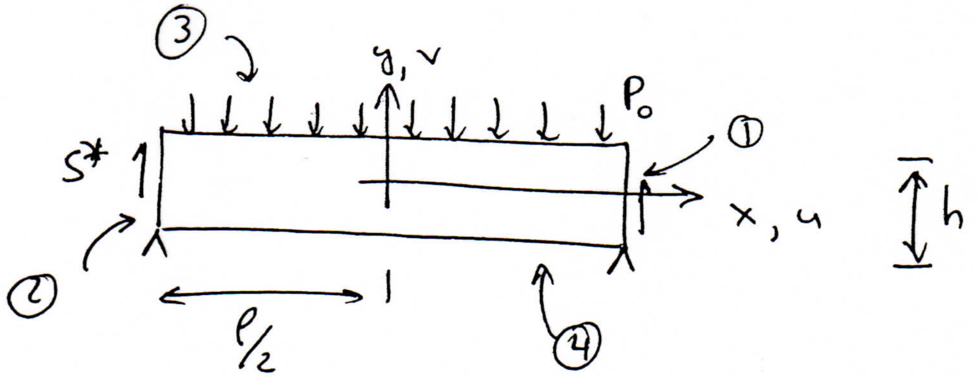
Rectangular domain



Plane strain
No $\underline{f}$.

Try: $$\phi(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn}\, x^n y^m$$

Note: 1.- Lower order terms $(m+n \le 3)$ satisfy biharmonic automatically

2.- Terms $a_{00} + a_{01} x + a_{10} y$ do not produce stress.

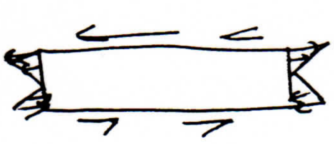
[e.g. (a) $\phi = A x y \Rightarrow \sigma_{xx} = \sigma_{yy} = 0$

$\sigma_{xy} = -A$ (pure shear)

(b) $\phi = A x y^3 \Rightarrow \sigma_{xx} = 6Axy$

$\sigma_{yy} = 0$

$\sigma_{xy} = -3Ay^2$]



· Use S.O.M. and symetry to guess m, n.

· From symetry here: $\sigma_x = \frac{\partial^2 \phi}{\partial y^2}$ must be even in x

$\sigma_y = \frac{\partial^2 \phi}{\partial x^2}$ " " " " x

$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$ " " odd " x

Try: $\phi(x,y) = ax^2 + bx^2y + cy^3 + dx^4 + eyx^4$
$+ fx^2y^3 + gy^5 + \ldots$

Biharmonic:

$\nabla^4 \phi = 0 \Rightarrow 24d + 24ey + 24fy + 120gy = 0$

Must hold $\forall (x,y) \Rightarrow \boxed{d = 0}$
and

$e + f + 5g = 0$

$\Rightarrow \boxed{f = -e - 5g}$

$$\therefore \ \phi(x,y) = ax^2 + bx^2y + cy^3 + e(x^4y - x^2y^3) + g(y^5 - 5x^2y^3)$$

Stresses become:

$$\sigma_x = 6cy - 6ex^2y + g(20y^3 - 30x^2y)$$

$$\sigma_y = 2a + 2by + e(12yx^2 - 2y^3) + g(-10y^3)$$

$$-\sigma_{xy} = 2bx + e(4x^3 - 6xy^2) + g(-30xy^2)$$

Boundary conditions:

On surface 3: $\quad y = \frac{h}{2}$

$$\sigma_y\left(x, \tfrac{h}{2}\right) = -P_o = 2a + 2b\frac{h}{2} + e\left(12\frac{h}{2}x^2 - 2\frac{h^3}{8}\right) - 10g\frac{h^3}{8}$$

$$\sigma_{xy}\left(x, \tfrac{h}{2}\right) = 0 = 2bx + e\left[4x^3 - 6x\left(\tfrac{h}{2}\right)^3\right] - \frac{15}{2}h^2gx$$

ust hold $\forall x \Rightarrow \boxed{e = 0}$

$$\therefore \left.\begin{aligned} -P_o &= 2a + bh - \frac{10}{8}gh^3 \\ 0 &= 2b - \frac{15}{2}h^2g \end{aligned}\right\} \text{— (1)}$$

On surface ④ : $y = -\frac{h}{2}$

$$\sigma_y(x, -\tfrac{h}{2}) = 0 = 2a - bh + 10g\left(\tfrac{h}{2}\right)^3$$

$$\sigma_{xy}(x, -\tfrac{h}{2}) = 0 = \sigma(x, +\tfrac{h}{2}) \quad \text{same as above}$$

$$\Rightarrow \quad 2a - bh + \frac{10}{8} h^3 g = 0 \qquad \text{— (2)}$$

(1), (2) $\Rightarrow$

$$\boxed{g = -\frac{P_0}{5h^3}}$$

$$\boxed{a = -\frac{1}{4} P_0}$$

check dimensions
to be sure.

$$\boxed{b = -\frac{3}{4} \frac{P_0}{h}}$$

Unfortunately we can only satisfy the b.c. on
①, ② forces on average

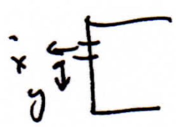
On surface ①

$$\sigma_x\left(\frac{\ell}{2}, y\right) = 6cy + g\left(20y^3 - \frac{15}{2}\ell^2 y\right)$$

$$-\sigma_{xy}\left(\frac{\ell}{2}, y\right) = b\ell + g\left(-15\ell y^2\right)$$

Use S.V. for justification of average b.c.

To satisfy "pinned" condition require no moment on end faces:



$$\int_{-h/2}^{h/2} y\,\sigma_x\,dy = 0$$

$$\Rightarrow \boxed{c = \frac{p_0}{10h}\left[1 - \frac{5}{2}\left(\frac{\ell}{h}\right)^2\right]}$$

∴ Solution becomes:

$$\frac{\sigma_x}{P_o} = \underbrace{-\frac{3}{2}\left\{\left(\frac{l}{h}\right)^2 - 4\left(\frac{x}{l}\right)^2\right\}\frac{y}{l}} + \left\{\frac{3}{5} - 4\left(\frac{y}{h}\right)^2\right\}\frac{y}{l}$$

$$\frac{\sigma_{xy}}{P_o} = \underbrace{\frac{3}{2}\left\{1 - 4\left(\frac{y}{x}\right)^2\right\}\frac{x}{h}}$$

s.o.m. solution.

$$\frac{\sigma_y}{P_o} = -\frac{1}{2} - \frac{3}{2}\frac{y}{h} + 2\left(\frac{y}{h}\right)^3$$

<u>Note:</u> Reaction forces on ① and ② come in through a <u>net</u> shear force S^*

$$S^* = \int_{-h/2}^{h/2} \sigma_{xy}\, dy$$

$$\Rightarrow S^* = \frac{3P_o}{2}\int_{-h/2}^{h/2}\left\{1 - 4\left(\frac{y}{h}\right)^2\right\}\cdot\frac{l}{2h}\, dy$$

$$\Rightarrow \boxed{S^* = \frac{1}{2}P_o l}$$