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Hyperfunction quantum field theory: Basic structural results

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The choice of the class E' of generalized functions on space-time in which to formulate general relativistic quantum field theory (QFT) is discussed. A first step is to isolate a set of conditions on E' that allows a formulation of QFT in otherwise the same way as the original proposal by Wightman [Ark. Fys. **28**, 129 (1965)], where E' is the class of tempered distributions. It is stressed that the formulation of QFT in which E' equals the class of Fourier hyperfunctions on space-time meets the following requirements: (A) Fourier hyperfunctions generalize tempered distributions thus allowing more singular fields as suggested by concrete models; (B) Fourier hyperfunction quantum fields are localizable both in space-time and in energy-momentum space thus allowing the physically indispensable standard interpretation of Poincaré covariance, local commutativity, and localization of energy-momentum spectrum; and (C) in Fourier hyperfunction quantum field theory almost all the basic structural results of "standard" QFT (existence of a PCT operator, spin-statistics theorems, existence of a scattering operator, etc.) hold. Finally, a short introduction to that part of Fourier hyperfunction theory needed in this context is given.

I. INTRODUCTION

A. Some motivation

A formulation of general relativistic quantum field theory (QFT) always has to start with a decision about the choice of the test-function space. For well known reasons the traditional choice for the test-function space E is

$$E = \mathscr{S}(\mathbb{R}^4,V) \simeq \mathscr{S}(\mathbb{R}^4) \otimes V, \tag{1.1}$$

where V is a finite-dimensional vector space and $\mathscr{S}(\mathbb{R}^4)$ is the Schwartz space of all C^∞ functions on space-time $\mathbb{R}^4$ that decay together with all their derivatives faster than any $(\text{polynomial})^{-1}$ (see Refs. 1–3).

Since the early days of QFT, for various reasons, there has been some discussion on this choice in the literature. Later we will discuss some of these proposals. The main reason for considering other test-function spaces are indications coming from model constructions that one has to admit (A) that there are more singular than tempered fields, respectively, stronger growth properties of the fields in energy-momentum space. This requirement is fulfilled by a test-function space E if the elements of E are "smoother" in coordinate space and decay more rapidly in energy-momentum space than those in $\mathscr{S}(\mathbb{R}^4)$. If one has a choice for the test-function space E that meets requirements (A) one usually gets into trouble with (B) an unambiguous and clear motion of localization in coordinate and momentum space, and accordingly not much is then known about (C) the permanence of the basic structural results known in QFT for tempered fields (more details follow later).

In this paper we want to show that there is a test-function space E that satisfies all three requirements (A)–(C). This test-function space

$$E = \underset{\sim}{\mathscr{O}}(\mathbb{D}^4,V) \tag{1.2}$$

is defined and described in Sec. II. Elements of its topological dual E' are called "Fourier hyperfunctions."

The suggestion to use a Fourier hyperfunction in quantum field theory has been made by Nagamachi and Mugibayashi in a series of papers.[4–7] This first suggestion is, at least for nonexperts in (Fourier) hyperfunctions, not always very transparent and clear, thus hiding in part its main achievements.

Accordingly one goal of this paper is to give a short but clear and complete introduction to QFT in terms of Fourier hyperfunctions. In particular, we present a more transparent (for nonexperts in hyperfunctions) account of the highly nontrivial fact that QFT in terms of Fourier hyperfunctions can deal very well in a "good physical understanding" with the localization problem in coordinate and momentum space [point (B) above] though the underlying space of test functions contains no elements of compact support, neither in coordinate nor in momentum space.

An important hint in favor of QFT in terms of Fourier hyperfunctions comes from the construction of concrete models. This has been discussed in more detail by Wightman.[8]

B. Quantum fields and their dependence on the test-function space

We begin by recalling the defining assumptions of general quantum field theory. For reasons that will become evident later we present here a variation of the set of assumptions proposed by Gårding and Wightman. In order to stress our point of view that the choice of the space of test functions is at one's disposal according to the problems at hand we start by isolating a list of conditions on a space E of functions

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on space-time in order that E be "admissible" as a test-function space of a relativistic quantum field theory.

(H_0) The test-function space E:

(a) The test-function space E is a locally convex topological vector space of functions on space-time $\mathbb{R}^4$.

(i) E admits the Fourier transformation $\mathscr{F}$ as an isomorphism of topological vector spaces.

(ii) For continuous linear functionals on E and on $\widetilde{E} = \mathscr{F}E$ the notion of support is available.

(b) On E and on $\widetilde{E}$ continuous involutions $f \to f^*$ are defined satisfying $(\mathscr{F}f)^* = \mathscr{F}(f^*)$, for all $f \in E$.

(c) The vector space E has a Z_2 grading and is accordingly decomposed into subspaces of "even" and "odd" elements:

$$E = E_0 \oplus E_1.$$

(d) The universal covering group $G = \mathrm{iSL}(2,\mathbb{C})$ of the Poincaré group acts on E by continuous linear maps α_g: $E \to E$, $g \in G$, such that, for all $f \in E$ and all $g \in G$,

(i) $\alpha_g(f)^* = \alpha_g(f^*)$,

(ii) $g \to \alpha_g(f)$ is a differentiable map $G \to E$,

(iii) α_g preserves the grading.

(H_1) Fields over E or fields with test-function space E: A field A over such a vector space E with state space $\mathscr{H}$, domain $\mathscr{D}$, and cyclic unit vector Φ_0 is specified in the following way.

(a) The state space is a (separable) complex Hilbert space $\mathscr{H}$.

(b) The domain $\mathscr{D}$ is a dense subspace of $\mathscr{H}$ containing the cyclic vector Φ_0.

(c) The field A is a linear map form E into the algebra $L(\mathscr{D},\mathscr{D})$ of linear operators $\mathscr{D} \to \mathscr{D}$ such that the following conditions hold.

(i) For all $\Phi,\Psi \in \mathscr{D}$, $f \to (\Phi, A(f)\Psi)$ is a continuous linear map $E \to \mathbb{C}$.

(ii) For each $f \in E$, the adjoint operator $A(f)^*$ of the densely defined operator $A(f)$ in $\mathscr{H}$ is an extension of $A(f^*)$:

$$A(f^*) \subset A(f)^*.$$

(iii) The linear span

$$\mathscr{D}_0 = \text{lin span}\{\Phi_0, A(f_{j_1}) \cdots A(f_{j_n})\Phi_0 \mid f_{j_i} \in E,\ n = 1,2,\ldots\}$$

is dense in $\mathscr{H}$.

(H_2) Poincaré covariance: A field $(A,\mathscr{H},\mathscr{D},\Phi_0)$ over E is said to be Poincaré covariant if and only if there is a unitary continuous representation U of $G = \mathrm{iSL}(2,\mathbb{C})$ on the Hilbert space $\mathscr{H}$ such that, for all $g \in G$ and all $f \in E$,

$$U(g)\mathscr{D} = \mathscr{D},$$

$$U(g)A(f)U(g)^* = A(\alpha_g f).$$

(H_3) Energy-momentum spectrum Σ: The energy-momentum spectrum Σ of the theory equals the spectrum $\sigma(P)$ of the infinitesimal generator $P = (P^0,P^1,P^2,P^3)$ of the time–space translations in the representation U, i.e.,

$$U(a,\mathbf{1}) = e^{ia\cdot P}, \quad a \in \mathbb{R}^4.$$

It is contained in the closed "forward light cone"

$$\overline{V}_+ = \{(q^0,\mathbf{q}) \in \mathbb{R}^4 \mid q^0 \geqslant |\mathbf{q}|, \quad \mathbf{q} \in \mathbb{R}^3\}$$

and contains the origin, i.e., $0 \in \Sigma \subset \overline{V}_+$.

(H_4) Locality (local commutativity): The restrictions A_α of A to E_α, $\alpha = 0,1$, satisfy, for all $\alpha, \beta \in \{0,1\}$,

$$\operatorname{supp}\langle A_\alpha, A_\beta\rangle \subset K,$$

where

$$K = \{(x,y) \in \mathbb{R}^4 \times \mathbb{R}^4 \mid y - x \in \overline{V} = \overline{V}_+ \cup (-\overline{V}_+)\}$$

and where $\langle A_\alpha, A_\beta\rangle$: $E_\alpha \times E_\beta \to L(\mathscr{D},\mathscr{D})$ is defined by

$$\langle A_\alpha, A_\beta\rangle(f_\alpha, f_\beta) = A_\alpha(f_\alpha)A_\beta(f_\beta) - \sigma_{\alpha\beta} A_\beta(f_\beta)A_\alpha(f_\alpha),$$

with $\sigma_{\alpha\beta} \in \{1, -1\}$, usually $\sigma_{\alpha\beta} = (-1)^{\alpha\beta}$.

(H_5) Uniqueness of the vacuum state: The subspace $\mathscr{H}_0$ of all translation invariant states in $\mathscr{H}$, i.e.,

$$\mathscr{H}_0 = \{\Psi \in \mathscr{H} \mid U(a,\mathbf{1})\Psi = \Psi, \text{ for all } a \in \mathbb{R}^4\},$$

is one dimensional and is generated by the cyclic unit vector Φ_0:

$$\mathscr{H}_0 = C\Phi_0.$$

Conditions (H_0)–(H_5) characterize a *relativistic quantum field A over E.*

Remark 1.1:

(a) For well known reasons the original choice for the test-function space E was $E = \mathscr{S}(\mathbb{R}^4, V)$, where V is some finite-dimensional vector space depending on the "type of fields" under discussion. Here the type of a field is specified by its transformation properties with respect to G, i.e., by α_g, $g \in G$. Clearly this test-function space satisfies condition (H_0).

(b) Notice that by an appropriate choice of the test-function space E (together with the action α of G on E and the involution $*$) the general case of a finite number of scalar, vector, tensor, and/or spinor fields as well as the case of non-Hermitian fields is covered by our formulation.

(c) Sometimes the spectral assumption (H_3) is strengthened by the requirement that the point $p = 0$ be isolated in Σ.

(d) The realization of the locality condition (H_4) [and the spectral condition (H_3)] depends on the test-function space E. If E contains functions on space-time with compact support this is understood in the obvious way. Otherwise an appropriate interpretation of this condition has to be given.

In any case, (H_4) says that the bilinear functional

$$(f,g) \to (\Psi, \langle A_\alpha, A_\beta\rangle(f,g)\Phi)$$

on $E \times E$ has its "support" in K for any $\Psi, \Phi \in \mathscr{D}$.

The value of $\sigma_{\alpha\beta} \in \{1, -1\}$ has to be specified according to the type of fields in agreement with the "spin and statistics" theorem.

(e) It is mainly part (ii) of condition (a) in the characterization (H_0) of an "admissible" test-function space that prevents an easy and/or obvious choice of E besides the traditional one [(1.1)].

Clearly one was well aware already at the beginning of general QFT that the choice of the underlying space of test functions is not only a technical assumption but also has implications of physical relevance.

(1) The allowed growth properties for a field and its

singular behavior depend on the test-function space [i.e., point (A)].

(2) Accordingly the class of interactions that can be controlled depends on the test-function space (distinction between "renormalizable" and "nonrenormalizable" interactions).

(3) The concrete realization of the locality and spectral condition depends on the test-function space.

For further details on points (1) and (2) we refer the reader to Refs. 8 and 9. Point (3) will be discussed in considerable detail in a later section. The localization problem in connection with the choice of the test-function space is also discussed in Sec. 15.5 of Ref. 3.

As a last but important point we want to recall that for the usual choice (1.1) of the test-function space E there are still no "nontrivial" models of relativistic quantum fields on physical space-time.

These are some reasons for considering QFT over test-function spaces other than the traditional one. Further reasons are presented in Refs. 8 and 9. Accordingly several attempts have been made in this direction, which we want to review briefly. Before doing this, however, we want to stress that any interesting modification of the test-function space E should still allow us to deduce all the structural results of QFT or at least most of them in order to meet requirement (C).

These structural results of QFT we have in mind here are (1) the existence of a PCT operator, (2) the connection between spin and statistics, (3) the existence of a scattering operator, and some further important but more technical results: (4) the cluster property, (5) analyticity results, (6) the global nature of local commutativity, (7) the general form of the two-point function, (8) the Borchers class of a field, (9) the Jost–Schroer theorem, (10) Euclidean reformulation, and (11) dispersion relations. The proofs of these results as given in the literature[1–3] usually seem to rely on the assumption of "temperedness" in an essential way. Nevertheless it is possible to prove some of these results also for various test-function spaces other than $E = \mathscr{S}(\mathbb{R}^4, V)$ as our review will show.

For the test-function space (1.2) for Fourier hyperfunctions we will prove the results (1), (2), and (4)–(9). The remaining points (3), (10), and (11) will be discussed in the last section.

The main sources of difficulties in proving these statements are (i) that there are no test functions of compact support and (ii) that continuous linear functionals on $\underset{\sim}{\mathscr{O}}(\mathbb{D}^4, V)$ we have a "support at infinity."

C. A short review

In 1967, Jaffe[10] seems to have been the first to consider the choice of test-function spaces for relativistic quantum fields systematically. In order to be able to realize the locality condition in the traditional way he determined a class of function spaces E_J on energy-momentum space $\mathbb{R}^4$ such that (i) $\mathscr{D}(\mathbb{R}^4) \subset E_J \subset \mathscr{S}(\mathbb{R}^4)$, and (ii) $\mathscr{F} E_J$, i.e., the space of Fourier transforms of elements in E_J, contains (enough) functions of compact support.

Somewhat later (1969) Iofa and Fainberg[11] proposed using a test-function space $E_I = E_I(\mathbb{R}^4)$ of entire functions that are polynomially decreasing in any strip $|\mathrm{Im}\, z_j| < \delta$, $\delta > 0$. Since such a space does not contain any function on space-time of compact support the localization of the fields is not possible in the usual sense. Accordingly they are called *nonlocalizable fields*. Clearly the locality condition (H_4) also cannot be formulated in a natural way for such fields. Nevertheless several structural properties [(4) and (5)] could be proved and some others [(1) and (2)] were indicated in such a theory.

In 1971, Constantinescu observed[2] that localizability of the fields in the above sense and locality of the fields according to (H_4) are different notions. He explained this on the level of two-point functions. However, this is not sufficient for the locality of the whole theory. Constantinescu proposed an inductive limit space $E_C(\mathbb{R}_p^4)$ of C^∞ functions on energy-momentum space such that $\mathscr{D}(\mathbb{R}^4) \subset E_C \subset \mathscr{S}(\mathbb{R}^4)$ and $\mathscr{F} E_C$ consists of test functions f holomorphic in some strip $|\mathrm{Im}\, z_j| < \delta$, $\delta = \delta(f) > 0$. He proves some structural properties of QFT and discusses some others.

Finally there is a series of papers by Lücke concerning the choice of the test-function space and the corresponding realization of the locality condition (H_4) as well as the structural properties (1)–(10). A recent source of information about this and further references is Ref. 13.

Lücke proposes to take the Gel'fand spaces $\mathscr{S}^s(\mathbb{R}^4)$, $0 \leqslant s < \infty$, on space-time, defined and studied in Chap. IV of Ref. 14 as test-function spaces for QFT. If $s > 1$, then $\mathscr{S}^s(\mathbb{R}^4)$ contains enough C^∞ functions of compact support; hence localization in the usual sense is possible and thus the usual realization of the locality condition (H_4). If, however, $s \leqslant 1$, then the space $\mathscr{S}^s(\mathbb{R}^4)$ consists of holomorphic functions (entire functions for $0 \leqslant s < 1$) and hence localizability is lost for such test functions. In this case fields are again called nonlocalizable fields.

The locality condition (H_4) is accordingly replaced by the assumption that the fields are "essentially local" which means that "sufficiently many" matrix elements of the (anti-) commutator of the field operators $[A(x_1), A(x_2)]_\pm$ are locally continuous on K with respect to $\mathscr{S}^s(\mathbb{R}^4)$.[15] Since permutation symmetry of the Wightman functions can be proved for essentially local fields,[16] some of the structural properties follow also for this class of fields over $\mathscr{S}^s(\mathbb{R}^4)$, $s \leqslant 1$.[13]

However, on one side it can be shown that there is no clear and unambiguous notion of support for $F \in \mathscr{S}^s(\mathbb{R}^4)'$, $0 \leqslant s < 1$.[13] On the other side we think it to be important that in QFT a sensitive mathematical formulation of the locality condition (H_4) has to realize the idea that the (anti-) commutator of the field operators has its "support" only inside K. Therefore we think that our point (B) above is really indispensable and accordingly explain this point for the test-function space (1.2) in some detail. In particular, we will explain that the notion of support for Fourier hyperfunctions (Sec. II D) used in its realization is a straightforward generalization of the notion of support for distributions and thus provides a genuine realization of the locality condition.

II. FOURIER HYPERFUNCTIONS

A. Introduction

This section introduces the notions and explains the results from Fourier hyperfunction theory that we are going to use. For proofs we clearly have to refer to the literature.[6,17,18]

Recall that the main original motivation for introducing distributions came from the theory of linear partial differential operators with constant coefficients.[19] Similarly hyperfunctions have proved to be an appropriate frame for the theory of linear partial differential operators with real analytic coefficients and those that have "regular singularities."[20]

Hyperfunctions are finite sums of boundary values of certain analytic functions.[21] Thus hyperfunctions generalize distributions. They admit the same basic operations (differentiation, integration, and convolution) as distributions. Just as distributions do, they have a "good notion" of *localization* (which agrees with the known localization properties of distribution if applied to them).

In contrast to distributions, hyperfunctions admit a canonical definition of a product (at least in the simplest case) and this may turn out to be of great importance in applications to QFT.

However, in general, hyperfunctions do not admit a canonical definition of Fourier transform as an isomorphism. Some "growth restrictions at infinity" are needed for this. A way to achieve this is to compactify the underlying space $\mathbb{R}^n$. The *radial compactification* $\mathbb{D}^n$ of $\mathbb{R}^n$ has proved to be very useful here. It is defined in a natural way as follows: Let S_∞^{n-1} be the $(n-1)$-dimensional sphere at infinity, which is homeomorphic to the unit sphere $S^{n-1} = \{x\in\mathbb{R}^n \mid |x| = 1\}$ by the mapping $x \to x_\infty$, where the point $x_\infty \in S_\infty^{n-1}$ lies on the ray connecting the origin with the point $x\in S^{n-1}$. The set $\mathbb{R}^n \cup S_\infty^{n-1}$ equipped with its "natural topology" (a fundamental system of the neighborhood of x_∞ is given by all open cones of arbitrary vertex generated by an arbitrary open neighborhoods of x_∞ in S_∞^{n-1}) is denoted by $\mathbb{D}^n$.[17]

A natural extension of hyperfunctions on $\mathbb{R}^n$ to $\mathbb{D}^n$ leads to Fourier hyperfunctions. It turns out that Fourier hyperfunctions have all the properties of hyperfunctions and, in addition, the Fourier transformation is an isomorphism for them.

The realization of the idea that hyperfunctions on $\mathbb{R}^n$ (and on $\mathbb{D}^n$) are finite sums of boundary values of analytic functions is immediate for the case of one variable $(n = 1)$. For $n\geqslant 2$ variables, however, we have to meet serious complications as a result of the considerably more complicated theory of analytic functions of more than one variable. New phenomena of analytic continuation cause the main complication in introducing an appropriate notion of boundary value in the $n\geqslant 2$ variable case. The first approach for a "good" notion of boundary values of holomorphic functions of several variables is due to Sato.[22] He introduced this notion by considering sheaves of germs of analytic functions and their relative cohomology.[23] Later it was realized how to introduce a hyperfunction without using cohomology theory.[24–27] According to Sec. I it is clear in QFT we prefer this second approach adapted to Fourier hyperfunctions in which Fourier hyperfunctions are defined as continuous linear functionals on some space of functions.

B. The test-function space for Fourier hyperfunctions

The spaces of test functions we are going to use later are

$$E = \underset{\sim}{\mathscr{O}}(\mathbb{D}^n, V) \simeq \underset{\sim}{\mathscr{O}}(\mathbb{D}^n) \otimes V, \tag{2.1}$$

with some finite-dimensional vector space V and an inductive limit space of functions on $\mathbb{D}^n$ for $n = 4$, which we describe now. By reasons that will become obvious soon we introduce and study the spaces

$$\underset{\sim}{\mathscr{O}}(K), \quad K \subset \mathbb{D}^n \text{ closed.} \tag{2.2}$$

Let $\{U_m, \quad m\in\mathbb{N}\}$ be a fundamental sequence of neighborhoods of K in $\mathbb{Q}^n = \mathbb{D}^n + i\mathbb{R}^n$. Then let $\mathscr{O}_c^m(U_m)$ be the Banach space of functions f analytic on $U_m \cap \mathbb{C}^n$ and continuous on $\overline{U}_m \cap \mathbb{C}^n$ such that

$$\|f\|_m = \sup_{z\in U_m \cap \mathbb{C}^n} |f(z)| e^{|z|/m} \tag{2.3}$$

is finite. The space $\underset{\sim}{\mathscr{O}}(K)$ is now defined as the inductive limit of the Banach spaces $\mathscr{O}_c^m(U_m)$:

$$\underset{\sim}{\mathscr{O}}(K) = \operatorname*{ind\,lim}_{m\to\infty} \mathscr{O}_c^m(U_m). \tag{2.4}$$

The following propositions collect some properties of the space $\underset{\sim}{\mathscr{O}}(K)$ of a *rapidly decreasing analytic function* on K that are used in QFT.

Proposition 2.1: (a) $\underset{\sim}{\mathscr{O}}(K)$ is a DFS space (a dual Fréchet–Schwartz space).

(b) $\underset{\sim}{\mathscr{O}}(\mathbb{D}^n)$ is nuclear and barreled.

(c) $\otimes^n \underset{\sim}{\mathscr{O}}(\mathbb{D})$ is dense in $\underset{\sim}{\mathscr{O}}(\mathbb{D}^n)$ and $\hat{\otimes}^n \underset{\sim}{\mathscr{O}}(\mathbb{D}) = \underset{\sim}{\mathscr{O}}(\mathbb{D}^n)$.

And, as a consequence, we have the following proposition.

Proposition 2.2: Let M be a separately continuous n-linear form on $\underset{\sim}{\mathscr{O}}(\mathbb{D}^m)^n = \underset{\sim}{\mathscr{O}}(\mathbb{D}^m) \times \cdots \times \underset{\sim}{\mathscr{O}}(\mathbb{D}^m)$. Then the following conditions hold.

(a) M: $\mathscr{O}(\mathbb{D}^m)^n \to \mathbb{C}$ is jointly continuous.

(b) There is a unique continuous linear form F on $\underset{\sim}{\mathscr{O}}(\mathbb{D}^{mn})$ such that, for all

$$f_i \in \underset{\sim}{\mathscr{O}}(\mathbb{D}^m),$$

$$M(f_1,\dots,f_n) = F(f_1 \otimes \cdots \otimes f_n).$$

For the proofs of these results we refer to Refs. 4 and 5. Note that the "kernel theorem" for this space [part (b) of Proposition 2.2] can be proved in an elementary way by using the explicit characterization of the topological dual $\underset{\sim}{\mathscr{O}}(\mathbb{D}^{mn})$, obtained in Sec. II C.

The following proposition is concerned with the Fourier transformation.

Proposition 2.3: The Fourier transformation $\mathscr{F}$ is well defined on $\underset{\sim}{\mathscr{O}}(\mathbb{D}^n)$ by

$$(\mathscr{F}f)(p) = (2\pi)^{-n/2} \int e^{ip\cdot x} f(x)\,dx, \tag{2.5}$$

where $\mathscr{F}$ is an isomorphism of the topological vector space $\underset{\sim}{\mathscr{O}}(\mathbb{D}^n)$ with inverse

$$(\bar{\mathscr{F}}f)(x) = (2\pi)^{-n/2} \int e^{-ip\cdot x} \tilde{f}(p)\,dp. \tag{2.6}$$

Proof: Clearly it suffices to consider the case $n = 1$. Here $f \in \underset{\sim}{\mathcal{O}}(\mathbb{D})$ means $f \in \mathcal{O}_c^m(U_m) \equiv \mathcal{O}_m$ for some m, where $U_m = \mathbb{D} + i(-1/m, 1/m)$. Then for all $k = p + iq \in \mathbb{C} \cap U_{m+1}$ we obtain, from (2.5),

$$|(\mathcal{F}f)(k)| \leqslant C_m \|f\|_m.$$

Now, if b is real, $|b| \leqslant 1/2m$, then the function f_b defined by $f_b(z) = f(z + ib)$, belongs to $\mathcal{O}_{2m}$ and satisfies

$$\|f_b\|_{2m} \leqslant C_m \|f\|_m, \quad b \in [-1/2m, 1/2m].$$

By analyticity, decay properties, and Cauchy's theorem one proves

$$(\mathcal{F}f_b)(k) = e^{-ik\cdot(ib)}(\mathcal{F}f)(k),$$

so that the above estimates imply, for all $|b| \leqslant 1/2m$,

$$|e^{kb}(\mathcal{F}f)(k)| \leqslant C_m \|f\|_m;$$

hence $\mathcal{F}f \in \mathcal{O}_{2m+1}$ and

$$\|\mathcal{F}f\|_{2m+1} \leqslant C_m \|f\|_m.$$

This proves $\mathcal{F}: \underset{\sim}{\mathcal{O}}(\mathbb{D}^n) \to \underset{\sim}{\mathcal{O}}(\mathbb{D}^n)$ to be a well defined continuous linear map.

Clearly the map $\overline{\mathcal{F}}$ has the same properties. And as usual one proves that $\overline{\mathcal{F}}\mathcal{F}$ is the identity on $\underset{\sim}{\mathcal{O}}(\mathbb{D}^n)$. Hence $\mathcal{F}$ is an isomorphism.

C. Fourier hyperfunctions

In our approach a *Fourier hyperfunction on* $\mathbb{D}^n$ is by definition an element of the topological dual $\underset{\sim}{\mathcal{O}}(\mathbb{D}^n)'$ of the space $\underset{\sim}{\mathcal{O}}(\mathbb{D}^n)$. In order to give an explicit characterization of $\underset{\sim}{\mathcal{O}}(\mathbb{D}^n)'$ and to relate this notion of Fourier hyperfunctions with its heuristic definition in Sec. II A as a finite sum of boundary values of holomorphic functions let us introduce the *sheaf* $\tilde{\mathcal{O}}$ *of slowly increasing holomorphic functions on* $\mathbb{Q}^n$.

For an open subset $\Omega \subset \mathbb{Q}^n$ denote by $\tilde{\mathcal{O}}(\Omega)$ the set of all analytic functions F on $\Omega \cap \mathbb{C}^n$ such that, for every $\epsilon > 0$ and every compact set $K \subset \Omega$,

$$\|F\|_{K,\epsilon} = \sup_{z \in K \cap \mathbb{C}^n} e^{-\epsilon|z|}|F(z)| \tag{2.7}$$

is finite. Here $\tilde{\mathcal{O}}(\Omega)$ is called the ($\mathbb{C}$-vector) space of slowly increasing analytic functions on Ω.

If Ω' is another open set contained in Ω we obviously have a well defined restriction map

$$\rho_{\Omega'\Omega}: \quad \tilde{\mathcal{O}}(\Omega) \to \tilde{\mathcal{O}}(\Omega'), \quad \rho_{\Omega'\Omega}(F) = F_{|\Omega'}, \quad F \in \tilde{\mathcal{O}}(\Omega), \tag{2.8}$$

such that

$$\rho_{\Omega''\Omega'} \cdot \rho_{\Omega'\Omega} = \rho_{\Omega''\Omega} \quad \text{and} \quad \rho_{\Omega\Omega} = \mathrm{id},$$

for all open sets $\Omega'' \subset \Omega' \subset \Omega$.

Thus with these restriction maps $\{\tilde{\mathcal{O}}(\Omega) | \Omega \subset \mathbb{Q}^n \text{ open}\}$ is a presheaf on $\mathbb{Q}^n$. This presheaf actually is a sheaf since furthermore the following localization properties are satisfied.

(L_1) If an open set Ω is covered by open sets Ω_α, $\Omega = \cup_\alpha \Omega_\alpha$, and if all the restrictions $F_{|\Omega_\alpha}$ of a function $F \in \tilde{\mathcal{O}}(\Omega)$ vanish then the function F itself vanishes.

(L_2) If any collection $\{\Omega_\alpha\}$ of open sets in $\mathbb{Q}^n$ is given together with a collection of functions $F_\alpha \in \tilde{\mathcal{O}}(\Omega_\alpha)$ satisfying

$$F_{\alpha|\Omega_\alpha \cap \Omega_\beta} = F_{\beta|\Omega_\alpha \cap \Omega_\beta},$$

for all α and β, then there exists a function $F \in \tilde{\mathcal{O}}(\cup_\alpha \Omega_\alpha)$ such that $F_{|\Omega_\alpha} = F_\alpha$, for all α.

For $j = 1, \ldots, n$, let us introduce the open subsets $W_j = \{z \in \mathbb{Q}^n | \operatorname{Im} z_j \neq 0\}$. The intersection

$$W = \bigcap_{j=1}^n W_j$$

of all these sets consists of 2^n open connected components separated by the "real points." Then

$$\hat{W}_k = \bigcap_{\substack{j=1 \\ j \neq k}}^n W_j$$

includes the real points in the kth variable.

In an obvious way we can consider

$$\sum_{k=1}^n \tilde{\mathcal{O}}(\hat{W}_k)$$

as a subspace of $\tilde{\mathcal{O}}(W)$. Thus the factor space

$$\mathcal{R} = \mathcal{R}(\mathbb{D}^n) = \tilde{\mathcal{O}}(W) \Big/ \Big(\sum_{k=1}^n \tilde{\mathcal{O}}(\hat{W}_k) \Big) \tag{2.9}$$

consists of equivalence classes $[F]$ of functions $f \in \tilde{\mathcal{O}}(W)$ where two functions F and F' define the same class if and only if

$$F' - F = \sum_{k=1}^n F_k, \quad F_k \in \tilde{\mathcal{O}}(\hat{W}_k). \tag{2.10}$$

The topological dual of $\underset{\sim}{\mathcal{O}}(\mathbb{D}^n)$ is now characterized in terms of this factor space as follows.

Proposition 2.4: The topological dual of $\underset{\sim}{\mathcal{O}}(\mathbb{D}^n)$ and the factor space (2.9) are isomorphic:

$$\mathcal{R}(\mathbb{D}^n) \simeq \underset{\sim}{\mathcal{O}}(\mathbb{D}^n)'.$$

This isomorphism and its inverse are given explicitly by the following formulas: For $\mu \in \underset{\sim}{\mathcal{O}}(\mathbb{D}^n)'$, define a function $\hat{\mu}$ on W by

$$\hat{\mu}(z) = \mu(h_z), \quad h_z(t) = \prod_{j=1}^n \frac{e^{-(t_j - z_j)^2}}{2\pi i(t_j - z_j)}, \quad z \in W; \tag{2.11}$$

then $\hat{\mu} \in \tilde{\mathcal{O}}(W)$ and thus $[\hat{\mu}] \in \mathcal{R}$. Conversely every equivalence class $[F] \in \mathcal{R}$ defines an element $\mu_{[F]} \in \underset{\sim}{\mathcal{O}}(\mathbb{D}^n)'$ by

$$\mu_{[F]}(f) = \int_{\Gamma_1} \cdots \int_{\Gamma_n} F(z_1, \ldots, z_n) f(z_1, \ldots, z_n) \times dz_1 \cdots dz_n \equiv \int_{\Gamma_1 \cdots \Gamma_n} F(z) f(z) dz, \tag{2.12}$$

where $F \in \tilde{\mathcal{O}}(W)$ is any representative of $[F]$ and where the paths $\Gamma_1, \ldots, \Gamma_n$ are chosen according to $f \in \underset{\sim}{\mathcal{O}}_c^m(U_m)$ for some m such that

$$\Gamma_1 \times \cdots \Gamma_n \subset U_m \cap W \cap \mathbb{C}^n, \text{ for instance,}$$

$$\Gamma_j = \Gamma_j^+ + \Gamma_j^-,$$

$$\Gamma_j^\pm = \{z_j | z_j = \pm x_j \pm i\delta_m, -\infty < x_j < \infty\}$$

with sufficiently small $\delta_m > 0$.

Proof[4]*:* The first part clearly relies on properties of the collection of functions h_z, $z \in W$. Those that are relevant here are contained in the following elementary lemma.

Lemma 2.5: (a) For every $z\in W\cap\mathbb{C}^n$, there is $m_0=m_0(z)$ such that h_z belongs to $\mathscr{O}_c^m(U_m)$ and

$$\|h_z\|_m \leqslant \text{const}\, e^{|z|/m}/\delta(z), \quad \delta(z)=\text{dist}(z,\mathbb{R}^n),$$

for all $m\geqslant m_0$.

(b) For every $z^0\in W\cap\mathbb{C}^n$ there is a polycircle

$$\mathscr{P}=\{z=(z_1,\ldots,z_n)\,|\,|z_j-z_j^0|<r_j,\quad j=1,\ldots,n\}$$

around z^0 such that $\overline{\mathscr{P}}\subset W$ and there are functions $\Delta_j:\mathscr{P}\to\underset{\sim}{\mathscr{O}}(\mathbb{D}^n)$, $j=1,\ldots,n$ such that, for all $z\in\mathscr{P}$,

$$h_z-h_{z^0}=\sum_{j=1}^n (z_j-z_j^0)\Delta_j(z),$$

with

$$\Delta_j(z)\to\Delta_j(z^0)\quad\text{in}\quad \underset{\sim}{\mathscr{O}}(\mathbb{D}^n)\quad \text{for } z\to z^0.$$

Now take any $\mu\in\underset{\sim}{\mathscr{O}}(\mathbb{D}^n)'$. Part (a) of the lemma implies immediately that $z\to\mu(h_z)$ is a well defined function on $W\cap\mathbb{C}^n$ and, according to part (b), this function has complex derivatives; hence $\hat{\mu}$ is analytic on $W\cap\mathbb{C}^n$.

Suppose a compact subset $K\subset W$ and a number $\epsilon>0$ to be given. Then $\delta=\text{dist}(K\cap\mathbb{C}^n,\ \mathbb{R}^n)>0$. There is $m_0=m_0(K)$ such that $\epsilon>1/m_0$ and $\text{dist}(U_m\cap\mathbb{C}^n, K\cap\mathbb{C}^n)\leqslant\delta/2$, for all $m\geqslant m_0$. Then, for fixed $m\geqslant m_0$, the collection of functions h_z, $z\in K\cap\mathbb{C}^n$, belongs to $\mathscr{O}_c^m(U_m)$ and the estimate of part (a) yields

$$\|\hat{\mu}\|_{K,\epsilon}\leqslant C_m \sup_{z\in K\cap\mathbb{C}^n} \|h_z\|e^{-\epsilon|z|}<\infty;$$

hence $\hat{\mu}$ is slowly increasing and therefore $\hat{\mu}\in\widetilde{\mathscr{O}}(W)$.

The growth restriction (2.7) for a function $F\in\widetilde{\mathscr{O}}(W)$ implies that the integral in Eq. (2.12) is well defined for all $f\in\underset{\sim}{\mathscr{O}}(\mathbb{D}^n)$; more precisely, for every $m\in\mathbb{N}$, there is $C_m=C_m(F)$ such that, for all $f\in\mathscr{O}_c^m(U_m)$, this integral is bounded in absolute value by

$$C_m\|f\|_m.$$

A function $F_k\in\widetilde{\mathscr{O}}(\widehat{W}_k)$ is, in particular, analytic in $z_k\in\mathbb{C}$. Hence if we rewrite the integral in (2.12) in the form

$$\int_{\Gamma_1}\cdots\left(\int_{\Gamma_k} F_k(z_1,\ldots,z_k,\ldots,z_n)f(z_1,\ldots,z_k,\ldots,z_n)\,dz_k\right)dz_1\cdots dz_{k-1}\,dz_{k+1}\cdots dz_n,$$

we see that this integral vanishes according to Cauchy's theorem and the growth restriction on F and f.

Therefore all elements F' in the equivalence class $[\,F\,]$ of $F\in\widetilde{\mathscr{O}}(W)$ define the same continuous linear functional on $\underset{\sim}{\mathscr{O}}(\mathbb{D}^n)$, that is, by (2.12), $\mathscr{R}(\mathbb{D}^n)$ is mapped linearly into $\underset{\sim}{\mathscr{O}}(\mathbb{D}^n)'$.

Another application of Cauchy's theorem together with the growth restrictions on F and f shows, by appropriate choice of the integration path: If $\mu_{[F]}(f)=0$, for all $f\in\underset{\sim}{\mathscr{O}}(\mathbb{D}^n)$, then

$$F\in\sum_{k=1}^n \widetilde{\mathscr{O}}(\widehat{W}_k),$$

i.e., $[\,F\,]=0$. Hence the mapping (2.12) $\mathscr{R}(\mathbb{D}^n)\to\underset{\sim}{\mathscr{O}}(\mathbb{D}^n)'$ is injective.

Since $h_z(\cdot)$ is a modified Cauchy kernel with appropriate decay properties, one knows, for all $f\in\underset{\sim}{\mathscr{O}}(\mathbb{D}^n)$ in a suitable complex neighborhood of $\mathbb{D}^n$,

$$\int_{\Gamma_1\times\cdots\times\Gamma_m} f(z)h_z\,dz=f(\cdot).$$

If $\mu\in\underset{\sim}{\mathscr{O}}(\mathbb{D}^n)'$ is applied to this equation one deduces

$$\mu_{[\hat{\mu}]}(f)=\mu(f);$$

hence the mapping (2.12) is an inverse of the mapping (2.11) and the proposition follows.

Via the isomorphism of Proposition 2.4 the heuristic definition of a Fourier hyperfunction as a finite sum of boundary values of slowly increasing holomorphic functions is easily given a precise meaning: The 2^n connected components of W can be described as

$$W(\alpha_1,\ldots,\alpha_n)=\{z\in\mathbb{Q}^n|\alpha_j\,\text{Im}\,z_j>0,\quad j=1,\ldots,n\},$$

$$\alpha_j\in\{1,-1\}.$$

Now define, for $\alpha=(\alpha_1,\ldots,\alpha_n)\in\{1,-1\}^n$,

$$\sigma(\alpha)=\prod_{j=1}^n \alpha_j,$$

and then, for $F\in\widetilde{\mathscr{O}}(W)$,

$$F_\alpha=\sigma(\alpha)F \text{ on } W(\alpha) \text{ and } F_\alpha=0 \text{ elsewhere.}$$

The *boundary value* of $F\in\widetilde{\mathscr{O}}(W)$ with respect to the cone $W(\alpha)$ is then defined by

$$\delta_{W(\alpha)}(F)=[\,F_\alpha\,]. \tag{2.13}$$

Clearly it follows that

$$[\,F\,]=\sum_{\alpha\in\{1,-1\}^n}\sigma(\alpha)\delta_{W(\alpha)}(F), \tag{2.14}$$

and hence, by Proposition 2.4, μ is the corresponding finite sum of boundary values of $\hat{\mu}\in\widetilde{\mathscr{O}}(W)$.

Next we establish the traditional view of boundary values as limits [in $\underset{\sim}{\mathscr{O}}(\mathbb{D}^n)'$] of slowly increasing functions. To this end we define, for $0<\delta_j<1/m$, a path $\Gamma_j(\alpha_j,\delta_j)$, $\alpha_j\in\{1,-1\}$, in the complex z_j plane by

$$\Gamma_j(\alpha_j,\delta_j)=\{z=\alpha_j x_j+i\alpha_j\delta_j\,|-\infty<x_j<\infty\},$$

so that

$$\Gamma(\alpha,\delta)=\Gamma_1(\alpha_1,\delta_1)\times\cdots\times\Gamma_n(\alpha_n,\delta_n)$$
$$\subset U_m\cap W(\alpha).$$

Then, for all $f\in\mathscr{O}_c^m(U_m)$ and all $F\in\widetilde{\mathscr{O}}(W)$,

$$\int_{\Gamma(\alpha,\delta)} F(z)f(z)dz=I(\alpha,\delta)$$

is independent of δ, $0<\delta_j<1/m$, and thus equals the limit $\delta_j\to 0$, $j=1,\ldots,n$, of this integral denoted by

$$\langle F_\alpha(x_1+i\alpha_1 0,\ldots,x_n+i\alpha_n 0),$$
$$f(x_1+i\alpha_1 0,\ldots,x_n+i\alpha_n 0)\rangle,$$

where the duality $\underset{\sim}{\mathscr{O}}(\mathbb{D}^n)'$, $\underset{\sim}{\mathscr{O}}(\mathbb{D}^n)$ is used. If we sum over all $\alpha\in\{1,-1\}^n$, we obtain

$$\sum_{\alpha\in\{1,-1\}^n} I(\alpha,\delta) = \int_{\Gamma(\delta)} F(z)f(z)dz.$$

Thus, by Proposition 2.4, every $\mu\in\underset{\sim}{\mathcal{O}}(\mathbb{D}^n)'$ is a finite sum of boundary values of slowly increasing functions.

D. Support of Fourier hyperfunctions

If $K \subset \mathbb{D}^n$ is any closed subset, then relations (2.3) and (2.4) easily imply that $\underset{\sim}{\mathcal{O}}(\mathbb{D}^n)$ is contained in $\underset{\sim}{\mathcal{O}}(K)$. For a Fourier hyperfunction μ on $\mathbb{D}^n$, denote by $C(\mu)$ the class of all closed subsets $K \subset \mathbb{D}^n$ such that there is a continuous extension μ_K of μ to $\underset{\sim}{\mathcal{O}}(K)$:

$$\mu_K\in\underset{\sim}{\mathcal{O}}(K)', \quad \mu_{K|\underset{\sim}{\mathcal{O}}(\mathbb{D}^n)} = \mu. \tag{2.15}$$

Any such subset $K\in C(\mu)$ is called a "carrier" of the Fourier hyperfunction μ. In contrast to a general analytic functional a Fourier hyperfunction μ has a smallest carrier, called the *support* of μ:

$$\operatorname{supp}\mu = \bigcap_{K\in C(\mu)} K. \tag{2.16}$$

This definition really works since one can prove[6] the following proposition.

Proposition 2.6: If $K_1,K_2\in C(\mu)$, then $K_1 \cap K_2\in C(\mu)$.

This result is by no means trivial. We give some hints. Having $K_j\in C(\mu)$ means that there are $\mu_j = \mu_{K_j}\in\underset{\sim}{\mathcal{O}}(K_j)'$ satisfying (2.15). Also, we have to define an extension to $\underset{\sim}{\mathcal{O}}(K_1 \cap K_2)$. Given $f\in\underset{\sim}{\mathcal{O}}(K_1 \cap K_2)$ there exist by the Mittag-Leffler theorem for rapidly decreasing functions[6] $f_j \in\underset{\sim}{\mathcal{O}}(K_j)$ such that

$$f = f_1 - f_2 \quad \text{on} \quad K_1 \cap K_2. \tag{2.17}$$

Now define a function μ:

$$\mu(f) = \mu_1(f_1) - \mu_2(f_2). \tag{2.18}$$

The right-hand side of Eq. (2.18) is independent of the special choice of the decomposition (2.17). Hence μ is well defined, and obviously μ is linear. One can prove continuity of μ by some general arguments.[6]

According to definition (2.16) the topological dual of $\underset{\sim}{\mathcal{O}}(K)$, $K \subset \mathbb{D}^n$ closed, is the set of Fourier hyperfunctions on $\mathbb{D}^n$ with support contained in K. With this interpretation in mind the space of *Fourier hyperfunctions on an open subset* V of $\mathbb{D}^n$ is naturally defined as the factor space of the space of all Fourier hyperfunctions on $\mathbb{D}^n$ with respect to the subspace of those Fourier hyperfunctions having support in the complement $V^c = \mathbb{D}^n - V$ of V:

$$\mathscr{R}(V) = \underset{\sim}{\mathcal{O}}(\mathbb{D}^n)'/\underset{\sim}{\mathcal{O}}(V^c)'. \tag{2.19}$$

It is known from the following proposition[17] that $\mathscr{R}(V)$ is isomorphic to

$$\mathscr{R}(V) \simeq \underset{\sim}{\mathcal{O}}(\overline{V})'/\underset{\sim}{\mathcal{O}}(\overline{V} - V)'.$$

Proposition 2.7: Let $K = \cup_{i=1}^{p} K_i$ be the union of p compact sets in $\mathbb{D}^n$. Suppose $\mu\in\underset{\sim}{\mathcal{O}}(K)'$; then there are $\mu_i\in\underset{\sim}{\mathcal{O}}(K_i)'$ such that $\mu = \Sigma_{i=1}^{p} \mu_i$.

Proof: Since the mapping $\underset{\sim}{\mathcal{O}}(K) \to \Pi_{i=p}^{p} \underset{\sim}{\mathcal{O}}(K_i)$, namely, $f \to \{f_{|K_i}\}_{i=1}^{p}$, is injective and of closed range, the mapping $\Pi_{i=1}^{p} \underset{\sim}{\mathcal{O}}(K_i)' \to \underset{\sim}{\mathcal{O}}(K)'$, namely, $\{\mu_i\}_{i=1}^{p} \to \Sigma_{i=1}^{p} \mu_i$, is accordingly surjective.

If V and W are open subsets in $\mathbb{D}^n$ with W contained in V, then we have $\underset{\sim}{\mathcal{O}}(V^c)' \subset \underset{\sim}{\mathcal{O}}(W^c)'$ and thus we get a restriction map

$$\rho_{WV}\colon \mathscr{R}(V) = \underset{\sim}{\mathcal{O}}(\mathbb{D}^n)'/\underset{\sim}{\mathcal{O}}(V^c)' \to \mathscr{R}(W) = \underset{\sim}{\mathcal{O}}(\mathbb{D}^n)'/\underset{\sim}{\mathcal{O}}(W^c)' \tag{2.20}$$

in a canonical way.

For our purposes it turns out to be important to have the following result.

Theorem 2.8: The assignment of the factor spaces $\mathscr{R}(V)$ with open subsets V of $\mathbb{D}^n$ according to (2.19) together with the canonical restriction maps $\rho_{WV}\colon \mathscr{R}(V) \to \mathscr{R}(W)$ for open subsets $W \subset V$ according to (2.20) is a flabby sheaf on $\mathbb{D}^n$, called the *sheaf* $\mathscr{R}$ *of Fourier hyperfunctions on* $\mathbb{D}^n$.

This means in particular that Fourier hyperfunctions have the following localization properties (L_1) and (L_2):

(L_1) if $V = \bigcup_\alpha V_\alpha$, $\quad V_\alpha \subset \mathbb{D}^n$ open, $\quad \mu\in\mathscr{R}(V)$,

$$\text{then } \mu_{|V_\alpha} = 0, \quad \text{for all } V_\alpha, \text{ implies } \mu = 0; \tag{2.21}$$

(L_2) if $V_\alpha \subset \mathbb{D}^n$ open and

if $\mu_\alpha\in\mathscr{R}(V_\alpha)$ satisfies $\mu_{\alpha|V_\alpha \cap V_\beta} = \mu_{\beta|V_\alpha \cap V_\beta}$

$$\text{then there is } \mu\in\mathscr{R}(V), \text{ such that } \mu_{|V_\alpha} = \mu_\alpha. \tag{2.22}$$

Furthermore, since $\mathscr{R}$ is flabby, any Fourier hyperfunction μ_V on any open subset $V \subset \mathbb{D}^n$ is the restriction of a Fourier hyperfunction $\mu\in\underset{\sim}{\mathcal{O}}(\mathbb{D}^n)'$ to V: $\mu_V = \mu_{|V}$.

Because of the localization properties of a sheaf the notion of *support of a Fourier hyperfunction* $\mu\in\underset{\sim}{\mathcal{O}}(\mathbb{D}^n)'$ can also be defined as the smallest closed subset $K \subset \mathbb{D}^n$ such that $\mu_{|K^c} = 0$. From (2.19) and (2.20) it is obvious that this notion of support agrees with the notion introduced previously in (2.15) and (2.16).

Proof of Theorem 2.8: First we assume $V = \cup_{\alpha\in I} V_\alpha$, $\mu\in\mathscr{R}(V)$. Let $\bar{\mu}\in\underset{\sim}{\mathcal{O}}(\mathbb{D}^n)'$ be a representative of μ. Then

$$\rho_{V_\alpha \mathbb{D}^n}(\bar{\mu}) = \rho_{V_\alpha V}\cdot\rho_{V\mathbb{D}^n}(\bar{\mu}) = \rho_{V_\alpha V}(\mu) = 0$$

implies $\operatorname{supp}\bar{\mu} \cap V_\alpha = \varnothing$, for all $\alpha\in I$, and hence $\operatorname{supp}\bar{\mu} \cap V = \varnothing$, which implies $\mu = 0$. Thus (L_1) is proved.

To prove (L_2) we begin with the case of just two open sets V_1 and V_2. Let $\bar{\mu}_\alpha\in\underset{\sim}{\mathcal{O}}(\overline{V}_\alpha)'$ be representatives of μ_α, for $\alpha = 1,2$. The support of $\mu_1 - \mu_2$ is contained in

$$(\overline{V}_1 \cup \overline{V}_2) - (\overline{V}_1 \cap \overline{V}_2) = (\overline{V}_1^c \cap \overline{V}_2) \cup (\overline{V}_1 \cap \overline{V}_2^c);$$

thus Proposition 2.7 gives a decomposition

$$\bar{\mu}_1 - \bar{\mu}_2 = \bar{\nu}_1 - \bar{\nu}_2,$$

$$\bar{\nu}_1\in\underset{\sim}{\mathcal{O}}(\overline{V}_1^c \cap \overline{V}_2)', \quad \bar{\nu}_2\in\underset{\sim}{\mathcal{O}}(\overline{V}_1 \cap \overline{V}_2^c)'.$$

Let

$$\mu = \bar{\mu}_1 - \bar{\nu}_1 = \bar{\mu}_2 - \bar{\nu}_2\in\underset{\sim}{\mathcal{O}}(\overline{V}_1 \cap \overline{V}_2)'.$$

Then we have $\mu_{|V_\alpha} = \mu_\alpha$ because $\operatorname{supp}(\mu - \mu_\alpha) \cap V_\alpha = \varnothing$.

In the general case (L_2) is proved by using some topological argument (see Theorem 4.17 of Ref. 6).

The existence of the representative $\bar{\mu}\in\underset{\sim}{\mathcal{O}}(\mathbb{D}^n)'$ of $\mu\in\mathscr{R}(V)$ implies the flabbiness of $\mathscr{R}$.

Remark 2.1: The restriction of a Fourier hyperfunction to $\mathbb{R}^n$ gives a hyperfunction. Since the sheaf $\mathscr{R}$ of a Fourier hyperfunction is flabby any hyperfunction on $\mathbb{R}^n$ can be extended to $\mathbb{D}^n$ as a Fourier hyperfunction. This extension, however, is not unique since there are Fourier hyperfunctions with support at "infinity" (support$\subset S_\infty^{n-1}$).[28]

Remark 2.2: For any $f\in\underset{\sim}{\mathscr{O}}(\mathbb{D}^n)$ we have $f_{|\mathbb{R}^n}\in\mathscr{S}(\mathbb{R}^n)$ by definition, and this injection of $\underset{\sim}{\mathscr{O}}(\mathbb{D}^n)$ into $\mathscr{S}(\mathbb{R}^n)$ is continuous. Hence Fourier hyperfunctions generalize tempered distributions. Thus for a tempered distribution we have defined the notion of support in the sense of Fourier hyperfunctions also. We now show that this notion of support agrees with that in the sense of distributions.

Suppose that the support of T in the sense of distributions is contained in some closed set $K\subset\mathbb{R}^n$. Let $\{U_m\}$ be a fundamental system of neighborhoods of $\overline{K}$ in $\mathbb{Q}^n$, the closure of K in $\mathbb{D}^n$; then for any U_m there exists a C^∞- function χ such that $\operatorname{supp}\chi\subset U_m\cap\mathbb{R}^n$ and $\chi=1$ on K. Since $\chi\cdot f\in\mathscr{S}(\mathbb{R}^n)$ for $f\in\underset{\sim}{\mathscr{O}}{}_c^m(U_m)$ and $T\cdot\chi(\varnothing)=T(\varnothing)$, for a $\varnothing\in\mathscr{S}(\mathbb{R}^n)$, T defines an element of $\underset{\sim}{\mathscr{O}}(\overline{K})'$. Hence the support of T in the sense of Fourier hyperfunctions is contained in $\overline{K}$.

Now consider the tempered distribution T as a Fourier hyperfunction and suppose that this Fourier hyperfunction has its support in a closed set $\overline{K}$ of $\mathbb{D}^n$, that is, $T\in\underset{\sim}{\mathscr{O}}(\overline{K})'$. Let $\varnothing\in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\varnothing\subset K^c$; then $\varnothing\in\underset{\sim}{\mathscr{O}}(\overline{K})$ and $T(\varnothing)=0$. This shows that the support of T in the sense of distributions is contained in K.

Remark 2.3: Let $n=n_1+n_2$. Note that $\mathbb{D}^n\neq\mathbb{D}^{n_1}\times\mathbb{D}^{n_2}$, but

$$\mathbb{Q}^n\cap\mathbb{R}^n=\mathbb{R}^n=(\mathbb{Q}^{n_1}\times\mathbb{Q}^{n_2})\cap\mathbb{R}^n$$

and

$$\underset{\sim}{\mathscr{O}}(\mathbb{D}^n)=\underset{\sim}{\mathscr{O}}(\mathbb{D}^{n_1}\times\mathbb{D}^{n_2}).$$

Let K_j $(j=1,2)$ be closed sets in $\mathbb{R}^{n_j}$, $\overline{K}_j$ be the closure of K_j in $\mathbb{D}^{n_j}$, and $\overline{K_1\times K_2}$ be the closure of $K_1\times K_2$ in $\mathbb{D}^n$. Then we have

$$\underset{\sim}{\mathscr{O}}(\mathbb{D}^n)\subset\underset{\sim}{\mathscr{O}}(\overline{K_1\times K_2})\subset\underset{\sim}{\mathscr{O}}(\overline{K}_1\times\overline{K}_2).$$

Thus we have

$$\underset{\sim}{\mathscr{O}}(\mathbb{D}^n)'\supset\underset{\sim}{\mathscr{O}}(\overline{K_1\times K_2})'\supset\underset{\sim}{\mathscr{O}}(\overline{K}_1\times\overline{K}_2)',$$

i.e., the elements of $\underset{\sim}{\mathscr{O}}(\overline{K}_1\times\overline{K}_2)'$ can be considered to be Fourier hyperfunctions.

E. Fourier, Fourier–Laplace transformation, and edge of the wedge theorem for (Fourier) hyperfunctions

According to Proposition 2.3 the Fourier transformation is an isomorphism of $\underset{\sim}{\mathscr{O}}(\mathbb{D}^n)$. Hence by duality we have the Fourier transform as an isomorphism for Fourier hyperfunctions:

$$(\mathscr{F}\mu)(f)=\mu(\mathscr{F}f),\qquad \text{for all } f\in\underset{\sim}{\mathscr{O}}(\mathbb{D}^n),\quad \mu\in\underset{\sim}{\mathscr{O}}(\mathbb{D}^n)'. \tag{2.23}$$

As in distribution theory, if appropriate support properties are available, the Fourier transformation has an extension to complex arguments to yield the "Fourier–Laplace transformation."

Proposition 2.9 (Paley–Wiener theorem for Fourier hyperfunctions)*:* Let Γ be a closed and strictly convex cone in $\mathbb{R}^n$ with its vertex at the origin such that $\Gamma\subset\{x\in\mathbb{R}^n|x\cdot e>0\}\cup\{0\}$ for some unit vector $e\in\mathbb{R}^n$. Let $\mathring{\Gamma}=\{\xi\,|x\cdot\xi>0,\text{ for all }x\in\Gamma\}$ be the polar set of Γ and let μ be a Fourier hyperfunction on $\mathbb{D}^n$.

(a) If $\operatorname{supp}\mu\subset\overline{\Gamma}=$ closure of Γ in $\mathbb{D}^n$, then the *Fourier–Laplace transform* of μ,

$$(\mathscr{L}\mu)(\zeta)=\mu(e_\zeta),\quad e_\zeta=(2\pi)^{-n/2}e^{i\zeta\cdot z}, \tag{2.24}$$

is well defined for $\zeta\in\mathbb{R}^n+i\mathring{\Gamma}$ and is a holomorphic function of its argument satisfying the following growth condition: For every relatively compact open cone $\Gamma_0\subset\mathring{\Gamma}$ and every $0<\epsilon<\epsilon_0(\Gamma_0)$ there is a constant $C=C(\epsilon,\Gamma_0)$ such that, for all $\zeta\in\mathbb{R}^n+i\Gamma_0$,

$$|(\mathscr{L}\mu)(\zeta)|\leqslant Ce^{\epsilon|\operatorname{Re}\zeta|+\chi_\epsilon(\operatorname{Im}\zeta)}, \tag{2.25}$$

where

$$\chi_\epsilon(\eta)=\sup_{x\in\Gamma-\epsilon e}\{\epsilon|x|-x\cdot\eta\}.$$

(b) Conversely if a holomorphic function F on $\mathbb{R}^n+i\mathring{\Gamma}$ satisfies the above growth condition then it is the Fourier–Laplace transform of a Fourier hyperfunction μ on $\mathbb{D}^n$ with support in $\overline{\Gamma}$.

This is proved in Ref. 17. And as for distributions there is an immediate connection with the Fourier transformation.

Corollary 2.10: If $\mu\in\underset{\sim}{\mathscr{O}}(\overline{\Gamma})'$, then the Fourier–Laplace transform $\mathscr{L}\mu$ is holomorphic in $\mathbb{R}^n+i\mathring{\Gamma}$ and its boundary value with respect to the open cone $\mathring{\Gamma}$ equals the Fourier transform of μ:

$$\delta_{\mathring{\Gamma}}(\mathscr{L}\mu)=\mathscr{F}\mu. \tag{2.26}$$

The proof is obvious from Proposition 2.9 and the definitions.

Finally we will use the *edge of the wedge theorem for hyperfunctions* as proved in Ref. 29 which generalizes Epstein's version of this result for distributions.

Proposition 2.11: Let Γ_1 and Γ_2 be two open convex cones in $\mathbb{R}^n$. For any open set U in $\mathbb{R}^n$ and its complex neighborhood V there exists a complex neighborhood W of U such that $W\subset V$ and the following holds: If the boundary values $\delta_{\Gamma_1}(F_1)$ and $\delta_{\Gamma_2}(F_2)$ of two functions F_j holomorphic in $V\cap T(\Gamma_j)$, $T(\Gamma_j)=\mathbb{R}^n+i\Gamma_j$, $j=1,2$, agree on U in the sense of hyperfunctions then there exists a function F holomorphic in $W\cap T(\operatorname{ch}(\Gamma_1\cup\Gamma_2))$ such that

$$F=F_j\quad\text{on}\quad W\cap T(\Gamma_j),\quad j=1,2,$$

where ch A denotes the convex hull of a subset A in $\mathbb{R}^n$.

An immediate consequence is the following corollary.

Corollary 3.11: Let Γ be some open convex cone in $\mathbb{R}^n$ and F some holomorphic function on the tube $T(\Gamma)=\mathbb{R}^n+i\Gamma$. If the boundary value $\delta_\Gamma(F)$ of F in the sense of hyperfunctions vanishes in some open nonempty subset $U\subset\mathbb{R}^n$ then the function F itself vanishes.

III. QFT IN TERMS OF FOURIER HYPERFUNCTIONS

A. The test-function space

A QFT over a test-function space

$$E=\underset{\sim}{\mathscr{O}}(\mathbb{D}^4,V),\quad \dim V<\infty, \tag{3.1}$$

as introduced in Sec. II, is called a *Fourier hyperfunction quantum field theory* (FHQFT). If a QFT is formulated over a test-function space F such that the space $\underset{\sim}{\mathscr{O}}(\mathbb{D}^4,V)$ is densely and continuously embedded into F we call such a theory a *special FHQFT* since then continuous linear functionals on F are special Fourier hyperfunctions.

However, according to Sec. I, a function space E has to meet several requirements in order to be "admissible" as a test-function space of a QFT. So we show here that $E = \underset{\sim}{\mathscr{O}}(\mathbb{D}^4, V)$ indeed is "admissible." For convenience we do this explicitly only for $E = \underset{\sim}{\mathscr{O}}(\mathbb{D}^4)$, e.g., dim $V = 1$.

Since

$$\underset{\sim}{\mathscr{O}}(\mathbb{D}^4) = \operatorname*{ind\,lim}_{m\to\infty} \underset{\sim}{\mathscr{O}}_c^m(U_m)$$

with a fundamental sequence of neighborhoods U_m of $\mathbb{D}^4$ in $\mathbb{Q}^4 = \mathbb{D}^4 + i\mathbb{R}^4$ such that U^m is invariant under complex conjugation $z \to \bar{z}$, a continuous involution $f \to f^*$ on $\underset{\sim}{\mathscr{O}}(\mathbb{D}^4)$ is well defined by

$$f^*(z) = \overline{f(\bar{z})}, \tag{3.2}$$

since on each $\mathscr{O}_c^m(U_m)$ we have $\|f^*\|_m = \|f\|_m$.

The action of the group $G = \mathrm{iSL}(2,\mathbb{C})$ on $\underset{\sim}{\mathscr{O}}(\mathbb{D}^4)$ is defined as usual by

$$(\alpha_{(a,A)} f)(z) = f(\Lambda(a)^{-1}(z-a)), \tag{3.3}$$

where $a \in \mathbb{R}^4$, $A \in \mathrm{SL}(2,\mathbb{C})$, $\Lambda(A) \in L^{\uparrow}_{+}$, and thus

$$\alpha_g(f^*) = \alpha_g(f)^* \tag{3.4}$$

follows easily. The differentiability of the map $G \to \underset{\sim}{\mathscr{O}}(\mathbb{D}^4)$,

$$g \to \alpha_g(f),$$

for each fixed $f \in \underset{\sim}{\mathscr{O}}(\mathbb{D}^4)$, is proved in Sec. V where it is actually used to prove the cluster property.

According to Sec. II the function space $\underset{\sim}{\mathscr{O}}(\mathbb{D}^4)$ admits the Fourier transformation as an isomorphism. Furthermore, for elements in $\underset{\sim}{\mathscr{O}}(\mathbb{D}^{4n})'$, $n = 1,2,\ldots$, a "good" notion of support is available, expressing the intuitive meaning of support in this mathematical frame. Hence we get an adequate formulation of the locality condition (H_4) if the notion of support is understood in the sense of Fourier hyperfunctions:

$$\mathrm{Supp}_{\mathrm{HF}} \langle A_\alpha, A_\beta \rangle \subset \overline{K} = \text{closure of } K \text{ in } \mathbb{D}^8. \tag{3.5}$$

With these specifications of the test-function space a *scalar relativistic quantum field in terms of Fourier hyperfunctions* is a field over

$$E = \{\underset{\sim}{\mathscr{O}}(\mathbb{D}^4), {}^*, \alpha_g, \quad g \in G \equiv \mathrm{iSL}(2,C)\}$$

satisfying (H_1)–(H_5).

B. HFQFT in terms of its n-point functions

In this subsection we briefly recall the description of a field in terms of the sequence of its n-point functions[1] and indicate, where necessary, the differences with respect to the "standard" approach as a result of the particularities of the test-function space $E = \underset{\sim}{\mathscr{O}}(\mathbb{D}^4)$.

Given a scalar field A over $E = \underset{\sim}{\mathscr{O}}(\mathbb{D}^4)$ satisfying (H_0) and (H_1) we can consider the sequence of separately continuous n-linear functionals on $E^n = E \times \cdots \times E$ (n times) defined by

$$(f_1, \ldots f_n) \to (\Phi_0, A(f_1) \cdots A(f_n) \Phi_0),$$

where Φ_0 denotes the cyclic unit vector. By Proposition 2.2 these functionals uniquely determine Fourier hyperfunctions $\mathscr{W}_n = \mathscr{W}_n^A \in \underset{\sim}{\mathscr{O}}(\mathbb{D}^{4n})'$ such that

$$\mathscr{W}_n(f_1 \otimes \cdots \otimes f_n) = (\Phi_0, A(f_1) \cdots A(f_n) \Phi_0), \tag{3.6}$$

for all $f_j \in \underset{\sim}{\mathscr{O}}(\mathbb{D}^4)$ and all $n = 1,2,\ldots$.

The sequence

$$\mathscr{W} = \mathscr{W}^A = \{1, \mathscr{W}_1^A, \mathscr{W}_2^A, \mathscr{W}_3^A, \ldots\} \tag{3.7}$$

of these *n-point functions* $\mathscr{W}_n^A$ of the field A is a *state* on the complete tensor algebra,

$$\underline{E} = \bigoplus_{n=0}^{\infty} E(n) \quad \text{(locally convex direct sum)},$$

$$E(0) = C, \quad E(n) = \underset{\sim}{\mathscr{O}}(\mathbb{D}^{4n}) = \hat{\otimes}^n \underset{\sim}{\mathscr{O}}(\mathbb{D}^4), \quad n \geqslant 1, \tag{3.8}$$

that is a continuous linear functional on $\underline{E}$, which is normalized according to

$$\mathscr{W}(\underline{1}) = 1, \quad \underline{1} = \{1,0,0,\ldots\}$$

and non-negative according to

$$\mathscr{W}(\underline{f}^* \cdot \underline{f}) \geqslant 0, \quad \text{for all } \underline{f} \in \underline{E}, \tag{3.9}$$

where the involution $*$ on $\underline{E}$ is given by canonical extension of the involution on E and the product is the usual product of tensor algebras. Conversely according to the well known reconstruction theorem[1-3] such a state $\mathscr{W}$ on $\underline{E}$ determines uniquely up to unitary equivalence a field A over $E = \underset{\sim}{\mathscr{O}}(\mathbb{D}^4)$ satisfying (H_1) and (3.6).

If the field A is covariant in the sense of condition (H_2) then the associated state $\mathscr{W} = \mathscr{W}^A$ on $\underline{E}$ is easily seen to be invariant under the action

$$\underline{\alpha}_g = \bigoplus_{n=0}^{\infty} \alpha_g^{\otimes n}, \quad \alpha_g^{\otimes 0} = 1, \tag{3.10}$$

of $G = \mathrm{iSL}(2,\mathbb{C})$ on $\underline{E}$.

Conversely if a state $\mathscr{W}$ on $\underline{E}$ is invariant under the action (3.10) of G it determines as above a field A over E and a continuous unitary representation U of G satisfying (H_1) and (H_2).

Next we translate the locality condition (H_4) into properties of the n-point functions $\mathscr{W}_n$. This condition says

$$\mathrm{supp}_{\mathrm{HF}}(\Phi, [A(\cdot), A(\cdot)]_\sigma \psi) \subset \overline{K}, \quad \text{for all } \Phi, \psi \in \mathscr{D}_0,$$

where $[A(f), A(g)]_\sigma = A(f)A(g) - \sigma A(g)A(f)$, for all $f,g \in E$, $\sigma = (-1)^\alpha$, $\alpha = 0$ or 1, that is, we assume $A = A_\alpha$ in (H_4). Hence if we introduce, for $0 \leqslant j \leqslant n$ and $n = 0,1,2,\ldots$, Fourier hyperfunctions $\mathscr{W}_{n,j} \in \underset{\sim}{\mathscr{O}}(\mathbb{D}^{4(n+2)})'$ by

$$\begin{aligned}\mathscr{W}_{n,j}&(x_1,\ldots,x_j,x,y,x_{j+1},\ldots,x_n)\\ &= \mathscr{W}_{n+2}(x_1,\ldots,x_j,x,y,x_{j+1},\ldots,x_n)\\ &\quad - \sigma \mathscr{W}_{n+2}(x_1,\ldots,x_j,y,x,x_{j+1},\ldots,x_n),\end{aligned} \tag{3.11}$$

we easily see that by definition of $\mathscr{D}_0$ the locality condition (H_4) is equivalent to

$$\mathrm{supp}_{\mathrm{HF}} \, \mathscr{W}_{n,j} \subset \overline{K}_{n,j} \tag{3.12}$$

or $\mathscr{W}_{n,j} \in \underset{\sim}{\mathscr{O}}(\overline{K}_{n,j})'$, for all $0 \leqslant j \leqslant n$ and all $n = 0,1,2,\ldots$, where $\overline{K}_{n,j}$ is the closure of $K_{n,j} = \mathbb{R}^{4j} \times K \times \mathbb{R}^{4(n-j)}$ in $\mathbb{D}^{4(n+2)}$.

In Sec. V the "cluster property" is proved to be equivalent to condition (H_5) (uniqueness of the vacuum state Φ_0).

Thus we are left with expressing the spectral condition (H_3) in terms of properties of the n-point functions. However, in HFQFT this is considerably more complicated than for the tempered field since also in energy-momentum space there are no test functions of compact support. Hence this point needs some additional arguments.

Suppose that (H_0)–(H_2) are satisfied. Then according to Proposition 2.2 there are continuous linear maps

$$\Phi_n\colon E(n)\to\mathscr{H},\ E(n)=\underset{\sim}{\mathscr{O}}(\mathbb{D}^{4n})$$

satisfying, for all $f_j\in E(1)=\underset{\sim}{\mathscr{O}}(\mathbb{D}^4)$,

$$\Phi_n(f_1\otimes\cdots\otimes f_n)=A(f_1)\cdots A(f_n)\Phi_0,\quad n=1,2,\ldots. \tag{3.13}$$

The covariance of the field under $G=\mathrm{iSL}(2,\mathbb{C})$ implies in particular the following transformation law for these maps Φ_n under translations:

$$\begin{aligned}&U(a)\Phi_n(f_n)=\Phi_n(f_{n,a}),\\&f_{n,a}(x_1,\ldots,x_n)=f_n(x_1-a,\ldots,x_n-a).\end{aligned}\tag{3.14}$$

The consequences of these transformation properties on the Fourier hyperfunctions Φ_n with values in the Hilbert space $\mathscr{H}$ are most conveniently analyzed if in its Fourier transform $\widetilde{\Phi}_n$ the following variables are introduced:

$$\begin{aligned}&(q_1,\ldots,q_n)=\chi_n^{-1}(p_1,\ldots,p_n),\quad q_k=\sum_{j=k}^{n}p_j,\\&\widetilde{Z}_n=\Phi_n\cdot\chi_n.\end{aligned}\tag{3.15}$$

Denote by P the generator of the translations $U(a)$, i.e.,

$$U(a)=e^{iaP}=\int e^{iak}E(dk).\tag{3.16}$$

The spectrum $\Sigma=\sigma(P)$ of the operator P is given by the support of the projection-valued measure E:

$$\Sigma=\sigma(P)=\operatorname{supp}E.\tag{3.17}$$

For any continuous bounded function h we know

$$h(P)=\int h(k)E(dk)=\int da\,\tilde{h}(a)U(a)\tag{3.18}$$

to be a bounded operator.

The transformation property (3.14) can now be expressed in the following way: For every $f\in E(1)\equiv\underset{\sim}{\mathscr{O}}(\mathbb{D}^4)$, every $g\in E(n-1)=\underset{\sim}{\mathscr{O}}(\mathbb{D}^{4(n-1)})$, and every function h in the multiplicator space of $\tilde{E}(1)$, one has

$$h(P)\widetilde{Z}_n(f\otimes g)=\widetilde{Z}_n(h\cdot f\otimes g).\tag{3.19}$$

This equation can be used to extend $\widetilde{Z}_n$ in its first argument f. For every $m\in\mathbb{N}$, define functions ρ_m and ψ_m by

$$\rho_m(q)=\prod_{i=0}^{3}\cosh\left(\frac{q_i}{m}\right)\quad\text{and}\quad\psi_m(q)=\rho_m(q)^{-1}.\tag{3.20}$$

It follows, for $m=1,2,\ldots$,

$$\psi_m\in\underset{\sim}{\mathscr{O}}(\mathbb{D}^4),\quad\text{and}\quad|\rho_m(q)|\leqslant Ce^{|q|/m}.$$

So we can rewrite Eq. (3.19) as

$$\widetilde{Z}_n(f\otimes g)=(\rho_m\cdot f)(P)\widetilde{Z}_n(\psi_m\otimes g),\tag{3.21}$$

and thus $f\to\widetilde{Z}_n(f\otimes g)$ can be extended to all those f for which $(\rho_m\cdot f)(P)$ is a bounded operator on $\mathscr{H}$, i.e., for which

$$\sup_{q\in\Sigma}|\rho_m(q)f(q)|\leqslant C|f|_{m,\Sigma}\equiv C\sup_{q\in\Sigma}e^{|q|/m}|f(q)|<\infty$$

is finite. This is in particular the case for all continuous functions f of compact support K in $\mathbb{R}^4$:

$$|f|_{m,\Sigma}\leqslant C_{K\cap\Sigma}|f|_{\infty,K\cap\Sigma}=C_{K\cap\Sigma}\sup_{q\in K\cap\Sigma}|f(q)|<\infty.$$

Hence we have proved the first part of the following proposition.

Proposition 3.1: (a) The vector-valued Fourier hyperfunctions $\widetilde{Z}_n$ of Eq. (3.15) can be extended to continuous linear maps

$$C_0(\mathbb{R}^4)\times\mathscr{O}(\mathbb{D}^{4(n-1)})\to\mathscr{H},$$

that is, $\widetilde{Z}_n$ is a Radon measure in q_1 and a Fourier hyperfunction in

$$q_2,\ldots,q_n,\quad n=2,3,\ldots.$$

(b) For every $g\in E(n-1)$ the measure $h\to\widetilde{Z}_n(h\otimes g)$ is slowly increasing and has its support $\Sigma_n(g)$ contained in Σ.

(c) The energy-momentum spectrum Σ of the theory is given by

$$\Sigma=\mathrm{cl}\left(\{0\}\cup\bigcup_{n=1}^{\infty}\Sigma_n\right),\quad\Sigma_n=\bigcup_{g\in E(n-1)}\Sigma_n(g),\tag{3.22}$$

where $\mathrm{cl}(A)$ denotes the closure of A in $\mathbb{R}^n$.

To complete the proof note that by Eq. (3.21) $Z_n(h\otimes g)$ extends to all functions

$$h\in F=\operatorname*{ind\,lim}_{m\to\infty}F_m,$$

where F_m is the Banach space of continuous functions on $\mathbb{R}^n$ such that $|h|_{m,\Sigma}$ is finite. Hence this measure is slowly increasing and has its support Σ_n contained in Σ.

Finally part (c) follows from the fact that

$$\{\Phi_0\},\ \{\widetilde{Z}_n(f\otimes g)\,|\,f\in E(1),\ g\in E(n-1),\ n=1,2,\ldots\}$$

is a total set of vectors in the representation space $\mathscr{H}$ of the unitary representation U.

The connection of the vector-valued Fourier hyperfunctions $\widetilde{Z}_n$ with the n-point functions of the theory is described by the following proposition.

Proposition 3.3: (a) Define Fourier hyperfunctions $\widetilde{W}_{n-1}$, $n=2,3,\ldots$, by

$$\begin{aligned}&\widetilde{W}_{n-1}(f)=(\Phi_0,\widetilde{Z}_n(\psi_m\otimes f)),\\&f\in E(n-1)=\mathscr{O}(\mathbb{D}^{4(n-1)}),\end{aligned}\tag{3.23}$$

where $m\in\mathbb{N}$ is arbitrary; then the Fourier transform $\widetilde{\mathscr{W}}_n$ of the n-point function $\mathscr{W}_n$ satisfies

$$\widetilde{\mathscr{W}}_n\cdot\chi_n(q_1,\ldots,q_n)=\delta(q_1)\widetilde{W}_{n-1}(q_2,\ldots,q_n).\tag{3.24}$$

(b) These Fourier hyperfunctions $\widetilde{W}_{n-1}$ allow the following decompositions:

$$\begin{aligned}&(\widetilde{Z}_j(f_j\otimes\cdots\otimes f_1),\widetilde{Z}_{j+1}(f_{j+1}\otimes\cdots\otimes f_n))\\&\quad=\widetilde{W}_{n-1}(\bar{f}_1\otimes\cdots\otimes\bar{f}_{j-1}\otimes\bar{f}_j\cdot f_{j+1}\otimes f_{j+2}\otimes\cdots\otimes f_n),\end{aligned}\tag{3.25}$$

for all $f_i\in\mathscr{O}(\mathbb{D}^4)$, $1\leqslant j\leqslant n-1$, $n=2,3,\ldots$.

Proof: (a) Define $\widetilde{W}_{n-1}$ by Eq. (3.23) with $m=1$. Then for arbitrary $m\in\mathbb{N}$ we use Eq. (3.21) to get

$$\widetilde{Z}_n(\psi_1\otimes f)=(\rho_m\cdot\psi_1)(P)\widetilde{Z}_n(\psi_m\otimes f),$$

and thus, since the cyclic unit vector Φ_0 is translation invariant and

$$(\rho_m \psi_1)(0) = 1,$$
$$(\Phi_0, \widetilde{Z}_n(\psi_1 \otimes f)) = (\rho_m \psi_1)(0)(\Phi_0, \widetilde{Z}_n(\psi_m \otimes f)) = (\Phi_0, \widetilde{Z}_n(\psi_m \otimes f)).$$

Hence the definition (3.23) is independent of $m \in \mathbb{N}$.

Similarly we have, for all $f_i \in \mathscr{O}(\mathbb{D}^4)$, according to Eqs. (3.6), (3.15), and (3.21),

$$\begin{aligned}\mathscr{W}_n \cdot \chi_n (f_1 \otimes \cdots \otimes f_n) &= (\Phi_0, \widetilde{Z}_n(f_1 \otimes \cdots \otimes f_n)) \\ &= (\rho_m \cdot f_1)(0)(\Phi_0, \widetilde{Z}_n(\psi_m \otimes \cdots \otimes f_n)) \\ &= f_1(0)\widetilde{W}_{n-1}(f_2 \otimes \cdots \otimes f_n),\end{aligned}$$

and this proves Eq. (3.24). Finally part (b) follows by straightforward calculations directly from the definitions.

Remark 3.1: Together with Proposition 3.1, Eq. (3.25) says that the Fourier hyperfunctions $\widetilde{W}_{n-1}$ can always be considered in one of its variables as slowly increasing Radon measure with support in Σ. In particular we have, for all $h \in E(1)$, $g \in E(n-1)$, $n = 1,2,\ldots,$

$$\widetilde{W}_{2n-1}(g^* \otimes \bar{h} \cdot h \otimes g) = (\widetilde{Z}_n(h \otimes g), \widetilde{Z}_n(h \otimes g)), \quad (3.26)$$

exhibiting positivity properties of these measures.

Finally we derive support properties of the Fourier hyperfunctions $\widetilde{W}_{n-1}$, $n = 2,3,\ldots,$ in all variables.

Denote by $\overline{\Sigma}$ the closure of Σ in $\mathbb{D}^4$ and introduce for $j = 1,\ldots,n-1$ the closed set

$$U_j = \{(q_1,\ldots,q_{n-1}) \in [\mathbb{D}^4]^{(n-1)} | q_j \in \overline{\Sigma}\}.$$

Then Eq. (3.25) and Proposition 3.1 imply

$$\operatorname{supp} \widetilde{W}_{n-1} \subset \overline{U}_j, \quad (3.27)$$

that is,

$$\widetilde{W}_{n-1|V_j} = 0, \quad (3.27')$$

for $V_j = \overline{U}_j^c = [\mathbb{D}^4]^{(n-1)} - \overline{U}_j$ and $j = 1,2,\ldots,n-1$.

The localization property (L_1) of the sheaf of Fourier hyperfunctions on energy-momentum space implies

$$\widetilde{W}_{n-1|\cup_{j=1}^{n-1} V_j} = 0$$

or

$$\operatorname{supp} \widetilde{W}_{n-1} \subset \left(\bigcup_{j=1}^{n-1} V_j\right)^c = \bigcap_{j=1}^{n-1} \overline{U}_j = \overline{\Sigma}^{n-1}. \quad (3.28)$$

Here we consider that $\widetilde{W}_{n-1}$ is defined on $[\mathbb{D}^4]^{n-1}$. Since $\mathscr{O}(\overline{\Sigma}^{n-1}) \supset \mathscr{O}(\overline{\Sigma^{n-1}})$, we have $\operatorname{supp} \widetilde{W}_{n-1} \subset \overline{\Sigma^{n-1}}$, if we consider that it is defined on $\mathbb{D}^{4(n-1)}$ (see Remark 2.3). By Proposition 3.2 this proves the following corollary.

Corollary 3.3: In a relativistic quantum field theory over $E = \mathscr{O}(\mathbb{D}^4)$ [only (H_1)–(H_3) have to be assumed] the Fourier transform $\widetilde{\mathscr{W}}_n$ of the n-point function $\mathscr{W}_n$ has its support contained in the closure of

$$\left\{(p_1,\ldots,p_n) \in \mathbb{R}^{4n} \middle| \sum_{j=1}^n p_j = 0, \left(\sum_{j=2}^n p_j, \sum_{j=3}^n p_j,\ldots,p_{n-1} + p_n, p_n\right) \in \Sigma^{(n-1)}\right\}$$

in $\mathbb{D}^{4n}$.

Therefore also in HFQFT a field A can be characterized in the usual way in terms of its n-point functions $\mathscr{W}_n = \mathscr{W}_n^A$ if the relevant support conditions (in coordinate and energy-momentum space) are interpreted in the sense of hyperfunctions. From (2.26) we have $\operatorname{supp} \widetilde{Z}_n \subset \overline{\Sigma}^n$ or $\operatorname{supp} \widetilde{Z}_n \subset \overline{\Sigma^n}$. The support properties of the Fourier transforms $\widetilde{\mathscr{W}}_n$ of the n-point function $\mathscr{W}_n$ together with the Paley–Wiener theorem for Fourier hyperfunctions (Proposition 2.8) allow us to derive the basic analyticity properties of the Wightman functions as easily as for tempered fields.[1-3]

Theorem 3.4: The n-point functions $\mathscr{W}_n$ of a relativistic quantum field over $E = \mathscr{O}(\mathbb{D}^4)$ [only (H_1)–(H_3) have to be assumed] are boundary values of $L^+(\mathbb{C})$-invariant holomorphic functions $\widehat{\mathscr{W}}_n$.

(a) $$\widehat{\mathscr{W}}_{n+1}(z_0,z_1,\ldots,z_n) = \widehat{W}_n(z_1 - z_0, z_2 - z_1,\ldots, z_n - z_{n-1}), \quad (3.29)$$

where $\widehat{W}_n$ is holomorphic and $L^+(\mathbb{C})$ invariant on the extended tube

$$\mathscr{T}'_n = \bigcup_{A \in L^+(\mathbb{C})} A\mathscr{T}^+_{n'} \quad \text{and} \quad \mathscr{T}^+_n = T(V^n_+)$$

is the forward tube.

(b) The restriction of $\widehat{W}_n$ to $\mathscr{T}^+_n$ is the Fourier–Laplace transform of the Fourier hyperfunction $\widetilde{W}$ defined in Proposition 3.2. As an identity for Fourier hyperfunctions we have, for fixed $y_j \in V_+$,

$$\mathscr{W}_{n+1}(x_0,x_1,\ldots,x_1) = \lim_{\epsilon \to +0} \widehat{W}_n(x_1 - x_0 + i\epsilon y_1,\ldots,x_n - x_{n-1} + i\epsilon y_n). \quad (3.30)$$

C. Characterization of locality, existence of PCT operator, and global nature of local commutativity

The locality condition (3.11) and (3.12) says that the n-point functions

$$\mathscr{W}_{n+2}(x_1,\ldots,x,y,x_{j+1},\ldots,x_n)$$

and

$$-(-1)^{\alpha\beta}\mathscr{W}_{n+2}(x_1,\ldots,x,y,x_{j+1},\ldots,x_n)$$

agree as Fourier hyperfunctions in particular on the subset

$$I_{n+2} = \{(x_1,\ldots,x,y,x_{j+1},\ldots,x_n) \in \mathbb{R}^{4(n+2)} | (x_i - x_j)^2 < 0 \ (i \neq j), (x-y)^2 < 0, \ (x_j - x)^2 < 0, \ (x_j - y)^2 < 0, \ j = 1,\ldots,n\},$$

which is open in $\mathbb{R}^{4(n+2)}$.

By Jost's characterization of the real points of the extended tube $\mathscr{T}'_{n+1}$ (see Ref. 2) it follows from Theorem 3.4 that I_{n+2} consists of real points of analyticity of the associated Wightman functions $\mathscr{W}_{n+2}(x_1,\ldots,x,y,x_{j+1},\ldots,x_n)$ and $\mathscr{W}_{n+2}(x_1,\ldots,y,x,x_{j+1},\ldots,x_n)$.

However, if two analytic functions agree, in the sense of Fourier hyperfunctions, on an open set of real points of analyticity they do so as analytic functions (Corollary 2.11). This implies now that we can argue as in the case of tempered fields and arrive at the following theorem.

Theorem 3.5: Consider the Wightman functions $\hat{\mathscr{W}}_n$ of a relativistic quantum field over $E=\mathscr{O}(\mathbb{D}^4)$ [satisfying only (H_1)–(H_3)] as given by Theorem 3.4. Then the locality condition (H_4) holds if and only if the $\mathscr{W}_n$ are analytic in

$$S_n^\pi=\{(z_1,...,z_n)\,|\,(z_{\pi(2)}-z_{\pi(1)},...,z_{\pi(n)}-z_{\pi(n-1)})\in\mathscr{T}'_{n-1},\quad \text{for some permutation } \pi \text{ of } (1,...,n)\},$$

and are permutation symmetric there:

$$\hat{W}_n(z_1,...,z_n)=\hat{\mathscr{W}}_n(z_{\pi(1)},...,z_{\pi(n)}),\quad n=2,3,... .$$

Remark 3.2: Without giving further details it should be clear from the above discussion on analyticity results that the PCT theorem[1–3] continues to hold in HFQFT.

Later we will have to use the following technical result that relies in an essential way on the analyticity properties of the Wightman functions.

Proposition 3.6: If $A=(A_1,...,A_M)$ is a relativistic quantum field over $E=\mathscr{O}(\mathbb{D}^4,V)$, $\dim V=M$, then $A_{j_0}(f)\Phi_0=0$, for all $f\in\mathscr{O}(\mathbb{D}^4)$ and some $j_0\in\{1,...,M\}$, implies $A_{j_0}=0$ (Φ_0 denotes the cyclic vacuum vector).

Proof: Since A is supposed to be local, the components $\{A_j\}$— of A are local relative to each other, that is,

$$\operatorname{supp}[A_i(\cdot),A_j(\cdot)]_{\sigma_{ij}}\subset\overline{K},\quad i,j=1,...,M.$$

At all points $(x_1,...,x,x_{k+1},...,x_n)$ such that

$$(x_2-x_1,...,x-x_k,x_{k+1}-x,...,x_n-x_{n-1})=(\xi_1,...,\xi_n)$$

is a Jost point we have, by repeated application of the locality condition as an identity for Fourier hyperfunctions,

$$\begin{aligned}W_n(\xi_1,...,\xi_n)&=(\Phi_0,A_{j_1}(x_1)\cdots A_{j_k}(x_k)A_{j_0}(x)A_{j_{k+1}}(x_{k+1})\cdots A_{j_n}(x_n)\Phi_0)\\&=\pm(\Phi_0,A_{j_1}(x_1)\cdots A_{j_n}(x_n)A_{j_0}(x)\Phi_0)=0.\end{aligned}$$

Thus Theorem 3.4 implies that the Wightman functions $\hat{W}_n\in\tilde{\mathscr{O}}(\mathscr{T}'_n)$ vanish on the open subset J_n of $\mathbb{R}^{4n}$. Hence by Corollary 2.11 $\tilde{W}_n$ vanishes identically. Therefore again by Theorem 3.4 the boundary value

$$(\Phi_0,A_{j_1}(x_1)\cdots A_{j_k}(x_k)A_{j_0}(x)A_{j_{k+1}}(x_{k+1})\cdots A_{j_n}(x_n)\Phi_0)$$

vanishes identically. And this holds for all $j_i\in\{1,...,M\}$, all $1\leqslant k<n$, and all $n=1,2,...$.

Thus $A_{j_0}(f)$ vanishes for all $f\in\mathscr{O}(\mathbb{D}^4)$ on the minimal domain $\mathscr{D}_0$, and we are done, since by cyclicity of Φ_0 the minimal domain is dense in the Hilbert space.

Remark 3.3: The proof of the "global nature of local commutativity" (Chap. 4.1 of Ref. 1) for tempered fields relies on analyticity properties of the Wightman functions and on arguments about analytic completion for special tube domains. The basic analyticity properties are provided by Theorem 3.4. The proofs of Theorem 3.5, Proposition 3.6, and Theorem 6.1 show that also in HFQFT the appropriate tools are available to imitate the proof given for tempered fields. Hence we conclude the following.

Theorem 3.7: Let A be a relativistic quantum field over $E=\mathscr{O}(\mathbb{D}^4,V)$, $\dim V<\infty$, satisfying (H_0)–(H_5) but the locality condition (H_4) only in the weaker form

$$\operatorname{supp}\langle A_\alpha,A_\beta\rangle\subset\overline{M},$$

with some closed subset $M\subset\mathbb{R}^4\times\mathbb{R}^4$ satisfying

$$K\subset M\quad\text{and}\quad M^c\neq\varnothing.$$

Then A satisfies $\operatorname{supp}\langle A_\alpha,A_\beta\rangle\subset\overline{K}$, i.e., A satisfies the locality condition (H_4).

IV. CLUSTER PROPERTY

The proof of the cluster property as given by Jost and Hepp[30] applies whenever the minimal domain $\mathscr{D}_0$ of the field, spanned by

$$\{\Phi_0,A(f_{j_1})\cdots A(f_{j_n})\Phi_0\,|\,f_{j_i}\in E,\ n=1,2,...\},$$

is invariant under the infinitesimal generators of $G=\mathrm{iSL}(2,\mathbb{C})$ in the given representation U [see (H_2)]. By the definition of the minimal domain and the action of $U(g)$, $g\in G$, on it this follows immediately from the invariance of the underlying space E of test functions under the infinitesimal generators of the action α of G on E according to (H_0). We give an explicit proof of this latter invariance for the test-function space $E=\mathscr{O}(\mathbb{D}^4)$ of rapidly decreasing holomorphic functions on $\mathbb{D}^4$ and prepare it by a sequence of lemmas. The technical details are given only for the more complicated case of the subgroup $\mathrm{SL}(2,\mathbb{C})$ of G. The corresponding proof for the subgroup of translations is left as an exercise.

Let $t\to\Lambda_t$ be a function on $\mathbb{R}$ with values in the space of $n\times n$ matrices such that

$$\begin{aligned}&\text{(i)}\quad \Lambda_0=I=\text{identity matrix},\\&\text{(ii)}\quad \Lambda_t=I+t\Sigma+o(t),\quad\text{for } |t|\to0,\\&\qquad\text{with some } n\times n \text{ matrix } \Sigma,\\&\qquad\text{i.e., } t\to\Lambda_t \text{ is differentiable at } t=0.\end{aligned}\tag{4.1}$$

For $x\in\mathbb{R}^n$ we introduce

$$y_t=\Lambda_t x-x\quad\text{and}\quad y=\Sigma x\tag{4.2}$$

and get immediately, with some constant $C\in\mathbb{R}_+$,

$$|y_t-ty|=o(t)|x|,\quad |y_t|\leqslant|t|C|x|.\tag{4.3}$$

Now let U_m be a neighborhood of $\mathbb{D}^n$, e.g.,

$$U_m=\{x+iy\in\mathbb{Q}^n|\ |\mathrm{Im}\,y|<1/m\}.$$

Take some fixed $f\in\mathcal{O}_c^m=\mathcal{O}_c^m(U_m)$, and apply Taylor's theorem for fixed x and $|t|\to 0$:

$$f(\Lambda_t x)-f(x)=\sum_{j=1}^{n} y_t^j\,\partial_j f(x)+\sum_{j,k=1}^{n} y_t^j y_t^k(\partial_j\partial_k f)(x+\theta_t y_t), \tag{4.4}$$

where y_t^j is the jth component of y_t and $\theta_t=\theta_t(x)$ some real number between 0 and 1. The terms on the right-hand side of this equation will be controlled by some lemmas.

Lemma 4.1: For $f\in\mathcal{O}_c^m$ and $\beta=(\beta_1,...,\beta_n)\in\mathbb{Z}_+^n$, we have $\partial_\beta f\in\mathcal{O}_c^{m'}$ for all $m'>m$.

Proof: For $m'>m$ there is $\delta>0$ such that the polycircles

$$C(z)=C_1\times\cdots\times C_n,\quad C_j=\{\zeta_j|\ |\zeta_j-z_j|=\delta\},\quad |\mathrm{Im}\, z|\leqslant 1/m',$$

are contained in $\{\zeta\ |\ |\mathrm{Im}\,\zeta|<1/m\}$. Hence, for $\zeta\in C(z)$ and $m'>m$, we know

$$|z|/m'-|\zeta|/m\leqslant\delta/m'.$$

By Cauchy's integral formula, $\partial_\beta f(z)$ is easily dominated according to

$$|\partial_\beta f(z)|=(2\pi)^{-n}\left|\int_{C(z)} f(\zeta)\prod_{j=1}^{n}(\zeta_1-z_j)^{-1-\beta}jd\zeta_j\right|\leqslant K_1\|f\|_m e^{-|\zeta|/m},$$

and thus

$$\|\partial_\beta f\|_{m'}=\sup_{|\mathrm{Im}\, z|\leq 1/m'} e^{|z|/m'}|\partial_\beta f(z)|\leq K_1\|f\|_m d e^{\delta/m'}.$$

Lemma 4.2: For fixed $g\in\mathcal{O}_c^m$ and all $m'>n.$,

$(1/t)\,y_t^j g$ converges for $|t|\to 0$ in $\mathcal{O}_c^{m'}$ to $y^j g$.

Proof: By (4.3), given $\epsilon>0$, there is $\delta>0$ such that, for all x and all $0<|t|<\delta$,

$$|y_t-ty|\leqslant\epsilon|t|\ |x|.$$

Hence, for all $m'>m$ and all $0<|t|<\delta$, we have, for $j=1,...,n$,

$$\|(1/t)\,y_t^j g-y^j g\|_{m'}=\sup_{|\mathrm{Im}\, x|\leqslant 1/m'}|(1/t)y_t^j-y^j|g(x)|e^{|x|/m'}\leqslant\sup_{|\mathrm{Im}\, x|\leqslant 1/m'}\epsilon|x|\ \|g\|_m e^{|x|(1/m'-1/m)}=\epsilon\|g\|_m C_m.$$

This implies the statement of the lemma.

Lemma 4.3: For fixed $g\in\mathcal{O}_c^m$ and all $m'>m$,

$(1/t)y_t^j\,y_t^k g(\cdot+\theta_t(\cdot)y_t)\to 0$ in $\mathcal{O}_c^{m'}$ as $|t|\to 0$.

Proof: Choose $\delta=(1-m/m')/C$, where the positive constant C is given by (4.3). Then we have by (4.3), for all $0<|t|<\delta$,

$$|\theta_t(x)y_t|\leqslant|y_t|\leqslant|x|,$$

and thus

$$|x+\theta_t y_t|\geqslant|x|(1-|t|C).$$

Furthermore, for all $0<|t|<\delta/2$,

$$\frac{1}{m'}-\frac{1-|t|C}{m}<\frac{1}{2}\left(\frac{1}{m'}-\frac{1}{m}\right)<0$$

is known. This implies, for $0<|t|<\delta/2$,

$$|(1/t)y_t^j y_t^k g(x+\theta_t(\cdot)y_t)|\leqslant C^2|t|\ |x|^2\|g\|_m e^{-|x+\theta_t(x)y_t|/m}\leqslant|t|C^2\|g\|_m|x|^2e^{-|x|(1-C|t|)/m}$$

and

$$\|(1/t)y_t^j y_t^k g(\cdot+\theta_t(\cdot)y_t)\|_{m'}\leqslant|t|C^2\|g\|_m\sup_{|\mathrm{Im} x|\leqslant 1/m}|x|^2e^{|x|[1/m'-(1-C|t|)/m]}\leqslant|t|\,\|g\|_m C_m,$$

for all $0<|t|<\delta/2$, follows easily.

Proposition 4.4: The space $\mathcal{O}(\mathbb{D}^n)$ is invariant under the infinitesimal generators of the induced action of (4.1) on $\mathcal{O}(\mathbb{D}^n)$, that is, for fixed $f\in\mathcal{O}(\mathbb{D}^n)$, one has

$$\lim_{|t|\to 0}\{f(\Lambda_t\cdot)-f(\cdot)\}=\hat{\Sigma}f\quad\text{in }\mathcal{O}(\mathbb{D}^n), \tag{4.5}$$

where

$$(\hat{\Sigma}f)(x)=\sum_{j=1}^{n}(\Sigma x)^j(\partial_j f)(x).$$

Proof: Since $\mathcal{O}(\mathbb{D}^n)$ is the inductive limit of the Banach spaces $\mathcal{O}_c^m=\mathcal{O}_c^m(\mathbb{D}^n)$ for $m\to\infty$ it suffices to show that, for fixed $f\in\mathcal{O}_c^m$, there is some $m'>m$ such that the above limit relation holds in $\mathcal{O}_c^{m'}$. According to Eq. (4.4),

$$(1/t)\{f(\Lambda_t x)-f(x)\}-(\hat{\Sigma}f)(x)=\sum_{i=1}^{n}\left(\frac{1}{t}y_t^j-y^j\right)\partial_j f(x)+\sum_{j,k=1}^{n}\frac{1}{t}y_t^j y_t^k(\partial_j\partial_k f)(x+\theta_t y_t).$$

Lemmas 4.1 and 4.2 imply that the first term of the right-hand side tends to zero in $\mathcal{O}_c^{m'}$ for $|t|\to 0$ for all $m'>m+1$. Similarly the second term converges to zero in $\mathcal{O}_c^{m'}$ for $|t|\to 0$ for all $m'>m+2$ by Lemmas 4.1 and 4.3. This proves (4.5).

Corollary 4.5: A test-function space of the form $E=\mathcal{O}(\mathbb{D}^4,V)$ with action α of $G=\mathrm{iSL}(2,\mathbb{C})$ on E specified by Eq. (3.3) is invariant under the infinitesimal generators of this action; i.e., E is invariant under the generators of the translations and the generators of the Lorentz transformations on E.

Proof: If $t\to A_t$ is a one-parameter subgroup of the Lie group $\mathrm{SL}(2,\mathbb{C})$, we take, in Proposition 4.4,

$$\Lambda_t=\Lambda(A_{-t}),\quad t\in\mathbb{R},$$

where Λ is the canonical homomorphism from $\mathrm{SL}(2,\mathbb{C})$ onto $L_+^\uparrow$. Since $t\to S(A_{-t})$ is easily seen to be differentiable (compare Sec. III A), Proposition 4.4 implies that

$$\lim_{|t|\to 0}(1/t)\{\alpha_{(0,A_t)}f-f\}=\lim_{|t|\to 0}(1/t)\{S(A_{-t})f(\Lambda_t\cdot)-f(\cdot)\}$$

exists in E and thus proves the invariance of this test-function space under the generators of the "Lorentz transformations" on E. The case of translations is even simpler. Consider the translation group in direction

$$e\in\mathbb{R}^4,\ |e|=1:\quad \alpha_{(te,1)}f(x)=f(x-te),\quad t\in\mathbb{R}.$$

If we identify $y_t=-te$ and $y=-e$ we have, instead of (4.3),

$$y_t-ty=0 \quad\text{and}\quad |y_t|=|t|, \tag{4.3'}$$

and thus the proof of Proposition 4.4 simplifies considerably. Finally this implies the invariance of E under the generators of the translations on E.

Theorem 4.6: In a relativistic quantum field theory over a test-function space $E=\mathscr{O}(\mathbb{D}^4,V)$, where the point $p=0$ is isolated in the energy momentum spectrum Σ, the following identity holds for arbitrary but fixed a, $a^2<0$ [only (H_0)–(H_3) are assumed]:

$$\operatorname*{w-lim}_{\lambda\to\infty} U(\lambda a,1)=Q_0\equiv E(\{0\}), \tag{4.6}$$

where Q_0 is the projection operator onto the subspace of translation invariant states.

Hence the theory has a unique vacuum state [i.e., condition (H_5) holds] if and only if the *cluster property*

$$\mathscr{W}(\underline{g}\cdot\underline{\alpha}_{\lambda a}\underline{f})\to\mathscr{W}(\underline{g})\mathscr{W}(\underline{f}),\quad \text{for } \lambda\to\infty,$$
$$\text{for all } \underline{g},\ \underline{f}\in\underline{E}, \tag{4.7}$$

is satisfied.

Proof: Corollary 4.5 assures that all assumptions for the "Jost–Hepp proof" of this statement[30] are satisfied. Thus we are finished.

V. CONNECTION BETWEEN SPIN AND STATISTICS

There is a set of results in QFT usually referred to as the spin-statistics theorem by which the form of the commutation relation for the field (used in the formulation of the locality condition) is related to the type of field (spinor or tensor).[1–3]

The main results in this respect are the theorem of Burgoyne, Lüders, and Zumino on one side and the theorem of Dell'Antonio on the other side. The proof of the result mentioned first relies on properties of the Lorentz group and its representations and on analyticity properties of the Wightman functions [analyticity and $L^+(\mathbb{C})$ covariance in the extended tubes, existence of Jost points, and the fact that $-\mathbf{1}_4\in L^+(\mathbb{C})$]. Since these properties are also available in HFQFT (see Sec. III) the theorem of Burgoyne, Lüders, and Zumino still holds in HFQFT. Dell'Antonio's theorem reads in its HFQFT version as follows:

Theroem 5.1: If a relativistic quantum field $A=(A_1,\ldots,A_M)$ over $E=\mathscr{Q}(\mathbb{D}^4,V)$, dim $V=M$, satisfies

$$\operatorname{supp}[A_k(x),A_j^*(y)]_-\subset\overline{K} \tag{5.1}$$

and

$$\operatorname{supp}[A_k(x),A_j(y)]_+\subset\overline{K}, \tag{5.2}$$

then either $A_j=0$ or $A_k=0$.

The same conclusion holds if in (5.1) and (5.2) the signs $+$ and $-$ are exchanged.

Compared to the situation in "standard" QFT this result is considerably harder to prove in HFQFT.

The starting point for a proof is the following elementary identity which holds for all test functions $f,g\in\mathscr{Q}(\mathbb{D}^4)$ and all $\lambda\geqslant 0$:

$$\begin{aligned}&\|A_j(f)A_k(g_\lambda)\Phi_0\|^2\\&\quad+(A_k(g)^*A_k(g)\Phi_0,U(\lambda a)A_j(f)^*A_j(f)\Phi_0)\\&=(A_k(g_\lambda)\Phi_0,A_j(f)^*[A_j(f),A_k(g_\lambda)]_+\Phi_0)\\&\quad+(A_k(g_\lambda)\Phi_0,[A_k(g_\lambda),A_j(f)^*]_-A_j(f)\Phi_0)\\&\equiv \mathrm{I}_\lambda+\mathrm{II}_\lambda,\end{aligned} \tag{5.3}$$

where

$$U(\lambda a)A_j(g)U(\lambda a)^{-1}=A_j(g_\lambda),\quad g_\lambda=g_{\lambda a},$$

with some spacelike vector $a=(0,\mathbf{a})$, $\mathbf{a}^2=1$.

If test functions f and g of compact support were available one could choose λ sufficiently large so that the functions f and g_λ would have spacelike separated supports. Then the assumptions (5.1) and (5.2) would easily imply that the right-hand side of Eq. (5.3) vanishes, and by the cluster property one would easily conclude the proof. In our case of HFQFT the control over the rhs of Eq. (5.3) needs considerably more preparation relating geometrical facts about a product of Minkowski spaces with the topology of the underlying test-function space $\mathscr{Q}(\mathbb{D}^4)$ as well as the precise formulation of the "locality conditions" (5.1) and (5.2).

Denote by $\overline{V}_+$ the closed forward light cone $\{\xi^3,\xi)\in\mathbb{R}^4|\xi^0\geqslant|\xi|\}$ and by $V=\overline{V}_+\cup\overline{V}_-$, $\overline{V}_-=-\overline{V}_+$ the closed light cone.

The set K of (5.1) and (5.2) decomposes into $\overline{K}=\overline{K}^4\cup\overline{K}^-$, where

$$K^\pm=\{z=(x,y)\in\mathbb{R}^4\times\mathbb{R}^4|x-y\in\overline{V}_\pm\}.$$

The following lemma establishes some facts about the separation of the set K from "spacelike" points. It is proved in the Appendix.

Lemma 5.2: For $a=(0,\mathbf{a})\in\mathbb{R}^4$, $\mathbf{a}^2=1$, denote $\hat{a}=(0,a)\in\mathbb{R}^4\times\mathbb{R}^4-K$. Then the following hold.

(a) The $\operatorname{dist}(K,\lambda\hat{a})=\lambda/2$, $\lambda\geqslant 0$, is attained at

$$\lambda a^\pm=\lambda((\mp 1,\mathbf{a}),(\pm 1,3\mathbf{a}))\in K.$$

(b) Denote by $e^\pm=a^\pm-\hat{a}$; then $z\cdot e^+\geqslant 0$, for all $z\in K^+$, and $z\cdot e^-\geqslant 0$, for all $z\in K^-$. Hence, for all $z\in K^\pm$,

$$|z-\lambda\hat{a}|\geqslant|z-\lambda a^\pm|.$$

(c) Given $\delta_0>0$, define $\lambda_0=\lambda_0(\delta_0)$ by

$$\lambda_0=[16|a^\pm|/(2|a^\pm|-1)]\delta_0,\quad |a^\pm|=\sqrt{3}/2;$$

then, for all $\lambda\geqslant\lambda_0$ and all $z\in K$,

$$|z-\lambda\hat{a}|\geqslant\epsilon_0|z|+\delta_0+\lambda/4,$$

where $\epsilon_0^{-1}=8|a^\pm|=4\sqrt{3}$.

Lemma 5.3: There exist a positive constant L_0 and a symmetric neighborhood $U=-U$ of $\overline{K}$ (the closure of K in $\mathbb{D}^8$) in $\mathbb{D}^8+i\mathbb{R}^8$ such that

$$\operatorname{dist}(U\cap\mathbb{C}^8,\lambda\hat{a})\geqslant\lambda/4,\quad\text{for all }\lambda\geqslant L_0.$$

Proof: Let $B(z,r)$ be the open ball of radius r and center $z\in\mathbb{R}^8$. For $0<\epsilon$, $0<\delta$, introduce the real neighborhood

$$K_{\epsilon,\delta}=\bigcup_{z\in K}B(z,\epsilon|z|+\delta)$$

of K and then the complex neighborhood

$$W_{\epsilon,\delta} = \{z\in\mathbb{C}^8 | \mathrm{Re}\, z\in K_{\epsilon,\delta}, \quad |\mathrm{Im}\, z| < \delta\}.$$

If we choose now $0 < \epsilon \leqslant \epsilon_0$ and $\delta_0 \leqslant 2\delta$ and apply part (c) of Lemma 5.2 we see easily that $|z - \lambda\hat{a}| \geqslant \lambda/4$ holds for all $z\in W_{\epsilon,\delta}$ and all $\lambda \geqslant \lambda_0(\delta_0)$.

By definition of $\mathbb{D}^8$ there exists a neighborhood U of $\overline{K}$ in $\mathbb{D}^8 + i\mathbb{R}^8$ such that $U \cup \mathbb{C}^8 = W_{\epsilon,\delta}$ holds.

Lemma 5.4: For any $f\in\mathcal{Q}(\mathbb{D}^8)$, define f_λ for $\lambda \geqslant 0$ by

$$f_\lambda(z) = f(z + \lambda\hat{a}), \quad z\in\mathbb{D}^8,$$

where $\hat{a}$ is defined in Lemma 5.2. Then

$$f_\lambda \to 0, \quad \text{for} \quad \lambda \to \infty \quad \text{in} \quad \mathcal{Q}(\overline{K}).$$

Proof: The space $\mathcal{Q}(\overline{K})$ is defined to be the inductive limit of a sequence of Banach spaces $\mathcal{O}_c^m(U_m)$, $m\in\mathbb{N}$, where $\{U_m | m\in\mathbb{N}\}$ is a fundamental system of neighborhoods of $\overline{K}$ in $\mathbb{Q}^8 = \mathbb{D}^8 + i\mathbb{R}^8$; hence we have to show that, for some m,

$$f_\lambda \to 0 \quad \text{for} \quad \lambda \to \infty \quad \text{in} \quad \mathcal{O}_c^m(U_m).$$

Since $f\in\mathcal{Q}(\mathbb{D}^8)$ there exist positive numbers δ and C such that

$$|f(z)| \leqslant Ce^{-\delta|z|} \quad \text{in} \quad \{z|\ |\mathrm{Im}\, z| \leqslant \delta\}.$$

With this $\delta > 0$, do the construction of Lemma 5.3 to obtain a neighborhood U of $\overline{K}$ in $\mathbb{Q}^8$ such that

$$U \cap \mathbb{C}^8 \subset \{z|\ |\mathrm{Im}\, z| < \delta\},$$

and, for sufficiently large λ,

$$\mathrm{dist}(U \cap \mathbb{C}^8, \lambda\hat{a}) \geqslant \lambda/4.$$

Then there is $M\in\mathbb{N}$ such that, for all $m \geqslant M$,

$$U_m \subset U \quad \text{and} \quad m\delta \geqslant 4.$$

Next observe that

$$z - \lambda\hat{a}\in U\cap\mathbb{C}^8 \quad \text{implies} \quad |z| \geqslant \lambda/4,$$

since then $\zeta = \lambda\hat{a} - z\in U\cap\mathbb{C}^8$ and $z = \lambda\hat{a} - \zeta$; thus, by Lemma 5.3,

$$|z| \geqslant \mathrm{dist}(U\cap\mathbb{C}^8, \lambda\hat{a}) \geqslant \lambda/4,$$

if $\lambda \geqslant L_0$.

Now fix $m \geqslant M$ and choose $\lambda \geqslant L_0$. Then the following chain of inequalities holds:

$$\|f_\lambda\|_m = \sup_{z\in U_m\cap\mathbb{C}^8} |f_\lambda(z)| e^{|z|/m}$$

$$\leqslant C \sup_{z - \lambda\hat{a}\in U_m\cap\mathbb{C}^8} e^{-\delta|z|} e^{|z-\lambda\hat{a}|/m}$$

$$\leqslant C \sup_{|z| \geqslant \lambda/4} e^{-(\delta - 1/m)|z| + \lambda/m} \leqslant Ce^{-\delta\lambda/8}.$$

Thus we conclude the proof of Lemma 5.4.

Proof of Theorem 5.1: Since $\|A_k(g_\lambda)\Phi_0\| = \|A_k(g)\Phi_0\|$ is known, the first term I_λ in Eq. (5.3) is dominated by

$$|\mathrm{I}_\lambda| \leqslant \|A_k(g)\Phi_0\|\ \|A_j(f)^*[A_j(f), A_k(g_\lambda)]_+\Phi_0\|.$$

Assumption (5.2) means that for any $\Phi,\Psi\in\mathcal{D}$ the functional

$$h_1 \times h_2 \to (A_j(f)\Phi, [A_k(h_1), A_j(h_2)]_+\Psi)$$

belongs to $\mathcal{Q}(\overline{K})'$. Since $\mathcal{Q}(\overline{K})$ is barreled it follows that

$$h_1 \times h_2 \to \|A_j(f)^*[A_k(h_1), A_j(h_2)]_+\Psi\|$$

$$= \sup_{\Phi\in\mathcal{D}, \|\Phi\| = 1} |(A_j(f)\Phi, [A_k(h_1), A_j(h_2)]_+\Psi)|$$

is a continuous seminorm on this space. By Lemma 5.4 we know $f \times g_\lambda \to 0$ for $\lambda \to \infty$ in $\mathcal{Q}(\overline{K})$, for every $f,g\in\mathcal{Q}(\mathbb{D}^4)$. This implies $\mathrm{I}_\lambda \to 0$ for $\lambda \to \infty$. Similarly assumption (5.1), barreledness of $\mathcal{Q}(\overline{K})$, and Lemma 5.4 are used to conclude $\mathrm{II}_\lambda \to 0$ for $\lambda \to \infty$.

Thus the right-hand side of Eq. (5.3) has a vanishing limit for $\lambda \to \infty$. By the cluster property (Theorem 4.6) the second term on the left-hand side of Eq. (5.3) has the limit

$$\|A_k(g)\Phi_0\|^2 \|A_j(f)\Phi_0\|^2,$$

for $\lambda \to \infty$. Hence $\|A_j(f)A_k(g_\lambda)\Phi_0\|^2$ also has a limit for $\lambda \to \infty$ and this limit is

$$-\|A_k(g)\Phi_0\|^2 \|A_j(f)\Phi_0\|^2.$$

We conclude $\|A_k(g)\Phi_0\|^2 \|A_j(f)\Phi_0\|^2 = 0$ and obtain either

$$A_k(g)\Phi_0 = 0 \quad \text{or} \quad A_j(f)\Phi_0 = 0, \quad \text{for all} \quad g, f\in\mathcal{Q}(\mathbb{D}^4).$$

If, for instance, $A_j(\cdot)\Phi_0 = 0$, then, by Proposition 3.6, $A_j = 0$ follows.

VI. SOME CHARACTERIZATIONS OF TRIVIAL QUANTUM FIELDS

A. Characterization of generalized free fields

Our first result here assures us that the well known characterization of generalized free fields in terms of commutator properties for the field operators still holds in HFQFT.

Theorem 6.1: For a relativistic quantum field A over $E = \mathcal{Q}(\mathbb{D}^4)$ with cyclic vacuum vector Φ_0 the following conditions are equivalent:

$$[A(f), A(g)] \subset \mathbb{W}_2(f\otimes g - g\otimes f)\mathbf{1}, \quad \text{for all } f,g\in E, \tag{6.1}$$

$$[A(f), A(g)]\Phi_0 = \mathbb{W}_2(f\otimes g - g\otimes f)\Phi_0, \quad \text{for all } f,g\in E. \tag{6.2}$$

Proof: (a) We only have to show that condition (6.2) implies (6.1). And by cyclicity of the vacuum state this follows from

$$\begin{aligned}
&\mathbb{W}_{n+m+2}(f_1\otimes\cdots\otimes f_n\otimes f\otimes g\otimes g_1\otimes\cdots\otimes g_m) \\
&\quad - \mathbb{W}_{n+m+2}(f_1\otimes\cdots\otimes f_n\otimes g\otimes f\otimes g_1\otimes\cdots\otimes g_m) \\
&\quad = \mathbb{W}_2([f,g])\mathbb{W}_{n+m}(f_1\otimes\cdots\otimes f_n\otimes g_1\otimes\cdots\otimes g_m),
\end{aligned} \tag{6.3}$$

for all $f,g,f_j,g_j\in E$ and $n,m = 0,1,2,\ldots$, with $[f,g] = f\otimes g - g\otimes f$. By Theorem 3.4 the hyperfunctions $\mathbb{W}_N$, $N = 2,3,\ldots$, are boundary values of analytic functions $\widehat{\mathbb{W}}_N$. So we study the analytic functions $\widehat{\mathbb{W}}^\pm$ on tubes $T(\Gamma_\pm)$ defined by

$$\mathbb{W}^+(\underline{z}_n, z, w, \underline{w}_m) = \widehat{\mathbb{W}}_{n+m+2}(\underline{z}_n, z, w, \underline{w}_m) - \widehat{\mathbb{W}}_2(z,w)\widehat{\mathbb{W}}_{n+m}(\underline{z}_n, \underline{w}_m),$$

$$\begin{aligned}
\Gamma_+ = \{(\underline{\xi}_n, \xi, \eta, \underline{\eta}_m) | &\xi_{j+1} - \xi_j\in V_+, \quad j \leqslant n-1, \\
&\xi - \xi_n\in V_+, \quad \eta - \xi\in V_+, \quad \eta_1 - \eta\in V_+, \\
&\eta_{j+1} - \eta_j\in V_+, \quad j \leqslant m-1\}
\end{aligned} \tag{6.4}$$

and

$$\hat{\mathbb{W}}^-(\underline{z}_n,z,w,\underline{w}_m) = \hat{\mathbb{W}}_{n+m+2}(\underline{z}_n,w,z,\underline{w}_m) - \hat{\mathbb{W}}_2(w,z)\hat{\mathbb{W}}_{n+m}(\underline{z}_n,\underline{w}_m),$$
$$\Gamma_- = \{(\underline{\xi}_n,\xi,\eta,\underline{\eta}_m)\,|\,\xi_{j+1}-\xi_j\in V_+,\ j\leqslant n-1,\ -\eta+\xi\in V_+,\ \eta-\xi_n\in V_+,\ \eta_1-\xi\in V_+,\ \eta_{j+1}-\eta_j\in V_+,\ j\leqslant m-1\}. \qquad (6.5)$$

(b) By Theorem 3.4 these analytic functions $\hat{\mathbb{W}}^\pm$ have boundary values $\mathbb{W}^+ = \delta_{\Gamma_+}\hat{\mathbb{W}}^+$ (resp. $\mathbb{W}^- = \delta_{\Gamma_-}\hat{\mathbb{W}}^-$) in the sense of hyperfunctions. By locality (H_4) and our assumption (6.2) these boundary values agree on the open subset U of $\mathbb{R}^{4(n+m+2)}$,

$$U = \{(\underline{x}_n,x,y,\underline{y}_m)\,|\,(x-y_j)^2<0,\ (y-y_j)^2<0,\ j=1,\dots,m\}.$$

Hence by the edge of the wedge theorem for hyperfunctions (Proposition 2.10) there exist a complex neighborhood V of U and a function

$$\hat{\mathbb{W}}\in\tilde{\mathscr{O}}(V\cap T(\mathrm{ch}(\Gamma_+\cup\Gamma_-))$$

such that
$$\hat{\mathbb{W}} = \hat{\mathbb{W}}^\pm \quad\text{on}\quad V\cap T(\Gamma_\pm). \qquad (6.6)$$

(c) Observe now that the complex cone

$$\Gamma_0 = \{(\underline{\xi}_n,\xi,\eta,\underline{\eta}_m)\,|\,\xi_{j+1}-\xi_j\in V_+,\ j\leqslant n-1,\ \times\xi-\xi_n\in V_+,\ \eta=\xi,\ \eta_1-\eta\in V_+,\ \times\eta_{j+1}-\eta_j\in V_+,\ j\leqslant m-1\} \qquad (6.7)$$

is contained in the convex hull $\mathrm{ch}(\Gamma_+\cup\Gamma_-)$ of $\Gamma_+\cup\Gamma_-$.

To prove this, suppose $(\underline{\xi}_n,\xi,\eta,\underline{\eta}_m)\in\Gamma_0$ to be given. Then there are $\zeta_j\in V_+$ such that $\zeta_1+\zeta_3-\zeta_2\in V_+$,

$$\xi-\xi_n = \zeta_1+\tfrac{1}{2}\zeta_3 \quad\text{and}\quad \eta_1-\eta = \zeta_2+\tfrac{1}{2}\zeta_3.$$

Write $\underline{\xi}_n = (\xi_1,\underline{\xi}'_n)$ and define

$$\underline{\zeta}^+ = (\underline{\xi}_n^+,\xi^+,\eta^+,\eta_1^+,\underline{\eta}_m'^+) \equiv (\underline{\xi}_n,\xi_n+\zeta_1,\zeta_1+\zeta_3+\xi_n,\zeta_1+\zeta_2+\zeta_3+\xi_n,\underline{\eta}'_m)$$

and

$$\underline{\zeta}^- = (\underline{\xi}_n^-,\xi^-,\eta^-,\eta_1^-,\underline{\eta}_m'^-) \equiv (\underline{\xi}_n,\xi+\tfrac{1}{2}\xi_3,\eta+\tfrac{1}{2}\xi_3-\zeta_2,\eta_1,\underline{\eta}'_m).$$

Then $\underline{\zeta}^\pm\in\Gamma_\pm$ and

$$(\underline{\xi}_n,\xi,\eta,\underline{\eta}_m) = (\underline{\zeta}^+ + \underline{\zeta}^-)/2\in\mathrm{ch}(\Gamma_+\cup\Gamma_-).$$

(d) By (6.6) and (6.7) we conclude that $\hat{\mathbb{W}}^+$ and $\hat{\mathbb{W}}^-$ are analytically continued with respect to the variables (z,w) to $\mathrm{Im}(z-w)=0$. Therefore $\mathbb{W}^\pm(\underline{x}_n,x,y,\underline{y}_m)$ can be considered to be hyperfunctions in the variables $\underline{x}_n,\underline{y}_m$ with real analytic parameters x,y. And these hyperfunctions are boundary values of functions $\hat{\mathbb{W}}^\pm(x,y;\underline{z}_n,\underline{w}_m)$ analytic in the tube $T(\Gamma)$,

$$\Gamma = \{(\underline{\xi}_n,\underline{\eta}_m)\,|\,\xi_{j+1}-\xi_j\in V_+,\ j\leqslant n-1,\ \xi_n\in V_+,\ \eta_1\in V_+,\ \eta_{j+1}-\eta_j\in V_+,\ j\leqslant m-1\},$$

hence

$$\hat{\mathbb{W}}(x,y,\underline{z}_n,\underline{w}_m) = \hat{\mathbb{W}}^+(x,y;\underline{z}_n,\underline{w}_m) - \hat{\mathbb{W}}^-(x,y;\underline{z}_n,\underline{w}_m)$$

is analytic in $T(\Gamma)$.

(e) If U_1 is some open bounded nonempty set in $\mathbb{R}^{2\cdot 4}$ there exists an open set U_2 in $\mathbb{R}^{4m}$ such that all $(x,y,\underline{y}_m)\in U_1\times U_2$ satisfy

$$(x-y_j)^2<0 \quad\text{and}\quad (y-y_j)^2<0, \quad\text{for } j=1,\dots,m.$$

For all $(x,y)\in U_1$ consider the boundary values

$$\mathbb{W}(x,y;\underline{x}_n,\underline{y}_m) = \delta_\Gamma\hat{\mathbb{W}}(x,y;\underline{z}_n,\underline{w}_m)$$

in $\mathbb{R}^{4n}\times U_2$ in the sense of hyperfunctions. As we have shown above in (b) all the boundary values vanish $(\mathbb{R}^{4n}\times U_1\times U_2\subset U)$. Hence by Corollary 2.11 all the analytic functions $\hat{\mathbb{W}}(x,y;\dots)$ on $T(\Gamma)$, $x,y\in U_1$, vanish. Since $\mathscr{W}(x,y;\dots)$ is real analytic in x,y, this function vanishes identically.

By definitions (6.4) and (6.5) this proves Eq. (6.3) and thus we have the theorem.

A relativistic quantum field A over $\mathscr{Q}(\mathbb{D}^4)$ that satisfies condition (6.1) or (6.2) is called a *generalized free field* over $\mathscr{Q}(\mathbb{D}^4)$. Such fields have been studied in some detail by Roberts.[31] Clearly as in the case of tempered fields relation (6.1) determines easily all n-point functions of the theory. The relevant formulas are given in the following corollary.

Corollary 6.2: If a relativistic quantum field A over $E=\mathscr{Q}(\mathbb{D}^4)$ has a vanishing one-point function $\mathbb{W}_1$ and satisfies (6.1) or (6.2) its n-point functions are

$$\mathbb{W}_{2n+1}=0, \quad\text{for } n=0,1,2,\dots,$$
$$\mathbb{W}_{2n}=\mathbb{W}_{2n}^0\equiv \text{Ⓢ}^n\mathbb{W}_2, \quad\text{for } n=1,2,\dots, \qquad (6.8)$$

where $\mathbb{W}_{2n}^0$ are recursively defined by

$$\mathbb{W}^0_{2(n+1)}(f_1\otimes\cdots\otimes f_{2n+2}) = \sum_{j=2}^{2n+2}\mathbb{W}^0_2(f_1\otimes f_j)\mathbb{W}^0_{2n}(f_1\otimes\cdots\otimes\hat{f}_j\otimes\cdots\otimes f_{2n+2}). \qquad (6.9)$$

B. Jost–Schroer theorem

According to Theorem 6.1 and Corollary 6.2 the four-point function can be used to decide whether a field over $E=\mathscr{Q}(\mathbb{D}^4)$ is a generalized free field or not. In the case of tempered fields Jost and Schroer[32] have observed that this result can be used to determine a scalar relativistic quantum field completely if its two-point function is known to have a special form (that of a free scalar field). For hyperfunction quantum fields this characterization continues to hold if we add a technical assumption on the support of the four-point function.

Theorem 6.3 (Jost–Schroer theorem for HFQFT): If the two-point function $\mathbb{W}_2$ of a relativistic quantum field A over $E=\mathscr{Q}(\mathbb{D}^4)$ with cyclic vacuum Φ_0 equals that of a free field of mass $m>0$ and if the four-point function has no "pathological support" (see Remark 6.1) in energy momentum space, then A is a free field of mass m.

Remark 6.1: According to the results of Sec. III B,

$$\tilde{g}\to\tilde{W}_3(\tilde{h}\otimes\tilde{f}\tilde{f}_1\otimes\tilde{g}) \qquad (6.10)$$

is a well defined Fourier hyperfunction with support in $\bar{\Sigma}$ (the closure of Σ in $\mathbb{D}^4$) for arbitrary $\tilde{h}\in\mathscr{Q}(\mathbb{D}^4)$ and $\tilde{f}\tilde{f}_1\in\mathscr{D}(\mathbb{R}^4)$. Then we say that $\tilde{\mathbb{W}}_4$ has no "pathological support" if the support of the Fourier hyperfunction (6.10) has no connected component contained in S^3_∞.

Remark 6.2: This technical assumption is actually known to be satisfied in some cases. If the theory is formulated over a certain slightly bigger test-function space $E_1 \supset E$ or if the four-point function has a continuous extension to this space then this support property can be shown to hold. A concrete example of such a space E_1 is described in Ref. 7.

Proof of Theorem 6.3: (a) By assumption we have, for all $f \in E$,

$$\|A((\Box + m^2)f)\Phi_0\|^2 = c\langle \delta_m^+(p), (-p^2+m^2)^2 |\tilde{f}(p)|^2\rangle = 0.$$

Hence the field B defined by $B = (\Box + m^2)A$ satisfies $B(f)\Phi_0$, for all $f \in E$. Since the local field B clearly is relatively local to the field A Proposition 3.6 implies $B = 0$, that is the field A solves the linear differential equation

$$(\Box + m^2)A(x)\Phi = 0, \quad \text{for} \quad \Phi \in \mathscr{D},$$

and this implies, for the Fourier transform $\tilde{A}\Phi$ of $A\Phi$,

$$\operatorname{supp} \tilde{A}\Phi \subset \overline{H}_m, \tag{6.11}$$

where H_m is the mass hyperboloid

$$H_m = H_m^+ \cup H_m^-, \quad H_m^\pm = \{(p_0, \mathbf{p}) | p_0 = \pm\sqrt{\mathbf{p}^2 + m^2}\},$$

and $\overline{H}_m$ denotes its closure in $\mathbb{D}^4$.

For such a field one obtains more refined support properties for the Fourier hyperfunction $\tilde{Z}_2$ introduced by Eqs. (3.13) and (3.15):

$$\operatorname{supp} \tilde{Z}_2 \subset \overline{\left(\bigcup_{p\in\Sigma}\{p\} \times T^+(p)\right)},$$

$$T^+(p) = \overline{H_m^+} \cap \overline{(p - H_m)} = [H_m^+ \cap (p - H_m)] \cup \overline{H_m^+} \cap S^3_\infty$$

by Proposition 3.2 and Corollary 3.3.

(b) In order to complete the proof it suffices, according to (6.11) and Theorem 6.1, to show Eq. (6.2). To this end we study the Fourier hyperfunction $[A(x_1), A(x_2)]\Phi_0$ in the coordinates

$$x = (x_1 + x_2)/2, \quad \xi = x_2 - x_1, \quad \Psi^+(x,\xi) = A(x_1)A(x_2)\Phi_0, \tag{6.12}$$

$$\Psi^-(x,\xi) = A(x_2)A(x_1)\Phi_0 = \Psi^+(x, -\xi).$$

The Fourier transform of Ψ^+ satisfies

$$\tilde{\Psi}^+(p,q) = \tilde{Z}_2(p, (q+p)/2),$$

$$\tilde{\Psi}^+(p,(q-p)/2) = \tilde{Z}_2(p,q), \tag{6.13}$$

and it follows, for $\tilde{\Psi} = \tilde{\Psi}^+ - \tilde{\Psi}^-$,

$$\operatorname{sup} \tilde{\Psi} \subset \left(\bigcup_{p\in\Sigma}\{p\} \times S^+(p)\right),$$

$$S(p) = S^+(p) \cup S^-(p), \quad S^-(p) = -S^+(p),$$

$$S^+(p) = \overline{(-p/2 + H_m^+)} \cap \overline{(p/2 - H_m)} = [(-p/2 + H_m^+) \cap (p/2 - H_m)] \times \cup \overline{H_m^+} \cap S^3_\infty. \tag{6.14}$$

Elementary geometry shows

(i) $S^+(0) = \overline{H}_m^+$,

(ii) if $B \subset \Sigma$ is compact in $\mathbb{R}^4$ and $0 \notin B$, then $\left(\bigcup_{p\in B}\{p\} \times S(p)\right) \subset B \times (\hat{B} \cup \overline{H}_m \cap S^3_\infty)$ with some compact set $\hat{B} \subset \mathbb{R}^4$. (6.15)

(c) In the same way as in Proposition 3.1 we can show that $\tilde{\Psi}(p,q)$ is a Radon measure in p and a Fourier hyperfunction in q. Thus we may choose $\tilde{f} \in \mathscr{D}(\mathbb{R}^4)$, $\operatorname{supp} \tilde{f} = B$ with $0 \notin B \subset \Sigma$ and know by Proposition 3.1 that $\tilde{g} \to \tilde{\Psi}(\tilde{f} \otimes \tilde{g})$ is a well defined Fourier hyperfunction, $\tilde{g} \in \mathscr{O}(\mathbb{D}^4)$. Relations (6.12) show

$$\tilde{\Psi}(\tilde{f} \otimes \tilde{g}) = \langle [A(x_1), A(x_2)]\Phi_0, f((x_1 + x_2)/2)g(x_2 - x_1)\rangle, \tag{6.16}$$

therefore by locality the Fourier transform of $\tilde{\Psi}(\tilde{f},q)$ vanishes for $\xi^2 < 0$. According to statement (6.15) the support of the Fourier hyperfunction $\tilde{\Psi}(\tilde{f},q)$ is contained in $\hat{B} \cup \overline{H}_m \cap S^3_\infty$. Now apply Eq. (3.26) for $\tilde{f} \in \mathscr{D}(\mathbb{R}^4)$ and $\tilde{f}_1, \tilde{g}, \tilde{g}_1 \in \mathscr{O}(\mathbb{D}^4)$ to get

$$(\tilde{Z}_2(\tilde{f}_1 \otimes \tilde{g}_1), \quad \tilde{\Psi}(\tilde{f} \otimes \tilde{g})) = \tilde{W}_3(\bar{\tilde{g}}_1 \otimes \bar{\tilde{f}}_1 \tilde{f} \otimes (\tilde{g} - \hat{\tilde{g}})),$$

where $\hat{\tilde{g}}(q) = \tilde{g}(-q)$. This is a Fourier hyperfunction with respect to $\tilde{g}$ with support in $\hat{B} \cup \overline{H}_m \cap S^3_\infty$ and its support is contained in $\hat{B}$ by Remark 6.1. Since $\tilde{f}_1, \tilde{g}_1 \in \mathscr{O}(\mathbb{D}^4)$ are arbitrary, the support of $\tilde{\Psi}(\tilde{f},q)$ is contained in $\hat{B}$. Hence its Fourier Laplace transform with respect to q is an entire analytic function of ζ that vanishes on the open subset $\xi^2 < 0$. Thus $\tilde{\Psi}(\tilde{f},q)$ vanishes and we deduce by choice of $\tilde{f}$

$$\operatorname{supp} \Psi \subset \{0\} \times S(0) = \{0\} \times \overline{H}_m. \tag{6.17}$$

(d) Denote by χ_ϵ the characteristic function of a ball of radius $\epsilon > 0$ and center $p = 0$. Proposition 3.1 and relation (6.17) imply that $\chi_\epsilon \tilde{\Psi}(p,q)$ is well defined and that, for all $\epsilon > 0$,

$$\chi_\epsilon \tilde{\Psi} = \tilde{\Psi}$$

holds. Thus we get for

$$\chi_\epsilon(P)[A(f), A(g)]\Phi_0 = [A(f), A(g)]\Phi_0,$$

$$[A(f), A(g)]\Phi_0 = \langle \tilde{\Psi}(p,q), \tilde{f}(p/2 - q)\tilde{g}(p/2 + q)\rangle.$$

But by uniqueness of the vacuum state (H_5) we know that

$$\chi_\epsilon(P) = \int \chi_\epsilon(p)\, dE(p)$$

converges strongly for $\epsilon \to 0$ to the projection operator $|\Phi_0\rangle\langle\Phi_0|$. This then proves Eq. (6.2).

C. Borchers classes

As with the result about the existence of a PCT operator for a QFT in terms of Fourier hyperfunctions we only indicate in this subsection that also for quantum fields over $E = \mathscr{O}(\mathbb{D}^4, V)$ the concept of the "Borchers class" of some field is available since the possibility for this concept relies exclusively on analyticity properties of the n-point functions and the existence of a PCT operator. These analyticity properties are provided by Theorem 3.4, and the techniques of the proof are very similar to those explained in detail in Secs. III C and VI A.

However, as expected, compared to the case of tempered fields the Borchers class of a field in HFQFT is considerably bigger. In order to see this recall that for tempered fields the Borchers class of a free massive field consists of all Wick polynomials including derivatives of that field.[33]

In Ref. 34 it has been shown that all power series

$$B(x) = \sum_{n=0}^{\infty} c_n :A^n:\frac{(x)}{n!}, \quad \lim_{n\to\infty} [|c_n|^2/n!]^{1/n} = 0, \tag{6.18}$$

define a relativistic quantum field over $E = \mathscr{O}(\mathbb{D}^4)$ and that the associated sequence of Wick polynomials

$$B_N(x) = \sum_{n=0}^{N} c_n :A^n:\frac{(x)}{n!}, \quad N = 1,2,..., \tag{6.19}$$

converges in the relevant topology to $B(x)$. Since all the $B_N(x)$ are known to be relatively local with respect to A it follows that B, too, is relatively local with respect to A.

Hence all entire function of A as described in (6.18) belong to the Borchers class of A.

VII. CONCLUSIONS

In order to give a comprehensive picture about QFT in terms of Fourier hyperfunctions we discuss here the status of the remaining points of the basic structural results of QFT mentioned in Sec. I. The existence of a scattering operator [point (3)] in HFQFT has been proved in Ref. 7; however, it has been proved only for a special class of Fourier hyperfunctions, that is, for a somewhat larger test-function space than $\mathscr{O}(\mathbb{D}^4)$. Though this is already quite a satisfactory result it would be preferable to have a scattering operator also for fields over $\mathscr{O}(\mathbb{D}^4)$. This point is under consideration.

In general form of the two-point function [point (7)] can also be determined in HFQFT. The result is the obvious generalization of the form given by Källen and Lehmann. If we combine the information provided by Propositions 3.1 and 3.2 with Eq. (3.26) for $n = 2$ we get immediately that the two-point function of a scalar field over $\mathscr{O}(\mathbb{D}^4)$ has the following general form:

$$\mathbb{W}_2(f\otimes g) = \int \tilde{t}(dp)\tilde{f}(-p)\tilde{g}(p),$$

with some $L^{\uparrow}_{+}$-invariant positive Radon measure t with support Σ, which is slowly increasing in the sense of Proposition 3.1(b). The structures of such measures are known[35]:

$$t(dp) = c\delta(p)d^4p + \int_0^{\infty} \rho(d\kappa)\delta_{\kappa}^{+}(p)d^4p, \quad c\geqslant 0,$$

with some positive slowly increasing measure ρ on $(0,\infty)$ that is not necessarily polynomially bounded as for tempered fields.

The possibility of a Euclidean reformulation [point (10)] of relativistic QFT in terms of Fourier hyperfunctions has been indicated to exist by Nagamachi and Mugibayashi in Ref. 5 shortly after Osterwalder and Schrader's solution of this problem in terms of distribution. At the price of introducing an even smaller test-function space $\tilde{\mathscr{O}}(\mathbb{D}^4) \subset \mathscr{O}(\mathbb{D}^4)$, Nagamachi and Mugibayashi[6] could actually prove a complete symmetry between the Euclidean and relativistic formulation of "HFQFT" without any additional growth restrictions as in the distributional setting.

However, the space $\tilde{\mathscr{O}}(\mathbb{D}^4)$ has some disadvantages as a space of test functions for QFT. So one might reconsider this problem for the test-function space $\mathscr{O}(\mathbb{D}^4)$. Admitting eventually similar additional growth restrictions as in the distributional setting the proof of equivalence between the Euclidean and relativistic formulation of HFQFT seems to be possible.

We have not tried to prove dispersion relations, which is quite an involved matter. However, we expect that it is possible to prove the necessary analyticity properties for the 2–2-particle scattering amplitude but not the necessary growth restrictions in order to be able to write a dispersion relation with a finite number of subtractions.

Finally we sum up the main points of this paper and give an outlook for further applications of HFQFT.

Since there is no *a priori* choice for the test-function space in relativistic quantum field theory we have isolated conditions on a space E of functions on space-time in order that E be "admissible" as the test-function space of a relativistic QFT [condition (H_0) in Sec. I]. As our short review shows it has been known since the early days of general QFT and has emerged more clearly later by considering model constructions that the traditional choice $E = \mathscr{S}(\mathbb{R}^4, V)$ has to be modified for various important reasons. And accordingly several attempts have been made in the past to generalize the notion of a "tempered relativistic quantum field." Most of these suggestions have considerable difficulties with an appropriate notion of localization in coordinate and/or momentum space. Though it might not have been so clear from the beginning, the only suggestion that has a precise notion of localization in both spaces has been that of Nagamachi and Mugibayashi.[4]

In this paper we have stressed the point of view that a sensible generalization of the notion of a tempered quantum field should not only have these localization properties but should also allow us to derive (hopefully) all the basic structural results of QFT known for tempered fields.

And accordingly in this paper we have presented a short introduction to QFT in terms of Fourier hyperfunctions and have shown that indeed most of the structural results of QFT continue to hold in this more general approach. We mention some further results of HFQFT that we think to be important for future applications.

The existence of entire functions of a free massive field A, for instance,

$$:e^{igA(x)}:, \quad g\in\mathbb{R},$$

can be used in the construction of concrete models.

For instance, the transformation

$$A(x) \to :e^{igA(x)}:$$

can be used for an easy "decoupling" of the interaction of the "derivative coupling model" and thus to obtain a solution of this model.[34] We expect that a renormalization theory based on Fourier hyperfunctions would admit a clearer and more powerful notion of "renormalizable interactions" than in the traditional approach based on (tempered) distributions. An example has been treated in Ref. 34.

One important reason for the great success of Euclidean methods in the construction of models in lower-dimensional space-time clearly is the fact that these methods allow us to take into account in a natural and powerful way the relevant positivity condition. Since the test-function space $E = \mathcal{Q}(\mathbb{D}^n, V)$ of HFQFT is a nuclear DFS space its topological dual E' has a well developed theory of Radon probability measures on it as for the standard case $\mathcal{S}'$ or $\mathcal{D}'$. This might turn out to be important for the construction of HFQFT models with nontrivial interactions according to the "functional integral point of view."

Thus we think that our paper clearly shows that the test-function space (1.2) of Fourier hyperfunctions provides quite a comprehensive realization of the requirements (A)–(C) of the Introduction. In any case this approach is much more natural with respect to the realization of the localization problems [requirement (B)] and is considerably more powerful in the realization of the structural results of QFT [requirement (C)] than any other approach. Furthermore as indicated above there are convincing prospects of further successful applications in model constructions.

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APPENDIX: PROOF OF LEMMA 5.2

(a) For $y \in V^c = \mathbb{R}^4 - V$ the distance to the light cone V is easily calculated to be

$$\operatorname{dist}(V, y) = (|\mathbf{y}| - |y_0|)/\sqrt{2} \tag{A1}$$

and it is attained at a point $e(y)$ of the boundary ∂V

$$e(y) = (|y_0| + |\mathbf{y}|)(\operatorname{sgn} y_0, \hat{\mathbf{y}})/2, \quad \hat{\mathbf{y}} = \mathbf{y}/|\mathbf{y}|. \tag{A2}$$

The points of K are parametrized by

$$K = \{z = (y - \xi, y) \,|\, y \in \mathbb{R}^4, \quad \xi \in V\}.$$

We calculate

$$\operatorname{dist}(K, \lambda\hat{a}) = \inf_{z \in K} |z - \lambda\hat{a}|$$

in two steps using (A1): The first step is simply

$$\inf_{\xi \in V, y \in V^c} |(y - \xi, y) - \lambda\hat{a}| = \inf_{y \in V} |y - \lambda a| = \lambda/2, \tag{A3}$$

and for the second we note

$$\inf_{\xi \in V,\ y \in V^c} |(y - \xi, y) - \lambda\hat{a}|^2$$

$$= \inf_{y \in V^c}\{\inf_{\xi \in V}[|y - \xi|^2 + |y - \lambda a|^2]\}.$$

For $\xi = e(y)$ this equals, according to (A1) and (A2),

$$\inf_{y \in V^c}\{(|\mathbf{y}| - |y_0|)^2/2 + |y - \lambda a|^2\}$$

$$= \inf_{|\mathbf{y}| > |y_0|}\{(|\mathbf{y}| - |y_0|)^2/2 + y_0^2 + (|\mathbf{y}| - \lambda)^2\}.$$

The last infimum is attained at

$$|y_0| = \lambda/4 \quad \text{and} \quad |\mathbf{y}| = 3\lambda/4$$

and equals $\lambda^2/4$; hence

$$\inf_{\xi \in V}\inf_{y \in V^c} |(y - \xi, y) - \lambda\hat{a}| = \lambda/2 \tag{A4}$$

and this is attained at

$$y_\lambda^\pm = \lambda(\pm 1, 3\mathbf{a})/4, \quad \xi = e(y_\lambda^\pm),$$

or at $\lambda a^\pm = (y_\lambda^\pm - e(y_\lambda^\pm), y_\lambda^\pm)$, i.e.,

$$\lambda a^\pm = \lambda((\mp 1, 3\mathbf{a}), (\pm 1, 3\mathbf{a})) \tag{A5}$$

and we conclude

$$\operatorname{dist}(K, \lambda\hat{a}) = |\lambda a^\pm - \lambda\hat{a}| = \lambda|e^\pm| = \lambda/2,$$

where

$$e^\pm = a^\pm - \hat{a} = (-\alpha^\pm, \alpha^\pm), \quad \alpha^\pm = (\pm 1, -\mathbf{a})/4. \tag{A6}$$

This proves part (a).

(b) For $z \in K^\pm$, that is, $\xi = y - x \in V^\pm$, we get

$$z \cdot e^\pm = \xi \cdot \alpha^\pm = (\pm \xi^0 - \xi \cdot \mathbf{a})/4 \geqslant (\pm \xi^0 - |\xi|)/4;$$

hence

$$\begin{aligned} &z \cdot e^+ \geqslant 0, \quad \text{for } z \in K^+, \\ &z \cdot e^- \geqslant 0, \quad \text{for } z \in K^-. \end{aligned} \tag{A7}$$

It follows that

$$\begin{aligned} &|z - \lambda\hat{a}| \geqslant |z - \lambda a^+|, \quad \text{for } z \in K^+, \\ &|z - \lambda\hat{a}| \geqslant |z - \lambda a^-|, \quad \text{for } z \in K^-. \end{aligned} \tag{A8}$$

Thus (b) follows.

(c) In order to prove part (c) we distinguish two cases. If $|z| < (\lambda/4 - \delta)\epsilon_0^{-1}$, then, by part (a),

$$\lambda/4 + \delta + \epsilon_0|z| < \lambda/4 + \delta + \lambda/4 - \delta = \lambda/2$$

$$= \operatorname{dist}(K, \lambda\hat{a}) \leqslant |z - \lambda\hat{a}|,$$

if z also belongs to K.

If, however, $|z| \leqslant (\lambda/4 - \delta)8|a^\pm|$, $z \in K$, we use (A8) to obtain

$$|z - \lambda\hat{a}| \geqslant |z - \lambda a^\pm| \geqslant |\,|z| - \lambda|a^\pm|\,| \geqslant |z| - \lambda|a^\mp|$$

$$\geqslant \epsilon_0|z| + \delta + \lambda/4 + \lambda(2|a^\pm| - 1)/2 - 8|a^\pm|\delta$$

$$\geqslant \epsilon_0|z| + \delta + \lambda/4,$$

since $\lambda \geqslant \lambda_0(\delta)$ is equivalent to $\lambda(2|a^\pm| - 1)/2 - 8|a^\pm|\delta \geqslant 0$.

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