AE 370 — Homework #5

1. **Confronting Runge's dastardly problem yet again!** We will approximate $\int_{-5}^{5} f(x)dx$, where $f(x) = \frac{1}{1+x^2}$, using polynomial interpolation.

   (a) Use global interpolation with Lagrange basis functions in uniformly spaced points to approximate the definite integral for $n = 2, 8, 16, 20$. Plot the error versus n. (You may modify the skeleton code on the next page. The skeleton code should also be highly applicable for part b).

   (b) Use global interpolation with Lagrange basis functions in Chebyshev points to approximate the definite integral for $n = 2, 8, 16, 20$. Plot the error versus n.

   (c) Use a composite trapezoid rule in uniformly spaced points to approximate the definite integral using $n = 2, 8, 16, 20$ sub-integrals. Plot the error versus n. You **may not** use trapz for this part.

   (d) *Extra credit (up to 5 points)*: Compare the results for these three different quadrature strategies. Which worked well and which did not? Justify your answer.

2. **Deriving a degree 0 composite rule**

   In class we derived the composite trapezoid rule by breaking $\int_a^b f(x)dx$ into several sub-integrals and representing $f(x)$ over each sub-integral using a degree one polynomial (a line). We also showed that the error in this approximation was $O(\Delta x^2)$, where Δx is the interval spacing of each sub-integral. In this problem you will perform an analogous set of steps, except you will use a degree zero polynomial.

   (a) Express $\int_a^b f(x)dx$ as n sub-integrals.

   (b) Approximate $f(x)$ over each sub-integral using a degree zero polynomial. Write this approximation in terms of $f(x)$ (possibly evaluated at specific points).

   (c) From this, derive a composite rule that involves a sum of n terms involving $f(x)$ (possibly evaluated at points) and Δx.

   (d) Derive the error associated with this quadrature rule in terms of Δx.

3. **A remarkable phenomenon involving the trapezoidal rule and infinitely differentiable periodic functions**

   We showed in class that, in general, the composite trapezoid rule has an error that decreases at a rate of $O(\Delta x^2)$. A very important exception to this exists: if the function being integrated is infinitely differentiable and periodic over the integration interval, the trapezoidal rule converges faster than any power of n! You will explore this behavior in this problem.

   (a) Approximate $\int_0^2 f(x)dx$, where $f(x) = \sin(10\pi x)$, using a composite trapezoidal rule for $n = 4, 16, 24, 48$. You **may not** use trapz for this part.

   (b) Plot the error associated with using the composite trapezoidal rule. Isn't this a remarkable result!

4. **Progressively improved approximations through Romberg integration**

   (a) Approximate $\int_0^2 f(x)dx$, where $f(x) = x^2 \sin(10x)$, using a composite trapezoidal rule for $n = 4, 16, 24, 48$. You **may not** use trapz for this part.

   (b) For the same values of n, approximate the same definite integral using Richardson extrapolation (You may modify the skeleton code on page 3).

   (c) Plot the error associated with using the composite trapezoidal rule and Richardson extrapolation.

Skeleton code for Q1a

```
clear all, close all, clc

%%

%a)

nvect = ???;

fcn = @(x) 1 ./ (1 + x.^2);

%exact answer
xs = sym( 'x' );
int_exact = int( fcn( xs ), xs, ???, ???);

err = zeros( length( nvect ), 1 );

for j = 1 : length( nvect )

    n = ???;

    %xjs at which to define the L_i
    xj = ???

    %define Lagrange basis vectors
    intval = 0;
    for i = 1 : n + 1
        L_i = ???

        %vector indexing can't start at zero, so go from 1 to n+1
        for k = 1 : n + 1
            if k ~= i
               %Don't forget to define L_i symbolically so that you
               % can do the integral exactly later...
                L_i = ??? .* L_i;
            end
        end

        Li_int = int( ???, xs, ???, ??? );

        intval = intval + ???;

    end

    err( j ) = abs( ??? );

end

%plot error
figure(100)
semilogy( nvect, err, 'k.', 'markersize', 26 )
```

Skeleton code for Q4b

```
%part a
nvect = [4, 16, 24, 48];
fcn = @(x) x.^2.*sin(10.*x);

%exact answer
xs = sym( 'x' );
int_exact = int( fcn( xs ), xs, 0, 2 );

%Initialize error
err = zeros( length( nvect ), 1 );

%Compute the approximation to the integral using a trapezoid rule with n intervals
%(interval spacing =h). Store results in a vector intvalh (size nvect x 1) that
%contains the trapezoid rule approximation for each n

intvalh = ?????
err = ????

%plot error
figure(100), hold on
semilogy( nvect, err, 'r.', 'markersize', 26 )


%part b
err = zeros( length( nvect ), 1 );

%--evaluate trap rule at h/2 and combine this with result from a) to get
%  Richardson extrapolated value.
intvalhb2 = zeros( length( nvect ), 1 );

for j = 1 : length( nvect )

    n = ????

    %Set interval size to be half of that in part a
    h = ??? %spacing between points

    %corresponding x points
    xj = 0 : h : 2;

    %compute trap approx for h/2 here!
   ????


    %Richardson extrapolated value
    int_rich = ????

    err( j ) = abs( ???? );

end


%plot error
figure(100), hold on
semilogy( nvect, err, 'r.', 'markersize', 26 )
```

```
%make plot pretty
h = legend( 'Comp Trap', 'Rich Integration' );
set( h, 'location', 'SouthWest', 'interpreter', 'latex', 'fontsize', 16)
xlabel( '$n$', 'interpreter', 'latex', 'fontsize', 16)
ylabel( 'max error', 'interpreter', 'latex', 'fontsize', 16)

set(gca, 'TickLabelInterpreter','latex', 'fontsize', 16 )

set(gcf, 'PaperPositionMode', 'manual')
set(gcf, 'Color', [1 1 1])
set(gca, 'Color', [1 1 1])
set(gcf, 'PaperUnits', 'centimeters')
set(gcf, 'PaperSize', [15 15])
set(gcf, 'Units', 'centimeters' )
set(gcf, 'Position', [0 0 15 15])
set(gcf, 'PaperPosition', [0 0 15 15])
```