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Calculating Pauli–Jordan function

Daqing Liu[1], Furui Chen[1], Shuyue Chen[1,3] and Ning Ma[2]

[1] School of Mathematics and Physics, Changzhou University, Changzhou, 213164, People's Republic of China

[2] Department of Physics, Taiyuan University of Technology, Taiyuan, 030024, People's Republic of China

E-mail: csyue2000@163.com

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Abstract

A derivation of the space-time expression of the Pauli–Jordan function (PJF) was shown and a more compact result was obtained. Compared to other derivations, the approach utilizes more physical image, such as a Lorentz invariance of the PJF. Furthermore, the approach does not need sophisticated mathematical techniques and is therefore more suitable for the beginner of quantum field theory, such as graduate students and senior undergraduates.

Keywords: Pauli–Jordan function, Lorentz invariance, time-like, space-like

(Some figures may appear in colour only in the online journal)

1. introduction

In the early development of quantum field theory, to determine the commutation or anticommutation relationship between field operators such as scalar fields, vector fields, etc, Pauli and Jordan introduced functions [1], nominated thereafter as Pauli–Jordan functions (PJFs) or sometimes nowadays Schwinger functions.

From the properties of a PJF, it can be easily verified that to quantize a scalar field correctly, commutation relations rather than anticommutation ones between scalar field operators are needed if one is to preserve micro-causality. Furthermore, from a simple deduction of a PJF one can also verify that to quantize a spinor, anticommutation relations between spinor operators are needed, and so on. In fact, there are many essential results in quantum field theory which stem from the properties of PJFs. In other words, PJFs and their associated concepts are the cornerstones of any quantum field theory, and there is much in the literature focusing on these functions or their generalizations, such as [2–7].

[3] Author to whom any correspondence should be addressed.

Although momentum-energy expressions of PJFs are direct, to relate PJFs to microcausality space-time expressions are needed. However, derivations are complicated and most of the literature only presents the explicit expression in special cases [8–10]. The authors of [11] presented a full derivation but with the sophisticated technique of Bessel functions and Hankel functions. The derivations are too tedious to be suitable for a beginner of quantum field theory. As for graduate students or senior undergraduates who are faced with quantum field theory for the first time, he/she would be better not to be confused by these mathematical techniques. In this manuscript we show another intuitive derivation of a PJF. The derivation does not use sophisticated techniques; instead, it uses a Lorentz invariance of a PJF, with the help of a subtraction function between a mass PJF and a massless PJF. We think the derivation is more suitable for graduate students or senior undergraduates in physics.

2. Calculating massless Pauli–Jordan function

A Lorentz-invariant Pauli–Jordan function is defined as

$$D^{(\pm)}(x^0, \mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{\pm(-iE_p x^0 + i\mathbf{p}\cdot\mathbf{x})}, \tag{1}$$

where m is the particle mass. Here we only study the positive frequency function, $D^{(+)}(x^0, \mathbf{x})$, in detail, since generalizing results of $D^{(+)}(x^0, \mathbf{x})$ to those of $D^{(-)}(x^0, \mathbf{x})$ is straightforward.

Notice that the Minkowski metric was used here, that is, the metric is $g_{\mu\nu} = diag(1, -1, -1, -1)$. Illustratively, for a four-dimensional space-time vector $(x^0, \mathbf{x})$ with time component x^0 and space component $\mathbf{x}$, and a four-dimensional energy-momentum vector $(E_p, \mathbf{p})$ with energy component E_p and momentum component $\mathbf{p}$, their Lorentz-invariant scalar product was defined as $x^0 E_p - \mathbf{x}\cdot\mathbf{p}$. Lorentz invariance also requires that the PJF only depends on m and $x^{02} - r^2$, where $r = |\mathbf{x}|$.

In general, the above integral is δ-divergent, which makes the direct calculation sophisticated [11]. Here we adopt a subtraction technique which is commonly used in quantum field theory to deal with integrals involving divergence, which can be seen, for instance, in [12]. We first consider massless PJF, which is divergent but is easy to get the expression. Then, we obtain the solution leveraging a subtraction function which has no singularity.

We now turn to the massless PJF. With $m = 0$, equation (1) reads

$$\begin{aligned} D_0^{(+)}(x^0, \mathbf{p}) &= \frac{e^{-ipx^0}}{2p}, \\ D_0^{(+)}(x^0, \mathbf{x}) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p} e^{i\mathbf{p}\cdot\mathbf{x}} = \frac{1}{4\pi^2 r} \int_0^\infty \sin pr e^{-ipx^0} dp \\ &= \frac{1}{8i\pi^2 r} \int_0^\infty [\cos p(r - x^0) - \cos p(r + x^0) \\ &\quad + i \sin p(r + x^0) + i \sin p(r - x^0)] dp. \end{aligned} \tag{2}$$

It is not difficult to obtain the integral, as shown in [8]:

$$D_0^{(+)}(x^0, \mathbf{x}) = \frac{1}{4\pi^2(r^2 - x^{02})} + \frac{i}{8\pi r}[\delta(r + x^0) - \delta(r - x^0)]. \tag{3}$$

Here we introduce a simpler method. Utilizing the odevity of the integrand, a Fourier transformation (FT) of unity and step function $\theta(x)$ can be written as

$$
\begin{aligned}
FT[1] &= \int_{-\infty}^{\infty} e^{-ipx}dx = 2\int_0^{\infty}\cos pxdx = 2\pi\delta(p) \\
FT[\theta(x)] &= \int_0^{\infty} e^{-ipx}dx = \int_0^{\infty}\cos pxdx - i\int_0^{\infty}\sin pxdx = \pi\delta(p) - \frac{i}{p},
\end{aligned} \tag{4}
$$

we have

$$
\begin{aligned}
&\int_0^{\infty}\cos pxdx = \pi\delta(p) \\
&\int_0^{\infty}\sin pxdx = \frac{1}{p}.
\end{aligned} \tag{5}
$$

Instituting the above equations into equation (2), one can easily obtain equation (3).

The real part of $D_0^{(+)}(x^0, \mathbf{x})$ is an even function of x^0. Meanwhile, the imaginary part is not only an odd function of x^0, but also a singular one on the forward/backward light cone. Obviously both the real part and imaginary part are Lorentz invariant. Furthermore, as expected, one can easily check that $\int D_0^{(+)}(x^0, \mathbf{x})e^{i\mathbf{p}\cdot\mathbf{x}}d^3x = \frac{e^{-ipx^0}}{2p}$.

3. Calculating mass Pauli–Jordan function

Now we turn to the calculation of PJF with $m \neq 0$. To eliminate the δ-divergence we define a subtraction function

$$
\Delta^{(+)}(m^2) = D^{(+)}(x^0, \mathbf{x}) - D_0^{(+)}(x^0, \mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\left(\frac{e^{-iE_px^0+i\mathbf{p}\cdot\mathbf{x}}}{2E_p} - \frac{e^{-ipx^0+i\mathbf{p}\cdot\mathbf{x}}}{2p}\right), \tag{6}
$$

which is the difference between mass PJF and massless PJF. It is obvious that $\Delta^{(+)}(0) = 0$. The integration is convergent, in other words, it is a well-defined function of x^0 and $\mathbf{x}$.

We first consider the space-like case. Noting that $\Delta^{(+)}(m^2)$ is Lorentz invariant, we perform the calculation in another reference frame, in which $x^{0'} = 0$ and $r' = \sqrt{r^2 - x^{02}}$. Integrating out equation (6) in this reference, one finds that

$$
\Delta^{(+)}(m^2) = \frac{1}{4\pi^2 r'}\int_0^{\infty}\left(\frac{p}{\sqrt{p^2 + m^2}} - 1\right)\sin pr'dp. \tag{7}
$$

We have, then,

$$
\frac{\partial}{\partial m^2}\Delta^{(+)}(m^2) = \frac{1}{4\pi^2 r'}\int_0^{\infty}\frac{p\sin pr'}{-2(m^2 + p^2)^{3/2}}dp = -\frac{1}{8\pi^2}K_0(mr'). \tag{8}
$$

In other words,

$$
\Delta^{(+)}(m^2) = \frac{1}{4\pi^2}\left[\frac{m}{\sqrt{r^2 - x^{02}}}K_1(m\sqrt{r^2 - x^{02}}) - \frac{1}{r^2 - x^{02}}\right], \tag{9}
$$

for the space-like situation since $\Delta^{(+)}(0) = 0$. With the space-like case, that is, $r^2 - x^{02} > 0$, the above expression is a real function. In the above equations $K_\nu(z)$ is a second-type modified Bessel function (or Macdonald function), which is defined as

$$
K_\nu(z) = \frac{\sqrt{\pi}}{\Gamma(\nu + \frac{1}{2})}\left(\frac{z}{2}\right)^{\nu}\int_0^{\infty} e^{-zt}(t^2 - 1)^{\nu - \frac{1}{2}}dt. \tag{10}
$$

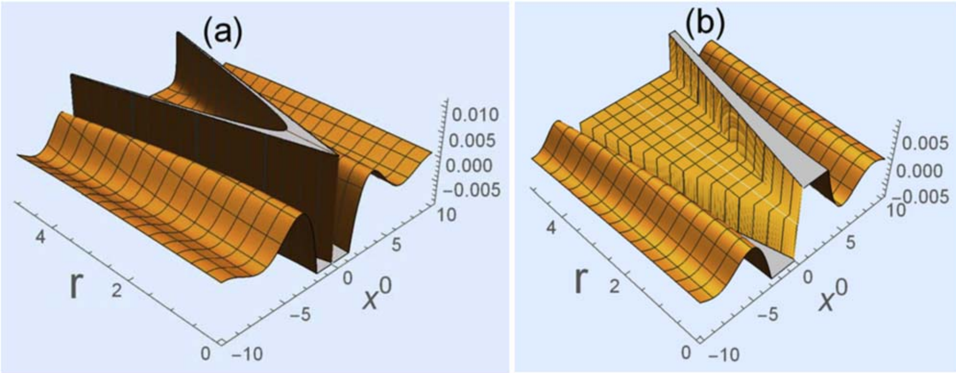

Figure 1. The real part (a) and the imaginary part (b) of $D^{(+)}(x^0, \mathbf{x})$ at $m = 1$, where $r = |\mathbf{x}|$. Here we ignored the δ functions.

As for the time-like case, that is, $x^{02} - r^2 > 0$, we also perform the calculation in another reference frame, in which $x^{0\prime} = \epsilon(x^0)\sqrt{x^{02} - r^2}$ and $r' = 0$, where $\epsilon(x^0) = \begin{cases} 1, & x^0 > 0 \\ -1, & x^0 < 0 \end{cases}$ is the sign function. Integrating out equation (6) in this reference, one finds that

$$\Delta^{(+)}(m^2) = \frac{1}{4\pi^2}\int_0^\infty dpp^2\left(\frac{e^{-iE_p x^{0\prime}}}{E_p} - \frac{e^{-ipx^{0\prime}}}{p}\right). \tag{11}$$

where $\Delta^{(+)}(0) = 0$ and $\Delta^{(+)}(m^2)$ is a well-defined function of $x^{0\prime}$.

We have, then,

$$\begin{aligned} \frac{\partial}{\partial x^{0\prime}}\Delta^{(+)}(m^2) &= \frac{-i}{4\pi^2}\left[\int_m^\infty t\sqrt{t^2 - m^2}\, e^{-itx^{0\prime}} dt - \int_0^\infty p^2 e^{-ipx^{0\prime}} dp\right] \\ &= -\frac{1}{4\pi^2}\left[\frac{m^2 K_2(imx^{0\prime})}{x^{0\prime}} + \frac{2}{x^{0\prime 3}}\right]. \end{aligned} \tag{12}$$

Utilizing $\Delta^{(+)}(0) = 0$, we get

$$\Delta^{(+)}(m^2) = \frac{1}{4\pi^2}\left[\frac{mK_1(imx^{0\prime})}{ix^{0\prime}} + \frac{1}{x^{0\prime 2}}\right], \tag{13}$$

where $x^{0\prime} = \pm\sqrt{x^{02} - r^2}$.

Combining equations (3), (6), (9) and (13), we finally obtain that

$$\begin{aligned} D^{(+)}(x^0, \mathbf{x}) &= \frac{i}{8\pi r}[\delta(r + x^0) - \delta(r - x^0)] + \frac{1}{4\pi^2}\frac{m\epsilon(x^0)}{\sqrt{r^2 - x^{02}}} K_1(m\epsilon(x^0)\sqrt{r^2 - x^{02}}) \\ &= \frac{i}{8\pi r}[\delta(r + x^0) - \delta(r - x^0)] \\ &\quad + \frac{1}{4\pi^2}\begin{cases} \frac{m}{\sqrt{r^2 - x^{02}}} K_1(m\sqrt{r^2 - x^{02}}), & x^0 > 0 \\ \left(\frac{m}{\sqrt{r^2 - x^{02}}} K_1(m\sqrt{r^2 - x^{02}})\right)^*, & x^0 < 0. \end{cases} \end{aligned} \tag{14}$$

Figure 1 shows the the real part and the imaginary part of $D^{(+)}(x^0, \mathbf{x})$ at $m = 1$, ignoring the δ functions. For the space-like case, the real part of $D^{(+)}(x^0, \mathbf{x})$ drops off exponentially as

r increases while the imaginary part is strictly equal to zero. However, for the time-like case with fixed r, both the real part and the imaginary part of $D^{(+)}(x^0, \mathbf{x})$ drop off oscillatingly as $|x^0|$ increases.

At first glance equation (14) is different from the result in [11] (see equations (16.10) and (16.11) therein). However, since for $|x| > 1$, one can verify that

$$\frac{m}{\sqrt{1-x^2}}K_1(m\sqrt{1-x^2}) = \frac{\pi}{2}\frac{m}{\sqrt{x^2-1}}[Y_1(m\sqrt{x^2-1}) + iJ_1(m\sqrt{x^2-1})].$$

These two results are indeed the same. Apparently, the expression in equation (14) is more compact.

4. Different types of Pauli–Jordan functions

The commutation/anticommutation functions are important functions since they are directly related to the quantization of the fields. As we know [9, 8], for a scalar field $\phi(x^0, \mathbf{x})$ with mass m, we have

$$[\phi(x^0, \mathbf{x}), \phi^\dagger(0)]_\pm = D^+(x^0, \mathbf{x}) \pm D^-(x^0, \mathbf{x}), \tag{15}$$

for a vector field $v^\mu(x^0, \mathbf{x})$ with mass m, we have

$$[v^\mu(x^0, \mathbf{x}), v^{\dagger\nu}(0)]_\pm = \left(g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right)[D^+(x^0, \mathbf{x}) \pm D^-(x^0, \mathbf{x})], \tag{16}$$

while for a spinor field $\psi(x^0, \mathbf{x})$ with mass m, we have

$$[\psi_a(x^0, \mathbf{x}), \bar{\psi}_b(0)]_\pm = (i\partial\!\!\!/ + m)_{ab}[D^+(x^0, \mathbf{x}) \mp D^-(x^0, \mathbf{x})], \tag{17}$$

where the subscripts $\pm$ in the left-hand side stand for the anticommutation or commutation relation, respectively. To quantize a field properly, the micro-causality should be preserved, that is, one should adopt a suitable commutation or anticommutation relation to make the relation vanish in the space-like case.

To this end, in the following we discuss the properties of commutation/anticommutation functions respectively.

4.1. Properties of commutation functions

A commutation function is defined as

$$-iD(x^0, \mathbf{x}) = D^{(+)}(x^0, \mathbf{x}) - D^{(-)}(x^0, \mathbf{x}). \tag{18}$$

It is easy to check that expression of $D(x^0, \mathbf{x})$ is as

$$D(x^0, \mathbf{x}) = \frac{\epsilon(x^0)}{2\pi}\delta(r^2 - x^{02}) - \frac{\epsilon(x^0)}{2\pi^2}\,\mathrm{Im}\frac{m}{\sqrt{r^2 - x^{02}}}K_1(m\sqrt{r^2 - x^{02}}). \tag{19}$$

Here we list properties of a Lorentz-invariant commutation function.

1. It is an odd function of x^0.
2. In the space-like case, $r^2 - x^{02} > 0$ and $D(x^0, \mathbf{x}) \equiv 0$. From equations (15)-(17), one finds that adoptions of commutation relations for scalar fields and vector fields and anticommutation relations for spinor fields are suitable since the adoptions do not violate micro-causality.
3. $\frac{\partial}{\partial x^0}D(x^0, \mathbf{x})$ is therefore an even function of x^0 and for $x^0 \to 0_+$ we have $\frac{\partial}{\partial x^0}D(x^0, \mathbf{x}) \to -\frac{\partial}{4\pi r\partial r}\delta_r'(0_+, r) \to -\delta(\mathbf{x})$.

4. Since $D(x^0, \mathbf{p}) = \frac{i}{2E_p}[e^{-iE_p x^0} - e^{iE_p x^0}]$, we have

$$D(p^0, \mathbf{p}) = \frac{2\pi i}{2E_p}[\delta(p^0 - E_p) - \delta(p^0 + E_p)] = 2\pi i\delta(p^{02} - E_p^2), \tag{20}$$

which means that $(p^{02} - E_p^2)D(p^0, \mathbf{p}) \equiv 0$, that is, $(\Box + m^2)D(x^0, \mathbf{x}) = 0$ where $\Box \equiv \frac{\partial^2}{\partial x^{02}} - \nabla^2$.

4.2. Properties of anticommutation functions

An anticommutation function is defined as

$$D_1(x^0, \mathbf{x}) = D^{(+)}(x^0, \mathbf{x}) + D^{(-)}(x^0, \mathbf{x}). \tag{21}$$

It is easy to check that expression of $D(x^0, \mathbf{x})$ is as

$$D_1(x^0, \mathbf{x}) = \frac{1}{2\pi^2}\mathrm{Re}\frac{m}{\sqrt{r^2 - x^{02}}}K_1(m\sqrt{r^2 - x^{02}}). \tag{22}$$

Here we list the properties of a Lorentz-invariant anticommutation function.

1. It is an even function of x^0.
2. $\frac{\partial}{\partial x^0}D_1(x^0, \mathbf{x})$ is therefore an odd function of x^0 and we have $\frac{\partial}{\partial x^0}D_1(0, \mathbf{x}) \equiv 0$.
3. In the space-like case, $r^2 - x^{02} > 0$ and $D_1(x^0, \mathbf{x}) \neq 0$. From equations (15)- (17), one finds that adoptions of anticommutation relations for scalar fields and vector fields and commutation relations for spinor fields are unsuitable since the adoptions do violate micro-causality.
4. Since $D_1(x^0, \mathbf{p}) = \frac{1}{2E_p}[e^{-iE_p x^0} + e^{iE_p x^0}]$, we have

$$D_1(p^0, \mathbf{p}) = \frac{2\pi}{2E_p}[\delta(p^0 - E_p) + \delta(p^0 + E_p)], \tag{23}$$

which means that we still have $(\Box + m^2)D_1(x^0, \mathbf{x}) = 0$.

5. Summary

In summary, in this paper a derivation of a space-time expression of a Pauli–Jordan function was shown and a more compact expression of a PJF was obtained. The idea of the derivation is as follows. Firstly we showed the expression of a massless PJF which is Lorentz invariant. Secondly, to eliminate the δ-divergence of the PJF we defined a subtraction function which is the difference between a mass PJF and massless PJF and is a well-defined function of space and time. Lastly, in space-like and time-like cases, an expression of the PJFs were both shown and then the full expression of the PJF was shown correspondingly utilizing the Lorentz invariance of the PJF. Comparing to the derivation in [11], our derivation places more emphasis on a physical background. Furthermore, since the derivation does not need sophisticated mathematical techniques, we think it is suitable to be introduced to graduate students or senior undergraduates.

As a basic application of the result, we also discussed the relationship between statistics and spin. It was found that in the field quantization, to preserve micro-causality for the vector field and scalar field, a commutation relation should be adopted while for the spinor field an anticommutation relation should be adopted.

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ORCID iDs

Daqing Liu https://orcid.org/0000-0001-7935-5944

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