Plane Elasticity in Cylindrical Coordinates

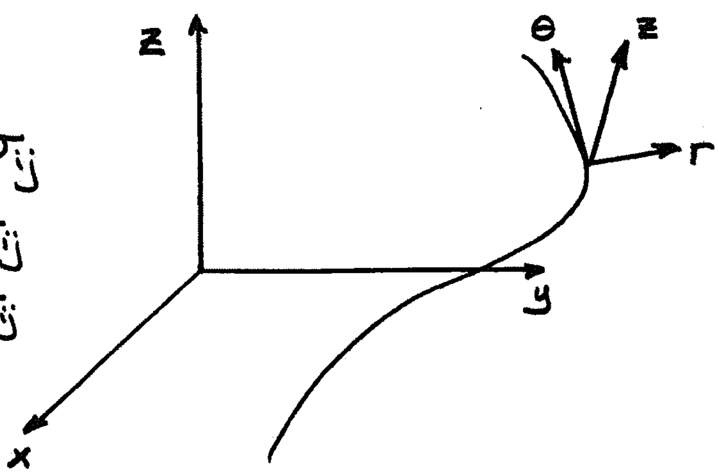
Airy's Stress Function:

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + V$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} + V$$

$$\sigma_{r\theta} = \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta}$$

$$\begin{cases} \sigma_{rr} = \alpha_{ri}\alpha_{rj}\sigma_{ij} \\ \sigma_{\theta\theta} = \alpha_{\theta i}\alpha_{\theta j}\sigma_{ij} \\ \sigma_{r\theta} = \alpha_{ri}\alpha_{\theta j}\sigma_{ij} \end{cases}$$



Laplacian Operator:

$$\nabla^2(\ ) = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

The biharmonic equation (no body forces) is:

$$\nabla^4\phi = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]\left[\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial \theta^2}\right] = 0$$

Now we will look at a special class of problems termed "<u>axisymmetric</u>." In cylindrical coordinates we often encounter problems that depend only on r and <u>not</u> on θ. In such cases the previous equations can be simplified.

$$\boxed{\text{Axisymmetric Problems}}$$

$$\begin{cases} \sigma_{rr} = \dfrac{1}{r}\dfrac{d\phi}{dr} + V \\ \sigma_{\theta\theta} = \dfrac{d^2\phi}{dr^2} + V \\ \sigma_{r\theta} = 0 \end{cases}$$

and the Laplacian operator becomes,

$$\nabla^2(\,) = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}$$

thus, the biharmonic equation is,

$$\nabla^4\phi = \left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right]\left[\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr}\right] = 0$$

$$= \frac{d}{dr}\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)\right)\right] = 0$$

This leaves the equation in a form that is readily integratable to get ϕ.

$$\int \frac{d}{dr}\left[r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)\right\}\right] = \int 0$$

$$r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)\right\} = c_1$$

integrate again

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = c_1 \ln r + c_2$$

$$\therefore \quad \frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = c_1 r \ln r + c_2 r$$

integrate

$$r\frac{d\phi}{dr} = c_1\frac{r^2}{2}(\ln r - 1) + c_2\frac{r^2}{2} + c_3$$

$$\therefore \quad \frac{d\phi}{dr} = c_1\frac{r}{2}(\ln r - 1) + c_2\frac{r}{2} + \frac{c_3}{r}$$

$$= \frac{c_1}{2} r \ln r - \frac{c_1}{2} r + \frac{c_2}{2} r + c_3\frac{1}{r}$$

integrate

$$\phi(r) = \frac{c_1}{2}\left\{\frac{r^2}{2}(\ln r - 1)\right\} - \frac{c_1 r^2}{4} + \frac{c_2 r^2}{4} + c_3 \ln r + c_4$$

$$= \frac{c_1}{4} r^2 \ln r - \frac{c_1 r^2}{4} - \frac{c_1 r^2}{4} + \frac{c_2 r^2}{4} + c_3 \ln r + c_4$$

$$\therefore \phi(r) = \bar{c}_1 r^2 \ln r + \bar{c}_2 r^2 + c_3 \ln r + c_4$$

where, $\bar{c}_1 = \frac{c_1}{4}$

$\bar{c}_2 = \frac{c_2}{4} - \frac{c_1}{2}$

dropping the overbars we have,

$$\boxed{\phi(r) = c_1 r^2 \ln r + c_2 r^2 + c_3 \ln r + c_4}$$

The stress can be obtained as,

$$\begin{cases} \sigma_{rr} = \dfrac{1}{r}\dfrac{d^2\phi}{dr^2} \\ \quad = c_1(1+2\ln r) + 2c_2 + \dfrac{c_3}{r^2} \\ \sigma_{\theta\theta} = \dfrac{d^2\phi}{dr^2} = \\ \quad = c_1(3+2\ln r) + 2c_2 - \dfrac{c_3}{r^2} \\ \sigma_{r\theta} = 0 \end{cases}$$

The strain-stress relations are,

plane stress
$$\begin{cases} \epsilon_{rr} = \dfrac{1}{E}\left[\sigma_{rr} - \nu\sigma_{\theta\theta}\right] \\ \epsilon_{\theta\theta} = \dfrac{1}{E}\left[\sigma_{\theta\theta} - \nu\sigma_{rr}\right] \\ \epsilon_{zz} = \dfrac{-\nu}{E}\left[\sigma_{rr} + \sigma_{\theta\theta}\right] \\ \epsilon_{r\theta} = \dfrac{1}{2G}\sigma_{r\theta} \\ \epsilon_{rz} = \epsilon_{\theta z} = 0 \end{cases}$$

$$
\text{plane strain}\quad
\begin{cases}
\epsilon_{rr} = \dfrac{1}{E}\left[\sigma_{rr} - \nu\left(\sigma_{\theta\theta} + \sigma_{zz}\right)\right] \\[2ex]
\epsilon_{\theta\theta} = \dfrac{1}{E}\left[\sigma_{\theta\theta} - \nu\left(\sigma_{rr} + \sigma_{\theta\theta}\right)\right] \\[2ex]
\epsilon_{zz} = 0 = \dfrac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{rr} + \sigma_{\theta\theta}\right)\right] \\[2ex]
\epsilon_{r\theta} = \dfrac{1}{2G}\sigma_{r\theta} \\[2ex]
\epsilon_{rz} = \epsilon_{\theta z} = 0
\end{cases}
$$

The strain-displacement relations in cylindrical coordinates are now,

$$
\begin{cases}
\epsilon_{rr} = \dfrac{\partial u_r}{\partial r} \\[2ex]
\epsilon_{\theta\theta} = \dfrac{u_r}{r} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta} \\[2ex]
\epsilon_{r\theta} = 0
\end{cases}
$$

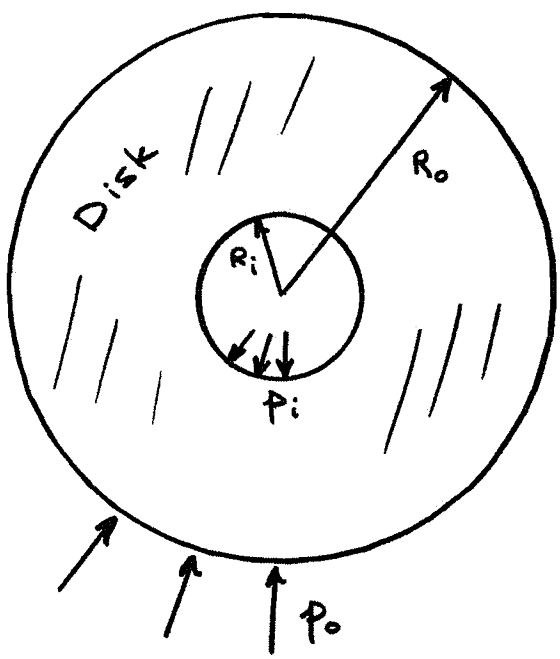
Integrating these equations gives the following displacements,

$$u_r = \frac{1}{E}\left\{ C_1 r\left[(1-\nu)(2\ln r - 1) - 2\nu\right] + 2C_2(1-\nu)r - C_3\left(\frac{1+\nu}{r}\right) + C_4\sin\theta + C_5\cos\theta \right\}$$

$$u_\theta = \frac{1}{E}\left\{ 4C_1 r\theta + C_4\cos\theta - C_5\sin\theta + C_6 r \right\}$$

Now for an example:

$$\boxed{\text{Thick-Walled Cylinder or Disk}}$$



boundary conditions:

$\sigma_{rr}(R_o) = -p_o$

$\sigma_{rr}(R_i) = -p_i$

From the first boundary condition,

$$-p_o = c_1\left(1 + 2\ln R_o\right) + 2c_2 + \frac{c_3}{R_o^2} \qquad (1)$$

and from the second,

$$-p_i = c_1\left(1 + 2\ln R_i\right) + 2c_2 + \frac{c_3}{R_i^2} \qquad (2)$$

However, we need another equation as there are 3 unknown constants. This comes from knowledge of the displacement u_θ.

Obviously, u_θ is periodic. That is,

$$u_\theta(0) = u_\theta(2\pi) = u_\theta(4\pi) \;\ldots\text{etc}\ldots \qquad (3)$$

Therefore, examining u_θ:

$$u_\theta = \frac{1}{E}\left\{4c_1 r\theta + c_4\cos\theta - c_5\sin\theta + c_6 r\right\} \qquad (4)$$

we see that for u_θ to be single-valued we must have,

$$C_1 \equiv 0 \tag{5}$$

Thus, we have 2 equations and 2 unknowns,

$$\begin{bmatrix} 2 & 1/R_i^2 \\ 2 & 1/R_o^2 \end{bmatrix} \begin{Bmatrix} C_2 \\ C_3 \end{Bmatrix} = - \begin{Bmatrix} p_i \\ p_o \end{Bmatrix} \tag{6}$$

Therefore,

$$C_2 = \frac{p_i R_i^2 - p_o R_o^2}{2 (R_o^2 - R_i^2)} \tag{7}$$

$$C_3 = \frac{R_i^2 R_o^2 (p_o - p_i)}{R_o^2 - R_i^2} \tag{8}$$

and the resulting stresses are,

$$\sigma_{rr} = \frac{p_i R_i^2 - p_o R_o^2}{R_o^2 - R_i^2} + \frac{R_i^2 R_o^2 (p_o - p_i)}{r^2 (R_o^2 - R_i^2)} \tag{9}$$

$$\sigma_{\theta\theta} = \frac{p_i R_i^2 - p_o R_o^2}{R_o^2 - R_i^2} - \frac{R_i^2 R_o^2 (p_o - p_i)}{r^2 (R_o^2 - R_i^2)} \qquad (10)$$

Normalizing by R_o^2 :

$$\sigma_{rr} = \frac{p_o\left[\left(\frac{R_i}{r}\right)^2 - 1\right] + p_i\left[\left(\frac{R_i}{R_o}\right)^2 - \left(\frac{R_i}{r}\right)^2\right]}{1 - \left(\frac{R_i}{R_o}\right)^2} \qquad (11)$$

$$\sigma_{\theta\theta} = \frac{-p_o\left[1 + \left(\frac{R_i}{r}\right)^2\right] + p_i\left[\left(\frac{R_i}{R_o}\right)^2 + \left(\frac{R_i}{r}\right)^2\right]}{1 - \left(\frac{R_i}{R_o}\right)^2} \qquad (12)$$

For the plane strain case we also have,

$$\sigma_{zz} = \nu\left(\sigma_{rr} + \sigma_{\theta\theta}\right)$$

$$= \frac{2\nu\left[p_i\left(\frac{R_i}{R_o}\right)^2 - p_o\right]}{1 - \left(\frac{R_i}{R_o}\right)^2} \qquad (13)$$

or for plane stress,

$$\epsilon_{zz} = \frac{-\nu}{E}\left(\sigma_{rr} + \sigma_{\theta\theta}\right)$$

$$= \frac{\frac{2\nu}{E}\left\{p_o - p_i\left(\frac{R_i}{R_o}\right)^2\right\}}{1-\left(\frac{R_i}{R_o}\right)^2} \tag{14}$$

Now let's look at a more complicated case.

→ $\boxed{\text{Thick-Walled cylinder with closed ends}}$ ←

Since we are dealing with a linear elastic theory we can superimpose solutions to other problems in order the find the solution to this problem.

We already have the solution to

the problem of an open-ended cylinder. Closing the ends of the cylinder will generate a uniformly distributed stress σ_{zz} which maintains overall equilibrium.

$$\text{Solution} = \left\{ \begin{array}{c} \text{thick-walled} \\ \text{plane} \\ \text{strain} \\ \text{Solution} \end{array} \right\} + \left\{ \begin{array}{c} \sigma_{zz} \text{ uniformly} \\ \text{distributed} \end{array} \right\}$$

$$F_z^* = \text{force in the } z\text{-direction}$$
$$= p_o \pi R_o^2 - p_i \pi R_i^2$$

$$\therefore \sigma_{zz}^* = \frac{F_z}{\pi (R_o^2 - R_i^2)} = \frac{p_i \left(\frac{R_i}{R_o}\right)^2 - p_o}{1 - \left(\frac{R_i}{R_o}\right)^2} \qquad (15)$$

Thus, the total solution will be

$$\sigma_{zz} = \text{equation (13)} \underset{\text{by superposition}}{+} \sigma_{zz}^* \text{ (equ. 15)}$$