AE 370 — Assignment 3 solutions

1. The spline interpolation for the **Runge's function**, which is given by,

$$f(x) = \frac{1}{1+x^2} \tag{1}$$

```
clear all, close all, clc, warning off all

%Polynomial degree
nvect = [5; 10; 25; 50];

%function to approx
f = @(x) 1./(1+x.^2);

%error vector:
err = zeros(size(nvect));

for j = 1 : length( nvect )

    %define current n
    n = nvect( j );

    %define interp points
    xj = (-5 : 10/n : 5)';

    %--build & solve lin system for the c_{i,k} (i = 1,...,n; j = 1,...,4)
    %--for natural splines

        A = zeros( 4*n ); %initialize matrix
        g = zeros( 4*n, 1 ); %initialize RHS vector

        %Build A matrix & f vector
        for jj = 1 : n

            ind = 4*(jj - 1) + 1;

            %condition (3) from spline recitation notes
            A( ind, ind ) = 1/6 * (xj(jj) - xj(jj+1))^2;
            A( ind, ind + 1 ) = 0;
            A( ind, ind + 2 ) = xj(jj);
            A( ind, ind + 3 ) = 1;

            g( ind ) = f( xj(jj) );

            %condition (4) from spline recitation notes
            A( ind+1, ind ) = 0;
            A( ind+1, ind + 1 ) = 1/6 * (xj(jj+1) - xj(jj))^2;
            A( ind+1, ind + 2 ) = xj(jj+1);
            A( ind+1, ind + 3 ) = 1;
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    g( ind + 1 ) = f( xj(jj + 1) );

    %derivative conditions
    %(careful here! index on derivs only goes to n-1...)

    if jj < n

        %condition (5) from spline recitation notes
        A( ind+2, ind ) = 0;
        A( ind+2, ind + 1 ) = 1/2 * (xj(jj+1) - xj(jj));
        A( ind+2, ind + 2 ) = 1;
        A( ind+2, ind + 3 ) = 0;

        A( ind+2, ind + 4 ) = -1/2 * (xj(jj+1) - xj(jj+2));
        A( ind+2, ind + 5 ) = -0;
        A( ind+2, ind + 6 ) = -1;
        A( ind+2, ind + 7 ) = -0;

        %condition (6) from spline recitation notes
        A( ind+3, ind ) = 0;
        A( ind+3, ind + 1 ) = 1;
        A( ind+3, ind + 2 ) = 0;
        A( ind+3, ind + 3 ) = 0;

        A( ind+3, ind + 4 ) = -1;
        A( ind+3, ind + 5 ) = -0;
        A( ind+3, ind + 6 ) = -0;
        A( ind+3, ind + 7 ) = -0;
    else

        %s_1''(x_0) = 0
        A( ind+2, 1 ) = 1;
        A( ind+2, 2 ) = 0;
        A( ind+2, 3 ) = 0;
        A( ind+2, 4 ) = 0;

        %s_n''(x_n) = 0
        A( ind+3, ind ) = 0;
        A( ind+3, ind + 1 ) = 1;
        A( ind+3, ind + 2 ) = 0;
        A( ind+3, ind + 3 ) = 0;

    end

    g( ind + 2 ) = 0;
    g( ind + 3 ) = 0;

end

%solve for coeffs:
c = A \ g;
```
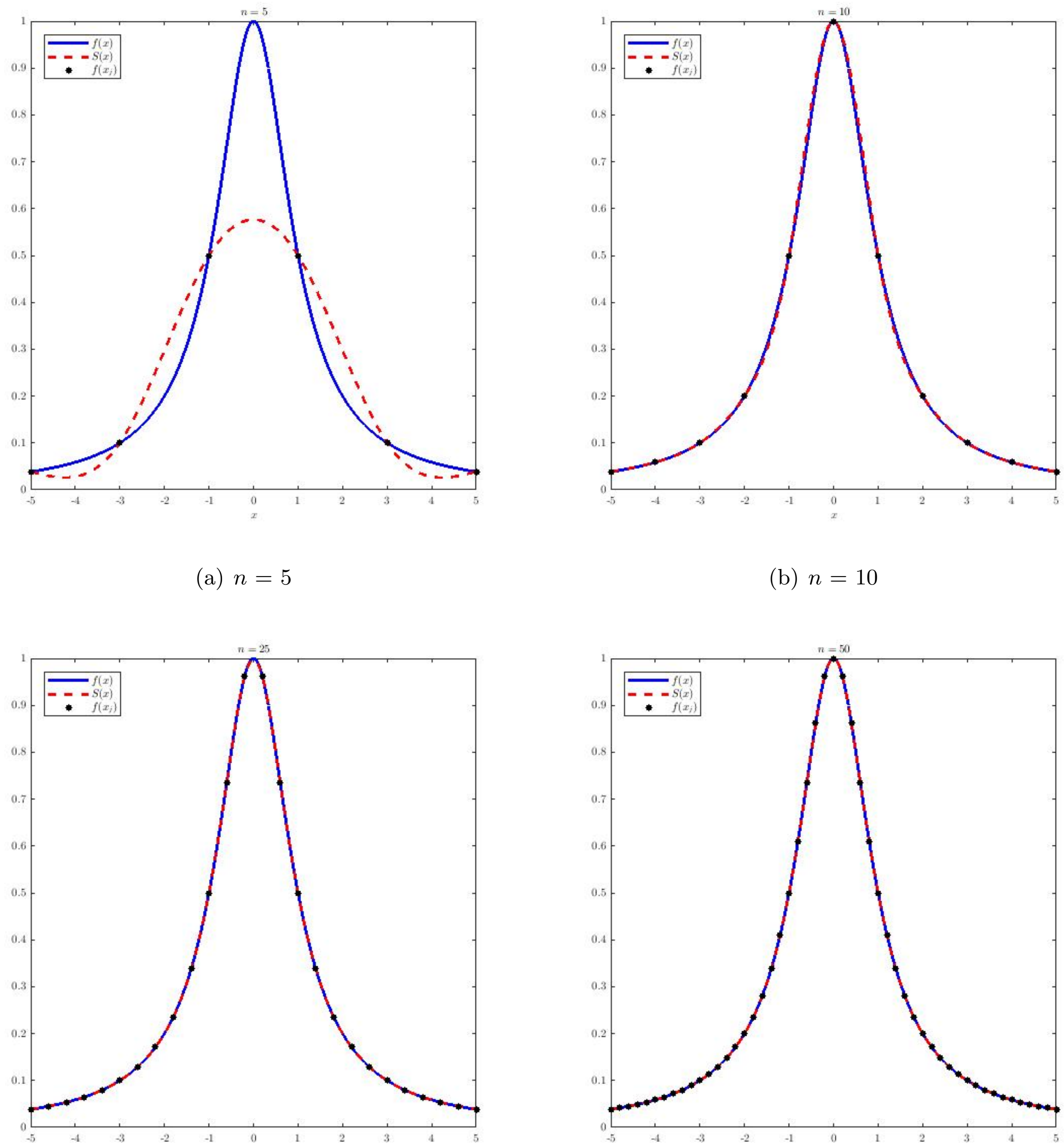

```matlab
%--

%--plot the spline interpolant S(x)

    xx = linspace( -5, 5, 10000 );

    S = zeros( size( xx ) );
    for jj = 1 : n

        ind = 4*(jj - 1) + 1;

        indxx = ( xx >= xj( jj ) & xx <= xj( jj + 1 ) );
        xxc = xx( indxx ~= 0 );

        S( indxx ~= 0 ) = c( ind ) * ( xxc - xj(jj+1) ).^3./ ...
            ( 6.*(xj(jj) - xj(jj+1)) ) + c( ind+1 ) * ( xxc - ...
            xj(jj) ).^3./( 6.*(xj(jj+1) - xj(jj)) ) + c( ind + 2 ) ...
            .* xxc + c( ind + 3 );

    end

    if j <= 4
        figure(j)
        plot( xx, f(xx), 'b-', 'linewidth', 2 ), hold on
        plot( xx, S, 'r--', 'linewidth', 2 )
        plot( xj, f(xj), 'k.', 'markersize', 16 )

        %make plot pretty
        title( ['$n = ', num2str( n ),'$'] ,'interpreter', 'latex',...
            'fontsize', 16)
        xlabel( '$x$', 'interpreter', 'latex', 'fontsize', 16)
        h = legend( '$f(x)$', '$S(x)$', '$f(x_j)$');
        set(h, 'location', 'NorthWest', 'Interpreter', 'Latex', 'fontsize', 16 )
        set(gca, 'TickLabelInterpreter','latex', 'fontsize', 16 )


        set(gcf, 'PaperPositionMode', 'manual')
        set(gcf, 'Color', [1 1 1])
        set(gca, 'Color', [1 1 1])
        set(gcf, 'PaperUnits', 'centimeters')
        set(gcf, 'PaperSize', [15 15])
        set(gcf, 'Units', 'centimeters' )
        set(gcf, 'Position', [0 0 15 15])
        set(gcf, 'PaperPosition', [0 0 15 15])

        svnm = ['pic_', num2str(j)];
        print( '-dpng', svnm, '-r200' )
    end

%--
```

```
    %--compute error
        err(j) = max(abs( f(xx) - S ) );


end


%plot error
figure(100)
semilogy( nvect, err, 'kx', 'markersize', 8, 'linewidth', 2 )

%make plot pretty
title( 'Maximum error' ,'interpreter', 'latex','fontsize', 16)
xlabel( '$n$', 'interpreter', 'latex', 'fontsize', 16)
ylabel( '$\max|f(x) - S(x)|$', 'interpreter', 'latex', 'fontsize', 16)

set(gca, 'TickLabelInterpreter','latex', 'fontsize', 16 )

set(gcf, 'PaperPositionMode', 'manual')
set(gcf, 'Color', [1 1 1])
set(gca, 'Color', [1 1 1])
set(gcf, 'PaperUnits', 'centimeters')
set(gcf, 'PaperSize', [15 15])
set(gcf, 'Units', 'centimeters' )
set(gcf, 'Position', [0 0 15 15])
set(gcf, 'PaperPosition', [0 0 15 15])

svnm = 'error';
print( '-dpng', svnm, '-r200' )
```



(a) $n = 5$ (b) $n = 10$

(c) $n = 25$ (d) $n = 50$

Figure 1: Comparison of Runge's function using spline interpolation.

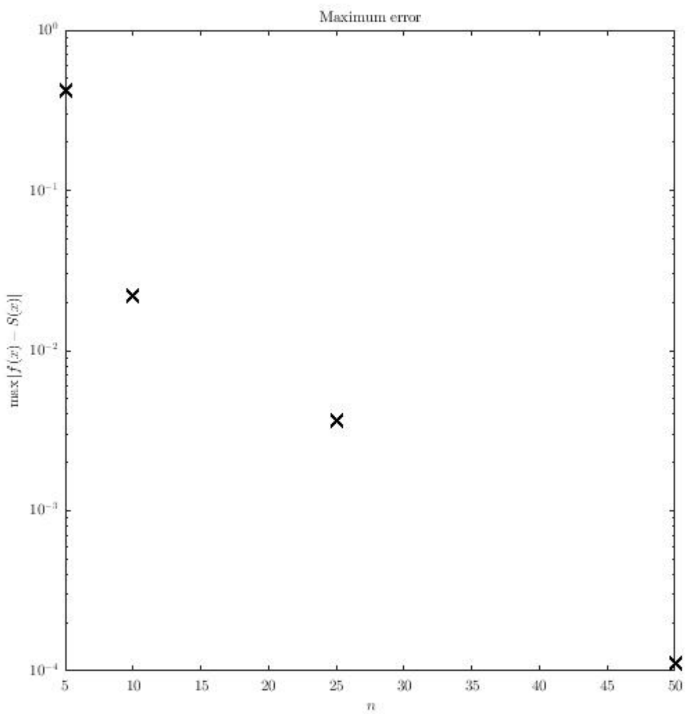
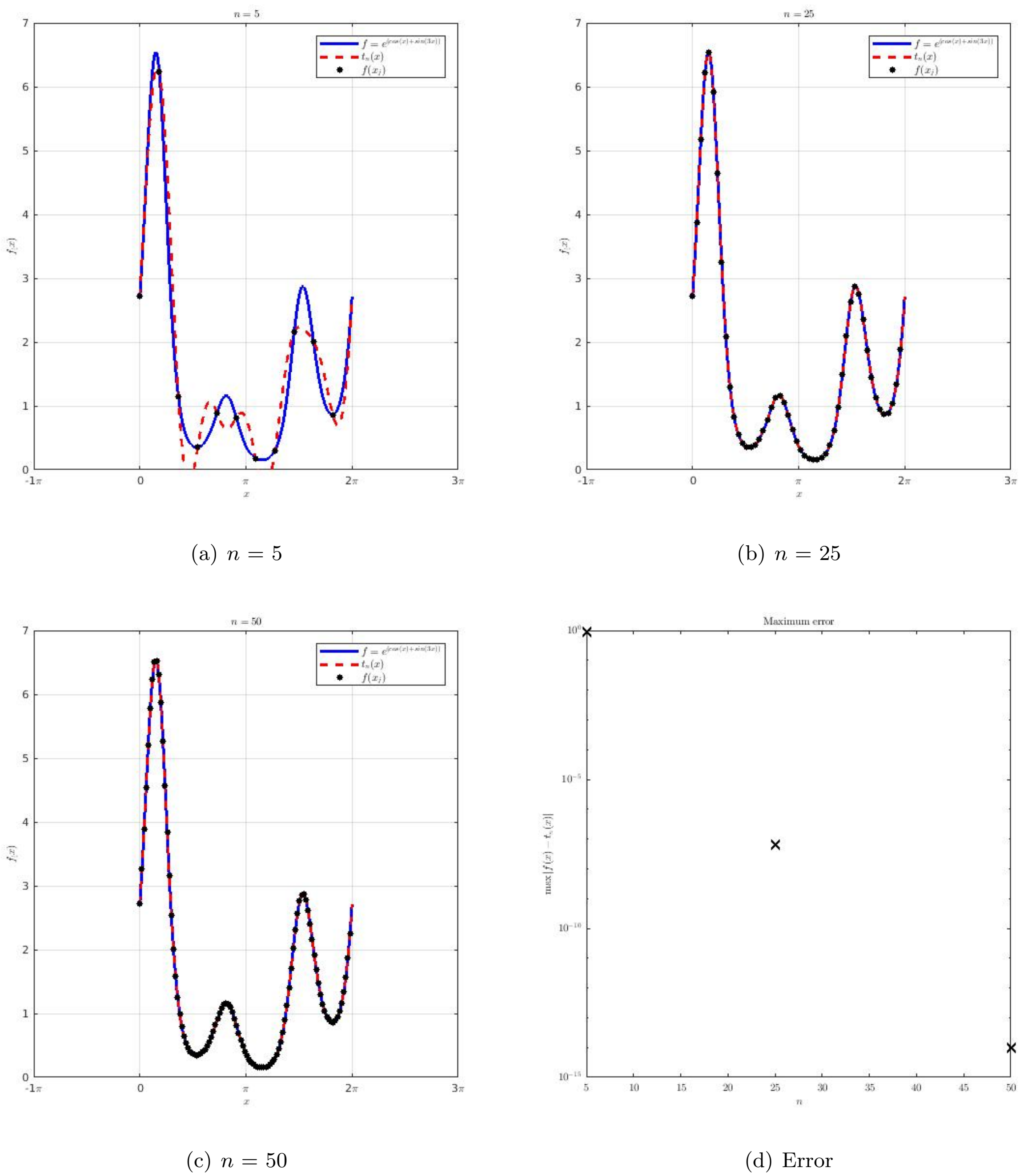


Figure 2: Error of the polynomial interpolant for different values of n.

2. The code for trigonometric interpolation for both the function is given below.

```
clear all, close all, clc, warning off all

%Polynomial degree
nvect = [5; 25; 50];

%function to approx
% f = @(x) exp(cos(x)+sin(3*x));
f = @(x) x;

%error vector:
err = zeros(size(nvect));

for j = 1 : length( nvect )

    %define current n
    n = nvect(j);

    %define interp points
    xj = (0 : 2*n)' * 2*pi /(2*n+1);

    %transpose to make column vector
    xj = xj';

    %define values of f at interp points
    fj = f(xj);

    %define coefficients
    cj = fft(fj)/(2*n+1);

    %--plot the trigonometric interpolant

        xx = linspace( 0, 2*pi, 1000 );

        tn = 0;
        for k = 1 : n + 1
            tn = tn + cj(k)*exp(1i*(k-1)*xx); % c_0, c_1, ... c_n terms
        end

        for k = n+2 : 2*n + 1
            tn = tn + cj(k)*exp(1i*(-2*n-2+k)*xx); % c_{-n}, ..., c_{-1} terms
        end

        figure(j)
        plot( xx, f(xx), 'b-', 'linewidth', 2 ), hold on
        plot( xx, tn, 'r--', 'linewidth', 2 )
        plot( xj, f(xj), 'k.', 'markersize', 16 )

        %make plot pretty
        axis([-pi 3*pi -1 7]);
%         lh = legend('$f = e^{(cos(x)+sin(3x))}$','$t_n(x)$','$f(x_j)$');
```

```
        lh = legend('$f = x$','$t_n(x)$','$f(x_j)$');
        set(lh,'interpreter','latex','fontsize',18);
        grid on;
        title(['$n = ', num2str( n ),'$'],'interpreter', 'latex','fontsize', 16);
        ylabel( '$f_(x)$', 'interpreter', 'latex', 'fontsize', 16);
        xticks([-pi 0 pi 2*pi 3*pi]);
        xticklabels({'-1\pi','0','\pi','2\pi','3\pi'});
        xlabel( '$x$','interpreter','latex','fontsize', 16);

        set(gcf, 'PaperPositionMode', 'manual')
        set(gcf, 'Color', [1 1 1])
        set(gca, 'Color', [1 1 1])
        set(gcf, 'PaperUnits', 'centimeters')
        set(gcf, 'PaperSize', [15 15])
        set(gcf, 'Units', 'centimeters' )
        set(gcf, 'Position', [0 0 15 15])
        set(gcf, 'PaperPosition', [0 0 15 15])

        svnm = ['pic_', num2str(j)];
        print( '-dpng', svnm, '-r200' )

    %--compute error
        err(j) = max(abs( f(xx) - tn ) );

end

%plot error
figure(100)
semilogy( nvect, err, 'kx', 'markersize', 8, 'linewidth', 2 )
axis([0 50 5.5 7.0])
%make plot pretty
title( 'Maximum error' ,'interpreter', 'latex','fontsize', 16)
xlabel( '$n$', 'interpreter', 'latex', 'fontsize', 16)
% yticks(5.5 6.0 6.5 7.0)
% yticklabels({'5.5','6.0','6.5','7.0'})
ylabel( '$\max|f(x) - t_n(x)|$', 'interpreter', 'latex', 'fontsize', 16)
set(gca, 'TickLabelInterpreter','latex', 'fontsize', 16 )
set(gcf, 'PaperPositionMode', 'manual')
set(gcf, 'Color', [1 1 1])
set(gca, 'Color', [1 1 1])
set(gcf, 'PaperUnits', 'centimeters')
set(gcf, 'PaperSize', [15 15])
set(gcf, 'Units', 'centimeters' )
set(gcf, 'Position', [0 0 15 15])
set(gcf, 'PaperPosition', [0 0 15 15])

svnm = 'error';
print( '-dpng', svnm, '-r200' )
```



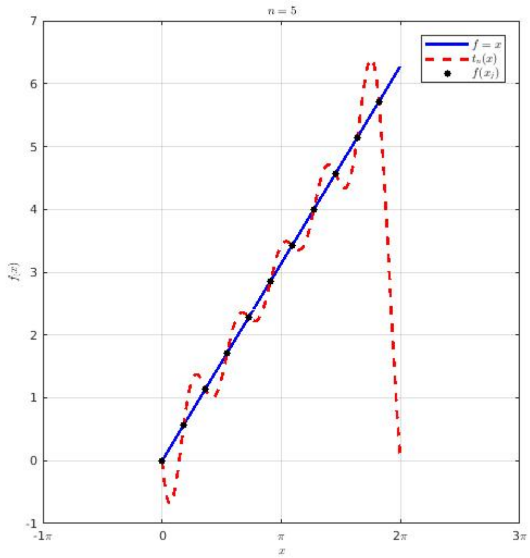
(a) $n = 5$

(b) $n = 25$

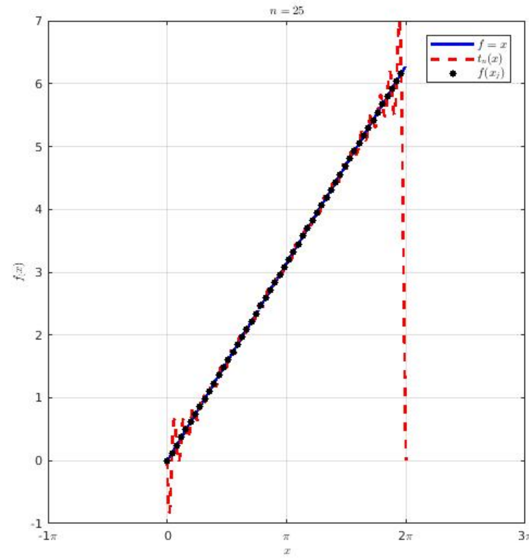
(c) $n = 50$

(d) Error

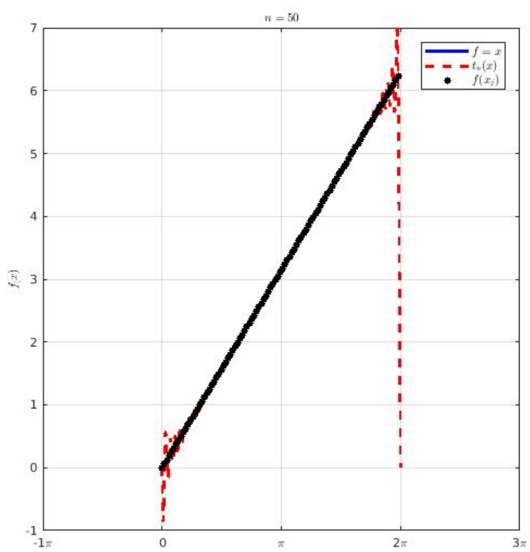
Figure 3: Comparison of trigonometric interpolation for function $f(x) = \exp(cos(x) + sin(3x))$.



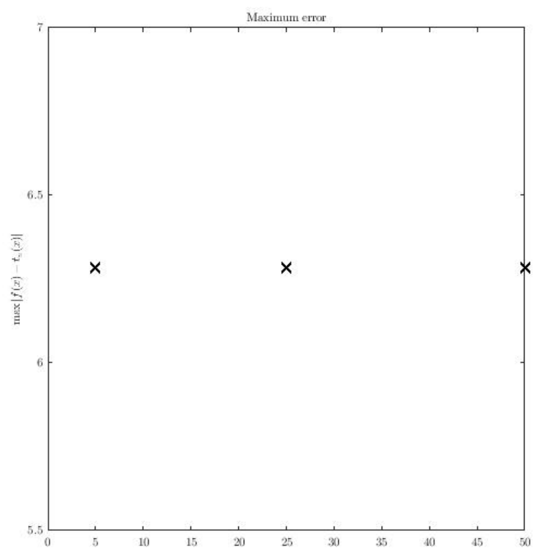
(a) $n = 5$



(b) $n = 25$



(c) $n = 50$



(d) Error

Figure 4: Comparison of trigonometric interpolation for function $f(x) = x$.

The trigonometric interpolation method is suitable for periodic functions. The first function,

$$f(x) = \exp(cos(x) + sin(3x)) \tag{2}$$

is periodic in the range $[0{,}2\pi]$, whereas, the second function, $f(x) = x$ is not periodic. This makes the interpolation of the second function not suitable for approximation using trigonometric interpolation.