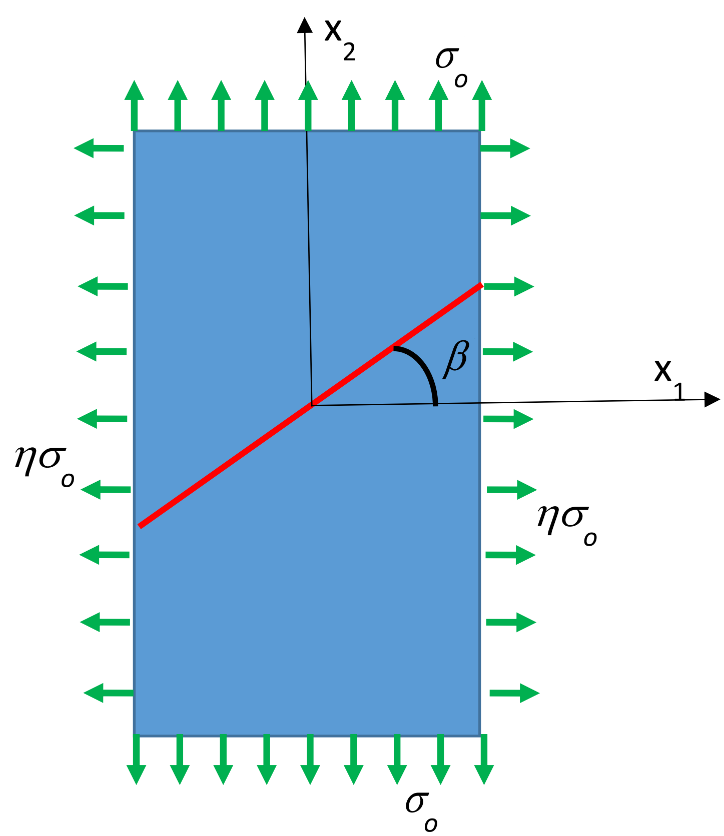
AE323 – Homework Assignment #1 – Spring 2019
Wednesday, Jan. 23, 2019
Due by Friday, Feb. 1, 2019 (In class)
Topics: Stresses and Cauchy relations

Problem 1.

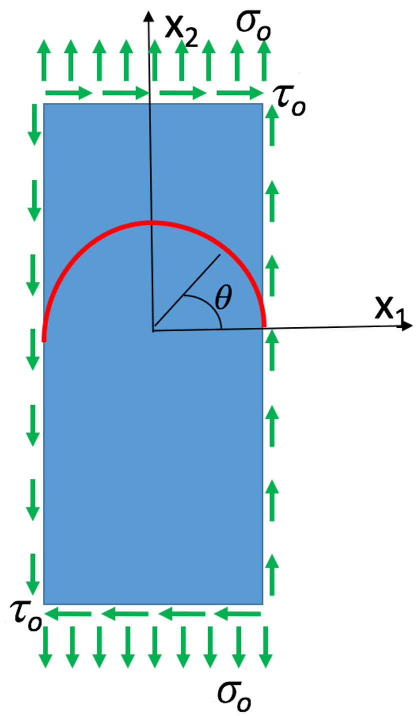
Consider the 2D problem of a rectangular linearly elastic solid (with stiffness E and Poisson's ratio ν) that is subjected to a biaxial state of stress (with amplitude σ_o and $\eta\sigma_o$ as indicated in the figure). The domain is cut in half along an inclined straight line (with angle β with respect to the horizontal axis, as indicated in the figure). The two halves are then glued together. We assume that the layer of adhesive is infinitely thin and does not contribute to the deformation of the rectangular domain. However, the manufacturer of the adhesive specifies that the bond will fail when the ratio of the shear traction to the normal traction exceeds a value of 2/3.

- At what angle critical angle β_c (as a function of η) will the bond fail?
- Show that $\beta_c = 33.7°$ when $\eta = 0$.
- Plot the dependence of β_c (in degrees) on η, for $0 \leq \eta \leq 0.25$. Don't forget to label the axes and to include your name in the title.



Problem 2

Consider the 2D problem of a rectangular linearly elastic solid (of width 2a and length 2L) that is subjected to a tensile traction of amplitude σ_o along the top and bottom edges, and a shear traction of amplitude $\tau_o = \beta\sigma_o$ along the entire boundary as indicated in the figure. A semi-circular cut is introduced in the rectangular domain as indicated and is then bonded again using an adhesive (similar to the first problem of this assignment).



1) What is the state of stress in the domain? Show that this state of stress satisfies the traction BC along all four edges of the rectangular domain.
2) Compute the normal traction T_n (as a function of σ_o, θ and β) along the boundary of the semi-circular cut.
3) At what angle θ^* (as a function of β) does the normal traction reach a maximum?
4) What should be the value of θ^* when $\beta = 0$? Check that the solution you found in 3) corresponds to your expectation.
5) Plot the dependence of θ^* (in degrees) on β (for $0 \le \beta \le 6$). Don't forget to label the axes and to include your name in the title.

Problem 3

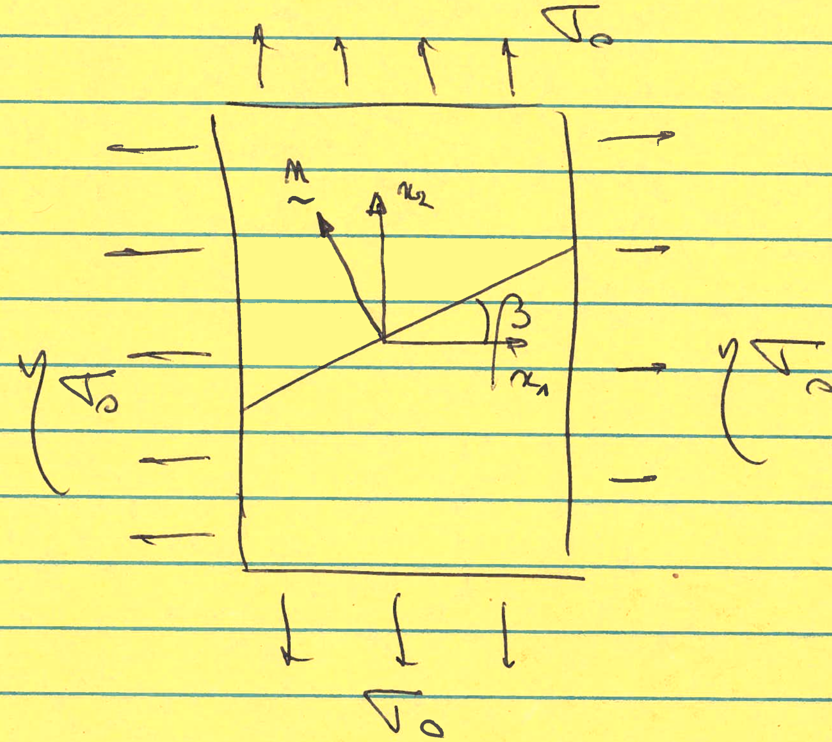
Consider the stress tensor $\sigma = \begin{pmatrix} 14 & -7 & 0 \\ -7 & 21 & 0 \\ 0 & 0 & 7 \end{pmatrix}$ kPa.

(a) Compute the traction vector $\boldsymbol{T}$ acting on a plane passing through the following three points A (1,0,0) B (0,2,0) C (0,0,3)
(b) Compute the norm of the traction vector and the normal traction
(c) Compute the norm of the tangential traction vector acting on the plane

Don't forget to indicate the units of the quantities you are computing.

AE323 – Homework 1 – Solution

Problem 1



State of stress

$$\underset{\sim}{\sigma} = \begin{pmatrix} \eta\sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}$$

Normal vector to interface $\quad \underset{\sim}{n} = \begin{pmatrix} -\sin\beta \\ \cos\beta \end{pmatrix}$

$\Rightarrow$ (Cauchy relation): traction $\underset{\sim}{T}$ along interface

$$\underset{\sim}{T} = \underset{\sim}{\sigma}\,\underset{\sim}{n} = \begin{pmatrix} \eta\sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}\begin{pmatrix} -\sin\beta \\ \cos\beta \end{pmatrix} = \begin{pmatrix} -\eta\sigma_0\sin\beta \\ \sigma_0\cos\beta \end{pmatrix}$$

Normal traction $\quad T_n = \underset{\sim}{T} \cdot \underset{\sim}{n} = \eta\,\sigma_0 \sin^2\beta + \sigma_0 \cos^2\beta$ $\quad (*)$

Shear traction

$$\underset{\sim}{T_s} = \underset{\sim}{T} - T_n \underset{\sim}{n}$$

$$= \begin{pmatrix} -\eta\,\sigma_0 \sin\beta \\ \sigma_0 \cos\beta \end{pmatrix} - \left[\eta\,\sigma_0 \sin^2\beta + \sigma_0 \cos^2\beta\right] \begin{pmatrix} -\sin\beta \\ \cos\beta \end{pmatrix}$$

$$= \begin{pmatrix} -\eta\,\sigma_0 \sin\beta + \eta\,\sigma_0 \sin^3\beta + \sigma_0 \cos^2\beta \sin\beta \\ \sigma_0 \cos\beta - \eta\,\sigma_0 \sin^2\beta \cos\beta - \sigma_0 \cos^3\beta \end{pmatrix}$$

$$= \begin{pmatrix} \eta\,\sigma_0 \sin\beta \left(\sin^2\beta - 1\right) + \sigma_0 \sin\beta \cos^2\beta \\ \sigma_0 \cos\beta \left(1 - \cos^2\beta\right) - \eta\,\sigma_0 \cos\beta \sin^2\beta \end{pmatrix}$$

$$= \begin{pmatrix} -\eta\,\sigma_0 \sin\beta \cos^2\beta + \sigma_0 \sin\beta \cos^2\beta \\ \sigma_0 \cos\beta \sin^2\beta - \eta\,\sigma_0 \cos\beta \sin^2\beta \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_0 \left(1-\eta\right) \sin\beta \cos^2\beta \\ \sigma_0 \left(1-\eta\right) \cos\beta \sin^2\beta \end{pmatrix}$$

$$= \frac{\sigma_0 \left(1-\eta\right)}{2} \sin 2\beta \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix}$$

$\sin 2\beta = 2 \sin\beta \cos\beta$

$$\hookrightarrow \quad |\underset{\sim}{T}_s| = \frac{\sigma_0 (1-\eta)}{2} \sin 2\beta \qquad (**)$$

Combining (*) and (**), we get

$$\frac{|\underset{\sim}{T}_s|}{T_n} = \frac{\frac{\sigma_0}{2}(1-\eta)\sin 2\beta}{\sigma_0 \cos^2\beta\,(1+\eta\tan^2\beta)} = \frac{(1-\eta)\tan\beta}{1+\eta\tan^2\beta}$$

The critical value of β $(=\beta_c)$ is such that

$$\frac{(1-\eta)\tan\beta_c}{1+\eta\tan^2\beta_c} = \frac{2}{3} \qquad (\square)$$

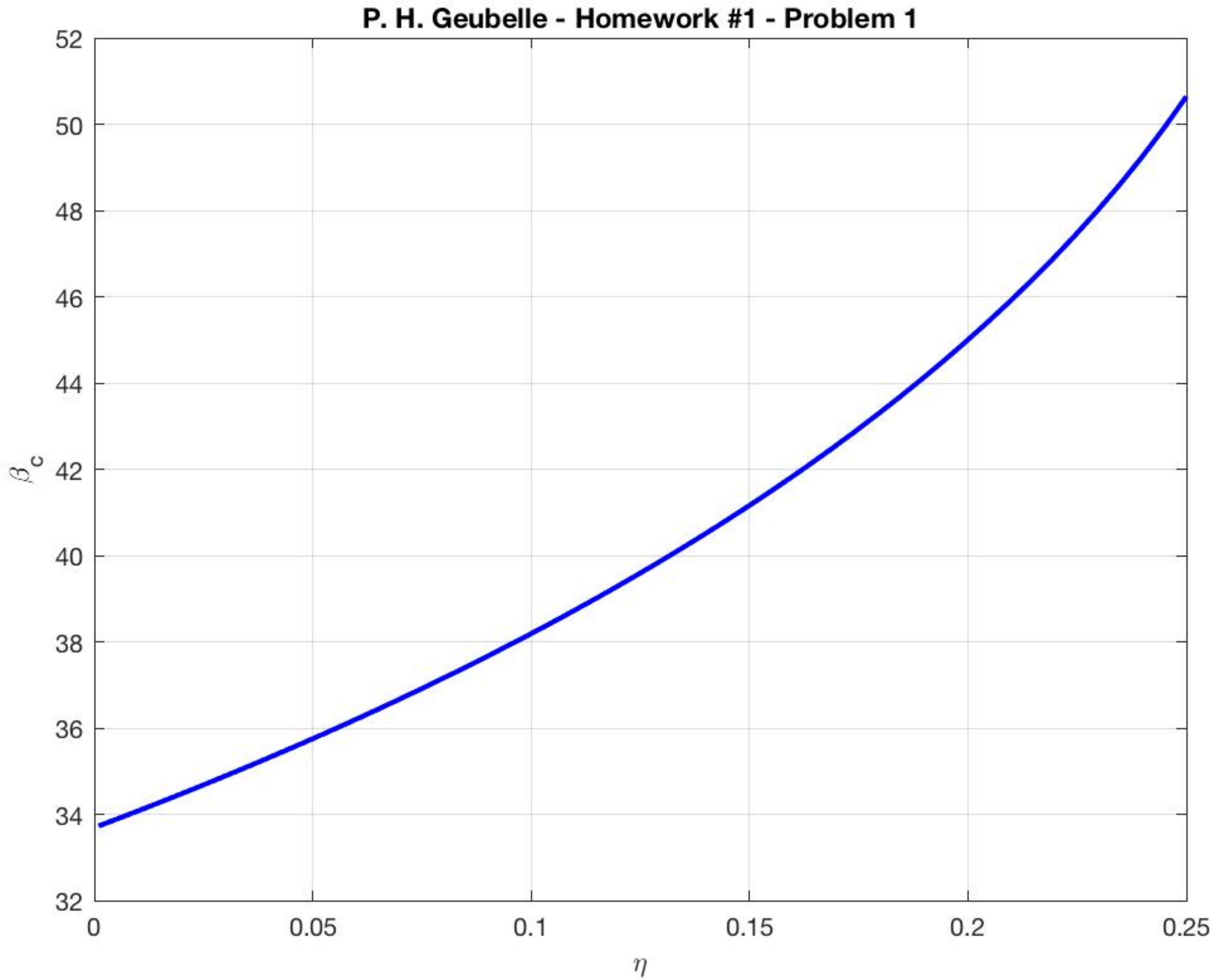
$$2\eta\tan^2\beta_c + 3(\eta-1)\tan\beta_c + 2 = 0$$

$$\tan\beta_c = \frac{3(1-\eta) \pm \sqrt{9(\eta-1)^2 - 4(2\eta)2}}{4\eta}$$

$$= \frac{3(1-\eta) \pm \sqrt{9\eta^2 - 34\eta + 9}}{4\eta}$$

When $\eta = 0$, $(\square)$ gives $\tan\beta_c = \frac{2}{3}$

$$\beta_c = 33.69^\circ$$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52
50
48
46
44
42
40
38
36
34
32
β_c
0
0.05
0.1
0.15
0.2
0.25
η

Problem 2.

(1) State of stress : $\underset{\sim}{\sigma} = \begin{pmatrix} 0 & \tau_0 \\ \tau_0 & \sigma_0 \end{pmatrix}$ with $\tau_0 = \beta\sigma_0$

Right edge $\underset{\sim}{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \underset{\sim}{T} = \underset{\sim}{\sigma}\,\underset{\sim}{n} = \begin{pmatrix} 0 \\ \tau_0 \end{pmatrix}$ ✓

Left edge $\underset{\sim}{n} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \Rightarrow \underset{\sim}{T} = \begin{pmatrix} 0 \\ -\tau_0 \end{pmatrix}$ ✓

Top edge $\underset{\sim}{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \underset{\sim}{T} = \begin{pmatrix} \tau_0 \\ \sigma_0 \end{pmatrix}$ ✓

Bottom edge $\underset{\sim}{n} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \Rightarrow \underset{\sim}{T} = \begin{pmatrix} -\tau_0 \\ -\sigma_0 \end{pmatrix}$ ✓

(2) Along semi-circular cut

$$\underset{\sim}{n} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\rightarrow \underset{\sim}{T} = \underset{\sim}{\sigma}\,\underset{\sim}{n} = \begin{pmatrix} 0 & \tau_0 \\ \tau_0 & \sigma_0 \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \tau_0 \sin\theta \\ \tau_0\cos\theta + \sigma_0\sin\theta \end{pmatrix}$$

$$= \sigma_0 \begin{pmatrix} \beta\sin\theta \\ \beta\cos\theta + \sin\theta \end{pmatrix}$$

$$T_n = \underset{\sim}{T}\cdot\underset{\sim}{n} = \sigma_0\left(2\beta\sin\theta\cos\theta + \sin^2\theta\right)$$

(3) θ^* s.t. $\left.\frac{\partial \tau_n}{\partial \theta}\right|_{\theta^*} = 0$

$$\Leftrightarrow \frac{\partial}{\partial \theta}\left(\tau_0 \left(\beta \sin 2\theta + \sin^2\theta \right) \right) = 0$$

$$2\beta \cos 2\theta^* + 2 \sin\theta^* \cos\theta^* = 0$$

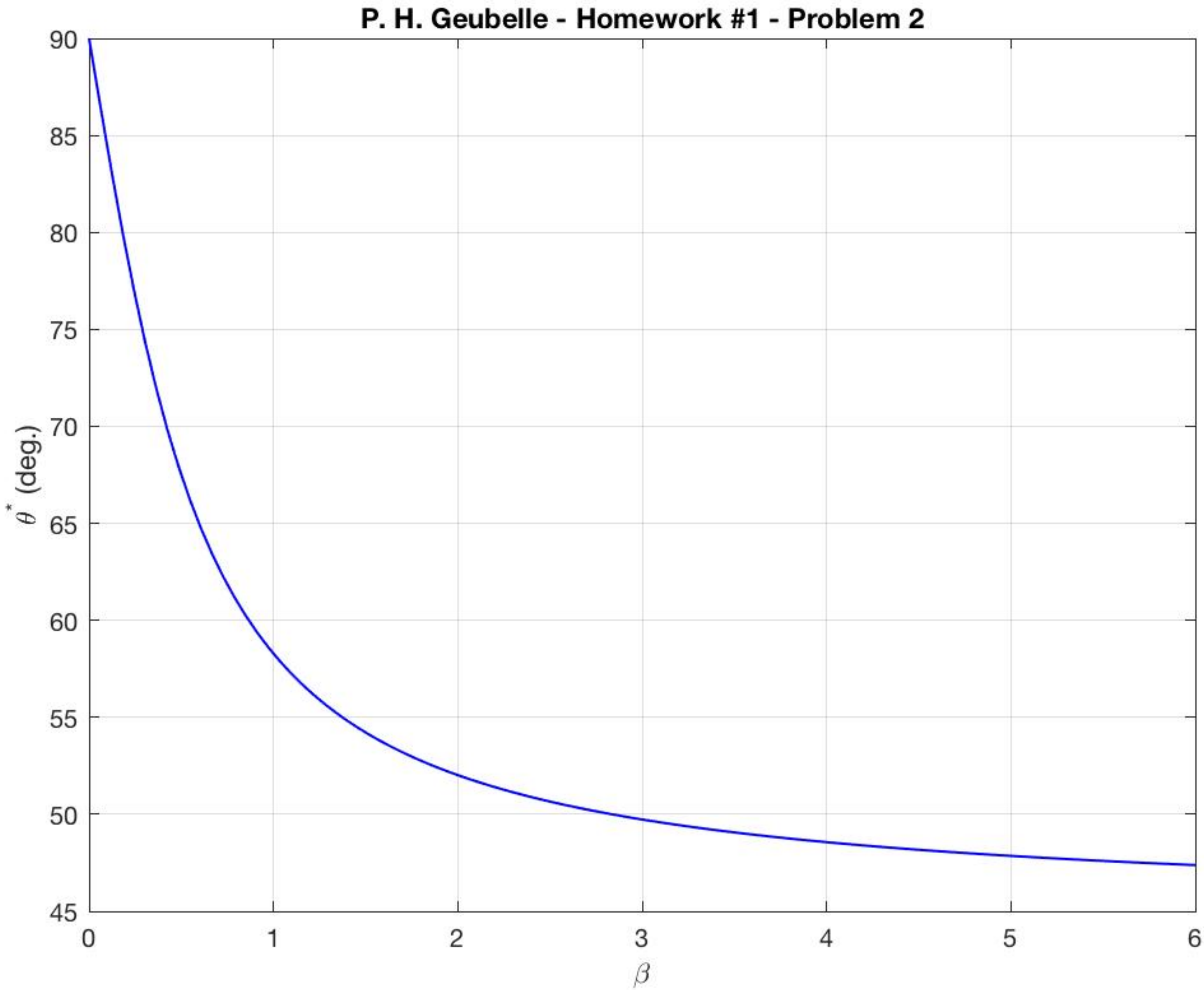
$$2\beta \cos 2\theta^* + \sin 2\theta^* = 0$$

$$\tan 2\theta^* = -2\beta$$

$$\theta^* = \frac{1}{2} \tan^{-1}\left(-2\beta\right) + \frac{\pi}{2} \qquad (*)$$

(4) When $\beta = 0$ (i.e., when $\tau_0 = 0$), we should get $\theta^* = \pi/2$

$(*) \Rightarrow$ ✓



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90
85
80
75
70
65
60
55
50
45
θ^* (deg.)
0
1
2
3
4
5
6
β

Problem 3

$$\underset{\sim}{T} = \begin{pmatrix} 12 & -6 & 0 \\ -6 & 8 & 0 \\ 0 & 0 & 15 \end{pmatrix} \text{ kPa}$$

A (1, 0, 0)
B (0, 2, 0)
C (0, 0, 3)

(a) First, we need to find the normal n to the plane defined by the 3 points A, B, C, or by the two vectors:

$$\overrightarrow{AB} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$$

$\hookrightarrow$ The vector product $\overrightarrow{AB} \times \overrightarrow{AC}$ will give a vector that is perpendicular to both vectors, i.e., that is perpendicular to the plane:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$$

§ 3.2

We need to normalize the normal vector

$$\underset{\sim}{n} = \frac{1}{\sqrt{6^2+3^2+2^2}} \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{49}} \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 6/7 \\ 3/7 \\ 2/7 \end{pmatrix} = \begin{pmatrix} 0.8571 \\ 0.4286 \\ 0.2857 \end{pmatrix}$$

Using the Cauchy relation, we get

$$\underset{\sim}{T} = \underset{\sim}{\sigma} \cdot \underset{\sim}{n} = \begin{pmatrix} 14 & -7 & 0 \\ -7 & 28 & 0 \\ 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} 6/7 \\ 3/7 \\ 2/7 \end{pmatrix}$$

$$= \begin{pmatrix} 9 \\ 3 \\ 2 \end{pmatrix} \text{ kPa}$$

(b) $$|\underset{\sim}{T}| = \sqrt{9^2+3^2+2^2} = 9.6954 \text{ kPa}$$

$$T_n = \underset{\sim}{T} \cdot \underset{\sim}{n} = 9.5714 \text{ kPa}$$

(c) $$\underset{\sim}{T_s} = \underset{\sim}{T} - T_n \underset{\sim}{n} = \begin{pmatrix} 9 \\ 3 \\ 2 \end{pmatrix} - 9.5714 \begin{pmatrix} 6/7 \\ 3/7 \\ 2/7 \end{pmatrix}$$

$$= \begin{pmatrix} +0.7959 \\ -1.1020 \\ -0.7347 \end{pmatrix} \text{ kPa} \qquad |\underset{\sim}{T_s}| = 1.5452 \text{ kPa}$$