AE 370 — Homework #1

1. Consider the 'first-order difference' matrix $\boldsymbol{A} \in \mathbb{R}^{n-1\times n}$ defined as

$$\boldsymbol{A} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & -1 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -1 & 1 \end{bmatrix} \tag{1}$$

We will encounter this matrix (and ones like it) later in class, and we already have tools to say a lot about the properties of this matrix. You will demonstrate that below.

   (a) Does the problem $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{b} \in \mathbb{R}^{n-1}$, have a solution? How do you know? *Hint*: it may be helpful to inspect this matrix for $n = 3$ or 4 and determine the answer for that case before considering the general n case.

   (b) If there is a solution, is it unique? Why or why not? If it is not unique, what vector can be added to the solution without affecting the solution. *Hint*: is there a vector that gets annihilated by taking differences of its components and their neighbors?)?

2. We derived the linear system for approximating a solution to $\boldsymbol{Ax} = \boldsymbol{b}$ using the fact that if $\boldsymbol{y}^* \in \mathcal{R}(\boldsymbol{A})$, then $(\boldsymbol{y}^* - \boldsymbol{b}, \boldsymbol{r}) = 0$ for *any* $\boldsymbol{r} \in \mathcal{R}(\boldsymbol{A})$.

   (a) Show that this is indeed true. *Hint*: any $\boldsymbol{w} \in \mathcal{R}(\boldsymbol{A})$ can be written as $\boldsymbol{w} = \boldsymbol{y}^* + \beta \boldsymbol{r}$ for some $\beta \in \mathbb{R}$ and $\boldsymbol{r} \in \mathcal{R}(\boldsymbol{A})$. Use this in conjunction with the fact that $||\boldsymbol{y}^* - \boldsymbol{b}|| \leq ||\boldsymbol{w} - \boldsymbol{b}||$.

   (b) Consider the matrix $\hat{\boldsymbol{A}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Determine the best approximation to the problem $\boldsymbol{Ax} = \boldsymbol{b}$ for $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and demonstrate the this solution indeed satisfies the required orthogonality relation.

3. An alien spacecraft has crash landed on Earth and biologists were able to measure concentrations of an unknown chemical as a function of radius from the crash-sight. The values of concentration ($\boldsymbol{y}$) and radius ($\boldsymbol{x}$) are as follows:

$$\boldsymbol{x} = \begin{bmatrix} 0.02 \\ 0.10 \\ 0.32 \\ 0.66 \\ 0.94 \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} 1.93 \\ 2.60 \\ 3.03 \\ 3.97 \\ 4.89 \end{bmatrix} \tag{2}$$

The biologists are turning to you to ask you to determine a line that best fits this data. That is, they want a best approximation to the problem

$$\boldsymbol{y} = a\boldsymbol{x} + b\boldsymbol{e} \tag{3}$$

where $a, b \in \mathbb{R}$ are the constants to be solved for and $\boldsymbol{e} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Can you help Planet Earth?!

Once you have solved for a, b, provide a plot that includes both the raw data (with circles or some comparable marker) as well as the best fit line through the data.

4. We showed in class that when the problem $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ (with $\boldsymbol{A} \in \mathbb{R}^{m \times n}, \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{b} \in \mathbb{R}^m$) does not have a solution, then we can approximate this solution using a basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ for $\mathcal{R}(\boldsymbol{A})$. Re-derive these equations assuming that the basis is orthogonal; *i.e.*, when $(\boldsymbol{v}_i, \boldsymbol{v}_j) = 0$ for $i \neq j$. What simplifications arise, and what are some benefits of these simplifications?