# Course Outline




**1. Fundamental Concepts of Elasticity** **(I, II)***

1.1 Stresses

1.2 Strains

1.3 Material Constitution

**2. Strength of Materials Analysis of Straight, Long Beams** **(III)**

2.1 Beam Bending/Extension

2.2 Beam Torsion

**3. Energy Methods** **(IV)**

3.1 Work and Potential Energy Principles

3.2 Analytical Solution of Static Problems

**4. Introduction to Buckling** **(III, V)**

4.1 Introduction

4.2 Beam Buckling using Euler-Bernoulli Theory

4.3 Beam Buckling using Energy Methods


*** Numbers in parentheses refer to chapters in the course textbook**



# Chapter 1: Fundamental Concepts of Elasticity




★ **1.1 Stresses** **(1, 2)**

1.1.1 Definition of traction and stress

1.1.2 Equations of equilibrium

**1.2 Strains** **(3, 4)**

1.2.1 Normal and shear strains

1.2.2 Compatibility

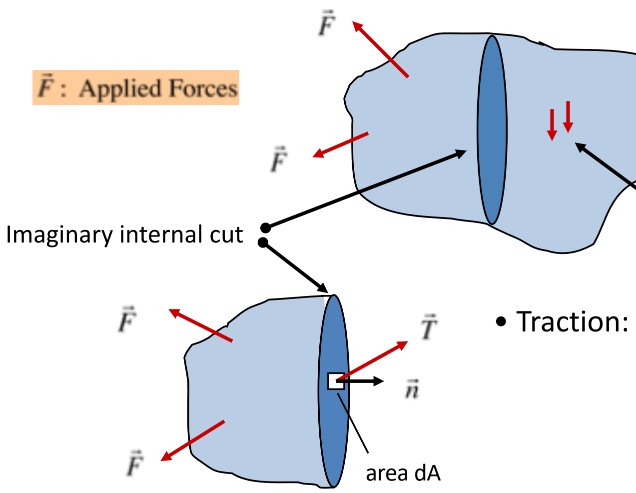
**1.3 Material Constitution** **(5, 6)**

1.3.1 Uniaxial material behavior

1.3.2 Generalized Hooke's law

# 1.1.1 Definition of traction (1.1)*

- Interaction between deformable bodies? ⇨ Surface loads




- Traction: $\vec{T} = \lim_{dA \to 0} \left( \frac{d\vec{F}}{dA} \right)$

Normal vs. shear

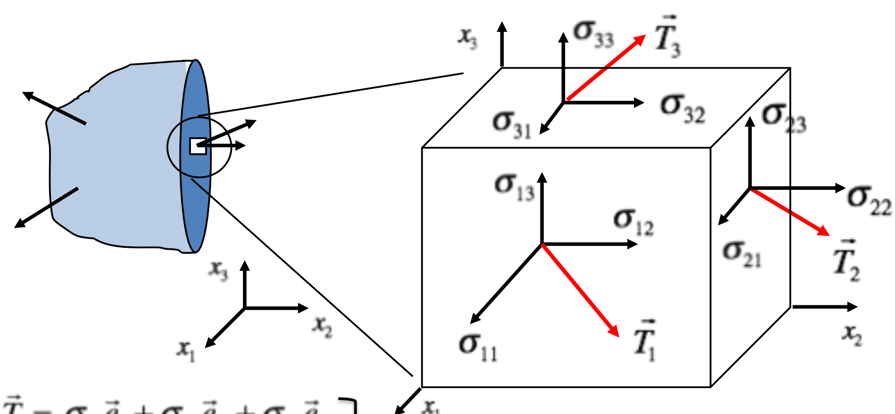
* The number in parentheses denotes the section in the textbook





# Definition of stress (1.1, 1.2)

- Cut out an infinitesimal cube at the point in question, and align with axes




$$\left.\begin{aligned} \vec{T}_1 &= \sigma_{11}\vec{e}_1 + \sigma_{12}\vec{e}_2 + \sigma_{13}\vec{e}_3 \\ \vec{T}_2 &= \sigma_{21}\vec{e}_1 + \sigma_{22}\vec{e}_2 + \sigma_{23}\vec{e}_3 \\ \vec{T}_3 &= \sigma_{31}\vec{e}_1 + \sigma_{32}\vec{e}_2 + \sigma_{33}\vec{e}_3 \end{aligned}\right\}$$

☺ **On coordinate planes !**

## Definition of stress (cont.)

- Coefficients $\sigma_{ij}$ are the **stress (tensor) components**

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

- Naming convention: $\sigma_{ij}$

Face on which component acts

Direction in which component acts

- Sign convention:
  – Normal (or extensional) components $\sigma_{11}, \sigma_{22}, \sigma_{33}$ $\begin{cases} >0 & \text{Tension} \\ <0 & \text{Compression} \end{cases}$

  –Shearing components $\sigma_{ij}$ $i \neq j$: positive if in positive direction on a positive face.





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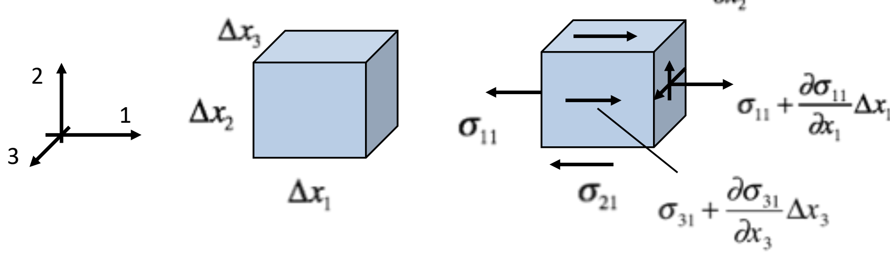
# 1.1.2 Equations of equilibrium (1.2)

- Consider unit cube aligned with coordinate axes:




- Apply fundamental laws of physics:
  - Conservation of linear momentum ⇨ $\sum \vec{F} = \vec{0}$
  - Conservation of angular momentum ⇨ $\sum \vec{M} = \vec{0}$
- Examples…





# Equations of equilibrium (cont.)

- Conservation of momentum:

LINEAR

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + f_1 = 0$$
$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + f_2 = 0$$
$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + f_3 = 0$$

ANGULAR

$$\sigma_{12} = \sigma_{21}$$
$$\sigma_{23} = \sigma_{32}$$
$$\sigma_{13} = \sigma_{31}$$

$\vec{f}$ = body force/unit volume

- At any given point there are six unknown stress components, that may depend on position:

$$\sigma_{11} = \sigma_{11}(x_1, x_2, x_3), \quad etc\ldots$$

- Statically determinate structure: one where stresses can be obtained from equilibrium only.

## Equilibrium at an "outer" boundary (1.3)

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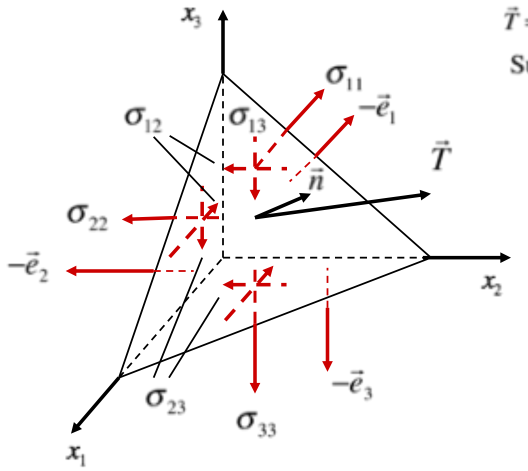
- Discretize an <u>arbitrary</u> surface with normal $\boldsymbol{n}$ by a tetrahedron:

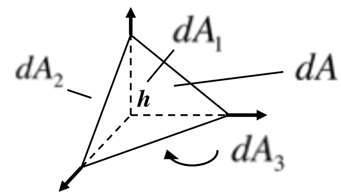



$\vec{T}$ = Traction acting on external surface

Surface : $\vec{n} = n_1\vec{e}_1 + n_2\vec{e}_2 + n_3\vec{e}_3$

- For equilibrium:

$$\sum \vec{F}_i = \vec{0}$$






## Equilibrium at an "outer" boundary (cont.)

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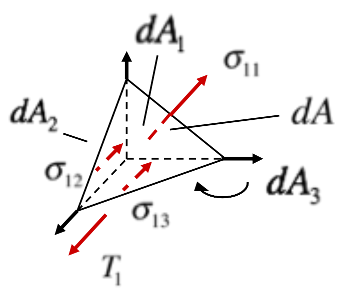



$$\vec{T}\,dA + \vec{T}_1 dA_1 + \vec{T}_2 dA_2 + \vec{T}_3 dA_3 + \vec{f} dV = \vec{0}$$

- Along $x_1$:

$$T_1 dA - \sigma_{11} dA_1 - \sigma_{12} dA_2 - \sigma_{13} dA_3 + \underbrace{f_1 dV}_{0} = 0$$

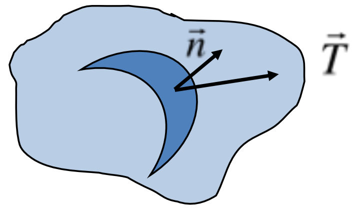
- But from geometry area relation is:

$$\left.\begin{aligned} dA_1 &= dA\cos(\vec{n}\cdot\vec{e}_1) = (dA)n_1 \\ dA_2 &= dA\cos(\vec{n}\cdot\vec{e}_2) = (dA)n_2 \\ dA_3 &= dA\cos(\vec{n}\cdot\vec{e}_3) = (dA)n_3 \end{aligned}\right\} \Rightarrow T_1 = \sigma_{11}n_1 + \sigma_{12}n_2 + \sigma_{13}n_3$$

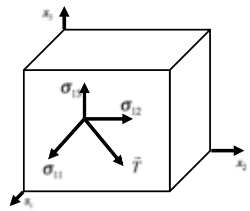
## The Cauchy relation



$$T_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$$
$$T_2 = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3$$
$$T_3 = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3$$




- Can be generalized to any "cut" inside the object
  - $T_i$: components of traction on a point of the "cut"
  - $n_i$: components of normal to the "cut"
  - $\sigma_{ij}$: stress components at point $x_1$, $x_2$, $x_3$
- "Special" case: cut along coordinate planes




- Net normal and shear traction components

$$\sigma_{norm} = \vec{T} \cdot \vec{n} \qquad \sigma_{shear} = \left| \vec{T} - \sigma_{norm} \vec{n} \right|$$

- Examples…



## 'On the shoulders of giants'

Augustin Louis Cauchy (1789-1857)

Stephen Timoshenko (1878-1972)

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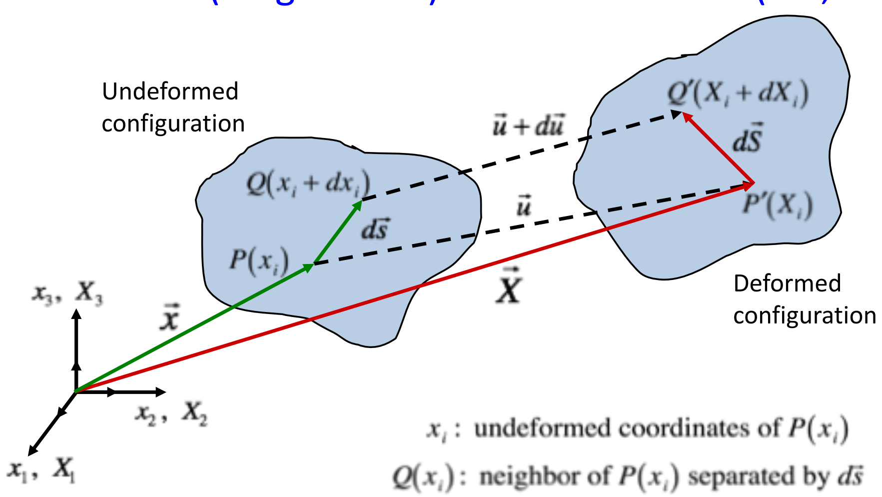
## 1.2.1 Normal (longitudinal) and shear strains (3.3,3.4)




$$\vec{X} = \vec{x} + \vec{u}$$

$x_i$ : undeformed coordinates of $P(x_i)$

$Q(x_i)$ : neighbor of $P(x_i)$ separated by $d\vec{s}$

$X_i$ : deformed coordinates of $P'(X_i)$

$Q'(X_i)$ : neighbor of $P'(X_i)$ separated by $d\vec{S}$

## Normal and shear strains (cont.)

- If *P* and *Q* move such that $\Rightarrow \; d\vec{s} = d\vec{S} \; \Rightarrow$ rigid motion
- Possible measures of **deformation**

$$dS - ds, \; \sqrt{dS} - \sqrt{ds}, \; f(dS) - f(ds)$$

with $ds = |d\vec{s}|$ , $dS = |d\vec{S}|$

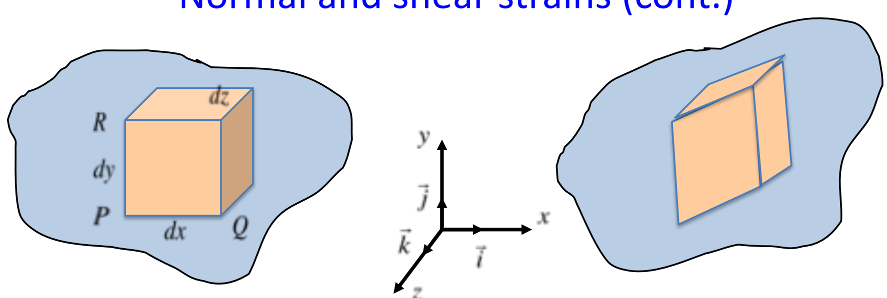
- Note: $\vec{u} = u_1\vec{e_1} + u_2\vec{e_2} + u_3\vec{e_3} = u\vec{e_1} + v\vec{e_2} + w\vec{e_3}$
- **Displacement field**: $u(x_1, x_2, x_3)$, $v(x_1, x_2, x_3)$, $w(x_1, x_2, x_3)$
- Choose as a measure of deformation: $\dfrac{dS - ds}{ds}$

**Lagrangian coordinates**

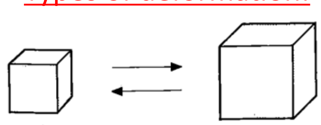





## Normal and shear strains (cont.)

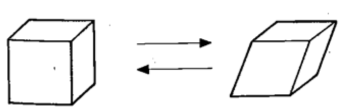



- Types of deformation:



- Dilatational deformation:
  – Volume change

$$\frac{\text{change of length}}{\text{original length}}$$



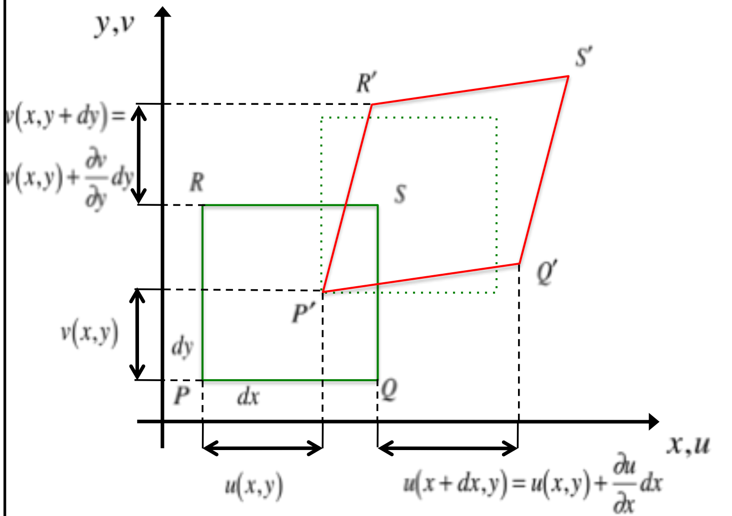
- Shearing deformation:
  – Shape change

$$\frac{1}{2}\,\text{change of angle from } 90^\circ$$

# Normal and shear strains (cont.)



• $u$, $v$, $w$ are the displacement field components. They must be continuous functions of position to ensure no holes or voids form in the object (compatibility).

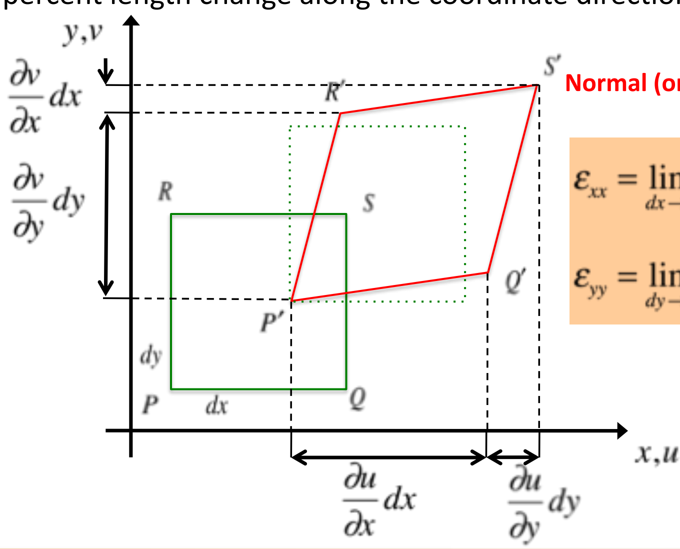





# Normal and shear strains (cont.)



• The **dilatational** (or **normal**, or **direct**) strains are defined as the percent length change along the coordinate directions




**Normal (or longitudinal) Strains**

$$\varepsilon_{xx} = \lim_{dx\to 0} \frac{P'Q' - PQ}{PQ} = \frac{\partial u}{\partial x}$$

$$\varepsilon_{yy} = \lim_{dy\to 0} \frac{P'R' - PR}{PR} = \frac{\partial v}{\partial y}$$

## Normal and shear strains (cont.)

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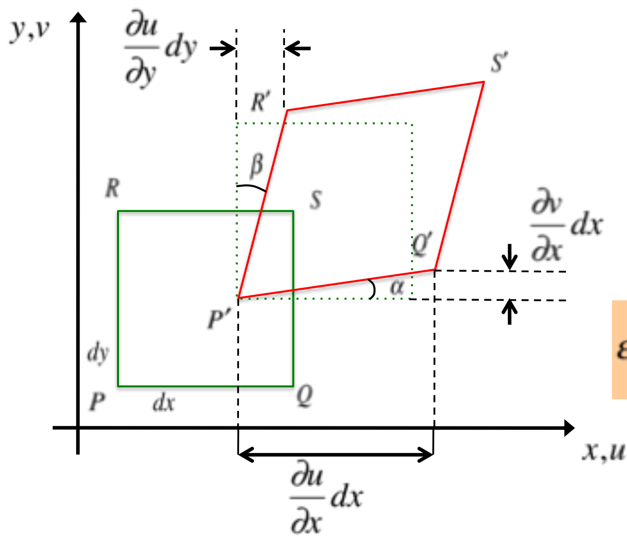
- The **shear** strains are defined as 1/2 the angle change from 90°




**Shear Strain**

$$\varepsilon_{xy} = \frac{1}{2}(\alpha+\beta)$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$$



## Normal and shear strains (cont.)

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- In general in 3D (and returning to $x_1, x_2, x_3$ notation) we have the **strain-displacement** equations (3.6):

$$\left.\begin{aligned} \varepsilon_{11} &= \frac{\partial u_1}{\partial x_1} \\ \varepsilon_{22} &= \frac{\partial u_2}{\partial x_2} \\ \varepsilon_{33} &= \frac{\partial u_3}{\partial x_3} \end{aligned}\right\} \text{Normal Strains}$$

$$\left.\begin{aligned} \varepsilon_{12} = \varepsilon_{21} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) \\ \varepsilon_{13} = \varepsilon_{31} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right) \\ \varepsilon_{23} = \varepsilon_{32} &= \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right) \end{aligned}\right\} \text{Shear Strains}$$

- **Infinitesimal strain tensor**:

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix}$$

$$[\varepsilon'] = [R][\varepsilon][R]^T$$

Strain rotation can also be performed with **Mohr's circle** (see AE321)

## Normal and shear strains (cont.)



- Sometimes (e.g., Timoshenko and Goodier) infinitesimal shear strain is defined with the ½ factor, e.g.:

$$\gamma_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

- But then strain is NOT a two-tensor

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ & \varepsilon_{22} & \varepsilon_{23} \\ sym. & & \varepsilon_{33} \end{bmatrix}$$

A tensor

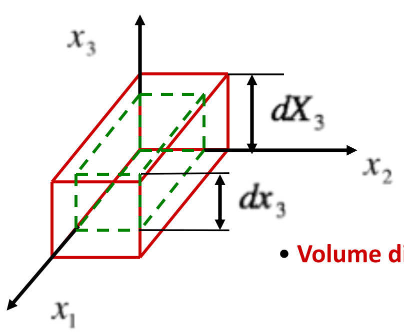
$$[\varepsilon] = \begin{bmatrix} \varepsilon_{11} & \gamma_{12} & \gamma_{13} \\ & \varepsilon_{22} & \gamma_{23} \\ sym. & & \varepsilon_{33} \end{bmatrix}$$

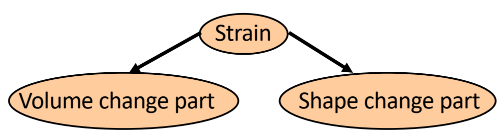
Not a tensor - **Engineering strain**



## Normal and shear strains (cont.)






- **Volume dilatation:** $\Delta = \dfrac{dV - dv}{dv} = \dfrac{\text{volume change}}{\text{original volume}}$

- Assuming no shear strain:

$$dv = dx_1 dx_2 dx_3 \quad \text{and} \quad dV = dX_1 dX_2 dX_3$$

But $\varepsilon_{11} = \dfrac{dX_1 - dx_1}{dx_1} \Rightarrow dX_1 = (1 + \varepsilon_{11}) dx_1$

## Normal and shear strains (cont.)



$$dv = dx_1 dx_2 dx_3$$

$$dV = (1+\varepsilon_{11})dx_1(1+\varepsilon_{22})dx_2(1+\varepsilon_{33})dx_3$$

$$\Delta = \frac{dv\left[(1+\varepsilon_{11})(1+\varepsilon_{22})(1+\varepsilon_{33})-1\right]}{dv}$$
$$= 1 + (\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + H.O.T + \ldots - 1$$
$$\approx \varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}$$

Volume dilatation: $\Delta = \varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33} = \varepsilon_1+\varepsilon_2+\varepsilon_3$



## Normal and shear strains (cont.)



- **Deviatoric** and **volumetric** strain

$$\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_m & 0 & 0 \\ 0 & \varepsilon_m & 0 \\ 0 & 0 & \varepsilon_m \end{bmatrix} + \begin{bmatrix} \varepsilon_{11}-\varepsilon_m & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22}-\varepsilon_m & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33}-\varepsilon_m \end{bmatrix}$$

Mean strain (volumetric)

$$\varepsilon_m = \frac{1}{3}(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) \Rightarrow \Delta = 3\varepsilon_m$$

Volume change with no shape change

Deviatoric strain

$$\Delta = 0$$

Shape change at constant volume

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**1.1 Stresses** **(1, 2)**

1.1.1 Definition of traction and stress

1.1.2 Equations of equilibrium

1.1.3 Principal stresses

★ **1.2 Strains** **(3, 4)**

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1.2.2 Compatibility

**1.3 Material Constitution** **(5,6)**

1.3.1 Uniaxial material behavior

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# 1.2.2 Compatibility (3.7)

- Given $u$ we can easily find $\varepsilon$
- Given $\varepsilon$, how can we find $u$?

6 independent components — 3 unknowns

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \qquad \varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2}, \qquad \varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3}, \qquad \varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)$$

- System is overdetermined
- Conditions of **Compatibility** are restrictions on strain so that **single valued** displacements are produced upon integration.
- Physically this restriction means that the displacement field cannot contain holes or voids.

# Compatibility (cont.)

• 6 independent compatibility conditions that are **necessary and sufficient** to ensure a unique displacement field:

$$\frac{\partial^2\varepsilon_{11}}{\partial x_2^2}+\frac{\partial^2\varepsilon_{22}}{\partial x_1^2}-2\frac{\partial^2\varepsilon_{12}}{\partial x_1\partial x_2}=0$$

$$\frac{\partial^2\varepsilon_{22}}{\partial x_3^2}+\frac{\partial^2\varepsilon_{33}}{\partial x_2^2}-2\frac{\partial^2\varepsilon_{23}}{\partial x_2\partial x_3}=0$$

$$\frac{\partial^2\varepsilon_{33}}{\partial x_1^2}+\frac{\partial^2\varepsilon_{11}}{\partial x_3^2}-2\frac{\partial^2\varepsilon_{13}}{\partial x_1\partial x_3}=0$$

$$\frac{\partial^2\varepsilon_{12}}{\partial x_1\partial x_3}+\frac{\partial^2\varepsilon_{13}}{\partial x_1\partial x_2}-\frac{\partial^2\varepsilon_{23}}{\partial x_1^2}-\frac{\partial^2\varepsilon_{11}}{\partial x_2\partial x_3}=0$$

$$\frac{\partial^2\varepsilon_{23}}{\partial x_2\partial x_1}+\frac{\partial^2\varepsilon_{21}}{\partial x_2\partial x_3}-\frac{\partial^2\varepsilon_{31}}{\partial x_2^2}-\frac{\partial^2\varepsilon_{22}}{\partial x_3\partial x_1}=0$$

$$\frac{\partial^2\varepsilon_{31}}{\partial x_3\partial x_2}+\frac{\partial^2\varepsilon_{32}}{\partial x_3\partial x_1}-\frac{\partial^2\varepsilon_{12}}{\partial x_3^2}-\frac{\partial^2\varepsilon_{33}}{\partial x_1\partial x_2}=0$$

• Examples...

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# 1.3.1 Uniaxial material behavior (5.2)

- Need to relate stress $\sigma$ to strain $\varepsilon$
- A material characteristic, i.e., external to the theory
  $\Rightarrow$ EXPERIMENTS
- Mechanical **constitutive equation**

$$\sigma = f\left(\varepsilon, \dot{\varepsilon}, T, \dot{T}, \ldots\right)$$

## Uniaxial material behavior (cont.)

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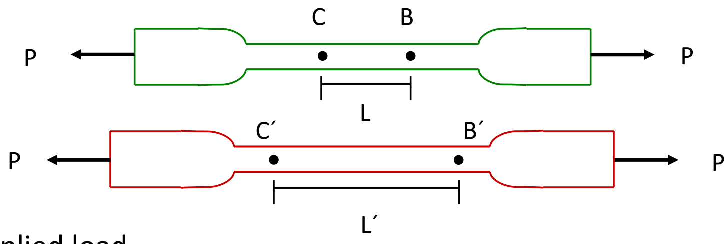


P: Applied load
L: Gauge section
ΔL: Change of gauge section length, L´-L

- Assume only axial tensile stress: $\sigma_{axial} = \dfrac{P}{A}$
- Strain: $\varepsilon_{axial} = \dfrac{\Delta L}{L}$

A: (initial) cross sectional area of gauge length



## Uniaxial material behavior (cont.)

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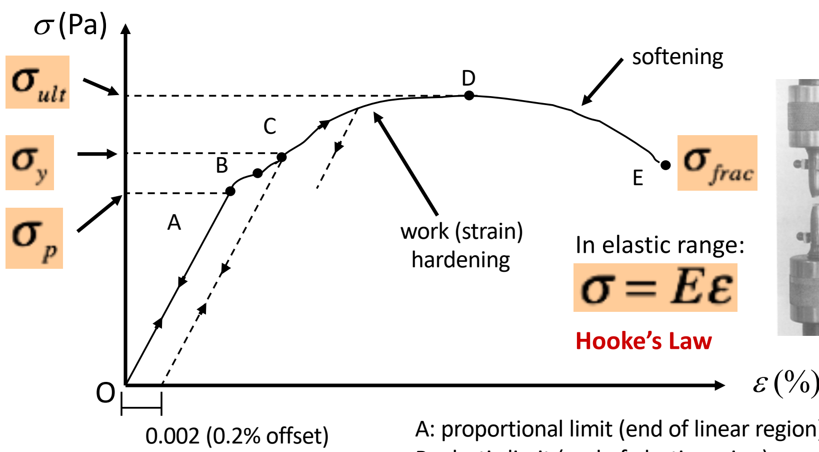
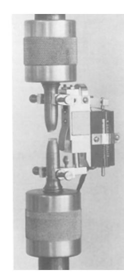
- Plot experimental **$\sigma$-$\varepsilon$ curve** (e.g. steel specimen in tension)




In elastic range:

$$\sigma = E\varepsilon$$

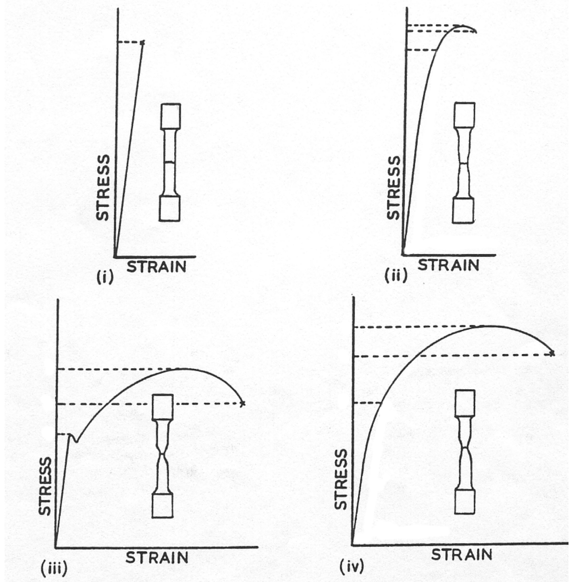
**Hooke's Law**



A: proportional limit (end of linear region)
B: elastic limit (end of elastic region)
C: yield stress at 0.2% offset
D: ultimate stress
E: fracture stress

# Uniaxial material behavior (cont.)



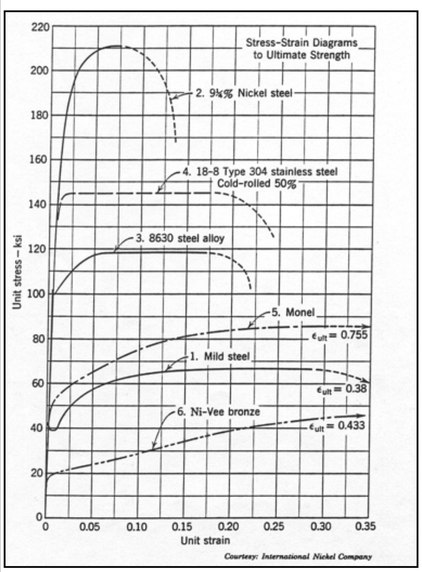


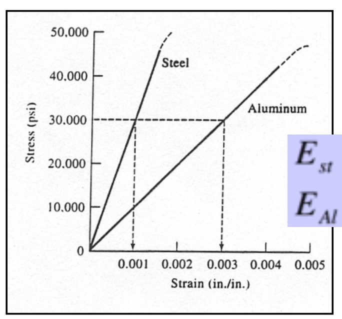

(i) Brittle
(ii) Brittle (or semi-ductile)
(iii), (iv) Ductile



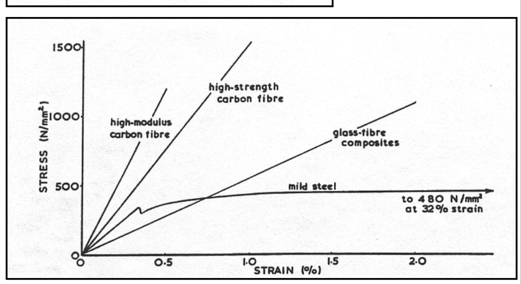
# Uniaxial material behavior (cont.)






$$E_{st} = 207GPa$$
$$E_{Al} = 70GPa$$



## Uniaxial material behavior (cont.)

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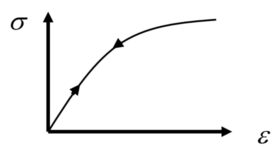
- Between O and the **proportional limit** (point A) the $\sigma$–$\varepsilon$ curve shows **linear elastic** behavior

Linear → straight line (i.e. $\sigma \sim \varepsilon$)
Elastic → unloading occurs along the same loading path (material returns to 0,0)

- Up to B, non-linear elastic material: e.g. rubber




Note: sometimes B and C coincide

- Upon continued loading we exceed the **elastic limit**. Then unloading follows a different path.



## Uniaxial material behavior (cont.)

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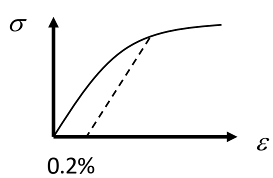
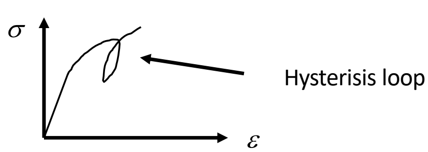
- After point C unloading produces a **permanent (residual) plastic strain** when $\sigma$ returns to 0.

$\sigma_C$: 0.2% offset stress (or flow stress)




- When unloading ductile metals in reality : **Hysterisis**



# Uniaxial material behavior (cont.)

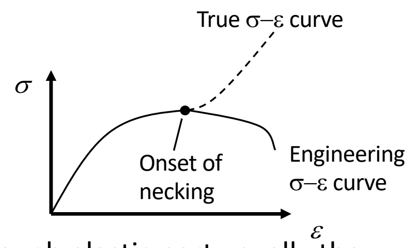


- The **Ultimate Tensile Strength (UTS)** (point D) is the maximum sustainable stress. At point D **necking** begins:

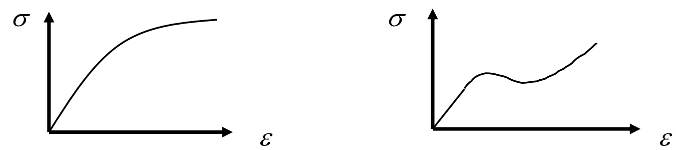
$A_{neck} < A_{initial}$

$$\sigma = \frac{P}{A_{initial}}$$ : engineering stress

$$\sigma = \frac{P}{A_{neck}}$$ : true stress




- **Compression** behavior may differ (although elastic part usually the same)

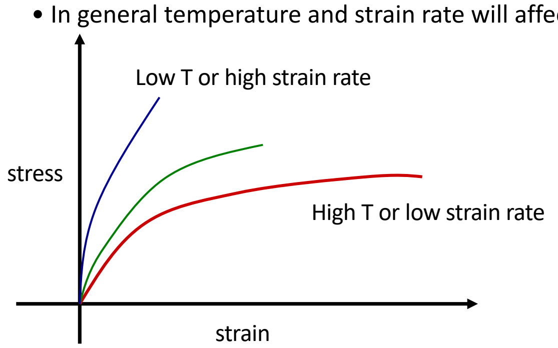


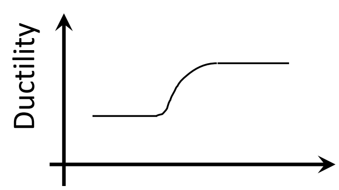



# Uniaxial material behavior (cont.)

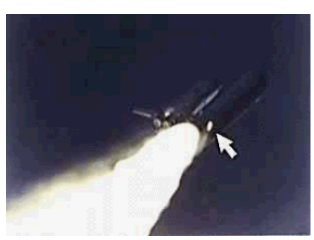


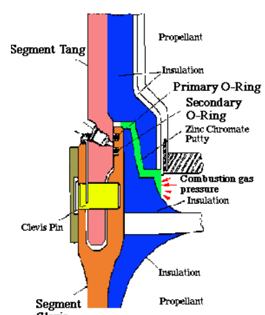
- In general temperature and strain rate will affect the material response:



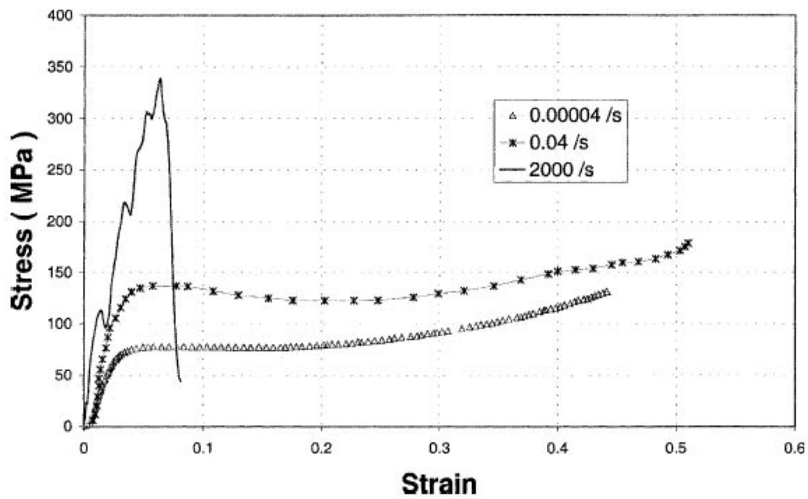

- Temperature effects (Challenger, 1/28/86):





## Uniaxial material behavior (cont.)






Effect of strain rate on σ–ε curve of PMMA (Plexiglas)



## Chapter 1: Fundamental Concepts of Elasticity




**1.1 Stresses** **(1, 2)**
- 1.1.1 Definition of traction and stress
- 1.1.2 Equations of equilibrium
- 1.1.3 Principal stresses

**1.2 Strains** **(3, 4)**
- 1.2.1 Normal and shear strains
- 1.2.2 Compatibility

★ **1.3 Material Constitution** **(5, 6)**
- 1.3.1 Uniaxial material behavior
- 1.3.2 Generalized Hooke's law

## 1.3.2 Generalized Hooke's Law (6.3)



- Stress σ, strain ε: 9 components each (6 independent)

⇒ Need multi-axial information

- In multi-axial experiments:

$$\begin{aligned} \textit{linear elastic} &\rightarrow \textit{proportional limit} \\ &\rightarrow \textit{yield} \\ &\rightarrow \textit{plastic reponse} \\ &\rightarrow \textit{necking} \\ &\rightarrow \textit{failure} \end{aligned}$$

- For 3D problems a linear elastic model is valid up to yield

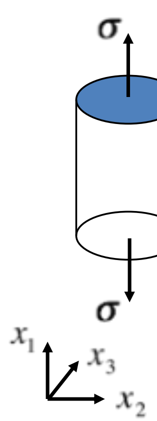
<u>Note:</u> Many (aerospace) structures are designed to operate well below the yield point (although made of ductile materials in order to exhibit progressive "failure").



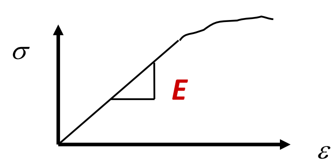
## Generalized Hooke's Law (cont.)



- Uniaxial tension: $\sigma_{11} = \sigma$, all other $\sigma = 0$ $\Rightarrow E\varepsilon_{11} = \sigma_{11}$




*E*: **Young's modulus (or elastic modulus)**




- Lateral strain, $\varepsilon_{22}$ and $\varepsilon_{33}$:

Lateral strain: $\Rightarrow \frac{\varepsilon_{22}}{\varepsilon_{11}} = \frac{\varepsilon_{33}}{\varepsilon_{11}} = -\nu$

$\nu$ : **Poisson's Ratio**

- How do we find $E$, $\nu$? ⇒ experiment

# “On the Shoulders of Giants”

0.54


Robert Hooke (1635-1703)

Simeon Denis Poisson (1781-1840)

(1678) Ceiiinosssttuv => Ut tensio, sic vis



# Generalized Hooke’s Law (cont.)

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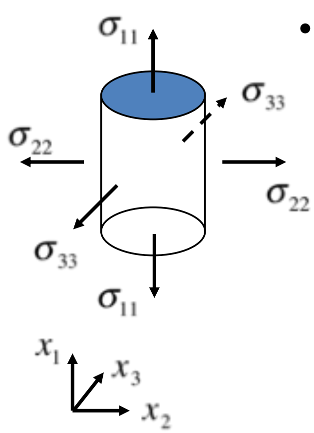
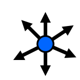
- What if an axial load is applied in all three directions?




- No directional or rotational dependence (i.e. same properties in <u>every</u> direction).
  - **Isotropic material**
  - Two independent constants



$$E\varepsilon_{11} = \underbrace{\sigma_{11}}_{\text{Axial effect}} - \underbrace{\nu(\sigma_{22}+\sigma_{33})}_{\text{Poisson effect}}$$

$$E\varepsilon_{22} = \sigma_{22} - \nu(\sigma_{11}+\sigma_{33})$$

$$E\varepsilon_{33} = \sigma_{33} - \nu(\sigma_{11}+\sigma_{22})$$

## Generalized Hooke's Law (cont.)

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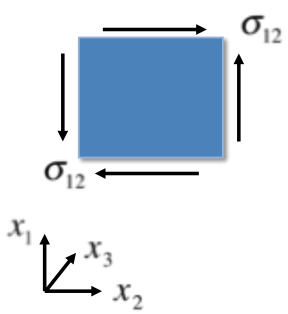
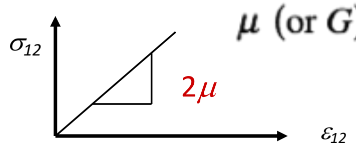
- What if we have **pure shear**? $\sigma_{12} = \sigma_{21}, \text{ all other } \sigma = 0$

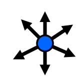



$$\therefore \ \sigma_{12} = 2\mu\varepsilon_{12}$$

$\mu$ (or $G$) : **Shear Modulus**




$$\varepsilon_{12} = \frac{1}{2\mu}\sigma_{12} = \frac{(1+\nu)}{E}\sigma_{12}$$



$$\varepsilon_{13} = \frac{1}{2\mu}\sigma_{13}$$

$$\varepsilon_{23} = \frac{1}{2\mu}\sigma_{23}$$



## Generalized Hooke's Law (cont.)

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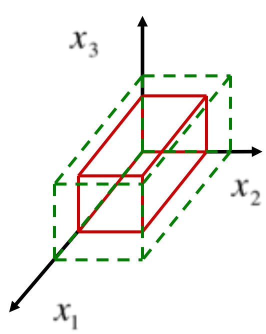


- Assume: $\sigma_{11} = \sigma_{22} = \sigma_{33} = -p \quad (p > 0)$

hydrostatic pressure

- But: $E\varepsilon_{11} = \sigma_{11} - \nu(\sigma_{22} + \sigma_{33}) = -p + 2\nu p$

$$E\varepsilon_{22} = \sigma_{22} - \nu(\sigma_{11} + \sigma_{33}) = -p + 2\nu p$$

$$E\varepsilon_{33} = \sigma_{33} - \nu(\sigma_{11} + \sigma_{22}) = -p + 2\nu p$$

$$E\underbrace{(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})}_{\frac{\Delta V}{V_0}} = -3p + 6\nu p \Rightarrow p = -\frac{E}{3(1-2\nu)}\frac{\Delta V}{V_0}$$

$$\Rightarrow \ k = \frac{E}{3(1-2\nu)} = \frac{1}{3}(2\mu + 3\lambda)$$

**Bulk Modulus**

## Generalized Hooke's Law (cont.)

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Of the elastic constants $E,\ \nu,\ \lambda,\ \mu,\ k$ only <u>two</u> can be taken as independent

| | $\lambda$ | $\mu$ | $E$ | $\nu$ | $k$ |
|---|---|---|---|---|---|
| $\lambda,\mu$ | $N/A$ | $N/A$ | $\mu(3\lambda+2\mu)/(\lambda+\mu)$ | $\lambda/2(\lambda+\mu)$ | $(3\lambda+2\mu)/3$ |
| $\lambda,E$ | $N/A$ | *irrational* | $N/A$ | *irrational* | *irrational* |
| $\lambda,\nu$ | $N/A$ | $\lambda(1-2\nu)/2\nu$ | $\lambda(1+\nu)(1-2\nu)/\nu$ | $N/A$ | $\lambda(1+\nu)/3\nu$ |
| $\lambda,k$ | $N/A$ | $3(k-\lambda)/2$ | $9k(k-\lambda)/(3k-\lambda)$ | $\lambda/(3k-\lambda)$ | $N/A$ |
| $\mu,E$ | $(2\mu-E)\mu/(E-3\mu)$ | $N/A$ | $N/A$ | $(E-2\mu)/2\mu$ | $\mu E/3(3\mu-E)$ |
| $\mu,\nu$ | $2\mu\nu/(1-2\nu)$ | $N/A$ | $2\mu(1+\nu)$ | $N/A$ | $2\mu(1+\nu)/3(1-2\nu)$ |
| $\mu,k$ | $(3k-2\mu)/3$ | $N/A$ | $9k\mu/(3k+\mu)$ | $(3k-2\mu)/2(3k+\mu)$ | $N/A$ |
| $E,\nu$ | $\nu E/(1+\nu)(1-2\nu)$ | $E/2(1+\nu)$ | $N/A$ | $N/A$ | $E/3(1-2\nu)$ |
| $E,k$ | | $3k(3k-E)/(9k-E)$ | $3kE/(9k-E)$ | $(3k-E)/6k$ | $N/A$ |
| $\nu,k$ | $3k\nu/(1+\nu)$ | $3k(1-2\nu)/(1+\nu)$ | $3k(1-2\nu)$ | $N/A$ | $N/A$ |



## Generalized Hooke's Law (cont.)

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- Energetic considerations: $E,\ k,\ \lambda,\ \mu > 0 \ \text{and} \ -1 \le \nu < \frac{1}{2}$
- From chart: $\nu \to \frac{1}{2} \ \Rightarrow \ \begin{cases} k \to \infty \\ \mu \to E/3 \end{cases}$ **Incompressible material**
- $\lambda$ has no obvious physical meaning
- Notions of **isotropy** and **homogeneity** are disjointed
  - — isotropic and homogeneous $\Rightarrow$ $E,\ \nu = const.$
  - — isotropic and inhomogeneous $\Rightarrow$ $E = E(x,y,z)$, $\nu = \nu(x,y,z)$

## Generalized Hooke's Law (cont.)

| | E (GPa) | ν | μ (GPa) | ρ ($kg/m^3$) | $\sigma_y$ (MPa) | $\sigma_{UTS}$ (MPa) |
|---|---|---|---|---|---|---|
| Steel AISI-4340 | 207 | 0.3 | 80 | 7833 | ≈ 800 | ≈ 1700 |
| Al 6061-T6 | 70 | 0.33 | 30 | 2700 | 275 | 310 |
| Ti-10V-2Fe-3Al | 110 | 0.32 | 40 | 4650 | ≈ 1350 | ≈ 1400(α) |
| Plexiglas (PMMA) | 3.25 | 0.35 | 1.2 | 1190 | 70 (tensile) | 70 |
| Glass | 70 | 0.22 | 30 | 2500 | | |





## Generalized Hooke's Law (cont.)

$$E\varepsilon_{11} = \sigma_{11} - \nu(\sigma_{22} + \sigma_{33})$$
$$E\varepsilon_{22} = \sigma_{22} - \nu(\sigma_{11} + \sigma_{33})$$
$$E\varepsilon_{33} = \sigma_{33} - \nu(\sigma_{11} + \sigma_{22})$$

$$\varepsilon_{12} = \frac{(1+\nu)}{E}\sigma_{12} = \frac{1}{2\mu}\sigma_{12}$$
$$\varepsilon_{13} = \frac{1}{2\mu}\sigma_{13} \qquad \varepsilon_{23} = \frac{1}{2\mu}\sigma_{23}$$

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{23}\\ \sigma_{31}\end{bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0\\ & 2\mu+\lambda & \lambda & 0 & 0 & 0\\ & & 2\mu+\lambda & 0 & 0 & 0\\ & & & 2\mu & 0 & 0\\ & sym. & & & 2\mu & 0\\ & & & & & 2\mu \end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{12}\\ \varepsilon_{23}\\ \varepsilon_{31}\end{bmatrix}$$

$\lambda,\ \mu$: Lamé moduli

## Elastic problem formulation



EQUILIBRIUM

$$\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2}+\frac{\partial\sigma_{13}}{\partial x_3}+f_1=0$$
$$\frac{\partial\sigma_{12}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}+\frac{\partial\sigma_{23}}{\partial x_3}+f_2=0$$
$$\frac{\partial\sigma_{13}}{\partial x_1}+\frac{\partial\sigma_{23}}{\partial x_2}+\frac{\partial\sigma_{33}}{\partial x_3}+f_3=0$$

STRAIN-DISPLACEMENT

$$\varepsilon_{11}=\frac{\partial u_1}{\partial x_1},\qquad \varepsilon_{12}=\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)$$
$$\varepsilon_{22}=\frac{\partial u_2}{\partial x_2},\qquad \varepsilon_{23}=\frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right)$$
$$\varepsilon_{33}=\frac{\partial u_3}{\partial x_3},\qquad \varepsilon_{13}=\frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right)$$

STRESS-STRAIN: normal

$$E\varepsilon_{11}=\sigma_{11}-\nu(\sigma_{22}+\sigma_{33})$$
$$E\varepsilon_{22}=\sigma_{22}-\nu(\sigma_{11}+\sigma_{33})$$
$$E\varepsilon_{33}=\sigma_{33}-\nu(\sigma_{11}+\sigma_{22})$$

STRESS-STRAIN: shear

$$2\mu\varepsilon_{12}=\sigma_{12}$$
$$2\mu\varepsilon_{13}=\sigma_{13}$$
$$2\mu\varepsilon_{23}=\sigma_{23}$$

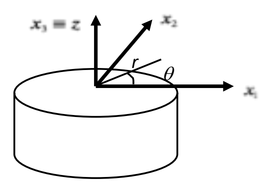


## Alternative coordinate systems



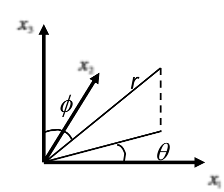
- Linear elastic, isotropic stress-strain law is the same in all coordinate systems:

  – Cylindrical coordinates




$$E\varepsilon_{rr}=\sigma_{rr}-\nu(\sigma_{\theta\theta}+\sigma_{zz})$$
$$E\varepsilon_{\theta\theta}=\sigma_{\theta\theta}-\nu(\sigma_{zz}+\sigma_{rr})$$
$$E\varepsilon_{zz}=\sigma_{zz}-\nu(\sigma_{\theta\theta}+\sigma_{rr})$$

  – Spherical coordinates




$$E\varepsilon_{rr}=\sigma_{rr}-\nu(\sigma_{\theta\theta}+\sigma_{\phi\phi})$$
$$E\varepsilon_{\theta\theta}=\sigma_{\theta\theta}-\nu(\sigma_{\phi\phi}+\sigma_{rr})$$
$$E\varepsilon_{\phi\phi}=\sigma_{\phi\phi}-\nu(\sigma_{\theta\theta}+\sigma_{rr})$$

# Cylindrical coordinates

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- Equilibrium:

$$\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{\partial\sigma_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}+R=0$$
$$\frac{\partial\sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{\partial\sigma_{\theta z}}{\partial z}+\frac{2\sigma_{r\theta}}{r}+\Theta=0$$
$$\frac{\partial\sigma_{zr}}{\partial z}+\frac{\partial\sigma_{\theta z}}{r\partial\theta}+\frac{\partial\sigma_z}{\partial z}+\frac{\sigma_{zr}}{r}+Z=0$$

*R, Θ, Z* body forces

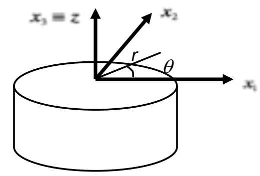
- Strain-Displacement:

$$\varepsilon_r=\frac{\partial u}{\partial r}\qquad \varepsilon_{r\theta}=\frac{1}{2}\left(\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{\partial u}{r\partial\theta}\right)$$
$$\varepsilon_\theta=\frac{u}{r}+\frac{\partial v}{r\partial\theta}\qquad \varepsilon_{\theta z}=\frac{1}{2}\left(\frac{\partial w}{r\partial\theta}+\frac{\partial v}{\partial z}\right)$$
$$\varepsilon_z=\frac{\partial w}{\partial z}\qquad \varepsilon_{rz}=\frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)$$

- Compatibility:

$$r\frac{\partial\varepsilon_r}{\partial r}-\frac{\partial^2\varepsilon_r}{\partial\theta^2}-\frac{\partial}{\partial r}\left[r\left(r\frac{\partial\varepsilon_\theta}{\partial r}-\frac{\partial\gamma_{r\theta}}{\partial\theta}\right)\right]=0$$
$$r\frac{\partial}{\partial z}\left[2\varepsilon_r-2\frac{\partial}{\partial r}(r\varepsilon_\theta)+\frac{\partial\gamma_{r\theta}}{\partial\theta}\right]+\frac{\partial}{\partial\theta}\left[\frac{\partial}{\partial r}(r\gamma_{\theta z})-\frac{\partial\gamma_{zr}}{\partial\theta}\right]=0$$
$$\frac{\partial}{\partial z}\left[2r\frac{\partial\varepsilon_r}{\partial\theta}-\frac{\partial}{\partial r}(r\gamma_{r\theta})\right]+r^2\frac{\partial}{\partial r}\left\{\frac{1}{r}\left[\frac{\partial}{\partial r}(r\gamma_{\theta z})-\frac{\partial\gamma_{zr}}{\partial\theta}\right]\right\}=0$$
$$r^2\frac{\partial^2\varepsilon_\theta}{\partial z^2}+r\frac{\partial\varepsilon_z}{\partial r}+\frac{\partial^2\varepsilon_z}{\partial\theta^2}-r\frac{\partial}{\partial z}\left(\frac{\partial\gamma_{\theta z}}{\partial\theta}+\gamma_{zr}\right)=0$$
$$2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varepsilon_z}{\partial\theta}\right)+\frac{\partial}{\partial z}\left[\frac{\partial\gamma_{r\theta}}{\partial z}-r\frac{\partial}{\partial r}\left(\frac{\gamma_{\theta z}}{r}\right)-\frac{1}{r}\frac{\partial\gamma_{zr}}{\partial\theta}\right]=0$$
$$\frac{\partial^2\varepsilon_r}{\partial z^2}+\frac{\partial^2\varepsilon_z}{\partial r^2}-\frac{\partial^2\gamma_{zr}}{\partial r\partial z}=0$$

$$r\rightarrow u$$
$$\theta\rightarrow v$$
$$z\rightarrow w$$






# Spherical coordinates

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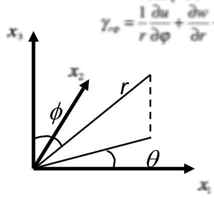
- Equilibrium:

$$\frac{\partial\sigma_r}{\partial r}+\frac{1}{r\sin\varphi}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{1}{r}\frac{\partial\tau_{\varphi r}}{\partial\varphi}+\frac{2\sigma_r-\sigma_\theta-\sigma_\varphi+\tau_{\varphi r}\cos\varphi}{r}+R=0$$
$$\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r\sin\varphi}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{1}{r}\frac{\partial\tau_{\theta\varphi}}{\partial\varphi}+\frac{3\tau_{r\theta}+2\tau_{\theta\varphi}\cot\varphi}{r}+\Theta=0$$
$$\frac{\partial\tau_{\varphi r}}{\partial r}+\frac{1}{r\sin\varphi}\frac{\partial\tau_{\theta\varphi}}{\partial\theta}+\frac{1}{r}\frac{\partial\sigma_\varphi}{\partial\varphi}+\frac{3\tau_{\varphi r}+(\sigma_\varphi-\sigma_\theta)\cot\varphi}{r}+\Phi=0$$

*R, Θ, Φ* body forces

- Strain-Displacement:

$$\varepsilon_r=\frac{\partial u}{\partial r}\qquad \gamma_{r\theta}=\frac{1}{r\sin\varphi}\frac{\partial u}{\partial\theta}+\frac{\partial v}{\partial r}-\frac{v}{r}$$
$$\varepsilon_\theta=\frac{1}{r\sin\varphi}\frac{\partial v}{\partial\theta}+\frac{u}{r}+\frac{\cot\varphi}{r}w\qquad \gamma_{\theta\varphi}=\frac{1}{r\sin\varphi}\frac{\partial w}{\partial\theta}+\frac{1}{r}\frac{\partial v}{\partial\varphi}-\frac{\cot\varphi}{r}v$$
$$\varepsilon_\varphi=\frac{1}{r}\frac{\partial w}{\partial\varphi}+\frac{u}{r}\qquad \gamma_{r\varphi}=\frac{1}{r}\frac{\partial u}{\partial\varphi}+\frac{\partial w}{\partial r}-\frac{w}{r}$$




$$r\rightarrow u$$
$$\theta\rightarrow v$$
$$\phi\rightarrow w$$

- Compatibility:

$$2\frac{\partial}{\partial\theta}\left(\frac{\partial\varepsilon_r}{\partial\varphi}-\varepsilon_r\cot\varphi\right)+\frac{\partial}{\partial r}\left(r\gamma_{r\theta}\cos\varphi-r\sin\varphi\frac{\partial\gamma_{r\theta}}{\partial\varphi}+r^2\sin\varphi\frac{\partial\gamma_{\theta\varphi}}{\partial r}-\frac{\partial\gamma_{\varphi r}}{\partial r}\right)=0$$
$$2\frac{\partial}{\partial\theta}\left(\varepsilon_r-r\frac{\partial\varepsilon_\varphi}{\partial r}+\frac{1}{2}\frac{\partial\gamma_{\varphi r}}{\partial\varphi}-\frac{1}{2}\gamma_{\varphi r}\cot\varphi\right)-r\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial\gamma_{r\theta}}{\partial\varphi}\right)+r\frac{\cos2\varphi}{\sin\varphi}\gamma_{r\theta}+$$
$$r^2\frac{\partial}{\partial r}\left(\sin\varphi\frac{\partial\gamma_{\theta\varphi}}{\partial\varphi}+2\gamma_{\theta\varphi}\cos\varphi\right)=0$$
$$2\sin^2\varphi\frac{\partial\varepsilon_r}{\partial\varphi}-2r\sin^2\varphi\frac{\partial}{\partial r}\left(\frac{\partial\varepsilon_\theta}{\partial\varphi}+(\varepsilon_\theta-\varepsilon_\varphi)\cot\varphi\right)+\frac{\partial^2}{\partial\theta\partial\varphi}(\gamma_{r\theta}\sin\varphi)+$$
$$r\sin\varphi\frac{\partial^2\gamma_{\theta\varphi}}{\partial\theta\partial r}-\frac{\partial^2\gamma_{\varphi r}}{\partial\theta^2}-2\gamma_{\varphi r}\sin^2\varphi=0$$
$$r\frac{\partial\varepsilon_r}{\partial r}-\frac{\partial^2\varepsilon_r}{\partial\varphi^2}-\frac{\partial}{\partial r}\left(r^2\frac{\partial\varepsilon_\varphi}{\partial r}-r\frac{\partial\gamma_{\varphi r}}{\partial\varphi}\right)=0$$
$$\frac{\partial^2\varepsilon_\varphi}{\partial\theta^2}+\frac{\partial}{\partial\varphi}\left(\sin^2\varphi\frac{\partial\varepsilon_\theta}{\partial\varphi}-\sin\varphi\frac{\partial\gamma_{\theta\varphi}}{\partial\theta}-\gamma_{\varphi r}\sin^2\varphi\right)-\sin\varphi\cos\varphi\left(\frac{\partial\varepsilon_\varphi}{\partial\varphi}+\gamma_{\varphi r}\right)+$$
$$\sin^2\varphi\left[r\frac{\partial}{\partial r}(\varepsilon_\theta+\varepsilon_\varphi)+2(\varepsilon_\varphi-\varepsilon_r)\right]-\sin\varphi\frac{\partial}{\partial\theta}\gamma_{r\theta}=0$$
$$r\sin^2\varphi\frac{\partial\varepsilon_r}{\partial r}-\frac{\partial^2\varepsilon_r}{\partial\theta^2}-\sin^2\varphi\frac{\partial}{\partial r}\left(r^2\frac{\partial\varepsilon_\theta}{\partial r}\right)+\sin\varphi\frac{\partial^2}{\partial\theta\partial r}(r\gamma_{r\theta})-$$
$$\sin\varphi\cos\varphi\left[\frac{\partial\varepsilon_r}{\partial\varphi}-\frac{\partial}{\partial r}(r\gamma_{\varphi r})\right]=0$$