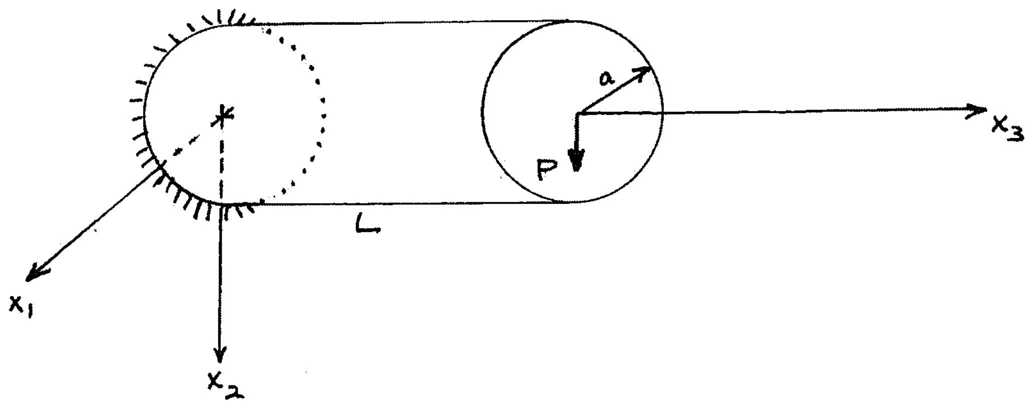
Transverse end loading

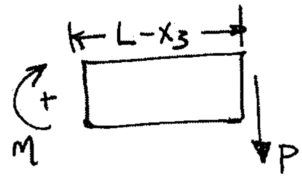


$$\sigma_{11} = \sigma_{22} = \sigma_{12} = 0$$

and from pure bending solution:

$$\sigma_{33} = \alpha X_2 = \frac{M}{I} X_2$$

what is M?



$$\Sigma M = 0$$

$$M + P(L - x_3) = 0$$

$$M = -P(L - x_3)$$

$$\therefore \sigma_{33} = \frac{-P}{I}(L - x_3) X_2$$

That leaves,

$$\sigma_{13} = ?$$

$$\sigma_{23} = ?$$

Equilibrium (no body forces):

$$\sigma_{ij,j} = 0$$

$$\cancelto{0}{\sigma_{11,1}} + \cancelto{0}{\sigma_{12,2}} + \sigma_{13,3} = 0$$

$$\boxed{\sigma_{13,3} = 0} \qquad (1)$$

$$\cancelto{0}{\sigma_{21,1}} + \cancelto{0}{\sigma_{22,2}} + \sigma_{23,3} = 0$$

$$\boxed{\sigma_{23,3} = 0} \qquad (2)$$

$$\sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} = 0$$

$$\boxed{\sigma_{13,1} + \sigma_{23,2} + \frac{PX_2}{I} = 0} \qquad (3)$$

Now we <u>define</u> a stress function $\phi(x_1, x_2)$:

$$\sigma_{13} \equiv \frac{\partial \phi}{\partial x_2} = \phi_{,2} \qquad (4)$$

$$\sigma_{23} \equiv -\frac{\partial \phi}{\partial x_1} - \frac{P}{2I} x_2^2 = -\phi_{,1} - \frac{Px_2^2}{2I} \qquad (5)$$

With this choice of stress function the equilibrium eqs. are <u>identically</u> satisfied:

e.g. $$\sigma_{13,3} = (\phi_{,2})_{,3} \stackrel{?}{=} 0$$

$$0 = 0$$

$$\sigma_{23,3} = \left(-\phi_{,1} - \frac{Px_2^{\,2}}{2I}\right)_{,3} \stackrel{?}{=} 0$$

$$-0 - 0 = 0$$

$$\sigma_{13,1} + \sigma_{23,2} + \frac{Px_2}{I} \stackrel{?}{=} 0$$

$$(\phi_{,2})_{,1} + \left(-\phi_{,1} - \frac{Px_2^{\,2}}{2I}\right)_{,2} + \frac{Px_2}{I} \stackrel{?}{=} 0$$

$$\phi_{,21} - \phi_{,12} - \frac{Px_2}{I} + \frac{Px_2}{I} = 0$$

We now have a stress "solution" to the problem which satisfies all equilibrium eqs.

Now calculate strains using Hooke's Law:

$$E\epsilon_{11} = \cancel{\sigma_{11}}^{\,0} - \nu(\cancel{\sigma_{22}}^{\,0} + \sigma_{33})$$

$$= \frac{\nu P}{I}(L - x_3)\, x_2$$

$$E\epsilon_{22} = \cancel{\sigma_{22}}^{\,0} - \nu(\cancel{\sigma_{11}}^{\,0} + \sigma_{33})$$

$$= \frac{\nu P}{I}(L - x_3)\, x_2$$

$$E\epsilon_{33} = \sigma_{33} - \nu(\cancel{\sigma_{11}}^{\,0} + \cancel{\sigma_{22}}^{\,0})$$

$$= \frac{-P}{I}(L - x_3)\, x_2$$

$$\cancel{2\mu}\,\epsilon_{12} = \cancel{\sigma_{12}}^{\,0} = 0$$

$$\cancel{2\mu}\,\epsilon_{13} = \phi_{,2} = \sigma_{13}$$

$$\cancel{2\mu}\,\epsilon_{23} = -\phi_{,1} - \frac{P x_2^2}{2I} = \sigma_{23}$$

Now we utilize compatability relations:

$$\epsilon_{11,22} + \epsilon_{22,11} - 2\epsilon_{12,12} = 0$$

$$0 + 0 - 0 = 0 \checkmark$$

$$\epsilon_{22,33} + \epsilon_{33,22} - 2\epsilon_{23,23} = 0$$

$$0 + 0 - 0 = 0 \checkmark$$

$$\epsilon_{33,11} + \epsilon_{11,33} - 2\epsilon_{13,31} = 0$$

$$0 + 0 - 0 = 0 \checkmark$$

$$\epsilon_{12,13} + \epsilon_{13,12} - \epsilon_{23,11} - \epsilon_{11,23} = 0$$

$$0 + \frac{1}{\cancel{2\mu}}\left(\phi_{,212}\right) - \frac{1}{\cancel{2\mu}}\left(-\phi_{,111}\right) - \left(\frac{-\nu P}{EI}\right) = 0$$

$$\phi_{,111} + \phi_{,221} = \frac{-\nu P}{EI}(2\mu)$$

$$\boxed{\left(\nabla^2\phi\right)_{,1} = \frac{-\nu P}{EI}(2\mu)} \qquad (6)$$

$$\epsilon_{23,21} + \epsilon_{21,23} - \epsilon_{31,22} - \epsilon_{22,31} = 0$$

$$-\phi_{,121} + 0 - \phi_{,222} - 0 = 0$$

$$\phi_{,112} + \phi_{,222} = 0$$

$$\boxed{\left(\nabla^2\phi\right)_{,2} = 0} \qquad (7)$$

$$\epsilon_{31,32} + \epsilon_{32,31} - \epsilon_{12,33} - \epsilon_{33,12} = 0$$

$$0 + 0 - 0 - 0 = 0 \checkmark$$

Integrating (6)+(7) yields,

$$\boxed{\nabla^2\phi = \frac{-\nu}{1+\nu}\left(\frac{P}{I}\right)x_1 + \underbrace{C}_{\text{constant}}} \qquad (8)$$

Get constant by considering ω_{12} (avg. rotation about x_3):

$$\omega_{12} = 0 \rightarrow \boxed{C = 0}$$

What about lateral b.c.s?

$$\sigma_{13}n_1 + \sigma_{23}n_2 = 0$$

After substitution this yields,

$$\frac{d\phi}{ds} = -\frac{1}{2}\frac{P}{I}x_2^2\frac{dx_1}{ds} \quad \text{on } \underbrace{S}_{\text{surface}} \qquad (9)$$

Eq. (8) subject to the condition of Eq (9) is the formal statement of the reduced b.v.p.

Solution to (8) can be expressed as the sum of a particular solution plus a harmonic function (homogeneous solution):

$$\phi(x_1,x_2) = \underbrace{f(x_1,x_2)}_{\substack{\text{harmonic} \\ (\text{i.e. } \nabla^2 f = 0)}} \underbrace{- \frac{1}{6}\frac{\nu}{1+\nu}\frac{P}{I}x_1^3}_{\text{particular}}$$

Easier to work in polar coordinates for this case. Thus,

$$\phi = f - \frac{1}{6}\frac{\nu}{1+\nu}\frac{P}{I}(r\cos\theta)^3 \qquad (10)$$

Subject to the b.c.s on the lateral surface:

$$\frac{1}{a}\frac{\partial \phi}{\partial \theta} = \frac{1}{2}\frac{P}{I}a^2 \sin^3\theta \quad \text{on } r=a \qquad (11)$$

These can be rewritten using trig identities as:

$$\begin{cases} \phi = \dfrac{P}{I}\left\{ f - \dfrac{1}{24}\dfrac{\nu}{1+\nu} r^3 (\cos 3\theta + 3\cos\theta) \right\} & (12) \\ \dfrac{\partial \phi}{\partial \theta} = \dfrac{1}{8}\dfrac{P}{I} a^3 (-\sin 3\theta + 3\sin\theta) \quad \text{on } r=a & (13) \end{cases}$$

We look for solutions of f such that;

$$f = \sum_n A_n r^n \cos n\theta$$

For example, take first 2 terms:

$$f = A_1 r\cos\theta + A_3 r^3 \cos 3\theta \qquad (14)$$

Combining (14) and (12),

$$\phi = \frac{P}{I}\left\{\left(A_1 r - \frac{\nu r^3}{8(1+\nu)}\right)\cos\theta + \left(A_3 - \frac{\nu}{24(1+\nu)}\right) r^3 \cos 3\theta\right\} \quad (15)$$

$$\therefore \frac{\partial\phi}{\partial\theta} = \frac{P}{I}\left\{\left(\frac{\nu r^3}{8(1+\nu)} - A_1 r\right)\sin\theta + 3r^3\left(\frac{\nu}{24(1+\nu)} - A_3\right)\sin 3\theta\right\} \quad (16)$$

Evaluate (16) for $r = a$ and equating to (13) yields,

$$\begin{cases} A_1 = \dfrac{-(3+2\nu)}{8(1+\nu)}a^2 & (17) \\[2ex] A_3 = \dfrac{1+2\nu}{24(1+\nu)} & (18) \end{cases}$$

Substituting (17) + (18) into (15), then reverting back to cartesian coordinates yields:

$$\boxed{\phi = \frac{P}{I}\left\{\frac{-(3+2\nu)}{8(1+\nu)}a^2 x_1 - \frac{1+2\nu}{8(1+\nu)}x_1 x_2^2 + \frac{1-2\nu}{24(1+\nu)}x_1^3\right\}} \quad (19)$$

The stresses can now be calculated as,

$$\sigma_{13} = \frac{-P}{4I}\left(\frac{1+2\nu}{1+\nu}\right) x_1 x_2$$

$$\sigma_{23} = \frac{P}{I}\left\{\frac{3+2\nu}{8(1+\nu)}\left[a^2 - x_2^2 - \left(\frac{1-2\nu}{3+2\nu}\right)x_1^2\right]\right\}$$

$$\sigma_{33} = \frac{-P}{I} x_2 (L - x_3)$$

where, $I = \frac{\pi}{4} a^4$