Chapter 5 - Formulation and Solution of Elasticity Problems

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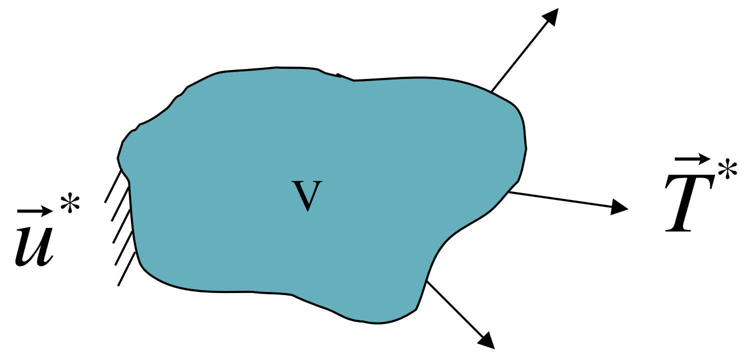
Formulation



In the volume

- Equilibrium:

$$\sigma_{ij,j} + f_i = 0 \qquad ①$$

- Constitutive relations:

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} \quad \text{or} \quad \sigma_{ij} = C_{ijkl}\varepsilon_{kl} \qquad ②$$

- Strain-displacement:

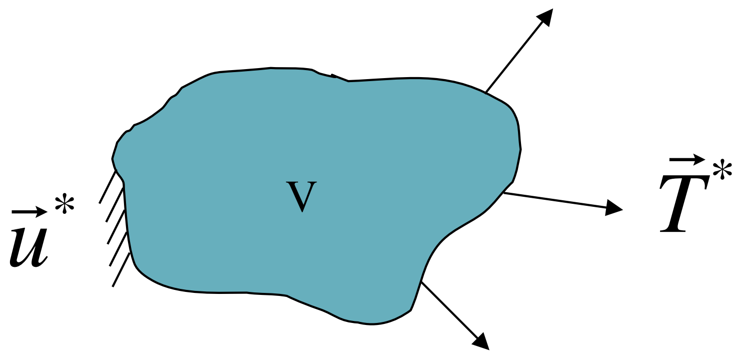
$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \qquad ③$$

- Note: assuming $\boldsymbol{\varepsilon}$
  ➔ compatibility:

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} = \varepsilon_{ik,jl} + \varepsilon_{jl,ik}$$

➔ system of 15 equations for 15 unknowns

Formulation (cont.)



On the surface

- On traction boundary:

$$T_i^* = \sigma_{ij} n_j$$

- On displacement boundary:

$$u_i = u_i^*$$

Field (volume) equations + boundary conditions = **Boundary Value Problem (BVP)**

- Two ways of formulation:
  - **Displacement based formulation**
  - **Stress based formulation**

Displacement Formulation

- Guess (?) displacement field (continuous and differentiable):

$$u_i \xrightarrow{③} \varepsilon_{ij} \xrightarrow{②} \sigma_{ij} \xrightarrow{①} \text{check equilibrium} \rightarrow \text{b.c.s}$$

$$\left.\begin{aligned} &② \quad \sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij} \\ &③ \quad \varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \end{aligned}\right\} \Rightarrow \quad \sigma_{ij} = \lambda \delta_{ij} u_{k,k} + \mu\left(u_{i,j} + u_{j,i}\right)$$

$$① \quad \sigma_{ij,j} + f_i = 0$$

$$\Rightarrow \quad \lambda \delta_{ij} u_{k,kj} + \mu\left(u_{i,jj} + u_{j,ij}\right) + f_i = 0$$

$$\boxed{\Rightarrow \quad (\lambda + \mu) u_{j,ij} + \mu u_{i,jj} + f_i = 0}$$

Lamé's equations

# of eqns ? # of unks ?

Stress Formulation

- Guess (?) stress (or strain) field:

$$\text{check equilibrium} \xrightarrow{①} \sigma_{ij} \xrightarrow{②} \varepsilon_{ij} \xrightarrow{\text{comp.}+③} u_i \rightarrow \text{b.c.s}$$

- Here we <u>need</u> compatibility:

$$\left.\begin{aligned} &② \quad \sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij} \\ &\varepsilon_{ij,kl} + \varepsilon_{kl,ij} = \varepsilon_{ik,jl} + \varepsilon_{jl,ik} \\ &① \quad \sigma_{ij,j} + f_i = 0 \end{aligned}\right\}$$

$$\boxed{\Rightarrow \quad \sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = \frac{-\nu}{1+\nu}\delta_{ij} f_{k,k} - \left(f_{i,j} + f_{j,i}\right)}$$

Beltrami-Mitchell eqns

# of eqns ? # of unks ?

$$\boxed{\Rightarrow \quad \sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = 0}$$

if f_i=const.

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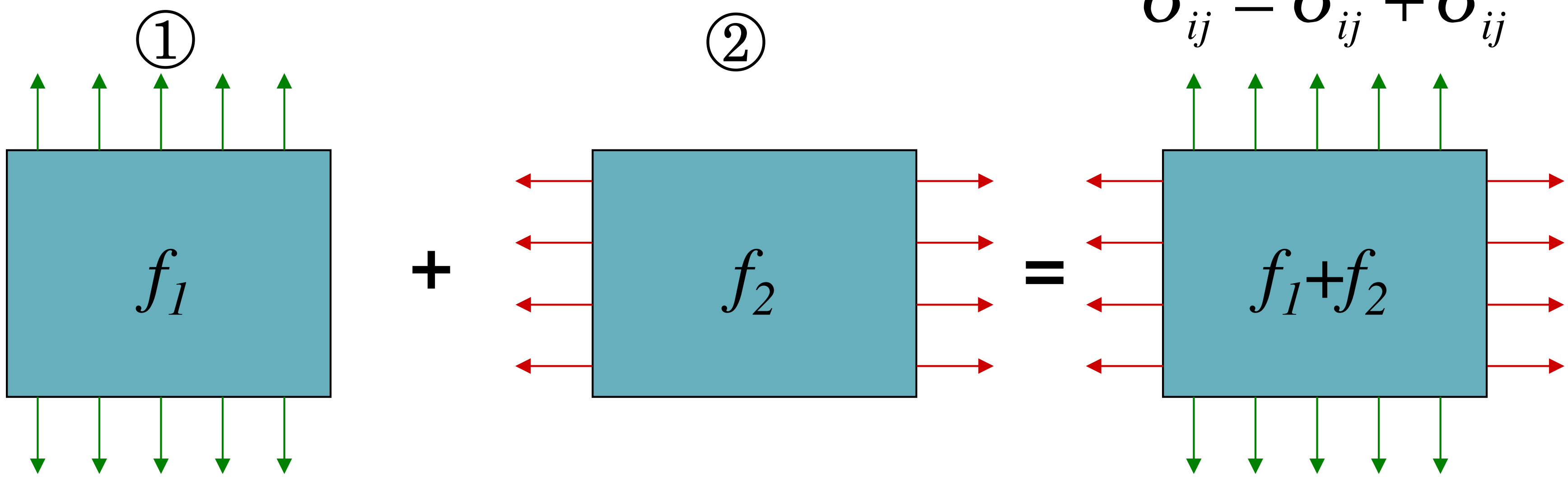
Superposition and Uniqueness

- Linear equations → **Superposition**

$$\sigma_{ij} = \sigma_{ij}^{1} + \sigma_{ij}^{2}$$



$$\text{e.g.,} \quad \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + f_1 =$$

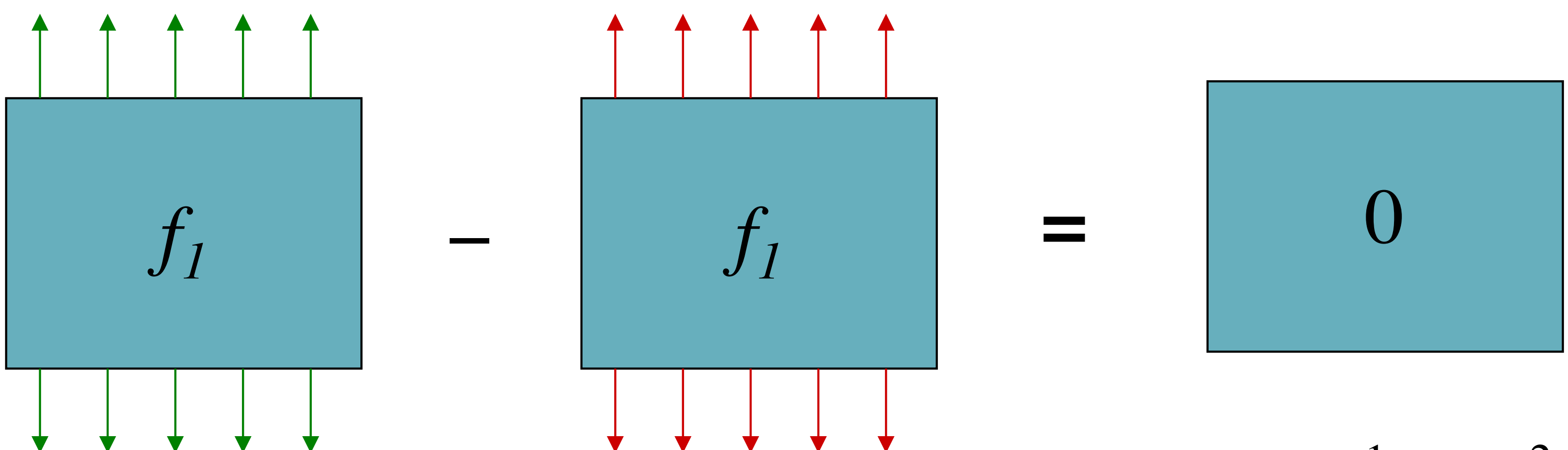
$$= \left[\frac{\partial \sigma_{11}^{1}}{\partial x_1} + \frac{\partial \sigma_{12}^{1}}{\partial x_2} + \frac{\partial \sigma_{13}^{1}}{\partial x_3} + f_1^{1} \right] + \left[\frac{\partial \sigma_{11}^{2}}{\partial x_1} + \frac{\partial \sigma_{12}^{2}}{\partial x_2} + \frac{\partial \sigma_{13}^{2}}{\partial x_3} + f_1^{2} \right] =$$

$$= 0 + 0$$

Superposition and Uniqueness (cont.)

- Super position does NOT hold if any one equation becomes **NONLINEAR!**
  — plastic constitutive law
  — Lagrangian/ Eulerian strains

- A solution to a linear elastic BVP is **UNIQUE!**



① ②

$$\sigma_{ij} = \sigma_{ij}^{1} - \sigma_{ij}^{2} = 0$$

Solution Strategies

- Direct method:
  - — integrate PDEs
  - — v. difficult
  - — usually done numerically

- Inverse method:
  - — guess $\boldsymbol{\sigma}$ or $\boldsymbol{u}$ satisfying some PDEs and/or b.c.s
  - — e.g. strength of materials
  - — most common

- Semi-Inverse method:
  - — start with part of solution
  - — integrate for remainder

Note: Uniqueness ➔ whatever way we use we have THE! solution