Hyperfunctions in Math and Physics

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1 Introduction and Background

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In physics, one often encounters Schwartz distributions. Perhaps the simplest and most common example is the Dirac delta function $\delta(x)$ with the defining property that for any smooth $f : \mathbb{R} \to \mathbb{R}$

$$\int_a^b \delta(x) f(x) dx = \begin{cases} f(0) & \text{if } 0 \in (a,b) \\ 0 & \text{else} \end{cases} \tag{1.2.1}$$

One thinks of δ as representing an idealized point mass at the origin. Unfortunately, it follows from basic rules of integration that no function $\mathbb{R} \to \mathbb{R}$ can possibly have this property! If, however, we allow ourselves to consider functions $\mathbb{C} \to \mathbb{C}$ the situation is better. By Cauchy's Theorem, we have that for an open ball $U \subseteq \mathbb{C}$ and for any holomorphic $f : \mathbb{C} \to \mathbb{C}$

$$\oint_{\partial U} \frac{f(z)}{2\pi i z} dz = \begin{cases} f(0) & \text{if } 0 \in U \\ 0 & \text{else} \end{cases} \tag{1.2.2}$$

where the boundary ∂U is oriented counter-clockwise. This suggests that $\delta(x)$ might somehow be analogous to $\frac{1}{2\pi i z}$. However, for any holomorphic $F : \mathbb{C} \to \mathbb{C}$ one has that $\oint_{\partial U} F(z) dz = 0$, and thus $\frac{1}{2\pi i z} + F(z)$ would also behave like $\delta(x)$ in the sense that

$$\oint_{\partial U} \left(\frac{1}{2\pi i z} + F(z) \right) f(z)\, dz = \begin{cases} f(0) & \text{if } 0 \in U \\ 0 & \text{else} \end{cases} \tag{1.2.3}$$

Therefore in order to represent $\delta(x)$ via holomorphic functions, we must pass to equivalence classes such that $[f + F] = [f]$ whenever F is holomorphic everywhere.

1.3 Sheaf Theory

Definition 1.3.1. Let (X, τ) be a topological space. Then a **pre-sheaf** on X consists of the following data

(i) For each $U \in \tau$, there is an associated vector space $\mathcal{F}(U)$, whose elements are called sections over U.

(ii) If $U, V \in \tau$ such that $U \subseteq V$, there is a linear map $\rho_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$ called the restriction map. The notation is also written $\rho_U^V(f) = f|_U$

These must satisfy the following axioms:

(P1) For all $U \in \tau$, the map $\rho_U^U : \mathcal{F}(U) \to \mathcal{F}(U)$ equals the identity map, i.e. $\rho_U^U = \mathrm{id}_{\mathcal{F}(U)}$

(P2) If $U, V, W \in \tau$ such that $U \subseteq V \subseteq W$, then $\rho_V^W \circ \rho_U^V = \rho_U^W$

Remark 1.3.1. More concisely, a pre-sheaf is a contravariant functor $\mathcal{F} : \tau \to \mathbf{Vect}_{\mathbb{C}}$, where τ is regarded as a poset with respect to inclusion. The definition can be generalized to take values in any abelian category, but complex vector spaces will suffice for the purpose of this paper.

Definition 1.3.2. A pre-sheaf $\mathcal{F}$ is called a **sheaf** if it additionally satisfies the sheaf axioms:

(S1) If $\{U_i\}_{i\in I}$ is an open cover of U and $f \in \mathcal{F}(U)$ such that $f|_{U_i} = 0$ for all $i \in I$, then $f = 0$.

(S2) If $\{U_i\}_{i\in I}$ is an open cover of U and $\{f_i \in \mathcal{F}(U_i)\}_{i\in I}$ is a family of sections which agree on overlap, i.e. $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for all $i, j \in I$, then there exists $f \in \mathcal{F}(U)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

Definition 1.3.3. Let $s \in \mathcal{F}(U)$. The the **support** of s is the complement in U of the largest open set on which s is zero. That is,

$$\mathrm{supp} s = U \setminus \bigcup \{V \subseteq U \text{ open } \mid s|_V = 0\}$$

Definition 1.3.4. Let X be a topological space and let $\mathcal{F}$ be a sheaf on X. Then $\mathcal{F}$ is called **flabby** if for every open subset $U \subseteq X$, the restriction mapping

$$\rho : \mathcal{F}(X) \to \mathcal{F}(U)$$

is surjective. This amounts to saying that any local section $s \in \mathcal{F}(U)$ can always be lifted to some global section $S \in \mathcal{F}(X)$ such that $S|_U = s$, though the lift is not necessarily unique.

1.4 Holomorphic Functions

We assume some familiarity with standard complex analysis in one variable. For reference, we record the most fundamental results.

Definition 1.4.1. For an open subset$U \subseteq \mathbb{C}$, a function $f : U \to \mathbb{C}$ is called **holomorphic** if for all $z \in U$ the limit

$$f'(z) := \lim_{\substack{\epsilon\to 0\\ \epsilon\in\mathbb{C}}} \frac{f(z+\epsilon) - f(z)}{\epsilon} \tag{1.4.1}$$

exists. If $U = \mathbb{C}$, then f is called **entire**.

Theorem 1.4.1 (Cauchy's Theorem)**.** *Suppose $U \subseteq \mathbb{C}$ is open and $f : U \to \mathbb{C}$ is holomorphic. Suppose $\gamma : [a, b] \to U$ is a piecewise-smooth curve with $\gamma(a) = \gamma(b)$. If γ is homotopic to a constant curve, then*

$$\oint_\gamma f(z)dz = 0 \tag{1.4.2}$$

Corollary 1.4.1 (Cauchy's Integral Formula)**.** *If γ encircles a point $z \in U$ in the counter-clockwise direction, then*

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta \tag{1.4.3}$$

$$\frac{d^n}{dz^n} f(z) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \tag{1.4.4}$$

Corollary 1.4.2. *Every holomorphic function is analytic.*

Theorem 1.4.2 (Maximum Modulus)**.** *Suppose $f : U \to \mathbb{C}$ is holomorphic. If the function $z \mapsto |f(z)|$ attains a maximum at some $p \in U$, then f is constant.*

Corollary 1.4.3. *If $K \subseteq U$ is compact, then* $\sup\{\ |f(z)|\ : z \in K\}$ *is attained on the boundary ∂K.*

Theorem 1.4.3 (Liouville's Theorem)**.** *If f is entire and bounded, then f is constant.*

Theorem 1.4.4 (Identity Theorem)**.** *Suppose f and g are holomorphic on U. If there exists a sequence $a : \mathbb{N} \to U$ which has a limit point and $f(a_n) = g(a_n)$ for all $n \in \mathbb{N}$, then $f = g$.*

2 Hyperfunctions of One Variable

2.1 Definition and Interpretation

Definition 2.1.1. Let $\Omega \subseteq \mathbb{R}$ be an open subset. A **complex neighborhood** of Ω is an open subset $U \subseteq \mathbb{C}$ such that $\Omega \subseteq U$ is a closed subset of U (with respect the the subspace topology).

Definition 2.1.2. For $U \subseteq \mathbb{C}$ open, define

$$\mathcal{O}(U) := \{f : U \to \mathbb{C} : \ f \text{ is holomorphic}\}$$

Note that $\mathcal{O}$ is a sheaf valued in the category $\mathbf{Vect}_{\mathbb{C}}$ of complex vector spaces.

Definition 2.1.3. Let $\Omega \subseteq \mathbb{R}$ be open, and let $U \subseteq \mathbb{C}$ be a complex neighborhood of Ω. Then the space of **hyperfunctions** on Ω is defined to be the quotient space

$$\mathcal{B}(\Omega) := \frac{\mathcal{O}(U \setminus \Omega)}{\mathcal{O}(U)}$$

Remark 2.1.1. This definition appears to depend on the choice of complex neighborhood U, but in fact any two complex neighborhoods yield isomorphic quotient spaces, so that $\mathcal{B}(\Omega)$ is well-defined. Before proving this result, let us mention 3 equivalent interpretations of the above quotient space.

(i) **Analytic Viewpoint**: For $K \subseteq \mathbb{R}$ compact, let $\mathcal{A}(K)$ denote the space of real analytic functions on K endowed with its usual inductive limit topology. In the case that $\Omega \subseteq \mathbb{R}$ is open and bounded, we can view a hyperfunction $[F] \in \mathcal{B}(\Omega)$ as a continuous linear functional on $\mathcal{A}(\overline{\Omega})$ in the following way:

$$\begin{aligned} \mathcal{B}(\Omega) &\to \big(\mathcal{A}(\overline{\Omega}) \to \mathbb{C}\big) \\ [F(z)] &\mapsto \left(\varphi \mapsto \oint_\gamma F(\zeta)\varphi(\zeta)d\zeta\right) \end{aligned}$$

where γ is a suitable curve encircling $\overline{\Omega}$ once counterclockwise. This viewpoint is very similar to that of distributions, with the main difference being that test functions are required to be analytic. The inverse of the above correspondence is given by

$$\begin{aligned} \big(\mathcal{A}(\overline{\Omega}) \to \mathbb{C}\big) &\to \mathcal{B}(\Omega) \\ \big(\varphi \mapsto T(\varphi)\big) &\mapsto \left[z \mapsto \frac{1}{2\pi i} T\left(\frac{1}{z-\zeta}\right)\right] \end{aligned}$$

Further, we note that the problem of considering unbounded Ω can be remedied by replacing $\mathbb{R}$ with the radial compactification $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, see [11].

(ii) **Boundary Value Viewpoint**: This perspective arises from the simple observation that $\mathbb{C} \setminus \mathbb{R}$ is a disjoint union $\mathbb{C}^+ \sqcup \mathbb{C}^-$, where $\mathbb{C}^\pm = \{z \in \mathbb{C} \mid \pm \mathrm{Im}(z) > 0\}$. Thus, a holomorphic function $F \in \mathcal{O}(\mathbb{C} \setminus \mathbb{R})$ is the same thing as two holomorphic functions: $F_+ \in \mathcal{O}(\mathbb{C}^+)$ and $F_- \in \mathcal{O}(\mathbb{C}^-)$. More explicitly, we can define $1^+, 1^- \in \mathcal{O}(\mathbb{C} \setminus \mathbb{R})$ by

$$1^+(z) = \begin{cases} 1 & \mathrm{Im}(z) > 0 \\ 0 & \mathrm{Im}(z) < 0 \end{cases} \qquad 1^-(z) = \begin{cases} 0 & \mathrm{Im}(z) > 0 \\ -1 & \mathrm{Im}(z) < 0 \end{cases}$$

Then $1^+ - 1^-$ is the constantly 1 function and hence represents the zero equivalence class in the quotient space. That is, $[1^+] = [1^-]$. Furthermore, any $F \in \mathcal{O}(\mathbb{C} \setminus \mathbb{R})$ can be decomposed as

$$F = F \cdot 1^+ - F \cdot 1^- \tag{2.1.1}$$

When passing to the equivalence classes, the traditional notation is $F(x \pm i\mathbf{0}) = [F \cdot 1^\pm]$ or sometimes $F(x \pm i\epsilon)$ so that

$$[F] = F(x + i\mathbf{0}) - F(x - i\mathbf{0}) = F(x + i\epsilon) - F(x - i\epsilon) \tag{2.1.2}$$

The use of the symbol $i\mathbf{0}$ or $i\epsilon$ is meant to emphasize that the complex neighborhood U can be taken to be arbitrarily small. It is in this way that one thinks of hyperfunctions as the (difference of) boundary values of holomorphic functions. Informally, the value of $[F]$ evaluated at $x \in \Omega$ is the limit $\lim_{\epsilon\to 0^+} F(x+i\epsilon) - F(x-i\epsilon)$, although this limit may not exist in the traditional sense. Instead, we must think of this limit in the distributional sense by integrating against an analytic test function (see Theorem 2.2.6).

(iii) **Sheaf Cohomology Viewpoint**: This perspective is much more abstract, but has the advantage that it generalizes easily to higher dimensions. One has the isomorphism

$$\frac{\mathcal{O}(U \setminus \Omega)}{\mathcal{O}(U)} \cong H^1_\Omega(U, \mathcal{O}) \tag{2.1.3}$$

where $H^1_\Omega(U,\mathcal{O})$ denotes degree one relative sheaf cohomology of the pair $(U, U \setminus \Omega)$ with coefficients in $\mathcal{O}$ (for the relevant definitions see, for example [2]). To understand where the isomorphism in equation (2.1.3) comes from, we start with the long exact sequence on relative cohomology

$$\begin{tikzcd} 0 \arrow[r] & H^0_\Omega(U,\mathcal{O}) \arrow[r] & H^0(U,\mathcal{O}) \arrow[r] & H^0(U\setminus\Omega,\mathcal{O}) \arrow[dll, "\delta"'] & \\ & H^1_\Omega(U,\mathcal{O}) \arrow[r] & H^1(U,\mathcal{O}) \arrow[r] & H^1(U\setminus\Omega,\mathcal{O}) \arrow[r] & \cdots \end{tikzcd}$$

First, recall that in degree zero we have

$$H^0(U,\mathcal{O}) = \mathcal{O}(U) \qquad\qquad H^0_\Omega(U,\mathcal{O}) = \{f \in \mathcal{O}(U) \mid \operatorname{supp}(f) \subseteq \Omega\} \tag{2.1.4}$$

Next we use the Identity Theorem (Thm 2.4.8, [14]) which implies that a holomorphic function with uncountably many zeros must be the constantly zero function. Then it follows that

$$H^0_\Omega(U,\mathcal{O}) = \{f \in \mathcal{O}(U) \mid \operatorname{supp}(f) \subseteq \Omega\} = 0 \tag{2.1.5}$$

Finally, it is a consequence of the Mittag-Leffler Theorem that $H^1(U,\mathcal{O}) = 0$ (see Theorem 2.2.1). Therefore we are left with the short exact sequence

$$0 \longrightarrow \mathcal{O}(U) \longrightarrow \mathcal{O}(U\setminus\Omega) \longrightarrow H^1_\Omega(U,\mathcal{O}) \longrightarrow 0$$

which proves (2.1.3) . For an open set $\Omega \subseteq \mathbb{R}^n$ and any complex neighborhood U of Ω, one can define hyperfunctions of n variables by $\mathcal{B}(\Omega) = H^n_\Omega(U,\mathcal{O})$.

Now that we have shown various ways to think about $\mathcal{B}(\Omega)$, let us turn to proving that (2.1.3) is well-defined.

2.2 Important Theorems and Examples

Theorem 2.2.1 (**Mittag-Leffler**). *Let $U_0, U_1 \subseteq \mathbb{C}$ be open sets with $U_0 \cap U_1 \neq \varnothing$. If $f \in \mathcal{O}(U_0 \cap U_1)$, then there exist $f_0 \in \mathcal{O}(U_0)$ and $f_1 \in \mathcal{O}(U_1)$ such that*

$$f = f_1 - f_0 \qquad \text{on } \; U_0 \cap U_1$$

Proof. First we prove the result in the special case that f is holomorphic on a bounded open set containing $\overline{U_0 \cap U_1}$. For $j \in \{0,1\}$, define $\gamma_j = \partial U_j \cap \overline{U_{1-j}}$ so that $\gamma_0 \cup \gamma_1 = \partial(U_0 \cap U_1)$. Orient them so that $\gamma_1 - \gamma_0$ encircles $U_0 \cap U_1$ in the positive (counter-clockwise) direction, see figure [ADD FIGURE HERE]. Now define $f_j \in \mathcal{O}(U_j)$ by

$$f_j(z) = \frac{1}{2\pi i}\int_{\gamma_j} \frac{f(\zeta)}{\zeta - z} d\zeta \tag{2.2.1}$$

Then for any $z \in U_0 \cap U_1$ we have by Cauchy's Theorem

$$(f_1 - f_0)(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta \; - \frac{1}{2\pi i} \int_{\gamma_0} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\partial(U_0 \cap U_1)} \frac{f(\zeta)}{\zeta - z} d\zeta = f(z) \tag{2.2.2}$$

To prove the general case, pick a sequence of bounded open subsets $U_{j,n} \subseteq U_j$ with the following properties

$$\text{Increasing} \qquad U_{j,0} \subset U_{j,1} \subset U_{j,2} \subset \cdots \subset U_j \tag{2.2.3}$$

$$\text{Uniformity} \qquad \overline{U_{j,k}} \subset U_{j,k+1} \tag{2.2.4}$$

$$\text{Exhaustive} \qquad \bigcup_{n \in \mathbb{N}} U_{j,n} = U_j \tag{2.2.5}$$

In the topology of $\mathbb{C}$, this is always possible. There is some $N \in \mathbb{N}$ such that the intersection $U_{0,n} \cap U_{1,n}$ is non-empty for all $n \geq N$. Let f_n denote the restriction of f to $U_{0,n} \cap U_{1,n}$. We can apply the above proof to yield $f_{j,n} \in \mathcal{O}(U_{j,n})$ such that $f_n = f_{1,n} - f_{0,n}$ on $U_{0,n} \cap U_{1,n}$. Now apply Runge's Approximation Theorem to $K_{j,n} = \overline{U_{j,n}}$ to conclude that $f_{j,n}$ converge uniformly to holomorphic functions $f_j \in \mathcal{O}(U_j)$ and that $f = f_1 - f_0$ on $U_0 \cap U_1$. For full details, see Lemma 1.5.1 in [8] □

Theorem 2.2.2. *Let $U, V \subseteq \mathbb{C}$ be complex neighborhoods of $\Omega \subseteq \mathbb{R}$. Then there is a vector space isomorphism*

$$\frac{\mathcal{O}(U \setminus \Omega)}{\mathcal{O}(U)} \cong \frac{\mathcal{O}(V \setminus \Omega)}{\mathcal{O}(V)}$$

Proof. First we will prove the result assuming that $U \supseteq V$. Since U and V are assumed to be complex neighborhoods of Ω, then $U \setminus \Omega$ and $V \setminus \Omega$ are open subsets of $\mathbb{C}$ and furthermore $U \setminus \Omega \supseteq V \setminus \Omega$. Since $\mathcal{O}$ is a sheaf, there is the restriction map $\rho : \mathcal{O}(U \setminus \Omega) \to \mathcal{O}(V \setminus \Omega)$. If $f \in \mathcal{O}(U)$, then $\rho(f) \in \mathcal{O}(V)$ because of the following commutative diagram

$$\begin{tikzcd}
\mathcal{O}(U) \arrow[r, "\rho^U_{U\setminus\Omega}"] \arrow[d, "\rho^U_V"'] & \mathcal{O}(U \setminus \Omega) \arrow[d, "\rho^{U\setminus\Omega}_{V\setminus\Omega}"] \\
\mathcal{O}(V) \arrow[r, "\rho^V_{V\setminus\Omega}"'] & \mathcal{O}(V \setminus \Omega)
\end{tikzcd}$$

Thus ρ descends to a well-defined map between the quotients $\tilde{\rho} : \mathcal{O}(U \setminus \Omega)/\mathcal{O}(U) \to \mathcal{O}(V \setminus \Omega)/\mathcal{O}(V)$ defined by $\tilde{\rho}[f] := [\rho(f)]$. This is our candidate isomorphism. To show that $\tilde{\rho}$ is injective, suppose $f \in \mathcal{O}(U \setminus \Omega)$ such that $\tilde{\rho}[f] = [0]$. Then $[\rho(f)] = 0$ so $\rho(f) \in \mathcal{O}(V)$. Since $\Omega \subseteq V$, then $f \in \mathcal{O}(U)$ and hence $[f] = [0]$, as desired.

Next we show that $\tilde{\rho}$ is surjective. To this end, suppose that $f \in \mathcal{O}(V \setminus \Omega)$. Apply Theorem (2.2.1) to $U_0 = V$ and $U_1 = U \setminus \Omega$. Then we obtain $f_1 \in \mathcal{O}(U \setminus \Omega)$ and $f_0 \in \mathcal{O}(V)$ such that $f_1 - f_0 = f$ on $U_0 \cap U_1 = V \setminus \Omega$. Then $\tilde{\rho}[f_1] = [\rho(f_1)] = [f + f_0] = [f]$. Therefore $\tilde{\rho}$ is an isomorphism.

Finally, for arbitrary complex neighborhoods U and V of Ω, the intersection $U \cap V$ is also a complex neighborhood of Ω. Therefore we can apply the above proof to the pairs $U \supseteq U \cap V$ and $V \supseteq U \cap V$ to conclude that

$$\mathcal{O}(U \setminus \Omega)/\mathcal{O}(U) \quad \cong \quad \mathcal{O}(U \cap V \setminus \Omega)/\mathcal{O}(U \cap V) \quad \cong \quad \mathcal{O}(V \setminus \Omega)/\mathcal{O}(V)$$

□

Remark 2.2.1. For any $\epsilon > 0$, the set $U_\epsilon = \{x + iy : x \in \Omega, |y| < \epsilon\}$ is a complex neighborhood of Ω. In light of Theorem 2.2.2, one often thinks of a complex neighborhood $U \supseteq \Omega$ as being taken to be arbitrarily narrow in the imaginary direction. Similarly, one can take U small enough in the real direction so that $U \cap \mathbb{R} = \Omega$, in which case the set difference $U \setminus \Omega$ is two disjoint pieces. For this reason we introduce a helpful notation.

Definition 2.2.1. Let $f \in \mathcal{B}(\Omega)$. There exists some $F \in \mathcal{O}(U \setminus \Omega)$ such that $f = [F]$. Then we define $F_\pm$ to be the restriction of F to the upper/lower half-plane $\mathbb{C}^\pm := \{z \in \mathbb{C} \mid \pm\text{Im}(z) > 0\}$ and we write

$$f = [F_+ \,,\, F_-]$$

where $F_\pm \in \mathcal{O}((U \setminus \Omega) \cap \mathbb{C}^\pm)$. If we wish to stress the fact f is a generalized function of a *real* variable while F is a function of a *complex* variable we may also write

$$f(x) = [F(z)] = [F_+(z)\ ,\ F_-(z)]$$

Theorem 2.2.3. *The assignment $\Omega \mapsto \mathcal{B}(\Omega)$ constitutes a sheaf on $\mathbb{R}$.*

Proof. Suppose $\Omega \supseteq \Omega'$ are open sets in $\mathbb{R}$. Let U, U' be complex neighborhoods of Ω and Ω' such that $U \setminus \Omega \supseteq U' \setminus \Omega'$. Then there is the natural restriction map $\rho : \mathcal{O}(U \setminus \Omega) \to \mathcal{O}(U' \setminus \Omega')$, which induces a map on the quotients $\tilde{\rho} : \mathcal{B}(\Omega) \to \mathcal{B}(\Omega')$. Since $\mathcal{O}$ is a presheaf, it follows that $\mathcal{B}$ is a presheaf. It remains to show the sheaf axioms:

($S1$) Let $f = [F] \in \mathcal{B}(\Omega)$ be represented by $F \in \mathcal{O}(U \setminus \Omega)$ for some complex neighborhood $U \supseteq \Omega$. Let $\{\Omega_i\}_{i\in I}$ be an open covering of Ω, and suppose $[F]|_{\Omega_i} = 0$ for all $i \in I$. Then for each $i \in I$, there exists a complex neighborhood $U_i \supseteq \Omega_i$ such that $F|_{U_i} \in \mathcal{O}(U_i)$. Then F is holomorphic on all of the union $\bigcup_{i\in I} U_i \supseteq \Omega$. Therefore $[F] = [0]$, as desired.

($S2$) Let $\{\Omega_i\}_{i\in I}$ be an open covering of Ω and let $f_i \in \mathcal{B}(\Omega_i)$ be represented by $F_i \in \mathcal{O}(U_i \setminus \Omega_i)$ for some complex neighborhood $U_i \supseteq \Omega_i$. Assume that $[F_i]\big|_{\Omega_i\cap\Omega_j} = [F_j]\big|_{\Omega_i\cap\Omega_j}$ for all $i, j \in I$. Then we have that $\left[F_i\big|_{\Omega_i\cap\Omega_j} - F_j\big|_{\Omega_i\cap\Omega_j}\right] = [0]$, or in other words $F_i - F_j \in \mathcal{O}(U_i \cap U_j)$. Therefore by Theorem 2.2.1, there exist $G_i \in \mathcal{O}(U_i)$ and $G_j \in \mathcal{O}(U_j)$ such that $F_i - F_j = G_i - G_j$ on $U_i \cap U_j$. Rearranging, we see that $F_i - G_i = F_j - G_j$ on $(U_i \setminus \Omega_i) \cap (U_j \setminus \Omega_j)$. Since $\mathcal{O}$ is a sheaf, we can apply the gluing axiom (S2) to yield $F \in \mathcal{O}(U \setminus \Omega)$, where $U = \bigcup_{i\in I} U_i$. Then $f = [F] \in \mathcal{B}(\Omega)$ is the desired global section since $[F]\big|_{\Omega_i} = [f_i - g_i] = [F_i] = f_i$. ☐

Theorem 2.2.4. $\mathcal{B}$ *is a flabby sheaf.*

Proof. Let $\Omega \subseteq \mathbb{R}$ be open, and let $f \in \mathcal{B}(\Omega)$. Define $U = \mathbb{C} - \partial\Omega$ (where $\partial\Omega$ denotes the boundary of Ω in $\mathbb{R}$, not in $\mathbb{C}$). Then U is a complex neighborhood of Ω, so by Theorem 2.2.2 we may represent $\mathcal{B}(\Omega)$ as

$$\mathcal{B}(\Omega) \cong \frac{\mathcal{O}(U \setminus \Omega)}{\mathcal{O}(U)} = \frac{\mathcal{O}(\mathbb{C} \setminus \overline{\Omega})}{\mathcal{O}(\mathbb{C} \setminus \partial\Omega)}$$

Write $f = [F]$ for some $F \in \mathcal{O}(\mathbb{C} \setminus \overline{\Omega})$. Evidently we have $\mathbb{C} \setminus \overline{\Omega} \supseteq \mathbb{C} \setminus \mathbb{R}$, so there is the restriction map $\rho : \mathcal{O}(\mathbb{C} \setminus \overline{\Omega}) \to \mathcal{O}(\mathbb{C} \setminus \mathbb{R})$. Then $[\rho(F)] \in \mathcal{B}(\mathbb{R})$ is the desired lift. ☐

Example 2.2.1. Some common examples of hyperfunctions include

(i)	$\delta(x) = \frac{i}{2\pi}\left[\frac{1}{z}\right]$	*Dirac's delta function*
(ii)	$\theta(x) = \frac{i}{2\pi}\left[\log(-z)\right]$	*Heavyside step function*
(iii)	$\mathrm{sgn}(x) = \frac{i}{2\pi}\left[\log(z) - \log(-z)\right]$	*Sign function*
(iv)	$\varphi(x + i\epsilon) = [\varphi(z)\ ,\ 0]$	*Real analytic function φ as a hyperfunction*
(v)	$\varphi(x - i\epsilon) = [0\ ,\ -\varphi(z)]$	
(vi)	p.v. $F(x) = \frac{1}{2}\left[F(z)\ ,\ -F(z)\right]$	*Cauchy principal value of meromorphic F*

Remark 2.2.2. Here and throughout, log will always denote the principle branch of the complex logarithm with branch cut along the negative real axis for which $-\pi < \mathrm{Im}(\log(z)) \leq \pi$

$$\begin{aligned} \log : \mathbb{C}^* &\to \mathbb{R} + i(-\pi, \pi] \\ re^{i\theta} &\mapsto \log(r) + i\theta \end{aligned}$$

The failure of log to be continuous on the branch cut is often thought of as an annoying technicality. It means that the complex logarithm does not obey the usual identity $\log(xy) = \log(x) + \log(y)$, but instead the more complicated

$$\log(z_1 z_2) = \log(z_1) + \log(z_2) + \begin{cases} 2\pi i, & \arg(z_1) + \arg(z_2) < -\pi \\ 0, & -\pi < \arg(z_1) + \arg(z_2) \leq \pi \\ -2\pi i, & \arg(z_1) + \arg(z_2) > \pi \end{cases} \tag{2.2.6}$$

From the perspective of hyperfunctions, however, this discontinuity is crucial. Consider the difference in boundary values of $\log(x + iy)$. If $x > 0$, then log is holomorphic at x, so

$$\lim_{y \to 0^+} \big(\log(x + iy) - \log(x - iy) \big) = \log(x) - \log(x) = 0$$

On the other hand if $x < 0$, then

$$\lim_{y \to 0^+} \big(\log(x + iy) - \log(x - iy) \big) = (\log(x) + i\pi) - (\log(x) - i\pi) = 2\pi i$$

This allows us to represent the Heavyside function $\theta(x)$ by composing with a reflection $z \mapsto -z$ and normalizing the jump discontinuity to be 1

$$\lim_{y \to 0^+} \frac{\log(-(x + iy)) - \log(-(x - iy))}{-2\pi i} = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

This explains the definition $\theta(x) = \frac{i}{2\pi}[\log(-z)]$.

[ADD GRAPH OF $\mathrm{Re}(\frac{i}{2\pi} \log(-z))$]

In the absence of this explanation, however, one might have had a hard time guessing that a step function should be represented as the equivalence class of the complex logarithm. This begs the question: how does one go about finding a representative for a particular hyperfunction? This is answered by the following theorem.

Theorem 2.2.5. *Let $f \in \mathcal{B}(\mathbb{R})$ have compact support $K = \operatorname{supp} f$, and define*

$$F(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x)}{x - z} dx = \frac{1}{2\pi i} \int_K \frac{f(x)}{x - z} dx$$

*Then $F \in \mathcal{O}(\mathbb{C} \setminus K)$ and $[F] = f$. F is called the **standard defining function** for f.*

Proof. [8] pg. 24 ☐

Corollary 2.2.1. *The sheaf L^1_{loc} of locally integrable functions embeds into $\mathcal{B}$ so that there are the inclusions*

$$\mathcal{A} \subset L^1_{loc} \subset \mathcal{B}$$

Proof. Suppose $f \in L^1_{\text{loc}}$ is locally integrable. Then we can write $f = \sum_{j \in J} f_j$ as a locally finite sum of functions $f_j \in L^1$ each having compact support, $\operatorname{supp} f = K_j$. Now define a map

$$\begin{aligned} \iota : L^1_{\text{loc}} &\to \mathcal{B} \\ \sum_{j \in J} f_j &\mapsto \left[\sum_{j \in J} \frac{1}{2\pi i} \int_{K_j} \frac{f_j(x)}{x - z} dx \right] \end{aligned}$$

A proof that ι is injective and does not depend on the choice of locally finite decomposition $f = \sum_{j \in J} f_j$ is given on pg. 25 of [8]. ☐

Remark 2.2.3. With a little extra care, we can use the same idea to show that the sheaf of Schwartz distributions $\mathcal{D}'$ is also a subsheaf of $\mathcal{B}$. That is, we have

$$\mathcal{A} \subset L^1_{\text{loc}} \subset \mathcal{D}' \subset \mathcal{B}$$

Theorem 2.2.6. *Let $T \in \mathcal{D}'(\mathbb{R})$ be a distribution. Then there exists $\hat{T} \in \mathcal{O}(\mathbb{C} \setminus \mathbb{R})$ such that for all $f \in \mathcal{D}$*

$$T(f) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \left[\hat{T}(x + i\epsilon) - \hat{T}(x - i\epsilon)\right] f(x) dx$$

Hence the map $T \mapsto [\hat{T}]$ is an embedding $\mathcal{D}' \subseteq \mathcal{B}$.

In the proof we will need the following lemma

Lemma 2.2.1. *For each $\epsilon > 0$, define the function $g_\epsilon(x) = \frac{\epsilon}{\pi(x^2+\epsilon^2)}$. Then for any Schwartz function $f \in \mathcal{D}$,*

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{R}} g_\epsilon(x) f(x) = f(0)$$

In other words, $\lim_{\epsilon \to 0^+} g_\epsilon = \delta$ as distributions.

Proof.

$$\begin{aligned} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}} g_\epsilon(x) f(x) &= \lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \frac{\epsilon f(x)}{\pi(x^2+\epsilon^2)} dx \\ &= \lim_{\epsilon \to 0^+} \frac{1}{\pi\epsilon} \int_{\mathbb{R}} \frac{f(x)}{(x/\epsilon)^2 + 1} dx \end{aligned}$$

Making the substitution $u = \frac{x}{\epsilon}$ and then using integration by parts, this becomes

$$\begin{aligned} \lim_{\epsilon \to 0^+} \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(\epsilon u)}{1+u^2} du &= \lim_{\epsilon \to 0^+} \frac{1}{\pi} \left[f(\epsilon u) \arctan(u) \Big|_{-\infty}^{\infty} - \epsilon \int_{\mathbb{R}} f'(\epsilon u) \arctan(u) du \right] \\ &= \frac{1}{\pi} \left[\frac{f(0)\pi}{2} - \frac{-f(0)\pi}{2} \right] = f(0) \end{aligned}$$

□

Now we prove the theorem.

Proof. First, we'll prove the result assuming $T \in \mathcal{E}'$ has compact support. In this case, for $z \in \mathbb{C} \setminus \mathbb{R}$ define

$$\hat{T}(z) = \frac{1}{2\pi i} T\left(x \mapsto \frac{1}{x-z}\right) = \frac{1}{2\pi i} \left\langle T(x), \frac{1}{x-z} \right\rangle$$

Now we compute the difference quotient

$$\begin{aligned} \frac{\hat{T}(z) - \hat{T}(w)}{z-w} &= \frac{1}{2\pi i} \frac{\left\langle T(x), \frac{1}{x-z} \right\rangle - \left\langle T(x), \frac{1}{x-w} \right\rangle}{z-w} \\ &= \frac{1}{2\pi i} \left\langle T(x), \frac{1}{z-w} \left(\frac{1}{x-z} - \frac{1}{z-w} \right) \right\rangle \\ &= \frac{1}{2\pi i} \left\langle T(x), \frac{1}{(x-z)(x-w)} \right\rangle \\ \hat{T}'(z) = \lim_{w \to z} \frac{\hat{T}(z) - \hat{T}(w)}{z-w} &= \frac{1}{2\pi i} \left\langle T(x), \frac{1}{(x-z)^2} \right\rangle \in \mathbb{C} \end{aligned}$$

Therefore $\hat{T}$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}$. Next we compute $\hat{T}(z) - \hat{T}(\overline{z})$ to be

$$\begin{aligned} \hat{T}(x+iy) - \hat{T}(x-iy) &= \frac{1}{2\pi i} \left\langle T(t), \frac{1}{t-x-iy} - \frac{1}{t-x+iy} \right\rangle \\ &= \left\langle T(t), \frac{1}{\pi} \frac{y}{(t-x)^2 + y^2} \right\rangle = \langle T(t), g_y(t-x) \rangle \end{aligned}$$

Using the definition of convolution and the fact that $f * \delta = f$ yields

$$\begin{aligned}\lim_{\epsilon\to 0^+}\int_{\mathbb{R}}\left[\hat{T}(x+i\epsilon)-\hat{T}(x-i\epsilon)\right]f(x)dx &= \lim_{\epsilon\to 0^+}\int_{\mathbb{R}}\langle T(t), g_\epsilon(t-x)\rangle f(x)dx \\ &= \lim_{\epsilon\to 0^+}\langle T(t), (g_\epsilon * f)(t)\rangle \\ &= \langle T(t), (\delta * f)(t)\rangle \\ &= \langle T(t), f(t)\rangle = T(f)\end{aligned}$$

Finally, for a general distribution $T \in \mathcal{D}'$, there exists a sequence of compactly supported distributions $T_n \in \mathcal{E}'$ such that $\lim_{n\to\infty} T_n = T$. Hence we can apply the above proof to each T_n so that $\hat{T} = \lim_{n\to\infty}\hat{T}_n$. $\square$

Example 2.2.2. As a distribution, δ is the map $f \mapsto f(0)$. So the theorem says that the hyperfunction representation of δ is given by

$$[\hat{\delta}] = \left[\delta\left(x \mapsto \frac{1}{2\pi i(x-z)}\right)\right] = \frac{i}{2\pi}\left[\frac{1}{z}\right]$$

which matches the definition of δ given previously.

Example 2.2.3. For an example where the distribution is not compactly supported, consider the Heavyside distribution defined by $\theta(f) = \int_0^\infty f(x)dx$. In this case supp $\theta = [0,\infty)$ and we can can write $\theta = \lim_{n\to\infty}\theta_n$, where $\theta_n(f) = \int_0^n f(x)dx$. Now we compute the hyperfunction representation of θ_n to be

$$[\hat{\theta}_n] = \left[\theta_n\left(x \mapsto \frac{1}{2\pi i(x-z)}\right)\right] = \left[\frac{1}{2\pi i}\int_0^n \frac{dx}{x-z}\right] = \frac{1}{2\pi i}\left[\log(n-z) - \log(-z)\right]$$

Note that if we take the limit $n \to \infty$ "inside the brackets", the integral does not converge. However, $\log(n-z)$ is holomorphic on $\mathbb{C}\setminus\mathbb{R}_{\geq n}$, where $\mathbb{R}_{\geq n} = \{x \in \mathbb{R} : x \geq n\}$. So when we restrict to $(-\infty, n)$, we have $[\log(n-z)]\big|_{(-\infty,n)} = [0]$. Then $[\hat{\theta}_n]\big|_{(-\infty,n)} = \frac{i}{2\pi}[\log(-z)]\big|_{(-\infty,n)}$, so in the limit we get

$$[\hat{\theta}] = \lim_{n\to\infty}[\hat{\theta}_n] = \lim_{n\to\infty}[\hat{\theta}_n]\Big|_{(-\infty,n)} = \frac{i}{2\pi}[\log(-z)]$$

2.3 Operations on Hyperfunctions

2.3.1 Differential $\mathcal{A}$-module Structure

For each operation on $\mathcal{O}$, we can extend to an operation on $\mathcal{B}$ by applying to the representatives (so long as doing so preserves the equivalence relation). For example, we have already noted that $\mathcal{O}$ is a complex vector space, so we can define the vector space operations on $\mathcal{B}$ in the following way.

Definition 2.3.1. Let $f, g \in \mathcal{B}(\Omega)$ be represented by $f(x) = [F(z)]$ and $g(x) = [G(z)]$, and $\lambda \in \mathbb{C}$. Then we define

$$\begin{aligned}(f+g)(x) &= [F(z)+G(z)] \\ \lambda f(x) &= [\lambda F(z)]\end{aligned}$$

In addition to being a vector space, $\mathcal{O}(U)$ has a multiplication. Using this multiplication on representatives leads us to the following definition.

Definition 2.3.2. Let $f = [F] \in \mathcal{B}(\Omega)$ and $\varphi \in \mathcal{A}(\Omega)$. Then we define

$$\varphi(x)\cdot f(x) = [\varphi(z)\cdot F(z)]$$

where $\varphi(z)$ is the extension of $\varphi(x)$ to a complex neighborhood U of Ω.

Taken together, these operations make $\mathcal{B}(\Omega)$ into an $\mathcal{A}(\Omega)$-module. Finally, we can define the derivative of a hyperfunction in a similar way.

Definition 2.3.3. Let $f = [F] \in \mathcal{B}(\Omega)$. Then the derivative of f is defined as

$$\frac{d}{dx} f(x) = \left[\frac{d}{dz} F(z)\right]$$

Remark 2.3.1. It is immediate to check that the above definitions are well-defined.

Example 2.3.1. The derivative of $\delta(x)$ is

$$\delta'(x) = \frac{d}{dx}\delta(x) = \left[\frac{d}{dz}\frac{-1}{2\pi i z}\right] = \frac{1}{2\pi i}\left[\frac{1}{z^2}\right]$$

More generally, the n^{th} derivative is given by

$$\delta^{(n)}(x) = \left[\frac{d^n}{dz^n}\frac{-1}{2\pi i z}\right] = \frac{(-1)^{n+1}}{2\pi i\ n!}\left[\frac{1}{z^{n+1}}\right]$$

Example 2.3.2. $\frac{d}{dx}\theta(x) = \frac{i}{2\pi}\left[\frac{d}{dz}\log(-z)\right] = \frac{i}{2\pi}\left[\frac{1}{z}\right] = \delta(x)$

By combining all of these together, we can act on hyperfunctions by differential operators with analytic coefficients. In the context of differential equations, flabbiness is a useful property because it means that hyperfunction solutions can always be extended to all of $\mathbb{R}$. This is in stark contrast to smooth or distributional solutions, which cannot always be extended past singularities on the boundary of their given domain.

Definition 2.3.4. Let $\Omega \subseteq \mathbb{R}$ be open and $a_0(x), \ldots, a_N(x) \in \mathcal{A}(\Omega)$ be a finite collection of analytic functions on Ω with $a_N \not\equiv 0$. Then we define the operator $P(D) = \sum a_n(x)D^n$ by

$$\sum_{n=0}^{N} a_n(x)D^n : \mathcal{B}(\Omega) \to \mathcal{B}(\Omega)$$
$$[F] \mapsto \left[\sum_{n=0}^{N} a_n(z)\frac{d^n}{dz^n}F(z)\right]$$

Unlike for Schwartz distributions, it is possible to extend this to infinite order differential operators. In this case, we must require that $\lim_{n\to\infty} \sqrt[n]{n! \cdot \sup_{x\in K}|a_n(x)|} = 0$ for any compact subset $K \subseteq \Omega$ in order that the series $\sum_{n=0}^{\infty} a_n(z)\frac{d^n}{dz^n}F(z)$ actually define a holomorphic function [THIS CONDITION IS STRONGER THAN NECESSARY? WHAT IS SUFFICIENT?].

Example 2.3.3 (ADD EXAMPLE OF TRANSLATION AS INFINITE DIFFERENTIAL OPERATOR)**.**

Proposition 2.3.1. *Let $P(D) = \sum a_n D^n$ be a differential operator as above. Then $P(D) : \mathcal{B} \to \mathcal{B}$ is a surjective sheaf homomorphism, i.e. for any hyperfunction $f \in \mathcal{B}(\Omega)$, there exists $g \in \mathcal{B}(\Omega)$ such that*

$$P(D)g = f \tag{2.3.1}$$

Moreover $P(D)$ commutes with restriction, meaning that if the coefficients $a_n(x)$ can be analytically continued to $\Omega' \supset \Omega$, then any solution of (2.3.1) can also be extended to Ω'.

Proof. This follows from Cauchy's existence theorem for differential equations with analytic coefficients. A full proof is on pg. 24 of [9] $\square$

2.3.2 Integration

Definition 2.3.5. Write $f = [F_+ \, , \, F_-]$, and let $(a, b) \subseteq \mathbb{R}$ be an open interval such that both F_+ and F_- are analytic at the endpoints a and b. Then we define the integral of f over the interval (a, b) by

$$\int_a^b f(x)dx = \int_{\gamma_+} F_+(z)dz - \int_{\gamma_-} F_-(z)dz$$

where γ_+ and γ_- are piecewise smooth curves starting at a and ending at b in the domain of F_+ and F_-, respectively. (See figure [ADD FIGURE])

Remark 2.3.2. This is well-defined because of Cauchy's Theorem.

Example 2.3.4. ADD EXAMPLE OR TWO

2.3.3 Linear Substitution

Suppose $f(x)$ is a hyperfunction and we want to make the linear substitution $x \mapsto ax + b$ with $a, b \in \mathbb{R}$ and $a \neq 0$. How should we define $f(ax + b)$? If we pick a representative $f(x) = [F(z)]$, then we are tempted to define $f(ax+b)$ simply by $[F(az+b)]$. It turns out that this will be correct only if $a > 0$. On the other hand if $a < 0$, we will pick up an extra negative sign. Perhaps the easiest way to see why is to use the boundary value notation $f(x) = [F_+(z), F_-(z)] = F_+(x + i\mathbf{0}) - F_-(x - i\mathbf{0})$

$$\begin{aligned} f(x) &= F_+(x + i\mathbf{0}) - F_-(x - i\mathbf{0}) \\ f(ax + b) &= F_+\big(a(x + i\mathbf{0}) + b\big) - F_-\big(a(x - i\mathbf{0}) + b\big) \\ &= F_+(ax + b + ai\mathbf{0}) - F_-(ax + b - ai\mathbf{0}) \end{aligned}$$

If $a > 0$, then $ai\mathbf{0} = i\mathbf{0}$ so this simplifies to the expected expression

$$\begin{aligned} f(ax + b) &= F_+(ax + b + i\mathbf{0}) - F_-(ax + b - i\mathbf{0}) \\ &= [F_+(az + b), F_-(az + b)] \end{aligned}$$

whereas if $a < 0$ then $ai\mathbf{0} = -i\mathbf{0}$ so the result is

$$\begin{aligned} f(ax + b) &= F_+(ax + b - i\mathbf{0}) - F_-(ax + b + i\mathbf{0}) \\ &= -\Big(F_-(ax + b + i\mathbf{0}) - F_+(ax + b - i\mathbf{0})\Big) \\ &= [-F_-(az + b), -F_+(az + b)] \end{aligned}$$

This is because the transformation $z \mapsto az + b$ sends $\mathbb{C}^+$ to $\mathbb{C}^-$ and vice versa. Consequently, $F_+(az + b)$ is holomorphic on the *lower* half plane while $F_-(az + b)$ is holomorphic on the *upper* half plane. Overall, we arrive at the definition

$$f(ax + b) := [\operatorname{sgn}(a)F(az + b)] \;\; = \begin{cases} \;\;[F_+(az + b), F_-(az + b)] & a > 0 \\ -[F_-(az + b), F_+(az + b)] & a < 0 \end{cases} \tag{2.3.2}$$

Example 2.3.5. Let us calculate $\delta(ax + b)$. Plugging in $\delta(x) = \frac{i}{2\pi}\left[\frac{1}{z}\right]$ to equation 2.3.2 yields

$$\begin{aligned} \delta(ax + b) &= \frac{i}{2\pi}\left[\frac{\operatorname{sgn}(a)}{az + b}\right] \\ &= \frac{i}{2\pi}\left[\frac{\operatorname{sgn}(a)}{a} \cdot \frac{1}{z + \frac{b}{a}}\right] \\ &= \frac{1}{|a|}\frac{i}{2\pi}\left[\frac{1}{z + \frac{b}{a}}\right] \end{aligned}$$

$$\delta(ax + b) = \frac{1}{|a|}\delta\left(x + \frac{b}{a}\right)$$

In the special case $a = -1$ and $b = 0$ we have the identity $\delta(-x) = \delta(x)$.

Example 2.3.6. Compute $\phi(x)\delta(x-a)$ where $a \in \mathbb{R}$ and $\phi(x)$ is real analytic.

$$\begin{aligned}
\phi(x)\delta(x-a) &= \frac{i}{2\pi}\left[\frac{\phi(z)}{z-a}\right] \\
&= \frac{i}{2\pi}\left[\frac{\phi(a)+\phi'(a)(z-a)+\frac{\phi''(a)}{2}(z-a)^2+\cdots}{z-a}\right] \\
&= \frac{i}{2\pi}\left[\frac{\phi(a)}{z-a}+\phi'(a)+\frac{\phi''(a)}{2}(z-a)+\cdots\right] \\
&= \frac{i}{2\pi}\left[\frac{\phi(a)}{z-a}\right]+[0]
\end{aligned}$$

$$\phi(x)\delta(x-a) = \phi(a)\delta(x-a)$$

In the last step, we use that $\phi'(a)+\frac{\phi''(a)}{2}(z-a)+\cdots = \frac{\phi(z)-\phi(a)}{z-a}$ is analytic so its equivalence class is zero.

Example 2.3.7. Calculate $\theta(ax+b)$

$$\begin{aligned}
\theta(ax+b) &= \frac{i}{2\pi}\left[\operatorname{sgn}(a)\log(-az-b)\right] \\
&= \frac{i}{2\pi}\operatorname{sgn}(a)\left[\log\left(|a|\Big(-\operatorname{sgn}(a)z-\frac{b}{|a|}\Big)\right)\right] \\
&= \frac{i}{2\pi}\operatorname{sgn}(a)\left[\log\left(-\operatorname{sgn}(a)z-\frac{b}{|a|}\right)+\log|a|\right] \\
&= \frac{i}{2\pi}\operatorname{sgn}(a)\left[\log\left(-\operatorname{sgn}(a)z-\frac{b}{|a|}\right)\right]+[0]
\end{aligned}$$

$$\theta(ax+b) = \theta\left(\operatorname{sgn}(a)x+\frac{b}{|a|}\right)$$

Again, in the last step that $\big[\log|a|\,\big] = [0]$ because the constant function $z \mapsto \log|a|$ is entire.

2.3.4 Conjugation

If $f(x) = [F(z)]$, how should we define $\overline{f(x)}$? One might guess that the answer would be $[\overline{F(z)}]$, but this doesn't make sense! The reason is that $\overline{F(z)}$ is (probably) not a holomorphic function. What is true though is that $\overline{F(\overline{z})}$ *is* holomorphic. However, we need to also account for the fact that $z \mapsto \overline{z}$ swaps the upper and lower half planes. Thus, for a similar reason as for linear substitutions $z \mapsto az+b$ with $a<0$, we will pick up an additional minus sign. Therefore we are led to the following definition.

Definition 2.3.6. Let $f(x) = [F(z)] \in \mathcal{B}(\mathbb{R})$. The the **conjugate hyperfunction** $\overline{f(x)} \in \mathcal{B}(\mathbb{R})$ is defined by

$$\overline{f(x)} := [-\overline{F(\overline{z})}] \tag{2.3.3}$$

2.3.5 General Substitutions

The question arises whether one can define composition with a non-linear function. Unfortunately, this is not possible in general. In essence, the problem concerns the domain of a composition of functions. For example, if $f(x) \in \mathcal{B}(\mathbb{R})$ consider trying to define $f(x^2)$. We can write $f(x) = [F(z)]$ for some holomorphic $F : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}$. Then $F(z^2)$ will not be defined if $z^2 \in \mathbb{R}$, which happens not only for $z \in \mathbb{R}$, but also when $z \in i\mathbb{R}$. Since the domain of $F(z^2)$ excludes $i\mathbb{R}$, it cannot possibly be a complex neighborhood of $\mathbb{R}$ (or any open neighborhood of zero) and thus the equivalence class $[F(z^2)]$ is not even an element of $\mathcal{B}(\mathbb{R})$. Hence we cannot hope to make sense of the hyperfunction $f(x^2)$ in any neighborhood of zero.

A more intuitive explanation for this is seen by using the boundary value notation along with the heuristic that $(i\mathbf{0})^2$ ought to be zero.

$$(x+i\mathbf{0})^2 = x^2 + 2xi\mathbf{0} + (i\mathbf{0})^2 = x^2 + 2xi\mathbf{0} = \begin{cases} x^2 + i\mathbf{0} & x > 0 \\ x^2 - i\mathbf{0} & x < 0 \\ 0 & x = 0 \end{cases}$$

This tells us that the upper and lower neighborhoods are preserved if $x > 0$ and they get swapped if $x < 0$ whereas when $x = 0$ equals zero the result is not a neighborhood whatsoever. To avoid this problematic point, we can start with a hyperfunction $f(x) = [F(z)] \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ and use the flabbiness of $\mathcal{B}$ to find an extension $\tilde{f}(x) \in \mathcal{B}(\mathbb{R})$. From the previous section, we know that swapping upper and lower neighborhoods introduces a factor of negative one, hence the restrictions of $[F(z^2)]$ are given by

$$[F(z^2)]\Big|_{\mathbb{R}^+} = [F(z^2)] \qquad\qquad [F(z^2)]\Big|_{\mathbb{R}^-} = -[F(z^2)]$$

Therefore the only possible extension to $\tilde{f}(x) = [\tilde{F}(z)] \in \mathcal{B}(\mathbb{R})$ must vanish near $x = 0$. In other words, it is necessary that $0 \notin \operatorname{supp}\tilde{f}$ in order that $\tilde{f}(x^2)$ be well-defined.

Example 2.3.8. Let $a \in \mathbb{R} \setminus \{0\}$ and consider $f(x) = \delta(x-a)$. Calculate $f(\phi(x))$ for $\phi(x) = x^2$.

Since supp $\delta(x-a) = \{a\}$, we can view $f(x)$ as an element of $\mathcal{B}(\Omega)$ for any neighborhood Ω of a. For instance we can choose $0 < \epsilon < a$ and define $\Omega := B_\epsilon(a) = \{x \in \mathbb{R} \mid |x-a| < \epsilon\}$ so that $0 \notin \Omega$. Since $f(x) \in \mathcal{B}(\Omega)$ then we will have $f(\phi(x)) \in \mathcal{B}(\phi^{-1}(\Omega))$. In this case we have

$$\phi^{-1}(\Omega) = \begin{cases} (-\sqrt{a+\epsilon}, -\sqrt{a-\epsilon}) \cup (\sqrt{a-\epsilon}, \sqrt{a+\epsilon}) & a > 0 \\ \varnothing & a < 0 \end{cases}$$

If $a < 0$, we see that $\delta(x^2 - a) \in \mathcal{B}(\varnothing) = \{0\}$ and therefore $\delta(x^2 - a) = 0$. If $a > 0$ then $\phi^{-1}(\Omega)$ is the disjoint union of two open intervals, thus by the sheaf property it suffices to calculate the restriction of $\delta(x^2 - a)$ on each interval individually. Let $\delta_+(x^2 - a)$ be the restriction to $\phi^{-1}(\Omega) \cap \mathbb{R}^+ = (\sqrt{a-\epsilon}, \sqrt{a+\epsilon})$ and likewise let $\delta_-(x^2 - a)$ denote the restriction to $\phi^{-1}(\Omega) \cap \mathbb{R}^- = (-\sqrt{a+\epsilon}, -\sqrt{a-\epsilon})$. Then by using partial fractions we see that

$$\begin{aligned} \delta_+(x^2 - a) = \frac{i}{2\pi}\left[\frac{1}{z^2 - a}\right]\Big|_{\phi^{-1}(\Omega)\cap\mathbb{R}^+} &= \frac{i}{2\pi}\frac{1}{2\sqrt{a}}\left[\frac{1}{z-\sqrt{a}} - \frac{1}{z+\sqrt{a}}\right]\Big|_{\phi^{-1}(\Omega)\cap\mathbb{R}^+} \\ &= \frac{i}{2\pi}\frac{1}{2\sqrt{a}}\left[\frac{1}{z-\sqrt{a}}\right] - [0] \\ &= \frac{\delta(x-\sqrt{a})}{2\sqrt{a}} \end{aligned}$$

where we used the fact that $\frac{1}{z+\sqrt{a}}$ is analytic away from $z = -\sqrt{a}$. As noted above, when we restrict to $\phi^{-1}(x) \cap \mathbb{R}^-$ there will be an additional negative sign due to the complex neighborhood being flipped, hence

$$\begin{aligned} \delta_-(x^2 - a) = \frac{i}{2\pi}\left[\frac{-1}{z^2 - a}\right]\Big|_{\phi^{-1}(\Omega)\cap\mathbb{R}^-} &= \frac{i}{2\pi}\frac{-1}{2\sqrt{a}}\left[\frac{1}{z-\sqrt{a}} - \frac{1}{z+\sqrt{a}}\right]\Big|_{\phi^{-1}(\Omega)\cap\mathbb{R}^-} \\ &= \frac{i}{2\pi}\frac{-1}{2\sqrt{a}}\left[0 - \frac{1}{z+\sqrt{a}}\right] \\ &= \frac{\delta(x+\sqrt{a})}{2\sqrt{a}} \end{aligned}$$

where again we use the fact that $\frac{1}{z-\sqrt{a}}$ is analytic away from $z = \sqrt{a}$. Because of the disjoint union $\phi^{-1}(\Omega) = (\phi^{-1}(\Omega) \cap \mathbb{R}^+) \sqcup (\phi^{-1}(\Omega) \cap \mathbb{R}^-)$, we must have that

$$\delta(x^2 - a) = \delta_+(x^2 - a) + \delta_-(x^2 - a) = \begin{cases} \frac{\delta(x-\sqrt{a}) + \delta(x+\sqrt{a})}{2\sqrt{a}} & a > 0 \\ 0 & a < 0 \end{cases}$$

This is commonly written in the more succinct form

$$\delta(x^2 - a^2) = \frac{\delta(x-a) + \delta(x+a)}{2|a|} \tag{2.3.4}$$

With this example in mind, we turn to defining $f(\phi(x))$ in full generality. Let $\phi : \Omega \to \Omega'$ be real analytic. Then there exists some extension of ϕ to a complex neighborhood of $U \supseteq \Omega$. Writing $z = x + iy$ and expanding around $z = x$ gives

$$\phi(z) = \phi(x) + \phi'(x)iy + O(y^2) \tag{2.3.5}$$

In the limit as $y \to 0$, we see from (2.3.5) that $\phi(x + i\mathbf{0}) = \phi(x) + \phi'(x)i\mathbf{0}$. Therefore in order that $\phi(U)$ is a complex neighborhood of Ω', we must have that $\phi'(x) \neq 0$ for all $x \in \Omega$. In the simple situation that Ω is connected, then $\phi'(x)$ must be either always positive so that $\phi(x \pm i\mathbf{0}) = \phi(x) \pm i\mathbf{0}$, or else $\phi'(x)$ is always negative in which case $\phi(x \pm i\mathbf{0}) = \phi(x) \mp i\mathbf{0}$. Any open subset of $\mathbb{R}$ is a disjoint union of connected intervals, so we can define the composition by its restriction to each connected component. We state this in the following way:

Definition 2.3.7. Suppose that $I \subseteq \mathbb{R}$ is an open interval and $\phi(x) : I \to \Omega$ is real analytic and let $f(x) = [F(z)] \in \mathcal{B}(\Omega)$. If $\phi'(x) \neq 0$ for all $x \in I$ then we define $f(\phi(x))$ by

$$\begin{aligned}
\phi_* : \mathcal{B}(\Omega) &\to \mathcal{B}(I) \\
f(\phi(x)) = \phi_* f(x) &:= \operatorname{sgn}(\phi')\big[\phi_* F(z)\big] \\
&= \begin{cases} \big[F_+(\phi(z))\,,\ F_-(\phi(z))\big] & \phi'(x) > 0, \forall x \in I \\ -\big[F_-(\phi(z))\,,\ F_+(\phi(z))\big] & \phi'(x) < 0, \forall x \in I \end{cases}
\end{aligned}$$

In the general situation that $\phi^{-1}(\Omega) = \coprod_j I_j$ is a disjoint union of intervals, define

$$\begin{aligned}
\phi_* : \mathcal{B}(\Omega) &\to \mathcal{B}(\phi^{-1}(\Omega)) \\
\phi_* f(x) &:= \sum_j \operatorname{sgn}(\phi'_j)\big[\phi_* F(z)\big]\big|_{I_j}
\end{aligned}$$

where ϕ'_j denotes the restriction of ϕ' to I_j.

Example 2.3.9. Compute $\delta(\phi(x))$, where $\phi(x)$ is real analytic such that $\phi'(a) \neq 0$ whenever $\phi(a) = 0$.

Case 1: If $\phi(x)$ has no zeros, then $\frac{1}{\phi(z)}$ is analytic, hence $\left[\frac{1}{\phi(z)}\right] = [0]$. Therefore $\delta(\phi(x)) = 0$.

Case 2: Suppose ϕ has a single real zero, say $\phi(a) = 0$. Then we can write

$$\begin{aligned}
\phi(z) &= \phi'(a)(z-a) + \frac{\phi''(a)}{2}(z-a)^2 + O((z-a)^3) \\
&= \phi'(a)(z-a)\left(1 + \frac{\phi''(a)}{2\phi'(a)}(z-a) + O((z-a)^2)\right)
\end{aligned}$$

It follows that

$$\frac{1}{\phi(z)} = \frac{1}{\phi'(a)(z-a)} + \psi(z)$$

where $\psi(z)$ is analytic near $z = a$. Then

$$\begin{aligned}
\delta(\phi(z)) &= \frac{i}{2\pi}\left[\frac{\operatorname{sgn}(\phi'(a))}{\phi(z)}\right] \\
&= \frac{i}{2\pi}\operatorname{sgn}(\phi'(a))\left[\frac{1}{\phi'(a)(z-a)} + \psi(z)\right] \\
&= \frac{i}{2\pi}\frac{1}{|\phi'(a)|}\left[\frac{1}{z-a} + 0\right]
\end{aligned}$$

$$\delta(\phi(x)) = \frac{\delta(x-a)}{|\phi'(a)|}$$

Case 3: Suppose $\phi(x)$ has multiple real zeros, say $\phi(a_j) = 0$ for all $j \in J$. Since $\phi \not\equiv 0$, the Identity Theorem implies that $\{a_j : j \in J\}$ is a discrete set. Thus there is a small enough neighborhood Ω of zero such that the pre-image $\phi^{-1}(\Omega) = \coprod \Omega_j$ is a disjoint union of intervals with $a_j \in \Omega_j$. Then the formula in Case 2 applies to ϕ_j, so we get

$$\delta(\phi_j(x)) = \frac{\delta(x - a_j)}{|\phi_j'(a_j)|} = \frac{\delta(x - a_j)}{|\phi'(a_j)|} \tag{2.3.6}$$

Gluing these together we arrive at the well-known formula

$$\delta(\phi(x)) = \sum_{\phi(a)=0} \frac{\delta(x - a)}{|\phi'(a)|} \tag{2.3.7}$$

2.4 Product of Hyperfunctions

3 Hyperfunctions of Multiple Variables

The theory of analytic functions in multiple variables is considerably more subtle than the single variable theory, and this leads to a corresponding subtlety in the theory of multi-variable hyperfunctions. One might try to imitate the one variable case by defining hyperfunctions on an open set $\Omega \subseteq \mathbb{R}^n$ supported in a compact set $K \subseteq \Omega$ to be the quotient $\mathcal{O}(U \setminus K)/\mathcal{O}(U)$, for some complex neighborhood $U \supseteq \Omega$. The problem is that for $n \geq 2$ this quotient is always zero! This is due to Hartog's Theorem.

Theorem 3.0.1 (Hartog's Theorem [7])**.** *Let $U \subset \mathbb{C}^n$ be open, let $K \subset U$ be compact and suppose $n \geq 2$. If $U \setminus K$ is connected and $f \in \mathcal{O}(U \setminus K)$, then there exists a unique $F \in \mathcal{O}(U)$ such that $F|_{U \setminus K} = f$.*

This is closely related to the fact that the Mittag-Leffler Theorem (Thm 2.2.1) no longer holds for all open sets in $\mathbb{C}^n$. In cohomological terminology, this says that $H^1(U, \mathcal{O})$ is no longer necessarily zero for every open set $U \subseteq \mathbb{C}^n$ when $n \geq 2$. What is true, however, is that the *relative* cohomology group with respect to $\mathbb{R}^n \subseteq \mathbb{C}^n$ vanishes in all dimensions except n.

Theorem 3.0.2 (Sato's Theorem [8])**.** *$\mathbb{R}^n$ is purely n-codimensional relative to the sheaf $\mathcal{O}$, i.e.*

$$H^p_{\mathbb{R}^n}(U, \mathcal{O}) = 0 \quad \textit{if} \quad p \neq n$$

This leads to the following definition:

Definition 3.0.1. Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $U \supseteq \Omega$ be a complex neighborhood. Then the space of **hyperfunctions** on Ω is defined to be

$$\mathcal{B}(\Omega) = H^n_\Omega(U, \mathcal{O})$$

Remark 3.0.1. Just as in the case of a single variable, the definition of $\mathcal{B}$ can be shown to be independent of the choice of complex neighborhood, and the assignment $\Omega \mapsto \mathcal{B}(\Omega)$ forms a flabby sheaf [8],[9]. This abstract definition is, however, regrettably unintuitive. Fortunately, it is still possible to interpret elements of $\mathcal{B}(\Omega)$ in terms of boundary values of holomorphic functions.

To motivate this, let's consider the simple example of a holomorphic function of two variables which happens to be the product of two holomorphic functions in one variable

$$H(z_1, z_2) = F(z_1)G(z_2) \tag{3.0.1}$$

If $f = [F]$, and $g = [G]$ then we can express f and g as sums of boundary values

$$f(x) = F(x + i\epsilon) - F(x - i\epsilon) \qquad g(x) = G(x + i\epsilon) - G(x - i\epsilon)$$

Formally multiplying $f(x_1)$ with $g(x_2)$ yields

$$\begin{aligned}
h(x_1, x_2) :&= f(x_1)g(x_2) \\
&= (F(x_1 + i\epsilon) - F(x_1 - i\epsilon))\,(G(x_2 + i\epsilon) - G(x_2 - i\epsilon)) \\
&= F(x_1 + i\epsilon)G(x_2 + i\epsilon) - F(x_1 - i\epsilon)G(x_2 + i\epsilon) - F(x_1 + i\epsilon)G(x_2 - i\epsilon) + F(x_1 - i\epsilon)G(x_2 - i\epsilon) \\
&= H(x_1 + i\epsilon, x_2 + i\epsilon) - H(x_1 - i\epsilon, x_2 + i\epsilon) - H(x_1 + i\epsilon, x_2 - i\epsilon) + H(x_1 - i\epsilon, x_2 - i\epsilon)
\end{aligned}$$

Writing $x = (x_1, x_2)$ and letting $\sigma = (\pm 1, \pm 1)$ run through the four possible choice of signs, we arrive at

$$h(x) = \sum_{\sigma} \operatorname{sgn}(\sigma)\, H(x + \sigma i\epsilon) \tag{3.0.2}$$

For example, the 2-dimensional δ function would be

$$\delta(x) = \delta(x_1)\delta(x_2) = \left(\frac{i}{2\pi}\right)^2 \left(\frac{1}{(x_1+i\epsilon)(x_2+i\epsilon)} - \frac{1}{(x_1+i\epsilon)(x_2-i\epsilon)} - \frac{1}{(x_1-i\epsilon)(x_2+i\epsilon)} + \frac{1}{(x_1-i\epsilon)(x_2-i\epsilon)}\right)$$

which corresponds to the cohomology class

$$\delta(x_1, x_2) = \left[\frac{-1}{4\pi^2 z_1 z_2}\right] \tag{3.0.3}$$

Extending this idea to n dimensions would lead to 2^n terms in the formal sum (3.0.2) corresponding to each possible choice of sign $\pm\mathrm{Im}(z_j) > 0$. Although this approach can be made rigorous, conditioning on the sign of the imaginary part of each coordinate paints only a partial picture of how hyperfunctions can be expressed as boundary values. After all, not all holomorphic functions of multiple variables split as a product as in Eq (3.0.1). In order to describe the more general case, we need to consider the multitude of ways that a path can approach $\mathbb{R}^n \subseteq \mathbb{C}^n$, not simply those along the imaginary axes. In principle, the crucial difference is that a limit $x \to 0$ in $\mathbb{R}$ can be approached from only 2 directions, while in $\mathbb{R}^{\geq 2}$ there are infinitely many diections to approach zero. To account for these additional directions, we introduce the following definitions.

Definition 3.0.2. A subset $\Gamma \subseteq \mathbb{R}^n$ is called a **cone** if for all $x \in \Gamma$, and for all $t \in \mathbb{R}$

$$t > 0 \implies tx \in \Gamma \tag{3.0.4}$$

If $\Gamma \subseteq \mathbb{R}^n$ is a cone and $E \subseteq \mathbb{R}^n$ is arbitrary, then a subset of $\mathbb{C}^n$ of the form $E + i\Gamma$ is called a **wedge** with edge E and opening Γ.

Definition 3.0.3. Let $\Gamma, \Gamma' \subseteq \mathbb{R}^n$ be cones. Then Γ is called a **proper subcone** of Γ', denoted $\Gamma \Subset \Gamma'$, if

$$\overline{\Gamma} \cap \mathbb{S}^{n-1} \subseteq \operatorname{int}(\Gamma') \tag{3.0.5}$$

where $\mathbb{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$. Intuitively this means that Γ is contained in Γ' within some positive margin of the boundary *except at the origin* (because doing so at the origin would contradict being a cone).

Definition 3.0.4. Let $\Omega \subseteq \mathbb{R}^n$ be open and $\Gamma \subseteq \mathbb{R}^n$ be an open cone. Then an open set $U \subseteq \mathbb{C}^n$ is called an **infinitessimal wedge** of type $\Omega + i\Gamma\mathbf{0}$ if

(1) $U \subseteq \Omega + i\Gamma$

(2) For every $\epsilon > 0$, for every proper subcone $\Gamma' \Subset \Gamma$, there exists $\delta > 0$ such that

$$U \supseteq \Omega_\epsilon + i\big(\Gamma' \cap \delta\,\mathbb{S}^{n-1}\big) \tag{3.0.6}$$

where $\Omega_\epsilon = \{x \in \Omega :\ \operatorname{dist}(x, \partial\Omega) > \epsilon\}$ and $\delta\,\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n :\ \|x\| < \delta\}$. Intuitively, this means that U is approximately a wedge with opening Γ when zoomed in near the edge Ω.

Definition 3.0.5. Let $\Omega \subseteq \mathbb{R}^n$ be open. Then a **(boundary value) hyperfunction** on Ω is a formal sum

$$\sum_{j=1}^{N} F_j(x + i\Gamma_j\mathbf{0}) \tag{3.0.7}$$

where $\Gamma_1, \ldots, \Gamma_N \subseteq \mathbb{R}^n$ is a finite collection of open cones, and $F_j \in \mathcal{O}(U_j)$ are holomorphic on some infinitesimal wedge U_j of type $\Omega + i\Gamma_j\mathbf{0}$. Furthermore, if $\Gamma_j \cap \Gamma_k \neq \varnothing$, then the formal sum (3.0.7) is subject to the relation

$$F_j(x + i\Gamma_j\mathbf{0}) + F_k(x + i\Gamma_k\mathbf{0}) = (F_j + F_k)\big(x + i(\Gamma_j \cap \Gamma_k)\mathbf{0}\big) \tag{3.0.8}$$

Theorem 3.0.3. *If $f(x) = F(x + i\Gamma\mathbf{0})$ is a hyperfunction with only one term, then*

$$f(x) = 0 \iff F(z) = 0 \tag{3.0.9}$$

4 Vector-Valued Hyperfunctions

4.1 Vector-Valued Holomorphic Functions

Upon examination of equation (1.4.1), we note that the definition of the derivative makes sense when $f(z)$ takes values in V for any topological vector space over $\mathbb{C}$ (TVS). Because the limit is taken with respect to the topology of V, it is often convenient to assume that V is Hausdorff and complete (or at least quasi-complete, meaning that any closed and bounded subset is complete). This leads to the following definition:

Definition 4.1.1. Let $U \subseteq \mathbb{C}$ be open and let V be a TVS. Then a function $f : U \to V$ is called **strongly holomorphic** if the limit

$$f'(z) := \lim_{\substack{\epsilon \to 0 \\ \epsilon \in \mathbb{C}}} \frac{f(z+\epsilon) - f(z)}{\epsilon}$$

exists for all $z \in U$.

Similarly, integration defined as a limit of Riemann sums easily generalizes to take values in a TVS.

Definition 4.1.2. Suppose $U \subseteq \mathbb{C}$ is open, let V be a TVS. Given $f : U \to V$ and a piecewise-smooth curve $\gamma : [a, b] \to U$, we define the **integral** of f along γ by

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \lim_{n\to\infty} \sum_{a=x_0<\cdots<x_n=b} f(\gamma(x_i))\,(x_i - x_{i-1})$$

The natural question is whether this generalized integral satisfies a vector-valued version Cauchy's Theorem. If it does, then many theorems from the scalar-valued theory can be carried over with little to no modification. In order for this to be true, the topology of V needs to be "nice" enough that the limits in definitions (4.1.1) and (4.1.2) behave like limits in $\mathbb{C}$. In general, this is not always the case (see example 4.1.1). One way to avoid thinking about limits in V is to post-compose f with functionals $\lambda : V \to \mathbb{C}$ to reduce back to the scalar-valued case.

Definition 4.1.3. Let $U \subseteq \mathbb{C}$ be open an V be a topological vector space over $\mathbb{C}$. Then a function $f : U \to V$ is called **weakly holomorphic** if for all continuous functionals $\lambda \in V^*$, the composition

$$\lambda \circ f : U \to \mathbb{C}$$

is holomorphic in the ordinary sense.

It is immediate to prove that strong holomorphy implies weak holomorphy, however the converse is not true in general. The problem is that V can have "not enough" functionals, as shown in the following example. A more detailed explanation including proofs can be found in [3], pg 112.

Example 4.1.1. Fix $0 < p < 1$ and define $V = L^p(0, 1)$ to be the set of (equivalence classes of) measurable functions $f : (0, 1) \to \mathbb{C}$ such that

$$((f))_p := \left(\int_0^1 |f(x)|^p dx \right)^{1/p} < \infty \tag{4.1.1}$$

Using the notation $((-))_p$ instead of $\| - \|_p$ is meant to emphasize that $((-))_p$ is not a norm when $0 < p < 1$. Despite this, it is still the case that $d(f, g) := ((f - g))_p$ is a metric on V, and it can be shown that V is a topological vector space when endowed with the metric topology.

Lemma 4.1.1. *If $U \subseteq V$ is non-empty, open, and convex, then $U = V$.*

Corollary 4.1.1. *$V^* = \{0\}$ and consequently* ***any*** *function $f : \mathbb{C} \to V$ is weakly holomorphic.*

Proof. Suppose that $\lambda : V \to \mathbb{C}$ is continuous and linear. For $\epsilon > 0$, let $B_\epsilon = \{ z \in \mathbb{C} \ : |z| < \epsilon\}$. Note that B_ϵ is a convex open neighborhood of $0 \in \mathbb{C}$. Since λ is assumed continuous, $\lambda^{-1}(B_\epsilon)$ is open in V. By linearity, $0 \in \lambda^{-1}(B_\epsilon) \neq \varnothing$. Furthermore $\lambda^{-1}(B_\epsilon)$ is convex. To see this, assume $v, w \in V$ such that $|\lambda(v)|, |\lambda(w)| < \epsilon$. Then for all $t \in [0, 1]$ we have

$$\begin{aligned} |t\lambda(v) + (1-t)\lambda(w)| &\leq |t\lambda(v)| + |(1-t)\lambda(w)| \\ &= t|\lambda(v)| + (1-t)|\lambda(w)| \\ &< t\epsilon + (1-t)\epsilon = \epsilon \end{aligned}$$

Hence the lemma implies that $\lambda^{-1}(B_\epsilon) = V$ for all $\epsilon > 0$. But then $\lambda = 0$ because

$$\ker(\lambda) = \lambda^{-1}\left(\bigcap_{\epsilon>0} B_\epsilon\right) = \bigcap_{\epsilon>0} \lambda^{-1}(B_\epsilon) = V$$

□

Due to this example, we need to make sure that V has enough functionals in order that the notion of weak holomorphy be meaningful. This is achieved by the relatively mild assumption that V is a locally convex space (LCS), for which the Hahn-Banach Theorem holds. In this case, the converse implication is true:

Theorem 4.1.1. *Let V be a quasi-complete Hausdorff LCS and suppose $f : U \to V$ is weakly holomorphic. Then*

1. *f is strongly holomorphic*
2. $\int_\gamma f(z)dz = 0$
3. $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta$
   Here, the curve γ is suitably chosen in the same way as the scalar-valued theory
4. *f is analytic, i.e. for all $z_0 \in U$, there is a sequence $\{v_n\}$ in V such that*

$$f(z) = \sum_{n=0}^{\infty} v_n(z - z_0)^n$$

for a sufficiently small neighborhood of z_0, and the coefficients are given by

$$v_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}d\zeta$$

Proof. [5] □

Remark 4.1.1. Once we have $\int_\gamma f(z)dz = 0$, many theorems such as Liouville's Theorem, Identity Theorem, etc. carry over with the proofs being identical. Commonly we will be in the situation that the values take place in a Banach space, in which case theorems which involve modulus such as Rouché's Theorem, Jensen's Formula, Maximum Modulus, etc. will also carry over by replacing $|f(z)|$ with $\|f(z)\|$.

(DO WE NEED TO TALK ABOUT MORE DIFFERENTIABILITY WHEN DOMAIN IS ALSO LCS? IF SO, EXAMPLE OF G-HOLOMORPHY NOT $\neq$ HOLOMORPHIC: pg 170 pdf of Dineen)

4.2 Hyperfunctions Valued in a Fréchet Space

4.3 Hyperfunctions valued in Unbounded Operators

Throughout this section, let $\mathcal{H}$ be a Hilbert space and $\mathcal{D}$ be a dense subset of $\mathcal{H}$.

Definition 4.3.1. Let $\mathcal{L}_+(\mathcal{D})$ be the set of all unbounded operators A on $\mathcal{H}$ such that

(i) $\mathrm{Dom}(A) = \mathcal{D}$

(ii) The adjoint operator A^* exists, and $\mathrm{Dom}(A^*) \supseteq \mathcal{D}$

(iii) $\mathcal{D}$ is invariant under both A and A^*, i.e. $A(\mathcal{D}) \cap A^*(\mathcal{D}) \subseteq \mathcal{D}$.

We shall put the following topology on $\mathcal{L}_+(\mathcal{D})$.

Theorem 4.3.1. *Define for all $\phi, \psi \in \mathcal{D}$ the function*

$$\begin{aligned} p_{\phi,\psi} : \mathcal{L}_+(\mathcal{D}) &\to \mathbb{R}_{\geq 0} \\ A &\mapsto |\langle \phi | A\psi \rangle| \end{aligned}$$

Then $p_{\phi,\psi}$ is a semi-norm and $\mathcal{L}_+(\mathcal{D})$ becomes a locally convex topological space when equipped with the family of semi-norms

$$\mathcal{P} = \{p_{\phi,\psi} \mid \phi, \psi \in \mathcal{D}\}$$

Theorem 4.3.2. *For $A \in \mathcal{L}_+(\mathcal{D})$, define $A^+ = A^*|_{\mathcal{D}}$. Then $A \mapsto A^+$ is an involution which makes $\mathcal{L}_+(\mathcal{D})$ into a *-algebra.*

Definition 4.3.2. Let $U \subseteq \mathbb{C}$ be open. A function $F : U \to \mathcal{L}_+(\mathcal{D})$ is called **holomorphic** if for all $\psi \in \mathcal{D}$, and all $\phi \in \mathcal{H}$ the function $z \mapsto \langle \phi | F(z)\psi \rangle$ is holomorphic in the ordinary sense. Let $\mathcal{O}(U, \mathcal{L}_+(\mathcal{D}))$ denote the set of all $\mathcal{L}_+(\mathcal{D})$-valued holomorphic functions.

$$\mathcal{O}(U, \mathcal{L}_+(\mathcal{D})) := \{F : U \to \mathcal{L}_+(\mathcal{D}) \mid \langle \phi, F(-)\psi \rangle \in \mathcal{O}(U), \quad \forall \psi \in \mathcal{D}, \forall \phi \in \mathcal{H}\}$$

Definition 4.3.3. Given $F \in \mathcal{O}(U, \mathcal{L}_+(\mathcal{D}))$ and $\psi \in \mathcal{D}$, we can define $F|\psi\rangle \in \mathcal{O}(U, \mathcal{H})$ in the obvious way:

$$\begin{aligned} F|\psi\rangle : U &\to \mathcal{H} \\ z &\mapsto F(z)\psi \end{aligned}$$

Theorem 4.3.3. *The function*

$$\begin{aligned} \varphi : \mathcal{O}(U, \mathcal{L}_+(\mathcal{D})) &\to \mathcal{L}(\mathcal{D}\,,\, \mathcal{O}(U, \mathcal{H})) \\ F \quad &\mapsto \ \big(\ \psi \mapsto F|\psi\rangle\ \big) \end{aligned}$$

is an injective linear map. Furthermore, φ is surjective if and only if $\mathcal{H}$ is finite dimensional.

Definition 4.3.4. For $F \in \mathcal{O}(U, \mathcal{L}_+(\mathcal{D}))$ and $G \in \mathcal{O}(V, \mathcal{L}_+(\mathcal{D}))$, define $F \otimes G \in \mathcal{O}(U \times V, \mathcal{L}_+(\mathcal{D}))$ by

$$F \otimes G(z, w) := F(z)G(w) \tag{4.3.1}$$

Definition 4.3.5. Let $\Omega \subseteq \mathbb{R}^n$ be open and suppose $F_j \in \mathcal{O}(U_j, \mathcal{L}_+(\mathcal{D}))$ where U_j is an infinitesimal wedge of type $\Omega + i\Gamma_j\mathbf{0}$. Then an $\mathcal{L}_+(\mathcal{D})$**-valued hyperfunction** on Ω is a formal sum

$$f(x) = \sum_{j=1}^{N} F_j(x + i\Gamma_j\mathbf{0}) \tag{4.3.2}$$

subject to the relation

$$\sum_{j=1}^{N} F_j(x + i\Gamma_j\mathbf{0}) = \sum_{j=1}^{M} G_j(x + i\Gamma_j\mathbf{0}) \quad \Longleftrightarrow \quad \sum_{j=1}^{N} F_j|\psi\rangle(x + i\Gamma_j\mathbf{0}) = \sum_{j=1}^{M} G_j|\psi\rangle(x + i\Gamma_j\mathbf{0}) \text{ for all } \psi \in \mathcal{D}$$

The set of all $\mathcal{L}_+(\mathcal{D})$-valued hyperfunctions on Ω is denoted $\mathcal{B}(\Omega, \mathcal{L}_+(\mathcal{D}))$

Theorem 4.3.4. *The assignment $\Omega \mapsto \mathcal{B}(\Omega, \mathcal{L}_+(\mathcal{D}))$ constitutes a sheaf.*

Proof. $\square$

We can now define some operations on $\mathcal{B}(\Omega, \mathcal{L}_+(\mathcal{D}))$.

Definition 4.3.6. Let $F \in \mathcal{O}(\Omega, \mathcal{L}_+(\mathcal{D}))$ and $A \in \mathcal{L}_+(\mathcal{D})$. Then we define AF and FA in $\mathcal{O}(\Omega, \mathcal{L}_+(\mathcal{D}))$ by:

$$(AF)(z) := A \circ (F(z)) \qquad (FA)(z) := (F(z)) \circ A$$

Remark 4.3.1. Notice that that composition $F \circ A$ does not make sense.

The operation of left and right multiplication extends to $\mathcal{B}(\Omega, \mathcal{L}_+(\mathcal{D}))$ in the obvious way.

Definition 4.3.7. Let $f(x) = \sum_j F_j(x + i\Gamma_j\mathbf{0}) \in \mathcal{B}(\Omega, \mathcal{L}_+(\mathcal{D}))$ and $A \in \mathcal{L}_+(\mathcal{D})$. Then we define $Af(x)$ and $f(x)A$ by

$$Af(x) := \sum_j \big(AF_j\big)(x + i\Gamma_j\mathbf{0}) \qquad f(x)A := \sum_j \big(F_jA)(x + i\Gamma_j\mathbf{0}) \tag{4.3.3}$$

Example 4.3.1. We will later be interested in the case where $U \in \mathfrak{B}(\mathcal{H})$ is unitary such that $U(\mathcal{D}) = \mathcal{D}$. In this situation, the restriction $U|_{\mathcal{D}}$ is an element of $\mathcal{L}_+(\mathcal{D})$. If we denote $V = U|_{\mathcal{D}}$, then the conjugation of $f(x)$ by U is

$$Uf(x)U^* = \sum_j \left(VF_jV^+\right)(x + i\Gamma_j\mathbf{0})$$

In this context, it is convenient to identify U with $U|_{\mathcal{D}}$ and simply write

$$Uf(x)U^* = \sum_j (UF_jU^*)(x + i\Gamma_j\mathbf{0})$$

Definition 4.3.8. There is a multiplication map

$$\mathcal{B}(\Omega_1, \mathcal{L}_+(\mathcal{D})) \times \mathcal{B}(\Omega_2, \mathcal{L}_+(\mathcal{D})) \to \mathcal{B}(\Omega_1 \times \Omega_2, \mathcal{L}_+(\mathcal{D}))$$

Explicitly, let $f(x) \in \mathcal{B}(\Omega_1, \mathcal{L}_+(\mathcal{D}))$, and $g(x) \in \mathcal{B}(\Omega_2, \mathcal{L}_+(\mathcal{D}))$ with representations $f(x) = \sum_j F_j(x + i\Gamma_j\mathbf{0})$ and $g(x) = \sum_k G_k(x + i\Gamma_k\mathbf{0})$. Then we define $f(x_1)g(x_2) \in \mathcal{B}(\Omega_1 \times \Omega_2, \mathcal{L}_+(\mathcal{D}))$ by

$$f(x_1)g(x_2) := \sum_{j,k} F_j \otimes G_k\big((x,y) + i(\Gamma_j \times \Gamma_k)\mathbf{0}\big) \tag{4.3.4}$$

where $F_j \otimes G_k$ is defined in (4.3.4).

Definition 4.3.9. For $f(x) \in \mathcal{B}(\Omega_1, \mathcal{L}_+(\mathcal{D}))$, and $g(x) \in \mathcal{B}(\Omega_2, \mathcal{L}_+(\mathcal{D}))$, we define their **commutator** to be

$$[f(x_1), g(x_2)] := f(x_1)g(x_2) - g(x_2)f(x_1) \quad \in \ \mathcal{B}(\Omega_1 \times \Omega_2, \mathcal{L}_+(\mathcal{D}))$$

and their **anti-commutator**

$$\{f(x_1), g(x_2)\} := f(x_1)g(x_2) + g(x_2)f(x_1) \quad \in \ \mathcal{B}(\Omega_1 \times \Omega_2, \mathcal{L}_+(\mathcal{D}))$$

Remark 4.3.2. Given $\phi(x) \in \mathcal{B}(\Omega, \mathcal{L}_+(\mathcal{D}))$, we would like to also have $\phi(x)^* \in \mathcal{B}(\Omega, \mathcal{L}_+(\mathcal{D}))$. If $\phi(x) = \sum_j \Phi_j(x + i\Gamma_j\mathbf{0})$, then one is tempted to define $\phi(x)^* = \sum_j \Phi_j(x + i\Gamma_j\mathbf{0})^+$. However, the problem is that $z \mapsto \Phi(z)^+$ is not necessarily holomorphic. For example, define $\Phi(z) = z \cdot \mathbb{1}$. Then $\Phi(z)^+ = \overline{z} \cdot \mathbb{1}$, so $\frac{\partial}{\partial \overline{z}}\Phi(z)^+ = \mathbb{1} \neq 0$. To remedy this, we need to also conjugate z so that $z \mapsto \Phi(\overline{z})^+$ is holomorphic.

Definition 4.3.10. Define a map

$$\begin{aligned} * : \mathcal{O}(\Omega + i\Gamma\mathbf{0}, \mathcal{L}_+(\mathcal{D})) &\to \mathcal{O}(\Omega - i\Gamma\mathbf{0}, \mathcal{L}_+(\mathcal{D})) \\ \Phi(z) &\mapsto \Phi^*(z) := \Phi(\overline{z})^+ \end{aligned}$$

Then for $\phi(x) = \sum_j \Phi_j(x + i\Gamma_j\mathbf{0}) \in \mathcal{B}(\Omega, \mathcal{L}_+(\mathcal{D}))$, we define

$$\phi(x)^* := \sum_j \Phi_j^*(x - i\Gamma_j\mathbf{0}) \tag{4.3.5}$$

5 Application of Hyperfunctions in Physics

5.1 The Dirac Formalism: Making Bra-Kets Rigorous

The Dirac formalism of quantum mechanics is used ubiquitously in physics. In this approach, quantum states are treated axiomatically rather than working over a specific Hilbert space. In the finite dimensional case, an orthonormal basis $\{e_1, \dots, e_n\}$ satisfies

$$\langle v_i, v_j \rangle = \delta_{i,j} \qquad\qquad v = \sum_{i=1}^{n} e_n \langle e_n, v \rangle \tag{5.1.1}$$

where $\delta_{i,j}$ is the Kronecker delta. Inspired by this, one generalizes to infinite dimensions by promoting $\delta_{i,j}$ to the Dirac delta $\delta(i-j)$, and also changing from $\sum$ to $\int dx$. That is, one postulates the existence of a continuous orthonormal basis of position states $\{\ |x\rangle : x \in \mathbb{R}^n\}$ satisfying the conditions

$$\langle x|x' \rangle = \delta(x - x') \qquad\qquad \int_{\mathbb{R}^n} |x\rangle\langle x| \; dx = \mathbb{1} \tag{5.1.2}$$

Furthermore, one postulates an orthonormal basis of momentum states $\{\ |p\rangle : p \in \mathbb{R}^n\}$

$$\langle p|p' \rangle = \delta(p - p') \qquad\qquad \int_{\mathbb{R}^n} |p\rangle\langle p| \; dp = \mathbb{1} \tag{5.1.3}$$

subject to the relation

$$\langle p|x \rangle = e^{2\pi i p\cdot x} \tag{5.1.4}$$

While this formal approach is no doubt useful, we cannot interpret $\langle x|x'\rangle$ as an inner product because $\delta(x-x') \notin \mathbb{C}$ when $x = x'$. The goal of this section is to show how the above formalism can be made mathematically rigorous using vector-valued hyperfunctions.

Remark 5.1.1. There are varying conventions for the Fourier transform, which lead to different relations for the momentum basis $|p\rangle$. We will use the unitary definition

$$\mathcal{F}[f(x)](p) = \hat{f}(p) := \int_{\mathbb{R}^n} e^{-2\pi i x\cdot p} f(x) dx \qquad\qquad \mathcal{F}^{-1}[f(p)](x) = \check{f}(x) := \int_{\mathbb{R}^n} e^{2\pi i x\cdot p} f(p) dp$$

It is common in physics to use natural units wherein $c = \frac{h}{2\pi} = 1$, which leads to using the the non-unitary definition

$$\tilde{f}(p) := \int_{\mathbb{R}^n} e^{-ix\cdot p} f(x) dx \qquad\qquad f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot p} \tilde{f}(p) dp$$

in which case equations (5.1.3) and (5.1.4) are replaced by

$$\langle p|p' \rangle = (2\pi)^n \delta(p - p') \qquad\qquad \int_{\mathbb{R}^n} |p\rangle\langle p| \; \frac{dp}{(2\pi)^n} = \mathbb{1} \qquad\qquad \langle p|x \rangle = e^{ipx}$$

Remark 5.1.2. In this section, the letter x will be reserved in order to match the traditional notation $|x\rangle$. To avoid confusion, we shall denote $f \in L^2(\mathbb{R}^n)$ by $\xi \mapsto f(\xi)$ rather than $x \mapsto f(x)$. Similarly, we will denote the momentum space variable $\omega \mapsto \hat{f}(\omega) \in \mathcal{F}\big(L^2(\mathbb{R}^n)\big)$ in order to use the standard notation $|p\rangle$.

Proposition 5.1.1. *The function*

$$\begin{aligned} C : \mathbb{C} \setminus \mathbb{R} &\to L^2(\mathbb{R}) \\ z &\mapsto \left(\xi \mapsto \frac{1}{2\pi i(\xi - z)}\right) \end{aligned}$$

is an element of $\mathcal{O}(\mathbb{C} \setminus \mathbb{R}; L^2(\mathbb{R}))$.

Proof. First we show that C is well-defined. If $z = a + ib \in \mathbb{C} \setminus \mathbb{R}$, then we have

$$\int_{\mathbb{R}} \left| \frac{1}{\xi - z} \right|^2 d\xi = \int_{\mathbb{R}} \left| \frac{1}{\xi - a - ib} \right|^2 d\xi = \int_{\mathbb{R}} \frac{1}{(\xi - a)^2 + b^2} d\xi = \int_{\mathbb{R}} \frac{1/b^2}{(\xi/b)^2 + 1} \, d\xi = \frac{\pi}{|b|} < \infty$$

It remains to show that C is holomorphic. It is clear that $\frac{d}{dz} \frac{1}{\xi - z} = \frac{1}{(\xi - z)^2}$, so a natural guess for $C'(z)$ is the map $\xi \mapsto \frac{1}{2\pi i(\xi - z)^2}$. First we check that this map is indeed an element of $L^2(\mathbb{R})$ for each $z \in \mathbb{C} \setminus \mathbb{R}$.

$$\int_{\mathbb{R}} \left| \frac{1}{(\xi - z)^2} \right|^2 d\xi = \int_{\mathbb{R}} \left| \frac{1}{(\xi - a - ib)^2} \right|^2 d\xi = \int_{\mathbb{R}} \frac{1}{((\xi - a)^2 + b^2)^2} d\xi = \frac{\pi}{2|b|^3} < \infty$$

Now we need to show that $\lim_{\epsilon \to 0} \left\| \frac{C(z+\epsilon) - C(z)}{\epsilon} - \frac{1}{2\pi i(\xi - z)^2} \right\| = 0$ in the norm topology of $L^2(\mathbb{R})$. Write $\epsilon = \alpha + i\beta$. If we take $\epsilon \in \mathbb{C}$ small enough so that $|\epsilon| < \frac{|b|}{2}$, then for all $\xi \in \mathbb{R}$ we have $|\xi - z - \epsilon|^2 = (\xi - a - \alpha)^2 + (b + \beta)^2 \geq (b - \frac{b}{2})^2 = \frac{b^2}{4}$. Hence we may use the bound $\frac{1}{|\xi - z - \epsilon|} \leq \frac{2}{|b|}$. Then the claim follows because

$$\begin{aligned} \left\| \frac{C(z+\epsilon) - C(z)}{\epsilon} - \frac{1}{2\pi i(\xi - z)^2} \right\| &= \frac{1}{2\pi} \left\| \frac{1}{(\xi - z)(\xi - z - \epsilon)} - \frac{1}{(x - z)^2} \right\| \\ &= \frac{1}{2\pi} \left\| \frac{\epsilon}{(\xi - z^2)(\xi - z - \epsilon)} \right\| \leq \frac{1}{\pi |b|} \left\| \frac{\epsilon}{(\xi - z)^2} \right\| \to 0 \end{aligned}$$

$\square$

Corollary 5.1.1. *For all $n \in \mathbb{N}$, the map $C_n : (\mathbb{C} \setminus \mathbb{R})^n \to L^2(\mathbb{R}^n)$ given by*

$$C_n(z_1, \ldots, z_n) := (\xi_1, \ldots, \xi_n) \mapsto \frac{1}{(2\pi i)^n} \frac{1}{(\xi - z_n) \cdots (\xi_n - z_n)}$$

is an element of $\mathcal{O}\big((\mathbb{C} \setminus \mathbb{R})^n; L^2(\mathbb{R}^n)\big)$.

Remark 5.1.3. For simplicity of presentation, we will generally assume $n = 1$. If desired, one can apply induction to extend to arbitrary $n \in \mathbb{N}$.

Definition 5.1.1. Define $c(x) \in \mathcal{B}(\mathbb{R}; L^2(\mathbb{R}))$ by

$$c(x) := [C(z)] = C(x + i\mathbf{0}) - C(x - i\mathbf{0})$$

What does the hyperfunction $c(x)$ represent? To find out, we need to see what happens upon integration.

$$\begin{aligned} \int_a^b c(x) dx &= -\oint_\gamma \left(\xi \mapsto \frac{1}{2\pi i(\xi - z)} \right) dz \\ &= \xi \mapsto \frac{1}{2\pi i} \oint_\gamma \frac{dz}{z - \xi} \\ &= \xi \mapsto \begin{cases} 1 & a < \xi < b \\ 0 & \text{else} \end{cases} \\ &= \chi_{(a,b)} \end{aligned}$$

But we can also write

$$\chi_{(a,b)}(\xi) = \int_a^b \delta(x - \xi) dx$$

Therefore we can think of $c(x)$ *without* the integral as being the generalized function

$$c(x) = \xi \mapsto \delta(x - \xi)$$

Remark 5.1.4. We would like to show that $c(x)$ satisfies the same properties as $|x\rangle$ in equation (5.1.2). To this end, we need to extend the inner product on $L^2(\mathbb{R})$ to hyperfunctions. First we'll consider the simpler case when only one of the terms is a hyperfunction. If $f(x) \in \mathcal{B}(\mathbb{R}; L^2(\mathbb{R}))$ and $\psi \in L^2(\mathbb{R})$, then we would expect the definition to be

$$\langle f(x)|\psi\rangle = \int_{\mathbb{R}} \overline{f(x)}\, \psi \; d\xi$$

Recalling the definition of conjugation for a hyperfunction (Def 2.3.6), we are led to the following.

Definition 5.1.2. Let $f(x) = [F(z)] \in \mathcal{B}(\mathbb{R}; L^2(\mathbb{R}))$ and $\psi \in L^2(\mathbb{R})$. Then we define $\langle f(x)|\psi\rangle$ and $\langle \psi|f(x)\rangle$ in $\mathcal{B}(\mathbb{R})$ by

$$\langle f(x)|\psi\rangle := \big[\langle -F(\overline{z})|\psi\rangle\big] \qquad\qquad \langle \psi|f(x)\rangle := \big[\langle \psi|F(z)\rangle\big] \tag{5.1.5}$$

Theorem 5.1.1. *Let $\iota : L^1_{loc} \hookrightarrow \mathcal{B}$ be the embedding of locally integrable functions into hyperfunctions defined in Corollary 2.2.1. Then*

$$\langle c(x)|\psi\rangle = \iota(\psi) \tag{5.1.6}$$

for all $\psi \in L^2(\mathbb{R})$.

Proof.

$$\begin{aligned}\langle c(x)|\psi\rangle &= \big[\langle -C(\overline{z}), \psi\rangle\big] \\ &= \left[\left\langle \frac{-1}{2\pi i(\xi - \overline{z})}, \psi(\xi)\right\rangle\right] \\ &= \left[\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\psi(\xi)}{\xi - z}\; d\xi\right] =: \iota(\psi)\end{aligned}$$

□

If we think of x as the hyperfunctional variable and ξ as the variable in $L^2(\mathbb{R})$, then we can write equation (5.1.6) as

$$\langle c(x)|\psi(\xi)\rangle = \psi(x) \tag{5.1.7}$$

We get the same result if we think of $c(x)$ as being the map $\xi \mapsto \delta(x - \xi)$ because in that case

$$\langle \delta(x-\xi)|\psi(\xi)\rangle = \int_{\mathbb{R}} \delta(x-\xi)\psi(\xi)dt = \psi(x) \tag{5.1.8}$$

In other words, Theorem 5.1.1 is the mathematically rigorous version of the the Dirac formalism $\langle x|\psi\rangle = \psi(x)$.

Remark 5.1.5. In order to define the inner product between two hyperfunctions, there is a problem similar to defining the product of two hyperfunctions. The naive guess $\langle f(x)|g(x)\rangle = \int_{\mathbb{R}} \overline{f(x)}g(x)dx$ does not make sense because the product $\overline{f(x)}g(x)$ might not make sense. To avoid this, we can think of $\overline{f(x_1)}g(x_2)$ as a hyperfunction of 2 variables.

Definition 5.1.3. Let $f(x), g(x) \in \mathcal{B}(\mathbb{R}; L^2(\mathbb{R}))$ with representatives $f(x) = [F(z)]$, $g(x) = [G(z)]$. Then we define $\langle f(x_1)|g(x_2)\rangle \in \mathcal{B}(\mathbb{R}^2)$ by

$$\langle f(x_1)|g(x_2)\rangle := [\langle -F(\overline{z_1})|G(z_2)\rangle] \tag{5.1.9}$$

Theorem 5.1.2.

$$\langle c(x)|c(y)\rangle = \delta(x-y) \tag{5.1.10}$$

Proof. First, we use Theorem 2.2.6 to write $\delta(x-y)$ as a hyperfunction. The action of $\delta(x-y)$ on a test function is

$$\begin{aligned}&\mathcal{S}(\mathbb{R}^2) \to \mathbb{C} \\ &\varphi \mapsto \int_{\mathbb{R}}\int_{\mathbb{R}} \delta(x-y)\varphi(x,y)dxdy = \int_{\mathbb{R}} \varphi(x,x)dx\end{aligned}$$

Then the representation of $\delta(x-y)$ as a hyperfunction is given by plugging in for φ the 2 dimensional Cauchy kernel $C_2(z_1,z_2) = \frac{1}{(2\pi i)^2}\frac{1}{(\xi_1-z_1)(\xi_2-z_2)}$

$$\delta(x-y) = \left[(z_1,z_2) \mapsto \frac{1}{(2\pi i)^2}\int_{\mathbb{R}} \frac{d\xi}{(\xi-z_1)(\xi-z_2)}\right]$$

By definition, this is the same as $\langle c(x)|c(y)\rangle$

$$\begin{aligned}\langle c(x)|c(y)\rangle &:= \left[\langle -C(\overline{z_1})\big|C(z_2)\rangle\right] \\ &= \left[\left\langle \frac{-1}{2\pi i(\xi-\overline{z_1})} \,\middle|\, \frac{1}{2\pi i(\xi - z_2)}\right\rangle\right] \\ &= \left[\frac{1}{(2\pi i)^2}\int_{\mathbb{R}} \frac{d\xi}{(\xi-z_1)(\xi-z_2)}\right]\end{aligned}$$

□

Theorem (5.1.10) is the rigorous version of Dirac's formal identity $\langle x|y\rangle = \delta(x-y)$; therefore we see that $c(x)$ behaves like $|x\rangle$. Next, we investigate the momentum basis. If $c(x)$ corresponds to $|x\rangle$, what is the hyperfunctional representation of $|p\rangle$? The obvious guess is that $|p\rangle$ corresponds to the Fourier transform $\widehat{c(x)}$.

Theorem 5.1.3. *The Fourier transform* $\widehat{C(z)}(\omega) = \int_{\mathbb{R}} \frac{e^{-2\pi i\xi\omega}}{2\pi i(\xi-z)}d\xi$ *is given by*

$$\widehat{C(z)}(\omega) = \begin{cases} e^{-2\pi i z\omega}\chi_{(-\infty,0)}(\omega) & \mathrm{Im} z > 0 \\ -e^{-2\pi i z\omega}\chi_{(0,\infty)}(\omega) & \mathrm{Im} z < 0\end{cases} = \begin{cases} e^{-2\pi i z\omega} & \mathrm{Im} z > 0,\ \omega < 0 \\ -e^{-2\pi i z\omega} & \mathrm{Im} z < 0,\ \omega > 0 \\ 0 & \text{otherwise}\end{cases}$$

Proof. Observe that if $\mathrm{Im}(z) > 0$ and $\omega < 0$, then $\mathrm{Re}(-2\pi i z\omega) < 0$. It follows that the function $F_+(z) := \omega \mapsto e^{-2\pi i z\omega}\chi_{(-\infty,0)}(\omega)$ is an element of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for all $z \in \mathbb{C}^+$. Then it's inverse Fourier transform converges in the ordinary sense and is given by

$$\mathcal{F}^{-1}[F_+(z)](\xi) := \int_{\mathbb{R}} e^{2\pi i\xi\omega}e^{-2\pi i z\omega}\chi_{(-\infty,0)}d\omega = \int_{-\infty}^{0} e^{2\pi i(\xi-z)\omega}d\omega = \frac{e^{2\pi i(\xi-z)\omega}}{2\pi i(\xi-z)}\Big|_{-\infty}^{0} = \frac{1}{2\pi i(\xi-z)}$$

In the case that $\mathrm{Im}(z) < 0$, the function $F_-(z) := \omega \mapsto -e^{-2\pi i z\omega}\chi_{(0,\infty)}(\omega)$ lies in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. It's inverse Fourier transform is given by

$$\mathcal{F}^{-1}[F_-(z)](\xi) := -\int_{\mathbb{R}} e^{2\pi i\xi\omega}e^{-2\pi i z\omega}\chi_{(0,\infty)}d\omega = -\int_{0}^{\infty} e^{2\pi i(\xi-z)\omega}d\omega = \frac{-e^{2\pi i(\xi-z)\omega}}{2\pi i(\xi-z)}\Big|_{0}^{\infty} = \frac{1}{2\pi i(\xi-z)}$$

By applying $\mathcal{F}$ to both sides, we see that

$$\widehat{C(z)} = \begin{cases} F_+(z) & z \in \mathbb{C}^+ \\ F_-(z) & z \in \mathbb{C}^-\end{cases}$$

as desired.

□

Definition 5.1.4. Define $\hat{c}(p) \in \mathcal{B}(\mathbb{R}; L^2(\mathbb{R}))$ by

$$\hat{c}(p) := \left[\widehat{C(z)}\right] = \hat{C}(p + i\mathbf{0}) - \hat{C}(p - i\mathbf{0}) \tag{5.1.11}$$

Remark 5.1.6. Informally we can think of $\hat{c}(p)$ as being the function $\omega \mapsto e^{-2\pi i\omega p}$ on momentum space, despite the fact that this function is not square-summable. Indeed, if we ignore the Hilbert space structure and simply take the point-wise limit, we get

$$\begin{aligned}\lim_{\epsilon\to 0^+}\big(\hat{C}(p+i\epsilon)(\omega)-\hat{C}(p-i\epsilon)(\omega)\big) &= \lim_{\epsilon\to 0^+}\Big(e^{-2\pi i\omega(p+i\epsilon)}\chi_{(-\infty,0)}(\omega)+e^{-2\pi i\omega(p-i\epsilon)}\chi_{(0,\infty)}(\omega)\Big)\\ &= e^{-2\pi i\omega p}\big(\chi_{(-\infty,0)}+\chi_{(0,\infty)}\big)(\omega)\\ &= e^{-2\pi i\omega p}\end{aligned}$$

Proposition 5.1.2. *Let $\psi \in L^2(\mathbb{R})$ and $\iota : L^2(\mathbb{R}) \hookrightarrow \mathcal{B}(\mathbb{R}; L^2(\mathbb{R}))$ be the canonical embedding. Then*

$$\langle \hat{c}(p)|\psi\rangle = \iota(\check{\psi}) \tag{5.1.12}$$

Proof. Using Theorem (5.1.1) and the fact that $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is unitary, we deduce

$$\langle \hat{c}(p)|\psi\rangle := \big[\langle -\widehat{C(\overline{z})}|\psi\rangle\big] = \big[\langle -\mathcal{F}C(\overline{z})|\psi\rangle\big] = \big[\langle -C(\overline{z})|\mathcal{F}^{-1}\psi\rangle\big] =: \langle c(x)|\check{\psi}\rangle = \iota(\check{\psi})$$

□

Theorem 5.1.4. *The hyperfunction $\langle \hat{c}(p)|\hat{c}(q)\rangle \in \mathcal{B}(\mathbb{R}^2)$ is given by*

$$\langle \hat{c}(p)|\hat{c}(q)\rangle = \delta(p-q) \tag{5.1.13}$$

Proof. Applying Theorem (5.1.2) and the unitary of $\mathcal{F}$, we obtain

$$\langle \hat{c}(p)|\hat{c}(q)\rangle := \big[-\langle\widehat{C(\overline{z})}|\widehat{C(w)}\rangle\big] = \big[-\langle C(\overline{z})|C(w)\rangle\big] =: \langle c(p)|c(q)\rangle = \delta(p-q)$$

□

Theorem 5.1.5. *Let $\varphi \in L^1_{\text{loc}}(\mathbb{R}^2)$ be given by $\varphi(\xi,\omega) = e^{2\pi i\xi\omega}$, and $\iota : L^1_{\text{loc}}(R^2) \hookrightarrow \mathcal{B}(\mathbb{R}^2)$ be the embedding. Then*

$$\langle \hat{c}(p)|c(x)\rangle = \iota(\varphi) \tag{5.1.14}$$

That is, $\langle \hat{c}(p)|c(x)\rangle = e^{2\pi ixp} \in \mathcal{B}(\mathbb{R}^2)$.

Proof. First, we prove that $\overline{\widehat{C(\overline{z})}}(\xi) = -\widehat{C(z)}(-\xi)$. By Theorem (5.1.3), we have

$$\widehat{C(z)}(\xi) = \begin{cases} e^{-2\pi iz\xi}\chi_{(-\infty,0)} & \text{Im}z > 0\\ -e^{-2\pi iz\xi}\chi_{(0,\infty)} & \text{Im}z < 0\end{cases}$$

$$\begin{aligned}\overline{\widehat{C(\overline{z})}}(\xi) &= \begin{cases} \overline{e^{-2\pi i\overline{z}\xi}\chi_{(-\infty,0)}} & \text{Im}\overline{z} > 0\\ \overline{-e^{-2\pi i\overline{z}\xi}\chi_{(0,\infty)}} & \text{Im}\overline{z} < 0\end{cases}\\ &= \begin{cases} e^{2\pi iz\xi}\chi_{(-\infty,0)} & \text{Im}z < 0\\ -e^{2\pi iz\xi}\chi_{(0,\infty)} & \text{Im}z > 0\end{cases}\\ &= -\widehat{C(z)}(-\xi)\end{aligned}$$

Therefore

$$\begin{aligned}\langle \hat{c}(p)|c(x)\rangle &:= \big[\langle -\widehat{C(\overline{z_1})}|C(z_2)\rangle\big]\\ &= \left[-\int_{\mathbb{R}} \overline{\widehat{C(\overline{z_1})}}(\xi)\ C(z_2)(\xi)\ d\xi\right]\\ &= \left[\int_{\mathbb{R}} \widehat{C(z_1)}(-\xi)\ C(z_2)(\xi)\ d\xi\right]\\ &= \left[\int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{2\pi i\omega\xi}C(z_1)(\omega)d\omega\right)\ C(z_1)(\xi)\ d\xi\right]\\ &= \left[\frac{1}{(2\pi i)^2}\int_{\mathbb{R}^2}\frac{e^{2\pi i\omega\xi}}{(\omega-z_1)(\xi-z_2)}\ d\omega\ d\xi\right] =: \iota(e^{2\pi i\omega\xi})\end{aligned}$$

□

To summarize the above theorems, we have now shown that

$$\langle c(x)|c(x')\rangle = \delta(x-x') \qquad \langle \hat{c}(p)|\hat{c}(p')\rangle = \delta(p-p') \qquad \langle \hat{c}(p)|c(x)\rangle = e^{2\pi i p x} \tag{5.1.15}$$

Hence we see that the hyperfunctions $c(x)$ and $\hat{c}(p)$ are none other than $|x\rangle$ and $|p\rangle$ of the Dirac formalism. Next, we shall address how to make to make rigorous the identity $\int_{\mathbb{R}} |x\rangle\langle x| dx = \mathbb{1}$ using the language of hyperfunctions.

5.2 The Resolvent Map and the Spectral Theorem

In finite dimensions, we can always find finitely many rank one projections $P_n = |x_n\rangle\langle x_n|$ such that $\sum_n |x_n\rangle\langle x_n| = \mathbb{1}$. In infinite dimensions, however, this approach runs into convergence issues. It does not work to sum up rank one projection $P_x = |x\rangle\langle x|$ for each $x \in \mathbb{R}$ for the same reason that one cannot evaluate an integral $\int_{\mathbb{R}} f(x)dx$ as the uncountably infinite sum $\sum_{x\in\mathbb{R}} f(x)$. Instead, we need to give a projection P_Δ for each measurable set $\Delta \subseteq \mathbb{R}$. With a suitable family $\{P_\Delta\}$ of projections, we can mimic the standard theory of integration with respect to a measure and make sense of the expression $\int_{\mathbb{R}} |x\rangle\langle x| dx$. The result is the famous Spectral Theorem.

There are several equivalent ways to state the Spectral Theorem, but perhaps the most natural for our purpose is that a self-adjoint operator A can be written as an integral

$$A = \int_{\mathbb{R}} \lambda \; dE(\lambda) \tag{5.2.1}$$

where dE is a projection-valued measure depending on A. More generally, any function of A is obtained by simply applying the function inside the integral

$$f(A) = \int_{\mathbb{R}} f(\lambda) \; dE(\lambda) \tag{5.2.2}$$

In this section, we will how explore how the projection valued measure dE can be viewed as a hyperfunction. In particular, equation (5.2.2) can be seen as a vector-valued generalization of the identity

$$f(a) = \int_{\mathbb{R}} f(x)\delta(x-a)dx$$

5.2.1 Resolvent and Spectrum

Throughout this section, let $\mathcal{H}$ be a complex Hilbert space with $\dim(\mathcal{H}) \geq 1$, and let A denote a closed linear operator on $\mathcal{H}$. Unless there is potential for confusion, we will denote a scalar multiple of the identity $z \cdot \mathbb{1}$ simply by z.

Definition 5.2.1. The **resolvent set** of A, denoted $\rho(A)$, is the set of complex numbers $z \in \mathbb{C}$ such that $A - z$ is invertible and bounded.

$$\rho(A) = \{z \in \mathbb{C} \mid (A-z)^{-1} \in \mathfrak{B}(\mathcal{H})\} \tag{5.2.3}$$

The **resolvent map** of A is the function $R(A) : \rho(A) \to \mathfrak{B}(\mathcal{H})$ given by $R(A,z) = (A-z)^{-1}$.

Remark 5.2.1. When the operator A is understood from context, we abbreviate $R = R(z) = R(A,z)$.

Theorem 5.2.1. *The resolvent has the following properties*

1. $R(A,z) - R(A,w) = (z-w)R(A,z)R(A,w)$
2. $R(A,z) - R(B,z) = R(A,z)(B-A)R(B,z)$
3. $R(A,z)R(A,w) = R(A,w)R(A,z)$
4. *$\rho(A)$ is an open subset of $\mathbb{C}$.*

5. *The function $z \mapsto R(A,z)$ is a $\mathfrak{B}(\mathcal{H})$-valued holomorphic function, i.e. $R(A) \in \mathcal{O}(\rho(A);\mathfrak{B}(\mathcal{H}))$. Moreover, $\frac{d^n}{dz^n}R(z) = n!R(z)^{n+1}$*

Proof. To prove (1), we compute

$$\begin{aligned} R(A,z) - R(A,w) &= (A-z)^{-1} - (A-w)^{-1} \\ &= (A-z)^{-1}\big[(A-w)(A-w)^{-1}\big] - \big[(A-z)^{-1}(A-z)\big](A-w)^{-1} \\ &= (A-z)^{-1}\big[(A-w)-(A-z)\big](A-w)^{-1} \\ &= (A-z)^{-1}(z-w)(A-w)^{-1} \\ &= (z-w)R(A,z)R(A,w) \end{aligned}$$

The proof of (2) is similar.

$$\begin{aligned} (A-z)^{-1} - (B-z)^{-1} &= (A-z)^{-1}\big[(B-z)(B-z)^{-1}\big] - \big[(A-z)^{-1}(A-z)\big](B-z)^{-1} \\ &= (A-z)^{-1}\big[(B-z)-(A-z)\big](B-z)^{-1} \\ &= (A-z)^{-1}(B-A)(B-z)^{-1} \end{aligned}$$

To prove (3), we first note that the statement is clearly true if $z = w$. If $z \neq w$, then by (1) we have

$$R(z)R(w) = \frac{R(z)-R(w)}{z-w} = \frac{R(w)-R(z)}{w-z} = R(w)R(z)$$

To show (4), we re-write (1) as follows

$$\begin{aligned} R(z) - R(w) &= (z-w)R(z)R(w) \\ R(z) &= R(w) + (z-w)R(z)R(w) \\ R(z) &= \big[1 + (z-w)R(z)\big]R(w) \\ R(z)\big[1+(z-w)R(z)\big]^{-1} &= R(w) \end{aligned}$$

Now suppose $z_0 \in \rho(A)$. If $|z - z_0| < \|R(z_0)\|^{-1}$, then we can apply the geometric series to the expression above to yield

$$R(z) = \sum_{n=0}^{\infty} R(z_0)^{n+1}(z-z_0)^n \tag{5.2.4}$$

Therefore $\{z \in \mathbb{C} : |z - z_0| < \|R(z_0)\|^{-1}\} \subseteq \rho(A)$ and hence $\rho(A)$ is open. Furthermore, the series (5.2.4) immediately proves (5). □

Definition 5.2.2. The **spectrum** of A, denoted $\sigma(A)$, is the complement of $\rho(A)$.

$$\sigma(A) = \mathbb{C} \setminus \rho(A) = \{z \in \mathbb{C} \mid (A-z)^{-1} \notin \mathfrak{B}(\mathcal{H})\} \tag{5.2.5}$$

Theorem 5.2.2. *If A is self-adjoint, then $\sigma(A) \subseteq \mathbb{R}$.*

Theorem 5.2.3. *Suppose $A \in \mathfrak{B}(\mathcal{H})$. Then $\sigma(A)$ is non-empty and compact.*

Proof. We have already shown $\rho(A)$ is open, so $\sigma(A)$ is closed. To prove compactness it suffices to show that $\sigma(A)$ is bounded. To this end, suppose $|z| > \|A\|$. Then $\left\|\frac{A}{z}\right\| < 1$, hence $1 - \frac{A}{z}$ is invertible by the geometric series. Then $A - z = -z(1 - \frac{A}{z})$ is also invertible since $z > 0$, therefore $\sigma(A) \subseteq \{z \in \mathbb{C} : |z| \leq \|A\|\}$.
Now suppose $\sigma(A) = \varnothing$. Then $R(A) : \mathbb{C} \to \mathfrak{B}(\mathcal{H})$ is entire. Furthermore, by applying the reverse triangle inequality we see that $R(A)$ is bounded.

$$\begin{aligned} \|A - z\| &\geq \big|\ \|A\| - |z|\ \big| \\ \|(A-z)^{-1}\| &\leq \left|\frac{1}{\|A\| - |z|}\right| \\ \lim_{|z|\to\infty} \|(A-z)^{-1}\| &\leq \lim_{|z|\to\infty} \left|\frac{1}{\|A\| - |z|}\right| = 0 \end{aligned}$$

By Liouville's Theorem $R(A)$ is constant, which contradicts $\dim(\mathcal{H}) > 0$. Therefore $\sigma(A) \neq \varnothing$. □

Remark 5.2.2. If A is unbounded, then $\sigma(A)$ may be unbounded or even empty. However this can be remedied by extending the domain of $R(A,z)$ from $\mathbb{C}$ to the Riemann sphere $\mathbb{C}\mathrm{P}^1 = \mathbb{C} \cup \infty$ and considering whether $R(A,z)$ is holomorphic at $z=\infty$ (recall that for a function $f(z)$ to be holomorphic at $z=\infty$ means that $f(\frac{1}{z})$ can be analytically continued to $z=0$.)

Definition 5.2.3. The **extended resolvent** set of A is

$$\tilde{\rho}(A) = \{z \in \mathbb{C}\mathrm{P}^1 : (A-z)^{-1} \in \mathfrak{B}(\mathcal{H})\} \tag{5.2.6}$$

and the **extended spectrum** of A is $\tilde{\sigma}(A) = \mathbb{C}\mathrm{P}^1 \setminus \tilde{\rho}(A)$

Theorem 5.2.4. *Let A be a closed linear operator on $\mathcal{H}$, and suppose that $\rho(A)$ contains the exterior of a circle, i.e. $\rho(A) \supseteq \{z \in \mathbb{C} : |z| > R\}$. Then exactly one of the following holds true:*

1. *$A \in \mathfrak{B}(\mathcal{H})$, $R(A)$ is holomorphic at $z=\infty$ and $R(A,\infty) = 0$.*
2. *$R(A)$ has an essential singularity at $z=\infty$.*

Proof. Theorem 6.13 of [10] on pg 176 □

In light of this theorem, we conclude

Corollary 5.2.1. *For any closed operator A,*

$$\tilde{\sigma}(A) = \begin{cases} \sigma(A) & A \in \mathfrak{B}(\mathcal{H}) \\ \sigma(A) \cup \{\infty\} & A \notin \mathfrak{B}(\mathcal{H}) \end{cases}$$

and therefore $\tilde{\sigma}(A)$ is always a non-empty, compact subset of $\mathbb{C}\mathrm{P}^1$.

5.2.2 Projection-Valued Measures

In this section, let (Ω, Σ) be a measurable space.

Definition 5.2.4. An operator $P \in \mathfrak{B}(\mathcal{H})$ is an **projection** if $P^2 = P$. If in addition $P = P^*$, then P is an **orthogonal projection**.

Definition 5.2.5. A **projection-valued** measure is a function $E : \Sigma \to \mathfrak{B}(\mathcal{H})$ such that

(i) $E(\Delta)$ is an orthogonal projection for all $\Delta \in \Sigma$

(ii) $E(\Omega) = \mathbb{1}$

(iii) For all $x, y \in \mathcal{H}$, the function $E_{x,y} : \Sigma \to \mathbb{C}$ defined by

$$E_{x,y}(\Delta) = \langle x, E(\Delta)y \rangle$$

is a complex measure.

Theorem 5.2.5. *Let $E : \Sigma \to \mathfrak{B}(\mathcal{H})$ be a function such that $E(\Delta)$ is an orthogonal projection for all $\Delta \in \Sigma$ and $E(\Omega) = \mathbb{1}$. Then E is a projection-valued measure if and only if for any sequence $\{\Delta_n\}_{n\in\mathbb{N}}$ of pairwise disjoint elements of Σ,*

$$E\left(\bigcup_{n\in\mathbb{N}} \Delta_n\right) = \lim_{N\to\infty} \sum_{n<N} E(\Delta_n)$$

where the limit is taken in the strong operator topology on $\mathfrak{B}(\mathcal{H})$.

Proof. [13] □

Corollary 5.2.2. *Let $E : \Sigma \to \mathfrak{B}(\mathcal{H})$ be a projection-valued measure. If $\{\Delta_n\}_{n\in\mathbb{N}}$ is a sequence in Σ such that*

$$\Delta_1 \subseteq \Delta_2 \subseteq \cdots \Delta_n \subseteq \cdots \quad \textit{and} \quad \Delta = \bigcup_{n\in\mathbb{N}} \Delta_n$$

then $E(\Delta_n) \to E(\Delta)$ in the strong operator topology. Dually, if

$$\Delta_1 \supseteq \Delta_2 \supseteq \cdots \Delta_n \supseteq \cdots \quad \textit{and} \quad \Delta = \bigcap_{n\in\mathbb{N}} \Delta_n$$

then $E(\Delta_n) \to E(\Delta)$ in the strong operator topology.

Proof. [13] □

Theorem 5.2.6. *Let $E : \Sigma \to \mathfrak{B}(\mathcal{H})$ be a projection-valued measure and let $f : \Omega \to \mathbb{C}$ be a bounded measurable function. Then there exists a unique bounded operator $I(f) \in \mathfrak{B}(\mathcal{H})$ such that*

$$\langle x, I(f)y\rangle = \int_\Omega f dE_{x,y}$$

In particular,

$$I(\chi_\Delta) = E(\Delta)$$

Proof. [13] □

Definition 5.2.6. The operator $I(f)$ of the above theorem is denoted

$$I(f) = \int_\Omega f dE = \int_\Omega f(\lambda) dE(\lambda)$$

The primary use of projection-valued measures is the famous Spectral Theorem

Theorem 5.2.7 (Spectral Theorem)**.** *Let A be a self-adjoint operator on $\mathcal{H}$, and let Σ denote the Borel σ-algebra of $\sigma(A)$. Then there exists a unique projection-valued measure $E : \Sigma \to \mathfrak{B}(\mathcal{H})$ such that*

$$A = \int_{\sigma(A)} \lambda \; dE(\lambda) \tag{5.2.7}$$

5.2.3 The Riesz Projectors

If A is self-adjoint, then the resolvent of A is a holomorphic function $R(A) : \mathbb{C} \setminus \mathbb{R} \to \mathfrak{B}(\mathcal{H})$. Therefore, it is the representative for a hyperfunction $[(A-z)^{-1}] \in \mathcal{B}(\mathbb{R}; \mathfrak{B}(\mathcal{H}))$. What could this hyperfunction be? In the simplest case that $\mathcal{H} = \mathbb{C}$, a self-adjoint operator $A \in \mathfrak{B}(\mathbb{C})$ is a map $x \mapsto ax$ for some $a \in \mathbb{R}$ and $\sigma(A) = \{a\}$. Under the identification $\mathfrak{B}(\mathbb{C}) \cong \mathbb{C}$, the operator A is viewed as simply the number a, and so $[(a-z)^{-1}] = 2\pi i \; \delta(x-a)$. Then we expect the hyperfunction $[\frac{1}{2\pi i} R(A,z)]$ to be $\delta(x - A)$, whatever that should mean.

Definition 5.2.7. Let A be a self-adjoint closed linear operator on $\mathcal{H}$. Then we define a hyperfunction in $\mathcal{B}(\mathbb{R}; \mathfrak{B}(\mathcal{H}))$ by

$$\delta(x - A) = \left[\frac{1}{2\pi i}(A - z)^{-1}\right] \tag{5.2.8}$$

Note that $\operatorname{supp}\big(\delta(x - A)\big) = \sigma(A)$

Theorem 5.2.8. *Suppose $A \in \mathfrak{B}(\mathcal{H})$ and $A = A^*$. Then*

$$\int_{\mathbb{R}} \delta(x - A) dx = \mathbb{1} \tag{5.2.9}$$

Proof. Note that $\int_{\mathbb{R}} \delta(x-A)dx = \int_{\sigma(A)} \delta(x-A)dx$ since $\operatorname{supp}\big(\delta(x-A)\big) = \sigma(A)$. Fix any $N > \|A\|$ and define $\gamma : [0, 2\pi] \to \rho(A)$ by $\gamma(t) = Ne^{it}$ so that γ encircles $\sigma(A)$ once in the counterclockwise direction. Then by the definition of integrating a hyperfunction, we have

$$\begin{aligned}
\int_{\sigma(A)} \delta(x-A)dx &= -\oint_\gamma \frac{1}{2\pi i}(A-z)^{-1}dz \\
&= \frac{-1}{2\pi i}\int_0^{2\pi} iNe^{it}(A-Ne^{it})^{-1}dt \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(1-\frac{A}{Ne^{it}}\right)^{-1} dt \\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=0}^{\infty}\frac{A^n e^{-int}}{N^n}dt \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{A}{N}\right)^0 dt \\
&= \mathbb{1}
\end{aligned}$$

where we used that $\|\frac{A}{N}\| < 1$ and also the fact that $\int_0^{2\pi} e^{int}dt = \begin{cases} 2\pi & n = 0 \\ 0 & n \in \mathbb{Z}\setminus\{0\}\end{cases}$ □

We remark that Theorem (5.2.8) is the mathematically rigorous version of Dirac's formal expression $\int_{\mathbb{R}} |x\rangle\langle x| dx = \mathbb{1}$. This is sometimes called a *resolution of the identity*. Thus, we may intuitively view $\delta(x-A)$ as being $|x\rangle\langle x|$ *without* the integral sign. Furthermore, with the notation $\delta(x-A)$, it is obvious that equation (5.2.9) is an operator-valued generalization of the identity $\int_{\mathbb{R}} \delta(x-a)dx = 1$. Motivated by the more general equation $\int_{\mathbb{R}} f(x)\delta(x-a)dx = f(a)$, we are led to define the functional calculus.

Definition 5.2.8. Suppose $A \in \mathfrak{B}(\mathcal{H})$ is self-adjoint and $f \in \mathcal{A}(\sigma(A))$ is holomorphic on a neighborhood of $\sigma(A)$. Then $f(A) \in \mathfrak{B}(\mathcal{H})$ is defined to be

$$f(A) := \int_{\mathbb{R}} f(x)\delta(x-A)dx = \int_{\sigma(A)} f(x)\delta(x-A)dx = \frac{-1}{2\pi i}\oint_\gamma f(z)(A-z)^{-1}dz \tag{5.2.10}$$

where $\gamma \subseteq \rho(A)$ encloses $\sigma(A)$ once in the counterclockwise direction.

Theorem 5.2.9 (Riesz Functional Calculus)**.** *Let $A \in \mathfrak{B}(\mathcal{H})$. Then*

1. *The map $f \mapsto f(A)$ is an algebra homomorphism $\mathcal{A}(\sigma(A)) \to \mathfrak{B}(\mathcal{H})$.*
2. *If $f(z) = \sum_{n=0}^\infty c_n z^n$ has radius of convergence $R > \|A\|$, then $f(A) = \sum_{n=0}^\infty c_n A^n$*
3. *If f_n is holomorphic on $U \supseteq \sigma(A)$ and $f_n \to f$ uniformly on compact subsets of U, then $f_n(A) \to f(A)$ in the strong operator topology.*

Proof. Theorem 4.7 in [3], pg 201. □

Note the special case $A = \int x\ \delta(x-A)dx$, and compare this with the Spectral Theorem (Thm 5.2.7). Because of the uniqueness of E, we are led to conclude that $dE = \delta(x-A)dx$, or in other words the projection associated to an open interval (a, b) must be given by $E\big((a,b)\big) = \int_a^b \delta(x-A)dx$. This is essentially true, except there is some subtlety involving the endpoints. For one thing, integration of a hyperfunction has only been defined when the representative is holomorphic at the endpoints. Another concern is that $E\big([a,b]\big)$ can differ from $E\big((a,b)\big)$, so how would $E\big([a,b]\big)$ be defined in terms of $\delta(x-A)$? Before dealing with these technicalities, let's first see how $\int_a^b \delta(x-A)dx$ defines a projection when the endpoint difficulties are not present.

Definition 5.2.9. Suppose $A \in \mathfrak{B}(\mathcal{H})$ is self-adjoint and $a, b \in \rho(A) \cap \mathbb{R}$ with $a < b$. Then the **Riesz projection** of A associated to the open interval $\Omega = (a, b) \subseteq \mathbb{R}$ is the operator

$$P_\Omega = \int_a^b \delta(x - A)dx \tag{5.2.11}$$

For any bounded open set, write $\Omega = \coprod \Omega_i$ as a disjoint union of open intervals and define $P_\Omega := \sum P_{\Omega_i}$. Lastly, if Ω is unbounded, set $\Omega_A = \Omega \cap \{x \in \mathbb{R} : |x| < \|A\|\}$ and define $P_\Omega := P_{\Omega_A}$.

Theorem 5.2.10. *P_Ω is an orthogonal projection.*

Proof. Define $\gamma : [0, 2\pi] \to \rho(A)$ by $\gamma(t) = \frac{b+a}{2} + \frac{b-a}{2} e^{it}$ so that $P_\Omega = \frac{-1}{2\pi i} \oint_\gamma (A - z)^{-1} dz$. Notice also that we picked γ in such a way that $\overline{\gamma(t)} = \gamma(2\pi - t)$, meaning that $\overline{\gamma}$ traces out the same circle, but with the opposite orientation. Now we calculate P_Ω^* to be

$$\begin{aligned}
P_\Omega^* &= \left(\frac{-1}{2\pi i} \int_\gamma (A - z)^{-1} dz \right)^* \\
&= \frac{1}{2\pi i} \int_\gamma (A^* - \overline{z})^{-1} d\overline{z} \\
&= \frac{1}{2\pi i} \int_{\overline{\gamma}} (A - z)^{-1} dz \\
&= \frac{-1}{2\pi i} \int_\gamma (A - z)^{-1} dz \\
&= P_\Omega
\end{aligned}$$

Next, we prove that $P_\Omega^2 = P_\Omega$. In order to avoid the contours overlapping, we need to enlarge one slightly. For $\epsilon > 0$, define $\gamma_\epsilon(t) = \frac{b+a}{2} + \left(\frac{b-a}{2} + \epsilon\right) e^{it}$. Since $\rho(A)$ is open, we can take ϵ sufficiently small so that $\gamma_\epsilon(t) \in \rho(A)$ and $\int_{\gamma_\epsilon} (A - z)^{-1} dz = \int_\gamma (A - z)^{-1} dz$ by Cauchy's Theorem. Then we have

$$\begin{aligned}
P_\Omega^2 &= \int_{a-\epsilon}^{b+\epsilon} \left(\int_a^b \delta(x - A)dx \right) \delta(y - A)dy \\
&= \left(\frac{-1}{2\pi i} \right)^2 \int_{\gamma_\epsilon} \int_\gamma R(z)R(w)\ dz\ dw \\
&= \frac{-1}{4\pi^2} \int_{\gamma_\epsilon} \int_\gamma \frac{R(z) - R(w)}{z - w}\ dz\ dw \\
&= \frac{-1}{4\pi^2} \int_{\gamma_\epsilon} \int_\gamma \frac{R(z)}{z - w} - \frac{R(w)}{z - w}\ dz\ dw \\
&= \frac{-1}{4\pi^2} \left(\int_\gamma R(z) \int_{\gamma_\epsilon} \frac{dw}{z - w}\ dz - \int_{\gamma_\epsilon} R(w) \int_\gamma \frac{dz}{z - w}\ dw \right) \\
&= \frac{-1}{2\pi i} \int_\gamma R(z)dz \ = P_\Omega
\end{aligned}$$

In the last step, we used that γ lies entirely within γ_ϵ, which implies that

$$\int_{\gamma_\epsilon} \frac{dw}{z - w} = -2\pi i \qquad \text{and} \qquad \int_\gamma \frac{dz}{z - w} = 0$$

□

Remark 5.2.3. Instead of defining the modified curve γ_ϵ, it would be equivalent to allow the contours to intersect and evaluate $\int_\gamma \int_\gamma R(z)R(w)\ dz\ dw$ in the sense of a Cauchy principle value. Then we would have

$$\text{P.V.} \int_\gamma \frac{dw}{z - w} = -\pi i \qquad \text{and} \qquad \text{P.V.} \int_\gamma \frac{dz}{z - w} = \pi i$$

which gives the same result.

Lemma 5.2.1. *If $v \in \mathcal{H}$ is an eigenvector of A with eigenvalue λ, then*

$$\delta(x - A)v = \delta(x - \lambda)v \tag{5.2.12}$$

Proof. Suppose $Av = \lambda v$. Then $(A - z)v = (\lambda - z)v$, and it follows that $(A - z)^{-1}v = \frac{1}{\lambda - z}\, v$. Passing to the equivalence classes gives

$$\delta(x - A)v = \left[\frac{1}{2\pi i} R(A, z)v\right] = \left[\frac{1}{2\pi i(\lambda - z)}\ v\right] = \delta(x - \lambda)v$$

□

Theorem 5.2.11. *Suppose $\Omega \subseteq \mathbb{R}$ such that $\Omega \cap \sigma(A) = \{\lambda\}$ consists of a single isolated eigenvalue of A, and denote by $E(\lambda) = \ker(A - \lambda)$ the eigenspace of λ. Then $E(\lambda) \subseteq P_\Omega(\mathcal{H})$.*

Proof. If $v \in E(\lambda)$, then

$$P_\Omega\ v = \left(\int_\Omega \delta(x - A)\ dx\right) v = \int_\Omega \delta(x - A)v\ dx = \int_\Omega \delta(x - \lambda)v\ dx = v$$

□

Remark 5.2.4. If $\dim\big(E(\lambda)\big) < \infty$, then P_Ω equals the projection onto $E(\lambda)$ but this isn't necessarily true if $\dim\big(E(\lambda)\big) = \infty$, see §III.5 in [10].

Corollary 5.2.3. *If A is compact with spectrum $\sigma(A) = \{\lambda_n : n \in \mathbb{N}\}$ then*

$$\delta(x - A) = \sum_{n \in \mathbb{N}} \delta(x - \lambda_n)P_n \tag{5.2.13}$$

where P_n is the projection onto $\ker(A - \lambda_n)$

So far we have seen that $\int_\Omega \delta(x - A)dx$ gives the same projection as the projection-valued measure $E(\Omega) = \int_\Omega dE$ from the Spectral Theorem in cases where $A \in \mathfrak{B}(\mathcal{H})$ and a suitable closed contour in can be chosen lying entirely within $\rho(A)$ which encloses Ω. These constraints make definitions and calculations considerably more convenient, however they are not actually necessary. To illustrate this, let us do an example where such a contour is impossible.

Example 5.2.1. Let $\mathcal{H} = L^2(\mathbb{R})$ and A be the multiplication operator $f(x) \mapsto xf(x)$. Then $A \notin \mathfrak{B}(\mathcal{H})$ and $\sigma(A) = \mathbb{R}$. The associated projection-valued measure is given by $E(\Omega) = f(x) \mapsto \begin{cases} f(x) & x \in \Omega \\ 0 & x \notin \Omega. \end{cases}$

Now consider $\int_\Omega \delta(x - A)dx$. For simplicity, put $\Omega = (-1, 1)$ and define the contour $\gamma(t) = e^{it}$. In principle, the integral

$$\int_{-1}^{1} \delta(x - A)dx = \frac{-1}{2\pi i}\int_\gamma R(A, z)dz = \frac{-1}{2\pi}\int_0^{2\pi} e^{it}(A - e^{it})^{-1}dt$$

is not well-defined because $(A - e^{it})^{-1}$ fails to exist when $t \in \{0, \pi, 2\pi\}$. This problem goes away however when we actually evaluate on a vector $f \in \mathcal{H}$. We have $(A - z)(f(x)) = (x - z)f(x)$, so $R(A, z)f(x) = \frac{f(x)}{x - z}$. Then

$$P_\Omega f(x) = \left(\int_{-1}^{1} \delta(x - A)dx\right)(f(x)) = \frac{-1}{2\pi i}\int_\gamma R(A, z)(f(x))dz = \frac{-1}{2\pi i}\int_\gamma \frac{f(x)}{x - z}dz = \begin{cases} f(x) & |x| < 1 \\ 0 & |x| > 1 \\ ? & |x| = 1 \end{cases}$$

What is the value at $x = \pm 1$? The answer is: it doesn't matter! Since $\{-1, 1\}$ has measure zero, we may choose $P_\Omega f(\pm 1)$ to be any value we like; the result will still define the same vector in $\mathcal{H}$, namely

$P_\Omega f = E(\Omega)f$. The fact that $\sigma(A)$ is unbounded is also not a problem for a similar reason. For example, we could define $P_\mathbb{R}$ to be the locally finite sum

$$P_\mathbb{R} = \sum_{n\in\mathbb{Z}} P_{(n,n+1)}$$

and the fact that $P_\mathbb{R} f(x)$ is ambiguously defined for $x \in \mathbb{Z}$ is irrelevant because $\mathbb{Z} \subseteq \mathbb{R}$ has measure zero. Since every self adjoint operator is unitarily equivalent to such a multiplication operator, this example helps to convince us that results about bounded operators could be extended to the unbounded setting with enough care.

Another way to obtain the spectral measure is by means of a spectral family.

Definition 5.2.10. A **spectral family** is a family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ of orthogonal projections in $\mathfrak{B}(\mathcal{H})$ such that

(i) $E_\lambda E_\mu = E_{\min\{\lambda,\mu\}}$

(ii) $E_\lambda = \lim\limits_{\epsilon\to 0^+} E_{\lambda+\epsilon}$

(iii) $\lim\limits_{\lambda\to-\infty} E_\lambda = 0$

(iv) $\lim\limits_{\lambda\to\infty} E_\lambda = \mathbb{1}$

where all limits are taken with respect to the strong operator topology.

Given a spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, one can obtain a projection-valued measure as follows. First, define $E\big((a,b]\big) := E_b - E_a$, and then extend this to all Borel sets of $\mathbb{R}$ by countable unions and intersections. For example, $(a,b) = \cup_{n\in\mathbb{N}}(a, b-\frac{1}{n}]$, so $E\big((a,b)\big) = \lim_{\epsilon\to0^+} E_{b-\epsilon} - E_a$. Conversely, given a projection-valued measure $E : \Sigma_\mathbb{R} \to \mathfrak{B}(\mathcal{H})$, one obtains a spectral family by defining $E_\lambda := E\big((-\infty,\lambda]\big)$. In terms of hyperfunctions, this can be expressed by a vector-valued generalization of the Heavyside function.

Definition 5.2.11. Let $A \in \mathfrak{B}(\mathcal{H})$ be self-adjoint. Then for all $z \in \mathbb{C}\setminus\mathbb{R}$ the operator $\log(A-z)$ is well-defined by the holomorphic calculus, and we define

$$\theta(x-A) := \int_{-\infty}^{x} \delta(t-A)dt = \left[\frac{-1}{2\pi i}\log(A-z)\right] \tag{5.2.14}$$

Theorem 5.2.12. *Suppose $A \in \mathfrak{B}(\mathcal{H})$ is self-adjoint and $\lambda,\mu \in \rho(A)$. Then*

(i) $\theta(\lambda-A)\theta(\mu-A) = \theta(\min\{\lambda,\mu\}-A)$

(ii) $\theta(\lambda-A) = \lim\ _{\epsilon\to0^+}\theta(\lambda+\epsilon-A)$

(iii) $\lim\limits_{\lambda\to-\infty}\theta(\lambda-A) = 0$

(iv) $\lim\limits_{\lambda\to\infty}\theta(\lambda-A) = \mathbb{1}$

Proof. To prove (i), first note that if $\lambda = \mu$ then we have $\theta(\lambda-A)^2 = P^2_{(-\infty,\lambda)} = P_{(-\infty,\lambda)} = \theta(\lambda-A)$ by Theorem (5.2.10). Now assume that $\mu < \lambda$ and pick appropriate contours Λ, M so that

$$\theta(\lambda-A) = \frac{-1}{2\pi i}\int_\Lambda R(A,z)dz \qquad \text{and} \qquad \theta(\mu-A) = \frac{-1}{2\pi i}\int_M R(A,z)dz$$

For example, pick any $N > \|A\|$ and put $\Lambda(t) = \frac{\lambda-N}{2} + \frac{\lambda+N}{2}e^{it}$, and similarly $M(t) = \frac{\mu-N}{2} + \frac{\mu+N}{2}e^{it}$. Now we calculate

$$
\begin{aligned}
\theta(\lambda - A)\theta(\mu - A) &= \int_{-\infty}^{\lambda} \left(\int_{-\infty}^{\mu} \delta(x - A)dx \right) \delta(y - A)dy \\
&= \left(\frac{-1}{2\pi i} \right)^2 \int_{\Lambda} \int_{M} R(z)R(w) \; dz\, dw \\
&= \frac{-1}{4\pi^2} \int_{\Lambda} \int_{M} \frac{R(z) - R(w)}{z - w} \; dz\, dw \\
&= \frac{-1}{4\pi^2} \int_{\Lambda} \int_{M} \frac{R(z)}{z - w} - \frac{R(w)}{z - w} \; dz\, dw \\
&= \frac{-1}{4\pi^2} \left(\int_{M} R(z) \int_{\Lambda} \frac{dw}{z - w} \; dz - \int_{\Lambda} R(w) \int_{M} \frac{dz}{z - w} \; dw \right) \\
&= \frac{-1}{2\pi i} \int_{M} R(z)dz \; = \theta(\mu - A)
\end{aligned}
$$

In the last step, we used that $\mu < \lambda$, which implies that

$$
\int_{\Lambda} \frac{dw}{z - w} = -2\pi i \qquad \text{and} \qquad \int_{M} \frac{dz}{z - w} = 0
$$

On the other hand, if $\lambda < \mu$ then

$$
\int_{\Lambda} \frac{dw}{z - w} = 0 \qquad \text{and} \qquad \int_{M} \frac{dz}{z - w} = -2\pi i
$$

in which case it follows that $\theta(\lambda - A)\theta(\mu - A) = \theta(\lambda - A)$.

To prove (ii), recall that $\rho(A)$ is an open set and invoke Cauchy's Theorem. Part (iii) was proved in Theorem (5.2.8). Finally, to prove (iv) take $\lambda < -\|A\|$ and observe that $R(A, z)$ is holomorphic on $(-\infty, \lambda]$. Then the result follows by Cauchy's Theorem. □

This shows that the family $\{\theta(\lambda - A)\}_{\lambda \in \rho(A)}$ satisfies the definition of a spectral family when restricted to $\rho(A)$. This is essentially the best we could hope for because $\theta(\lambda - A)$ does not make sense as an element of $\mathfrak{B}(\mathcal{H})$ when $\lambda \in \sigma(A)$ for the same reason that $\theta(0)$ does not make sense as an element of $\mathbb{R}$ in the scalar-valued case. Nevertheless, we attempt to extend the family to all of $\mathbb{R}$.

Definition 5.2.12. For $A \in \mathfrak{B}(\mathcal{H})$ self-adjoint, we define a function

$$
\begin{aligned}
\overline{\theta} &: \mathbb{R} \to \mathfrak{B}(\mathcal{H}) \\
&\lambda \mapsto \lim_{\epsilon \to 0^+} \text{P.V.} \int_{-\infty}^{\lambda+\epsilon} \delta(x - A)dx
\end{aligned}
$$

where P.V. indicates that the contour integral $\int_{-\infty}^{\lambda+\epsilon} \delta(x - A)dx = \frac{-1}{2\pi i} \int_{\gamma} (A - z)^{-1} dz$ should be taken in the sense of a Cauchy principle value.

We conjecture that $\overline{\theta}$ has the following properties:

1. $\overline{\theta}$ is well-defined
2. $\overline{\theta}(\lambda) = \theta(\lambda - A)$ if $\lambda \in \rho(A)$
3. $\{\overline{\theta}(\lambda)\}_{\lambda \in \mathbb{R}}$ is a spectral family

5.3 Operator-valued Hyperfunctions in QFT

5.3.1 Whightman Axioms

In Whiteman's axiomatic formulation of quantum field theory, a field operator is a function

$$\phi : \mathcal{S}(\mathbb{R}^4) \to \mathcal{L}(\mathcal{H}) \tag{5.3.1}$$

which associates to every Schwartz function on space-time an unbounded operator on a Hilbert space in such a way that composing with the inner product

$$\begin{aligned} \mathcal{S}(\mathbb{R}^4) &\to \mathbb{C} \\ f &\mapsto \langle v \mid \phi(f) w \rangle \end{aligned}$$

defines a tempered distribution [15]. Since every distribution is a hyperfunction, it is possible to express the Whiteman axioms in the language of hyperfunctions.

Definition 5.3.1. Let $d \in \mathbb{N}$. A **Wightman hyperfunction quantum field theory** of space-time dimension $d+1$ consists of

(i) A Hilbert space $\mathcal{H}$ with a dense linear subspace $\mathcal{D} \subseteq \mathcal{H}$

(ii) A unit vector $|0\rangle \in \mathbb{S}(\mathcal{H}) \cap \mathcal{D}$ called the *vacuum vector*

(iii) A finite-dimension representation $\varrho : \widetilde{SO}^{\uparrow}(1,d) \to \mathrm{GL}_n(\mathbb{C})$ of the universal cover of the proper orthochronous Lorentz group $SO^{\uparrow}(1,d)$

(iv) A unitary representation $U : \widetilde{P^{\uparrow}_{+}}(d+1) \to \mathcal{U}(\mathcal{H})$, where $\widetilde{P^{\uparrow}_{+}}(d+1)$ denotes the universal cover of the proper orthochronous Poincaré group $P^{\uparrow}_{+}(d+1) \cong \mathbb{R}^{d+1} \rtimes SO^{\uparrow}(1,d)$

(v) A natural number $n \in \mathbb{N}$ and a collection $\{\phi^j(x)\}_{j \in n} \subseteq \mathcal{B}(\mathbb{R}^{d+1}, \mathcal{L}_{+}(\mathcal{D}))$ of operator-valued hyperfunctions called *field operators*

which satisfy the following axioms:

(W1) (Invariance of domain)

$$U(a,\Lambda)\mathcal{D} \subseteq \mathcal{D}, \quad \text{for all} \quad (a,\Lambda) \in \widetilde{P^{\uparrow}_{+}}(d+1)$$

(W2) (Transformation Law)

For all $(a,\Lambda) \in \widetilde{P^{\uparrow}_{+}}(d+1)$, the fields $\phi^j(x)$ transform via

$$U(a,\Lambda)\phi^j(x)U(a,\Lambda)^{-1} = \sum_{k \in n} \varrho(\Lambda^{-1})^j_k \, \phi^k(a + \pi(\Lambda)x)$$

where $\pi : \widetilde{SO}^{+}(1,d) \to SO^{+}(1,d)$ is the covering projection. The explicit description of the map π for $d = 3$ is detailed in Appendix A.3.

(W3) (Causality) For all $j, k \in n$ with $j \neq k$, the field operators satisfy either

$$(x-y)^2 < 0 \quad \Longrightarrow \quad [\phi^j(x), \phi^k(y)] = [\phi^j(x), \phi^j(y)^*] = 0 \tag{5.3.2}$$

in which case the fields are called *bosonic*, or else they satisfy

$$(x-y)^2 < 0 \quad \Longrightarrow \quad \{\phi^j(x), \phi^k(y)\} = \{\phi^j(x), \phi^j(y)^*\} = 0 \tag{5.3.3}$$

in which case the fields are called *fermionic*.

(W4) (Cyclicity of the Vacuum) Suppose $\psi \in \mathcal{H}$ is such that for all possible compositions $\phi^{j_1}(x_1)\cdots\phi^{j_m}(x_m)$ of the field operators and their adjoints, ψ satisfies the equation

$$\langle 0|\phi^{j_1}(x_1)\cdots\phi^{j_m}(x_m)|\psi\rangle = 0 \tag{5.3.4}$$

Then $\psi = 0$ (not $|0\rangle$) .

Remark 5.3.1. Note that the equalities in (5.3.2)-(5.3.4) are between different types of hyperfunctions. For example, $[\phi^j(x), \phi^k(y)]$ and $[\phi^j(x), \phi^j(y)^*]$ are both elements of $\mathcal{B}(\mathbb{R}^D \times \mathbb{R}^D, \mathcal{L}_+(\mathcal{D}))$, and (W3) says that their restrictions to the open set $S := \{(x,y) \in \mathbb{R}^D \times \mathbb{R}^D \mid (x-y)^2 < 0\}$ coincide with the zero hyperfunction $0 \in \mathcal{B}(S, \mathcal{L}_+(\mathcal{D}))$. On the other hand, equation (5.3.4) says that $\langle 0|\phi^{j_1}(x_1)\cdots\phi^{j_m}(x_m)|\psi\rangle = 0 \in \mathcal{B}((\mathbb{R}^D)^m)$.

5.3.2 The Free Scalar Field

The goal of this section is to describe the free scalar field in the language of hyperfunctions. To start, we introduce the following notation. Fix $d \in \mathbb{N}$ and let $D := d+1$. For $x, y \in \mathbb{R}^D$, denote $\langle x, y\rangle_M = x_0y_0 - x_1y_1 - \cdots - x_dy_d$ simply by xy and similarly $x^2 := \langle x, x\rangle_M$. We will also write $x = (x_0, \vec{x})$ so that $x^2 = x_0^2 - |\vec{x}|^2$. This notation also extends to complex numbers by linearity. Denote the cone of forward time-like vectors by

$$V^+ := \{x \in \mathbb{R}^D \mid x^2 > 0 \text{ and } x_0 > 0\}$$

Fix $m > 0$ and define

$$V_m^+ := \{x \in \mathbb{R}^D \mid x^2 = m^2 \text{ and } x_0 > 0\}$$

Let $\Omega = \Omega_m$ be the measure on V_m^+ defined by

$$\int_{V_m^+} f(p_0, \vec{p})\ d\Omega(p) = \int_{\mathbb{R}^d} \frac{f(\sqrt{|\vec{p}|^2 + m^2}, \vec{p})}{2\sqrt{|\vec{p}|^2 + m^2}}\ d\vec{p} \tag{5.3.5}$$

The measure $d\Omega$ is defined this way so that it is Lorentz invariant, meaning that $d\Omega(\Lambda p) = d\Omega(p)$ for all $\Lambda \in SO^\uparrow(1,d)$. Our 1 particle Hilbert space will be

$$\mathcal{H}_1 := L^2(V_m^+, d\Omega) \tag{5.3.6}$$

On this Hilbert space, there is a unitary representation of $\widetilde{P_+^\uparrow}(d+1)$

Theorem 5.3.1. *Define the function*

$$\begin{aligned} U : \widetilde{P_+^\uparrow}(d+1) &\to (\mathcal{H} \to \mathcal{H}) \\ (a, \Lambda) &\mapsto \big(\ \psi(p) \mapsto e^{ipa}\psi(\pi(\Lambda)^{-1}p)\ \big) \end{aligned}$$

Then U is a unitary representation.

Now we introduce an $\mathcal{H}_1$-valued holomorphic function which will be important for constructing the free field.

Definition 5.3.2. Define functions $E : \mathbb{R}^D + iV^+ \to \mathcal{H}_1$ and $E^* : \mathbb{R}^D - iV^+ \to \mathcal{H}_1$ by

$$\begin{aligned} E : \mathbb{R}^D + iV^+ &\to \mathcal{H}_1 \\ z &\mapsto \big(p \mapsto e^{ipz}\big) \end{aligned} \qquad\qquad \begin{aligned} E^* : \mathbb{R}^D - iV^+ &\to \mathcal{H}_1 \\ z &\mapsto \big(p \mapsto e^{-ipz}\big) \end{aligned}$$

Theorem 5.3.2. *$E \in \mathcal{O}(\mathbb{R}^D - iV^+, \mathcal{H}_1)$ and $E^* \in \mathcal{O}(\mathbb{R}^D - iV^+, \mathcal{H}_1)$*

Proof. First, we show the claim for E. Write $z = x + iy$, with $x \in \mathbb{R}^D$ and $y \in V^+$. Then $ipz = -py + ipx$. Since $y \in V^+$, it follows that $-py < 0$ for all $p \in V_m^+$. Hence

$$\begin{aligned}
\|E(z)\|^2 &= \int_{V_m^+} \left|e^{-py+ipx}\right|^2 \, d\Omega(p) \\
&= \int_{V_m^+} e^{-2py} \, d\Omega(p) \\
&= \int_{\mathbb{R}^d} \frac{e^{-2y_0\sqrt{|\vec{p}|^2+m^2}+\vec{p}\cdot\vec{y}}}{\sqrt{|\vec{p}|^2+m^2}} \, d\vec{p} \\
&\leq \frac{1}{m}\int_{\mathbb{R}^d} e^{-2y_0\sqrt{|\vec{p}|^2+m^2}+\vec{p}\cdot\vec{y}} \, d\vec{p} \; < \infty
\end{aligned}$$

The convergence in the last line is concluded by noting that the denominator of the integrand is bounded below by $\sqrt{0+m^2}$, while the numerator decays exponentially since $-py < 0$.

Now, let $\psi \in \mathcal{H}_1$ and $j \in \{0, \ldots, d\}$. Then we have

$$\begin{aligned}
\frac{\partial}{\partial \overline{z_j}}\langle \psi | E(z)\rangle &= \frac{\partial}{\partial \overline{z_j}} \int_{V_m^+} \overline{\psi(p)} e^{ipz} \, d\Omega(p) \\
&= \int_{V_m^+} \overline{\psi(p)} \left(\frac{\partial}{\partial \overline{z_j}} e^{ipz}\right) \, d\Omega(p) \; = 0
\end{aligned}$$

Therefore E is $\mathcal{H}_1$-holomorphic. The claim now also follows for E^* by noting that $E^*(z) = \overline{E(\overline{z})}$. $\square$

Next we'll define the Hilbert space for n particles. Define

$$\mathcal{H}_n := \mathrm{Sym}\left(\widehat{\bigotimes_{j=1}^{n}}\mathcal{H}_1\right) = \left\{\psi(p_1,\ldots,p_n) \in L^2\left((V_m^+)^n, \frac{d\Omega^n}{n!}\right) \middle| \psi(p_{\sigma(1)},\ldots,p_{\sigma(n)}) = \psi(p_1,\ldots,p_n) \text{ for all } \sigma \in S_n\right\} \tag{5.3.7}$$

In other words, an element $\psi \in \mathcal{H}_n$ is a function defined on $(V_m^+)^n$ which is unchanged by a permutation of the input variables and is square-summable with respect to the inner product

$$\langle \varphi|\psi\rangle_n := \frac{1}{n!}\int_{V_m^+}\cdots\int_{V_m^+} \overline{\varphi(p_1,\ldots,p_n)}\psi(p_1,\ldots,p_n) d\Omega(p_1)\cdots d\Omega(p_n) \tag{5.3.8}$$

For the special case $n = 0$, we define $\mathcal{H}_0 := \mathbb{C}$ with the standard inner product $\langle z_1|z_2\rangle_0 := \overline{z_1}z_2$. Finally, the entire Hilbert space will be given by

$$\mathcal{H} := \widehat{\bigoplus_{n=0}^{\infty}}\mathcal{H}_n \qquad\qquad \langle\varphi|\psi\rangle = \sum_{n=0}^{\infty}\langle\varphi_n|\psi_n\rangle_n \tag{5.3.9}$$

The unitary representation extends in a natural way to $\mathcal{H}_n$, and hence to all of $\mathcal{H}$

$$U(a,\Lambda)\psi_n(p_1,\ldots,p_n) = e^{i\sum_{j=1}^n p_j a}\psi_n(\pi(\Lambda)^{-1}p_1,\ldots,\pi(\Lambda)^{-1}p_n) \tag{5.3.10}$$

$$U(a,\Lambda)\big(\psi_0,\psi_1(p_1),\psi_2(p_1,p_2),\ldots\big) = \big(\psi_0 \;,\; e^{ip_1a}\psi_1(\pi(\Lambda)^{-1}p_1) \;,\; e^{i(p_1+p_2)a}\psi_2(\pi(\Lambda)^{-1}p_1 \;,\; \pi(\Lambda)^{-1}p_2) \;,\; \ldots\big) \tag{5.3.11}$$

For the dense subspace $\mathcal{D} \subseteq \mathcal{H}$, we will put

$$\mathcal{D} := \bigoplus_{n=0}^{\infty}\mathcal{H}_n = \big\{\psi = (\psi_n) \in \mathcal{H} \;\big|\; \text{all but finitely many } \psi_n \neq 0\big\} \tag{5.3.12}$$

and further, we set $|0\rangle := (1, 0, 0, \ldots) \in \mathcal{D}$. Next, we introduce the hyperfunction versions of the annihilation and creation operators. It will be convenient to recall the definition of the "partial bra-", reviewed in Appendix A.1.

Definition 5.3.3. Let $\mathcal{H}$ be any Hilbert space, and $\varphi \in \mathcal{H}$. For $n \in \mathbb{N}$, the **partial bra-** $\langle\phi|_n : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n-1}$ is defined on pure tensors by

$$\begin{aligned} \langle\varphi|_n : \mathcal{H}^{\otimes n} &\to \mathcal{H}^{\otimes n-1} \\ \psi_1 \otimes \cdots \otimes \psi_n &\mapsto \langle\varphi|\psi_1\rangle \psi_2 \otimes \cdots \otimes \psi_n \end{aligned}$$

and extended by linearity. The adjoint of this map, denoted by $|\varphi\rangle_n$, is called the **partial ket**. On pure tensors it is given by

$$\begin{aligned} |\varphi\rangle_n : \mathcal{H}^{\otimes n-1} &\to \mathcal{H}^{\otimes n} \\ \psi_1 \otimes \cdots \otimes \psi_{n-1} &\mapsto \varphi \otimes \psi_1 \otimes \cdots \otimes \psi_{n-1} \end{aligned}$$

and extended by linearity.

We will need to know how the map $\langle\varphi|_n$ behaves with respect to a change of basis.

Lemma 5.3.1. *Let $\mathcal{H}$ be any Hilbert space, $\varphi \in \mathcal{H}$, and $U \in \mathcal{U}(\mathcal{H})$ a unitary operator. Then the following diagram commutes*

$$\begin{tikzcd} \mathcal{H}^{\otimes n} \arrow[r, "\langle\varphi|_n"] \arrow[d, "U^{\otimes n}"'] & \mathcal{H}^{\otimes n-1} \arrow[d, "U^{\otimes n-1}"] \\ \mathcal{H}^{\otimes n} \arrow[r, "\langle U\varphi|_n"'] & \mathcal{H}^{\otimes n-1} \end{tikzcd}$$

By making the number n implicit, this can be written as

$$U \ \langle\varphi| \ U^* = \langle U\varphi| \tag{5.3.13}$$

Proof. Let $\psi = \psi_1 \otimes \cdots \psi_n \in \mathcal{H}^{\otimes n}$. Then we have

$$\begin{aligned} \langle U\varphi|_n U^{\otimes n} \ \psi &= \langle U\varphi|_n U\psi_1 \otimes \cdots \otimes U\psi_n \\ &= \langle U\varphi|U\psi_1\rangle U\psi_2 \otimes \cdots \otimes U\psi_n \\ &= \langle\varphi|\psi_1\rangle U\psi_2 \otimes \cdots \otimes U\psi_n \\ &= U^{\otimes n-1}\big(\ \langle\varphi|\psi_1\rangle\psi_2 \otimes \cdots \otimes \psi_n \ \big) \\ &= U^{\otimes n-1}\langle\varphi|_n \ \psi \end{aligned}$$

□

In our case, $\mathcal{H}_n$ does not equal $\mathcal{H}_1^{\otimes n}$, but rather the symmetric subspace $\mathrm{Sym}\left(\mathcal{H}_1^{\otimes n}\right)$. The symmetrized version of $|\varphi\rangle_n$ looks slightly different, so for clarity we will describe explicitly the maps $\langle\varphi|_n$ and $|\varphi\rangle_n$ for the specific Hilbert space $\mathcal{H}_n$ defined in equation (5.3.7).

Definition 5.3.4. For $\varphi \in \mathcal{H}_1$ and $n \geq 1$ define the map

$$\begin{aligned} \langle\varphi|_n : \mathcal{H}_n &\longrightarrow \mathcal{H}_{n-1} \\ \langle\varphi|_n \ \psi_n &:= \left((p_1, \ldots, p_{n-1}) \mapsto \int_{V_m^+} \overline{\varphi(p)} \psi_n(p, p_1, \ldots, p_{n-1}) d\Omega(p) \right) \end{aligned}$$

For the case $n = 0$, define $\langle\varphi|_0 : \mathcal{H}_0 \to \{0\}$ to be to zero map. Together, these extend to a map on all of $\mathcal{H}$ given by

$$\begin{aligned} \langle\varphi| : \mathcal{H} &\to \mathcal{H} \\ \big(\psi_0, \psi_1(p_1), \psi_2(p_1, p_2), \ldots\big) &\mapsto \big(\langle\varphi|_1\psi_1, \langle\varphi|_2\psi_2(p_1), \langle\varphi|_3\psi_3(p_1, p_2), \ldots\big) \end{aligned}$$

Although the maps $\langle\varphi|_n$ and $\langle\varphi|_m$ are technically different functions for $n \neq m$, the correct domain can always be determined by which element it is acting on. Therefore, if it is clear from context that $\varphi \in \mathcal{H}_1$ and $\psi_n \in \mathcal{H}_n$, we can drop the subscript and simplify the notation to

$$\langle\varphi|\psi = \langle\varphi|\big(\psi_0, \psi_1, \psi_2, \ldots\big) = \big(\ \langle\varphi|\psi_1 \ , \ \langle\varphi|\psi_2 \ , \ \ldots\big)$$

Definition 5.3.5. For all $\varphi \in \mathcal{H}_1$, the map $\langle\varphi|_n$ has an adjoint, the "partial ket" $|\varphi\rangle_n$, which for $n \geq 1$ is given by

$$|\varphi\rangle_n : \mathcal{H}_{n-1} \to \mathcal{H}_n \tag{5.3.14}$$

$$\psi_{n-1} \mapsto \left((p_1, \ldots, p_n) \mapsto \sum_{j=1}^{n} \varphi(p_j)\psi_{n-1}(p_1, \ldots, \check{p_j}, \ldots, p_n) \right) \tag{5.3.15}$$

where the notation $\check{p_j}$ indicates that the variable p_j is omitted.

Just as before, we can extend this to a map $\mathcal{H} \to \mathcal{H}$, and again we will avoid writing the subscript unless confusion would arise.

$$|\varphi\rangle : \mathcal{H} \to \mathcal{H}$$
$$\psi \mapsto |\varphi\rangle\psi = \big(0,\ |\varphi\rangle\psi_0\ ,\ |\varphi\rangle\psi_1\ ,\ \ldots\big)$$

Definition 5.3.6. Define a function $A : \mathbb{R}^D - iV^+ \to \mathcal{L}_+(\mathcal{D})$ by

$$A(z) : \mathcal{D} \to \mathcal{D}$$
$$\psi \mapsto \left\langle \frac{E(\overline{z})}{(2\pi)^{d/2}} \right| \ \psi$$

Explicitly, this means

$$(A(z)\psi)_n := \left((p_1, \ldots, p_n) \mapsto \frac{1}{(2\pi)^{d/2}} \int_{V_m^+} e^{-ipz}\psi_{n+1}(p, p_1, \ldots, p_n) d\Omega(p) \right) \tag{5.3.16}$$

Analogously, we define $A^* : \mathbb{R}^D + iV^+ \to \mathcal{L}_+(\mathcal{D})$ by

$$A^*(z) : \mathcal{D} \to \mathcal{D}$$
$$\psi \mapsto \left| \frac{E(z)}{(2\pi)^{d/2}} \right\rangle \ \psi$$

Explicitly,

$$(A^*(z)\psi)_n := \left((p_1, \ldots, p_n) \mapsto \frac{1}{(2\pi)^{d/2}} \sum_{j=1}^{n} e^{ip_j z}\psi(p_1, \ldots, \hat{p_j}, \ldots, p_n) \right) \tag{5.3.17}$$

Definition 5.3.7. Define the following hyperfunctions in $\mathcal{B}(\mathbb{R}^D, \mathcal{L}_+(\mathcal{D}))$

$$a(x) := A(x - iV^+\mathbf{0}) \tag{5.3.18}$$
$$a^*(x) := A^*(x + iV^+\mathbf{0}) \tag{5.3.19}$$
$$\phi(x) := a(x) + a^*(x) \tag{5.3.20}$$

We call $a(x)$ the **annihilation operator**, $a^*(x)$ is the **creation operator**, and $\phi(x)$ is the **field operator**.

To prove the main theorem of this section, we shall need the following facts about the complexification of $SO(1,d)$, whose elements satisfy the same defining property but are allowed to have entries in $\mathbb{C}$.

$$SO^+_{\mathbb{C}}(1,d) := \{\Lambda \in SL_D(\mathbb{C}) \mid (\Lambda z)(\Lambda w) = zw \text{ for all } z, w \in \mathbb{C}^D\} \tag{5.3.21}$$

Lemma 5.3.2. *Let $\varrho : \widetilde{SO}^+(1,d) \to GL_n(\mathbb{C})$ be a matrix representation, and suppose there exists a finite collection of holomorphic functions $\{f_j\} \subseteq \mathcal{O}((\mathbb{R}^D - iV^+)^n)$ satisfying the following transformation law:*

$$f_j(\pi(\Lambda)z_1, \cdots, \pi(\Lambda)z_n) = \sum_k \varrho(\Lambda)_{jk} f_k(z_1, \ldots, z_n) \qquad \textit{for all } \Lambda \in \widetilde{SO}^+(1,d) \tag{5.3.22}$$

Then each f_j can be analytically continued to the extended domain

$$D_n := \{(Az_1, \dots, Az_n) \mid z_j \in \mathbb{R}^D - iV^+, A \in SO^+_{\mathbb{C}}(1,d)\} \tag{5.3.23}$$

and this extension satisfies the same transformation law (5.3.22) for all $\Lambda \in \widetilde{SO^+_{\mathbb{C}}}(1,d)$

Proof. The proof can be found on pg. 66 of [15], but we sketch the main idea here. The covering group $\widetilde{SO}^+(1,d)$ can be represented faithfully by matrices. The first few are $\widetilde{SO}^+(1,1) \cong GL_1(\mathbb{R})$, $\widetilde{SO}^+(1,2) \cong SL_2(\mathbb{R})$, $\widetilde{SO}^+(1,3) \cong SL_2(\mathbb{C})$, and so on. A detailed derivation of the matrix representation of $\widetilde{SO}^+(p,q)$ can be found in [6]. Next, for each fixed $(z_1, \dots, z_n) \in (\mathbb{R}^D - iV^+)^n$, we can view the matrix $\Lambda \in \widetilde{SO}^+(1,d)$ as a function of $\frac{d(d-1)}{2}$ real parameters. Moreover, it is an *analytic* function of these parameters since the defining equation $(\Lambda z)(\Lambda w) = zw$ is a system of polynomials (with respect to the parameters of Λ while holding z_j, w_j fixed). Therefore, we can analytically extend the domain to allow for complex values and the Identity Theorem (Thm 1.4.4) ensures that the transformation law will remain true. Finally, the fact that the analytic extension is single-valued (without choosing a branch cut) follows from the fact that, by definition, $\widetilde{SO}^+(1,d)$ is connected and simply connected. □

Lemma 5.3.3. *A real point $(x_1, \dots, x_n) \in (\mathbb{R}^D)^N$ belongs to the extended domain (5.3.23) if and only if the following condition holds:*

$$\left(\sum_{j=1}^n \lambda_j x_j\right)^2 < 0 \qquad \text{for all } \lambda_j \geq 0 \text{ with } \sum_j \lambda_j > 0 \tag{5.3.24}$$

In other words, if and only if all convex combinations of the x_j space-like.

Proof. pg 71 of [15] □

In the case that $n = 1$, the above statements can be simplified dramatically.

Corollary 5.3.1. *Suppose $f \in \mathcal{O}(\mathbb{R}^D - iV^+)$ is such that*

$$f(\pi(\Lambda)z) = f(z) \qquad \text{for all } \Lambda \in \widetilde{SO}^+(1,d)$$

Then f can be analytically continued to the open set of space-like vectors $\{x \in \mathbb{R}^{1,d} \mid x^2 < 0\}$

Theorem 5.3.3. *The data $(\mathcal{H}, \mathcal{D}, |0\rangle, \phi(x), U, \varrho)$ as constructed in this section satisfies the Wightman axioms (Def 5.3.1), where $\varrho : \widetilde{SO(1,d)} \to GL_1(\mathbb{C})$ is the trivial representation.*

Proof. (W1) Suppose $\psi \in \mathcal{D}$, and $(a, \Lambda) \in \widetilde{P^\uparrow_+}(d+1)$. By definition of $\mathcal{D}$, there exists $N \in \mathbb{N}$ such that $\psi_n = 0$ for all $n \geq N$. Then $U(a,\Lambda)\psi_n = 0$ for all $n \geq N$ as well, hence $U(a,\Lambda)(\mathcal{D}) \subseteq \mathcal{D}$.

(W2) Let $(\lambda, \Lambda) \in \widetilde{P^\uparrow_+}(d+1)$, and $z \in \mathbb{R}^D - iV^+$. Then by (A.1.2) we have

$$U(\lambda,\Lambda)\ \langle E(z)|\ U^*(\lambda,\Lambda) = \langle U(\lambda,\Lambda)E(z)|$$

Using the definition of U and E and the fact that $(\Lambda x)(\Lambda y) = xy$ yields

$$\begin{aligned} U(\lambda,\Lambda)E(z) &= U(\lambda,\Lambda)e^{ipz} \\ &= e^{i\lambda p} e^{i(\pi(\Lambda)^{-1}p)z} \\ &= e^{i\lambda p} e^{ip(\pi(\Lambda)z)} \\ &= e^{ip(\lambda + \pi(\Lambda)z)} \\ &= E(\lambda + \pi(\Lambda)z) \end{aligned}$$

Thus, we have

$$U(\lambda,\Lambda)\ A(z)\ U^*(\lambda,\Lambda) = A(\lambda+\pi(\Lambda)z), \quad \text{for all} \ \ z\in\mathbb{R}^D - iV^+ \tag{5.3.25}$$

By taking adjoints of both sides, we obtain

$$U(\lambda,\Lambda)\ A^*(z)\ U^*(\lambda,\Lambda) = A^*(\lambda+\pi(\Lambda)z), \quad \text{for all} \ \ z\in\mathbb{R}^D + iV^+ \tag{5.3.26}$$

Passing to the boundary values, we conclude that

$$U(\lambda,\Lambda)\ \phi(x)\ U^*(\lambda,\Lambda) = \phi(\lambda+\pi(\Lambda)x) \tag{5.3.27}$$

(W3) First we note that since $n=1$ and $\phi(x)^* = \phi(x)$, (W3) simplifies to the implication

$$(x-y)^2 \quad \Longrightarrow \quad [\phi(x),\phi(y)] = 0 \tag{5.3.28}$$

To prove this, we expand using the definition of $\phi(x)$

$$[\phi(x),\phi(y)] = [a(x),a(y)] + [a^*(x),a(y)] + [a(x),a^*(y)] + [a^*(x),a^*(y)]$$

First we'll show that $[a(x),a(y)] = [a^*(x),a^*(y)] = 0$ (without any constraint on x and y). Let $\psi\in\mathcal{H}$ and $z,w\in\mathbb{R}^D - iV^+$, Then

$$\begin{aligned}
\big(A(z)A(w)\psi\big)_n(p_1,\dots,p_n) &= \frac{1}{(2\pi)^{d/2}}\int_{V_m^+} e^{-ipz}\big(A(w)\psi\big)_{n+1}(p\,,p_1,\dots,p_n)d\Omega(p) \\
&= \frac{1}{(2\pi)^{d}}\int_{V_m^+}\int_{V_m^+} e^{-i(pz+qw)}\psi_{n+2}(p,q,p_1,\dots,p_n)d\Omega(p)d\Omega(q) \\
&= \frac{1}{(2\pi)^{d}}\int_{V_m^+}\int_{V_m^+} e^{-i(qw+pz)}\psi_{n+2}(q,p,p_1,\dots,p_n)d\Omega(q)d\Omega(p) \\
&= \frac{1}{(2\pi)^{d/2}}\int_{V_m^+} e^{-iqw}\big(A(z)\psi\big)_{n+1}(q\,,p_1,\dots,p_n)d\Omega(q) \\
&= \big(A(w)A(z)\psi\big)_n(p_1,\dots,p_n)
\end{aligned}$$

where in the third line we exchanged the order of integration and used the fact that ψ_{n+2} is unchanged by a permutation of variables. Now that we know $[A(z),A(w)]=0$, we take the adjoint to obtain

$$0 = [A(z),A(w)]^* = [A(w)^*,A(z)^*] = [A^*(\overline{w}),A^*(\overline{z})]$$

Next, we calculate $[a(x),a^*(y)]$. Let $\psi\in\mathcal{H}$ and $z\in\mathbb{R}^D - iV^+$ be as before, but this time assume

$w \in \mathbb{R}^D + iV^+$. Then

$$
\begin{aligned}
\big(A(z)A^*(w)\psi\big)_n(p_1,\ldots,p_n) &= \frac{1}{(2\pi)^{d/2}} \int_{V_m^+} e^{-ipz} \big(A^*(w)\psi\big)_{n+1}(p,p_1,\ldots,p_n)d\Omega(p) \\
&= \frac{1}{(2\pi)^d} \int_{V_m^+} e^{-ipz} \left(e^{ipw}\psi_n(p_1,\ldots,p_n) + \sum_{j=1}^n e^{ip_jw}\psi_n(p,p_1,\ldots,\hat{p_j},\ldots,p_n) \right) d\Omega(p) \\
&= \frac{1}{(2\pi)^d} \int_{V_m^+} e^{-ip(z-w)}\psi_n(p_1,\ldots,p_n)d\Omega(p) \\
&\quad + \frac{1}{(2\pi)^d} \sum_{j=1}^n \int_{V_m^+} e^{-i(pz-p_jw)}\psi_n(p,p_1,\ldots,\hat{p_j},\ldots,p_n)d\Omega(p)
\end{aligned}
$$

$$
\begin{aligned}
\big(A^*(w)A(z)\psi\big)_n(p_1,\ldots,p_n) &= \frac{1}{(2\pi)^{d/2}} \sum_{j=1}^n e^{ip_jw} \left(A(z)\psi\right)_{n-1}(p_1,\ldots,\hat{p_j},\ldots,p_n) \\
&= \frac{1}{(2\pi)^d} \sum_{j=1}^n e^{ip_jw} \int_{V_m^+} e^{-ipz}\psi_n(p,p_1,\ldots,\hat{p_j},\ldots,p_n)d\Omega(p) \\
&= \frac{1}{(2\pi)^d} \sum_{j=1}^n \int_{V_m^+} e^{-i(pz-p_jw)}\psi_n(p,p_1,\ldots,\hat{p_j},\ldots,p_n)d\Omega(p)
\end{aligned}
$$

Subtracting the two, we see that

$$
\begin{aligned}
\left([A(z),A^*(w)]\psi\right)_n(p_1,\ldots,p_n) &= \frac{1}{(2\pi)^d} \int_{V_m^+} e^{-ip(z-w)}\psi_n(p_1,\ldots,p_n)d\Omega(p) \\
&= \frac{1}{(2\pi)^d} \int_{V_m^+} e^{-ip(z-w)}d\Omega(p) \quad \psi_n(p_1,\ldots,p_n)
\end{aligned}
$$

Therefore $[A(z),A^*(w)]$ is a scalar multiple of the identity

$$
[A(z),A^*(w)] = \frac{1}{(2\pi)^d} \int_{V_m^+} e^{-ip(z-w)}d\Omega(p) \ \cdot \ \mathbb{1} \tag{5.3.29}
$$

Now we will use Lemma (5.3.2) and (5.3.3) to show that $[A(z),A^*(w)]$ can be extended to real points (x,y) whenever $(x-y)^2 < 0$. To this end let $\varphi,\psi \in \mathcal{H}$ be fixed and define the following function

$$
\begin{aligned}
F : (\mathbb{R}^D - iV^+)^2 &\to \mathbb{C} \\
(z_1,z_2) &\mapsto \langle\varphi|\ [A(z_1),A^*(-z_2)]\ |\psi\rangle
\end{aligned}
$$

Then $F \in \mathcal{O}\big((\mathbb{R}^D - iV^+)^2\big)$ and for all $\Lambda \in SO^+(1,d)$ we have

$$
\begin{aligned}
F(\Lambda z_1,\Lambda z_2) &= \langle\varphi|\ \frac{1}{(2\pi)^d} \int_{V_m^+} e^{-ip(\Lambda z_1+\Lambda z_2)}d\Omega(p)\ |\psi\rangle \\
&= \langle\varphi|\ \frac{1}{(2\pi)^d} \int_{V_m^+} e^{-i(\Lambda^{-1}p)(z_1+z_2)}d\Omega(p)\ |\psi\rangle \\
&= \langle\varphi|\ \frac{1}{(2\pi)^d} \int_{V_m^+} e^{-ip(z_1+z_2)}d\Omega(p)\ |\psi\rangle \\
&= F(z_1,z_2)
\end{aligned}
$$

Then by Lemma (5.3.3) F can be analytically continued to a real point $(x,y) \in \mathbb{R}^D \times \mathbb{R}^D$ if $(x+y)^2 < 0$. This occurs precisely when $\big(\mathrm{Re}(z_1) - \mathrm{Re}(z_2)\big)^2 < 0$. Furthermore, since φ and ψ were arbitrary, this extension can be lifted to the level of operators. That is, the operator-valued hyperfunction $[a(x),a^*(y)]$

has a well-defined value whenever $(x-y)^2 < 0$, and this value is given by

$$[a(x), a^*(y)] = \lim_{\substack{\eta_1,\eta_2\to 0 \\ \eta_1,\eta_2\in V^+}} [A(x-i\eta_1), A^*(y+i\eta_2)] = \frac{1}{(2\pi)^d}\int_{V_m^+} e^{-ip(x-y)}\ d\Omega(p)\ \cdot \mathbb{1} \tag{5.3.30}$$

Now suppose $(x,y) \in \mathbb{R}^D \times \mathbb{R}^D$ such that $(x-y)^2 < 0$. First, assume that $x - y = (0, \vec{x} - \vec{y})$ has zero time component. Then using the substitution $p \mapsto -p$, we obtain (leaving $\cdot\mathbb{1}$ implicit)

$$\begin{aligned}
[a(x), a^*(y)] &= \frac{1}{(2\pi)^d}\int_{V_m^+} e^{-ip(x-y)}\ d\Omega(p) \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^3} \frac{e^{i\vec{p}\cdot(\vec{x}-\vec{y})}}{2\sqrt{|\vec{p}|^2+m^2}}\ d^3\vec{p} \\
&= \frac{1}{(2\pi)^d}\int_{-\mathbb{R}^3} \frac{e^{i(-\vec{p})\cdot(\vec{x}-\vec{y})}}{2\sqrt{|-\vec{p}|^2+m^2}}\ d^3\,(-\vec{p}) \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^3} \frac{e^{i\vec{p}\cdot(\vec{y}-\vec{x})}}{2\sqrt{|\vec{p}|^2+m^2}}\ d^3\vec{p} \\
&= \frac{1}{(2\pi)^d}\int_{V_m^+} e^{-ip(y-x)}\ d\Omega(p) =\ [a(y), a^*(x)]
\end{aligned}$$

Hence we have in this special case

$$0 = [a(x), a^*(y)] - [a(y), a^*(x)] = [a(x), a^*(y)] + [a^*(x), a(y)]$$

The general case then follows by Lorentz invariance since for a general pair (x, y) of space-like separated points there exists some $\Lambda \in SO(1,d)$ such that $\Lambda(x-y) = (0, \vec{v})$ has zero time component.

(W4) Suppose that $\psi = (\psi_n)_{n\in\mathbb{N}} \in \mathcal{H}$ has the property that

$$\langle 0|\phi(x_1)\cdots\phi(x_m)|\psi\rangle = 0 \qquad\qquad \forall M \in \mathbb{N} \tag{5.3.31}$$

In particular, by setting $M = 0$ we see that $\psi_0 = \langle 0|\psi\rangle = 0$. Next we will show that $\psi_M = 0$ by induction. Suppose that we have already shown that $\psi_k = 0$ for all $k < M$. Consider the expansion

$$\langle 0|\phi(x_1)\cdots\phi(x_M)|\psi\rangle = \langle 0|a(x_1)\cdots a(x_m)|\psi\rangle + \langle 0|a(x_1)a^*(x_2)\cdots a(x_M)|\psi\rangle + \cdots + \langle 0|a^*(x_1)\cdots a^*(x_M)|\psi\rangle$$

where the sum is taken over all the 2^M (non-commuting) monomials of length M in $a(x_j)$ and $a^*(x_k)$. By the induction hypothesis, any of the terms in which with the number of annihilation operators $a(x_j)$ is less than M vanish so that

$$\langle 0|\phi(x_1)\cdots\phi(x_M)|\psi\rangle = \langle 0|a(x_1)\cdots a(x_M)|\psi\rangle$$

Since ψ is assumed to satisfy (5.3.31), we know that $\langle 0|a(x_1)\cdots a(x_M)|\psi\rangle = 0$. This is a hyperfunction with only one term, so by Theorem 3.0.3 the holomorphic function $\langle 0|A(z_1)\cdots A(z_m)|\psi\rangle$ is identically equal to zero for all $z_j \in \mathbb{R}^D - iV^+$. To make plugging in the definitions more manageable, we recall the traditional shorthand $\omega_p := \sqrt{|\vec{p}|^2+m^2}$. With this notation, we have

$$\begin{aligned}
0 &= \langle 0|A(z_1)\cdots A(z_m)|\psi\rangle \\
&= \frac{1}{(2\pi)^{dM/2}}\int_{V_m^+}\cdots\int_{V_m^+} e^{-i\sum_{j=0}^M p_j z_j}\psi_M(p_1,\dots,p_M)d\Omega(p_1)\cdots d\Omega(p_M) \\
&= \frac{1}{(2\pi)^{dM/2}}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} e^{\sum_{j=0}^M\left(-i\omega_{p_j}(z_j)_0 + i\vec{p_j}\cdot\vec{z_j}\right)}\psi_M\Big((\omega_{p_1},\vec{p}_1),\cdots,(\omega_{p_M},\vec{p}_M)\Big)\left(\frac{d\vec{p}_1}{2\omega_{p_1}}\right)\cdots\left(\frac{d\vec{p}_M}{2\omega_{p_M}}\right) \\
&= \mathcal{F}_{dM}^{-1}\left[\frac{e^{-i\sum_{j=1}^M \omega_{p_j}(z_j)_0}\psi_M\Big((\omega_{p_1},\vec{p}_1),\cdots,(\omega_{p_M},\vec{p}_M)\Big)}{2^M\prod_{j=1}^M \omega_{p_j}}\right](\vec{z}_1,\dots,\vec{z}_M)
\end{aligned}$$

$$\Longrightarrow 0 = \psi_M(p_1,\dots,p_M)$$

where in the last step we applied $\mathcal{F}$ to both sides divided through by the non-zero terms. This completes the proof.

$\square$

5.3.3 Pauli-Jordan Commutator Hyperfunction as Equivalence Class

Due to Poincaré invariance, the commutator $[\phi(x), \phi(y)]$ depends only on the relative difference $x - y$:

$$[\phi(x), \phi(y)] = \frac{1}{(2\pi)^3} \int_{V_m^+} e^{-ip(x-y)} - e^{ip(x-y)} \; d\Omega(p) = -iD_m(x-y) \tag{5.3.32}$$

where $D_m(x)$ is called the **Pauli-Jordan commutator function** and given by

$$D_m(x) = \frac{1}{2\pi}\epsilon(x_0)\delta(x^2) - \frac{m}{4\pi\sqrt{x^2}}\epsilon(x_0)\theta(x^2)J_1(m\sqrt{x^2}) \tag{5.3.33}$$

where $\epsilon(x)$ is the sign function, $\delta(x)$ is the Dirac delta function, $\theta(x)$ is the Heavyside step function, $J_1(x)$ is the Bessel function (see appendix A.2), and $d = 3$ so that $x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$. The factor of $-i$ is a convention so that $D_m(x)$ is real-valued. Corresponding to $[a(x), a^*(y)]$ and $[a^*(x), a(y)]$, there are also the positive/negative frequency commutator functions

$$\begin{aligned}
[a(x), a^*(y)] &= \frac{1}{(2\pi)^3} \int_{V_m^+} e^{-ip(x-y)} \; d\Omega(p) = -iD_m^-(x-y) \\
[a^*(x), a(y)] &= \frac{-1}{(2\pi)^3} \int_{V_m^+} e^{ip(x-y)} \; d\Omega(p) = -iD_m^+(x-y)
\end{aligned}$$

which are defined to satisfy $D_m^+(x) + D_m^-(x) = D_m(x)$ can can be shown to equal [12],[1]

$$D_m^{\pm}(x) = \frac{1}{4\pi}\epsilon(x_0)\delta(x^2) \pm \frac{m\;\theta(x^2)\big(Y_1(m\sqrt{x^2}) \mp i\epsilon(x_0)J_1(m\sqrt{x^2})\big)}{8\pi i\sqrt{x^2}} \pm \frac{im\;\theta(-x^2)K_1(m\sqrt{-x^2})}{4\pi^2\sqrt{-x^2}} \tag{5.3.34}$$

See appendix A.2 for the definitions of $Y_1(x)$ and $K_1(x)$. We will see that the enormous expression (5.3.34) simplifies dramatically when expressed as a hyperfunction rather than as a distribution. In this section we will express $D_m(x)$ as a hyperfunction via the equivalence class definition. In the following section, we'll express $D_m(x)$ as a hyperfunction via the boundary value definition and show that the results are equivalent.

To start, let's first write $\epsilon(\epsilon_0)\delta(x^2)$ as a hyperfunction.

Lemma 5.3.4. *As a distribution, the action of $\epsilon(x_0)\delta(x^2)$ on a test function is*

$$\begin{aligned}
\epsilon(x_0)\delta(x^2) : \mathcal{S}(\mathbb{R}^4) &\to \mathbb{C} \\
f &\mapsto \int_{\mathbb{R}^3} \frac{f(|\vec{x}|, \vec{x}) - f(-|\vec{x}|, \vec{x})}{2|\vec{x}|} d^3\vec{x}
\end{aligned}$$

Proof. Recalling from (2.3.4) that $\delta(x^2 - a^2) = \frac{\delta(x-a)+\delta(x+a)}{2|a|}$, we have

$$\begin{aligned}
\int_{\mathbb{R}^4} \epsilon(x_0)\delta(x^2)f(x)d^4x &= \int_0^\infty \int_{\mathbb{R}^3} \delta(x^2)f(x)d^3\vec{x}\; dx_0 \;-\int_{-\infty}^0 \int_{\mathbb{R}^3} \delta(x^2)f(x)d^3\vec{x}\; dx_0 \\
&= \int_0^\infty \int_{\mathbb{R}^3} \frac{\delta(x_0 - |\vec{x}|) + \cancel{\delta(x_0 + |\vec{x}|)}}{2|x|} f(x)d^3\vec{x}\; dx_0 \;- \int_{-\infty}^0 \int_{\mathbb{R}^3} \frac{\cancel{\delta(x_0 - |\vec{x}|)} + \delta(x_0 + |\vec{x}|)}{2|x|} f(x)d^3\vec{x}\; dx_0 \\
&= \int_0^\infty \int_{\mathbb{R}^3} \frac{\delta(x_0 - |\vec{x}|)}{2|x|} f(x)d^3\vec{x}\; dx_0 \;- \int_{-\infty}^0 \int_{\mathbb{R}^3} \frac{\delta(x_0 + |\vec{x}|)}{2|x|} f(x)d^3\vec{x}\; dx_0 \\
&= \int_{\mathbb{R}^3} \frac{f(|\vec{x}|, \vec{x}) - f(-|\vec{x}|, \vec{x})}{2|\vec{x}|} d^3\vec{x}
\end{aligned}$$

$\square$

Proposition 5.3.1. *Considered as a hyperfunction, $\epsilon(x_0)\delta(x^2)$ is the boundary value of the holomorphic function of four complex variables*

$$F(z) = \left(\frac{1}{2\pi i}\right)^4 \int_{\mathbb{R}^3} \frac{d^3\vec{x}}{(|\vec{x}|^2 - z_0^2)(\vec{x} - \vec{z})} \tag{5.3.35}$$

where $z = (z_0, \vec{z}) = (z_0, z_1, z_2, z_3)$ and $(\vec{x} - \vec{z}) := (x_1 - z_1)(x_3 - z_3)(x_3 - z_3)$.

Proof. To find the representation as a hyperfunction, we need to plug in the Cauchy kernel $C(x)$ into the action on a Schwartz function, where

$$C(x) = C(x_0, \vec{x}) = \left(\frac{1}{2\pi i}\right)^4 \frac{1}{(x_0 - z_0)(\vec{x} - \vec{z})} := \left(\frac{1}{2\pi i}\right)^4 \frac{1}{(x_0 - z_0)(x_1 - z_1)(x_3 - z_3)(x_3 - z_3)} \tag{5.3.36}$$

We find that

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{C(|\vec{x}|, \vec{x}) - C(-|\vec{x}|, \vec{x})}{2|\vec{x}|} d^3\vec{x} &= \left(\frac{1}{2\pi i}\right)^4 \int_{\mathbb{R}^3} \frac{1}{2|\vec{x}|(\vec{x} - \vec{z})} \left(\frac{1}{|\vec{x}| - z_0} - \frac{1}{-|\vec{x}| - z_0}\right) d^3\vec{x} \\
&= \left(\frac{1}{2\pi i}\right)^4 \int_{\mathbb{R}^3} \frac{d^3\vec{x}}{(|\vec{x}|^2 - z_0^2)(\vec{x} - \vec{z})}
\end{aligned}$$

□

Next, we can apply the same process to write $\epsilon(x_0)\theta(x^2)$ as the boundary value of a holomorphic function.

Lemma 5.3.5. *As a distribution, the action of $\epsilon(x_0)\theta(x^2)$ on a test function is given by*

$$\begin{aligned}
\epsilon(x_0)\theta(x^2) : \mathcal{S}(\mathbb{R}^4) &\to \mathbb{C} \\
f &\mapsto \int_{\mathbb{R}^3} \int_{|\vec{x}|}^{\infty} f(x_0, \vec{x}) - f(-x_0, \vec{x}) \; dx_0 \; d^3\vec{x}
\end{aligned}$$

Proof. Using the fact that $\theta(x^2) = \theta(x_0 - |\vec{x}|) + \theta(-x_0 - |\vec{x}|)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^4} \epsilon(x_0)\theta(x^2)f(x) \; dx &= \int_{\mathbb{R}^4} \epsilon(x_0)\theta(x_0 - |\vec{x}|)f(x) \; dx + \int_{\mathbb{R}^4} \epsilon(x_0)\theta(-x_0 - |\vec{x}|)f(x) \; dx \\
&= \int_{\mathbb{R}^3} \int_{|\vec{x}|}^{\infty} \epsilon(x_0)f(x_0, \vec{x}) \; dx_0 \; d^3\vec{x} + \int_{\mathbb{R}^3} \int_{-\infty}^{-|\vec{x}|} \epsilon(x_0)f(x_0, \vec{x}) \; dx_0 \; d^3\vec{x} \\
&= \int_{\mathbb{R}^3} \int_{|\vec{x}|}^{\infty} f(x_0, \vec{x}) \; dx_0 \; d^3\vec{x} - \int_{\mathbb{R}^3} \int_{-\infty}^{-|\vec{x}|} f(x_0, \vec{x}) \; dx_0 \; d^3\vec{x} \\
&= \int_{\mathbb{R}^3} \int_{|\vec{x}|}^{\infty} f(x_0, \vec{x}) \; dx_0 \; d^3\vec{x} - \int_{\mathbb{R}^3} \int_{\infty}^{|\vec{x}|} f(-x_0, \vec{x}) \; d(-x_0) \; d^3\vec{x} \\
&= \int_{\mathbb{R}^3} \int_{|\vec{x}|}^{\infty} f(x_0, \vec{x}) - f(-x_0, \vec{x}) \; dx_0 \; d^3\vec{x}
\end{aligned}$$

□

Proposition 5.3.2. *As a hyperfunction, $\epsilon(x_0)\theta(x^2)$ is the boundary value of*

$$F(z) = \left(\frac{1}{2\pi i}\right)^4 \int_{\mathbb{R}^3} \frac{-\log(|\vec{x}|^2 - z_0^2)}{\vec{x} - \vec{z}} \; d^3\vec{x} \tag{5.3.37}$$

where $z = (z_0, \vec{z}) = (z_0, z_1, z_2, z_3)$ and $(\vec{x} - \vec{z}) := (x_1 - z_1)(x_3 - z_3)(x_3 - z_3)$.

Remark 5.3.2. The integral (5.3.37) does not converge in the ordinary sense. To make sense of this divergent integral, one must work "modulo analytic functions." It is, however, convenient to write $F(z)$ in this form because it is seen to be the standard defining function of $\frac{-1}{2\pi i}\log(|\vec{x}|^2 - z_0) \in \mathcal{A}(\mathbb{R}^3)$, c.f. Definition (2.2.5) and Corollary (2.2.1)

Proof. As before, the hyperfunction representation is given by applying the action on a test function to the Cauchy kernel $C(x)$ defined in equation (5.3.36). In this case, we obtain

$$\begin{aligned} F(z) &= \int_{\mathbb{R}^3} \int_{|\vec{x}|}^{\infty} C(x_0, \vec{x}) - C(-x_0, \vec{x}) \; dx_0 \; d^3\vec{x} \\ &= \left(\frac{1}{2\pi i}\right)^4 \int_{\mathbb{R}^3} \frac{1}{\vec{x} - \vec{z}} \int_{|\vec{x}|}^{\infty} \frac{1}{x_0 - z_0} + \frac{1}{x_0 + z_0} \; dx_0 \; d^3\vec{x} \end{aligned}$$

Notice that the inner integral does not converge so $F(z)$ is not a well-defined function. This reflects the fact that $\epsilon(x_0)\theta(x^2)$ does not have compact support. In order to remedy this, recall that a real analytic function does not contribute to the boundary value. Then the way to eliminate the divergence is by finding a sequence of compactly supported distributions which converge to $\epsilon(x_0)\theta(x^2)$ and discarding any real analytic terms first before taking the limit. To this end, write $\epsilon(x_0)\theta(x^2) = \lim_{N\to\infty} \epsilon(x_0)\theta(x^2)\theta(x_0 - N)$. For any $N > 0$, $\epsilon(x_0)\theta(x^2)\theta(x_0 - N)$ is given as the boundary value of the holomorphic function

$$\begin{aligned} F_N(z) &= \left(\frac{1}{2\pi i}\right)^4 \int_{\mathbb{R}^3} \frac{1}{\vec{x} - \vec{z}} \int_{|\vec{x}|}^{N} \frac{1}{x_0 - z_0} + \frac{1}{x_0 + z_0} \; dx_0 \; d^3\vec{x} \\ &= \left(\frac{1}{2\pi i}\right)^4 \int_{\mathbb{R}^3} \frac{1}{\vec{x} - \vec{z}} \left(\log(N - z_0) - \log(|\vec{x}| - z_0) + \log(N + z_0) - \log(|\vec{x}| + z_0)\right) \; d^3\vec{x} \\ &= \left(\frac{1}{2\pi i}\right)^4 \int_{\mathbb{R}^3} \frac{\log(N^2 - z_0^2) - \log(|\vec{x}|^2 - z_0^2)}{\vec{x} - \vec{z}} \; d^3\vec{x} \end{aligned}$$

In the last line, the sum rule for logarithms is justified since $\left|\arg(a - z_0) + \arg(a + z_0)\right| < \pi$, for any $a \in \mathbb{R}$. Next, note that $\log(N^2 - z_0^2)$ is real analytic on the open interval $(-N, N) \subseteq \mathbb{R}$. Then for any fixed $y \in \mathbb{R}$, we can guarantee that $\log(N^2 - z_0^2)$ is analytic at y by choosing $N > |y|$. In the limit $N \to \infty$, $\log(N^2 - z_0^2)$ does not contribute to the boundary value at any point, hence we may safely remove that summand. We conclude that $\epsilon(x_0)\theta(x^2)$ is the boundary value of

$$F(z) = \left(\frac{1}{2\pi i}\right)^4 \int_{\mathbb{R}^3} \frac{-\log(|\vec{x}|^2 - z_0^2)}{\vec{x} - \vec{z}} \; d^3\vec{x}$$

□

Remark 5.3.3. The above propositions express both $\epsilon(x_0)\delta(x^2)$ and $\epsilon(x_0)\theta(x^2)$ as boundary values of an integral of the form

$$\left(\frac{1}{2\pi i}\right)^3 \int_{\mathbb{R}^3} \frac{f_{z_0}(\vec{x})}{\vec{x} - \vec{z}} \; d^3\vec{x}$$

where $f_{z_0}(\vec{x}) \in \mathcal{A}(\mathbb{R}^3)$ for any fixed $z_0 \in \mathbb{C} \setminus \mathbb{R}$. Hence it is natural to view $\epsilon(x_0)\delta(x^2)$ and $\epsilon(x_0)\theta(x^2)$ as being hyperfunctions of the variable x_0, while considering $\vec{x}$ to be a parameter.

Corollary 5.3.2. *Let $\vec{x} \in \mathbb{R}^3$ be fixed. Considered as a hyperfunction of one variable x_0, $\epsilon(x_0)\delta(x^2)$ is represented by the equivalence class*

$$\epsilon(x_0)\delta(x^2) = \frac{-1}{2\pi i} \left[\frac{1}{z_0^2 - |\vec{x}|^2}\right] \tag{5.3.38}$$

and similarly $\epsilon(x_0)\theta(x^2)$ is represented by the equivalence class

$$\epsilon(x_0)\theta(x^2) = \frac{-1}{2\pi i} \left[\log(|\vec{x}|^2 - z_0^2)\right] \tag{5.3.39}$$

Theorem 5.3.4. *Let $\vec{x} \in \mathbb{R}^3$ be fixed, and consider the Pauli-Jordan commutator function $D_m(x_0, \vec{x})$ to be a hyperfunction of the variable x_0. Then $D_m(x_0, \vec{x})$ is given by the equivalence class*

$$D_m(x_0, \vec{x}) = \frac{-1}{4\pi^2 i} \left[\frac{1}{z_0^2 - |\vec{x}|^2} - \frac{mJ_1\left(m\sqrt{z_0^2 - |\vec{x}|^2}\right)\log(|\vec{x}|^2 - z_0^2)}{2\sqrt{z_0^2 - |\vec{x}|^2}}\right] \tag{5.3.40}$$

Proof. First, we use Cor. (5.3.2) to write

$$\begin{aligned}
D_m(x) &= \frac{1}{2\pi}\epsilon(x_0)\delta(x^2) - \frac{m}{4\pi\sqrt{x^2}}\epsilon(x_0)\theta(x^2)J_1(m\sqrt{x^2}) \\
&= \frac{-1}{4\pi^2 i}\left[\frac{1}{z_0^2-|\vec{x}|^2}\right] + \frac{mJ_1(m\sqrt{x^2})}{8\pi^2 i\sqrt{x^2}}\left[\log(|\vec{x}|^2 - z_0^2)\right]
\end{aligned}$$

Next, we use the fact that $\frac{J_1(\sqrt{x})}{\sqrt{x}}$ is real analytic on all of $\mathbb{R}$ with power series given by $\frac{J_1(\sqrt{x})}{\sqrt{x}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+1)!}\left(\frac{x}{2}\right)^n$. Recalling the $\mathcal{A}(\mathbb{R})$-module structure on $\mathcal{B}(\mathbb{R})$ (see Def. 2.3.2), we can pull this inside the brackets to obtain

$$\begin{aligned}
D_m(x) &= \frac{-1}{4\pi^2 i}\left[\frac{1}{z_0^2-|\vec{x}|^2}\right] + \frac{mJ_1(m\sqrt{x^2})}{8\pi^2 i\sqrt{x^2}}\left[\log(|\vec{x}|^2 - z_0^2)\right] \\
&= \left[\frac{-1}{4\pi^2 i}\frac{1}{z_0^2-|\vec{x}|^2} + \frac{mJ_1\left(m\sqrt{z_0^2-|\vec{x}|^2}\right)\log\left(|\vec{x}|^2-z_0^2\right)}{8\pi^2 i\sqrt{z_0^2-|\vec{x}|^2}}\right] \\
&= \frac{-1}{4\pi^2 i}\left[\frac{1}{z_0^2-|\vec{x}|^2} - \frac{mJ_1\left(m\sqrt{z_0^2-|\vec{x}|^2}\right)\log(|\vec{x}|^2-z_0^2)}{2\sqrt{z_0^2-|\vec{x}|^2}}\right]
\end{aligned}$$

□

Remark 5.3.4. In order to write $D_m(x)$ as a hyperfunction of 4 variables rather than treating $\vec{x}$ as a parameter, we are tempted to simply replace $z_0^2 - |\vec{x}|^2$ with $z^2 := z_0^2 - z_1^2 - z_2^2 - z_3^2$ so that

$$D_m(x) \;=\; \frac{-1}{4\pi^2 i}\left[\frac{1}{z^2} - \frac{mJ_1\left(m\sqrt{z^2}\right)\log(-z^2)}{2\sqrt{z^2}}\right] \tag{5.3.41}$$

This is correct *provided that* we remember the brackets in equation (5.3.40) and the brackets in equation (5.3.41) denote equivalence classes with respect to 2 different equivalence relations defined on 2 different sets. Furthermore, it does not suffice to replace the set of representatives $\mathcal{O}(\mathbb{C}\setminus\mathbb{R})$ with $\mathcal{O}((\mathbb{C}\setminus\mathbb{R})^4)$ because the function in (5.3.41) has singularities away from the real axis. For example $\frac{1}{z^2} = \frac{1}{z_0^2-z_1^2-z_2^2-z_3^2}$ is singular at $z = iy$ whenever $y \in \mathbb{R}^4$ lies on the light-cone, i.e. when $y^2 = 0$. In order to address this subtlety, it will be much more illuminating to use the boundary-value representation of hyperfunctions.

5.3.4 Pauli-Jordan Commutator Hyperfunction as Boundary Value

The goal of this section is to express $D_m(x) = \sum_j F_j(x + i\Gamma_j\mathbf{0})$ as the boundary value of holomorphic functions. To this end, we will use the explicit construction of $\phi(x)$ given in section 5.3.2 to calculate $[\phi(x),\phi(y)]$ directly.

Recall that

$$\begin{aligned}
[\phi(x),\phi(y)] &= [a(x),a^*(y)] - [a(y),a^*(x)] \\
&= [A(x - iV^+\mathbf{0}), A^*(y + iV^+\mathbf{0})] - [A(y - iV^+\mathbf{0}), A^*(x + iV^+\mathbf{0})]
\end{aligned}$$

Therefore it suffices to determine the function $\mathbf{D}(z-w) := [A(z), A^*(w)]$.

Definition 5.3.8. Define the holomorphic functions $\mathbf{D}^{\pm}(z) \in \mathcal{O}(\mathbb{R}^4 \pm iV^+)$ by

$$\mathbf{D}^+(x+iy) := \frac{1}{(2\pi)^3}\int_{V_m^+} e^{ip(x+iy)}d\Omega(p) \qquad \mathbf{D}^-(x-iy) := \frac{1}{(2\pi)^3}\int_{V_m^+} e^{-ip(x-iy)}d\Omega(p) \tag{5.3.42}$$

Then by equation (5.3.29), we can write $[\phi(x), \phi(y)]$ as a boundary value of

$$[A(z_1), A^*(z_2)] - [A(z_2), A^*(z_1)] = \frac{1}{(2\pi)^3} \int_{V_m^+} e^{-ip(z_1-z_2)} - e^{-ip(z_2-z_1)}\ d\Omega(p) \ \cdot \mathbb{1}$$
$$= (\mathbf{D}^-(z_1 - z_2) - \mathbf{D}^+(z_1 - z_2)) \ \cdot \mathbb{1}$$

In order to calculate the integrals $\mathbf{D}^\pm(z)$, we will need the Bessel/Hankel functions and some of their properties. See appendix (A.2) for a summary. Notice that $\mathbf{D}^-(z) = \overline{\mathbf{D}^+(\overline{z})}$, so we will focus our attention only on computing $\mathbf{D}^+(z)$.

Theorem 5.3.5.

$$\mathbf{D}^+(z) = \begin{cases} \frac{mH_1^{(1)}(m\sqrt{z^2})}{8\pi i\ \sqrt{z^2}}, & \sqrt{z^2} \in \mathbb{C}^+ \\ \\ \frac{-mH_1^{(2)}(m\sqrt{z^2})}{8\pi i\ \sqrt{z^2}}, & \sqrt{z^2} \in \mathbb{C}^- \end{cases} \tag{5.3.43}$$

Remark 5.3.5. In order for (5.3.43) to give the correct value when $\mathrm{Re}(z)$ is space-like, the value of $\sqrt{z^2}$ along the branch cut should be defined by $\sqrt{-x} := i\sqrt{x}$ when $x > 0$. Since $\mathbf{D}^-(z) = \overline{\mathbf{D}^+(\overline{z})}$, this implies that $\sqrt{-x} = -i\sqrt{x}$ in the formula for $\mathbf{D}^-(z)$. This convention could equivalently be expressed as

$$\mathbf{D}^+(z) = \begin{cases} \frac{mH_1^{(1)}(m\sqrt{z^2})}{8\pi i\ \sqrt{z^2}}, & z^2 \in \mathbb{C}^+ \\ \\ \frac{-mH_1^{(1)}(im\sqrt{-z^2})}{8\pi\ \sqrt{-z^2}}, & z^2 \in \mathbb{R}^- \\ \\ \frac{-mH_1^{(2)}(m\sqrt{z^2})}{8\pi i\ \sqrt{z^2}}, & z^2 \in \mathbb{C}^- \end{cases} \qquad \mathbf{D}^-(z) = \begin{cases} \frac{mH_1^{(1)}(m\sqrt{z^2})}{8\pi i\ \sqrt{z^2}}, & z^2 \in \mathbb{C}^+ \\ \\ \frac{+mH_1^{(2)}(-im\sqrt{-z^2})}{8\pi\ \sqrt{-z^2}}, & z^2 \in \mathbb{R}^- \\ \\ \frac{-mH_1^{(2)}(m\sqrt{z^2})}{8\pi i\ \sqrt{z^2}}, & z^2 \in \mathbb{C}^- \end{cases} \tag{5.3.44}$$

The case $z^2 \geq 0$ is impossible when $\mathrm{Im}(z)$ is time-like. In order to avoid such complicated piecewise expressions, let's determine when $\sqrt{z^2} \in \mathbb{C}^\pm$.

Lemma 5.3.6. *Suppose that* $z = x + iy \in \mathbb{R}^4 + iV^+$. *Then*

(i) $z^2 \in \mathbb{C}^+ \iff (\overline{z})^2 \in \mathbb{C}^-$

(ii) $z^2 \in \mathbb{R} \implies z^2 < 0$

(iii) $x^2 \geq 0 \implies \sqrt{z^2} \in \mathbb{C}^{\epsilon(x_0)}$

(iv) $x^2 < 0 \implies \lim_{y \to 0} \big(\mathbf{D}^+(z) - \mathbf{D}^-(\overline{z}) \big) = 0$

Proof. 1. Suppose $z^2 \in \mathbb{C}^+$. The equation $z^2 = (x^2 - y^2) + i(2xy)$ still holds true for the Minkowski product, so we know that $xy > 0$. Hence $(\overline{z})^2 = x^2 - (-y)^2 + 2ix(-y) = x^2 - y^2 - 2ixy \in \mathbb{C}^-$. The reverse implication follows from $\overline{\overline{z}} = z$.

2. Suppose $xy = 0$. We know that $y^2 > 0$, and also that the product of time-like vectors is non-zero. Therefore it must be that $x^2 \leq 0$ and thus $x^2 - y^2 < 0$.

3. First, consider the case when $x = 0$. With the square root defined on the branch cut as above, we have

$$\sqrt{z^2} = \sqrt{-y^2} = i\sqrt{y^2} \in \mathbb{C}^+$$

Now assume $x \neq 0$. Since $x^2 \geq 0$ and $y \in V^+$, we know that

$$\begin{aligned} x_0^2 - |\vec{x}|^2 &\geq 0 \\ |x_0| &\geq |\vec{x}| \\ \epsilon(x_0)x_0 &\geq |\vec{x}| \end{aligned} \qquad\qquad \begin{aligned} y_0^2 - |\vec{y}|^2 &> 0 \\ |y_0| &> |\vec{y}| \\ y_0 &> |\vec{y}| \end{aligned}$$

We also know by the Cauchy Schwartz inequality that $|\vec{x} \cdot \vec{y}| \leq |\vec{x}||\vec{y}|$. Combining these yields

$$|\vec{x} \cdot \vec{y}| < \epsilon(x_0) x_0 y_0$$
$$-\epsilon(x_0) x_0 y_0 < \vec{x} \cdot \vec{y} < \epsilon(x_0) x_0 y_0$$

If $x_0 < 0$, then the left inequality says that $x_0 y_0 < \vec{x} \cdot \vec{y}$ whereas if $x_0 > 0$ then the right inequality says that $\vec{x} \cdot \vec{y} < x_0 y_0$. Overall, we see that $\epsilon(xy) = \epsilon(x_0)$. Then $z^2 = x^2 - y^2 + i\epsilon(x_0)|2xy| \in \mathbb{C}^{\epsilon(x_0)}$, hence also $\sqrt{z^2} \in \mathbb{C}^{\epsilon(x_0)}$.

4. This is true because $\phi(x)$ satisfies Wightman axiom (W3), which was proved in Theorem 5.3.3.

$\square$

Remark 5.3.6. If x is space-like, then whether $\sqrt{z^2}$ lies in $\mathbb{C}^+$ or $\mathbb{C}^-$ will depend on y. So in the region $x^2 < 0$, whether $\mathbf{D}^+(z)$ is expressed in terms of $H_1^{(1)}(z)$ or in terms of $H_1^{(2)}(z)$ changes depending on both x and y. However, part (4) above tells us that we need not worry about this ambiguity because the difference of boundary values $\mathbf{D}^+ - \mathbf{D}^-$ will vanish regardless. Therefore for the purpose of determining $D_m(x)$, it suffices to only consider the restriction of $\mathbf{D}^+$ to the region $\mathrm{Re}(z) \in \overline{V} = \{x \in \mathbb{R}^4 \mid x^2 \geq 0\}$.

Lemma 5.3.7. *Let $L_+(\mathbb{C}) := SO^+_{\mathbb{C}}(1,3)$ be the complex Lorentz group defined in (5.3.21). Then we have*

$$\mathbf{D}^\pm(\Lambda z) = \mathbf{D}^\pm(z) \qquad \textit{for all } \Lambda \in L_+(\mathbb{C}) \tag{5.3.45}$$

Proof. This is a special case of Lemma (5.3.2) where ϱ is the trivial representation. $\square$

Lemma 5.3.8. *Suppose that $z \in \mathbb{R}^4 + iV^+$ Then there exists $\Lambda \in L_+(\mathbb{C})$ such that*

$$\Lambda z = (\sqrt{z^2}, 0, 0, 0) \tag{5.3.46}$$

Proof. We will use the Pauli vector $\sigma(z) \in M_2(\mathbb{C})$ and the covering map $\Pi : SL_2(\mathbb{C}) \otimes SL_2(\mathbb{C}) \to L_+(\mathbb{C})$ defined in the appendix (A.3.4) to find $\Lambda \in L_+(\mathbb{C})$ with the desired property. To this end, define

$$A := \sigma\left(\frac{z}{\sqrt{z^2}}\right)^{-1} = \sqrt{z^2}\,\sigma(z)^{-1} = \frac{1}{\sqrt{z^2}}\begin{pmatrix} z_0 - z_3 & -z_1 + iz_2 \\ -z_1 - iz_2 & z_0 + z_3 \end{pmatrix}$$

Remark 5.3.7. $\sigma(z)^{-1}$ is the matrix inverse of $\sigma(z)$, not to be confused with the map $\sigma^{-1} : M_2(\mathbb{C}) \to \mathbb{C}^4$.

This is well-defined since $\sqrt{z^2} \neq 0$, and it is immediate to check that $\det(A) = 1$, i.e. $A \in SL_2(\mathbb{C})$. Now, define $\Lambda := \Pi(A, I)$, where I is the 2×2 identity matrix. Then by definition of Π and σ, we have

$$\begin{aligned} \Lambda z &= \Pi(A, I) z \\ &= \sigma^{-1}\big(A\ \sigma(z)\ I\big) \\ &= \sigma^{-1}\left(\sqrt{z^2}\ \sigma(z)^{-1}\sigma(z)\right) \\ &= \sigma^{-1}\left(\sqrt{z^2}\ I\right) \\ &= (\sqrt{z^2}, 0, 0, 0) \end{aligned}$$

$\square$

Now we turn to proving Theorem 5.3.5.

Proof. Suppose $z \in \mathbb{R}^4 + iV^+$. By Lemma 5.3.7 and Lemma 5.3.8 we may assume without loss of generality that $z = (\sqrt{z^2}, 0, 0, 0)$. Let's define the shorthand $\lambda := \sqrt{z^2}$. Then the integral is

$$\mathbf{D}^+(z) = \frac{1}{(2\pi)^3}\int_{V_m^+} e^{ipz}\ d\Omega(p) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{e^{i\lambda\sqrt{|\vec{p}|^2+m^2}}}{2\sqrt{|\vec{p}|^2+m^2}}\ d^3\vec{p}$$

Next, we convert to spherical coordinates and integrate over the angle variables:

$$
\begin{aligned}
\mathbf{D}^{+}(z) &= \frac{1}{2(2\pi)^3} \int_0^{\infty} \int_0^{\pi} \int_0^{2\pi} \frac{\rho^2 \sin\varphi \; e^{i\lambda\sqrt{\rho^2+m^2}}}{\sqrt{\rho^2+m^2}} \; d\theta \; d\varphi \; d\rho \\
&= \frac{1}{(2\pi)^2} \int_0^{\infty} \frac{\rho^2 \; e^{i\lambda\sqrt{\rho^2+m^2}}}{\sqrt{\rho^2+m^2}} \; d\rho
\end{aligned}
$$

Now we'll make the substitution $\rho = m\sinh(t)$. Recalling the hyperbolic trig identities $\cosh^2 t - \sinh^2 = 1$, $\frac{d}{dt}\sinh t = \cosh t$, and $\frac{d}{dt}\cosh t = \sinh t$, our integral becomes

$$
\begin{aligned}
\mathbf{D}^{+}(z) &= \frac{1}{(2\pi)^2} \int_0^{\infty} \frac{\rho^2 \; e^{i\lambda\sqrt{\rho^2+m^2}}}{\sqrt{\rho^2+m^2}} \; d\rho \\
&= \frac{1}{(2\pi)^2} \int_0^{\infty} \frac{m^2\sinh^2 t \; e^{i\lambda\sqrt{m^2\sinh^2 t+m^2}}}{\sqrt{m^2\sinh^2 t+m^2}} \; d(m\sinh t) \\
&= \frac{m^2}{(2\pi)^2} \int_0^{\infty} \frac{\sinh^2 t \quad e^{i\lambda \; m\cosh t}}{\cancel{m\cosh t}} \; (\cancel{m\cosh t} \; dt) \\
&= \frac{m^2}{(2\pi)^2} \int_0^{\infty} (\cosh^2 - 1) \; e^{im\lambda\cosh t} \; dt
\end{aligned}
$$

In order to evaluate this, we'll use the integral representation (A.2.14) of the Hankel functions. For $\nu = 0$ this is

$$
\int_{-\infty}^{\infty} e^{i\zeta\cosh t} dt = \begin{cases} \pi i \; H_0^{(1)}(\zeta) \;, & \zeta \in \mathbb{C}^+ \\ \\ -\pi i \; H_0^{(2)}(\zeta) \;, & \zeta \in \mathbb{C}^- \end{cases} \tag{5.3.47}
$$

In order to write this piecewise expression in one line, let's denote $\sigma := \epsilon(\mathrm{Im}(\zeta))$ and relabel the superscripts in the Hankel functions $H_\nu^1 := H_\nu^{(1)}$, $H_\nu^{-1} := H_\nu^{(2)}$. With this notation, (5.3.47) reads

$$
\int_{-\infty}^{\infty} e^{i\zeta\cosh t} dt = \pi i^{\sigma} H_0^{\sigma}(\zeta) \tag{5.3.48}
$$

To make the integrands match, define $\zeta := m\lambda$ and notice that

$$
\begin{aligned}
\frac{\partial^2}{\partial\zeta^2} \; e^{i\zeta\cosh t} &= (i\cosh t)^2 e^{i\zeta\cosh t} \\
\left(-\frac{\partial^2}{\partial\zeta^2} - 1\right) e^{i\zeta\cosh t} &= (\cosh^2 - 1) \; e^{i\zeta\cosh t}
\end{aligned}
$$

Plugging this in above and using the fact that $\cosh(-t) = \cosh(t)$, we obtain

$$
\begin{aligned}
\mathbf{D}^{+}(z) &= \frac{m^2}{(2\pi)^2} \int_0^{\infty} (\cosh^2 - 1) \; e^{im\lambda\cosh t} \; dt \\
&= \frac{m^2}{(2\pi)^2} \int_0^{\infty} \left(-\frac{\partial^2}{\partial\zeta^2} - 1\right) e^{i\zeta\cosh t} \; dt \\
&= \frac{-m^2}{(2\pi)^2} \left(\frac{\partial^2}{\partial\zeta^2} + 1\right) \int_0^{\infty} e^{i\zeta\cosh t} \; dt \\
&= \frac{-m^2}{(2\pi)^2} \left(\frac{\partial^2}{\partial\zeta^2} + 1\right) \; \frac{1}{2} \int_{-\infty}^{\infty} e^{i\zeta\cosh t} \; dt \\
&= \frac{m^2}{8\pi i^{\sigma}} \left(\frac{\partial^2}{\partial\zeta^2} + 1\right) \; H_0^{\sigma}(\zeta)
\end{aligned}
$$

We can now apply the derivative relations (A.2.10) and (A.2.11) to calculate that for both $\sigma = \pm 1$ we have

$$\begin{aligned}
\frac{\partial}{\partial\zeta} H_0^\sigma(\zeta) &= -H_1^\sigma(\zeta) \\
\frac{\partial^2}{\partial\zeta^2} H_0^\sigma(\zeta) &= \frac{H_1^\sigma(\zeta)}{\zeta} - H_0^\sigma(\zeta) \\
\left(\frac{\partial^2}{\partial\zeta^2} + 1\right) \ H_0^\sigma(\zeta) &= \frac{H_1^\sigma(\zeta)}{\zeta}
\end{aligned}$$

Putting it all together, we finally arrive at (5.3.5):

$$\begin{aligned}
\mathbf{D}^+(z) &= \frac{m^2}{8\pi i^\sigma} \left(\frac{\partial^2}{\partial\zeta^2} + 1\right) \ H_0^\sigma(\zeta) \\
&= \frac{m^2}{8\pi i^\sigma} \ \frac{H_1^\sigma(\zeta)}{\zeta} \\
&:= \begin{cases} \frac{mH_1^{(1)}(m\sqrt{z^2})}{8\pi i \ \sqrt{z^2}} , & \sqrt{z^2} \in \mathbb{C}^+ \\ \\ \frac{-mH_1^{(2)}(m\sqrt{z^2})}{8\pi i \ \sqrt{z^2}} , & \sqrt{z^2} \in \mathbb{C}^- \end{cases}
\end{aligned}$$

By taking into account the special case $\sqrt{z^2} \in i\mathbb{R}^+$ (i.e. when $z^2 \in \mathbb{R}^-$), we obtain

$$\mathbf{D}^+(z) = \begin{cases} \frac{mH_1^{(1)}(m\sqrt{z^2})}{8\pi i \ \sqrt{z^2}} , & z^2 \in \mathbb{C}^+ \\ \\ \frac{-mH_1^{(1)}(im\sqrt{-z^2})}{8\pi \ \sqrt{-z^2}} , & z^2 \in \mathbb{R}^- \\ \\ \frac{-mH_1^{(2)}(m\sqrt{z^2})}{8\pi i \ \sqrt{z^2}} , & z^2 \in \mathbb{C}^- \end{cases} \tag{5.3.49}$$

□

Corollary 5.3.3. *Let $z \in \mathbb{R}^4 - iV^+$. Then*

$$\mathbf{D}^-(z) = \begin{cases} \frac{mH_1^{(1)}(m\sqrt{z^2})}{8\pi i \ \sqrt{z^2}} , & z^2 \in \mathbb{C}^+ \\ \\ \frac{+mH_1^{(2)}(-im\sqrt{-z^2})}{8\pi \ \sqrt{-z^2}} , & z^2 \in \mathbb{R}^- \\ \\ \frac{-mH_1^{(2)}(m\sqrt{z^2})}{8\pi i \ \sqrt{z^2}} , & z^2 \in \mathbb{C}^- \end{cases} \tag{5.3.50}$$

Proof. Use the fact that $\mathbf{D}^-(z) = \overline{\mathbf{D}^+(\overline{z})}$. The result follows by expressing $\overline{\mathbf{D}^+(\overline{z})}$ via (5.3.49) and applying Lemma 5.3.6 (i) as well as the reflection formulae for the Hankel functions (A.2.7)-(A.2.8). □

Subtracting the two, we conclude that the boundary value representation of $D_m(x)$ is given by

$$\mathbf{D}^+\big(x + iV^+\mathbf{0}\big) - \mathbf{D}^-\big(x - iV^+\mathbf{0}\big) = iD_m(x) \tag{5.3.51}$$

where the extra factor of i stems from the convention $[\phi(x), \phi(y)] = -iD_m(x-y)$.

Remark 5.3.8. Note that the piecewise expression (5.3.49) displays how $\mathbf{D}^+(z)$ changes behavior depending on the imaginary part of z^2. In combination with Lemma 5.3.6, we see that $\mathrm{Im}((x+iy)^2)$ is determined by the signs of x_0 and x^2. This dependence is also evident in the formulation as a distribution (5.3.34) by virtue of the factors of $\epsilon(x_0)$, $\theta(x^2)$, and $\theta(-x^2)$.

Although the piecewise expression (5.3.49) gives insight into the behavior of $\mathbf{D}^+(x+iV^+)$ for the case when x is time-like versus when x is space-like, it is also rather unwieldy. Next, we will show that the formula for $\mathbf{D}^+(z)$ can be simplified dramatically using identities between the Bessel functions. Intuitively, the reason we need piecewise expressions arises because the Hankel functions have a branch cut along the negative real axis. However, the complex square root $\sqrt{z^2}$ also involves a branch cut along the negative real axis. So by composing them, the need for a piecewise expression "cancels out," so to speak. Of course, this intuitive argument needs to be made precise.

Theorem 5.3.6. *Let $U = \{z \in \mathbb{C}^4 \mid z^2 \notin \mathbb{R}_{\geq 0}\}$ and define a the holomorphic function $\mathbf{D} \in \mathcal{O}(U)$ by*

$$\mathbf{D}(z) := \frac{m}{4\pi^2} \frac{K_1(m\sqrt{-z^2})}{\sqrt{-z^2}} \tag{5.3.52}$$

Then $\mathbf{D}^+(z)$ and $\mathbf{D}^-(z)$ are equal to the restrictions of $\mathbf{D}$

$$\mathbf{D}^+ = \mathbf{D}\big|_{V^+} \qquad\qquad \mathbf{D}^- = \mathbf{D}\big|_{V^-}$$

Lemma 5.3.9. *Let $z \in \mathbb{C}\setminus\{0\}$, and let $\sqrt{z} := e^{\frac{1}{2}\log(z)}$ be defined using the principal branch of $\log(z)$. Then*

$$\log(z) = \begin{cases} \log(-z) + \pi i \ , & z \in \mathbb{C}^+ \cup \mathbb{R}^- \\ \log(-z) - \pi i \ , & z \in \mathbb{C}^- \cup \mathbb{R}^+ \end{cases} \tag{5.3.53}$$

Consequently, $\sqrt{z}$ and $\sqrt{-z}$ are related by

$$\sqrt{z} = \begin{cases} i\sqrt{-z} \ , & z \in \mathbb{C}^+ \cup \mathbb{R}^- \\ -i\sqrt{-z} \ , & z \in \mathbb{C}^- \cup \mathbb{R}^+ \end{cases} \tag{5.3.54}$$

Proof. Write $z = re^{i\theta}$, where $r > 0$ and $\theta = \arg(z) \in (-\pi, \pi]$. Then for any $n \in \mathbb{Z}$ we have

$$-z = -re^{i\theta} = re^{i(\theta+\pi+2\pi n)}$$

In order to make sure that the exponent fall in the necessary range $(-\pi,\pi]$, we must set $n = \begin{cases} -1 \ , & \theta \in (0,\pi] \\ 0 \ , & \theta \in (-\pi, 0] \end{cases}$

In other words, we can write $-z$ as

$$-z = re^{i\arg(-z)} = \begin{cases} re^{i(\theta-\pi)} \ , & z \in \mathbb{C}^+ \cup \mathbb{R}^- \\ re^{i(\theta+\pi)} \ , & z \in \mathbb{C}^- \cup \mathbb{R}^+ \end{cases}$$

Applying the principal logarithm, we see that

$$\begin{aligned} \log(z) - \log(-z) &= \log\left(re^{i\arg(z)}\right) - \log\left(re^{i\arg(-z)}\right) \\ &= \cancel{\log(r)} + i\arg(z) - \cancel{\log(r)} - i\arg(-z) \\ &= i\theta - \begin{cases} i(\theta-\pi) \ , & z \in \mathbb{C}^+ \cup \mathbb{R}^- \\ i(\theta+\pi) \ , & z \in \mathbb{C}^- \cup \mathbb{R}^+ \end{cases} \\ &= \begin{cases} \pi i \ , & z \in \mathbb{C}^+ \cup \mathbb{R}^- \\ -\pi i \ , & z \in \mathbb{C}^- \cup \mathbb{R}^+ \end{cases} \end{aligned}$$

which proves (5.3.53). Plugging this into the definition of $\sqrt{z}$, we immediately arrive at (5.3.54):

$$\begin{aligned} \sqrt{z} :&= e^{\frac{1}{2}\log(z)} \\ &= \begin{cases} e^{(\log(-z)+\pi i)/2} \ , & z \in \mathbb{C}^+ \cup \mathbb{R}^- \\ e^{(\log(-z)-\pi i)/2} \ , & z \in \mathbb{C}^- \cup \mathbb{R}^+ \end{cases} \\ &= \begin{cases} i\sqrt{-z} \ , & z \in \mathbb{C}^+ \cup \mathbb{R}^- \\ -i\sqrt{-z} \ , & z \in \mathbb{C}^- \cup \mathbb{R}^+ \end{cases} \end{aligned}$$

□

Now we can prove the theorem.

Proof. Suppose $\zeta \in \mathbb{C} \setminus \{0\}$ and recall the identities (A.2.21)-(A.2.22)

$$K_1(\zeta) = \begin{cases} -\frac{\pi}{2} H_1^{(1)}(i\zeta) \ , & \arg(\zeta) \in [-\pi, \frac{\pi}{2}] \\ -\frac{\pi}{2} H_1^{(2)}(-i\zeta) \ , & \arg(\zeta) \in [-\frac{\pi}{2}, \pi] \end{cases}$$

Plugging in $\sqrt{\zeta}$ gives

$$K_1\left(\sqrt{\zeta}\right) = \begin{cases} -\frac{\pi}{2} H_1^{(1)}\left(i\sqrt{\zeta}\right) \ , & \arg(\sqrt{\zeta}) \in [-\pi, \frac{\pi}{2}] \\ -\frac{\pi}{2} H_1^{(2)}\left(-i\sqrt{\zeta}\right) \ , & \arg(\sqrt{\zeta}) \in [-\frac{\pi}{2}, \pi] \end{cases}$$

Here, we observe that the conditions on the right-hand side are *always* satisfied. This is because $\arg(\zeta) \in (-\pi, \pi]$, hence $\arg(\sqrt{\zeta}) = \frac{1}{2}\arg(\zeta) \in (-\frac{\pi}{2}, \frac{\pi}{2}]$. So in fact we have the general identity

$$K_1\left(\sqrt{\zeta}\right) \ = \ -\frac{\pi}{2} H_1^{(1)}\left(i\sqrt{\zeta}\right) \ = \ -\frac{\pi}{2} H_1^{(2)}\left(-i\sqrt{\zeta}\right) \qquad , \quad \zeta \neq 0 \tag{5.3.55}$$

Now let $z \in \mathbb{R}^4 + iV^+$. Then $z^2 \neq 0$ by Lemma 5.3.6, so we may substitute $\zeta = -m^2 z^2$ which gives

$$K_1\left(m\sqrt{-z^2}\right) \ = \ -\frac{\pi}{2} H_1^{(1)}\left(im\sqrt{-z^2}\right) \ = \ -\frac{\pi}{2} H_1^{(2)}\left(-im\sqrt{-z^2}\right)$$

Finally, we apply Lemma 5.3.9 to re-write $\frac{m}{4\pi^2} \frac{K_1\left(m\sqrt{-z^2}\right)}{\sqrt{-z^2}}$ in terms of $\sqrt{z^2}$:

$$\begin{aligned} \frac{m}{4\pi^2} \ \frac{K_1\left(m\sqrt{-z^2}\right)}{\sqrt{-z^2}} &= \frac{m}{4\pi^2} \begin{cases} -\frac{\pi}{2} \ \frac{H_1^{(1)}(m\sqrt{z^2})}{-i\sqrt{z^2}} \ , & z^2 \in \mathbb{C}^+ \cup \mathbb{R}^- \\ -\frac{\pi}{2} \ \frac{H_1^{(2)}(m\sqrt{z^2})}{i\sqrt{z^2}} \ , & z^2 \in \mathbb{C}^- \cup \cancel{\mathbb{R}^+} \end{cases} \\ &= \begin{cases} \frac{m}{8\pi i} \ \frac{H_1^{(1)}(m\sqrt{z^2})}{\sqrt{z^2}} \ , & \sqrt{z^2} \in \mathbb{C}^+ \\ \frac{-m}{8\pi i} \ \frac{H_1^{(2)}(m\sqrt{z^2})}{\sqrt{z^2}} \ , & \sqrt{z^2} \in \mathbb{C}^- \end{cases} \\ &= \mathbf{D}^+(z) \end{aligned}$$

Note that in the last step, we used that $z^2 \notin \mathbb{R}^+$ which was proved in Lemma 5.3.6.

If instead we assume that $z \in \mathbb{R}^4 - iV^+$, then the formula for $\mathbf{D}^-(z)$ is obtained by the reflection formula (A.2.24)

$$\mathbf{D}^-(z) = \overline{\mathbf{D}^+(\overline{z})} = \overline{\left(\frac{m}{4\pi^2} \ \frac{K_1\left(m\sqrt{-\overline{z}^2}\right)}{\sqrt{-\overline{z}^2}} \right)} = \frac{m}{4\pi^2} \ \frac{K_1\left(m\sqrt{-z^2}\right)}{\sqrt{-z^2}}$$

□

We have now shown that the Pauli Jordan function can be written as a boundary value

$$iD_m(x) = \mathbf{D}(x + iV^+\mathbf{0}) - \mathbf{D}(x - iV^+\mathbf{0}) \tag{5.3.56}$$

However, neither formula (5.3.5) nor (5.3.52) seem to agree with the representation of $D_m(x)$ as an equivalence class (5.3.41). In fact, these two different representations are equivalent which we will now show.

Theorem 5.3.7. *Consider $\mathbf{D}(z)$ to be a function of the single complex variable $\zeta := z^2$ and define a one variable hyperfunction $D(x) := \left[\frac{\mathbf{D}(z)}{i}\right] = \left[\frac{m}{4\pi^2 i} \ \frac{K_1\left(m\sqrt{-\zeta}\right)}{\sqrt{-\zeta}}\right]$. Then $D(x)$ is the same equivalence class (5.3.41) calculated in the previous section.*

$$D(x) = \left[\frac{m}{4\pi^2 i} \ \frac{K_1\left(m\sqrt{-z^2}\right)}{\sqrt{-z^2}}\right] \ = \ \frac{-1}{4\pi^2 i} \left[\frac{1}{z^2} - \frac{m\log(-z^2) J_1\left(m\sqrt{z^2}\right)}{2\sqrt{z^2}}\right] \tag{5.3.57}$$

Proof. First, let's show that $\frac{I_1(\sqrt{-\zeta})}{\sqrt{-\zeta}} = \frac{J_1(\sqrt{\zeta})}{\sqrt{\zeta}}$, for all $\zeta \in \mathbb{C}$ (the singularity at $\zeta = 0$ is removable). Using Lemma 5.3.9 and the definition of $I_1(\zeta)$ (Def. A.2.17), we have

$$\frac{I_1\left(\sqrt{-\zeta}\right)}{\sqrt{-\zeta}} = \frac{-iJ_1\left(i\sqrt{-\zeta}\right)}{\sqrt{-\zeta}} = \begin{cases} \frac{-iJ_1\left(i(-i\sqrt{z})\right)}{-i\sqrt{z}}, & z \in \mathbb{C}^+ \cup \mathbb{R}^- \\ \frac{-iJ_1\left(i(i\sqrt{z})\right)}{i\sqrt{z}}, & z \in \mathbb{C}^- \cup \mathbb{R}^+ \end{cases} = \begin{cases} \frac{J_1\left(\sqrt{z}\right)}{\sqrt{z}}, & z \in \mathbb{C}^+ \cup \mathbb{R}^- \\ \frac{-J_1\left(-\sqrt{z}\right)}{\sqrt{z}}, & z \in \mathbb{C}^- \cup \mathbb{R}^+ \end{cases}$$

Then the result follows from using the fact that $J_1(\zeta)$ is odd. Next, set $\zeta = z^2$ and use the series (A.2.25) for $K_1(\zeta)$. After reducing modulo entire functions, we obtain

$$\begin{aligned} \left[\frac{m}{4\pi^2 i}\ \frac{K_1\left(m\sqrt{-\zeta}\right)}{\sqrt{-\zeta}}\right] &= \left[\frac{m}{4\pi^2 i}\ \frac{\frac{1}{m\sqrt{-\zeta}} + \log(m\sqrt{-\zeta})I_1(m\sqrt{-\zeta})}{\sqrt{-\zeta}}\right] \\ &= \frac{m}{4\pi^2 i}\left[\frac{1}{-m\zeta} + \cancel{\frac{\log(m)I_1(m\sqrt{-\zeta})}{\sqrt{-\zeta}}} + \frac{\log(\sqrt{-\zeta})I_1(m\sqrt{-\zeta})}{\sqrt{-\zeta}}\right] \\ &= \frac{-1}{4\pi^2 i}\left[\frac{1}{\zeta} - \frac{m\log(-\zeta)J_1(m\sqrt{\zeta})}{2\sqrt{\zeta}}\right] \end{aligned}$$

□

5.3.5 The Free Dirac Field

In this section, we'll construct a quantum field $\psi(x)$ which satisfies the Dirac equation

$$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi(x) = 0 \tag{5.3.58}$$

which describes a spin $\frac{1}{2}$ particle of mass m, where γ^μ are the gamma matrices defined in Appendix A.4.

When we constructed the free field $\phi(x)$, the finite dimensional representation $\varrho : \widetilde{SO}^{\uparrow}(1,d) \to GL_n(\mathbb{C})$ (see Def. 5.3.1) was forced to be trivial since $n = 1$. The Dirac field and has 4 components $\psi^1(x), \psi^2(x), \psi^3(x), \psi^4(x)$, which allows for a non-trivial representation of the (universal cover of) the proper Lorentz group. We start by specifying all the necessary data for a hyperfunction quantum field before defining the Dirac field and showing that it satisfies the Wightman Axioms. Throughout this section let $m > 0$ be fixed, and set $d = 3$, $n = 4$ in the notation of Def. (5.3.1).

First, we define the finite dimensional representation ϱ. We shall use that fact that $\widetilde{SO}^{\uparrow}(1,3) \cong SL_2(\mathbb{C})$ (see Appendix A.3).

Definition 5.3.9. Define a 4-dimensional representation $\varrho : SL_2(\mathbb{C}) \to GL_4(\mathbb{C})$ by

$$\varrho(A) := \begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix} \tag{5.3.59}$$

The inner product of he Hilbert space uses a modified version of the adjoint, called the *Dirac adjoint.*

Definition 5.3.10. The **Dirac adjoint** of $v \in \mathbb{C}^4$ is defined to be

$$\tilde{v} := v^*\gamma^0 \tag{5.3.60}$$

where $v^* = \overline{v^T}$ denotes the adjoint of bounded operators when regarding $\mathbb{C}^4$ as $\mathfrak{B}(\mathbb{C}, \mathbb{C}^4)$, and γ^0 is defined by equation (A.4.2).

Remark 5.3.9. The Dirac adjoint is commonly denoted $\overline{v}$, but we've used the non-standard notation $\tilde{v}$ to avoid confusion with the complex conjugate.

Now we can define the Hilbert space.

Definition 5.3.11. Recall that V_m^+ is the upper-sheeted hyperbola of mass m

$$V_m^+ = \{p \in \mathbb{R}^4 \ \mid \ p^2 = m^2 \ , \ p_0 > 0\}$$

and Ω is the Lorentz invariant measure on V_m^+ defined by

$$\int_{V_m^+} f(p_0, \vec{p}) \ d\Omega(p) = \int_{\mathbb{R}^4} \frac{f(\sqrt{|\vec{p}|^2 + m^2}, \vec{p})}{2\sqrt{|\vec{p}|^2 + m^2}} \ d\vec{p}$$

Now, define the single particle Hilbert space $\mathcal{H}_1$ whose underlying set is given by

$$\mathcal{H}_1 := \{ \ f : V_m^+ \to \mathbb{C}^4 \ \mid \ \langle f|f\rangle_{\frac{1}{2}} < \infty \ \} \tag{5.3.61}$$

where the inner product is given by

$$\langle f|g\rangle_1 := \frac{1}{m} \int_{V_m^+} \widetilde{f(p)} \ \not{p} \ g(p) \, d\Omega(p) \ = \ \int_{V_m^+} f(p)^* \left(\frac{\gamma^0 \not{p}}{m} \right) g(p) \, d\Omega(p) \tag{5.3.62}$$

where $\not{p}$ is the Feynman slash notation (Definition A.4.2).

Definition 5.3.12. The n particle Hilbert space $\mathcal{H}_n$ is defined to be the n-fold anti-symmetric tensor product of $\mathcal{H}_1$ with itself (see Appendix A.1):

$$\mathcal{H}_n := A\big(\mathcal{H}_1^{\hat{\otimes} n}\big) \cong \{f : (V_m^+)^n \to (\mathbb{C}^4)^{\otimes n} \mid f(p_{\sigma(0)}, \ldots, p_{\sigma(n)}) = \operatorname{sgn}(\sigma) \, f(p_1, \ldots, p_n), \ \forall \sigma \in S_n\} \tag{5.3.63}$$

where the inner product is given by

$$\langle f|g\rangle_n := \frac{1}{n!} \int_{V_m^+} \cdots \int_{V_m^+} f(p)^* \left(\frac{\gamma^0 \not{p}}{m} \otimes \cdots \otimes \frac{\gamma^0 \not{p}}{m} \right) g(p) \, d\Omega(p) \tag{5.3.64}$$

(If $n = 0$, this reduces to $\mathcal{H}_0 := \mathbb{C}$ with its standard inner product $\langle f_0|g_0\rangle = \overline{f_0} g_0$).
The total Hilbert space $\mathcal{H}$ is then defined to be the direct sum

$$\mathcal{H} := \bigoplus_{n=0}^{\infty} \mathcal{H}_n \tag{5.3.65}$$

whose inner product is given by

$$\langle \, (f_n) \, | \, (g_n) \, \rangle := \sum_{n=0}^{\infty} \langle f_n|g_n\rangle_n \tag{5.3.66}$$

The dense subspace $\mathcal{D} \subseteq \mathcal{H}$ is defined to be the set of all vectors having finitely many non-zero components:

$$\mathcal{D} := \{(f_n) \in \bigoplus_{n=0}^{\infty} \mathcal{H}_n \ \mid \ \exists N \in \mathbb{N}, n \geq N \Rightarrow f_n = 0\} \tag{5.3.67}$$

and the vacuum vector is $\langle 0| := (1, 0, 0, 0, \ldots)$

Next, we define the unitary representation of $\widetilde{\mathcal{P}_+^\uparrow} \cong \mathbb{R}^4 \ltimes SL_2(\mathbb{C})$ in $\mathcal{H}$.

Definition 5.3.13. Define the map $U_1 : \mathbb{R}^4 \ltimes SL_2(\mathbb{C}) \to \mathcal{U}(\mathcal{H}_1)$ by

$$\begin{aligned} U_1(a, A) : \mathcal{H}_1 &\to \mathcal{H}_1 \\ f(p) &\mapsto e^{ipa} \varrho(A) f(\pi(A^{-1})p) \end{aligned}$$

Extend this to the entire space in the natural way by defining $U_n(a, A) = U_1(a, A)^{\otimes n}$, and $U(a, A) = \bigoplus_{n=0}^{\infty} U_n(a, A)$. Explicitly, this means

$$\big(U(a, A)f\big)_n(p_1, \ldots, p_n) = e^{ia \sum_{k=1}^n p_k} \varrho(A)^{\otimes n} f_n\big(\pi(A^{-1}) \, p_1 \, , \, \ldots \, , \, \pi(A^{-1}) \, p_n \big) \tag{5.3.68}$$

To show that this map is indeed unitary, we'll need to use some properties of the representation ϱ.

Lemma 5.3.10. *For all $A \in SL_2(\mathbb{C})$ and for all $p \in \mathbb{R}^4$, the following relations hold:*

1. $\gamma^0 \varrho(A)^* \gamma^0 = \varrho(A)^{-1}$
2. $\gamma^0 \not{p}\, \gamma^0 = \not{p}^{\,*} = \cancel{\begin{pmatrix} p_0 \\ -\vec{p} \end{pmatrix}}$
3. $\varrho(A) \not{p}\, \varrho(A)^{-1} = \cancel{(\pi(A)p)}$

Proof. 1.

$$\gamma^0 \varrho(A)^* \gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} A^* & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \begin{pmatrix} A^{-1} & 0 \\ 0 & A^* \end{pmatrix} = \varrho(A)^{-1}$$

2.

$$\gamma^0 \not{p}\, \gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma(p) \\ \overline{\sigma}(p) & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \overline{\sigma}(p) \\ \sigma(p) & 0 \end{pmatrix} = \not{p}^{\,*}$$

3. By definition of π (Def A.3.3), we have $\sigma(\pi(A)p) = A\,\sigma(p)A^*$. Using Theorem (A.3.1.v), we also have $\overline{\sigma}(\pi(A)p) = \overline{\sigma}\big(\sigma^{-1}(A\,\sigma(p)A^*)\big) = A^{*-1}\overline{\sigma}(p)A^{-1}$. Now we compute

$$\begin{aligned} \varrho(A) \not{p}\, \varrho(A)^{-1} &= \begin{pmatrix} A & 0 \\ 0 & A^{*-1} \end{pmatrix} \begin{pmatrix} 0 & \sigma(p) \\ \overline{\sigma}(p) & 0 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A^* \end{pmatrix} \\ &= \begin{pmatrix} 0 & A\,\sigma(p)A^* \\ A^{*-1}\overline{\sigma}(p)A^{-1} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \sigma\big(\pi(A)p\big) \\ \overline{\sigma}\big(\pi(A)p\big) & 0 \end{pmatrix} = \cancel{(\pi(A)p)} \end{aligned}$$

□

Proposition 5.3.3. *For all $(a, A) \in \mathbb{R}^4 \ltimes SL_2(\mathbb{C})$, $U_1(a, A)$ is unitary.*

Proof. Let $f, g \in \mathcal{H}_1$, $a \in \mathbb{R}^4$ and $A \in SL_2(\mathbb{C})$. Using the Lemma and fact that $d\Omega(p) = d\Omega(\pi(A)p)$, we see that

$$\begin{aligned} \langle U(a,A)f | U(a,A)g \rangle &= \frac{1}{m} \int_{V_m^+} \left(e^{ipa} \varrho(A) f(\pi(A^{-1})p)\right)^* \gamma^0 \not{p}\, \left(e^{ipa} \varrho(A) g(\pi(A^{-1})p)\right) d\Omega(p) \\ &= \frac{1}{m} \int_{V_m^+} f(\pi(A^{-1})p)^* \varrho(A)^* \gamma^0 \not{p}\, \varrho(A) g(\pi(A^{-1})p)\, d\Omega(p) \\ &= \frac{1}{m} \int_{V_m^+} f(p)^* \gamma^0\, \varrho(A)^{-1} \cancel{(\pi(A)p)} \varrho(A)\, g(p)\, d\Omega(\pi(A)p) \\ &= \frac{1}{m} \int_{V_m^+} f(p)^* \gamma^0 \not{p}\, g(p)\, d\Omega(p) \ = \langle f | g \rangle \end{aligned}$$

□

Now we turn to defining the field operators. We will need to use the fact that each $A \in SL_2(\mathbb{C})$ is conjugate to the transpose of its inverse, i.e. there exists $J \in SL_2(\mathbb{C})$ such that $JAJ^{-1} = (A^T)^{-1}$.

Lemma 5.3.11. *Define the matrix $J \in SL_2(\mathbb{C})$ by*

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{5.3.69}$$

Then the following identities hold

1. For all $A \in SL_2(\mathbb{C})$,

$$JA^T J^{-1} = A^{-1} \tag{5.3.70}$$

2. $[\gamma^0, \varrho(J)] = 0$

3. $\varrho(J) \not{p}\, \varrho(J)^{-1} = \not{p}^{\,T}$

Proof. 1. Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then we have

$$JA^T J^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \det(A) \cdot A^{-1} = A^{-1}$$

2. From (Lemma 5.3.10), we know that $\gamma^0 \varrho(J) \gamma^0 = \varrho(J^{*-1})$. The claim follows since $J^* = J^{-1} = -J$.

3. From (Lemma 5.3.10), we know that $\varrho(J) \not{p}\, \varrho(J)^{-1} = \cancel{(\pi(J)p)}$. Since $J^* = J^{-1}$, we can use part (1) to calculate $\pi(J)$:

$$\begin{aligned} \pi(J)p &= \sigma^{-1}\left(J\sigma(p)J^{-1}\right) \\ &= \sigma^{-1}\left(\det(\sigma(p))\,\sigma(p)^{-1\,T}\right) \\ &= \sigma^{-1}\left(p^2 \cdot \frac{\overline{\sigma}(p)^T}{p^2}\right) \\ &= \sigma^{-1}\left(\sigma(p_0, -\vec{p})^T\right) \\ &= \sigma^{-1}\left(\sigma(p_0, -p_1, p_2, -p_3)\right) = (p_0, -p_1, p_2, -p_3) \end{aligned}$$

$$\Rightarrow \qquad \pi(J) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The claim follows because $\not{p}^{\,T} = \overline{\not{p}^{\,*}} = \cancel{\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} p\right)}$.

$\square$

Definition 5.3.14. Let $(\mathbf{e}_j)_{j\in 4}$ denote the standard basis of $\mathbb{C}^4$, and let J denote the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. For each $j \in 4$, define the functions $E_+^j, E_-^j : \mathbb{R}^4 + iV^+ \to \mathcal{H}_1$ by

$$E_+^j(z) \;:=\; \frac{e^{ipz}\,(\not{p} + m)\,\gamma^0 \mathbf{e}_j}{\sqrt{2m(2\pi)^3}} \tag{5.3.71}$$

$$E_-^j(z) \;:=\; \frac{e^{ipz}\,(\not{p} - m)\,\varrho(J)\mathbf{e}_j}{\sqrt{2m(2\pi)^3}} \tag{5.3.72}$$

Theorem 5.3.8. *For each $j \in 4$, the functions $E_+^j(z)$ and $E_-^j(z)$ are $\mathcal{H}_1$-holomorphic.*

Proof. First, let's check that these functions are in fact square integrable. Using the fact that $\not{p}^{\,2} = m^2 \cdot \mathbb{1}$ for all $p \in V_m^+$ and that $\gamma^0 \not{p}\, \gamma^0 = \not{p}^{\,*}$, it follows that $\gamma^0(\not{p} + m)^* \left(\frac{\gamma^0 \not{p}}{m}\right)(\not{p} + m) = 2m(\not{p} + m)$. Then we have

$$\begin{aligned} \|E_+^j\|^2 &= \frac{1}{2m(2\pi)^3} \int_{V_m^+} e^{-ip\overline{z} + ipz} \mathbf{e}_j^* \gamma^0 (\not{p} + m)^* \left(\frac{\gamma^0 \not{p}}{m}\right) (\not{p} + m)\gamma^0 \mathbf{e}_j \; d\Omega(p) \\ &= \frac{1}{(2\pi)^3} \int_{V_m^+} e^{-2p(\mathrm{Im}\,z)} \mathbf{e}_j^* (\not{p} + m)\gamma^0 \mathbf{e}_j \; d\Omega(p) \end{aligned}$$

Since z is assumed to be in $\mathbb{R}^4 + iV^+$, we know that $p\,(\mathrm{Im} z) > 0$, for all $p \in V_m^+$. Then it follows that the above integral converges since $e^{-2p(\mathrm{Im} z)}$ decays exponentially, while each entry of the matrix $(\not{p} + m)\gamma^0$ is a polynomial in the variables $p_0, \ldots, p_3$. For the same reason, we see that $\frac{\partial}{\partial z_k} E_+^j(z) = ip_k E_+^j(z)$ is also square integrable, hence $E_+^j(z) \in \mathcal{O}(\mathbb{R}^4 + iV^+; \mathcal{H}_1)$. Similarly, $E_-^j(z) \in \mathcal{O}(\mathbb{R}^4 + iV^+; \mathcal{H}_1)$ because

$$
\begin{aligned}
\|E_-^j(z)\|^2 &= \frac{1}{2m(2\pi)^3} \int_{V_m^+} e^{-ip\overline{z}+ipz} \mathbf{e}_j^* \varrho(J)^* (\not{p} - m)^* \left(\frac{\gamma^0 \not{p}}{m}\right) (\not{p} - m) \varrho(J) \mathbf{e}_j \; d\Omega(p) \\
&= \frac{1}{(2\pi)^3} \int_{V_m^+} e^{-ip\overline{z}+ipz} \mathbf{e}_j^* \gamma^0 \varrho(J)^{-1} (\not{p} - m) \varrho(J) \mathbf{e}_j \; d\Omega(p) \\
&= \frac{1}{(2\pi)^3} \int_{V_m^+} e^{-2p(\mathrm{Im} z)} \mathbf{e}_j^* \gamma^0 (\not{p} - m)^T \mathbf{e}_j \; d\Omega(p)
\end{aligned}
$$

□

Definition 5.3.15. For each j, define $B_\pm^j \in \mathcal{O}(\mathbb{R}^4 - iV^+; \mathcal{L}_+(\mathcal{D}))$ to be the anti-symmetric partial bra- determined by $E_\pm^j$

$$
B_\pm^j(z) := \left\langle E_\pm^j(\overline{z}) \right| \tag{5.3.73}
$$

Similarly, define $B_\pm^{j\,*} \in \mathcal{O}(\mathbb{R}^4 + iV^+; \mathcal{L}_+(\mathcal{D}))$ to be the anti-symmetric partial ket

$$
B_\pm^{j\,*}(z) := B_\pm^j(\overline{z})^* = \left| E_\pm^j(z) \right\rangle \tag{5.3.74}
$$

Remark 5.3.10. The partial bra-, partial ket, and their restriction to symmetric/anti-symmetric tensors (Def. A.1.4) are detailed in appendix A.1. For convenience, we will write out the explicit formula for the specific case of the Hilbert space constructed in this section (Eq. 5.3.65). The partial bra- determined by an element $v \in \mathcal{H}_1$ is an operator $\langle v| \in \mathfrak{B}(\mathcal{H})$ which acts on $(\Phi_n)_{n\geq 0} \in \mathcal{H}$ by

$$
(\, \langle v|\, \Phi \,)_n \,(p_1, \ldots, p_n) = \int_{V_m^+} \left(\frac{v(p)^* \gamma^0 \not{p}}{m} \otimes \mathbb{1}^{\otimes n} \right) \Phi_{n+1}(p, p_1, \ldots, p_n) \, d\Omega(p)
$$

The partial ket determined by v is the adjoint of $\langle v|$ and denoted $|v\rangle$. Its action on $(\Phi_n)_{n\geq 0} \in \mathcal{H}$ is given by

$$
(\, |v\rangle\, \Phi \,)_n \,(p_1, \ldots, p_n) = \sum_{k=1}^{n} (-1)^{k+1} v(p_k) \otimes \Phi_{n-1}(p_1, \ldots, \hat{p_k}, \ldots, p_n)
$$

Definition 5.3.16. Define the j^{th} **particle annihilation** and **particle creation** operators to be the hyperfunctions $b_+^j(x), b_+^{j\,*}(x) \in \mathcal{B}(\mathbb{R}^4; \mathcal{L}_+(\mathcal{D}))$

$$
b_+^j(x) := B_+^j(x - iV^+\mathbf{0}) \qquad\qquad b_+^{j\,*}(x) := B_+^{j\,*}(x + iV^+\mathbf{0})
$$

Similarly, define the j^{th} **anti-particle annihilation** and **anti-particle creation** operators $b_-^j(x), b_-^{j\,*}(x) \in \mathcal{B}(\mathbb{R}^4; \mathcal{L}_+(\mathcal{D}))$ by

$$
b_-^j(x) := B_-^j(x - iV^+\mathbf{0}) \qquad\qquad b_-^{j\,*}(x) := B_-^{j\,*}(x + iV^+\mathbf{0})
$$

Finally, define the j^{th} **Dirac field** operator to be the hyperfunction

$$
\psi^j(x) := b_+^j(x) + b_-^{j\,*}(x) \tag{5.3.75}
$$

Theorem 5.3.9. *The collection $\psi(x) = \{\psi^0(x), \psi^1(x), \psi^2(x), \psi^3(x)\}$ is a quantum field in the sense that $(\mathcal{H}, \mathcal{D}, \langle 0|\,, \psi(x), U, \varrho)$ satisfies the Wightman Axioms (Def. 5.3.1)*

Proof. (W1) If $(\Phi_n)_{n\geq 0} \in \mathcal{D}$, then there exists $N \in \mathbb{N}$ such that $\Phi_n = 0$ whenever $n \geq N$. It is then evident from the definition of U that choosing the same value of N implies that $(U(a, A)\Phi)_n = 0$ for all $n \geq N$ and for all $(a, A) \in \mathbb{R}^4 \ltimes SL_2(\mathbb{C})$.

(W2) Let $(a, A) \in \mathbb{R}^4 \ltimes SL_2(\mathbb{C})$ and $j \in 4$. Start by expanding out

$$\begin{aligned} U(a,A)\psi^j(x)U(a,A)^* &= U(a,A)\big(\, b_+^j(x) + b_-^{j\,*}(x)\,\big)U(a,A)^* \\ &= U(a,A)B_+^j(x - iV^+\mathbf{0})U(a,A)^* + U(a,A)B_-^{j\,*}(x+iV^+\mathbf{0})U(a,A)^* \\ &= U(a,A)\left\langle E_+^j(x+iV^+\mathbf{0})\right| U(a,A)^* + U(a,A)\left|E_-^j(x+iV^+\mathbf{0})\right\rangle U(a,A)^* \end{aligned}$$

Suppose $z \in \mathbb{R}^4 + iV^+$. Then by proposition (A.1.1) we have

$$U(a,A)\left\langle E_+^j(z)\right| U(a,A)^* + U(a,A)\left|E_-^j(z)\right\rangle U(a,A)^* = \left\langle U(a,A)E_+^j(z)\right| + \left|U(a,A)E_-^j(z)\right\rangle$$

Plugging in the definitions (and denoting the normalizing constant by $(16\pi^3 m)^{-1/2} = C$) , the first term is

$$\begin{aligned} \left\langle U(a,A)E_+^j(z)\right| &= \left\langle U(a,A)\, C\, e^{ipz}(\not{p}+m)\gamma^0\mathbf{e}_j\right| \\ &= \left\langle C\, e^{ipa}\varrho(A)e^{i\left(\pi(A^{-1})p\right)z}\left(\cancel{(\pi(A^{-1})p)} + m\right)\gamma^0\mathbf{e}_j\right| \\ &= \left\langle C\, e^{ip\left(a+\pi(A)z\right)}\varrho(A)\left(\cancel{(\pi(A^{-1})p)} + m\right)\gamma^0\mathbf{e}_j\right| \end{aligned}$$

Next, we use Lemma (5.3.10) to move $\varrho(A)$ to the right

$$\begin{aligned} \left\langle C\, e^{ip\left(a+\pi(A)z\right)}\varrho(A)\left(\cancel{(\pi(A^{-1})p)} + m\right)\gamma^0\mathbf{e}_j\right| &= \left\langle C\, e^{ip\left(a+\pi(A)z\right)}\left(\not{p}+m\right)\varrho(A)\gamma^0\mathbf{e}_j\right| \\ &= \left\langle C\, e^{ip\left(a+\pi(A)z\right)}\left(\not{p}+m\right)\gamma^0\varrho(A^{*\,-1})\mathbf{e}_j\right| \end{aligned}$$

Finally we make use of the conjugate-linearity of $\langle - |$ as well as the fact that any matrix $M \in M_4(\mathbb{C})$ can be expressed in the standard basis via $M\mathbf{e}_j = \sum_k M_j^k \mathbf{e}_k = \sum_k \left(M^T\right)_k^j \mathbf{e}_k$.

$$\begin{aligned} \left\langle C\, e^{ip\left(a+\pi(A)z\right)}\left(\not{p}+m\right)\gamma^0\varrho(A^{*\,-1})\mathbf{e}_j\right| &= \left\langle C\, e^{ip\left(a+\pi(A)z\right)}\left(\not{p}+m\right)\gamma^0\sum_{k\in 4}\varrho\left(\overline{A^{-1}}\right)_k^j\mathbf{e}_k\right| \\ &= \sum_{k\in 4}\varrho(A^{-1})_k^j\left\langle C\, e^{ip\left(a+\pi(A)z\right)}\left(\not{p}+m\right)\gamma^0\mathbf{e}_k\right| \end{aligned}$$

$$\implies \qquad \left\langle U(a,A)E_+^j(z)\right| = \sum_{k\in 4}\varrho(A^{-1})_k^j\left\langle E_+^k\left(a+\pi(A)z\right)\right|$$

The second term is similar:

$$\begin{aligned} \left|U(a,A)E_-^j(z)\right\rangle &= \left|U(a,A)\, C\, e^{ipz}(\not{p}-m)\varrho(J)\mathbf{e}_j\right\rangle \\ &= \left|C\, e^{ipa}\varrho(A)e^{i\left(\pi(A^{-1})p\right)z}\left(\cancel{(\pi(A^{-1})p)} - m\right)\varrho(J)\mathbf{e}_j\right\rangle \\ &= \left|C\, e^{ip\left(a+\pi(A)z\right)}\left(\not{p}-m\right)\varrho(A)\varrho(J)\mathbf{e}_j\right\rangle \end{aligned}$$

However this time we instead use Lemma (5.3.11) to move $\varrho(A)$ to the right of $\varrho(J)$

$$\begin{aligned} \left|C\, e^{ip\left(a+\pi(A)z\right)}\left(\not{p}-m\right)\varrho(A)\varrho(J)\mathbf{e}_j\right\rangle &= \left|C\, e^{ip\left(a+\pi(A)z\right)}\left(\not{p}-m\right)\varrho(JA^{T\,-1}J^{-1})\varrho(J)\mathbf{e}_j\right\rangle \\ &= \left|C\, e^{ip\left(a+\pi(A)z\right)}\left(\not{p}-m\right)\varrho(J)\varrho(A^{T\,-1})\mathbf{e}_j\right\rangle \\ &= \left|C\, e^{ip\left(a+\pi(A)z\right)}\left(\not{p}-m\right)\varrho(J)\sum_{k\in 4}\varrho(A^{-1})_k^j\mathbf{e}_k\right\rangle \\ \left|U(a,A)E_-^j(z)\right\rangle &= \sum_{k\in 4}\varrho(A^{-1})_k^j\left|E_-^k\left(a+\pi(A)z\right)\right\rangle \end{aligned}$$

Since z was arbitrary, the same transformation law holds for the boundary values $b_+^j(x), b_-^{j\,*}(x)$ and hence also for $\psi^j(x)$.

(W3) First, we compute the anti-commutator $\{\psi^j(x), \psi^k(y)\}$. This is the boundary value of

$$\left\{ \left\langle E_+^j(z)\right| + \left|E_-^j(z)\right\rangle \,,\, \left\langle E_+^k(w)\right| + \left|E_-^k(w)\right\rangle \right\}$$

where $z, w \in \mathbb{R}^4 + iV^+$. Next we expand by bilinearity of $\{\cdot,\cdot\}$ and use Theorem (A.1.1).

$$\begin{array}{ccccccc} \left\{ \left\langle E_+^j(z)\right| , \left\langle E_+^k(w)\right| \right\} & + \left\{ \left\langle E_+^j(z)\right| , \left|E_-^k(w)\right\rangle \right\} & + \left\{ \left|E_-^j(z)\right\rangle , \left\langle E_+^k(w)\right| \right\} & + \left\{ \left|E_-^j(z)\right\rangle , \left|E_-^k(w)\right\rangle \right\} \\ = \quad 0 & + \quad \langle E_+^j(z)|E_-^k(w)\rangle & + \quad \langle E_+^k(w)|E_-^j(z)\rangle & + \quad 0 \end{array}$$

Now plugging in the definition we get

$$\begin{aligned} \langle E_+^j(z)|E_-^k(w)\rangle &= \frac{1}{2m(2\pi)^3}\int_{V_m^+} \mathbf{e}_j^*\gamma^0(\not{p}+m)^* e^{-ip\overline{z}} \left(\frac{\gamma^0\not{p}}{m}\right) e^{ipw}(\not{p}-m)\varrho(J)\mathbf{e}_k \; d\Omega(p) \\ &= \frac{1}{2m(2\pi)^3}\int_{V_m^+} e^{-ip(\overline{z}-w)}\mathbf{e}_j^*(\not{p}+m)(m-\not{p})\varrho(J)\mathbf{e}_k \; d\Omega(p) \\ &= \frac{1}{2m(2\pi)^3}\int_{V_m^+} e^{-ip(\overline{z}-w)}\mathbf{e}_j^*(m^2-\not{p}^2)\varrho(J)\mathbf{e}_k \; d\Omega(p) \quad \equiv 0 \end{aligned}$$

where we have used the identities $\gamma^0\not{p}\gamma^0 = \not{p}^*$ and the fact that $\not{p}^2 = m^2 \cdot \mathbb{1}_4$ for all $p \in V_m^+$. Thus, $\langle E_+^k(w)|E_-^j(z)\rangle$ is also identically zero by interchanging z with w and j with k.

Next, we compute the anti-commutator $\{\psi^j(x), \psi^{k\,*}(y)\}$, which is the boundary value of

$$\left\{ \left\langle E_+^j(z)\right| + \left|E_-^j(z)\right\rangle \,,\, \left\langle E_-^k(w)\right| + \left|E_+^k(w)\right\rangle \right\} \qquad z, w \in \mathbb{R}^4 + iV^+$$

With another application of Theorem (A.1.1), this simplifies to a scalar multiple of the identity:

$$\left\{ \left\langle E_+^j(z)\right| + \left|E_-^j(z)\right\rangle \,,\, \left\langle E_-^k(w)\right| + \left|E_+^k(w)\right\rangle \right\} = \langle E_+^j(z)|E_+^k(w)\rangle + \langle E_-^k(w)|E_-^j(z)\rangle$$

The first term is

$$\begin{aligned} \langle E_+^j(z)|E_+^k(w)\rangle &= \frac{1}{2m(2\pi)^3}\int_{V_m^+} e^{-ip(\overline{z}-w)}\mathbf{e}_j^*\gamma^0(\not{p}+m)^* \left(\frac{\gamma^0\not{p}}{m}\right) (\not{p}+m)\gamma^0\mathbf{e}_k \; d\Omega(p) \\ &= \frac{1}{2m(2\pi)^3}\int_{V_m^+} e^{-ip(\overline{z}-w)}\mathbf{e}_j^*(\not{p}+m)(\not{p}+m)\gamma^0\mathbf{e}_k \; d\Omega(p) \\ &= \frac{1}{(2\pi)^3}\int_{V_m^+} e^{-ip(\overline{z}-w)}\mathbf{e}_j^*(\not{p}+m)\gamma^0\mathbf{e}_k \; d\Omega(p) \end{aligned}$$

If we reduce the indices j, k modulo 4, this can be written more succinctly as

$$\langle E_+^j(z)|E_+^k(w)\rangle = \frac{1}{(2\pi)^3}\int_{V_m^+} e^{-ip(\overline{z}-w)}(\not{p}+m)^j_{k+2} \; d\Omega(p) \tag{5.3.76}$$

For the second term, we can use Lemma (5.3.11) and the fact that $J^* = J^{-1} = -J$ to calculate

$$\begin{aligned} \langle E_-^k(w)|E_-^j(z)\rangle &= \frac{1}{2m(2\pi)^3}\int_{V_m^+} e^{ip(z-\overline{w})}\mathbf{e}_k^*\varrho(J)^*(\not{p}-m)^* \left(\frac{\gamma^0\not{p}}{m}\right) (\not{p}-m)\varrho(J)\mathbf{e}_j \; d\Omega(p) \\ &= \frac{1}{2m(2\pi)^3}\int_{V_m^+} e^{ip(z-\overline{w})}\mathbf{e}_k^*\gamma^0\varrho(J)^*(\not{p}-m)\,(m-\not{p})\varrho(J)\mathbf{e}_j \; d\Omega(p) \\ &= \frac{1}{(2\pi)^3}\int_{V_m^+} e^{ip(z-\overline{w})}\mathbf{e}_k^*\gamma^0\varrho(J)(\not{p}-m)\varrho(J)^{-1}\mathbf{e}_j \; d\Omega(p) \\ &= \frac{1}{(2\pi)^3}\int_{V_m^+} e^{ip(z-\overline{w})}\mathbf{e}_k^*\gamma^0(\not{p}^T-m)\mathbf{e}_j \; d\Omega(p) \end{aligned}$$

Taking the transpose and interpreting the indices modulo 4, we have

$$\langle E_-^k(w)|E_-^j(z)\rangle = \frac{1}{(2\pi)^3}\int_{V_m^+} e^{ip(z-\overline{w})}(\not{p}-m)^j_{k+2}\, d\Omega(p) \tag{5.3.77}$$

Taking the sum of (5.3.76) and (5.3.77) we obtain

$$\begin{aligned}
\langle E_+^j(z)|E_+^k(w)\rangle + \langle E_-^k(w)|E_-^j(z)\rangle &= \frac{1}{(2\pi)^3}\int_{V_m^+} e^{-ip(\overline{z}-w)}(\not{p}+m)^j_{k+2} + e^{ip(z-\overline{w})}(\not{p}-m)^j_{k+2}\, d\Omega(p)\\
&= \frac{1}{(2\pi)^3}\int_{V_m^+} \big(i\not{\partial}_{\overline{z}}+m\big)^j_{k+2} e^{-ip(\overline{z}-w)} + \big(-i\not{\partial}_z - m\big)^j_{k+2} e^{ip(z-\overline{w})}\, d\Omega(p)\\
&= \Big(i\big(\not{\partial}_{\overline{z}}+\not{\partial}_z\big)+m\Big)^j_{k+2} \frac{1}{(2\pi)^3}\int_{V_m^+} e^{-ip(\overline{z}-w)} + e^{ip(z-\overline{w})}\, d\Omega(p)\\
&= \Big(i\not{\partial}_x + m\Big)^j_{k+2} \big(\mathbf{D}^-(\overline{z}-\overline{w}) - \mathbf{D}^+(z-w)\big)
\end{aligned}$$

where $\mathbf{D}^{\pm}(z)$ are the holomorphic functions (Def 5.3.8) whose boundary value gives the commutator of the free field (Eq 5.3.51). Passing to the boundary values, we arrive at

$$\{\psi^j(x), \psi^{k\,*}(y)\} = \Big(i\not{\partial}_x + m\Big)^j_{k+2} [\phi(x), \phi(y)] \tag{5.3.78}$$

We know by Theorem (5.3.3) that $[\phi(x), \phi(y)]$ vanishes for all $(x-y)^2 < 0$, hence so too must any linear combinations of its partial derivatives.

(W4) The argument is similar to that for the free scalar field (c.f. Thm. 5.3.3). If $\Phi = (\Phi_n) \in \mathcal{H}$ satisfies $\langle 0|\Phi\rangle = 0$, then it immediately follows that $\Phi_0 = 0$. If $\langle 0|\, \psi^j(x)\, |\Phi\rangle = 0$, then

$$\begin{aligned}
0 &= \langle 0|\, b_+^j(x) + b_-^{j\,*}(x)\, |\Phi\rangle\\
&= \langle 0|\, b_+^j(x)\, |\Phi\rangle \; + \; 0
\end{aligned}$$

This implies that the holomorphic function $z \mapsto \langle 0|\, B_+^j(z)\, |\Phi\rangle$ is identically zero for all $z \in \mathbb{R}^4 - iV^+$. But then we have

$$\begin{aligned}
0 &= \langle 0|\, B_+^j(z)\, |\Phi\rangle\\
&= \int_{V_m^+} e^{-ipz}\mathbf{e}_j^*(\not{p}+m)\Phi_1(p)\, d\Omega(p)
\end{aligned}$$

Since this is true for all j and for all z, the invertibility of the Fourier transform implies that $(\not{p}+m)\Phi_1(p) = 0$. Similarly, the condition $\langle 0|\, \psi^{j\,*}\, |\Phi\rangle = 0$ implies that $(\not{p}-m)\Phi_1(p) = 0$ also holds true. However we know that $\ker(\not{p}+m) \cap \ker(\not{p}-m) = \{0\}$ since $\pm m$ are distinct eigenvalues of $\not{p}$. Therefore, we must have $\Phi_1 = 0$. By the same induction argument as for the free field, we conclude that $\Phi_n = 0$ for all n.

$\square$

A Appendix

A.1 The Partial Bra- and Partial Ket

Given a vector space V and $n \in \mathbb{N}$, there is a natural isomorphism $V \otimes V^{\otimes n} \cong V^{\otimes n+1}$ given by

$$\begin{aligned} V \otimes V^{\otimes n} &\to V^{\otimes n+1} \\ v \otimes \big(v_1 \otimes \cdots \otimes v_n\big) &\mapsto v \otimes v_1 \otimes \cdots \otimes v_n \end{aligned}$$

At first glance, this map seems less than interesting. However, we can apply the Tensor-Hom adjunction, which states that

$$\operatorname{Hom}(X \otimes Y, Z) \cong \operatorname{Hom}(X, \operatorname{Hom}(Y, Z)) \tag{A.1.1}$$

where $\operatorname{Hom}(X, Y)$ denotes the vector space of all linear maps from X to Y. Under this equivalence, we can re-express the map as

$$\begin{aligned} &V \to (V^{\otimes n} \to V^{\otimes n+1}) \\ &v \mapsto \big(v_1 \otimes \cdots \otimes v_n \mapsto v \otimes v_1 \otimes \cdots \otimes v_n\big) \end{aligned}$$

In the special case that $V = \mathcal{H}$ is a Hilbert space, the linear map $v_1 \otimes \cdots \otimes v_n \mapsto v \otimes v_1 \otimes \cdots \otimes v_n$ is continuous for all $v \in \mathcal{H}$ and therefore has an adjoint. Combinations of this map and its adjoint for various choices of v lead to non-trivial applications in quantum physics such as the creation/annihilation operators and the notion of partial trace. In this section, we'll review the preliminary definitions and some basic properties before considering the necessary modifications in the case of symmetric/anti-symmetric tensor products. All tensor products are understood to be the (complete) Hilbert space tensor product $\otimes = \hat{\otimes}$. For $n \in \mathbb{Z}$, we shall use the convention

$$\mathcal{H}^{\otimes n} = \begin{cases} \bigotimes_{i=1}^{n} \mathcal{H} & n > 0 \\ \mathbb{C} & n = 0 \\ 0 & n < 0 \end{cases}$$

Definition A.1.1. Let $\mathcal{H}$ be any Hilbert space, $\varphi \in \mathcal{H}$ and $n \in \mathbb{Z}$. Then the **partial bra-** of φ is the linear map $\langle \phi|_n : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n-1}$ defined on pure tensors by

$$\begin{aligned} \langle \varphi|_n : \mathcal{H}^{\otimes n} &\to \mathcal{H}^{\otimes n-1} \\ \psi_1 \otimes \cdots \otimes \psi_n &\mapsto \langle \varphi|\psi_1\rangle \psi_2 \otimes \cdots \otimes \psi_n \end{aligned}$$

and extended by linearity . The adjoint of this map is called the **partial ket** of φ and is denoted $|\varphi\rangle_n$. On pure tensors it is given by

$$\begin{aligned} |\varphi\rangle_n \ : \ \mathcal{H}^{\otimes n-1} &\to \mathcal{H}^{\otimes n} \\ \psi_1 \otimes \cdots \otimes \psi_{n-1} &\mapsto \varphi \otimes \psi_1 \otimes \cdots \otimes \psi_{n-1} \end{aligned}$$

and extended by linearity. It is convenient to omit the subscript n if the domain can be inferred from context.

Remark A.1.1. We have chosen to act in the left-most component, but one can of course define analogous maps which tensor/inner product with φ in the j^{th} component. The choice of j will become less arbitrary when we consider symmetric and anti-symmetric tensor products.

The following proposition describes how the partial bra- and partial ket behave under a change of (orthonormal) basis.

Proposition A.1.1. *Let $\mathcal{H}$ be Hilbert space, $\varphi \in \mathcal{H}$, $n \in \mathbb{Z}$, and $U \in \mathcal{U}(\mathcal{H})$ a unitary operator. Then the following diagram commutes*

$$\begin{tikzcd}
\mathcal{H}^{\otimes n} \arrow[r, "\langle\varphi|_n"] \arrow[d, "U^{\otimes n}"'] & \mathcal{H}^{\otimes n-1} \arrow[d, "U^{\otimes n-1}"] \\
\mathcal{H}^{\otimes n} \arrow[r, "\langle U\varphi|_n"'] & \mathcal{H}^{\otimes n-1}
\end{tikzcd}$$

By making the number n implicit, this can be written as

$$U\,\langle\varphi|\,U^* = \langle U\varphi| \tag{A.1.2}$$

Similarly,

$$U\,|\varphi\rangle\,U^* = |U\varphi\rangle \tag{A.1.3}$$

Proof. Let $\psi = \psi_1 \otimes \cdots \psi_n \in \mathcal{H}^{\otimes n}$. Then we have

$$\begin{aligned}
\langle U\varphi|_n U^{\otimes n}\ \psi &= \langle U\varphi|_n U\psi_1 \otimes \cdots \otimes U\psi_n \\
&= \langle U\varphi|U\psi_1\rangle U\psi_2 \otimes \cdots \otimes U\psi_n \\
&= \langle\varphi|\psi_1\rangle U\psi_2 \otimes \cdots \otimes U\psi_n \\
&= U^{\otimes n-1}\big(\ \langle\varphi|\psi_1\rangle\psi_2 \otimes \cdots \otimes \psi_n\ \big) \\
&= U^{\otimes n-1}\langle\varphi|_n\ \psi
\end{aligned}$$

which shows that $U\langle\varphi|U^* = \langle U\varphi|$. By taking the adjoint of both sides, we obtain $U|\varphi\rangle U^* = |U\varphi\rangle$. □

Definition A.1.2. Define the **symmetrization operator** $\mathrm{Sym}_n \in \mathfrak{B}(\mathcal{H}^{\otimes n})$ by

$$\begin{aligned}
\mathrm{Sym}_n^+ &: \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n} \\
v_1 \otimes \cdots \otimes v_n &\mapsto \frac{1}{n!}\sum_{\sigma\in S_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}
\end{aligned}$$

Similarly, the **anti-symmetrization operator** $\mathrm{Sym}_n^- \in \mathfrak{B}(\mathcal{H}^{\otimes n})$ is defined by

$$\begin{aligned}
\mathrm{Sym}_n^- &: \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n} \\
v_1 \otimes \cdots \otimes v_n &\mapsto \frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)\, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}
\end{aligned}$$

where S_n denotes the symmetric group and $\mathrm{sgn} : S_n \to \{1, -1\}$ is the sign homomorphism.

Proposition A.1.2. *The operators $\mathrm{Sym}_n^\pm$ have the following properties:*

1. *For all n, the operators $\mathrm{Sym}_n^\pm$ are projections*

$$\left(\mathrm{Sym}_n^\pm\right)^2 = \left(\mathrm{Sym}_n^\pm\right)^* = \mathrm{Sym}_n^\pm$$

2. *For $n \geq 1$, $\mathrm{Sym}_n^\pm$ is related to $\mathrm{Sym}_{n-1}^\pm$ by*

$$\mathrm{Sym}_n^\pm(v_1 \otimes \cdots \otimes v_n) = \frac{1}{n}\sum_{k=1}^{n}(\pm 1)^{k+1} v_k \otimes S_{n-1}^\pm(v_1 \otimes \cdots \otimes \hat{v_k} \otimes \cdots \otimes v_n)$$

where the notation $\hat{v_k}$ indicates that the vector v_k is to be omitted from the tensor product.

We will be interested in the subspaces $\mathrm{Sym}_n^\pm(\mathcal{H}^{\otimes n})$ of symmetric and anti-symmetric tensors. It is convenient to collect them all into one Hilbert space called Fock space by taking the direct sum over n. When doing this, the factor of $\frac{1}{n!}$ should be absorbed into the inner product so this scaling factor remains relevant between different n.

Definition A.1.3. Let $(\mathcal{H}, \langle\cdot|\cdot\rangle)$ be a Hilbert space. For $n \geq 0$, define $(\mathcal{H}_n, \langle\cdot|\cdot\rangle_n)$ to be Hilbert space given by

$$\mathcal{H}_n := \mathcal{H}^{\otimes n} \qquad\qquad \langle v_1 \otimes \cdots \otimes v_n \,|\, w_1 \otimes \cdots \otimes w_n\rangle_n := \frac{\langle v_1|w_1\rangle \cdots \langle v_n|w_n\rangle}{n!}$$

Define subspaces of (anti-)symmetric tensors $\mathcal{H}_n^+, \mathcal{H}_n^- \subseteq \mathcal{H}_n$ by

$$\mathcal{H}_n^{\pm} := \mathrm{Sym}_n^{\pm}(\mathcal{H}_n)$$

Then the **(anti-)symmetric Fock space** on $\mathcal{H}$ is defined to be the direct sum

$$\mathcal{F}^{\pm}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}_n^{\pm} \tag{A.1.4}$$

The partial bra- can be extended to Fock space in the natural way.

Definition A.1.4. For $\varphi \in \mathcal{H}$, and $\sigma \in \{+,-\}$ define the **(anti-)symmetric partial bra-** $\langle\varphi|^{\sigma} : \mathcal{F}^{\sigma}(\mathcal{H}) \to \mathcal{F}^{\sigma}(\mathcal{H})$ by

$$\begin{aligned} \langle\varphi|^{\sigma} &: \mathcal{F}^{\sigma}(\mathcal{H}) \to \mathcal{F}^{\sigma}(\mathcal{H}) \\ \sum_{n=0}^{\infty} v_n &\mapsto \sum_{n=0}^{\infty} \langle\varphi|_n \, v_n \qquad\qquad v_n \in \mathcal{H}_n^{\sigma} \end{aligned}$$

The **(anti-)symmetric partial ket** $|\varphi\rangle^{\sigma} : \mathcal{F}^{\sigma}(\mathcal{H}) \to \mathcal{F}^{\sigma}(\mathcal{H})$ is then defined to be its adjoint

$$|\varphi\rangle^{\sigma} := (\langle\varphi|^{\sigma})^*$$

Proposition A.1.3. *For $\sigma \in \{+,-\}$, the (anti-)symmetric partial ket $|\varphi\rangle^{\sigma}$ is explicitly given by*

$$|\varphi\rangle^{\sigma} = \bigoplus_{n=0}^{\infty} n \,\mathrm{Sym}_n^{\sigma} \, |\varphi\rangle_n$$

Proof. Since $(A \oplus B)^* = A^* \oplus B^*$, it suffices to determine the adjoint of $\langle\varphi|_n^{\sigma} : \mathcal{H}_n^{\sigma} \to \mathcal{H}_{n-1}^{\sigma}$. First, we'll determine the adjoint of $\langle\varphi|_n : \mathcal{H}_n \to \mathcal{H}_{n-1}$. Let $v_1 \otimes \cdots \otimes v_n \in \mathcal{H}_n$ and $w_1 \otimes \cdots \otimes w_{n-1} \in \mathcal{H}_{n-1}$. Then

$$\begin{aligned} \Big\langle \langle\varphi|_n \; v_1 \otimes \cdots \otimes v_n \,\Big|\, w_1 \otimes \cdots \otimes w_{n-1} \Big\rangle_{n-1} &= \frac{1}{(n-1)!}\Big\langle \langle\varphi|v_1\rangle v_2 \otimes \cdots \otimes v_n \,\Big|\, w_1 \otimes \cdots \otimes w_{n-1} \Big\rangle \\ &= \frac{\langle v_1|\varphi\rangle}{(n-1)!}\Big\langle v_2 \otimes \cdots \otimes v_n \,\Big|\, w_1 \otimes \cdots \otimes w_{n-1} \Big\rangle \\ &= \frac{1}{(n-1)!}\Big\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \,\Big|\, \varphi \otimes w_1 \otimes \cdots \otimes w_{n-1} \Big\rangle \\ &= n\Big\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \,\Big|\, |\varphi\rangle_n \; w_1 \otimes \cdots \otimes w_{n-1} \Big\rangle_n \end{aligned}$$

Therefore $(\langle\varphi|_n)^* = n\,|\varphi\rangle_n$ when considered as a map $\mathcal{H}_{n-1} \to \mathcal{H}_n$ with the scaled inner product (Def A.1.3). Then we can determine the restriction to (anti-)symmetric tensors by using the fact that Sym_n^{σ} is a projection (Prop. A.1.2).

$$\begin{aligned} &\Big\langle \langle\varphi|_n \; \mathrm{Sym}_n^{\sigma}(v_1 \otimes \cdots \otimes v_n) \,\Big|\, \mathrm{Sym}_n^{\sigma}(w_1 \otimes \cdots \otimes w_{n-1}) \Big\rangle_{n-1} \\ &= \Big\langle \mathrm{Sym}_n^{\sigma}(v_1 \otimes \cdots \otimes v_n) \,\Big|\, n\,|\varphi\rangle_n \, \mathrm{Sym}_n^{\sigma}(w_1 \otimes \cdots \otimes w_{n-1}) \Big\rangle_n \\ &= \Big\langle \mathrm{Sym}_n^{\sigma\,*}\mathrm{Sym}_n^{\sigma}(v_1 \otimes \cdots \otimes v_n) \,\Big|\, n\,|\varphi\rangle_n \, \mathrm{Sym}_n^{\sigma}(w_1 \otimes \cdots \otimes w_{n-1}) \Big\rangle_n \\ &= \Big\langle \mathrm{Sym}_n^{\sigma}(v_1 \otimes \cdots \otimes v_n) \,\Big|\, n\,\mathrm{Sym}_n^{\sigma}\,|\varphi\rangle_n \; \mathrm{Sym}_n^{\sigma}(w_1 \otimes \cdots \otimes w_{n-1}) \Big\rangle_n \end{aligned}$$

This shows that $(\langle\varphi|_n^{\sigma})^* = n\,\mathrm{Sym}_n^{\sigma}\,|\varphi\rangle$. ∎

Now we can state the main result of this section.

Theorem A.1.1. *Let $\varphi, \psi \in \mathcal{H}_1$ and $\sigma \in \{+1, -1\}$. Then the (anti-)commutator between (anti-)symmetric partial bra-'s and ket's is always a scalar multiple of the identity. In paticular,*

1.
$$\langle\varphi|^{\sigma}\ |\psi\rangle^{\sigma} - \sigma\, |\psi\rangle^{\sigma}\ \langle\varphi|^{\sigma} = \langle\varphi|\psi\rangle \cdot \mathbb{1} \tag{A.1.5}$$

2.
$$\langle\varphi|^{\sigma}\ \langle\psi|^{\sigma} - \sigma\, \langle\psi|^{\sigma}\ \langle\varphi|^{\sigma} = 0 \tag{A.1.6}$$

3.
$$|\varphi\rangle^{\sigma}\ |\psi\rangle^{\sigma} - \sigma\, |\psi\rangle^{\sigma}\ |\varphi\rangle^{\sigma} = 0 \tag{A.1.7}$$

Proof. 1. Let $n \geq 0$ and $v \in \mathcal{H}_n^{\sigma}$. Then there exist $v_1, \ldots, v_n \in \mathcal{H}_1$ such that $v = \mathrm{Sym}_n^{\sigma}(v_1, \otimes \cdots \otimes v_n)$. Using Propositions (A.1.2 - A.1.3), we calculate $\langle\varphi|^{\sigma}\, |\psi\rangle^{\sigma}\, v$ to be

$$\begin{aligned}
\langle\varphi|^{\sigma}\, |\psi\rangle^{\sigma}\, v &= \langle\varphi|_{n+1}^{\sigma}\ |\psi\rangle_{n+1}^{\sigma}\ \mathrm{Sym}_n^{\sigma}(v_1, \otimes \cdots \otimes v_n) \\
&= \langle\varphi|_{n+1}^{\sigma}\, (n+1)\mathrm{Sym}_{n+1}^{\sigma}\big(\psi \otimes \mathrm{Sym}_n^{\sigma}(v_1 \otimes \cdots \otimes v_n)\big) \\
&= \langle\varphi|_{n+1}^{\sigma}\, (n+1)\left(\frac{1}{n+1}\left(\psi \otimes \mathrm{Sym}_n^{\sigma}(v_1 \otimes \cdots \otimes v_n) + \sum_{k=1}^{n} \sigma^k\, v_k \otimes \mathrm{Sym}_n^{\sigma}(\psi \otimes v_1 \otimes \cdots \hat{v_k} \cdots \otimes v_n)\right)\right) \\
&= \langle\varphi|\psi\rangle\, v + \sum_{k=1}^{n} \sigma^k\, \langle\varphi|v_k\rangle \mathrm{Sym}_n^{\sigma}(\psi \otimes v_1 \otimes \cdots \hat{v_k} \cdots \otimes v_n)
\end{aligned}$$

Next, we compute the term $\sigma\, |\psi\rangle^{\sigma}\, \langle\varphi|^{\sigma}\, v$ to be

$$\begin{aligned}
\sigma\, |\psi\rangle^{\sigma}\, \langle\varphi|^{\sigma}\, v &= \sigma\, |\psi\rangle^{\sigma}\, \langle\varphi|^{\sigma}\, \mathrm{Sym}_n^{\sigma}(v_1 \otimes \cdots \otimes v_n) \\
&= \sigma\, n\, \mathrm{Sym}_n^{\sigma}\, |\psi\rangle_n\, \langle\varphi|_n \left(\frac{1}{n}\sum_{k=1}^{n} \sigma^{k+1} v_k \otimes \mathrm{Sym}_{n-1}^{\sigma}(v_1 \otimes \cdots \hat{v_k} \cdots \otimes v_n)\right) \\
&= \mathrm{Sym}_n^{\sigma}\, |\psi\rangle_n \sum_{k=1}^{n} \sigma^{k+2} \langle\varphi|v_k\rangle \mathrm{Sym}_{n-1}^{\sigma}(v_1 \otimes \cdots \hat{v_k} \cdots \otimes v_n) \\
&= \mathrm{Sym}_n^{\sigma}\left(\sum_{k=1}^{n} \langle\varphi|v_k\rangle\, \psi \otimes \mathrm{Sym}_{n-1}^{\sigma}(v_1 \otimes \cdots \hat{v_k} \cdots \otimes v_n)\right) \\
&= \sum_{k=1}^{n} \sigma^k\, \langle\varphi|v_k\rangle \mathrm{Sym}_n^{\sigma}(\psi \otimes v_1 \otimes \cdots \hat{v_k} \cdots \otimes v_n)
\end{aligned}$$

Subtracting the two, we see that

$$\langle\varphi|^{\sigma}\, |\psi\rangle^{\sigma}\, v - \sigma\, |\psi\rangle^{\sigma}\, \langle\varphi|^{\sigma}\, v = \langle\varphi|\psi\rangle v$$

2.

$$\begin{aligned}
\langle\varphi|^{\sigma}\, \langle\psi|^{\sigma}\, \mathrm{Sym}_n^{\sigma}(v_1 \otimes \cdots \otimes v_n) &= \langle\varphi|_{n-1}\, \langle\psi|_n \left(\frac{1}{n}\sum_{k=1}^{n} \sigma^{k+1} v_k \otimes \mathrm{Sym}_{n-1}(v_1 \otimes \cdots \hat{v_k} \cdots \otimes v_n)\right) \\
&= \frac{1}{n}\, \langle\varphi|_{n-1} \sum_{k=1}^{n} \sigma^{k+1} \langle\psi|v_k\rangle \mathrm{Sym}_{n-1}(v_1 \otimes \cdots \hat{v_k} \cdots \otimes v_n)
\end{aligned}$$

In order to use Prop. (A.1.2) a second time, we need to break the summation into two parts. This is because the index needs to be shifted by one if the second omitted vector occurs after v_k.

$$\begin{aligned}
(n-1) \cdot \mathrm{Sym}_{n-1}(v_1 \otimes \cdots \hat{v_k} \cdots \otimes v_n) = &\sum_{1 \leq j < k} \sigma^{j+1} v_j \otimes \mathrm{Sym}_{n-2}(v_1 \otimes \cdots \hat{v_j} \cdots \hat{v_k} \cdots \otimes v_n) \\
&+ \sum_{k < j \leq n} \sigma^{j}\ \ v_j \otimes \mathrm{Sym}_{n-2}(v_1 \otimes \cdots \hat{v_k} \cdots \hat{v_j} \cdots \otimes v_n)
\end{aligned}$$

Substituting this in above yields

$$n(n-1)\cdot \langle\varphi|^{\sigma} \langle\psi|^{\sigma} \mathrm{Sym}_n^{\sigma}(v_1\otimes\cdots\otimes v_n) = \Bigg(\sum_{1\leq j<k\leq n} \sigma^{j+k} \quad \langle\varphi|v_j\rangle\langle\psi|v_k\rangle \mathrm{Sym}_{n-2}(v_1\otimes\cdots\hat{v_j}\cdots\hat{v_k}\cdots\otimes v_n)$$

$$+\sum_{1\leq k<j\leq n} \sigma^{j+k+1}\langle\varphi|v_j\rangle\langle\psi|v_k\rangle \mathrm{Sym}_{n-2}(v_1\otimes\cdots\hat{v_k}\cdots\hat{v_j}\cdots\otimes v_n)\Bigg)$$

Interchanging the roles of ψ and φ corresponds to interchanging j with k. However, the same effect can also be achieved by multiplying the right hand side by σ (since $\sigma^2 = 1$). Therefore, it must be the case that $\langle\varphi|^{\sigma} \langle\psi|^{\sigma} v = \sigma \langle\psi|^{\sigma} \langle\varphi|^{\sigma} v$.

3. This is obtained by taking the adjoint of equation (A.1.6) and multiplying through by $-\sigma$.

□

A.2 Bessel Functions

The Bessel functions (also called cylinder functions) naturally arise when solving Laplace's equation $\nabla^2 F = 0$ in cylindrical coordinates. There are *many* known identities involving the Bessel functions, but we shall only record in this appendix those which are necessary for our purpose. For a full account, see [4].

Definition A.2.1. For any $\nu \in \mathbb{C}$ the functions $J_\nu(z)$ and $Y_\nu(z)$, called the **Bessel function of the first (resp. second) kind** are linearly independent solutions to the differential equation

$$z^2F''(z) + zF'(z) + (z^2 - \nu^2)F(z) = 0 \tag{A.2.1}$$

The function $J_\nu(z)$ is explicitly given by

$$J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu} \tag{A.2.2}$$

The function $Y_\nu(z)$ is given by

$$Y_\nu(z) := \lim_{\alpha\to\nu} \frac{J_\alpha(z)\cos(\pi\alpha) - J_{-\alpha}(z)}{\sin(\pi\alpha)} \tag{A.2.3}$$

The **Hankel functions** (also called **Bessel functions of the third kind**) are defined by

$$H_\nu^{(1)}(z) := J_\nu(z) + iY_\nu(z) \qquad\qquad H_\nu^{(2)}(z) := J_\nu(z) - iY_\nu(z) \tag{A.2.4}$$

Theorem A.2.1. *The functions $J_\nu(z)$, $Y_\nu(z)$, $H_\nu^{(1)}(z)$, and $H_\nu^{(2)}(z)$ have the following properties:*

(i) If $n \in \mathbb{Z}$, then $J_n(z) \in \mathcal{O}(\mathbb{C})$ is entire. If $\nu \in \mathbb{C}\setminus\mathbb{Z}$, then $J_\nu(z) \in \mathcal{O}\big(\mathbb{C}\setminus(-\infty,0]\big)$ is holomorphic except for a branch cut along the negative real axis.

(ii) For any $\nu \in \mathbb{C}$, $Y_\nu(z) \in \mathcal{O}\big(\mathbb{C}\setminus(-\infty,0]\big)$ has a branch cut along the negative real axis.

(iii) Under the Schwartz reflection $F(z) \mapsto \overline{F(\overline{z})}$, the Bessel and Hankel functions transform via

$$\overline{J_\nu(\overline{z})} = J_{\overline{\nu}}(z) \tag{A.2.5}$$

$$\overline{Y_\nu(\overline{z})} = Y_{\overline{\nu}}(z) \tag{A.2.6}$$

$$\overline{H_\nu^{(1)}(\overline{z})} = H_{\overline{\nu}}^{(2)}(z) \tag{A.2.7}$$

$$\overline{H_\nu^{(2)}(\overline{z})} = H_{\overline{\nu}}^{(1)}(z) \tag{A.2.8}$$

(iv) Let $F_\nu(z) = \alpha J_\nu(z) + \beta Y_\nu(z)$, $\alpha, \beta \in \mathbb{C}$ be any linear combination of the Bessel functions (notice this includes $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ as special cases). Then the following recurrence relations hold:

$$F_{\nu-1}(z) + F_{\nu+1}(z) = \frac{2\nu\ F_\nu(z)}{z} \tag{A.2.9}$$

$$\frac{d}{dz}F_\nu(z) = F_{\nu-1}(z) - \frac{\nu\ F_\nu(z)}{z} \tag{A.2.10}$$

In particular,

$$\frac{d}{dz}F_0(z) = -F_1(z) \tag{A.2.11}$$

(v) Let $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ denote the digamma function. Then for all $n \in \mathbb{N}$ the function $Y_n(z)$ can be expressed as the following series:

$$Y_n(z) = \frac{-1}{\pi}\sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-n} + \frac{2}{\pi}\log\left(\frac{z}{2}\right)J_n(z) - \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\psi(k+1)+\psi(n+k+1)\right)}{k!(n+k)!}\left(\frac{z}{2}\right)^{2k+n} \tag{A.2.12}$$

In particular, for $n = 1$ there exists an entire function $F \in \mathcal{O}(\mathbb{C})$ such that

$$Y_1(z) = \frac{2}{\pi}\left(\log(z)J_1(z) - \frac{1}{z}\right) \ + \ F(z) \tag{A.2.13}$$

(vi) If $\mathrm{Im}(z) > 0$ *then*

$$H_\nu^{(1)}(z) = \frac{e^{\nu\pi i/2}}{\pi i} \int_{-\infty}^{\infty} e^{iz\cosh t - \nu t} dt \tag{A.2.14}$$

(vii) If $\mathrm{Im}(z) < 0$*, then*

$$H_\nu^{(2)}(z) = \; - \; \frac{e^{\nu\pi i/2}}{\pi i} \int_{-\infty}^{\infty} e^{-iz\cosh t - \nu t} dt \tag{A.2.15}$$

Along with the Bessel functions, there are also the *modified* Bessel functions. The relation between the modified Bessel functions and the (un-modified) Bessel functions is analogous to the relationship between hyperbolic trigonometric functions and standard trigonometric functions.

Definition A.2.2. For $\nu \in \mathbb{C}$, the **modified Bessel functions of the first (resp. second) kind**, denoted $I_\nu(z)$ and $K_\nu(z)$ are linearly independent solutions to the differential equation

$$z^2 F''(z) + zF'(z) - (z^2 + \nu^2)F(z) = 0 \tag{A.2.16}$$

The function $I_\nu(z)$ is given by

$$I_\nu(z) := e^{-\nu\pi i/2} J_\nu(iz) = \sum_{n=0}^{\infty} \frac{1}{n!\;\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu} \tag{A.2.17}$$

and the function $K_\nu(z)$ is given by

$$K_\nu(z) = \lim_{\alpha\to\nu} \frac{\pi}{2} \; \frac{I_{-\alpha}(z) - I_\alpha(z)}{\sin(\pi\alpha)} \tag{A.2.18}$$

Theorem A.2.2. *The modified Bessel functions* $I_\nu(z)$ *and* $K_\nu(z)$ *satisfy the following properties:*

1. *If* $\mathrm{Re}(z) \geq 0$*, then* $J_\nu(z)$ *and* $Y_\nu(z)$ *can be expressed in terms of* $K_\nu(z)$ *via*

$$J_\nu(z) = \frac{e^{-\nu\pi i/2} K_\nu(-iz) - e^{\nu\pi i/2} K_\nu(iz)}{\pi i} \tag{A.2.19}$$

$$Y_\nu(z) = \frac{e^{-\nu\pi i/2} K_\nu(-iz) + e^{\nu\pi i/2} K_\nu(iz)}{-\pi} \tag{A.2.20}$$

2. $K_\nu(z)$ *can be written in terms of* $H_\nu^{(1)}(z)$ *or* $H_\nu^{(1)}(z)$*, depending on the phase of* z*.*
   If $-\pi \leq \arg(z) \leq \frac{\pi}{2}$*, then*

$$K_\nu(z) = \frac{\pi i}{2} \; e^{\nu\pi i/2} H_\nu^{(1)}(iz) \tag{A.2.21}$$

   If $-\frac{\pi}{2} \leq \arg(z) \leq \pi$*, then*

$$K_\nu(z) = \frac{-\pi i}{2} \; e^{-\nu\pi i/2} H_\nu^{(2)}(-iz) \tag{A.2.22}$$

3. *Under the reflection* $F(z) \mapsto \overline{F(\overline{z})}$*, the modified Bessel functions satisfy*

$$\overline{I_\nu(\overline{z})} = I_{\overline{\nu}}(z) \tag{A.2.23}$$
$$\overline{K_\nu(\overline{z})} = K_{\overline{\nu}}(z) \tag{A.2.24}$$

4. *Let* $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ *denote the digamma function. Then for all* $n \in \mathbb{N}$ *the function* $K_n(z)$ *can be expressed as the following series:*

$$\begin{aligned} K_n(z) =& \frac{1}{2}\left(\frac{z}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{-z^2}{4}\right)^k + (-1)^{n+1} \log\left(\frac{z}{2}\right) I_n(z) \\ &+ \frac{(-1)^n}{2} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{\psi(k+1) + \psi(n+k+1)}{k!(n+k)!} \left(\frac{z^2}{4}\right)^k \end{aligned}$$

In particular, for $n = 1$ there exists an entire function $F \in \mathcal{O}(\mathbb{C})$ such that

$$K_1(z) = \log(z) I_1(z) + \frac{1}{z} \; + \; F(z) \tag{A.2.25}$$

A.3 The Universal Covers of $L_+^\uparrow$ and $L_+(\mathbb{C})$

It is well known that the universal cover of the proper Lorentz group $L_+^\uparrow$ is the group $SL_2(\mathbb{C})$ of 2×2 complex matrices. In this section, we'll describe the explicit covering map $SL_2(\mathbb{C}) \twoheadrightarrow L_+^\uparrow$ and use a similar technique to show that $SL_2(\mathbb{C})\times SL_2(\mathbb{C})$ is the universal cover of the complex Lorentz group $L_+(\mathbb{C})$. For $z \in \mathbb{C}^4$, we will use the notation $z^2 := z_0^2 - z_1^2 - z_2^2 - z_3^2$.

Definition A.3.1. The **Pauli matrices** $\sigma_0, \sigma_1, \sigma_2, \sigma_3 \in M_2(\mathbb{C})$ are defined to be to following four matrices:

$$\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.3.1}$$

Definition A.3.2. Define a function σ by

$$\begin{aligned} &\sigma : \mathbb{C}^4 \to M_2(\mathbb{C}) \\ &\begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto z_0\sigma_0 + z_1\sigma_1 + z_2\sigma_2 + z_3\sigma_3 \; = \begin{pmatrix} z_0 + z_3 & z_1 - iz_2 \\ z_1 + iz_2 & z_0 - z_3 \end{pmatrix} \end{aligned}$$

The matrix $\sigma(z)$ is also typically denoted $z \cdot \sigma$ and called the **Pauli vector**. We also define a second **conjugate Pauli vector** by

$$\overline{\sigma}(z) := z_0\sigma_0 - z_1\sigma_1 - z_2\sigma_2 - z_3\sigma_3 = \begin{pmatrix} z_0 - z_3 & -z_1 + iz_2 \\ -z_1 - iz_2 & z_0 + z_3 \end{pmatrix}$$

Theorem A.3.1. *The map σ has the following properties:*

(i) The map σ is an isomorphism of complex vector spaces $\mathbb{C}^4 \cong M_2(\mathbb{C})$. Moreover, the inverse is given by

$$(\sigma^{-1} A)_\mu = \frac{1}{2} \mathrm{tr}\,(A\,\sigma_\mu)$$

(ii) Let $M_2(\mathbb{C})_{s.a.}$ denote the subset of all self-adjoint matrices:

$$M_2(\mathbb{C})_{s.a.} := \{A \in M_2(\mathbb{C}) \mid A^* = A\}$$

Then the restriction of σ to $\mathbb{R}^4$ is an isomorphism of real vector spaces $\mathbb{R}^4 \cong M_2(\mathbb{C})_{s.a.}$.

(iii) The composition $\det \circ\, \sigma : \mathbb{C}^4 \to \mathbb{C}$ is given by

$$\det \sigma(z) = z^2 \tag{A.3.2}$$

(iv) For all $z \in \mathbb{C}^4$,

$$\sigma(z)\;\overline{\sigma}(z) = \overline{\sigma}(z)\;\sigma(z)\; = z^2 \cdot \mathbb{1}_2 \tag{A.3.3}$$

In particular, if $z^2 \neq 0$ then

$$\sigma(z)^{-1} = \frac{\overline{\sigma}(z)}{z^2} = \overline{\sigma}\left(\frac{z}{z^2}\right)$$

(v) For all $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C})$*, we have*

$$\overline{\sigma}(\sigma^{-1}(A)) \;=\; \sigma_2\, A^T\, \sigma_2 = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{A.3.4}$$

If A *is invertible, this can be expressed as*

$$\overline{\sigma}(\sigma^{-1}(A)) = \det(A)\, A^{-1} \tag{A.3.5}$$

Proof. Straightforward □

Now we can describe the covering map $SL_2(\mathbb{C}) \twoheadrightarrow L_+^\uparrow$

Definition A.3.3. Define a function π by

$$\begin{aligned} \pi : SL_2(\mathbb{C}) &\to (\mathbb{R}^4 \to \mathbb{R}^4) \\ A &\mapsto \Big(x \mapsto \sigma^{-1}\big(A\ \sigma(x)\ A^*\big)\Big) \end{aligned}$$

Theorem A.3.2. *The function* π *has the following properties:*

1. π *is well-defined, and its image is* $\pi(SL_2(\mathbb{C})) = L_+^\uparrow$.
2. $\pi(AB) = \pi(A)\pi(B)$
3. $\pi(A) = \pi(B) \iff A = B$ *or* $A = -B$.
4. $SL_2(\mathbb{C})$ *is the universal cover of* $L_+^\uparrow$ *and* $\pi : SL_2(\mathbb{C}) \to L_+^\uparrow$ *is the 2-sheeted covering map.*

Proof. □

The universal cover of $L_+(\mathbb{C})$ is found in a similar way.

Definition A.3.4. Define a function Π by

$$\begin{aligned} \Pi : SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) &\to (\mathbb{C}^4 \to \mathbb{C}^4) \\ (A, B) &\mapsto \Big(z \mapsto \sigma^{-1}\big(A\ \sigma(z)\ B^T\big)\Big) \end{aligned}$$

Evidently, Π is a generalization of π because the two are related by $\pi(A) = \Pi(A, \overline{A})$.

Theorem A.3.3. *The function* Π *has the following properties:*

1. Π *is well-defined and it's image is* $\Pi\big(SL_2(\mathbb{C}) \times SL_2(\mathbb{C})\big) = L_+(\mathbb{C})$.
2. $\Pi(A_1A_2, B_1B_2) = \Pi(A_1, B_1)\Pi(A_2, B_2)$
3. $\Pi(A_1, B_2) = \Pi(A_2, B_2) \iff (A_1, B_1) = (A_2, B_2)$ *or* $(A_1, B_1) = (-A_2, -B_2)$
4. $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ *is the universal cover of* $L_+(\mathbb{C})$ *and* Π *is the 2-sheeted covering map.*

Proof. □

A.4 Gamma Matrices and Feynman Slash

The gamma matrices are a collection $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\} \subseteq GL(4, \mathbb{C})$ of 4×4 complex matrices which satisfy the following anti-commutation relations:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \mathbb{1}_4 \tag{A.4.1}$$

where $\mathbb{1}_4$ denotes the 4×4 identity matrix and η is the Minkowski metric with signature $(+\ -\ -\ -)$

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Any set of matrices satisfying (A.4.1) can be called gamma matrices and their important properties are independent of the choice of representation. We will use the **chiral** representation.

Definition A.4.1. The **(chiral) gamma matrices** are defined by

$$\gamma^\mu := \begin{pmatrix} 0 & \eta^{\mu\mu}\sigma_\mu \\ \sigma_\mu & 0 \end{pmatrix} \tag{A.4.2}$$

where σ_μ are the Pauli matrices (Def A.3.1). More explicitly, this means

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Remark A.4.1. Another commonly used representation of the gamma matrices is the *Pauli Representation*, in which γ^0 is diagonal:

$$\gamma^0_P = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} \qquad \gamma^j_P = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}$$

These two representations of the gamma matrices are unitary equivalent.

$$\gamma^\mu_P = U\gamma^\mu U^{-1} \qquad U = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_0 & \sigma_0 \\ \sigma_0 & -\sigma_0 \end{pmatrix} \tag{A.4.3}$$

Recall we defined the conjugate Pauli vector $\overline{\sigma}(z) = z_0\sigma_0 - z_1\sigma_1 - z_2\sigma_2 - z_3\sigma_3$ which associates to each $z \in \mathbb{C}^4$ a 2×2 matrix. We can do something similar with the gamma matrices, assigning to each $z \in \mathbb{C}^4$ the 4×4 matrix $z_0\gamma^0 - z_1\gamma^1 - z_2\gamma^2 - z_3\gamma^3$. This is called the *Feynman Slash*.

Definition A.4.2. Define a function

$$\begin{aligned} \gamma : \mathbb{C}^4 &\to M_4(\mathbb{C}) \\ z &\mapsto z_0\gamma^0 - z_1\gamma^1 - z_2\gamma^2 - z_3\gamma^3 \end{aligned}$$

More explicitly, this means

$$\gamma(z) = \begin{pmatrix} 0 & \sigma(z) \\ \overline{\sigma}(z) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & z_0 + z_3 & z_1 - iz_2 \\ 0 & 0 & z_1 + iz_2 & z_0 - z_3 \\ z_0 - z_3 & -z_1 + iz_2 & 0 & 0 \\ -z_1 - iz_2 & z_0 + z_3 & 0 & 0 \end{pmatrix}$$

The matrix $\gamma(z)$ is called the **Feynman Slash**, and is traditionally written with the infix notation $\not{z} = \gamma(z)$.

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