Chapter 2 - Kinematics: Theory of Strain

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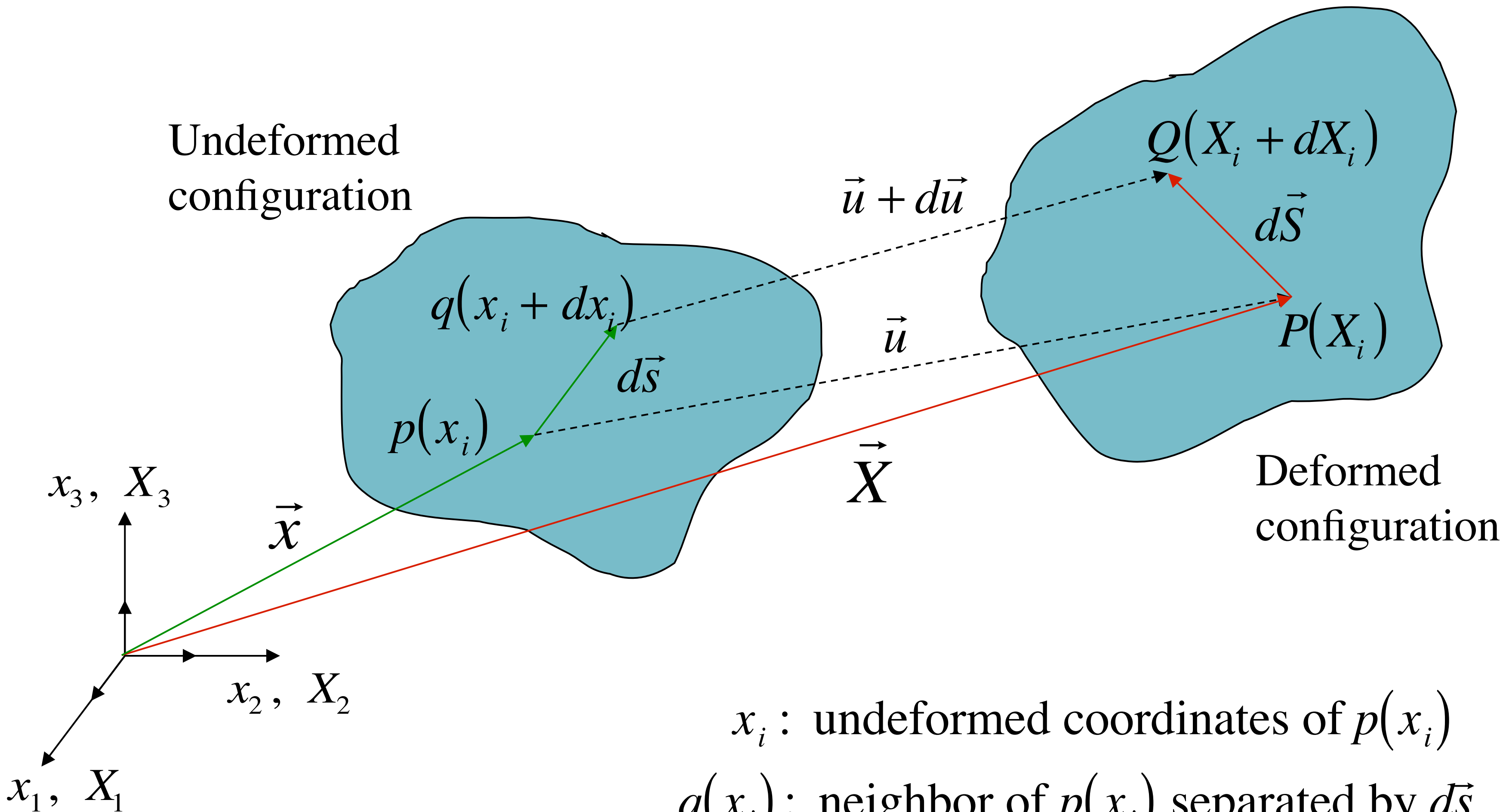
Measures of Deformation



x_i : undeformed coordinates of $p(x_i)$

$q(x_i)$: neighbor of $p(x_i)$ separated by $d\vec{s}$

X_i : deformed coordinates of $P(X_i)$

$Q(X_i)$: neighbor of $P(X_i)$ separated by $d\vec{S}$

$$\boxed{\vec{X} = \vec{x} + \vec{u}}$$

Measures of Deformation (cont.)

- If p and q move such that $\Rightarrow \quad d\vec{s} = d\vec{S}$

$\Rightarrow$ rigid motion

$\Rightarrow$ no deformation

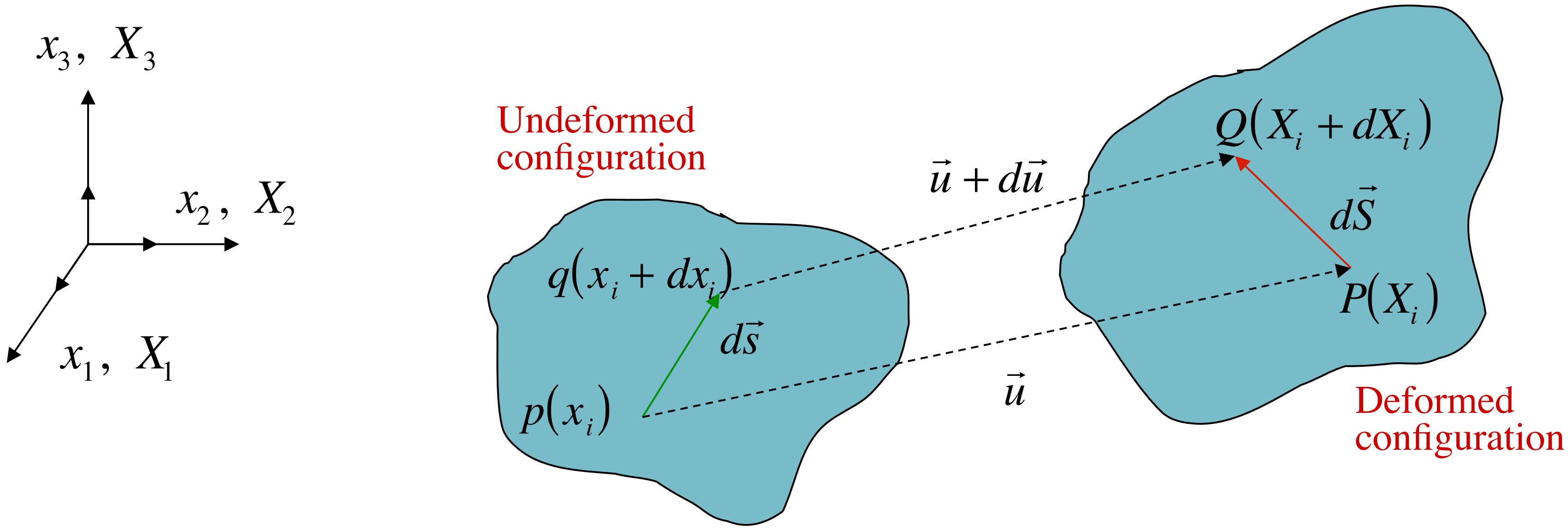
- Possible measures of deformation

$$dS - ds,\ \sqrt{dS} - \sqrt{ds},\ f(dS) - f(ds)$$

with $\quad ds = \left|d\vec{s}\right| \ ,\ dS = \left|d\vec{S}\right|$

- Choose: $\quad \boxed{dS^2 - ds^2}$

Lagrangian Strain Tensor



$$ds = \left|d\vec{s}\right| = \sqrt{dx_1^2 + dx_2^2 + dx_3^2} = \sqrt{dx_i dx_i}$$

$$dS = \left|d\vec{S}\right| = \sqrt{dX_1^2 + dX_2^2 + dX_3^2} = \sqrt{dX_i dX_i}$$

$$\Rightarrow \quad dS^2 - ds^2 = dX_i dX_i - dx_i dx_i$$

Lagrangian Strain Tensor (cont.)

$dS^2 - ds^2 = dX_i dX_i - dx_i dx_i$ is a field (i.e. depends on coordinate)

- Choose **undeformed configuration** x_i as independent variable

$$X_i = X_i(x_1, x_2, x_3) = X_i(x_i)$$

$$\Rightarrow \quad dX_i = \frac{\partial X_i}{\partial x_1} dx_1 + \frac{\partial X_i}{\partial x_2} dx_2 + \frac{\partial X_i}{\partial x_3} dx_3$$

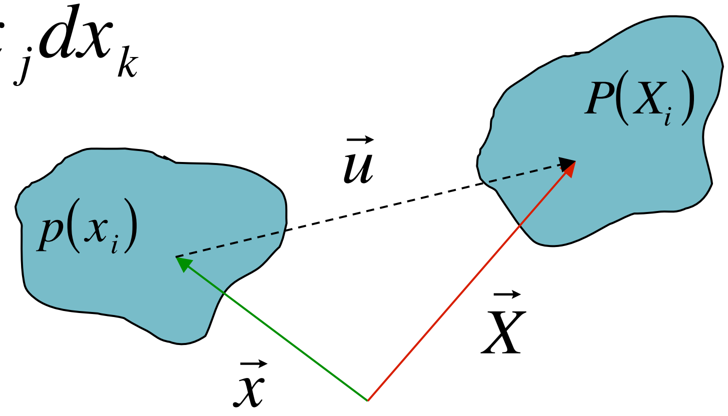
$$\Rightarrow \quad dX_i = X_{i,j} dx_j$$

$$\Rightarrow \quad dS^2 - ds^2 = \left(X_{i,j} dx_j\right)\left(X_{i,k} dx_k\right) - dx_i dx_i$$

$$\therefore \quad dS^2 - ds^2 = \left(X_{i,j} X_{i,k} - \delta_{ij}\delta_{ik}\right) dx_j dx_k$$

Lagrangian Strain Tensor (cont.)

$$dS^2 - ds^2 = \left(X_{i,j}X_{i,k} - \delta_{ij}\delta_{ik}\right)dx_j dx_k$$



$$\vec{X} = \vec{x} + \vec{u} \quad \Rightarrow \quad X_i = x_i + u_i$$

$$dS^2 - ds^2 = \left[\left(x_i + u_i\right)_{,j}\left(x_i + u_i\right)_{,k} - \delta_{jk}\right]dx_j dx_k$$

- Recall: $$x_{i,j} = \frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

Lagrangian Strain Tensor (cont.)

$$dS^2 - ds^2 = \left[\left(\delta_{ij} + u_{i,j}\right)\left(\delta_{ik} + u_{i,k}\right) - \delta_{jk}\right] dx_j dx_k$$

$$= \left(\delta_{ij}\delta_{ik} + \delta_{ij}u_{i,k} + u_{i,j}\delta_{ik} + u_{i,j}u_{i,k} - \delta_{jk}\right) dx_j dx_k$$

$$= \left(\delta_{jk} + u_{j,k} + u_{k,j} + u_{i,j}u_{i,k} - \delta_{jk}\right) dx_j dx_k$$

$$= \left(u_{j,k} + u_{k,j} + u_{i,j}u_{i,k}\right) dx_j dx_k$$

- Define:

$$\boxed{E_{ij} = \varepsilon_{ij}^{L} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right)}$$

- So that:

$$dS^2 - ds^2 = 2E_{ij}dx_i dx_j$$

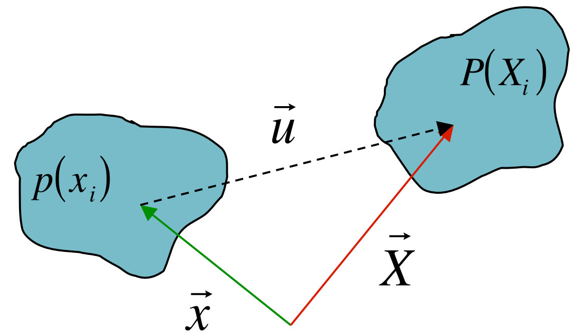
Lagrangian Strain Tensor (cont.)

$$E_{ij} = \varepsilon_{ij}^{L} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right)$$

- E_{ij} are the components of the **Lagrangian** or **Green's** or **material** strain tensor
- Lagrangian: is expressed in terms of undeformed (or Lagrangian coordinates)
- By definition $E_{ij} = E_{ji}$, i.e. symmetric
- No assumptions on loading, material or amount of deformation

Eulerian Strain Tensor

- Recall $\boxed{dS^2 - ds^2 = dX_i dX_i - dx_i dx_i}$



- Choose deformed X_i as independent variable

$$x_i = x_i(X_1, X_2, X_3) = x_i(X_i)$$

$$\Rightarrow \quad dx_i = \frac{\partial x_i}{\partial X_1} dX_1 + \frac{\partial x_i}{\partial X_2} dX_2 + \frac{\partial x_i}{\partial X_3} dX_3$$

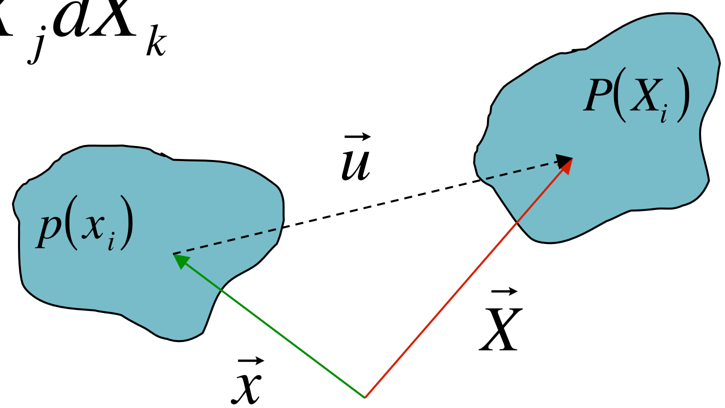
$$\Rightarrow \quad dx_i = x_{i,J} dX_J \longleftarrow$$ Differentiation wrt X

$$\Rightarrow \quad dS^2 - ds^2 = dX_i dX_i - (x_{i,J} dX_J)(x_{i,K} dX_K)$$

$$\therefore \quad dS^2 - ds^2 = (\delta_{ij}\delta_{ik} - x_{i,J} x_{i,K}) dX_J dX_K$$

Eulerian Strain Tensor (cont.)

$$dS^2 - ds^2 = \left(\delta_{ij}\delta_{ik} - x_{i,J}x_{i,K}\right)dX_j dX_k$$



$$\vec{X} = \vec{x} + \vec{u} \quad \Rightarrow \quad x_i = X_i - u_i$$

$$dS^2 - ds^2 = \left[\delta_{jk} - \left(X_i - u_i\right)_{,J}\left(X_i - u_i\right)_{,K}\right]dx_j dx_k$$

- Recall: $X_{i,J} = \dfrac{\partial X_i}{\partial X_j} = \delta_{ij}$

Eulerian Strain Tensor (cont.)

$$dS^2 - ds^2 = \left[\delta_{jk} - \left(\delta_{ij} - u_{i,J}\right)\left(\delta_{ik} - u_{i,K}\right)\right] dX_j dX_k$$

$$= \left(\delta_{jk} - \delta_{ij}\delta_{ik} + \delta_{ij}u_{i,K} + u_{i,J}\delta_{ik} - u_{i,J}u_{i,K}\right) dX_j dX_k$$

$$= \left(\delta_{jk} - \delta_{jk} + u_{j,K} + u_{k,J} - u_{i,J}u_{i,K}\right) dX_j dX_k$$

$$= \left(u_{j,K} + u_{k,J} - u_{i,J}u_{i,K}\right) dX_j dX_k$$

- Define: $$\boxed{e_{ij} = \varepsilon_{ij}^{E} = \frac{1}{2}\left(u_{i,J} + u_{j,I} - u_{k,I}u_{k,J}\right)}$$

Differentiation wrt X_i

- So that: $$dS^2 - ds^2 = 2e_{ij} dX_i dX_j$$

Eulerian Strain Tensor (cont.)

$$e_{ij} = \varepsilon_{ij}^{E} = \frac{1}{2}\left(u_{i,J} + u_{j,I} - u_{k,I}u_{k,J}\right)$$

Differentiation wrt X_i

- e_{ij} are the components of the **Eulerian** or **Almansi** or **spatial** strain tensor
- Eulerian: is expressed in terms of deformed (or Eulerian coordinates)
- By definition $e_{ij} = e_{ji}$, i.e. symmetric
- No assumptions on loading, material or amount of deformation
- Differentiation is with respect to X_i

Infinitesimal Strain

- Lagrangian strain: $$E_{ij} = \varepsilon_{ij}^{L} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right)$$
- Eulerian strain: $$e_{ij} = \varepsilon_{ij}^{E} = \frac{1}{2}\left(u_{i,J} + u_{j,I} - u_{k,I}u_{k,J}\right)$$
- Assumptions:

(a) If $\left|u_{i,j}\right| << 1$ then $u_{k,i}u_{k,j}$ can be ignored wrt $u_{i,j}$

(b) If $x_i \approx X_i$ then $\dfrac{\partial(\ )}{\partial x_i} \approx \dfrac{\partial(\ )}{\partial X_i}$

$$E_{ij} \approx e_{ij} \approx \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \equiv \varepsilon_{ij}$$

Infinitesimal Strain (cont.)

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

- ε_{ij} are the components of the **infinitesimal strain tensor**
- By definition $\varepsilon_{ij} = \varepsilon_{ji}$, i.e. symmetric
- Condition (a) implies small displacement gradients. Condition (b) implies small displacements
- Above equations are the **strain-displacement relations**
- # of equations vs. # of unknowns
- Dimensions?

Infinitesimal Strain (cont.)

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

- Expanding:

$$\left.\begin{aligned} \varepsilon_{11} &= \frac{\partial u_1}{\partial x_1} \\ \varepsilon_{22} &= \frac{\partial u_2}{\partial x_2} \\ \varepsilon_{33} &= \frac{\partial u_3}{\partial x_3} \end{aligned}\right\} \text{Normal Strains}$$

$$\left.\begin{aligned} \varepsilon_{12} = \varepsilon_{21} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \\ \varepsilon_{13} = \varepsilon_{31} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \varepsilon_{23} = \varepsilon_{32} &= \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \end{aligned}\right\} \text{Shear Strains}$$

Infinitesimal Strain (cont.)

$$[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ & \varepsilon_{22} & \varepsilon_{23} \\ sym. & & \varepsilon_{33} \end{bmatrix}$$

- Since infinitesimal strain is a two-tensor it transforms (rotates) according to:

$$\varepsilon'_{ij} = \alpha_{ik}\alpha_{jl}\varepsilon_{kl}$$

or

$$[\underline{\varepsilon}'] = [R][\underline{\varepsilon}][R]^T$$

Engineering Strain

- Sometimes (e.g. Timoshenko and Goodier) infinitesimal shear strain is defined:

$$\gamma_{ij} = \left(u_{i,j} + u_{j,i}\right) \qquad i \neq j$$

- But then strain is NOT a two-tensor

$$\left[\varepsilon_{ij}\right] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ & \varepsilon_{22} & \varepsilon_{23} \\ sym. & & \varepsilon_{33} \end{bmatrix}$$

A tensor

$$\left[\varepsilon_{ij}\right] = \begin{bmatrix} \varepsilon_{11} & \gamma_{12} & \gamma_{13} \\ & \varepsilon_{22} & \gamma_{23} \\ sym. & & \varepsilon_{33} \end{bmatrix}$$

<u>Not</u> a tensor - **Engineering strain**

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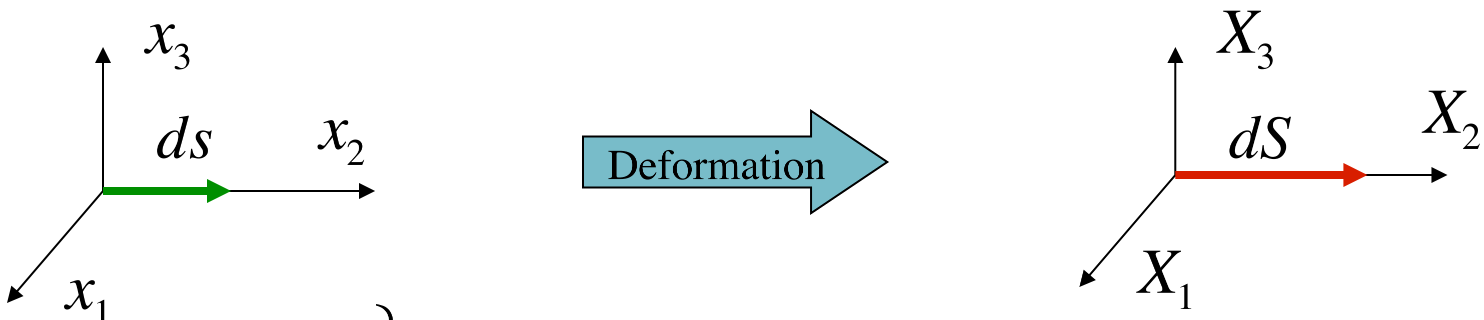
Normal components

Take $d\vec{s}$ // dx_2



Here

$$\left.\begin{aligned} dx_1 &= 0 \\ dx_2 &= ds \\ dx_3 &= 0 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} dS^2 - ds^2 &= 2E_{ij}dx_i dx_j \\ &= 2E_{22}ds^2 \end{aligned}$$

$$\Rightarrow \quad dS^2 = \left(1 + 2E_{22}\right)ds^2$$

$$\Rightarrow \quad dS = \sqrt{\left(1 + 2E_{22}\right)}ds$$

$$\therefore \quad \frac{dS}{ds} = \frac{\text{final length}}{\text{original length}} = \sqrt{1 + 2E_{22}}$$

Stretch ratio along x_2

Normal components (cont.)

- Extension ratio: $$\frac{dS-ds}{ds}=\frac{\text{change of length}}{\text{initial length}}=\sqrt{1+2E_{22}}-1$$

$$\therefore \quad E_{22} \;\propto\; \Delta L/L \text{ along } x_2$$

- For small strains $E_{22} << 1$: $$\frac{dS-ds}{ds}=\sqrt{1+2E_{22}}-1$$

$$=\left(1+2E_{22}\right)^{1/2}-1$$

$$=1+\frac{1}{2}\left(2E_{22}\right)+\ldots-1$$

$$\approx E_{22}=\varepsilon_{22}$$

$$\therefore \text{ normal strains } \varepsilon_{ii} \;(nosum)\; \text{ represent } \frac{\text{change of length}}{\text{initial length}} \text{ along } x_i$$

Shear Components

$$
\begin{aligned}
d\vec{S}^{1} \cdot d\vec{S}^{3} &= dS_i^1 dS_i^3 \\
&= dS^1 dS^3 \cos\theta \\
&= dS^1 dS^3 \cos\left(\frac{\pi}{2} - \gamma_1 - \gamma_3\right) \\
&= dS^1 dS^3 \cos\left(\frac{\pi}{2} - \gamma\right)
\end{aligned}
$$

with $\gamma = \gamma_1 + \gamma_3$

$$
\boxed{\Rightarrow \quad dS_i^1 dS_i^3 = dS^1 dS^3 \cos\left(\frac{\pi}{2} - \gamma\right)}
$$

Shear Components (cont.)

- Recall: $X_i = x_i + u_i$

$$\therefore \quad dS_i = dX_i = X_{i,j} dx_j = \left(\delta_{ij} + u_{i,j}\right) dx_j$$

$$\Rightarrow \quad dS_i^1 dS_i^3 = \left(\delta_{ij} + u_{i,j}\right) dx_j^1 \left(\delta_{ik} + u_{i,k}\right) dx_k^3$$

$$= \left(\delta_{jk} + u_{j,k} + u_{k,j} + u_{i,j} u_{i,k}\right) dx_j^1 dx_k^3$$

- For this choice of geometry:

$$dx_1^1 = ds^1 \qquad dx_2^1 = dx_3^1 = 0$$

$$dx_2^3 = dx_2^3 = 0 \qquad dx_3^3 = ds^3$$

Shear Components (cont.)

- Then j=1 and k=3 survive:

$$\begin{aligned} dS_i^1 dS_i^3 &= \left(\delta_{13} + u_{1,3} + u_{3,1} + u_{i,1}u_{i,3}\right) ds^1 ds^3 \\ &= 2E_{13} ds^1 ds^3 \end{aligned}$$

E_{13} Lagrangian strain component

- Substitute into ★
- Using the normal result:

$$\left.\begin{aligned} dS_i^1 dS_i^3 &= dS^1 dS^3 \cos\left(\frac{\pi}{2} - \gamma\right) \\ &= 2E_{13} ds^1 ds^3 \\ \frac{dS^1}{ds^1} &= \sqrt{1+2E_{11}}, \quad \frac{dS^3}{ds^3} = \sqrt{1+2E_{33}} \end{aligned}\right\} \Rightarrow$$

Shear Components (cont.)

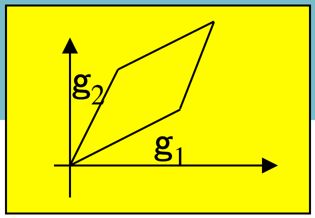
$$\Rightarrow \quad \cos\left(\frac{\pi}{2}-\gamma\right)=\sin\gamma=\frac{2E_{13}}{\sqrt{1+2E_{11}}\sqrt{1+2E_{33}}}$$

- Under small strain conditions: $E_{11} \ll 1$, $E_{33} \ll 1$ and γ small

$$\sin\gamma \approx \gamma, \quad \sqrt{1+2E_{11}} \approx 1, \quad \sqrt{1+2E_{33}} \approx 1, \quad E_{13} \approx \varepsilon_{13}$$

$$\therefore \quad \gamma \approx 2\varepsilon_{13} \quad \Rightarrow \quad \varepsilon_{13} \approx \frac{1}{2}\gamma = \frac{1}{2}\left(\gamma_1+\gamma_3\right)$$

$\therefore$ shear strains ε_{ij} $(i \neq j)$ represent $\frac{1}{2}\left(\text{change of angle from } \frac{\pi}{2}\right)$ between x_i and x_j



Remarks

- Under small displacement gradients ε_{ij} attain physical meaning

$$\varepsilon_{ii}(nosum) = \frac{\text{change of length}}{\text{intial length}}$$

$$\varepsilon_{ij}(i \neq j) = \frac{1}{2}\left(\text{change of angle from } \frac{\pi}{2}\right)$$

- Sometimes shear strain is taken as "total change from π/2". This is the Engineering Strain, but is NOT a tensor.

- Note that: $u_{i,j} = \varepsilon_{ij} + \omega_{ij}$ where $\omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right)$

Clearly: $\omega_{ij} = 0 \;\; (i = j)$ and $\omega_{ij} = -\omega_{ji} \;\; (i \neq j)$

ω is a skew-symmetric tensor called the **rotation tensor**, e.g.

$$\omega_{13} = \frac{\gamma_1 - \gamma_3}{2}, \quad \text{average rotation about } x_2$$

Volume and Shape Changes

- Principal strains: $\left(\varepsilon_{ij} - \lambda\delta_{ij}\right)n_j = 0 \quad \Rightarrow \quad \det\left[\varepsilon_{ij} - \lambda\delta_{ij}\right] = 0$

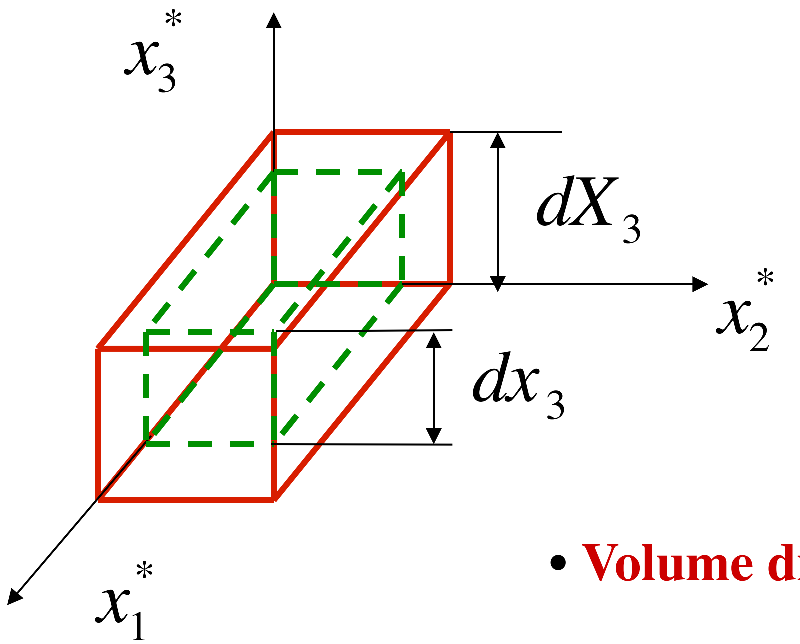
$$\Rightarrow \left.\begin{array}{l} \lambda_1 = \varepsilon_1 \\ \lambda_2 = \varepsilon_2 \\ \lambda_3 = \varepsilon_3 \end{array}\right\} \text{ principal strains}$$

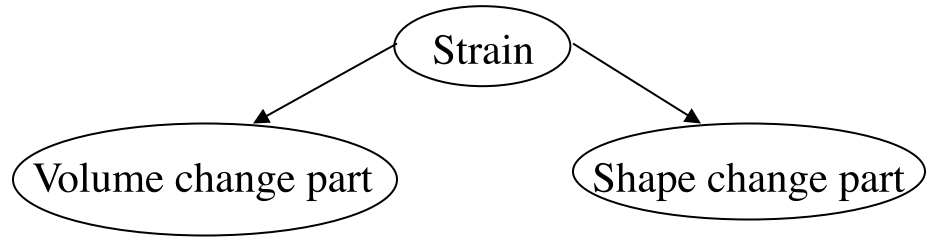
- In a principal frame for strain:

$$\left[\varepsilon_{ij}\right] = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$$

- Principal frames for strain and stress coincide only for an **isotropic** material

Volume and Shape Changes (cont.)





- **Volume dilatation:** $\Delta = \dfrac{dV - dv}{dv} = \dfrac{\text{volume change}}{\text{original volume}}$

$$dv = dx_1 dx_2 dx_3 \quad \text{and} \quad dV = dX_1 dX_2 dX_3$$

But $\quad dX_k = (1 + \varepsilon_k) dx_k \quad$ (no sum)

principal strains

Volume and Shape Changes (cont.)

$$dv = dx_1 dx_2 dx_3$$

$$dV = (1+\varepsilon_1)dx_1(1+\varepsilon_2)dx_2(1+\varepsilon_3)dx_3$$

$$\begin{aligned}\Delta &= \frac{dv\left[(1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3)-1\right]}{dv} \\ &= 1+(\varepsilon_1+\varepsilon_2+\varepsilon_3)+H.O.T+\ldots-1 \\ &\approx \varepsilon_1+\varepsilon_2+\varepsilon_3 \\ &= \varepsilon_{ii}\end{aligned}$$

Volume dilatation: $\boxed{\Delta = \varepsilon_{ii} = tr(\underline{\varepsilon})}$

Volume and Shape Changes (cont.)

- Deviatoric and volumetric strain

$$[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_m & 0 & 0 \\ 0 & \varepsilon_m & 0 \\ 0 & 0 & \varepsilon_m \end{bmatrix} + \begin{bmatrix} \varepsilon_{11} - \varepsilon_m & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} - \varepsilon_m & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} - \varepsilon_m \end{bmatrix}$$

Mean strain (volumetric)

$$\varepsilon_m = \frac{1}{3}\varepsilon_{ii} \quad \Rightarrow \quad \Delta = 3\varepsilon_m = \varepsilon_{ii}$$

Volume change with no shape change

Deviatoric strain

$$\Delta = 0$$

Shape change at constant volume

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Compatibility

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

- Given u_i can find ε_{ij}
- Given ε_{ij} , $u_i = ?$

$$\varepsilon_{ij} \rightarrow u_i$$

6 components 3 unknowns

- System is overdetermined
- Conditions of **Compatibility** are restrictions on ε_{ij} so that **single valued** u_i are produced

Compatibility (cont.)

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

$$\varepsilon_{ij,kl} = \frac{\partial^2}{\partial x_k \partial x_l}\left[\frac{1}{2}\left(u_{i,j} + u_{j,i}\right)\right] = \frac{1}{2}\left(u_{i,jkl} + u_{j,ikl}\right)$$

$$\Rightarrow \quad \varepsilon_{ij,kl} + \varepsilon_{kl,ij} = \frac{1}{2}\left(u_{i,jkl} + u_{j,ikl}\right) + \frac{1}{2}\left(u_{k,lij} + u_{l,kij}\right)$$

Also:

$$\varepsilon_{ik,jl} + \varepsilon_{jl,ik} = \frac{1}{2}\left(u_{i,kjl} + u_{k,ijl}\right) + \frac{1}{2}\left(u_{j,lik} + u_{l,jik}\right)$$

Differentiation order immaterial

$$\therefore \quad \varepsilon_{ij,kl} + \varepsilon_{kl,ij} = \varepsilon_{ik,jl} + \varepsilon_{jl,ik}$$

Compatibility equations

\# of eqns. =?

Compatibility (cont.)

- Only 6 are independent (why?):

$$\varepsilon_{11,22} + \varepsilon_{22,11} - 2\varepsilon_{12,12} = 0$$
$$\varepsilon_{22,33} + \varepsilon_{33,22} - 2\varepsilon_{23,23} = 0$$
$$\varepsilon_{33,11} + \varepsilon_{11,33} - 2\varepsilon_{13,13} = 0$$
$$\varepsilon_{12,13} + \varepsilon_{13,12} - \varepsilon_{23,11} - \varepsilon_{11,23} = 0$$
$$\varepsilon_{23,21} + \varepsilon_{21,23} - \varepsilon_{31,22} - \varepsilon_{22,31} = 0$$
$$\varepsilon_{31,32} + \varepsilon_{32,31} - \varepsilon_{12,33} - \varepsilon_{33,12} = 0$$

- These are **necessary and sufficient** to ensure a unique u_i

- Examples…

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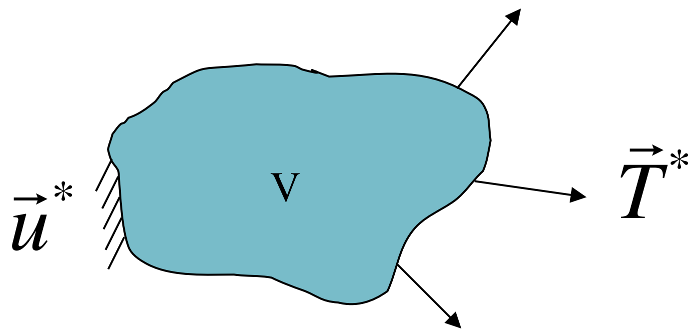
3.4 Summary of Stress and Strain



In the volume

On the surface

- Equilibrium:

$$\sigma_{ij,j} + f_i = 0$$

- Strain-displacement:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

- On traction boundary:

$$T_i^* = \sigma_{ij} n_j$$

- On displacement boundary:

$$u_i = u_i^*$$

# of unknowns= ?

# of equations= ?

- Need 6 additional equations: Must relate $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$

END OF CHAPTER 3