Chapter 4 - Material Behavior

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Uniaxial Response

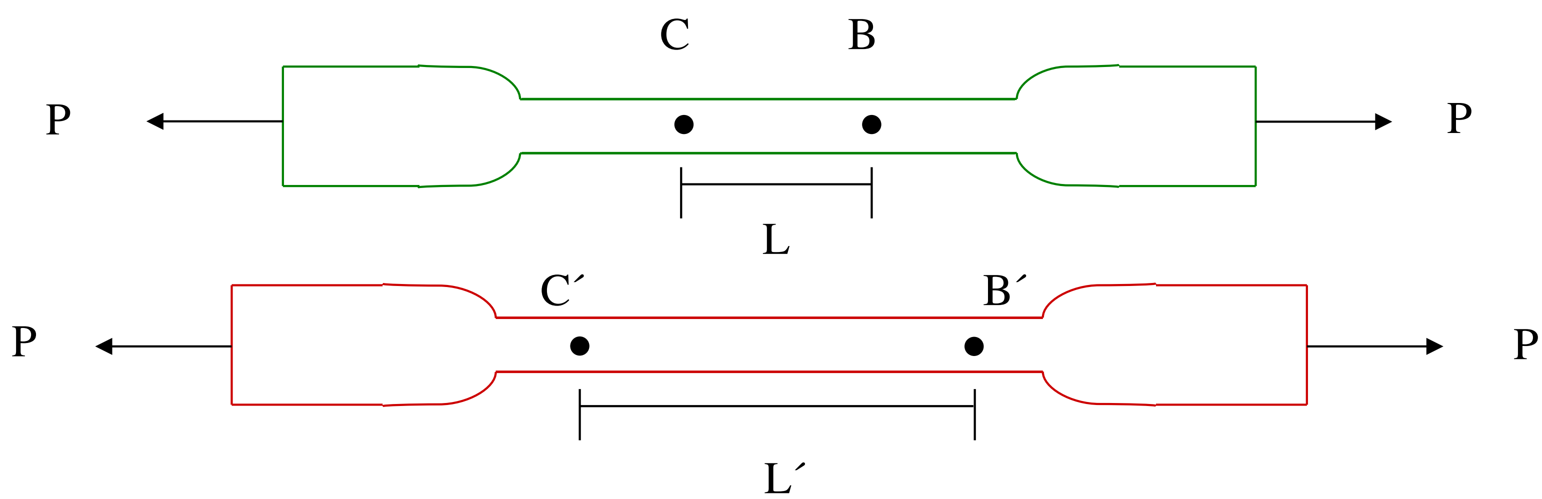
- Need to relate σ_{ij} to ε_{ij}
- A material characteristic, i.e., external to the theory

  ⇒ EXPERIMENTS

- Mechanical **constitutive equation**

$$\sigma_{ij} = f\left(\varepsilon_{ij}, \dot{\varepsilon}_{ij}, T, \dot{T}, \ldots\right)$$

Tensile Test



P: Applied load
L: Gauge section
ΔL: Change of gauge section length, L´-L

- Assume only axial tensile stress:

$$\sigma_{axial} = \frac{P}{A}$$

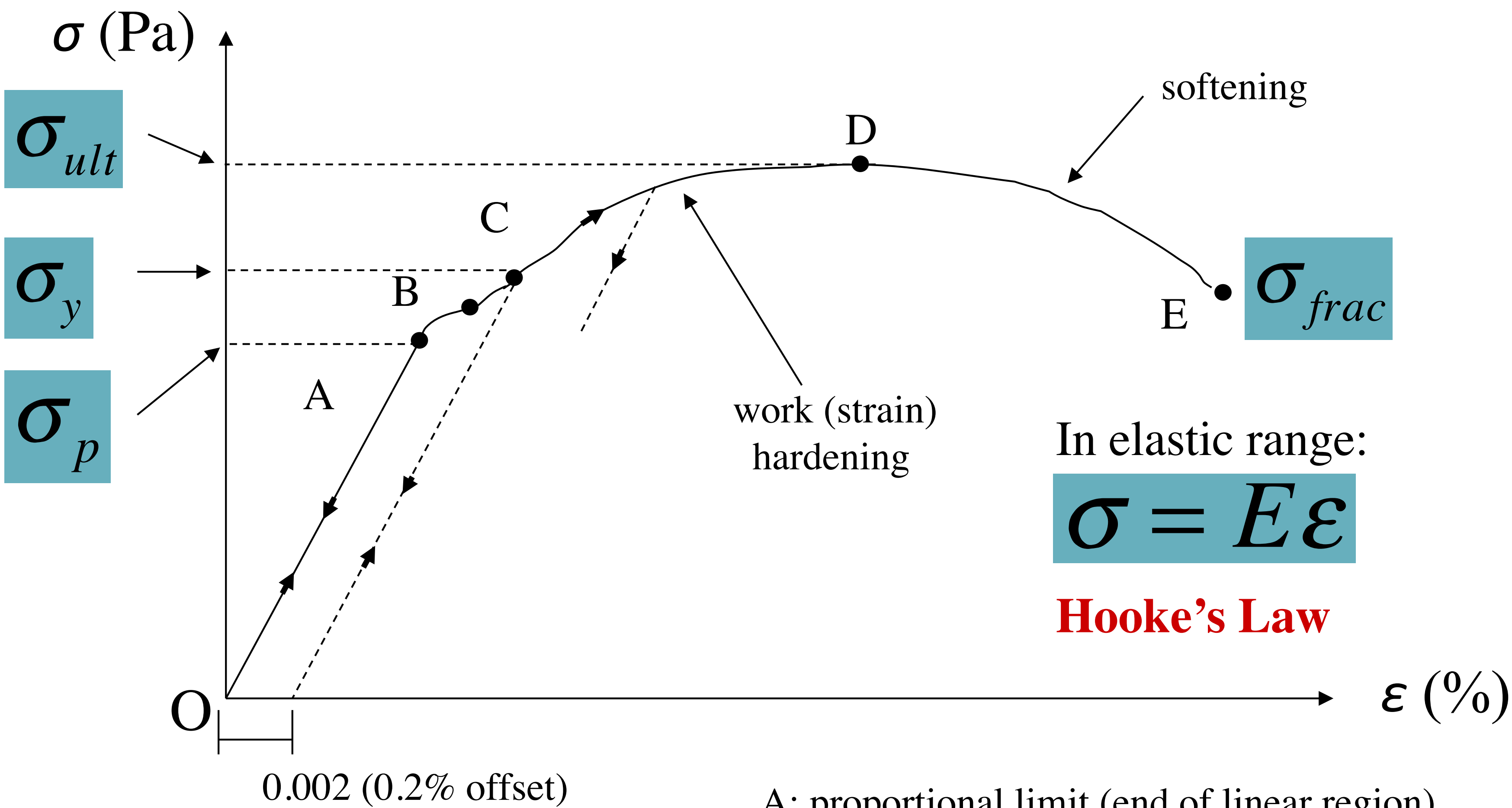
- Strain:

$$\varepsilon_{axial} = \frac{\Delta L}{L}$$

A: (initial) cross sectional area of gauge length

Tensile Test (cont.)

- Plot experimental ***σ-ε* curve** (e.g. steel specimen in tension)



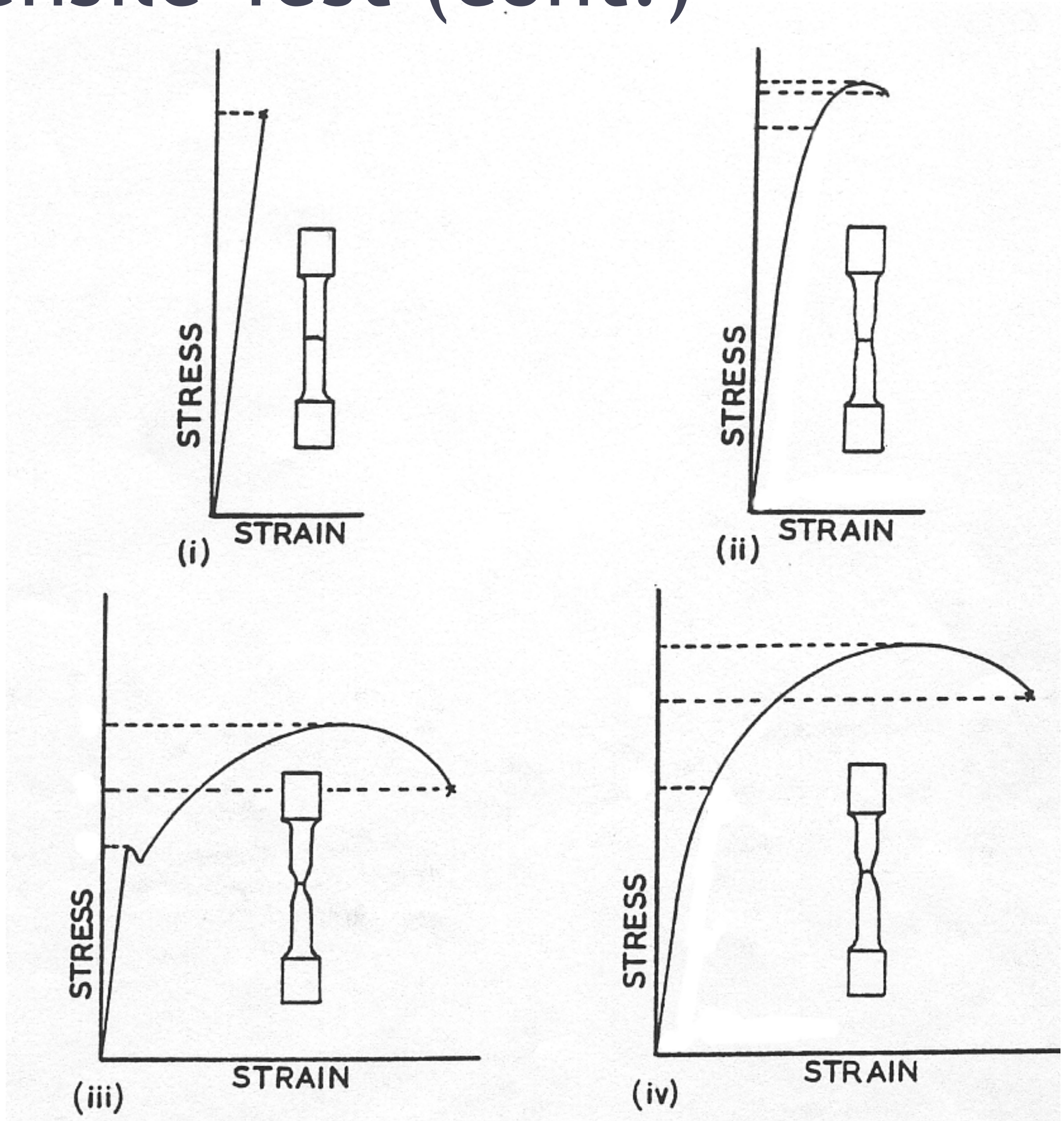
In elastic range:

$$\sigma = E\varepsilon$$

Hooke's Law

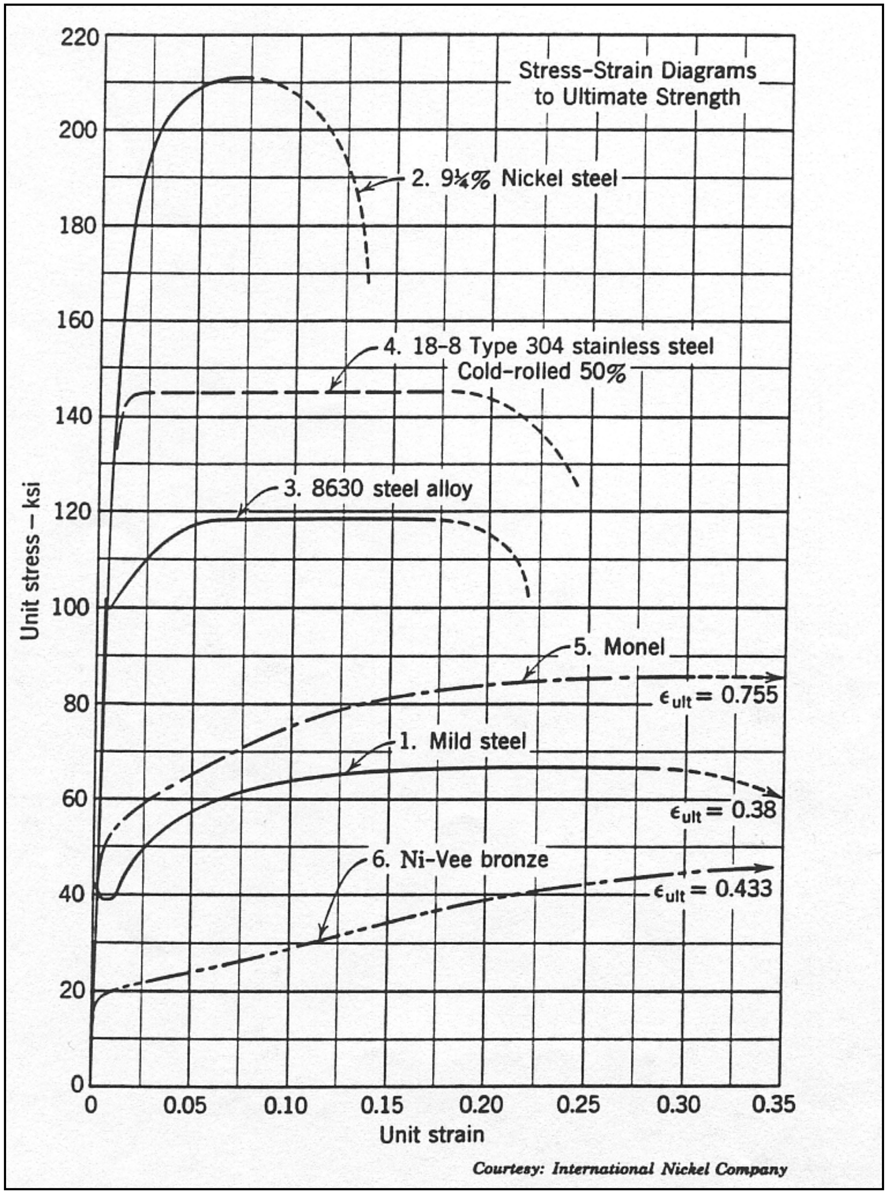
A: proportional limit (end of linear region)
B: elastic limit (end of elastic region)
C: yield stress at 0.2% offset
D: ultimate stress
E: fracture stress

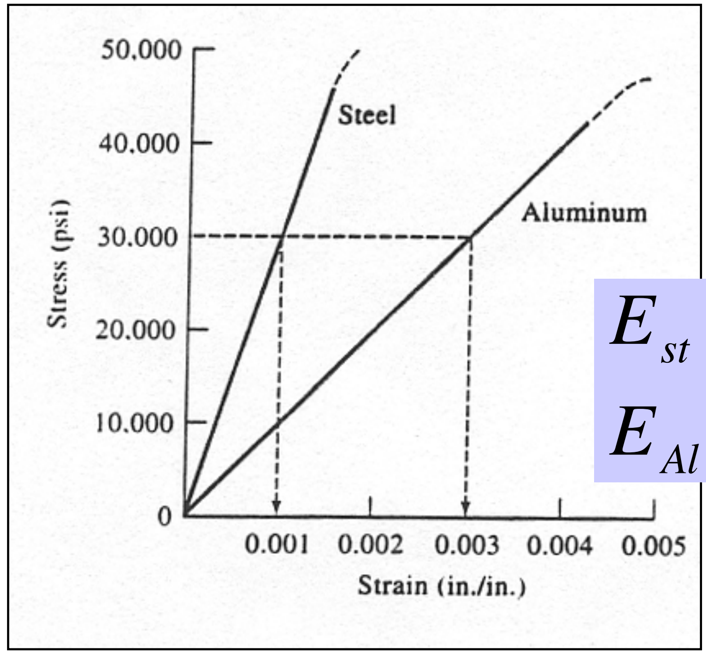
Tensile Test (cont.)



(i) Brittle
(ii) Brittle (or semi-ductile)
(iii), (iv) Ductile

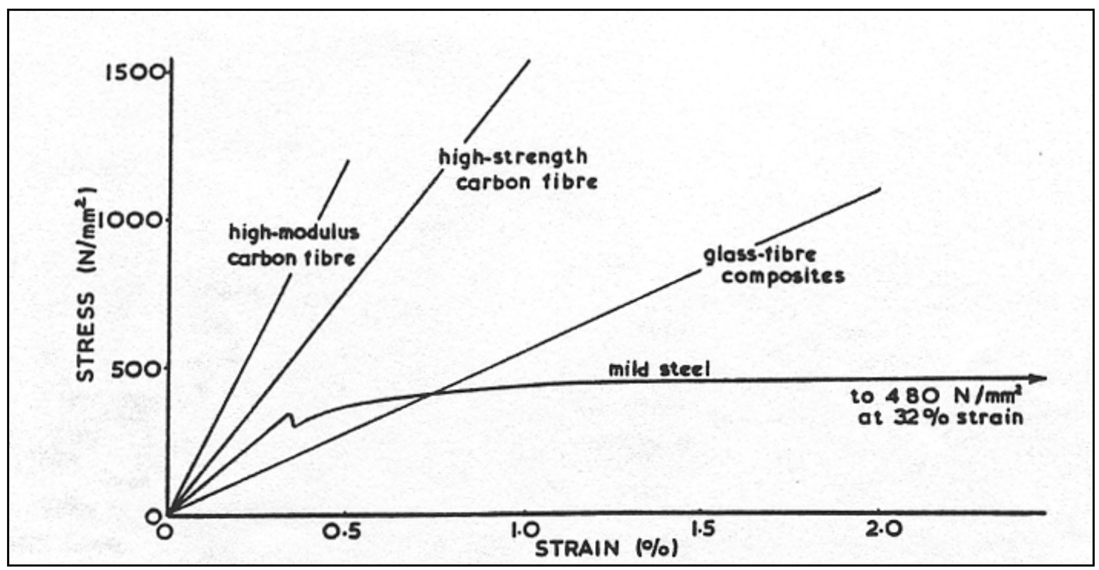
Tensile Test (cont.)





$$E_{st} = 207\ GPa$$

$$E_{Al} = 70\ GPa$$

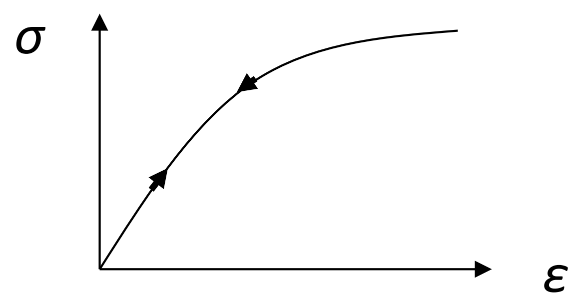


Tensile Test (cont.)

- Between O and the **proportional limit** (point A) the σ–ε curve shows **linear elastic** behavior

Linear → straight line (i.e. $\sigma \sim \varepsilon$)
Elastic → unloading occurs along the same loading path (material returns to 0,0)

- Up to B, non-linear elastic material: e.g. rubber



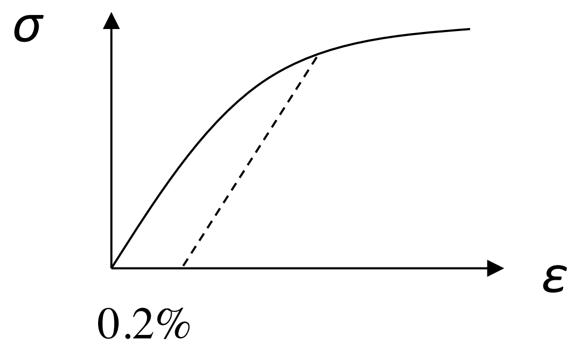
Note: sometimes B and C coincide

- Upon continued loading we exceed the **elastic limit**. Then unloading follows a different path.

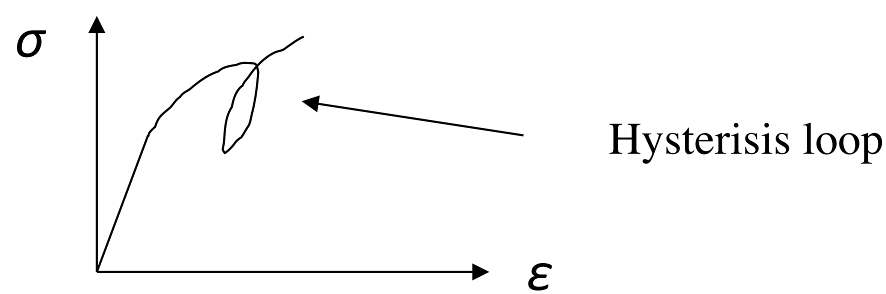
Tensile Test (cont.)

• After point C unloading produces a **permanent (residual) plastic strain** when σ returns to 0.

σ_C: 0.2% offset stress (or flow stress)



• When unloading ductile metals in reality : **Hysterisis**



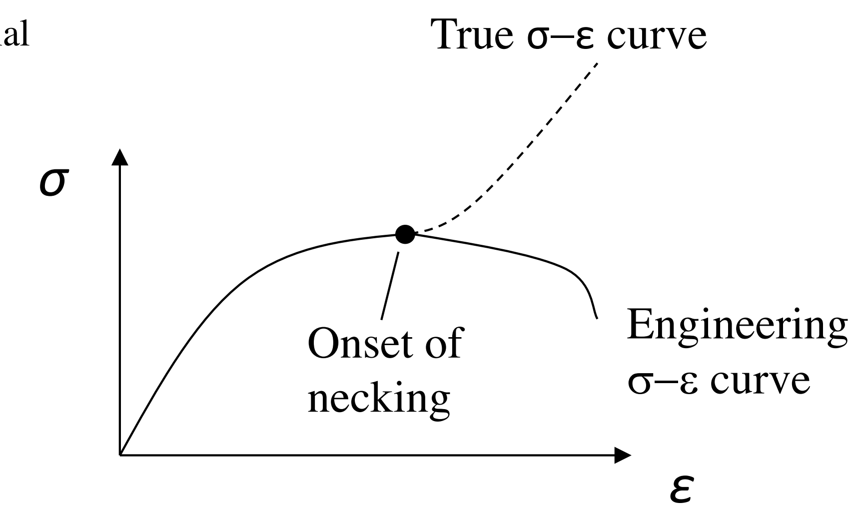
Tensile Test (cont.)

- The **Ultimate Tensile Strength (UTS)** (point D) is the maximum sustainable stress. At point D **necking** begins:

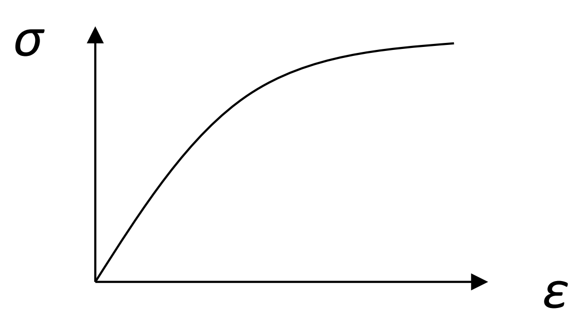
$$A_{neck} < A_{initial}$$

$$\sigma = \frac{P}{A_{initial}}$$: engineering stress

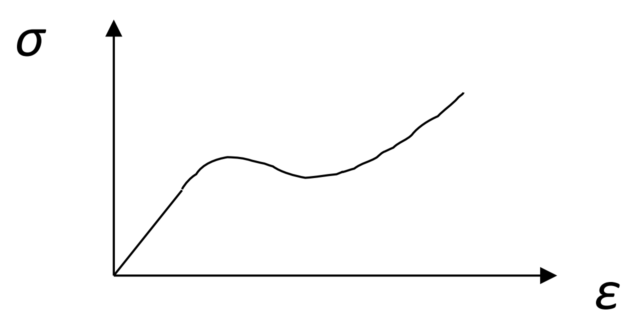
$$\sigma = \frac{P}{A_{neck}}$$: true stress



- **Compression** behavior may differ (although elastic part usually the same)



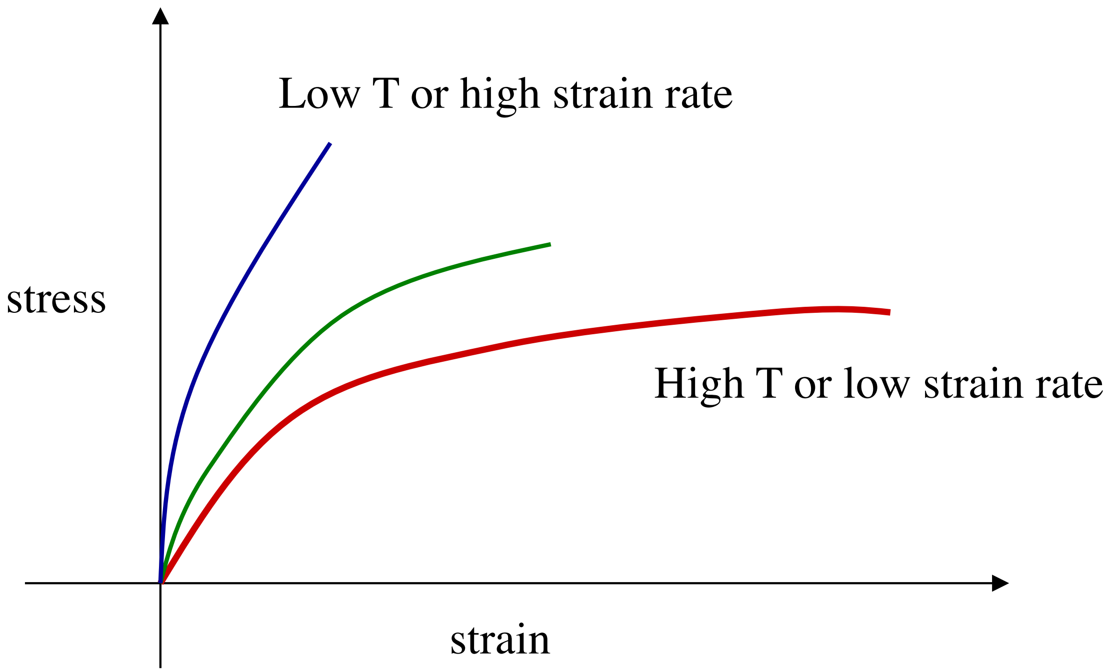
tension

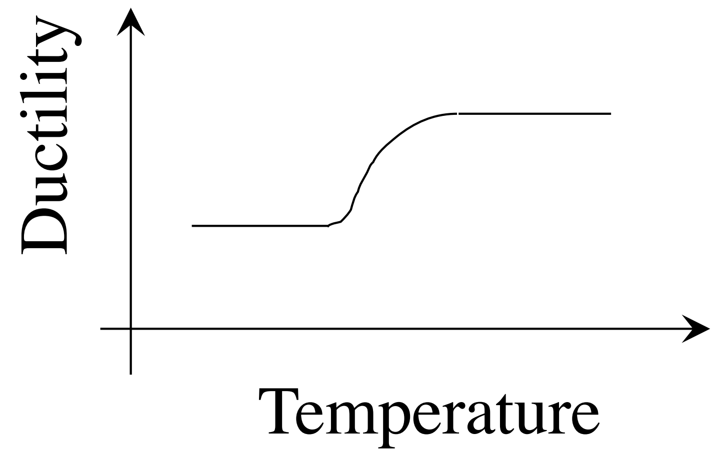


compression

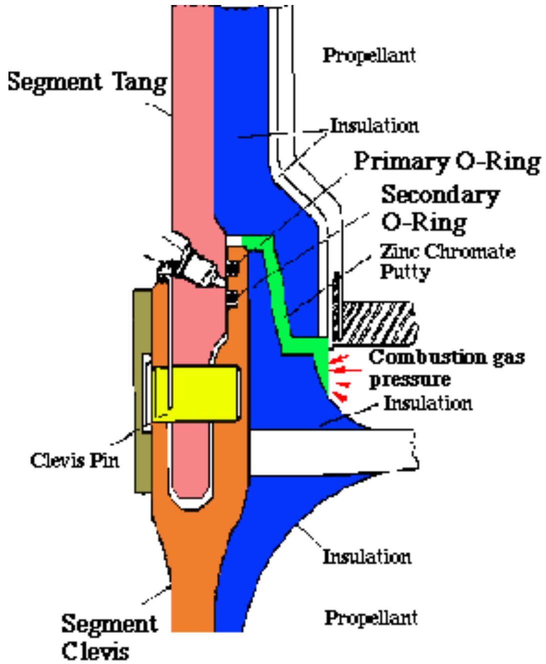
Tensile Test (cont.)

- In general temperature and strain rate will affect the material response:

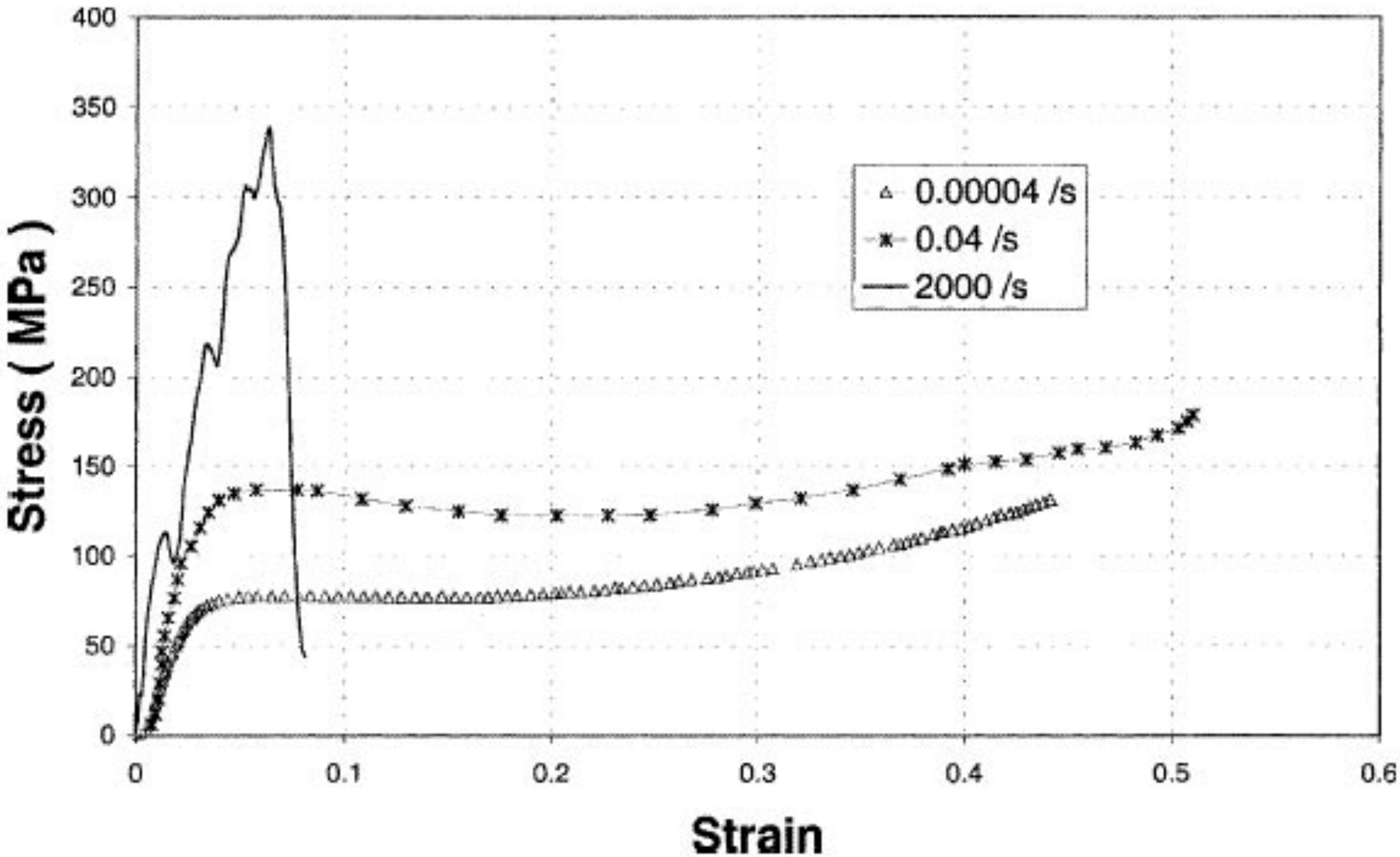




- Temperature effects (Challenger, 1/28/86):



Tensile Test (cont.)



Effect of strain rate on σ–ε curve of PMMA (Plexiglas)

Tensile Test (cont.)

- Stress σ_{ij}, strain ε_{ij}: 9 components each

$\Rightarrow$ Need multi-axial information

- In multi-axial experiments:

$$
\begin{aligned}
\textit{linear elastic} &\rightarrow \textit{proportional limit} \\
&\rightarrow \textit{yield} \\
&\rightarrow \textit{plastic reponse} \\
&\rightarrow \textit{necking} \\
&\rightarrow \textit{failure}
\end{aligned}
$$

- For 3D problems a linear elastic model is valid up to yield

Note: Many (aerospace) structures are designed to operate well below the yield point (although made of ductile materials in order to exhibit progressive “failure”).

Chapter 4 - Material Behavior

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Generalized Hooke's Law

- Assume that stress is the gradient of a potential:

$$\boxed{\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}}$$

$W(\varepsilon_{ij})$: strain energy density (quadratic)

- For an **inhomogeneous** solid $W\left(\varepsilon_{ij}(\vec{x}), \vec{x}\right)$

Explicit spatial dependence

- For stability $W \geq 0$. W(0)=0 unloaded state.

Generalized Hooke's Law (cont.)

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \quad \text{or} \quad \underline{\sigma} = \underline{\underline{C}}\,\underline{\varepsilon}$$

C_{ijkl} : Elastic constants (stiffness) = 81 components

C_{ijkl} is a four tensor $\qquad C'_{pqrs} = \alpha_{pi}\alpha_{qj}\alpha_{rk}\alpha_{sl}C_{ijkl}$

$$\left.\begin{array}{l}\sigma_{ij} = \sigma_{ji} \\ \varepsilon_{ij} = \varepsilon_{ji}\end{array}\right\} \Rightarrow C_{ijkl} = C_{jikl} = C_{ijlk=\ldots}$$

Minor symmetry

Also, $C_{ijkl} = C_{klij}$ Major symmetry

81 components reduce to **21 independent** constants

Generalized Hooke's Law (cont.)

- Expanding

$$
\begin{aligned}
\sigma_{11} &= C_{11kl}\varepsilon_{kl} \\
&= C_{111l}\varepsilon_{1l} + C_{112l}\varepsilon_{2l} + C_{113l}\varepsilon_{3l} \\
&= \left(C_{1111}\varepsilon_{11} + \underline{C_{1112}\varepsilon_{12}} + \underline{C_{1113}\varepsilon_{13}}\right) + \\
&\quad + \left(\underline{C_{1121}\varepsilon_{21}} + C_{1122}\varepsilon_{22} + \underline{C_{1123}\varepsilon_{23}}\right) + \\
&\quad + \left(\underline{C_{1131}\varepsilon_{31}} + \underline{C_{1132}\varepsilon_{32}} + C_{1133}\varepsilon_{33}\right)
\end{aligned}
$$

- Only 6 terms are independent

Generalized Hooke's Law (cont.)

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}
=
\begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & 2C_{1112} & 2C_{1123} & 2C_{1131} \\
 & C_{2222} & C_{2233} & 2C_{2212} & 2C_{2223} & 2C_{2231} \\
 & & C_{3333} & 2C_{3312} & 2C_{3323} & 2C_{3331} \\
 & & & 2C_{1212} & 2C_{1223} & 2C_{1231} \\
 & sym. & & & 2C_{2323} & 2C_{2331} \\
 & & & & & 2C_{3131}
\end{bmatrix}
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}
$$

Generalized Hooke's Law (cont.)

- Simplifying notation: $\{\boldsymbol{\sigma}\} = [\boldsymbol{C}]\{\boldsymbol{\varepsilon}\}$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & sym. & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

Stiffness matrix

Generalized Hooke's Law (cont.)

Notes:

- $[C]$ is not a tensor. Only C_{ijkl} is a tensor.

- 21 independent elastic constants ⇒ **anisotropic material** (most general elastic material)

- Coupling between normal stresses and shear strains (and vice versa), e.g.

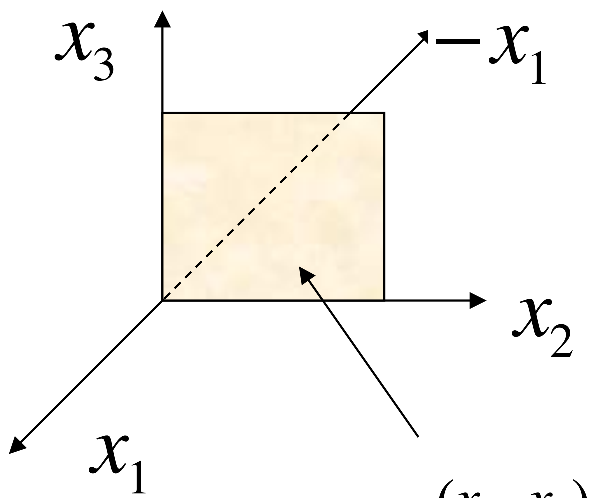
$$\sigma_{11} = \underbrace{\boxed{C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{13}\varepsilon_{33}}}_{\text{normal strain dependence}} + \underbrace{\boxed{C_{14}\varepsilon_{12} + C_{15}\varepsilon_{23} + C_{16}\varepsilon_{13}}}_{\text{shear strain dependence}}$$

⇒ principal axes of stress and strain do NOT coincide

- For a linear elastic solid: $\boxed{W = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}}$

Material Symmetry

- Assume one plane of symmetry



Monoclinic material

(x_2, x_3) plane of symmetry

⇨ change of x_1 to -x_1 should not change properties

- Coordinate transformation:

$$x'_1 = -x_1$$
$$x'_2 = x_2$$
$$x'_3 = x_3$$

Material Symmetry (cont.)

- With $x_1' = -x_1$ use $\sigma_{ij}' = \alpha_{ik}\alpha_{jl}\sigma_{kl}$

  $x_2' = x_2$ $\quad\varepsilon_{ij}' = \alpha_{ik}\alpha_{jl}\varepsilon_{kl}$

  $x_3' = x_3$

  $[C'] = [C]$ symmetry

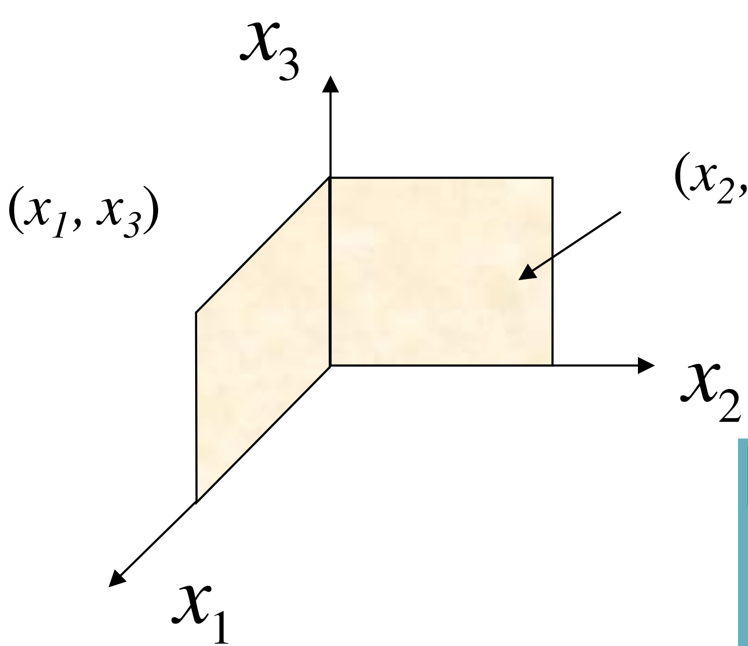
$$\therefore \quad C_{14} = C_{24} = C_{34} = C_{16} = C_{26} = C_{36} = C_{45} = C_{56} = 0$$

- $[C]$ ⇒ 13 independent constants

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & C_{15} & 0 \\ & C_{22} & C_{23} & 0 & C_{25} & 0 \\ & & C_{33} & 0 & C_{35} & 0 \\ & & & C_{44} & 0 & C_{46} \\ & sym. & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

Material Symmetry (cont.)

- Assume two orthogonal planes of symmetry



Orthotropic material

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & sym. & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

- 9 independent elastic constants

Material Symmetry (cont.)

<u>Notes on orthotropic solid:</u>

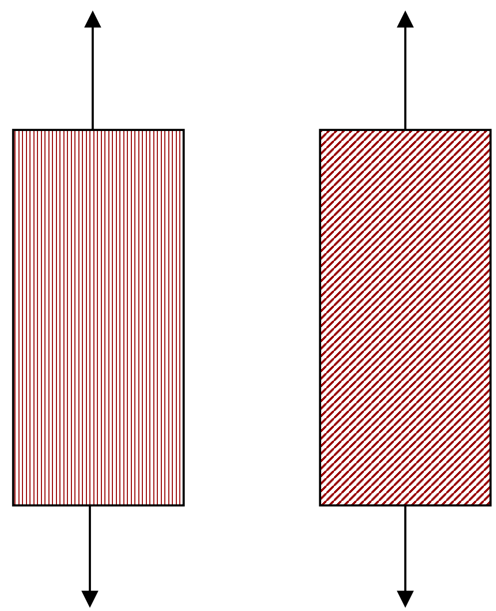
- The x_1, x_2, x_3 axes in which [C] is as above are the **material axes**

- In any other frame [C] becomes more populated (looks like monoclinic).

- If loading axes and material axes coincide,

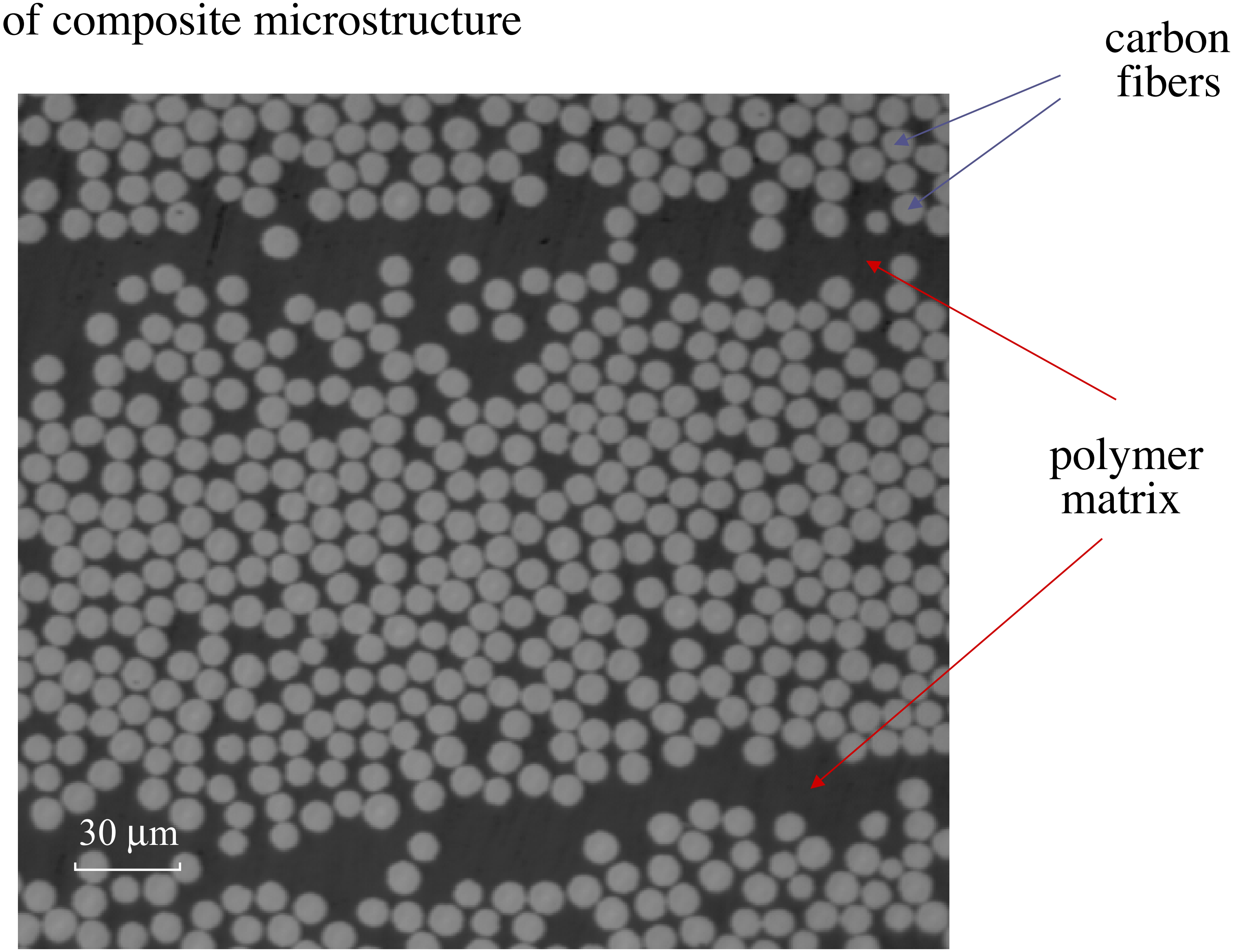
  ⇒ no normal/shear coupling

- Examples:
  - — fiber reinforced composites
  - — skin/stringer structures
  - — reinforced concrete



Material Symmetry (cont.)

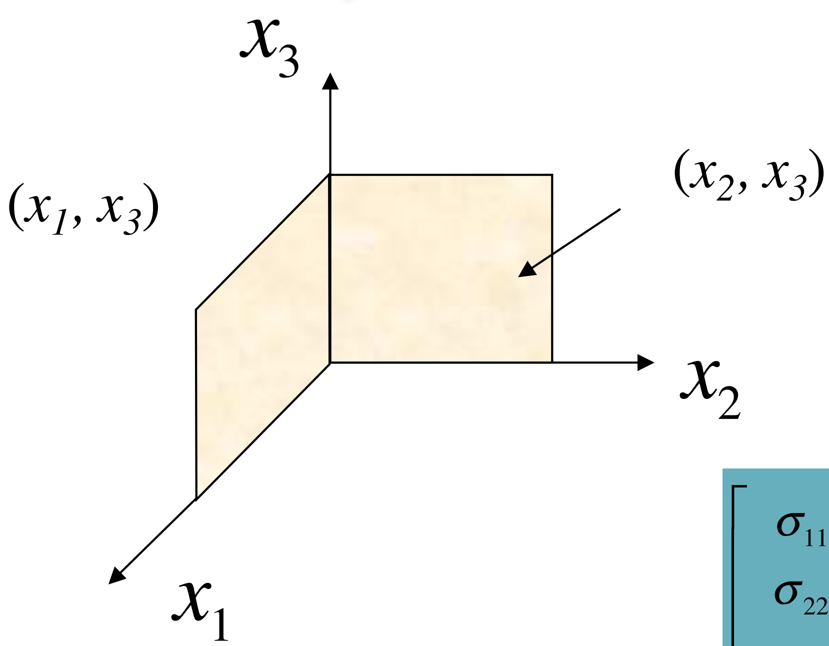
- Example of composite microstructure



Material Symmetry (cont.)

- Directional independence: what if x_2 and x_3 direction properties are the same?

  ⇒ i.e. invariant under rotation w.r.t. x_1 axis



Transversely isotropic material

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{22} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & sym. & & & (C_{22} - C_{23}) & 0 \\ & & & & & C_{44} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

- 5 independent elastic constants

Material Symmetry (cont.)

- With rotational independence w.r.t. two axes

  ⇒ 3 independent constants

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & sym. & & & C_{44} & 0 \\ & & & & & C_{44} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}
$$

Cubic material

- In all cases [C] becomes more populated under coordinate rotation - recall it is NOT a tensor.

- But the number of <u>independent constants</u> does not change with rotation

Chapter 4 - Material Behavior

Uniaxial response-Tensile test

Generalized Hooke's law
- Anisotropic solid
- Material symmetry

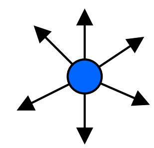
★ **Isotropic solid**
- Elastic constants
- Thermal change

Viscoelastic materials

Isotropic Material - Elastic Constants

- No directional or rotational dependence (i.e. same properties in every direction).

→ **isotropic material** - 2 independent constants



$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ & & 2\mu+\lambda & 0 & 0 & 0 \\ & & & 2\mu & 0 & 0 \\ & sym. & & & 2\mu & 0 \\ & & & & & 2\mu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

$\lambda, \ \mu:$ Lamé constants

Elastic Constants (cont.)

- Indicial notation: $\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk}$

or

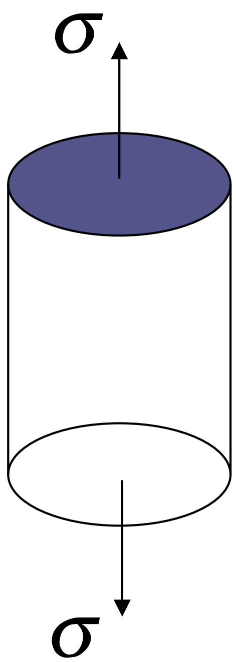
$$\begin{cases} \sigma_{11} = (2\mu + \lambda)\varepsilon_{11} + \lambda\varepsilon_{22} + \lambda\varepsilon_{33} \\ \sigma_{22} = \lambda\varepsilon_{11} + (2\mu + \lambda)\varepsilon_{22} + \lambda\varepsilon_{33} \\ \sigma_{33} = \lambda\varepsilon_{11} + \lambda\varepsilon_{22} + (2\mu + \lambda)\varepsilon_{33} \\ \sigma_{12} = 2\mu\varepsilon_{12} \\ \sigma_{13} = 2\mu\varepsilon_{13} \\ \sigma_{23} = 2\mu\varepsilon_{23} \end{cases}$$

- Inverting: $\varepsilon_{ij} = \dfrac{1}{2\mu}\sigma_{ij} - \dfrac{\lambda}{2\mu(2\mu + 3\lambda)}\delta_{ij}\sigma_{kk}$

- How do we find λ, μ? ➡ experiment

Elastic Constants (cont.)

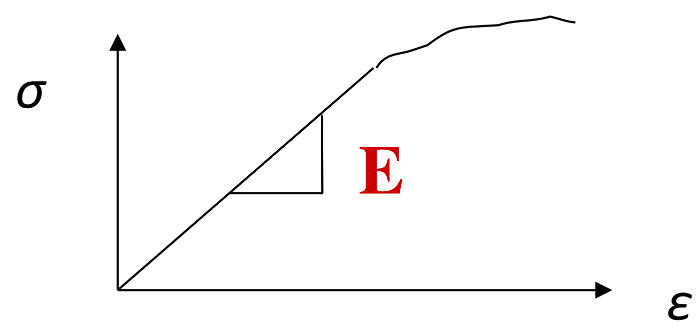
- Uniaxial tension: $\sigma_{11} = \sigma, \;\; \text{all other } \sigma_{ij} = 0$



$$\left.\begin{aligned}
\sigma_{11} &= \sigma = (2\mu+\lambda)\varepsilon_{11} + \lambda\varepsilon_{22} + \lambda\varepsilon_{33} \\
\sigma_{22} &= 0 = \lambda\varepsilon_{11} + (2\mu+\lambda)\varepsilon_{22} + \lambda\varepsilon_{33} \\
\sigma_{33} &= 0 = \lambda\varepsilon_{11} + \lambda\varepsilon_{22} + (2\mu+\lambda)\varepsilon_{33}
\end{aligned}\right\}$$

$$\Rightarrow \quad \sigma = \frac{\mu(2\mu+3\lambda)}{\mu+\lambda}\varepsilon_{11} \equiv E\varepsilon_{11}$$

E: **Young's modulus**
(or elastic modulus)



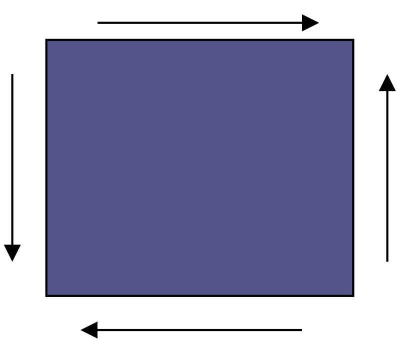
Elastic Constants (cont.)

$$\left.\begin{aligned}\sigma_{11} &= \sigma = (2\mu+\lambda)\varepsilon_{11} + \lambda\varepsilon_{22} + \lambda\varepsilon_{33} \\ \sigma_{22} &= 0 = \lambda\varepsilon_{11} + (2\mu+\lambda)\varepsilon_{22} + \lambda\varepsilon_{33} \\ \sigma_{33} &= 0 = \lambda\varepsilon_{11} + \lambda\varepsilon_{22} + (2\mu+\lambda)\varepsilon_{33}\end{aligned}\right\}$$

ν : **Poisson's Ratio**

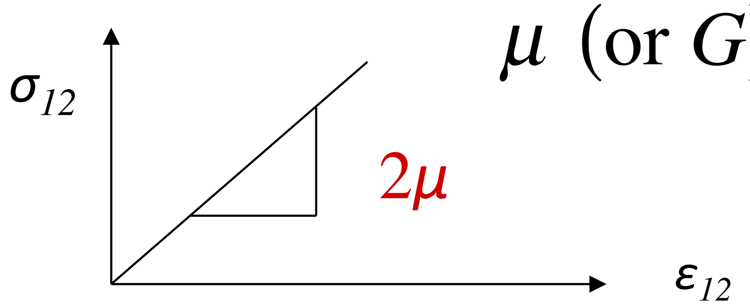
Lateral strain: $\Rightarrow \dfrac{\varepsilon_{22}}{\varepsilon_{11}} = -\dfrac{\lambda}{2(\lambda+\mu)} \equiv -\nu$

- Pure shear: $\sigma_{12} = \sigma_{21} = \sigma,\ \text{all other } \sigma_{ij} = 0$

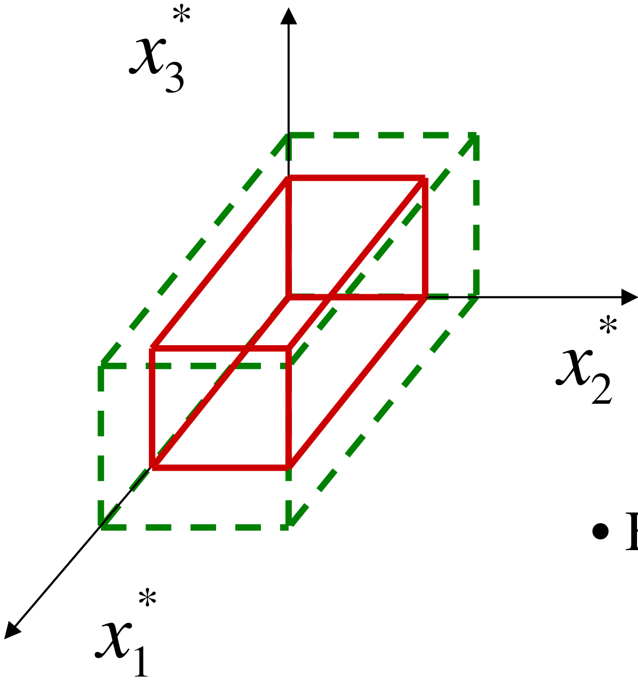


$$\therefore\ \sigma_{12} = \sigma = 2\mu\varepsilon_{12}$$

μ (or G) : **Shear Modulus**



Elastic Constants (cont.) - Volume Change



- Assume: $\sigma_{ij} = -p\delta_{ij} \quad (p > 0)$

$$\Rightarrow \quad \sigma_{kk} = -3p$$

hydrostatic pressure

- But: $\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk}$

$$\Rightarrow \quad \sigma_{kk} = (2\mu + 3\lambda)\varepsilon_{kk}$$

and

$$\Rightarrow \quad -3p = (2\mu + 3\lambda)\varepsilon_{kk} = (2\mu + 3\lambda)\frac{\Delta V}{V_0}$$

$$\Rightarrow \quad p = -k\frac{\Delta V}{V_0} \qquad \boxed{\Rightarrow \quad k = \frac{1}{3}(2\mu + 3\lambda)}$$

Bulk Modulus

Of the elastic constants E, ν, λ, μ, k only <u>two</u> can be taken as independent

	λ	μ	E	ν	k
λ,μ	N/A	N/A	$\mu(3\lambda+2\mu)/(\lambda+\mu)$	$\lambda/2(\lambda+\mu)$	$(3\lambda+2\mu)/3$
λ,E	N/A	*irrational*	N/A	*irrational*	*irrational*
λ,ν	N/A	$\lambda(1-2\nu)/2\nu$	$\lambda(1+\nu)(1-2\nu)/\nu$	N/A	$\lambda(1+\nu)/3\nu$
λ,k	N/A	$3(k-\lambda)/2$	$9k(k-\lambda)/(3k-\lambda)$	$\lambda/(3k-\lambda)$	N/A
μ,E	$(2\mu-E)\mu/(E-3\mu)$	N/A	N/A	$(E-2\mu)/2\mu$	$\mu E/3(3\mu-E)$
μ,ν	$2\mu\nu/(1-2\nu)$	N/A	$2\mu(1+\nu)$	N/A	$2\mu(1+\nu)/3(1-2\nu)$
μ,k	$(3k-2\mu)/3$	N/A	$9k\mu/(3k+\mu)$	$(3k-2\mu)/2(3k+\mu)$	N/A
E,ν	$\nu E/(1+\nu)(1-2\nu)$	$E/2(1+\nu)$	N/A	N/A	$E/3(1-2\nu)$
E,k	$3k(3k-E)/(9k-E)$	$3kE/(9k-E)$	N/A	$(3k-E)/6k$	N/A
ν,k	$3k\nu/(1+\nu)$	$3k(1-2\nu)/2(1+\nu)$	$3k(1-2\nu)$	N/A	N/A

Elastic Constants (cont.)

- Energetic considerations (see 11.3): $E,\ k,\ \lambda,\ \mu > 0 \ \text{ and } \ -1 \le \nu < \frac{1}{2}$

- From chart: $\nu \to \frac{1}{2} \ \Rightarrow \begin{cases} k \to \infty \\ \mu \to E/3 \end{cases}$ **Incompressible material**

- λ has no obvious physical meaning

- Notions of **isotropy** and **homogeneity** are disjointed
  - anisotropic and homogeneous ⇨ $C_{ijkl} = const.$
  - anisotropic and inhomogeneous ⇨ $C_{ijkl} = C_{ijkl}(\vec{x})$
  - isotropic and homogeneous ⇨ $\lambda,\ \mu = const.$
  - isotropic and inhomogeneous ⇨ $\lambda = \lambda(\vec{x}),\ \mu = \mu(\vec{x})$

Elastic Constants (cont.)

- Alternative form:

$$\sigma_{ij} = \frac{E}{1+\nu}\varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)}\delta_{ij}\varepsilon_{kk}$$

$$E\varepsilon_{ij} = (1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}$$

- Expanding:

$$E\varepsilon_{11} = \underbrace{\sigma_{11}}_{\text{Axial effect}} - \underbrace{\nu(\sigma_{22}+\sigma_{33})}_{\text{Poisson effect}}$$

$$E\varepsilon_{22} = \sigma_{22} - \nu(\sigma_{11}+\sigma_{33})$$

$$E\varepsilon_{33} = \sigma_{33} - \nu(\sigma_{11}+\sigma_{22})$$

$$\varepsilon_{12} = \frac{(1+\nu)}{E}\sigma_{12} = \frac{1}{2\mu}\sigma_{12}$$

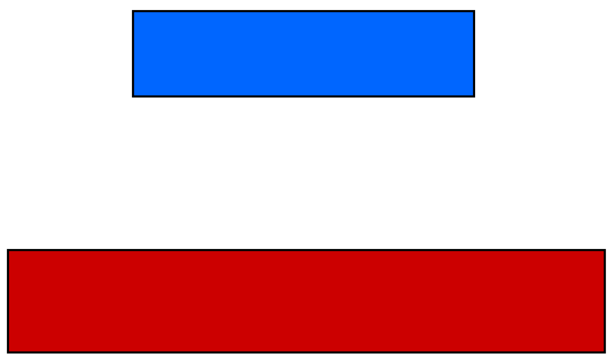
$$\varepsilon_{13} = \frac{1}{2\mu}\sigma_{13}$$

$$\varepsilon_{23} = \frac{1}{2\mu}\sigma_{23}$$

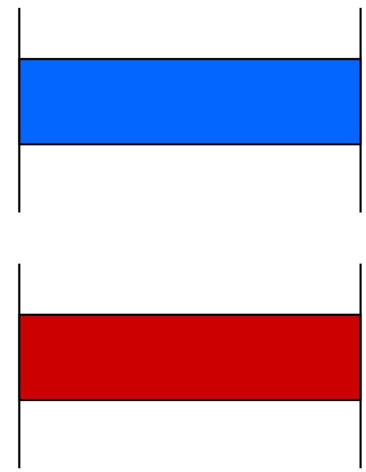
Elastic Constants (cont.) - Typical Values

	E (GPa)	ν	μ (GPa)	ρ (kg/m^3)	σ_y (MPa)	σ_{UTS} (MPa)
Steel AISI-4340	**207**	**0.3**	**80**	**7833**	**≈ 800**	**≈ 1700**
Al 6061-T6	**70**	**0.33**	**30**	**2700**	**275**	**310**
Ti-10V-2Fe-3Al	**110**	**0.32**	**40**	**4650**	**≈ 1350**	**≈ 1400(α)**
Plexiglas (PMMA)	**3.25**	**0.35**	**1.2**	**1190**	**70 (tensile)**	**70**
Glass	**70**	**0.22**	**30**	**2500**		

Thermal Change



$$\varepsilon_{ii} = \alpha\Delta T \quad (no \quad sum)$$



$$\varepsilon_{ii} = 0 \quad (no \quad sum)$$

α: coefficient of thermal expansion

$\Rightarrow$ Thermal Stresses

$$\Delta T = T - T_0$$

$$E(\varepsilon_{11} - \alpha\Delta T) = \sigma_{11} - \nu(\sigma_{22} + \sigma_{33})$$

$$E(\varepsilon_{22} - \alpha\Delta T) = \sigma_{22} - \nu(\sigma_{11} + \sigma_{33})$$

$$E(\varepsilon_{33} - \alpha\Delta T) = \sigma_{33} - \nu(\sigma_{11} + \sigma_{22})$$

$$2\mu\varepsilon_{ij} = \sigma_{ij} \qquad i \neq j$$

Chapter 4 - Material Behavior

Uniaxial response-Tensile test

Generalized Hooke's law

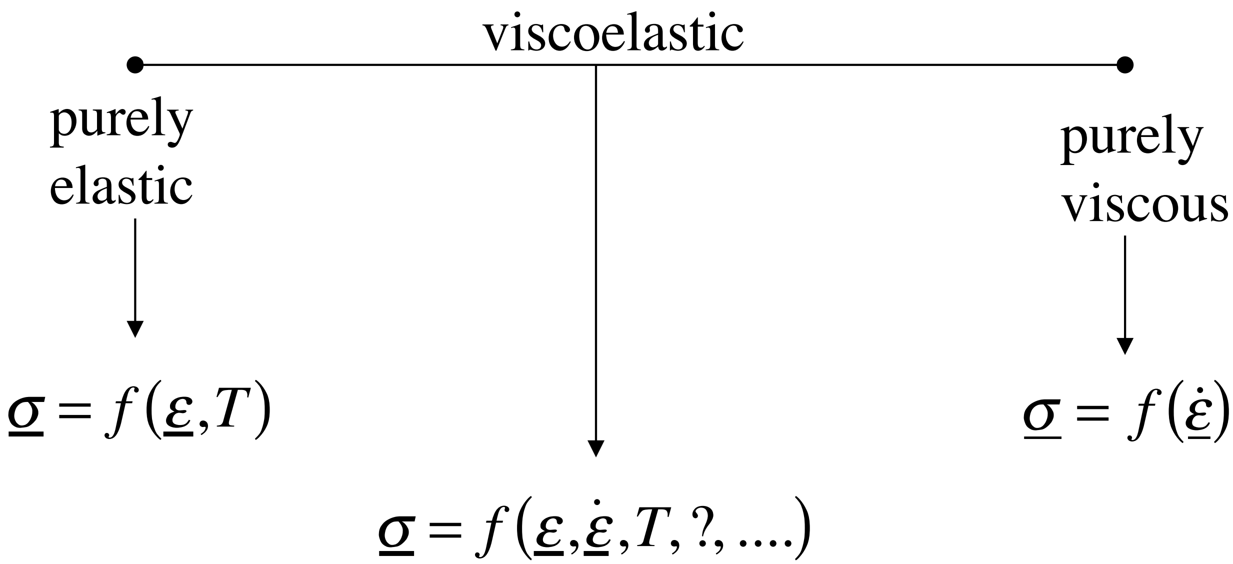
- Anisotropic solid
- Material symmetry

Isotropic solid

- Elastic constants
- Thermal change

★ **Viscoelastic materials**

Viscoelastic Materials



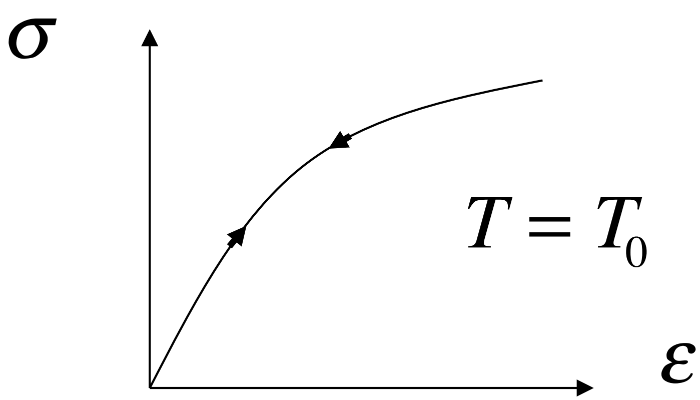
• <u>All materials</u> exhibit viscoelasticity to some degree, depending upon strain rate and temperature.

Viscoelastic Materials (cont.)

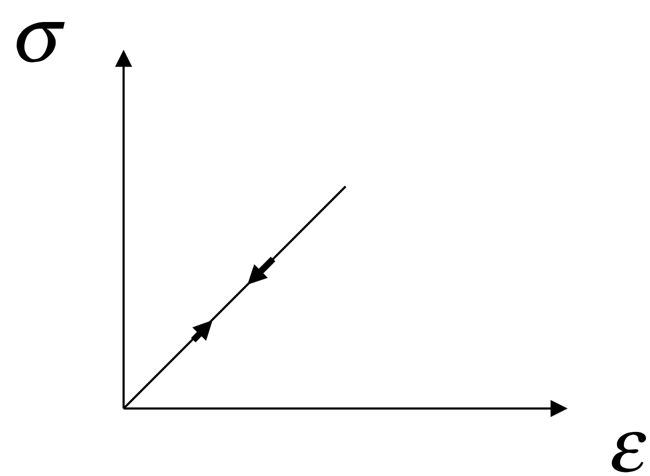
❶ Purely Elastic Behavior

$$\sigma_{ij} = f\left(\varepsilon_{ij}, T\right)$$

$\underline{\sigma}$ depends on instanteneous $(\underline{\varepsilon}, T)$



• Special case: Linear elastic



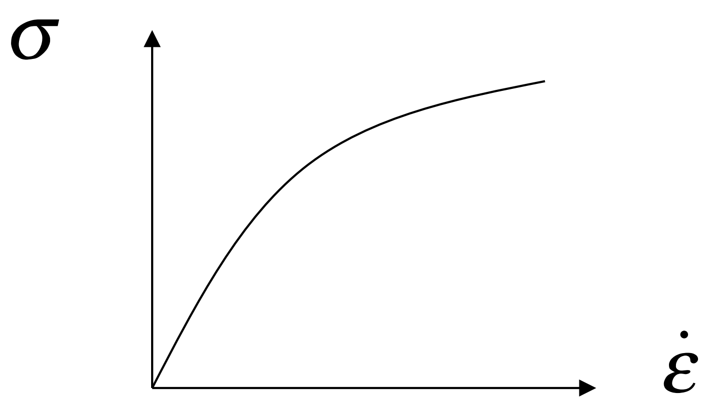
$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk} - \alpha\delta_{ij}(3\lambda + 2\mu)(T - T_0)$$

Viscoelastic Materials (cont.)

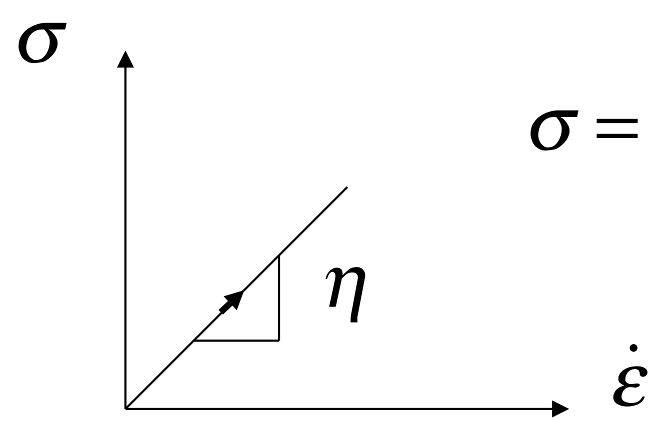
❷ Purely Viscous Behavior

$$\sigma_{ij} = f\left(\dot{\varepsilon}_{ij}\right)$$

$\underline{\sigma}$ depends on instanteneous strain rate



- Special case: Linear dependence



$$\sigma = \eta\dot{\varepsilon}$$

- **Newtonian Solid**: completely dissipates all input energy

Viscoelastic Materials (cont.)

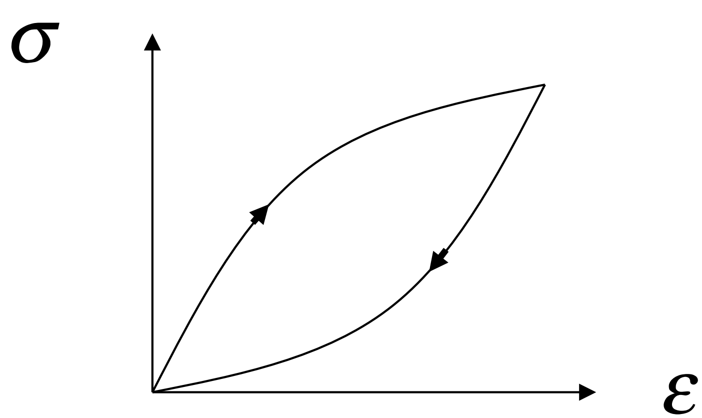
❸ Viscoelastic Behavior

• Combining elastic and viscous materials

$\Rightarrow$ $\underline{\sigma}$ depends on instanteneous $\underline{\varepsilon}, \underline{\dot{\varepsilon}}, T$ and past history of $\underline{\varepsilon}$

$$\sigma_{ij} = f\left(\varepsilon_{ij}(t), \varepsilon_{ij}(t-\Delta t), \varepsilon_{ij}(t-2\Delta t), \ldots, T\right)$$

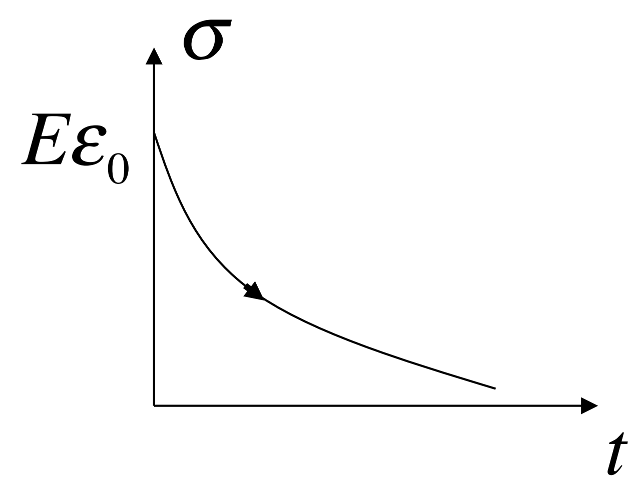
$$= f\left(\varepsilon_{ij}(t), T, \int \varepsilon dt\right)$$



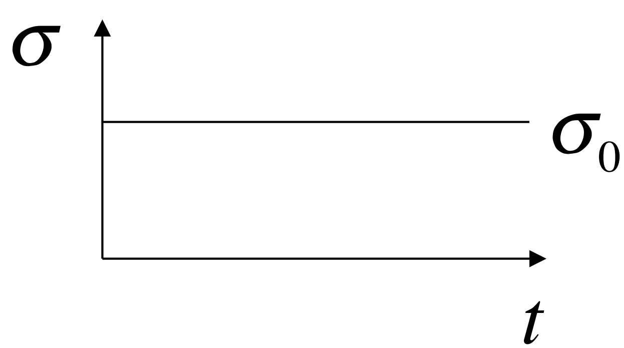
• Dissipates some energy, but returns to (0,0)
e.g., polymers, polymer composites, metals at high temperatures

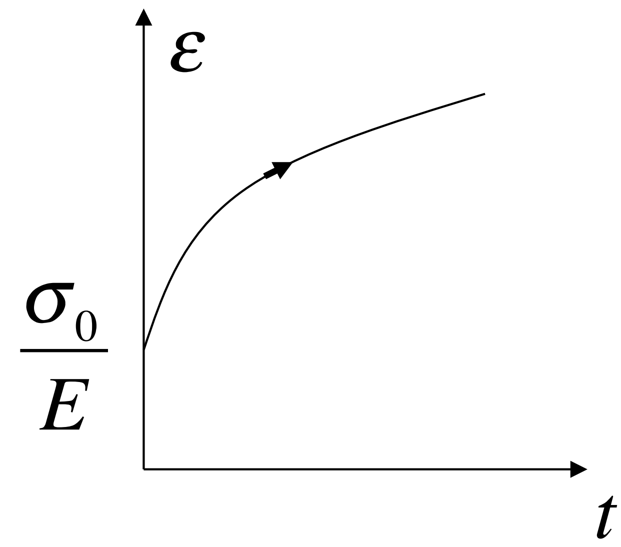
Viscoelastic Materials (cont.)- Modeling

- **Stress relaxation:**



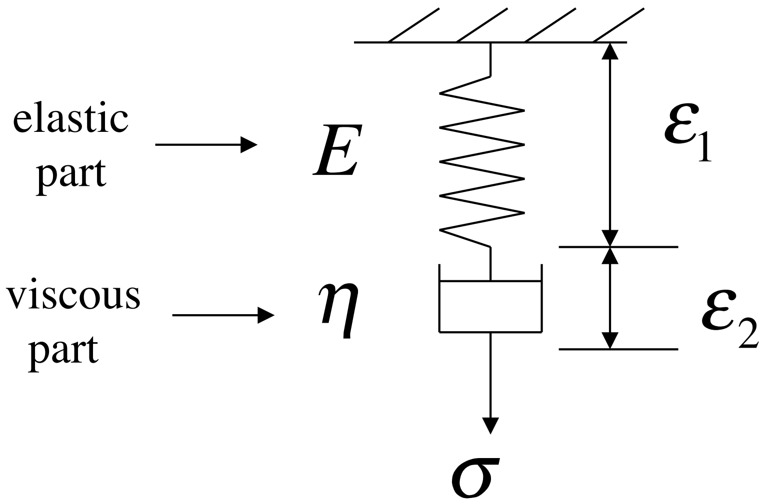
- **Creep:**





Viscoelastic Materials (cont.)- Modeling

- Maxwell Model (series):



— Same σ
— Different ε

$$\varepsilon_{tot} = \varepsilon = \varepsilon_1 + \varepsilon_2 \quad \Rightarrow \quad \dot{\varepsilon} = \dot{\varepsilon}_1 + \dot{\varepsilon}_2$$

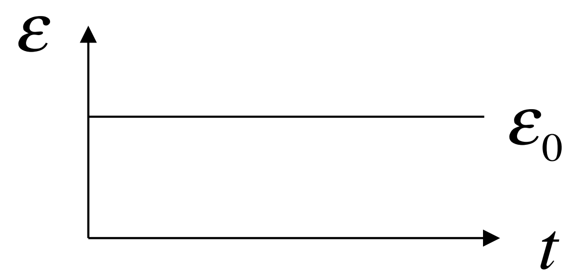
but, $$\varepsilon_1 = \frac{\sigma}{E}, \qquad \dot{\varepsilon}_2 = \frac{\sigma}{\eta}$$

$$\boxed{\Rightarrow \quad \dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}}$$

Viscoelastic Materials (cont.)- Modeling

- For relaxation $\varepsilon(t) = \varepsilon_0 H(t)$

  $H(t)$: step function



$$\left.\begin{aligned} \dot{\varepsilon} &= \frac{d}{dt}\left[\varepsilon_0 H(t)\right] = 0 \\ \text{and} \quad \dot{\varepsilon} &= \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} \end{aligned}\right\}$$

$$\Rightarrow \frac{1}{E}\frac{d\sigma}{dt} + \frac{\sigma}{\eta} = 0$$

$$\Rightarrow \frac{d\sigma}{\sigma} = -\frac{E}{\eta}dt$$

$$\Rightarrow \ln\sigma = -\frac{E}{\eta}t + C$$

$$\Rightarrow \sigma = C\exp\left(-\frac{E}{\eta}t\right)$$

Viscoelastic Materials (cont.)- Modeling

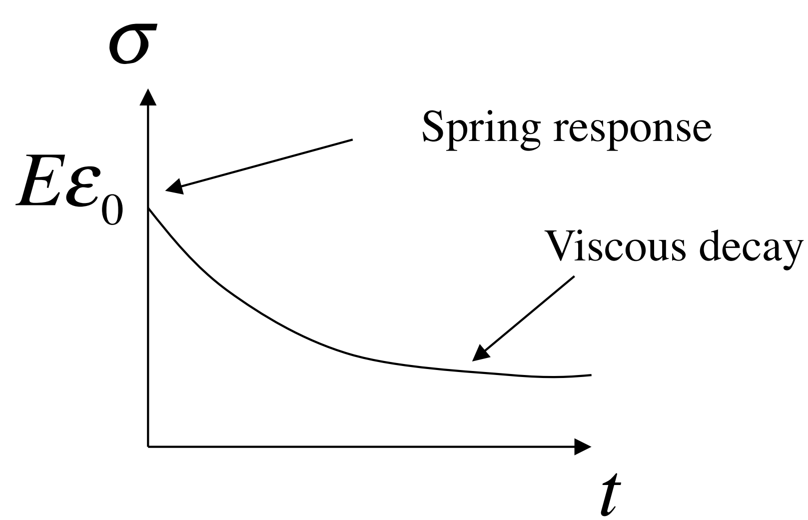
$$\sigma = C \exp\left(-\frac{E}{\eta} t\right) \qquad \text{At } t = 0, \quad \sigma = \sigma_0 = E\varepsilon_0$$

instantaneous elastic response

$$\Rightarrow \quad C = E\varepsilon_0$$

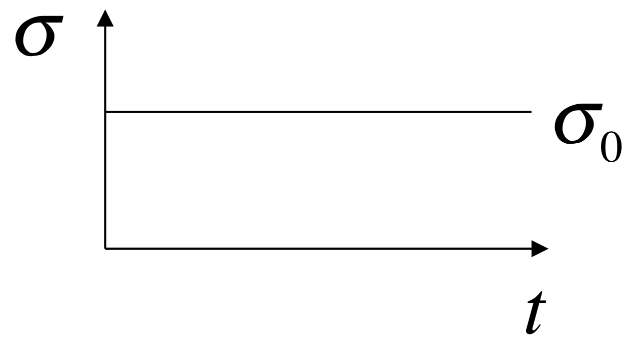
- Call: $\tau = \dfrac{\eta}{E}$ **relaxation time**

$$\boxed{\therefore \quad \sigma = E\varepsilon_0 \exp\left(-\frac{t}{\tau}\right)}$$



Viscoelastic Materials (cont.)- Modeling

- For creep $\sigma(t) = \sigma_0 H(t)$



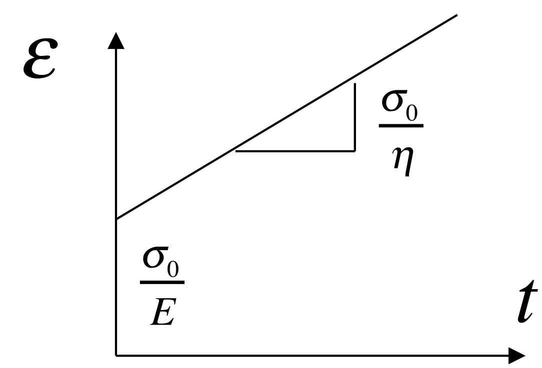
$$\dot{\varepsilon} = \frac{d\varepsilon}{dt} = \cancelto{0}{\frac{\dot{\sigma}}{E}} + \frac{\sigma}{\eta} \quad \Rightarrow \quad \varepsilon = \frac{\sigma_0}{\eta} t + C$$

$$\text{At } t = 0, \quad \varepsilon = \varepsilon_0 = \frac{\sigma_0}{E}$$

instantaneous elastic response

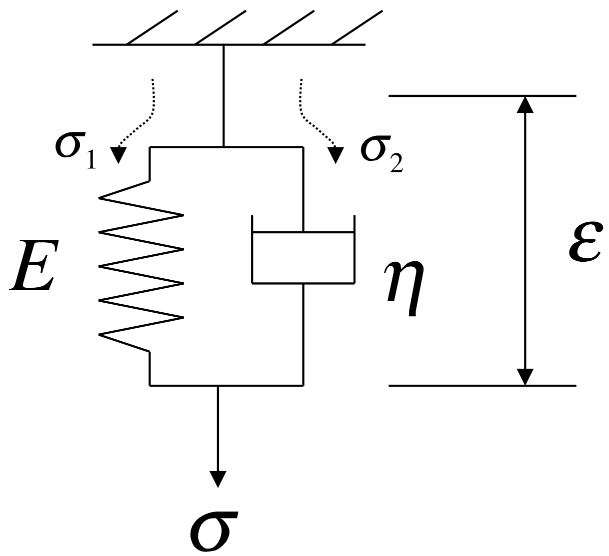
$$\Rightarrow \quad C = \frac{\sigma_0}{E}$$

$$\therefore \quad \boxed{\varepsilon = \frac{\sigma_0}{E}\left(\frac{t}{\tau} + 1\right)}$$



Viscoelastic Materials (cont.)- Modeling

- Kelvin-Voigt Model (parallel):



— Same ε
— Different σ

$$\left.\begin{array}{l}\sigma_{tot} = \sigma = \sigma_1 + \sigma_2 \\ \\ \text{but,} \quad \begin{array}{l}\sigma_1 = E\varepsilon \\ \sigma_2 = \eta\dot{\varepsilon}\end{array}\end{array}\right\} \quad \boxed{\Rightarrow \quad \sigma = E\varepsilon + \eta\dot{\varepsilon}}$$

Viscoelastic Materials (cont.)- Modeling

- For relaxation $\varepsilon(t) = \varepsilon_0 H(t)$

$$\Rightarrow \quad \sigma = E\varepsilon + \eta 0$$

$$\Rightarrow \quad \sigma = E\varepsilon_0$$

i.e., does NOT predict relaxation

- For creep $\sigma(t) = \sigma_0 H(t) \quad \Rightarrow \quad \sigma = \sigma_0 = E\varepsilon + \eta\dot{\varepsilon}$

Homogeneous part: $$\eta \frac{d\varepsilon}{dt} + E\varepsilon = 0 \Rightarrow \varepsilon = C\exp(-t/\tau)$$

Particular solution: Try $\varepsilon = const. = A \quad \Rightarrow \quad \sigma_0 = EA + 0 \Rightarrow A = \sigma_0/E$

$$\Rightarrow \quad \varepsilon = C\exp(-t/\tau) + \sigma_0/E$$

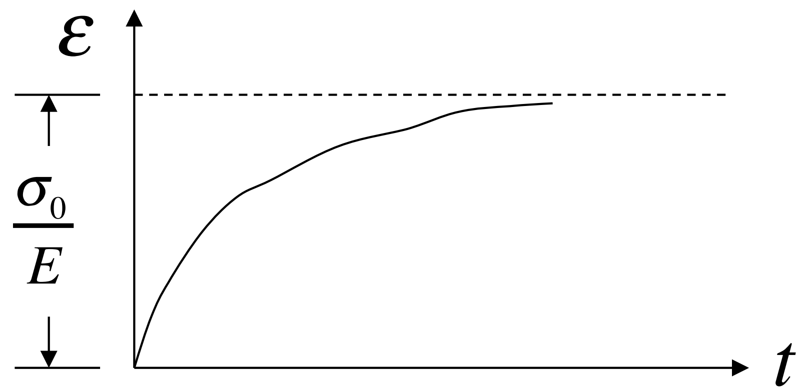
Viscoelastic Materials (cont.)- Modeling

At $t = 0, \quad \varepsilon = 0$ $\quad \Rightarrow \; 0 = C + \sigma_0 / E$

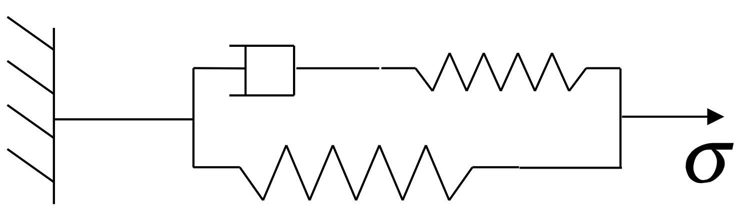
Instantaneous response:

$$\Rightarrow \; C = -\sigma_0 / E$$

$$\boxed{\therefore \; \varepsilon = \frac{\sigma_0}{E}\left[1 - e^{(-t/\tau)}\right]}$$



• Can generate more complex viscoelastic models by combining springs and dashpots, e.g.



Standard linear solid

END OF CHAPTER 4