VECTOR-VALUED FOURIER HYPERFUNCTIONS AND BOUNDARY VALUES

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To the memory of Professor Paweł Domański.

ABSTRACT. This work is dedicated to the development of the theory of Fourier hyperfunctions in one variable with values in a complex non-necessarily metrisable locally convex Hausdorff space E. Moreover, necessary and sufficient conditions are described such that a reasonable theory of E-valued Fourier hyperfunctions exists. In particular, if E is an ultrabornological PLS-space, such a theory is possible if and only if E satisfies the so-called property (PA). Furthermore, many examples of such spaces having (PA) resp. not having (PA) are provided. We also prove that the vector-valued Fourier hyperfunctions can be realized as the sheaf generated by equivalence classes of certain compactly supported E-valued functionals and interpreted as boundary values of slowly increasing holomorphic functions.

1. INTRODUCTION

The aim of the present work, which is the main result of the author's PhD thesis [46] with some improvements, is the development of the theory of Fourier hyperfunctions in one variable with values in a complex non-necessarily metrisable locally convex Hausdorff space E and to find necessary and sufficient conditions such that a reasonable theory of E-valued Fourier hyperfunctions is possible. In particular, we show that, if E is an ultrabornological PLS-space, such a theory exists if and only if E satisfies the so-called property (PA). It turns out that the vector-valued Fourier hyperfunctions can be realised as the sheaf generated by equivalence classes of certain compactly supported E-valued functionals and interpreted as boundary values of slowly increasing holomorphic functions.

Scalar-valued Fourier hyperfunctions $\mathcal{R}$, indicated by Sato [69](1958), were introduced by Kawai [36] in 1970. He constructed them as a flabby sheaf on D^d, where D^d means the radial compactification of $\mathbb{R}^d$, $d \in \mathbb{N}$, using cohomology theory and Hörmander's L^2-estimates [16]. He proved that the global sections are stable under Fourier transformation $\mathscr{F}$, i.e. $\mathscr{F} \colon \mathcal{R}(D^d) \to \mathcal{R}(D^d)$ is an isomorphism. This sheaf is a generalisation of the sheaf $\mathcal{B}$ of hyperfunctions on $\mathbb{R}^d$, which was developed by Sato [70] (and [71]); in particular, $\mathcal{R}_{|\mathbb{R}^d} = \mathcal{B}$ holds. Hyperfunctions emerged as an useful tool in the theory of partial differential equations (see [39]), in particular, in the solution of the abstract Cauchy problem. Komatsu developed the theory of Laplace hyperfunctions, a theory of operator-valued generalised functions with a suitable Laplace transform, more precisely, for operators in Banach spaces, and the abstract Cauchy problem was solved by a condition on the resolvent of the operator which characterised the generators of hyperfunction semigroups (see [41], [42], [43] and [44]). This theory was improved and extended beyond operators in

Banach spaces by Domański and Langenbruch (see [8], [9]). Since some partial differential equations can be taken as ordinary vector-valued equations (e.g. [67], [68]), the question arose whether there was a vector-valued counterpart for the theory of (Fourier) hyperfunctions. Whereas Schwartz achieved this in the analogous theory of distributions by tensor products [73], one faces a crucial problem in the development of such a theory of vector-valued, in short, E-valued where E is a locally convex Hausdorff space over $\mathbb{C}$, (Fourier) hyperfunctions, namely, the lack of a natural linear Hausdorff topology on the scalar-valued (Fourier) hyperfunctions (with the exception of the space of global sections in the case of Fourier hyperfunctions). Despite of this difficulty, Ion and Kawai [18](1975) developed a theory of hyperfunctions with values in Fréchet spaces, Ito and Nagamachi [30], [31](1975) a theory of Fourier hyperfunctions with values in separable Hilbert spaces (see [27] for general Hilbert spaces), which was used by Mugibayashi and Nagamachi ([65], [66]) for an axiomatic formulation of quantum field theory in terms of Fourier hyperfunctions, and Junker [33](1979) a theory of Fourier hyperfunctions with values in Fréchet spaces (cf. [20], [21], [22], [23], [24], [25], [26]). Since Fourier hyperfunctions with values in non-metrisable spaces E like the space of distributions, are of interest as well, there were some efforts to extend the theory of Fourier hyperfunctions to non-Fréchet spaces E (see [29]). However, to the best of our knowledge the present paper is the only fully correct theory of E-valued Fourier hyperfunctions including non-Fréchet spaces E (see Remark 5.3, Remark 5.13).

Domański and Langenbruch [7](2008) not only overcame these obstacles and developed a theory of vector-valued hyperfunctions beyond the class of Fréchet spaces, but also found natural limits of this kind of theory. They characterised in a large natural class of locally convex Hausdorff spaces those spaces for which a reasonable theory of E-valued hyperfunctions exists at all (see [7, Theorem 8.9, p. 1139]). To be more precise: they state that a reasonable theory of E-valued hyperfunctions should generate a flabby sheaf with the property that the set of sections supported by a compact subset $K \subset \mathbb{R}^d$ should coincide with $L(\mathscr{A}(K), E)$, the space of continuous linear operators from $\mathscr{A}(K)$ to E where $\mathscr{A}(K)$ denotes the space of germs of real analytic functions on K. Transferring this condition to the theory of Fourier hyperfunctions, we are convinced that a reasonable theory of E-valued Fourier hyperfunctions (in one variable) should produce a flabby sheaf such that the set of sections supported by a compact subset $K \subset \overline{\mathbb{R}}$ should coincide with „the space of E-valued $\mathcal{P}_*$-functionals" $L(\mathcal{P}_*(K), E)$ where $D^1 = \overline{\mathbb{R}}$ is the radial compactification of $\mathbb{R}$ and $\mathcal{P}_*(K)$ the space of rapidly decreasing holomorphic germs near K (see Proposition 3.5). If one restricts such a sheaf to $\mathbb{R}$, the restricted sheaf fulfils the condition of Domański and Langenbruch for a reasonable theory of E-valued hyperfunctions, since $\mathcal{P}_*(K) = \mathscr{A}(K)$ for compact $K \subset \mathbb{R}$, which is desirable in the spirit of the property $\mathcal{R}_{|\mathbb{R}} = \mathcal{B}$ of the scalar-valued case. Furthermore, the global sections of such a sheaf are stable under Fourier transformation (see Corollary 3.10). This implies that for those spaces E, for which a reasonable theory of E-valued hyperfunctions is impossible, a reasonable theory of E-valued Fourier hyperfunctions is impossible as well. A long list of examples of spaces E for which a reasonable theory of E-valued Fourier hyperfunctions is possible resp. impossible can be found in Example 4.4 resp. Example 4.5.

In the approach of Domański and Langenbruch the existence of an E-valued sheaf of hyperfunctions is deeply connected with the solvability of the E-valued Laplace equation; namely, if the $(d+1)$-dimensional Laplace operator

$$\Delta_{d+1} \colon \mathcal{C}^\infty(\Omega, E) \to \mathcal{C}^\infty(\Omega, E)$$

is surjective for every open set $\Omega \subset \mathbb{R}^{d+1}$ where $\mathcal{C}^\infty(\Omega, E)$ is the space of smooth E-valued functions on Ω, then a reasonable theory of E-valued hyperfunctions on

$\mathbb{R}^d$ is possible (see [7, Theorem 6.9, p. 1125]). For E-valued Fourier hyperfunctions in one variable the corresponding counterpart is the following. A complex locally convex Hausdorff space E is called *admissible* if the Cauchy-Riemann operator

$$\overline{\partial}\colon \mathcal{E}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E) \to \mathcal{E}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E)$$

is surjective for any compact set $K \subset \overline{\mathbb{R}}$ where $\overline{\mathbb{C}} := \overline{\mathbb{R}} + \mathrm{i}\mathbb{R}$ and $\mathcal{E}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E)$ is, roughly speaking, the space of slowly increasing smooth E-valued functions outside K (see Definition 3.1). E is called *strictly admissible* if E is admissible and, in addition,

$$\overline{\partial}\colon \mathcal{C}^{\infty}(\Omega, E) \to \mathcal{C}^{\infty}(\Omega, E)$$

is surjective for any open set $\Omega \subset \mathbb{C}$. We prove that E being strictly admissible yields to the existence of a reasonable theory of E-valued Fourier hyperfunctions in one variable (see Theorem 5.9).

The outline of the present paper is as follows. In Section 2 we introduce some notations and preliminaries needed to phrase our concepts. In Section 3 we define the spaces $\mathcal{E}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E)$, its subspace $\mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E)$ of holomorphic functions and $\mathcal{P}_*(K)$. Further, we recall some of their properties and a kind of Silva-Köthe-Grothendieck duality (see Theorem 3.6), give a boundary value representation of $L_b(\mathcal{P}_*(\overline{\mathbb{R}}), E)$ and define the Fourier transformation on this space. In Section 4 we collect some results on strict admissibility (see Theorem 4.3) and give many examples of strictly admissible spaces E. In correspondence with the scalar-valued case, the E-valued Fourier hyperfunctions are defined in Section 5 from two different points of view for a strictly admissible space E. On the one hand, as the sheaf generated by equivalence classes of E-valued $\mathcal{P}_*$-functionals, and on the other, as the sheaf of boundary values of the elements of $\mathcal{O}^{exp}(U \smallsetminus \overline{\mathbb{R}}, E)$. This is, to put it roughly, the space of holomorphic E-valued slowly increasing functions on U outside an open set $\Omega \subset \overline{\mathbb{R}}$ where U is an open set in $\overline{\mathbb{C}}$ with $U \cap \overline{\mathbb{R}} = \Omega$ (see Definition 5.6). The construction of these sheaves benefits from our kind of Silva-Köthe-Grothendieck duality and it turns out that both sheaves are flabby and isomorphic (see Theorem 5.9), solving two problems of Ito (see Lemma 5.2, Remark 5.3, Corollary 5.10). At the end of the fifth section, we show that, if E is an ultrabornolgical PLS-space, a reasonable theory of E-valued Fourier hyperfunctions in one variable exists if and only if E satisfies the property (PA) (see Theorem 5.12).

2. NOTATION AND PRELIMINARIES

The notation and preliminaries are essentially the same as in [47, 52, 55, Section 2]. We denote by $|\cdot|$ the Euclidean norm on $\mathbb{R}^2$ and $\mathbb{C}$, identify $\mathbb{R}^2$ and $\mathbb{C}$ as (normed) vector spaces, write $\mathbb{D}_r(z) := \{w \in \mathbb{C} \mid |w - z| < r\}$ for the open ball with radius $r > 0$ around $z \in \mathbb{C}$ and denote the restriction of a function $f\colon M \to \mathbb{C}$ to $K \subset M \subset \mathbb{C}$ by $f_{|K}$. We define the distance of two subsets $M_0, M_1 \subset \mathbb{R}^2$ w.r.t. $|\cdot|$ on $\mathbb{R}^2$ via

$$\mathrm{d}(M_0, M_1) := \begin{cases} \inf_{x \in M_0,\, y \in M_1} |x - y| & ,\ M_0, M_1 \neq \varnothing, \\ \infty & ,\ M_0 = \varnothing \text{ or } M_1 = \varnothing, \end{cases}$$

and write $\mathrm{d}(z, M_1) := \mathrm{d}(\{z\}, M_1)$ for $z \in \mathbb{R}^2$. We denote by $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ the radial compactifaction of $\mathbb{R}$, i.e. we equip $\overline{\mathbb{R}}$ with the following topology. A set $\Omega \subset \overline{\mathbb{R}}$ is called open if $\Omega \cap \mathbb{R}$ is open in $(\mathbb{R}, |\cdot|)$ and, in addition, there exists $a \in \mathbb{R}$ such that $[-\infty, a] \subset \Omega$ resp. $[a, \infty] \subset \Omega$ if $-\infty \in \Omega$ resp. $\infty \in \Omega$. $\overline{\mathbb{R}}$ becomes a compact space with this topology. We set $\overline{\mathbb{C}} := \overline{\mathbb{R}} + \mathrm{i}\mathbb{R}$ and equip it with the product topology. For a topological space X we denote the complement of a subset $M \subset X$ by $M^C := X \smallsetminus M$, the closure of M in X by $\overline{M}$ and the boundary of M by ∂M.

By E we always denote a non-trivial locally convex Hausdorff space over the field $\mathbb{C}$ ($\mathbb{C}$-lcHs) equipped with a directed fundamental system of seminorms $(p_\alpha)_{\alpha\in\mathfrak{A}}$. If $E = \mathbb{C}$, then we set $(p_\alpha)_{\alpha\in\mathfrak{A}} \coloneqq \{|\cdot|\}$. Further, we denote by $L(F,E)$ the space of continuous linear maps from a locally convex Hausdorff space F to E and sometimes use the notation $\langle T,f\rangle \coloneqq T(f)$, $f \in F$, for $T \in L(F,E)$. If $E = \mathbb{C}$, we write $F' \coloneqq L(F,\mathbb{C})$ for the dual space of F. We denote by $L_t(F,E)$ the space $L(F,E)$ equipped with the locally convex topology of uniform convergence on the absolutely convex compact subsets of F if $t = \kappa$, and on the bounded subsets of F if $t = b$. The ε-product of Schwartz [73, Chap. I, §1, Définition, p. 18] is defined by

$$F\varepsilon E \coloneqq L_e(F'_\kappa, E)$$

where $L(F'_\kappa, E)$ is equipped with the topology of uniform convergence on equicontinuous subsets of F'. By $F\widehat{\otimes}_\pi E$ we denote the completion of the projective tensor product $F \otimes_\pi E$. The space $F\widehat{\otimes}_\pi E$ is topologically isomorphic to $F\varepsilon E$ if F and E are complete and one of them is nuclear.

We recall the following well-known definitions concerning continuous partial differentiability of vector-valued functions (cf. [48, p. 4]). A function $f\colon\Omega \to E$ on an open set $\Omega \subset \mathbb{R}^2$ to E is called continuously partially differentiable (f is $\mathcal{C}^1$) if for the nth unit vector $e_n \in \mathbb{R}^2$ the limit

$$\partial^{e_n} f(x) \coloneqq \lim_{\substack{h\to 0 \\ h\in\mathbb{R}, h\neq 0}} \frac{f(x + he_n) - f(x)}{h}$$

exists in E for every $x \in \Omega$ and $\partial^{e_n} f$ is continuous on Ω ($\partial^{e_n} f$ is $\mathcal{C}^0$) for every $n \in \{1,2\}$. For $k \in \mathbb{N}$ a function f is said to be k-times continuously partially differentiable (f is $\mathcal{C}^k$) if f is $\mathcal{C}^1$ and all its first partial derivatives are $\mathcal{C}^{k-1}$. A function f is called infinitely continuously partially differentiable (f is $\mathcal{C}^\infty$) if f is $\mathcal{C}^k$ for every $k \in \mathbb{N}$. The linear space of all functions $f\colon\Omega \to E$ which are $\mathcal{C}^\infty$ is denoted by $\mathcal{C}^\infty(\Omega, E)$ and we write $\mathcal{C}^\infty(\Omega) \coloneqq \mathcal{C}^\infty(\Omega, \mathbb{C})$. Let $f \in \mathcal{C}^\infty(\Omega, E)$. For $\beta = (\beta_n) \in \mathbb{N}_0^2$ we set $\partial^{\beta_n} f \coloneqq f$ if $\beta_n = 0$, and

$$\partial^{\beta_n} f \coloneqq \underbrace{\partial^{e_n}\cdots\partial^{e_n}}_{\beta_n\text{-times}} f$$

if $\beta_n \neq 0$ as well as

$$\partial^\beta f \coloneqq \partial^{\beta_1}\partial^{\beta_2} f.$$

Due to the vector-valued version of Schwarz' theorem $\partial^\beta f$ is independent of the order of the partial derivatives on the right-hand side and we call $|\beta| \coloneqq \beta_1 + \beta_2$ the order of differentiation.

A function $f\colon\Omega \to E$ on an open set $\Omega \subset \mathbb{C}$ to E is called holomorphic if the limit

$$\partial^1_\mathbb{C} f(z_0) \coloneqq \lim_{\substack{h\to 0 \\ h\in\mathbb{C}, h\neq 0}} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists in E for every $z_0 \in \Omega$. As before we define derivatives of higher order recursively, i.e. for $n \in \mathbb{N}_0$ we set $\partial^0_\mathbb{C} f \coloneqq f$ and $\partial^n_\mathbb{C} f \coloneqq \partial^1_\mathbb{C}\partial^{n-1}_\mathbb{C} f$, $n \geq 1$, if the corresponding limits exist. The linear space of all functions $f\colon\Omega \to E$ which are holomorphic is denoted by $\mathcal{O}(\Omega, E)$ and we write $\mathcal{O}(\Omega) \coloneqq \mathcal{O}(\Omega, \mathbb{C})$. If E is locally complete and $f \in \mathcal{O}(\Omega, E)$, then $\partial^n_\mathbb{C} f(z_0)$ exists in E for every $z_0 \in \Omega$ and $n \in \mathbb{N}_0$ by [12, 2.2 Theorem and Definition, p. 18] and [12, 5.2 Theorem, p. 35]. E is called locally compete if every closed disk in E is a Banach disk (see [32, 10.2.1 Proposition, p. 197]). In particular, every sequentially complete space is locally complete.

For the convenience of the reader we recall the definition of a (pre)sheaf and a flabby sheaf.

2.1. **Definition** ((pre)sheaf, [2, 1.1 Definition, p. 1, 1.7, p. 6])**.** Let X be a topological space and for every open $U \subset X$ let there be a vector space $\mathcal{F}(U)$ such that $\mathcal{F}(\varnothing) = 0$ and for every pair $V \subset U$ of open sets in X let there be a linear map $R_{U,V} \colon \mathcal{F}(U) \to \mathcal{F}(V)$. Let $\mathcal{F} \coloneqq \{\mathcal{F}(U) \mid U \subset X \text{ open}\}$ and $R \coloneqq R^{\mathcal{F}} \coloneqq \{R_{U,V} \mid U, V \subset X \text{ open}\}$. The tuple $(\mathcal{F}, R)$ is called a *presheaf* on X and the maps in R *restrictions* if:

    (i) $R_{U,U} = \mathrm{id}$ for every open U, and
    (ii) $R_{V,W} \circ R_{U,V} = R_{U,W}$ for every open $W \subset V \subset U$.

A presheaf $(\mathcal{F}, R)$ is a *sheaf* on X if for every family of open sets $\{U_j \mid j \in J\}$ with $U \coloneqq \bigcup_{j \in J} U_j$ the following is valid:

    (S1) If $f \in \mathcal{F}(U)$ is such that $R_{U,U_j}(f) = 0$ for all $j \in J$, then $f = 0$.
    (S2) Let $f_j \in \mathcal{F}(U_j)$, $j \in J$, be given such that for every pair $(j, i) \in J^2$

$$R_{U_j, U_j \cap U_i}(f_j) = R_{U_i, U_j \cap U_i}(f_i)$$

    holds. Then there is $f \in \mathcal{F}(U)$ such that $R_{U,U_j}(f) = f_j$.

2.2. **Definition** (flabby, [2, 5.1 Definition, p. 47])**.** Let X be a topological space. A sheaf $(\mathcal{F}, R)$ on X is called *flabby* if $R_{X,U} \colon \mathcal{F}(X) \to \mathcal{F}(U)$ is surjective for every open set $U \subset X$.

    The following simple observation will turn out to be a useful tool in the proof of Theorem 5.9 c).

2.3. **Proposition** ([46, 6.6 Proposition, p. 115])**.** *Let X be a topological space, $(\mathcal{G}, R^{\mathcal{G}})$ a presheaf and $(\mathcal{F}, R^{\mathcal{F}})$ a sheaf on X. Let $h \colon \mathcal{G} \to \mathcal{F}$ be a homomorphism of presheaves such that $h_\Omega \colon \mathcal{G}(\Omega) \to \mathcal{F}(\Omega)$ is an isomorphism for every open set $\Omega \subset X$. Then $(\mathcal{G}, R^{\mathcal{G}})$ is a sheaf (and h an isomorphism of sheaves).*

Proof. First, we remark that $h \colon \mathcal{G} \to \mathcal{F}$ is a homomorphism of presheaves (see [2, p. 8]), i.e. the diagram

$$
\begin{array}{ccc}
\mathcal{G}(\Omega) & \xrightarrow{\ h_\Omega\ } & \mathcal{F}(\Omega) \\
{\scriptstyle R^{\mathcal{G}}_{\Omega,\Omega_1}} \downarrow & & \downarrow {\scriptstyle R^{\mathcal{F}}_{\Omega,\Omega_1}} \\
\mathcal{G}(\Omega_1) & \xrightarrow[\ h_{\Omega_1}\ ]{} & \mathcal{F}(\Omega_1)
\end{array}
$$

commutes for open sets $\Omega_1 \subset \Omega \subset X$. Let $f \in \mathcal{F}(\Omega)$. Since h_Ω and h_{Ω_1} are isomorphisms by our assumption, we have

$$h_{\Omega_1}^{-1}(f_{|\Omega_1}) = h_{\Omega_1}^{-1}\big(h_\Omega(h_\Omega^{-1}(f))_{|\Omega_1}\big) = h_{\Omega_1}^{-1}\big(h_{\Omega_1}(h_\Omega^{-1}(f)_{|\Omega_1})\big) = h_\Omega^{-1}(f)_{|\Omega_1}$$

since h is a homomorphism of presheaves, which means that the diagram

$$
\begin{array}{ccc}
\mathcal{G}(\Omega) & \xleftarrow{\ h_\Omega^{-1}\ } & \mathcal{F}(\Omega) \\
{\scriptstyle R^{\mathcal{G}}_{\Omega,\Omega_1}} \downarrow & & \downarrow {\scriptstyle R^{\mathcal{F}}_{\Omega,\Omega_1}} \\
\mathcal{G}(\Omega_1) & \xleftarrow[\ h_{\Omega_1}^{-1}\ ]{} & \mathcal{F}(\Omega_1)
\end{array}
$$

commutes as well, so h^{-1} is homomorphism of presheaves.

    (S1): Let $\{\Omega_j \mid j \in J\}$ be a familiy of open subsets of X and $\Omega \coloneqq \bigcup_{j \in J} \Omega_j$. Let $f \in \mathcal{G}(\Omega)$ such that $f_{|\Omega_j} = 0$ for all $j \in J$. Then $h_\Omega(f) \in \mathcal{F}(\Omega)$ and

$$h_\Omega(f)_{|\Omega_j} = h_\Omega(f_{|\Omega_j}) = h_\Omega(0) = 0$$

for all $j \in J$ due to the assumption and since h is a homomorphism of presheaves. As $\mathcal{F}$ is a sheaf, hence satisifies $(S1)$, we obtain $h_\Omega(f) = 0$. Due to the injectivity of h_Ω, we get $f = 0$.

$(S2)$: Let $\{\Omega_j \mid j \in J\}$ and Ω be like above. Let $f_j \in \mathcal{G}(\Omega_j)$ such that $f_{j|\Omega_j \cap \Omega_k} = f_{k|\Omega_j \cap \Omega_k}$ for all $j, k \in J$. Then $h_{\Omega_j}(f_j) \in \mathcal{F}(\Omega_j)$ and

$$h_{\Omega_j}(f_j)_{|\Omega_j \cap \Omega_k} - h_{\Omega_k}(f_k)_{|\Omega_j \cap \Omega_k} = h_{\Omega_j \cap \Omega_k}(f_{j|\Omega_j \cap \Omega_k}) - h_{\Omega_j \cap \Omega_k}(f_{k|\Omega_j \cap \Omega_k}) = 0$$

for all $j, k \in J$ by the assumption and since h is a homomorphism of presheaves. As $\mathcal{F}$ is a sheaf, hence satisfies $(S2)$, there exists $G \in \mathcal{G}(\Omega)$ such that $G_{|\Omega_j} = h_{\Omega_j}(f_j)$ for every $j \in J$. Now, we define $F := h_\Omega^{-1}(G) \in \mathcal{F}(\Omega)$. By virtue of the remark in the beginning, we gain

$$F_{|\Omega_j} = h_\Omega^{-1}(G)_{|\Omega_j} = h_{\Omega_j}^{-1}(G_{|\Omega_j}) = h_{\Omega_j}^{-1}(h_{\Omega_j}(f_j)) = f_j$$

for all $j \in J$. Therefore, $\mathcal{G}$ is a sheaf and thus h an isomorphism of sheaves. $\qquad\square$

For the notions not explained we refer the reader to the literature. For the classical theory of (Fourier) hyperfunctions we refer the reader to [11], [17], [35], [64], [72] or [75], for the sheaf theory to [2] or [56], for the theory of locally convex spaces to [10], [32] or [62], for PLS-spaces to [6] and for the theory of ε-products and tensor products to [4], [32], [34] or [47].

3. Silva-Köthe-Grothendieck duality, boundary values and Fourier transformation

This section is devoted to a duality theorem, a resulting boundary value representation and the Fourier transformation. We recall the well-known topological Silva-Köthe-Grothendieck isomorphism

$$\mathcal{O}(\mathbb{C} \smallsetminus K, E)/\mathcal{O}(\mathbb{C}, E) \cong L_b(\mathscr{A}(K), E) \tag{1}$$

for a quasi-complete $\mathbb{C}$-lcHs E and compact $\varnothing \neq K \subset \mathbb{R}$ (see e.g. [74, p. 6], [13, Proposition 1, p. 46], [76, Satz 9, p. 90], [45, §27.4, p. 375-378], [64, Theorem 2.1.3, p. 25]). Here $\mathcal{O}(\mathbb{C} \smallsetminus K, E)$ is equipped with the compact-open topology, the quotient space with the induced quotient topology and $\mathscr{A}(K)$ is the space of germs of real analytic functions on K with its inductive limit topology. We introduce the spaces $\mathcal{E}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E)$, $\mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E)$ and $\mathcal{P}_*(K)$ for a compact set $K \subset \overline{\mathbb{R}}$ in this section which will be used in the counterpart of the Silva-Köthe-Grothendieck isomorphism with $\mathcal{O}(\mathbb{C} \smallsetminus K, E)$ and $\mathscr{A}(K)$ replaced by $\mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E)$ and $\mathcal{P}_*(K)$, respectively. Then we come to a boundary value representation of $L_b(\mathcal{P}_*(\overline{\mathbb{R}}, E)$ and define the Fourier transformation on it.

For a compact set $K \subset \overline{\mathbb{R}}$ and $t \in \mathbb{R}$, $t \geq 1$, we define the open sets

$$U_t(K) := \{z \in \mathbb{C} \mid \mathrm{d}(z, K \cap \mathbb{C}) < \tfrac{1}{t}\}$$

$$\cup \begin{cases} \varnothing & , K \subset \mathbb{R}, \\ (t, \infty) + \mathrm{i}(-\tfrac{1}{t}, \tfrac{1}{t}) & , \infty \in K, -\infty \notin K, \\ (-\infty, -t) + \mathrm{i}(-\tfrac{1}{t}, \tfrac{1}{t}) & , \infty \notin K, -\infty \in K, \\ ((-\infty, -t) \cup (t, \infty)) + \mathrm{i}(-\tfrac{1}{t}, \tfrac{1}{t}) & , \pm\infty \in K, \end{cases}$$
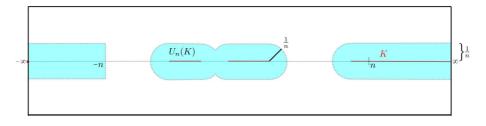

Figure 1. $U_n(K)$ for $\pm\infty \in K$

and

$$S_t(K) := \left(\overline{U_t(K)}\right)^C \cap \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < t\}, \quad t > 1,$$

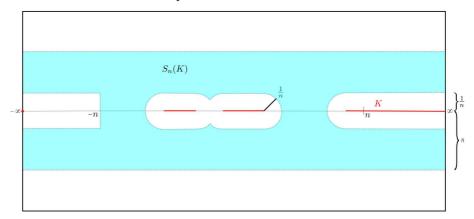where the closure and the complement are taken in $\mathbb{C}$.

FIGURE 2. $S_n(K)$ for $\pm\infty \in K$

3.1. **Definition** ([46, 3.2 Definition, p. 12-13]). Let $K \subset \overline{\mathbb{R}}$ be a compact set and E a $\mathbb{C}$-lcHs. We define the space of E-valued slowly increasing smooth functions outside K by

$$\mathcal{E}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E) := \{f \in \mathcal{C}^\infty(\mathbb{C} \smallsetminus K, E) \mid \forall\, n \in \mathbb{N},\, n \geq 2,\, m \in \mathbb{N}_0,\, \alpha \in \mathfrak{A} : |f|_{K,n,m,\alpha} < \infty\}$$

where

$$|f|_{K,n,m,\alpha} := \sup_{\substack{z \in S_n(K) \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} p_\alpha(\partial^\beta f(z)) e^{-(1/n)|\mathrm{Re}(z)|}.$$

We define the space of E-valued slowly increasing holomorphic functions outside K by

$$\mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E) := \{f \in \mathcal{O}(\mathbb{C} \smallsetminus K, E) \mid \forall\, n \in \mathbb{N},\, n \geq 2,\, \alpha \in \mathfrak{A} : |f|_{K,n,\alpha} < \infty\}$$

where

$$|f|_{K,n,\alpha} := |f|_{K,n,0,\alpha} = \sup_{z \in S_n(K)} p_\alpha(f(z)) e^{-(1/n)|\mathrm{Re}(z)|}.$$

Further, we set $\mathcal{E}^{exp}(\overline{\mathbb{C}} \smallsetminus K) := \mathcal{E}^{exp}(\overline{\mathbb{C}} \smallsetminus K, \mathbb{C})$ and $\mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus K) := \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus K, \mathbb{C})$.

We exclude the case $n = 1$ because

$$\left(\overline{U_1(\overline{\mathbb{R}})}\right)^C \cap \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < 1\} = \varnothing$$

but we could also include the case $n = 1$ with definition $S_1(K) := S_{t_0}(K)$ for some $t_0 \in (1, 2)$ which would not change the spaces $\mathcal{E}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E)$ and $\mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E)$. In the literature the symbols $\widetilde{\mathcal{E}}(\overline{\mathbb{C}} \smallsetminus K, E)$ resp. $\widetilde{\mathcal{O}}(\overline{\mathbb{C}} \smallsetminus K, E)$ are also used for $\mathcal{E}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E)$ resp. $\mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E)$ (see [33, 1.2 Definition, p. 5]).

Several times we will use the following useful relation between real and complex partial derivatives of a holomorphic function.

3.2. **Proposition** ([53, 3.23 Proposition, p. 19]). *Let E be a locally complete $\mathbb{C}$-lcHs, $\Omega \subset \mathbb{C}$ open and $f \in \mathcal{O}(\Omega, E)$. Then $f \in \mathcal{C}^\infty(\Omega, E)$ and*

$$\partial^\beta f(z) = \mathrm{i}^{\beta_2} \partial_{\mathbb{C}}^{|\beta|} f(z), \quad z \in \Omega,\, \beta \in \mathbb{N}_0^2. \tag{2}$$

For complete E this was already observed in [46, 3.4 Lemma, p. 17].

3.3. **Proposition.** *Let E be a locally complete $\mathbb{C}$-lcHs, $a_1 \leq a_2 < 0$, $\Omega_1 \subset \Omega_2 \subset \mathbb{C}$ be open and let there exist $0 < r \leq 1$ such that $\overline{\mathbb{D}_r(z)} \subset \Omega_2$ for all $z \in \Omega_1$. Then for all $f \in \mathcal{O}(\Omega_2, E)$, $m \in \mathbb{N}_0$ and $\alpha \in \mathfrak{A}$ it holds that*

$$\sup_{\substack{z \in \Omega_1 \\ k \in \mathbb{N}_0, k \leq m}} p_\alpha(\partial_\mathbb{C}^k f(z)) e^{a_1 |\operatorname{Re}(z)|} \leq e^{-a_1 r} \frac{m!}{r^m} \sup_{z \in \Omega_2} p_\alpha(f(z)) e^{a_2 |\operatorname{Re}(z)|}.$$

Proof. a) Let $m \in \mathbb{N}_0$. We note that

$$a_1 |\operatorname{Re}(z)| \leq a_1 |\operatorname{Re}(\zeta)| - a_1 |\operatorname{Re}(z) - \operatorname{Re}(\zeta)| \leq a_2 |\operatorname{Re}(\zeta)| - a_1 r, \quad z \in \Omega_1, \ \zeta \in \overline{\mathbb{D}_r(z)}.$$

By Cauchy's inequality [53, 3.14 Corollary, p. 14] we have

$$p_\alpha(\partial_\mathbb{C}^k f(z)) \leq \frac{k!}{r^k} \sup_{\zeta \in \mathbb{C}, |\zeta - z| = r} p_\alpha(f(\zeta)), \quad z \in \Omega_1,$$

for all $f \in \mathcal{O}(\Omega_2, E)$ and $\alpha \in \mathfrak{A}$, which implies

$$\sup_{\substack{z \in \Omega_1 \\ k \in \mathbb{N}_0, k \leq m}} p_\alpha(\partial_\mathbb{C}^k f(z)) e^{a_1 |\operatorname{Re}(z)|} \leq \sup_{\substack{z \in \Omega_1 \\ k \in \mathbb{N}_0, k \leq m}} \frac{k!}{r^k} \sup_{\zeta \in \mathbb{C}, |\zeta - z| = r} p_\alpha(f(\zeta)) e^{a_1 |\operatorname{Re}(z)|}$$

$$\leq e^{-a_1 r} \frac{m!}{r^m} \sup_{\zeta \in \Omega_2} p_\alpha(f(\zeta)) e^{a_2 |\operatorname{Re}(\zeta)|}.$$

$$\square$$

Proposition 3.3 in combination with (2) implies that the topology of $\mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E)$ coincides with the induced topology of $\mathcal{E}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E)$ for compact $K \subset \overline{\mathbb{R}}$ and locally complete E. For Fréchet spaces E this can also be found in [33, 1.4 Lemma (1), p. 5] and for complete E in [46, 3.6 Theorem (4), p. 21].

3.4. **Remark.** Let $K \subset \overline{\mathbb{R}}$ be compact.
   a) The spaces $\mathcal{E}^{exp}(\overline{\mathbb{C}} \smallsetminus K)$ and $\mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus K)$ are nuclear Fréchet spaces, e.g. by [48, 3.7 Proposition, p. 7] and [50, Theorem 3.1, p. 13] combined with [50, 2.7 Remark, p. 5] and [50, 2.8 Example (ii), p. 5].
   b) We have the topological isomorphisms

$$\mathcal{E}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E) \cong \mathcal{E}^{exp}(\overline{\mathbb{C}} \smallsetminus K) \varepsilon E \quad \text{and} \quad \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E) \cong \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus K) \varepsilon E$$

   for a locally complete $\mathbb{C}$-lcHs E by [49, 3.23 Corollary c), p. 16] and thus

$$\mathcal{E}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E) \cong \mathcal{E}^{exp}(\overline{\mathbb{C}} \smallsetminus K) \widehat{\otimes}_\pi E \quad \text{and} \quad \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E) \cong \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus K) \widehat{\otimes}_\pi E$$

   for a complete $\mathbb{C}$-lcHs E due to nuclearity.

Part a) of the preceding remark can also be found in [33, 1.4 Lemma (2), p. 5] and [33, 1.6 Folgerung, p. 7] as well as [46, 3.6 Theorem (2), p. 21] and [46, 3.7 Theorem, p. 23]. Part b) for Fréchet spaces E is also given in [33, 1.7 Satz, p. 8-9] and for complete spaces E in [46, 3.11 Theorem, p. 31] and [46, 3.12 Corollary, p. 35].

3.5. **Proposition.** *Let $K \subset \overline{\mathbb{R}}$ be a non-empty compact set. For $n \in \mathbb{N}$ let*

$$\mathcal{O}_n(\overline{U_n(K)}) := \{ f \in \mathcal{O}(U_n(K)) \cap \mathcal{C}^0(\overline{U_n(K)}) \mid \|f\|_{K,n} < \infty \}$$

where

$$\|f\|_{K,n} := \sup_{z \in \overline{U_n(K)}} |f(z)| e^{(1/n) |\operatorname{Re}(z)|}$$

and the spectral maps for $n, k \in \mathbb{N}$, $n \leq k$, be given by the restrictions

$$\pi_{n,k} : \mathcal{O}_n(\overline{U_n(K)}) \to \mathcal{O}_k(\overline{U_k(K)}), \quad \pi_{n,k}(f) := f_{|U_k(K)}.$$

Then the space of rapidly decreasing holomorphic germs near $K \neq \varnothing$ given by the inductive limit

$$\mathcal{P}_*(K) := \varinjlim_{n \in \mathbb{N}} \mathcal{O}_n(\overline{U_n(K)})$$

exists and is a DFS-space. If $K = \varnothing$, we set $\mathcal{P}_*(\varnothing) := 0$.

The preceding proposition is a special case of [54, 3.3 Proposition a), p. 6-7]. It is already mentioned in [36, p. 469] resp. proved in [33, 1.11 Satz, p. 11] and [46, 3.5 Theorem, p. 17] that $\mathcal{P}_*(K)$ is a DFS-space. In the literature the symbol $\mathcal{Q}(K)$ is also used for $\mathcal{P}_*(K)$ and the symbol $\mathcal{P}_*$ for the special case $\mathcal{P}_*(\overline{\mathbb{R}})$ (see [36, Definition 1.1.3, p. 468-469]). If $\varnothing \neq K \subset \mathbb{R}$ is compact, then $\mathcal{P}_*(K) = \mathscr{A}(K)$. The counterpart of the Silva-Köthe-Grothendieck isomorphism (1) for vector-valued slowly increasing holomorphic functions outside a non-empty compact set $K \subset \overline{\mathbb{R}}$ reads as follows.

3.6. Theorem ([54, 3.15 Corollary (iii), p. 21, Eq. (6), p. 14, Eq. (12), p. 20]). *Let $K \subset \overline{\mathbb{R}}$ be a non-empty compact set and E a sequentially complete $\mathbb{C}$-lcHs. Then the map*

$$H_K : \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E)/\mathcal{O}^{exp}(\overline{\mathbb{C}}, E) \to L_b(\mathcal{P}_*(K), E)$$

given by

$$H_K(f)(\varphi) := \int_{\gamma_{K,n,r}} F(\zeta)\varphi(\zeta)\mathrm{d}\zeta$$

for $f = [F] \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E)/\mathcal{O}^{exp}(\overline{\mathbb{C}}, E)$ and $\varphi \in \mathcal{O}_n(\overline{U_n(K)})$, $n \in \mathbb{N}$, where the integral is a Pettis-integral and $\gamma_{K,n,r}$ a suitable path along K in $\overline{U_n(K)}$, is a topological isomorphism.
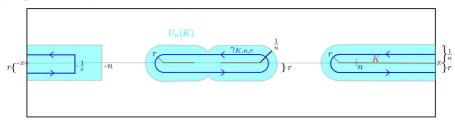

FIGURE 3. Path $\gamma_{K,n,r}$ for $\pm\infty \in K$

Its inverse

$$H_K^{-1} : L_b(\mathcal{P}_*(K), E) \to \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E)/\mathcal{O}^{exp}(\overline{\mathbb{C}}, E)$$

is given by

$$H_K^{-1}(T) := \Big[\mathbb{C} \smallsetminus K \ni z \longmapsto \frac{1}{2\pi\mathrm{i}}\langle T, \frac{e^{-(z-\cdot)^2}}{z-\cdot}\rangle\Big], \quad T \in L_b(\mathcal{P}_*(K), E),$$

In addition, for all non-empty compact sets $K_1 \subset K$ it holds that

$$H_{K|\mathcal{O}^{exp}(\overline{\mathbb{C}}\smallsetminus K_1, E)/\mathcal{O}^{exp}(\overline{\mathbb{C}}, E)} = H_{K_1} \tag{3}$$

on $\mathcal{P}_*(K)$ and

$$H_K^{-1}(T) = H_{\overline{\mathbb{R}}}^{-1}(T), \quad T \in L(\mathcal{P}_*(K), E). \tag{4}$$

The topological isomorphism $\mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E)/\mathcal{O}^{exp}(\overline{\mathbb{C}}, E) \cong L_b(\mathcal{P}_*(K), E)$ in Theorem 3.6 is already known for special cases. For $E = \mathbb{C}$ it can be found in [36, Theorem 3.2.1, p. 480], and if E is a Fréchet space in [33, 3.9 Satz, p. 41] but the proof is different. For an interval $K = [a, \infty]$, $a \in \mathbb{R}$, and $E = \mathbb{C}$ the duality in Theorem 3.6 was proved in [63, Theorem 3.3, p. 85-86] and was the starting point to prove Theorem 3.6 for complete E in [46, 4.1 Theorem, p. 41]. The map Θ_K is also called (weighted) *Cauchy transformation* (see [63, p. 84]).

As nuclearity is inherited by quotient spaces, we derive from Remark 3.4 a), Theorem 3.6 with $E = \mathbb{C}$ and the reflexivity of $\mathcal{P}_*(K)$ that $\mathcal{P}_*(K)$ is nuclear for every compact set $K \subset \overline{\mathbb{R}}$ (cf. [33, 1.11 Satz, p. 11]). By [36, Theorem 2.2.1, p. 474] $\mathcal{P}_*(\overline{\mathbb{R}})$ is dense in $\mathcal{P}_*(K)$ for a non-empty compact set $K \subset \overline{\mathbb{R}}$. So for different compact sets $K, J \subset \overline{\mathbb{R}}$ we may identify elements of $L(\mathcal{P}_*(K), E)$ and $L(\mathcal{P}_*(J), E)$ by means of their restrictions to $\mathcal{P}_*(\overline{\mathbb{R}})$. Then the following result defining the support of a vector-valued $\mathcal{P}_*$-functional is valid, whose counterpart for compact subsets of $\mathbb{R}$ is given in [7, Proposition 5.3, p. 1121].

3.7. Proposition (support). *Let $K \subset \overline{\mathbb{R}}$ be compact and E a sequentially complete $\mathbb{C}$-lcHs.*

   a) If $J \subset \overline{\mathbb{R}}$ is compact and $K \cap J \neq \varnothing$, then
$$L(\mathcal{P}_*(K), E) \cap L(\mathcal{P}_*(J), E) = L(\mathcal{P}_*(K \cap J), E).$$

   b) For any $T \in L(\mathcal{P}_*(K), E)$ there is a minimal compact set $J \subset K$ such that $T \in L(\mathcal{P}_*(J), E)$. The set J is called the support *of T.*

Proof.   a) „$\subset$": Let $T \in L(\mathcal{P}_*(K), E) \cap L(\mathcal{P}_*(J), E)$. Then $H_K^{-1}(T) = H_J^{-1}(T)$ by (4) and
$$H_K^{-1}(T) \in \left(\mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E)/\mathcal{O}^{exp}(\overline{\mathbb{C}}, E)\right) \cap \left(\mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus J, E)/\mathcal{O}^{exp}(\overline{\mathbb{C}}, E)\right)$$
$$= \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus (K \cap J), E)/\mathcal{O}^{exp}(\overline{\mathbb{C}}, E)$$

and $T = (H_{K \cap J} \circ H_K^{-1})(T) \in L(\mathcal{P}_*(K \cap J), E)$ by Theorem 3.6 and (4). The other inclusion is obvious.

   b) This is clear by Theorem 3.6 since for any $f \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E)$ there is a minimal compact set J such that $f \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus J, E)$.

$\hfill\square$

3.8. Remark. Let E be a $\mathbb{C}$-lcHs. Proposition 3.7 improves [46, 4.3 Proposition, p. 50] from complete E to sequentially complete E. If $K, J \subset \mathbb{R}$ are compact sets with $K \cap J \neq \varnothing$, then Proposition 3.7 is still valid for locally complete E since Theorem 3.6 holds in this case as well by [54, 3.15 Corollary (i), p. 21].

For $K = \overline{\mathbb{R}}$ we look at the duality Theorem 3.6 once again, but from a different point of view. Let $f \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\mathbb{R}}, E)$. In the spirit of [57] and [72, Chap. II, p. 77-97] we assign the boundary value

$$\langle R(f), \varphi \rangle \coloneqq \lim_{t,\, t' \searrow 0} \langle R_{t,t'}(f), \varphi \rangle \coloneqq \lim_{t,\, t' \searrow 0} \int_{\mathbb{R}} (f(x + it) - f(x + it'))\varphi(x)\mathrm{d}x, \quad \varphi \in \mathcal{P}_*(\overline{\mathbb{R}}),$$

to this function, if the limit in E of this Pettis-integral exists. Furthermore, we define the upper boundary value by

$$\langle R^+(f), \varphi \rangle \coloneqq \lim_{t \searrow 0} \langle R_t^+(f), \varphi \rangle \coloneqq \lim_{t \searrow 0} \int_{\mathbb{R}} f(x + it)\varphi(x)\mathrm{d}x, \quad \varphi \in \mathcal{P}_*(\overline{\mathbb{R}}),$$

and the lower boundary value by

$$\langle R^-(f), \varphi \rangle \coloneqq \lim_{t \searrow 0} \langle R_t^-(f), \varphi \rangle \coloneqq \lim_{t \searrow 0} \int_{\mathbb{R}} f(x - it)\varphi(x)\mathrm{d}x, \quad \varphi \in \mathcal{P}_*(\overline{\mathbb{R}}),$$

if the limit in E of these Pettis-integrals exists.

3.9. Theorem. *Let E be a sequentially complete $\mathbb{C}$-lcHs.*

   a) $(R_{t,t'}(f))$, $(R_t^+(f))$ and $(R_t^-(f))$ converge to $R(f)$, $R^+(f)$ and $R^-(f)$ in $L_b(\mathcal{P}_*(\overline{\mathbb{R}}), E)$, respectively, and
$$R(f) = R^+(f) - R^-(f) = -H_{\overline{\mathbb{R}}}([f])$$
   for all $f \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\mathbb{R}}, E)$.

b) *The map* $[f] \mapsto R(f)$ *from* $\mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \mathbb{R}, E)/\mathcal{O}^{exp}(\overline{\mathbb{C}}, E)$ *to* $L_b(\mathcal{P}_*(\overline{\mathbb{R}}), E)$ *is a topological isomorphism.*

Proof. a) Let $f \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \mathbb{R}, E)$. We fix $t > 0$ and observe that $f(\cdot \pm it)\varphi$ is Pettis integrable for every $\varphi \in \mathcal{P}_*(\overline{\mathbb{R}})$ on every compact set $K \subset \mathbb{R}$ by [51, 4.7 Lemma, p. 14], i.e. there is $e_K^{\pm} = e_K^{\pm}(\pm t, f, \varphi) \in E$ such that

$$\langle e', e_K^{\pm} \rangle = \int_K \langle e', f(x \pm it)\varphi(x) \rangle \mathrm{d}x, \quad e' \in E'.$$

We set $a_{k,-}^- := e_{[-k,0]}^-$ and $a_{k,+}^- := e_{[0,k]}^-$ for $k \in \mathbb{N}$. Let $\alpha \in \mathfrak{A}$, $n \in \mathbb{N}$ and $\varphi \in \mathcal{O}_n(U_n(\overline{\mathbb{R}}))$. We choose $m \in \mathbb{N}$ with $m > 2\max(n,t)$ and observe that for $k, p \in \mathbb{N}$, $k > p$,

$$p_\alpha(a_{k,-}^- - a_{p,-}^-) \le \int_{[-k,-p]} p_\alpha(f(x \pm it))|\varphi(x)|\mathrm{d}x \le |f|_{\overline{\mathbb{R}},m,\alpha} \|\varphi\|_{\overline{\mathbb{R}},n} \int_{-k}^{-p} e^{\frac{1}{m}|x-it|-\frac{1}{n}|x|}\mathrm{d}x$$

$$\le e^{\frac{1}{2n}t}|f|_{\overline{\mathbb{R}},m,\alpha} \|\varphi\|_{\overline{\mathbb{R}},n} \int_{-k}^{-p} e^{-\frac{1}{2n}|x|}\mathrm{d}x.$$

We derive that $(a_{k,-}^-)$ and, analogously, $(a_{k,+}^-)$ are Cauchy sequences in the sequentially complete space E. Hence they have limits a_-^- resp. a_+^- in E and it is easy to check that

$$\langle e', a_-^- + a_+^- \rangle = \int_{\mathbb{R}} \langle e', f(x - it)\varphi(x) \rangle \mathrm{d}x, \quad e' \in E',$$

which means that $f(\cdot - it)\varphi$ is Pettis-integrable on $\mathbb{R}$ with $R_t^-(f)(\varphi) = a_-^- + a_+^-$. In the same way, it follows that $f(\cdot + it)\varphi$ is Pettis-integrable on $\mathbb{R}$. Further, we obtain

$$p_\alpha(R_t^{\pm}(f)(\varphi)) \le 2e^{\frac{1}{2n}t}|f|_{\overline{\mathbb{R}},m,\alpha} \|\varphi\|_{\overline{\mathbb{R}},n} \int_0^\infty e^{-\frac{1}{2n}x}\mathrm{d}x = 4ne^{\frac{1}{2n}t}|f|_{\overline{\mathbb{R}},m,\alpha} \|\varphi\|_{\overline{\mathbb{R}},n} < \infty.$$

Thus $R_t^{\pm}(f) \in L(\mathcal{O}_n(U_n(\overline{\mathbb{R}})), E)$ for every $n \in \mathbb{N}$, implying $R_t^{\pm}(f) \in L(\mathcal{P}_*(\overline{\mathbb{R}}), E)$.

Now, we set $\varphi_t^{\pm}(x) := \varphi(x \pm it)$. Then the functions

$$t \mapsto R_t^{\pm}(f)(\varphi_t^{\pm}) = \int_{\mathbb{R}} f(x \pm it)\varphi(x \pm it)\mathrm{d}x \tag{5}$$

are defined for $\varphi \in \mathcal{O}_n(U_n(\overline{\mathbb{R}}))$, $n \in \mathbb{N}$, on $(0, \frac{1}{n})$ and constant by Cauchy's integral theorem (see the proof of [54, 3.7 Proposition c), p. 10]). Thus the limits $\lim_{t \searrow 0} R_t^{\pm}(f)(\varphi_t^{\pm})$ exist in E for every $\varphi \in \mathcal{P}_*(\overline{\mathbb{R}})$.

Let $\alpha \in \mathfrak{A}$, $n \in \mathbb{N}$, and $\varphi \in \mathcal{O}_n(U_n(\overline{\mathbb{R}}))$. For $0 < t < \frac{1}{3n}$ and $z \in \overline{U_{3n}(K)}$ we have

$$|\varphi(z) - \varphi(z \pm it)|e^{\frac{1}{3n}|\operatorname{Re}(z)|}$$

$$= |\int_{[z \pm it, z]} \varphi'(w)\mathrm{d}w|e^{\frac{1}{3n}|\operatorname{Re}(z)|} \le t \sup_{w \in [z \pm it, z]} |\varphi'(w)|e^{\frac{1}{3n}|\operatorname{Re}(z)|}$$

$$\le t \sup_{w \in [z \pm it, z]} 6n \max_{|\zeta - w| = \frac{1}{6n}} |\varphi(\zeta)|e^{\frac{1}{3n}|\operatorname{Re}(z)|}$$

$$\le 6ne^{\frac{1}{18n^2}}t \sup_{w \in [z \pm it, z]} \max_{|\zeta - w| = \frac{1}{6n}} |\varphi(\zeta)|e^{\frac{1}{3n}|\operatorname{Re}(\zeta)|} \le 6ne^{\frac{1}{18n^2}} \|\varphi\|_{\overline{\mathbb{R}},n}t$$

by Cauchy's inequality where $[z \pm it, z]$ is the line segment from $z \pm it$ to z. Hence we get

$$\|\varphi - \varphi(\cdot \pm it)\|_{\overline{\mathbb{R}},3n} \le 6ne^{\frac{1}{18n^2}} \|\varphi\|_{\overline{\mathbb{R}},n}t. \tag{6}$$

Further, we have for $0 < t < \frac{1}{3n}$ and $x \in \mathbb{R}$

$$|\operatorname{Im}(x \pm \mathrm{i}\tfrac{1}{3n})| = \tfrac{1}{3n} \quad \text{plus} \quad 6n > \tfrac{1}{n} > |\operatorname{Im}(x \pm t \pm \mathrm{i}\tfrac{1}{3n})| = t + \tfrac{1}{3n} > \tfrac{1}{6n}.$$

Due to Cauchy's integral theorem we obtain for all $0 < t < \frac{1}{3n}$

$$p_\alpha\big(R_t^\pm(f)(\varphi) - R_t^\pm(f)(\varphi_t^\pm)\big) = p_\alpha\Big(\int_{\mathbb{R}} f(x \pm \mathrm{i}t)(\varphi(x) - \varphi(x \pm \mathrm{i}t))\mathrm{d}x\Big)$$

$$= p_\alpha\Big(\int_{\mathbb{R}} f(x \pm \mathrm{i}t \pm \mathrm{i}\tfrac{1}{3n})(\varphi(x \pm \mathrm{i}\tfrac{1}{3n}) - \varphi(x \pm \mathrm{i}t \pm \mathrm{i}\tfrac{1}{3n}))\mathrm{d}x\Big)$$

$$\leq |f|_{\mathbb{R},6n,\alpha}\|\varphi - \varphi(\cdot \pm \mathrm{i}t)\|_{\mathbb{R},3n} \int_{-\infty}^{\infty} e^{\frac{1}{6n}|x \pm \mathrm{i}t| - \frac{1}{3n}|x|}\mathrm{d}x$$

$$\leq 12ne^{\frac{1}{6n}t}|f|_{\mathbb{R},6n,\alpha}\|\varphi - \varphi(\cdot \pm \mathrm{i}t)\|_{\mathbb{R},3n}$$

$$\underset{(6)}{\leq} \big(72n^2 e^{\frac{1}{18n^2}}\|\varphi\|_{\mathbb{R},n}|f|_{\mathbb{R},6n,\alpha}\big)e^{\frac{1}{6n}t}t \underset{t \searrow 0}{\to} 0.$$

Since the limits $\lim_{t\searrow 0} R_t^\pm(f)(\varphi_t^\pm)$ exist in E for every $\varphi \in \mathcal{P}_*(\overline{\mathbb{R}})$, we deduce that the limits $\langle R^\pm(f), \varphi\rangle = \lim_{t\searrow 0} R_t^\pm(f)(\varphi)$ exist in E, more precisely,

$$\langle R^\pm(f), \varphi\rangle = \lim_{t\searrow 0} R_t^\pm(f)(\varphi) = \lim_{t\searrow 0} R_t^\pm(f)(\varphi_t^\pm).$$

The space $\mathcal{P}_*(\overline{\mathbb{R}})$ is a DFS-space by Proposition 3.5 and hence a Montel space. Thus it is barrelled and by the Banach-Steinhaus theorem [10, 10.3.4 Satz, p. 53] we obtain that $R^\pm(f) \in L_b(\mathcal{P}_*(\overline{\mathbb{R}}), E)$ and $(R_t^\pm(f))$ converges to $R^\pm(f)$ in $L_b(\mathcal{P}_*(\overline{\mathbb{R}}), E)$ as $t \searrow 0$. Furthermore, we get

$$\langle R(f), \varphi\rangle = \lim_{t,\,t'\searrow 0}\big(R_t^+(f)(\varphi) - R_{t'}^-(f)(\varphi)\big) = \lim_{t\searrow 0} R_t^+(f)(\varphi) - \lim_{t\searrow 0} R_t^-(f)(\varphi)$$

$$= \langle R^+(f), \varphi\rangle - \langle R^-(f), \varphi\rangle = \lim_{t\searrow 0}\big(R_t^+(f)(\varphi_t^+) - R_t^-(f)(\varphi_t^-)\big)$$

$$= \lim_{t\searrow 0}\Big(\int_{\mathbb{R}} f(x + \mathrm{i}t)\varphi(x + \mathrm{i}t)\mathrm{d}x - \int_{\mathbb{R}} f(x - \mathrm{i}t)\varphi(x - \mathrm{i}t)\mathrm{d}x\Big) = -H_{\overline{\mathbb{R}}}([f])(\varphi)$$

$$\tag{7}$$

for every $\varphi \in \mathcal{P}_*(\overline{\mathbb{R}})$ by Theorem 3.6 and [54, 3.7 Proposition c), p. 10]. In particular, this means that $R_{t,t'}(f)$ converges to $R(f)$ in $L_b(\mathcal{P}_*(\overline{\mathbb{R}}), E)$ as $t, t' \searrow 0$ for every $f \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\mathbb{R}}, E)$.

b) By the first part the considered map coincides with $-H$ and the statement follows directly by Theorem 3.6. $\qquad\square$

Theorem 3.9 improves [46, 4.5 Theorem, p. 51] from complete E to sequentially complete E. In particular, this theorem contains, at least in one variable, [36, Theorem 3.2.9, p. 483-484] for $E = \mathbb{C}$ and [33, Satz 3.13, p. 44] for Fréchet spaces E, where it is stated that the map

$$\widetilde{R}\colon \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\mathbb{R}}, E)/\mathcal{O}^{exp}(\overline{\mathbb{C}}, E) \to L_b(\mathcal{P}_*(\overline{\mathbb{R}}), E),$$

defined by

$$\widetilde{R}([f])(\varphi) := R_t^+(f)(\varphi_t^+) - R_t^-(f)(\varphi_t^-)$$

for $f \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\mathbb{R}}, E)$ and $\varphi \in \mathcal{P}_*(\overline{\mathbb{R}})$ and fixed t small enough, is a topological isomorphism. This result is contained since the functions in (5) are constant and due to (7).

Finally, we define the Fourier transformation on $L_b(\mathcal{P}_*(\overline{\mathbb{R}}), E)$. By [36, Proposition 3.2.4, p. 483] (cf. [35, Proposition 8.2.2, p. 376]) the Fourier transformation

$\mathscr{F}\colon\mathcal{P}_*(\overline{\mathbb{R}})\to\mathcal{P}_*(\overline{\mathbb{R}})$ defined by

$$\mathscr{F}(\varphi)(\zeta)\coloneqq\widehat{\varphi}(\zeta)\coloneqq\int_{\mathbb{R}}\varphi(x)e^{\mathrm{i}x}\mathrm{d}x,\quad \varphi\in\mathcal{O}_n(U_n(\overline{\mathbb{R}})),\ \zeta\in U_k(\overline{\mathbb{R}}),\ k>n,$$

is a topological isomorphism. The Fourier transformation on $L_b(\mathcal{P}_*(\overline{\mathbb{R}}),E)$ is now defined by transposition (see e.g. [33, 3.14 Folgerung, p. 45], [36, Definition 3.2.5, p. 483], [46, 4.6 Theorem, p. 53]).

3.10. **Corollary.** *Let E be a $\mathbb{C}$-lcHs. The Fourier transformation*

$$\mathscr{F}_\star\colon L_b(\mathcal{P}_*(\overline{\mathbb{R}}),E)\to L_b(\mathcal{P}_*(\overline{\mathbb{R}}),E),$$

$$\mathscr{F}_\star(T)(\varphi)\coloneqq\langle T,\mathscr{F}(\varphi)\rangle,\quad T\in L_b(\mathcal{P}_*(\overline{\mathbb{R}}),E),\ \varphi\in\mathcal{P}_*(\overline{\mathbb{R}}),$$

is a topological isomorphism with inverse given by $\mathscr{F}_\star^{-1}(T)(\varphi)\coloneqq\langle T,\mathscr{F}^{-1}(\varphi)\rangle$, for $T\in L_b(\mathcal{P}_*(\overline{\mathbb{R}}),E),\ \varphi\in\mathcal{P}_*(\overline{\mathbb{R}})$.

This follows directly from the fact that $\mathscr{F}$ is a topological isomorphism.

4. Strict admissibility

In this section we recall some results on the notion of strict admissibility from the introduction.

4.1. **Definition** ((strictly) admissible, [46, p. 55]). Let E be a $\mathbb{C}$-lcHs. We call E *admissible* if the Cauchy-Riemann operator

$$\overline{\partial}\coloneqq\tfrac{1}{2}(\partial^{e_1}+\mathrm{i}\partial^{e_2})\colon\mathcal{E}^{exp}(\overline{\mathbb{C}}\smallsetminus K,E)\to\mathcal{E}^{exp}(\overline{\mathbb{C}}\smallsetminus K,E)$$

is surjective for every compact set $K\subset\overline{\mathbb{R}}$. We call E *strictly admissible* if E is admissible and

$$\overline{\partial}\colon\mathcal{C}^\infty(U,E)\to\mathcal{C}^\infty(U,E)$$

is surjective for every open set $U\subset\mathbb{C}$.

Using that $E=\mathbb{C}$ is admissible (see e.g. [46, 5.16 Theorem, p. 80] or [52, 4.8 Theorem, p. 20]), it follows from Grothendieck's classical theory of tensor products [14] and Remark 3.4 that Fréchet spaces E are admissible, from Vogt's splitting theory for Fréchet spaces that $E\coloneqq F_b'$, where F is a Fréchet space satisfying the condition (DN), is admissible by [78, Theorem 2.6, p. 174], and from Bonet's and Domański's splitting theory for PLS-spaces that an ultrabornological PLS-space E having the property (PA) is admissible by [7, Corollary 3.9, p. 1112] since $\mathcal{O}^{exp}(\overline{\mathbb{C}}\smallsetminus K)=\ker\overline{\partial}$ in $\mathcal{E}^{exp}(\overline{\mathbb{C}}\smallsetminus K)$ has the property (Ω) (see [62, Chap. 29, Definition, p. 367]) by [46, 5.20 Theorem, p. 85] or [55, 4.5 Corollary, p. 13] if $K=\varnothing$ and [46, 5.22 Theorem, p. 92] or [54, 4.9 Corollary (ii), p. 29] if $K\neq\varnothing$.

We recall that a Fréchet space $(F,(\|\cdot\|_k)_{k\in\mathbb{N}})$ satisfies (DN) by [62, Chap. 29, Definition, p. 359] if

$$\exists\,p\in\mathbb{N}\ \forall\,k\in\mathbb{N}\ \exists\,n\in\mathbb{N},\ C>0\ \forall\,x\in F:\ \|x\|_k^2\leq C\|x\|_p\|x\|_n.$$

A *PLS-space* is a projective limit $X=\varprojlim_{N\in\mathbb{N}}X_N$, where the inductive limits $X_N=\varinjlim_{n\in\mathbb{N}}(X_{N,n},\|\cdot\|_{N,n})$ are DFS-spaces, and it satisfies (PA) if

$$\forall\,N\ \exists\,M\ \forall\,K\ \exists\,n\ \forall\,m\ \forall\,\eta>0\ \exists\,k,C,r_0>0\ \forall\,r>r_0\ \forall\,x'\in X_N':$$

$$\left\|x'\circ i_N^M\right\|_{M,m}^*\leq C\Big(r^\eta\big\|x'\circ i_N^K\big\|_{K,k}^*+\frac{1}{r}\|x'\|_{N,n}^*\Big)$$

where $\|\cdot\|^*$ denotes the dual norm of $\|\cdot\|$ (see [1, Section 4, Eq. (24), p. 577]). By [55, 5.5 Remark, p. 20] a Fréchet-Schwartz space F satisfies (DN) if and only if the DFS-space $E\coloneqq F_b'$ satisfies (PA).

4.2. Theorem ([52, 4.10 Example a), p. 22], [54, 5.3 Corollary (ii), p. 32], [55, 5.7 Corollary, p. 21]). *Let $K \subset \overline{\mathbb{R}}$ be a compact set. If*

a) E is a Fréchet space over $\mathbb{C}$, or if

b) $E \coloneqq F_b'$ where F is a Fréchet space over $\mathbb{C}$ satisfying (DN), or if

c) E is an ultrabornological PLS-space over $\mathbb{C}$ satisfying (PA),

then E is admissible, i.e.

$$\overline{\partial}: \mathcal{E}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E) \to \mathcal{E}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E)$$

is surjective for every compact set $K \subset \overline{\mathbb{R}}$.

This result can also be found in [46, 5.17 Theorem, p. 82] and [46, 5.24 Theorem, p. 85]. In the non-weighted case the Cauchy-Riemann operator

$$\overline{\partial}: \mathcal{C}^{\infty}(U, E) \to \mathcal{C}^{\infty}(U, E) \tag{8}$$

is surjective for every open set $U \subset \mathbb{C}$ if $E = \mathbb{C}$ by [16, Theorem 1.4.4, p. 12]. Furthermore, $\mathcal{O}(U)$ and $\mathcal{C}^{\infty}(U)$, both equipped with the topology of uniform convergence on compact subsets (of partial derivatives of any order in the latter case), are nuclear Fréchet spaces by [62, Examples 5.18 (3), (4), p. 42], [62, Examples 28.9 (1), (4), p. 349-350] and we have the topological isomorphisms

$$\mathcal{C}^{\infty}(U, E) \cong \mathcal{C}^{\infty}(U)\varepsilon E \cong \mathcal{C}^{\infty}(U)\widehat{\otimes}_{\pi} E$$

plus

$$\mathcal{O}(U, E) \cong \mathcal{O}(U)\varepsilon E \cong \mathcal{O}(U)\widehat{\otimes}_{\pi} E$$

by [77, Theorem 44.1, p. 449] resp. [32, 16.7.5 Corollary, p. 366] for open $U \subset \mathbb{C}$ and complete E. By [34, Theorem 10.10, p. 240] the $\overline{\partial}$-operator in (8) is surjective if E is a Fréchet space. If $E \coloneqq F_b'$ where F is a Fréchet space satisfying (DN) or E is an ultrabornological PLS-space satisfying (PA), this holds due to [78, 2.6 Theorem, p. 174] resp. [7, Corollary 3.9, p. 1112] as well because $\mathcal{O}(U) = \ker \overline{\partial}$ in $\mathcal{C}^{\infty}(U)$ has the property (Ω) by [78, Proposition 2.5 (b), p. 173] for every open $U \subset \mathbb{C}$. In combination with Theorem 4.2 this means:

4.3. Theorem ([46, 5.25 Theorem, p. 98]). *If*

a) E is a Fréchet space over $\mathbb{C}$, or if

b) $E \coloneqq F_b'$ where F is a Fréchet space over $\mathbb{C}$ satisfying (DN), or if

c) E is an ultrabornological PLS-space over $\mathbb{C}$ satisfying (PA),

then E is strictly admissible.

To close this section we provide some examples of ultrabornological PLS-spaces satisfying (PA) and spaces of the form $E \coloneqq F_b'$ where F is a Fréchet space satisfying (DN). Most of them are already contained in [7, Corollary 4.8, p. 1116].

4.4. Example ([55, 5.8 Example, p. 21]). a) The following spaces are ultrabornological PLS-spaces with property (PA) and also strong duals of a Fréchet space satisfying (DN), hence are strictly admissible:

- the strong dual of a power series space of inifinite type $\Lambda_{\infty}(\alpha)_b'$,
- the strong dual of any space of holomorphic functions $\mathcal{O}(U)_b'$ where U is a Stein manifold with the strong Liouville property (for instance, for $U = \mathbb{C}^d$),
- the space of germs of holomorphic functions $\mathcal{O}(K)$ where K is a completely pluripolar compact subset of a Stein manifold (for instance K consists of one point),
- the space of tempered distributions $\mathcal{S}(\mathbb{R}^d)_b'$ and the space of Fourier ultra-hyperfunctions $\mathcal{P}_{**}'$ (with the strong dual topology),

- the weighted distribution spaces $(K\{pM\})'_b$ of Gelfand and Shilov if the weight M satisfies

$$\sup_{|y| \le 1} M(x+y) \le C \inf_{|y| \le 1} M(x+y), \quad x \in \mathbb{R}^d,$$

- $\mathcal{D}(K)'_b$ for any compact set $K \subset \mathbb{R}^d$ with non-empty interior,
- $\mathcal{C}^\infty(\overline{U})'_b$ for any non-empty open bounded set $U \subset \mathbb{R}^d$ with $\mathcal{C}^1$-boundary.

b) The following spaces are ultrabornological PLS-spaces with property (PA):

- an arbitrary Fréchet-Schwartz space,
- a PLS-type power series space $\Lambda_{r,s}(\alpha, \beta)$ whenever $s = \infty$ or $\Lambda_{r,s}(\alpha, \beta)$ is a Fréchet space,
- the spaces of distributions $\mathcal{D}(U)'_b$ and ultradistributions of Beurling type $\mathcal{D}_{(\omega)}(U)'_b$ for any open set $U \subset \mathbb{R}^d$,
- the kernel of any linear partial differential operator with constant coefficients in $\mathcal{D}(U)'_b$ or in $\mathcal{D}_{(\omega)}(U)'_b$ when $U \subset \mathbb{R}^d$ is open and convex,
- the space $L_b(X, Y)$ where X has (DN), Y has (Ω) and both are nuclear Fréchet spaces. In particular, $L_b(\Lambda_\infty(\alpha), \Lambda_\infty(\beta))$ if both spaces are nuclear.

c) The following spaces are strong duals of a Fréchet space satisfying (DN), hence are strictly admissible:

- the strong dual F'_b of any Banach space F,
- the strong dual $\lambda^2(A)'_b$ of the Köthe space $\lambda^2(A)$ with a Köthe matrix $A = (a_{j,k})_{j,k \in \mathbb{N}_0}$ satisfying

$$\exists \, p \in \mathbb{N}_0 \; \forall \, k \in \mathbb{N}_0 \; \exists \, n \in \mathbb{N}_0, C > 0 : \; a_{j,k}^2 \le C a_{j,p} a_{j,n}.$$

4.5. **Example** ([7, Corollary 4.9, p. 1117]). a) The following ultrabornological PLS-spaces do not have (PA):

- the strong dual of power series space of finite type $\Lambda_0(\alpha)'_b$,
- the space of ultradifferentiable functions of Roumieu type $\mathcal{E}_{\{\omega\}}(U)$ where ω is a non-quasianalytic weight and $U \subset \mathbb{R}^d$ is an arbitrary open set,
- the strong dual of any space of holomorphic functions $\mathcal{O}(U)'_b$ where U is a Stein manifold which does not have the strong Liouville property (for instance, $U = \mathbb{D}^d$ the polydisc, $U = \mathbb{B}_d$ the unit ball etc.),
- the space of germs of holomorphic functions $\mathcal{O}(K)$ where K is compact and not completely pluripolar (for instance, $K = \overline{\mathbb{D}}^d$ or $K = \overline{\mathbb{B}}_d$),
- the space of distributions $\mathcal{E}'(U)$ and ultradistributions $\mathcal{E}'_{(\omega)}(U)$ (with the strong dual topology) with compact support for $U \subset \mathbb{R}^d$ open,
- the space of real analytic functions $\mathscr{A}(U)$ for any open set $U \subset \mathbb{R}^d$.

b) For the following LFS-spaces E the map (8) is not surjective and hence E is not strictly admissible:

- the space of test functions $\mathcal{D}(U)$ (with its inductive limit topology) where $U \subset \mathbb{R}^d$ is any open set,
- the space of test functions for ultradistributions $\mathcal{D}_{(\omega)}(U)$, the space of ultradistributions of Roumieu type with compact support $\mathcal{E}_{\{\omega\}}(U)'_b$ where ω is a non-quasianalytic weight, $U \subset \mathbb{R}^d$ is any open set,
- the strong dual $\mathscr{A}(U)'_b$ for any open set $U \subset \mathbb{R}^d$.

5. DUALITY METHOD

In this section we present the main results of [46, Chapter 6]. We construct E-valued Fourier hyperfunctions in one variable as the sheaf generated by equivalence classes of compactly supported E-valued $\mathcal{P}_*$-functionals and show that they form

a flabby sheaf under the condition that E is strictly admissible. This construction relies on the Silva-Köthe-Grothendieck duality Theorem 3.6 and the method, which goes back to Martineau [58], is sometimes called *duality method* (see [7] and [28]). Furthermore, a description of E-valued Fourier hyperfunctions as boundary values of slowly increasing holomorphic functions is provided and finally the necessity of the conditions that are used for the construction of vector-valued Fourier hyperfunctions will be examined.

5.1. **Definition** (Fourier hyperfunctions)**.** For an non-empty open set $\Omega \subset \overline{\mathbb{R}}$ and $\mathbb{C}$-lcHs E we define the space of E-valued *Fourier hyperfunctions* on Ω by

$$\mathcal{R}(\Omega, E) \coloneqq L(\mathcal{P}_*(\overline{\Omega}), E)/L(\mathcal{P}_*(\partial\Omega), E)$$

and $\mathcal{R}(\varnothing, E) \coloneqq 0$. For $T \in L(\mathcal{P}_*(\overline{\Omega}), E)$ we denote by $[T]$ the corresponding element of $\mathcal{R}(\Omega, E)$. If the set Ω is equipped with an index, then we sometimes do the same with the corresponding equivalence class in order to distinguish between different classes. Further, we use the notation $\mathcal{R}(\Omega) \coloneqq \mathcal{R}(\Omega, \mathbb{C})$.

We observe that $L(\mathcal{P}_*(\varnothing), E) = L(0, E) = 0$ and hence $\mathcal{R}(\overline{\mathbb{R}}, E) = L(\mathcal{P}_*(\overline{\mathbb{R}}), E)$ (more precisely, we identify $L(\mathcal{P}_*(\overline{\mathbb{R}}), E)$ and $\{\{T\} \mid T \in L(\mathcal{P}_*(\overline{\mathbb{R}}), E)\}$). Thus there is a reasonable locally convex Hausdorff topology on $\mathcal{R}(\overline{\mathbb{R}}, E)$. For $\Omega \neq \overline{\mathbb{R}}$ there is no reasonable locally convex Hausdorff topology on $\mathcal{R}(\Omega, E)$ by [33, 3.10 Bemerkung, p. 41-42].

Let us first take a look at the scalar case. Let $\Omega \subset \Omega_1 \subset \overline{\mathbb{R}}$ be open. It is straightforward to prove that the canonical injective (by Proposition 3.7 a)) linear map

$$I \colon \mathcal{P}_*(\overline{\Omega})'/\mathcal{P}_*(\partial\Omega)' \to \mathcal{P}_*(\overline{\Omega}_1)'/\mathcal{P}_*(\overline{\Omega}_1 \smallsetminus \Omega)'.$$

is surjective (see [46, p. 101-102]), thus an algebraic isomorphism. Therefore the restrictions and a sheaf structure may be defined on $\mathcal{R}_{\Omega_1} \coloneqq \{\mathcal{R}_\Omega \mid \Omega \subset \Omega_1 \text{ open}\}$ like in Definition 5.4. It is not known whether the corresponding map I in the vector-valued case is always an algebraic isomorphism (see Remark 5.3). But this holds if we additionally assume that

$$\overline{\partial} \colon \mathcal{E}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E) \to \mathcal{E}^{exp}(\overline{\mathbb{C}} \smallsetminus K, E)$$

is surjective for any compact set $K \subset \overline{\mathbb{R}}$, i.e. that E is *admissible*. Let us turn to the already indicated statement whose counterpart for hyperfunctions is given in [7, Lemma 6.2, p. 1122].

5.2. **Lemma.** *Let E be admissible, $\Omega_2 \subset \Omega_1 \subset \overline{\mathbb{R}}$ be open and $\Omega_2 \neq \varnothing$. Then the canonical map*

$$I \colon L(\mathcal{P}_*(\overline{\Omega}_2), E)/L(\mathcal{P}_*(\partial\Omega_2), E) \to L(\mathcal{P}_*(\overline{\Omega}_1), E)/L(\mathcal{P}_*(\overline{\Omega}_1 \smallsetminus \Omega_2), E),$$

$$[T]_2 \mapsto [T],$$

is an algebraic isomorphism.

Proof. This map is well-defined, in particular, independent of the choice of the representative since $\mathcal{P}_*(\overline{\Omega}_1)$ is continuously and densely embedded in $\mathcal{P}_*(\overline{\Omega}_2)$ (see the remark right above Proposition 3.7) and thus the embedding of $L(\mathcal{P}_*(\overline{\Omega}_2), E)$ into $L(\mathcal{P}_*(\overline{\Omega}_1), E)$ is defined as well as the map of $L(\mathcal{P}_*(\partial\Omega_2), E)$ into $L(\mathcal{P}_*(\overline{\Omega}_1 \smallsetminus \Omega_2), E)$ in this manner.

If $\mathbb{R} \subset \Omega_2$, then $\overline{\Omega}_2 = \overline{\Omega}_1 = \overline{\mathbb{R}}$ and therefore $\overline{\Omega}_1 \smallsetminus \Omega_2 = \partial\Omega_2$. Hence the statement is obviously true.

Now, let $\mathbb{R} \not\subset \Omega_2$. Let $T \in L(\mathcal{P}_*(\overline{\Omega}_2), E)$ with $[T] = 0$. Then we get by Proposition 3.7 a)

$$T \in L(\mathcal{P}_*(\overline{\Omega}_2), E) \cap L(\mathcal{P}_*(\overline{\Omega}_1 \smallsetminus \Omega_2), E) = L(\mathcal{P}_*(\overline{\Omega}_2 \cap (\overline{\Omega}_1 \smallsetminus \Omega_2)), E) = L(\mathcal{P}_*(\partial\Omega_2), E)$$

and thus $[T]_2 = 0$, implying the injectivity of I.

The surjectivity of I is equivalent to the surjectivity of the map

$$I_0 \colon L(\mathcal{P}_*(\overline{\Omega}_1 \smallsetminus \Omega_2), E) \times L(\mathcal{P}_*(\overline{\Omega}_2), E) \to L(\mathcal{P}_*(\overline{\Omega}_1), E), \ (T_1, T_2) \mapsto T_1 + T_2.$$

By Theorem 3.6 the surjectivity of I_0 is equivalent to the surjectivity of

$$I_1 \colon \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus (\overline{\Omega}_1 \smallsetminus \Omega_2), E)/\mathcal{O}^{exp}(\overline{\mathbb{C}}, E) \times \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\Omega}_2, E)/\mathcal{O}^{exp}(\overline{\mathbb{C}}, E)$$

$$\to \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\Omega}_1, E)/\mathcal{O}^{exp}(\overline{\mathbb{C}}, E), \ (f_1, f_2) \mapsto f_1 + f_2,$$

and thus to the surjectivity of

$$I_2 \colon \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus (\overline{\Omega}_1 \smallsetminus \Omega_2), E) \times \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\Omega}_2, E) \to \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\Omega}_1, E), \ (f_1, f_2) \mapsto f_1 + f_2.$$

The proof is now done in several steps, beginning with the construction of a cut-off function. We restrict to the case that $\pm\infty \in \overline{\Omega}_1$, $-\infty \in \Omega_2$ and $\infty \notin \overline{\Omega}_2$. For the similar treatment of the other cases we refer to the proof of [46, 6.2 Lemma, p. 103].

(i) There is $x_0 \in \mathbb{R}$ such that $[x_0, \infty] \subset \overline{\Omega}_2^{C}$ since $\overline{\Omega}_2^{C} \subset \overline{\mathbb{R}}$ is open and $\infty \notin \Omega_2$. Further, there is $\widetilde{x}_1 \in \mathbb{R}$ such that $[-\infty, \widetilde{x}_1] \subset \Omega_2$, since Ω_2 is open and $-\infty \in \Omega_2$, $[-\infty, \widetilde{x}_1] \subset (\overline{\Omega}_1 \smallsetminus \Omega_2)^C$. We define the sets

$$F_0 \coloneqq (\mathbb{R} \smallsetminus \Omega_2) \cup ([x_0 + 2, \infty) \times [-1, 1]) \ \text{ and } \ F_1 \coloneqq (\mathbb{R} \cap \overline{\Omega}_2) \cup ((-\infty, \widetilde{x}_1 - 2] \times [-1, 1]).$$

The sets F_0 and F_1 are non-empty and closed in $\mathbb{R}^2$, $F_0 \cap \mathbb{R} = \mathbb{R} \smallsetminus \Omega_2$, $F_1 \cap \mathbb{R} = \overline{\Omega}_2 \cap \mathbb{R}$ and $F_0 \cap F_1 = \partial\Omega_2 \cap \mathbb{R}$. By [15, Corollary 1.4.11, p. 31] there exists $\varphi \in \mathcal{C}^\infty((F_0 \cap F_1)^C) = \mathcal{C}^\infty(\mathbb{R}^2 \smallsetminus \partial\Omega_2)$, $0 \le \varphi \le 1$, such that $\varphi = 0$ on V_0 and $\varphi = 1$ on V_1 where V_0, $V_1 \subset \mathbb{R}^2$ are open and

$$V_0 \supset F_0 \smallsetminus (F_0 \cap F_1) = F_0 \smallsetminus \partial\Omega_2 \supset (\mathbb{R} \smallsetminus \overline{\Omega}_2) \ \text{ and } \ V_1 \supset F_1 \smallsetminus (F_0 \cap F_1) = F_1 \smallsetminus \partial\Omega_2 \supset (\mathbb{R} \cap \Omega_2).$$
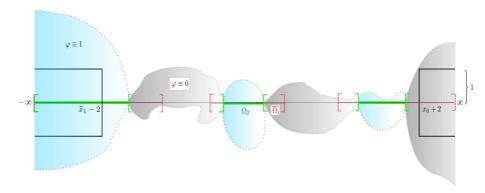

FIGURE 4. case: $\pm\infty \in \overline{\Omega}_1$, $-\infty \in \Omega_2$, $\infty \notin \overline{\Omega}_2$

Furthermore,

$$|\varphi^{(k)}(z; y^1, \cdots, y^k)| \le C^k |y^1| \cdots |y^k| \frac{\mathrm{d}(z)^{-k}}{d_1 \cdots d_k} \tag{9}$$

for all $z \in \mathbb{R}^2 \smallsetminus \partial\Omega_2$ and all $y^i \in \mathbb{R}^2$, $1 \le i \le k$, $k \in \mathbb{N}$, where $\varphi^{(k)}$ denotes the differential of order k of φ, $C > 0$ is a constant independent of z, y^i and k,

$$\mathrm{d}(z) \coloneqq \max\big(\mathrm{d}(z, F_0), \mathrm{d}(z, F_1)\big) = \max\big(\min_{x \in F_0} |z - x|, \min_{x \in F_1} |z - x|\big)$$

and $(d_n)_{n \in \mathbb{N}}$ is any decreasing sequence with $\sum_{n=1}^\infty d_n = 1$, e.g. $d_n \coloneqq (\frac{1}{2})^n$. We observe that for $\beta = (\beta_1, \beta_2) \in \mathbb{N}_0^2$ the relation

$$\partial^\beta \varphi(z) = \varphi^{(|\beta|)}\big(z; \underbrace{e_1, \cdots, e_1}_{\beta_1-\text{times}}, \underbrace{e_2, \cdots, e_2}_{\beta_2-\text{times}}\big)$$

holds between the differential of order $|\beta|$ and the βth partial derivative where e_j, $j = 1, 2$, is the jth unit vector in $\mathbb{R}^2$. Thus we obtain from (9) the estimate

$$|\partial^\beta \varphi(z)| \le C^{|\beta|} \frac{\mathrm{d}(z)^{-|\beta|}}{d_1 \cdots d_{|\beta|}}, \quad z \in \mathbb{R}^2 \smallsetminus \partial\Omega_2, \ \beta \in \mathbb{N}_0^2, \tag{10}$$

where we set $d_1 \cdots d_{|0|} := 1$ which is consistent with $|\partial^0 \varphi| = \varphi \le 1$. Let us take a closer look at the right-hand side of this inequality. For $z \in \mathbb{R}^2 \smallsetminus \partial\Omega_2$ there is $z_i \in F_i$ such that $\mathrm{d}(z, F_i) = |z - z_i|$, $i = 0, 1$, since F_0 and F_1 are closed. Let $n \in \mathbb{N}$, $n \ge 2$. We claim that

$$\mathrm{d}(z) \ge \frac{1}{n}, \quad z \in S_n(\partial\Omega_2).$$

Let $z \in S_n(\partial\Omega_2)$.

case z_0, $z_1 \in \mathbb{R}$: Let us assume that $\mathrm{d}(z) < \frac{1}{n}$. The definition of the set $S_n(\partial\Omega_2)$ implies $z_i \notin \partial\Omega_2 \cap \mathbb{R}$, $i = 0, 1$. Thus we get by the definition of the sets F_i that $z_0 \in \mathbb{R} \smallsetminus \overline{\Omega}_2$ and $z_1 \in \mathbb{R} \cap \Omega_2$, in particular, $z_0 \ne z_1$. W.l.o.g. $z_0 < z_1$. Then $O_0 := (z_0, z_1) \cap (\mathbb{R} \smallsetminus \overline{\Omega}_2)$ and $O_1 := (z_0, z_1) \cap (\mathbb{R} \cap \Omega_2)$ are disjoint, open sets in $\mathbb{R}$. Assume that there is no $\widetilde{z} \in \partial\Omega_2 \cap \mathbb{R}$ with $z_0 < \widetilde{z} < z_1$. Due to this assumption, we obtain

$$O_0 \cup O_1 = (z_0, z_1) \cap \left[(\mathbb{R} \smallsetminus \overline{\Omega}_2) \cup (\mathbb{R} \cap \Omega_2) \right] = (z_0, z_1) \cap (\mathbb{R} \smallsetminus \partial\Omega_2) = (z_0, z_1)$$

and hence, as (z_0, z_1) is connected, $(z_0, z_1) \subset O_0$ or $(z_0, z_1) \subset O_1$. If $(z_0, z_1) \subset O_0$, we get $z_1 \notin \mathbb{R} \cap \Omega_2$, and if $(z_0, z_1) \subset O_1$, we have $z_0 \notin \mathbb{R} \smallsetminus \overline{\Omega}_2$, which is a contradiction. So there must be a $\widetilde{z} \in \partial\Omega_2 \cap \mathbb{R}$ with $z_0 < \widetilde{z} < z_1$. The convexity of $\mathbb{D}_{\mathrm{d}(z)}(z)$ implies $\widetilde{z} \in (z_0, z_1) \subset \mathbb{D}_{\mathrm{d}(z)}(z)$, but then the following is valid

$$\frac{1}{n} < |z - \widetilde{z}| \le \max(|z - z_0|, |z - z_1|) = \mathrm{d}(z) < \frac{1}{n},$$

which is again a contradiction.

case $(z_0 \notin \mathbb{R}, z_1 \in \mathbb{R})$ or $(z_0 \in \mathbb{R}, z_1 \notin \mathbb{R})$: We only consider the first case, the latter one is analogous. We have $z_1 < x_0$ and $\mathrm{Re}(z_0) \ge x_0 + 2$. Therefore, we get

$$|z_1 - \mathrm{Re}(z_0)| \ge |x_0 - (x_0 + 2)| = 2.$$

If $|z - z_0| < \frac{1}{n}$, we obtain by the estimate above

$$\mathrm{d}(z) \ge |z - z_1| \ge |\mathrm{Re}(z) - z_1| \ge |z_1 - \mathrm{Re}(z_0)| - |\mathrm{Re}(z_0) - \mathrm{Re}(z)| > |z_1 - \mathrm{Re}(z_0)| - \frac{1}{n}$$

$$\ge 2 - \frac{1}{n} \ge \frac{1}{n}.$$

case $z_i \notin \mathbb{R}$, $i = 0, 1$: If $|z - z_0| < \frac{1}{n}$, we have

$$\mathrm{Re}(z_1) \le \widetilde{x}_1 - 2 < \widetilde{x}_1 < x_0 < x_0 + 2 - \frac{1}{n} \le \mathrm{Re}(z)$$

and thus we get

$$\mathrm{d}(z) \ge |z - z_1| \ge |\mathrm{Re}(z) - \mathrm{Re}(z_1)| \ge 4 - \frac{1}{n} > \frac{1}{n}.$$

Hence the claim is proved and via (10) we obtain

$$|\partial^\beta \varphi(z)| \le C^{|\beta|} \frac{n^{|\beta|}}{d_1 \cdots d_{|\beta|}}, \quad z \in S_n(\partial\Omega_2). \tag{11}$$

(ii) Let $f \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\Omega}_1, E)$. Due to the choice of φ, the function $\overline{\partial}(\varphi f)$ may be regarded as an element of $\mathcal{C}^\infty(\mathbb{R}^2 \smallsetminus \partial\Omega_2, E)$ by $\mathcal{C}^\infty$-extension via $\overline{\partial}(\varphi f) := 0$ on

$\mathbb{R} \smallsetminus \partial\Omega_2$. Furthermore,

$$\overline{\partial}(\varphi f)(z) = \begin{cases} 0 & , \; z \in V_0 \cup V_1, \\ (\overline{\partial}\varphi)(z)f(z) & , \; \text{else}, \end{cases}$$

is valid.

Let $n \in \mathbb{N}$, $n \geq 2$, $m \in \mathbb{N}_0$ and $\alpha \in \mathfrak{A}$. We define the set $S(n) \coloneqq S_n(\partial\Omega_2) \smallsetminus (V_0 \cup V_1)$. By applying the Leibniz rule (see e.g. [48, 3.9 Proposition, p. 7]), we obtain

$$|\overline{\partial}(\varphi f)|_{\partial\Omega_2, n, m, \alpha}$$

$$= \sup_{\substack{z \in S_n(\partial\Omega_2) \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} p_\alpha(\partial^\beta \overline{\partial}(\varphi f)(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|}$$

$$\leq \sup_{\substack{z \in S_n(\partial\Omega_2) \smallsetminus (V_0 \cup V_1) \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} |\partial^{\beta-\gamma}\overline{\partial}\varphi(z)| p_\alpha(\partial^\gamma f(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|}$$

$$\underset{(2)}{\leq} (m!)^2 \sum_{|\gamma| \leq m+1} \sup_{z \in S(n)} |\partial^\gamma \varphi(z)| \underbrace{\sup_{\substack{z \in S(n) \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} p_\alpha(\partial_{\mathbb{C}}^{|\beta|} f(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|}}_{=:C(f)}$$

$$\underset{(11)}{\leq} (m!)^2 C(f) \sum_{|\gamma| \leq m+1} C^{|\gamma|} \frac{n^{|\gamma|}}{d_1 \cdots d_{|\gamma|}}$$

$$\leq (m!)^2 \frac{[\max(C,1)]^{m+1}}{d_1 \cdots d_{m+1}} C(f) \sum_{|\gamma| \leq m+1} n^{|\gamma|} \qquad (12)$$

where we used the properties of (d_j), which imply $0 < d_j < 1$ for all $j \in \mathbb{N}$, in the last estimate.

Now, we have to take a closer look at $C(f)$. We decompose the set $S(n)$ in the following manner:

$$S(n) = \underbrace{\left[S(n) \cap \left\{ z \in \mathbb{C} \,\middle|\, |\operatorname{Im}(z)| > \tfrac{1}{2n} \right\} \right]}_{\subset S_{2n}(\overline{\Omega}_1)} \cup \underbrace{\left[S(n) \smallsetminus \left\{ z \in \mathbb{C} \,\middle|\, |\operatorname{Im}(z)| > \tfrac{1}{2n} \right\} \right]}_{=:M}$$
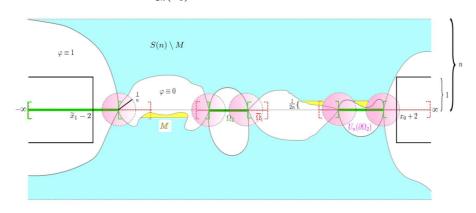

FIGURE 5. case: $\pm\infty \in \overline{\Omega}_1$, $-\infty \in \Omega_2$, $\infty \notin \overline{\Omega}_2$

Due to Proposition 3.3, we get for $r \coloneqq \frac{1}{2}\left(\frac{1}{2n} - \frac{1}{3n}\right)$

$$C(f) \leq \sup_{\substack{z \in S_{2n}(\overline{\Omega}_1) \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} p_\alpha(\partial_{\mathbb{C}}^{|\beta|} f(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|} + \sup_{\substack{z \in M \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} p_\alpha(\partial_{\mathbb{C}}^{|\beta|} f(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|}$$

$$\leq e^{\frac{r}{n}}\frac{m!}{r^m}|f|_{\overline{\Omega}_1,3n,\alpha} + \sup_{\substack{z\in M \\ \beta\in\mathbb{N}_0^2,|\beta|\leq m}} p_\alpha(\partial_{\mathbb{C}}^{|\beta|}f(z))e^{-\frac{1}{n}|\operatorname{Re}(z)|}. \tag{13}$$

Let us turn our attention to the set M. First, we observe that

$$\mathbb{R}\subset\Big[\underbrace{V_0\cup V_1}_{\supset\mathbb{R}\smallsetminus\partial\Omega_2}\cup\bigcup_{x\in\partial\Omega_2\cap\mathbb{R}}\mathbb{D}_{1/n}(x)\Big]=:V.$$

$V\subset\mathbb{R}^2$ is open as the union of open sets and so we get by the definition of M that

$$\overline{M}\subset\overline{V^C}=V^C\subset(\mathbb{R}^2\smallsetminus\mathbb{R}). \tag{14}$$

We claim that M is bounded. As $|\operatorname{Im}(z)|\leq\frac{1}{2n}$ for every $z\in M$, it suffices to prove that there is $C_1>0$ such that $|\operatorname{Re}(z)|\leq C_1$ for every $z\in M$. The choice of the sets F_0 and F_1 gives $C_1:=\max(|\widetilde{x}_1-2|,|x_0+2|)$. Hence $\overline{M}$ is compact and we have by (14) and the continuity of $\partial_{\mathbb{C}}^{|\beta|}f$ on $\mathbb{R}^2\smallsetminus\mathbb{R}$ for all $\beta\in\mathbb{N}_0^2$ that

$$\sup_{\substack{z\in M \\ \beta\in\mathbb{N}_0^2,|\beta|\leq m}} p_\alpha(\partial_{\mathbb{C}}^{|\beta|}f(z))e^{-\frac{1}{n}|\operatorname{Re}(z)|} \leq \sup_{\substack{z\in\overline{M} \\ \beta\in\mathbb{N}_0^2,|\beta|\leq m}} p_\alpha(\partial_{\mathbb{C}}^{|\beta|}f(z))e^{-\frac{1}{n}|\operatorname{Re}(z)|} < \infty.$$

Thus $C(f)<\infty$ by (13) and therefore $|\overline{\partial}(\varphi f)|_{\partial\Omega_2,n,m,\alpha}<\infty$ for all $n\in\mathbb{N}$, $n\geq2$, $m\in\mathbb{N}_0$ and $\alpha\in\mathfrak{A}$ by (12), implying $\overline{\partial}(\varphi f)\in\mathcal{E}^{exp}(\overline{\mathbb{C}}\smallsetminus\partial\Omega_2,E)$. As E is admissible, there is $g\in\mathcal{E}^{exp}(\overline{\mathbb{C}}\smallsetminus\partial\Omega_2,E)$ such that

$$\overline{\partial}g=\overline{\partial}(\varphi f). \tag{15}$$

(iii) We set $f_1:=(1-\varphi)f+g$ and $f_2:=\varphi f-g$. It remains to be proved that $f_1\in\mathcal{O}^{exp}(\overline{\mathbb{C}}\smallsetminus(\overline{\Omega}_1\smallsetminus\Omega_2),E)$ and $f_2\in\mathcal{O}^{exp}(\overline{\mathbb{C}}\smallsetminus\overline{\Omega}_2,E)$. The proof is quite similar to part (ii). f_1 is defined on $\mathbb{C}\smallsetminus(\overline{\Omega}_1\smallsetminus\Omega_2)$ (by setting $(1-\varphi)f:=0$ on $\Omega_2\cap\mathbb{R}$) and can be regarded as an element of $\mathcal{O}(\mathbb{C}\smallsetminus(\overline{\Omega}_1\smallsetminus\Omega_2),E)$ due to (15).

Let $n\in\mathbb{N}$, $n\geq2$, and set $S(n):=S_n(\overline{\Omega}_1\smallsetminus\Omega_2)\smallsetminus V_1$. Remark that $S_n(\overline{\Omega}_1\smallsetminus\Omega_2)\subset S_n(\partial\Omega_2)$ and

$$S(n)=\underbrace{\Big[S(n)\cap\{z\in\mathbb{C}\,|\,|\operatorname{Im}(z)|>\tfrac{1}{2n}\}\Big]}_{\subset S_{2n}(\overline{\Omega}_1)}\cup\underbrace{\Big[S(n)\smallsetminus\{z\in\mathbb{C}\,|\,|\operatorname{Im}(z)|>\tfrac{1}{2n}\}\Big]}_{=:M}.$$
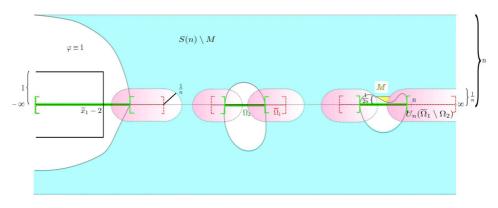

FIGURE 6. case: $\pm\infty\in\overline{\Omega}_1$, $-\infty\in\Omega_2$, $\infty\notin\overline{\Omega}_2$

For $\alpha\in\mathfrak{A}$ we have by the choice of φ

$$|f_1|_{\overline{\Omega}_1\smallsetminus\Omega_2,n,\alpha}=\sup_{z\in S_n(\overline{\Omega}_1\smallsetminus\Omega_2)}p_\alpha(f_1(z))e^{-\frac{1}{n}|\operatorname{Re}(z)|}$$

$$\leq \underbrace{\sup_{z \in S_n(\partial \Omega_2)} p_\alpha(g(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|}}_{=|g|_{\partial \Omega_2, n, 0, \alpha}} + \sup_{z \in S_n(\overline{\Omega}_1 \smallsetminus \Omega_2)} p_\alpha((1-\varphi)f(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|}$$

$$= |g|_{\partial \Omega_2, n, 0, \alpha} + \sup_{z \in S(n)} \underbrace{|1 - \varphi(z)|}_{\leq 1} p_\alpha(f(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|}$$

$$\leq |g|_{\partial \Omega_2, n, 0, \alpha} + \underbrace{\sup_{z \in S_{2n}(\overline{\Omega}_1)} p_\alpha(f(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|}}_{=|f|_{\overline{\Omega}_1, 2n, \alpha}} + \sup_{z \in M} p_\alpha(f(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|}$$

$$= |g|_{\partial \Omega_2, n, 0, \alpha} + |f|_{\overline{\Omega}_1, 2n, \alpha} + \sup_{z \in M} p_\alpha(f(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|}. \tag{16}$$

Again, we have to take a closer look at the set M. First, we observe that

$$\mathbb{R} \cap \overline{\Omega}_1 \subset \Big[\underbrace{V_1}_{\supset \mathbb{R} \cap \Omega_2} \cup \bigcup_{x \in (\overline{\Omega}_1 \smallsetminus \Omega_2) \cap \mathbb{R}} \mathbb{D}_{\frac{1}{n}}(x) \Big] =: V.$$

$V \subset \mathbb{R}^2$ is open and so we get by the definition of the set M

$$\overline{M} \subset \overline{V^C} = V^C \subset (\mathbb{R}^2 \smallsetminus \overline{\Omega}_1).$$

Like in part (ii) the set M is bounded because the real part is bounded with $|\operatorname{Re}(z)| \leq \max(|\widetilde{x}_1 - 2|, n)$ for all $z \in M$. Since f is continuous on $\mathbb{R}^2 \smallsetminus \overline{\Omega}_1$, we obtain

$$\sup_{z \in M} p_\alpha(f(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|} < \infty.$$

Thus we get $|f_1|_{\overline{\Omega}_1 \smallsetminus \Omega_2, n, \alpha} < \infty$ for every $n \in \mathbb{N}$ and $\alpha \in \mathfrak{A}$ by (16), implying $f_1 \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus (\overline{\Omega}_1 \smallsetminus \Omega_2), E)$.

f_2 is defined on $\mathbb{C} \smallsetminus \overline{\Omega}_2$ (by setting $\varphi f := 0$ on $\overline{\Omega}_1 \smallsetminus \overline{\Omega}_2$) and can be regarded as an element of $\mathcal{O}(\mathbb{C} \smallsetminus \overline{\Omega}_2, E)$ due to (15). Let $n \in \mathbb{N}$, $n \geq 2$. We set $S(n) := S_n(\overline{\Omega}_2) \smallsetminus V_0$ and remark that $S_n(\overline{\Omega}_2) \subset S_n(\partial \Omega_2)$ as well as

$$S(n) = \underbrace{\Big[S(n) \cap \{z \in \mathbb{C} \mid |\operatorname{Im}(z)| > \tfrac{1}{2n}\} \Big]}_{\subset S_{2n}(\overline{\Omega}_1)} \cup \underbrace{\Big[S(n) \smallsetminus \{z \in \mathbb{C} \mid |\operatorname{Im}(z)| > \tfrac{1}{2n}\} \Big]}_{=:M}.$$
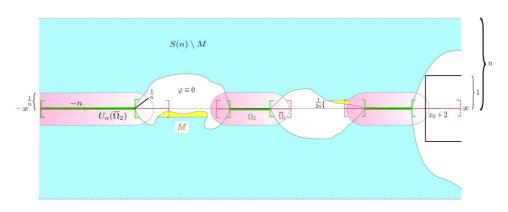

FIGURE 7. case: $\pm\infty \in \overline{\Omega}_1$, $-\infty \in \Omega_2$, $\infty \notin \overline{\Omega}_2$

For $\alpha \in \mathfrak{A}$ we have by the choice of φ

$$|f_2|_{\overline{\Omega}_2, n, \alpha} = \sup_{z \in S_n(\overline{\Omega}_2)} p_\alpha(f_2(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|}$$

$$\leq \underbrace{\sup_{z\in S_n(\partial\Omega_2)} p_\alpha(g(z))e^{-\frac{1}{n}|\operatorname{Re}(z)|}}_{=|g|_{\partial\Omega_2,n,0,\alpha}} + \sup_{z\in S_n(\overline{\Omega}_2)} p_\alpha(\varphi f(z))e^{-\frac{1}{n}|\operatorname{Re}(z)|}$$

$$= |g|_{\partial\Omega_2,n,0,\alpha} + \sup_{z\in S(n)} \underbrace{|\varphi(z)|}_{\leq 1} p_\alpha(f(z))e^{-\frac{1}{n}|\operatorname{Re}(z)|}$$

$$\leq |g|_{\partial\Omega_2,n,0,\alpha} + \underbrace{\sup_{z\in S_{2n}(\overline{\Omega}_1)} p_\alpha(f(z))e^{-\frac{1}{n}|\operatorname{Re}(z)|}}_{=|f|_{\overline{\Omega}_1,2n,\alpha}} + \sup_{z\in M} p_\alpha(f(z))e^{-\frac{1}{n}|\operatorname{Re}(z)|}$$

$$= |g|_{\partial\Omega_2,n,0,\alpha} + |f|_{\overline{\Omega}_1,2n,\alpha} + \sup_{z\in M} p_\alpha(f(z))e^{-\frac{1}{n}|\operatorname{Re}(z)|}. \tag{17}$$

Again, we have to take a closer look at the set M and observe that

$$\mathbb{R}\subset\Big[\underbrace{V_0}_{\supset\mathbb{R}\smallsetminus\overline{\Omega}_2}\cup\bigcup_{x\in\overline{\Omega}_2\cap\mathbb{R}}\mathbb{D}_{\frac{1}{n}}(x)\Big]=:V.$$

$V\subset\mathbb{R}^2$ is open and so we get by the definition of the set M

$$\overline{M}\subset\overline{V^C}=V^C\subset(\mathbb{R}^2\smallsetminus\mathbb{R}).$$

Like before the set M is bounded because the real part is bounded with $|\operatorname{Re}(z)|\leq \max(|-n|,|x_0+2|)$ for all $z\in M$. Again, we gain

$$\sup_{z\in M} p_\alpha(f(z))e^{-\frac{1}{n}|\operatorname{Re}(z)|}<\infty$$

and thus get $|f_2|_{\overline{\Omega}_2,n,\alpha}<\infty$ for every $n\in\mathbb{N}$, $n\geq 2$, and $\alpha\in\mathfrak{A}$ by (17), implying $f_2\in\mathcal{O}^{exp}(\overline{\mathbb{C}}\smallsetminus\overline{\Omega}_2,E)$. Obviously $f_1+f_2=f$, completing the proof by part (i). $\square$

Ito (see [29, p. 15, l. 16]) states that Lemma 5.2 is valid for any $\mathbb{C}$-lcHs E, but he does not prove that I is surjective. Nevertheless, he states as an open problem (see [29, Problem A, p. 17]) whether for two compact sets $K_1,K_2\subset\overline{\mathbb{R}}$ the map

$$\mathcal{L}:L(\mathcal{P}_*(K_1),E)\times L(\mathcal{P}_*(K_2),E)\to L(\mathcal{P}_*(K_1\cup K_2),E),$$

given by $\mathcal{L}(T_1,T_2):=T_1-T_2$, is surjective for non-Fréchet spaces E.

5.3. Remark. Let $\Omega_2\subset\Omega_1\subset\overline{\mathbb{R}}$ be open and E an $\mathbb{C}$-lcHs. Then the following assertions are equivalent:

a) The canonical map

$$I:L(\mathcal{P}_*(\overline{\Omega}_2),E)/L(\mathcal{P}_*(\partial\Omega_2),E)\to L(\mathcal{P}_*(\overline{\Omega}_1),E)/L(\mathcal{P}_*(\overline{\Omega}_1\smallsetminus\Omega_2),E)$$

is an algebraic isomorphism.

b) The map

$$\mathcal{L}:L(\mathcal{P}_*(\overline{\Omega}_1\smallsetminus\Omega_2),E)\times L(\mathcal{P}_*(\overline{\Omega}_2),E)\to L(\mathcal{P}_*(\overline{\Omega}_1),E)$$

is surjective.

Proof. I is obviously surjective if and only if $\mathcal{L}$ is surjective. Moreover, I is always linear and injective by Proposition 3.7 a). $\square$

The corresponding issue in Ito's paper [28] on vector-valued hyperfunctions (see [28, p. 34, l. 2, Problem A, p. 35]) was pointed out by Domański and Langenbruch in [7, Remark 6.3, p. 1123]. Using Lemma 5.2, we can define the restrictions on $\mathcal{R}(\Omega,E)$, if E is admissible, as follows.

5.4. Definition. Let E be admissible. For open sets $\Omega_2 \subset \Omega_1 \subset \overline{\mathbb{R}}$, $\Omega_2 \neq \varnothing$, we denote by

$$q \colon L(\mathcal{P}_*(\overline{\Omega}_1), E)/L(\mathcal{P}_*(\partial\Omega_1), E) \to L(\mathcal{P}_*(\overline{\Omega}_1), E)/L(\mathcal{P}_*(\overline{\Omega}_1 \setminus \Omega_2), E)$$

the canonical quotient map. We define the restriction maps via Lemma 5.2 by

$$R_{\Omega_1,\Omega_2} \colon \mathcal{R}(\Omega_1, E) \to \mathcal{R}(\Omega_2, E), \ R_{\Omega_1,\Omega_2}([T]) := [T]_{|\Omega_2} := I^{-1}\big(q([T])\big),$$

and for an open set $\Omega_1 \subset \overline{\mathbb{R}}$

$$R_{\Omega_1,\varnothing} \colon \mathcal{R}(\Omega_1, E) \to \mathcal{R}(\varnothing, E), \ R_{\Omega_1,\varnothing}([T]) := [T]_{|\varnothing} := 0.$$

The next lemma is the counterpart of [7, Lemma 6.5, p. 1124].

5.5. Lemma. *Let E be admissible, $\Omega \subset \overline{\mathbb{R}}$ open and set $\mathcal{R}_\Omega(E) := \{\mathcal{R}(\omega, E) \mid \omega \subset \Omega \text{ open}\}$. Then $\mathcal{R}_\Omega(E)$, equipped with the restrictions of Definition 5.4, forms a presheaf on Ω, satisfying the condition (S1):*
For every family of open sets $\{\omega_j \subset \Omega \mid j \in J\}$ with $\omega := \bigcup_{j \in J} \omega_j$ holds: If $[T] \in \mathcal{R}(\omega, E)$ such that $R_{\omega,\omega_j}([T]) = 0$ for all $j \in J$, then $[T] = 0$.

Proof. (i) We begin with the proof that $\mathcal{R}_\Omega(E)$ with its restrictions is a presheaf. We clearly have $R_{\omega,\omega} = \mathrm{id}_{\mathcal{R}(\omega,E)}$. Let $\omega_3 \subset \omega_2 \subset \omega_1 \subset \Omega$ be open. We have to show that $R_{\omega_2,\omega_3} \circ R_{\omega_1,\omega_2} = R_{\omega_1,\omega_3}$ is valid. This is obvious if one of the sets is empty, so let them all be non-empty. Let $T \in L(\mathcal{P}_*(\overline{\omega}_1), E)$. Let $T_0 \in L(\mathcal{P}_*(\overline{\omega}_3), E)$ be a representative of $R_{\omega_1,\omega_3}([T]_1)$, let $T_1 \in L(\mathcal{P}_*(\overline{\omega}_2), E)$ be a representative of $R_{\omega_1,\omega_2}([T]_1)$ and $T_2 \in L(\mathcal{P}_*(\overline{\omega}_3), E)$ a representative of $R_{\omega_2,\omega_3} \circ R_{\omega_1,\omega_2}([T]_1) = R_{\omega_2,\omega_3}([T_1]_2)$. By the definition of the restrictions the following is true:

(1) $T_0 - T \in L(\mathcal{P}_*(\overline{\omega}_1 \setminus \omega_3), E)$,
(2) $T_1 - T \in L(\mathcal{P}_*(\overline{\omega}_1 \setminus \omega_2), E)$,
(3) $T_2 - T_1 \in L(\mathcal{P}_*(\overline{\omega}_2 \setminus \omega_3), E)$.

First, we observe that

$$T_0 - T_2 \in L(\mathcal{P}_*(\overline{\omega}_3), E). \tag{18}$$

It remains to be shown that $T_0 - T_2 \in L(\mathcal{P}_*(\partial\omega_3), E)$. The equality

$$T_0 - T_2 = (T_0 - T) + (T - T_1) + (T_1 - T_2)$$

holds on $\mathcal{P}_*(\overline{\mathbb{R}})$ and the right-hand side is an element of

$$L(\mathcal{P}_*(\overline{\omega}_1 \setminus \omega_3) \cap \mathcal{P}_*(\overline{\omega}_1 \setminus \omega_2) \cap \mathcal{P}_*(\overline{\omega}_2 \setminus \omega_3), E) = L(\mathcal{P}_*(\overline{\omega}_1 \setminus \omega_3), E)$$

by (1)-(3) and as $\omega_3 \subset \omega_2 \subset \omega_1$. So due to the remark above Proposition 3.7, $T_0 - T_2$ can also be regarded as an element of $L(\mathcal{P}_*(\overline{\omega}_1 \setminus \omega_3), E)$ and thus we get by Proposition 3.7 a) and (18)

$$T_0 - T_2 \in L(\mathcal{P}_*(\overline{\omega}_3), E) \cap L(\mathcal{P}_*(\overline{\omega}_1 \setminus \omega_3), E) = L(\mathcal{P}_*(\overline{\omega}_3 \cap (\overline{\omega}_1 \setminus \omega_3)), E)$$

$$= L(\mathcal{P}_*(\partial\omega_3), E).$$

(ii) Let T be like in (S1) and $j \in J$. Then for a representative T_j of $R_{\omega,\omega_j}([T])$ it holds $T_j \in L(\mathcal{P}_*(\partial\omega_j), E)$, since $R_{\omega,\omega_j}([T]) = 0$, and $T - T_j \in L(\mathcal{P}_*(\overline{\omega} \setminus \omega_j), E)$ by the definition of the restriction. Again, the equality

$$T = (T - T_j) + T_j$$

holds on $\mathcal{P}_*(\overline{\mathbb{R}})$ and the right-hand side is an element of

$$L(\mathcal{P}_*(\overline{\omega} \setminus \omega_j) \cap \mathcal{P}_*(\partial\omega_j), E) = L(\mathcal{P}_*(\overline{\omega} \setminus \omega_j), E).$$

By the same argument as in part (i), we can regard T as an element of $L(\mathcal{P}_*(\overline{\omega} \setminus \omega_j), E)$ and get

$$\operatorname{supp} T \subset \overline{\omega} \setminus \omega_j$$

where the support $\operatorname{supp} T$ is meant in the sense of Proposition 3.7 b). Since this is valid for all $j \in J$, we obtain

$$\operatorname{supp} T \subset \bigcap_{j \in J} (\overline{\omega} \smallsetminus \omega_j) = \overline{\omega} \smallsetminus \bigcup_{j \in J} \omega_j = \overline{\omega} \smallsetminus \omega = \partial\omega$$

and thus $T \in L(\mathcal{P}_*(\partial\omega), E)$, i.e. $[T] = 0$. $\qquad\square$

For the special case $\Omega = \overline{\mathbb{R}}$ we use the notation $\mathcal{R}(E) \coloneqq \mathcal{R}_{\overline{\mathbb{R}}}(E)$. We will see that the presheaf $\mathcal{R}_\Omega(E)$, which satisfies $(S1)$, is already a sheaf, so satisfies, in addition, the sheaf condition $(S2)$ if we assume that E is not only admissible, but strictly admissible. For this purpose we introduce a boundary value representation of $\mathcal{R}(E)$ in the following way. Let $\Omega \subset \overline{\mathbb{R}}$, $\Omega \neq \varnothing$, be an open set and we define

$$\mathcal{U}(\Omega) \coloneqq \{U \mid U \subset \overline{\mathbb{C}} \text{ open}, \ U \cap \overline{\mathbb{R}} = \Omega\}.$$

Now, we define, similar to Definition 3.1, spaces of vector-valued slowly increasing holomorphic functions on $U \smallsetminus \overline{\mathbb{R}}$ resp. U for $U \in \mathcal{U}(\Omega)$.

If $-\infty \in \Omega$ or $\infty \in \Omega$, we define

$$\mathcal{O}^{exp}(U \smallsetminus \overline{\mathbb{R}}, E) \coloneqq \{f \in \mathcal{O}((U \smallsetminus \overline{\mathbb{R}}) \cap \mathbb{C}, E) \mid \forall\, n \in \mathbb{N},\, n \geq 2,\, \alpha \in \mathfrak{A}: \ \|f\|_{U^*, n, \alpha} < \infty\}$$

where

$$\|f\|_{U^*, n, \alpha} \coloneqq \sup_{z \in S_n(U)} p_\alpha(f(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|}$$

and

$S_n(U)$

$$\coloneqq \begin{cases} U \cap \{z \in \mathbb{C} \mid \frac{1}{n} < |\operatorname{Im}(z)| < n, \ \operatorname{Re}(z) > -n, \ \mathrm{d}(z, \mathbb{C} \cap \partial U) > \frac{1}{n}\} & , \ -\infty \notin \Omega, \ \infty \in \Omega, \\ U \cap \{z \in \mathbb{C} \mid \frac{1}{n} < |\operatorname{Im}(z)| < n, \ \operatorname{Re}(z) < n, \ \mathrm{d}(z, \mathbb{C} \cap \partial U) > \frac{1}{n}\} & , \ -\infty \in \Omega, \ \infty \notin \Omega, \\ U \cap \{z \in \mathbb{C} \mid \frac{1}{n} < |\operatorname{Im}(z)| < n, \ \mathrm{d}(z, \mathbb{C} \cap \partial U) > \frac{1}{n}\} & , \ \pm\infty \in \Omega. \end{cases}$$
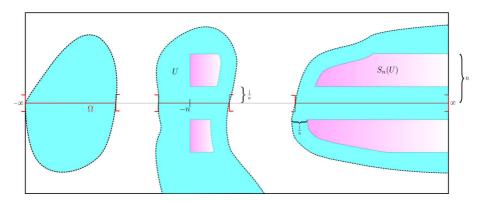

FIGURE 8. $S_n(U)$ for $\infty \in \Omega$, $-\infty \notin \Omega$

If $\pm\infty \notin \Omega$, we define

$$\mathcal{O}^{exp}(U \smallsetminus \overline{\mathbb{R}}, E) \coloneqq \mathcal{O}((U \smallsetminus \overline{\mathbb{R}}) \cap \mathbb{C}, E).$$

If $-\infty \in \Omega$ or $\infty \in \Omega$, we define

$$\mathcal{O}^{exp}(U, E) \coloneqq \{f \in \mathcal{O}(U \cap \mathbb{C}, E) \mid \forall\, n \in \mathbb{N},\, n \geq 2,\, \alpha \in \mathfrak{A}: \ \|f\|_{U, n, \alpha} < \infty\}$$

where

$$\|f\|_{U, n, \alpha} \coloneqq \sup_{z \in T_n(U)} p_\alpha(f(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|}$$

and

$T_n(U)$

$$:= \begin{cases} U \cap \{z \in \mathbb{C} \mid |\operatorname{Im}(z)| < n, \ \operatorname{Re}(z) > -n, \ \mathrm{d}(z, \mathbb{C} \cap \partial U) > \frac{1}{n}\} & , \ -\infty \notin \Omega, \ \infty \in \Omega, \\ U \cap \{z \in \mathbb{C} \mid |\operatorname{Im}(z)| < n, \ \operatorname{Re}(z) < n, \ \mathrm{d}(z, \mathbb{C} \cap \partial U) > \frac{1}{n}\} & , \ -\infty \in \Omega, \ \infty \notin \Omega, \\ U \cap \{z \in \mathbb{C} \mid |\operatorname{Im}(z)| < n, \ \mathrm{d}(z, \mathbb{C} \cap \partial U) > \frac{1}{n}\} & , \ \pm\infty \in \Omega. \end{cases}$$

If $\pm\infty \notin \Omega$, we define

$$\mathcal{O}^{exp}(U, E) := \mathcal{O}(U \cap \mathbb{C}, E).$$

We remark that $\mathcal{O}^{exp}(U \smallsetminus \overline{\mathbb{R}}, E)$ and $\mathcal{O}^{exp}(U, E)$ for complete E are complete $\mathbb{C}$-lcHs by [48, 3.7 Proposition, p. 7] if $-\infty \in \Omega$ or $\infty \in \Omega$. If $\pm\infty \notin \Omega$, then this is obviously valid for the corresponding spaces as well if equipped with the topology of uniform convergence on compact subsets. Moreover, if $U = \overline{\mathbb{C}}$, so $\Omega = \overline{\mathbb{R}}$, then the definition of $\mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\mathbb{R}}, E)$ and $\mathcal{O}^{exp}(\overline{\mathbb{C}}, E)$ in the just introduced sense coincides with the one in the sense of Definition 3.1 (and therefore the spaces have the same symbol).

5.6. Definition. For an open set $\Omega \subset \overline{\mathbb{R}}$, $\Omega \neq \varnothing$, and $U \in \mathcal{U}(\Omega)$ we define the space of boundary values by

$$bv(\Omega, E) := \mathcal{O}^{exp}(U \smallsetminus \overline{\mathbb{R}}, E) / \mathcal{O}^{exp}(U, E)$$

and $bv(\varnothing, E) := 0$.

The counterpart of the next observation in the context of vector-valued hyperfunctions can be found in [7, Lemma 6.7, p. 1124].

5.7. Lemma. *Let $\Omega \subset \overline{\mathbb{R}}$ be non-empty and open and E admissible. The definition of $bv(\Omega, E)$ is independent of the choice of $U \in \mathcal{U}(\Omega)$ and for every $f \in bv(\Omega, E)$ there is $F \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\Omega}, E)$ such that $f = [F]$.*

Proof. Let $U, U_1 \in \mathcal{U}(\Omega)$, w.l.o.g. $U_1 := (\overline{\mathbb{C}} \smallsetminus \overline{\mathbb{R}}) \cup \Omega$. Then $U \subset U_1$. The canonical map

$$J \colon \mathcal{O}^{exp}(U_1 \smallsetminus \overline{\mathbb{R}}, E) / \mathcal{O}^{exp}(U_1, E) \to \mathcal{O}^{exp}(U \smallsetminus \overline{\mathbb{R}}, E) / \mathcal{O}^{exp}(U, E), \ [f] \mapsto [f_{|(U \smallsetminus \overline{\mathbb{R}}) \cap \mathbb{C}}],$$

is well-defined since $\mathcal{O}^{exp}(U_1, E) \subset \mathcal{O}^{exp}(U, E)$.

Let $f \in \mathcal{O}^{exp}(U_1 \smallsetminus \overline{\mathbb{R}}, E)$ with $[f_{|(U \smallsetminus \overline{\mathbb{R}}) \cap \mathbb{C}}] = 0$, i.e. $f_{|(U \smallsetminus \overline{\mathbb{R}}) \cap \mathbb{C}} \in \mathcal{O}^{exp}(U, E)$. Then

$$f \in \mathcal{O}^{exp}((U_1 \smallsetminus \overline{\mathbb{R}}) \cup U, E) = \mathcal{O}^{exp}(U_1, E)$$

and therefore $[f] = 0$, yielding the injectivity of J.

The proof of surjectivity resembles the one of Lemma 5.2, but it is sometimes necessary to use two cut-off functions. We restrict to the case that $\infty \in \Omega$ and $-\infty \in \partial\Omega$. For the similar treatment of the other cases we refer to the proof of [46, 6.8 Lemma, p. 118].

(i) There are $\widetilde{x}_0 \in \mathbb{R}$, w.l.o.g. $\widetilde{x}_0 \geq 0$, and $\varepsilon_0 > 0$ such that $[\widetilde{x}_0, \infty] \subset \Omega$ and $[\widetilde{x}_0, \infty] \times [-\varepsilon_0, \varepsilon_0] \subset U$ since $\infty \in \Omega$, Ω is open and $U \in \mathcal{U}(\Omega)$. We define the sets

$$F_0 := (U^C \cap \mathbb{R}^2) \cup \left[\mathbb{R} \times \left(\mathbb{R} \smallsetminus \left(-\tfrac{\varepsilon_0}{2}, \tfrac{\varepsilon_0}{2}\right)\right)\right]$$

and

$$F_1 := (\mathbb{R} \cap \overline{\Omega}) \cup \left([\widetilde{x}_0 + 2, \infty) \times \left[-\tfrac{\varepsilon_0}{4}, \tfrac{\varepsilon_0}{4}\right]\right).$$

The sets F_0 and F_1 are non-empty and closed in $\mathbb{R}^2$ and $F_0 \cap F_1 = \mathbb{R} \cap \partial\Omega$. By [15, Corollary 1.4.11, p. 31] there exists $\varphi_0 \in \mathcal{C}^\infty((F_0 \cap F_1)^C) = \mathcal{C}^\infty(\mathbb{R}^2 \smallsetminus \partial\Omega)$, $0 \leq \varphi_0 \leq 1$, such that $\varphi_0 = 0$ on V_0 and $\varphi_0 = 1$ on V_1 where V_0, $V_1 \subset \mathbb{R}^2$ are open and

$$V_0 \supset F_0 \smallsetminus (F_0 \cap F_1) = F_0 \smallsetminus \partial\Omega \supset (\mathbb{R} \smallsetminus \overline{\Omega})$$

and

$$V_1 \supset F_1 \smallsetminus (F_0 \cap F_1) = F_1 \smallsetminus \partial\Omega \supset (\mathbb{R} \cap \Omega)$$

as well as

$$|\partial^\beta \varphi_0(z)| \le C^{|\beta|} \frac{\mathrm{d}(z)^{-|\beta|}}{d_1 \cdots d_{|\beta|}}, \quad z \in \mathbb{R}^2 \smallsetminus \partial\Omega, \, \beta \in \mathbb{N}_0^2, \tag{19}$$

where C, d and (d_n) with $d_1 \cdots d_{|0|} := 1$ are like in part (i) of the proof of Lemma 5.2.
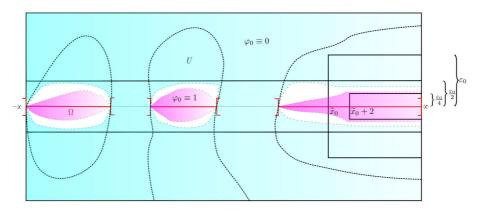

FIGURE 9. case: $\infty \in \Omega, \, -\infty \in \partial\Omega$

Furthermore, we define the sets $K_0 := \{(x,y) \in \mathbb{R}^2 \mid y \le -2e^{-|x|} \text{ or } y \ge 2e^{-|x|}\}$ and $K_1 := \{(x,y) \in \mathbb{R}^2 \mid -e^{-|x|} \le y \le e^{-|x|}\}$ as well as

$$\widetilde{F}_0 := K_0 \cup ([0,\infty) \times [\mathbb{R} \smallsetminus (-2,2)]) \quad \text{and} \quad \widetilde{F}_1 := K_1 \cup ([0,\infty) \times [-1,1]).$$

The sets $\widetilde{F}_0$ and $\widetilde{F}_1$ are non-empty and closed in $\mathbb{R}^2$ and $\widetilde{F}_0 \cap \widetilde{F}_1 = \varnothing$. Like above there is $\varphi_1 \in \mathcal{C}^\infty((\widetilde{F}_0 \cap \widetilde{F}_1)^C) = \mathcal{C}^\infty(\mathbb{R}^2)$, $0 \le \varphi_1 \le 1$, such that $\varphi_1 = 0$ on W_0 and $\varphi_1 = 1$ on W_1 where $W_0, W_1 \subset \mathbb{R}^2$ are open and

$$W_0 \supset \widetilde{F}_0 \smallsetminus (\widetilde{F}_0 \cap \widetilde{F}_1) = \widetilde{F}_0 \quad \text{and} \quad W_1 \supset \widetilde{F}_1 \smallsetminus (\widetilde{F}_0 \cap \widetilde{F}_1) = \widetilde{F}_1$$

as well as

$$|\partial^\beta \varphi_1(z)| \le \widetilde{C}^{|\beta|} \frac{\widetilde{\mathrm{d}}(z)^{-|\beta|}}{d_1 \cdots d_{|\beta|}}, \quad z \in \mathbb{R}^2, \, \beta \in \mathbb{N}_0^2, \tag{20}$$

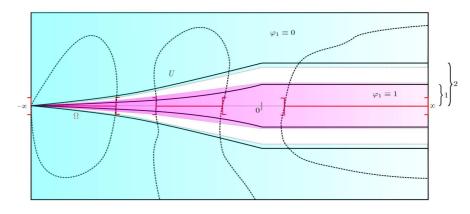where $\widetilde{C}$, $\widetilde{\mathrm{d}}$ and (d_n) are like above.

FIGURE 10. case: $\infty \in \Omega, \, -\infty \in \partial\Omega$

Again, we take a closer look at the right-hand side of (19) resp. (20) and claim that

$$B := \inf_{z \in S_n(\partial\Omega)} \mathrm{d}(z) > 0 \tag{21}$$

and

$$D := \inf_{z \in S_n(\partial\Omega)} \widetilde{\mathrm{d}}(z) > 0 \tag{22}$$

for all $n \in \mathbb{N}$, $n \geq 2$. We begin with (21).

(21.1) For $z \in S_n(\partial\Omega)$ with $\mathrm{Re}(z) \leq -n$ we have

$$\mathrm{d}(z) = \max(\mathrm{d}(z, F_0), \mathrm{d}(z, F_1)) \geq \mathrm{d}(z, F_1) \geq \min\Big(n + \widetilde{x} + 2 - \frac{1}{n}, \frac{1}{n}\Big) \geq \frac{1}{n}.$$

For $z \in S_n(\partial\Omega)$ with $\mathrm{Re}(z) \geq \widetilde{x}_0 + 2$ we have

$$\mathrm{d}(z) \geq \begin{cases} \frac{\varepsilon_0}{4} & , \ z \in F_0, \\ \min(\frac{1}{2}\frac{\varepsilon_0}{4}, 2) & , \ z \notin F_0, \ z \notin F_1, \\ \min(\frac{\varepsilon_0}{4}, 2) & , \ z \in F_1, \end{cases} \geq \min\Big(\frac{\varepsilon_0}{8}, 2\Big).$$

(21.2) For $z \in S_n(\partial\Omega)$ with $\mathrm{Re}(z) \leq \widetilde{x}_0$ and $|\mathrm{Im}(z)| \geq \frac{1}{n}$ we get

$$\mathrm{d}(z) \geq \mathrm{d}(z, F_1) \geq \min\Big(2, \frac{1}{n}\Big) = \frac{1}{n}.$$

(21.3) By [54, 3.5 Remark a), p. 9] the set $U_n(\partial\Omega)$ has finitely many components Z_j, so there exists $k \in \mathbb{N}$ with $U_n(\partial\Omega) = \bigcup_{j=1}^{k} Z_j$. Since $-\infty \in \partial\Omega$ and $\infty \notin \Omega$, all but one Z_j are bounded. Denote by Z_1 the unbounded component and by Z_j, $2 \leq j \leq k$, the bounded components if $k \geq 2$. Let $a_j := \min(Z_j \cap \partial\Omega)$ and $b_j := \max(Z_j \cap \partial\Omega)$, $2 \leq j \leq k$, if $k \geq 2$. If $\max(Z_1 \cap \partial\Omega)$ exists, we set $b_1 := \max(Z_1 \cap \partial\Omega)$, and if $\max(Z_1 \cap \partial\Omega)$ does not exist, implying $\partial\Omega = \{-\infty\}$, $\Omega = (-\infty, \infty]$ and $Z_1 = (-\infty, n] \times [-\frac{1}{n}, \frac{1}{n}]$, then we set $b_1 := -n - \frac{1}{n}$. We observe that $b_k = \max_{1 \leq j \leq k} b_j < \widetilde{x}_0$.

Let $k \geq 2$. W.l.o.g. $a_j < a_{j+1}$ for $2 \leq j \leq k$ (otherwise renumber). Due to [54, 3.5 Remark b)(i), (iii), (v), p. 9] there is $0 < r_j < \frac{1}{n}$ such that $\{z \in \mathbb{C} \mid \mathrm{d}(z, (-\infty, b_1]) \leq r_1\} \subset Z_1$ $\{z \in \mathbb{C} \mid \mathrm{d}(z, [a_j, b_j]) \leq r_j\} \subset Z_j$ for all $2 \leq j \leq k$ (see Figure 3). Since $k \geq 2$, we have $a_1 := -n < b_k$. Let $z \in S_n(\partial\Omega)$ such that $a_1 = -n < \mathrm{Re}(z) < b_k$ and $|\mathrm{Im}(z)| < \frac{1}{n}$. If $a_j < b_j$ for some $2 \leq j \leq k$, we obtain for z with $a_j < \mathrm{Re}(z) < b_j$

$$\mathrm{d}(z) \geq \mathrm{d}(z, F_1) \geq r_j.$$

Now, we consider $z \in S_n(\partial\Omega)$ with $b_j < \mathrm{Re}(z) < a_{j+1}$ for $1 \leq j \leq k-1$ and $|\mathrm{Im}(z)| < \frac{1}{n}$. If $\mathrm{d}(z) \leq \frac{1}{2n}$, we have with $N_0 := \{w \in \mathbb{C} \mid |\mathrm{Im}(w)| > \frac{3}{2n}\}$ and $N_1 := \{w \in \mathbb{C} \mid \mathrm{d}(\mathbb{R} \cap \partial\Omega) < \frac{1}{3n}\}$ that

$$\mathrm{d}(z, F_1) = \mathrm{d}(z, F_1 \smallsetminus N_1) = \mathrm{d}\big(z, \underbrace{([b_j, a_{j+1}] \cap \overline{\Omega}) \smallsetminus N_1}_{=:K_{1,j}}\big)$$

and

$$\begin{aligned} \mathrm{d}(z, F_0) &= \mathrm{d}(z, F_0 \smallsetminus (N_0 \cup N_1)) \\ &= \mathrm{d}\big(z, \underbrace{(F_0 \cap \{w \in \mathbb{C} \mid b_j \leq \mathrm{Re}(w) \leq a_{j+1}\}) \smallsetminus (N_0 \cup N_1)}_{=:K_{0,j}}\big) \end{aligned}$$

because $\frac{1}{n} - \frac{1}{3n} > \frac{1}{2n}$ and $\frac{1}{n} + \frac{1}{2n} = \frac{3}{2n}$. The sets $K_{0,j}$ and $K_{1,j}$ are bounded closed sets in $\mathbb{R}^2$, thus compact, and disjoint. Hence $c_j := \mathrm{d}(K_{0,j}, K_{1,j}) > 0$, yielding to

$$\mathrm{d}(z) = \max_{i \in \{0,1\}} \mathrm{d}(z, K_{i,j}) \geq \frac{c_j}{2} > 0$$

for all $1 \leq j \leq k - 1$. Combining these results, we obtain

$$\mathrm{d}(z) \geq \min\Big(\min_{1 \leq j \leq k, a_j \neq b_j} r_j, \ \min_{1 \leq j \leq k-1} \frac{c_j}{2}, \frac{1}{2n} \Big) > 0$$

for $z \in S_n(\partial\Omega)$ with $|\mathrm{Im}(z)| \leq \frac{1}{n}$ and $a_1 < \mathrm{Re}(z) < b_k$ with $k \geq 2$.

(21.4) Let $k \in \mathbb{N}$, $z \in S_n(\partial\Omega)$ such that $b_k \leq \operatorname{Re}(z) < \widetilde{x}_0 + 2$. If $\mathrm{d}(z) \leq \frac{1}{2n}$, we get with $N_2 := \{w \in \mathbb{C} \mid \operatorname{Re}(w) > \widetilde{x}_0 + 2 + \frac{1}{2n}\}$ and $N_3 := \{w \in \mathbb{C} \mid |\operatorname{Im}(w)| > n + \frac{1}{2n}$ or $\operatorname{Re}(w) < b_k\}$

$$\mathrm{d}(z, F_1) = \mathrm{d}(z, F_1 \smallsetminus (\mathbb{D}_{\frac{1}{3n}}(b_k) \cup N_2))$$
$$= \mathrm{d}(z, \underbrace{\{w \in F_1 \mid \operatorname{Re}(w) \geq b_k\} \smallsetminus (\mathbb{D}_{\frac{1}{3n}}(b_k) \cup N_2)}_{=: \widetilde{K}_1})$$

as well as

$$\mathrm{d}(z, F_0) = \mathrm{d}(z, \underbrace{F_0 \smallsetminus (\mathbb{D}_{\frac{1}{3n}}(b_k) \cup N_2 \cup N_3)}_{=: \widetilde{K}_0}).$$

$\widetilde{K}_0$ and $\widetilde{K}_1$ are compact and disjoint. Thus we have $c_0 := \mathrm{d}(\widetilde{K}_0, \widetilde{K}_1) > 0$, implying

$$\mathrm{d}(z) = \max_{i \in \{0,1\}} \mathrm{d}(z, \widetilde{K}_i) \geq \frac{c_0}{2} > 0.$$

(21.5) Merging (21.1)-(21.4), we gain

$$B = \inf_{z \in S_n(\partial\Omega)} \mathrm{d}(z) \geq \min\left(\frac{1}{n}, \min\left(\frac{\varepsilon_0}{8}, 2\right), \min\left(\min_{j \in J} r_j, \min_{1 \leq j \leq k-1} \frac{c_j}{2}, \frac{1}{2n}\right), \frac{c_0}{2}\right) > 0$$

if $k \geq 2$ and $J := \{j \in \mathbb{N} \mid j \leq k, a_j < b_j\} \neq \varnothing$. If $J = \varnothing$ resp. $k = 1$, then the $\min_{j \in J}$-term resp. the $\min_{1 \leq j \leq k-1}$-term does not appear in the estimate above.

Let us turn to (22).

(22.1) For $z \in S_n(\partial\Omega)$ with $\operatorname{Re}(z) \geq 1$ we have

$$\widetilde{\mathrm{d}}(z) = \max(\mathrm{d}(z, \widetilde{F}_0), \mathrm{d}(z, \widetilde{F}_1)) \geq \left\{ \begin{array}{ll} 1 & , z \in \widetilde{F}_0, \\ \min(\frac{1}{2}, 1) & , z \notin \widetilde{F}_0, z \notin \widetilde{F}_1, \\ \min(1, 1) & , z \in \widetilde{F}_1, \end{array} \right\} \geq \frac{1}{2}.$$

(22.2) Let $z \in S_n(\partial\Omega)$ such that $0 \leq \operatorname{Re}(z) < 1$. If $\widetilde{\mathrm{d}}(z) \leq \frac{1}{2n}$, then

$$\mathrm{d}(z, \widetilde{F}_0) = \mathrm{d}(z, \widetilde{F}_0 \smallsetminus (N_0 \cup N_1)) \quad \text{and} \quad \mathrm{d}(z, \widetilde{F}_1) = \mathrm{d}(z, \widetilde{F}_1 \smallsetminus N_1)$$

where $N_0 := \{w \in \mathbb{C} \mid |\operatorname{Im}(w)| > n + \frac{1}{2n}\}$ and $N_1 := \{w \in \mathbb{C} \mid \operatorname{Re}(w) < -\frac{1}{n}$ or $\operatorname{Re}(w) > 1 + \frac{1}{n}\}$. The sets $\widetilde{F}_0 \smallsetminus (N_0 \cup N_1)$ and $\widetilde{F}_1 \smallsetminus N_1$ are compact and disjoint, thus we gain $c_0 := \mathrm{d}(\widetilde{F}_0 \smallsetminus (N_0 \cup N_1), \widetilde{F}_1 \smallsetminus N_1) > 0$ and therefore

$$\widetilde{\mathrm{d}}(z) \geq \frac{c_0}{2} > 0.$$

(22.3) Let $z \in S_n(\partial\Omega)$ with $\operatorname{Re}(z) < 0$. If $\widetilde{\mathrm{d}}(z) \leq \frac{1}{2n}$, then

$$\mathrm{d}(z, \widetilde{F}_0) = \mathrm{d}(z, \widetilde{F}_0 \smallsetminus (N_0 \cup N_2)) \quad \text{and} \quad \mathrm{d}(z, \widetilde{F}_1) = \mathrm{d}(z, \widetilde{F}_1 \smallsetminus N_2)$$

with N_0 from (22.2) and $N_2 := \{w \in \mathbb{C} \mid (|\operatorname{Im}(w)| < \frac{1}{3n}$ and $\operatorname{Re}(w) < -n - \frac{1}{2n})$ or $\operatorname{Re}(w) > \frac{1}{n}\}$. The sets $\widetilde{F}_0 \smallsetminus (N_0 \cup N_2)$ and $\widetilde{F}_1 \smallsetminus N_2$ are compact and disjoint, so we obtain $c_1 := \mathrm{d}(\widetilde{F}_0 \smallsetminus (N_0 \cup N_2), \widetilde{F}_1 \smallsetminus N_2) > 0$ and hence $\widetilde{\mathrm{d}}(z) \geq \frac{c_1}{2} > 0$.

(22.4) By combining these results, we have

$$D = \inf_{z \in S_n(\partial\Omega)} \widetilde{\mathrm{d}}(z) \geq \min\left(\frac{1}{2}, \frac{1}{2n}, \frac{c_0}{2}, \frac{c_1}{2}\right) > 0.$$

(ii) Let $f \in \mathcal{O}^{exp}(U \smallsetminus \overline{\mathbb{R}}, E)$. By the choice of φ_0 and φ_1 the function $\overline{\partial}(\varphi_1 \varphi_0 f)$ may be regarded as an element of $\mathcal{C}^\infty(\mathbb{R}^2 \smallsetminus \partial\Omega, E)$ by $\mathcal{C}^\infty$-extension via $\overline{\partial}(\varphi_1 \varphi_0 f) := 0$ on $[(U^C \cap \mathbb{R}^2) \cup \mathbb{R}] \smallsetminus \partial\Omega$. Moreover, with the definition

$$V := (V_0 \cup W_0) \cup (V_1 \cap W_1),$$

the equation

$$\overline{\partial}(\varphi_1\varphi_0 f)(z) = \begin{cases} 0 & , z \in V, \\ [(\overline{\partial}\varphi_1)\varphi_0 f + (\overline{\partial}\varphi_0)\varphi_1 f](z) & , \text{else,} \end{cases}$$

is valid.

The next step is similar to (12). Let $n \in \mathbb{N}$, $n \geq 2$, $m \in \mathbb{N}_0$ and $\alpha \in \mathfrak{A}$. We define the set $S(n) \coloneqq S_n(\partial\Omega) \setminus V$ and the cardinality $C_m \coloneqq |\{\gamma \in \mathbb{N}_0^2 \mid |\gamma| \leq m\}|$. By applying the Leibniz rule twice, we obtain

$$|\overline{\partial}(\varphi_1\varphi_0 f)|_{\partial\Omega,n,m,\alpha}$$

$$= \sup_{\substack{z \in S_n(\partial\Omega) \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} p_\alpha(\partial^\beta\overline{\partial}(\varphi_1\varphi_0 f)(z))e^{-\frac{1}{n}|\operatorname{Re}(z)|}$$

$$\underset{(2)}{\leq} (m!)^2 \sup_{\substack{z \in S(n) \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} \sum_{\gamma \leq \beta} |\partial^{\beta-\gamma}[(\overline{\partial}\varphi_1)\varphi_0 + (\overline{\partial}\varphi_0)\varphi_1](z)| \underbrace{\sup_{\substack{z \in S(n) \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} p_\alpha(\partial_\mathbb{C}^{|\beta|}f(z))e^{-\frac{1}{n}|\operatorname{Re}(z)|}}_{=:C(f)}$$

$$\leq (m!)^4 C(f) \sup_{\substack{z \in S(n) \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} \sum_{\gamma \leq \beta}\sum_{\tau \leq \beta-\gamma} |\partial^\tau(\overline{\partial}\varphi_1)(z)\partial^{\beta-\gamma-\tau}\varphi_0(z) + \partial^\tau(\overline{\partial}\varphi_0)(z)\partial^{\beta-\gamma-\tau}\varphi_1(z)|$$

$$\leq (m!)^4 C(f) \sum_{\substack{|\gamma| \leq m \\ |\tau| \leq m+1}} \sup_{z \in S(n)}|\partial^\tau\varphi_1(z)| \sup_{\substack{z \in S(n) \\ v \in \mathbb{N}_0^2, |v| \leq m}}|\partial^v\varphi_0(z)| + \sup_{z \in S(n)}|\partial^\tau\varphi_0(z)| \sup_{\substack{z \in S(n) \\ v \in \mathbb{N}_0^2, |v| \leq m}}|\partial^v\varphi_1(z)|$$

$$\underset{\substack{(19), \\ (20)}}{\leq} (m!)^4 C_m C(f) \sum_{|\tau| \leq m+1} \widetilde{C}^{|\tau|} \sup_{z \in S(n)} \frac{\widetilde{\mathrm{d}}(z)^{-|\tau|}}{d_1\cdots d_{|\tau|}} \sup_{\substack{z \in S(n) \\ v \in \mathbb{N}_0^2, |v| \leq m}} C^{|v|}\frac{\mathrm{d}(z)^{-|v|}}{d_1\cdots d_{|v|}}$$

$$+ C^{|\tau|} \sup_{z \in S(n)}\frac{\mathrm{d}(z)^{-|\tau|}}{d_1\cdots d_{|\tau|}} \sup_{\substack{z \in S(n) \\ v \in \mathbb{N}_0^2, |v| \leq m}}\widetilde{C}^{|v|}\frac{\widetilde{\mathrm{d}}(z)^{-|v|}}{d_1\cdots d_{|v|}}$$

$$\underset{\substack{(21), \\ (22)}}{\leq} (m!)^4 C_m \frac{[\max(C,\widetilde{C},1)]^{m+1}}{(d_1\cdots d_{m+1})^2} C(f) \sum_{|\tau| \leq m+1} D^{-|\tau|}\sup_{\substack{v \in \mathbb{N}_0^2 \\ |v| \leq m}} B^{-|v|} + B^{-|\tau|}\sup_{\substack{v \in \mathbb{N}_0^2 \\ |v| \leq m}} D^{-|v|}.$$

$$(23)$$

Now, we have to take a closer look at $C(f)$. First of all, we remark that

$$[(U^C \cup \overline{\mathbb{R}}) \cap \mathbb{R}^2] = ([(U^C \cap \mathbb{R}^2) \cup (\mathbb{R} \cap \overline{\Omega})] \setminus \partial\Omega) \cup (\partial\Omega \cap \mathbb{R})$$

$$\subset \Big[(V_0 \cup W_0) \cup (V_1 \cap W_1) \cup \bigcup_{x \in \mathbb{R} \cap \partial\Omega} \mathbb{D}_{\frac{1}{n}}(x)\Big]$$

$$= V \cup \bigcup_{x \in \mathbb{R} \cap \partial\Omega} \mathbb{D}_{\frac{1}{n}}(x) =: W.$$

W is an open set in $\mathbb{R}^2$ as the union of open sets and we get

$$\overline{S(n)} = \overline{[S_n(\partial\Omega) \setminus V]} \subset \overline{W^C} = W^C \subset [(U \setminus \overline{\mathbb{R}}) \cap \mathbb{R}^2]. \qquad (24)$$

In the following we prove that there are $k \in \mathbb{N}$, $k \geq 2$, $M_0 \subset S(n)$ bounded and $M_1 \subset S_k(U)$ such that

$$S(n) \subset (M_0 \cup M_1).$$

As $|\operatorname{Im}(z)| \leq \frac{1}{n}$ for every $z \in S(n)$, it suffices to prove that there is $C_1 > 0$ such that $|\operatorname{Re}(z)| \leq C_1$ for every $z \in M_0$. We define the set $M \coloneqq \{z \in \mathbb{C} \mid \operatorname{Re}(z) > \widetilde{x}_0 + 2\}$ and decompose

$$S(n) = \underbrace{[S(n) \setminus M]}_{=:M_0} \cup \underbrace{[S(n) \cap M]}_{=:M_1}.$$
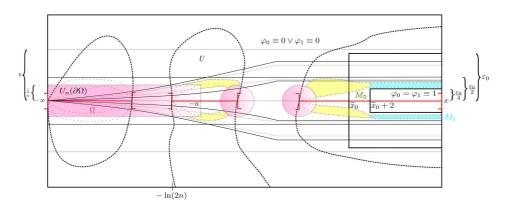
FIGURE 11. case: $\infty \in \Omega$, $-\infty \in \partial\Omega$

We observe that the inequality $\frac{1}{n} \geq 2e^{-|x|}$ is equivalent to $\ln(2n) \leq |x|$ for all $x \in \mathbb{R}$. Hence M_0 is bounded since

$$|\operatorname{Re}(z)| \leq \max(|-n|, |-\ln(2n)|, |\widetilde{x}_0 + 2|)$$

for all $z \in M_0$. Let

$$r := \frac{1}{2}\min\left(2, \frac{\varepsilon_0}{2}, \frac{\varepsilon_0}{4}\right) = \min\left(1, \frac{\varepsilon_0}{8}\right),$$

choose $k \in \mathbb{N}$ with $k > \max(n, \varepsilon_0)$ with $\frac{1}{k} < \frac{\varepsilon_0}{8}$ and $-k < \widetilde{x}_0$. Then

$$\frac{\varepsilon_0}{8} \leq \frac{\varepsilon_0}{4} - r \quad \text{and} \quad \frac{1}{k} < \min\left(\frac{1}{n}, \frac{\varepsilon_0}{8}\right)$$

is valid and thus we have for all $z \in M_1$

$$\begin{aligned}
\overline{\mathbb{D}_r(z)} &\subset \{w \in \mathbb{C} \mid \mathrm{d}(w, M_1) \leq r\} \\
&\subset \left([\widetilde{x}_0 + 2 - r, \infty) \times [-\tfrac{\varepsilon_0}{2} - r, \tfrac{\varepsilon_0}{2} + r]\right) \smallsetminus \{w \in \mathbb{C} \mid |\operatorname{Im}(w)| < \tfrac{\varepsilon_0}{4} - r\} \\
&\subset \left([\widetilde{x}_0 + 1, \infty) \times [-\tfrac{5\varepsilon_0}{8}, \tfrac{5\varepsilon_0}{8}]\right) \smallsetminus \{w \in \mathbb{C} \mid |\operatorname{Im}(w)| < \tfrac{\varepsilon_0}{8}\} \subset S_k(U). \quad (25)
\end{aligned}$$

From Proposition 3.3 follows that

$$\sup_{\substack{z \in M_1 \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} p_\alpha(\partial_{\mathbb{C}}^{|\beta|} f(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|} \leq e^{\frac{r}{n}} \frac{m!}{r^m} \sup_{\zeta \in S_k(U)} p_\alpha(f(\zeta)) e^{-\frac{1}{k}|\operatorname{Re}(\zeta)|}$$

$$= e^{\frac{r}{n}} \frac{m!}{r^m} \|f\|_{U^*, k, \alpha}.$$

Since $\overline{M}_0 \subset [(U \smallsetminus \overline{\mathbb{R}})] \cap \mathbb{R}^2$ by (24), $\overline{M}_0$ is compact and $f \in \mathcal{O}^{exp}(U \smallsetminus \mathbb{R}, E)$, we have

$$C(f) \leq \sup_{\substack{z \in \overline{M}_0 \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} p_\alpha(\partial_{\mathbb{C}}^{|\beta|} f(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|} + e^{\frac{r}{n}} \frac{m!}{r^m} \|f\|_{U^*, k, \alpha} < \infty.$$

Due to (23) this implies that $|\overline{\partial}(\varphi_1\varphi_0 f)|_{\partial\Omega, n, m, \alpha} < \infty$ for all $n \in \mathbb{N}$, $n \geq 2$, $m \in \mathbb{N}_0$ and $\alpha \in \mathfrak{A}$ and thus $\overline{\partial}(\varphi_1\varphi_0 f) \in \mathcal{E}^{exp}(\overline{\mathbb{C}} \smallsetminus \partial\Omega, E)$. As E is admissible, there exists $g \in \mathcal{E}^{exp}(\overline{\mathbb{C}} \smallsetminus \partial\Omega, E)$ such that

$$\overline{\partial}g = \overline{\partial}(\varphi_1\varphi_0 f). \quad (26)$$

(iii) We set $F := \varphi_1\varphi_0 f - g$. The next step is to show that $F \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\Omega}, E)$, which implies $F \in \mathcal{O}^{exp}(U_1 \smallsetminus \overline{\mathbb{R}}, E)$ since $\mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\Omega}, E) \subset \mathcal{O}^{exp}(U_1 \smallsetminus \overline{\mathbb{R}}, E)$, and that $f - F \in \mathcal{O}^{exp}(U, E)$. F is defined on $\mathbb{C} \smallsetminus \overline{\Omega}$ (by setting $\varphi_1\varphi_0 f := 0$ on $[(U^C \cup \overline{\Omega}) \smallsetminus \partial\Omega] \cap \mathbb{C}$) and can be regarded as an element of $\mathcal{O}(\mathbb{C} \smallsetminus \overline{\Omega}, E)$ due to (26). Let $n \in \mathbb{N}$,

$n \geq 2$. We set $V := V_0 \cup W_0$, $S(n) := S_n(\overline{\Omega}) \smallsetminus V$ and remark that $S_n(\overline{\Omega}) \subset S_n(\partial\Omega)$. For $\alpha \in \mathfrak{A}$ we have by the choice of φ_i, $i = 1, 2$,

$$
\begin{aligned}
|F|_{\overline{\Omega},n,\alpha} &= \sup_{z \in S_n(\overline{\Omega})} p_\alpha(F(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|} \\
&\leq \underbrace{\sup_{z \in S_n(\partial\Omega)} p_\alpha(g(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|}}_{=|g|_{\partial\Omega,n,0,\alpha}} + \sup_{z \in S_n(\overline{\Omega})} p_\alpha(\varphi_1\varphi_0 f(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|} \\
&= |g|_{\partial\Omega,n,0,\alpha} + \sup_{z \in S(n)} \underbrace{|(\varphi_1\varphi_0)(z)|}_{\leq 1} p_\alpha(f(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|} \\
&\leq |g|_{\partial\Omega,n,0,\alpha} + \sup_{z \in S(n)} p_\alpha(f(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|}.
\end{aligned}
\tag{27}
$$

First, we observe that

$$
(U^C \cup \overline{\mathbb{R}}) \cap \mathbb{C} \subset \left[V_0 \cup \bigcup_{x \in \overline{\Omega} \cap \mathbb{R}} \mathbb{D}_{\frac{1}{n}}(x) \right] =: W.
$$

$W \subset \mathbb{C}$ is open and so we get by the definition of the set $S(n)$ that

$$
\overline{S(n)} \subset \overline{W^C} = W^C \subset (U \smallsetminus \overline{\mathbb{R}}) \cap \mathbb{C}.
$$

Again, we claim that there are $M_0 \subset S(n)$ bounded, $k \in \mathbb{N}$, $k \geq 2$, and $M_1 \subset S_k(U)$ such that $S(n) = M_0 \cup M_1$. For the boundedness we just have to prove that there is $C_1 > 0$ such that $|\operatorname{Re}(z)| \leq C_1$ for every $z \in M_0$. We choose $k \in \mathbb{N}$ such that $k > n$, $\frac{1}{k} < \frac{\varepsilon_0}{2} < k$ and $-k < \widetilde{x}_0 + 2$. Then we decompose the set $S(n)$ as follows

$$
S(n) = \underbrace{\left[S(n) \smallsetminus S_k(U) \right]}_{=:M_0} \cup \underbrace{\left[S(n) \cap S_k(U) \right]}_{=:M_1}.
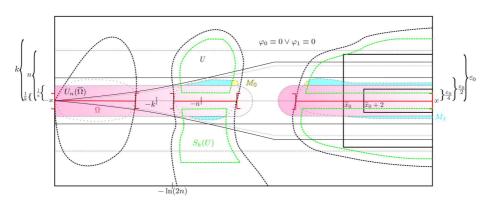$$

FIGURE 12. case: $\infty \in \Omega$, $-\infty \in \partial\Omega$

Obviously $M_1 \subset S_k(U)$ and $\overline{M}_0 \subset \overline{S(n)} \subset (U \smallsetminus \overline{\mathbb{R}}) \cap \mathbb{C}$. By the choice of V_0 we have

$$
M_0 = \left[S(n) \smallsetminus S_k(U) \right] \subset (S_n(\overline{\Omega}) \smallsetminus V_0) \subset \left\{ z \in \mathbb{C} \mid |\operatorname{Im}(z)| < \frac{\varepsilon_0}{2} \right\}
\tag{28}
$$

and by the choice of W_0

$$
M_0 \subset (S_n(\overline{\Omega}) \smallsetminus W_0) \subset \{ z \in \mathbb{C} \mid \operatorname{Re}(z) > \min(-n, -\ln(2n)) \}.
\tag{29}
$$

Let $z \in S(n)$ with $|\operatorname{Im}(z)| < \frac{\varepsilon_0}{2}$ and $\operatorname{Re}(z) \geq \widetilde{x}_0 + 2$. Then

$$
z \in \left([\widetilde{x}_0 + 2, \infty) \times \left[-\frac{\varepsilon_0}{2}, \frac{\varepsilon_0}{2} \right] \right) \subset \left([\widetilde{x}_0, \infty) \times [-\varepsilon_0, \varepsilon_0] \right) \subset U
$$

and therefore

$$\mathrm{d}\big(z, \mathbb{C} \cap \partial U\big) \geq \min\big(2, \tfrac{\varepsilon_0}{2}\big) > \tfrac{1}{k}$$

by the choice of k. Furthermore,

$$k > n > |\mathrm{Im}(z)| > \tfrac{1}{n} > \tfrac{1}{k}$$

as $[\widetilde{x}_0, \infty] \subset \Omega$ and due to the choice of k. In addition, $\mathrm{Re}(z) \geq \widetilde{x}_0 + 2 > -k$ and $z \in U$ by the choice of k and since $z \in S(n) \subset U$. Hence we obtain $z \in S_k(U)$. So it follows from (28) that

$$M_0 = [S(n) \smallsetminus S_k(U)] \subset \{z \in \mathbb{C} \mid \mathrm{Re}(z) < \widetilde{x}_0 + 2\}$$

and due to (29) we gain the claim with $C_1 := \max(n, \ln(2n), |\widetilde{x}_0 + 2|)$. By the same arguments as in part (ii) we get $\sup_{z \in S(n)} p_\alpha(f(z)) e^{-\frac{1}{n}|\mathrm{Re}(z)|} < \infty$ and by (27) that $F \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\Omega}, E)$.

(iv) $f - F$ is defined on $U \cap \mathbb{C}$ (by the setting in the beginning of part (iii)) and can be regarded as an element of $\mathcal{O}(U \cap \mathbb{C}, E)$ due to (26). Let $n \in \mathbb{N}$, $n \geq 2$. We set $V := V_1 \cap W_1$ and $T(n) := T_n(U) \smallsetminus V$. With $R := \{z \in \mathbb{C} \mid \mathrm{Re}(z) \leq -n\}$ we have

$$\big[\overline{U_n(\partial\Omega)} \cup \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| \geq n\}\big] \subset \big[R \cup \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| \geq n\} \cup \bigcup_{x \in \mathbb{C} \cap \partial U} \overline{\mathbb{D}_{\frac{1}{n}}(x)}\big] =: \widetilde{R}$$

and thus

$$T_n(U) \subset \big[(U \cap \mathbb{C}) \smallsetminus \widetilde{R}\big] \subset \big(\mathbb{C} \smallsetminus [\overline{U_n(\partial\Omega)} \cup \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| \geq n\}]\big) = S_n(\partial\Omega). \quad (30)$$

For $\alpha \in \mathfrak{A}$ we have by the choice of φ_i, $i = 1, 2$,

$$\|f - F\|_{U,n,\alpha} = \sup_{z \in T_n(U)} p_\alpha\big([(1 - \varphi_1\varphi_0)f + g](z)\big) e^{-\frac{1}{n}|\mathrm{Re}(z)|}$$

$$\underset{(30)}{\leq} \underbrace{\sup_{z \in S_n(\partial\Omega)} p_\alpha(g(z)) e^{-\frac{1}{n}|\mathrm{Re}(z)|}}_{=|g|_{\partial\Omega,n,0,\alpha}} + \sup_{z \in T_n(U)} p_\alpha\big((1 - \varphi_1\varphi_0)f(z)\big) e^{-\frac{1}{n}|\mathrm{Re}(z)|}$$

$$= |g|_{\partial\Omega,n,0,\alpha} + \sup_{z \in T(n)} \underbrace{|1 - (\varphi_1\varphi_0)(z)|}_{\leq 1} p_\alpha(f(z)) e^{-\frac{1}{n}|\mathrm{Re}(z)|}$$

$$\leq |g|_{\partial\Omega,n,0,\alpha} + \sup_{z \in T(n)} p_\alpha(f(z)) e^{-\frac{1}{n}|\mathrm{Re}(z)|}. \quad (31)$$

We choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \min(\frac{1}{n}, \frac{\varepsilon_0}{4})$. First, we observe that

$$(U^C \cup \overline{\mathbb{R}}) \cap \mathbb{C} \subset \big[V \cup \bigcup_{x \in U^C \cap \mathbb{C}} \mathbb{D}_{\frac{1}{n}}(x)\big] =: W.$$

The set $W \subset \mathbb{C}$ is open and thus we get by the definition of the set $T(n)$

$$T(n) = T_n(U) \smallsetminus V = \underbrace{\big[T_n(U) \smallsetminus \big(\bigcup_{x \in U^C \cap \mathbb{C}} \mathbb{D}_{\frac{1}{n}}(x)\big)\big]}_{=T_n(U)} \smallsetminus V \subset W^C$$

and so

$$\overline{T(n) \smallsetminus S_k(U)} \subset \overline{T(n)} \subset \overline{W^C} = W^C \subset (U \smallsetminus \overline{\mathbb{R}}) \cap \mathbb{C}. \quad (32)$$

Then we can decompose the set $T(n)$ in the following manner

$$T(n) = \underbrace{[T(n) \smallsetminus S_k(U)]}_{=:M_0} \cup \underbrace{[T(n) \cap S_k(U)]}_{=:M_1}.$$
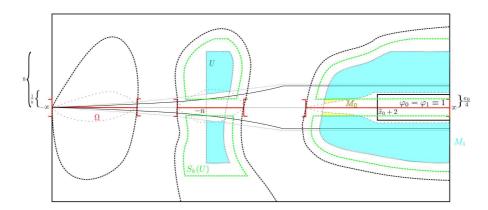
FIGURE 13. case: $\infty \in \Omega$, $-\infty \in \partial\Omega$

We claim that the set M_0 is bounded. Again, we just have to prove that there is $C_1 > 0$ such that $|\operatorname{Re}(z)| \le C_1$ for every $z \in M_0$. By the choice of k and the definition of V_1 and W_1 we have $\operatorname{Re}(z) \in [-n, \max(0, \widetilde{x}_0 + 2)]$ for every $z \in M_0$, proving the claim. Therefore, $\overline{M_0}$ is compact and by (32) we get $\overline{M_0} \subset (U \smallsetminus \overline{\mathbb{R}}) \cap \mathbb{C}$. Then

$$\sup_{z \in T(n)} p_\alpha(f(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|} \le \sup_{z \in \overline{M_0}} p_\alpha(f(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|} + \sup_{z \in M_1} p_\alpha(f(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|}$$

$$\le \sup_{z \in \overline{M_0}} p_\alpha(f(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|} + \|f\|_{U^*, k, \alpha} < \infty$$

for all $n \in \mathbb{N}$, $n \ge 2$, and $\alpha \in \mathfrak{A}$ since $f \in \mathcal{O}^{exp}(U \smallsetminus \overline{\mathbb{R}}, E)$. Hence we obtain by (31) that $f - F \in \mathcal{O}^{exp}(U, E)$.

So we have found $F \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\Omega}, E) \subset \mathcal{O}^{exp}(U_1 \smallsetminus \mathbb{R}, E)$ such that $[F_{|(U \smallsetminus \overline{\mathbb{R}}) \cap \mathbb{C}}] = [f]$, proving the surjectivity of J. For arbitrary U, $U_0 \in \mathcal{U}(\Omega)$ we have, with U_1 from the proof,

$$\mathcal{O}^{exp}(U \smallsetminus \overline{\mathbb{R}}, E)/\mathcal{O}^{exp}(U, E) \cong \mathcal{O}^{exp}(U_1 \smallsetminus \overline{\mathbb{R}}, E)/\mathcal{O}^{exp}(U_1, E)$$

$$\cong \mathcal{O}^{exp}(U_0 \smallsetminus \overline{\mathbb{R}}, E)/\mathcal{O}^{exp}(U_0, E)$$

algebraically, yielding the general statement. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

By virtue of Lemma 5.7 we may define restrictions in $bv(\Omega, E)$ in the following manner.

5.8. Definition. Let E be admissible and $\Omega, \Omega_1 \subset \overline{\mathbb{R}}$, $\Omega_1 \subset \Omega$, be open. For $\Omega_1 \ne \varnothing$ let $[f] \in bv(\Omega, E) = \mathcal{O}^{exp}(U \smallsetminus \overline{\mathbb{R}}, E)/\mathcal{O}^{exp}(U, E)$ where $U \in \mathcal{U}(\Omega)$. Setting $U_1 := U \cap (\Omega_1 \times \mathbb{R})$, we may define the restriction map by

$$R_{\Omega, \Omega_1}([f]) := [f]_{|\Omega_1} := [f_{|(U_1 \smallsetminus \overline{\mathbb{R}}) \cap \mathbb{C}}] \in \mathcal{O}^{exp}(U_1 \smallsetminus \overline{\mathbb{R}}, E)/\mathcal{O}^{exp}(U_1, E) = bv(\Omega_1, E).$$

In addition, we define for an open set $\Omega \subset \overline{\mathbb{R}}$

$$R_{\Omega, \varnothing} : bv(\Omega, E) \to bv(\varnothing, E), \ R_{\Omega, \varnothing}([f]) := [f]_{|\varnothing} := 0.$$

We denote the family $\{bv(\Omega, E) \mid \Omega \subset \overline{\mathbb{R}} \text{ open}\}$ by $bv(E)$.

5.9. Theorem. *Let E be strictly admissible.*

a) $bv(E)$, equipped with the restrictions from Definition 5.8, is a sheaf on $\overline{\mathbb{R}}$.
b) $bv(E)$ is flabby, i.e. $R_{\overline{\mathbb{R}}, \Omega}$ is surjective for any open $\Omega \subset \overline{\mathbb{R}}$.
c) $bv(E)$ is isomorphic to $\mathcal{R}(E) = \mathcal{R}_{\overline{\mathbb{R}}}(E)$. In particular, $\mathcal{R}(E)$ is a sheaf.

Proof. a)(i) For open $\Omega \subset \overline{\mathbb{R}}$ the map $R_{\Omega,\Omega}$ can be regarded as $\mathrm{id}_{bv(\Omega,E)}$ by Lemma 5.7. Let $\Omega_3 \subset \Omega_2 \subset \Omega_1 \subset \overline{\mathbb{R}}$ be open. We have to prove that $R_{\Omega_2,\Omega_3} \circ R_{\Omega_1,\Omega_2} = R_{\Omega_1,\Omega_3}$ holds. This is obviously true if one of the sets is empty, so let them all be non-empty. Let $[f] \in bv(\Omega_1, E) = \mathcal{O}^{exp}(U_1 \smallsetminus \overline{\mathbb{R}}, E)/\mathcal{O}^{exp}(U_1, E)$ where $U_1 \in \mathcal{U}(\Omega_1)$. With $U_2 := U_1 \cap (\Omega_2 \times \mathbb{R})$ and

$$U_3 := U_2 \cap (\Omega_3 \times \mathbb{R}) = [U_1 \cap (\Omega_2 \times \mathbb{R})] \cap (\Omega_3 \times \mathbb{R}) \underset{\Omega_3 \subset \Omega_2}{=} U_1 \cap (\Omega_3 \times \mathbb{R}) \qquad (33)$$

we get

$$R_{\Omega_2,\Omega_3} \circ R_{\Omega_1,\Omega_2}([f]) = R_{\Omega_2,\Omega_3}([f_{|(U_2 \smallsetminus \overline{\mathbb{R}}) \cap \mathbb{C}}]) \underset{U_3 \subset U_2}{=} [f_{|(U_3 \smallsetminus \overline{\mathbb{R}}) \cap \mathbb{C}}] \underset{(33)}{=} R_{\Omega_1,\Omega_3}([f]).$$

(ii) $(S1)$: Let $\{\Omega_j \subset \overline{\mathbb{R}} \mid j \in J\}$ be a family of open sets and $\Omega := \bigcup_{j \in J} \Omega_j$. Let $[f] \in bv(\Omega, E) = \mathcal{O}^{exp}(U \smallsetminus \overline{\mathbb{R}}, E)/\mathcal{O}^{exp}(U, E)$, where $U \in \mathcal{U}(\Omega)$, such that $R_{\Omega,\Omega_j}([f]) = 0$ for all $j \in J$. The assumption $R_{\Omega,\Omega_j}([f]) = 0$ is equivalent to $f \in \mathcal{O}^{exp}(U_j, E)$ for every $j \in J$ where $U_j := U \cap (\Omega_j \times \mathbb{R})$. Thus we obtain

$$f \in \mathcal{O}^{exp}([U \smallsetminus \overline{\mathbb{R}}] \cup \bigcup_{j \in J} \Omega_j, E) = \mathcal{O}^{exp}([U \smallsetminus \overline{\mathbb{R}}] \cup \Omega, E) \underset{U \in \mathcal{U}(\Omega)}{=} \mathcal{O}^{exp}(U, E)$$

and hence $[f] = 0$.

(iii) $(S2)$: Let $(\Omega_j)_{j \in J}$ and Ω be like in part (ii). Let $[f_j] \in bv(\Omega_j, E) = \mathcal{O}^{exp}(U_j \smallsetminus \overline{\mathbb{R}}, E)/\mathcal{O}^{exp}(U_j, E)$, where $U_j \in \mathcal{U}(\Omega_j)$, such that $[f_j]_{|\Omega_j \cap \Omega_k} = [f_k]_{|\Omega_j \cap \Omega_k}$. Hence we have, using that $bv(\Omega_j \cap \Omega_k, E)$ does not depend on the choice of the open neighbourhood in $\mathbb{C}$ of $\Omega_j \cap \Omega_k$ by Lemma 5.7, that

$$g_{jk} := f_{j|[(U_j \cap U_k) \smallsetminus \overline{\mathbb{R}}] \cap \mathbb{C}} - f_{k|[(U_j \cap U_k) \smallsetminus \overline{\mathbb{R}}] \cap \mathbb{C}} \in \mathcal{O}^{exp}(U_j \cap U_k, E)$$

and $g_{jk} = -g_{kj}$ as well as $g_{jk} + g_{kl} + g_{lj} = 0$ on $U_j \cap U_k \cap U_l$ by a simple calculation.

(iii.1) If $\pm\infty \notin \Omega$ and thus $\pm\infty \notin \Omega_j$, then exactly like in [16, Theorem 1.4.5, p. 13], where one uses that E is *strictly* admissible instead of [16, Theorem 1.4.4, p. 12], there are $g_j \in \mathcal{O}(U_j \cap \mathbb{C}, E)$ such that $g_{jk} = g_k - g_j$ on $U_j \cap U_k \cap \mathbb{C}$ (here the adjunct *strictly* is needed). The setting $F_j := f_j + g_j$ defines a function $F \in \mathcal{O}((U \smallsetminus \overline{\mathbb{R}}) \cap \mathbb{C}, E) = \mathcal{O}^{exp}(U \smallsetminus \overline{\mathbb{R}}, E)$ since

$$F_j - F_k = f_j + g_j - f_k - g_k = f_j - f_k + g_j - g_k = g_{jk} - g_{jk} = 0$$

on $U_j \cap U_k \cap \mathbb{C}$ such that $[F]_{|\Omega_j} = [f_j]$ for any $j \in J$.

(iii.2) Now, let $-\infty \in \Omega$ or $\infty \in \Omega$, i.e. there exists $j \in J$ such that $-\infty \in \Omega_j$ or $\infty \in \Omega_j$. We only consider the case that there are $j_0, j_1 \in J$ such that $-\infty \in \Omega_{j_0}$ and $\infty \in \Omega_{j_1}$. For the other two cases the proof is analogous. Then there are $x_0, x_1 \in \mathbb{R}$ and $\varepsilon_0, \varepsilon_1 > 0$ such that $[-\infty, x_0] \times [-\varepsilon_0, \varepsilon_0] \subset U_{j_0}$ and $[x_1, \infty] \times [-\varepsilon_1, \varepsilon_1] \subset U_{j_1}$. Now, let $x := \max(|x_0|, |x_1|)$ and $\varepsilon := \min(\varepsilon_0, \varepsilon_1)$. We define the sets

$$G_0 := [(-\infty, -x - 1) \times (-\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{2})]^C, \quad H_0 := (-\infty, -x - 2] \times [-\tfrac{\varepsilon}{4}, \tfrac{\varepsilon}{4}]$$

as well as

$$G_1 := [(x + 1, \infty) \times (-\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{2})]^C, \quad H_1 := [x + 2, \infty) \times [-\tfrac{\varepsilon}{4}, \tfrac{\varepsilon}{4}].$$

By the proof of [15, Theorem 1.4.1, p. 25] there are $\varphi_i \in \mathcal{C}^\infty(\mathbb{R}^2)$, $i = 0, 1$, such that $0 \leq \varphi_i \leq 1$ and $\varphi_i = 0$ near G_i plus $\varphi_i = 1$ near H_i as well as $|\partial^\beta \varphi_i| \leq C_{i,\beta} \widetilde{\varepsilon}^{-|\beta|}$ for all $\beta \in \mathbb{N}_0^2$ where $\widetilde{\varepsilon} := \tfrac{1}{4} \min(\tfrac{\varepsilon}{4}, 1)$ and $C_{i,\beta} > 0$.
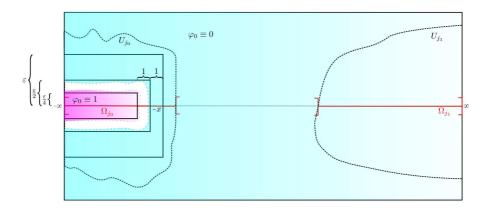
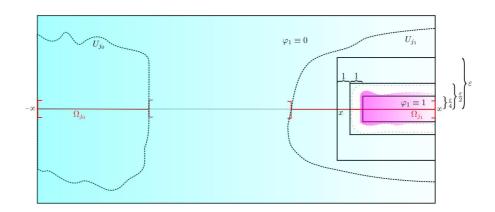FIGURE 14. case: $-\infty \in \Omega_{j_0}$, $\infty \in \Omega_{j_1}$

FIGURE 15. case: $-\infty \in \Omega_{j_0}$, $\infty \in \Omega_{j_1}$

Due to case (iii.1) there is $F \in \mathcal{O}((U \setminus \overline{\mathbb{R}}) \cap \mathbb{C}, E)$ such that $[F]_{|\Omega_j \cap \mathbb{R}} = [f_j]_{|\Omega_j \cap \mathbb{R}}$ for every $j \in J$. By Lemma 5.7 there exists $\widetilde{F} \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \setminus \overline{\Omega}, E)$ with $F - \widetilde{F} \in \mathcal{O}(U \cap \mathbb{C}, E)$. Thus we obtain

$$f_j - \widetilde{F} = \underbrace{(f_j - F)}_{\in \mathcal{O}(U_j \cap \mathbb{C}, E)} + \underbrace{(F - \widetilde{F})}_{\in \mathcal{O}(U \cap \mathbb{C}, E)} \in \mathcal{O}(U_j \cap \mathbb{C}, E) \tag{34}$$

for all $j \in J$. So by the choice of φ_i we can regard $\overline{\partial}(\varphi_0(f_{j_0} - \widetilde{F}) + \varphi_1(f_{j_1} - \widetilde{F}))$ as an element of $\mathcal{C}^\infty(\mathbb{R}^2, E)$ (set $\varphi_i(f_{j_i} - \widetilde{F}) \coloneqq 0$ outside U_{j_i}). Let $n \in \mathbb{N}$, $n \geq 2$, $m \in \mathbb{N}_0$ and $\alpha \in \mathfrak{A}$. Then we obtain by applying the Leibniz rule and the choice of φ_i like in (12) resp. (23)

$$|\overline{\partial}(\varphi_0(f_{j_0} - \widetilde{F}) + \varphi_1(f_{j_1} - \widetilde{F}))|_{\varnothing, n, m, \alpha}$$
$$= \sup_{\substack{z \in S_n(\varnothing) \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} p_\alpha(\partial^\beta \overline{\partial}(\varphi_0(f_{j_0} - \widetilde{F}) + \varphi_1(f_{j_1} - \widetilde{F}))(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|}$$
$$\leq (m!)^2 \sum_{i=0,1} \sup_{\substack{z \in S_n(\varnothing) \setminus (G_i \cup H_i) \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} \sum_{\gamma \leq \beta} |\partial^{\beta - \gamma}(\overline{\partial}\varphi_i)(z)| p_\alpha(\partial^\gamma(f_{j_i} - \widetilde{F})(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|}$$

$$\underset{(2)}{\leq} (m!)^2 \sum_{\substack{i=0,1 \\ |\gamma| \leq m+1}} \underbrace{\sup_{z \in S_n(\varnothing) \smallsetminus (G_i \cup H_i)} |\partial^\gamma \varphi_i(z)|}_{\leq C_{i,\gamma} \widetilde{\varepsilon}^{-|\gamma|}} \underbrace{\sup_{\substack{z \in S_n(\varnothing) \smallsetminus (G_i \cup H_i) \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} p_\alpha(\partial_{\mathbb{C}}^{|\beta|}(f_{j_i} - \widetilde{F})(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|}}_{=:C(f_{j_i} - \widetilde{F})}$$

$$\leq (m!)^2 (C(f_{j_0} - \widetilde{F}) + C(f_{j_1} - \widetilde{F})) \sum_{|\gamma| \leq m+1} (C_{0,\gamma} + C_{1,\gamma}) \widetilde{\varepsilon}^{-|\gamma|}. \tag{35}$$

Now, we have to take a closer look at $C(f_{j_i} - \widetilde{F})$. By the choice of the sets G_i and H_i

$$S_n(\varnothing) \smallsetminus (G_0 \cup H_0) \subset \underbrace{\left\{ z \in \mathbb{C} \mid |\operatorname{Im}(z)| < \tfrac{\varepsilon}{2},\ -x - 2 \leq \operatorname{Re}(z) < -x - 1 \right\}}_{=:N_0}$$

$$\cup \underbrace{\left\{ z \in \mathbb{C} \mid \tfrac{\varepsilon}{4} < |\operatorname{Im}(z)| < \tfrac{\varepsilon}{2},\ \operatorname{Re}(z) < -x - 2 \right\}}_{=:M_0}$$

and

$$S_n(\varnothing) \smallsetminus (G_1 \cup H_1) \subset \underbrace{\left\{ z \in \mathbb{C} \mid |\operatorname{Im}(z)| < \tfrac{\varepsilon}{2},\ x + 1 < \operatorname{Re}(z) \leq x + 2 \right\}}_{=:N_1}$$

$$\cup \underbrace{\left\{ z \in \mathbb{C} \mid \tfrac{\varepsilon}{4} < |\operatorname{Im}(z)| < \tfrac{\varepsilon}{2},\ \operatorname{Re}(z) > x + 2 \right\}}_{=:M_1}$$
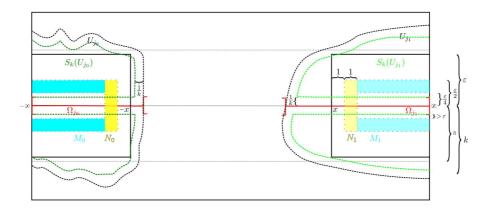
is valid.

FIGURE 16. case: $-\infty \in \Omega_{j_0}$, $\infty \in \Omega_{j_1}$

The sets N_i are clearly bounded and $\overline{N}_0 \subset U_{j_0}$ as well as $\overline{N}_1 \subset U_{j_1}$. This implies

$$\sup_{\substack{z \in N_i \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} p_\alpha(\partial_{\mathbb{C}}^{|\beta|}(f_{j_i} - \widetilde{F})(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|} < \infty, \quad i = 0, 1, \tag{36}$$

by (34). If we set

$$r := \frac{1}{2} \min\left(2, \frac{\varepsilon}{2}, \frac{\varepsilon}{4}\right) = \min\left(1, \frac{\varepsilon}{8}\right)$$

and choose $k \in \mathbb{N}$ with $k > \max(n, \varepsilon)$ and $\frac{1}{k} < \frac{\varepsilon}{8}$ and, in addition, $-k < x$, if $\infty \notin \Omega_{j_0}$ resp. $-\infty \notin \Omega_{j_1}$, then

$$\overline{\mathbb{D}}_r(z) \subset S_k(U_{j_i}) \subset S_k(\overline{\Omega}), \quad i = 0, 1,$$

holds for all $z \in M_i$ like in (25). Due to Proposition 3.3 we have for $i = 0, 1$

$$\sup_{\substack{z \in M_i \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} p_\alpha(\partial_{\mathbb{C}}^{|\beta|}(f_{j_i} - \widetilde{F})(z))e^{-\frac{1}{n}|\operatorname{Re}(z)|} \leq e^{\frac{r}{n}} \frac{m!}{r^m}(\|f_{j_i}\|_{U_{j_i}^*, k, \alpha} + |\widetilde{F}|_{\overline{\Omega}, k, \alpha}) < \infty. \quad (37)$$

So we get $C(f_{j_i} - \widetilde{F}) < \infty$, $i = 0, 1$, by (36) and (37), implying $\overline{\partial}(\varphi_0(f_{j_0} - \widetilde{F}) + \varphi_1(f_{j_1} - \widetilde{F})) \in \mathcal{E}^{exp}(\overline{\mathbb{C}}, E)$ by virtue of (35). Since E is admissible, there is $g \in \mathcal{E}^{exp}(\overline{\mathbb{C}}, E)$ such that

$$\overline{\partial} g = \overline{\partial}(\varphi_0(f_{j_0} - \widetilde{F}) + \varphi_1(f_{j_1} - \widetilde{F})). \quad (38)$$

(iii.3) We set $h := \varphi_0(f_{j_0} - \widetilde{F}) + \varphi_1(f_{j_1} - \widetilde{F}) - g$. Then $h \in \mathcal{O}(\mathbb{C}, E)$ by (38) and for all $n \in \mathbb{N}$, $n \geq 2$, and $\alpha \in \mathfrak{A}$ we have

$$|h|_{\{\pm\infty\}, n, \alpha}$$
$$\leq \sum_{i=0,1} \sup_{z \in S_n(\{\pm\infty\}) \smallsetminus G_i} p_\alpha(\varphi_i(f_{j_i} - \widetilde{F})(z))e^{-\frac{1}{n}|\operatorname{Re}(z)|} + \underbrace{\sup_{z \in S_n(\varnothing)} p_\alpha(g(z))e^{-\frac{1}{n}|\operatorname{Re}(z)|}}_{=|g|_{\varnothing, n, \alpha}}$$
$$\leq \sum_{i=0,1} \sup_{z \in S_n(\{\pm\infty\}) \smallsetminus G_i} p_\alpha((f_{j_i} - \widetilde{F})(z))e^{-\frac{1}{n}|\operatorname{Re}(z)|} + |g|_{\varnothing, n, \alpha}. \quad (39)$$

Furthermore, if we choose $k \in \mathbb{N}$ such that $k > n$ and $\frac{1}{k} < \min(1, \frac{\varepsilon}{2})$ and, in addition, $-k < x + 1$, if $\infty \notin \Omega_{j_0}$ resp. $-\infty \notin \Omega_{j_1}$, then $[S_n(\{\pm\infty\}) \smallsetminus G_i] \subset [M_i \cup S_k(U_{j_i})]$, $i = 0, 1$, where

$$M_i := \begin{cases} \varnothing & , n \leq x + 1, \\ \{z \in \mathbb{C} \mid -n < \operatorname{Re}(z) < -x - 1, |\operatorname{Im}(z)| \leq \frac{1}{n}\} & , n > x + 1, i = 0, \\ \{z \in \mathbb{C} \mid x + 1 < \operatorname{Re}(z) < n, |\operatorname{Im}(z)| \leq \frac{1}{n}\} & , n > x + 1, i = 1, \end{cases}$$

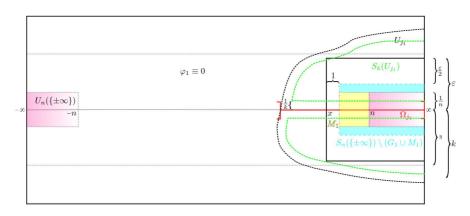and its closure $\overline{M}_i$ is a compact subset of $U_{j_i} \cap \mathbb{C}$.

FIGURE 17. case: $-\infty \in \Omega_{j_0}$, $\infty \in \Omega_{j_1}$, $n > x + 1$, $i = 1$

In addition, $S_k(U_{j_i}) \subset S_k(\overline{\Omega})$ and hence, keeping (34) in mind,

$$\sup_{z \in S_n(\{\pm\infty\}) \smallsetminus G_i} p_\alpha((f_{j_i} - \widetilde{F})(z))e^{-\frac{1}{n}|\operatorname{Re}(z)|}$$
$$\leq \sum_{i=0,1} \sup_{z \in M_i} p_\alpha((f_{j_i} - \widetilde{F})(z))e^{-\frac{1}{n}|\operatorname{Re}(z)|} + \underbrace{\sup_{z \in S_k(U_{j_i})} p_\alpha((f_{j_i})(z))e^{-\frac{1}{n}|\operatorname{Re}(z)|}}_{=\|f_{j_i}\|_{U_{j_i}^*, k, \alpha}}$$

$$+ 2 \underbrace{\sup_{z \in S_k(\overline{\Omega})} p_\alpha(\widetilde{F}(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|}}_{= |\widetilde{F}|_{\overline{\Omega}, n, \alpha}}$$

$$\leq 2|\widetilde{F}|_{\overline{\Omega}, n, \alpha} + \sum_{i=0,1} \|f_{j_i}\|_{U_{j_i}^*, k, \alpha} + \sum_{i=0,1} \sup_{z \in \overline{M}_i} p_\alpha((f_{j_i} - \widetilde{F})(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|} < \infty.$$

So we gain $h \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \{\pm\infty\}, E)$ by (39).

(iii.4) Now, we define the function $F^* := \widetilde{F} + h$. Then we have

$$F^* = \widetilde{F} + h \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\Omega}, E) \subset \mathcal{O}^{exp}(U \smallsetminus \overline{\mathbb{R}}, E).$$

The last step is to prove that F^* has the desired property, i.e. $[F^*]_{|\Omega_j} = [f_j]$ for all $j \in J$. If $j \in J$ with $\pm\infty \notin \Omega_j$, then

$$f_j - F^* = (f_j - \widetilde{F}) - h \in \mathcal{O}(U_j \cap \mathbb{C}, E)$$

by (34) and since $\mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \{\pm\infty\}, E) \subset \mathcal{O}(\mathbb{C}, E)$. Thus we have $[F^*]_{|\Omega_j} = [f_j]$.

Let $j \in J$ such that $-\infty \in \Omega_j$ or $\infty \in \Omega_j$. Then we have for $n \in \mathbb{N}$, $n \geq 2$, and $\alpha \in \mathfrak{A}$

$$\|f_j - F^*\|_{U_j, n, \alpha}$$

$$= \sup_{z \in T_n(U_j)} p_\alpha((f_j - \widetilde{F} - \varphi_0(f_{j_0} - \widetilde{F}) - \varphi_1(f_{j_1} - \widetilde{F}) + g)(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|}$$

$$\leq \sum_{i=0,1} \sup_{z \in T_n(U_j) \cap H_i} p_\alpha((f_j - f_{j_i})(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|} + \underbrace{\sup_{z \in S_n(\varnothing)} p_\alpha(g(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|}}_{= |g|_{\varnothing, n, \alpha}}$$

$$+ \sup_{z \in T_n(U_j) \smallsetminus (H_0 \cup H_1)} p_\alpha((f_j - \widetilde{F} - \varphi_0(f_{j_0} - \widetilde{F}) - \varphi_1(f_{j_1} - \widetilde{F}))(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|} \quad (40)$$

where we used $T_n(U_j) \subset S_n(\varnothing)$ plus

$$H_0 \subset G_1 \quad \text{and} \quad H_1 \subset G_0. \quad (41)$$

Moreover, the following estimate holds

$$\sup_{z \in T_n(U_j) \smallsetminus (H_0 \cup H_1)} p_\alpha((f_j - \widetilde{F} - \varphi_0(f_{j_0} - \widetilde{F}) - \varphi_1(f_{j_1} - \widetilde{F}))(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|}$$

$$\leq \sup_{z \in T_n(U_j) \smallsetminus (H_0 \cup H_1)} p_\alpha((f_j - \widetilde{F})(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|}$$

$$+ \sum_{i=0,1} \sup_{z \in T_n(U_j) \smallsetminus (H_i \cup G_i)} p_\alpha((f_{j_i} - \widetilde{F})(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|} \quad (42)$$

by (41) and the properties of φ_i. Choose $k \in \mathbb{N}$ such that $k > \max(n, \frac{\varepsilon}{2})$ and $\frac{1}{k} < \frac{\varepsilon}{4}$ and, in addition, $-k < x + 1$, if $\infty \notin \Omega_{j_0}$ resp. $-\infty \notin \Omega_{j_1}$. We remark that

$$T_n(U_j) \smallsetminus (H_i \cup G_i) \subset \begin{cases} [(-\infty, -x-1) \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})] \smallsetminus ((-\infty, -x-2] \times [-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}]) & , i = 0, \\ [(x+1, \infty) \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})] \smallsetminus ([x+2, \infty) \times [-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}]) & , i = 1, \end{cases}$$

$$\subset S_k(U_{j_i}) \cup M_i, \quad i = 0, 1,$$

with

$$M_i := \begin{cases} \{z \in \mathbb{C} \mid -x-2 < \operatorname{Re}(z) < -x-1, |\operatorname{Im}(z)| \leq \frac{1}{k}\} & , i = 0, \\ \{z \in \mathbb{C} \mid x+1 < \operatorname{Re}(z) < x+2, |\operatorname{Im}(z)| \leq \frac{1}{k}\} & , i = 1, \end{cases}$$
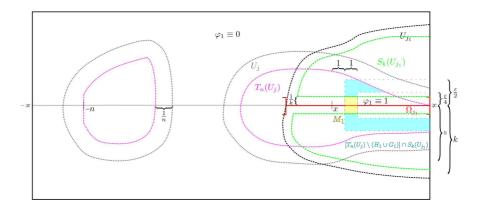
by the choice of k.

FIGURE 18. case $i = 1$: $\infty \in \Omega_j$, $-\infty \notin \Omega_j$, $\infty \in \Omega_{j_1}$, $-\infty \notin \Omega_{j_1}$

The sets M_i, $i = 0, 1$, are obviously bounded and $\overline{M}_i \subset (U_{j_i} \cap \mathbb{C})$. Further, we define the set

$$M_2 := [T_n(U_j) \setminus (H_0 \cup H_1)] \setminus S_k(U_j)$$

which is bounded, since $M_2 \subset \{z \in \mathbb{C} \mid -x - 2 < \mathrm{Re}(z) < x + 2, |\mathrm{Im}(z)| \leq \frac{1}{k}\}$ due to the choice of k, and we have $\overline{M}_2 \subset \overline{T_n(U_j)} \subset (U_j \cap \mathbb{C})$.
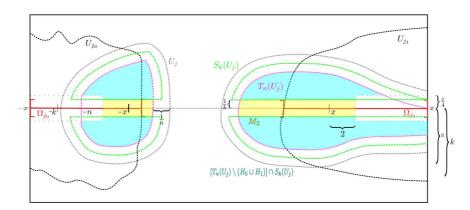

FIGURE 19. case: $\infty \in \Omega_j$, $-\infty \notin \Omega_j$, $-\infty \in \Omega_{j_0}$, $\infty \in \Omega_{j_1}$

These results yield to

$$\sup_{z \in T_n(U_j) \setminus (H_i \cup G_i)} p_\alpha((f_{j_i} - \widetilde{F})(z)) e^{-\frac{1}{n}|\mathrm{Re}(z)|}$$

$$\leq \|f_{j_i}\|_{U_{j_i}^*, k, \alpha} + |\widetilde{F}|_{\overline{\Omega}, k, \alpha} + \sup_{z \in \overline{M}_i} p_\alpha((f_{j_i} - \widetilde{F})(z)) e^{-\frac{1}{n}|\mathrm{Re}(z)|} < \infty$$

for $i = 0, 1$ and

$$\sup_{z \in T_n(U_j) \setminus (H_0 \cup H_1)} p_\alpha((f_j - \widetilde{F})(z)) e^{-\frac{1}{n}|\mathrm{Re}(z)|}$$

$$\leq \|f_j\|_{U_j^*, k, \alpha} + |\widetilde{F}|_{\overline{\Omega}, k, \alpha} + \sup_{z \in \overline{M}_2} p_\alpha((f_j - \widetilde{F})(z)) e^{-\frac{1}{n}|\mathrm{Re}(z)|} < \infty$$

by (34). Thus the right-hand side of (42) is bounded from above.

Let us turn to the still pending estimates in (40), so we have to take a look at the sets $T_n(U_j) \cap H_i$, $i = 0, 1$.
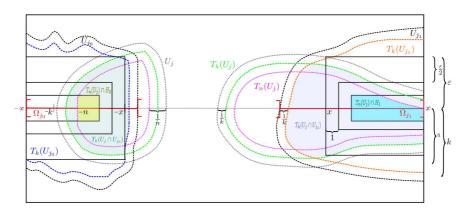
FIGURE 20. case: $\infty \in \Omega_j$, $-\infty \notin \Omega_j$, $\infty \notin \Omega_{j_0}$, $-\infty \in \Omega_{j_0}$, $\infty \in \Omega_{j_1}$, $-\infty \notin \Omega_{j_1}$

We choose $k \in \mathbb{N}$ such that $k > n$ and $\frac{1}{k} < \min(1, \frac{\varepsilon}{2})$ and, in addition, $-k < x+1$, if $\infty \notin \Omega_{j_0}$ resp. $-\infty \notin \Omega_{j_1}$. Let $z \in H_i$, $i = 0, 1$, with $|\mathrm{Im}(z)| < k$. Then $z \in U_{j_i}$ and

$$\mathrm{Re}(z) \leq -x-2 < k, \text{ if } i = 0, \infty \notin \Omega_{j_0}, \quad \text{resp.} \quad \mathrm{Re}(z) \geq x+2 > -k, \text{ if } i = 1, -\infty \notin \Omega_{j_1},$$

by the choice of k as well as

$$\mathrm{d}(z, \mathbb{C} \cap \partial U_{j_i}) \geq \min\left(1, \frac{\varepsilon}{2}\right) > \frac{1}{k},$$

implying $z \in T_k(U_{j_i})$. Since $k > n$, we have $T_n(U_j) \subset T_k(U_j)$ and thus $(T_n(U_j) \cap H_i) \subset [T_k(U_j) \cap T_k(U_{j_i})]$. Now, let $(T_n(U_j) \cap H_i) \neq \varnothing$ for some $i = 0, 1$ (in the case „$= \varnothing$" we have $\sup_{z \in T_n(U_j) \cap H_i} \ldots = -\infty$ in (40)). Let $z \in T_k(U_j) \cap T_k(U_{j_i})$, which is a non-empty set. Then $z \in U_j \cap U_{j_i}$ and $|\mathrm{Im}(z)| < k$. Since $\mathbb{C} \cap \partial(U_j \cap U_{j_i})$ is closed, there is $z_0 \in \mathbb{C} \cap \partial(U_j \cap U_{j_i})$ with

$$\mathrm{d}(z, \mathbb{C} \cap \partial(U_j \cap U_{j_i})) = |z - z_0|$$

if $\mathbb{C} \cap \partial(U_j \cap U_{j_i}) \neq \varnothing$. Moreover,

$$[\mathbb{C} \cap \partial(U_j \cap U_{j_i})] \subset [(\mathbb{C} \cap \partial U_j) \cup (\mathbb{C} \cap \partial U_{j_i})]$$

and thus we obtain

$$\mathrm{d}(z, \mathbb{C} \cap \partial(U_j \cap U_{j_i})) = |z - z_0| \geq \begin{cases} \mathrm{d}(z, \mathbb{C} \cap \partial U_j) & , z_0 \in \mathbb{C} \cap \partial U_j, \\ \mathrm{d}(z, \mathbb{C} \cap \partial U_{j_i}) & , z_0 \in \mathbb{C} \cap \partial U_{j_i}, \end{cases} > \frac{1}{k}$$

if $\mathbb{C} \cap \partial(U_j \cap U_{j_i}) \neq \varnothing$. In the case $\mathbb{C} \cap \partial(U_j \cap U_{j_i}) = \varnothing$ we note that $\mathrm{d}(z, \mathbb{C} \cap \partial(U_j \cap U_{j_i})) = \infty > \frac{1}{k}$. If $\pm\infty \notin \Omega_j \cap \Omega_{j_i}$, we have in addition $-k < \mathrm{Re}(z) < k$. Therefore, $T_k(U_j) \cap T_k(U_{j_i})$ is bounded and its closure is a subset of $U_j \cap U_{j_i} \cap \mathbb{C}$ if $\pm\infty \notin \Omega_j \cap \Omega_{j_i}$, and $[\overline{T_k(U_j) \cap T_k(U_{j_i})}] \subset T_k(U_j \cap U_{j_i})$ if $-\infty \in \Omega_j \cap \Omega_{j_i}$ or $\infty \in \Omega_j \cap \Omega_{j_i}$. Because $f_j - f_{j_i} \in \mathcal{O}^{exp}(U_j \cap U_{j_i}, E)$, this yields to

$$\sup_{z \in T_n(U_j) \cap H_i} p_\alpha((f_j - f_{j_i})(z)) e^{-\frac{1}{n}|\mathrm{Re}(z)|}$$

$$\leq \begin{cases} \sup_{z \in \overline{T_k(U_j) \cap T_k(U_{j_i})}} p_\alpha((f_j - f_{j_i})(z)) e^{-\frac{1}{k}|\mathrm{Re}(z)|} & , \pm\infty \notin \Omega_j \cap \Omega_{j_i}, \\ \|f_j - f_{j_i}\|_{U_j \cap U_{j_i}, k, \alpha} & , \text{else}, \end{cases}$$

$$< \infty.$$

Combining our results, we conclude $\|f_j - F^*\|_{U_j, n, \alpha} < \infty$ for all $n \in \mathbb{N}$, $n \geq 2$, and $\alpha \in \mathfrak{A}$ by (40) and thus $f_j - F^* \in \mathcal{O}^{exp}(U_j, E)$, i.e. $[F^*]_{|\Omega_j} = [f_j]$.

b) Let $[f] \in bv(\Omega, E) = \mathcal{O}^{exp}(U \smallsetminus \overline{\mathbb{R}}, E)/\mathcal{O}^{exp}(U, E)$ where $U \in \mathcal{U}(\Omega)$ and $\Omega \subset \overline{\mathbb{R}}$ open. By Lemma 5.7 there is $F \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\Omega}, E) \subset \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\mathbb{R}}, E)$ such that $f - F \in \mathcal{O}^{exp}(U, E)$. Hence $[F] \in bv(\overline{\mathbb{R}}, E)$ is an extension of $[f]$ to $\overline{\mathbb{R}}$.

c)(i) For an open set $\Omega \subset \overline{\mathbb{R}}$, $\Omega \ne \varnothing$, we have the following (algebraic) isomorphisms

$$\mathcal{R}(\Omega, E) = L(\mathcal{P}_*(\overline{\Omega}), E)/L(\mathcal{P}_*(\partial\Omega), E) \cong \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\Omega}, E)/\mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \partial\Omega, E)$$

$$\cong \mathcal{O}^{exp}((\Omega \times \mathbb{R}) \smallsetminus \overline{\mathbb{R}}, E)/\mathcal{O}^{exp}(\Omega \times \mathbb{R}, E) = bv(\Omega, E).$$

(ii) The first isomorphism is due to Theorem 3.6 and given by the map

$$G_\Omega \colon L(\mathcal{P}_*(\overline{\Omega}), E)/L(\mathcal{P}_*(\partial\Omega), E) \to \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\Omega}, E)/\mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \partial\Omega, E),$$

$$[T] \mapsto [\widetilde{T}]_\sim, \quad \text{with } [\widetilde{T}]_{\overline{\Omega}} = H_{\overline{\Omega}}^{-1}(T),$$

where $H_{\overline{\Omega}}$ is the isomorphism from Theorem 3.6 and we denote by $[\cdot]$ the equivalence classes in $L(\mathcal{P}_*(\overline{\Omega}), E)/L(\mathcal{P}_*(\partial\Omega), E)$, by $[\cdot]_\sim$ the ones in $\mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\Omega}, E)/\mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \partial\Omega, E)$ and by $[\cdot]_{\overline{\Omega}}$ the ones in $\mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\Omega}, E)/\mathcal{O}^{exp}(\overline{\mathbb{C}}, E)$.

well-defined: Let $T_0, T_1 \in L(\mathcal{P}_*(\overline{\Omega}), E)$ such that $[T_0] = [T_1]$, i.e. $T_0 - T_1 \in L(\mathcal{P}_*(\partial\Omega), E)$. Then

$$H_{\overline{\Omega}}^{-1}(T_0 - T_1) = H_{\partial\Omega}^{-1}(T_0 - T_1)$$

by (4) and

$$[\widetilde{T}_0 - \widetilde{T}_1]_{\overline{\Omega}} = [\widetilde{T}_0]_{\overline{\Omega}} - [\widetilde{T}_1]_{\overline{\Omega}} = H_{\overline{\Omega}}^{-1}(T_0) - H_{\overline{\Omega}}^{-1}(T_1) = H_{\overline{\Omega}}^{-1}(T_0 - T_1)$$

$$= H_{\partial\Omega}^{-1}(T_0 - T_1) \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \partial\Omega, E)/\mathcal{O}^{exp}(\overline{\mathbb{C}}, E)$$

holds. Thus $\widetilde{T}_0 - \widetilde{T}_1 \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \partial\Omega, E)$, i.e. $[\widetilde{T}_0 - \widetilde{T}_1]_\sim = 0$. On the other hand, let $T \in L(\mathcal{P}_*(\overline{\Omega}), E)$ and $\widetilde{T}_0, \widetilde{T}_1 \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\Omega}, E)$ such that $[\widetilde{T}_0]_{\overline{\Omega}} = [\widetilde{T}_1]_{\overline{\Omega}} = H_{\overline{\Omega}}^{-1}(T)$. Then $\widetilde{T}_0 - \widetilde{T}_1 \in \mathcal{O}^{exp}(\overline{\mathbb{C}}, E) \subset \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \partial\Omega, E)$ and hence $[\widetilde{T}_0 - \widetilde{T}_1]_\sim = 0$.

injectivity: Let $T \in L(\mathcal{P}_*(\overline{\Omega}), E)$ with $G_\Omega(T) = [\widetilde{T}]_\sim = 0$. Then $\widetilde{T} \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \partial\Omega, E)$ and thus

$$H_{\overline{\Omega}}^{-1}(T) = [\widetilde{T}]_{\overline{\Omega}} \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \partial\Omega, E)/\mathcal{O}^{exp}(\overline{\mathbb{C}}, E).$$

Therefore, we get

$$T = H_{\overline{\Omega}}(H_{\overline{\Omega}}^{-1}(T)) = H_{\partial\Omega}(H_{\overline{\Omega}}^{-1}(T)) \in L(\mathcal{P}_*(\partial\Omega), E)$$

by (3) and so $[T] = 0$.

surjectivity: Let $T_0 \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\Omega}, E)$. Then we have $H_{\overline{\Omega}}([T_0]_{\overline{\Omega}}) \in L(\mathcal{P}_*(\overline{\Omega}), E)$ by Theorem 3.6. We define $T := H_{\overline{\Omega}}([T_0]_{\overline{\Omega}})$ and get

$$H_{\overline{\Omega}}^{-1}(T) = H_{\overline{\Omega}}^{-1}(H_{\overline{\Omega}}([T_0]_{\overline{\Omega}})) = [T_0]_{\overline{\Omega}}$$

by Theorem 3.6 again. This means that $G_\Omega([T]) = [T_0]_\sim$.

(iii) The second isomorphism is defined by the map

$$J_\Omega \colon \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\Omega}, E)/\mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \partial\Omega, E) \to \mathcal{O}^{exp}((\Omega \times \mathbb{R}) \smallsetminus \overline{\mathbb{R}}, E)/\mathcal{O}^{exp}(\Omega \times \mathbb{R}, E),$$

$$[f]_\sim \mapsto [f_{|((\Omega \times \mathbb{R}) \smallsetminus \overline{\mathbb{R}}) \cap \mathbb{C}}]_\Omega,$$

where $[\cdot]_\Omega$ denotes the equivalence classes in $\mathcal{O}^{exp}((\Omega \times \mathbb{R}) \smallsetminus \overline{\mathbb{R}}, E)/\mathcal{O}^{exp}(\Omega \times \mathbb{R}, E)$. This map is well-defined since $\mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \partial\Omega, E) \subset \mathcal{O}^{exp}(\Omega \times \mathbb{R}, E)$.

injectivity: Let $f \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\Omega}, E)$ with $J_\Omega([f]_\sim) = 0$, i.e. $f \in \mathcal{O}^{exp}(\Omega \times \mathbb{R}, E)$. Then it follows that $f \in \mathcal{O}(\mathbb{C} \smallsetminus \partial\Omega, E)$. Further, the estimate

$$|f|_{\partial\Omega, n, \alpha} \le \underbrace{\sup_{z \in S_n(\overline{\Omega})} p_\alpha(f(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|}}_{=|f|_{\overline{\Omega}, n, \alpha}} + \sup_{z \in S_n(\partial\Omega) \smallsetminus S_n(\overline{\Omega})} p_\alpha(f(z)) e^{-\frac{1}{n}|\operatorname{Re}(z)|} \quad (43)$$

holds for all $n \in \mathbb{N}$, $n \geq 2$, and $\alpha \in \mathfrak{A}$. Let us examine the set $S_n(\partial\Omega) \smallsetminus S_n(\overline{\Omega})$. We have for $z \in S_n(\partial\Omega) \smallsetminus S_n(\overline{\Omega})$

$$\mathrm{Re}(z) \in \begin{cases} [\min(\mathbb{R} \cap \partial\Omega), \max(\mathbb{R} \cap \partial\Omega)] & , \ \pm\infty \notin \overline{\Omega}, \\ [-n, n] & , \ \pm\infty \in \partial\Omega, \\ (-\infty, n] & , \ -\infty \in \Omega, \ \infty \in \partial\Omega, \\ [-n, \infty) & , \ -\infty \in \partial\Omega, \ \infty \in \Omega, \\ \mathbb{R} & , \ \pm\infty \in \Omega, \\ [-n, \max(\mathbb{R} \cap \partial\Omega)] & , \ -\infty \in \partial\Omega, \ \infty \notin \overline{\Omega}, \\ (-\infty, \max(\mathbb{R} \cap \partial\Omega)] & , \ -\infty \in \Omega, \ \infty \notin \overline{\Omega}, \\ [\min(\mathbb{R} \cap \partial\Omega), n] & , \ -\infty \notin \overline{\Omega}, \ \infty \in \partial\Omega, \\ [\min(\mathbb{R} \cap \partial\Omega), \infty) & , \ -\infty \notin \overline{\Omega}, \ \infty \in \Omega, \end{cases}$$

and $|\mathrm{Im}(z)| \leq \frac{1}{n}$. Furthermore, we observe that $W \coloneqq \bigcup_{x \in \mathbb{R} \cap \partial\Omega} \mathbb{D}_{\frac{1}{n}}(x)$ is open and

$$\overline{S_n(\partial\Omega) \smallsetminus S_n(\overline{\Omega})} = \left(\overline{[S_n(\partial\Omega) \smallsetminus S_n(\overline{\Omega})]} \smallsetminus W\right) \subset \overline{W^C} = W^C \subset \mathbb{C} \smallsetminus \partial\Omega. \qquad (44)$$

So, if $\pm\infty \notin \Omega$, then $\overline{S_n(\partial\Omega) \smallsetminus S_n(\overline{\Omega})}$ is a compact subset of $\mathbb{C} \smallsetminus \partial\Omega$. Due to (43) and since $f \in \mathcal{O}(\mathbb{C} \smallsetminus \partial\Omega, E)$, we get $|f|_{\partial\Omega, n, \alpha} < \infty$ in this case. Let $-\infty \in \Omega$ or $\infty \in \Omega$. Then there are $x_i \in \mathbb{R}$, $i = 0, 1$, such that $[-\infty, x_0] \subset \Omega$ resp. $[x_1, \infty] \subset \Omega$. We choose $k \in \mathbb{N}$ such that $k > n$ and, in addition,

$$k > x_0, \ \text{if } -\infty \in \Omega, \ \infty \notin \Omega, \quad \text{resp.} \quad -k < x_1, \ \text{if } -\infty \notin \Omega, \ \infty \in \Omega.$$

Then we obtain for $z \in \left[S_n(\partial\Omega) \smallsetminus S_n(\overline{\Omega})\right] \smallsetminus T_k(\Omega \times \mathbb{R}) =: M$

$$|\mathrm{Re}(z)| \leq \begin{cases} \max(|x_0|, n) & , \ -\infty \in \Omega, \ \infty \in \partial\Omega, \\ \max(|x_1|, n) & , \ -\infty \in \partial\Omega, \ \infty \in \Omega, \\ \max(|x_0|, |x_1|) & , \ \pm\infty \in \Omega, \\ \max(|x_0|, |\max(\mathbb{R} \cap \partial\Omega)|) & , \ -\infty \in \Omega, \ \infty \notin \overline{\Omega}, \\ \max(|\min(\mathbb{R} \cap \partial\Omega)|, |x_1|) & , \ -\infty \notin \overline{\Omega}, \ \infty \in \Omega, \end{cases}$$

by the choice of k and as $\partial\Omega \subset \Omega^C$. Hence M is bounded, thus $\overline{M}$ compact, and $\overline{M} \subset (\mathbb{C} \smallsetminus \partial\Omega)$ by (44). Therefore, we gain

$$\sup_{z \in S_n(\partial\Omega) \smallsetminus S_n(\overline{\Omega})} p_\alpha(f(z)) e^{-\frac{1}{n}|\mathrm{Re}(z)|}$$

$$\leq \underbrace{\sup_{z \in T_k(\Omega \times \mathbb{R})} p_\alpha(f(z)) e^{-\frac{1}{k}|\mathrm{Re}(z)|}}_{=\|f\|_{\Omega \times \mathbb{R}, k, \alpha}} + \sup_{z \in \overline{M}} p_\alpha(f(z)) e^{-\frac{1}{n}|\mathrm{Re}(z)|} < \infty$$

since $f \in \mathcal{O}^{exp}(\Omega \times \mathbb{R}, E)$ and $f \in \mathcal{O}(\mathbb{C} \smallsetminus \partial\Omega, E)$. By (43) we have $|f|_{\partial\Omega, n, \alpha} < \infty$ for all $n \in \mathbb{N}$, $n \geq 2$, and $\alpha \in \mathfrak{A}$ in this case as well and thus $f \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \partial\Omega, E)$, proving the injectivity of J_Ω.

surjectivity: Let $[f]_\Omega \in \mathcal{O}^{exp}((\Omega \times \mathbb{R}) \smallsetminus \overline{\mathbb{R}}, E) / \mathcal{O}^{exp}(\Omega \times \mathbb{R}, E)$. By Lemma 5.7 there is $F \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus \overline{\Omega}, E)$ such that $f - F \in \mathcal{O}^{exp}(\Omega \times \mathbb{R}, E)$, i.e. $J_\Omega([F]_\sim) = [f]_\Omega$.

(iv) The last step is to prove that these isomorphisms, which we denote by $h_\Omega \coloneqq J_\Omega \circ G_\Omega$, are compatible with the respective restrictions, i.e. that for open sets

$\Omega_1 \subset \Omega \subset \overline{\mathbb{R}}$ the diagram

$$
\begin{array}{ccc}
\mathcal{R}(\Omega, E) & \xrightarrow{\;h_\Omega\;} & bv(\Omega, E) \\
{\scriptstyle R^{\mathcal{R}}_{\Omega,\Omega_1}} \Big\downarrow & & \Big\downarrow {\scriptstyle R^{bv}_{\Omega,\Omega_1}} \\
\mathcal{R}(\Omega_1, E) & \xrightarrow[\;h_{\Omega_1}\;]{} & bv(\Omega_1, E)
\end{array}
$$

commutes. Let $T \in L(\mathcal{P}_*(\overline{\Omega}), E)$. We choose a representative T_0 of $R^{\mathcal{R}}_{\Omega,\Omega_1}([T])$. By the definition of the restriction

$$T_0 - T \in L(\mathcal{P}_*(\overline{\Omega} \smallsetminus \Omega_1), E) \tag{45}$$

is valid. Let $\widetilde{T}_0$ be a representative of $H^{-1}_{\overline{\Omega}}(T_0)$. Then we have

$$(h_{\Omega_1} \circ R^{\mathcal{R}}_{\Omega,\Omega_1})([T]) = h_{\Omega_1}([T_0]_1) = (J_{\Omega_1} \circ G_{\Omega_1})([T_0]_1) = [\widetilde{T}_{0|((\Omega_1 \times \mathbb{R}) \smallsetminus \overline{\mathbb{R}}) \cap \mathbb{C}}]_{\Omega_1}.$$

On the other hand, let $\widetilde{T}$ be a representative of $H^{-1}_{\overline{\Omega}}(T)$. Then we get

$$(R^{bv}_{\Omega,\Omega_1} \circ h_\Omega)([T]) = R^{bv}_{\Omega,\Omega_1}([\widetilde{T}_{|((\Omega \times \mathbb{R}) \smallsetminus \overline{\mathbb{R}}) \cap \mathbb{C}}]_\Omega) = [\widetilde{T}_{|((\Omega_1 \times \mathbb{R}) \smallsetminus \overline{\mathbb{R}}) \cap \mathbb{C}}]_{\Omega_1}.$$

Further,

$$[\widetilde{T}_0 - \widetilde{T}]_{\overline{\Omega}} = H^{-1}_{\overline{\Omega}}(T_0 - T) = H^{-1}_{\overline{\Omega} \smallsetminus \Omega_1}(T_0 - T) \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus (\overline{\Omega} \smallsetminus \Omega_1), E)/\mathcal{O}^{exp}(\overline{\mathbb{C}}, E)$$

by (45) and (4). Therefore, $\widetilde{T}_0 - \widetilde{T} \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus (\overline{\Omega} \smallsetminus \Omega_1), E) \subset \mathcal{O}^{exp}(\Omega_1 \times \mathbb{R}, E)$, which implies $(h_{\Omega_1} \circ R^{\mathcal{R}}_{\Omega,\Omega_1})([T]) = (R^{bv}_{\Omega,\Omega_1} \circ h_\Omega)([T])$. By virtue of Proposition 2.3 it follows that $\mathcal{R}(E)$ is a sheaf which is isomorphic to $bv(E)$. $\qquad\square$

Theorem 5.9 a), b) for $E = \mathbb{C}$ can be found in [36, Corollary 3.2.3, p. 482], and Theorem 5.9 a)-c) for Fréchet spaces E in [33, 3.8 Folgerung, p. 40], [33, 3.12 Satz, p. 44] and [33, Satz, p. 45-46]. The counterpart of Theorem 5.9 in the theory of vector-valued hyperfunctions is [7, Theorem 6.9, p. 1125] and immediately we get the following corollary whose counterpart for hyperfunctions is [7, Corollary 6.10, p. 1126].

5.10. **Corollary.** *Let E be strictly admissible and $\Omega \subset \overline{\mathbb{R}}$ open. $\{\mathcal{R}(\omega, E) \mid \omega \subset \Omega \text{ open}\}$, equipped with the restrictions of Definition 5.4, forms a flabby sheaf.*

Corollary 5.10 provides an answer to a problem stated by Ito, at least for E-valued Fourier hyperfunctions in one variable (see [29, Problem B, p. 18]).

Now, we want to describe the sections with support in a given compact set $K \subset \overline{\mathbb{R}}$. We recall the definition of the support of a section of a sheaf (see [2, 1.10 Definition, p. 7]). Let Ω be a topological space, $(\mathcal{F}, R^{\mathcal{F}})$ a sheaf on Ω and $f \in \mathcal{F}(\Omega)$ a section of a sheaf. Then the *support* of f, denoted by $\mathrm{supp}_{\mathcal{F}} f$ or shortly $\mathrm{supp}\, f$, is the complement of the largest open subset of Ω on which $f = 0$, i.e.

$$\mathrm{supp}\, f = \Omega \smallsetminus \bigcup_{V \in Z_f} V$$

where $Z_f := \{V \mid V \subset \Omega \text{ open}, f_{|V} = 0\}$ (condition $(S1)$ is used in this definition). This directly yields to the following description of the support of an element of $bv(\Omega, E)$ for an open set $\Omega \subset \overline{\mathbb{R}}$ and a strictly admissible space E. Namely, let $f = [F] \in \mathcal{O}^{exp}(U \smallsetminus \overline{\mathbb{R}}, E)/\mathcal{O}^{exp}(U, E)$, where $U \in \mathcal{U}(\Omega)$, and $\Omega_1 \subset \Omega$ is open. If $-\infty \in \Omega$ or $\infty \in \Omega$, we define the set

$$S_n(U, \Omega_1) := \{z \in U \cap \mathbb{C} \mid \mathrm{d}(z, (\overline{\Omega} \cap \mathbb{R}) \smallsetminus \Omega_1) > \tfrac{1}{n}, \mathrm{d}(z, \mathbb{C} \cap \partial U) > \tfrac{1}{n}, |\mathrm{Im}(z)| < n\}$$

$$\cap \begin{cases} \mathbb{C} & , \pm\infty \in \Omega, \\ \{z \in \mathbb{C} \mid \mathrm{Re}(z) > -n\} & , \infty \in \Omega, -\infty \notin \Omega, \\ \{z \in \mathbb{C} \mid \mathrm{Re}(z) < n\} & , \infty \notin \Omega, -\infty \in \Omega, \end{cases}$$

$$\diagdown \begin{cases} [(-\infty, -n] \cup [n, \infty)] + \mathrm{i}[-\frac{1}{n}, \frac{1}{n}] & , \pm\infty \notin \Omega_1, \\ (-\infty, -n] + \mathrm{i}[-\frac{1}{n}, \frac{1}{n}] & , -\infty \notin \Omega_1, \infty \in \Omega_1, \\ [n, \infty) + \mathrm{i}[-\frac{1}{n}, \frac{1}{n}] & , \infty \notin \Omega_1, -\infty \in \Omega_1, \\ \varnothing & , \pm\infty \in \Omega_1, \end{cases}$$

for $n \in \mathbb{N}$, $n \geq 2$.

If $-\infty \in \Omega$ or $\infty \in \Omega$, then $f_{|\Omega_1} = 0$ is equivalent to

a) F can be extended to a holomorphic function on $[(U \smallsetminus \overline{\mathbb{R}}) \cup \Omega_1] \cap \mathbb{C}$ if $\pm\infty \notin \Omega_1$, or

b) F can be extended to a holomorphic function on $[(U \smallsetminus \overline{\mathbb{R}}) \cup \Omega_1] \cap \mathbb{C}$ and

$$|F|_{U,\Omega_1,n,\alpha} := \sup_{z \in S_n(U,\Omega_1)} p_\alpha(F(z)) e^{-\frac{1}{n}|\mathrm{Re}(z)|} < \infty \tag{46}$$

for every $n \in \mathbb{N}$, $n \geq 2$, and $\alpha \in \mathfrak{A}$ if $-\infty \in \Omega_1$ or $\infty \in \Omega_1$.

We remark that (46) is valid in (a) as well. If $\pm\infty \notin \Omega$, then $f_{|\Omega_1} = 0$ is equivalent to statement a).

Observing that

$$\left[(U \smallsetminus \overline{\mathbb{R}}) \cup \bigcup_{V \in Z_f} V\right] \cap \mathbb{C} = [(U \smallsetminus \overline{\mathbb{R}}) \cup (\Omega \smallsetminus \mathrm{supp}\, f)] \cap \mathbb{C} = (U \smallsetminus \mathrm{supp}\, f) \cap \mathbb{C},$$

since $U \in \mathcal{U}(\Omega)$, and

$$(\overline{\Omega} \cap \mathbb{R}) \smallsetminus \bigcup_{V \in Z_f} V = (\mathrm{supp}\, f \cup \partial\Omega) \cap \mathbb{R},$$

where the closure and the boundary are taken in $\overline{\mathbb{R}}$, we get $F \in \mathcal{O}((U \smallsetminus \mathrm{supp}\, f) \cap \mathbb{C}, E)$ and, if $-\infty \in \Omega_1$ or $\infty \in \Omega_1$, in addition,

$$|F|_{U,\bigcup_{V \in Z_f} V,n,\alpha} = \sup_{z \in S_n(U,\bigcup_{V \in Z_f} V)} p_\alpha(F(z)) e^{-\frac{1}{n}|\mathrm{Re}(z)|} < \infty$$

for every $n \in \mathbb{N}$, $n \geq 2$, and $\alpha \in \mathfrak{A}$ where we have

$$\mathrm{d}\big(z, (\overline{\Omega} \cap \mathbb{R}) \smallsetminus \bigcup_{V \in Z_f} V\big) = \mathrm{d}(z, (\mathrm{supp}\, f \cup \partial\Omega) \cap \mathbb{R})$$

in the definition of $S_n(U, \bigcup_{V \in Z_f} V)$.

Now, let $K \subset \Omega$ be compact, set

$$bv_K(\Omega, E) := \{f \in bv(\Omega, E) \mid \mathrm{supp}\, f \subset K\}$$

and for $U \in \mathcal{U}(\Omega)$

$$\mathcal{O}^{exp}(U \smallsetminus K, E) := \{f \in \mathcal{O}((U \smallsetminus K) \cap \mathbb{C}, E) \mid \forall\, n \in \mathbb{N}, n \geq 2, \alpha \in \mathfrak{A}: |F|_{U,\Omega \smallsetminus K,n,\alpha} < \infty\}$$

if $-\infty \in \Omega$ or $\infty \in \Omega$, resp.

$$\mathcal{O}^{exp}(U \smallsetminus K, E) := \mathcal{O}((U \smallsetminus K) \cap \mathbb{C}, E)$$

if $\pm\infty \notin \Omega$. Due to the considerations above and Lemma 5.7 we gain the following description of $bv_K(\Omega, E)$ whose special cases that $E = \mathbb{C}$ or more general that E is a Fréchet space are given in [36, Theorem 3.2.1, p. 480] and [33, 3.6 Satz, p. 37].

5.11. Lemma. *Let E be strictly admissible, $\Omega \subset \overline{\mathbb{R}}$ open and $K \subset \Omega$ compact. For any $U \in \mathcal{U}(\Omega)$ we have the (algebraic) isomorphism*

$$bv_K(\Omega, E) \cong \mathcal{O}^{exp}(U \smallsetminus K, E)/\mathcal{O}^{exp}(U, E).$$

In particular, we have

$$bv_K(\overline{\mathbb{R}}, E) \cong \mathcal{O}^{exp}(\overline{\mathbb{C}} \smallsetminus K)/\mathcal{O}^{exp}(\overline{\mathbb{C}}, E) \cong L(\mathcal{P}_*(K), E).$$

Proof. Using Lemma 5.7, we represent $bv(\Omega, E)$ by $\mathcal{O}^{exp}(U \setminus \overline{\mathbb{R}}, E)/\mathcal{O}^{exp}(U, E)$. Then the identity-map

$$\{[F] \in \mathcal{O}^{exp}(U \setminus \overline{\mathbb{R}}, E)/\mathcal{O}^{exp}(U, E) \mid \text{supp}[F] \subset K\} \to \mathcal{O}^{exp}(U \setminus K, E)/\mathcal{O}^{exp}(U, E),$$
$$[F] \mapsto [F],$$

is (well-)defined and surjective by the considerations above and obviously injective.

Now, let $\Omega := \overline{\mathbb{R}}$, set $\Omega_1 := \overline{\mathbb{R}} \setminus K$ and choose $U := \overline{\mathbb{C}}$. We claim that the definition of the space $\mathcal{O}^{exp}(\overline{\mathbb{C}} \setminus K, E)$ in the sense above and in the sense of Definition 3.1 coincide (and therefore the spaces have the same symbol). Let $n \in \mathbb{N}$, $n \geq 2$. Then

$$\text{d}(z, (\overline{\Omega} \cap \mathbb{R}) \setminus \Omega_1) = \text{d}(z, K \cap \mathbb{R})$$

and

$$\text{d}(z, \mathbb{C} \cap \partial U) = \text{d}(z, \varnothing) = \infty > \frac{1}{n}$$

holds for $z \in \mathbb{C}$. Further,

$$\left. \begin{array}{r} \pm\infty \notin \overline{\mathbb{R}} \setminus K \\ -\infty \notin \overline{\mathbb{R}} \setminus K \\ \infty \notin \overline{\mathbb{R}} \setminus K \\ \pm\infty \in \overline{\mathbb{R}} \setminus K \end{array} \right\} \quad \text{is equivalent to} \quad \left\{ \begin{array}{l} \pm\infty \in K \\ -\infty \in K \\ \infty \in K \\ \pm\infty \notin K \end{array} \right.$$

and hence we obtain $S_n(\overline{\mathbb{C}}, \overline{\mathbb{R}} \setminus K) = S_n(K)$. Thus the claim is proved. Therefore,

$$bv_K(\overline{\mathbb{R}}, E) \cong \mathcal{O}^{exp}(\overline{\mathbb{C}} \setminus K, E)/\mathcal{O}^{exp}(\overline{\mathbb{C}}, E) \cong L(\mathcal{P}_*(K), E)$$

holds by Theorem 3.6, which proves the endorsement. $\square$

We remark that this isomorphism induces a reasonable locally convex Hausdorff topology on $bv_K(\overline{\mathbb{R}}, E)$ since $L(\mathcal{P}_*(K), E)$ has such a topology.

As already mentioned, we are convinced that a reasonable theory of E-valued Fourier hyperfunctions (in one variable) should produce a flabby sheaf $\mathcal{F}$ on $\overline{\mathbb{R}}$ such that the set of sections supported by a compact subset $K \subset \overline{\mathbb{R}}$ coincides, in the sense of being isomorphic, with $L(\mathcal{P}_*(K), E)$ since the restricted sheaf $\mathcal{F}_{|\mathbb{R}}$ then satisfies the conditions of Domański and Langenbruch for a reasonable theory of E-valued hyperfunctions. In addition, the map $\mathfrak{F} \colon \mathcal{F}(\overline{\mathbb{R}}) \to \mathcal{F}(\overline{\mathbb{R}})$, defined by $\mathfrak{F} := J^{-1} \circ \mathscr{F}_\star \circ J$, where $J \colon \mathcal{F}(\overline{\mathbb{R}}) \to L(\mathcal{P}_*(\overline{\mathbb{R}}), E)$ is an isomorphism existing by assumption and $\mathscr{F}_\star$ the Fourier transformation of Corollary 3.10, can be regarded as the Fourier transformation on the space of global sections and is an isomorphism.

If E is strictly admissible, the sheaves $bv(E)$ and $\mathcal{R}(E)$ satisfy this condition for a reasonable theory of E-valued Fourier hyperfunctions by Theorem 5.9 and Lemma 5.11 (for $\mathcal{R}(E)$ remark that sheaf isomorphisms preserve supports, so the definition of a support in Proposition 3.7 b) was well-chosen). The next theorem confirms that the sufficient condition of E being strictly admissible is also necessary for a reasonable theory of E-valued Fourier hyperfunctions in one variable if E is an ultrabornological PLS-space and describes further equivalent sufficient and necessary conditions. We use its counterpart for vector-valued hyperfunctions [7, Theorem 8.9, p. 1139] in the proof.

5.12. **Theorem.** *Let E be a complex ultrabornological PLS-space. Then the following assertions are equivalent:*

a) There is a flabby sheaf $\mathcal{F}$ on some open set $\varnothing \neq \Omega \subset \overline{\mathbb{R}}$ such that

$$\mathcal{F}_K(\Omega) := \{T \in \mathcal{F}(\Omega) \mid \text{supp}_{\mathcal{F}}(T) \subset K\}$$
$$\cong L(\mathcal{P}_*(K), E) \quad \text{for any compact } K \subset \Omega.$$

b) *There is a flabby sheaf $\mathcal{F}$ on $\overline{\mathbb{R}}$ such that*

$$\mathcal{F}_K(\overline{\mathbb{R}}) := \{T \in \mathcal{F}(\overline{\mathbb{R}}) \mid \operatorname{supp}_{\mathcal{F}}(T) \subset K\}$$
$$\cong L(\mathcal{P}_*(K), E) \quad \text{for any compact } K \subset \overline{\mathbb{R}}.$$

c) *E is strictly admissible.*
d) *$P(D): \mathcal{C}^{\infty}(U,E) \to \mathcal{C}^{\infty}(U,E)$ is surjective for some (any) elliptic linear partial differential operator $P(D)$ and some (any) open set $U \subset \mathbb{R}^d$ and some (any) $d \in \mathbb{N}$, $d \geq 2$.*
e) *E has (PA).*

Proof. $e) \Leftrightarrow d)$: [7, Corollary 4.1, p. 1113] resp. [7, Corollary 3.9, p. 1112]
$e) \Rightarrow c)$: Theorem 4.3 c)
$c) \Rightarrow b)$: Theorem 5.9 and Lemma 5.11
$b) \Rightarrow a)$: Obvious with $\Omega := \overline{\mathbb{R}}$.
$a) \Rightarrow e)$: Let there be a flabby sheaf $\mathcal{F}$ on some open set $\varnothing \neq \Omega \subset \overline{\mathbb{R}}$ such that

$$\mathcal{F}_K(\Omega) = \{T \in \mathcal{F}(\Omega) \mid \operatorname{supp}_{\mathcal{F}}(T) \subset K\}$$
$$\cong L(\mathcal{P}_*(K), E) \quad \text{for any compact } K \subset \Omega.$$

Then the restriction $\mathcal{F}_{|\Omega \cap \mathbb{R}}$ of $\mathcal{F}$ to $\Omega \cap \mathbb{R}$ is a flabby sheaf as well such that

$$(\mathcal{F}_{|\Omega \cap \mathbb{R}})_K(\Omega \cap \mathbb{R}) = \{T \in \mathcal{F}_{|\Omega \cap \mathbb{R}}(\Omega \cap \mathbb{R}) \mid \operatorname{supp}_{\mathcal{F}_{|\Omega \cap \mathbb{R}}}(T) \subset K\}$$
$$\cong L(\mathscr{A}(K), E) \quad \text{for any compact } K \subset (\Omega \cap \mathbb{R})$$

since $\mathcal{P}_*(K) = \mathscr{A}(K)$ for every compact set $K \subset \mathbb{R}$. By virtue of [7, Theorem 8.9, p. 1139] this implies that E has (PA). $\qquad \square$

Due to the preceding theorem a reasonable theory of E-valued Fourier hyperfunctions (in one variable) does not exist for the ultrabornological PLS-spaces E from Example 4.5 a).

5.13. Remark. It follows from Theorem 5.12 „$d) \Leftrightarrow e)$" with $P(D) = \overline{\partial}$ and the fact that ultrabornological PLS-spaces are complete, in particular, quasi-complete that [19, Theorem 3.1, p. 989] (Dolbeaut-Grothendieck resolution of $^E\widetilde{\mathcal{O}}^p$, cf. [20, 2.1.3 Theorem, p. 76]) is not correct for $p = n = 1$ and ultrabornological PLS-spaces E without (PA), for instance, for the spaces E from Example 4.5 a).

Clearly, there are still some open problems.

5.14. Problem.   (i) Is *strict admissibility* a necessary condition for the existence of a reasonable theory of E-valued Fourier hyperfunctions for general $\mathbb{C}$-lcHs E? In particular, does such a reasonable theory exist for the spaces E from Example 4.5 b)?
 (ii) Are strict admissibility and admissibility equivalent?
(iii) Is strict admissibility of E equivalent to belonging to the classes of spaces from Theorem 4.3?
(iv) Do the results for E-valued Fourier hyperfunctions in one variable ($d = 1$) carry over to several variables ($d \geq 2$)?

One way to tackle Problem 5.14 (iv) might be to adapt the approach from vector-valued hyperfunctions [7] as decsribed in [46, Chapter 7, p. 153-155]. Maybe, another way is to use the heat method developed by Matsuzawa in [59], [60] and [61], namely, to represent $\mathbb{C}$-valued hyperfunctions as boundary values of solutions of the heat equation, which was transferred to $\mathbb{C}$-valued Fourier hyperfunctions in [3], [5], [37] and [38].

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