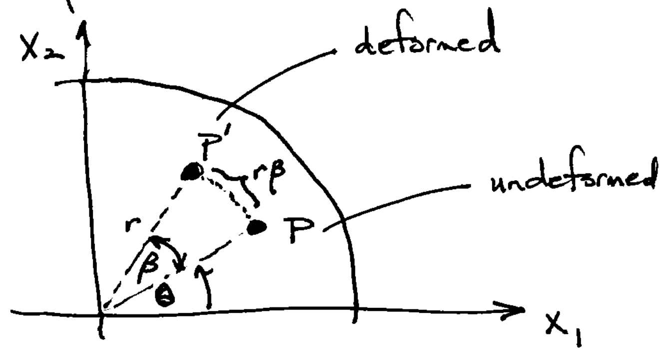
Observations:

1) each section rotates as a <u>rigid body</u> about the center axis,

2) for small deformations, rotation is a <u>linear</u> function of axial distance,

3) because of symmetry, cross-section remains planar after deformation.

Assume a displacement field:



$$\therefore \quad u_1 = -r\beta \sin\theta = -\beta x_2$$

$$u_2 = r\beta\cos\theta = \beta x_1$$

$$u_3 = 0 \qquad \text{(no warping)}$$

but we also realize that rotations are linear function of axial coordinate,

$$\underbrace{\beta}_{\text{angle of twist per unit length}} = \alpha x_3$$

$$\therefore \quad \boxed{\begin{aligned} u_1 &= -\alpha x_2 x_3 \\ u_2 &= \alpha x_1 x_3 \\ u_3 &= 0 \end{aligned}}$$

$\Rightarrow$ candidate displacement field

Strains:

$$\epsilon_{ij} = \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

$$\epsilon_{11} = u_{1,1} = 0$$

$$\epsilon_{22} = u_{2,2} = 0$$

$$\epsilon_{33} = u_{3,3} = 0$$

$$\epsilon_{12} = \frac{1}{2}\left(u_{1,2} + u_{2,1}\right) = \frac{1}{2}\left(-\alpha x_3 + \alpha x_3\right) = 0$$

$$\epsilon_{13} = \frac{1}{2}\left(u_{1,3} + u_{3,1}\right) = \frac{1}{2}\left(-\alpha x_2 + 0\right) = -\frac{\alpha x_2}{2}$$

$$\epsilon_{23} = \frac{1}{2}\left(u_{2,3} + u_{3,2}\right) = \frac{1}{2}\left(\alpha x_1 + 0\right) = \frac{\alpha x_1}{2}$$

$$\epsilon_{ij} = \begin{bmatrix} 0 & 0 & -\frac{\alpha x_2}{2} \\ 0 & 0 & \frac{\alpha x_1}{2} \\ -\frac{\alpha x_2}{2} & \frac{\alpha x_1}{2} & 0 \end{bmatrix}$$

Stresses:

$$\sigma_{ij} = \lambda \delta_{ij} \cancelto{0}{\epsilon_{kk}} + \cancel{2}\mu \epsilon_{ij}$$

$$\left.\begin{aligned} \sigma_{13} &= -\cancel{\mu}\alpha x_2 \\ \sigma_{23} &= \cancel{\mu}\alpha x_1 \end{aligned}\right\}$$

all other $\sigma_{ij} = 0$

Check equilibrium:

$$\sigma_{ij,j} = 0 \qquad \text{(no body forces)}$$

$$\cancelto{0}{\sigma_{11,1}} + \cancelto{0}{\sigma_{12,2}} + \cancel{\sigma_{13,3}} = 0$$
$$0 = 0 \quad \checkmark$$

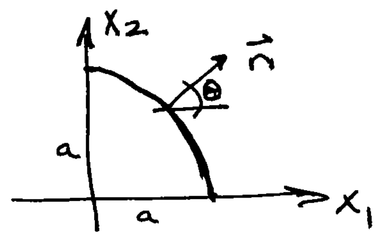
$$\cancelto{0}{\sigma_{12,1}} + \cancelto{0}{\sigma_{22,2}} + \cancel{\sigma_{23,3}} = 0$$
$$0 = 0 \quad \checkmark$$

$$\sigma_{13,1} + \sigma_{23,2} + \cancelto{0}{\sigma_{33,3}} = 0$$
$$0 + 0 = 0 \quad \checkmark$$

Now, boundary conditions:

later surface:

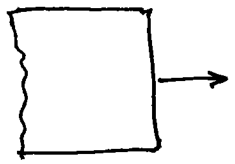
$$T_i = \sigma_{ij} n_j$$



$$\vec{n} = \{\cos\theta,\ \sin\theta,\ 0\} = \left\{\frac{x_1}{a},\ \frac{x_2}{a},\ 0\right\}$$

$$T_1 = \cancelto{0}{\sigma_{11}} n_1 + \cancelto{0}{\sigma_{12}} n_2 + \sigma_{13} \cancelto{0}{n_3} = 0 \quad \checkmark$$
$$T_2 = \cancelto{0}{\sigma_{12}} n_1 + \cancelto{0}{\sigma_{22}} n_2 + \sigma_{23} \cancelto{0}{n_3} = 0 \quad \checkmark$$
$$T_3 = \sigma_{13} n_1 + \sigma_{23} n_2 + \cancelto{0}{\sigma_{33}} \cancelto{0}{n_3}$$
$$= -\mu\alpha x_2\left(\frac{x_1}{a}\right) + \mu\alpha x_1\left(\frac{x_2}{a}\right) = 0 \quad \checkmark$$

end faces :



$\vec{n} = \{0, 0, 1\}$ $\quad T_i = \sigma_{ij} n_j$

$$T_1 = \sigma_{11} \cancel{n_1} + \sigma_{12} \cancel{n_2} + \sigma_{13} n_3$$

$$= \sigma_{13} \quad ?$$

$$T_2 = \sigma_{23}$$

$$T_3 = \sigma_{33} = 0 \quad \text{(satisfied exactly)}$$

what about (T_1, T_2) ?

$\longrightarrow$ Invoke St. Venant's Principle

$\therefore$ want $F_1^{net} = 0 = \int_A T_1 \, dA = \int_A \sigma_{13} \, dA$

$$\int_A \sigma_{13} \, dA = -\alpha\mu \int_A x_2 \, dA$$

but $\int_A = 0$ for centroidal axes

$$\left.\begin{array}{l} \therefore \; F_1^{net} = 0 \\ \text{similarly, } \; F_2^{net} = 0 \end{array}\right\} \checkmark$$

$$M_3^{net} = \text{Avg. Torque} = \underbrace{T}_{\text{applied torque}}$$

$$T = \int_A \left(\sigma_{23} x_1 - \sigma_{13} x_2 \right) dA$$

$$= \int_A \left(\mu \alpha x_1^2 + \mu \alpha x_2^2 \right) dA$$

$$= \mu \alpha \underbrace{\int_A \left(x_1^2 + x_2^2 \right) dA}_{J:\ \text{polar moment of area}}$$

$$\therefore \boxed{T = \mu \alpha J} \Rightarrow \boxed{\alpha = \frac{T}{\mu J}}$$

Complete solution:

$$[\sigma] = \begin{bmatrix} 0 & 0 & -\frac{T x_2}{J} \\ 0 & 0 & \frac{T x_1}{J} \\ -\frac{T x_2}{J} & \frac{T x_1}{J} & 0 \end{bmatrix}$$

$$[\epsilon] = \begin{bmatrix} 0 & 0 & -\frac{T x_2}{2\mu J} \\ 0 & 0 & \frac{T x_1}{2\mu J} \\ -\frac{T x_2}{2\mu J} & \frac{T x_1}{2\mu J} & 0 \end{bmatrix}$$

$$\{u\} = \begin{Bmatrix} -\frac{T}{\mu J} x_2 x_3 \\ \frac{T}{\mu J} x_1 \ x_3 \\ 0 \end{Bmatrix}$$

Observations:

1. For a circular section $J = \frac{\pi}{2} a^4$

2. Deformed shape is a cylinder,

   $\epsilon_{KK} = 0 \Rightarrow$ volume preserving deformation

3. Since we applied St. Venant, our solution is good only for long cylinders $(L \gg D)$