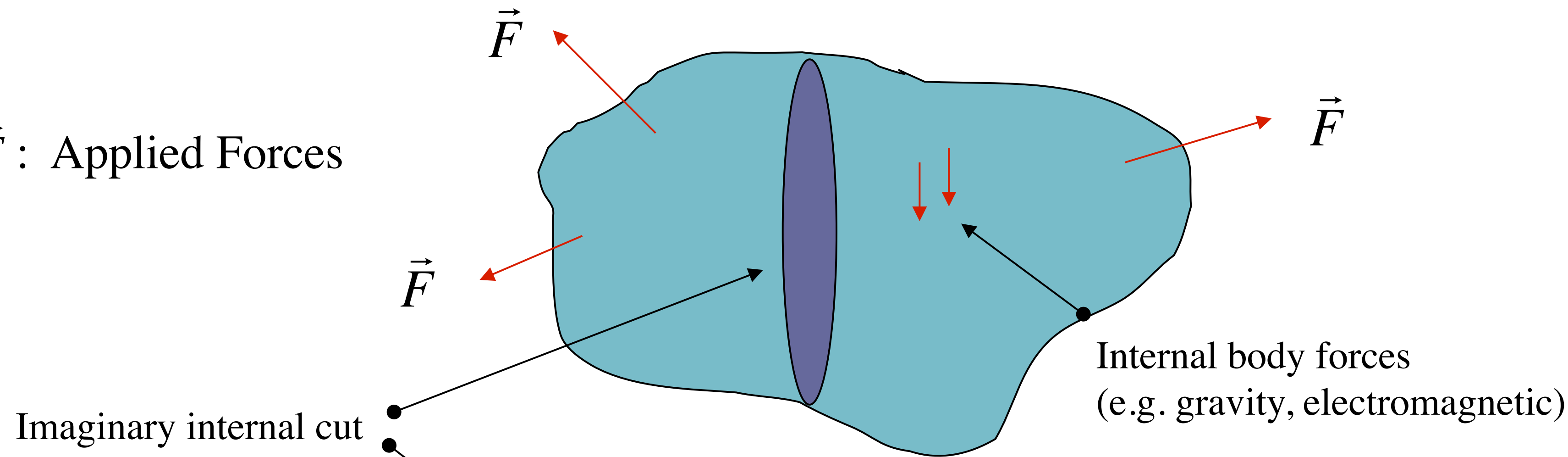
Chapter 3 - Kinetics: Traction, Stress and Equilibrium

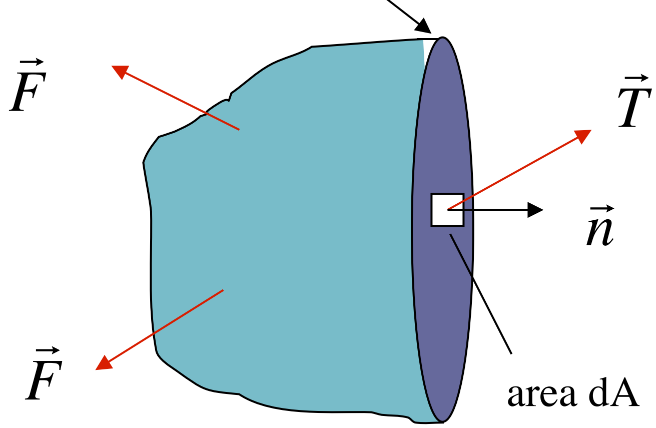
- Table of contents
  - **The stress tensor**
    - Traction components
    - Relation between traction and stress - Cauchy's equation
    - Normal and tangential components
  - **Equilibrium equations**
    - Heuristic approach
    - Conservation of linear momentum
    - Conservation of angular momentum
  - **Principal stresses**
    - Principal stresses
    - Principal directions
    - Stress transformation
    - Special cases

Traction Components

- Interaction between deformable bodies? → Surface loads

$\vec{F}$: Applied Forces





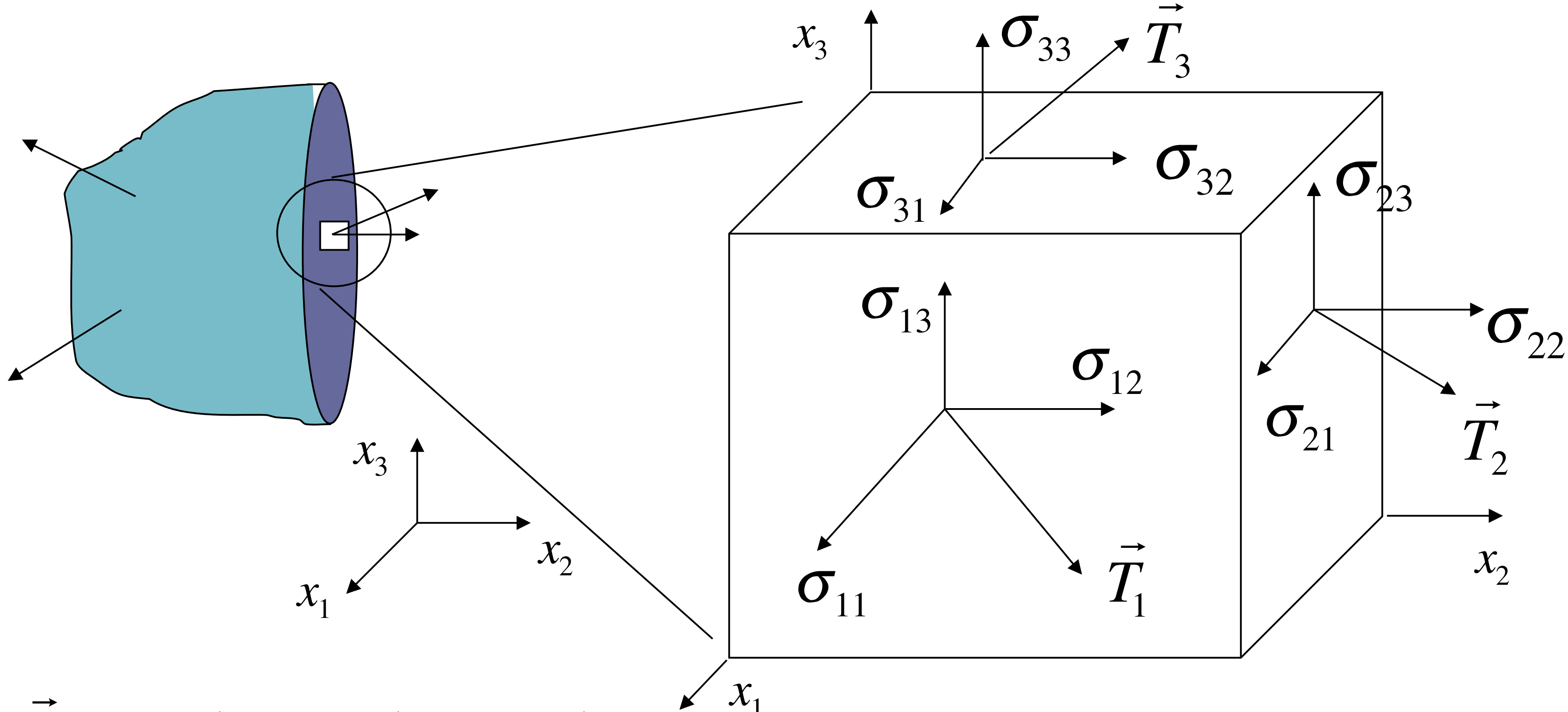
- Traction:

$$\vec{T} = \lim_{dA \to 0} \left(\frac{d\vec{F}}{dA} \right)$$

Normal vs. shear

Relation between Traction and Stress

- Cut out an infinitesimal cube at the point in question, and align with axes



$$\vec{T}_1 = \sigma_{11}\vec{e}_1 + \sigma_{12}\vec{e}_2 + \sigma_{13}\vec{e}_3$$

$$\vec{T}_2 = \sigma_{21}\vec{e}_1 + \sigma_{22}\vec{e}_2 + \sigma_{23}\vec{e}_3$$

$$\vec{T}_3 = \sigma_{31}\vec{e}_1 + \sigma_{32}\vec{e}_2 + \sigma_{33}\vec{e}_3$$

or $\vec{T}_i = \sigma_{ij}\vec{e}_j$

Traction and Stress (cont.)

- Coefficients σ_{ij} are the **stress (tensor) components**

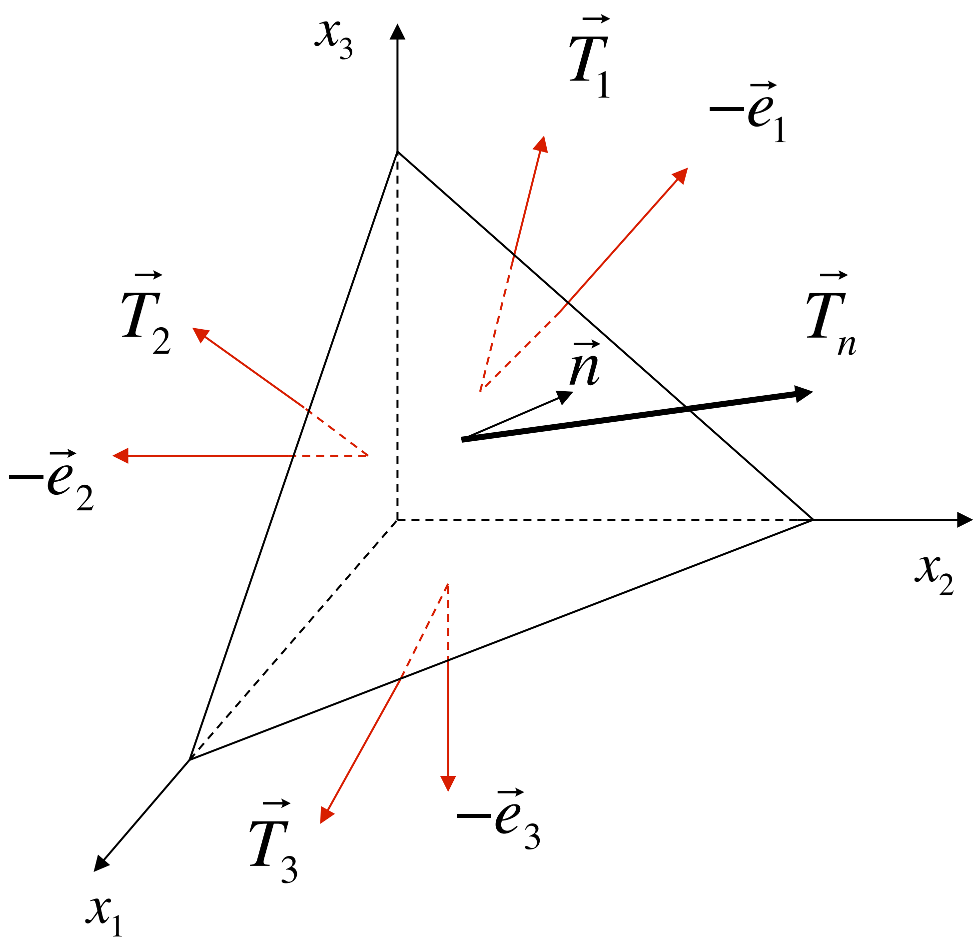
- Naming convention: σ_{ij}

  Face on which component acts → i; Direction in which component acts → j

- Sign convention:
  - Normal (or extensional) components $\sigma_{11},\ \sigma_{22},\ \sigma_{33} \begin{cases} > 0 & \text{Tension} \\ < 0 & \text{Compression} \end{cases}$
  - Shearing components $\sigma_{ij}\quad i \neq j$: positive if in positive direction on a positive face.

Traction and Stress (cont.)

- What about traction on an arbitrary surface with normal **n** ?



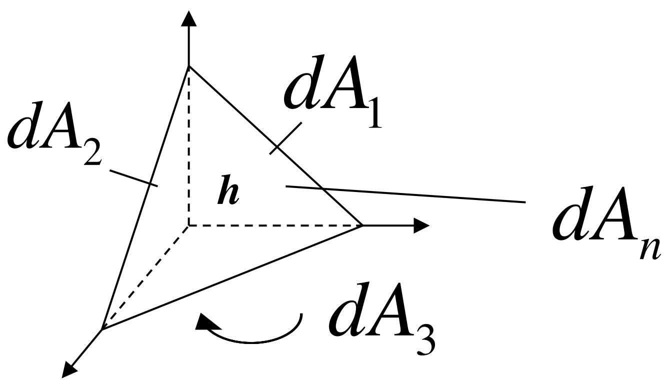
$\vec{T}_n$ = Traction acting on cut plane

$\vec{n} = n_1\vec{e}_1 + n_2\vec{e}_2 + n_3\vec{e}_3$ Cut plane

- For equilibrium:

$$\sum \vec{F}_i = \vec{0}$$

Traction and Stress (cont.)



Force on cut plane $= \vec{T}_n dA_n$ (no sum)

Force on other faces $= \vec{T}_i dA_i$ (no sum)

$$\left.\begin{array}{l} \sum \vec{F}_i = \vec{0} \quad \Rightarrow \quad \vec{T}_n dA_n + \vec{T}_i dA_i + \vec{f} dV = \vec{0} \quad \text{(no sum on n)} \\[1em] \bullet \text{ Area relation:} \quad dA_i = dA_n \cos\left(\vec{n} \cdot \vec{e}_i\right) = -(dA_n) n_i \end{array}\right\} \Rightarrow$$

$\vec{f}$ = body force/unit volume

Traction and Stress (cont.)

$$\Rightarrow \quad \vec{T}_n dA_n - \vec{T}_i n_i dA_n + \vec{f}\left(\frac{h}{3}\right) dA_n = \vec{0} \quad \text{(no sum on n)}$$

$$\Rightarrow \quad \vec{T}_n - \vec{T}_i n_i + \vec{f}\left(\frac{h}{3}\right) = \vec{0}$$

But, as a vector: $\vec{T}_n = T_i^n \vec{e}_i$

$$\underset{h \to 0}{\Rightarrow} \quad T_i^n \vec{e}_i = \vec{T}_i n_i$$

$$= \vec{T}_j n_j$$

dummy

but $\vec{T}_j = \sigma_{ji} \vec{e}_i$

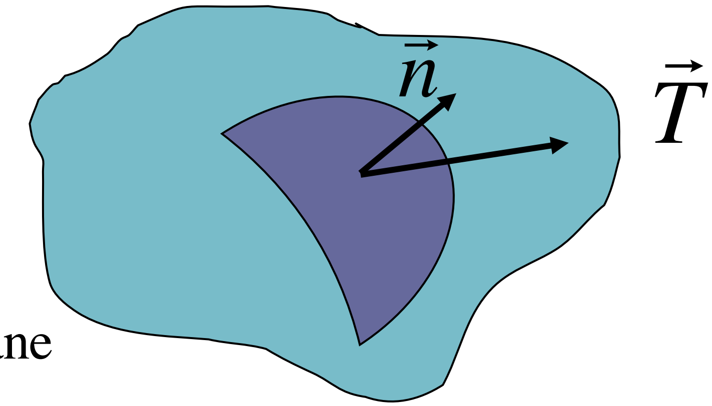
$$\Rightarrow \quad T_i^n \vec{e}_i = \sigma_{ji} \vec{e}_i n_j$$

$$\boxed{\therefore \quad T_i^n = \sigma_{ji} n_j}$$

Cauchy's Equation

Cauchy's Equation

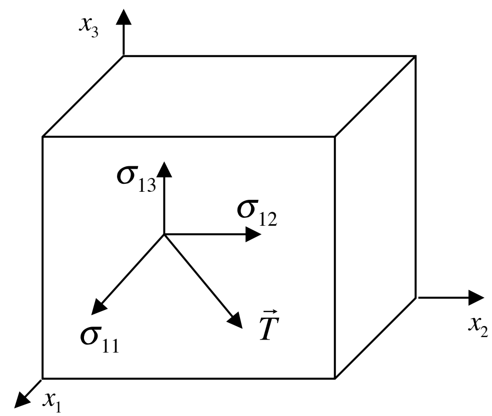
$$T_i = \sigma_{ji} n_j$$



T_i : components of traction on point in cut plane

n_j : componets of normal to cut plane

σ_{ji} : stress components at point



• In expanded form:

$$T_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$$
$$T_2 = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3$$
$$T_3 = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3$$

The Stress Tensor

$$T_i = \sigma_{ji} n_j$$

• It can be shown that since T_i and n_i are vector components, σ_{ij} **must** form the components of a second order tensor.

• Stress is therefore a second order tensor and transforms as:

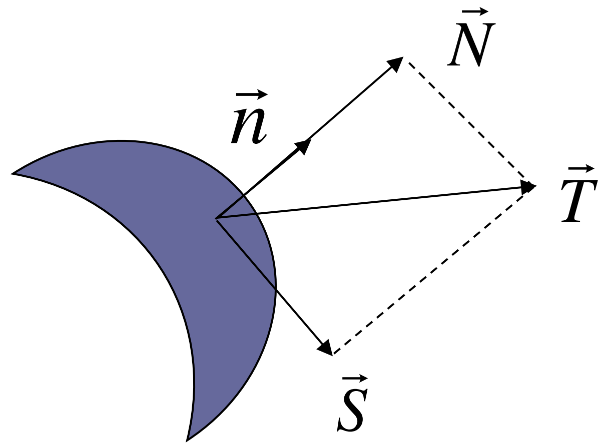
$$\sigma'_{ij} = \alpha_{ik}\alpha_{jl}\sigma_{kl} \quad or \quad [\sigma'] = [R][\sigma][R]^T$$

• Thus σ_{ij} and geometry (n_i) are sufficient to define the internal stress state of an object at a point - Note σ_{ij} will in general be a tensor field, i.e. it will depend on (x_1, x_2, x_3).

Normal and Tangential Components

$\vec{n}$: unit nornal

$\vec{T}$: traction



- Resolve $\vec{T}$ along a normal and a tangential component, i.e. // and $\perp$ to $\vec{n}$
- Normal component: $\boxed{\vec{N} = \sigma_{norm}\vec{n}}$

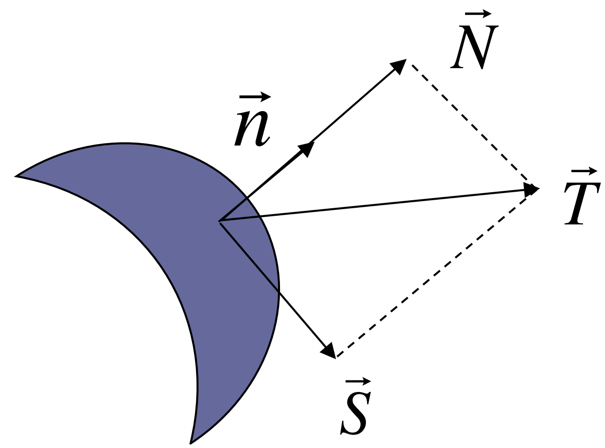
$$\sigma_{norm} = \vec{T} \cdot \vec{n}$$
$$= T_i n_i$$
$$= (\sigma_{ji} n_j) n_i$$

Cauchy's eqn.

Normal and Tangential Components (cont.)

- Shear (tangential) component:

$$\vec{S} = \vec{T} - \vec{N}$$



- From Pythagorean theorem:

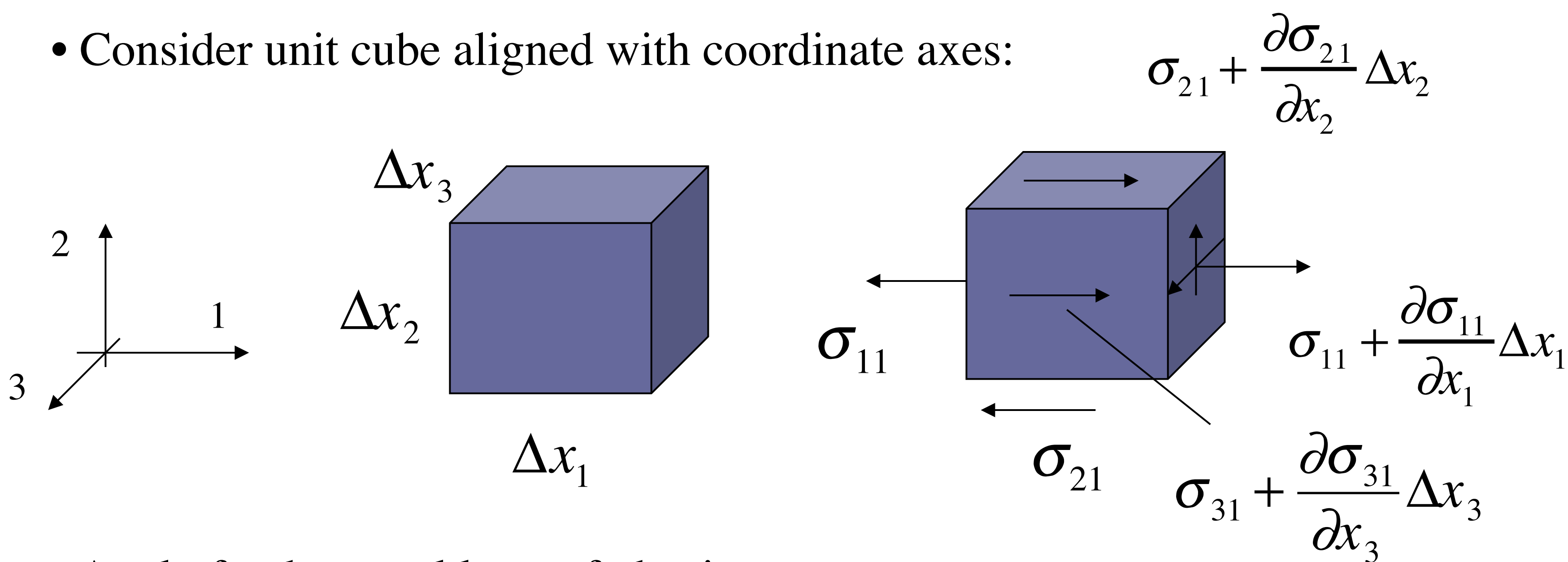
$$\sigma_{sh} = \sqrt{T_i T_i - \sigma_{norm}^2}$$

- Examples: Cartesian, Polar...

Chapter 3 - Kinetics: Traction, Stress and Equilibrium

Heuristic Approach

- Consider unit cube aligned with coordinate axes:

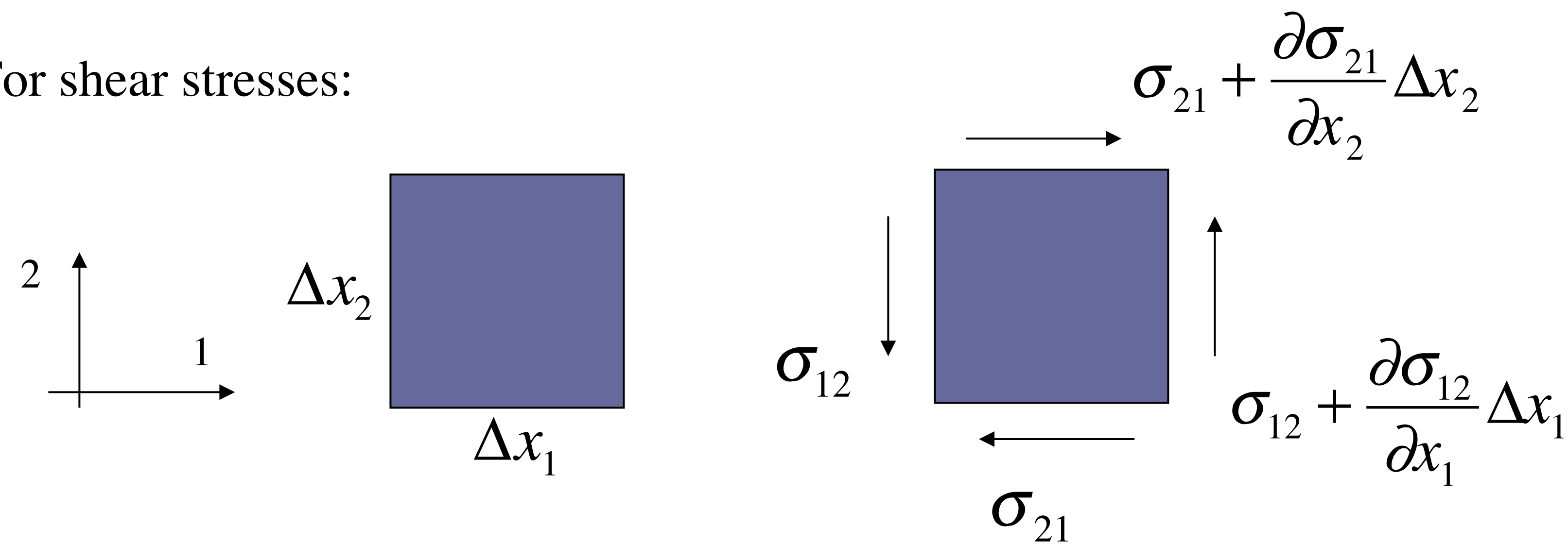


- Apply fundamental laws of physics:

$$\boxed{\sigma_{11,1} + \sigma_{21,2} + \sigma_{31,3} + f_x = 0}$$

Heuristic Approach (cont.)

- For shear stresses:



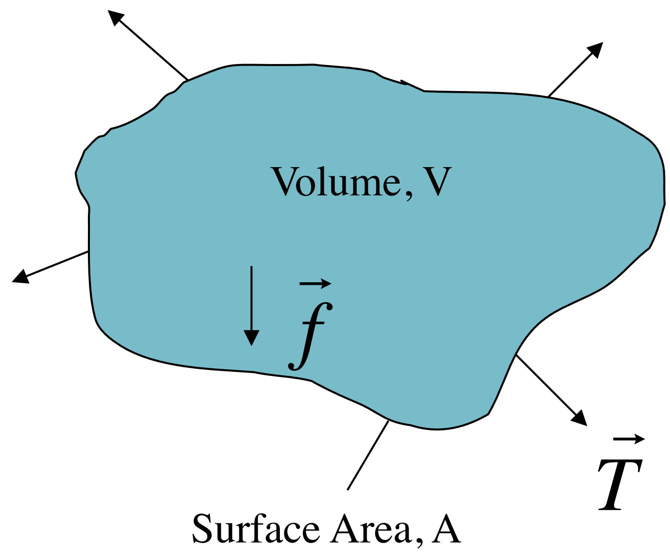
- Apply fundamental laws of physics:

$$\boxed{\sigma_{12} = \sigma_{21}}$$

Conservation of Linear Momentum

- What governs distribution of stress in a body? → Conservation principles

- Apply conservation of linear momentum:



$$\int_V \vec{f}\,dV + \int_A \vec{T}\,dA = \int_V \rho\ddot{\vec{u}}\,dV$$

$\vec{f}$: body force density ρ : mass density

$\vec{T}$: surface traction $\ddot{\vec{u}}$: acceleration vector

Conservation of Linear Momentum (cont.)

- For static problems: $\ddot{\vec{u}} = 0$

- In component form: $\int_V \vec{f}\, dV + \int_A \vec{T}\, dA = 0$

$$\int_V f_i \vec{e}_i dV + \int_A T_i \vec{e}_i dA = 0$$

Cauchy's Equation

$$T_i = \sigma_{ji} n_j$$

$$\Rightarrow \int_V f_i dV + \int_A \sigma_{ji} n_j dA = 0$$

Conservation of Linear Momentum (cont.)

- Use divergence theorem

$$\int_A \vec{G} \cdot \vec{n} dA = \int_V \nabla \cdot \vec{G} dV$$

$$\int_A G_i n_i dA = \int_V G_{i,i} dV$$

- Apply to

$$\int_V f_i dV + \int_A \sigma_{ji} n_j dA = 0$$

$$\Rightarrow \int_V f_i dV + \int_V \sigma_{ji,j} dV = 0$$

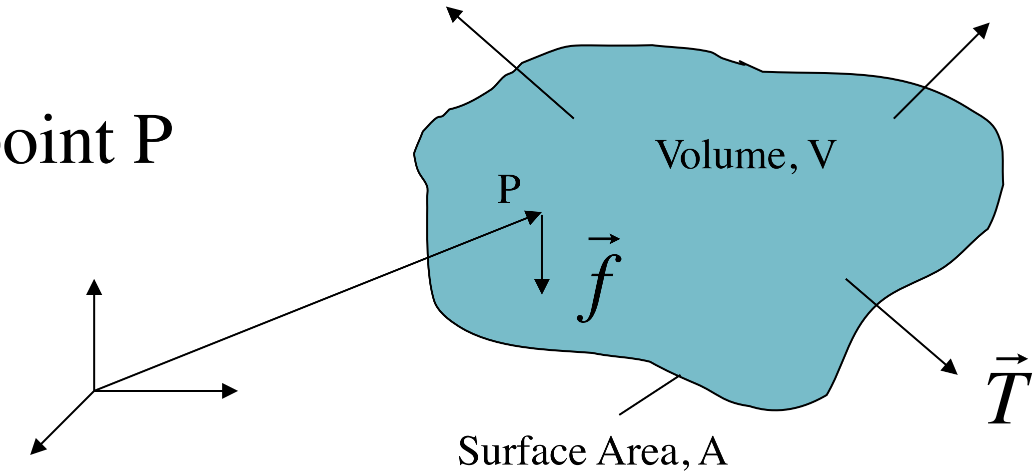
$$\Rightarrow \int_V (f_i + \sigma_{ji,j}) dV = 0 \qquad \forall \ (x_1, x_2, x_3)$$

$$\Rightarrow \boxed{\sigma_{ji,j} + f_i = 0}$$

Equilibrium Equations!

Conservation of Angular Momentum

$\vec{r}$: Position vector to point P



$$\int_V \left(\vec{r} \times \vec{f}\right) dV + \int_A \left(\vec{r} \times \vec{T}\right) dA = \int_V \left(\vec{r} \times \rho \ddot{\vec{u}}\right) dV$$

$$\int_V \varepsilon_{ijk} x_j f_k dV + \int_A \varepsilon_{ijk} x_j \sigma_{lk} n_l dA = 0$$

Cauchy's Equation

Conservation of Angular Momentum (cont.)

- Apply divergence theorem:

$$\int_V \varepsilon_{ijk} x_j f_k dV + \int_V \left(\varepsilon_{ijk} x_j \sigma_{lk}\right)_{,l} dV = 0$$

- Also

$$\left.\begin{aligned} &\left(\varepsilon_{ijk} x_j \sigma_{lk}\right)_{,l} = \varepsilon_{ijk} x_{j,l} \sigma_{lk} + \varepsilon_{ijk} x_j \sigma_{lk,l} \\ &x_{j,l} = \frac{\partial x_j}{\partial x_l} = \begin{Bmatrix} 1 & \text{if } j = l \\ 0 & \text{if } j \neq l \end{Bmatrix} = \delta_{jl} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \int_V \varepsilon_{ijk} \left(x_j f_k + \delta_{jl} \sigma_{lk} + x_j \sigma_{lk,l}\right) dV = 0$$

Conservation of Angular Momentum (cont.)

$$\int_V \varepsilon_{ijk}\left(x_j f_k + \delta_{jl}\sigma_{lk} + x_j \sigma_{lk,l}\right) dV = 0$$

$$0$$

$$\Rightarrow \quad \int_V \varepsilon_{ijk}\delta_{jl}\sigma_{lk}\, dV = 0$$

$$\Rightarrow \quad \int_V \varepsilon_{ijk}\sigma_{jk}\, dV = 0 \quad \forall\ \left(x_1, x_2, x_3\right) \quad \Rightarrow \quad \varepsilon_{ijk}\sigma_{jk} = 0$$

- Expanding:

$$i = 1: \quad \sigma_{23} - \sigma_{32} = 0$$

$$i = 2: \quad \sigma_{31} - \sigma_{13} = 0$$

$$i = 3: \quad \sigma_{12} - \sigma_{21} = 0$$

$$\boxed{\therefore \quad \sigma_{ij} = \sigma_{ji}}$$

Symmetric stress tensor!

Conservation of Angular Momentum (cont.)

Since $\sigma_{ij} = \sigma_{ji}$

- Cauchy's Equation $$T_i = \sigma_{ij} n_j$$

- Equilibrium Equations $$\sigma_{ij,j} + f_i = 0$$

$$\sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} + f_1 = 0$$
$$\sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} + f_2 = 0$$
$$\sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} + f_3 = 0$$

Notes:

- # of equations ?
- # of unknowns ?
- With no body forces all constant stress fields satisfy equilibrium

- Examples...

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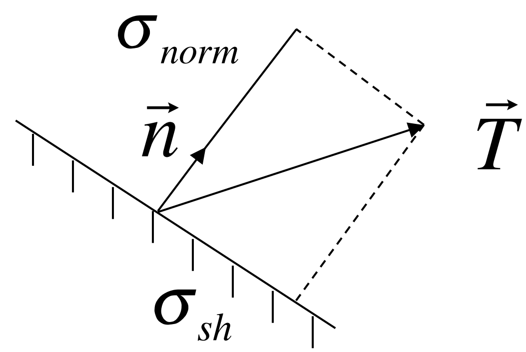
  - ★ **Principal stresses**
    - Principal stresses
    - Principal directions
    - Stress transformation
    - Special cases

Principal Stresses

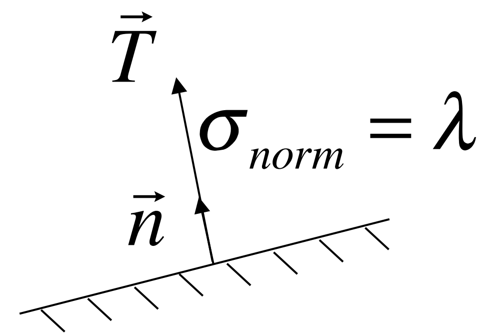


- Recall: $\vec{T} = \vec{N} + \vec{S}$

  $\vec{T}$: Surface traction

  $\vec{N}$: Normal component, $\sigma_{norm} = \left|\vec{N}\right|$

  $\vec{S}$: Shear component, $\sigma_{sh} = \left|\vec{S}\right|$



- Seek to find planes on which: $\sigma_{sh} = 0$

$$\Rightarrow \quad \vec{T} = \lambda \vec{n}$$

$$\lambda : \text{ scalar}$$

But in general $\vec{T} = \sigma \vec{n}$

$$\Rightarrow \quad \sigma \vec{n} = \lambda \vec{n}$$

$$\Rightarrow \quad \boxed{\left(\sigma_{ij} - \lambda \delta_{ij}\right) n_j = 0}$$

an eigenvalue problem

Principal Stresses (cont.)

$$\left(\sigma_{ij} - \lambda\delta_{ij}\right)n_j = 0$$

- 3 homogeneous algebraic eqns. with unknowns $\lambda, \quad n_j$

Principal stresses

Principal directions

$$\begin{bmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \lambda \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Principal Stresses (cont.)

- For non-trivial solutions: ?

$$\begin{vmatrix} \sigma_{11}-\lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\lambda & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\lambda \end{vmatrix} = 0$$

- Cubic equation for λ: $\lambda^3 - Q_1\lambda^2 + Q_2\lambda - Q_3 = 0$

- Is solution guaranteed? **σ** is symmetric $\Rightarrow$ Yes! Always 3 real roots

$$\lambda_i = \sigma_i$$ Principal stresses

$$\vec{n}^{(k)} = n_i^{(k)}\vec{e}_i$$ Principal unit directions (one for each p.s.)

Principal Stresses (cont.)

- Characteristic equation $\lambda^3 - Q_1\lambda^2 + Q_2\lambda - Q_3 = 0$

$$Q_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_{ii} = tr(\underline{\sigma})$$

$$Q_2 = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{13} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{23} & \sigma_{33} \end{vmatrix} = \frac{1}{2}\varepsilon_{mik}\varepsilon_{mjl}\sigma_{ij}\sigma_{kl}$$

$$= \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}\right) = \frac{1}{2}\left[(tr\boldsymbol{\sigma})^2 - tr\boldsymbol{\sigma}^2\right]$$

$$Q_3 = \det(\underline{\sigma}) = \frac{1}{6}\varepsilon_{mik}\varepsilon_{njl}\sigma_{ij}\sigma_{kl}\sigma_{mn}$$

Q_i : Invariants of $\underline{\sigma}$, i.e. do NOT depend on frame

Principal Directions

- Solve cubic equation to find eigenvalues (i.e. principal stresses) λ_k

- For principal directions need to solve:

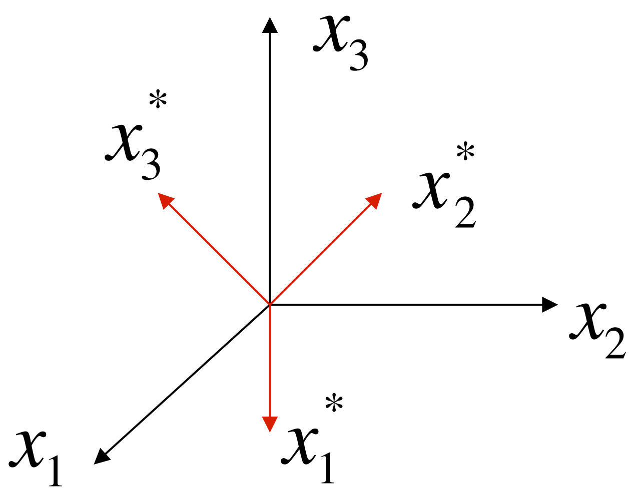
$$\left(\sigma_{ij} - \lambda_k \delta_{ij}\right) n_j^{(k)} = 0 \qquad \text{no sum}$$

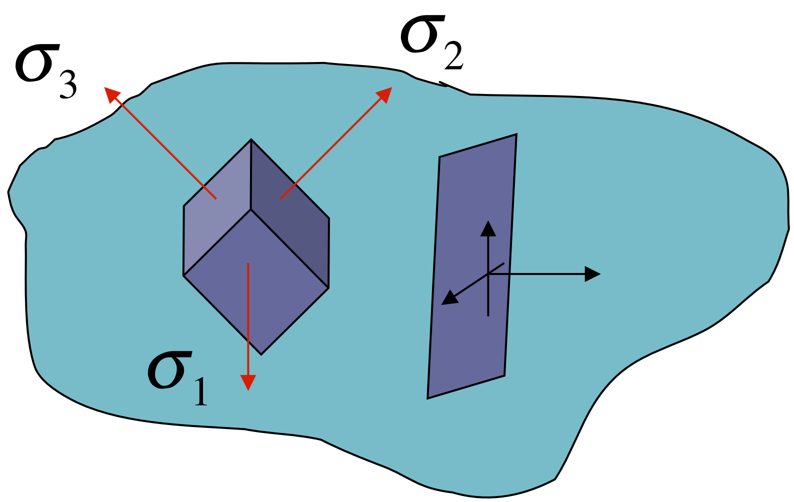
Constraint: $n_j^{(k)} n_j^{(k)} = 1 \qquad \text{no sum on } k$

- Notes:
  - Since the stress tensor is symmetric, eigenvalues are always real
  - Two (or three) of the principal stresses may be equal
  - The principal directions (i.e. eigenvectors) are always orthogonal
  - Usually we order the principal stresses as $\sigma_1 > \sigma_2 > \sigma_3$

- Examples...

Stress Transformation





- Stress transformation $\sigma'_{ij} = \alpha_{ik}\alpha_{jl}\sigma_{kl}$ $\quad [\sigma'] = [R][\sigma][R]^T$

$$\alpha_{ij} = \cos\left(\vec{e}'_i, \vec{e}_j\right)$$

i.e. coefficients of each eigenvector

Stress Transformation (cont.)

- To rotate to a **principal frame** use:

$$[R^*] = \begin{bmatrix} n_1^{(1)} & n_2^{(1)} & n_3^{(1)} \\ n_1^{(2)} & n_2^{(2)} & n_3^{(2)} \\ n_1^{(3)} & n_2^{(3)} & n_3^{(3)} \end{bmatrix}$$

$n_i^{(k)}$: directional cosines of eigen vectors

- Stress tensor components in a principal frame:

$$[\underline{\sigma}^*] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

- No shear stress in a p.f.
- Are p.s. frame dependent?
- Invariants:

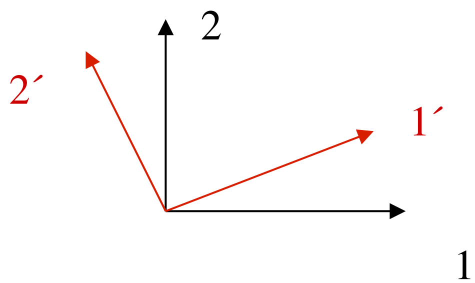
$$Q_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$Q_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

$$Q_3 = \sigma_1\sigma_2\sigma_3$$

Stress Transformation (cont.)

- In two dimensions we can use **Mohr's Circle**:



$$\sigma'_{11} = \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta + \sigma_{12}\sin 2\theta$$

$$\sigma'_{22} = \sigma_{11}\sin^2\theta + \sigma_{22}\cos^2\theta - \sigma_{12}\sin 2\theta$$

$$\sigma'_{12} = \frac{1}{2}(\sigma_{22} - \sigma_{11})\sin 2\theta + \sigma_{12}\cos 2\theta$$

- If (1,2) are principal axes: $\sigma_{11} = \sigma_1, \quad \sigma_{22} = \sigma_2, \quad \sigma_{12} = 0$

$$\sigma'_{11} = \sigma_1\cos^2\theta + \sigma_2\sin^2\theta$$

$$\sigma'_{22} = \sigma_1\sin^2\theta + \sigma_2\cos^2\theta$$

$$\sigma'_{12} = \frac{1}{2}(\sigma_2 - \sigma_1)\sin 2\theta$$

$$\left\{ \begin{aligned} \cos^2\theta &= \frac{1}{2}(1 + \cos 2\theta) \\ \sin^2\theta &= \frac{1}{2}(1 - \cos 2\theta) \end{aligned} \right.$$

Stress Transformation (cont.)

$$\sigma'_{11} = \sigma = \frac{\sigma_1}{2} + \frac{\sigma_2}{2} + \frac{\sigma_1}{2}\cos 2\theta - \frac{\sigma_2}{2}\cos 2\theta$$

$$= \frac{1}{2}(\sigma_1 + \sigma_2) + \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta$$

and

$$\sigma'_{12} = \tau = \frac{1}{2}(\sigma_2 - \sigma_1)\sin 2\theta$$

$$\Rightarrow \boxed{\left[\sigma - \left(\frac{\sigma_1 + \sigma_2}{2}\right)\right]^2 + \tau^2 = \left(\frac{\sigma_1 - \sigma_2}{2}\right)^2}$$

$$\sin^2 2\theta + \cos^2 2\theta = 1$$

- Equation of a circle in (σ, τ) → **Mohr's Circle**

- In 2D:

$$\sigma_{1,2} = \frac{\sigma_{11} + \sigma_{22}}{2} \pm \sqrt{\sigma_{12}^2 + \left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2}$$

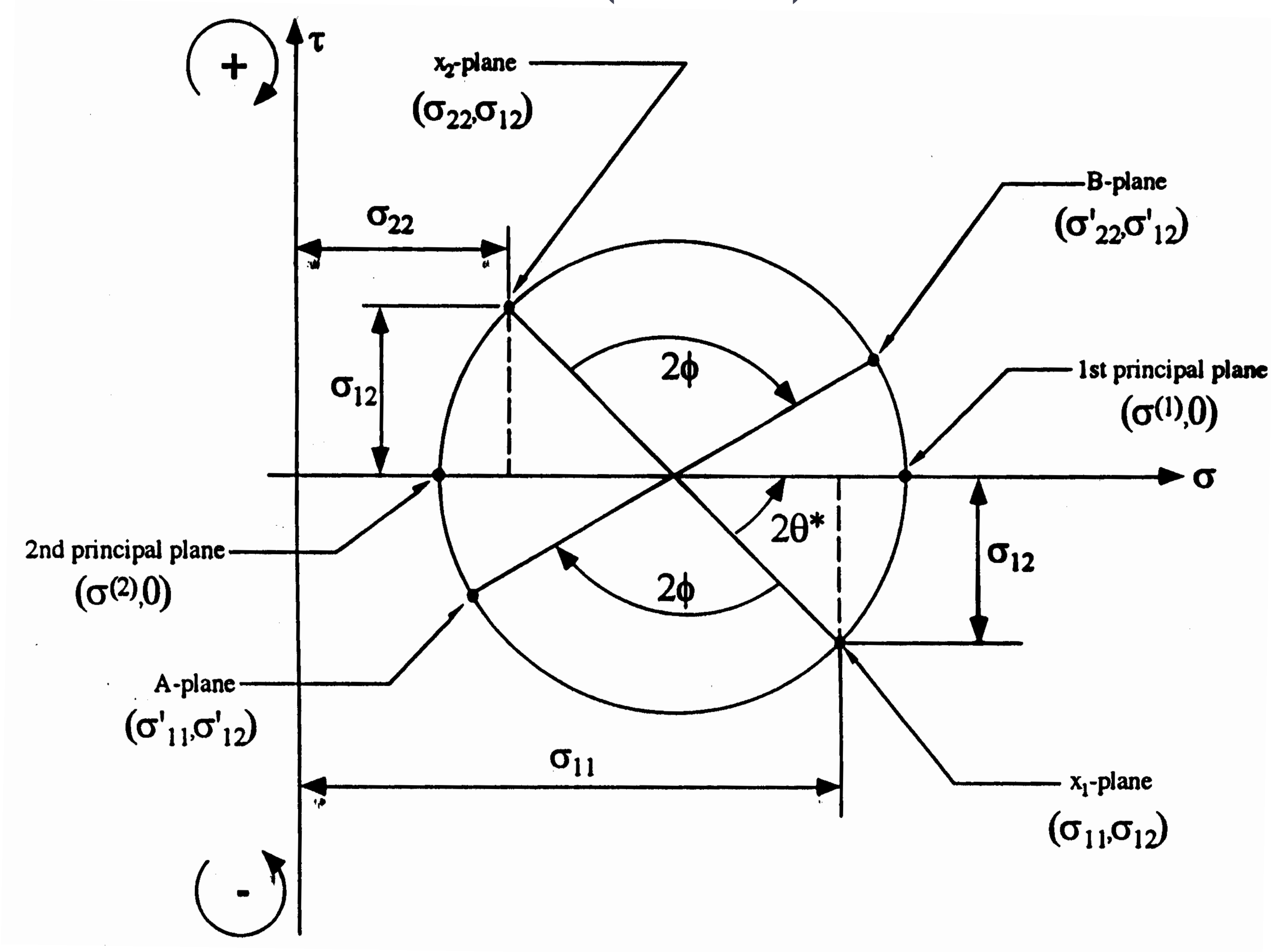
$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2) = \sqrt{\sigma_{12}^2 + \left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2}$$

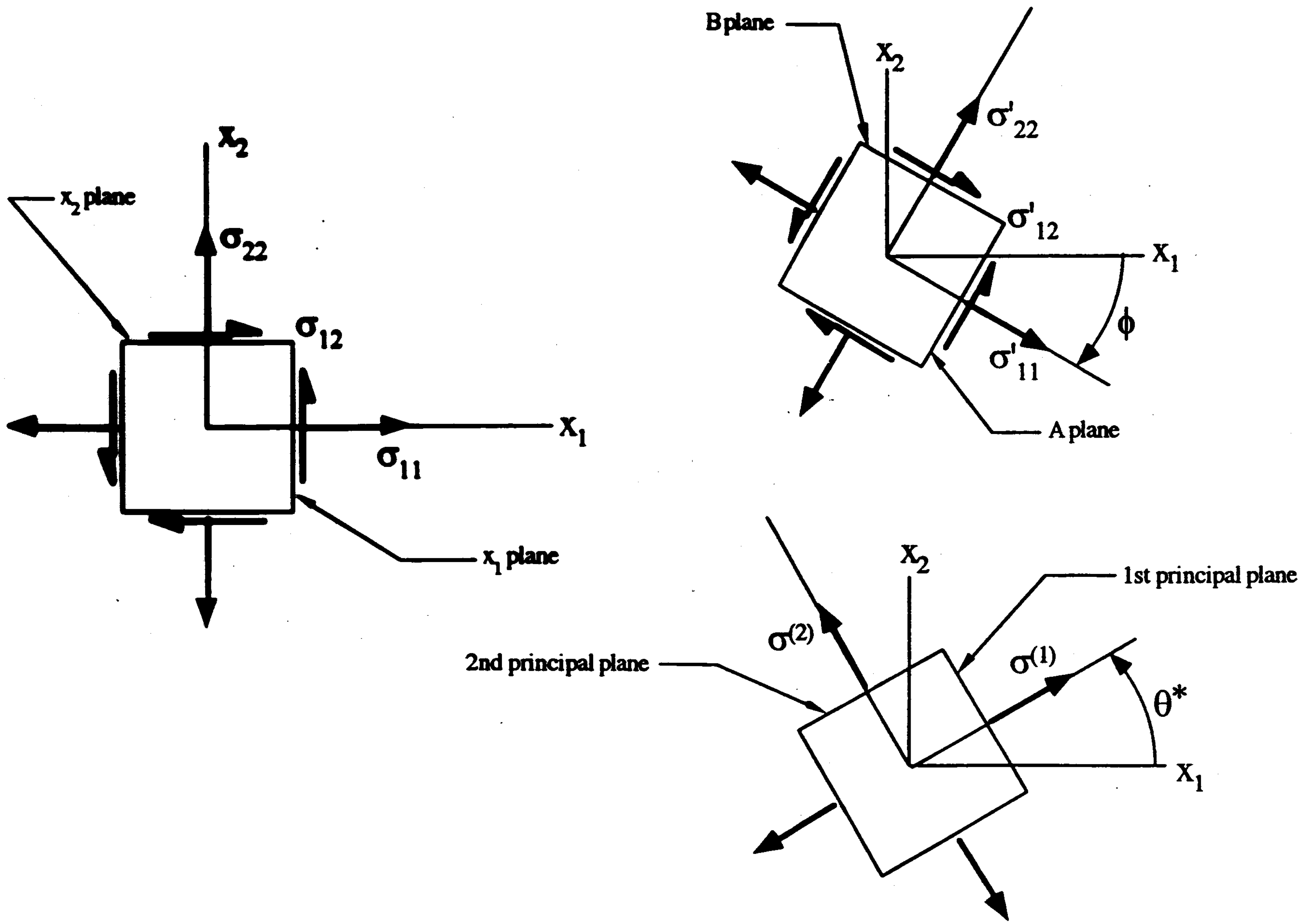
Stress Transformation (cont.)

$$\left[\sigma-\left(\frac{\sigma_1+\sigma_2}{2}\right)\right]^2+\tau^2=\left(\frac{\sigma_1-\sigma_2}{2}\right)^2 \quad \Rightarrow \quad [\sigma-C]^2+\tau^2=R^2$$

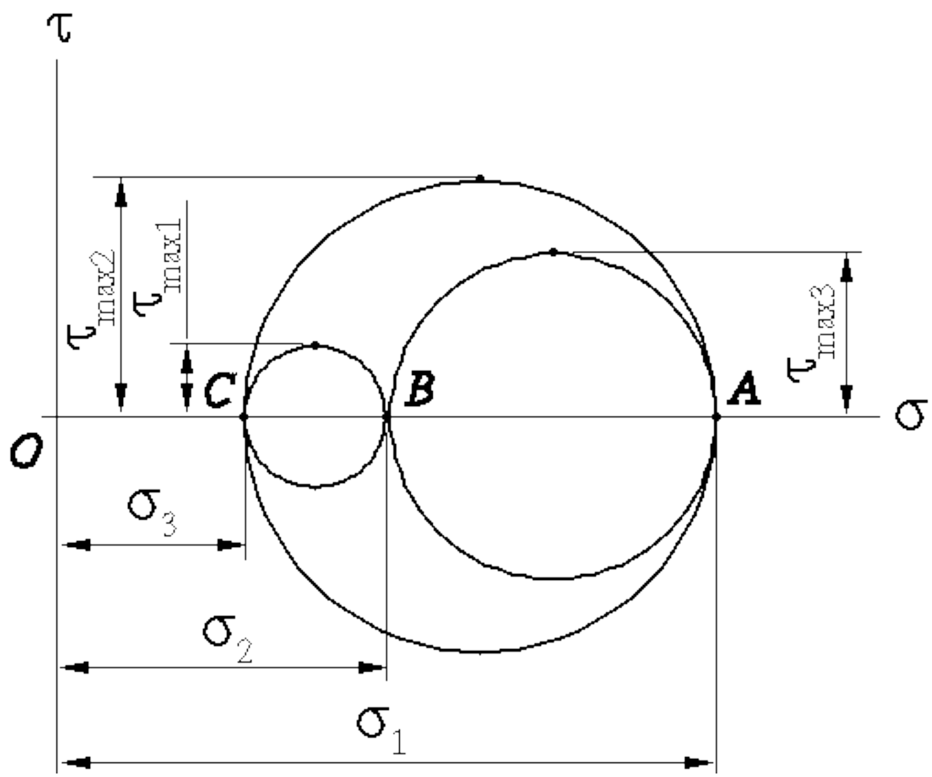
- Procedure:
  - Draw σ and τ axes
  - Plot the two points (σ_{11}, σ_{12}) and (σ_{22}, σ_{12})
  - Plot shear stresses above σ-axis if they produce a clockwise couple, below if counterclockwise
  - Connect the points with a straight line
  - Locate the center at (C,0)
  - Draw a circle of radius R and centered at (C,0)
  - Find the principal stresses as the intersections of the circle with the σ-axis
  - Find the angle $2\Theta^*$ between the x_1-face and the principal plane
- Notes:
  - Angles on the Mohr's circle are double angles, i.e. 2Θ, $2\Theta^*$
  - Points on the Mohr's circle represent stress states on "cutting" planes of the original x_1-x_2 system
  - Angles are positive if CCW, negative if CW
- Examples...

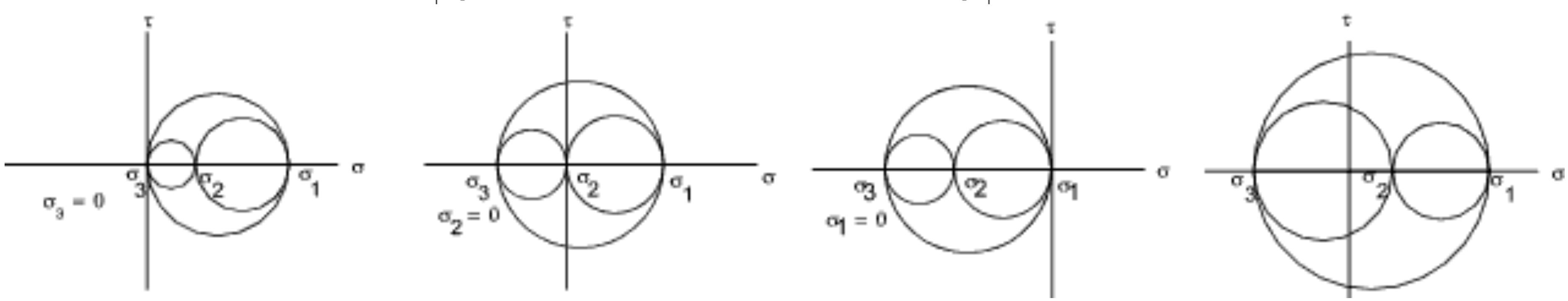
Stress Transformation (cont.)



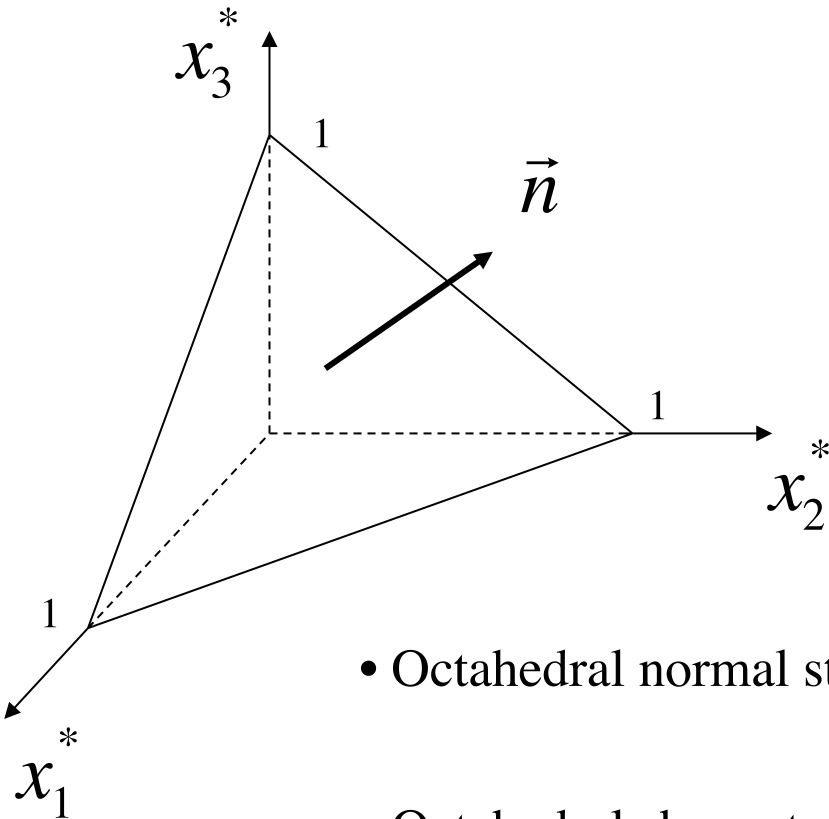

x_2
x_2 plane
σ_{22}
σ_{12}
x_1
σ_{11}
x_1 plane
B plane
x_2
σ'_{22}
σ'_{12}
x_1
ϕ
σ'_{11}
A plane
x_2
1st principal plane
$\sigma^{(2)}$
$\sigma^{(1)}$
2nd principal plane
θ^*
x_1

Stress Transformation (cont.)





Special Cases: Octahedral stresses



$(x_1^*,\ x_2^*,\ x_3^*)$ is a principal frame

- Octahedral plane: $n_1 = n_2 = n_3 = \dfrac{1}{\sqrt{3}}$

- Octahedral normal stress: $\sigma_{norm}^{oct} = \dfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \dfrac{1}{3}\sigma_{ii}$

- Octahedral shear stress:

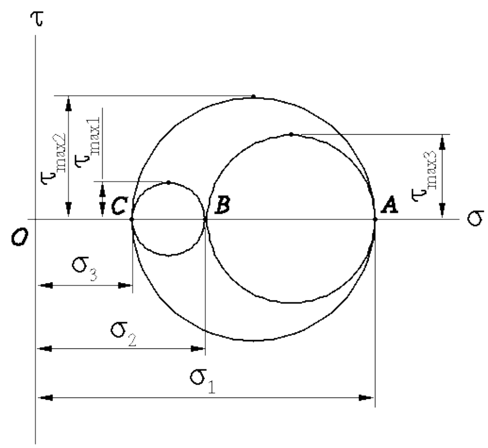
$$\sigma_{sh}^{oct} = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2}$$

$$\sigma_{sh}^{oct} = \sqrt{\frac{2}{9}\left[Q_1^2 - 3Q_2\right]}$$

Special Cases (cont.): Maximum shear stress

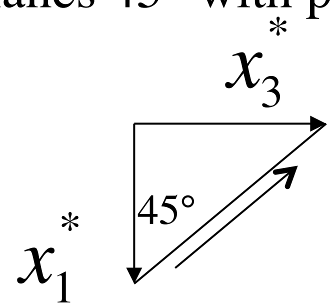
- When is the shear stress a maximum?

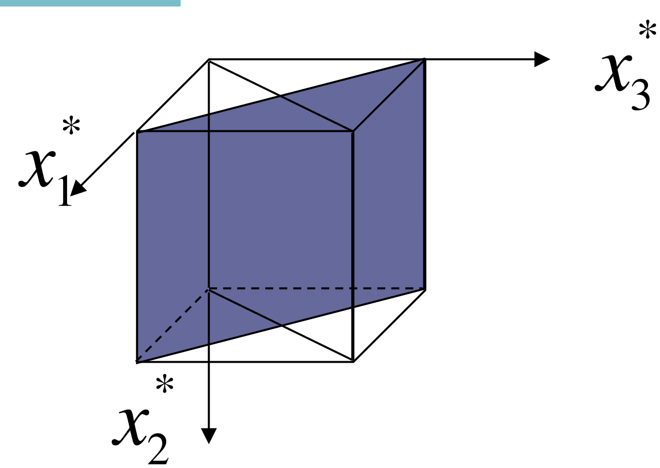
$$\frac{d\sigma_{sh}}{d\vec{n}} = 0 \Rightarrow \quad \text{3 roots} \qquad \sigma_{sh} = \begin{cases} \frac{1}{2}\left|\sigma_1 - \sigma_2\right| \\ \frac{1}{2}\left|\sigma_2 - \sigma_3\right| \\ \frac{1}{2}\left|\sigma_3 - \sigma_1\right| \end{cases}$$



- Absolute maximum: $\boxed{\sigma_{sh}^{\max} = \frac{1}{2}\left(\sigma_1 - \sigma_3\right)}$

- Acts on planes 45° with p. axes:





Special Cases (cont.): Stress deviator

- Decomposition of stress tensor as

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} + \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix}$$

Isotropic (or hydrostatic) stress state

Deviatoric stress state

with: $p = \frac{1}{3} tr(\underline{\sigma}) = \frac{1}{3}\sigma_{kk}$ and $S_{ii} = 0$

$$\boxed{\therefore \quad \sigma_{ij} = \frac{1}{3}\sigma_{kk}\delta_{ij} + S_{ij}}$$

Special Cases (cont.)

- Plane stress: $\sigma_{3i} = 0$

$$[\underline{\sigma}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Uniaxial stress: $\sigma_2 = \sigma_3 = 0$

$$[\underline{\sigma}] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Pure shear stress: $\sigma_{ii} = 0 \quad \text{no sum}$

$$\sigma_1 = \sigma \quad \sigma_2 = 0 \quad \sigma_3 = -\sigma$$

$$[\underline{\sigma}] = \begin{bmatrix} 0 & \sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Hydrostatic stress: $\sigma_{ij} = p\delta_{ij}$

$$\sigma_1 = p \quad \sigma_2 = p \quad \sigma_3 = p$$

$$[\underline{\sigma}] = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}$$

Appendix - Solution of cubic equations

- Method 1: Graeffe's root-squaring method

Find the roots of: $\lambda^3 - 6\lambda^2 + 5\lambda - 1 = 0$

Let, $\Delta(\lambda) = \lambda^3 - 6\lambda^2 + 5\lambda - 1$

$$= a_0\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3$$

The next polynomial Δ_2(l) is found by using the coefficients of the preceding polynomial (a_i) to calculate the new coefficients ($a_i´$): $a_i' = a_i^2 - 2a_{i-1}a_{i+1}$

$$e.g.,\quad a_0' = a_0^2 - 2a_{-1}a_{+1} = 1^2 - 2(0)(6) = 1$$

$$a_1' = a_1^2 - 2a_0a_2 = 6^2 - 2(1)(5) = 26$$

$$a_2' = 13$$

$$a_3' = 1$$

$$\therefore \quad \Delta_2(\lambda) = \lambda^3 - 26\lambda^2 + 13\lambda - 1$$

Solution of cubic equations (cont.)

Repeating: $\Delta_4(\lambda) = \lambda^3 - 650\lambda^2 + 117\lambda - 1$

$$\Delta_8(\lambda) = \lambda^3 - 422266\lambda^2 + 12389\lambda - 1$$

$$\vdots$$

$$\Delta_{32}(\lambda) = \lambda^3 - 3.17941\times10^{22}\lambda^2 + 2.329953\times10^{16}\lambda - 1$$

The roots are obtained from (where n is the equation number):

$$\lambda_1^n = a_1, \quad \lambda_2^n = \frac{a_2}{a_1}, \quad \lambda_3^n = \frac{a_3}{a_2}$$

Each iteration will increase the accuracy of the roots. However, D8 or even D4 can give reasonable results.

$$e.g., \quad \Delta_2 \Rightarrow \lambda_1 = 5.0990, \quad \lambda_2 = 0.70711, \quad \lambda_3 = 0.27735$$

$$\Delta_4 \Rightarrow \lambda_1 = 5.0493, \quad \lambda_2 = 0.65136, \quad \lambda_3 = 0.30406$$

$$\Delta_8 \Rightarrow \lambda_1 = 5.0489, \quad \lambda_2 = 0.64333, \quad \lambda_3 = 0.30787$$

$$\Delta_{16}, \Delta_{32} \Rightarrow \lambda_1 = 5.0489, \quad \lambda_2 = 0.64310, \quad \lambda_3 = 0.30798$$

Solution of cubic equations (cont.)

- Method 2: Stress deviator method $\lambda^3 - Q_1\lambda^2 + Q_2\lambda - Q_3 = 0$

Let,

$$\alpha = \frac{3Q_2 - Q_1^2}{9}$$

$$\beta = \frac{-9Q_1Q_2 + 27Q_3 + 2Q_1^3}{54}$$

$$\gamma = \alpha^3 + \beta^2$$

For real roots g<0, then $\cos\theta = \frac{\beta}{\sqrt{-\alpha^3}}$

and the roots are

$$\lambda_1 = 2\sqrt{-\alpha}\cos(\theta/3) + Q_1/3$$

$$\lambda_2 = 2\sqrt{-\alpha}\cos(\theta/3 + 120°) + Q_1/3$$

$$\lambda_3 = 2\sqrt{-\alpha}\cos(\theta/3 + 240°) + Q_1/3$$

Solution of cubic equations (cont.)

Example, $\lambda^3 - 6\lambda^2 + 5\lambda - 1 = 0$

$$\alpha = -2.333, \quad \beta = 3.5, \quad \gamma = -0.4537 < 0$$

$$\cos\theta = 0.98198 \;\Rightarrow\; \theta = 10.893^\circ \;\Rightarrow\; \begin{aligned} \lambda_1 &= 5.0489 \\ \lambda_2 &= 0.30798 \\ \lambda_3 &= 0.64310 \end{aligned}$$

- Method 3: After finding one root (e.g numerically), factor it out and solve a quadratic

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0 \;\Rightarrow\; (\lambda - \lambda_1)\left(a\lambda^2 + b\lambda - c\right) = 0$$

$$\Rightarrow \quad \lambda_{2,3} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

END OF CHAPTER 3