AE 370 — Homework #4

1. **A piece of mathematical magic: generating orthogonal polynomial basis functions**

   A hugely powerful tool in applied mathematics is the Gram-Schmidt process, which enables an orthogonal basis to be generated using a recursive procedure. That is, the resulting basis defined by the functions $\{\phi_0, \phi_1, \dots, \phi_n\}$ satisfies $(\phi_i, \phi_j) = 0$ for $i \neq j$. When dealing with $\mathcal{P}_n$ (the space of all polynomials of degree n or less), the recursion relation can be written as:

$$\phi_0 = \frac{\hat{\phi}_0}{\sqrt{\left(\hat{\phi}_0, \hat{\phi}_0\right)}}, \quad \text{where } \hat{\phi}_0 = \frac{1}{(1,1)} \tag{1}$$

$$\phi_1 = \frac{\hat{\phi}_1}{\sqrt{\left(\hat{\phi}_1, \hat{\phi}_1\right)}}, \quad \text{where } \hat{\phi}_1 = x - \frac{(x,1)}{(1,1)} \tag{2}$$

$$\phi_k = \frac{\hat{\phi}_k}{\sqrt{\left(\hat{\phi}_k, \hat{\phi}_k\right)}}, \quad \text{where } \hat{\phi}_k = x\phi_{k-1} - \frac{(x\phi_{k-1}, \phi_{k-1})}{(\phi_{k-1}, \phi_{k-1})}\phi_{k-1} - \frac{(x\phi_{k-1}, \phi_{k-2})}{(\phi_{k-2}, \phi_{k-2})}\phi_{k-2}, \quad k \geq 2 \tag{3}$$

   where $(\cdot, \cdot)$ is an appropriately defined inner product.

   (a) Show that ϕ_0 and ϕ_1 are orthogonal

   (b) Show that ϕ_2 is orthogonal to both ϕ_0 and ϕ_1

   (c) Using these facts, show that ϕ_k is orthogonal to ϕ_j for $j \neq k$.

   (d) What is (ϕ_i, ϕ_i) for $i = 0, \dots, n$?

2. **Re-re-visiting Runge's problem.** We will approximate $f(x) = \frac{1}{1+x^2}$ over the interval $-5 \leq x \leq 5$. This time, as opposed to interpolating $f(x)$ at specified points, we will compute the best least-squares approximation to $f(x)$ onto $\mathcal{P}_n$ (the space of all polynomials of degree n or less).

   (a) Recall that we showed that the least-squares approximation problem could be written as a linear system $\boldsymbol{Gc} = \boldsymbol{b}$. Re-derive the result by representing the function and basis functions symbolically as $f(x)$ and $\{\phi_0, \dots, \phi_n\}$, respectively. You may leave your results in terms of a generic inner product. Do **not** assume the basis is orthogonal for this part of the problem.

   (b) Assuming you use a basis constructed using the recursion procedure from problem 1, what does $\boldsymbol{G}$ look like for this choice of basis? Again, don't plug in for the basis functions yet, just write $\boldsymbol{G}$ symbolically in terms of a generic inner product.

   (c) Solve the problem $\boldsymbol{Gc} = \boldsymbol{b}$ for $n = [5, 10, 25, 50]$ using the basis described using the recursion relation in problem 1. Also provide plots of $f(x)$ and $p_n(x)$ (the best least-squares polynomial approximation to $f(x)$) for each value of n. Use the inner product defined by $(f, g) = \int_{-5}^{5} f(x)g(x)dx$ in your computations. You may evaluate all inner products numerically using the trapz command in Matlab (be sure to use enough points in the interval to get an accurate approximation!)

   (d) Does your solution suffer from the same issues as global polynomial interpolants on uniformly spaced points?