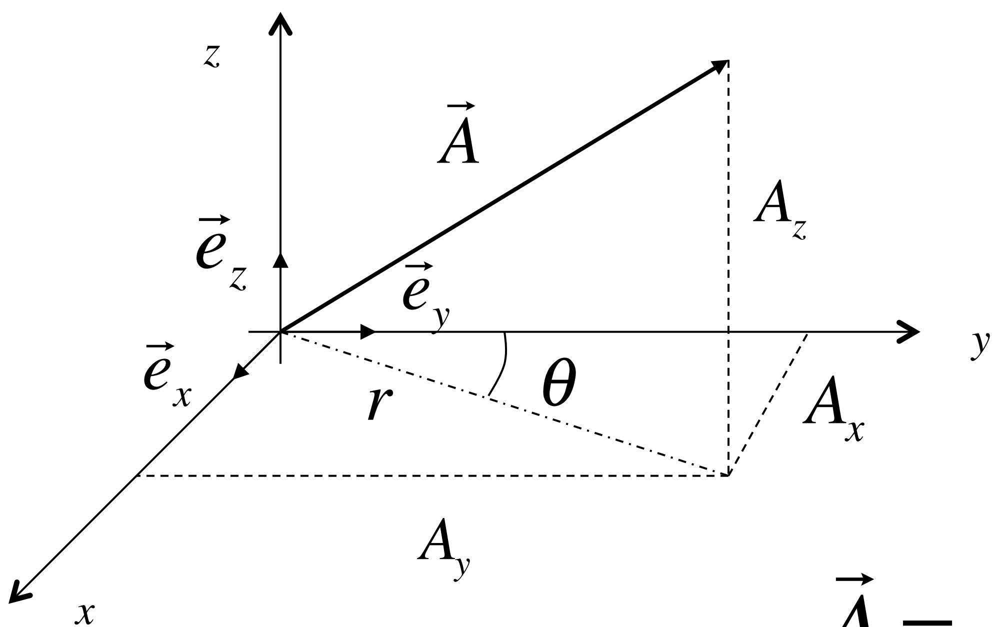
Chapter 1 - Mathematical Preliminaries

Table of contents

1.1.1 Vector Algebra

- **Vector:** Directed line segment ⇒ Magnitude and direction



$\vec{e}_x,\ \vec{e}_y,\ \vec{e}_z$: unit basis vectors for (x,y,z) or (x_1, x_2, x_3)

$$\vec{A} = A_x\vec{e}_x + A_y\vec{e}_y + A_z\vec{e}_z$$
$$= r\vec{e}_r + A_z\vec{e}_z$$

Vector Algebra (cont.)

$$\vec{A} = A_x\vec{e}_x + A_y\vec{e}_y + A_z\vec{e}_z \qquad \vec{B} = B_x\vec{e}_x + B_y\vec{e}_y + B_z\vec{e}_z$$

- Magnitude: $A = \left|\vec{A}\right| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

- Vector multiplication:
  - Scalar (inner or dot) product: $C = \vec{A}\cdot\vec{B}$

$$= A_xB_x + A_yB_y + A_zB_z$$
$$= AB\cos\theta_{AB}$$

  - Vector (outer or cross) product: $\vec{C} = \vec{A}\times\vec{B}$

$$= \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

1.1.2 Scalar and Vector Fields

Field: (physical) quantity that possess spatial variation, i.e. depends on x, y, z

- Scalar field: A scalar quantity expressed as a function of Cartesian coordinates

$$f(x_1, x_2, x_3) = const.$$

e.g. Temperature at a point, density at a point

- Vector field: vector components depend on Cartesian coordinates

$$\vec{A}(x_1, x_2, x_3) = A_1(x_1, x_2, x_3)\vec{e}_1 + A_2(x_1, x_2, x_3)\vec{e}_2 + A_3(x_1, x_2, x_3)\vec{e}_3$$

e.g. Velocity of a particle

Vector Calculus

- Gradient of a scalar, f: $$\nabla f = \frac{\partial f}{\partial x_1}\vec{e}_1 + \frac{\partial f}{\partial x_2}\vec{e}_2 + \frac{\partial f}{\partial x_3}\vec{e}_3$$

- Divergence of a vector, **A**: $$\nabla \cdot \vec{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

- Curl of a vector, **A**: $$\nabla \times \vec{A} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

- Gradient of a vector: **?**

Chapter 1 - Mathematical Preliminaries

Table of contents

1.2 Indicial Notation

- A mathematical shorthand useful in Mechanics

$$\vec{A} = \left(A_1, A_2, A_3\right) = A_i \quad \textit{index} \quad i = 1,2,3$$

$$[B] = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} = B_{ij} \quad \textit{indices} \quad \begin{matrix} i = 1,2,3 \\ j = 1,2,3 \end{matrix}$$

$$A \Rightarrow 1 \quad \textit{term}$$

$$B_i \Rightarrow 3 \ \textit{individual} \ \ \textit{terms}$$

$$C_{ij} \Rightarrow 9 \ \textit{individual} \ \ \textit{terms}$$

$$D_{ijk} \Rightarrow 27 \ \textit{individual} \ \ \textit{terms}$$

$$E_{ij\ldots n} \Rightarrow 3^n \ \textit{individual} \ \ \textit{terms}$$

Summation Convention

- Dot product of vectors:

$$\vec{A} \cdot \vec{B} = A_1 B_1 + A_2 B_2 + A_3 B_3$$

$$= \sum_{i=1}^{3} A_i B_i$$

$$\Rightarrow \quad \vec{A} \cdot \vec{B} = A_i B_i \quad (implied \;\; i = 1,2,3)$$

- **Einstein's Summation Convention:** repeated indices (only twice) are summed over 1, 2, 3:

$$A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3$$

$$D_{ii} = D_{11} + D_{22} + D_{33}$$

Summation Convention (cont.)

- Two types of indices:
  - **Free index** = appears only **once** per term and ranges from 1 to 3 (i.e. 3 separate equations)
  - **Dummy (repeated) index** = appears **twice** per term and implies summation (also ranges from 1 to 3)

- Examples:

$$D_{ii} = D_{11} + D_{22} + D_{33} = D_{jj} = D_{kk}$$

$$A_i B_i C_{ii} = \quad \textbf{INCORRECT !}$$

$$A_i B_i C_{kl} = \quad ?$$

$$A_{ij} x_j = C_i \quad \Rightarrow \quad ?$$

Summation Convention (cont.)

- No sum condition:

$$A_{ii} + B_{ii} = C_{ii} \quad no \;\; sum \quad \Rightarrow \quad \begin{aligned} A_{11} + B_{11} &= C_{11} \\ A_{22} + B_{22} &= C_{22} \\ A_{33} + B_{33} &= C_{33} \end{aligned}$$

- Derivative notation: partial differentiation wrt x_i is abbreviated with a comma followed by i

$$\frac{\partial A_i}{\partial x_i} = A_{i,i}$$ **Divergence** (# of terms = ?)

$$\frac{\partial D_{ij}}{\partial x_j} = D_{ij,j} = D_{i1,1} + D_{i2,2} + D_{i3,3}$$ (# of terms = ?)

$$\frac{\partial A_i}{\partial x_j} = A_{i,j}$$ **Gradient** (# of terms = ?)

Applications

1) How many terms are represented by

a) A_{ij} b) $B_{ij,kl}$ c) $C_{ij,ji}$ d) D_{ijk} e) $A_{pq}B_{qr}$

2) How many equations are represented by

a) $A_{ij}B_{jk}C_{kl} = D_{il}$ b) $A_{ij,j} = 0$ c) $A_{ijk,l} + B_{ij,kl} = C_{ijkl}$

Kronecker Delta: δ_{ij}

- Definition:

$$\delta_{ij} = 1 \quad if \quad i = j$$
$$= 0 \quad if \quad i \neq j$$

$$\left[\delta_{ij}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

e.g. $$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

- Effect of d_{ij}: changes indices by multiplication

$$A_i \delta_{ij} = A_j$$

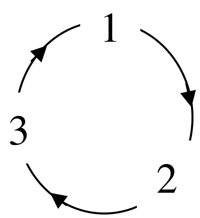
$$D_{ij} \delta_{jk} = D_{ik}$$

$$C_{ijk} \delta_{ij} = C_{jjk} = C_{11k} + C_{22k} + C_{33k}$$

Alternator: e_{ijk} or ε_{ijk}

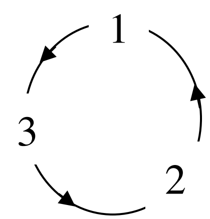
- Definition:

$$
\begin{aligned}
e_{ijk} &= 1 \quad \textit{if} \quad i,j,k \;\; \textit{are an even permutation of} \;\; 1,2,3 \\
&= -1 \quad \textit{if} \quad i,j,k \;\; \textit{are an odd permutation of} \;\; 1,2,3 \\
&= 0 \quad \textit{otherwise}
\end{aligned}
$$



even permutation

$$e_{123} = e_{231} = e_{312} = 1$$



odd permutation

$$e_{132} = e_{213} = e_{321} = -1$$

- Useful in vector cross product:

$$
\begin{aligned}
\vec{A} \times \vec{B} &= e_{ijk} \vec{e}_i A_j B_k \\
&= (A_2 B_3 - A_3 B_2)\vec{e}_1 + (A_3 B_1 - A_1 B_3)\vec{e}_2 + (A_1 B_2 - A_2 B_1)\vec{e}_3
\end{aligned}
$$

Some useful relations

- Transpose: $$u^T_{ij} = u_{ji}$$
- Symmetry: $$u_{ij} = u_{ji}$$
- Skew (anti) symmetry: $$u_{ij} = -u_{ji}$$
- Inverse: $$u^{-1}_{ij} u_{jk} = \delta_{ik}$$
- Orthogonality: $$u^T_{ij} = u^{-1}_{ij}$$

Applications

1) Compute

a) $\delta_{ij}\delta_{ij} = ?$

b) $\delta_{ij}\delta_{jk}\delta_{ki} = ?$

2) Knowing that

$$\varepsilon_{ijk}\varepsilon_{pqr} = \det\begin{bmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{bmatrix},$$

compute/simplify

(a) $\varepsilon_{ijk}\varepsilon_{iqr} =$

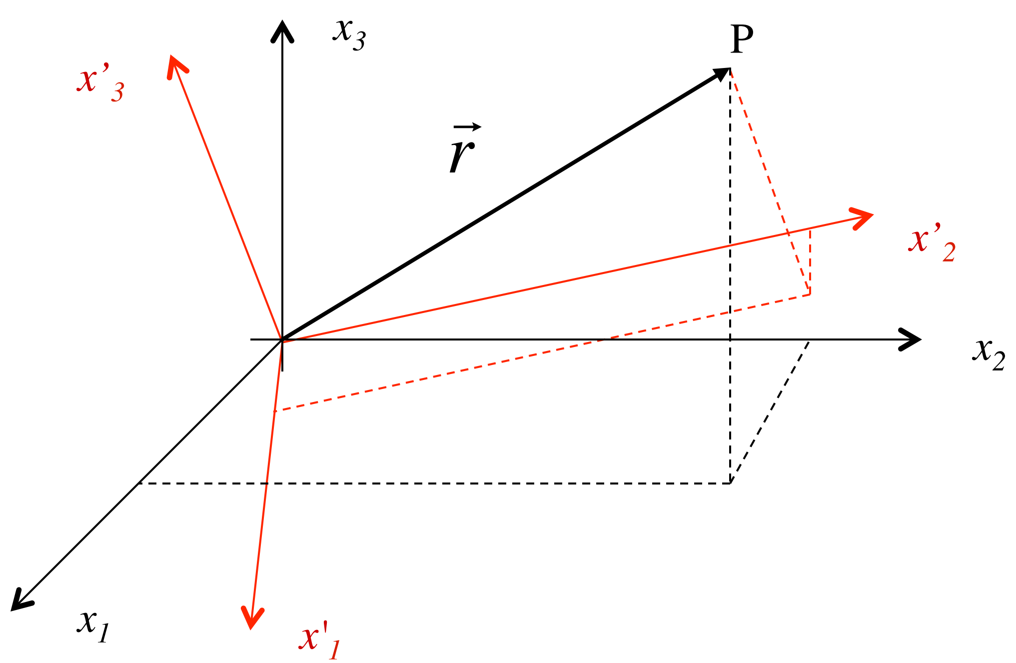
(b) $\varepsilon_{ijk}\varepsilon_{ijr} =$

(c) $\varepsilon_{ijk}\varepsilon_{ijk} =$

Chapter 1 - Mathematical Preliminaries

Table of contents

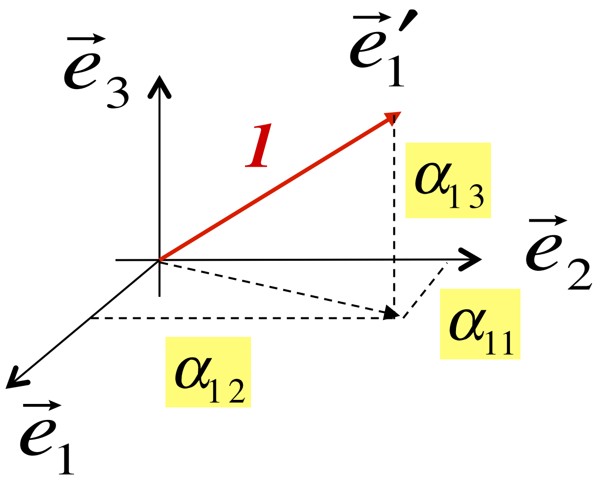
1.3.1 Coordinate Transformation



- Position vector of point P:

$$\vec{r} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$$
$$= x_1'\vec{e}_1' + x_2'\vec{e}_2' + x_3'\vec{e}_3'$$

Coordinate Transformation (cont.)



- **α_{ij} : directional cosines** (9 terms) – cosine of angle between ith-primed and jth-unprimed axis

$$\alpha_{ij} = \cos\left(x_i', x_j\right) = \cos\left(\vec{e}_i', \vec{e}_j\right) = \vec{e}_i' \cdot \vec{e}_j = \frac{\partial x_i'}{\partial x_j}$$

- Coordinate transformation for unit vectors:

$$\vec{e}_1' = \alpha_{11}\vec{e}_1 + \alpha_{12}\vec{e}_2 + \alpha_{13}\vec{e}_3$$
$$\vec{e}_2' = \alpha_{21}\vec{e}_1 + \alpha_{22}\vec{e}_2 + \alpha_{23}\vec{e}_3$$
$$\vec{e}_3' = \alpha_{31}\vec{e}_1 + \alpha_{32}\vec{e}_2 + \alpha_{33}\vec{e}_3$$

or, more simply: $\vec{e}_i' = \alpha_{ij}\vec{e}_j$ or, for coordinates: $x_i' = \alpha_{ij}x_j$

Rotation matrix [R]

- In matrix form: $\{x'\} = [R]\{x\}$

$$\{x'\} = \begin{Bmatrix} x_1' \\ x_2' \\ x_3' \end{Bmatrix}, \quad \{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}, \quad [R] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}$$

- [R]: rotation matrix from $(x_1, x_2 x_3)$ to $(x_1', x_2' x_3')$

- Inverse transformation: $x_i = \alpha_{ji} x_j'$

<u>Note :</u> $\alpha_{ij} \neq \alpha_{ji}$,

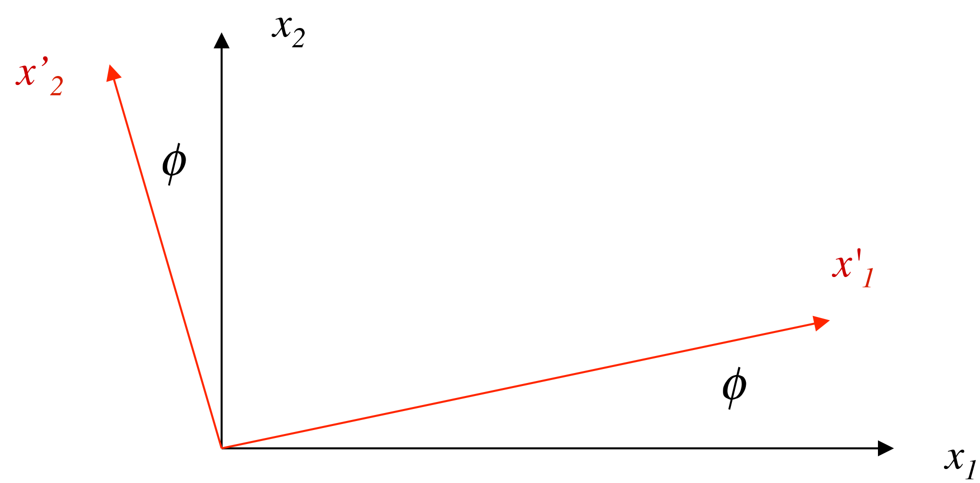
i.e. not symmetric

- **Definition of a vector:** Any quantity which obeys the transformation law

$$\{x'\} = [R]\{x\} \qquad x_i' = \alpha_{ij} x_j$$

is a defined as a vector

Example: Pure rotation in 1-2 plane



$$\alpha_{11} = \cos\phi \qquad \alpha_{12} = \cos(90 - \phi) = \sin\phi$$

$$\alpha_{21} = -\sin\phi \qquad \alpha_{22} = \cos\phi$$

$$\Rightarrow \quad [R] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Properties of α_{ij}

$$\delta_{ij} = \vec{e}_i \cdot \vec{e}_j$$

$$= \vec{e}_i' \cdot \vec{e}_j'$$

$$= \quad ?$$

Properties of α_{ij}

$$\delta_{ij} = \vec{e}_i \cdot \vec{e}_j$$

$$= \vec{e}_i' \cdot \vec{e}_j'$$

$$= (\alpha_{ik}\vec{e}_k) \cdot (\alpha_{jk}\vec{e}_k) = (\alpha_{ik}\vec{e}_k) \cdot (\alpha_{jl}\vec{e}_l)$$

$$= \alpha_{ik}\alpha_{jl}(\vec{e}_k \cdot \vec{e}_l)$$

$$= \alpha_{ik}\alpha_{jl}\delta_{kl}$$

$$\Rightarrow \quad \delta_{ij} = \alpha_{ik}\alpha_{jk}$$

Properties of α_{ij} (cont.)

$$\boxed{\delta_{ij} = \alpha_{ik}\alpha_{jk}}$$

i.e. a_{ij} is orthogonal

- Consequence #1: Let i=j (no sum)

$$1 = \alpha_{ik}\alpha_{ik} \qquad no \quad sum \quad on \quad i$$

$$\Rightarrow \quad 1 = \alpha_{i1}^2 + \alpha_{i2}^2 + \alpha_{i3}^2$$

$$\Rightarrow \begin{cases} 1 = \alpha_{11}^2 + \alpha_{12}^2 + \alpha_{13}^2 \\ 1 = \alpha_{21}^2 + \alpha_{22}^2 + \alpha_{23}^2 \\ 1 = \alpha_{31}^2 + \alpha_{32}^2 + \alpha_{33}^2 \end{cases}$$

- i.e. **normality property** of directional cosines

Properties of α_{ij} (cont.)

$$\boxed{\delta_{ij} = \alpha_{ik}\alpha_{jk}}$$

- Consequence #2: Let i≠j

$$\begin{aligned} 0 &= \alpha_{ik}\alpha_{jk} \qquad\qquad i \neq j \\ &= \alpha_{i1}\alpha_{j1} + \alpha_{i2}\alpha_{j2} + \alpha_{i3}\alpha_{j3} \end{aligned}$$

- Expanding:

$$\begin{aligned} 0 &= \alpha_{11}\alpha_{21} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{23} \quad , \; i = 1, \;\; j = 2 \\ 0 &= \alpha_{21}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{23}\alpha_{33} \quad , \; i = 2, \;\; j = 3 \\ &etc\ldots \end{aligned}$$

- i.e. **orthogonality property** of direction cosines

1.3.2 Cartesian Tensors

- A tensor of order n is a set of 3^n quantities which transform from a coordinate system $(x_1, x_2 x_3)$ to another $(x_1', x_2' x_3')$ according to the relation:

n	order	transf. law
0	zero (scalar)	$A_i' = A_i$
1	one (vector)	$A_i' = \alpha_{ij} A_j$
2	two (two - tensor)	$A_{ij}' = \alpha_{ik} \alpha_{jl} A_{kl}$
3	three	$A_{ijk}' = \alpha_{il} \alpha_{jm} \alpha_{kn} A_{lmn}$
4	four	$A_{ijkl}' = \alpha_{im} \alpha_{jn} \alpha_{kp} \alpha_{lq} A_{mnpq}$

- Second order tensors are very important in Elasticity. In matrix form:

$$[A'] = [R]\ [A]\ [R]^T$$

Tensor Algebra

- Tensor arithmetic and algebra similar to matrix operations wrt addition, equality and multiplication with a scalar.

- Multiplication (one form) of two tensors of order m and n produces a tensor of order m+n:

$$A_i B_{jk} = C_{ijk} \qquad A_{ij} \otimes B_{kl} = C_{ijkl}$$

- Two-tensor multiplication with one contraction (summed index):

$$A_{ij} B_{jk} = C_{ik}$$

- Two-tensor multiplication with two contractions (inner or dot product):

$$\underline{A} \cdot \underline{B} = A_{ij} B_{ij}, \quad \text{a scalar}$$

Tensor (Field) Calculus

- Gradient of a vector field, v, is a second order tensor field:

$$(\nabla \vec{v})_{ij} = v_{i,j} \quad \text{9 components}$$

- Can perform div, grad, curl operations on a second order tensor, A, as follows:

$$(div\,\underline{A})_i = (\nabla \cdot \underline{A})_i = A_{ij,j}$$

i.e. 3 numbers ---> a vector

$$(curl\,\underline{A})_{ij} = (\nabla \times \underline{A})_{ij} = e_{ipq} A_{jq,p}$$

i.e. 9 numbers ---> a second order tensor

$$(grad\,\underline{A})_{ijk} = (\nabla \underline{A})_{ijk} = A_{ij,k}$$

i.e. 27 numbers ---> a third order tensor

Applications

1) If **a**, **b** and **c** are vectors, use indicial notations to prove

a) $(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c}=\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c})$

b) $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=(\mathbf{a}\bullet\mathbf{c})\mathbf{b}-(\mathbf{b}\bullet\mathbf{c})\mathbf{a}$

2) Let the vector $\mathbf{A}$ be defined as $\mathbf{A}=2\mathbf{e}_1+4\mathbf{e}_2+2\mathbf{e}_3$.

(a) Find the components of $\mathbf{A}$ after a counter-clockwise rotation of 45° around the x_2 axis.

(b) What is the magnitude of $\mathbf{A}$ in each case?

3) Let $\mathbf{A}$ be a symmetric two-tensor and $\mathbf{B}$ a skew-symmetric one. Show that $\mathbf{A}\bullet\mathbf{B}=0$.

4) Using indicial notation, show that the gradient of a vector is a second-order tensor.

5) Let a, $\mathbf{u}$ and $\mathbf{W}$ be a scalar, vector and two-tensor field, respectively.
Using indicial notation, prove that

a) $\nabla\times\nabla a=0$

b) $\nabla\times(\nabla\times\mathbf{u})=\nabla(\nabla\bullet\mathbf{u})-\nabla^2\mathbf{u}$

End of Chapter 1