Equation sheet – AE 323
Final Exam

1) Theory of elasticity

1.1) 3D theory

Equilibrium equations: $\sigma_{ij,j} + f_i = 0$; $\sigma_{ij} = \sigma_{ji}$

Cauchy's relation: $T_i = \sigma_{ji} n_j$

Kinematic: $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$

Compatibility: $\varepsilon_{ij,kl} + \varepsilon_{kl,ij} = \varepsilon_{ik,jl} + \varepsilon_{jl,ik}$

Constitutive (isotropic): $\sigma_{ij} = \lambda \varepsilon_{mm} \delta_{ij} + 2\mu \varepsilon_{ij}$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad \mu = G = \frac{E}{2(1+\nu)}$$

1.2) 2D theory (plane stress)

Equilibrium: $\sigma_{\alpha\beta,\beta} + f_\alpha = 0; \quad \sigma_{\alpha\beta} = \sigma_{\beta\alpha} \qquad (\alpha, \beta = 1,2)$

Kinematic: $\varepsilon_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right)$

Compatibility: $2\varepsilon_{12,12} = \varepsilon_{11,22} + \varepsilon_{22,11}$

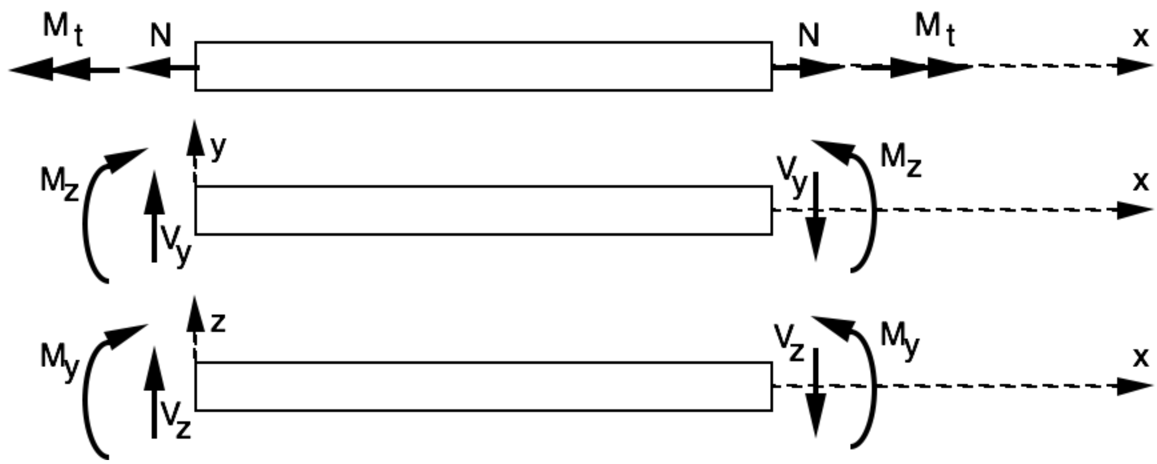
Constitutive (isotropic): $\sigma_{11} = \frac{E}{1-\nu^2}\left[\varepsilon_{11} + \nu\varepsilon_{22}\right], \quad \sigma_{22} = \frac{E}{1-\nu^2}\left[\varepsilon_{22} + \nu\varepsilon_{11}\right], \quad \sigma_{12} = \frac{E}{1+\nu}\varepsilon_{12}$

$$\varepsilon_{33} = -\frac{\nu}{1-\nu}\left(\varepsilon_{11} + \varepsilon_{22}\right), \qquad \sigma_{33} = 0$$

For plane strain: replace E by $E/\left(1-\nu^2\right)$; ν by $\nu/(1-\nu)$; $\varepsilon_{33} = 0$; $\sigma_{33} = \nu\left(\sigma_{11} + \sigma_{22}\right)$

2) Bending and extension of beams

2.1) Conventions for positive force and moment resultants



$$N(x) = +\iint_A \sigma_{xx}(x,y,z)\,dy\,dz; \qquad V_y(x) = -\iint_A \sigma_{xy}(x,y,z)\,dy\,dz; \qquad V_z(x) = -\iint_A \sigma_{xz}(x,y,z)\,dy\,dz$$

$$M_y(x) = -\iint_A z\sigma_{xx}(x,y,z)\,dy\,dz; \qquad M_z(x) = -\iint_A y\sigma_{xx}(x,y,z)\,dy\,dz; \qquad M_t(x) = +\iint_A \left[y\sigma_{xz} - z\sigma_{xy}\right]dy\,dz$$

2.2) Euler-Bernouilli assumptions

- a planar cross-section remains planar
- a planar cross-section remains perpendicular to the (deformed) neutral axis
- a planar cross-section retains its original size and shape

2.3) Beam displacements

$$u(x,y,z) = u(x) - y\,v'(x) - z\,w'(x); \qquad v(x,y,z) = v(x); \qquad w(x,y,z) = w(x)$$

2.4) Beam stresses (without thermal stresses)

$$\sigma_{xx}(x,y,z) = E\left[u'(x) - yv''(x) - zw''(x)\right]$$

$$N(x) = E_o A^* u'(x) \quad ; \quad M_y(x) = E_o I_{yz}^* v''(x) + E_o I_{yy}^* w''(x) \quad ; \quad M_z(x) = E_o I_{zz}^* v''(x) + E_o I_{yz}^* w''(x)$$

$$\sigma_{xx}(x,y,z) = \frac{E(x,y,z)}{E_o}\left[\frac{N(x)}{A^*} - y\left(\frac{M_z(x)I_{yy}^* - M_y(x)I_{yz}^*}{I_{yy}^* I_{zz}^* - I_{yz}^{*\,2}}\right) - z\left(\frac{M_y(x)I_{zz}^* - M_z(x)I_{yz}^*}{I_{yy}^* I_{zz}^* - I_{yz}^{*\,2}}\right)\right]$$

where

$$I_{yy}^* = \iint_A z^2 \frac{E}{E_o}\,dA; \qquad I_{zz}^* = \iint_A y^2 \frac{E}{E_o}\,dA; \qquad I_{yz}^* = \iint_A yz \frac{E}{E_o}\,dA; \qquad A^* = \iint_A \frac{E}{E_o}\,dA$$

2.5) Beam equilibrium equations

$$\left(E_o A^* u'\right)' = -f_x(x)$$

$$\left(E_o I_{zz}^* v''\right)'' + \left(E_o I_{yz}^* w''\right)'' = f_y(x) - m_z'(x)$$

$$\left(E_o I_{yz}^* v''\right)'' + \left(E_o I_{yy}^* w''\right)'' = f_z(x) + m_y'(x)$$

2.6) Boundary conditions

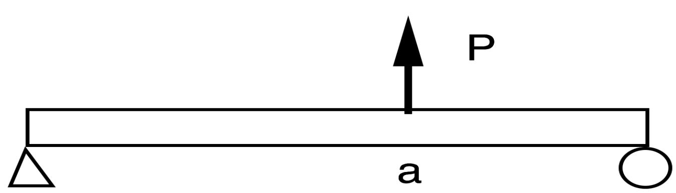
Differential Equation	**Either**	**Or**
u	Axial deflection $u(b) = \tilde{u}(b)$	Axial force $E_o A^*(b) u'(b) = \tilde{N}(b)$
v	Lateral deflection $v(b) = \tilde{v}(b)$	Shear force $\left[E_o I_{zz}^* v''(b)\right]' + \left[E_o I_{yz}^* w''(b)\right]' + m_z(b) = \tilde{V}_y(b)$
	Bending slope $v'(b) = \tilde{v}'(b)$	Bending moment $E_o I_{zz}^* v''(b) + E_o I_{yz}^* w''(b) = \tilde{M}_z(b)$
w	Lateral deflection $w(b) = \tilde{w}(b)$	Shear force $\left[E_o I_{yz}^* v''(b)\right]' + \left[E_o I_{yy}^* w''(b)\right]' - m_y(b) = \tilde{V}_z(b)$
	Bending slope $w'(b) = \tilde{w}'(b)$	Bending moment $E_o I_{yz}^* v''(b) + E_o I_{yy}^* w''(b) = \tilde{M}_y(b)$

2.7) Elastic boundary conditions

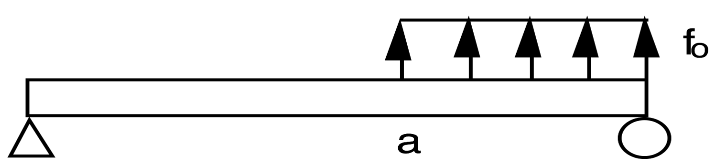
Basic principle: (linear and/or torsional) springs oppose the deflections of the beam

2.8) Concentrated and partial span loads

$$f_z(x) = P\,\delta[x-a]$$

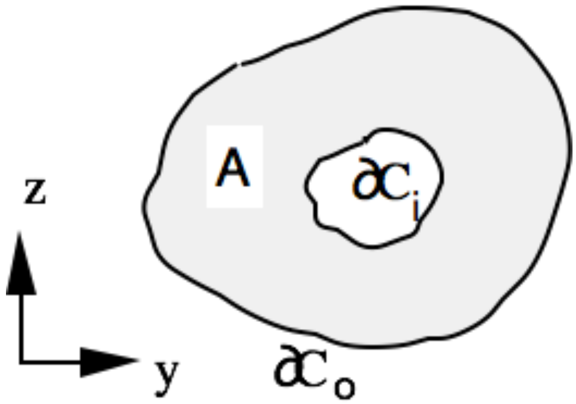


$$f_z(x) = f_o\; stp[x-a]$$



3) Uniform torsion of beams

3.1) Basic relations using Prandtl stress function Ψ



$$\nabla^2\Psi = -2\mu\theta$$
$$\Psi = 0 \quad \text{on } \partial C_o$$
$$\Psi = \Psi_i \quad \text{on } \partial C_i \qquad \text{with} \qquad \Psi_i \text{ such that } \int_{C_i} \frac{\partial \Psi}{\partial n} ds = 2A_i \mu\theta$$
$$M_t = 2\iint_A \Psi(y,z)\, dA + 2\sum_i \psi_i A_i$$
$$\sigma_{xy} = \frac{\partial \Psi}{\partial z}; \quad \sigma_{xz} = -\frac{\partial \Psi}{\partial y}$$
$$v(x,y,z) = -xz\theta$$
$$w(x,y,z) = xy\theta$$
$$u(x,y,z) = \theta\phi(y,z) \qquad \text{with} \begin{cases} \dfrac{\partial \phi}{\partial y} = \dfrac{1}{\mu\theta}\dfrac{\partial \Psi}{\partial z} + z \\ \dfrac{\partial \phi}{\partial z} = -\dfrac{1}{\mu\theta}\dfrac{\partial \Psi}{\partial y} - y \end{cases}$$

3.2) The membrane analogy

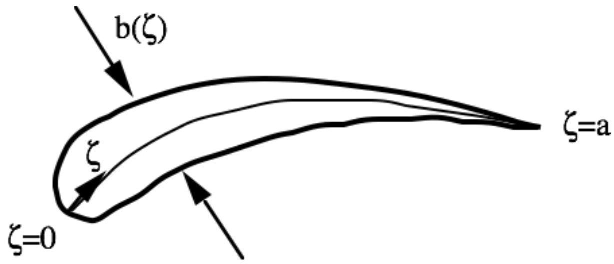
The Prandtl stress function $\Psi(y,z)$ can be visualized as the deformed shape of an elastic membrane subjected to a uniform pressure. Then,

- M_t is proportional to the volume of the deformed membrane
- the shearing stresses are proportional to the slope of the deformed membrane

Using this analogy, we obtain, for a narrow rectangular bar with cross-section $a*b$ (with $a>>b$)

$$M_t \approx \frac{\mu\theta ab^3}{3}; \qquad J = \frac{M_t}{\mu\theta} \approx \frac{ab^3}{3}$$

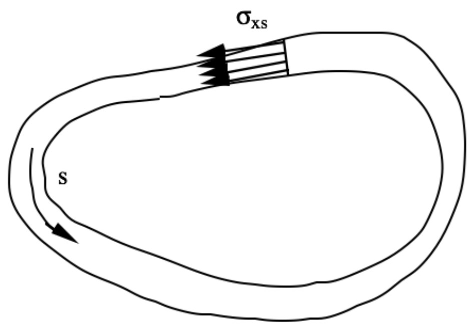
3.3) Approximate method for thin-walled open cross-sections



$$J = \frac{M_t}{\mu\theta} \cong \frac{1}{3}\int_0^a b^3(\zeta)d\zeta$$

3.4) Approximate method for thin-walled closed cross-sections

a) Single cell



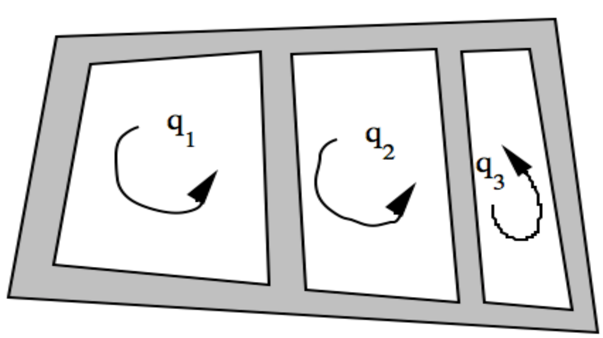
- assumptions
  * no stress variation through the thickness (σ_{xs} =constant)
  * shear stress acts in a direction tangent to the median line through the wall thickness

- shear flow = $q = \sigma_{xs} t$ = constant at all points of a given cross-section

- Bredt-Batho equations : $q = M_t / 2\hat{A}$ where $\hat{A}$ = “swept” area

$$J^* = \frac{M_t}{\mu_o\theta} = \frac{4\hat{A}^2}{\oint \frac{ds}{t^*(s)}} \text{ where } t^*(s) = \frac{\mu(s)}{\mu_o}t(s)$$

b) Multiple cell



$$M_t = \sum_{i=1}^{N} M_{t_i} = 2\sum_{i=1}^{N} q_i\hat{A}_i$$

Assuming that the cross-section retains its shape:

$\theta_1 = \theta_2 = ... = \theta_N$, which yields

$$\frac{1}{2\mu_o\hat{A}_i}\oint_{\partial C_i}\frac{q(s)}{t^*(s)}ds = \frac{1}{2\mu_o\hat{A}_1}\oint_{\partial C_1}\frac{q(s)}{t^*(s)}ds, \quad (i = 2,3,...,N)$$

3.5) Non-uniform torsion of beams with variable geometry

Differential equation: $\left(\mu_o J^*(x)\phi'(x)\right)' = -m_t(x)$

Boundary conditions: <u>either</u> the angle of twist ϕ is imposed at $x=b$: $\phi(b) = \tilde{\phi}(b)$

<u>or</u> the twisting moment $\boldsymbol{M}_t$ is imposed at $x=b$: $\mu_o J^*(b)\phi'(b) = \tilde{M}_t(b)$

4) Work and potential energy principles

4.1) Work and potential energy

The <u>work</u> done by a force $\boldsymbol{F}$ traveling along a path from point A to point B is $W = \int_A^B \mathbf{F} \bullet d\mathbf{r}$

If W is path independent (i.e., depends only on the initial and final positions), $\boldsymbol{F}$ is <u>conservative</u>. In that case, $\boldsymbol{F} = -\nabla V$, where the potential V is called the <u>potential energy</u>.

4.2) Virtual work and virtual potential energy

A virtual displacement $\delta \boldsymbol{r}$ - is consistent with the geometrical constraints
- has no effect on the forces acting on the system

The virtual work $\delta W = \mathbf{F} \bullet \delta\mathbf{r}$

The virtual potential energy (for conservative systems) $\delta V = \nabla V \bullet \delta\mathbf{r}$

4.3) Variational operator *δ()*

Is very similar to the differential operator *d( )* except

- δ(independent variable)$=0$. E.g., $\delta x = \delta y = \delta z = 0$
- A force is considered as an independent operator for the operator δ, i.e., $\delta \boldsymbol{F} = \boldsymbol{0}$

Basic properties:

$$\delta\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(\delta u)$$

$$\delta \iint F(\quad)dx\,dy = \iint \delta F(\quad)dx\,dy$$

4.4) Principle of Virtual Work (PVW)

a) Rigid systems

"A particle is in equilibrium if and only if the virtual work done by all forces on the particle is zero during any arbitrary virtual displacement"

b) Deformable systems

The PVW is written as $\delta W = \delta W_{in} + \delta W_{ex} = 0$, where

$\delta W_{ex} = \iint_{S_T}\left(T_x\delta u + T_y\delta v + T_z\delta w\right)dS + \iiint_V \rho\left(B_x\delta u + B_y\delta v + B_z\delta w\right)dVol$ is <u>the virtual work done by the external loads</u>

$\delta W_{in} = -\iiint_{Vol}\langle\sigma\rangle\{\delta\gamma\}dVol$ is the opposite of the <u>virtual work done by the internal stresses</u>

c) For conservative systems (PMPE)

The PVW becomes the Principle of Minimum Potential Energy (PMPE)

$\delta\Pi = \delta U + \delta V = 0$,

where

$\Pi = U + V$ is the total potential energy of the system,

$U = \frac{1}{2}\iiint_{Vol} \langle\gamma\rangle[E]\{\gamma\}\,dVol = \iiint_{Vol} U_o\, dVol$ is the total strain energy stored in the linearly elastic body

V is the potential energy associated with the external loads

U_o is the strain energy density (area under the stress-strain curve)

4.5) Principle of Complementary Virtual Work (PCVW)

Here, the tractions, body forces and stresses are the dependent variables, while the displacements and strains are the independent variables. To be acceptable virtual quantities, the stresses, tractions and body forces must

- be in equilibrium with each other
- satisfy the Cauchy relations

PCVW : $\delta W^* = \delta W^*_{in} + \delta W^*_{ex} = 0$

where

$\delta W^*_{ex} = \iint_{S_U} \left(u\delta T_x + v\delta T_y + w\delta T_z\right)dS + \iiint_V \rho\left(u\delta B_x + v\delta B_y + w\delta B_z\right)dVol$ is the complementary virtual work associated with the external loads

$\delta W^*_{in} = -\iiint_{Vol} \langle\gamma\rangle\langle\delta\sigma\rangle dVol$ is the opposite of the complementary virtual work done by the internal stresses

For conservative systems, the PCVW becomes the PMCPE

$$\delta\Pi^* = \delta U^* + \delta V^* = 0,$$

where

$\Pi^* = U^* + V^*$ is the total complementary potential energy of the system,

$U^* = \frac{1}{2}\iiint_{Vol} \langle\sigma\rangle[S]\{\sigma\}\,dVol = \iiint_{Vol} U^*_o\, dVol$ is the total complementary strain energy stored in the linearly elastic body

V^* is the complementary potential energy associated with the external loads

U^*_o is the complementary strain energy density (area above the stress-strain curve)

Notations : δq_i are the generalized virtual displacements (e.g., displacements, slopes, ...)

δQ_i are the generalized virtual loads (e.g., forces, moments, ...)

5) Analytical solution of static problems using energy methods

5.1) (Complementary) strain energy for simple systems

• Axially loaded bar $\quad U = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx$

$$U^* = \frac{1}{2}\int_0^L \frac{N^2}{EA}\, dx$$

• Beam bending/extension $\quad U = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx + \frac{1}{2}\int_0^L EI_{zz}\left(\frac{d^2v}{dx^2}\right)^2 dx + \frac{1}{2}\int_0^L EI_{yy}\left(\frac{d^2w}{dx^2}\right)^2 dx$

$$U^* = \frac{1}{2}\int_0^L \frac{N^2}{EA}dx + \frac{1}{2}\int_0^L \frac{M_z^2}{EI_{zz}}dx + \frac{1}{2}\int_0^L \frac{M_y^2}{EI_{yy}}dx$$

• Beam torsion

$$U = \frac{1}{2}\int_0^L \mu J\left(\frac{d\phi}{dx}\right)^2 dx$$

$$U^* = \frac{1}{2}\int_0^L \frac{M_t^2}{\mu J}dx$$

5.2) Castigliano's theorems

• First theorem : $PMPE \quad \Rightarrow \quad \frac{\partial U}{\partial q_i} = Q_i$

• Second theorem : $PMCPE \quad \Rightarrow \quad \frac{\partial U^*}{\partial Q_i} = q_i$

For linearly elastic structures ($U = U^*$), thus $PMCPE \quad \Rightarrow \quad \frac{\partial U}{\partial Q_i} = q_i$

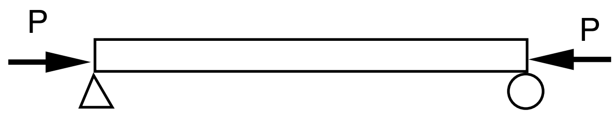
i.e., "the generalized displacement at a location in an elastic structure can be found by taking the derivative of the total strain energy of the system with respect to the corresponding generalized force acting at that location".

Note : if there is no concentrated force or moment at the location of interest, one must add a dummy load P_D. The corresponding displacement q is then obtained by $q = \lim_{P_D \to 0}\left(\partial U / \partial P_D\right)$.

For statically determinate systems, the (complementary) strain energy can be computed directly.
For statically indeterminate systems,

- first determine the degree of redundancy
- then remove the constraints to make the system statically determinate (by assuming that the corresponding reactions are externally applied loads)
- solve the problem as if it were a statically determinate system
- obtain the unknown reactions by reimposing the constraints

6. Buckling of beams



6.1) Euler-Bernouilli analysis of beam buckling

GDE: $$\left(EIw'' + Pw\right)'' = 0$$

General solution: $$w(x) = A\sin(\lambda x) + B\cos(\lambda x) + Cx + D$$

with $$\lambda = \sqrt{\frac{P}{EI}}$$

The buckling load will depend on the boundary conditions: e.g., for a simply supported beam:

$$P_{crit,n} = \frac{n^2\pi^2 EI}{L^2} \qquad (n = 1,2,3,...)$$

6.2) Energy-based analysis of beam buckling

$$\Pi = \frac{1}{2}\int_0^L \left\{ EI\left(\frac{d^2w}{dx^2}\right)^2 - P\left(\frac{dw}{dx}\right)^2 \right\} dx$$