Chapter 6 - Extension, Bending and Torsion

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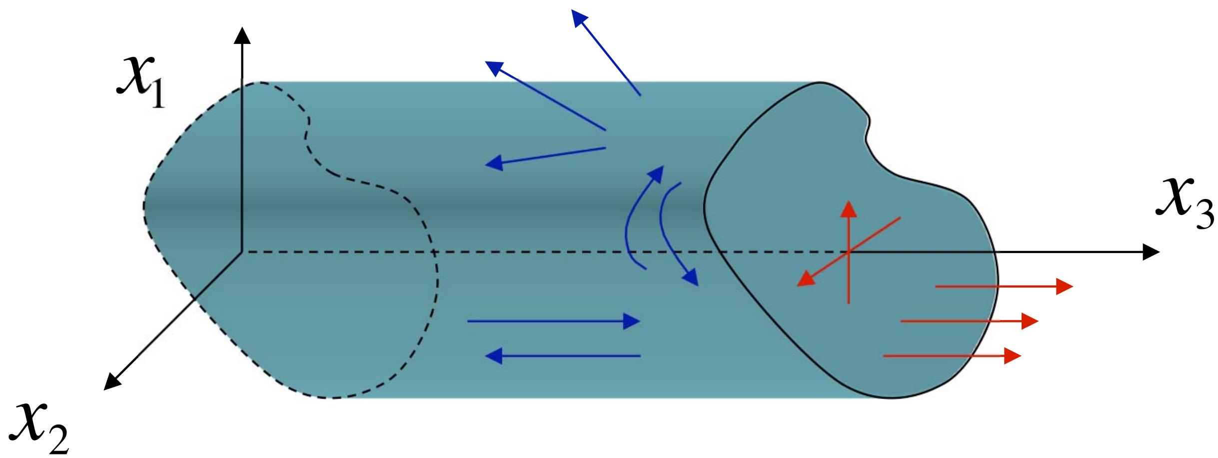
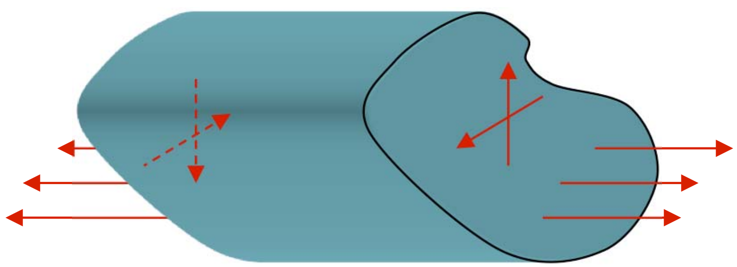
- Cylindrical bodies are common in engineering:



- Superposition



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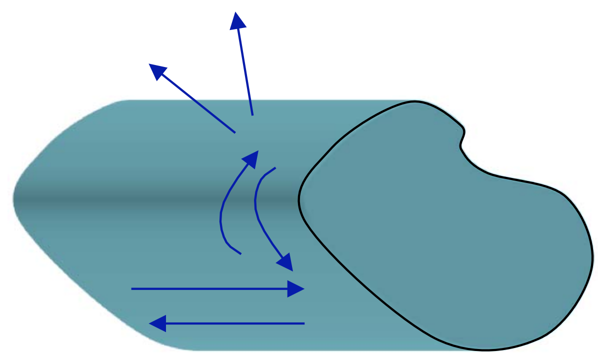


Load on end faces

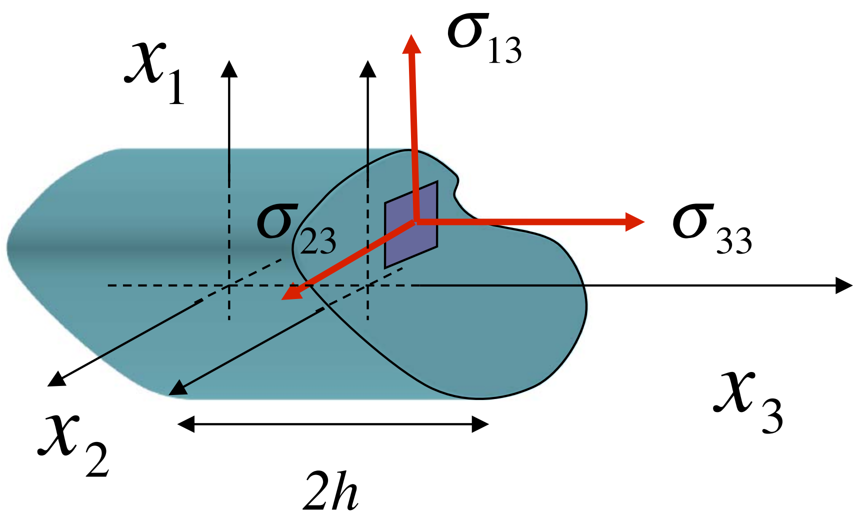
A

Load on lateral surface

B

Plane stress, plane strain, anti-plane shear (later)

Stress resultants



- Tractions on end faces $x_3=\pm h$:

$$T_1 = \sigma_{13}(x_1,x_2)$$
$$T_2 = \sigma_{23}(x_1,x_2)$$
$$T_3 = \sigma_{33}(x_1,x_2)$$

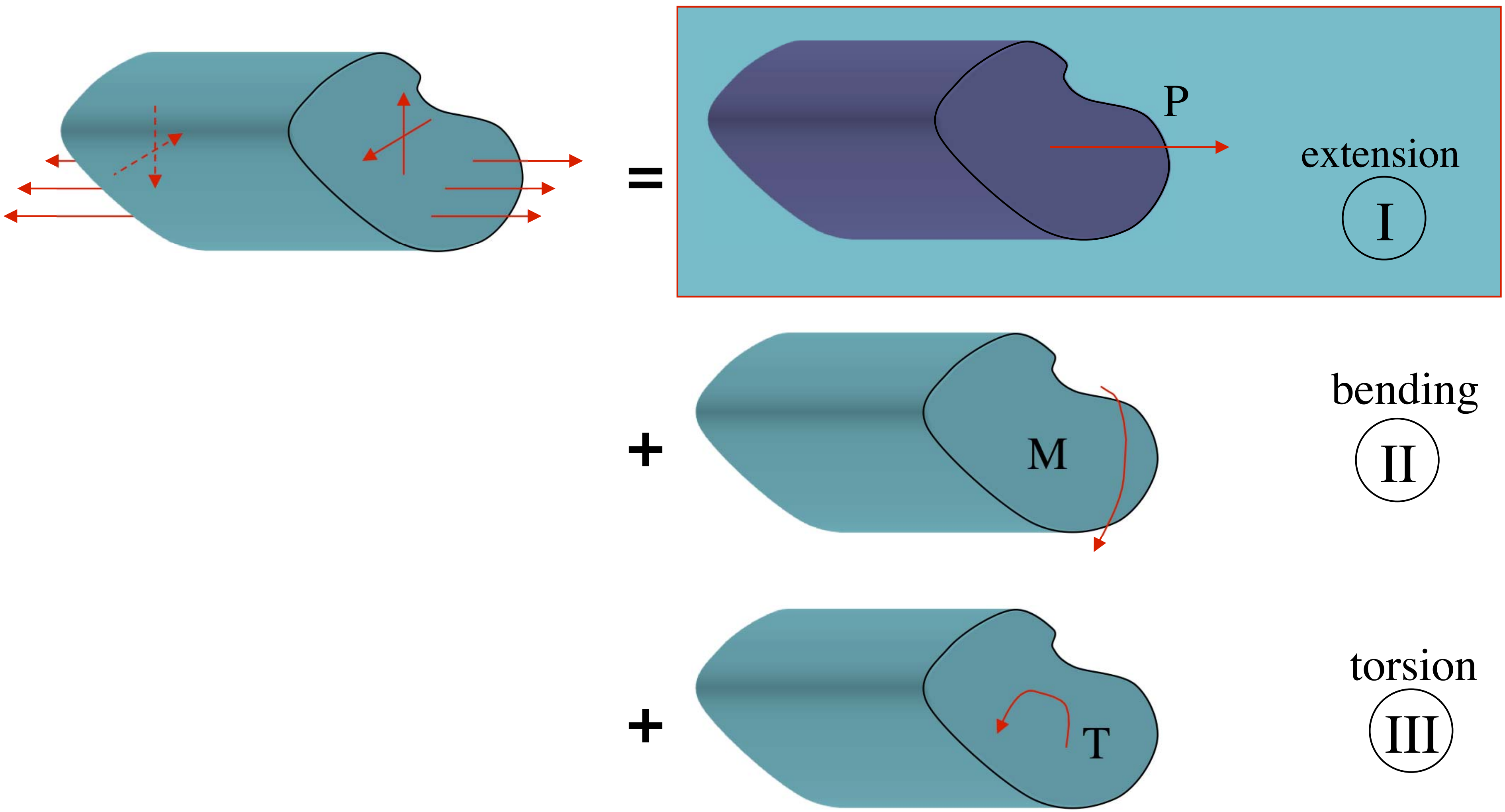
- There are three net, or resultant, forces (one each along x_1, x_2, x_3 axes) and three net moments (one each about x_1, x_2, x_3 axes):

$$F_1^{net} = \int_A T_1 dA = \int_A \sigma_{13} dA \qquad M_1^{net} = \int_A \sigma_{33} x_2 dA$$

$$F_2^{net} = \int_A T_2 dA = \int_A \sigma_{23} dA \qquad M_2^{net} = \int_A \sigma_{33} x_1 dA$$

$$F_3^{net} = \int_A T_3 dA = \int_A \sigma_{33} dA \qquad M_3^{net} = \int_A (\sigma_{32} x_1 - \sigma_{31} x_2) dA$$

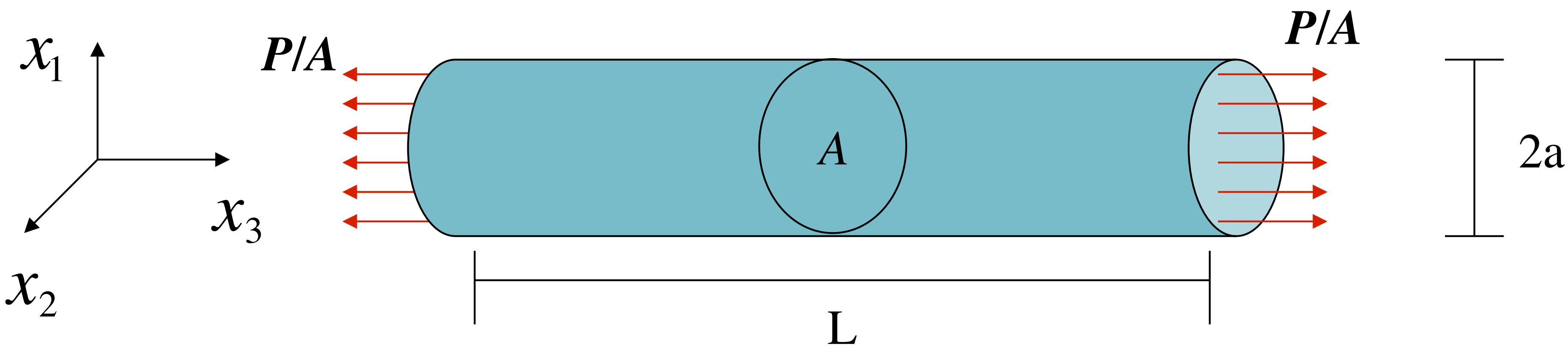
- Problem A :



6.1.1 Uniform Applied Traction (9.2)

- Problem I Extension:

- Assume a circular cylinder



$$P(x_1, x_2) = P = const.$$

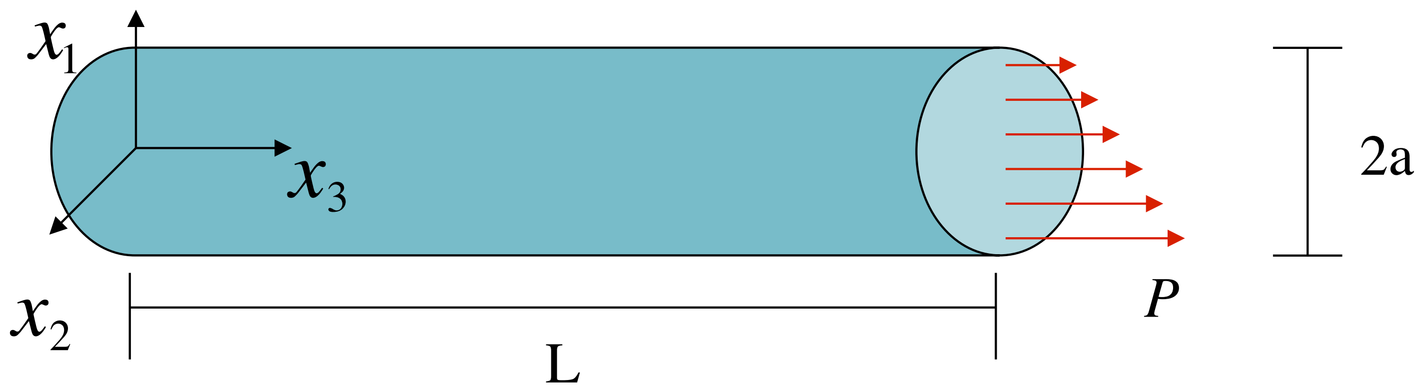
$$\sigma_0 = \frac{P}{A}$$

- Solution (Done in class…)

6.1.2 Non-Uniform Applied Traction (9.2)

- Question: What if applied traction is NOT constant? e.g.

$$T_3(x_1, x_2, L) = P(x_1, x_2) = ax_1 + bx_2$$



- Solution as before except b.c.s on end faces

$$\text{On } x_3 = L \quad T_1 = T_2 = 0$$

$$\text{but } T_3 = \frac{P}{A} = const \neq (ax_1 + bx_2)/A$$

Non-Uniform Applied Traction (cont.) (5.6)

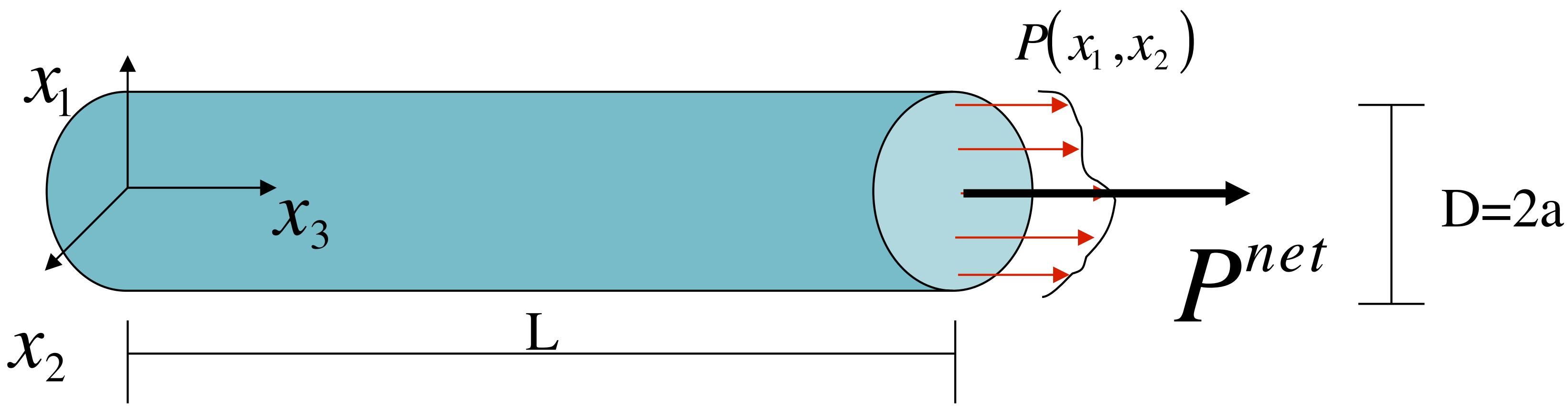
- **St. Venant's Principle** (Baron de St. Venant, 1855): For two traction distributions with the same resultant, then the difference of the two solutions at large distances from the point of application of the tractions, decays to zero.

$\Rightarrow$ Exact distribution on end not important. Only resultant is important

- Large distances ? $\Rightarrow$ 4-5 characteristic lengths

- Note: In real situations we seldom have control over the precise distribution of applied loads.

Non-Uniform Applied Traction (cont.)



- Satisfy average (resultant) b.c. on end faces

$$\int_A T_3 dA = \int_A \sigma_{33} dA = \int_A \frac{P(x_1, x_2)}{A} dA = P^{net}$$

- Solution for non-uniform P: $\boxed{\sigma_{ij} = \delta_{3i}\delta_{3j} \dfrac{P^{net}}{A}}$

- How far away? (3-5)D → **Need L >> D**

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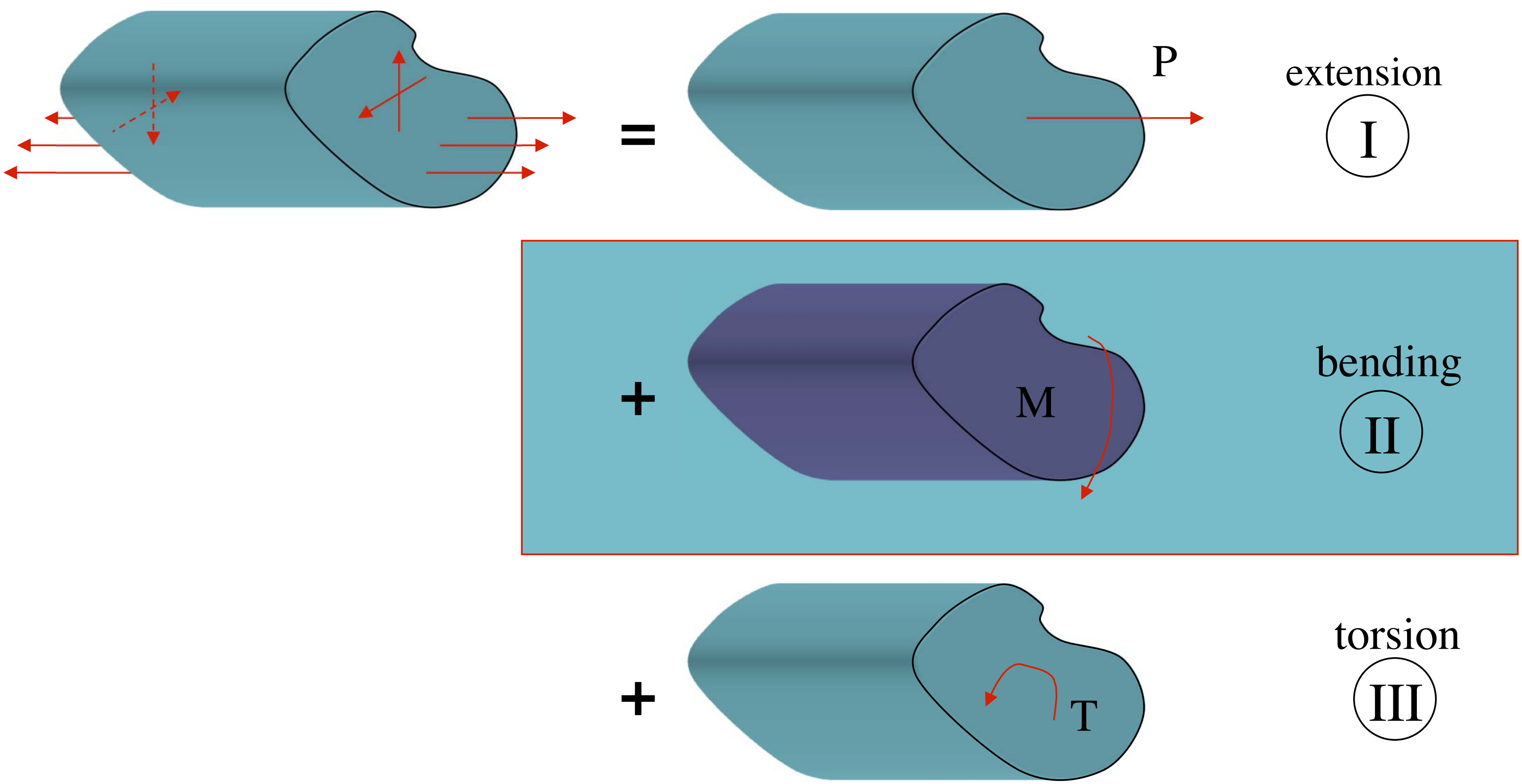
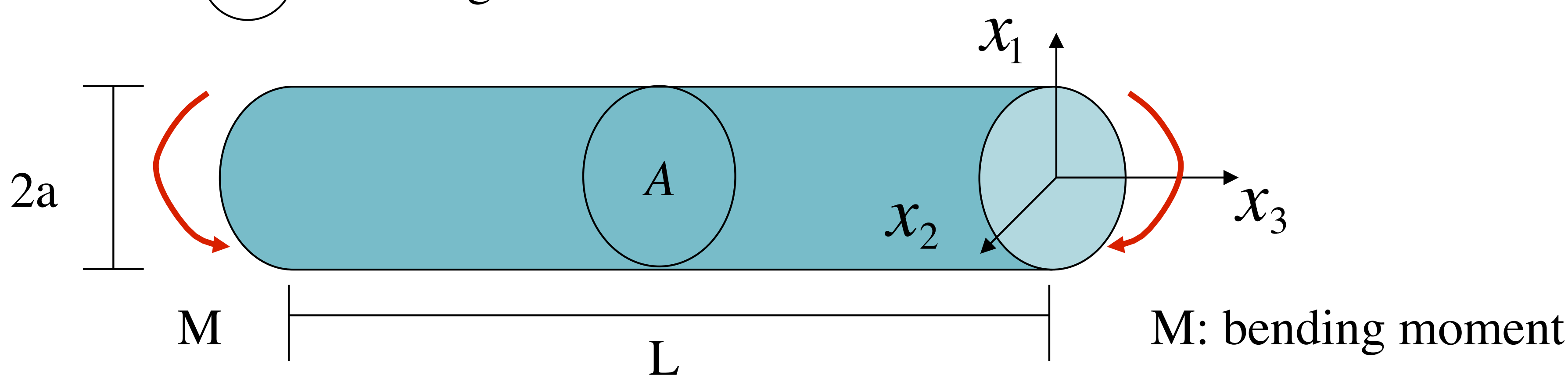
6.2 Bending

- Problem A :



Bending (cont.)

- Problem II Bending:



- Solution (Done in class…)

Bending (cont.)

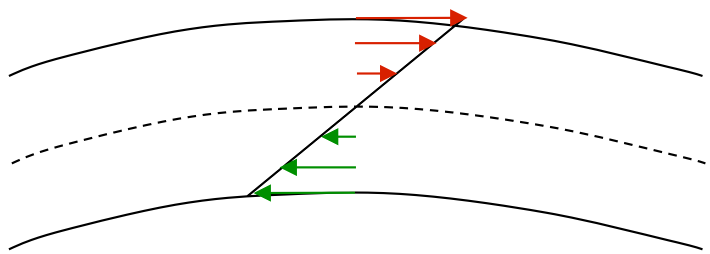
$$[\sigma_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Mx_1/I \end{bmatrix} \quad [\varepsilon_{ij}] = \begin{bmatrix} -\nu Mx_1/EI & 0 & 0 \\ 0 & -\nu Mx_1/EI & 0 \\ 0 & 0 & Mx_1/EI \end{bmatrix}$$

$$\begin{aligned} u_1 &= \underbrace{-\frac{\alpha x_3^2}{2E} - \frac{\nu\alpha x_1^2}{2E} + \frac{\nu\alpha x_2^2}{2E}}_{\text{stress generating terms}} \underbrace{- a_1 x_3 + r_1 x_2 + d_2}_{\text{Rigid rotation and translation}} \\ u_2 &= -\frac{\nu\alpha x_1 x_2}{E} - b_1 x_3 - r_1 x_1 + d_3 \\ u_3 &= \frac{\alpha x_1 x_3}{E} + a_1 x_1 + b_1 x_2 + d_1 \end{aligned}$$

Bending (cont.)

- In "Strength of Materials" approach:

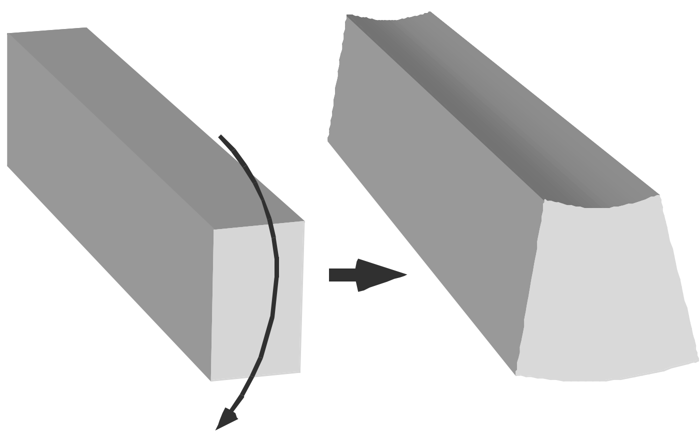
$$\sigma_{33} = \frac{Mx_1}{I}$$



but

$$u_2 = 0$$
$$u_1 \sim x_3^2$$
$$u_3 \sim x_1 x_3$$

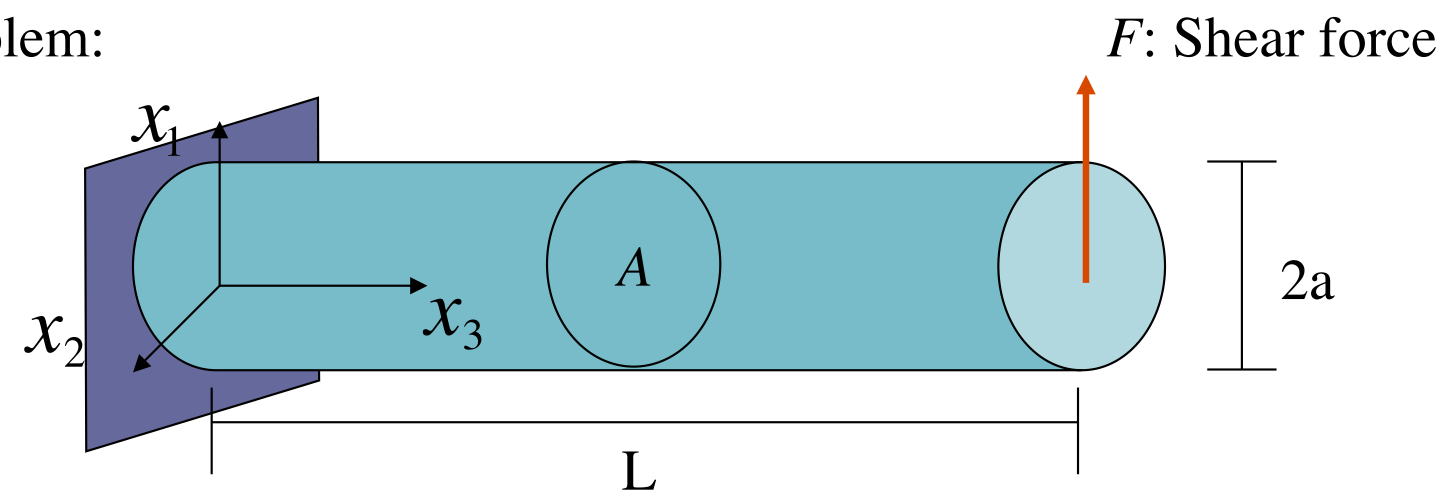
- Square cross section beam: Same solution, different *I*. Deformed shape:



anti-clastic curvature

Bending (cont.) (9.9)

- Problem:



- Solution (Done in class)...

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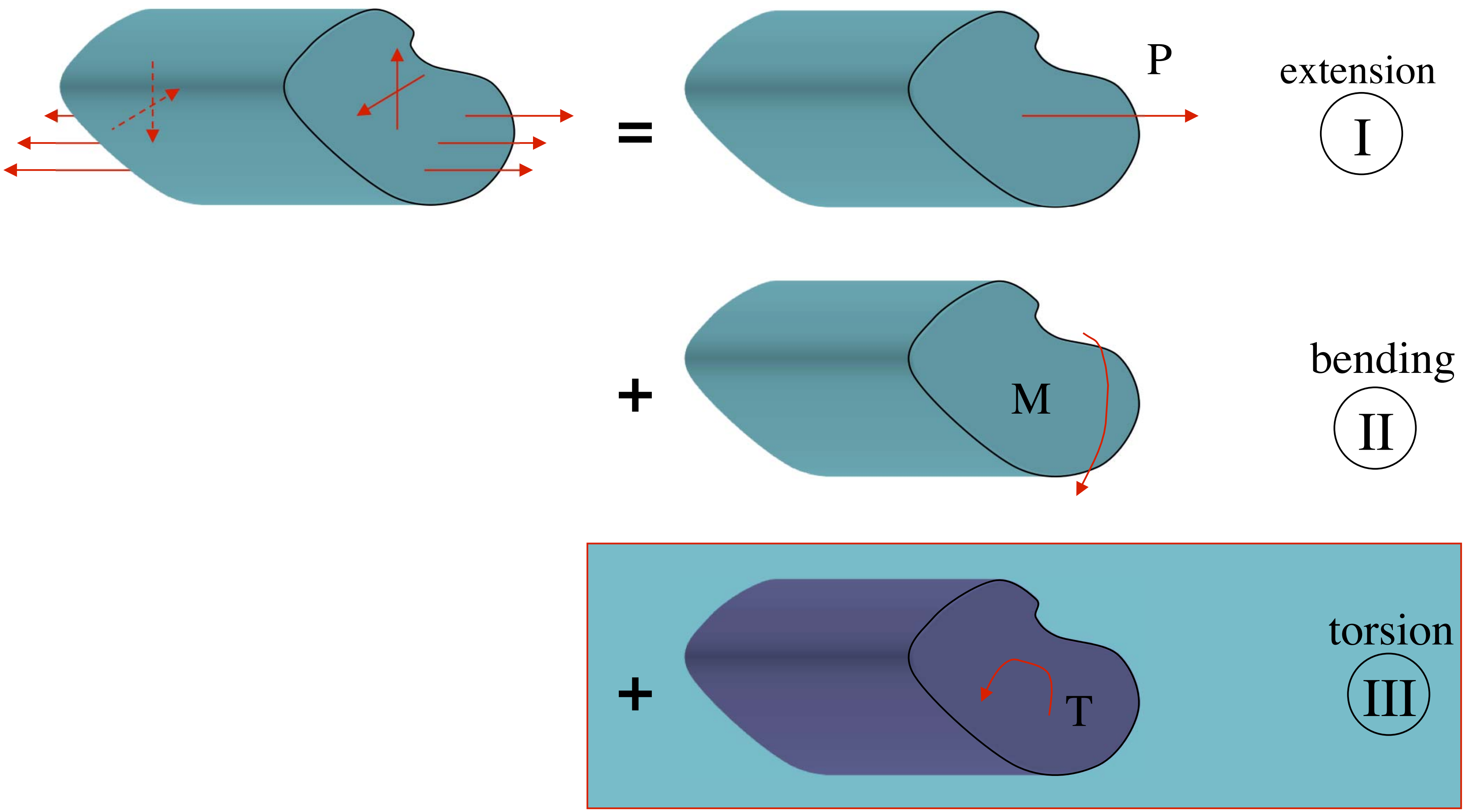
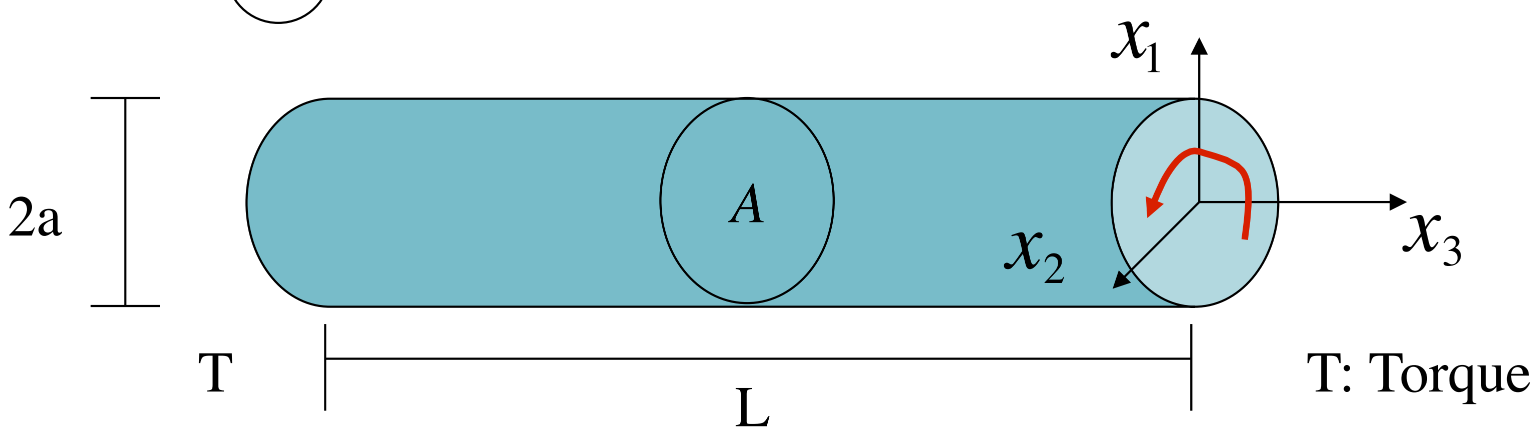
6.3 Torsion

- Problem A :



6.3.1 Torsion of circular sections

- Problem III Torsion:



- Solution (Done in class…)

Torsion of circular sections (cont.)

$$[\sigma_{ij}] = \begin{bmatrix} 0 & 0 & -\frac{Tx_2}{J} \\ 0 & 0 & \frac{Tx_1}{J} \\ -\frac{Tx_2}{J} & \frac{Tx_1}{J} & 0 \end{bmatrix}$$

$$[\varepsilon_{ij}] = \begin{bmatrix} 0 & 0 & -\frac{Tx_2}{2\mu J} \\ 0 & 0 & \frac{Tx_1}{2\mu J} \\ -\frac{Tx_2}{2\mu J} & \frac{Tx_1}{2\mu J} & 0 \end{bmatrix}$$

$$\{u_i\} = \begin{Bmatrix} -\frac{T}{\mu J} x_2 x_3 \\ \frac{T}{\mu J} x_1 x_3 \\ 0 \end{Bmatrix}$$

$$\boxed{\alpha = \frac{T}{(\mu J)}}$$

torsional rigidity

Torsion of circular sections (cont.)

- For a circular section $J = \dfrac{\pi}{2} a^4$

- Deformed shape $\Longrightarrow$ cylinder

$$\varepsilon_{kk} = ? \quad \Longrightarrow \quad \text{volume preserving deformation}$$

- Have assumed St. Venant's principle $\Longrightarrow$ need L >> D

Torsion of circular sections (cont.)

Cylindrical Coordinates (2.7, 3.7)

- Equilibrium:

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + \Theta = 0$$

$$\frac{\partial \sigma_{zr}}{\partial z} + \frac{\partial \sigma_{\theta z}}{r\partial \theta} + \frac{\partial \sigma_{z}}{\partial z} + \frac{\sigma_{zr}}{r} + Z = 0$$

R, Θ, Z body forces

- Strain-Displacement:

$$\varepsilon_r = \frac{\partial u}{\partial r} \qquad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{\partial u}{r\partial \theta}\right)$$

$$\varepsilon_\theta = \frac{u}{r} + \frac{\partial v}{r\partial \theta} \qquad \varepsilon_{\theta z} = \frac{1}{2}\left(\frac{\partial w}{r\partial \theta} + \frac{\partial v}{\partial z}\right)$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)$$

- Compatibility:

$$r\frac{\partial \varepsilon_r}{\partial r} - \frac{\partial^2 \varepsilon_r}{\partial \theta^2} - \frac{\partial}{\partial r}\left[r\left(r\frac{\partial \varepsilon_\theta}{\partial r} - \frac{\partial \gamma_{r\theta}}{\partial \theta}\right)\right] = 0$$

$$r\frac{\partial}{\partial z}\left[2\varepsilon_r - 2\frac{\partial}{\partial r}(r\varepsilon_\theta) + \frac{\partial \gamma_{r\theta}}{\partial \theta}\right] + \frac{\partial}{\partial \theta}\left[\frac{\partial}{\partial r}(r\gamma_{\theta z}) - \frac{\partial \gamma_{zr}}{\partial \theta}\right] = 0$$

$$\frac{\partial}{\partial z}\left[2r\frac{\partial \varepsilon_r}{\partial \theta} - \frac{\partial}{\partial r}(r\gamma_{r\theta})\right] + r^2\frac{\partial}{\partial r}\left\{\frac{1}{r}\left[\frac{\partial}{\partial r}(r\gamma_{\theta z}) - \frac{\partial \gamma_{zr}}{\partial \theta}\right]\right\} = 0$$

$$r^2\frac{\partial^2 \varepsilon_\theta}{\partial z^2} + r\frac{\partial \varepsilon_z}{\partial r} + \frac{\partial^2 \varepsilon_z}{\partial \theta^2} - r\frac{\partial}{\partial z}\left(\frac{\partial \gamma_{\theta z}}{\partial \theta} + \gamma_{zr}\right) = 0$$

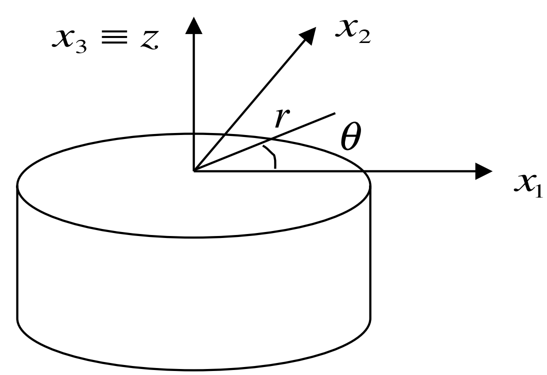
$$2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \varepsilon_z}{\partial \theta}\right) + \frac{\partial}{\partial z}\left[\frac{\partial \gamma_{r\theta}}{\partial z} - r\frac{\partial}{\partial r}\left(\frac{\gamma_{\theta z}}{r}\right) - \frac{1}{r}\frac{\partial \gamma_{zr}}{\partial \theta}\right] = 0$$

$$\frac{\partial^2 \varepsilon_r}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial r^2} - \frac{\partial^2 \gamma_{zr}}{\partial r \partial z} = 0$$

$$r \rightarrow u$$

$$\theta \rightarrow v$$

$$z \rightarrow w$$



Torsion of circular sections (cont.)

Spherical Coordinates (2.7, 3.7)

- Equilibrium:

$$\frac{\partial \sigma_r}{\partial r}+\frac{1}{r\sin\varphi}\frac{\partial \tau_{r\theta}}{\partial \theta}+\frac{1}{r}\frac{\partial \tau_{\varphi r}}{\partial \varphi}+\frac{2\sigma_r-\sigma_\theta-\sigma_\varphi+\tau_{\varphi r}\cos\varphi}{r}+R=0$$

$$\frac{\partial \tau_{r\theta}}{\partial r}+\frac{1}{r\sin\varphi}\frac{\partial \sigma_\theta}{\partial \theta}+\frac{1}{r}\frac{\partial \tau_{\theta\varphi}}{\partial \varphi}+\frac{3\tau_{r\theta}+2\tau_{\theta\varphi}\cot\varphi}{r}+\Theta=0$$

$$\frac{\partial \tau_{\varphi r}}{\partial r}+\frac{1}{r\sin\varphi}\frac{\partial \tau_{\theta\varphi}}{\partial \theta}+\frac{1}{r}\frac{\partial \sigma_\varphi}{\partial \varphi}+\frac{3\tau_{\varphi r}+(\sigma_\varphi-\sigma_\theta)\cot\varphi}{r}+\Phi=0$$

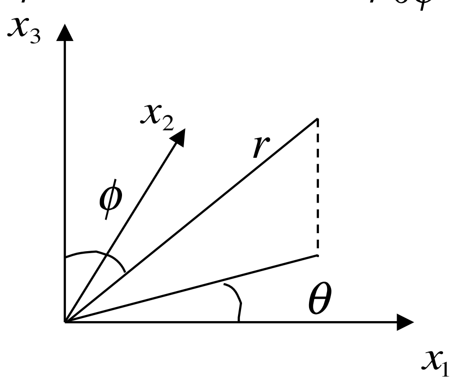
R, Θ, Φ body forces

- Strain-Displacement:

$$\varepsilon_r=\frac{\partial u}{\partial r}\qquad \gamma_{r\theta}=\frac{1}{r\sin\varphi}\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial r}-\frac{v}{r}$$

$$\varepsilon_\theta=\frac{1}{r\sin\varphi}\frac{\partial v}{\partial \theta}+\frac{u}{r}+\frac{\cot\varphi}{r}w\qquad \gamma_{\theta\varphi}=\frac{1}{r\sin\varphi}\frac{\partial w}{\partial \theta}+\frac{1}{r}\frac{\partial v}{\partial \varphi}-\frac{\cot\varphi}{r}v$$

$$\varepsilon_\varphi=\frac{1}{r}\frac{\partial w}{\partial \varphi}+\frac{u}{r}\qquad \gamma_{r\varphi}=\frac{1}{r}\frac{\partial u}{\partial \varphi}+\frac{\partial w}{\partial r}-\frac{w}{r}$$



$$r\rightarrow u$$
$$\theta\rightarrow v$$
$$\phi\rightarrow w$$

- Compatibility:

$$2\frac{\partial}{\partial \theta}\left(\frac{\partial \varepsilon_r}{\partial \varphi}-\varepsilon_r\cot\varphi\right)+\frac{\partial}{\partial r}\left(r\gamma_{r\theta}\cos\varphi-r\sin\varphi\frac{\partial \gamma_{r\theta}}{\partial \varphi}+r^2\sin\varphi\frac{\partial \gamma_{\theta\varphi}}{\partial r}-\frac{\partial \gamma_{\varphi r}}{\partial r}\right)=0$$

$$2\frac{\partial}{\partial \theta}\left(\varepsilon_r-r\frac{\partial \varepsilon_\varphi}{\partial r}+\frac{1}{2}\frac{\partial \gamma_{\varphi r}}{\partial \varphi}-\frac{1}{2}\gamma_{\varphi r}\cot\varphi\right)-r\frac{\partial}{\partial \varphi}\left(\sin\varphi\frac{\partial \gamma_{r\theta}}{\partial \varphi}\right)+r\frac{\cos 2\varphi}{\sin\varphi}\gamma_{r\theta}+$$
$$r^2\frac{\partial}{\partial r}\left(\sin\varphi\frac{\partial \gamma_{\theta\varphi}}{\partial \varphi}+2\gamma_{\theta\varphi}\cos\varphi\right)=0$$

$$2\sin^2\varphi\ \frac{\partial \varepsilon_r}{\partial \varphi}-2r\sin^2\varphi\frac{\partial}{\partial r}\left(\frac{\partial \varepsilon_\theta}{\partial \varphi}+(\varepsilon_\theta-\varepsilon_\varphi)\cot\varphi\right)+\frac{\partial^2}{\partial \theta\partial \varphi}(\gamma_{r\theta}\sin\varphi)+$$
$$r\sin\varphi\frac{\partial^2 \gamma_{\theta\varphi}}{\partial \theta\partial r}-\frac{\partial^2 \gamma_{\varphi r}}{\partial \theta^2}-2\gamma_{\varphi r}\sin^2\varphi=0$$

$$r\frac{\partial \varepsilon_r}{\partial r}-\frac{\partial^2 \varepsilon_r}{\partial \varphi^2}-\frac{\partial}{\partial r}\left(r^2\frac{\partial \varepsilon_\varphi}{\partial r}-r\frac{\partial \gamma_{\varphi r}}{\partial \varphi}\right)=0$$

$$\frac{\partial^2 \varepsilon_\varphi}{\partial \theta^2}+\frac{\partial}{\partial \varphi}\left(\sin^2\varphi\frac{\partial \varepsilon_\theta}{\partial \varphi}-\sin\varphi\frac{\partial \gamma_{\theta\varphi}}{\partial \theta}-\gamma_{\varphi r}\sin^2\varphi\right)-\sin\varphi\cos\varphi\left(\frac{\partial \varepsilon_\varphi}{\partial \varphi}+\gamma_{\varphi r}\right)+$$
$$\sin^2\varphi\left[r\frac{\partial}{\partial r}(\varepsilon_\theta+\varepsilon_\varphi)+2(\varepsilon_\varphi-\varepsilon_r)\right]-\sin\varphi\frac{\partial}{\partial \theta}\gamma_{r\theta}=0$$

$$r\sin^2\varphi\frac{\partial \varepsilon_r}{\partial r}-\frac{\partial^2 \varepsilon_r}{\partial \theta^2}-\sin^2\varphi\frac{\partial}{\partial r}\left(r^2\frac{\partial \varepsilon_\theta}{\partial r}\right)+\sin\varphi\frac{\partial^2}{\partial \theta\partial r}(r\gamma_{r\theta})-$$
$$\sin\varphi\cos\varphi\left[\frac{\partial\ \varepsilon_r}{\partial \varphi}-\frac{\partial}{\partial r}(r\gamma_{\varphi r})\right]=0$$

Torsion of circular sections (cont.)

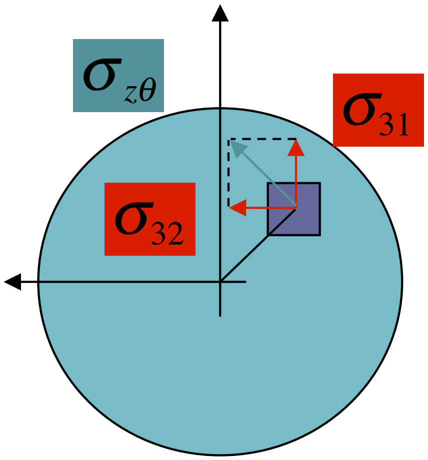
- Cylindrical coordinates

- Constitutive equations unchanged, i.e.,

$$E\varepsilon_{rr} = \sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz}) \qquad 2\mu\varepsilon_{r\theta} = \sigma_{r\theta}$$

$$E\varepsilon_{\theta\theta} = \sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz}) \qquad 2\mu\varepsilon_{rz} = \sigma_{rz}$$

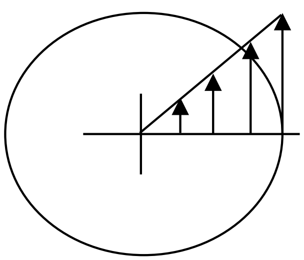
$$E\varepsilon_{zz} = \sigma_{zz} - \nu(\sigma_{rr} + \sigma_{\theta\theta}) \qquad 2\mu\varepsilon_{z\theta} = \sigma_{z\theta}$$



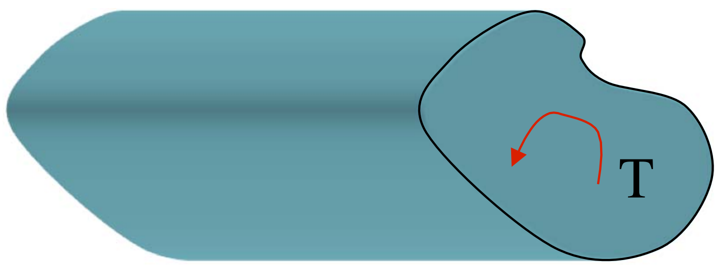
- Resultant is $\sigma_{z\theta} = \mu\alpha\sqrt{x_1^2 + x_2^2}$

$$\Rightarrow \quad \sigma_{z\theta} = \mu\alpha r$$

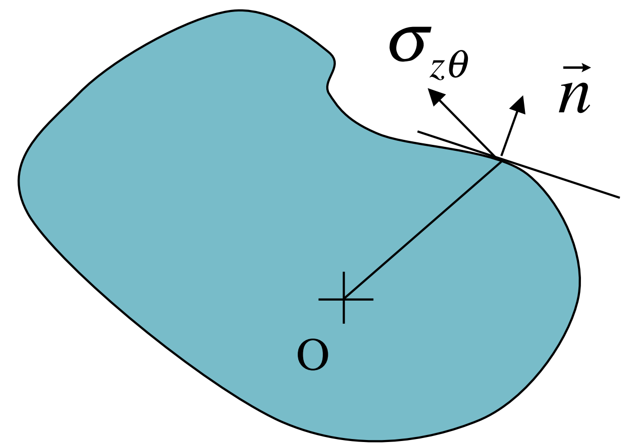
$$\alpha = \frac{T}{\mu J}$$



6.3.2 Torsion of non-circular sections (9.3)



- Radial stress NOT solution



- How is u_i affected?

$\Longrightarrow$ Warping (St. Venant)

- Try:

$$\left.\begin{aligned} u_1 &= -\alpha x_2 x_3 \\ u_2 &= \alpha x_1 x_3 \\ u_3 &= \alpha w(x_1, x_2) \end{aligned}\right\} \quad w(x_1, x_2): \text{ **Warping Function**}$$

Torsion of non-circular sections (cont.) (9.3.2)

- Strains and Stresses:

$$\left.\begin{aligned}\varepsilon_{13} &= \frac{1}{2}\alpha\left(w_{,1} - x_2\right) \\ \varepsilon_{23} &= \frac{1}{2}\alpha\left(w_{,2} + x_1\right)\end{aligned}\right\} \Rightarrow \begin{aligned}\sigma_{13} &= \mu\alpha\left(w_{,1} - x_2\right) \\ \sigma_{23} &= \mu\alpha\left(w_{,2} + x_1\right)\end{aligned}$$

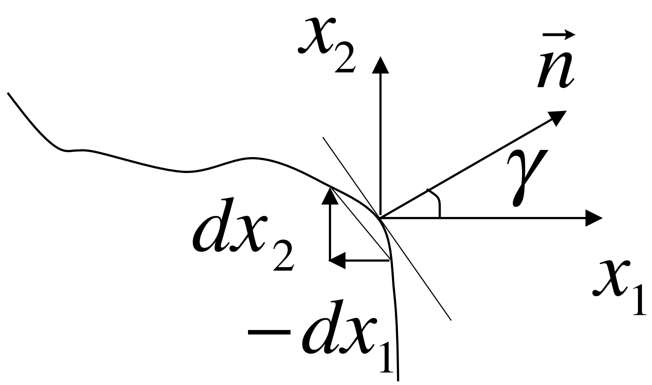
- Equilibrium:

$$\begin{aligned}\sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} &= 0 \\ \sigma_{12,1} + \sigma_{22,2} + \sigma_{23,3} &= 0 \\ \sigma_{13,1} + \sigma_{23,2} + \sigma_{33,3} &= \mu\alpha w_{,11} + \mu\alpha w_{,22} = 0\end{aligned}$$

$$\Rightarrow \boxed{w_{,11} + w_{,22} = \nabla^2 w\left(x_1, x_2\right) = 0}$$

Torsion of non-circular sections (cont.)

- Lateral surface b.c.s



$$\{\vec{n}\} = \{\cos\gamma, \sin\gamma, 0\}$$

$$= \left\{ \frac{dx_2}{ds}, -\frac{dx_1}{ds}, 0 \right\}$$

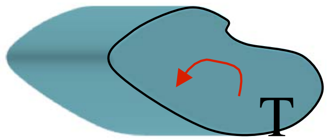
$$T_1 = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 = 0$$

$$T_2 = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3 = 0$$

$$T_3 = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3 = \mu\alpha\left(w_{,1} - x_2\right)\cos\gamma + \mu\alpha\left(w_{,2} + x_1\right)\sin\gamma$$

$$\boxed{\therefore\ \mu\alpha\left(w_{,1} - x_2\right)\cos\gamma + \mu\alpha\left(w_{,2} + x_1\right)\sin\gamma = 0}$$

Torsion of non-circular sections (cont.) (9.3.1)



- Alternative solution approach: Stress formulation (Prandtl)

- Assume **Stress Function**, $\Phi(x_1,x_2)$, such that:

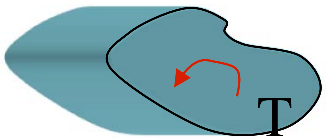
$$\left.\begin{aligned}\sigma_{13} &= \Phi_{,2}\\ \sigma_{23} &= -\Phi_{,1}\end{aligned}\right\} \Rightarrow$$

Equilibrium Automatically satisfied

$$\varepsilon_{13} = \frac{1}{2\mu}\Phi_{,2} \qquad \varepsilon_{23} = -\frac{1}{2\mu}\Phi_{,1}$$

- Compatibility yields: $\boxed{\Phi_{,11} + \Phi_{,22} = -2\mu\alpha}$

Torsion of non-circular sections (cont.)



- Relation to warping function:

$$\sigma_{13} = \Phi_{,2} = \mu\alpha\left(w_{,1} - x_2\right)$$

$$\sigma_{23} = -\Phi_{,1} = \mu\alpha\left(w_{,2} + x_1\right)$$

- Boundary condition - Lateral surface:

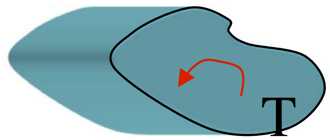
$$\mu\alpha\left(w_{,1} - x_2\right)\cos\gamma - \mu\alpha\left(w_{,2} - x_1\right)\sin\gamma = 0$$

$$\Rightarrow \quad \Phi_{,2}\frac{dx_2}{ds} + \Phi_{,1}\frac{dx_1}{ds} = 0 \quad \Rightarrow \frac{d\Phi}{ds} = 0 \quad \Rightarrow \quad \boxed{\Phi\left(x_1, x_2\right) = \text{const.}}$$

- Boundary condition - End Face:

$$T = \int_A \left(\sigma_{32}x_1 - \sigma_{31}x_2\right)dA \quad \Rightarrow \quad \boxed{T = 2\int_A \Phi\left(x_1, x_2\right)dA}$$

Torsion of non-circular sections (cont.)



- Need to solve:

compatibility

$$\Phi_{,11} + \Phi_{,22} = -2\mu\alpha$$

lateral bc

$$\Phi(x_1, x_2) = \text{const.}$$

$$T = 2\int_A \Phi(x_1, x_2)\,dA$$

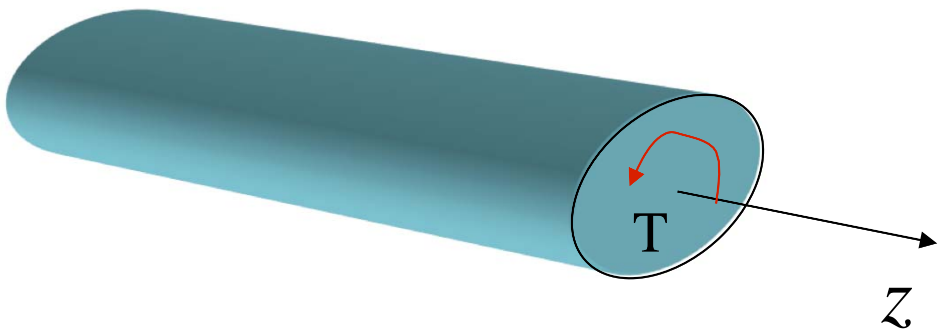
end face bc

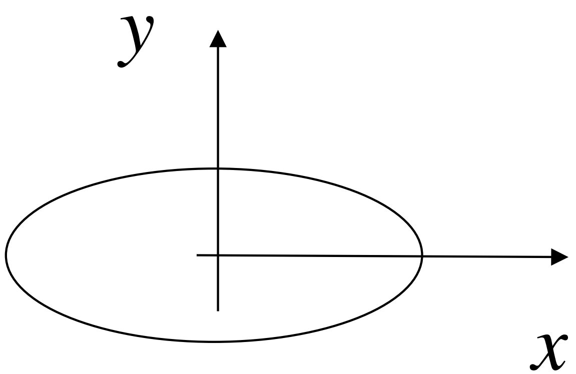
$$\left.\begin{aligned} &\Phi_{,11} + \Phi_{,22} = -2\mu\alpha \\ &\Phi(x_1, x_2) = \text{const.} \\ &T = 2\int_A \Phi(x_1, x_2)\,dA \end{aligned}\right\} \Rightarrow$$

Usual procedure to find *Φ*

Torsion of non-circular sections (cont.)

- Example: Elliptical cross section





$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

- Solution (Done in class… Example 9-1 in textbook)
- See also examples 9-2 to 9-5

END OF CHAPTER 6