The Geometry of Classical and Quantum Fields

Markus J. Pflaum

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Authors

The following people have contributed to this work, in alphabetical order:

Jonathan Belcher
Markus J. Pflaum (main author)
Daniel Spiegel

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Contents

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The main author of this work is Markus J. Pflaum.

Jonathan Belcher contributed notes of lectures held by M.J. Pflaum. From these notes the following material has been incorporated:

- Remark 3.1.8,
- in Section 3.2, the statement of Theorem 3.2.5 (Wigner's theorem), and Theorem 3.2.8 (Bargmann's theorem),
- Proposition 15.3.3.

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Introduction

Classical and quantum mechanical systems are mathematically described in a different way. For finitely many degrees of freedom, differential geometry, notably symplectic and Poisson geometry, provides the language in which classical mechanical systems are described, whereas functional analysis and in particular the theory of Hilbert spaces is the appropriate language in which quantum mechanics is formulated. The mathematics is well understood in both situations, and one even has a powerful tool for the passage from the classical to the quantum mechanical description of a corresponding system, namely quantization theory.

In their book (*Mathematical Concepts of Quantum Mechanics*, Gustafson & Sigal, 2011), the authors depict the situation by the following diagram, where $d \to \infty$ denotes the passage from finitely to infinitely many degrees of freedom.

$$\begin{tikzcd} \mathrm{CM} \arrow[r, "\text{quantization}"] \arrow[d, "q\to\infty"'] & \mathrm{QM} \arrow[d, "q\to\infty"] \\ \mathrm{CFT} \arrow[r, "\text{quantization}"'] & \mathrm{QFT} \end{tikzcd}$$

The key ingrediants for the description of a physical system are the mathematical objects which encode its state space, the observable space, and its dynamics. These objects should depend in some functorial on the system and usually come from quite distinct categories, depending on whether the system is classical or quantum, has finitely or infinitely many degrees of freedom.

Part I.

Classical Field Theory

1. Variational calculus

1.1. Euler–Lagrange equations

Regular domains

Let M be a smooth manifold of finite type which means that M is diffeomorphic to the interior of a compact manifold-with-boundary $\overline{M}$. Denote by d the dimension of M and assume that $d > 0$. By a *regular domain* in M we understand a non-empty open connected subset $\Omega \subset M$ such that its closure $\bar{\Omega}$ in $\overline{M}$ possesses a piecewise smooth triangulation $\kappa : |K| \to \bar{\Omega}$, where $|K|$ is the underlying topological space of a finite (geometric) simplicial complex. It is further assumed that $\kappa^{-1}(\partial\Omega)$ is a simplicial subcomplex of K of dimension $< d$ where $\partial\Omega$ denotes the (topological) boundary of Ω. Piecewise smoothness of the triangulation κ hereby means that for every simplex $\sigma \in K$ the restriction $\kappa|_\sigma : \sigma \to \kappa(\sigma)$ is a diffeomorphism onto its image. More precisely, $\kappa|_\sigma$ being a diffeomorphism just says that for every smooth f defined on an open neighborhood of $\kappa(\sigma)$ the pullback $\kappa|_\sigma^* f$ can be extended to a smooth function on the euclidean space $\mathbb{R}^n$ in which the simplicial complex K lies and that for every smooth g defined on an open neighborhood of the simplex $\sigma \subset \mathbb{R}^n$ the pullback $\left(\kappa|_\sigma^{-1}\right)^* g$ has a smooth extension to $\overline{M}$.

In most applications and in particular in those which we will consider in this section, Ω will be the interior of a submanifold-with-corners of $\overline{M}$ (see **??** and **?** for details on manifolds-with-corners and submanifolds in this category). In the proofs in this section involving regular domains we will therefore consider mostly this particular case and only briefly indicate how the argument in the more general situation goes. For ease of presentation we will call a regular domain $\Omega \subset M$ such that $\bar{\Omega} \subset M$ is a submanifold-with-corners a *strongly regular domain*.

In most cases we choose M to coincide with the euclidean space $\mathbb{R}^d$. Note that $\mathbb{R}^d$ is a manifold of finite type and that it is diffeomorphic to the interior of the d-disc $\mathbb{D}^d$. A diffeomorphism between the euclidean space $\mathbb{R}^d$ and the interior of the disc of the same dimension is given by the smooth map $\varphi : \mathbb{R}^d \to \mathring{\mathbb{D}}^d$, $x \mapsto \frac{1}{\sqrt{1+\|x\|^2}} x$. It has inverse $\psi : \mathring{\mathbb{D}}^d \to \mathbb{R}^d$, $y \mapsto \frac{1}{\sqrt{1-\|y\|^2}} y$ as the following two equalities show.

$$\varphi(\psi(y)) = \frac{1}{\sqrt{1 + \frac{\|y\|^2}{1-\|y\|^2}}} \frac{1}{\sqrt{1 - \|y\|^2}} y = y$$

$$\psi(\varphi(x)) = \frac{1}{\sqrt{1 - \frac{\|x\|^2}{1+\|x\|^2}}} \frac{1}{\sqrt{1 + \|x\|^2}} x = x$$

In the remainder of this section we will identify $\mathbb{R}^d$ with its image in the d-disc and call $\mathbb{D}^d$ the *radial compactification* of $\mathbb{R}^d$. The boundary $\partial\mathbb{D}^d = \mathbb{D}^d \backslash \varphi(\mathbb{R}^d)$ will be called the $d-1$-sphere at infinity and denoted by the symbol $\mathbb{S}^{d-1}_\infty$.

The local case

Assume that $\Omega \subset \mathbb{R}^d$ is an open subset which can be identified with the interior of a compact submanifold-with-boundary $\bar{\Omega} \subset \mathbb{D}^d$ of the disc $\mathbb{D}^d$ under the above diffeomorphism $\varphi : \mathbb{R}^d \to \mathring{\mathbb{D}}^d$. Later we will relax this assumption and allow Ω to be a regular domain in $\mathbb{R}^d$. By assumption, the boundary $\partial\Omega$ is a submanifold of $\mathbb{D}^d$. We interpret the preimages $\varphi^{-1}(\bar{\Omega})$ and $\varphi^{-1}(\partial\Omega)$ as intersections $\bar{\Omega} \cap \mathbb{R}^d$ and $\partial\Omega \cap \mathbb{R}^d$, respectively. Note that both of these spaces are submanifolds of $\mathbb{R}^d$, the first one possibly with boundary. Let $(x^1, \ldots, x^d) : \Omega \to \mathbb{R}^d$ be the canonical coordinates of Ω. Observe that $\bar{\Omega} \cap \mathbb{R}^d$ and Ω are oriented by the restriction of the canonical volume form $dx^1 \wedge \ldots \wedge dx^d$ to $\bar{\Omega} \cap \mathbb{R}^d$. We denote that restriction by ω.

Next assume that $\pi : E = \bar{\Omega} \times F \to \bar{\Omega}$ is a trivial smooth fiber bundle with fiber F being a connected open subset of some euclidean space $\mathbb{R}^n$. The canonical fiber coordinates will be denoted by $(u^1, \ldots, u^n) : F \to \mathbb{R}^n$. The canonical charts of the interior of the base and the fiber give rise to a fibered chart $(x, u) : E = \Omega \times F \to \mathbb{R}^d \times \mathbb{R}^n$.

Finally, we assume to be given a *lagrangian function* $L \in \mathcal{C}^\infty_{\mathrm{loc}}\left(\mathsf{J}^\infty\pi\right)$. Since L is a local function on the jet bundle, it can be regarded as an element of $\mathcal{C}^\infty\left(\mathsf{J}^k\pi\right)$ for some natural k. Let $o = o(L)$ be the smallest of such numbers and call it the *order* of the langragian function. The canonical volume form ω together with the lagrangian L give rise to the *lagrangian density* $\mathcal{L} = L\,\omega$ on the jet bundle $\mathsf{J}^\infty\pi$. Before we can write down the action functional induced by the lagrangian density $\mathcal{L}$ we need to fix some boundary conditions. For now, we will restrict to the so-called *Dirichlet boundary conditions* (*with compact support*). These are encoded by smooth sections $b : \partial\Omega \to E$ with support being compact and contained in $\partial\Omega \cap \mathbb{R}^d$. More precisely, the *space of Dirichlet boundary conditions* over the regular domain Ω is defined by

$$\mathcal{BC}^\infty_{\mathrm{D}}(\Omega; E) = \Gamma^\infty_0(\partial\Omega; E) = \left\{ b \in \Gamma^\infty(\partial\Omega; E) \;\middle|\; \operatorname{supp} b \subset\subset \partial\Omega \cap \mathbb{R}^d \right\} .$$

Given an element $b \in \mathcal{BC}^\infty_{\mathrm{D}}(\Omega; E)$ we single out the space X_b of *allowable sections* of E:

$$\mathrm{X}_b = \left\{ s \in \Gamma^\infty_0(\bar{\Omega}; E) \;\middle|\; s|_{\partial\Omega} = b \right\} ,$$

where $\Gamma^\infty_0(\bar{\Omega}; E)$ denotes the space of all smooth sections $s : \bar{\Omega} \to E$ having compact support contained in $\bar{\Omega} \cap \mathbb{R}^d$. In other words, X_b consists of all smooth sections $s : \bar{\Omega} \to E$ which fulfill the Dirichlet boundary condition $s|_{\partial\Omega} = b$. Observe that X_b is an affine space over the vector space $\mathrm{V} = \left\{ s \in \Gamma^\infty_0(\bar{\Omega}; E) \;\middle|\; s|_{\partial\Omega} = 0 \right\}$. That space carries a natural locally convex topology. More precisely, its locally convex structure is given as the colimit of the strict inductive system of Fréchet spaces $\mathrm{V}_N = \left\{ s \in \mathrm{V} \;\middle|\; \operatorname{supp} s \subset \bar{\Omega} \cap \bar{\mathbb{B}}_N(0, \mathbb{R}^d) \right\}$, $N \in \mathbb{N}$. The affine space X_b inherits the locally convex topology from V and thus becomes a manifold globally modeled on V. The tangent bundle of X_b then is canonically isomorphic to the product manifold $\mathrm{X}_b \times \mathrm{V}$.

Now we can write down the *action functional* associated to the lagrangian density $\mathcal{L}$:

$$\mathrm{S} : \mathrm{X}_b \to \mathbb{R},\; s \mapsto \int_{\bar{\Omega} \cap \mathbb{R}^d} \left(\mathsf{j}^\infty s\right)^* \mathcal{L} = \int_{\bar{\Omega} \cap \mathbb{R}^d} \left(L \circ \mathsf{j}^\infty s\right) \omega \; . \tag{1.1.1}$$

Note that even though the domain of integration might be unbounded, the integral is well-defined for every $s \in \mathrm{X}_b$ since $L \circ \mathsf{j}^\infty s$ has compact support contained in $\bar{\Omega} \cap \mathbb{R}^d$ whenever s has that property.

1.1.1 Proposition *Assume that $\Omega \subset \mathbb{R}^d$ is an open subset and $\pi : E = \bar{\Omega} \times F \to \bar{\Omega}$ a trivial fiber bundle which fulfill the above assumptions. Let b be an element of the space $\mathcal{BC}^\infty_{\mathrm{D}}(\Omega; E)$ of Dirichlet boundary conditions. Then the action functional $\mathrm{S} : \mathrm{X}_b \to \mathbb{R}$ associated to a lagrangian density $\mathcal{L}$ is continuous on the space X_b of allowable sections. Moreover, S is Gateaux differentiable and the functional derivative $\delta S : T\mathrm{X}_b = \mathrm{X}_b \times \mathrm{V} \to \mathbb{R}$ is given by*

Proof. We first show that the functional S is continuous with respect to the locally convex topology on X_b. So choose a convergent sequence $(s_k)_{k\in\mathbb{N}}$ in X_b and let $s \in \mathrm{X}_b$ its limit. By definition of the topology of X_b there exists a positive natural N such that $s - b \in V_N$ and $s_k - b \in V_N$ for all $k \in \mathbb{N}$. Since the support of b is compact in $\partial\Omega \cap \mathbb{R}^d$, we can assume after possibly increasing N that $\operatorname{supp} b \subset \overline{\mathbb{B}}_N(0, \mathbb{R}^d)$. Hence the supports of s and each s_k are contained in $\bar{\Omega} \cap \overline{\mathbb{B}}_N(0, \mathbb{R}^d)$, and for every $\alpha \in \mathbb{N}^d$ the sequence $\left(\frac{\partial^{|\alpha|} s_k}{\partial x^\alpha}\right)_{k\in\mathbb{N}}$ converges uniformly on $\bar{\Omega} \cap \overline{\mathbb{B}}_N(0, \mathbb{R}^d)$ to $\frac{\partial^{|\alpha|} s}{\partial x^\alpha}$. Since the lagrangian function L has finite order, the compositions $L \circ \mathrm{j}^\infty s_k$ and $L \circ \mathrm{j}^\infty s$ also have compact support contained in $\bar{\Omega} \cap \overline{\mathbb{B}}_N(0, \mathbb{R}^d)$, and the sequence $(L \circ \mathrm{j}^\infty s_k)_{k\in\mathbb{N}}$ converges uniformly on $\bar{\Omega} \cap \overline{\mathbb{B}}_N(0, \mathbb{R}^d)$ to $L \circ \mathrm{j}^\infty s$. Hence the sequence of integrals $\int_{\bar{\Omega}\cap\mathbb{R}^d} \left(L \circ \mathrm{j}^\infty s_k\right) \omega$ converges to $S(s) = \int_{\bar{\Omega}\cap\mathbb{R}^d} \left(L \circ \mathrm{j}^\infty s\right) \omega$, and the action functional S is continuous.

To prove Gateaux differentiability, let $s \in \mathrm{X}_b$, $v \in \mathrm{V}$ and compute, where o is the order of the langrangian and $\mathcal{I} = \{1, \ldots, d\}$:

$$\begin{aligned}
\lim_{t\to 0} \frac{S(s+tv) - S(s)}{t} &= \lim_{t\to 0} \int_{\bar{\Omega}\cap\mathbb{R}^d} \frac{L \circ \mathrm{j}^\infty(s+tv) - L \circ \mathrm{j}^\infty s}{t} \omega = \\
&= \int_{\bar{\Omega}\cap\mathbb{R}^d} \left(\lim_{t\to 0} \frac{L \circ \mathrm{j}^\infty(s+tv) - L \circ \mathrm{j}^\infty s}{t} \right) \omega = \\
&= \sum_{i=1}^{d} \int_{\bar{\Omega}\cap\mathbb{R}^d} \omega \sum_{a=1}^{n} \sum_{I\in\mathcal{I}^\bullet}
\end{aligned}$$

□

1.1.2 We now want to find the extremal points s_0 of the functional S, if such exist. To this end we first derive a necessary condition for $s_0 \in \mathrm{X}_b$ to be an extremal point of S.

1.2. The variational bicomplex

The Cartan distribution

1.2.1 We start with a smooth fiber bundle $\pi : E \to M$ over a d-dimensional manifold M. The typical fiber is denoted F and assumed to have dimension n. Consider the *infinite jet bundle* $\mathsf{J}^\infty E \to M$ and recall that $(\mathsf{J}^\infty E, \mathcal{C}^\infty)$ is the pro-manifold defined as the limit of the (cofiltered) diagram

$$\left(E, \mathcal{C}^\infty\right) \xleftarrow{\pi_{0,1}} \left(\mathsf{J}^1 E, \mathcal{C}^\infty\right) \xleftarrow{\pi_{1,2}} \ldots \xleftarrow{\pi_{k-1,k}} \left(\mathsf{J}^k E, \mathcal{C}^\infty\right) \xleftarrow{\pi_{k,k+1}} \ldots \, . \tag{1.2.1}$$

in the category of commutative locally $\mathbb{R}$-ringed spaces. This means that in the category of topological spaces $\mathsf{J}^\infty E$ coincides with $\lim\limits_{k\in\mathbb{N}} \mathsf{J}^k E$ and that the structure sheaf $\mathcal{C}^\infty_{\mathsf{J}^\infty E}$ is given by $\operatorname*{colim}\limits_{k\in\mathbb{N}} \pi^*_{k,\infty} \mathcal{C}^\infty_{\mathsf{J}^k E}$, where the $\pi_{k,\infty} : \mathsf{J}^\infty E \to \mathsf{J}^k E$ are the natural maps from the (topological) limit to the objects of the diagram. Recall also that $\mathcal{C}^\infty_{\mathrm{loc},\mathsf{J}^\infty E}$, or just $\mathcal{C}^\infty_{\mathrm{loc}}$ when no confusion can arise, stands for the presheaf of *local functions* on the infinite jet bundle. Its space of sections over some open $U \subset \mathsf{J}^\infty E$ consists of all continuous maps $f : U \to \mathbb{R}$ for which there exists a $k \in \mathbb{N}$, an open $U_k \subset \mathsf{J}^k E$ and a smooth function $f_k : U_k \to \mathbb{R}$ such that $U \subset \pi^{-1}_{k,\infty}(U_k)$ and $f = f_k \circ \pi_{k,\infty}|_U$.

Next observe that the diagram Equation (1.2.1) of jet bundles of finite order induces another filtered diagram by taking tangent bundles and tangent maps:

$$\big(TE, \mathcal{C}^\infty\big) \xleftarrow{T\pi_{0,1}} \big(T\mathsf{J}^1 E, \mathcal{C}^\infty\big) \xleftarrow{T\pi_{1,2}} \ldots \xleftarrow{T\pi_{k-1,k}} \big(T\mathsf{J}^k E, \mathcal{C}^\infty\big) \xleftarrow{T\pi_{k,k+1}} \ldots \, . \tag{1.2.2}$$

The resulting limit in the category of commutative locally $\mathbb{R}$-ringed spaces is called the *tangent bundle* of the pro-manifold $(\mathsf{J}^\infty E, \mathcal{C}^\infty)$ and is denoted $(T\mathsf{J}^\infty E, \mathcal{C}^\infty)$. One writes $T\pi_{k,\infty} : T\mathsf{J}^\infty E \to T\mathsf{J}^k E$ for the natural maps of the limit and finally observes that the family of canonical projections $\pi_k : \mathsf{J}^k E \to M$ is compatible with the diagram Equation (1.2.1) in the sense that $\pi_l = \pi_k \circ \pi_{k,l}$ for all $k \leqslant l$. Hence one obtains a smooth map $\pi_\infty : \mathsf{J}^\infty E \to M$ uniquely determined by the property that $\pi_\infty = \pi_k \circ \pi_{k,\infty}$ for all $k \in \mathbb{N}$. Its tangent map $T\pi_\infty : T\mathsf{J}^\infty E \to TM$ obviously then satisfies $T\pi_\infty = T\pi_k \circ T\pi_{k,\infty}$ for all $k \in \mathbb{N}$ and is also uniquely determined by that property.

Now we have all the tools to define the main object of this section, the Cartan distribution.

1.2.2 Let p be a point of the base manifold M. Choose an open contractible neighborhood $U \subset M$ of p over which there exists a coordinate system $x : U \to \mathbb{R}^d$. Denote by $\mathcal{E}_U$ the space of smooth sections of the bundle $\pi : E \to M$ over U and by I_ε the open interval $(-\varepsilon, \varepsilon)$ around 0. By Borel's theorem, the jet map $\mathsf{j}^\infty_q : \mathcal{E}_U \to \mathsf{J}^\infty_q E$ is surjective for every $q \in U$. Call a smooth path

$$\gamma = (\sigma, m) : I_\varepsilon \to \mathcal{E}_U \times U,\ t \mapsto (\sigma_t, m_t)$$

with $m_0 = p$ *vertical over* p if m is a constant path and *horizontal over* p if σ is a constant path. Smoothness of σ hereby means that $\sigma^\vee : I_\varepsilon \times U \to E,\ (t,q) \mapsto \sigma_t(q)$ is smooth. The composition

$$\mathsf{j}^\infty \circ \gamma : I_\varepsilon \to \mathsf{J}^\infty E,\ t \mapsto \mathsf{j}^\infty_{m_t}(\sigma_t)$$

then is a smooth path in the jet bundle and the derivative $(\mathsf{j}^\infty \circ \gamma)'(0) = \frac{d}{dt}(\mathsf{j}^\infty \circ \gamma)|_{t=0}$ an element of the tangent space $T_\theta \mathsf{J}^\infty E$ over the footpoint $\theta = \mathsf{j}^\infty_p(\sigma_0)$. In case γ is vertical, the path $\pi_\infty \circ \mathsf{j}^\infty \circ \gamma$ is constant with value p which implies that the tangent vector $(\mathsf{j}^\infty \circ \gamma)'(0)$ has to be an element of the vertical bundle $\mathsf{V}\pi_\infty = \ker T\pi_\infty \subset T\mathsf{J}^\infty E$. Let us show that every vertical tangent vector with footpoint θ can be obtained that way. So let $v \in \mathsf{V}_\theta \pi_\infty$ be represented by a smooth path $\varrho : (-\varepsilon, \varepsilon) \to \mathsf{J}^\infty E$ such that $\pi_\infty(\varrho(t)) = p$ for all t. After possibly shrinking U and ε one can assume that there exists a fibered chart $(x, u) : \widetilde{U} \to \mathbb{R}^d \times \mathbb{R}^n$ over some open $\widetilde{U} \subset E$ such that $\pi(\widetilde{U}) = U$, (x, u) is trivialising in the sense that its image coincides with the cartesian product of $x(U)$ and an open $V \subset \mathbb{R}^n$ and such that $\pi_{0,\infty}(\gamma(t)) \in \widetilde{U}$ for all t. One obtains a family of smooth real valued functions $u^a \circ \varrho, u^a_i \circ \varrho, \ldots, u^a_I \circ \varrho, \ldots$, where the index a runs through $1, \ldots, n$, the index i through $1, \ldots, d$, and I through the all (Roman) multiindices over $\{1, \ldots, d\}$ (of order $\geqslant 2$). By Borel's Theorem with parameters (Kriegl & Michor, 1997, 15.4) there exists a smooth function $s = (s^1, \ldots, s^n) : I_\varepsilon \times U \to V$ such that $\frac{\partial^{|I|} s^a}{\partial x^I}(t, p) = u^a_I \circ \varrho(t)$ for all t. Let $s : I_\varepsilon \to \mathcal{E}_U$ be the smooth path of sections $t \mapsto \sigma(t,-)$, $m : I_\varepsilon \to M$ the constant path at p and let $\gamma = (\sigma, m)$. Then γ is vertical and, by construction,

$$(\mathsf{j}^\infty \circ \gamma)'(0) = \varrho'(0) = v \ .$$

This shows the claim.

Next assume to be given a jet $\theta \in \mathsf{J}^\infty_p E$. Define the *horizontal space* at that jet by

$$\mathsf{C}_\theta \mathsf{J}^\infty E = \left\{ (\mathsf{j}^\infty \circ \gamma)'(0) \in T_\theta \mathsf{J}^\infty E \;\middle|\; \gamma = (\sigma, m) \text{ is horizontal over } p \text{ and } \mathsf{j}^\infty \sigma_0 = \theta \right\} .$$

One calls $\mathsf{CJ}^\infty E = \bigcup_{\theta \in \mathsf{J}^\infty E} \mathsf{C}_\theta \mathsf{J}^\infty E$ the *Cartan distribution* on the jet bundle $\mathsf{J}^\infty E$. In the following we will study its properties and will show that it is an involutive distribution on the jet bundle which is complementary to the vertical bundle.

1.2.3 Lemma *Let $\theta \in \mathsf{J}^\infty_p E$ be a jet and choose a trivialising fibered chart $(x, u) : \widetilde{U} \to \mathbb{R}^d \times \mathbb{R}^n$ around an open neighborhood of $e = \pi_{1,\infty}(\theta)$. Let $m : I_\varepsilon \to U$ be a smooth path with $m_0 = p$, $\sigma : I_\varepsilon \to \mathcal{E}_U$ a smooth path of sections and finally $\tau : U \to E$ a smooth section such that that the images of all σ_t and τ are in $\widetilde{U}$ and such that $\mathsf{j}^\infty_p(\sigma_0) = \mathsf{j}^\infty_p(\tau) = \theta$. Denote by m^i the composition $x^i \circ m$ and by σ^a and τ^a the compositions $u^a \circ \sigma$ and $u^a \circ \tau$, respectively. Then the tangent vector of the vertical path (σ, p) is given by*

$$\left(\mathsf{j}^\infty_p \sigma_t\right)'(0) = \sum_{a=1}^{n} \sum_{I} \frac{\partial^{|I|} (\sigma^a)'(0)}{\partial x^I}(p) \frac{\partial}{\partial u^a_I} \tag{1.2.3}$$

and the tangent vector of the horizontal path (τ, m) by

$$\left(\mathsf{j}^\infty_{m_t} \tau\right)'(0) = \sum_{i=1}^{d} (m^i)'(0) \left(\frac{\partial}{\partial x^i} + \sum_{a=1}^{n} \sum_{I} \frac{\partial^{|I|+1} \tau^a}{\partial x^i \partial x^I}(p) \frac{\partial}{\partial u^a_I} \right) . \tag{1.2.4}$$

In these formulas, I runs through all Roman multiindices over $\{1, \ldots, d\}$.

Proof. Let $\gamma = (\sigma, m)$. Then in the selected fibered chart

$$x^i \circ \mathsf{j}^\infty \circ \gamma = m^i \quad \text{and} \quad (u^a_I \circ \mathsf{j}^\infty \circ \gamma)_t = \frac{\partial^{|I|} \sigma^a_t}{\partial x^I}(m_t) \ ,$$

from which the claim follows by specialization to $m_t = p$ respectively $\sigma_t = \tau$ and the chain rule.□

1.2.4 Lemma *Let $\theta \in \mathsf{J}^\infty_p E$ be a jet and $\sigma, \tau : U \to E$ two smooth sections such that $\mathsf{j}^\infty_p(\sigma) = \mathsf{j}^\infty_p(\tau) = \theta$. Then for every smooth path $m : I_\varepsilon \to M$ with $m_0 = p$ the equality*

$$\left(\mathsf{j}^\infty_{m_t} \sigma\right)'(0) = \left(\mathsf{j}^\infty_{m_t} \tau\right)'(0)$$

holds true, where $'$ denotes the derivative with respect to the parameter t. Hence

$$\begin{aligned} \mathsf{C}_\theta \mathsf{J}^\infty E &= \left\{ \left(\mathsf{j}^\infty_{m_t} \sigma\right)'(0) \in T_\theta \mathsf{J}^\infty E \;\middle|\; m \in \mathcal{C}^\infty(I_\varepsilon, M) \ \& \ m_0 = p \right\} \\ &= \left\{ \left(\mathsf{j}^\infty_{m_t} \tau\right)'(0) \in T_\theta \mathsf{J}^\infty E \;\middle|\; m \in \mathcal{C}^\infty(I_\varepsilon, M) \ \& \ m_0 = p \right\} . \end{aligned} \tag{1.2.5}$$

1.2.5 Remark The lemma implies in particular that the horizontal space $\mathsf{C}_\theta \mathsf{J}^\infty E$ does not depend on the choice of a section representing θ.

Proof. After possibly shrinking U and ε choose a trivialising fibered chart $(x,u) : \widetilde{U} \to \mathbb{R}^d \times \mathbb{R}^n$ around an open neighborhood of $\sigma(p) = \tau(p)$ as above. Moreover, we can assume after possible shrinking U and ε again that both $\sigma(U)$ and $\tau(U)$ are contained in $\widetilde{U}$. Then compute

$$\begin{aligned}\left(\mathsf{j}^\infty_{m_t}\sigma\right)'(0) &= \sum_{i=1}^d (m^i)'(0)\left(\frac{\partial}{\partial x^i} + \sum_{a=1}^n \sum_I \frac{\partial^{|I|+1}\sigma^a}{\partial x^i \partial x^I}(p)\frac{\partial}{\partial u^a_I}\right)\\ &= \sum_{i=1}^d (m^i)'(0)\left(\frac{\partial}{\partial x^i} + \sum_{a=1}^n \sum_I \frac{\partial^{|I|+1}\tau^a}{\partial x^i \partial x^I}(p)\frac{\partial}{\partial u^a_I}\right) = \left(\mathsf{j}^\infty_{m_t}\tau\right)'(0)\ ,\end{aligned}$$

where $m^i = x^i \circ m$, $\sigma^a = u^a \circ \sigma$, $\tau^a = u^a \circ \tau$ and where I runs through the Roman multiindices over the index set $\{1, \ldots, d\}$. This proves the claim. $\square$

1.2.6 Lemma *For every section $\sigma \in \mathcal{E}_U$ the map*

$$T_pM \to T_pM,\ m'(0) \mapsto \left(\pi_\infty \circ \mathsf{j}^\infty_{m_t}(\sigma)\right)'(0)$$

is the identity map, where tangent vectors at p are represented as derivatives of smooth paths $m : I_\varepsilon \to M$ at p.

Proof. This is trivial, since $\pi_\infty \circ \mathsf{j}^\infty_{m_t}(\sigma) = m_t$ for all $t \in I_\varepsilon$. $\square$

Despite the lemma being trivial, some of its consequences are not.

1.2.7 Proposition *For every smooth fiber bundle $\pi : E \to M$ the Cartan distribution is a smooth involutive vector subbundle of the tangent bundle on $\mathsf{J}^\infty E$. The Cartan distribution has fiber dimension $d = \dim M$. In a fibered chart $(x,u) : \widetilde{U} \to \mathbb{R}^d \times \mathbb{R}^n$ a local frame for the Cartan distribution is given by the family of vector fields*

$$D_i = \frac{\partial}{\partial x^i} + \sum_{a=1}^n \sum_I u^a_{Ii} \frac{\partial}{\partial u^a_I}\ ,\ i = 1, \ldots, d\ ,$$

where the right summation is taken over all Roman multiindices I.

Proof. By Lemma 1.2.6 it is clear that $\dim \mathsf{C}_\theta \mathsf{J}^\infty E = d$ for every $\theta \in \mathsf{J}^\infty E$.

2. Semi-riemannian geometry

2.1. Causal structures

2.1.1 In this section, we let (M, g) denote a connected lorentzian manifold of dimension $D = d+1$, $d \in \mathbb{N}_{>0}$. In particular this means that the signature of the semi-riemannian structure g is $(1, d)$ or, in different notation, $(+, -, \ldots, -)$. At each point $p \in M$ the tangent space T_pM then canonically carries the structure of a D-dimensional lorentzian vector space. Denote by $q_{\mathrm{L}} : TM \to \mathbb{R}$ the *Lorentz quadratic form* $v \mapsto g(v, v)$. With these notational agreements in mind we now make the following definition.

2.1.2 Definition A tangent vector $v \in TM$ is called

(i) *lightlike* or *null* if $v \neq 0$ and $q_{\mathrm{L}}(v) = 0$,

(ii) *timelike* if $q_{\mathrm{L}}(v) > 0$,

(iii) *spacelike* if $v = 0$ or $q_{\mathrm{L}}(v) < 0$, and

(iv) *causal* (or *non-spacelike*) if $v \neq 0$ and $q_{\mathrm{L}}(v) \geqslant 0$.

A piecewise differentiable curve $\gamma : [a, b] \to M$, $-\infty \leqslant a < b \leqslant \infty$, is called *lightlike*, *timelike*, or *spacelike* if each of its tangent vectors is so, respectively.

2.1.3 Proposition

2.1.4

From now on, we assume that M is *temporally orientable* that is that

Part II.

Quantum Mechanics

3. The postulates of quantum mechanics

3.1. The geometry of projective Hilbert spaces

3.1.1 Let $\mathfrak{H}$ be a Hilbert space over the field $\mathbb{K} = \mathbb{R}$ or $= \mathbb{C}$. The associated *projective Hilbert space* $\mathbb{P}\mathfrak{H}$ then is defined as the space of all *rays* in $\mathfrak{H}$ that is as the space

$$\mathbb{P}\mathfrak{H} = \left\{ \ell \in \mathcal{P}(\mathfrak{H}) \;\middle|\; \ell \text{ is a 1-dimensional } \mathbb{K}\text{-linear subspace of } \mathfrak{H} \right\} .$$

It carries a natural topology which we now describe. Consider $\mathfrak{H}\backslash\{0\}$ with its subspace topology. Then one has a natural map

$$\pi : \mathfrak{H}\backslash\{0\} \to \mathbb{P}\mathfrak{H},\; v \mapsto \mathbb{K}v$$

which obviously is surjective. One endows $\mathbb{P}\mathfrak{H}$ with the final topology with respect to π. Next let us introduce an equivalence relation $\sim$ on $\mathfrak{H}\backslash\{0\}$ by defining $v \sim w$ if there exists a $\lambda \in \mathbb{K}^\times = \mathbb{K}\backslash\{0\}$ such that $v = \lambda w$. Obviously $\sim$ is reflexive, since $1 \in \mathbb{K}^\times$, symmetric, since with $\lambda \in \mathbb{K}^\times$ the inverse λ^{-1} is in $\mathbb{K}^\times$ as well, and transitive, since the product of two elements of $\mathbb{K}^\times$ is in $\mathbb{K}^\times$. Hence $\sim$ is an equivalence relation indeed. Denote by $\hat{v}$ the equivalence class of an element $v \in \mathfrak{H}\backslash\{0\}$. Let $\widehat{\mathfrak{H}}$ be the quotient space $(\mathfrak{H}\backslash\{0\})/\!\sim$ and $\widehat{\pi} : \mathfrak{H}\backslash\{0\} \to \widehat{\mathfrak{H}}$ the quotient map.

3.1.2 Lemma *The map $\pi : \mathfrak{H}\backslash\{0\} \to \mathbb{P}\mathfrak{H}$ factors through a unique homeomorphism $\kappa : \widehat{\mathfrak{H}} \to \mathbb{P}\mathfrak{H}$ which means that the diagram*

$$\begin{tikzcd}
\mathfrak{H}\backslash\{0\} \arrow[d, "\widehat{\pi}"'] \arrow[dr, "\pi"] & \\
\widehat{\mathfrak{H}} \arrow[r, "\kappa"'] & \mathbb{P}\mathfrak{H}
\end{tikzcd}$$

commutes and that κ is uniquely determined by this condition.

Proof. If $v \sim w$, then the lines through v and through w coincide, hence π factors through a unique continuous map $\kappa : \widehat{\mathfrak{H}} \to \mathbb{P}\mathfrak{H}$ by the universal property of the quotient space. By surjectivity of π, κ is surjective, too. By definition, κ maps $\hat{v}$ to $\mathbb{K}v$, hence if $\mathbb{K}v = \mathbb{K}w$, then v and w are linearly dependant, and $v \sim w$. So κ is injective. Continuity of the inverse $\kappa^{-1} : \mathbb{P}\mathfrak{H} \to \widehat{\mathfrak{H}}$ is a consequence of the fact that $\mathbb{P}\mathfrak{H}$ carries the final topology with respect to π. Uniqueness of κ follows from $\widehat{\pi}$ being surjective. □

3.1.3 Lemma *The projection map $\pi : \mathfrak{H}\backslash\{0\} \to \mathbb{P}\mathfrak{H}$ and its restriction $\pi|_{\mathbb{S}\mathfrak{H}} : \mathbb{S}\mathfrak{H} \to \mathbb{P}\mathfrak{H}$ to the sphere of $\mathfrak{H}$ are open.*

Proof. By the preceding lemma it suffices to show that $\widehat{\pi} : \mathfrak{H}\backslash\{0\} \to \widehat{\mathfrak{H}}$ is open. Let $U \subset \mathfrak{H}\backslash\{0\}$ be open. Then

$$\widehat{\pi}^{-1}\big(\widehat{\pi}(U)\big) = \bigcup_{\lambda \in \mathbb{K}^\times} \lambda \cdot U ,$$

which is again open and the first part of the claim is proved. The second part follows in the same way, since

$$\widehat{\pi}|_{\mathbb{S}\mathfrak{H}}^{-1}\big(\widehat{\pi}|_{\mathbb{S}\mathfrak{H}}(U)\big) = \bigcup_{\lambda\in\mathbb{S}(\mathbb{K})} \lambda \cdot U$$

is open for all $U \subset \mathbb{S}\mathfrak{H}$ open. □

3.1.4 Remark Strictly speaking, the projective space $\mathbb{P}\mathfrak{H}$ depends on the ground field $\mathbb{K}$. If $\mathfrak{H}$ is a complex Hilbert space one therefore sometimes writes $\mathbb{R}\mathbb{P}\mathfrak{H}$ or $\mathbb{C}\mathbb{P}\mathfrak{H}$ to denote that the projective space of all real respectively all complex lines is meant. In this work we agree that for $\mathfrak{H}$ complex $\mathbb{P}\mathfrak{H}$ always stands for the projective space of complex lines in $\mathfrak{H}$. If we want to consider the projective space of real lines in some complex Hilbert space $\mathfrak{H}$ instead, we write $\mathbb{R}\mathbb{P}\mathfrak{H}$.

3.1.5 The inner product on the underlying Hilbert space $\mathfrak{H}$ induces the *projective inner product* or *ray inner product*

$$\langle\!|\cdot,\cdot|\!\rangle : \mathbb{P}\mathfrak{H} \times \mathbb{P}\mathfrak{H} \to [0,1],\ (\mathbb{K}v, \mathbb{K}w) \mapsto \langle\!|\mathbb{K}v, \mathbb{K}w|\!\rangle = \frac{|\langle v, w\rangle|}{\|v\|\,\|w\|}, \quad \text{where } v, w \in \mathfrak{H}\backslash\{0\},$$

on the associated projective space. Note that the projective inner product is well-defined, since $\frac{|\langle v,w\rangle|}{\|v\|\,\|w\|}$ is homogeneous of degree 0 both in v and w.

Now we can formulate the first postulate of quantum mechanics.

(QM1) The *state space* of a quantum mechanical system is accomplished by a projective space $\mathbb{P}\mathfrak{H}$ associated to a complex separable Hilbert space $\mathfrak{H}$. The elements $v \in \mathfrak{H}\backslash\{0\}$ are called *state vectors*, the rays $\ell \in \mathbb{P}\mathfrak{H}$ are the *pure states*.

If a quantum mechanical system is prepared so that it is in the state $\ell \in \mathbb{P}\mathfrak{H}$, the probability that a measurement detects the system to be in the state $\hbar \in \mathbb{P}\mathfrak{H}$ is given by the *transition probability* $\langle\!|\hbar, \ell|\!\rangle^2$.

Because of their appearance in the first postulate of quantum mechanics we want to study projective Hilbert spaces in some more depth. We will use topological, geometric and analytic tools for that endeavor. A first result is the following.

3.1.6 Theorem *Let $\mathbb{P}\mathfrak{H}$ be the projective space of a Hilbert space of dimension $\geqslant 2$ over the field $\mathbb{K}$ of real or complex numbers. Then the following holds true:*

(i) *The projective Hilbert space $\mathbb{P}\mathfrak{H}$ is a completely metrizable topological space.*

(ii) *A complete metric inducing the topology on $\mathbb{P}\mathfrak{H}$ is given by*

$$d : \mathbb{P}\mathfrak{H} \times \mathbb{P}\mathfrak{H} \to \mathbb{R}_{\geqslant 0},\ (\hbar, \ell) \mapsto \inf\left\{\|v - w\| \;\middle|\; v \in \hbar,\ w \in \ell \ \&\ \|v\| = \|w\| = 1\right\}\ .$$

(iii) *The metric d and the transition amplitudes satisfy the relation*

$$d^2(\hbar, \ell) = 2\big(1 - \langle\!|\hbar, \ell|\!\rangle\big) \geqslant 1 - \langle\!|\hbar, \ell|\!\rangle^2 \quad \textit{for all } \hbar, \ell \in \mathbb{P}\mathfrak{H}\ . \tag{3.1.1}$$

(iv) *The* Fubini–Study distance

$$d_{\mathrm{FS}} : \mathbb{P}\mathfrak{H} \times \mathbb{P}\mathfrak{H} \to \mathbb{R}_{\geqslant 0},\ (\mathscr{k}, \mathscr{l}) \mapsto \arccos \langle\!| \mathscr{k}, \mathscr{l} |\!\rangle$$

is a metric on $\mathbb{P}\mathfrak{H}$ *which is equivalent to the metric* d*. More precisely*

$$d(\mathscr{k}, \mathscr{l}) \leqslant d_{\mathrm{FS}}(\mathscr{k}, \mathscr{l}) \leqslant \sqrt{2} d(\mathscr{k}, \mathscr{l}) \quad \textit{for all } \mathscr{k}, \mathscr{l} \in \mathbb{P}\mathfrak{H} \ . \tag{3.1.2}$$

The diameter of $\mathbb{P}\mathfrak{H}$ *with respect to the Fubini–Study distance equals* $\frac{\pi}{2}$.

(v) *The mapping* $P : \mathbb{P}\mathfrak{H} \to \mathfrak{B}(\mathfrak{H})$ *which associates to every ray* $\mathscr{k}$ *the orthogonal projection onto it is a bi-Lipschitz embedding. The* gap metric

$$d_{\mathrm{gap}} : \mathbb{P}\mathfrak{H} \times \mathbb{P}\mathfrak{H} \to \mathbb{R}_{\geqslant 0},\ (\mathscr{k}, \mathscr{l}) \mapsto \| P(\mathscr{k}) - P(\mathscr{l}) \|$$

obtained by restricting the operator norm distance to $\mathbb{P}\mathfrak{H}$ *is equivalent to* d *and satisfies*

$$\frac{1}{\sqrt{2}} d(\mathscr{k}, \mathscr{l}) \leqslant d_{\mathrm{gap}}(\mathscr{k}, \mathscr{l}) = \sqrt{1 - \langle\!| \mathscr{k}, \mathscr{l} |\!\rangle^2} \leqslant d(\mathscr{k}, \mathscr{l}) \quad \textit{for all } \mathscr{k}, \mathscr{l} \in \mathbb{P}\mathfrak{H} \ . \tag{3.1.3}$$

Proof. ad (ii)Let us first show that the map d is a metric indeed. By definition, d is non-negative and symmetric. Assume $d(\mathscr{k}, \mathscr{l}) = 0$ for two rays $\mathscr{k}, \mathscr{l}$. For given unit vectors $v \in \mathscr{k}$ and $w \in \mathscr{l}$ there then exists a sequence $(\sigma_k)_{k \in \mathbb{N}} \subset \mathbb{S}^1$ such that

$$\lim_{k \to \infty} \| v - \sigma_k w \| = 0 \ .$$

By compactness of $\mathbb{S}^1$ we can assume that the sequence $(\sigma_k)_{k \in \mathbb{N}}$ converges after possibly passing to a subsequence. Let $\sigma \in \mathbb{S}^1$ be its limit. Then $\| v - \sigma w \| = 0$, hence $\mathscr{k} = \mathscr{l}$. Now let $\mathscr{k}, \mathscr{l}, \mathscr{j} \in \mathbb{P}\mathfrak{H}$ and $z \in \mathscr{j}$ a representing unit vector. Then

$$\begin{aligned} d(\mathscr{k}, \mathscr{l}) &= \inf \left\{ \| v - w \| \;\middle|\; v \in \mathscr{k},\ w \in \mathscr{l} \ \&\ \| v \| = \| w \| = 1 \right\} \leqslant \\ &\leqslant \inf \left\{ \| v - z \| + \| z - w \| \;\middle|\; v \in \mathscr{k},\ w \in \mathscr{l} \ \&\ \| v \| = \| w \| = 1 \right\} = \\ &= \inf \left\{ \| v - z \| \;\middle|\; v \in \mathscr{k} \ \&\ \| v \| = 1 \right\} + \inf \left\{ \| z - w \| \;\middle|\; w \in \mathscr{l} \ \&\ \| w \| = 1 \right\} = \\ &= d(\mathscr{k}, \mathscr{j}) + d(\mathscr{j}, \mathscr{l}) \ , \end{aligned}$$

hence d satisfies the triangle inequality, and therefore is a metric.

Next we prove that the metric topology of d coincides with the quotient topology of π. Let $v, w \in \mathbb{S}\mathfrak{H}$. By definition of the metric d one then has

$$d(\mathbb{K}v, \mathbb{K}w) \leqslant \| v - w \| \ .$$

This implies that for all $\varepsilon > 0$

$$\pi \left(\mathbb{B}_{\mathbb{S}\mathfrak{H}}(v, \varepsilon) \right) \subset \mathbb{B}_{\mathbb{P}\mathfrak{H}}(\mathbb{K}v, \varepsilon) \ ,$$

where $\mathbb{B}_{\mathbb{S}\mathfrak{H}}(v, \varepsilon)$ denotes the ε-ball around v in the sphere with respect to the norm and $\mathbb{B}_{\mathbb{P}\mathfrak{H}}(\mathbb{K}v, \varepsilon)$ the ε-ball around $\mathbb{K}v$ in the projective Hilbert space with respect to the metric d. Hence the quotient topology on $\mathbb{P}\mathfrak{H}$ is finer than the metric topology. If for given $\varepsilon > 0$ a $\delta > 0$ is chosen so that $\delta < \varepsilon$, then for every ray ℓ with $d(\mathbb{K}v, \ell) < \delta$ there exists an element $w \in \ell \cap \mathbb{S}\mathfrak{H}$ such that $\|v - w\| < \epsilon$ which means that $\ell = \pi(w) \in \pi\big(B(v, \varepsilon)\big)$. Hence

$$\mathbb{B}_{\mathbb{P}\mathfrak{H}}(\mathbb{K}v, \delta) \subset \pi\left(\mathbb{B}_{\mathbb{S}\mathfrak{H}}(v, \varepsilon)\right)$$

and the quotient topology on $\mathbb{P}\mathfrak{H}$ is coarser than the metric topology. So d induces the topology on $\mathbb{P}\mathfrak{H}$ as claimed.

It remains to verify that d is a complete metric. To this end observe first that for every $v \in \mathbb{S}\mathfrak{H}$ and ray ℓ there exists a representative $w \in \ell \cap \mathbb{S}\mathfrak{H}$ such that $\langle v, w\rangle = \langle\!|\mathbb{K}v, \ell|\!\rangle$. We will call such a representative of ℓ *distinguished with respect to* v. Now let $(\ell_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of rays. Then there exists an increasing sequence of natural numbers $n_0 < \ldots < n_k < n_{k+1} < \ldots$ such that

$$d(\ell_n, \ell_m) < \frac{1}{2^{k+1}} \quad \text{for all } n, m \geqslant n_k \ .$$

Choose a representative $v_0 \in \ell_{n_0} \cap \mathbb{S}\mathfrak{H}$ and let $v_1 \in \mathbb{S}\mathfrak{H}$ be a representative of ℓ_{n_1} distinguished with respect to v_0. Then

$$\|v_1 - v_0\| = \sqrt{2(1 - \mathfrak{Re}\langle v_0, v_1\rangle)} = \sqrt{2(1 - \langle\!|\ell_{n_0}, \ell_{n_1}|\!\rangle)} = d(\ell_{n_0}, \ell_{n_1}) < \frac{1}{2} \ .$$

Now assume we have constructed $v_0, \ldots, v_k \in \mathbb{S}\mathfrak{H}$ such that $\mathbb{K}v_l = \ell_{n_l}$ for $l = 0, \ldots, k$ and such that for $l = 0, \ldots, k-1$

$$\|v_{l+1} - v_l\| < \frac{1}{2^{l+1}} \ . \tag{3.1.4}$$

Let $v_{k+1} \in \mathbb{S}\mathfrak{H}$ be a representative of $\ell_{n_{k+1}}$ distinguished with respect to v_k. Then

$$\|v_{k+1} - v_k\| = \sqrt{2(1 - \mathfrak{Re}\langle v_{k+1}, v_k\rangle)} = \sqrt{2(1 - \langle\!|\ell_{n_{k+1}}, \ell_{n_k}|\!\rangle)} = d(\ell_{n_{k+1}}, \ell_{n_k}) < \frac{1}{2^{k+1}} \ .$$

We thus obtain a sequence $(v_k)_{k\in\mathbb{N}}$ in $\mathfrak{H}$ such that (3.1.4) is fulfilled for all $l \in \mathbb{N}$. The sequence $(v_k)_{k\in\mathbb{N}}$ is even a Cauchy sequence since for $n \geqslant m \geqslant k$

$$\|v_n - v_m\| \leqslant \sum_{k=m}^{n-1} \|v_{k+1} - v_k\| < \sum_{k=m}^{n-1} \frac{1}{2^{k+1}} < \frac{1}{2^m} \ .$$

Let $v \in \mathfrak{H}$ be its limit. Then

$$\lim_{k\to\infty} d(\mathbb{K}v, \ell_{n_k}) \leqslant \lim_{k\to\infty} \|v - v_k\| = 0 \ .$$

Hence the sequence of rays $(\ell_n)_{n\in\mathbb{N}}$ converges to the ray $\mathbb{K}v$ and $\mathbb{P}\mathfrak{H}$ is complete with respect to the metric d. Claim (i) is now proved as well.

ad (iii)Let $\hbar, \ell$ be rays in $\mathfrak{H}$ and $v \in \hbar$, $w \in \ell$ representing unit vectors. Let $\lambda \in \mathbb{S}^1$ such that $\langle v, w\rangle = \lambda\, \langle\!|\hbar, \ell|\!\rangle$ and $\sigma \in \mathbb{S}^1$ arbitrary. Then compute

$$\|v - \sigma w\|^2 = 2\big(1 - \mathfrak{Re}\langle v, \sigma w\rangle\big) = 2\big(1 - \langle\!|\hbar, \ell|\!\rangle\, \mathfrak{Re}\, \overline{\sigma}\lambda\big) \geqslant 2\big(1 - \langle\!|\hbar, \ell|\!\rangle\big) \ .$$

For $\sigma = \lambda$, equality holds, hence

$$d^2(\hbar, \ell) = \inf\left\{ \|v - \sigma w\|^2 \ \middle|\ \sigma \in \mathbb{S}^1 \right\} = 2\big(1 - \langle\!|\hbar, \ell|\!\rangle\big) \ .$$

With $\hbar, \ell, v, w$ as before and $\delta = d(\hbar, \ell)$, the claimed inequality now follows immediately:

$$d^2(\hbar, \ell) \geqslant \delta^2 \left(1 - \frac{1}{4}\delta^2\right) = 2\big(1 - \langle\!|\hbar, \ell|\!\rangle\big) \left(1 - \frac{1}{2}\big(1 - \langle\!|\hbar, \ell|\!\rangle\big)\right) = 1 - \langle\!|\hbar, \ell|\!\rangle^2 \ .$$

ad (iv)The map d_{FS} is symmetric by symmetry of the projective inner product. By the assumption $\dim \mathfrak{H} \geqslant 2$, the image of $\langle\!|\cdot,\cdot|\!\rangle$ is the whole interval $[0,1]$, since $\mathbb{P}\mathfrak{H}$ is connected, $\langle\!|\cdot,\cdot|\!\rangle$ is bounded by 1, $\langle\!|\ell,\ell|\!\rangle = 1$ for every ray ℓ and since there exist orthogonal rays. The image of d_{FS} therefore coincides with $[0,\frac{\pi}{2}]$ which already entails the claim about the diameter. By strict monotony of arccos, $d_{\mathrm{FS}}(\mathscr{k},\ell) = 0$ if and only if $\langle\!|\mathscr{k},\ell|\!\rangle = 1$. By (3.1.1) this is the case if and only if $d(\mathscr{k},\ell) = 0$ which means if and only if $\mathscr{k} = \ell$. Let us now show that d_{FS} satisfies the triangle inequality. To this end let $\mathscr{k},\ell,\jmath$ be rays in $\mathfrak{H}$. If the Fubini–Study distance between any two of these rays is zero, the triangle inequality obviously holds true, so we exclude that case. Choose representatives $v \in \mathscr{k}$, $w \in \ell$, $z \in \jmath$ such that all have norm 1. After possibly multiplying v and z by elements of $\mathbb{S}^1 \cap \mathbb{K}$ one can achieve that

$$\langle v,w\rangle = \langle\!|\mathscr{k},\ell|\!\rangle \quad \text{and} \quad \langle w,z\rangle = \langle\!|\ell,\jmath|\!\rangle \ .$$

Let $\theta = \arccos\langle v,w\rangle$ and $\varphi = \arccos\langle w,z\rangle$. Then $\theta = d_{\mathrm{FS}}(\mathscr{k},\ell)$ and $\varphi = d_{\mathrm{FS}}(\ell,\jmath)$. Now let x be a unit vector in the plane through v and w which is orthogonal to w and y a unit vector in the plane through w and z which is orthogonal to w. After possibly multiplying x and y by elements of $\mathbb{S}^1 \cap \mathbb{K}$ one can achieve that $\langle v,x\rangle, \langle z,y\rangle \in [0,1]$. Then

$$v = \langle v,w\rangle\, w + \langle v,x\rangle\, x \quad \text{and} \quad z = \langle z,w\rangle\, w + \langle z,y\rangle\, y \ .$$

By $\theta,\varphi \in \left[0,\frac{\pi}{2}\right]$ and $\langle v,x\rangle, \langle z,y\rangle \geqslant 0$ one concludes

$$v = \cos\theta\, w + \sin\theta\, x \quad \text{and} \quad z = \cos\varphi\, w + \sin\varphi\, y \ .$$

Hence, by the triangle inequality for the absolute value and the Cauchy–Schwarz inequality

$$|\langle v,z\rangle| = |\cos\theta\, \cos\varphi + \sin\theta\, \sin\varphi\, \langle x,y\rangle| \geqslant \cos\theta\, \cos\varphi - \sin\theta\, \sin\varphi = \cos(\theta+\varphi) \ .$$

Since arccos is monotone decreasing, one obtains

$$d_{\mathrm{FS}}(\mathscr{k},\jmath) = \arccos|\langle v,z\rangle| \leqslant \theta + \varphi = d_{\mathrm{FS}}(\mathscr{k},\ell) + d_{\mathrm{FS}}(\ell,\jmath) \ .$$

So the Fubini–Study distance satisfies the triangle inequality and is a metric indeed.

Last we need to prove that the Fubini–Study distance is equivalent to d. To this end consider the functions

$$f : [0,\sqrt{2}] \to \mathbb{R},\ s \mapsto \arccos\left(1 - \frac{s^2}{2}\right) \quad \text{and} \quad g : [0,\frac{\pi}{2}] \to \mathbb{R},\ t \mapsto \sqrt{2(1-\cos t)} \ .$$

Then both functions are continuous and differentiable on the interior of their domains. Now observe that $f(0) = g(0) = 0$ and compute

$$f'(s) = \frac{s}{\sqrt{1 - \left(1 - \frac{s^2}{2}\right)^2}} = \frac{s}{\sqrt{s^2 - \frac{s^4}{4}}} = \frac{2}{\sqrt{4 - s^2}} \leqslant \sqrt{2} \quad \text{for } s \in (0,\sqrt{2})$$

and

$$g'(t) = \frac{\sqrt{2}}{2}\, \frac{\sin t}{\sqrt{1-\cos t}} = \frac{\sqrt{2}}{2}\sqrt{1+\cos t} \leqslant 1 \quad \text{for } t \in (0,\frac{\pi}{2}) \ .$$

By definition of d_{FS} and (3.1.1), the mean-value theorem then entails

$$d(\mathscr{k},\ell) = g\left(d_{\mathrm{FS}}(\mathscr{k},\ell)\right) \leqslant d_{\mathrm{FS}}(\mathscr{k},\ell) = f\left(d(\mathscr{k},\ell)\right) \leqslant \sqrt{2}\, d(\mathscr{k},\ell) \quad \text{for all } \mathscr{k},\ell \in \mathbb{P}\mathfrak{H} \ .$$

Hence the estimate (3.1.2) is proved and the metrics d and d_{FS} are equivalent.

ad (v)Recall that the operator norm distance of $P(\mathscr{k})$ and $P(\ell)$ is given by

$$\|P(\mathscr{k}) - P(\ell)\| = \sup\left\{ \left\|\big(P(\mathscr{k}) - P(\ell)\big)z\right\| \;\middle|\; z \in \mathbb{S}\mathfrak{H} \right\} . \tag{3.1.5}$$

Choose normalized representatives $v \in \mathscr{k}$ and $w \in \ell$. After possibly multiplying w by a complex number of modulus 1 we can assume that $\langle v, w\rangle = \langle\!| \mathscr{k}, \ell |\!\rangle \geqslant 0$. If $\langle v, w\rangle = 1$ or in other words if v and w are linearly dependent then $\mathscr{k}$ and ℓ coincide and the claim is trivial, so we assume that v and w are linearly independent. First we want to show that

$$\left\|\big(P(\mathscr{k}) - P(\ell)\big)z\right\| \leqslant 1 - |\langle v, w\rangle|^2 \quad \text{for all } z \in \mathbb{S}\mathfrak{H} . \tag{3.1.6}$$

To this end expand $z = z^{\|} + z^{\perp}$, where $z^{\|}$ lies in the plane spanned by v and w and $z^{\perp}$ is perpendicular to that plane. Then

$$\big(P(\mathscr{k}) - P(\ell)\big)z = \langle z, v\rangle v - \langle z, w\rangle w = \langle z^{\|}, v\rangle v - \langle z^{\|}, w\rangle w = \big(P(\mathscr{k}) - P(\ell)\big)z^{\|} .$$

Hence it suffices to verify (3.1.6) for $z \in \mathbb{S}\mathfrak{H} \cap \operatorname{Span}(v, w)$. Observe that there exist unique elements $\varphi \in [0, \frac{\pi}{2}]$ and $\mu \in \mathbb{S}^1 \cap \mathbb{K}$ such that $\langle z, v\rangle = \overline{\mu}\cos\varphi$. One can then find a normalized vector $w^{\perp} \in \operatorname{Span}(v, w)$ perpendicular to v such that

$$\mu z = \cos\varphi\, v + \sin\varphi\, w^{\perp} .$$

Note that with this

$$w = \langle v, w\rangle v + \langle w, w^{\perp}\rangle w^{\perp} \quad \text{and} \quad |\langle w, w^{\perp}\rangle|^2 = 1 - |\langle v, w\rangle|^2 .$$

Now compute

$$\begin{aligned}
\left\|\big(P(\mathscr{k}) - P(\ell)\big)z\right\|^2 &= \left\|\big(P(\mathscr{k}) - P(\ell)\big)\mu z\right\|^2 = \|\langle z, v\rangle v - \langle z, w\rangle w\|^2 = \\
&= |\langle \mu z, v\rangle|^2 - 2\langle v, w\rangle\, \mathfrak{Re}\left(\langle \mu z, v\rangle \langle \mu z, w\rangle\right) + |\langle \mu z, w\rangle|^2 = \\
&= \cos^2\varphi - 2\cos\varphi\, \langle v, w\rangle \left(\cos\varphi\, \langle v, w\rangle + \sin\varphi\, \mathfrak{Re}\langle w, w^{\perp}\rangle\right) \\
&\quad + \cos^2\varphi\, |\langle v, w\rangle|^2 + 2\cos\varphi\, \langle v, w\rangle \sin\varphi\, \mathfrak{Re}\langle w, w^{\perp}\rangle + \sin^2\varphi\, |\langle w, w^{\perp}\rangle|^2 = \\
&= 1 - |\langle v, w\rangle|^2 .
\end{aligned}$$

This proves (3.1.6), but also implies by (3.1.5) that

$$\|P(\mathscr{k}) - P(\ell)\|^2 = 1 - |\langle v, w\rangle|^2 = 1 - \langle\!| \mathscr{k}, \ell |\!\rangle^2 .$$

The claim now follows by (iii) and the theorem is proved. □

After having examined some topological properties we come now to the geometry of projective Hilbert spaces.

3.1.7 Theorem *The projective Hilbert space $\mathbb{P}\mathfrak{H}$ of a Hilbert space of dimension $\geqslant 2$ over $\mathbb{K}$ has the following differential geometric properties:*

(i) *$\mathbb{P}\mathfrak{H}$ carries a natural structure of an analytic manifold modelled on a Hilbert space isomorphic to each of the Hilbert spaces $\mathrm{V}_w = (\mathbb{K}w)^{\perp}$, where $w \in \mathfrak{H}$ is a unit vector.*

(ii) *Let $\mathbb{S}\mathfrak{H} \subset \mathfrak{H}\backslash\{0\}$ be the sphere in $\mathfrak{H}$. Then the restriction*

$$\pi|_{\mathbb{S}\mathfrak{H}} : \mathbb{S}\mathfrak{H} \to \mathbb{P}\mathfrak{H},\ v \mapsto \mathbb{K}v$$

is a real analytic fiber bundle with typical fiber $\mathbb{S}^1$ in the complex case and typical fiber $\mathbb{Z}/2$ in the real case.

(iii) *Endow $\mathbb{S}\mathfrak{H}$ with the riemannian metric g inherited from the ambient Hilbert space. Then there exists a unique riemannian metric g_{FS} on $\mathbb{P}\mathfrak{H}$ such that $\pi|_{\mathbb{S}\mathfrak{H}} : \mathbb{S}\mathfrak{H} \to \mathbb{P}\mathfrak{H}$ becomes a riemannian submersion. This metric is called the* Fubini–Study *metric. Its geodesic distance coincides with the Fubini–Study distance d_{FS}.*

(iv) *In case $\mathfrak{H}$ is a complex Hilbert space, the projective space $\mathbb{P}\mathfrak{H}$ carries in a natural way the structure of a Kähler manifold. Its complex structure is the one inherited from $\mathfrak{H}$, and its riemannian metric is the Fubini–Study metric.*

Proof. ad (i) For a given unit vector $w \in \mathbb{S}\mathfrak{H}$ consider the linear form $w^\flat : \mathfrak{H} \to \mathbb{K},\ v \mapsto \langle v, w\rangle$. Let $\mathrm{V}_w = \ker w^\flat = (\mathbb{K}w)^\perp$ and $U_w = \pi\big(\mathfrak{H}\backslash\mathrm{V}_w\big)$. Then, by Theorem 12.2.3, one has the orthogonal decomposition $\mathfrak{H} = \mathrm{V}_w \oplus \mathbb{K}w$ which gives rise to the orthogonal projection $\mathrm{pr}_{\mathrm{V}_w} : \mathfrak{H} \to \mathrm{V}_w$. Next observe that $U_w \subset \mathbb{P}\mathfrak{H}$ is open since $\pi^{-1}(U_w) = \mathfrak{H}\backslash\mathrm{V}_w$ is open and $\mathbb{P}\mathfrak{H}$ carries the quotient topology with respect to π. Now we can define a chart $h_w : U_w \to \mathrm{V}_w$ by

$$h_w(\mathbb{K}v) = \mathrm{pr}_{\mathrm{V}_w}\left(\frac{v}{\langle v, w\rangle}\right) = \frac{v}{\langle v, w\rangle} - w \quad \text{for } v \in \mathfrak{H}\backslash\mathrm{V}_w\ .$$

The map h_w is well-defined since $\langle v, w\rangle \neq 0$ for all $v \in \mathfrak{H}\backslash\mathrm{V}_w$ and since $\frac{v}{\langle v,w\rangle} = \frac{\lambda v}{\langle \lambda v,w\rangle}$ for all $\lambda \in \mathbb{K}^\times$. Moreover, h_w is continuous by continuity of the composition $h_w \circ \pi|_{\mathfrak{H}\backslash\mathrm{V}_w}$. If $h_w(\mathbb{K}v) = h_w(\mathbb{K}v')$, then

$$\mathrm{pr}_{\mathrm{V}_w}\left(\frac{v}{\langle v, w\rangle} - \frac{v'}{\langle v', w\rangle}\right) = 0 \quad \text{and} \quad \left\langle \frac{v}{\langle v, w\rangle} - \frac{v'}{\langle v', w\rangle}, w \right\rangle = 0,$$

hence $\mathbb{K}v = \mathbb{K}v'$, so h_w is injective. The map $\mathrm{V}_w \to U_w,\ y \mapsto \pi(y + w)$ is obviously continuous and inverse to h_w since $h_w\big(\pi(y + w)\big) = y$ for all $y \in \mathrm{V}_w$ and since h_w is injective. So we have proved that $h_w : U_w \to \mathrm{V}_w$ is a homeomorphism.

Next observe that all the Hilbert spaces V_w, $w \in \mathbb{S}\mathfrak{H}$ are pairwise isomorphic since each of them has codimension 1 in $\mathfrak{H}$. After this observation we show that for all $v, w \in \mathbb{S}\mathfrak{H}$

$$h_w(U_w \cap U_v) = \mathrm{V}_w\backslash\left(-\,\mathrm{pr}_{\mathbb{K}v}\, w + \mathrm{V}_w \cap \mathrm{V}_v\right)\ . \tag{3.1.7}$$

Assume that $y \in \mathrm{V}_w$. The relation $v \notin \left(-\,\mathrm{pr}_{\mathbb{K}v}\, w + \mathrm{V}_w \cap \mathrm{V}_v\right)$ then is equivalent to $\mathrm{pr}_{\mathbb{K}v}(y + w) \neq 0$, which on the other hand is equivalent to the existence of some $\lambda \in \mathbb{K}^\times$ and $x \in \mathrm{V}_v$ such that $y + w = \lambda(x + v)$. Since $h_w^{-1}(y) = \pi(y + w)$, the latter is equivalent to the existence of an $x \in \mathrm{V}_v$ such that $h_w^{-1}(y) = \pi(x + v)$. But that is equivalent to $h_w^{-1}(y) \in U_w \cap U_v$. This proves (3.1.7).

The transition map between the chart h_w and the chart h_v is now given by

$$h_v \circ h_w^{-1} : \mathrm{V}_w\backslash\left(-\,\mathrm{pr}_{\mathbb{K}v}\, w + \mathrm{V}_w \cap \mathrm{V}_v\right) \to \mathrm{V}_v\backslash\left(-\,\mathrm{pr}_{\mathbb{K}w}\, v + \mathrm{V}_w \cap \mathrm{V}_v\right),\ y \mapsto \mathrm{pr}_{\mathrm{V}_v} \frac{y + w}{\langle y + w, v\rangle}\ .$$

But this map is analytic as a composition of analytic maps, hence any two charts are $\mathcal{C}^\omega$-compatible. Since $\mathbb{P}\mathfrak{H}$ is obviously covered by the open domains U_w, $w \in \mathbb{S}\mathfrak{H}$, the projective Hilbert space $\mathbb{P}\mathfrak{H}$ becomes an analytic manifold locally modelled on a Hilbert space isomorphic to each of the V_w, $w \in \mathbb{S}\mathfrak{H}$.

ad (ii) Fix a unit vector $w \in \mathbb{S}\mathfrak{H}$, let $\mathrm{V}_w = (\mathbb{K}w)^\perp$ as before and and put

$$\widetilde{\mathrm{V}}_w = \begin{cases} \mathrm{V}_w & \text{if } \mathbb{K} = \mathbb{R} , \\ \mathrm{V}_w \oplus \mathrm{i}\mathbb{R}w & \text{if } \mathbb{K} = \mathbb{C} . \end{cases}$$

Then $\widetilde{\mathrm{V}}_w$ is the orthogonal complement of the real line $\mathbb{R}w$ with respect to the real inner product $\mathfrak{Re}\langle -, - \rangle$ on $\mathfrak{H}$. Hence any vector $v \in \mathfrak{H}$ can be uniquely represented in the form $v = v_0 w + \hat{v}$ where $v_0 = \mathfrak{Re}\langle v, w \rangle \in \mathbb{R}$ and $\hat{v} = \mathrm{pr}_{\widetilde{\mathrm{V}}_w}(v) \in \widetilde{\mathrm{V}}_w$. Put $N_w = \mathbb{S}\mathfrak{H} \backslash \{-w\}$. The stereographic projection

$$g_w : N_w \to \widetilde{\mathrm{V}}_w, \ v \mapsto \frac{2}{1+v_0} \hat{v}$$

then is a chart for $\mathbb{S}\mathfrak{H}$ with inverse

$$g_w^- : \widetilde{\mathrm{V}}_w \to N_w, \ z \mapsto \frac{1}{4 + \|z\|^2} \left((4 - \|z\|^2) w + 4z \right) .$$

Since $\frac{4-r}{4+r} > -1$ for all $r \geqslant 0$ and

$$\|g_w^-(z)\|^2 = \frac{1}{(4+\|z\|^2)^2} \left((4 - \|z\|^2)^2 + (4\|z\|)^2 \right) = 1 \quad \text{for all } z \in \widetilde{\mathrm{V}}_w ,$$

the map g_w^- has image in N_w, indeed. Moreover, for $z \in \widetilde{\mathrm{V}}_w$,

$$g_w \circ g_w^-(z) = \frac{2}{1 + \frac{4-\|z\|^2}{4+\|z\|^2}} \frac{4}{4+\|z\|^2} z = z$$

and for $v \in N_w$ by application of the equality $|v_0|^2 + \|\hat{v}\|^2 = 1$,

$$\begin{aligned} g_w^- \circ g_w(v) = g_w^2 \left(\frac{2}{1+v_0} \hat{v} \right) &= \frac{1}{4 + \frac{4}{(1+v_0)^2} \|\hat{v}\|^2} \left(4 - \frac{4}{(1+v_0)^2} \|\hat{v}\|^2 w + \frac{8}{1+v_0} \hat{v} \right) = \\ &= \frac{1}{(1+v_0)^2 + \|\hat{v}\|^2} \left(\left((1+v_0)^2 - \|\hat{v}\|^2 \right) w + 2(1+v_0)\hat{v} \right) = \\ &= \frac{1}{2(1+v_0)} \left(2v_0(1+v_0) w + 2(1+v_0) \hat{v} \right) = v_0 w + \hat{v} = v . \end{aligned}$$

Therefore, g_w are g_w^- mutually inverse as claimed. Observe that for $v \in \mathbb{S}\mathfrak{H} \backslash \{w\}$ the transition map $g_w \circ g_v^- : \widetilde{\mathrm{V}}_v \backslash \{g_v(-w)\} \to \widetilde{\mathrm{V}}_w \backslash \{g_w(-v)\}$ is given by

$$\begin{aligned} z \mapsto g_w & \left(\frac{1}{4+\|z\|^2} \left((4 - \|z\|^2) v + 4z \right) \right) = \\ &= \frac{2}{1 + \frac{1}{4+\|z\|^2} \left(\mathfrak{Re}\langle (4 - \|z\|^2) v + 4z, w \rangle \right)} \mathrm{pr}_{\widetilde{\mathrm{V}}_w} \left(\frac{1}{4+\|z\|^2} \left((4 - \|z\|^2) v + 4z \right) \right) = \\ &= \frac{2}{4 + \|z\|^2 + \mathfrak{Re}\langle (4 - \|z\|^2) v + 4z, w \rangle} \left((4 - \|z\|^2) v + 4z - \mathfrak{Re}\langle (4 - \|z\|^2) v + 4z, w \rangle w \right) , \end{aligned}$$

which is real analytic. Since the open sets N_w with $w \in \mathbb{S}\mathfrak{H}$ cover the sphere $\mathbb{S}\mathfrak{H}$ it thus becomes a real analytic manifold modelled on a possibly infinite dimensional real Hilbert space. Now consider the composition

$$\widetilde{\mathrm{V}}_w \backslash 2\mathbb{S}\mathrm{V}_w \to \mathrm{V}_w, \quad z \mapsto h_w \circ \pi \circ g_w^-(z) = \mathrm{pr}_{\mathrm{V}_w}\left(\frac{(4-\|z\|^2)w + 4z}{4 - \|z\|^2 + 4\langle z, w\rangle}\right) = \frac{4\,(z - \langle z, w\rangle w)}{4 - \|z\|^2 + 4\langle z, w\rangle} \ .$$

This is a real analytic map for every $w \in \mathbb{S}\mathfrak{H}$, so $\pi|_{\mathbb{S}\mathfrak{H}}$ is real analytic. Let us show that it is a principal fiber bundle. To this end put $G = \mathbb{Z}/2$ in the real case and $G = \mathbb{S}^1$ in the complex case and note that G acts smoothly on $\mathbb{S}\mathfrak{H}$ by scalar multiplication. Since G is abelian, we can write this also as a right action $\cdot : \mathbb{S}\mathfrak{H} \times G \to \mathbb{S}\mathfrak{H}$. By definition of the projective Hilbert space this right action is free and transitive on the fibers of the projection $\pi : \mathbb{S}\mathfrak{H} \to \mathbb{P}\mathfrak{H}$ which therefore are homeomorphic to G. For each $w \in \mathbb{S}\mathfrak{H}$ the map

$$f_w : \mathbb{S}\mathfrak{H}\backslash \mathrm{V}_w \to U_w \times G \subset \mathbb{P}\mathfrak{H} \times G, \ v \mapsto \left(\mathbb{K}v, \frac{\langle v, w\rangle}{|\langle v, w\rangle|}\right)$$

now is a bundle trivialization as the following argument shows. By construction, f_w is real analytic with inverse

$$f_w^- : U_w \times G \to \mathbb{S}\mathfrak{H}\backslash \mathrm{V}_w, \ (\mathbb{K}v, \lambda) \mapsto \lambda \frac{h_w(\mathbb{K}v) + w}{\|h_w(\mathbb{K}v) + w\|} \ .$$

Indeed, f_w is obviuously surjective and

$$f_w^- \circ f_w(v) = \frac{\langle v, w\rangle}{|\langle v, w\rangle|} \frac{h_w(\mathbb{K}v) + w}{\|h_w(\mathbb{K}v) + w\|} = \frac{\langle v, w\rangle}{|\langle v, w\rangle|} \frac{\frac{v}{\langle v, w\rangle}}{\frac{\|v\|}{|\langle v, w\rangle|}} = v \quad \text{for all } v \in \mathbb{S}\mathfrak{H}\backslash \mathrm{V}_w \ .$$

Observe that $\mathrm{pr}_2 f_w(v \cdot \lambda) = (\mathrm{pr}_2 f_w(v))\lambda$ for all $v \in \mathbb{S}\mathfrak{H}\backslash \mathrm{V}_w$ and $\lambda \in G$, where pr_2 denotes projection onto the second coordinate. Finally note that for $v, w \in \mathbb{S}\mathfrak{H}$ and $z \in \mathbb{S}\mathfrak{H}\backslash(\mathrm{V}_v \cup \mathrm{V}_w)$,

$$f_v \circ f_w^-(\mathbb{K}z, \lambda) = f_v\left(\lambda \frac{h_w(\mathbb{K}z) + w}{\|h_w(\mathbb{K}z) + w\|}\right) = \left(\mathbb{K}z, \lambda \frac{\langle z, v\rangle}{\langle z, w\rangle}\right) = (\mathbb{K}z, \lambda) \cdot \frac{\langle z, v\rangle}{\langle z, w\rangle} \ ,$$

where $\cdot : (\mathbb{P}\mathfrak{H} \times G) \times G \to \mathbb{P}\mathfrak{H} \times G$ denotes the right action $((\ell, \mu), \lambda)) \mapsto (\ell, \mu) \cdot \lambda = (\ell, \mu\lambda)$. Hence $\pi|_{\mathbb{S}\mathfrak{H}} : \mathbb{S}\mathfrak{H} \to \mathbb{P}\mathfrak{H}$ is a real analytic G-principal bundle with local trivializations f_w, $w \in \mathbb{S}\mathfrak{H}$. □

3.1.8 Remark Notice that the chart h_w in the proof of (i) can be written as

$$h_w(\mathbb{K}v) = \frac{v}{\langle v, w\rangle} - w \ .$$

This is the same as for the charts of finite dimensional projective space $\mathbb{K}\mathbb{P}^n$. Indeed, we can choose w as a basis element, say e_k, $k = 0, \ldots, n$ and we have a line

$$[v_0 : \ldots : v_k : \ldots : v_n] \in \mathbb{K}\mathbb{P}^n$$

represented by the vector $v = (v_0, \ldots, v_k, \ldots, v_n)$, where $v_k \neq 0$. Then the standard chart is obtained as follows. First normalize the vector representing the line in the k-th coordinate, i.e. divide by $\langle v, w\rangle$:

$$\left[\frac{v_0}{v_k} : \ldots : 1 : \ldots : \frac{v_n}{v_k}\right] \ ,$$

and then map this to $\mathbb{K}^n$ via dropping the 1 in the k-th coordinate:

$$\left[\frac{v_0}{v_k}:\ldots:1:\ldots:\frac{v_n}{v_k}\right] \mapsto \left(\frac{v_0}{v_k},\ldots,\frac{v_{k-1}}{v_k},\frac{v_{k+1}}{v_k},\ldots,\frac{v_n}{v_k}\right) .$$

3.2. Quantum mechanical symmetries

Automorphisms of the projective Hilbert space and Wigner's theorem

3.2.1 Assume that a quantum mechanical system is described by the projective Hilbert space $\mathbb{P}\mathfrak{H}$ and that two observers $\mathcal{O}$ and $\mathcal{O}'$ observe the system. While observer $\mathcal{O}$ describes the states the system is in by rays $\mathscr{k},\mathscr{l},\mathscr{l}_i,,\ldots \in \mathbb{P}\mathfrak{H}$, observer $\mathcal{O}'$ describes them by possibly different rays $\mathscr{k}',\mathscr{l}',\mathscr{l}'_i,\ldots \in \mathbb{P}\mathfrak{H}$. In other words this means that from the point of physics the rays are not invariant under observer change. Rather does the observer change give rise to a map $A:\mathbb{P}\mathfrak{H}\to\mathbb{P}\mathfrak{H}$, $\mathscr{l}\mapsto A\mathscr{l}=\mathscr{l}'$. This map has to be invertible because the observer change is reversible. Even though rays describing the states of the system do change under an observer change, the corresponding transition probabilities remain invariant by the paradigm that the laws of (quantum) physics do not change from one observer to another. Mathematically this can be expressed by

$$\langle\!|A\mathscr{k},A\mathscr{l}|\!\rangle^2 = \langle\!|\mathscr{k},\mathscr{l}|\!\rangle^2 \quad \text{for all } \mathscr{k},\mathscr{l}\in\mathbb{P}\mathfrak{H} .$$

This leads us to the following definition.

3.2.2 Definition Let $\mathbb{P}\mathfrak{H}$, $\mathbb{P}\mathfrak{H}_1$ and $\mathbb{P}\mathfrak{H}_2$ denote projective Hilbert spaces. One then calls a map $A:\mathbb{P}\mathfrak{H}_1\to\mathbb{P}\mathfrak{H}_2$ an *isometry*, if

$$\langle\!|A\mathscr{k},A\mathscr{l}|\!\rangle = \langle\!|\mathscr{k},\mathscr{l}|\!\rangle \quad \text{for all } \mathscr{k},\mathscr{l}\in\mathbb{P}\mathfrak{H}_1 .$$

A bijective isometry $A:\mathbb{P}\mathfrak{H}\to\mathbb{P}\mathfrak{H}$ is called an *isometric automorphism*, a *Wigner automorphism* or just an *automorphism*.

In quantum mechanics, an automorphism of a projective Hilbert space $\mathbb{P}\mathfrak{H}$ is called a *symmetry* of the quantum mechanical system described by $\mathbb{P}\mathfrak{H}$.

3.2.3 Because the composition of isometric maps between projective Hilbert spaces is an isometric map and the identity map on a projective Hilbert space is isometric the projective Hilbert spaces as objects and the isometric maps as morphisms form a category which we call the *Wigner category* denoted it by Wig. The Wigner automorphisms are then the automorphisms of that category.

The automorphisms of a projective Hilbert space $\mathbb{P}\mathfrak{H}$ form a group denoted by $\mathsf{Aut}(\mathbb{P}\mathfrak{H})$.

3.2.4 From now on in this section let the symbol $\mathfrak{H}$ stand for a complex Hilbert space of dimension $\geqslant 2$. We want to examine what maps on $\mathfrak{H}$ induce automorphisms of the corresponding projective Hilbert space.

If $S:\mathfrak{H}\to\mathfrak{H}$ is a unitary operator that is $S\in\mathsf{GL}(\mathfrak{H})$ and $\langle Sv,Sw\rangle=\langle v,w\rangle$ for all $v,w\in\mathfrak{H}$, then $\hat{S}:\mathbb{P}\mathfrak{H}\to\mathbb{P}\mathfrak{H}$, $\mathbb{C}v\mapsto\mathbb{C}Sv$ is well-defined and an automorphism of $\mathbb{P}\mathfrak{H}$. But not every automorphism of $\mathbb{P}\mathfrak{H}$ is of the form $\hat{S}$ with $S \in \mathsf{U}(\mathfrak{H})$. Namely let $T : \mathfrak{H} \to \mathfrak{H}$ be an anti-unitary map that is $T \in \mathsf{GL}(\mathfrak{H}, \mathbb{R})$, $T(\lambda v) = \overline{\lambda} Tv$ for all $v \in \mathfrak{H}$, $\lambda \in \mathbb{C}$ and $\langle Tv, Tw \rangle = \overline{\langle v, w \rangle} = \langle w, v \rangle$ for all $v, w \in \mathfrak{H}$. Then $\hat{T} : \mathbb{P}\mathfrak{H} \to \mathbb{P}\mathfrak{H}$, $\mathbb{C}v \mapsto \mathbb{C}Tv$ is also well-defined, invertible and preserves transition probabilities. Therefore $\hat{T} \in \mathsf{Aut}(\mathbb{P}\mathfrak{H})$. We will later see that $\hat{T}$ is not equal to any of the automorphisms $\hat{S}$ with $S \in \mathsf{U}(\mathfrak{H})$. Observe also that by the dimension assumption on $\mathfrak{H}$ there exists an anti-unitary transformation, for example the real linear map $T : \mathfrak{H} \to \mathfrak{H}$ which acts on some initially chosen Hilbert basis $(v_j)_{j \in J}$ by $T(v_j) = v_j$ and $T(\mathrm{i}v_j) = -\mathrm{i}v_j$.

One easily checks that the products ST and TS of a unitary operator $S : \mathfrak{H} \to \mathfrak{H}$ and an anti-unitary operator $T : \mathfrak{H} \to \mathfrak{H}$ are anti-unitary. If $T_1, T_2 : \mathfrak{H} \to \mathfrak{H}$ are both anti-unitary, then the product $T_1 T_2$ is unitary. Hence we obtain a new group $\mathsf{AU}(\mathfrak{H})$ consisting of all unitary and anti-unitary operators on $\mathfrak{H}$. The map

$$\pi : \mathsf{AU}(\mathfrak{H}) \to \mathsf{Aut}(\mathbb{P}\mathfrak{H}),\ S \mapsto \hat{S}$$

then is a group homomorphism. Its kernel coincides with $\mathsf{U}(1) \cong \mathbb{S}^1$. To see this let $\pi(S) = \mathrm{id}_{\mathbb{P}\mathfrak{H}}$. Then for every ray ℓ there exists a complex number μ_ℓ such that $Sv = \mu_{\mathbb{C}v} v$ for all $v \in \ell$. By unitarity $|\mu_\ell| = 1$. Let $v, w \in \mathfrak{H}$ be two linearly independant vectors of norm 1. Since

$$\mu_{\mathbb{C}(w-v)}(w - v) = S(w - v) = \mu_{\mathbb{C}w} w - \mu_{\mathbb{C}v} v\ ,$$

one has $0 = (\mu_{\mathbb{C}(w-v)} - \mu_{\mathbb{C}w}) w + (\mu_{\mathbb{C}v} - \mu_{\mathbb{C}(w-v)}) v$ which implies $\mu_{\mathbb{C}w} = \mu_{\mathbb{C}(w-v)} = \mu_{\mathbb{C}v}$ by linear independence of v and w. Hence all the $\mu_{\mathbb{C}v}$ coincide and $S = \mu\, \mathrm{id}_{\mathfrak{H}}$ for some complex number $\mu \in \mathsf{U}(1) \cong \mathbb{S}^1$. A consequence of this observation is also that the homomorphism $\pi|_{\mathsf{U}(\mathfrak{H})} : \mathsf{U}(\mathfrak{H}) \to \mathsf{Aut}(\mathbb{P}\mathfrak{H})$, $S \mapsto \hat{S}$ is not surjective because for every anti-unitary T and unitary S the product TS^{-1} is anti-unitary, hence can not be an element of $\mathsf{U}(1)$. We denote the image of $\mathsf{U}(\mathfrak{H})$ under π by $\mathsf{U}(\mathbb{P}\mathfrak{H})$ and call its elements the *unitary automorphisms* of $\mathbb{P}\mathfrak{H}$.

3.2.5 Theorem (Wigner's theorem, Wigner (1944)) *Let $\mathfrak{H}$ be a complex Hilbert space of dimension $\geqslant 2$. Then the sequence of group homomorphisms*

$$1 \longrightarrow \mathsf{U}(1) \longrightarrow \mathsf{AU}(\mathfrak{H}) \xrightarrow{\pi} \mathsf{Aut}(\mathbb{P}\mathfrak{H}) \longrightarrow 1$$

is exact.

3.2.6 Remark Wigner's theorem was first stated in Wigner (1944), but with an incomplete proof. Only several years later complete and independent proofs of Wigner's result were given by Uhlhorn (1962), Lomont & Mendelson (1963), and Bargmann (1964).

Proof. Wigner's theorem is an immediate consequence of the precedinmg considerations and the following more general result. □

3.2.7 Theorem (Optimal version of Wigner's theorem, Gehér (2014)) *Let $\mathfrak{H}$ be a complex Hilbert space of dimension $\geqslant 2$. Then for every isometry $A : \mathbb{P}\mathfrak{H} \to \mathbb{P}\mathfrak{H}$ there exists a linear or conjugate-linear isometry $S : \mathfrak{H} \to \mathfrak{H}$ such that $A = \hat{S}$, where $\hat{S}$ is the isometry on $\mathbb{P}\mathfrak{H}$ which maps the ray $\mathbb{C}v$ with $v \in \mathfrak{H} \backslash \{0\}$ to the ray $\mathbb{C}Sv$.*

Proof. To prove the claim we will follow the elementary argument by Gehér (2014). □

Lifting of projective representations and Bargmann's theorem

3.2.8 Theorem (Bargmann's Theorem) *Let $\mathfrak{H}$ be a complex Hilbert space and G a connected and simply connected Lie group with $H^2(\mathfrak{g},\mathbb{R}) = 0$. Then every projective representation $\tau : G \to \mathsf{U}(\mathbb{P}\mathfrak{H})$ can be lifted to a unitary representation $\sigma : G \to \mathsf{U}(H)$ that is $\pi \circ \sigma = \tau$, where $\pi : \mathsf{U}(\mathfrak{H}) \to \mathsf{U}(\mathbb{P}\mathfrak{H})$ is the canonical projection.*

3.2.9 Remark The lifting theorem was proved first in Bargmann (1954). The short proof we present here goes back to Simms (1971). We closely follow his argument.

Proof of the theorem. Let E be the fibered product of π and τ with the canonical homomorphisms $\tilde{\tau} : E \to \mathsf{U}(\mathfrak{H})$ and $\pi^E : E \to G$. For the resulting commutative diagram of groups with two exact rows

$$\begin{tikzcd}
1 \arrow[r] & \mathsf{U}(1) \arrow[r] \arrow[d, "\mathrm{id}"'] & E \arrow[r, "\pi^E"] \arrow[d, "\tilde{\tau}"'] & G \arrow[r] \arrow[d, "\tau"] & 1 \\
1 \arrow[r] & \mathsf{U}(1) \arrow[r] & \mathsf{U}(\mathfrak{H}) \arrow[r, "\pi"'] & \mathsf{U}(\mathbb{P}\mathfrak{H}) \arrow[r] & 1
\end{tikzcd}$$

we want to construct a section $s : G \to E$ of $\pi^E : E \to G$ which is a splitting meaning that s is a group homomorphism and $\pi^E \circ s = \mathrm{id}_G$. With the construction of such an s we are done because then the unitary representation $\tilde{\tau} \circ s$ is a lifting of the projective representation $\tau : G \to \mathsf{U}(\mathbb{P}\mathfrak{H})$.

Observe that E is a Lie group by Kuranishi's theorem, see (Montgomery & Zippin, 1955, §4.3), since E is central extension of a Lie group, hence locally compact, and there exist local continuous sections $\sigma : U \to E$ that is $U \subset G$ is open and $\pi^E \circ \sigma = \mathrm{id}_U$.

The short exact sequence of Lie groups

$$1 \longrightarrow \mathsf{U}(1) \longrightarrow E \xrightarrow{\pi^E} G \longrightarrow 1$$

induces a short exact sequence of Lie algebras

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{e} \xrightarrow{T\pi^E} \mathfrak{g} \longrightarrow 0 \ , \tag{3.2.1}$$

where $\mathfrak{e}$ is the Lie algebra of E and $\mathfrak{g}$ the one of G. Observe that $T\pi^E$ is surjective with kernel $\mathbb{R}$ being in the center of $\mathfrak{e}$. Choose a linear map $\lambda : \mathfrak{g} \to \mathfrak{e}$ such that $\pi^E \circ \lambda = \mathrm{id}_{\mathfrak{g}}$. Put $\Theta(x,y) = [\lambda(x),\lambda(y)] - \lambda([x,y])$ for all $x,y \in \mathfrak{g}$. Then

$$T\pi^E \circ \Theta(x,y) = [T\pi^E \circ \lambda(x), T\pi^E \circ \lambda(y)] - T\pi^E \circ \lambda([x,y]) = [x,y] - [x,y] = 0 \ .$$

Hence $\Theta(x,y)$ is in the kernel of $T\pi^E$ which means that Θ is a map $\mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. By definition, $\Theta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is skew symmetric. Let us show that it satisfies the Jacobi identity. Compute, using the Jacobi identity for the Lie algebra bracket and the fact that Θ has image in the center of $\mathfrak{e}$,

$$\begin{aligned}
\Theta([x,y]),z)+ & \Theta([y,z],x) + \Theta([z,x],y) = \\
& = [\lambda([x,y]),\lambda(z)] + [\lambda([y,z]),\lambda(x)] + [\lambda([z,x]),\lambda(y)]- \\
& \quad - \lambda([[x,y],z]) - \lambda([[y,z],x]) - \lambda([[z,x],y]) = \\
& = [[\lambda(x),\lambda(y)],\lambda(z)] + [[\lambda(y),\lambda(z)],\lambda(x)] + [[\lambda(z),\lambda(x)],\lambda(y)]- \\
& \quad - [\Theta([x,y]),\lambda(z)] - [\Theta([y,z]),\lambda(x)] - [\Theta([z,x]),\lambda(y)] = 0 \ .
\end{aligned}$$

Therefore, Θ is a Lie algebra 2-cocycle. By $H^2(\mathfrak{g}, \mathbb{R}) = 0$, there exists a linear $\theta : \mathfrak{g} \to \mathbb{R}$ such that $\Theta(x, y) = \theta([x, y])$ for all $x, y \in \mathfrak{g}$. Put $\mu(x) = \lambda(x) + \theta(x)$. Then, since θ has values in the center of $\mathfrak{e}$,

$$\begin{aligned}[\mu(x), \mu(y)] &= [\lambda(x) + \theta(x), \lambda(y) + \theta(y)] = [\lambda(x), \lambda(y)] = \\ &= \Theta(x, y) + \lambda([x, y]) = \theta([x, y]) + \lambda([x, y]) = \mu([x, y]) \ .\end{aligned}$$

Hence $\mu : \mathfrak{g} \to \mathfrak{e}$ is a Lie-Algebra homomorphism and fulfills

$$T\pi^E \circ \mu(x) = T\pi^E(\lambda(x) + \theta(x)) = T\pi^E(\lambda(x)) = x \quad \text{for all } x \in \mathfrak{g} \ .$$

So μ is also a section of $T\pi^E$ which shows that the short exact sequence of Lie algebras (3.2.1) is split.

By $\pi_1(G) = 1$, the Lie algebra homomorphism $\mu : \mathfrak{g} \to \mathfrak{e}$ has a lifting to a group homomorphism $s : G \to E$ such that $\pi^E \circ s = \mathrm{id}_{\mathfrak{g}}$. The proof is finished. $\square$

4. Deformation quantization

4.1. Fedosov's construction of star products

The various Weyl algebras of a Poisson vector space

4.1.1 Definition By a *Poisson vector space* over the field $\mathbb{K}$ of real or complex numbers one understands a pair (V, Π) where V is a finite dimensional vector space over $\mathbb{K}$ and $\Pi \in \Lambda^2 V$ is an antisymmetric bivector.

Given two Poisson vector spaces (V, Π) and (W, Ξ), a linear map $f : V \to W$ is called a *morphism of Poisson vector spaces* if $f_* \Pi := (f \otimes f)\Pi = \Xi$.

Poisson vector spaces together with their morphisms obviously form a category which we denote by $\mathsf{PVec}_{\mathbb{K}}$.

4.1.2 Example Let $V = \mathbb{R}^{2n}$ or $V = \mathbb{R}^{2n+1}$. Then V together with the bivector $\Pi_{\mathrm{can}} = \sum_{k=1}^{n} \frac{\partial}{\partial x_{k+n}} \wedge \frac{\partial}{\partial x_k}$ is a Poisson vector space. One calls Π_{can} the canonical (constant) Poisson structure on V.

4.1.3 Let $\operatorname{rk}\Pi$ be the rank of Π that is the dimension of the image of the musical map

$$\Pi^\sharp : V^* \to V, \ \alpha \mapsto \alpha \lrcorner \Pi \ ,$$

where

$$\alpha \lrcorner : \Lambda^k V \to \Lambda^{k-1} V, \ \sum_{i=1}^{N} v_{i,1} \wedge \ldots \wedge v_{i,k} \mapsto \sum_{i=1}^{N} \sum_{j=1}^{k} (-1)^{j+1} \langle \alpha, v_{i,j} \rangle \wedge v_{i,1} \wedge \ldots \wedge \widehat{v_{i,j}} \wedge \ldots \wedge v_{i,k}$$

denotes the interior product of a 1-form with an alternating k-vector. Then $\operatorname{rk}\Pi$ is even dimensional, and (V, Π) isomorphic as a Poisson vector space to the product of $(\mathbb{R}^{\operatorname{rk}\Pi}, \Pi_{\mathrm{can}})$ with $(\mathbb{R}^{\dim V - \operatorname{rk}\Pi}, 0)$.

4.1.4 Remark The category $\mathsf{PVec}_{\mathbb{K}}$ is dual to the category $\mathsf{PSVec}_{\mathbb{K}}$ of presymplectic vector spaces over $\mathbb{K}$ that is the category of all finite dimensional $\mathbb{K}$-vector spaces W together with a (constant) 2-form $\omega \in \Lambda^2 W^*$.

A contravariant isomorphism between these two categories is given by the dualization functor $^* :$ $\mathsf{PVec}_{\mathbb{K}} \to \mathsf{PSVec}_{\mathbb{K}}$ which maps $V \mapsto V^*$ and the bivector Π on V to the 2-form $\omega : V^* \times V^* \to \mathbb{K}$, $(\alpha, \beta) \mapsto \beta \lrcorner (\alpha \lrcorner \Pi)$. Its inverse is again given by dualization.

4.1.5 The bivector Π of a Poisson vector space (V, Π) turns V into a Poisson manifold with bracket $\{-,-\} : \mathcal{C}^\infty(V) \times \mathcal{C}^\infty(V) \to \mathcal{C}^\infty(V)$ given by

$$\{f, g\} = dg \lrcorner (df \lrcorner \Pi) \quad \text{for } f, g \in \mathcal{C}^\infty(V) \ .$$

By construction, $\{-,-\}$ is antilinear and a derivation in each component. Since for all linear functions $\lambda, \mu : V \to \mathbb{K}$ the Poisson bracket $\{\lambda, \mu\}$ is constant, the Poisson bracket $\{\{\lambda, \mu\}, \nu\}$ of three linear functions vanishes, hence the Jacobi identity holds for linear and affine functions. This implies that the Jacobi identity is satisfied for all smooth functions, hence $\{-,-\}$ is a Poisson bracket on V indeed. We call it the *constant Poisson structure* associated to Π.

4.1.6 Definition The *Weyl algebra* of a Poisson vector space (V, Π) is defined by

$$\mathsf{A}(V, \Pi) = \mathsf{T}^\bullet V^* / (\alpha \otimes \beta - \beta \otimes \alpha - \beta \lrcorner (\alpha \lrcorner \Pi) \mid \alpha, \beta \in V^*) \ ,$$

where (X) stands for the ideal generated by $X \subset \mathsf{T}^\bullet V^*$.

4.1.7 Remarks (a) To a presymplectic vector space (W, ω) one associates the *Weyl algebra*

$$\mathsf{A}(W, \omega) = \mathsf{T}^\bullet W / (v \otimes w - w \otimes v - w \llcorner (v \llcorner \omega) \mid v, w \in W) \ ,$$

where $\llcorner$ denotes the interior product of a vector with a k-form. If (W, ω) is the dual of a Poisson vector space (V, Π), then the two Weyl algebras $\mathsf{A}(V, \Pi)$ and $\mathsf{A}(W, \omega)$ coincide by definition. We will silently make use of this fact in the following considerations.

(b) Let $\mathbb{K}$ be a field of characteristic 0 and $\mathbb{K}[x_1, \ldots, x_n]$ the polynomial ring over $\mathbb{K}$ in n (commuting) indetereminates. The *n-th Weyl algebra* $\mathsf{A}_n(\mathbb{K})$ over $\mathbb{K}$ is then defined as the subalgebra of the endomorphism ring $\mathrm{End}_{\mathbb{K}}(\mathbb{K}[x_1, \ldots, x_n])$ generated by the elements

$$\widehat{x}_k : \mathbb{K}[x_1, \ldots, x_n] \to \mathbb{K}[x_1, \ldots, x_n], \ p \mapsto x_k \cdot p$$

and

$$\partial_k : \mathbb{K}[x_1, \ldots, x_n] \to \mathbb{K}[x_1, \ldots, x_n], \ p \mapsto \frac{\partial p}{\partial x_k} \ ,$$

where k runs through $1, \ldots, n$. The commutation relations for these operators are, using the Kronecker delta,

$$[\widehat{x}_k, \widehat{x}_l] = 0, \quad [\partial_k, \partial_l] = 0, \quad [\partial_k, \widehat{x}_l] = \delta_{k,l} \ . \tag{4.1.1}$$

Recall that $\mathsf{A}_n(\mathbb{K})$ coincides with the ring of differential operators on $\mathbb{K}[x_1, \ldots, x_n]$ in the sense of Grothendieck. For a proof see Coutinho (1995).

(c) Let ω be a the canonical symplectic form on $\mathbb{R}^{2n}$. The Weyl algebra $\mathsf{A}(\mathbb{R}^{2n}, \omega)$ then coincides naturally with the algebra of differential operators on $\mathbb{R}^n$ with polynomial coefficients. To see this denote the canonical basis of $\mathbb{R}^{2n}$ by $(Q_1, \ldots, Q_n, P_1, \ldots, P_n)$ and the corresponding coordinate functions by $(q_1, \ldots, q_n, p_1, \ldots, p_n)$. The commutators of these basis elements in the Weyl algebra are

$$[Q_k, Q_l] = 0, \quad [P_k, P_l] = 0, \quad [P_k, Q_l] = \delta_{k,l} \ . \tag{4.1.2}$$

Therefore, any element of $\mathsf{A}(\mathbb{R}^{2n}, \omega)$

Next consider the *symmetric* (*covariant*) *tensor algebra* $\mathsf{S}^\bullet V^*$ over V. Recall that it is defined as the algebra with underlying vector space

$$\mathsf{S}^\bullet V^* = \bigoplus_{k\in\mathbb{N}} \mathsf{S}^k V^* \ , \tag{4.1.3}$$

where $\mathsf{S}^k V^* \subset \bigotimes^k V^*$ denotes the space of all symmetric (covariant) k-tensors in V. An element $t \in \mathsf{S}^k V^*$ is called *homogenous* of *symmetric degree* $\deg_{\mathrm{s}} t = k$. It can be written in the form

$$t = \sum_{i\in I} t_{i,1} \otimes \ldots \otimes t_{i,k} \ ,$$

where I is a finite index set, and $t_{i,1}, \ldots, t_{i,k}$ are elements of the dual V^*.

The bundle of formal Weyl algebras

Let M be a smooth manifold. Recall the notion of the *symmetric* (*covariant*) *tensor algebra bundle* $\mathsf{S}^\bullet T^*M$ over M. It is defined by

$$\mathsf{S}^\bullet T^*M = \bigoplus_{k\in\mathbb{N}} \mathsf{S}^k T^*M \ , \tag{4.1.4}$$

where $\mathsf{S}^k T^*M = \bigcup_{p\in M} \mathsf{S}^k T^*_p M \subset \bigotimes^k T^*M$ is the bundle of all symmetric (covariant) k-tensors. Note that we have a canonical (fiberwise) isomorphism $\mathsf{S}^k T^*M \cong \big(\mathsf{S}^k TM\big)^*$ which leads to the canonical identifications

$$\mathsf{S}^\bullet M = \bigoplus_{k\in\mathbb{N}} \mathsf{S}_k T^*M \cong \bigoplus_{k\in\mathbb{N}} \mathsf{S}^k TM = \mathsf{S}^\bullet TM \ .$$

An element $t \in \mathsf{S}^k T^*M$ is called *homogenous* of *symmetric degree* $\deg_{\mathrm{s}} t = k$. It can be written in the form

$$t = \sum_{i\in I} t_{1,i} \otimes \ldots \otimes t_{k,i} \ ,$$

where I is a finite index set, and $t_{1,i}, \ldots, t_{k,i}$ are elements of the cotangent bundle T^*M having the same footpoint as t. Every element of the symmetric tensor algebra bundle $\mathsf{S}^\bullet M$ can be expanded as a finite sum of homogeneous symmetric tensors.

The (fiberwise) *symmetric product* $\vee : \mathsf{S}M \times_M \mathsf{S}M \to \mathsf{S}M$ is constructed by defining it, for each $p \in M$, first on homogeneous elements $t = \sum_{i\in I} t_{1,i} \otimes \ldots \otimes t_{k,i} \in \mathsf{S}^k_p M$ and $s = \sum_{j\in J} s_{k+1,j} \otimes \ldots \otimes s_{k+l,j} \in \mathsf{S}^l_p M$ by

$$\begin{aligned} \vee(t,s) = t \vee s &= \frac{1}{k!\,l!} \sum_{\sigma\in S_{k+l}} \sum_{i\in I, j\in J} v_{\sigma(1),ij} \otimes \ldots \otimes v_{\sigma(k+l),ij} \ , \text{ where} \\ v_{m,ij} &= \begin{cases} t_{m,i} & \text{if } 1 \leqslant m \leqslant k \ , \\ s_{m,j} & \text{if } k < m \leqslant k+l \ , \end{cases} \end{aligned} \tag{4.1.5}$$

and then extending it linearly in each component to the whole fiber $\mathsf{S}^\bullet_p M \times \mathsf{S}^\bullet_p M$. Using the canonical symmetrization operator

$$\mathsf{S} : \mathsf{T}^\bullet M = \mathsf{T}^\bullet TM \to \mathsf{S}M,\ t = \sum_{i \in I} t_{1,i} \otimes \ldots \otimes t_{k,i} \mapsto \sum_{\sigma \in S_k} \sum_{i \in I} t_{\sigma(1),i} \otimes \ldots \otimes t_{\sigma(k),i}$$

we can also write

$$t \vee s = \binom{k+l}{k} \mathsf{S}(t \otimes s). \tag{4.1.6}$$

Together with the symmetric product $\mathsf{S}M$ now becomes a graded algebra. Note that it is canonically isomorphic to the algebra $\mathcal{C}^\infty_{\mathrm{pol}}(TM)$ of smooth functions on TM which are polynomial in the fibers of TM.

Let us define an action of an antisymmetric bivector field $B = \sum_\iota B^1_\iota \otimes B^2_\iota \in \Omega^2 M$ on $\mathsf{S}M \otimes \mathsf{S}M$ by

$$\mathsf{S}M \otimes \mathsf{S}M \ni t \otimes s \mapsto B\,(t \otimes s) = \sum_\iota B^1_\iota \lrcorner\, t \otimes B^2_\iota \lrcorner\, s \in \mathsf{S}M \otimes \mathsf{S}M. \tag{4.1.7}$$

Under the isomorphism $\mathsf{S}M \to \mathcal{C}^\infty_{\mathrm{pol}}(TM)$ the bivector field B acts as a bidifferential operator, i.e. we have for $f, g \in \mathcal{C}^\infty_{\mathrm{pol}}(TM)$, $v, w \in T_x M$ and $x \in M$

$$B\,(f \otimes g)(v,w) = \sum_\iota B^1_\iota f(v) \otimes B^2_\iota g(w) = \sum_\iota \frac{d}{dt} f(v + tB^1_\iota)\Big|_{t=0} \frac{d}{ds} g(w + sB^2_\iota)\Big|_{s=0}. \tag{4.1.8}$$

With these preparations in mind we are now able to define Fedosov's notion of the bundle of formal Weyl algebras.

4.1.8 Definition Let (M, ω) be a symplectic manifold of dimension $2n$ and Π the corresponding Poisson bivector. The *formal Weyl algebra* $\mathsf{A}M$ of M is then defined as the space $\mathsf{S}M[[\hbar]]$ of formal power series with coefficients in $\mathsf{S}M$ together with the *Moyal product* $\circ$ given by

$$f \circ g = \sum_{k \in \mathbb{N}} \frac{1}{k!} \left(\frac{i\,\hbar}{2}\right)^k \vee \left(\Pi^k\,(f \otimes g)\right) = \vee \left(\exp\left(\frac{i\,\hbar}{2}\Pi\right) f \otimes g\right). \tag{4.1.9}$$

Note that in this definition all operations on $\mathsf{S}M$ were naturally extended to $\mathsf{S}M[[\hbar]]$.

On the Weyl algebra bundle $\mathsf{A}M$ we introduce the *Fedosov filtration*

$$\mathsf{A}M = \mathsf{A}^0 M \supset \mathsf{A}^1 M \supset \mathsf{A}^2 M \supset \ldots \supset \mathsf{A}^k M \supset \ldots \tag{4.1.10}$$

by defining

$$\mathsf{A}^k M = \left\{ t = \sum_{l,r \in \mathbb{N}} t_{rl}\,\hbar^l \in \mathsf{S}M[[\hbar]] \;:\; t_{rl} \in \mathsf{S}^r M \;\&\; t_{rl} = 0 \text{ for } r + 2l < k \right\}. \tag{4.1.11}$$

The topology generated by this filtration is called the *F-topology*. Furthermore we define the *F-degree* $\deg_{\mathrm{F}} t$ of an element $t \in \mathsf{A}M$ as the supremum of all $k \in \mathbb{N}$ with $t \in \mathsf{A}^k M$.

By definition $\deg_{\mathrm{F}}\, 0 = \infty$, $\deg_{\mathrm{F}}\, \hbar = 2$ and $\deg_{\mathrm{F}}\, \lambda = m$ for any covariant m-tensor field λ.

We have to show that $\circ$ is a well-defined product on $\mathsf{S}M[[\hbar]]$ and that the $\mathsf{A}^k M$ define a filtration on the algebra $\mathsf{A}M$ indeed. It suffices to show that $\circ$ is associative and that $\mathsf{A}^k M \circ \mathsf{A}^l M \subset \mathsf{A}^{k+l} M$ holds for all $k, l \in \mathbb{N}$. Associativity of $\circ$ follows from the following chain of equalities:

$$\begin{aligned}
(f \circ g) \circ h &= \sum_{k\in\mathbb{N}} \sum_{l\in\mathbb{N}} \left(\frac{i\hbar}{2}\right)^{k+l} \vee \Pi^k \left(\vee \Pi^l (f \otimes g) \otimes h\right) \\
&= \sum_{r\in\mathbb{N}} \sum_{k+l=r} \left(\frac{i\hbar}{2}\right)^{r} \vee \Pi^k \left(\vee \Pi^l (f \otimes g) \otimes h\right) \\
&= \sum_{r\in\mathbb{N}} \sum_{k+l+m=r} \left(\frac{i\hbar}{2}\right)^{r} \vee \left(\Pi^k_{13}\, \Pi^l_{23}\, \Pi^m_{12}\, (f \otimes g \otimes h)\right) \\
&= \sum_{r\in\mathbb{N}} \sum_{k+l+m=r} \left(\frac{i\hbar}{2}\right)^{r} \vee \left(\Pi^k_{13}\, \Pi^m_{12}\, \Pi^l_{23}\, (f \otimes g \otimes h)\right) \\
&= \sum_{r\in\mathbb{N}} \sum_{k+l=r} \left(\frac{i\hbar}{2}\right)^{r} \vee \Pi^l \left(f \otimes \vee \Pi^k (g \otimes h)\right) \\
&= f \circ (g \circ h).
\end{aligned} \tag{4.1.12}$$

Here we have denoted by $\Pi_{\iota\kappa}\,(a \otimes b \otimes c)$ the natural action of Π on the ι, κ-factors of $a \otimes b \otimes c$ and have used the Jacobi-identity for the Poisson bivector Π. The second claim follows immediately from Eq. (4.1.9) and the definition of $\mathsf{A}^k M$.

to do: By definition $\mathsf{S}M$ is a graded $\mathcal{C}^\infty(M)$-module.

Besides $\mathsf{A}M$ we will consider in the following differential forms with values in $\mathsf{A}M$, i.e. we will consider the space $\Omega\mathsf{A}M := \mathsf{A}M \otimes \Omega M \cong (\mathsf{S}M \otimes \Omega M)[[\hbar]] = (\mathsf{S}M \otimes \Omega M)^{\mathbb{N}}$. By $\circ$ and the exterior product on ΩM this vector space carries a multiplicative structure which also will be denoted by $\circ$. A second multiplicative structure, which we denote by $\cdot$, comes from the symmetric product on $\mathsf{S}M$ and the exterior product on ΩM. The filtration on $\mathsf{A}M$ induces one on $\Omega\mathsf{A}M$ by

$$\Omega\mathsf{A}M \subset \mathsf{A}^1 M \otimes \Omega M \subset \ldots \subset \mathsf{A}^k M \otimes \Omega M \subset \ldots\,; \tag{4.1.13}$$

thus making $(\Omega\mathsf{A}M, \circ)$ into a filtered algebra. Additionally $\Omega\mathsf{A}M$ posseses a graduation coming from ΩM:

$$\Omega\mathsf{A}M = \bigoplus_{1 \leqslant q \leqslant 2n} \mathsf{A}M \otimes \Omega^q M. \tag{4.1.14}$$

The corresponding degree function $\Omega\mathsf{A}M \to \mathbb{R}^{2n}$ will be denoted by $\deg_{\mathrm{a}}$, the *antisymmetric degree*. Together with the symmetric degree $\Omega\mathsf{A}M$ now becomes a bigraded vector space. Therefore we have for any element $a \in \Omega\mathsf{A}M$ a decomposition

$$a = \sum_{pq} a_{pq}, \tag{4.1.15}$$

where a_{pq} is the unique homogeneous component of a with symmetric degree p and antisymmetric degree q or in other words with bidegree (p, q). With respect to the product $\cdot$, but not $\circ$, $\Omega\mathsf{A}M$

becomes a bigraded algebra. Nevertheless $(\Omega\mathsf{A}M, \circ)$ is a graded algebra with respect to the antisymmetric degree.

Next we introduce the $\circ$-supercommutator $[-,-]$ on $\Omega\mathsf{A}M$ as the unique bilinear map such that for two elements $a, b \in \Omega\mathsf{A}M$ being homogeneous with respect to the antisymmetric degree the equation

$$[a,b] = a \circ b \; - \; (-1)^{\deg_{\mathrm{a}} a \cdot \deg_{\mathrm{a}} b}\, b \circ a \tag{4.1.16}$$

holds. The supercommutator induces for every $a \in \Omega\mathsf{A}M$ an adjoint map

$$\mathrm{ad}\,(a): \; \Omega\mathsf{A}M \to \Omega\mathsf{A}M, \; b \mapsto [a,b]. \tag{4.1.17}$$

Moreover $\frac{1}{\hbar}\,\mathrm{ad}\,(a)$ is a well-defined map on $\Omega\mathsf{A}M$ and comprises a superderivation of $\Omega\mathsf{A}M$. The symplectic form $\omega = \sum_{ij} \omega_{ij}\, dx_i \otimes dx_j$ can be interpreted as an element of $\mathsf{A}M \otimes \Omega^1 M$. Thus it gives rise to the inner superderivation

$$\delta = -\frac{i}{\hbar}\,\mathrm{ad}\,(\omega) \tag{4.1.18}$$

of $\Omega\mathsf{A}M$. Let us denote for any smooth vector field $V \in \mathcal{C}^\infty(TM)$ and every element $f \otimes \alpha \in \mathsf{A}M \otimes \Omega M$ the insertion $(V \llcorner f) \otimes \alpha$ (resp. $f \otimes (V \llcorner \alpha)$) of V in the symmetric (resp. antisymmetric) part of $f \otimes \alpha$ by $V \llcorner_{\mathrm{s}} (f \otimes \alpha)$ (resp. $V \llcorner_{\mathrm{a}} (f \otimes \alpha)$). With this notation we get the following expansion of δ in local coordinates:

$$\begin{aligned} \delta(a) &= -\frac{i}{\hbar}\left(\omega \circ a - (-1)^k a \circ \omega\right) \\ &= -\frac{i}{\hbar}\underbrace{\left(\omega \cdot a - (-1)^k a \cdot \omega\right)}_{=0} + \\ &\quad + \frac{1}{2}\sum_{kl} \Pi_{kl}\, \omega\left(\frac{\partial}{\partial x_k}, -\right) \cdot \left(\frac{\partial}{\partial x_l}\right) \llcorner\, a - (-1)^k \Pi_{kl}\left(\frac{\partial}{\partial x_k} \llcorner a\right) \cdot \omega\left(\frac{\partial}{\partial x_l}, -\right) \\ &= \sum_l \,(1 \otimes dx_l) \cdot \left(\frac{\partial}{\partial x_l} \llcorner\, a\right). \end{aligned} \tag{4.1.19}$$

Here we have used the local expansion

$$\Pi = \sum_{kl} \Pi_{kl}\, \frac{\partial}{\partial x_k} \otimes \frac{\partial}{\partial x_l} \tag{4.1.20}$$

and the fact that

$$\sum_k \Pi_{kl}\, \omega\left(\frac{\partial}{\partial x_k}, -\right) = dx_l. \tag{4.1.21}$$

Analogously we can define a second operator δ^* on $\Omega\mathsf{A}M$ by setting locally

$$\delta^*\,(a) = \sum_l (dx_l \otimes 1) \cdot \left(\frac{\partial}{\partial x_l} \llcorner_{\mathrm{a}}\, a\right). \tag{4.1.22}$$

$\delta^*(a)$ is well-defined, as it can be written in the form

$$\delta^*(a) = e(\vee \otimes \llcorner)a, \tag{4.1.23}$$

where $e \in \mathcal{C}^\infty(TM)$ is the *Euler tensor field* which locally is given by $e = \sum_l dx_l \otimes \frac{\partial}{\partial x_l}$. Note that δ^* is not a superderivation of $\Omega\mathsf{A}M$.

4.1.9 Proposition *The operators δ and δ^* are homogeneous of symmetric degree -1 (resp. 1) and antisymmetric degree 1 (resp. -1). Moreover they fulfill the following two relations:*

$$\delta^2 = (\delta^*)^2 = 0, \tag{4.1.24}$$

$$(\delta\,\delta^* + \delta^*\delta)(f \otimes \alpha) = (p+q)(f \otimes \alpha), \tag{4.1.25}$$

where $f \in \mathsf{A}M$ is homogeneous of symmetric degree p and $\alpha \in \Omega^q M$.

Proof. The first property follows from the local expressions for δ and δ^*:

$$\delta^2(f \otimes \alpha) = \sum_{kl} \left(\frac{\partial}{\partial x_k} \vee \frac{\partial}{\partial x_l} \right) \lrcorner\, f \otimes dx_k \wedge dx_l \wedge \alpha = 0, \tag{4.1.26}$$

$$\delta^{*2}(f \otimes \alpha) = \sum_{kl} (dx_k \vee dx_l) \otimes \left(\frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial x_l} \right) \lrcorner\, \alpha = 0, \tag{4.1.27}$$

as both sums are symmetric and antisymetric with respect to the indices k, l. The second property is also a direct consequence of the local expressions for δ and δ^*. □

Denote by $\delta^- : \Omega\mathsf{A}M \to \Omega\mathsf{A}M$ the operator

$$\Omega\mathsf{A}M \ni a = \sum_{pq} a_{pq} \mapsto \delta^-(a) = \sum_{p+q>0} \frac{1}{p+q}\, \delta^* a_{pq} \in \Omega\mathsf{A}M. \tag{4.1.28}$$

Then the above proposition entails a kind of Hodge-De Rham decomposition in $\Omega\mathsf{A}M$, namely the relation

$$a = \delta\,\delta^-(a) + \delta^-\,\delta(a) + a_{00}. \tag{4.1.29}$$

for every $a \in \Omega\mathsf{A}M$.

In the following the notion of the $\circ$-*center* $\mathsf{Z}_{()}\circ M$ of $\Omega\mathsf{A}M$ will be very useful. It is defined as the kernel of the family $\big(\operatorname{ad}(a)\big)_{a\in\Omega\mathsf{A}M}$ and obviously fulfills the equation

$$\mathsf{Z}_{()}\circ M = \mathsf{S}^0 M \otimes \Omega M = \{a \in \Omega\mathsf{A}M :\ \deg_{\mathrm{s}} a = 0\}. \tag{4.1.30}$$

There are two canonical projections from $\Omega\mathsf{A}M$ in $\mathsf{Z}_{()}\circ M$, namely

$$\pi_{00} : \Omega\mathsf{A}M \to \Omega\mathsf{A}M, \quad a = \sum_{pq} a_{pq} \mapsto a_{00} \tag{4.1.31}$$

and

$$\pi_0 : \Omega\mathsf{A}M \to \Omega\mathsf{A}M, \quad a = \sum_{pq} a_{pq} \mapsto \sum_q a_{0q}. \tag{4.1.32}$$

Connections on the formal Weyl algebra

We now want to give $\Omega\mathsf{A}M$ some more differential geometric structure. To achieve this let us choose a symplectic connection ∇ on M, i.e. a connection ∇ fulfilling $\nabla\omega = 0$. Then ∇ gives rise to a connection ∇ on $\Omega\mathsf{A}M$ by defining

$$\nabla(f \otimes \alpha) = \nabla f \cdot \alpha + f \otimes d\alpha \tag{4.1.33}$$

for $f \in \mathsf{A}M$ and $\alpha \in \Omega M$. Hereby we naturally regard ∇f as an element of $\Omega^1\mathsf{A}M$. As ∇ is supposed to be torsionfree, we have $d\alpha = \nabla\alpha$, so $\nabla : \Omega\mathsf{A}M \to \Omega\mathsf{A}M$ is a connection on $\Omega\mathsf{A}M$ indeed, i.e. it fulfills

$$\nabla(\varphi a) = (1 \otimes d\varphi) \cdot a + \varphi\, Da \tag{4.1.34}$$

for every $a \in \Omega\mathsf{A}M$ and $\varphi \in \mathcal{C}^\infty(M)$. Moreover, ∇ is a homogeneous superderivation of $(\Omega\mathsf{A}M, \cdot)$ with bidegree $(0, 1)$, as the equation

$$\begin{aligned}\nabla\left((f \otimes \alpha) \cdot (g \otimes \beta)\right) &= \nabla\left((f \vee g) \otimes (\alpha \wedge \beta)\right) \\ &= (\nabla f \cdot g + f \cdot \nabla g) \cdot (\alpha \wedge \beta) + (f \vee g) \cdot \Big(d\alpha \wedge \beta + (-1)^{\deg_{\mathrm{a}} \alpha}\, \alpha \wedge d\beta\Big) \\ &= (\nabla f \cdot \alpha + f \otimes d\alpha) \cdot (g \otimes \alpha) + (-1)^{\deg_{\mathrm{a}} \alpha}\, (f \otimes \alpha) \cdot (\nabla g \cdot \beta + g \otimes d\beta) \\ &= (\nabla(f \otimes \alpha)) \cdot (g \otimes \beta) + (-1)^{\deg_{\mathrm{a}} \alpha}\, (f \otimes \alpha) \cdot (\nabla(g \otimes \beta))\end{aligned} \tag{4.1.35}$$

holds for homogeneous $f \otimes \alpha, g \otimes \beta \in \Omega\mathsf{A}M$. With respect to $*$, the connection ∇ is a homogeneous superderivation of antisymmetric degree 1 as well. To prove this first recall that $\nabla\Pi = 0$, hence

$$\nabla(f * g) = (\nabla f) * g + f * (\nabla g). \tag{4.1.36}$$

But then

$$\begin{aligned}\nabla\left(f \otimes \alpha) * (g \otimes \beta)\right) &= \nabla(f * g) \cdot (\alpha \wedge \beta) + (f * g) \otimes d(\alpha \wedge \beta) \\ &= ((\nabla f) * g + f * (\nabla g)) \cdot (\alpha \wedge \beta) + (f * g) \otimes \Big(d\alpha \wedge \beta + (-1)^{\deg_{\mathrm{a}} \alpha} \alpha \wedge d\beta\Big) \\ &= (\nabla f \cdot \alpha + f \otimes d\alpha) * (g \otimes \alpha) + (-1)^{\deg_{\mathrm{a}} \alpha}\, (f \otimes \alpha) * (\nabla g \cdot \beta + g \otimes d\beta) \\ &= (\nabla(f \otimes \alpha)) * (g \otimes \beta) + (-1)^{\deg_{\mathrm{a}} \alpha}\, (f \otimes \alpha) * (\nabla(g \otimes \beta))\end{aligned} \tag{4.1.37}$$

which gives the claim.

4.1.10 Proposition *The $*$-superderivation ∇ fulfills the following relations:*

$$[\nabla, \delta] = \nabla\delta + \delta\nabla = 0, \tag{4.1.38}$$

$$[\nabla, \nabla] = 2\nabla^2 = 2\frac{i}{\hbar}\, \mathrm{ad}\, (\tilde{R}), \tag{4.1.39}$$

where $\tilde{R} \in \mathsf{S}T^*M \otimes \Omega^2 M$ is the contraction $\tilde{R} = \omega \llcorner R$ of the curvature tensor R of ∇. Furthermore the contracted curvature $\tilde{R}$ satisfies the relation

$$\nabla\tilde{R} = \delta\tilde{R} = 0. \tag{4.1.40}$$

Proof. By $\nabla\omega = 0$ we have

$$[\nabla, \delta] = -\frac{i}{\hbar}[\nabla, \operatorname{ad}(\omega)] = -\frac{i}{\hbar}(\nabla\omega) = 0. \tag{4.1.41}$$

As ∇ has antisymmetric degree 1, the supercommutator of ∇ with itself is equal to $2\nabla^2$. But now we have in local coordinates

$$\begin{aligned} 2\nabla^2(f \otimes \alpha) &= 2\nabla(\nabla f \cdot \alpha + f \otimes d\alpha) = 2\nabla^2 f \cdot \alpha \\ &= \sum_{rs} (\nabla_{\partial_r}\nabla_{\partial_s} - \nabla_{\partial_s}\nabla_{\partial_r}) f \otimes dx_r \wedge dx_s \wedge \alpha \\ &= -\sum_{klrs} {R^k}_{lrs}\, dx_l \vee (\partial_k \llcorner f) \otimes dx_r \wedge dx_s \wedge \alpha \end{aligned} \tag{4.1.42}$$

and

$$\begin{aligned} 2\frac{i}{\hbar}\operatorname{ad}(\tilde{R})(f \otimes \alpha) &= 2\frac{i}{\hbar}\left(\tilde{R} * (f \otimes \alpha) - (f \otimes \alpha) * \tilde{R}\right) = \\ &= -\frac{1}{2}\left(\sum_{klmrs} \Pi_{mk}\, \tilde{R}_{mlrs}\, dx_l \vee (\partial_k \llcorner f) \otimes dx_r \wedge dx_s \wedge \alpha - \right. \\ &\qquad \left. - \sum_{klmrs} \Pi_{mk}\, \tilde{R}_{klrs}\, (\partial_m \llcorner f) \vee dx_l \otimes \alpha \wedge dx_r \wedge dx_s \right) \\ &= -\sum_{klrs} {R^k}_{lrs}\, dx_l \vee (\partial_k \llcorner f) \otimes dx_r \wedge dx_s \wedge \alpha, \end{aligned} \tag{4.1.43}$$

which gives the second equation. The relation $\nabla\tilde{R} = 0$ is nothing else but the first Bianchi identity for the connection ∇. Last we have

$$\delta\tilde{R} = \frac{1}{2}\sum_{klrs} R_{klrs}\, dx_l \otimes dx_k \wedge dx_r \wedge dx_s = 0, \tag{4.1.44}$$

as R_{klrs} is cyclic with respect to the indices (l, r, s). □

Besides ∇ we will also consider more general connections on $\Omega\mathsf{A}M$, in particular connections $D : \Omega\mathsf{A}M \to \Omega\mathsf{A}M$ of the form

$$D = \nabla + \frac{i}{\hbar}\operatorname{ad}(\gamma), \tag{4.1.45}$$

where γ is an element of $\Omega^1\mathsf{A}M$, uniquely determined by D up to a central one-form. We call such a D a *Weyl connection* and attach to it a now unique one-form γ_D fulfilling Eq. (4.1.45) and the normalization condition

$$\pi_0(\gamma_D) = 0. \tag{4.1.46}$$

The two-form

$$\Omega = \tilde{R} + \nabla\gamma_D + \frac{i}{\hbar}\gamma_D * \gamma_D \tag{4.1.47}$$

will then be called the *Weyl curvature* of D. Furthermore a Weyl connection D will be called *abelian*, if its Weyl curvature is a central form or, using the following proposition, if

$$D^2 = \frac{i}{\hbar}\operatorname{ad}(\tilde{\Omega}) = 0. \tag{4.1.48}$$

4.1.11 Proposition *Let D be a Weyl connection on $\Omega\mathsf{A}M$ and Ω its Weyl curvature. Then Ω fulfills the Bianchi-identity*

$$D\Omega = 0 \tag{4.1.49}$$

and the relation

$$D^2 = \frac{i}{\hbar}\,\mathrm{ad}\,(\Omega). \tag{4.1.50}$$

Proof. The Bianchi-identity follows from

$$\begin{aligned} D\Omega &= \nabla\Omega + \frac{i}{\hbar}\,[\gamma_D, \Omega] = \\ &= \nabla\tilde{R} + \nabla^2\gamma_D + \frac{i}{\hbar}\,[\nabla\gamma_D, \gamma_D] + \frac{i}{\hbar}\,[\gamma_D, \tilde{R}] + \frac{i}{\hbar}\,[\gamma_D, \nabla\gamma_D] + \frac{i}{\hbar}\,[\gamma_D, \gamma_D^2]. \end{aligned} \tag{4.1.51}$$

By the Bianchi identity for ∇ the first term vanishes, the last one as γ_D commutes with γ_D^2. By Eq. (4.1.39) the second and the fourth term cancel each other, hence $D\Omega = 0$. Using Proposition 4.1.10 the second equation follows immediately:

$$\begin{aligned} D^2 &= \nabla^2 + \frac{i}{\hbar}\,\mathrm{ad}\,(\nabla\gamma_D) + \frac{1}{2}\left(\frac{i}{\hbar}\right)^2 \mathrm{ad}\,([\gamma_D, \gamma_D]) \\ &= \frac{i}{\hbar}\,\mathrm{ad}\,\left(\tilde{R} + \nabla\gamma_D + \frac{i}{\hbar}\gamma_D * \gamma_D\right). \end{aligned} \tag{4.1.52}$$

□

We now will look for Abelian D or in other words for conditions on γ_D which guarantee D to be Abelian. To achieve this let us write γ_D in the form

$$\gamma_D = \omega + r, \tag{4.1.53}$$

where $r \in \Omega^1\mathsf{A}M$. Then we have

$$\Omega = \tilde{R} + \nabla r + \frac{i}{\hbar} r * r + \frac{i}{\hbar}\,\mathrm{ad}\,(\omega)(r) - 1 \otimes \omega, \tag{4.1.54}$$

as $\omega * \omega = i\hbar\, 1 \otimes \omega$. If now r fulfills

$$\delta(r) = \tilde{R} + \nabla r + \frac{i}{\hbar} r * r, \tag{4.1.55}$$

then $\Omega = -1 \otimes \omega$, hence D will be Abelian.

4.1.12 Lemma *An element $r \in \Omega^1\mathsf{A}M$ with $\deg_{\mathrm{F}}\, r \geqslant 2$ fulfills $\delta^- r = 0$ and Eq. (4.1.55) if and only if*

$$r = \delta^-\tilde{R} + \delta^-\left(\nabla r + \frac{i}{\hbar} r * r\right). \tag{4.1.56}$$

Proof. If the first condition is satisfied, (4.1.56) follows easily from $(\delta^-\delta + \delta\delta^-)r = r$. Let us show the converse and suppose (4.1.56) to be true. Then obviously $\delta^- r = 0$ by $(\delta^-)^2 = 0$. Let D be the Weyl connection on $\Omega\mathsf{A}M$ with $\gamma_D = \omega + r$. To prove (4.1.55) it then suffices to show $\Omega = -1 \otimes \omega$. We have

$$\delta^-(\Omega + 1 \otimes \omega) = \delta^-\left(\tilde{R} + \nabla r + \frac{i}{\hbar} r * r\right) - \delta^-\delta r = r - \delta^-\delta r = \delta\delta^- r = 0, \tag{4.1.57}$$

hence by the Bianchi identity $D\Omega = 0$ and $D(1 \otimes \omega) = 1 \otimes d\omega = 0$ the relation

$$\delta(\Omega + 1 \otimes \omega) = (D + \delta)(\Omega + 1 \otimes \omega) \tag{4.1.58}$$

is true. Using the Hodge-deRham decomposition in $\Omega \mathsf{A} M$ this entails

$$\Omega + 1 \otimes \omega = \delta^{-}(D + \delta)(\Omega + 1 \otimes \omega) = \delta^{-}\left(\nabla + \frac{i}{\hbar}\,\mathrm{ad}\,(r)\right)(\Omega + 1 \otimes \omega). \tag{4.1.59}$$

As the operator $\delta^{-}\left(\nabla + \frac{i}{\hbar}\,\mathrm{ad}\,(r)\right)$ raises the F-degree by 1, we must have $\Omega + 1 \otimes \omega = 0$. But this gives the claim. $\square$

5. Quantum spin systems

5.1. The quasi-local algebra of a spin lattice model

5.1.1 By a *Bravais lattice* or briefly just a *lattice* one understands a subgroup Λ of the additive group $\mathbb{R}^d$ of the form

$$\Lambda = \left\{ \sum_{i=1}^{d} \lambda_i a_i \;\middle|\; \lambda_i \in \mathbb{Z} \text{ for } i = 1, \ldots, d \right\} ,$$

where $(a_1, \ldots, a_d)$ is a basis of $\mathbb{R}^d$. We then say that Λ is the lattice *induced* by the basis $(a_1, \ldots, a_d)$. The length d of an inducing basis will be called the *dimension* of the lattice. Note that the dimension is uniquely determined by a given lattice but that there might be several bases by which the lattice is induced. The lattice $\mathbb{Z}^d$ will be called the *standard* or *cubic lattice* in dimension d. It is induced by the standard basis $(e_1, \ldots, e_d)$ of $\mathbb{R}^d$.

5.1.2 Let Λ be a lattice of dimension d, and denote by $\mathcal{P}_{\text{fin}}(\Lambda)$ the set of all finite subsets of Λ. Fix a natural number $N \geqslant 1$ and call $\frac{N}{2}$ the *spin degree* of the spin lattice model we are going to define. For each $x \in \Lambda$ let $\mathfrak{H}_x$ be the $N+1$-dimensional complex Hilbert space $\mathbb{C}^{N+1}$. Now put for $\mathcal{O} \in \mathcal{P}_{\text{fin}}(\Lambda)$

$$\mathfrak{H}_{\mathcal{O}} = \bigotimes_{x \in \mathcal{O}} \mathfrak{H}_x$$

and define the *local algebra* over $\mathcal{O}$ as the C^*-algebra

$$\mathfrak{A}_{\mathcal{O}} = \mathfrak{B}(\mathfrak{H}_{\mathcal{O}}) \ .$$

Note that due to their finite dimensionality the tensor product of finitely many Hilbert spaces $\mathfrak{H}_x$ coincides here with their Hilbert tensor product. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two finite subsets of λ such that $\mathcal{O}_1$ is a subset of $\mathcal{O}_2$, then one has the natural embedding

$$\alpha_{\mathcal{O}_1,\mathcal{O}_2} : \mathfrak{A}_{\mathcal{O}_1} \hookrightarrow \mathfrak{A}_{\mathcal{O}_2}$$

which, under the natural identification $\mathfrak{A}_{\mathcal{O}} \cong \bigotimes_{x \in \mathcal{O}} \mathfrak{B}(\mathfrak{H}_x)$, maps a tensor of the form $\otimes_{x \in \mathcal{O}_1} A_x$ with $A_x \in \mathfrak{B}(\mathfrak{H}_x)$ for all $x \in \mathcal{O}_1$ to the simple tensor $\otimes_{x \in \mathcal{O}_2} A_x$, where A_x is defined to be $\mathbb{1}_{\mathfrak{H}_x}$ whenever $x \in \mathcal{O}_2 \backslash \mathcal{O}_1$. In more abstract terms, $\alpha_{\mathcal{O}_1,\mathcal{O}_2}$ is the unique linear map making the diagram

commute where $\pi_{\mathcal{O}} : \prod_{x \in \mathcal{O}} \mathfrak{A}_x \to \bigotimes_{x \in \mathcal{O}} \mathfrak{A}_x$ is the canonical projection mapping the family $(A_x)_{x \in \mathcal{O}}$ to $\otimes_{x \in \mathcal{O}} A_x$ and $\overline{\alpha}_{\mathcal{O}_1,\mathcal{O}_2}$ is the map

$$\overline{\alpha}_{\mathcal{O}_1,\mathcal{O}_2} : \prod_{x \in \mathcal{O}_1} \mathfrak{A}_x \to \bigotimes_{x \in \mathcal{O}_2} \mathfrak{A}_x, \ (A_x)_{x \in \mathcal{O}_1} \mapsto (\otimes_{x \in \mathcal{O}_1} A_x) \otimes (\otimes_{x \in \mathcal{O}_2 \backslash \mathcal{O}_1} \mathbb{1}_{\mathfrak{H}_x}) \ .$$

6. Molecular quantum mechanics

6.1. The von Neumann–Wigner no-crossing rule

6.1.1 Theorem (von Neumann & Wigner (1929)) *For any positive integer n let*

$$\mathfrak{Herm}(n) = \{A \in \mathfrak{gl}(n,\mathbb{C}) \mid A^* = A\}$$

be the space of all (complex) hermitian $n \times n$ matrices and

$$\mathfrak{Sym}(n) = \{A \in \mathfrak{gl}(n,\mathbb{R}) \mid A^{\mathrm{t}} = A\}$$

the space of all (real) symmetric $n \times n$ matrices. Then $\mathfrak{Herm}(n)$ and $\mathfrak{Sym}(n)$ are real vector space of dimension n^2 and $\frac{n(n+1)}{2}$, respectively. The subspaces $\mathfrak{Herm}_{\mathrm{dgt}}(n) \subset \mathfrak{Herm}(n)$ and $\mathfrak{Sym}_{\mathrm{dgt}}(n) \subset \mathfrak{Sym}(n)$ of hermitian respectively symmetric $n \times n$ matrices having at least one degenerate eigenvalue are (real) algebraic varieties of codimension 3 and 2, respectively.

6.1.2 Remark Recall that an eigenvalue of a real or complex $n \times n$ matrix is called *degenerate* if its algebraic multiplicity is at least 2. For hermitian or symmetric matrices this is equivalent to the geometric multiplicity of the eigenvalue being $\geqslant 2$.

Proof. Since the diagonal elements of a hermitian matrix $A = (a_{ij})_{1\leqslant i,j\leqslant n}$ are all real and $a_{ij} = \overline{a_{ji}}$ for $i \neq j$, the (real) dimension of $\mathfrak{Herm}(n)$ is given as the sum of the number of diagonal elements of A and twice the number of its upper diagonal elements. So one obtains

$$\dim \mathfrak{Herm}(n) = n + 2\sum_{k=1}^{n-1} k = n + (n-1)n = n^2 \ .$$

In the real symmetric case, one needs to count the number of diagonal or upper diagonal elements, hence

$$\dim \mathfrak{Sym}(n) = \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \ .$$

□

The eigenvalues of a complex hermitian or real symmetric matrix A coincide with the zeros of its characteristic polynomial $\chi_A = \det(A - \lambda I_n) \in \mathbb{C}[\lambda]$. Let $\mathbb{D}(\chi_A)$ be the discriminant of the characteristic polynomial; see (Cohen, 1993, Sec. 3.3.2) for the definition and properties of the discriminant. Then $\mathbb{D}(\chi_A)$ is a polynomial in the coefficients of χ_A and vanishes if and only if χ_A has a multiple root. Since the coefficients of χ_A are polynomials in the entries of A, the set of hermitian (respectively symmetric) $n \times n$ matrices with a degenerate eigenvalue is a real algebraic variety in $\mathfrak{Herm}(n)$ (respectively $\mathfrak{Sym}(n)$).

Next let us determine the codimension of the variety $\mathfrak{Herm}_{\mathrm{dgt}}(n)$. To this end recall that a hermitian matrix A can be written in the form $A = UDU^{-1}$, where D is a diagonal matrix having the eigenvalues of A as its entries and where U is a complex unitary $n \times n$ matrix. The diagonal matrix $D = (d_{ij})_{1 \leqslant i,j \leqslant n}$ is uniquely determined when one requires that its diagonal entries are linearly ordered so that $d_{11} \leqslant \ldots \leqslant d_{nn}$. The matrix U is uniquely up to a unitary matrix V commuting with D. In case A has n different eigenvalues, the only unitary matrices commuting with D are diagonal matrices with entries from $\mathsf{U}(1)$. Since $\dim \mathsf{U}(n) = n^2$ Hence the codimension of $\mathfrak{Herm}_{\mathrm{dgt}}(n)$ in $\mathfrak{Herm}(n)$ is

Part III.

Quantum Field Theory

7. Representations of the Lorentz and Poincaré groups

7.1. The Lorentz invariant measure on a mass hyperboloid

7.1.1 Consider Minkowski space of space dimension d that is $\mathbb{R}^{1+d}$ endowed with the Minkowski inner product

$$\langle \cdot, \cdot \rangle_{\mathrm{M}} : \mathbb{R}^{1+d} \times \mathbb{R}^{1+d} \to \mathbb{R},\ (p, q) \mapsto -p^0 q^0 + \langle \vec{p}, \vec{q} \rangle = p^0 q^0 - \sum_{i=1}^{d} p^i q^i \ .$$

Note that $\langle \cdot, \cdot \rangle$ stands here for the euclidean inner product, and $\vec{p}$ is the *spacial vector* $(p^1, \ldots, p^d)$ associated to the *space-time vector* $p \in \mathbb{R}^{1+d}$. We sometimes will denote space-time dimension $1 + d$ by D. For $m > 0$ let

$$H_m^+ = \{p \in \mathbb{R}^D \mid \langle p, p \rangle_{\mathrm{M}} = m^2 \ \&\ p^0 > 0\}$$

be the positive mass hyperboloid of mass m. Observe that

$$\chi^+ : \mathbb{R}^d \to H_m^+,\ \mathsf{p} \mapsto \big(E(\mathsf{p}), \mathsf{p}\big) \quad \text{with } E(\mathsf{p}) = \sqrt{m^2 + \langle \mathsf{p}, \mathsf{p} \rangle}$$

is a global chart of the mass hyperboloid. Its inverse is given by

$$\vec{\ } : H_m^+ \to \mathbb{R}^d,\ p = (p^0, p^1, \ldots, p^d) \mapsto \vec{p} = (p^1, \ldots, p^d) \ .$$

Note that $E(\vec{p}) = p^0$ for all $p \in H_m^+$.

Now let λ denote Lebesgue measure on $\mathbb{R}^d$. We will show that the pushforward measure $\Omega_m = \chi_*^+ \left(\frac{1}{E}\lambda\right)$ is a Lorentz invariant measure on H_m^+ that is $\Lambda_* \Omega_m = \Omega_m$ for all $\Lambda \in \mathsf{SO}^\uparrow(1, d)$. Note that we have used here that Λ leaves H_m^+ invariant.

7.1.2 Lemma *For $\Lambda \in \mathsf{SO}^\uparrow(1, d)$ let Ψ_Λ denote the map*

$$\Psi_\Lambda : \mathbb{R}^d \to \mathbb{R}^d,\ \mathsf{p} \mapsto \Psi_\Lambda(\mathsf{p}) = \overrightarrow{\Lambda \chi^+(\mathsf{p})} \ .$$

Then Ψ_Λ is a diffeomorphism and the following holds true:

(i) *The map $\mathsf{SO}^\uparrow(1, d) \to \mathsf{Diff}(\mathbb{R}^d),\ \Psi : \Lambda \mapsto \Psi_\Lambda$ is a homomorphism that is*

$$\Psi_{\Lambda_1 \Lambda_2} = \Psi_{\Lambda_1} \Psi_{\Lambda_2} \quad \textit{for all } \Lambda_1, \Lambda_2 \in \mathsf{SO}^\uparrow(1, d) \ .$$

(ii) *The jacobian of Ψ_Λ is given by*

$$J_{\Psi_\Lambda} = |D\Psi_\Lambda| = \det \circ D\Psi_\Lambda = \frac{E \circ \Psi_\Lambda}{E} \ .$$

Proof. ad (i). Let $\Lambda_1, \Lambda_2 \in \mathsf{SO}^\uparrow(1,d)$, $\mathsf{p} \in \mathbb{R}^d$ and compute

$$\Psi_{\Lambda_1}\Psi_{\Lambda_2}(\mathsf{p}) = \Psi_{\Lambda_1}\left(\overrightarrow{\Lambda_2\chi^+(\mathsf{p})}\right) = \overrightarrow{\Lambda_1\Lambda_2\chi^+(\mathsf{p})} = \Psi_{\Lambda_1\Lambda_2}(\mathsf{p}) \ .$$

This implies in particular that Ψ_Λ is a diffeomorphism with inverse $\Psi_{\Lambda^{-1}}$.

ad (ii). Assume first that $\Lambda \in \mathsf{SO}^\uparrow(1,d)$ is a rotation that is $\Lambda = \left(\begin{smallmatrix} 1 & 0 \\ 0 & R \end{smallmatrix}\right)$ for some $R \in \mathsf{SO}(d)$. Then observe that $\Psi_\Lambda = R$ and compute for $\mathsf{p} \in \mathbb{R}^d$

$$E(\Psi_\Lambda \mathsf{p}) = E(R\mathsf{p}) = \sqrt{m^2 + \langle R\mathsf{p}, R\mathsf{p}\rangle} = \sqrt{m^2 + \langle \mathsf{p}, \mathsf{p}\rangle} = E(\mathsf{p}) \ .$$

Hence

$$\det\left(D\Psi_\Lambda(\mathsf{p})\right) = 1 = \frac{E(\Psi_\Lambda \mathsf{p})}{E(\mathsf{p})} \ .$$

Next let Λ be a Lorentz boost in the direction of p^1 that is let $\Lambda = \left(\begin{smallmatrix} \cosh\tau & \sinh\tau & 0 \\ \sinh\tau & \cosh\tau & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ where $\tau \in \mathbb{R}$ and 1 denotes the identity matrix over $\mathbb{R}^{d-1}$.

Then compute with $\Psi_{\Lambda,i}$ for $i = 1, \ldots, d$ denoting the i-th component of $\Psi_\Lambda : \mathbb{R}^d \to \mathbb{R}^d$:

$$\Psi_{\Lambda,i}(\mathsf{p}) = \Lambda_{i0}E(\mathsf{p}) + \sum_{i=1}^{d} \Lambda_{ij}\mathsf{p}^j = \begin{cases} \sinh\tau \cdot E(\mathsf{p}) + \cosh\tau \cdot \mathsf{p}^1 & \text{for } i = 1, \\ \mathsf{p}^i & \text{for } i = 2, \ldots, d, \end{cases}$$

$$\frac{\partial \Psi_{\Lambda,i}}{\partial \mathsf{p}^j}(\mathsf{p}) = \begin{cases} \sinh\tau \cdot \frac{\mathsf{p}^1}{E(\mathsf{p})} + \cosh\tau & \text{for } i = j = 1, \\ \sinh\tau \cdot \frac{\mathsf{p}^j}{E(\mathsf{p})} & \text{for } i = 1 \text{ and } j = 2, \ldots, d, \\ 0 & \text{for } i = 2, \ldots, d \text{ and } j = 1, \\ \delta_{ij} & \text{for } i, j = 2, \ldots, d, \end{cases}$$

and

$$\begin{aligned} E^2(\Psi_\Lambda \mathsf{p}) &= \sinh^2\tau \cdot E^2(\mathsf{p}) + 2\sinh\tau\cosh\tau \cdot E(\mathsf{p}) \cdot \mathsf{p}^1 + \cosh^2\tau \cdot (\mathsf{p}^1)^2 + \sum_{i=2}^{d} \left(\mathsf{p}^i\right)^2 + m^2 = \\ &= (\sinh^2\tau + 1) \cdot E^2(\mathsf{p}) + 2\sinh\tau\cosh\tau \cdot E(\mathsf{p}) \cdot \mathsf{p}^1 + (\cosh^2\tau - 1) \cdot (\mathsf{p}^1)^2 = \\ &= \left(\cosh\tau \cdot E(\mathsf{p}) + \sinh\tau \cdot \mathsf{p}^1\right)^2 \ . \end{aligned}$$

This entails the equality

$$\det\left(D\Psi_\Lambda(\mathsf{p})\right) = \sinh\tau \cdot \frac{\mathsf{p}^1}{E(\mathsf{p})} + \cosh\tau = \frac{E(\Psi_\Lambda \mathsf{p})}{E(\mathsf{p})} \ .$$

Since $\mathsf{SO}^\uparrow(1,d)$ is generated by the rotations and Lorentz boosts in direction p^1 and since by (i)

$$\det\left(D\Psi_{\Lambda_1\Lambda_2}(\mathsf{p})\right) = \det\left(D\Psi_{\Lambda_1}(\Psi_{\Lambda_2}\mathsf{p})\right) \cdot \det\left(D\Psi_{\Lambda_2}(\mathsf{p})\right)$$

the claim follows. □

7.1.3 Proposition *With notation as above the pushforward measure $\Omega_m = \chi^+_*\left(\frac{1}{\omega}\lambda\right)$ is a Lorentz invariant measure on the positive mass hyperboloid H^+_m that is*

$$\int_{H^+_m} f(\Lambda p)\, d\Omega_m(p) = \int_{H^+_m} f(p)\, d\Omega_m(p) \tag{7.1.1}$$

for all $f \in L^1(H^+_m, \Omega_m)$ and $\Lambda \in \mathsf{SO}^\uparrow(1,d)$.

Proof. By definition of the pushforward measure Ω_m is the unique Borel measure on H^+_m such that for all $f \in \mathcal{C}_{\mathrm{cpt}}(H^+_m)$

$$\int_{H^+_m} f(p)\, d\Omega_m(p) = \int_{\mathbb{R}^d} f(\chi^+\mathsf{p})\, \frac{1}{E(\mathsf{p})} d\lambda(\mathsf{p}) \ .$$

The claim follows from this observation since for all $\Lambda \in \mathsf{SO}^\uparrow(1,d)$ the equality

$$\begin{aligned} \int_{\mathbb{R}^d} f(\Lambda\chi^+\mathsf{p}) \frac{1}{E(\mathsf{p})} d\lambda(\mathsf{p}) &= \int_{\mathbb{R}^d} f(\chi^+\Psi_\Lambda\mathsf{p}) \frac{1}{E(\mathsf{p})} d\lambda(\mathsf{p}) = \\ &= \int_{\mathbb{R}^d} f(\chi^+\Psi_\Lambda\mathsf{p}) \frac{1}{E(\Psi_\Lambda\mathsf{p})} \det\left(D\Psi_\Lambda(\mathsf{p})\right) d\lambda(\mathsf{p}) = \int_{\mathbb{R}^d} f(\chi^+\mathsf{p}) \frac{1}{E(\mathsf{p})} d\lambda(\mathsf{p}) \ . \end{aligned}$$

holds true by Lemma 7.1.2 (ii) . $\square$

8. Axiomatic quantum field theory à la Wightman and Gårding

8.1. Wightman axioms

8.1.1 The Wightman axioms were first introduced in the paper Wightman & Gårding (1964), and then explained in more detail in the textbooks Jost (1965) and (Streater & Wightman, 2000, Sec. 3.1). The latter is still the main reference for the axiomatic treatment of quantum field theory in the spirit of Wightman and Gårding. See also (Schottenloher, 2008, Sec. 8.3) for a more modern formulation which we follow here.

8.1.2 Definition A *Wightman quantum field theory* of *space-time dimension* $D = d+1$, $d \in \mathbb{N}_{>0}$, consists of the following data:

- the *state space* of the theory given by the projective space $\mathbb{P}(\mathfrak{H})$ associated to a separable complex Hilbert space $\mathfrak{H}$,
- a distinguished state $\omega_\circ = \mathbb{C}v_\circ \in \mathbb{P}(\mathfrak{H})$ called the *vacuum state* together with the choice of a normalized representing vector $v_\circ \in \mathfrak{H}$ called *vacuum vector*,
- a unitary representation $U : \widetilde{\mathsf{P}^\uparrow_+}(d+1) \to \mathsf{U}(\mathfrak{H})$ of the universal cover
$$\widetilde{\mathsf{P}^\uparrow_+}(d+1) \cong \mathbb{R}^{d+1} \rtimes \widetilde{\mathsf{SO}^\uparrow}(1,d)$$
of the proper orthochronous Poincaré group $\mathsf{P}^\uparrow_+(d+1) = \mathbb{R}^{1+d} \rtimes \mathsf{SO}^\uparrow(1,d)$,
- and finally a family $(\Phi^j)_{1\leqslant j\leqslant n}$, $n \in \mathbb{N}_{>0}$, of so-called *field operators*
$$\Phi^j : \mathcal{S}(\mathbb{R}^{d+1}) \to \mathfrak{L}_{\mathrm{u}}(\mathfrak{H})$$
which are defined on the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$ and map to the space of unbounded linear operators on the Hilbert space $\mathfrak{H}$.

These data are assumed to fulfill the following axioms, the so-called *Wightman axioms*:

(W1) (*Assumptions about the domain and the continuity of the field*)
There exists a dense linear subspace $\mathcal{D} \subset \mathfrak{H}$ containing $v_\circ$ such that $\mathcal{D}$ is contained in the domain of all the operators $\Phi^j(f)$ and their adjoints $\Phi^j(f)^*$, where $f \in \mathcal{S}(\mathbb{R}^{d+1})$ and $j = 1, \ldots, k$. Moreover, the unitary representation U and the operators $\Phi^j(f)$ and $\Phi^j(f)^*$ leave $\mathcal{D}$ invariant that is
$$U(a, A)\mathcal{D} \subset \mathcal{D}, \quad \Phi^j(f)\mathcal{D} \subset \mathcal{D},\ \Phi^j(f)^*\mathcal{D} \subset \mathcal{D}$$

for all $(a, A) \in \widetilde{\mathsf{P}^{\uparrow}_{+}}(1, d)$, $f \in \mathcal{S}(\mathbb{R}^{d+1})$ and $j = 1, \ldots, k$. Finally, for every $v \in \mathcal{D}$, $w \in \mathfrak{H}$ and $j = 1, \ldots, n$ the maps

$$\mathcal{S}(\mathbb{R}^{d+1}) \to \mathbb{C}, \ f \mapsto \langle w, \Phi^j(f)v \rangle$$

are tempered distributions.

(W2) (*Transformation law of the field*)
For all $(a, A) \in \widetilde{\mathsf{P}^{\uparrow}_{+}}(d+1)$ and all $f \in \mathcal{S}(\mathbb{R}^{d+1})$ the equation

$$U(a, A)\Phi^j(f)U(a, A)^{-1} = \sum_{k=1}^{n} \varrho^{jk}\big(A^{-1}\big)\Phi^k((a, A)f)$$

is valid over the domain D, where $\varrho : \widetilde{\mathsf{SO}^{\uparrow}}(1, d) \to \mathsf{GL}(n, \mathbb{C})$ is a finite dimensional representation of the universal cover of the proper orthochronous Lorentz group $\mathsf{SO}^{\uparrow}(1, d)$ and the action of $\widetilde{\mathsf{P}}(d+1)$ on $\mathcal{S}(\mathbb{R}^{d+1})$ is given by

$$\begin{aligned} &\widetilde{\mathsf{P}}(d+1) \times \mathcal{S}(\mathbb{R}^{d+1}) \to \mathcal{S}(\mathbb{R}^{d+1}), \\ &\qquad \big((a, A), f\big) \mapsto (a, A)f = \Big(\mathbb{R}^{d+1} \ni x \mapsto f\big(A^{-1}(x-a)\big) \in \mathbb{C}\Big) \ . \end{aligned}$$

(W3) (*Local commutativity or microscopic causality*)
If the support of test functions $f, g \in \mathcal{S}(\mathbb{R}^{d+1})$ is space-like separated that is if $f(x)\, g(y) = 0$ for all $x, y \in \mathbb{R}^{d+1}$ with $\langle x - y, x - y \rangle_{\mathrm{M}} \geqslant 0$, then for all $j, k = 1, \ldots, n$ the relation

$$[\Phi^j(f), \Phi^k(g)]_- = [\Phi^j(f), \Phi^j(g)^*]_- = 0$$

or the relation

$$[\Phi^j(f), \Phi^k(g)]_+ = [\Phi^j(f), \Phi^j(g)^*]_+ = 0$$

holds true over the domain $\mathcal{D}$. Hereby, $[S, T]_-$ denotes the *commutator*

$$[S, T]_- : \ \mathcal{D} \to \mathfrak{H}, \quad v \mapsto STv - TSv$$

and $[S, T]_+$ the *anti-commutator*

$$[S, T]_+ : \ \mathcal{D} \to \mathfrak{H}, \quad v \mapsto STv + TSv$$

of two operators $S, T \in \mathfrak{L}_{\mathrm{u}}(\mathfrak{H})$ which are both assumed to be defined on the domain $\mathcal{D}$ and to leave it invariant.

(W4) (*Cyclicity of the vacuum vector*)
The linear span of the set of all elements $v \in \mathfrak{H}$ of the form

$$v = \Phi^{j_1}(f_1) \ \ldots \ \Phi^{j_m}(f_m)v_\circ \ ,$$

where $m \in \mathbb{N}$, $1 \leqslant j_1, \ldots, j_m \leqslant n$, and $f_1, \ldots, f_m \in \mathcal{S}(\mathbb{R}^{d+1})$, is dense in $\mathfrak{H}$.

8.1.3 Remarks (a) The vacuum vector $v_\circ$ being normalized just means that $\|v_\circ\| = 1$. This implies that the vacuum state $\omega_\circ$ determines $v_\circ$ only up to a factor $z \in S^1 \subset \mathbb{C}$. The physically measurable quantities of the quantum field theory such as expectation values or transition amplitudes do not depend on that choice.

(b) The field operators Φ^j are operator valued distributions. This reflects the fact that only the "smeared" fields $\Phi^j(f)$ can be interpreted physically as observable. The notation $\Phi^j(x)$ for a field evaluated at a space-time point $x \in \mathbb{R}^{1,3}$ therefore does not make sense, neither mathematically nor physically. Nevertheless it is often used for reasons of convenience, in particular in the physics literature. The smeared field $\Phi^j(f)$ then is interpreted, again imprecisely, as the integral

$$\Phi^j(f) = \int_{\mathbb{R}^{d+1}} f(x)\, \Phi^j(x)\, dx \ .$$

We will avoid the notation of pointwise evaluated fields in the formulation of definitions and theorems, but occasionally use it as a heuristic.

For example, Axiom (W3) can heuristically be interpreted as saying that the (anti-) commutation relations

$$[\Phi^j(x), \Phi^k(y)]_\mp = [\Phi^j(x), \Phi^j(y)^*]_\mp = 0$$

hold true for $x, y \in \mathbb{R}^{1,d}$ space-like separated which means for the situation when

$$\langle x - y, x - y \rangle_{\mathrm{M}} < 0 \ .$$

8.2. Fock space

8.2.1 Recall from Section 12.4 that the Hilbert tensor product $\mathfrak{H}_1 \widehat{\otimes} \mathfrak{H}_2$ of two Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ is defined as the completion of the the algebraic tensor product $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ endowed with the inner product

$$\langle \cdot, \cdot \rangle : \big(\mathfrak{H}_1 \otimes \mathfrak{H}_2\big) \times \big(\mathfrak{H}_1 \otimes \mathfrak{H}_2\big) \to \mathbb{K},\ \big(v_1 \otimes v_2, w_1 \otimes w_2\big) \mapsto \langle v_1, w_1 \rangle \cdot \langle v_2, w_2 \rangle \ .$$

The norm of an element $v_1 \otimes v_2 \in \mathfrak{H}_1 \widehat{\otimes} \mathfrak{H}_2$ is then given by $\|v_1 \otimes v_2\| = \|v_1\| \cdot \|v_2\|$, and every element $v \in \mathfrak{H}_1 \widehat{\otimes} \mathfrak{H}_2$ can be written as the sum of a square summable family $\big(v_{i1} \otimes v_{i2}\big)_{i \in I}$ that is as

$$v = \sum_{i \in I} v_{i1} \otimes v_{i2} \quad \text{where } \|v\|^2 = \sum_{i \in I} \|v_{i1}\|^2 \cdot \|v_{i2}\|^2 < \infty \ .$$

If $(e_i)_{i \in I}$ is Hilbert basis for $\mathfrak{H}_1$ and $(f_j)_{j \in J}$ one of $\mathfrak{H}_2$, the family $(e_i \otimes f_j)_{(i,j) \in I \times J}$ is a Hilbert basis of $\mathfrak{H}_1 \widehat{\otimes} \mathfrak{H}_2$. Moreover, the canonical map $\tau : \mathfrak{H}_1 \times \mathfrak{H}_2 \to \mathfrak{H}_1 \widehat{\otimes} \mathfrak{H}_2$, $(v_1, v_2) \mapsto v_1 \otimes v_2$ is bilinear and weakly Hilbert–Schmidt that means that there exists a $C \geqslant 0$ such that for all Hilbert bases $(e_i)_{i \in I}$ of $\mathfrak{H}_1$, all Hilbert bases $(f_j)_{j \in J}$ of $\mathfrak{H}_2$, and all $w \in \mathfrak{H}_1 \widehat{\otimes} \mathfrak{H}_2$

$$\sum_{(i,j) \in I \times J} |\langle \tau(e_i, f_j), w \rangle|^2 \leqslant C \|w\|^2 \ .$$

Note that if this condition holds for one Hilbert basis of $\mathfrak{H}_1$ and one of $\mathfrak{H}_2$, it holds for all. The Hilbert tensor product, which in the following we will only call tensor product, satisfies the following universal property.

(HTensor) For every Hilbert space $\mathfrak{H}$ and every weakly Hilbert–Schmidt bilinear map $\mu : \mathfrak{H}_1 \times \mathfrak{H}_2 \to \mathfrak{H}$ there exists a unique bounded linear map $\widehat{\mu} : \mathfrak{H}_1 \widehat{\otimes} \mathfrak{H}_2 \to \mathfrak{H}$ such that the diagram

$$\begin{tikzcd} \mathfrak{H}_1 \times \mathfrak{H}_2 \arrow[r, "\mu"] \arrow[d, "\tau"'] & \mathfrak{H} \\ \mathfrak{H}_1 \widehat{\otimes} \mathfrak{H}_2 \arrow[ur, "\widehat{\mu}"'] & \end{tikzcd}$$

commutes.

For a proof of the universal property see Section 12.4 or (Kadison & Ringrose, 1997, Sec. 2.6.). Note that by its universal property the Hilbert tensor product $\widehat{\otimes}$ is a bifunctor on the category Hilb of Hilbert spaces and bounded maps. Moreover, Hilb equipped with the bifunctor $\widehat{\otimes}$ becomes a monoidal category. See Section 12.4 for details and proofs.

8.2.2 Now let us fix a Hilbert space $\mathfrak{H}$ and consider the higher Hilbert tensor product powers $\mathfrak{F}^n(\mathfrak{H}) = \mathfrak{H}^{\widehat{\otimes}\, n}$ for natural n. These are recursively defined by

$$\mathfrak{H}^{\widehat{\otimes}\, 0} = \mathbb{K}, \quad \mathfrak{H}^{\widehat{\otimes}\, n+1} = \mathfrak{H} \widehat{\otimes} \left(\mathfrak{H}^{\widehat{\otimes}\, n}\right) .$$

The *Fock space* of $\mathfrak{H}$ now is defined as the Hilbert space direct sum

$$\mathfrak{F}(\mathfrak{H}) = \widehat{\bigoplus_{n\in\mathbb{N}}} \mathfrak{F}^n(\mathfrak{H}) = \widehat{\bigoplus_{n\in\mathbb{N}}} \mathfrak{H}^{\widehat{\otimes}\, n} .$$

Its elements are families $(v_n)_{n\in\mathbb{N}}$ of vectors $v_n \in \mathfrak{H}^{\widehat{\otimes}\, n}$ such that $\sum_{n\in\mathbb{N}} \|v_n\|^2 < \infty$. The inner product of two such families $v = (v_n)_{n\in\mathbb{N}}, w = (w_n)_{n\in\mathbb{N}} \in \mathfrak{F}(\mathfrak{H})$ is given, according to **??**, by

$$\langle v, w\rangle = \sum_{n\in\mathbb{N}} \langle v_n, w_n\rangle .$$

8.2.3 Remark The construction of the Fock space resembles the one of the tensor algebra. Recall that the tensor algebra of $\mathfrak{H}$ is the vector space $\mathsf{T}(\mathfrak{H}) = \bigoplus_{n\in\mathbb{N}} \mathsf{T}^n(\mathfrak{H})$ where $\mathsf{T}^n(\mathfrak{H})$ is defined as the algebraic tensor product power $\mathfrak{H}^{\otimes n}$. The *completed tensor algebra* of $\mathfrak{H}$ now is the ℓ^1-completion

$$\widehat{\mathsf{T}}(\mathfrak{H}) = \ell_1\text{-}\widehat{\bigoplus_{n\in\mathbb{N}}} \widehat{\mathsf{T}}^n(\mathfrak{H}) ,$$

where $\widehat{\mathsf{T}}^n(\mathfrak{H}) = \mathfrak{F}^n(\mathfrak{H}) = \mathfrak{H}^{\widehat{\otimes}\, n}$. The completed tensor algebra lies densely in Fock space. To verify this observe that, regarded in the category of Banach spaces, Fock space (including its norm) coincides with the ℓ_2-direct sum of the spaces Banach spaces $\widehat{\mathsf{T}}^n(\mathfrak{H})$ and $\widehat{\mathsf{T}}(\mathfrak{H})$ with their ℓ_1-direct sum. Since for every summable family $v = (v_n)_{\in\mathbb{N}}$ with $v_n \in \widehat{\mathsf{T}}^n(\mathfrak{H})$ the relation

$$\|v\| = \|v\|_2 = \sqrt{\sum_{n\in\mathbb{N}} \|v_n\|^2} \leqslant \sqrt{\sum_{n\in\mathbb{N}} \|v_n\|} \cdot \sqrt{\sup_{n\in\mathbb{N}} \|v_n\|} \leqslant \sum_{n\in\mathbb{N}} \|v_n\| = \|v\|_1$$

holds true by Hölders inequality for series, $\widehat{\mathsf{T}}(\mathfrak{H})$ is contained in $\mathfrak{F}(\mathfrak{H})$. It is also dense in Fock space because the (algebraic) direct sum $\bigoplus_{n\in\mathbb{N}} \widehat{\mathsf{T}}^n(\mathfrak{H})$ is already so.

Unlike Fock space in the case $\dim \mathfrak{H} = \infty$, the completed tensor algebra $\widehat{\mathsf{T}}(\mathfrak{H})$ always carries a canonical algebra structure. To define the product of two summable families $v = (v_n)_{n\in\mathbb{N}}$ and $w = (w_n)_{n\in\mathbb{N}}$ one puts for all natural n

$$z_n = \sum_{k=0}^{n} v_k \otimes w_{n-k} \ .$$

Then $z_n \in \mathsf{T}^n(\mathfrak{H})$ for all $n \in \mathbb{N}$, and the family $z = (z_n)_{n\in\mathbb{N}}$ is absolutely summable again since

$$\sum_{n\in\mathbb{N}} \|z_n\| = \lim_{N\to\infty} \sum_{n=0}^{N} \|z_n\| \leqslant \lim_{N\to\infty} \sum_{n=0}^{N} \sum_{k=0}^{n} \|v_k\| \, \|w_{n-k}\| \leqslant \lim_{N\to\infty} \sum_{k=0}^{N} \sum_{l=0}^{N} \|v_k\| \, \|w_l\| \leqslant \|v\|_1 \, \|w\|_1 \ .$$

Hence $z = (z_n)_{n\in\mathbb{N}}$ is an element $\widehat{\mathsf{T}}(\mathfrak{H})$ which we call the *product* of v and w. It will be denoted by $v \otimes w$. By the preceding estimate we thus obtain a continuous map

$$\otimes : \widehat{\mathsf{T}}(\mathfrak{H}) \times \widehat{\mathsf{T}}(\mathfrak{H}) \to \widehat{\mathsf{T}}(\mathfrak{H}), \ (v, w) \mapsto v \otimes w$$

such that

$$\|v \otimes w\|_1 \leqslant \|v\|_1 \, \|w\|_1 \quad \text{for all } v, w \in \widehat{\mathsf{T}}(\mathfrak{H}) \ .$$

The restriction of $\otimes$ to the (uncompleted) tensor algebra $\mathsf{T}(\mathfrak{H}) = \bigoplus_{n\in\mathbb{N}} \mathsf{T}^n(\mathfrak{H})$ is associative, so by density one concludes that $\otimes$ on $\widehat{\mathsf{T}}(\mathfrak{H})$ is associative as well. Hence $\widehat{\mathsf{T}}(\mathfrak{H})$ is a Banach algebra.

Even though $\mathfrak{F}(\mathfrak{H})$ might not possess a compatible Banach algebra structure, it carries the structure of a $\widehat{\mathsf{T}}(\mathfrak{H})$ left and right module with the left and right actions being continuous. Let us show this for the left module structure in some more detail. The right module case is analogous. So assume $v = (v_n)_{n\in\mathbb{N}} \in \widehat{\mathsf{T}}(\mathfrak{H})$, $w = (w_n)_{n\in\mathbb{N}} \in \mathfrak{F}(\mathfrak{H})$, and let $z = (z_n)_{n\in\mathbb{N}}$ where as before $z_n = \sum_{k=0}^{n} v_k \otimes w_{n-k}$. Put $w_k = 0$ for $k < 0$. Then compute using the triangle and Hölder's inequality

$$\begin{aligned}
\|z\|_2^2 &= \lim_{N\to\infty} \sum_{n=0}^{N} \|z_n\|^2 = \lim_{N\to\infty} \sum_{n=0}^{N} \left\| \sum_{k=0}^{n} v_k \otimes w_{n-k} \right\|^2 \leqslant \\
&\leqslant \lim_{N\to\infty} \sum_{n=0}^{N} \left(\sum_{k=0}^{N} \left(\|v_k\|^{1/2} \, \|w_{n-k}\| \right) \, \|v_k\|^{1/2} \right)^2 \leqslant \\
&\leqslant \lim_{N\to\infty} \sum_{n=0}^{N} \left(\sum_{k=0}^{N} \|v_k\| \, \|w_{n-k}\|^2 \right) \left(\sum_{k=0}^{N} \|v_k\| \right) \leqslant \\
&\leqslant \lim_{N\to\infty} \|v\|_1 \sum_{k=0}^{N} \left(\|v_k\| \sum_{n=0}^{N} \|w_{n-k}\|^2 \right) \leqslant \\
&\leqslant \lim_{N\to\infty} \|v\|_1 \sum_{k=0}^{N} \left(\|v_k\| \sum_{n=0}^{N} \|w_n\|^2 \right) = \|v\|_1^2 \, \|w\|_2^2 \ .
\end{aligned}$$

Hence $z \in \mathfrak{F}(\mathfrak{H})$, and the product $\otimes : \widehat{\mathsf{T}}(\mathfrak{H}) \times \widehat{\mathsf{T}}(\mathfrak{H}) \to \widehat{\mathsf{T}}(\mathfrak{H})$ has a unique continuous extension to a left action

$$\otimes : \widehat{\mathsf{T}}(\mathfrak{H}) \times \mathfrak{F}(\mathfrak{H}) \to \mathfrak{F}(\mathfrak{H}), \ (v, w) \mapsto v \otimes w$$

such that

$$\|v \otimes w\|_2 \leqslant \|v\|_1 \, \|w\|_2 \quad \text{for all } v \in \widehat{\mathsf{T}}(\mathfrak{H}), \ w \in \mathfrak{F}(\mathfrak{H}) \ .$$

8.2.4 Next we will show that associating to a Hilbert space its Fock space can be extended to a functor on the category Hilb_1 of Hilbert spaces and linear contractions between them. Recall that by a linear contraction one understands a bounded linear operator with norm $\leqslant 1$. So assume that $A : \mathfrak{H}_1 \to \mathfrak{H}_2$ is a contraction between Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$. By functoriality of the algebraic tensor product one obtains for each $n \in \mathbb{N}_{>0}$ a linear map

$$A^{\otimes n} : \mathfrak{H}_1^{\otimes n} \to \mathfrak{H}_2^{\otimes n},\ v_1 \otimes \ldots \otimes v_n \mapsto Av_1 \otimes \ldots \otimes Av_n \ .$$

By Proposition 12.4.5 or (Kadison & Ringrose, 1997, Prop. 2.6.12 & Eq. 2.6.(16)) this operator has norm $\|A\|^n$ and extends uniquely to a bounded linear operator $\mathfrak{F}^n(A) : \mathfrak{F}^n(\mathfrak{H}_1) \to \mathfrak{F}^n(\mathfrak{H}_2)$ having the same norm. Since by assumption $\|A\| \leqslant 1$, one concludes that $\|\mathfrak{F}^n(A)\| \leqslant 1$ for all $n \in \mathbb{N}_{>0}$. One further puts $\mathfrak{F}^0(A) = \mathrm{id}_{\mathbb{K}}$ and observes that then $\sup_{n\in\mathbb{N}} \|\mathfrak{F}^n(A)\| = 1$. Hence, by construction of the operators $\mathfrak{F}^n(A)$ and definition of the Hilbert direct sum the map

$$\mathfrak{F}(A) : \mathfrak{F}(\mathfrak{H}_1) \to \mathfrak{F}(\mathfrak{H}_2),\ v = (v_n)_{n\in\mathbb{N}} \mapsto \big(\mathfrak{F}^n(A)(v_n)\big)_{n\in\mathbb{N}}$$

is well-defined and a bounded linear operator of norm 1. Note that hereby we have again used the (silent) agreement that $v = (v_n)_{n\in\mathbb{N}}$ denotes a square-integrable family with $v_n \in \mathfrak{F}^n(\mathfrak{H}_1)$ for all $n \in \mathbb{N}$. By construction it is immediate that $\mathfrak{F}(\mathrm{id}_{\mathfrak{H}}) = \mathrm{id}_{\mathfrak{F}(\mathfrak{H})}$ for every Hilbert space $\mathfrak{H}$ and that for linear contractions $A : \mathfrak{H}_1 \to \mathfrak{H}_2$ and $B : \mathfrak{H}_2 \to \mathfrak{H}_3$ between Hilbert spaces the relation

$$\mathfrak{F}(BA) = \mathfrak{F}(B)\,\mathfrak{F}(A)$$

holds true. Hence we obtain as promised a (covariant) functor $\mathfrak{F}$ from the category Hilb to itself. One sometimes calls $\mathfrak{F}$ the *functor of second quantization*.

8.2.5 Particularly important for quantum field theory is the observation going back to Cook (1953) that every closed densely defined linear operator on a Hilbert space has an extension to Fock space which again is closed and densely defined. Let us explain this in some more detail. We essentially follow the approach by Cook (1953); see also Emch (2009).

Let $(\mathfrak{H})_{i=1}^n$ be a finite family of Hilbert spaces and $(A_i)_{i=1}^n$ a family of closed densely defined unbounded linear operators $A_i : \mathrm{Dom}(A_i) \subset \mathfrak{H}_i \to \mathfrak{H}_i$, $i = 1, \ldots, n$ over the same index set. Hence the adjoint A_i^* of A_i is a closed densely defined unbounded linear operator on $\mathfrak{H}_i$ for every index $i = 1, \ldots, n$. Put $\mathfrak{D}_i = \mathrm{Dom}(A_i)$ and $\mathfrak{D}_i^* = \mathrm{Dom}(A_i^*)$ and note that then $\mathfrak{D}_i$ and $\mathfrak{D}_i^*$ are dense in $\mathfrak{H}_i$ by assumption and the preceding observation.

8.3. The free scalar field

8.3.1 Here we want to show that a model of the Wightman axioms is given by the free scalar field of mass $m > 0$ in space-time dimension $D = d + 1$ for $d \in \mathbb{N}_{>0}$. The Hilbert space $\mathfrak{H}$ of the free scalar field is the symmetric Fock space $\mathfrak{F}_{\mathrm{s}}\big(L^2(H_m^+, \Omega_m)\big)$ over the 1-particle Hilbert space $L^2(H_m^+, \Omega_m)$ of square-integrable functions on the positive mass hyperboloid $H_m^+ \subset \mathbb{R}^D$ equipped with the lorentz-invariant measure Ω_m which has been defined in Section 7.1. By definition, Ω_m coincides with the pushforward measure $\chi_*^+(\frac{1}{E}\lambda)$, where λ denotes Lebesgue measure on $\mathbb{R}^d$, $E(\mathsf{p}) = \sqrt{m^2 + \langle \mathsf{p}, \mathsf{p}\rangle}$ for all $\mathsf{p} \in \mathbb{R}^d$, and $\chi^+ : \mathbb{R}^d \to H_m^+$ is the chart of the positive mass hyperpoloid which maps $\mathsf{p} \in \mathbb{R}^d$ to $\big(E(\mathsf{p}), \mathsf{p}\big) \in H_m^+$. In this section we will often denote the 1-particle Hilbert space of the free scalar field by $\mathfrak{H}^{(1)}$.

9. Algebraic quantum field theory à la Haag–Kastler

9.1. The Haag–Kastler axioms

Mathematical Toolbox

10. Topological Vector Spaces

10.1. Topological division rings and fields

10.1.1 Vector spaces with a compatible topology can not only defined for vector spaces over the ground fields $\mathbb{R}$ and $\mathbb{C}$ but also over fields $\mathbb{K}$ carrying an absolute value $|\cdot| : \mathbb{K} \to \mathbb{R}_{\geqslant 0}$. This endows the ground field with a topology which will be needed in the definition of a topological vector space. We therefore give here a brief introduction to topological division rings and fields first.

10.1.2 Definition Let R be a division ring. By an *absolute value* on R one understands a map $|\cdot| : R \to \mathbb{R}_{\geqslant 0}$ such that the following axioms hold true.

(VDR1) The function $|\cdot|$ is multiplicative that is

$$|xy| = |x|\,|y| \quad \text{for all } x, y \in R \ .$$

(VDR2) The triangle inequality is satisfied which means that

$$|x + y| \leqslant |x| + |y| \quad \text{for all } x, y \in R \ .$$

(VDR3) For all $x \in R$ the relation $|x| = 0$ holds true if and only if $x = 0$.

A division ring or field endowed with an absolute value is called a *valued division ring* respectively a *valued field*. An absolute value $|\cdot|$ on a division ring R and the corresponding valued division ring $(R, |\cdot|)$ are called *non-archimedean* if the *strong triangle inequality* is satisfied that is if

(VDR4) $|x + y| \leqslant \max\{|x|, |y|\}$ for all $x, y \in R$.

Otherwise $|\cdot|$ and $(R, |\cdot|)$ are called *archimedean*.

10.1.3 Lemma *Let $(R, |\cdot|)$ be a valued division ring. Then*

(i) $|1| = 1$,

(ii) $|-x| = |x|$ *for all* $x \in R$, *and*

(iii) $\big||x| - |y|\big| \leqslant |x - y| \leqslant |x| + |y|$ *for all* $x, y \in R$.

Proof. (i) holds true since $|1| = |1^2| = |1|^2$ and $|1| \neq 0$ by $1 \neq 0$. To verify (ii) it suffices to show that $|-1| = 1$. But that holds true since $|-1|^2 = |(-1)^2| = 1$ and $|-1| \geqslant 0$. The last claim follows by

$$-|x-y| = |x| - (|y-x| + |x|) \leqslant |x| - |y| \leqslant |x-y| + |y| - |y| = |x-y|$$

and

$$|x-y| = |x + (-y)| \leqslant |x| + |-y| = |x| + |y| \, . \qquad \square$$

10.1.4 Examples (a) Obviously, the *standard absolute values*

$$|\cdot|_\infty : \mathbb{Q}, \mathbb{R} \to \mathbb{R}_{\geqslant 0}, \ x \mapsto \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases} \quad \text{and} \quad |\cdot|_\infty : \mathbb{C} \to \mathbb{R}_{\geqslant 0}, \ z \mapsto \sqrt{z\overline{z}}$$

are absolute values on the fields $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. These absolute values are all archimedean since $|1+1|_\infty = 2 > 1$. Unless mentioned differently, we always assume $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ to be equipped with the standard absolute values. If no confusion can arise we usually write $|\cdot|$ instead of $|\cdot|_\infty$.

(b) The *standard absolute value* on the quaternions

$$|\cdot|_\infty : \mathbb{H} \to \mathbb{R}_{\geqslant 0}, \ q = a + b\,\mathsf{i} + c\,\mathsf{j} + d\,\mathsf{k} \mapsto \sqrt{\overline{q}q} = \sqrt{a^2 + b^2 + c^2 + d^2} \ ,$$

where a, b, c, d are real, is an archimedean absolute value. Usually it is briefly denoted $|\cdot|$.

(c) For every division ring R the map

$$|\cdot| : R \to \mathbb{R}, \ x \mapsto \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{else} \end{cases}$$

is a non-archimedean absolute value. It is called the *trivial absolute value* on R.

(d) An absolute value $|\cdot| : \mathbb{F} \to \mathbb{R}_{\geqslant 0}$ defined on a finite field $\mathbb{F}$ has to be trivial. To see this observe that for each $x \in \mathbb{K}^\times$ there exists an $n \in \mathbb{N}$ such that $x^n = 1$. This entails $|x|^n = 1$, hence $|x| = 1$ for all $x \in \mathbb{K}^\times$. So $|\cdot|$ is trivial.

(e) The field of formal Laurent power series $\mathbb{K}((X))$ over a field $\mathbb{K}$ can be equipped with an absolute value as follows. Choose $0 < \varepsilon < 1$ and define the absolute value $\left|\sum_{k\in\mathbb{Z}} a_k X^k\right|$ of an element $\sum_{n\in\mathbb{Z}} a_n X^n \in \mathbb{K}((X))$ as ε^n, where n is the minimal integer such that $a_n \neq 0$.

(f) Let p be prime number. For every integer $m \neq 0$ let $\nu_p(m)$ be the exponent of p in the prime factor decomposition of m that is $m = p^{\nu_p(n)} n$ where n is relatively prime to p. For $m \in \mathbb{Z}$ and $n \in \mathbb{N}_{>0}$ one defines the *p-adic absolute value* of the rational number $x = \frac{m}{n}$ by

$$|x|_p = \begin{cases} 0 & \text{if } m = 0 \ , \\ p^{-\nu_p(m) + \nu_p(n)} & \text{else} \ . \end{cases}$$

Note that $|x|_p$ does not depend on the particular representation of x as the quotient of integers m and n. By definition it is immediately clear that the p-adic absolute value is an absolute value on $\mathbb{Q}$ indeed. It is non-archimedean.

10.1.5 Proposition *A valued division ring $(R, |\cdot|)$ is non-archimedean if and only if the image of $\mathbb{Z}$ under the canonical map $\mathbb{Z} \to R$ is bounded.*

Proof. Assume that $(R, |\cdot|)$ is a non-archimedean valued division ring. Then, $|0 \cdot 1| = |0| = 0$ and, under the assumption that $|(n-1) \cdot 1| \leqslant 1$ for some $n \in \mathbb{N}_{>0}$, $|n \cdot 1| = |(n-1) \cdot 1 + 1| = \max\{|(n-1) \cdot 1|, 1\} = 1$. Hence by induction and since $|-1| = 1$ one obtains that $|n \cdot 1| \leqslant 1$ for all $n \in \mathbb{Z}$, and the image of $\mathbb{Z}$ in R is bounded.

To show the converse assume that the image of $\mathbb{Z}$ in R is bounded by some constant $C > 0$. Then, for all $x, y \in R$ and $n \in \mathbb{N}_{>0}$ by the binomial formula and the triangle inequality

$$|x+y|^n = \left| \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \right| \leqslant (n+1)\, C \max\{|x|, |y|\}^n \ .$$

Taking the n-th root gives $|x+y| \leqslant \big((n+1)C\big)^{1/n} \max\{|x|, |y|\}$ which after passing to the limit $n \to \infty$ entails $|x+y| \leqslant \max\{|x|, |y|\}$ since $\lim\limits_{n\to\infty} \big((n+1)C\big)^{1/n} = 1$. Hence $(R, |\cdot|)$ is non-archimedean. □

10.1.6 Proposition *Let $|\cdot|$ be an absolute value on the division ring R. Then for every $\tau > 0$ with $\tau \leqslant 1$ the map $|\cdot|^\tau : R \to \mathbb{R}_{\geqslant 0}$ is an absolute value on R as well. It is archimedean if and only if $|\cdot|$ is archimedean.*

Proof. To prove that $|\cdot|^\tau$ is an absolute value it suffices to show that $(a+b)^\tau \leqslant a^\tau + b^\tau$ for all $a, b \geqslant 0$. Without loss of generality we may assume $a \geqslant b > 0$. By dividing through b^τ one sees that the claim is equivalent to $(t+1)^\tau \leqslant t^\tau + 1$ for all $t \geqslant 1$. For $t = 1$ this is certainly true. The derivative of the function $h : [1, \infty) \to \mathbb{R}$, $t \mapsto (t+1)^\tau - t^\tau$ now is given by $h'(t) = \tau\big((t+1)^{\tau-1} - t^{\tau-1}\big)$ which is negative since $\tau - 1 \leqslant 0$ and $1 + t > t \geqslant 1$. Hence h is monotone decreasing and $(t+1)^\tau - t^\tau \leqslant 1$ for all $t \geqslant 1$.

Since $(0, \infty) \to \mathbb{R}$, $t \mapsto t^\tau$ is strictly increasing and unbounded, the image of $\mathbb{Z}$ in R is unbounded with respect to $|\cdot|$ if and only if it is with respect to $|\cdot|^\tau$. □

10.1.7 An absolute value $|\cdot| : R \to \mathbb{R}_{\geqslant 0}$ on a division ring R induces the metric $d : R \times R \to \mathbb{R}_{\geqslant 0}$, $(x, y) \mapsto |x - y|$ which then gives rise to a topology on R. This topology has the following properties:

(TDR1) Addition $+ : R \times R \to R$ is continuous.

(TDR2) Multiplication $\cdot : R \times R \to R$ is continuous.

(TDR3) Inversion $(\,\cdot\,)^{-1} : R^\times \to R^\times$ is continuous, where $R^\times$ denotes the set of units in R i.e. $R^\times = R \backslash \{0\}$.

Proof. Addition is continuous since for all $a, b, x, y \in R$ by the triangle inequality

$$d(x+y, a+b) = |x + y - (a+b)| \leqslant |x - a| + |y - b| = d(x, a) + d(y, b) \ .$$

Actually, this even shows that addition is Lipschitz continuous. Now fix $a, b \in R$ and let $C = \max\{|a|, |b|\} + 1$. Then for all $x, y \in R$ with $d(y, b) < 1$

$$d(x \cdot y, a \cdot b) = |(x \cdot y - a \cdot y) + (a \cdot y - a \cdot b)| \leqslant |x - a|\,|y| + |a|\,|y - b| \leqslant C\big(d(x, a) + d(y, b)\big) \ .$$

Hence multiplication is continuous. Finally, fix $a \in R^\times$ and let $x \in R^\times$ with $d(x,a) < \frac{|a|}{2}$. Then $|x| \geqslant |a| - d(x,a) > \frac{|a|}{2} > 0$ and

$$d\left(x^{-1}, a^{-1}\right) = \left|x^{-1} - a^{-1}\right| = \left|x^{-1} \cdot a^{-1}\right| |x - a| = \frac{1}{|x|\,|a|} d(x,a) < \frac{2}{|a|^2} d(x,a) \ .$$

So inversion is also continuous. □

10.1.8 Definition A division ring or field R which is equipped with a topology so that (TDR1), (TDR2) and (TDR3) are satisfied is called a *topological division ring* or a *topological field*, respectively.

10.1.9 Lemma *If $|\cdot|$ is a non-trivial absolute value on the division ring R, then there exists an element $t \in R^\times$ such that the sequence $(t^n)_{n\in\mathbb{N}}$ converges to 0. Furthermore in this case every 0-neighborhood in R contains infinitely many elements.*

Proof. By non-triviality of $|\cdot|$ there exists $t \in R^\times$ such that $|t| \neq 1$. By possibly passing to t^{-1} we can assume $|t| < 1$. Since then $\lim\limits_{n\to\infty} |t|^n = 0$, the sequence $(t^n)_{n\in\mathbb{N}}$ converges to 0. This implies in particular that for every $\varepsilon > 0$ the open ball $\mathbb{B}(0,\varepsilon) = \{t \in R \mid |t| < \varepsilon\}$ contains infinitely many elements. So the lemma is proved. □

10.1.10 Definition Two absolute values $|\cdot|$ and $|\cdot|'$ on a division ring R are called *equivalent* if they induce the same topology on R.

10.1.11 Theorem *Let $|\cdot|$ and $|\cdot|'$ be two absolute values on the division ring R. Then they are equivalent if and only if there exists $e > 0$ such that $|\cdot|' = |\cdot|^\tau$. In particular the trivial absolute value is the only one inducing the discrete topology on R.*

Proof. Let us first show the following proposition.

(A) If $|\cdot|$ and $|\cdot|'$ are equivalent, then the relation $|x| < 1$ holds true for $x \in R^\times$ if and only if $|x|' < 1$.

Since $\left|x^{-1}\right| = \frac{1}{|x|}$ and $\left|x^{-1}\right|' = \frac{1}{|x|'}$ for all $x \in R^\times$, (A) implies that $|x| > 1$ if and only if $|x|' > 1$ and that $|x| = 1$ if and only if $|x|' = 1$. To verify claim (A) assume now that $0 < |x| < 1$. Then $\lim\limits_{n\to\infty} |x^n| = 0$, hence $(x^n)_{n\in\mathbb{N}}$ converges to 0. By assumption, $\lim\limits_{n\to\infty} |x^n|' = 0$ then holds as well which implies that $|x|' < 1$. By switching $|\cdot|$ and $|\cdot|'$ the converse holds true, so (A) is proved.

Next we show that $|\cdot|$ is trivial if and only if the induced topology on R is discrete. Namely, if $|\cdot|$ is non-trivial, then there exists $x \in R^\times$ such that $|x| \neq 1$. After possibly passing to $\frac{1}{x}$ we can achieve that $|x| < 1$. So $\lim\limits_{n\to\infty} |x^n| = 0$, which means that $(x^n)_{n\in\mathbb{N}}$ is a sequence of non-zero elements of R converging to 0. But this implies that the singleton $\{0\}$ is not open in the topology induced by $|\cdot|$, hence this topology is non-discrete. Since obviously the trivial absolute value induces the discrete topology on R the second claim of the theorem is proved.

Now assume that $|\cdot|' = |\cdot|^\tau$ for some $\tau > 0$. Then a subset $B \subset R$ is a metric open ball with respect to $|\cdot|$ if and only if it is one with respect to $|\cdot|'$ since for $x \in R$ and $\varepsilon > 0$

$$\left\{y \in R \;\middle|\; |y-x| < \varepsilon\right\} = \left\{y \in R \;\middle|\; |y-x|' < \varepsilon^\tau\right\} \text{ and}$$
$$\left\{y \in R \;\middle|\; |y-x|' < \varepsilon\right\} = \left\{y \in R \;\middle|\; |y-x| < \varepsilon^{1/\tau}\right\} \ .$$

Hence the open sets with respect to the metric defined by $|\cdot|$ coincide with those defined by $|\cdot|'$ and the two absolute values are equivalent.

Let us finally show the other direction and assume that $|\cdot|$ and $|\cdot|'$ are equivalent. By the already proven second claim of the theorem we can restrict to the case where the induced topology is non-discrete which means to the case where both $|\cdot|$ and $|\cdot|'$ are non-trivial. We show that there exists $\tau > 0$ such that $|x|' = |x|^\tau$ for all $x \in R^\times$ with $|x| > 1$. This is sufficient, since if $|x| = 1$, then $|x|' = 1 = |x|^\sigma$ for any $\sigma > 0$ by (A), and since if $x \in R^\times$ with $|x| < 1$ then $|x^{-1}| > 1$ and

$$|x|' = \frac{1}{|x^{-1}|'} = \frac{1}{|x^{-1}|^\tau} = |x|^\tau \ .$$

The existence of a $\tau > 0$ with the claimed property is equivalent to the function

$$R^\times \to \mathbb{R},\ x \mapsto \frac{\ln|x|'}{\ln|x|}$$

being constant. Assume that that is not the case. Then there exist $x, y \in R^\times$ with $|x|, |y| > 1$ such that $\frac{\ln|x|'}{\ln|x|} \neq \frac{\ln|y|'}{\ln|y|}$. By possibly switching x and y we can assume $\frac{\ln|x|'}{\ln|x|} < \frac{\ln|y|'}{\ln|y|}$. But that implies $\frac{\ln|x|'}{\ln|y|'} < \frac{\ln|x|}{\ln|y|}$ since the logarithms are positive by assumptions on x and y and (A). Hence there exists a rational number $\frac{p}{q}$ with $p, q \in \mathbb{N}_{>0}$ such that

$$\frac{\ln|x|'}{\ln|y|'} < \frac{p}{q} < \frac{\ln|x|}{\ln|y|} \ .$$

Then $|x^q|' < |y^p|'$ and $|y^p| < |x^q|$ which entails

$$\left|\frac{x^q}{y^p}\right|' < 1 \text{ and } \left|\frac{x^q}{y^p}\right| > 1 \ .$$

This contradicts (A) and the theorem is proved. □

10.1.12 Remarks (a) By Ostrowski's theorem (Ostrowski, 1916, p. 276), see also (Gouvêa, 1997, Thm. 3.1.3), every non-trivial absolute value on the field $\mathbb{Q}$ of rational numbers is either equivalent to the standard absolute value $|\cdot|_\infty$ or to a p-adic absolute value $|\cdot|_p$ for some prime number p. Observe that for different primes p and q the absolute values $|\cdot|_p$ and $|\cdot|_q$ are not equivalent.

(b) Another theorem of Ostrowski (Ostrowski, 1916, p. 284), sometimes called big Ostrowski's theorem, tells that for every archimedean valued field $(\mathbb{K}, |\cdot|)$ there exists an embedding $\iota : \mathbb{K} \hookrightarrow \mathbb{C}$ into the field of complex numbers with its standard absolute value and a positive real number $\tau \leqslant 1$ such that

$$|x| = |\iota(x)|_\infty^\tau \quad \text{for all } x \in \mathbb{K} \ .$$

In particular this means that every complete archimedean valued field is isomorphic to either $(\mathbb{R}, |\cdot|_\infty^\tau)$ or $(\mathbb{C}, |\cdot|_\infty^\tau)$ for some positive $\tau \leqslant 1$.

(c) The p-adic absolute values on $\mathbb{Q}$ have extensions to $\mathbb{R}$ by (Lang, 2002, XII, §4, Thm. 4.1). This is a highly non-obvious result. To prove it one has to check first that $|\cdot|_p$ can be extended to an absolute value $|\cdot|$ on the field $\Bbbk$ of real numbers algebraic over $\mathbb{Q}$. This extended absolute value is, and that turns out to be crucial, again non-archimedean. Now one observes that $|\cdot|$ can be extended to the polynomial ring $\Bbbk[X]$ by the *Gauß norm* $|p(X)| = \max_{0\leqslant i\leqslant n}\{a_i\}$ where $p(X) = a_nX^n + \ldots + a_1X + a_0 \in \Bbbk[X]$. The Gauß norm obviously extends to an absolute value on the fraction field $\Bbbk(X)$. Again, this extension is non-archimedean. Now one recalls that $\mathbb{R}$ is a purely transcendental field extension of $\Bbbk$ and uses a transfinite induction type argument involving the just constructed Gauß norm to extend $|\cdot|$ from $\mathbb{K}$ to $\mathbb{R}$. The thus obtained extension of the p-adic absolute value to $\mathbb{R}$ is not unique. In its construction, the axiom of choice is used, so one can not even give an explicit formula for such an extension.

10.2. The category of topological vector spaces

Vector space topologies

10.2.1 Definition Let R be a topological division ring. A topology $\mathcal{T}$ on a vector space E over R is called a *vector space topology* if the following axioms hold true:

(TVS1) Addition $+ : \mathrm{E}\times\mathrm{E}\to\mathrm{E}$ is continuous.

(TVS2) Multiplication by scalars $\cdot : R\times\mathrm{E}\to\mathrm{E}$ is continuous.

The topology $\mathcal{T}$ on E is called *translation invariant* if for every $w\in\mathrm{E}$ the linear map $\ell_w : \mathrm{E}\to\mathrm{E}$, $v\mapsto v+w$ is a homeomorphism.

A vector space E endowed with a vector space topology on it is called a *topological vector space* (*over* R), for short a tvs

10.2.2 Remark Let us recall at this point some notation from linear algebra. Assume that V is a left vector space over the divison ring R. If $A, B\subset\mathrm{V}$ are two non-empty subsets, then $A+B$ is the set of all $v\in\mathrm{V}$ for which there exist $x\in A$ and $y\in B$ such that $v = x+y$. If A or B is empty, then $A+B$ is defined as the empty set. In case A is a singleton that is if $A=\{x\}$, then we often write $x+B$ instead of $\{x\}+B$. If $\mathcal{B}\subset\mathcal{P}(\mathrm{V})$ is a non-empty set of subsets of V, then we denote by $A+\mathcal{B}$ and $x+\mathcal{B}$ the sets $\{A+B\in\mathcal{P}(\mathrm{V}) \mid B\in\mathcal{B}\}$ and $\{x+B\in\mathcal{P}(\mathrm{V}) \mid B\in\mathcal{B}\}$, respectively. If $\mathcal{A}\subset\mathcal{P}(\mathrm{V})$ is a second non-empty set of subsets of V, then $\mathcal{A}+\mathcal{B}$ stands for the set of all sets of the form $A+B$, where $A\in\mathcal{A}$ and $B\in\mathcal{B}$.

In case C is a subset of the ground ring R, then $C\cdot A$ is defined as the set of all $v\in\mathrm{V}$ for which there exist $r\in C$ and $x\in A$ such that $v = r\cdot x$. If $r\in R$ we write $r\cdot A$ for $\{r\}\cdot A$. Likewise, if $x\in\mathrm{V}$, $C\cdot x$ stands for $C\cdot\{x\}$. Analogously as for addition the sets $\mathcal{C}\cdot A$, $C\cdot\mathcal{A}$ and $\mathcal{C}\cdot\mathcal{A}$ are defined when $\mathcal{C}\subset\mathcal{P}(R)$ and $\mathcal{A}\subset\mathcal{P}(\mathrm{V})$ are non-empty.

10.2.3 Proposition *Let* E *be a tvs over a topological division ring* R. *Then the following holds true:*

(i) *For every* $r\in R^\times$ *and* $w\in\mathrm{E}$ *the homothety* $\ell_{r,w} : \mathrm{E}\to\mathrm{E}$, $v\mapsto rv+w$ *is a homeomorphism with inverse* $\ell_{r^{-1},-r^{-1}w}$.

(ii) *Let* w *be an element of* E *and* $r\in R^\times$. *A filter base* $\mathcal{B}$ *on* E *then is a filter base for the zero neighborhoods if and only if* $w+r\mathcal{B}$ *is a filter base for the neighborhoods of* w.

(iii) *If $\mathcal{B}$ is a filter base of the filter of zero neighborhoods, then the closure of any non-empty $A \subset \mathrm{E}$ is given by*
$$\bar{A} = \bigcap_{U \in \mathcal{B}} A + U \ .$$

(iv) *Let $A \subset \mathrm{E}$ be open and $B \subset \mathrm{E}$. Then the set $A + B$ is open.*

(v) *Let $A, B \subset \mathrm{E}$ be closed and assume that A is quasi-compact that is that any filter on A has a cluster point. Then the set $A + B$ is closed.*

(vi) *The space* E *is* **??** *or, equivalently, each point of* E *possesses a neighborhood base consisting of closed subsets.*

Proof. ad (i). The homothety $\ell_{r,w}$ is continuous since addition and multiplication by a scalar are continuous maps on a tvs Since for all $v \in V$
$$\ell_{r^{-1},-r^{-1}w} \circ \ell_{r,w}(v) = r^{-1}(rv + w) - r^{-1}w = v, \text{ and}$$
$$\ell_{r,w} \circ \ell_{r^{-1},-r^{-1}w}(v) = r(r^{-1}v - r^{-1}w) + w = v$$
the homothety $\ell_{r,w}$ is invertible, and its inverse is $\ell_{r^{-1},-r^{-1}w}$.

ad (ii). This follows since $\ell_{r,w}$ is a homeomorphism.

ad (iii). Let $B = \bigcap_{U \in \mathcal{B}} A + U$. Let v be an element of the closure of A. Then, for $U \in \mathcal{B}$, there exists an element $a \in A \cap v - U$ by (ii) and since $-U$ is a zero neighborhood. Hence $v \in a + U$, and $\bar{A} \subset B$ follows. Now let $v \in B$ and V be a neighborhood of v. Then there exists $U \in \mathcal{B}$ such that $v - U \subset V$. By definition of B there exists an element $a \in A$ such that $v \in a + U$. Hence $a \in v - U \subset V$ which implies that $v \in \bar{A}$. So $B \subset \bar{A}$.

ad (iv). The set $A + B$ is either empty or coincides with the union $\bigcup_{v \in B} v + A$. In the latter case, each of the sets $v + A$ is non-empty and open by continuity of addition. So $A + B$ is open under the assumptions made.

ad (v). We can assume that A and B are non-empty because the claim is trivial otherwise. Assume that $A + B$ is not closed. Then there exists an element $v \in \mathrm{E} \backslash (A + B)$ such that each neighborhood of v meets $A+B$. This means in particular that the restriction of the neighborhood filter $\mathcal{U}$ of v to $A + B$ is a filter base. Consequently, $(-B + \mathcal{U}) \cap A$ is a filter base on A, hence possesses an accummulation point $x \in A$. For each neighborhood $V \in \mathcal{U}$ the point x is then contained in the closure of $-B + V$. Hence, by (iii), x is contained in $v - B + U + U$ for every zero neighborhood U. Since by continuity of addition $U + U$ runs through a base of zero neighborhoods when U runs through the zero neighborhoods, $x \in v - \bar{B} = v - B$ follows. Since $x \in A$ this contradicts the assumption $v \in A + B$ and $A + B$ has to be closed.

ad (vi). Let $v \in \mathrm{E}$, $A \subset \mathrm{E}$ closed, and assume $v \notin A$. Choose an open neighborhood V of v such that $V \cap A = \varnothing$. Then there exists an open zero neighborhood U such that $v + U + U \subset V$. By possibly passing to $U \cap (-U)$ we can assume that $U = -U$. Now $v+U$ is an open neighborhood of v and $A+U$ one of A. These neighborhoods are disjoint because if the intersection $v+U \cap A+U$ is non-empty, then there exists an element $w \in v + U + U \cap A$ since $-U = U$. This contradicts $V \cap A = \varnothing$, so $v + U$ and $A + U$ are disjoint neighborhoods of v and A, respectively. Hence E satisfies **??**. □

10.2.4 Corollary *Every vector space topology on a vector space over a topological division ring is translation invariant.*

Proof. This follows immediately by Proposition 10.2.3 (i). □

10.2.5 Definition A subset C of a vector space E over a valued division ring $(R, |\cdot|)$ is called

(i) *symmetric* if $-v \in C$ for all $v \in C$,

(ii) *circled* or *balanced* if $rv \in C$ for all $v \in C$ and $r \in R$ with $|r| \leqslant 1$.

10.2.6 Remark Symmetry of a subset of a vector space of a division ring is even defined when the underlying division ring does not carry an absolute value.

10.2.7 Lemma *Let C be a subset of a topological vector space E over a valued division ring $(R, |\cdot|)$ and $r \in R$.*

(i) *If C is symmetric, then the closure $\bar{C}$ and the interior $\mathring{C}$ are symmetric.*

(ii) *If C is circled, then the closure $\bar{C}$ and the union $\mathring{C} \cup \{0\}$ are circled.*

(iii) *The set rC is symmetric (respectively circled) if C has that property.*

Proof. Without loss of generality we can assume $C \neq \varnothing$. Claim (i) then follows immediately since multiplication by -1 is a homeomorphism. To prove claim (ii) assume that C is circled. Let $s \in R$ with $|s| \leqslant 1$. Assume $v \in \bar{C}$ and consider sv. We have to show that $sv \in \bar{C}$. If $s = 0$ then $sv = 0 \in C \subset \bar{C}$ since C is circled. So we can assume $s \neq 0$ and need to show that for every neighborhood V of sv the intersection $C \cap V$ is non-empty. Since $|s| > 0$, the homothety $\ell_s : \mathrm{E} \to \mathrm{E}$, $w \mapsto sw$ is a homeomorphism with inverse $\ell_{s^{-1}}$. Hence $s^{-1}V$ is a neighborhood of v. Since v lies in the closure of C there exists an element $w \in C \cap s^{-1}V$. Hence $sw \in C \cap V$ by assumption on C and $\bar{C}$ is circled.

If $v \in \mathring{C} \cup \{0\}$ then $0 = 0 \cdot v \in \mathring{C} \cup \{0\}$. It remains to show that $sv \in \mathring{C} \cup \{0\}$ for $s \in R$ with $0 < |s| \leqslant 1$ and $v \in \mathring{C} \backslash \{0\}$. Under this assumption the homothety ℓ_s is a homeomorphism, so $s\mathring{C}$ is an open subset of C since C is circled. Hence $sv \in s\mathring{C} \subset \mathring{C}$, and $\mathring{C} \cup \{0\}$ is circled as well.

Claim (iii) follows immediately from the observation that for $v \in C$ and $s \in R$ the relation $srv \in rC$ holds true if $sv \in C$. □

10.2.8 Proposition and Definition *The intersection of a non-empty family $(C_i)_{i \in I}$ of symmetric (respectively circled) subsets $C_i \subset \mathrm{E}$, $i \in I$ of a topological vector space E over a valued division ring $(R, |\cdot|)$ is symmetric (respectively circled). In particular, if $A \subset \mathrm{E}$ is a subset, then the sets*

$$\operatorname{Sym} A = \bigcap_{\substack{A \subset B \subset \mathrm{E} \\ B \text{ is symmetric}}} B \quad \textit{and} \quad \operatorname{Circ} A = \bigcap_{\substack{A \subset B \subset \mathrm{E} \\ B \text{ is circled}}} B$$

are symmetric and circled, respectively. They have the property that $\operatorname{Sym} A$ *is the smallest symmetric and* $\operatorname{Circ} A$ *the smallest circled subsets of* E *containing A. They are called the* symmetric *and the* circled hull *of A, respectively. Analogously,*

$$\overline{\operatorname{Sym}} A = \bigcap_{\substack{A \subset B = \bar{B} \subset \mathrm{E} \\ B \text{ is symmetric}}} B \quad \textit{and} \quad \overline{\operatorname{Circ}} A = \bigcap_{\substack{A \subset B = \bar{B} \subset \mathrm{E} \\ B \text{ is circled}}} B$$

are called the closed symmetric *and the* closed circled hull *of A, respectively. They have the property that* $\overline{\mathrm{Sym}}\, A$ *is the smallest closed symmetric and* $\overline{\mathrm{Circ}}\, A$ *the smallest closed circled subset of* E *containing* A*.*

Proof. Note first that all the hulls in the proposition are well-defined since E is closed and circled. Let C denote the intersection of the family $(C_i)_{i\in I}$. Assume that for some $r \in R$ with $|r| \leqslant 1$ the inclusion $rC_i \subset C$ holds true for all $i \in I$. Then $rC \subset C$, hence if all C_i are symmetric (respectively circled), so is C. This observation now entails that $\mathrm{Sym}\, A$ is symmetric, Circ is circled, $\overline{\mathrm{Sym}}\, A$ is closed and symmetric, and finally that $\overline{\mathrm{Circ}}\, A$ is closed and circled. Moreover, all those sets contain A. The minimality properties of these sets are clear by construction. □

10.2.9 Remark Observe that by the proposition A is symmetric if and only if $\mathrm{Sym}\, A = A$ and circled if and only if $\mathrm{Circ}\, A = A$. Analogously, $\overline{\mathrm{Sym}}\, A = A$ if and only if A is closed symmetric and $\overline{\mathrm{Circ}}\, A = A$ if and only if A is closed and circled.

10.2.10 Lemma *Let* E *be a topological vector space over the valued division ring* $(R, |\cdot|)$ *and* $A \subset \mathrm{E}$ *non-empty. Then*

$$\mathrm{Sym}\, A = A \cup -A \quad \textit{and} \quad \mathrm{Circ}\, A = \bigcup_{r\in R,\, |r|\leqslant 1} rA \ .$$

For the closed hulls one has

$$\overline{\mathrm{Sym}}\, A = \overline{\mathrm{Sym}\, A} \quad \textit{and} \quad \overline{\mathrm{Circ}}\, A = \overline{\mathrm{Circ}\, A} \ .$$

Proof. Since $A \cup -A$ is symmetric by definition, contains A, and is contained in $\mathrm{Sym}\, A$, the equality $\mathrm{Sym}\, A = A \cup -A$ holds true. Similarly, $\bigcup_{r\in R,\, |r|\leqslant 1} rA$ is circled by definition, contains A, and is contained in $\mathrm{Circ}\, A$ by definition of the circled hull. Hence $\mathrm{Circ}\, A = \bigcup_{r\in R,\, |r|\leqslant 1} rA$. The remainder of the claim follows from Lemma 10.2.7. □

10.2.11 Definition Assume that B, C are subsets of a vector space E over the valued division ring $(R, |\cdot|)$. Then one says that

(i) C *absorbes* B if there exists a real number $t \in \mathbb{R}_{\geqslant 0}$ such that $B \subset rC$ for all $r \in R$ with $|r| \geqslant t$,

(ii) C is *absorbing* or *absorbent* if C absorbes every one-point set of E that is if for every $v \in \mathrm{E}$ there exists $t \in \mathbb{R}_{\geqslant 0}$ such that $v \in rC$ for all $r \in R$ with $|r| \geqslant t$.

If the vector space E carries in addition a vector space topology, then one says that

(iii) the subset $B \subset \mathrm{E}$ is *bounded* if it is absorbed by every zero neighborhood.

10.2.12 Lemma *Let* E *be a vector space over the valued division ring* $(R, |\cdot|)$*. Then the following holds true.*

(i) *If* $C_1, \ldots, C_n$ *are absorbing subset of* E*, then the intersection* $C_1 \cap \ldots \cap C_n$ *is absorbing.*

(ii) *If* C *is an absorbing subset of* E*, then* rC *is absorbing for every* $r \in R^\times$*.*

Proof. ad (i). Let $v \in \mathrm{E}$ and choose $t_1, \ldots, t_n \in \mathbb{R}_{\geqslant 0}$ such that $v \in rC_i$ for $|r| \geqslant t_i$. Put $t = \max\{t_1, \ldots, t_n\}$. Then $v \in r(C_1 \cap \ldots \cap C_n)$ for $|r| \geqslant t$, hence $C_1 \cap \ldots \cap C_n$ is absorbing.

ad (*ii*). Choose $t \in \mathbb{R}_{\geqslant 0}$ such that $v \in sC$ for all $s \in R$ with $|s| \geqslant t$. Then one has $|sr| \geqslant t$ for all $s \in R$ with $|s| \geqslant \frac{t}{|r|}$, hence $v \in s(rC)$ for all such s. Therefore rC is absorbing. □

10.2.13 Proposition *The filter of zero neighborhoods of a topological vector space* E *over* $(R, |\cdot|)$ *has a filter base* $\mathcal{B}$ *with the following properties:*

(i) *For each* $V \in \mathcal{B}$ *there exists* $U \in \mathcal{B}$ *such that* $U + U \subset V$.

(ii) *Every element* $V \in \mathcal{B}$ *is circled and absorbing.*

(iii) *There exists an element* $r \in R^\times$ *with* $0 < |r| < 1$ *such that* $V \in \mathcal{B}$ *implies* $rV \in \mathcal{B}$.

Conversely, if $\mathcal{B}$ *is a filter base on an* R*-vector space* E *such that* (i) *to* (iii) *hold true, then there exists a unique vector space topology on* E *such that* $\mathcal{B}$ *is a neighborhood base at the origin. In case the ground ring* R *is archimedean, a filter base on* E *which satisfies* (i) *and* (ii) *already induces a unique vector space topology having* $\mathcal{B}$ *as a neighborhood base at* 0. *In either of these two cases, the thus constructed topology coincides with the coarsest translation invariant topology for which* $\mathcal{B}$ *is a set of zero neighborhoods.*

Proof. Assume that E is a tvs Let $\mathcal{B}$ be the set of circled neighborhoods of 0. We show first that $\mathcal{B}$ is a base of the filter $\mathcal{U}_0$ of zero neighborhoods. Let $W \in \mathcal{U}_0$. By Axiom (TVS2) there exists an $\varepsilon > 0$ and an open zero neighborhood U such that $sU \subset W$ for all $s \in R$ with $|s| < \varepsilon$. Then $V = \bigcup_{s \in R^\times \,\&\, |s| < \varepsilon} sU$ is a zero neighborhood since by Lemma 10.1.9 the set of $s \in R^\times$ with $|s| < \varepsilon$ is non-empty. By construction V is contained in W and circled, so $V \in \mathcal{B}$. Hence $\mathcal{B}$ is a filter base of $\mathcal{U}_0$.

Next recall that there exists $r \in R^\times$ with $0 < |r| < 1$ since the absolute value $|\cdot|$ is non-trivial. Let $V \in \mathcal{B}$. Then $sV \subset V$ for all $s \in R$ with $|s| \leqslant 1$ which entails $srV \subset rV$ for all such s. Hence rV is circled and an element of $\mathcal{B}$ as well. This proves (iii). Since addition on E is continuous, there exist for given $V \in \mathcal{B}$ open neighborhoods U_1, U_2 of the origin such that $U_1 + U_2 \subset V$. Choose $U \in \mathcal{B}$ such that $U \subset U_1 \cap U_2$. Then $U + U \subset V$ and (i) is proved. To show that any $V \in \mathcal{B}$ is absorbing let $v \in$ E. By continuity of scalar multiplication there exists $\varepsilon > 0$ such that $sv \in V$ for all $s \in R$ with $|s| < \varepsilon$. By Proposition 10.2.3 (i) this entails $v \in sV$ for all $s \in R$ with $|s| > \varepsilon$ and V is absorbing.

Now assume that E is an R-vector space and that $\mathcal{B}$ is a filter base that satisfies (i), (ii) and, if $|\cdot|$ is non-archimedean, (iii). Since $\mathcal{B}$ consists of non-empty circled sets, $0 \in V$ for all $V \in \mathcal{B}$. Let $\mathcal{T} \subset \mathcal{P}(\mathrm{E})$ be the set of all $U \subset$ E such that for each $v \in U$ there exists $V \in \mathcal{B}$ with $v + V \subset U$. By definition and since $\mathcal{B}$ is a filter base, $\mathcal{T}$ is a topology on E. By construction, $\mathcal{T}$ is also the coarsest translation invariant topology for which $\mathcal{B}$ is a set of zero neighborhoods. We show that $\mathcal{B}$ is a base of the filter $\mathcal{U}_0$ of zero neighborhoods. By definition of $\mathcal{T}$ there exists for each $U \in \mathcal{U}_0$ a $V \in \mathcal{B}$ such that $V \subset U$. So it remains to show that each $V \in \mathcal{B}$ is a zero neighborhood. To this end let U be the set of all $v \in V$ for which there exists a $W \in \mathcal{B}$ with $v + W \subset V$. Since $0 + V \subset V$ one has $0 \in U$. The relation $U \subset V$ holds because $0 \in W$ for all $W \in \mathcal{B}$. Now let $v \in U$. By (i) there exists W' such that $v + W' + W' \subset V$ which entails $v + W' \subset U$. Hence $U \in \mathcal{T}$ and V is a zero neighborhood. Next we verify that $\mathcal{T}$ is a vector space topology. We start with continuity of addition. Let W be an open neighborhood of $v + w$, where $v, w \in$ E. Then there exists $V \in \mathcal{B}$ such that $v + w + V \subset W$. Choose $U \in \mathcal{B}$ such that $U + U \subset V$. Then $v + U$ and

$w+U$ are neighborhoods of v and w, respectively, and $(v+U)+(w+U) \subset v+w+V \subset W$. So addition is continuous. We continue with scalar multiplication. Let W be an open neighborhood of rv, where $r \in R$ and $v \in \mathrm{E}$. Then there exists $V \in \mathcal{B}$ such that $rv + V + V \subset W$. Since V is absorbing by (ii) there exists $\varepsilon > 0$ such that $(s-r)v \in V$ for all $s \in R$ with $|s-r| < \varepsilon$. Now if $|\cdot|$ is non-archimedean choose $t \in R^\times$ according to (iii), and put $V_n = t^n V$ for all $n \in \mathbb{N}$. In the archimedean case let $t = \frac{1}{2}$ and use (i) to construct recursively a sequence $(V_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{B}$ such that $2^n V_n = V_n + \ldots + V_n \subset V$, where the sum has 2^n summands. In either of these cases, choose $N \in \mathbb{N}$ large enough so that $|t|^N < \frac{1}{|r|+\varepsilon}$. Then $V_N \in \mathcal{B}$ and $v + V_N$ is a neighborhood of v. Moreover, for $w \in v + V_N$ there exists an element $x \in V$ such that $w - v = t^N x$. Then the relation $s(w-v) = st^N x \in V$ holds whenever $|s-r| < \varepsilon$ since V_N is circled. Hence for such w and s

$$sw = rv + s(w-v) + (s-r)v \in rv + V + V \subset W \ .$$

This means that scalar multiplication is continuous, and the proof is finished. □

Morphisms of topological vector spaces

10.2.14 Definition By a *morphism* of topological vector spaces over the topological division ring R one understands a continuous R-linear map $f : \mathrm{E} \to \mathrm{F}$ between two topological vector spaces E and F over R. The space of morphisms between E and F will be denoted $\mathrm{Hom}_{R\text{-}\mathsf{TVS}}(\mathrm{E},\mathrm{F})$ or just $\mathrm{Hom}_R(\mathrm{E},\mathrm{F})$ or $\mathrm{Hom}(\mathrm{E},\mathrm{F})$ if now confusion can arrise.

10.2.15 Theorem *The topological vector spaces over a topological division ring R as objects together with their morphisms form an additive category which we denote by R-TVS. More precisely, R-TVS is a category enriched over the category of R-vector spaces where addition and scalar multiplication on the hom-spaces* $\mathrm{Hom}(\mathrm{E},\mathrm{F})$ *are given by*

$$+ : \mathrm{Hom}(\mathrm{E},\mathrm{F}) \times \mathrm{Hom}(\mathrm{E},\mathrm{F}) \to \mathrm{Hom}(\mathrm{E},\mathrm{F}),\ (f,g) \mapsto f+g = (\mathrm{E} \ni v \mapsto f(v)+g(v) \in \mathrm{F}) \ ,$$
$$\cdot : R \times \mathrm{Hom}(\mathrm{E},\mathrm{F}) \to \mathrm{Hom}(\mathrm{E},\mathrm{F}),\ (r,f) \mapsto r\cdot f = (\mathrm{E} \ni v \mapsto r \cdot f(v) \in \mathrm{F}) \ .$$

Proof. Observe first that the identity map id_E on a topological vector space E is linear and continuous and so is the composition $g \circ f$ of two morphisms of topological vector spaces $f : \mathrm{E} \to \mathrm{F}$ and $g : \mathrm{F} \to \mathrm{G}$. Hence topological vector spaces over R together with linear and continuous maps between them form a category.

Next check that the hom-space $\mathrm{Hom}(\mathrm{E},\mathrm{F})$ is an abelian group. Associativity and commutativity of addition follow from the respective properties on F. The zero element is the constant map $\mathrm{E} \to \mathrm{F}$, $v \mapsto 0$ and the inverse of a morphism $f : \mathrm{E} \to \mathrm{F}$ is given by $-f : \mathrm{E} \to \mathrm{F}$, $v \mapsto -f(v)$. Similarly one checks that multiplication by scalars on $\mathrm{Hom}(\mathrm{E},\mathrm{F})$ is associative and distributes from the left and from the right over addition since scalar multiplication on F has these properties. Finally, the unit of R acts as identity on $\mathrm{Hom}(\mathrm{E},\mathrm{F})$ since it does so on F. Hence $\mathrm{Hom}(\mathrm{E},\mathrm{F})$ carries the structure of an R left vector space.

Composition of morphisms $\mathrm{Hom}(\mathrm{E},\mathrm{F}) \times \mathrm{Hom}(\mathrm{F},\mathrm{G}) \to \mathrm{Hom}(\mathrm{E},\mathrm{G})$, $(f,g) \to g \circ f$ is an R-bilinear map as the following equalities for $f, f_1, f_2 \in \mathrm{Hom}(\mathrm{E},\mathrm{F})$, $g, g_1, g_2 \in \mathrm{Hom}(\mathrm{F},\mathrm{G})$, $r \in R$, and $v \in \mathrm{E}$

show:

$$\begin{aligned}
&(f \circ (g_1 + g_2))(v) = f((g_1 + g_2)(v)) = f(g_1(v) + g_2(v)) = \\
&\quad = f \circ g_1(v) + f \circ g_2(v) = (f \circ g_1 + f \circ g_2)(v) \ , \\
&(f \circ (rg))(v) = f((rg)(v)) = f(rg(v)) = rf(g(v)) = (r(f \circ g))(v) \ , \\
&((f_1 + f_2) \circ g)(v) = (f_1 + f_2)(g(v)) = f_1(g(v)) + f_2(g(v)) = \\
&\quad = f_1 \circ g(v) + f_2 \circ g(v) = (f_1 \circ g + f_2 \circ g)(v) \ , \\
&((rf) \circ g)(v) = (rf)(g(v)) = r(f(g(v))) = r(f \circ g(v)) = (r(f \circ g))(v) \ .
\end{aligned}$$

Hence R-TVS is a category enriched over the category of R-vector spaces. In particular, R-TVS then is an additive category. □

10.2.16 Example For every tvs E and non-zero element t of the ground ring R the map $\ell_t : \mathrm{E} \to \mathrm{E}$, $v \mapsto tv$ is an isomorphism of topological vector spaces by Proposition 10.2.3 (i).

10.2.17 Proposition and Definition *A linear map $f : \mathrm{E} \to \mathrm{F}$ between topological vector spaces over a valued division ring $(R, |\cdot|)$ maps symmetric sets to symmetric sets and circled sets to circled sets. If in addition f is continuous, then f is* bounded *that means it maps bounded subsets of* E *to bounded subsets of* F.

Proof. Since by linearity $f(tv) = tf(v)$ for all $v \in \mathrm{E}$ and $t \in R$, $f(C)$ is symmetric (respectively circled) if the subset $C \subset \mathrm{E}$ is.

To verify the second claim let $B \subset \mathrm{E}$ be bounded and $V \subset \mathrm{F}$ a zero neighborhood. Then $f^{-1}(V)$ is a zero neighborhood in E by continuity of f. Hence there exists an $r \in \mathbb{R}_{\geqslant 0}$ such that $B \subset tf^{-1}(V)$ for all $t \in R$ with $|t| \geqslant r$. By linearity of f one obtains $f(B) \subset tV$ for all such t, so f is bounded. □

10.2.18 Remark By the proposition continuity of a linear map between topological vector spaces implies the map to be bounded. As we will see later in this monograph, the converse does in general not hold true unless the underlying topological vector spaces are for example normable.

Normed real division algebras and local convexity

10.2.19 The major class of topological divison rings over which topological vector spaces are defined is formed by valued division rings $(R, |\cdot|)$ which carry the structure of an $\mathbb{R}$-algebra such that for all $r \in \mathbb{R}$ and $x \in R$ the equality

$$|rx| = |r|_\infty \cdot |x|$$

holds true. We will therefore given them a particular name and call them *normed real division algebras*. Note that the field of real numbers can be embedded into a normed real division algebra R by the natural map $\mathbb{R} \mapsto R$, $r \mapsto r \cdot 1$. Since $\mathbb{R}$ with its standard absolute value is archimedean, so is every normed real division algebra. By the Frobenius theorem, Frobenius (1878), there exist only three finite dimensional real division algebras, namely the field of real numbers $\mathbb{R}$, the field of complex numbers $\mathbb{C}$, and the quaternions $\mathbb{H}$.

10.2.20 Definition Under the assumption that R is a normed real division algebra one calls a subset $C \subset \mathrm{E}$ of an R-vector space

(i) *convex* if $tv + (1-t)w \in C$ for all $v, w \in C$ and $t \in \mathbb{R}$ with $0 \leqslant t \leqslant 1$,

(ii) *absolutely convex* if $rv + sw \in C$ for all $v, w \in C$ and $r, s \in R$ such that $|r| + |s| \leqslant 1$,

(iii) a *cone* if $tv \in C$ for all $v \in C$ and $t \in \mathbb{R}$ with $0 \leqslant t \leqslant 1$.

10.2.21 Lemma *Let R be a normed real division algebra. A subset C of an R-vector space E then is absolutely convex if and only if it is circled and convex.*

Proof. The claim is trivial when $C = \varnothing$, so we assume that C is nonempty.

Let C be absolutely convex. Since C contains at least one element v one has $0 = 0 \cdot v + 0 \cdot v \in C$. Hence $rv = (1 - |r|) \cdot 0 + rv \in C$ for all $v \in C$ and $r \in R$ with $|r| \leqslant 1$. So C is circled. By definition of absolute convexity C is convex.

If C is circled and convex, then it contains with elements v, w also $rv + sw$ if $|r| + |s| \leqslant 1$. To see this observe first that $\varrho v \in C$ and $\sigma w \in C$ where the elements $\varrho, \sigma \in R$ have been chosen so that $|\varrho| = |\sigma| = 1$, $r = |r| \cdot \varrho$ and $s = |s| \cdot \sigma$. Now if $|r| + |s| = 0$, then $rv + sw = 0 \in C$ since C is circled. If $|r| + |s| > 0$, then

$$rv + sw = (|r| + |s|) \left(\frac{|r|}{|r| + |s|} \varrho v + \frac{|s|}{|r| + |s|} \sigma w \right) \in C$$

since C is convex and circled. Hence C is absolutely convex. □

10.2.22 Lemma *A linear map $f : \mathrm{E} \to \mathrm{F}$ between vector spaces over a normed real divison algebra R maps convex sets to convex sets, absolutely convex sets to absolutely convex sets, and cones to cones.*

Proof. This an immediate consequence of the linearity of f. □

10.2.23 Lemma *Let E be a tvs over a normed real division algebra R, let $C, D \subset \mathrm{E}$ be convex and $r \in R$. Then the following holds true.*

(i) *The closure $\bar{C}$ and the interior $\mathring{C}$ are convex.*

(ii) *The sets $C + D$ and rC are convex.*

(iii) *If C is absolutely convex, then so are $\bar{C}$ and $\mathring{C}$.*

(iv) *If C is absolutely convex, then so is rC for each $r \in R^\times$.*

Proof. We consider only the cases $C, D \neq \varnothing$ because otherwise the claim is trivial.

ad (i). Let $t \in (0, 1)$. Then $t\bar{C} + (1-t)\bar{C} \subset \bar{C}$ by continuity of the map $\mathrm{E} \times \mathrm{E} \to \mathrm{E}$, $(v, w) \mapsto tv + (1-t)w$. Hence $\bar{C}$ is convex. Now let v, w be points of the interior of C and $z = tv + (1-t)w$. Then $z \in C$, and there exists a zero neighborhood U such that $v + U \subset C$ and $w + U \subset C$. Let $u \in U$ and compute

$$z + u = tv + (1-t)w + tu + (1-t)u = t(v + u) + (1-t)(w + u) \ .$$

Since both $v + u$ and $w + u$ are elements of C so is $z + u$ by convexity of C. Hence $z + U \subset C$ and z lies in the interior of C.

ad (ii). If $v, w \in C$, $x, y \in D$ and $t \in (0,1)$, then by convexity of C and D

$$t(v+x) + (1-t)(w+y) = \big(tv + (1-t)w\big) + \big(tx + (1-t)y\big) \in C + D \ .$$

Hence $C + D$ is convex. Similarly,

$$t(rv) + (1-t)(rw) = r\big(tv + (1-t)w\big) \in rC \ ,$$

so rC is convex as well.

ad (iii). Let C be absolutely convex. If $\mathring{C} \neq \emptyset$, then $0 \in \frac{1}{2}\mathring{C} - \frac{1}{2}\mathring{C} \subset C$, hence $0 \in \mathring{C}$. By Lemma 10.2.7 and (i) the claim now follows.

ad (iv). By (ii), rC is convex, so it remains to show that rC is circled. Assume that $v \in rC$. Then $v = rw$ for a unique $w \in C$. Since C is circled, $tw \in C$ for every $t \in R$ with $|t| \leqslant 1$. Hence $tv = r(tw) \in rC$ for such t and rC is circled. □

10.2.24 Proposition and Definition *The intersection of a non-empty family $(C_i)_{i \in I}$ of convex (respectively absolutely convex) subsets $C_i \subset \mathrm{E}$, $i \in I$ of a topological vector space* E *over a normed real division algebra R is convex (respectively absolutely convex). In particular, if $A \subset \mathrm{E}$ is a subset, then the sets*

$$\operatorname{Conv} A = \bigcap_{\substack{A \subset B \subset \mathrm{E} \\ B \text{ is convex}}} B \quad \textit{and} \quad \operatorname{AConv} A = \bigcap_{\substack{A \subset B \subset \mathrm{E} \\ B \text{ is absolutely convex}}} B$$

are convex and absolutely convex, respectively. The set Conv A *is called the* convex hull *of A and is the smallest convex set containing A. Similarly,* AConv A *is the smallest absolutely convex set containing A. It is called the* absolutely convex hull *of A. The* closed convex hull $\overline{\operatorname{Conv}}\, A$ *and the* closed absolutely convex hull $\overline{\operatorname{AConv}}\, A$ *of A are defined by*

$$\overline{\operatorname{Conv}}\, A = \bigcap_{\substack{A \subset B = \bar{B} \subset \mathrm{E} \\ B \text{ is convex}}} B \quad \textit{and} \quad \overline{\operatorname{AConv}}\, A = \bigcap_{\substack{A \subset B = \bar{B} \subset \mathrm{E} \\ B \text{ is absolutely convex}}} B \ .$$

These sets have the property that $\overline{\operatorname{Conv}}\, A$ is the smallest closed convex subset and $\overline{\operatorname{AConv}}\, A$ the smallest closed absolutely convex subset of E *containing A.*

Proof. Let C be the intersection $\bigcap_{i \in I} C_i$ and assume that each C_i is absolutely convex. Let $v, w \in C$ and $r, s \in R$ with $|r| + |s| \leqslant 1$. Then $v, w \in C_i$, hence $rv + sw \in C_i$ for all $i \in I$. Therefore $rv + sw \in C$ and C is absolutely convex. This argument also shows that C is convex if all C_i are convex. The rest of the claim follows as in the proof of Proposition and Definition 10.2.8. □

10.2.25 Remark The proposition in particular entails that A is convex if and only if $\operatorname{Conv} A = A$ and absolutely convex if and only if $\operatorname{AConv} A = A$. Analogously, $\overline{\operatorname{Conv}}\, A = A$ if and only if A is closed and convex, and $\overline{\operatorname{AConv}}\, A = A$ if and only if A is closed and absolutely convex.

10.2.26 Lemma *Let $A \subset \mathrm{E}$ be a non-empty subset of a tvs E over a normed real division algebra R. Then*

$$\mathrm{Conv}\, A = \left\{ \sum_{i=1}^k t_i v_i \in \mathrm{E} \;\middle|\; k \in \mathbb{N}_{>0},\ v_1, \ldots v_k \in A,\ t_1 \ldots, t_k \in \mathbb{R}_{\geqslant 0},\ \sum_{i=1}^k t_i = 1 \right\} , \tag{10.2.1}$$

$$\mathrm{AConv}\, A = \left\{ \sum_{i=1}^k r_i v_i \in \mathrm{E} \;\middle|\; k \in \mathbb{N}_{>0},\ v_1, \ldots v_k \in A,\ r_1 \ldots, r_k \in R,\ \sum_{i=1}^k |r_i| \leqslant 1 \right\} . \tag{10.2.2}$$

For the closed hulls one has

$$\overline{\mathrm{Conv}}\, A = \overline{\mathrm{Conv}\, A} \quad \textit{and} \quad \overline{\mathrm{AConv}}\, A = \overline{\mathrm{AConv}\, A} \ .$$

Finally, if A is circled, then

$$\mathrm{AConv}\, A = \mathrm{Conv}\, A \ .$$

Proof. By definition, the right hand side of Eq. (10.2.1) is convex and contains A, hence it contains $\mathrm{Conv}\, A$. Conversely, one shows by induction on $k \in \mathbb{N}_{>0}$ and convexity of $\mathrm{Conv}\, A$ that each element of the form $\sum_{i=1}^k t_i v_i$ with $v_1, \ldots, v_k \in A$ and $t_1, \ldots, t_k \in \mathbb{R}_{\geqslant 0}$ such that $\sum_{i=1}^k t_i = 1$ is in $\mathrm{Conv}\, A$. This proves Eq. (10.2.1). The proof of Eq. (10.2.2) is similar. Observe that the right hand side of Eq. (10.2.2) is absolutely convex and contains A. Hence it contains $\mathrm{AConv}\, A$. An argument using induction on $k \in \mathbb{N}_{>0}$ and absolute convexity of $\mathrm{AConv}\, A$ shows that each element of the form $\sum_{i=1}^k r_i v_i$ with $v_1, \ldots v_k \in A$ and $r_1 \ldots, r_k \in R$ such that $\sum_{i=1}^k |r_i| \leqslant 1$ is in $\mathrm{Conv}\, A$. So Eq. (10.2.2) holds true as well. The claim about the closed hulls is a consequence of Lemma 10.2.23. For the proof of the last claim it suffices to show that $\mathrm{Conv}\, A$ is circled if A is. To this end let $v \in \mathrm{Conv}\, A$ and $r \in R$ with $|r| \leqslant 1$. Then one can write v in the form $v = \sum_{i=1}^k t_i v_i$ with $v_1, \ldots, v_k \in A$ and $t_1, \ldots, t_k \in \mathbb{R}_{\geqslant 0}$, where $\sum_{i=1}^k t_i = 1$. Hence $rv = \sum_{i=1}^k t_i (r v_i)$, which is in $\mathrm{Conv}\, A$, since $r v_i \in A$ for all i because A is circled. □

10.2.27 Lemma *Let $A \subset \mathrm{E}$ be a non-empty subset of a tvs E over a normed real division algebra R.*

(i) *If A is convex and $t_1, \ldots, t_k \in \mathbb{R}_{\geqslant 0}$ with $k \in \mathbb{N}_{>0}$, then*

$$\sum_{i=1}^k t_i A = \left(\sum_{i=1}^k t_i \right) A \ .$$

(ii) *If A is absolutely convex and $r_1, \ldots, r_k \in R$ with $k \in \mathbb{N}_{>0}$, then*

$$\sum_{i=1}^k r_i A = \left(\sum_{i=1}^k |r_i| \right) A \ .$$

Proof. ad (i). Obviously $\sum_{i=1}^k t_i A \supset \left(\sum_{i=1}^k t_i \right) A$. Let us show the converse inclusion. Without loss of generality we can assume that $t_i > 0$ for all i. Then $t = \sum_{i=1}^k t_i > 0$, so, after division by t, we can reduce the claim to showing that $\sum_{i=1}^k t_i A \subset A$ for $t_1, \ldots, t_k \in \mathbb{R}_{>0}$ such that $\sum_{i=1}^k t_i = 1$. But $\sum_{i=1}^k t_i A \subset \mathrm{Conv}\, A = A$ by Lemma 10.2.26 and convexity of A.

ad (*ii*). Since by absolute convexity $r_i A = |r_i| A$ for $i = 1, \ldots, k$, the claim follows from (i). $\square$

10.2.28 Lemma *Let* $\mathbb{K}$ *be one of the division rings* $\mathbb{C}$ *or* $\mathbb{H}$ *with their standard absolute values and let* E *be a vector space over* $\mathbb{K}$*. Then a convex subset* $C \subset \mathrm{E}$ *is absorbent in* E *if and only if it is absorbent in the realification* $\mathrm{E}^{\mathbb{R}}$*.*

Proof. It suffices to show the non-trivial direction. So assume that C is convex and absorbent in the realification $\mathrm{E}^{\mathbb{R}}$. Denote by $u_1, \ldots, u_n$ the standard basis of $\mathbb{K}$ over $\mathbb{R}$ with $n = 2$ or $n = 4$ depending on $\mathbb{K}$. In particular this means $u_1 = 1$. For given $v \in \mathrm{E}$ there now exists $t \in \mathbb{R}_{\geqslant 0}$ such that

$$\pm \frac{1}{u_1} v, \ldots, \pm \frac{1}{u_n} v \in rC \quad \text{for all } r \geqslant t \ .$$

Without loss of generality we can assume $t \geqslant 1$. Let $z \in \mathbb{K}$ with $|z| \geqslant nt$. Then the vectors $c_1 = \operatorname{sgn} z_1 \frac{n}{|z|\, u_1} v, \ldots, c_n = \operatorname{sgn} z_n \frac{n}{|z|\, u_n} v$ are elements of C. By convexity of C and since $0 \in C$ one has $\frac{|z_1|}{|z|} c_1, \ldots, \frac{|z_n|}{|z|} c_n \in C$. Again by convexity one concludes

$$\frac{1}{z} v = \sum_{i=1}^{n} \frac{z_i}{|z|^2\, u_i} v = \sum_{i=1}^{n} \frac{|z_i|}{n|z|} c_i \in C \ .$$

Hence C is absorbing and the claim is proved. $\square$

10.2.29 Definition A topological vector space E over a normed real division algebra R for which Axiom LCVS below holds true is called a *locally convex topological vector space*, a *locally convex vector space* or shortly a *locally convex tvs.*

(LCVS) The vector space topology on E has a base consisting of convex sets.

10.2.30 Remark For better readability, we often say *locally convex topology* instead of *locally convex vector space topology.*

10.2.31 Proposition *The locally convex topological vector spaces over a normed real division algebra* R *together with the continuous linear maps between them form a full subcategory of the category* R-TVS *of topological* R*-vector spaces. It is denoted* R-LCVS*.*

Proof. This is clear by definition. $\square$

10.2.32 Proposition and Definition *The filter of zero neighborhoods of a locally convex topological vector space* E *over a normed real divison algebra* R *has a filter base* $\mathcal{B}$ *with the following properties:*

(i) *For each* $V \in \mathcal{B}$ *there exists* $U \in \mathcal{B}$ *such that* $U + U \subset V$*.*

(ii) *Every element of* $\mathcal{B}$ *is a* barrel *that means is absolutely convex, closed and absorbing.*

(iii) *Let* $r \in R^{\times}$*. Then* $V \in \mathcal{B}$ *if and only if* $rV \in \mathcal{B}$*.*

Conversely, if $\mathcal{B}$ *is a filter base on an* R*-vector space* E *such that* (i) *holds true and such that each element of* $\mathcal{B}$ *is absolutely convex and absorbing, then there exists a unique locally convex topology on* E *such that* $\mathcal{B}$ *is a neighborhood base of the origin. It is the coarsest among all translation invariant topologies for which* $\mathcal{B}$ *is a set of zero neighborhoods and is called the* locally convex topology generated *or* induced by $\mathcal{B}$.

Proof. Let E be a locally convex tvs. Let $\mathcal{B}$ be the collection of all barrels which are at the same time zero neighborhoods. Let V be an element of $\mathcal{U}_0$, the filter of zero neighborhoods. Since E is (T3) by Proposition 10.2.3, there exists a closed zero neighborhood V_a such that $V_a \subset V$. By local convexity of E there exists a convex zero neighborhood V_b with $V_b \subset V_a$. By Proposition 10.2.13 there exists a circled zero neighborhood V_c with $V_c \subset V_b$. The closed convex hull $U = \overline{\mathrm{Conv}}\, V_c$ then is a barrel contained in V. Since it is a zero neighborhood it is an element of $\mathcal{B}$, and $\mathcal{B}$ is a filter base of $\mathcal{U}_0$. This proves (ii).

To verify (i), let $V \in \mathcal{B}$ and observe that by continuity of addition there exist zero neighborhoods U_1 and U_2 such that $U_1 + U_2 \subset V$. Choose $U \in \mathcal{B}$ such that $U \subset U_1 \cap U_2$. Then $U + U \subset V$.

Claim (iii) holds true since multiplication by an element $r \in R^\times$ is a homeomorphism which preserves circled and convex sets.

The remaining claim follows immediately from Proposition 10.2.13 and the observation that a real division algebra is archimedean. □

10.2.33 Corollary *Let $\mathcal{S}$ be a non-empty set of absolutely convex and absorbent subsets of a vector space* E *over a normed real divison algebra R. Then the set*

$$\mathcal{B} = \Big\{ r \bigcap_{B \in \mathcal{F}} B \in \mathcal{P}(\mathrm{E}) \;\Big|\; \mathcal{F} \in \mathcal{P}_{\mathrm{fin}}(\mathcal{S}),\ \mathcal{F} \neq \varnothing \ \&\ r \in R^\times \Big\}$$

consists of absolutely convex and absorbent subsets of V *and is a base of the filter of zero neighborhoods of a locally convex topology $\mathcal{T}$ on* E *uniquely determined by that property. This topology is the coarsest among all vector space topologies for which $\mathcal{S}$ is a set of zero neighborhoods. The topology $\mathcal{T}$ is called the* locally convex topology generated *or* induced *by $\mathcal{S}$.*

Proof. The intersection of finitely many absolutely convex and absorbing sets is non-empty and again absolutely convex and absorbing by Lemma 10.2.12 (i) and Proposition and Definition 10.2.24. By Lemma 10.2.12 (ii) and Lemma 10.2.23, the scalar multiple of an absolutely convex and absorbing set again has these properties whenever the scalar is invertible. Hence each element of $\mathcal{B}$ is absolutely convex and absorbing. Given two elements $C, D \in \mathcal{B}$ there exist non-empty $\mathcal{F}, \mathcal{G} \in \mathcal{P}_{\mathrm{fin}}(\mathcal{S})$ and $r, s \in R^\times$ such that $C = r \bigcap\limits_{B \in \mathcal{F}} B$ and $D = s \bigcap\limits_{B \in \mathcal{G}} B$. Without loss of generality one can assume that $|r| \leqslant |s|$. Then $A = r \bigcap\limits_{B \in \mathcal{F} \cup \mathcal{G}} B \in \mathcal{B}$ and $A = C \cap rs^{-1} D \subset C \cap D$ since D is balanced and $|rs^{-1}| \leqslant 1$. Hence $\mathcal{B}$ is a filter base consisting of absolutely convex and absorbent sets. Moreover, $\frac{1}{2}C + \frac{1}{2}C \subset C$ for every $C \in \mathcal{B}$ by absolut convexity. By Proposition 10.2.32 the filter base $\mathcal{B}$ therefore generates a unique locally convex topology $\mathcal{T}$ for which $\mathcal{B}$ is a base of the filter of zero neighborhoods. Moreover, $\mathcal{T}$ is the coarsest translation invariant topology so that $\mathcal{B}$ is a set of zero neighborhoods. This implies in particular that $\mathcal{S}$ is a set of zero neighborhoods for $\mathcal{T}$. Now let $\mathcal{T}'$ be a vector topology such that each element of $\mathcal{S}$ is a zero neighborhood. Then finite intersections of elements of $\mathcal{S}$ are zero neighborhoods with respect to $\mathcal{T}'$ and therefore also all elements of $\mathcal{B}$. Since $\mathcal{T}'$ is translation invariant one concludes that $\mathcal{T}$ is coarser than $\mathcal{T}'$ and the claim is proved. □

10.3. Seminorms and gauge functionals

10.3.1 Throughout the rest of this chapter the symbol $\mathbb{K}$ will always stand for the field of real numbers $\mathbb{R}$, the field of complex numbers $\mathbb{C}$ or the division algebra of quaternions $\mathbb{H}$. We assume these division algebras to be equipped with their standard absolute values $|\cdot|$. Moreover, vector spaces are assumed to be defined over the ground field $\mathbb{K}$ unless mentioned differently and are always assumed to be left vector spaces.

Seminorms and induced vector space topologies

10.3.2 Definition By a *seminorm* on a vector space E one understands a map $p : \mathrm{E} \to \mathbb{R}$ with the following properties:

(N0) The map p is *positive* that is $p(v) \geqslant 0$ for all $v \in \mathrm{E}$.

(N1) The map p is *absolutely homogeneous* that means

$$p(rv) = |r|\, p(v) \quad \text{for all } v \in \mathrm{E} \text{ and } r \in \mathbb{K}.$$

(N2) The map p is *subadditive* or in other words satisfies the *triangle inequality*

$$p(v + w) \leqslant p(v) + p(w) \quad \text{for all } v, w \in \mathrm{E}.$$

A seminorm is called a *norm* if in addition the following axiom is satisfied:

(N3) For all $v \in \mathrm{E}$ the relation $p(v) = 0$ holds true if and only if $v = 0$.

A vector space E equipped with a norm $\|\cdot\| : \mathrm{E} \to \mathbb{R}_{\geqslant 0}$ is called a *normed vector space.*

10.3.3 Let us introduce some useful further properties a map $p : \mathrm{E} \to \mathbb{R}$ can have. One calls such a map p

(1) *positively homogeneous* if $p(tv) = t\, p(v)$ for all $t \in \mathbb{R}_{>0}$ and all $v \in \mathrm{E}$,

(2) *sublinear* if $p(tv + sw) \leqslant t\, p(v) + s\, p(w)$ for all $t, s \in \mathbb{R}_{\geqslant 0}$ and all $v, w \in \mathrm{E}$, and

(3) *convex* if $p(tv + sw) \leqslant t\, p(v) + s\, p(w)$ for all $t, s \in \mathbb{R}_{\geqslant 0}$ with $t + s = 1$ and all $v, w \in \mathrm{E}$.

10.3.4 Lemma *For a real-valued map $p : \mathrm{E} \to \mathbb{R}$ on a vector space E the following are equivalent:*

(i) *p is sublinear.*

(ii) *p is positively homogeneous and convex.*

(iii) *p is positively homogeneous and subadditive.*

Proof. Let p be sublinear. Then p is subadditive by definition. Subadditivity implies $p(0) \leqslant p(0) + p(0)$, hence $p(0) \geqslant 0$. By sublinearity

$$p(0) = p(0 \cdot 0 + 0 \cdot 0) \leqslant 0 \cdot p(0) + 0 \cdot p(0) = 0 \ ,$$

so $p(0) = 0$. We show that p is positively homogeneous. Applying sublinearity again one checks for $v \in \mathrm{E}$ and $t \geqslant 0$ that

$$p(tv) = p(tv + 0 \cdot 0) \leqslant tp(v) + 0 \cdot p(0) = tp(v) \ ,$$

so p is positively homogeneous and the implication (i) $\implies$ (iii) follows. If p is positively homogeneous and subadditive, then for $v, w \in \mathrm{E}$ and $t, s > 0$ with $t + s = 1$

$$p(tv + sw) \leqslant p(tv) + p(sw) \leqslant tp(v) + sp(w),$$

so p is convex. This gives the implication (iii) $\implies$ (ii). If p is positively homogeneous and convex, then one computes for $v, w \in \mathrm{E}$ and $t, s \geqslant 0$ with $t + s > 0$

$$p(tv + sw) = (t+s)\, p\left(\frac{t}{t+s}v + \frac{s}{t+s}w\right) \leqslant (t+s)\left(\frac{t}{t+s}p(v) + \frac{s}{t+s}p(w)\right) = tp(v) + sp(w) \ .$$

Since $p(0) = \lim\limits_{t \searrow 0} p(t0) = \lim\limits_{t \searrow 0} t\, p(0) = 0$ by positive homogeneity, p then has to be sublinear and one obtains the implication (ii) $\implies$ (i). ☐

10.3.5 Lemma *Let $p : \mathrm{E} \to \mathbb{R}$ be a real-valued map defined on a vector space E over $\mathbb{K}$.*

(i) *If $p : \mathrm{E} \to \mathbb{R}$ is positively homogeneous, then $p(0) = 0$.*

(ii) *If $p : \mathrm{E} \to \mathbb{R}$ is subadditive, then $p(0) \geqslant 0$ and for all $v, w \in \mathrm{E}$*

$$|p(v) - p(w)| \leqslant \max\{p(v - w), p(w - v)\} \ .$$

(iii) *If $p : \mathrm{E} \to \mathbb{R}$ is convex, then the sets $\mathbb{B}_{p,\varepsilon} := \{v \in \mathrm{E} \mid p(v) < \varepsilon\}$ and $\overline{\mathbb{B}}_{p,\varepsilon} := \{v \in \mathrm{E} \mid p(v) \leqslant \varepsilon\}$ are convex for all $\varepsilon > 0$.*

(iv) *If p is sublinear, then $\mathbb{B}_{p,\varepsilon}$ and $\overline{\mathbb{B}}_{p,\varepsilon}$ are convex and absorbent for all $\varepsilon > 0$.*

Proof. *ad (i).* As already observed, $p(0) = \lim\limits_{t \searrow 0} p(t0) = \lim\limits_{t \searrow 0} t\, p(0) = 0$.

ad (ii). Note that by subadditivity

$$p(0) \leqslant p(0) + p(0), \quad p(v) - p(w) \leqslant p(v - w), \quad \text{and} \quad p(w) - p(v) \leqslant p(w - v) \ .$$

This entails (ii).

ad (iii). Let $v, w \in \{v \in \mathrm{E} \mid p(v) < \varepsilon\}$ and $0 \leqslant t \leqslant 1$. Then, by convexity of p,

$$p\big(tv + (1-t)w\big) \leqslant tp(v) + (1-t)p(w) < t\varepsilon + (1-t)\varepsilon = \varepsilon \ .$$

Hence $tv + (1-t)w \in \{v \in \mathrm{E} \mid p(v) < \varepsilon\}$. The proof for $\{v \in \mathrm{E} \mid p(v) \leqslant \varepsilon\}$ is analogous.

ad (iv). Convexity of the sets $\mathbb{B}_{p,\varepsilon}$ and $\overline{\mathbb{B}}_{p,\varepsilon}$ holds by (iii). Moreover, $\mathbb{B}_{p,\varepsilon} \subset \overline{\mathbb{B}}_{p,\varepsilon}$ by definition. Hence it suffices by Lemma 10.2.28 to show that $\mathbb{B}_{p,\varepsilon}$ is absorbent in the realification $\mathrm{E}^{\mathbb{R}}$. Since p is positively homogenous by Lemma 10.3.4 and $0 \leqslant p(v) + p(-v)$ for all $v \in \mathrm{E}$, one concludes that for all $t \in \mathbb{R}$ and $v \in \mathrm{E}$

$$|p(tv)| \leqslant |t| \max\{p(v), p(-v)\} \ .$$

Hence $tv \in \mathbb{B}_{p,\varepsilon}$ if $0 < t < \frac{\varepsilon}{\max\{p(v),p(-v)\}+1}$, and $\mathbb{B}_{p,\varepsilon}$ is absorbent in $\mathrm{E}^{\mathbb{R}}$. □

10.3.6 Definition If $p : \mathrm{E} \to \mathbb{R}$ is a seminorm on a vector space E, we denote for every $v \in \mathrm{E}$ and $\varepsilon > 0$ by $\mathbb{B}_{p,\varepsilon}(v)$ the (*open*) *ε-ball associated with* p and *with center* v that is the set

$$\mathbb{B}_{p,\varepsilon}(v) = \left\{ w \in \mathrm{E} \;\middle|\; p(w - v) < \varepsilon \right\} \ .$$

The *closed ε-ball associated with* p and *with center* v is defined as

$$\overline{\mathbb{B}}_{p,\varepsilon}(v) = \left\{ w \in \mathrm{E} \;\middle|\; p(w - v) \leqslant \varepsilon \right\} \ .$$

The positive number ε is called the *radius* of the ball. In case the center of the ball is the origin, we often write $\mathbb{B}_{p,\varepsilon}$ and $\overline{\mathbb{B}}_{p,\varepsilon}$ for $\mathbb{B}_{p,\varepsilon}(0)$ and $\overline{\mathbb{B}}_{p,\varepsilon}(0)$, respectively. If in addition the radius equals 1, then we usually write only $\mathbb{B}_p$ and $\overline{\mathbb{B}}_p$ and call these sets the *open* respectively the *closed unit ball*. More generally, for the particular radius 1 we denote the corresponding balls by $\mathbb{B}_p(v)$ and $\overline{\mathbb{B}}_p(v)$ and call them the *open* respectively *closed unit balls with center v*. When by the context it is clear which seminorm p a ball is associated with we often do not mention p explicitely. This is in particular the case when the underlying vector space is a normed vector space.

If P is a finite set or a finite family of seminorms on E we define the *open* and *closed ε-multiballs with center v* by

$$\mathbb{B}_{P,\varepsilon}(v) = \left\{ w \in \mathrm{E} \;\middle|\; p(w - v) < \varepsilon \text{ for all } p \in P \right\}$$

and

$$\overline{\mathbb{B}}_{P,\varepsilon}(v) = \left\{ w \in \mathrm{E} \;\middle|\; p(w - v) \leqslant \varepsilon \text{ for all } p \in P \right\} ,$$

respectively. As before, we abbreviate $\mathbb{B}_{P,\varepsilon} = \mathbb{B}_{P,\varepsilon}(0)$ and $\overline{\mathbb{B}}_{P,\varepsilon} = \overline{\mathbb{B}}_{P,\varepsilon}(0)$.

10.3.7 Remark For convenience, we will also use the symbols $\mathbb{B}_{p,\varepsilon}$ and $\overline{\mathbb{B}}_{p,\varepsilon}$ to denote the sets $\left\{ v \in \mathrm{E} \;\middle|\; p(v) < \varepsilon \right\}$ and $\left\{ v \in \mathrm{E} \;\middle|\; p(v) \leqslant \varepsilon \right\}$, respectively, when $p : \mathrm{E} \to \mathbb{R}$ is just a real-valued convex map on the vector space E. Note that for such a p the set $\left\{ v \in \mathrm{E} \;\middle|\; p(v) < 0 \right\}$ might be non-empty. But as we have shown in Lemma 10.3.5 the sets $\mathbb{B}_{p,\varepsilon}$ and $\overline{\mathbb{B}}_{p,\varepsilon}$ associated to a convex p share with the the balls associated to a seminorm several nice properties like convexity.

10.3.8 Proposition *Let* E *be a* $\mathbb{K}$*-vector space, and* P *a finite set of seminorms on* E*. Then, for every* $\varepsilon > 0$ *and* $v \in \mathrm{E}$*, the* ε*-multiballs* $\mathbb{B}_{P,\varepsilon}(v)$ *and* $\overline{\mathbb{B}}_{P,\varepsilon}(v)$ *are convex. The* ε*-multiballs* $\mathbb{B}_{P,\varepsilon}$ *and* $\overline{\mathbb{B}}_{P,\varepsilon}$ *centered at the origin are absolutely convex and absorbent.*

Proof. Axiom (N1) immediately entails that $\mathbb{B}_{P,\varepsilon}$ and $\overline{\mathbb{B}}_{P,\varepsilon}$ are circled. Axiom (N2) together with (N1) entails that the sets $\mathbb{B}_{P,\varepsilon}(v)$ and $\overline{\mathbb{B}}_{P,\varepsilon}(v)$ are convex. Namely, if $w_1, w_2 \in \mathbb{B}_{P,\varepsilon}(v)$ and $t \in [0, 1]$, then one has for all seminorms $p \in P$

$$p\,(tw_1 + (1 - t)w_2 - v) \leqslant t\,p\,(w_1 - v) + (1 - t)\,p\,(w_2 - v) < t\,\varepsilon + (1 - t)\,\varepsilon = \varepsilon$$

and likewise $p\left(tw_1 + (1-t)w_2 - v\right) \leqslant \varepsilon$ for all $w_1, w_2 \in \overline{\mathbb{B}}_{P,\varepsilon}(v)$ and $p \in P$.

Now let $v \in \mathrm{E}$ and $\varepsilon > 0$ be given. Put $t_p = \frac{p(v)+1}{\varepsilon}$ for every $p \in P$ and $t_0 = \max\{t_p \mid p \in P\}$. Then one has for all $t \in \mathbb{K}$ with $|t| \geqslant t_0$ and for all $p \in P$

$$p\left(\frac{1}{t}v\right) \leqslant \frac{\varepsilon}{p(v)+1}\, p(v) < \varepsilon \ ,$$

hence $v \in t\,\mathbb{B}_{P,\varepsilon}$. So $\mathbb{B}_{P,\varepsilon}$ is absorbing. Since $\overline{\mathbb{B}}_{P,\varepsilon}$ contains the absorbing set $\mathbb{B}_{P,\varepsilon}$, it is absorbing as well. □

10.3.9 Proposition and Definition *Assume to be given a set Q of seminorms on a vector space* E. *Let $\mathcal{P}_{\mathrm{fin}}(Q)$ be the collection of all finite subsets of Q. A base of a topology on* E *then is given by*

$$\mathcal{B} = \left\{ \mathbb{B}_{P,\varepsilon}(v) \,\middle|\, P \in \mathcal{P}_{\mathrm{fin}}(Q),\ v \in \mathrm{E},\ \varepsilon > 0 \right\} .$$

The topology $\mathcal{T}$ generated by $\mathcal{B}$ is called the topology generated, induced *or* defined *by Q. Moreover, $\mathcal{T}$ is a locally convex vector space topology on* E. *It coincides with the coarsest translation invariant topology on* E *such that each seminorm in Q is continuous.*

Proof. Consider the set $\mathcal{B}_0$ of all multiballs $\mathbb{B}_{P,\varepsilon}$ with $P \in \mathcal{P}_{\mathrm{fin}}(Q)$ and $\varepsilon > 0$ centered at the origin. Clearly, $\mathcal{B}_0$ is a filter base since for $P_1, P_2 \in \mathcal{P}_{\mathrm{fin}}(Q)$ and $\varepsilon_1, \varepsilon_2 > 0$ the multiball $\mathbb{B}_{P_1 \cup P_2, \min\{\varepsilon_1,\varepsilon_2\}}$ is contained in $\mathbb{B}_{P_1,\varepsilon_1} \cap \mathbb{B}_{P_2,\varepsilon_2}$. Moreover it consists of absolutely convex and absorbing sets by Proposition 10.3.8.

By a similar argument one shows that $\mathcal{B}$ is base of a topology. Let $\mathbb{B}_{P_1,\varepsilon_1}(v_1), \mathbb{B}_{P_2,\varepsilon_2}(v_2) \in \mathcal{B}$ and $v \in \mathbb{B}_{P_1,\varepsilon_1}(v_1) \cap \mathbb{B}_{P_2,\varepsilon_2}(v_2)$. Let ε be the minium of the numbers $\varepsilon_1 - p_1(v - v_1)$ and $\varepsilon_2 - p_2(v - v_2)$, where p_1 runs through the elements of P_1 and p_2 through the ones of P_2. Then $\varepsilon > 0$ and $\mathbb{B}_{P_1 \cup P_2,\varepsilon}(v) \subset \mathbb{B}_{P_1,\varepsilon_1}(v_1) \cap \mathbb{B}_{P_2,\varepsilon_2}(v_2)$, and $\mathcal{B}$ is a base for a topology $\mathcal{T}$ indeed. By construction, $\mathcal{B}_0$ then is a base for the filter of zero neighborhoods and each element of $\mathcal{B}_0$ is open in $\mathcal{T}$. Moreover, each closed multiball $\overline{\mathbb{B}}_{P,\varepsilon}(v)$ is closed in $\mathcal{T}$ since the complement $\mathrm{E}\backslash\overline{\mathbb{B}}_{P,\varepsilon}(v)$ contains with w also the open multiball $\mathbb{B}_{P,\delta}(w)$, where $\delta = \min\{p(v-w) - \varepsilon | p \in P\}$.

We now prove continuity of addition with respect to $\mathcal{T}$. Let $v_1, v_2 \in \mathrm{E}$, $P \in \mathcal{P}_{\mathrm{fin}}(Q)$, and $\varepsilon > 0$. Since the triangle inequality holds for every seminorm in F, one has

$$\mathbb{B}_{P,\frac{\varepsilon}{2}}(v_1) + \mathbb{B}_{P,\frac{\varepsilon}{2}}(v_2) \subset \mathbb{B}_{P,\varepsilon}(v_1 + v_2) \ ,$$

which entails continuity of addition at each $(v_1, v_2) \in \mathrm{E} \times \mathrm{E}$. Next consider multiplication by scalars and let $\lambda \in \mathbb{K}$ and $v \in \mathrm{E}$. Again let $P = \{p_1, \ldots, p_n\} \in \mathcal{P}_{\mathrm{fin}}(Q)$ and $\varepsilon > 0$. Let $C_1 = \sup\{p_j(v) \mid 1 \leqslant j \leqslant n\} + 1$, $C_2 = |\lambda| + 1$ and put $\delta_1 = \min\{1, \frac{\varepsilon}{2\,C_1}\}$ and $\delta_2 = \frac{\varepsilon}{2\,C_2}$. Then one obtains by absolute homogeneity and subadditivity of each seminorm

$$p_j(\mu w - \lambda v) \leqslant |\mu|\, p_j(w - v) + |\mu - \lambda|\, p_j(v) \quad \text{for all } \mu \in \mathbb{K} \text{ and } w \in \mathrm{E},$$

hence

$$\mathbb{B}_{\delta_1}(\lambda) \cdot \mathbb{B}_{P,\delta_2}(v) \subset \mathbb{B}_{P,\varepsilon}(\lambda \cdot v) \ ,$$

where $\mathbb{B}_{\delta_1}(\lambda) = \{\mu \in \mathbb{K} \mid |\mu - \lambda| < \delta_1\}$. This shows continuity of scalar multiplication at each $(\lambda, v) \in \mathbb{K} \times \mathrm{E}$, and $\mathcal{T}$ is a vector space topology.

Since each of the base elements $\mathbb{B}_{P,\varepsilon} \in \mathcal{B}_0$ is convex, Axiom LCVS holds true as well and the topology $\mathcal{T}$ is locally convex.

Every seminorm $p \in Q$ is continuous with respect to the topology $\mathcal{T}$ since for all $a < b$ the preimage $p^{-1}((a,b)) = \mathbb{B}_{p,b} \setminus \overline{\mathbb{B}}_{p,a}$ is open in $\mathcal{T}$. Now let $\mathcal{T}'$ be a translation invariant topology on E for which every seminorm $p \in Q$ is continuous. In that topology $\mathcal{B}_0$ is a set of zero neighborhoods. As shown before, every element $B \in \mathcal{B}_0$ is absolutely convex, absorbing and satisfies $\frac{1}{2}B + \frac{1}{2}B \subset B$. Hence by Proposition and Definition 10.2.32 the topology $\mathcal{T}'$ is finer than the locally convex topology generated by $\mathcal{B}_0$. But the latter topology coincides with $\mathcal{T}$ by construction. This shows the last part of the claim and the proof is finished. □

Gauge functionals and induced seminorms

10.3.10 As we have seen, any vector space with a topology defined by a family of seminorms on it is a locally convex topological vector space. The converse also holds true. The fundamental notion needed for the proof of this is the following.

10.3.11 Definition Let E be a vector space and $A \subset \mathrm{E}$ absorbent. Then the map

$$p_A : \mathrm{E} \to \mathbb{R}_{\geqslant 0},\ v \mapsto p_A(v) = \inf \left\{ t \in \mathbb{R}_{>0} \;\middle|\; v \in tA \right\}$$

is called the *gauge functional*, the *Minkowski functional* or the *Minkowski gauge* of A.

10.3.12 Remark By definition of an absorbent set, $\left\{ t \in \mathbb{R}_{>0} \;\middle|\; v \in tA \right\}$ is non-empty whenever $A \subset \mathrm{E}$ is absorbent. Hence p_A is well-defined for such A.

10.3.13 Proposition *The Minkowski gauge $p_A : \mathrm{E} \to \mathbb{R}_{\geqslant 0}$ of an absorbent subset A of a vector space* E *has the following properties.*

(i) *The gauge functional is positively homogeneous that is $p_A(tv) = t\, p_A(v)$ for all $t \in \mathbb{R}_{>0}$ and all $v \in \mathrm{E}$.*

(ii) *If A is convex, then p_A is subadditive and*

$$\mathbb{B}_p(v) = \bigcup_{0<t<1} tA \subset A \subset \bigcap_{1<t} tA = \overline{\mathbb{B}}_p(v)\ .$$

(iii) *If A is absolutely convex, then p_A is a seminorm on* E.

Proof. If $t > 0$, then $tv \in sA$ for some $s > 0$ if and only if $v \in \frac{s}{t}A$. Hence $\left\{ s \in \mathbb{R}_{>0} \;\middle|\; tv \in sA \right\}$ and $t\left\{ s \in \mathbb{R}_{>0} \;\middle|\; v \in sA \right\}$ coincide for all $t > 0$, so (i) follows.

Assume that A is convex. Let $v, w \in \mathrm{E}$ and $\varepsilon > 0$. Then there exist $t > p_A(v)$ and $s > p_A(w)$ such that $v \in tA$, $w \in sA$, $t < p_A(v) + \frac{\varepsilon}{2}$ and $s < p_A(w) + \frac{\varepsilon}{2}$. By convexity of A and Lemma 10.2.27, $v + w \in tA + sA = (t+s)A$. Hence $p_A(v+w) \leqslant (t+s) < p_A(v) + p_A(w) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $p_A(v+w) \leqslant p_A(v) + p_A(w)$ and p_A is subadditive. If $v \in tA$ for some t with $0 < t < 1$, then $p_A(v) \leqslant t < 1$ by definition. Conversely, if $p_A(v) < 1$, then there exists a $t > 0$ such that $t < 1$ and $v \in tA$. Hence the equality $\mathbb{B}_p(v) = \bigcup_{0<t<1} tA$ follows. Since A is absorbing, 0 is an element of A. By convexity of A one therefore concludes $tA = (1-t)\{0\} + tA \subset A$ whenever $0 <$

$t < 1$. For $t > 1$ this shows $\frac{1}{t}A \subset A$, hence $A \subset tA$. So the relation $\bigcup_{0<t<1} tA \subset A \subset \bigcap_{1<t} tA$ is proved. Now assume that $v \in tA$ for all $t > 1$. Then $p_A(v) \leqslant 1$ by definition. If conversely $p_A(v) \leqslant 1$, then there exists for each $\varepsilon > 0$ an $s \geqslant 0$ such that $p_A(v) \leqslant s$, $v \in sA$ and $s < 1 + \varepsilon$. Hence, for $t \geqslant 1 + \varepsilon$ by Lemma 10.2.27 and $0 \in A$,

$$v \in sA = sA + (t - s)\{0\} \subset sA + (t - s)A = tA \ .$$

Since $\varepsilon > 0$ was arbitrary, $v \in tA$ for all $t > 1$ follows. So one obtains the equality $\bigcap_{1<t} tA = \overline{\mathbb{B}}_p(v)$, and (ii) is proved.

To verify (iii) recall that A is circled whenever A is absolutely convex. This entails for $r \in \mathbb{K}$, $v \in \mathrm{E}$ and absolutely convex A

$$p_A(rv) = \inf\left\{t \in \mathbb{R}_{>0} \;\middle|\; rv \in tA\right\} = \inf\left\{t \in \mathbb{R}_{>0} \;\middle|\; |r|v \in tA\right\} = p_A(|r|v) = |r|p_A(v) \ ,$$

where for the last equality we have used (i). □

10.3.14 Lemma *Let A and B be absorbent subsets of a vector space* E*. Then the following holds true.*

(i) $p_{tA}(v) = p_A(t^{-1}v)$ *for all* $t \in \mathbb{K}^\times$ *and* $v \in \mathrm{E}$.

(ii) *If* $B \subset A$*, then* $p_A \leqslant p_B$.

(iii) *If A is convex, then* $v \in tA$ *for all* $v \in \mathrm{E}$ *and* $t > p_A(v)$.

(iv) *If A and B are convex, then the intersection $A \cap B$ is absorbent and convex and* $p_{A\cap B} = \sup\{p_A, p_B\}$*, where* $\sup\{p_A, p_B\}(v) = \sup\{p_A(v), p_B(v)\}$ *for all* $v \in \mathrm{E}$.

Proof. ad (i). If $t \in \mathbb{K}$ is invertible, then $v \in tA$ if and only if $t^{-1}v \in A$.

ad (ii). Let $v \in \mathrm{E}$ and $\varepsilon > 0$. Then there exists t with $p_B(v) \leqslant t < p_B(v) + \varepsilon$ such that $v \in tB$. By $B \subset A$ this implies $v \in tA$, hence $p_A(v) \leqslant t < p_B(v) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, the estimate $p_A \leqslant p_B$ follows.

ad (iii). By definition of the Minkowski gauge there exists $s \in \mathbb{R}$ such that $p_A(v) < s < t$ and $v \in sA$. By convexity of A one concludes $\frac{s}{t}v = \frac{s}{t}v + \left(1 - \frac{s}{t}\right) \cdot 0 \in sA$, hence $v \in tA$.

ad (iv). The intersection of convex sets is convex, so $A \cap B$ is convex. Let $v \in \mathrm{E}$ and choose $r_A \geqslant 0$ and $r_B \geqslant 0$ such that $v \in tA$ for all $t \geqslant r_A$ and $v \in sB$ for all $s \geqslant r_B$. Then $v \in (tA) \cap (tB) = t(A \cap B)$ for all $t \geqslant \max\{r_A, r_B\}$, so $A \cap B$ is absorbent. One has $p_{A\cap B} \geqslant \sup\{p_A, p_B\}$ by (ii). To show the converse inequality assume that $v \in \mathrm{E}$ and $t > \sup\{p_A(v), p_B(v)\}$. Then $v \in tA \cap tB = t(A \cap B)$, which implies $p_{A\cap B}(v) \leqslant t$. Hence $p_{A\cap B}(v) \leqslant \sup\{p_A(v), p_B(v)\}$ since $t > \sup\{p_A(v), p_B(v)\}$ was arbitrary. □

10.3.15 Lemma *Let* $p : \mathrm{E} \to \mathbb{R}$ *be a sublinear map on a vector space* E *and* $A \subset \mathrm{E}$ *convex. If*

$$\mathbb{B}_p \subset A \subset \overline{\mathbb{B}}_p \ ,$$

then the gauge functional p_A *coincides with* $\sup\{p, 0\}$*. If p is even a seminorm, then* $p = p_A$*.*

Proof. Let $p : \mathrm{E} \to \mathbb{R}$ be sublinear. Observe that then $\mathbb{B}_p$ is absorbent by Lemma 10.3.5 (iv). Hence A must also be absorbent by assumption, so the associated Minkowski gauge p_A is positively homogeneous by Proposition 10.3.13 (i).

Assume now that there exists $v \in \mathrm{E}$ such that $\max\{p(v), 0\} < p_A(v)$. By positive homogeneity of p and p_A one can achive by possibly multiplying v by a positive real number that $\max\{p(v), 0\} < 1 < p_A(x)$. The first inequality entails $v \in \mathbb{B}_p$, the second $v \notin \overline{\mathbb{B}}_p$ which is a contradiction. Next assume that there exists $v \in \mathrm{E}$ with $p_A(v) < \max\{p(v), 0\}$. As before one can then achieve that $p_A(v) < 1 < \max\{p(v), 0\}$ for some $v \in \mathrm{E}$. By the first inequality one concludes $v \in A$, by the second $v \notin A$. This is a contradiction. So the equality $\max\{p(v), 0\} = p_A(v)$ holds for all $v \in \mathrm{E}$.

In case p is a seminorm, then $p(v) \geqslant 0$ for all $v \in \mathrm{E}$ and the second claim follows by the first. $\square$

10.3.16 Proposition *Let* E *be a topological vector space, and* $p : \mathrm{E} \to \mathbb{R}$ *be sublinear. Then the following are equivalent.*

(i) *The map p is continuous in the origin.*

(ii) *The map p is uniformly continuous.*

(iii) *The map p is continuous.*

(iv) *The unit ball* $\mathbb{B}_p$ *is a zero neighborhood.*

Proof. Let us first show (i) $\Longrightarrow$ (ii). To this end fix $\varepsilon > 0$. By assumption there exists a zero neighborhood $V \subset \mathrm{E}$ such that $|p(v)| < \varepsilon$ for all $v \in V$. By possibly passing to $V \cap (-V)$ one can assume that V is symmetric. Lemma 10.3.5 (ii) now implies

$$|p(v) - p(w)| < \varepsilon \quad \text{for all } v, w \in V \ .$$

Hence p is uniformly continuous. The implications (ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (iv) are trivial. It remains to prove (iv) $\Longrightarrow$ (i). Assume that $\mathbb{B}_p(0,1)$ is a zero neighborhood. Then there exists a symmetric zero neighborhood V contained in $\mathbb{B}_p(0,1)$. Since $p(0) = 0$ one concludes by Lemma 10.3.5 (ii)

$$|p(v)| < \max\{p(v), p(-v)\} < 1 \quad \text{for all } v \in V \ .$$

But this implies $|p(v)| < \varepsilon$ for all $v \in \varepsilon V$ and $\varepsilon > 0$, so p is continuous at the origin. $\square$

Normability

10.3.17 Definition A topological vector space E is called *seminormable* if its topology is generated by a single seminorm $p : \mathrm{E} \to \mathbb{R}_{\geqslant 0}$. If the topology on E coincides with the vector space topology generated by a norm $\|\cdot\|$, then one calls E *normable.*

10.3.18 Theorem (Kolmogorov's normability criterion) *A topological vector space* E *is normable if and only if it is a* **??** *space and possesses a bounded convex neighborhood of the origin.*

10.4. Function spaces and their topologies

10.4.1 Proposition *Let X be a topological space and (Y,d) a metric space. Then the following holds true.*

(i) *The space*

$$\mathcal{B}(X,Y) = \big\{ f : X \to Y \;\big|\; \exists y_0 \in Y \, \exists C > 0 \, \forall x \in X : \, d\big(f(x), y_0\big) \leqslant C \big\}$$

of bounded functions from X to Y is a metric space with metric

$$\varrho : \mathcal{B}(X,Y) \times \mathcal{B}(X,Y) \to \mathbb{R}_{\geqslant 0}, \; (f,g) \mapsto \sup_{x \in X} d\big(f(x), g(x)\big) \; .$$

(ii) *If (Y,d) is complete, then $(\mathcal{B}(X,Y), \varrho)$ is so, too.*

(iii) *The space*

$$\mathcal{C}_{\mathrm{b}}(X,Y) = \mathcal{C}(X,Y) \cap \mathcal{B}(X,Y)$$

of continuous bounded functions from X to Y is a closed subspace of $\mathcal{B}(X,Y)$.

Proof. Note first that by the triangle inequality there exists for every $f \in \mathcal{B}(X,Y)$ and $y \in Y$ a real number $C_{f,y} > 0$ such that

$$d\big(f(x), y\big) \leqslant C_{f,x} \quad \text{for all } x \in X \; .$$

ad (i). Before verifying the axioms of a metric for ϱ we need to show that ϱ is well-defined meaning that $\sup_{x \in X} d\big(f(x), g(x)\big) < \infty$ for all $f, g \in \mathcal{B}(X,Y)$. To this end fix some $y \in Y$ and observe using the triangle inequality that

$$d\big(f(x), g(x)\big) \leqslant d\big(f(x), y\big) + d\big(y, g(x)\big) \leqslant C_{f,y} + C_{g,y} \quad \text{for all } x \in X \; .$$

Since furthermore $d\big(f(x), g(x)\big) \geqslant 0$ for all $x \in X$, the map ϱ is well-defined indeed with image in $\mathbb{R}_{\geqslant 0}$. If $\varrho(f,g) = 0$, then $d\big(f(x), g(x)\big) = 0$ for all $x \in X$, hence $f = g$. Obviously, ϱ is symmetric since d is symmetric. Finally, let $f, g, h \in \mathcal{B}(X,Y)$ and check using the triangle inequality for d:

$$\begin{aligned} \varrho(f,g) &= \sup_{x \in X} d\big(f(x), g(x)\big) \leqslant \sup_{x \in X} \Big(d\big(f(x), h(x)\big) + d\big(h(x), g(x)\big) \Big) \leqslant \\ &\leqslant \sup_{x \in X} d\big(f(x), h(x)\big) + \sup_{x \in X} d\big(h(x), g(x)\big) = d(f,h) + d(h,g) \; . \end{aligned}$$

Hence ϱ is a metric.

ad (ii). Assume (Y,d) to be complete and let $(f_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{B}(X,Y)$. Let $\varepsilon > 0$ and choose $N_\varepsilon \in \mathbb{N}$ so that

$$\varrho(f_n, f_m) < \varepsilon \quad \text{for all } n, m \geqslant N \; .$$

Then for every $x \in X$ the relation

$$d\big(f_n(x), f_m(x)\big) < \varepsilon \quad \text{for all } n, m \geqslant N_\varepsilon \tag{10.4.1}$$

holds true, so $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in Y. By completeness of (Y,d) it has a limit which we denote by $f(x)$. By passing to the limit $m \to \infty$ in (10.4.1) one obtains that

$$d\big(f(x), f_n(x)\big) \leqslant \varepsilon \quad \text{for all } x \in X \text{ and } n \geqslant N_\varepsilon \ . \tag{10.4.2}$$

Using the triangle inequality one infers from this for an element $y \in Y$ which we now fix that

$$d\big(f(x), y)\big) \leqslant d\big(f(x), f_{N_1}(x))\big) + d\big(f_{N_1}(x), y\big) \leqslant 1 + C_{f_{N_1},y} \ .$$

Hence f is a bounded function. Moreover, (10.4.2) entails that

$$\varrho(f, f_n) = \sup_{x\in X} d\big(f(x), f_n(x)\big) \leqslant \varepsilon \quad \text{for all } n \geqslant N_\varepsilon \ ,$$

so $(f_n)_{n\in\mathbb{N}}$ converges to f.

ad (iii). We have to show that the limit f of a sequence $(f_n)_{n\in}$ of functions $f_n \in \mathcal{C}_\mathrm{b}(X,Y)$ which converges in $(\mathcal{B}(X,Y), \varrho)$ has to be continuous. To this end let $\varepsilon > 0$ and choose $N_\varepsilon \in \mathbb{N}$ so that

$$\varrho(f_n, f) < \frac{\varepsilon}{3} \quad \text{for all } n \geqslant N_\varepsilon \ .$$

Let $x_0 \in X$. By continuity of f_{N_ε} there exists a neighborhood $U \subset X$ of x so that

$$d\big(f_{N_\varepsilon}(x), f_{N_\varepsilon}(x_0)\big) < \frac{\varepsilon}{3} \quad \text{for all } x \in U \ .$$

By the triangle inequality one concludes that

$$d\big(f(x), f(x_0)\big) \leqslant d\big(f(x), f_{N_\varepsilon}(x)\big) + d\big(f_{N_\varepsilon}(x), f_{N_\varepsilon}(x_0)\big) + d\big(f_{N_\varepsilon}(x_0), f(x_0)\big) < \varepsilon$$

for all $x \in U$. Hence f is continuous at x_0. Since $x_0 \in X$ was arbitrary f, is a continuous map, hence an elemnt of $\mathcal{C}_\mathrm{b}(X,Y)$.

□

10.4.2 Proposition *Let X be a topological space and $\mathbb{K}$ the division algebra of real or complex numbers or of quaternions. Then the following holds true.*

(i) *The space $\mathcal{B}(X,\mathbb{K})$ of bounded $\mathbb{K}$-valued functions on X can be expressed as*

$$\mathcal{B}(X,\mathbb{K}) = \big\{ f : X \to \mathbb{K} \ \big| \ \exists C > 0 \, \forall x \in X : \ |f(x)| \leqslant C \big\} \ . \tag{10.4.3}$$

It carries the structure of a $\mathbb{K}$-algebra by pointwise addition and multiplication of functions and becomes a Banach algebra when equipped with the supremums-norm

$$\|\cdot\|_\infty : \ \mathcal{B}(X,\mathbb{K}) \to \mathbb{K}, \quad f \mapsto \sup_{x\in X} |f(x)| \ .$$

(ii) *The subspace $\mathcal{C}_\mathrm{b}(X,\mathbb{K}) \subset \mathcal{B}(X,\mathbb{K})$ of bounded continuous $\mathbb{K}$-valued functions on X is a closed subalgebra of $\big(\mathcal{B}(X,\mathbb{K}), \|\cdot\|_\infty\big)$, so a Banach algebra as well when endowed with the supremums-norm. For X compact this means in particular that the algebra $\big(\mathcal{C}(X,\mathbb{K}), \|\cdot\|_\infty\big)$ is a Banach algebra.*

Proof. Eq. (10.4.3) is obvious since the distance of two elements $a, b \in \mathbb{K}$ is given by $d(a,b) = |a-b|$, so in particular $d(a,0) = |a|$. Let $f, g \in \mathcal{B}(X,\mathbb{K})$ and choose $C_f, C_g \geqslant 0$ so that $|f(x)| \leqslant C_f$ and $|g(x)| \leqslant C_g$ for all $x \in X$. Then, by the triangle inequality and absolute homogeneity of the absolute value,

$$|f(x) + g(x)| \leqslant C_f + C_g, \quad |a\, f(x)| \leqslant |a|\, C_f, \quad \text{and} \quad |f(x) \cdot g(x)| \leqslant C_f \cdot C_g \ .$$

Hence the sum and the product of two bounded functions are bounded and so is any scalar multiple of a bounded function. Therefore, $\mathcal{B}(X,\mathbb{K})$ is an algebra over $\mathbb{K}$. Using the triangle inequality and absolute homogeneity of the absolute value again one verifies that $\|f\|_\infty$ is a norm on $\mathcal{B}(X,\mathbb{K})$ indeed and that it fulfills $\|fg\|_\infty \leqslant \|f\|_\infty \cdot \|g\|_\infty$ for all $f, g \in \mathcal{B}(X,\mathbb{K})$. Furthermore, by definition, $\|f\|_\infty = \varrho(f,0)$ for all $f \in \mathcal{B}(X,\mathbb{K})$, where ϱ is defined as in Proposition 10.4.1. Since $(\mathcal{B}(X,\mathbb{K}),\varrho)$ is a complete metric space, $(\mathcal{B}(X,\mathbb{K}),\|\cdot\|_\infty)$ therefore is a Banach algebra. This proves the first claim.

For the second observe that for $f, g \in \mathcal{C}_\mathrm{b}(X,\mathbb{K})$ and $a \in \mathbb{K}$ the sum $f+g$, the scalar multiple af, and the product $f \cdot g$ are elements of $\mathcal{C}_\mathrm{b}(X,\mathbb{K})$ again. To verify this let $x \in X$ and $\varepsilon > 0$. Choose neighborhoods U_1 and U_2 of x so that

$$|f(y) - f(x)| < \min\left\{\frac{\varepsilon}{2}, \frac{\varepsilon}{|a|+1}, \frac{\varepsilon}{2(|g(x)|+1)}\right\} \quad \text{for } y \in U_1$$

and

$$|g(y) - g(x)| < \left\{1, \frac{\varepsilon}{2}, \frac{\varepsilon}{2(|f(x)|+1)}\right\} \quad \text{for } y \in U_2 \ .$$

Then for all $y \in U_1 \cap U_2$

$$\begin{aligned}
|(f+g)(y) - (f+g)(x)| &\leqslant |f(y) - f(x)| + |g(y) - g(x)| < \varepsilon \ , \\
|(af)(y) - (af)(x)| &\leqslant |a| \cdot |f(y) - f(x)| < \varepsilon \ , \\
|(f \cdot g)(y) - (f \cdot g)(x)| &\leqslant |g(y)| \cdot |f(y) - f(x)| + |f(x)| \cdot |(g(y) - g(x)| < \varepsilon \ .
\end{aligned}$$

This means that $f+g$, af and fg are continuous in x, hence elements of $\mathcal{C}_\mathrm{b}(X,\mathbb{K})$ since $x \in X$ was arbitrary. So $\mathcal{C}_\mathrm{b}(X,\mathbb{K})$ is a subalgebra of $\mathcal{B}(X,\mathbb{K})$. By Proposition 10.4.1 one knows that $\mathcal{C}_\mathrm{b}(X,\mathbb{K})$ is a closed subspace of $\mathcal{B}(X,\mathbb{K})$. The rest of the claim is obvious. □

10.4.3 As the next step, we introduce seminorms and their topologies on spaces of differentiable functions defined over an open set $\Omega \subset \mathbb{R}^n$. We agree that from now on Ω will always denote in this section an open subset of $\mathbb{R}^n$. For any differentiability order $m \in \mathbb{N} \cup \{\infty\}$ the symbol $\mathcal{C}^m(\Omega)$ stands for the space of m-times continuously differentiable complex valued functions on Ω. For $i = 1, \ldots, n$ we denote by $x^i : \mathbb{R}^n \to \mathbb{R}$ the i-th coordinate function and, if $m \geqslant 1$, by $\partial_i : \mathcal{C}^m(\Omega) \to \mathcal{C}^{m-1}(\Omega)$ the operator which maps $f \in \mathcal{C}^m(\Omega)$ to the partial derivative $\frac{\partial f}{\partial x^i}$. More generally, if $\alpha \in \mathbb{N}^n$ is a multiindex satisfying $|\alpha| = \alpha_1 + \ldots \alpha_n \leqslant m$, then we write $\partial^\alpha : \mathcal{C}^m(\Omega) \to \mathcal{C}^{m-|\alpha|}(\Omega)$ for the higher order partial derivative which maps $f \in \mathcal{C}^m(\Omega)$ to $\frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdot \ldots \cdot \partial x_n^{\alpha_n}}$. Recall that the sum and the product of two m-times differentiable functions and scalar multiples of m-times differentiable functions are again m-times differentiable, hence $\mathcal{C}^m(\Omega)$ forms a $\mathbb{C}$-algebra. Now we define $\overline{\mathcal{C}}^m(\Omega)$ to be the space of continuous functions on the closure $\overline{\Omega}$ which are m-times continuosly differentiable on Ω so that each of its partial derivatives of order

$\leqslant m$ has a continuos extension to $\bar{\Omega}$. Since the operators ∂_i are linear and also derivations by the Leibniz rule, $\bar{\mathcal{C}}^m(\Omega)$ is a subalgebra of $\mathcal{C}^m(\Omega)$. In general, these algebras do not coincide as for example the function $\frac{1}{x}$ on $\mathbb{R}_{>0}$ shows. It is an element of $\mathcal{C}^\infty(\mathbb{R}_{>0})$ but can not be extended to a continuous function on $\mathbb{R}_{\geqslant 0}$, so is not an element of $\bar{\mathcal{C}}^\infty(\mathbb{R}_{>0})$.

If $X \subset \mathbb{R}^n$ is locally closed which means that X is the intersection of an open and a closed susbet of $\mathbb{R}^n$, then define $\mathcal{C}^m(X)$ as the quotient space $\mathcal{C}^m(\Omega)/\mathcal{J}_X(\Omega)$, where $\Omega \subset \mathbb{R}^n$ open is chosen so that $X = \bar{X} \cap \Omega$ and where $\mathcal{J}_X$ denotes the ideal sheaf of all m-times continuously differentiable functions vanishing on X that is

$$\mathcal{J}_X(\Omega) = \left\{ f \in \mathcal{C}^m(\Omega) \;\middle|\; f|_X = 0 \right\} .$$

Using a smooth partition of unity type of argument one shows that $\mathcal{C}^m(X)$ does not depend on the particular choice of the neighborhood Ω in which X is relatively closed and that $\mathcal{C}^m(X)$ can be naturally identified with the space of continuous functions on X which have an extension to an element of $\mathcal{C}^m(\Omega)$.

10.4.4 Proposition *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $m \in \mathbb{N}_{>0}$. Then $\bar{\mathcal{C}}^m(\Omega)$ equipped with the norm*

$$\| \cdot \|_{\Omega,m} : \bar{\mathcal{C}}^m(\Omega) \to \mathbb{R}_{\geqslant 0}, \quad f \mapsto$$

10.5. Summability

10.5.1 Definition Assume to be given a locally convex topological vector space V over the field $\mathbb{K}$ of real or complex numbers. Let $(v_i)_{i \in I}$ be a family of elements of V. Let $\mathcal{F}(I)$ be the set of finite subsets of I and note that it is filtered by set-theoretic inclusion. The family $(v_i)_{i \in I}$ then gives rise to the net $\left(\sum_{i \in J} v_i \right)_{J \in \mathcal{F}(I)}$. One calls the family $(v_i)_{i \in I}$ *summable* to an element $v \in \mathrm{V}$ if the net $\left(\sum_{i \in J} v_i \right)_{J \in \mathcal{F}(I)}$ converges to v. In other words this means that for every convex zero neighborhood $U \subset \mathrm{V}$ and $\varepsilon > 0$ there exists an element $J_{U,\varepsilon} \in \mathcal{F}(I)$ such that for all finite sets J with $J_{U,\varepsilon} \subset J \subset I$

$$p_U \left(v - \sum_{i \in J} v_i \right) < \varepsilon .$$

As before, p_U denotes here the gauge of U. If V is Hausdorff, the limit v of a summable family $(v_i)_{i \in I}$ is uniquely determined, and one writes in this situation

$$v = \sum_{i \in I} v_i .$$

We denote the space of summable families in V over the given index set I by $\ell^1(I, \mathrm{V})$. For $E = \mathbb{C}$ we just write $\ell^1(I)$ instead of $\ell^1(I, \mathbb{C})$. If in addition the index set coincides with $\mathbb{N}$, we briefly denote $\ell^1(\mathbb{N})$ by ℓ^1.

10.5.2 Proposition (Cauchy criterion for summability) *Let V be a complete locally convex topological vector space. A family $(v_i)_{i \in I}$ of elements of V then is summable to some $v \in \mathrm{V}$ if and only if it satisfies the following Cauchy condition:*

(C) *For every convex zero neighborhood $U \subset \mathrm{V}$ and $\varepsilon > 0$ there exists an element $J_{U,\varepsilon} \in \mathcal{F}(I)$ such that for all $K \in \mathcal{F}(I)$ with $K \cap J_{U,\varepsilon} = \emptyset$ the relation*

$$p_U\left(\sum_{i\in K} v_i\right) < \varepsilon$$

holds true.

Proof. By completeness of V it suffices to verify that the net $\left(\sum_{i\in J} v_i\right)_{J\in\mathcal{F}(I)}$ is a Cauchy net if and only if condition (C) is satisfied. Recall that one calls $\left(\sum_{i\in J} v_i\right)_{J\in\mathcal{F}(I)}$ a Cauchy net if for every convex zero neighborhood $U \subset \mathrm{V}$ all $\varepsilon > 0$ there exists an element $J_{U,\varepsilon} \in \mathcal{F}(I)$ such that for all $J, J' \in \mathcal{F}(I)$ containing $J_{U,\varepsilon}$ as a subset the relation

$$p_U\left(\sum_{i\in J} v_i - \sum_{i\in J'} v_i\right) < \varepsilon$$

holds true. But that is clearly equivalent to condition (C). □

10.5.3 Several other notions of summability have been introduced in the analysis and functional analysis literature. These are mainly either used to establish summability criteria or are used in the study of topological tensor products and nuclearity of locally convex topological vector spaces, see Grothendieck (1955); Pietsch (1972). In the following we define these further notions of summability and study their properties. The symbol V hereby always stands for a locally convex tvs, I always denotes a nonempty index set, and $\mathcal{F}(I)$ the set of its finite subsets.

10.5.4 Definition A family $(v_i)_{i\in I}$ in V is called *weakly summable* to $v \in \mathrm{V}$ if for every continuous linear form $\alpha : \mathrm{V} \to \mathbb{K}$ the net $\left(\sum_{i\in J} \alpha(v_i)\right)_{J\in\mathcal{F}(I)}$ converges in $\mathbb{K}$ to $\alpha(v)$. In other words this means that for every $\alpha \in \mathrm{V}'$ and $\varepsilon > 0$ there exists a finite set $J_{\alpha,\varepsilon} \subset I$ such that for all finite sets J with $J_{\alpha,\varepsilon} \subset J \subset I$

$$\left|\alpha(v) - \sum_{j\in J} \alpha(v_i)\right| < \varepsilon \ .$$

The set of all weakly summable families in V with index set I is denoted $\ell^1[I, \mathrm{V}]$.

10.5.5 Definition A family $(v_i)_{i\in I}$ in V is called *absolutely summable* if for every circled convex zero neighborhood $U \subset \mathrm{V}$ there exists some $C \geqslant 0$ such that

$$\sum_{i\in J} p_U(v_i) \leqslant C \quad \text{for all } J \in \mathcal{F}(I) \ .$$

We denote the set of all absolutely summable families in V by $\ell^1\{I, \mathrm{V}\}$.

10.5.6 Proposition *A family $(v_i)_{i\in I} \subset \mathrm{V}$ is absolutely summable if and only if for every element U of a basis of circled convex zero neighborhoods there exists a $C \geqslant 0$ such that*

$$\sum_{i\in J} p_U(v_i) \leqslant C \quad \textit{for all } J \in \mathcal{F}(I) \ .$$

Proof.

10.5.7 Definition A family $(v_i)_{i\in I}$ in V is called *totally summable* if there exists a bounded absolutely convex subset $B \subset \mathrm{V}$ and a $C \geqslant 0$ such that

$$\sum_{i\in J} p_B(v_i) \leqslant C \quad \text{for all } J \in \mathcal{F}(I) .$$

We write $\ell^1\langle I, \mathrm{V}\rangle$ for the set of all totally summable families in V.

Summable families of complex numbers

10.5.8 Lemma (cf. (Pietsch, 1972, Lem. 1.1.2)) *Let $(z_i)_{i\in I}$ be a family of complex numbers for which there exists a positive real number $C > 0$ such that*

$$\left|\sum_{i\in J} z_i\right| \leqslant C \quad \textit{for all } J \in \mathcal{F}(I) .$$

Then one has the estimate

$$\sum_{i\in J} |z_i| \leqslant 4C \quad \textit{for all } J \in \mathcal{F}(I) .$$

Proof. We assume first that all z_i are real. Then let I^+ the set of all indices $i \in I$ such that $z_i \geqslant 0$, and I^- the set of all $i \in I$ such that $z_i < 0$. Then, for all finite $J \subset I$

$$\sum_{i\in J} |z_i| = \sum_{i\in J\cap I^+} |z_i| + \sum_{i\in J\cap I^-} |z_i| = \left|\sum_{i\in J\cap I^+} z_i\right| + \left|\sum_{i\in J\cap I^-} z_i\right| \leqslant 2C .$$

In the general case decompose z_i into real and imaginary parts $x_i = \mathfrak{Re} z_i$ and $y_i = \mathfrak{Im} z_i$. By the triangle inequality one obtains for all finite $J \subset I$

$$\sum_{i\in J} |z_i| \leqslant \sum_{i\in J} |x_i| + \sum_{i\in J} |y_i| \leqslant 4C .$$

□

10.5.9 Proposition *For a family $(z_i)_{i\in I}$ of complex numbers the following are equivalent.*

(i) *The family $(z_i)_{i\in I}$ is summable.*

(ii) *The family $(|z_i|)_{i\in I}$ is summable.*

(iii) *The family $(z_i)_{i\in I}$ is absolutely summable.*

(iv) *There exists some $C > 0$ such that $\sum_{i\in J} |z_i| \leqslant C$ for all $J \in \mathcal{F}(I)$.*

In case that one hence all of the conditions are fulfilled, the estimate

$$\left|\sum_{i\in I} z_i\right| \leqslant \sum_{i\in I} |z_i|$$

holds true.

Proof. Assume that $(z_i)_{i\in I}$ is absolutely summable. Since $\mathbb{C}$ is normed with norm given by the absolut value this just means that there exists some $C > 0$ such that $\sum_{i\in J} |z_i| \leqslant C$ for all $J \in \mathcal{F}(I)$. Hence the supremum $c = \sup \{\sum_{i\in J} |z_i| \mid J \in \mathcal{F}(I)\}$ exists and is $\leqslant C$. For given $\varepsilon > 0$ choose $J_\varepsilon \in \mathcal{F}(I)$ such that

$$c - \varepsilon \leqslant \sum_{i\in J_\varepsilon} |z_i| \leqslant c \ .$$

Then one has for all $K \in \mathcal{F}(I)$ with $K \cap J_\varepsilon = \varnothing$

$$\left| \sum_{i\in K} z_i \right| \leqslant \sum_{i\in K} |z_i| \leqslant \varepsilon \ .$$

Hence $\left(\sum_{i\in J} z_i\right)_{J\in\mathcal{F}(I)}$ is a Cauchy net, so has to converges by completeness of $\mathbb{C}$. This proves summability of $(z_i)_{i\in I}$.

Vice versa, assume now that $(z_i)_{i\in I}$ is summable. Then $\left(\sum_{i\in J} z_i\right)_{J\in\mathcal{F}(I)}$ is a Cauchy net. Hence there exists an element $J_1 \in \mathcal{F}(I)$ such that for all $K \in \mathcal{F}(I)$ with $K \cap J_1 = \varnothing$ the inequality

$$\left| \sum_{i\in K} z_i \right| < 1$$

holds true. Let $C = \sum_{i\in J_1} |z_i|$. Then one has for all $J \in \mathcal{F}(I)$

$$\left| \sum_{i\in J} z_i \right| \leqslant \left| \sum_{i\in J\setminus J_1} z_i \right| + \left| \sum_{i\in J\cap J_1} z_i \right| \leqslant 1 + C \ .$$

By the preceding lemma the set of partial sums $\sum_{i\in J} |z_i|$, where J runs through the finite subsets of I, is then bounded by $4 + 4C$, hence $(z_i)_{i\in I}$ is absolutely summable. □

Summability in Banach spaces

10.5.10 Proposition *Let* V *be a normed vector space. For a family* $(v_i)_{i\in I}$ *of elements in* V *the following are equivalent:*

(i) *The family* $(v_i)_{i\in I}$ *is absolutely summable.*

(ii) *The family* $(\|v_i\|)_{i\in I}$ *is summable.*

(iii) *There exists some* $C > 0$ *such that* $\sum_{i\in J} \|v_i\| \leqslant C$ *for all* $J \in \mathcal{F}(I)$.

If V *is even a Banach space, these conditions are all equivalent to*

(iv) *The family* $(v_i)_{i\in I}$ *is summable.*

Proof. (ii) and (iii) are equivalent by Proposition 10.5.9 Assume now that (i) holds true. □

to do: Carl Neumann series

Properties of and relations between the various summability types

10.5.11 Theorem *Let I be a non-empty index set. Then the spaces $\ell^1(I, \mathrm{V})$ of summable families, $\ell^1[I, \mathrm{V}]$ of weakly summable families, $\ell^1\{I, \mathrm{V}\}$ of absolutely summable families and $\ell^1\langle I, \mathrm{V}\rangle$ of totally summable families in E are all subvector spaces of the product vector space $E^I = \Pi_{i\in I} E$. Furthermore one has the following chain of inclusions:*

$$\ell^1\langle I, \mathrm{V}\rangle \subset \ell^1\{I, \mathrm{V}\} \quad \text{and} \quad \ell^1(I, \mathrm{V}) \subset \ell^1[I, \mathrm{V}] \ .$$

If E is complete, then one even has

$$\ell^1\{I, \mathrm{V}\} \subset \ell^1(I, \mathrm{V})$$

Proof. Now let (v_i) be a summable family and $\alpha : \mathrm{V} \to \mathbb{K}$ a continuous linear form.

Let U be an absolutely convex zero neighborhood. Then U absorbes B, so there exists $r > 0$ such that $B \subset rU$. Hence □

10.6. Topological tensor products

10.6.1 Definition (cf. (Grothendieck, 1955, Chap. I, § 3, nº 3)) Let V and W be two locally convex topological vector spaces over the ground field $\mathbb{K}$. A locally convex vector topology τ on the (algebraic) tensor product $\mathrm{V} \otimes \mathrm{W}$ is called *compatible with the tensor product structure*, an *admissible tensor product topology* or just *admissible* if the following conditions hold true:

(ATPT1) The canonical map $\mathrm{V} \times \mathrm{W} \to \mathrm{V} \otimes_\tau \mathrm{W}$ is seperately continuous that is for each $v \in \mathrm{V}$ and each $w \in \mathrm{W}$ the linear maps

$$\mathrm{W} \to \mathrm{V} \otimes_\tau \mathrm{W}, y \mapsto v \otimes y \quad \text{and} \quad \mathrm{V} \to \mathrm{V} \otimes_\tau \mathrm{W}, x \mapsto x \otimes w$$

are continuous where $\mathrm{V} \otimes_\tau \mathrm{W}$ denotes the vector space $\mathrm{V} \otimes \mathrm{W}$ equipped with τ.

(ATPT2) For all linear maps $\alpha \in \mathrm{V}'$ and $\beta \in \mathrm{W}'$ the canonical linear map map $\alpha \otimes \beta : \mathrm{V} \otimes_\tau \mathrm{W} \to \mathbb{K}$ is continuous.

(ATPT3) For every equicontinuous subset $A \subset \mathrm{V}'$ and equicontinuous subset $B \subset \mathrm{W}'$ the set $\{\alpha \otimes \beta \mid \alpha \in A \,\&\, \beta \in B\}$ is an equicontinuous subset of the topological dual of $\mathrm{V} \otimes_\tau \mathrm{W}$.

The locally convex vector topology τ is called *strongly compatible with the tensor product structure*, a *strongly admissible tensor product topology* or briefly *strongly admissible* if it satisfies:

(sATPT) The canonical map $\mathrm{V} \times \mathrm{W} \to \mathrm{V} \otimes_\tau \mathrm{W}$ is continuous where $\mathrm{V} \times \mathrm{W}$ carries the product topology.

10.6.2 The admissible respectively strongly admissible vector topologies on $\mathrm{V} \otimes \mathrm{W}$ are obviously partially ordered by set-theoretic inclusion. Therefore, the following definition makes sense.

10.6.3 Definition

11. Distributions and Fourier transform

11.1. Schwartz distributions

11.2. Pullback of distributions

11.2.1 Let M and N be smooth manifolds and $f : M \to N$ a smooth map. One then has a continuous pullback map $f^* : \mathcal{C}^\infty(N) \to \mathcal{C}^\infty(M)$ which maps an element $h \in \mathcal{C}^\infty(N)$ to the composition $h \circ f : M \to \mathbb{R}$ which obviously is a smooth function on M. The Faà-di-Bruno formula from Theorem 17.1.10 tells that f^* is continuous indeed. In this section we want to establish criteria under which the pullback of functions can be extended to a pullback of distributions. We also will study continuity properties of the distributional pullback operation

Let us start with the following observation.

11.2.2 Lemma *Let $f : U_1 \to U_2$ be a diffeomorphism between two open subsets $U_1, U_2 \subset \mathbb{R}^n$ and λ the Lebesgue measure on $\mathbb{R}^n$. Then for every $u \in \mathcal{C}(U_2)$ and $\varphi \in \mathcal{D}(U_1)$ the equality*

$$\int_{U_1} \varphi\, f^* u \, d\lambda = \int_{U_2} (\varphi \circ f^{-1})\, u \,|\det Df^{-1}|\, d\lambda$$

holds true.

Proof. The claim is an immediate consequence of the change-of-variables formula. □

11.2.3 Using the lemma as guideline we now extend the pullback of functions to distributions. Denote for $U \subset \mathbb{R}^n$ by $\langle \cdot, \cdot \rangle$ the pairing between $\mathcal{D}'(U)$ and $\mathcal{D}(U)$. Under the assumptions of the lemma assume u to be a distribution on U_1 that is an element of $\mathcal{D}'(U_2)$. Then the map

$$f^* u : \mathcal{D}(U_1) \to \mathbb{R}, \quad \varphi \mapsto \langle u, |\det Df^{-1}|\, (f^{-1})^* \varphi \rangle \, .$$

is an element of the distribution space $\mathcal{D}'(U_1)$ since the map

$$\mathcal{D}(U_1) \to \mathcal{D}(U_2), \quad \varphi \mapsto |\det Df^{-1}|\, \varphi \circ f^{-1}$$

is linear and continuous with respect to the LF-topologies on $\mathcal{D}(U_1)$ and $\mathcal{D}(U_2)$. One calls $f^* u$ the *pullback* of the distribution u under f. By Lemma 11.2.2, this pullback operation extends the one for continuous functions and it is obviously uniquely determined by that property.

We continue with another observation.

11.2.4 Lemma *Assume that $U \subset \mathbb{R}^n$ is open, $f : U \to \mathbb{R}$ a submersion and $\varphi \in \mathcal{D}(U)$ a test function. Then the map*

$$f_*\varphi : \mathbb{R} \to \mathbb{R},\ t \mapsto \frac{d}{dt} \int_{\{x \in U \mid f(x) < t\}} \varphi(x) dx$$

is well-defined, smooth and has compact support.

Proof. Let us assume first that the map $\Psi : U \to \mathbb{R}^n$, $(x_1, \ldots, x_n) \mapsto ()$ is $\square$

todo Possibly assume that M is orientable and carries a volume form.

11.3. Hyperfunctions of a single variable

11.3.1 Let us introduce some notation. For every open intervall $I \subset \mathbb{R}$ call an open subset $U \subset \mathbb{C}$ such that $I = U \cap \mathbb{R}$ a *complex neighborhood* of I. Denote by $\mathbb{C}^+$ the upper complex half-plane $\{z \in \mathbb{C} \mid \mathfrak{Im} z > 0\}$ and by $\mathbb{C}^-$ the lower complex half-plane $\{z \in \mathbb{C} \mid \mathfrak{Im} z < 0\}$. More generally, put $U^+ = U \cap \mathbb{C}^+$ and $U^- = U \cap \mathbb{C}^-$ for every open subset $U \subset \mathbb{C}$.

12. Hilbert Spaces

12.1. Inner product spaces

12.1.1 Let us first remind the reader that as before $\mathbb{K}$ stands for the field of real or of complex numbers. We will keep this notational agreement throughout the whole chapter.

12.1.2 Definition By a *sesquilinear form* on a $\mathbb{K}$-vector space V one understands a map $\langle\cdot,\cdot\rangle : \mathrm{V}\times\mathrm{V}\to\mathbb{K}$ with the following two properties:

(SF1) The map $\langle\cdot,\cdot\rangle$ is *conjugate-linear* in its first coordinate which means that

$$\langle v_1+v_2,w\rangle = \langle v_1,w\rangle + \langle v_2,w\rangle \quad \text{and} \quad \langle rv,w\rangle = \overline{r}\langle v,w\rangle$$

for all $v,v_1,v_2,w\in\mathrm{V}$ and $r\in\mathbb{K}$.

(SF2) The map $\langle\cdot,\cdot\rangle$ is *linear* in its second coordinate which means that

$$\langle v,w_1+w_2\rangle = \langle v,w_1\rangle + \langle v,w_2\rangle \quad \text{and} \quad \langle v,rw\rangle = r\langle v,w\rangle$$

for all $v,w,w_1,w_2\in\mathrm{V}$ and $r\in\mathbb{K}$.

A *hermitian form* is a sesquilinear form $\langle\cdot,\cdot\rangle$ on V with the following additional property:

(SF3) The map $\langle\cdot,\cdot\rangle$ is *conjugate-symmetric* which means that

$$\langle v,w\rangle = \overline{\langle w,v\rangle} \quad \text{for all } v,w\in\mathrm{V}\ .$$

A sesquilinear form $\langle\cdot,\cdot\rangle$ is called *weakly-nondegenerate* if it satisfies axiom

(SF4w) For every $v\in\mathrm{V}$, the map $\mathrm{V}\to\mathbb{K}$, $w\to\langle w,v\rangle$ is the zero map if and only if $v=0$.

Finally, one calls a hermitian form $\langle\cdot,\cdot\rangle$ on V *positive semidefinite* if

(SF5s) $\langle v,v\rangle\geqslant 0$ for all $v\in\mathrm{V}$.

12.1.3 Remark Recall that a map $\langle\cdot,\cdot\rangle : \mathrm{V}\times\mathrm{V}\to\mathbb{K}$ is called *bilinear*, if it satisfies (SF2) and

(BF1) The map $\langle\cdot,\cdot\rangle$ is *linear* in its first coordinate which means that

$$\langle v_1+v_2,w\rangle = \langle v_1,w\rangle + \langle v_2,w\rangle \quad \text{and} \quad \langle rv,w\rangle = r\langle v,w\rangle$$

for all $v,v_1,v_2,w\in\mathrm{V}$ and $r\in\mathbb{K}$.

If the underlying ground field $\mathbb{K}$ coincides with the field of real numbers, a sesquilinear form is by definition the same as a bilinear form, and a hermitian form the same as a symmetric bilinear form.

12.1.4 Given a positive semidefinite hermitian form $\langle\cdot,\cdot\rangle$ on a $\mathbb{K}$-vector space V, one calls two vectors $v, w \in \mathrm{V}$ *orthogonal* if $\langle v, w\rangle = 0$. Since the hermitian form $\langle\cdot,\cdot\rangle$ is assumed to be positive semidefinite, the map

$$\|\cdot\| : \mathrm{V} \to \mathbb{R}_{\geqslant 0},\ v \mapsto \|v\| = \sqrt{\langle v, v\rangle}$$

is well-defined. We will later see that $\|\cdot\|$ is a seminorm on V and therefore call the map $\|\cdot\|$ the *seminorm associated to* $\langle\cdot,\cdot\rangle$. The following formulas are immediate consequences of the properties defining a positive semidefinite hermitian form and the definition of the associated seminorm:

$$\|v + w\|^2 = \|v\|^2 + 2\,\mathfrak{Re}\,\langle v, w\rangle + \|w\|^2 \quad \text{for all } v, w \in \mathrm{V}\ , \tag{12.1.1}$$

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2 \quad \text{for all orthogonal } v, w \in \mathrm{V}\ , \tag{12.1.2}$$

$$\|v + w\|^2 + \|v - w\|^2 = 2\big(\|v\|^2 + \|w\|^2\big) \quad \text{for all } v, w \in \mathrm{V}\ , \tag{12.1.3}$$

$$\|rv\| = \sqrt{|r|^2\langle v, v\rangle} = |r|\|v\| \quad \text{for all } v, w \in \mathrm{V} \text{ and } r \in \mathbb{K}\ . \tag{12.1.4}$$

Formula (12.1.2) is an abstract version of the *pythagorean theorem*, Equation (12.1.3) is called the *parallelogram identity.* The triangle inequality for the map $\|\cdot\|$ will turn out to be a consequence of the following result.

12.1.5 Proposition (Cauchy–Schwarz inequality) *Given a positive semidefinite hermitian form $\langle\cdot,\cdot\rangle$ on a $\mathbb{K}$-vector space* V *the following inequality holds true:*

$$|\langle v, w\rangle| \leqslant \|v\|\|w\| \quad \textit{for all } v, w \in \mathrm{V}. \tag{12.1.5}$$

Equality holds if v and w are linearly dependant. In case $\langle\cdot,\cdot\rangle$ is positive definite, the converse holds true as well.

Proof. First consider the case where $\|v\| = \|w\| = 0$. Then put $c = -\langle v, w\rangle$ and compute

$$0 \leqslant \|cv + w\|^2 = 2\,\mathfrak{Re}\big(\overline{c}\,\langle v, w\rangle\big) = -2|\langle v, w\rangle|^2\ . \tag{12.1.6}$$

This entails $\langle v, w\rangle = 0$ and the Cauchy–Schwarz inequality is proved for $\|v\| = \|w\| = 0$.

If $\|v\| \neq 0$ or $\|w\| \neq 0$, we can assume without loss of generality that $\|v\| \neq 0$. Then put

$$c = -\frac{\langle v, w\rangle}{\|v\|^2}$$

and compute

$$\begin{aligned} 0 \leqslant \|cv + w\|^2 &= |c|^2\|v\|^2 + 2\,\mathfrak{Re}\big(\overline{c}\,\langle v, w\rangle\big) + \|w\|^2 = \\ &= \frac{|\langle v, w\rangle|^2}{\|v\|^2} - 2\frac{|\langle v, w\rangle|^2}{\|v\|^2} + \|w\|^2 = \|w\|^2 - \frac{|\langle v, w\rangle|^2}{\|v\|^2}\ . \end{aligned} \tag{12.1.7}$$

Hence

$$|\langle v, w\rangle|^2 \leqslant \|v\|^2\|w\|^2$$

which entails the Cauchy–Schwarz inequality.

In case v, w are linearly dependant nonzero elements of V, then there exists a nonzero scalar $a \in \mathbb{K}$ such that $v = aw$. Hence

$$|\langle v, w\rangle| = |a| \, \|w\|^2 = \|v\| \|w\| .$$

If one of v or w is 0, then both sides of the Cauchy–Schwarz inequality are 0.

In the positive definite case, equality in (12.1.5) entails by Equation (12.1.7) that $cv + w = 0$ whenever $v \neq 0$. If $v = 0$, then $v = 0 \cdot w$. In either case this means that v and w are linearly dependant. □

12.1.6 Lemma *A positive semidefinite hermitian form $\langle \cdot, \cdot \rangle$ on a $\mathbb{K}$-vector space* V *is weakly-nondegenerate if and only if it is* positive definite *that is if and only if*

(SF5p) $\langle v, v\rangle > 0$ *for all* $v \in \mathrm{V}\backslash\{0\}$.

Proof. A positive definite real bilinear or complex hermitian form $\langle \cdot, \cdot \rangle$ is weakly-nondegenerate since for every $v \in \mathrm{V}\backslash\{0\}$ the linear form $\langle v, -\rangle : \mathrm{V} \to \mathbb{K}$ is nonzero by $\langle v, v\rangle > 0$.

Conversely, if $\langle v, -\rangle : \mathrm{V} \to \mathbb{K}$ is nonzero for all $v \in \mathrm{V}\backslash\{0\}$, then there exists an element $w \in \mathrm{V}$ such that $\langle w, v\rangle \neq 0$. The Cauchy–Schwarz inequality entails

$$0 < |\langle w, v\rangle|^2 \leqslant \langle w, w\rangle \langle v, v\rangle ,$$

which implies $\langle v, v\rangle > 0$. Hence $\langle \cdot, \cdot \rangle$ is positive definite. □

12.1.7 Proposition *The map*

$$\| \cdot \| : V \to \mathbb{R}_{\geqslant 0}, \; v \mapsto \|v\| = \sqrt{\langle v, v\rangle}$$

associated to a positive semidefinite hermitian form $\langle \cdot, \cdot \rangle$ on a $\mathbb{K}$-vector space V *is a seminorm. If the hermitian form is positive definite, then $\| \cdot \|$ is even a norm.*

Proof. Absolute homogeneity (N1) is given by Eq. (12.1.4). The triangle inequality is a consequence of the Cauchy–Schwarz inequality:

$$\|v + w\|^2 = \|v\|^2 + 2\,\mathfrak{Re}\,\langle v, w\rangle + \|w\|^2 \leqslant \|v\|^2 + 2\,\|v\|\,\|w\| + \|w\|^2 = \big(\|v\| + \|w\|\big)^2 .$$

Finally, if $\langle \cdot, \cdot \rangle$ is positive definite, then $\|v\| = \sqrt{\langle v, v\rangle} > 0$ for all $v \in \mathrm{V}\backslash\{0\}$, so $\| \cdot \|$ is a norm. □

12.1.8 Definition By an *inner product* or a *scalar product* on a $\mathbb{K}$-vector space $\mathfrak{H}$ one understands a positive definite hermitian form on $\mathfrak{H}$. A $\mathbb{K}$-vector space $\mathfrak{H}$ endowed with an inner product $\langle \cdot, \cdot \rangle : \mathfrak{H} \times \mathfrak{H} \to \mathbb{K}$ is called an *inner product space* or a *pre-Hilbert space.*

A hermitian form on a $\mathbb{K}$-vector space $\mathfrak{H}$ which is only positive semidefinite is called a *semi-inner product* or a *semi-scalar product.*

A *Hilbert space* is an inner product space $(\mathfrak{H}, \langle \cdot, \cdot \rangle)$ which is complete as a normed vector space. In other words, a Hilbert space is Banach space where the norm on the space is induced by an inner product.

12.1.9 Examples (a) The vector space $\mathbb{R}^n$ with the *euclidean inner product*

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},\ \big((v_1, \ldots, v_n), (w_1, \ldots, w_n)\big) \mapsto \sum_{i=1}^n v_i w_i$$

is a real Hilbert space. Obviously, $\langle \cdot, \cdot \rangle$ is linear in the first argument, symmetric, and positive definite, hence a real inner product. The associated norm is the *euclidean norm*. We have seen before that $\mathbb{R}^n$ with the euclidean norm is complete.

(b) The vector space $\mathbb{C}^n$ together with the hermitian form

$$\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C},\ \big((v_1, \ldots, v_n), (w_1, \ldots, w_n)\big) \mapsto \sum_{i=1}^n \overline{v}_i w_i$$

is a complex Hilbert space. One immediately verifies that $\langle \cdot, \cdot \rangle$ is linear in the second argument, conjugate-symmetric, and positive definite. Hence $\langle \cdot, \cdot \rangle$ is a complex inner product which we sometimes call the *standard hermitian inner product* on $\mathbb{C}^n$. Its associated norm is again the euclidean norm, so by completeness of $\mathbb{C}^n \cong \mathbb{R}^{2n}$ with respect to the euclidean norm one obtains the claim.

(c) The set

$$\ell^2 = \left\{ (z_k)_{k \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \;\middle|\; \sum_{k=0}^{\infty} |z_k|^2 < \infty \right\}$$

of square summable sequences of complex numbers is a complex Hilbert space with inner product

$$\langle \cdot, \cdot \rangle : \ell^2 \times \ell^2 \to \mathbb{C},\ \big((z_k)_{k \in \mathbb{N}}, (w_k)_{k \in \mathbb{N}}\big) \mapsto \sum_{k=0}^{\infty} \overline{z}_k w_k \ .$$

To prove this one needs to first verify that ℓ^2 is a subvector space of $\mathbb{C}^{\mathbb{N}}$. For $z = (z_k)_{k \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ denote by $\|z\|$ the *extended norm* $\sqrt{\sum_{k=0}^{\infty} |z_k|^2} = \sup\limits_{K \in \mathbb{N}} \sqrt{\sum_{k=0}^{K} |z_k|^2} \in [0, \infty]$. Then $z \in \ell^2$ if and only if $\|z\| < \infty$. Now let $a \in \mathbb{C}$ and $z \in \ell^2$ and compute

$$\|az\| = \sqrt{\sum_{k=0}^{\infty} |a z_k|^2} = |a| \sqrt{\sum_{k=0}^{\infty} |z_k|^2} = |a| \cdot \|z\| < \infty \ .$$

Hence $az \in \ell^2$. If $z, w \in \ell^2$, denote for each $K \in \mathbb{N}$ by $z_{(K)}$ and $w_{(K)}$ the "cut-off" vectors $(z_0, \ldots, z_K) \in \mathbb{C}^{K+1}$ and $(w_0, \ldots, w_K) \in \mathbb{C}^{K+1}$, respectively. By the triangle inequality for the norm on the Hilbert space $\mathbb{C}^{K+1}$ one concludes

$$\sqrt{\sum_{k=0}^{K} |z_k + w_k|^2} = \|z_{(K)} + w_{(K)}\| \leqslant \|z_{(K)}\| + \|w_{(K)}\| \leqslant \|z\| + \|w\| < \infty \ .$$

Therefore, the sequence of partial sums $\sum_{k=0}^{K} |z_k + w_k|^2$, $K \in \mathbb{N}$, is bounded, so convergent by the the monotone convergence theorem. One obtains

$$\|z + w\| = \lim_{K \to \infty} \sqrt{\sum_{k=0}^{K} |z_k + w_k|^2} \leqslant \|z\| + \|w\| < \infty \ .$$

Hence $z+w$ is square summable and ℓ^2 a vector subspace of $\mathbb{C}^{\mathbb{N}}$ indeed. Note that our argument also shows that the restriction of the extended norm to ℓ^2 is a norm.

We need to show that $\langle \cdot, \cdot \rangle$ is well-defined. To this end it suffices to prove that for all $z, w \in \ell^2$ the family $(z_k \overline{w}_k)_{k\in\mathbb{N}}$ is absolutely summable or in other words that $\sum_{k=0}^{\infty} |z_k \overline{w}_k| < \infty$. One concludes by the Hölder inequality for sums

$$\sum_{k=0}^{K} |\overline{z}_k w_k| = \sum_{k=0}^{K} |z_k w_k| \leqslant \|z_{(K)}\| \, \|w_{(K)}\| \leqslant \|z\| \, \|w\| \ .$$

So the left hand side has an upper bound uniform in K which by the monotone convergence theorem entails convergence of the partial sums and the estimate

$$\sum_{k=0}^{\infty} |\overline{z}_k w_k| \leqslant \|z\| \, \|w\| < \infty \ .$$

By definition it is clear that $\langle \cdot, \cdot \rangle$ is linear in the second argument, conjugate-symmetric and positive definite, hence a complex inner product. Note that the norm associated to $\langle \cdot, \cdot \rangle$ coincides with the above defined map $\| \cdot \|$.

It remains to be shown that ℓ^2 is complete. Let $(z^n)_{n\in\mathbb{N}}$ with $z^n = (z^n_k)_{k\in\mathbb{N}} \in \ell^2$ for all $n \in \mathbb{N}$ be a Cauchy sequence in ℓ^2. For $\varepsilon > 0$ choose $N_\varepsilon \in \mathbb{N}$ so that

$$\|z^n - z^m\| < \varepsilon \quad \text{for all } n, m \geqslant N_\varepsilon \ .$$

For each fixed $k \in \mathbb{N}$ one therefore has

$$|z^n_k - z^m_k| \leqslant \|z^n - z^m\| < \varepsilon \quad \text{for all } n, m \geqslant N_\varepsilon \ . \tag{12.1.8}$$

By completeness of $\mathbb{C}$ there exist $z_k \in \mathbb{C}$ such that $\lim_{n\to\infty} z^n_k = z_k$ for all $k \in \mathbb{N}$. We claim that $z = (z_k)_{k\in\mathbb{N}}$ is an element of ℓ^2 and that $(z^n)_{n\in\mathbb{N}}$ converges to z. To verify this observe that for all $\varepsilon > 0$, $K \in \mathbb{N}$ and $n \geqslant N_\varepsilon$

$$\sum_{k=0}^{K} |z_k - z^n_k|^2 = \lim_{m\to\infty} \sum_{k=0}^{K} |z^m_k - z^n_k|^2 \leqslant \sup_{m \geqslant N_\varepsilon} \sum_{k=0}^{K} |z^m_k - z^n_k|^2 \leqslant \sup_{m \geqslant N_\varepsilon} \|z^m - z^n\|^2 \leqslant \varepsilon^2 \ .$$

This implies by the triangle inequality and the fact that the Cauchy sequence $(z^n)_{n\in\mathbb{N}}$ is bounded in norm by some $C > 0$ that for all $K \in \mathbb{N}$ and $N = N_1$

$$\sqrt{\sum_{k=0}^{K} |z_k|^2} = \|z_{(K)}\| \leqslant \|z_{(K)} - z^N_{(K)}\| + \|z^N_{(K)}\| \leqslant \|z_{(K)} - z^N_{(K)}\| + \|z^N\| \leqslant 1 + C \ .$$

Hence $\|z\| = \sqrt{\sum_{k=0}^{\infty} |z_k|^2} \leqslant 1 + C$ and $z \in \ell^2$. In addition one obtains

$$\|z - z^n\| = \lim_{K\to\infty} \sqrt{\sum_{k=0}^{K} |z_k - z^n_k|^2} \leqslant \varepsilon \quad \text{for all } n \geqslant N_\varepsilon \ .$$

This means that z is the limit of the sequence $(z^n)_{n\in\mathbb{N}}$ and ℓ^2 is complete.

(d) Denote by λ the Lebesgue measure and let

$$\mathcal{L}^2(\mathbb{R}^d) = \left\{ f : \mathbb{R}^d \to \mathbb{C} \;\middle|\; f \text{ is Lebesgue measurable and } \|f\|_2 := \sqrt{\int_{\mathbb{R}^d} |f|^2 d\lambda} < \infty \right\}$$

be the space of Lebesgue square integrable functions on $\mathbb{R}^d$. Then $\mathcal{L}^2(\mathbb{R}^d)$ is a linear subspace of the space of all measurable functions by Minkowski's inequality which reads

$$\|f + g\|_p \leqslant \|f\|_p + \|g\|_p \quad \text{for all measurable } f, g : \mathbb{R}^d \to \mathbb{C} \ .$$

Hereby, $\|f\|_p$ denotes for $p \in [1, \infty)$ the $\mathcal{L}^p$-seminorm $\left(\int_{\mathbb{R}^d} |f|^p d\lambda\right)^{1/p}$ of a measurable function $f : \mathbb{R}^d \to \mathbb{C}$. Note that $\|f\|_p$ can attain the value ∞, namely when f is not in the space $\mathcal{L}^p(\mathbb{R}^d)$. By Hölder's inequality, the product fg is Lebesgue integrable for $f, g \in \mathcal{L}^2(\mathbb{R}^d)$ and one has the estimate

$$\int_{\mathbb{R}^d} |fg| d\lambda = \|fg\|_1 \leqslant \|f\|_2 \, \|g\|_2 \ .$$

Hence the map

$$\langle \cdot, \cdot \rangle : \mathcal{L}^2(\mathbb{R}^d) \times \mathcal{L}^2(\mathbb{R}^d) \to \mathbb{C}, \ (f, g) \mapsto \int_{\mathbb{R}^d} \overline{f} g \, d\lambda$$

is well-defined and a positive semidefinite hermitian form on $\mathcal{L}^2(\mathbb{R}^d)$. By construction, the associated seminorm is the $\mathcal{L}^2$-seminorm $\|\cdot\|_2$. Modding out $\mathcal{L}^2(\mathbb{R}^d)$ by the kernel

$$\mathcal{N} := \operatorname{Ker}(\|\cdot\|_2) = \left\{ f \in \mathcal{L}^2(\mathbb{R}^d) \;\middle|\; \int_{\mathbb{R}^d} |f|^2 d\lambda = 0 \right\}$$

gives the Lebesgue space

$$L^2(\mathbb{R}^d) := \mathcal{L}^2(\mathbb{R}^d)/\mathcal{N} \ .$$

The hermitian form $\langle \cdot, \cdot \rangle$ vanishes on $\mathcal{N} \times \mathcal{L}^2(\mathbb{R}^d)$ and $\mathcal{L}^2(\mathbb{R}^d) \times \mathcal{N}$ by the Cauchy–Schwarz inequality, hence descends to a hermitian form

$$\langle \cdot, \cdot \rangle : L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to \mathbb{C}, \ (f + \mathcal{N}, g + \mathcal{N}) \mapsto \int_{\mathbb{R}^d} \overline{f} g \, d\lambda \ .$$

That hermitian form is positive definite, since $\langle f + \mathcal{N}, f + \mathcal{N} \rangle = 0$ means $\int_{\mathbb{R}^d} |f|^2 d\lambda = 0$, hence $f \in \mathcal{N}$. Let us show that $L^2(\mathbb{R}^d)$ is complete with respect to the L^2-norm $\|\cdot\|_2$ induced by the inner product. Note that on the quotient space $\|\cdot\|_2$ is a norm indeed by construction. So let $(f_n + \mathcal{N})_{n \in \mathbb{N}}$ be a Cauchy sequence in $L^2(\mathbb{R}^d)$. Choose a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ such that

$$\|f_{n_k} - f_{n_{k-1}}\|_2 < \frac{1}{2^k} \quad \text{for all } k \in \mathbb{N}_{>0}$$

and put

$$g_n(x) = \sum_{k=1}^{n} |f_{n_k}(x) - f_{n_{k-1}}(x)| \quad \text{for } x \in \mathbb{R}^d \text{ and } n \in \mathbb{N} \ .$$

The limit function

$$g : \mathbb{R}^d \to [0, \infty], \ x \mapsto \lim_{n \to \infty} g_n(x) = \liminf_{n \to \infty} g_n(x)$$

then exists even though it might not be finite everyhwere. Minkowski's inequality for the $\mathcal{L}^2$-norm entails that $\|g_n\|_2 \leqslant 1$ for all $n \in \mathbb{N}$, hence g is measurable and $\|g\|_2 \leqslant \liminf_{n\to\infty} \|g_n\|_2 \leqslant 1$ by Fatou's lemma. Therefore, $g(x)$ is finite for all x up to a set $Z \subset \mathbb{R}^d$ of measure 0, and for those x the series with partial sums $g_n(x)$ converges absolutely. For all $x \in \mathbb{R}^d \backslash Z$ the limit

$$f(x) = \lim_{k\to\infty} f_{n_k}(x) = f_{n_0} + \lim_{k\to\infty} \sum_{j=1}^{k} (f_{n_j}(x) - f_{n_{j-1}}(x))$$

therefore exists in $\mathbb{C}$. Put $f(x) = 0$ for all $x \in Z$, and let $\chi_Z : \mathbb{R}^d \to \mathbb{R}$ be the characteristic function of Z. Then the sequence of functions $(\chi_Z f_{n_k})_{k\in\mathbb{N}}$ converges pointwise to f, and each of the functions $\chi_Z f_n$ is measurable, actually even square integrable. Since

$$|\chi_Z f_{n_k}| \leqslant |\chi_Z f_{n_0}| + g_k \leqslant |\chi_Z f_{n_0}| + g \quad \text{for all } k \in \mathbb{N}$$

and since $|\chi_Z f_{n_0}| + g$ is square integrable by Minkowski's inequality, the pointwise limit f is square integrable by Lebesgue's dominated convergence theorem, and $f + \mathcal{N}$ is in $L^2(\mathbb{R}^d)$. It remains to show that $(f_n + \mathcal{N})_{n\in\mathbb{N}}$ converges to $f + \mathcal{N}$ in the norm $\|\cdot\|_2$. To this end let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $\|f_n - f_m\|_2 < \varepsilon$ for $n, m \geqslant N$. By Fatou's lemma one obtains

$$\int_{\mathbb{R}^d} |f_n - f|^2 d\lambda \leqslant \liminf_{m\to\infty} \int_{\mathbb{R}^d} |f_n - f_m|^2 d\lambda \leqslant \varepsilon^2 \quad \text{for all } n \geqslant N \ ,$$

hence $\lim_{n\to\infty} \|f_n - f\|_2 = 0$ and $L^2(\mathbb{R}^d)$ endowed with the inner product $\langle\cdot,\cdot\rangle$ is a Hilbert space. It is called the *Hilbert space of square-integrable functions* on $\mathbb{R}^d$. Note that for every complete measure space (Ω, μ) one obtains in the same way the Hilbert space $L^2(\Omega, \mu)$ of square-integrable functions on (Ω, μ).

12.1.10 Theorem *Let* V *be a normed* $\mathbb{K}$*-vector space. Then the norm* $\|\cdot\| : \mathrm{V} \to \mathbb{R}_{\geqslant 0}$ *is associated to an inner product* $\langle\cdot,\cdot\rangle : \mathrm{V} \times \mathrm{V} \to \mathbb{K}$ *if and only if the* parallelogram identity

$$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$$

holds true for all $v, w \in \mathrm{V}$. *In this case, the inner product of two elements* $v, w \in \mathrm{V}$ *can be expressed by the* polarization identity for $\mathbb{K} = \mathbb{R}$

$$\langle v, w\rangle = \frac{1}{4}\left(\|v + w\|^2 - \|v - w\|^2\right) = \frac{1}{2}\left(\|v + w\|^2 - \|v\|^2 - \|w\|^2\right) \tag{12.1.9}$$

respectively by the polarization identity for $\mathbb{K} = \mathbb{C}$

$$\langle v, w\rangle = \frac{1}{4}\sum_{k=1}^{4} \mathsf{i}^k \,\|w + \mathsf{i}^k\, v\|^2 \ . \tag{12.1.10}$$

Proof. The forward direction is a consequence of 12.1.4, Eq. 12.1.3. To show the backward direction we consider two cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$ separately.

1. Case. Given the norm $\|\cdot\|$ define $\langle\cdot,\cdot\rangle : \mathrm{V} \times \mathrm{V} \to \mathbb{R}$ by real polarization

$$\langle v, w\rangle = \frac{1}{4}\left(\|v + w\|^2 - \|v - w\|^2\right), \quad \text{where } v, w \in \mathrm{V} \ .$$

Note that the parallelogram identity entails

$$\frac{1}{4}\left(\|v+w\|^2-\|v-w\|^2\right)=\frac{1}{2}\left(\|v+w\|^2-\|v\|^2-\|w\|^2\right) .$$

Observe that by definition $\langle v,w\rangle=\langle w,v\rangle$ and $\|v\|=\sqrt{\langle v,v\rangle}$. Let us show additivity in the first variable. Let $v_1,v_2,w\in \mathrm{V}$ and compute using the parallelogram identity

$$\begin{aligned}\|v_1+v_2+w\|^2&=2\|v_1+w\|^2+2\|v_2\|^2-\|v_1+w-v_2\|^2 ,\\ \|v_1+v_2+w\|^2&=2\|v_2+w\|^2+2\|v_1\|^2-\|v_2+w-v_1\|^2 .\end{aligned}$$

Hence

$$\|v_1+v_2\pm w\|^2=\|v_1\pm w\|^2+\|v_2\pm w\|^2+\|v_1\|^2+\|v_2\|^2-\|v_1\pm w-v_2\|^2-\|v_2\pm w-v_1\|^2 .$$

Subtracting the $-$ version from the $+$ version of this equation entails

$$\begin{aligned}\langle v_1+v_2,w\rangle&=\frac{1}{4}\left(\|v_1+v_2+w\|^2-\|v_1+v_2-w\|^2\right)=\\&=\frac{1}{4}\left(\|v_1+w\|^2+\|v_2+w\|^2-\|v_1-w\|^2-\|v_2-w\|^2\right)=\langle v_1,w\rangle+\langle v_2,w\rangle ,\end{aligned}$$

so additivity in the first variable is proved. By induction one derives from this that for all natural n

$$\langle nv,w\rangle=n\langle v,w\rangle \quad \text{for all } v,w\in \mathrm{V} . \tag{12.1.11}$$

Since then $\langle -nv,w\rangle-n\langle v,w\rangle=\langle -nv+nv,w\rangle=0$ for all $n\in\mathbb{N}$, Eq. (12.1.11) also holds for $n\in\mathbb{Z}$. Now let $p\in\mathbb{Z}$ and $q\in\mathbb{N}_{>0}$. Then $q\langle \frac{p}{q}v,w\rangle=\langle pv,w\rangle=p\langle v,w\rangle$, hence one has for rational r

$$\langle rv,w\rangle=r\langle v,w\rangle \quad \text{for all } v,w\in \mathrm{V} . \tag{12.1.12}$$

Since addition, multiplication by scalars and the norm are continuous, the function

$$\mathbb{R}\to\mathbb{R},\ r\mapsto\langle rv,w\rangle-r\langle v,w\rangle=\frac{1}{4}\left(\|rv+w\|^2+r\|v-w\|^2-\|rv-w\|^2-r\|v+w\|^2\right)$$

is continuous. Since it vanishes over $\mathbb{Q}$, it has to coincide with the zero map. Therefore, Eq. (12.1.12) holds for all $r\in\mathbb{R}$. So $\langle\cdot,\cdot\rangle$ is linear in the first coordinate. By symmetry, it is so too in the second coordinate. Hence $\langle\cdot,\cdot\rangle$ is a symmetric bilinear form inducing $\|\cdot\|$.

2. Case. In the case $\mathbb{K}=\mathbb{C}$ use complex polarization and put

$$\langle v,w\rangle=\frac{1}{4}\sum_{k=1}^{4}\mathrm{i}^k\,\|w+\mathrm{i}^k\,v\|^2 \quad \text{for all } v,w\in \mathrm{V} .$$

Then $\langle\cdot,\cdot\rangle$ is conjugate-symmetric, since

$$\overline{\langle v,w\rangle}=\frac{1}{4}\sum_{k=1}^{4}(-\mathrm{i})^k\,\|w+\mathrm{i}^k\,v\|^2=\frac{1}{4}\sum_{k=1}^{4}(-\mathrm{i})^k\,\|(-\mathrm{i})^k\,w+v\|^2=\langle w,v\rangle .$$

Next compute

$$\mathfrak{Re}\,\langle v,w\rangle=\frac{1}{4}\left(\|w+v\|^2-\|w-v\|^2\right)$$

and

$$\mathfrak{Im}\,\langle v, w\rangle = \frac{1}{4}\left(\|w + \mathrm{i}v\|^2 - \|w - \mathrm{i}v\|^2\right) \ .$$

By the first case one concludes that $\mathfrak{Re}\langle\cdot,\cdot\rangle$ and $\mathfrak{Im}\langle\cdot,\cdot\rangle$ are both $\mathbb{R}$-linear in the first and the second coordinate. Moreover,

$$\mathfrak{Re}\,\langle v, \mathrm{i}w\rangle = \frac{1}{4}\left(\|\mathrm{i}w + v\|^2 - \|\mathrm{i}w - v\|^2\right) = \frac{1}{4}\left(\|w - \mathrm{i}v\|^2 - \|w + \mathrm{i}v\|^2\right) = -\mathfrak{Im}\,\langle v, w\rangle = \mathfrak{Re}\,\mathrm{i}\langle v, w\rangle$$

and

$$\mathfrak{Im}\,\langle v, \mathrm{i}w\rangle = \frac{1}{4}\left(\|\mathrm{i}w + \mathrm{i}v\|^2 - \|\mathrm{i}w - \mathrm{i}v\|^2\right) = \mathfrak{Re}\,\langle v, w\rangle = \mathfrak{Im}\,\mathrm{i}\langle v, w\rangle \ ,$$

hence $\langle\cdot,\cdot\rangle$ is complex linear in the second coordinate. Finally,

$$\mathfrak{Re}\,\langle v, v\rangle = \|v\|^2 \quad \text{and} \quad \mathfrak{Im}\,\langle v, v\rangle = \frac{1}{4}\left(\|v + \mathrm{i}v\|^2 - \|v - \mathrm{i}v\|^2\right) = 0 \ .$$

This finishes the proof that $\langle\cdot,\cdot\rangle$ is a complex inner product inducing the norm $\|\cdot\|$. □

12.1.11 Next we will turn Hilbert spaces into a category. To this end one needs to know what morphisms in this category should be. There are two options each giving rise to a category of Hilbert spaces. These categories just differ by their morphism classes. The first one is to have as morphisms linear maps $A : \mathfrak{H}_1 \to \mathfrak{H}_2$ preserving the inner products which means that they fulfill

$$\langle Av_1, Av_2\rangle = \langle v_1, v_2\rangle \quad \text{for all } v_1, v_2 \in \mathfrak{H}_1 \ .$$

By Theorem 12.1.10 this property is equivalent to

$$\|Av\| = \|v\| \quad \text{for all } v \in \mathfrak{H}_1 \ ,$$

that is to A being *norm preserving* or *isometric*. Obviously, the identity map between two Hilbert spaces is isometric and the composition of two composable isometric linear maps between Hilbert spaces is again isometric and linear. Hence Hilbert spaces together with norm preserving linear maps between them form a category which we denote by $\mathsf{Hilb}_{\mathrm{np}}$. The isomorphisms in this category are the surjective and inner product preserving linear maps between Hilbert spaces. Such maps are called *unitary*. The condition of a linear map being norm preserving is pretty restrictive, so the category $\mathsf{Hilb}_{\mathrm{np}}$ contains only few morphisms. This can be healed by allowing all *bounded* linear maps between Hilbert spaces to be morphisms that is of all linear $A : \mathfrak{H}_1 \to \mathfrak{H}_2$ for which there exists a $C \geqslant 0$ such that

$$\|Av\| \leqslant C\|v\| \quad \text{for all } v \in \mathfrak{H}_1 \ .$$

The smallest such C is called the *operator norm* of A and is denoted $\|A\|$. Equivalently, the operator norm is given by

$$\|A\| = \sup\left\{\|Av\| \;\middle|\; v \in \mathfrak{H}_1,\ \|v\| \leqslant 1\right\} = \sup\left\{\|Av\| \;\middle|\; v \in \mathfrak{H}_1,\ \|v\| = 1\right\} \ .$$

Every norm preserving linear map is bounded with operator norm 1. In particular the identity map on a Hilbert space is bounded. Moreover, if $A : \mathfrak{H}_1 \to \mathfrak{H}_2$ and $B : \mathfrak{H}_2 \to \mathfrak{H}_3$ are bounded linear operators between Hilbert spaces, then the composition $BA : \mathfrak{H}_1 \to \mathfrak{H}_3$ is bounded with operator norm $\leqslant \|B\| \, \|A\|$ since for all $v \in \mathfrak{H}_1$ with $\|v\| \leqslant 1$

$$\|BAv\| \leqslant \|B\| \, \|Av\| \leqslant \|B\| \, \|A\| \ .$$

Hence Hilbert spaces as objects together with bounded linear maps as morphisms form a category which we denote by Hilb and call the *category of Hilbert spaces*. Note that the morphisms in this category appear to "forget" the inner product and just preserve the linear and the topological structure. John Baez (Baez, 1997, p. 133) has explained how to heal this apparent defect by showing that Hilb carries a so-called $*$-structure given by the adjoint map on bounded linear operators. We will come back to this point later when we introduce adjoint operators.

12.1.12 Last in this section we will introduce bounded bilinear and sesquilinear maps. We define them for normed vector spaces. Their main application lies in the operator theory on Hilbert spaces, so we introduce them here.

12.1.13 Definition Let V be a vector space over $\mathbb{K}$ with norm $\| \cdot \| : \mathrm{V} \to \mathbb{R}_{\geqslant 0}$. A bilinear or sesquilinear form $b : \mathrm{V} \times \mathrm{V} \to \mathbb{K}$ is called *bounded* if there exists a $C > 0$ such that

$$|b(v,w)| \leqslant C \, \|v\| \, \|w\| \quad \text{for all } v, w \in \mathrm{V} \ .$$

In this case,

$$\|b\| := \sup \left\{ |b(v,w)| \;\middle|\; v, w \in \mathrm{V} \ \& \ \|v\| = \|w\| = 1 \right\}$$

exists and is called the *norm* of the form b.

12.1.14 Example The inner product on a (pre-) Hilbert space is bounded by the Cauchy–Schwarz inequality and has norm 1.

12.1.15 Proposition *A bounded bilinear or sesquilinear form $b : \mathrm{V} \times \mathrm{V} \to \mathbb{K}$ on a normed vector space V over $\mathbb{K}$ is continuous. Vice versa, if V is complete, then continuity of $b : \mathrm{V} \times \mathrm{V} \to \mathbb{K}$ implies boundedness.*

Proof. If b is bounded, then

$$\begin{aligned} \left|b(v,w) - b(v',w')\right| &\leqslant \left|b(v,w) - b(v',w)\right| + \left|b(v',w) - b(v',w')\right| \leqslant \\ &\leqslant \|b\| \, \big(\|w\| \, \|v - v'\| + \|v'\| \, \|w - w'\| \big) \end{aligned}$$

for all $v, v', w, w' \in \mathrm{V}$. Hence b is locally Lipschitz continuous, so in particular continuous.

Now assume that V is a Banach space and that b is continuous. Then one can find $\delta > 0$ such that for all $v, w \in V$ of norm less than δ the relation $|b(v,w)| < 1$ holds true. But that entails for all non-zero v, w

$$|b(v,w)| = \frac{4 \, \|v\| \, \|w\|}{\delta^2} \cdot b\left(\delta \frac{v}{2\|v\|}, \delta \frac{w}{2\|w\|} \right) \leqslant \frac{4}{\delta^2} \|v\| \, \|w\| \ .$$

Hence b is bounded. $\square$

12.2. Orthogonal decomposition and the Riesz representation theorem

12.2.1 One of the issues with infinite-dimensional analysis is that a closed subspace of an infinite dimensional Banach space might not have a closed complement. Fortunately, the situation in Hilbert space theory is not so grim because every closed subspace of a Hilbert space admits an orthogonal complement. This is one of the four crucial properties which distinguish Hilbert spaces from Banach spaces and which are stated in the following.

In this section $\mathfrak{H}$ will always denote a Hilbert space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The symbol $\langle \cdot, \cdot \rangle$ will stand for the inner product of $\mathfrak{H}$.

12.2.2 Theorem (Best approximation theorem) *Every closed convex nonempty subset C of a Hilbert space $\mathfrak{H}$ has a unique element of minimal norm.*

Proof. Let $d = \inf\{\|v\| \mid v \in C\}$ which is a non-negative real number. We claim there exists a unique $v_0 \in C$ with $\|v_0\| = d$. For uniqueness, consider two vectors v_0, v_1 satisfying the desired property, and let $v = \frac{1}{2}(v_0 + v_1)$ be their midpoint. Then

$$\|v\| = \frac{1}{2}\|v_0 + v_1\| \leqslant \frac{1}{2}(\|v_0\| + \|v_1\|) = d$$

By minimality of d this entails $\|v\| = d$. By the parallelogram identity

$$\left\|\frac{1}{2}(v_0 + v_1)\right\|^2 + \left\|\frac{1}{2}(v_0 - v_1)\right\|^2 = 2\left\|\frac{v_0}{2}\right\|^2 + 2\left\|\frac{v_1}{2}\right\|^2 = d^2 ,$$

hence

$$\left\|\frac{1}{2}(v_0 - v_1)\right\|^2 \leqslant d^2 - \|v\|^2 = 0 ,$$

proving $v_0 = v_1$.

For the proof of existence observe that by definition of d there exists a sequence $(v_n)_{n\in\mathbb{N}} \subset C$ such that $\lim_{n\to\infty} \|v_n\| = d$. By convexity

$$\frac{1}{2}(v_n + v_m) \in C$$

for all $n, m \in \mathbb{N}$, hence $\frac{1}{4}\|v_n + v_m\|^2 \geqslant d^2$. The parallelogram equality entails

$$0 \leqslant \|v_n - v_m\|^2 = 2\|v_n\|^2 + 2\|v_m\|^2 - \|v_n + v_m\|^2 \leqslant 2\|v_n\|^2 + 2\|v_m\|^2 - 4d^2 .$$

Since $\lim_{n\to\infty} \|v_n\| = d$ there exists for given $\varepsilon > 0$ an $N \in \mathbb{N}$ such that $\|v_n\|^2 - d^2 \leqslant \frac{1}{4}\varepsilon^2$ for all $n \geqslant N$. Hence, for $n, m \geqslant N$

$$0 \leqslant \|v_n - v_m\| \leqslant \varepsilon ,$$

and $(v_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence, so convergent by completeness of $\mathfrak{H}$. Put $v_0 := \lim_{n\to\infty} v_n$. Then $v_0 \in C$ since C is closed and $\|v_0\| = \lim_{n\to\infty} \|v_n\| = d$. The existence claim follows and the proof is finished. $\square$

12.2.3 Theorem and Definition (Orthogonal decomposition theorem) *Let* $\mathrm{V} \subset \mathfrak{H}$ *be a closed subspace of the Hilbert space* $\mathfrak{H}$. *Then the* orthogonal complement

$$\mathrm{V}^{\perp} = \left\{ w \in \mathfrak{H} \,\middle|\, \langle v, w \rangle = 0 \text{ for each } v \in \mathrm{V} \right\}$$

is a closed subspace of $\mathfrak{H}$ *and* $\mathfrak{H} = \mathrm{V} \oplus \mathrm{V}^{\perp}$. *The map* $\mathrm{pr}_{\mathrm{V}} : \mathfrak{H} \to \mathrm{V}$ *which maps* $w \in \mathfrak{H}$ *to the unique* $w_1 \in \mathrm{V}$ *such that* $w - w_1 \in \mathrm{V}^{\perp}$ *is called the* orthogonal projection *onto* V. *It satisfies* $\|w - \mathrm{pr}_{\mathrm{V}}(w)\| = d(w, \mathrm{V}) := \inf\left\{ \|v - w\| \,\middle|\, v \in \mathrm{V} \right\}$ *that is* $\mathrm{pr}_{\mathrm{V}}(w)$ *is the unique element of* V *having shortest distance from* w.

Proof. For $v \in \mathfrak{H}$ define $v^{\flat} : \mathfrak{H} \to \mathbb{R}$ by $v^{\flat}(w) = \langle w, v \rangle$. Recall that this map is continuous and linear. Hence the kernel $(v^{\flat})^{-1}(0)$ is a closed linear subspace of $\mathfrak{H}$ and

$$\mathrm{V}^{\perp} = \bigcap_{v \in \mathrm{V}} (v^{\flat})^{-1}(0)$$

is a closed linear subspace. To show $\mathrm{V} \cap \mathrm{V}^{\perp} = \{0\}$, consider $v \in \mathrm{V} \cap \mathrm{V}^{\perp}$. Then $\|v\|^2 = \langle v, v \rangle = 0$. Now, given some $w \in \mathfrak{H}$, it can be written as $w = w_1 + w_2$ with $w_1 \in \mathrm{V}$ and $w_2 \in \mathrm{V}^{\perp}$. To see this put $C = w - \mathrm{V}$. Then C is closed and convex. By the best approximation theorem there exists a unique element $w_2 \in C$ of minimal norm. Let w_1 be the unique element of V such that $w_2 = w - w_1$. It remains to show $w_2 \in \mathrm{V}^{\perp}$. Since w_2 has minimal norm among the elements of $w - \mathrm{V}$ the following inequality holds for all vectors $v \in \mathrm{V}$:

$$\|w_2\|^2 \leqslant \|w_2 + v\|^2 = \|w_2\|^2 + 2\,\mathfrak{Re}\langle w_2, v \rangle + \|v\|^2 .$$

Hence

$$0 \leqslant 2\,\mathfrak{Re}\langle w_2, v \rangle + \|v\|^2 \quad \text{for all } v \in \mathrm{V} .$$

Now assume that $\|v\| = 1$ and choose $\varphi \in \mathbb{R}$ such that $e^{i\varphi}\langle w_2, v \rangle \in \mathbb{R}$. Setting $v' = e^{-i\varphi}v$, one obtains for all $\lambda \in \mathbb{R}$ by the last inequality

$$0 \leqslant 2\langle w_2, \lambda v' \rangle + \|\lambda v'\|^2 = 2\lambda\langle w_2, x' \rangle + \lambda^2 .$$

For $\lambda = -\langle w_2, v' \rangle$ this entails the estimate

$$\|\langle w_2, v' \rangle\|^2 = -\left(-2\|\langle w_2, v \rangle\|^2 + \|\langle w_2, v' \rangle\|^2 \right) = -\left(2\lambda\langle w_2, x' \rangle + \lambda^2 \right) \leqslant 0 .$$

Hence $\langle w_2, v \rangle = 0$ for all unit vectors $v \in \mathrm{V}$, therefore $w_2 \in \mathrm{V}^{\perp}$.

The remainder of the claim is now a consequence of the construction of w_1 from the given w and the observation that $\mathrm{pr}_{\mathrm{V}}(w) = w_1$. □

12.2.4 Corollary *For every closed subspace* $\mathrm{V} \subset \mathfrak{H}$ *of a Hilbert space* $\mathfrak{H}$ *the relation*

$$V = (V^{\perp})^{\perp}$$

holds true.

Proof. One has $V \subset (V^{\perp})^{\perp}$ by definition of the orthogonal complement. Since

$$\mathfrak{H} = V \oplus V^{\perp} = (V^{\perp})^{\perp} \oplus V^{\perp}$$

by the preceding theorem, the claim follows. □

12.2.5 Theorem (Riesz representation theorem for Hilbert spaces) *Let $\mathfrak{H}$ be a Hilbert space and $\mathfrak{H}'$ its topological dual. Then the* musical map

$$^\flat : \mathfrak{H} \to \mathfrak{H}', \quad v \mapsto v^\flat = (\mathfrak{H} \ni w \mapsto \langle w, v\rangle \in \mathbb{K})$$

is an isometric isomorphism which is linear in the real case and conjugate-linear in the complex case.

Proof. Obviously, $^\flat$ is linear if the ground field $\mathbb{K}$ equals $\mathbb{R}$ and conjugate-linear if $\mathbb{K} = \mathbb{C}$. Now observe that for all $v \in \mathfrak{H}$ by the Cauchy–Schwarz inequality

$$\|v^\flat\| = \sup\left\{ |\langle w, v\rangle| \;\middle|\; w \in \mathfrak{H} \;\&\; \|w\| = 1 \right\} = \|v\| \ ,$$

hence $^\flat$ is an isometry, so in particular injective. It remains to show surjectivity. So assume that $\alpha : \mathfrak{H} \to \mathbb{K}$ is a nontrivial continuous linear form. Let V be its kernel. Then V is a closed linear subspace of $\mathfrak{H}$. Since α is nontrivial, the orthogonal complement $\mathrm{V}^\perp$ is nontrivial, too. Hence $\mathrm{V}^\perp \cong \mathfrak{H}/\mathrm{V}$ is isomorphic to $\operatorname{im}\alpha = \mathbb{K}$ and there exists a vector $v \in \mathrm{V}^\perp \setminus \{0\}$ such that $\alpha(v) = 1$. Since v spans $\mathrm{V}^\perp$ there exists for every $w \in \mathfrak{H}$ a unique $\lambda_w \in \mathbb{K}$ such that $w = \mathrm{pr}_V(w) + \lambda_w v$. Then compute

$$\alpha(w) = \alpha(\lambda_w v) = \lambda_w \quad \text{and} \quad \left(\frac{v}{\|v\|^2}\right)^\flat (w) = \frac{1}{\|v\|^2}\langle w, v\rangle = \frac{\lambda_w}{\|v\|^2}\langle v, v\rangle = \lambda_w \ .$$

This entails $\alpha = \left(\frac{v}{\|v\|^2}\right)^\flat$, and $^\flat$ is surjective. □

12.2.6 Remark Sometimes, and we will follow that convention, the inverse of the musical isomorphism $^\flat : \mathfrak{H} \to \mathfrak{H}'$ is denoted $^\sharp : \mathfrak{H}' \to \mathfrak{H}$.

12.2.7 Corollary *Every Hilbert space $\mathfrak{H}$ is* reflexive *that is the canonical map*

$$H \to H'', \ v \mapsto \big(H' \ni \lambda \mapsto \lambda(v) \in \mathbb{K}\big)$$

is an isometric isomorphism.

Proof. By the Riesz Representation Theorem, the dual $\mathfrak{H}'$ is a Hilbert space with inner product

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle : \mathfrak{H}' \times \mathfrak{H}' \to \mathbb{K}, \ (\lambda, \mu) \mapsto \langle\!\langle \lambda, \mu \rangle\!\rangle = \langle \mu^\sharp, \lambda^\sharp \rangle \ .$$

Hence, by applying the Riesz Representation Theorem twice, the map $^\flat \circ {}^\flat : \mathfrak{H} \to \mathfrak{H}''$ is an isometric linear isomorphism. Now compute for $v \in \mathfrak{H}$ and $\lambda \in \mathfrak{H}'$

$$(v^\flat)^\flat(\mu) = \langle\!\langle \lambda, v^\flat \rangle\!\rangle = \langle v, \lambda^\sharp \rangle = \lambda(v) \ .$$

Hence $^\flat \circ {}^\flat$ coincides with the canonical map above and the claim follows. □

12.2.8 Corollary *Let $b : \mathfrak{H} \times \mathfrak{H} \to \mathbb{K}$ be a bounded sesquilinear form on a Hilbert space $\mathfrak{H}$. Then there exists unique bounded linear map $A : \mathfrak{H} \to \mathfrak{H}$ such that*

$$b(v, w) = \langle Av, w \rangle \quad \textit{for all } v, w \in \mathfrak{H} \ .$$

Moreover, the operator norm $\|A\|$ coincides with $\|b\|$.

Proof. First let us show uniqueness. So let $A, B : \mathfrak{H} \to \mathfrak{H}$ be bounded and linear so that

$$b(v,w) = \langle Av, w\rangle = \langle Bv, w\rangle \quad \text{for all } v, w \in \mathfrak{H} \ .$$

Then $\|(A-B)v\|^2 = \langle Av - Bv, (A-B)v\rangle = b(v,(A-B)v) - b(v,(A-B)v) = 0$ for all $v \in \mathfrak{H}$ which entails equality of A and B.

To prove existence observe that for every $v \in \mathfrak{H}$ the map $\mathfrak{H} \to \mathbb{K}$, $w \mapsto \overline{b(v,w)}$ is bounded an linear, so by the Riesz representation theorem there exists for every v an element $Av \in \mathfrak{H}$ such that $\langle w, Av\rangle = \overline{b(v,w)}$ for all $w \in \mathfrak{H}$. Let us show that the map A is linear. For $v_1, v_2 \in \mathfrak{H}$ check that

$$\begin{aligned}\langle w, A(v_1+v_2)\rangle &= \overline{b(v_1+v_2,w)} = \overline{b(v_1,w)} + \overline{b(v_2,w)} = \\ &= \langle w, Av_1\rangle + \langle w, Av_2\rangle = \langle w, Av_1 + Av_2\rangle \quad \text{for all } w \in \mathfrak{H} \ .\end{aligned}$$

But that implies $A(v_1+v_2) = Av_1 + Av_2$. Given $r \in \mathbb{K}$ and $v \in \mathfrak{H}$ one verifies

$$\langle w, A(rv)\rangle = \overline{b(rv,w)} = \overline{rb(v,w)} = \overline{r}\,\overline{b(v,w)} = \overline{r}\langle w, Av\rangle = \langle w, rAv\rangle \quad \text{for all } w \in \mathfrak{H} \ .$$

Hence $A(rv) = rAv$ and linearity of A is proved.

For the operator norm compute

$$\begin{aligned}\|A\| &= \sup\big\{ |\langle Av, w\rangle| \;\big|\; v, w \in \mathfrak{H} \,\&\, \|v\| = \|w\| = 1\big\} = \\ &= \sup\big\{ |b(v,w)| \;\big|\; v, w \in \mathfrak{H} \,\&\, \|v\| = \|w\| = 1\big\} = \|b\| \ .\end{aligned}$$ □

12.2.9 Last in this section we will examine the *Hilbert direct sum* or just *Hilbert sum* of a family $(\mathfrak{H}_i)_{i\in I}$ of Hilbert spaces. It is defined by

$$\begin{aligned}\widehat{\bigoplus_{i\in I}}\mathfrak{H}_i &= \left\{ (v_i)_{i\in I} \in \prod_{i\in I}\mathfrak{H}_i \;\middle|\; \big(\|v_i\|^2\big)_{i\in I} \text{ is summable}\right\} = \\ &= \left\{ (v_i)_{i\in I} \in \prod_{i\in I}\mathfrak{H}_i \;\middle|\; \exists C \geqslant 0\, \forall J \in \mathcal{F}(I) : \sum_{i\in J}\|v_i\|^2 \leqslant C\right\} \ ,\end{aligned}$$

where, as usual, $\mathcal{F}(I) \subset \mathcal{P}(I)$ denotes the set of all finite subsets of I.

12.2.10 Proposition *Let $(\mathfrak{H}_i)_{i\in I}$ be a family of Hilbert spaces. Then the Hilbert direct sum $\widehat{\bigoplus_{i\in I}}\mathfrak{H}_i$ is a Hilbert space with inner product given by*

$$\langle -, -\rangle : \widehat{\bigoplus_{i\in I}}\mathfrak{H}_i \times \widehat{\bigoplus_{i\in I}}\mathfrak{H}_i \to \mathbb{K}, \quad \big((v_i)_{i\in I}, (w_i)_{i\in I}\big) \mapsto \sum_{i\in I}\langle v_i, w_i\rangle \ .$$

Proof. We show first that $\widehat{\bigoplus_{i\in I}}\mathfrak{H}_i$ is a subvector space of the direct product $\prod_{i\in I}\mathfrak{H}_i$. Let $z \in \mathbb{K}$ and $(v_i)_{i\in I}, (w_i)_{i\in I} \in \widehat{\bigoplus_{i\in I}}\mathfrak{H}_i$. Choose $C, D \geqslant 0$ such that

$$\sum_{i\in J}\|v_i\|^2 \leqslant C \quad \text{and} \quad \sum_{i\in J}\|w_i\|^2 \leqslant D \quad \text{for all} J \in \mathcal{J} \ .$$

Then

$$\sum_{i\in J}\|zv_i\|^2 = |z|\sum_{i\in J}\|v_i\|^2 \leqslant |z|\,C \quad \text{for all } J\in\mathcal{J}\ , \tag{12.2.1}$$

so $(zv_i)_{i\in I}\in\widehat{\bigoplus_{i\in I}}\mathfrak{H}_i$. Moreover, by Minkowski's inequality for finite sums,

$$\sum_{i\in J}\|v_i+w_i\|^2 \leqslant \left(\sqrt{\sum_{i\in J}\|v_i\|^2}+\sqrt{\sum_{i\in J}\|w_i\|^2}\right)^2 \leqslant \left(\sqrt{C}+\sqrt{D}\right)^2 \quad \text{for all } J\in\mathcal{J}\ . \tag{12.2.2}$$

Hence the family $\left(\|v_i+w_i\|^2\right)_{i\in I}$ is summable and $(v_i+w_i)_{i\in I}\in\widehat{\bigoplus_{i\in I}}\mathfrak{H}_i$.

Next observe that the map

$$\|-\| : \widehat{\bigoplus_{i\in I}}\mathfrak{H}_i \to \mathbb{K},\ (v_i)_{i\in I}\mapsto \left\|(v_i)_{i\in I}\right\| = \sqrt{\sum_{i\in I}\|v_i\|^2}$$

is well-defined by definition of the Hilbert direct sum. It is even a norm by (12.2.1) and (12.2.2).

Now we need to show that the inner product on $\widehat{\bigoplus_{i\in I}}\mathfrak{H}_i$ is well-defined which means that the family $(\langle v_i,w_i\rangle)_{i\in I}$ is summable for all $(v_i)_{i\in I},(w_i)_{i\in I}\in\widehat{\bigoplus_{i\in I}}\mathfrak{H}_i$. To this end let $J\subset I$ be a finite subset. Then, by the triangle inequality, the Cauchy–Schwarz inequality on the Hilbert spaces $\mathfrak{H}_i$ and the Cauchy–Schwarz inequality for finite sums,

$$\left|\sum_{i\in J}\langle v_i,w_i\rangle\right| \leqslant \sum_{i\in J}|\langle v_i,w_i\rangle| \leqslant \sum_{i\in J}\|v_i\|\,\|w_i\| \leqslant \sqrt{\sum_{i\in J}\|v_i\|^2}\cdot\sqrt{\sum_{i\in J}\|w_i\|^2} \leqslant \left\|(v_i)_{i\in I}\right\|\left\|(w_i)_{i\in I}\right\|\ .$$

Hence the family $(\langle v_i,w_i\rangle)_{i\in I}$ is absolutely summable, so in particular summable, and the inner product is well-defined.

By definition and since all the inner products on the Hilbert spaces $\mathfrak{H}_i$ are conjugate symmetric and positive definite, the map $\langle -,-\rangle$ on $\widehat{\bigoplus_{i\in I}}\mathfrak{H}_i$ has to be conjugate symmetric and positive definite as well. It remains to show linearity in the first argument. Denote for $(v_i)_{i\in I},(w_i)_{i\in I}\in\prod_{i\in I}\mathfrak{H}_i$ and $J\in\mathcal{F}(I)$ by $\langle (v_i)_{i\in I},(w_i)_{i\in I}\rangle_J$ the finite sum $\sum_{i\in J}\langle v_i,w_i\rangle$. Observe that the net $\left(\langle (v_i)_{i\in I},(w_i)_{i\in I}\rangle_J\right)_{J\in\mathcal{F}(I)}$ converges to $\langle (v_i)_{i\in I},(w_i)_{i\in I}\rangle$ in case both $(v_i)_{i\in I}$ and $(w_i)_{i\in I}$ are in $\widehat{\bigoplus_{i\in I}}\mathfrak{H}_i$. Now let $z\in\mathbb{K}$ and $(v_i)_{i\in I},(v_i')_{i\in I},(w_i)_{i\in I}\in\widehat{\bigoplus_{i\in I}}\mathfrak{H}_i$. Then

$$\begin{aligned}\langle (v_i)_{i\in I}+(v_i')_{i\in I},(w_i)_{i\in I}\rangle_J &= \langle (v_i)_{i\in I},(w_i)_{i\in I}\rangle_J + \langle (v_i')_{i\in I},(w_i)_{i\in I}\rangle_J \quad \text{and}\\ \langle z(v_i)_{i\in I},(w_i)_{i\in I}\rangle_J &= z\langle (v_i)_{i\in I},(w_i)_{i\in I}\rangle_J\ .\end{aligned}$$

By convergence of all the nets $\left(\langle (v_i)_{i\in I},(w_i)_{i\in I}\rangle_J\right)_{J\in\mathcal{F}(I)}$, linearity in the first argument follows.

By construction, the norm associated to the inner product $\langle -,-\rangle$ on $\widehat{\bigoplus_{i\in I}}\mathfrak{H}_i$ coincides with the above defined norm $\|-\|$. It remains to show that $\widehat{\bigoplus_{i\in I}}\mathfrak{H}_i$ equipped with the norm $\|-\|$ is complete.

To this end observe that for every finite $J \subset I$ the map

$$\| - \|_J : \prod_{i \in I} \mathfrak{H}_i \to \mathbb{R}_{\geqslant 0},\ (v_i)_{i\in I} \mapsto \sqrt{\langle (v_i)_{i\in I}, (v_i)_{i\in I} \rangle_J} = \sqrt{\sum_{i\in J} \|v_i\|^2}$$

is a seminorm and that $(v_i)_{i\in I} \in \prod_{i\in I} \mathfrak{H}_i$ lies in the Hilbert direct sum $\widehat{\bigoplus_{i\in I}} \mathfrak{H}_i$ if and only if the family $\left(\|(v_i)_{i\in I}\|_J\right)_{J\in\mathcal{F}(I)}$ is bounded. Now let $((v_i^n)_{i\in I})_{n\in\mathbb{N}}$ be a Cauchy sequence. Let $\varepsilon > 0$ and choose $N_\varepsilon \in \mathbb{N}$ such that

$$\left\|(v_i^m)_{i\in I} - (v_i^n)_{i\in I}\right\| < \varepsilon \quad \text{for all } n, m \geqslant N_\varepsilon\ . \tag{12.2.3}$$

Hence

$$\left\|(v_i^m)_{i\in I} - (v_i^n)_{i\in I}\right\|_J < \varepsilon \quad \text{for all } J \in \mathcal{F}(I) \text{ and } n, m \geqslant N_\varepsilon\ . \tag{12.2.4}$$

Taking $J = \{j\}$ for $j \in I$ this implies that the sequence $(v_j^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Hilbert space $\mathfrak{H}_j$. Let $v_j \in \mathfrak{H}_j$ be its limit. The family $(v_i)_{i\in I}$ then is an element of $\widehat{\bigoplus_{i\in I}} \mathfrak{H}_i$. To verify this put $N = N_1$ and observe that by (12.2.4) for all finite $J \subset I$

$$\begin{aligned} \left\|(v_i)_{i\in I}\right\|_J &\leqslant \left\|(v_i^N)_{i\in I}\right\|_J + \left\|(v_i)_{i\in I} - (v_i^N)_{i\in I}\right\|_J = \\ &= \left\|(v_i^N)_{i\in I}\right\|_J + \lim_{m\to\infty} \left\|(v_i^m)_{i\in I} - (v_i^N)_{i\in I}\right\|_J \leqslant \left\|(v_i^N)_{i\in I}\right\| + 1. \end{aligned}$$

Hence the family $\left(\|(v_i)_{i\in I}\|_J\right)_{J\in\mathcal{F}(I)}$ is bounded and $(v_i)_{i\in I}$ lies in the Hilbert direct sum of the spaces $\mathfrak{H}_i$, $i \in I$. Moreover, (12.2.4) entails that

$$\left\|(v_i)_{i\in I} - (v_i^n)_{i\in I}\right\|_J = \lim_{m\to\infty} \left\|(v_i^m)_{i\in I} - (v_i^n)_{i\in I}\right\|_J \leqslant \varepsilon \quad \text{for all } J \in \mathcal{F}(I) \text{ and } n \geqslant N_\varepsilon\ .$$

Since $\left\|(v_i)_{i\in I} - (v_i^n)_{i\in I}\right\|$ is the limit of the net $\left(\|(v_i)_{i\in I} - (v_i^n)_{i\in I}\|_J\right)_{J\in\mathcal{F}(I)}$, the estimate

$$\left\|(v_i)_{i\in I} - (v_i^n)_{i\in I}\right\| \leqslant \varepsilon \quad \text{for all } n \geqslant N_\varepsilon$$

follows and the sequence $((v_i^n)_{i\in I})_{n\in\mathbb{N}}$ convergence to $(v_i)_{i\in I}$. This finishes the proof. □

12.3. Orthonormal bases in Hilbert spaces

12.3.1 Definition A (possibly empty) subset S of a Hilbert space $\mathfrak{H}$ is called an *orthogonal system* or just *orthogonal* if for any two different elements $v, w \in S$ the relation $\langle v, w\rangle = 0$ holds true. If in addition $\|v\| = 1$ for all elements $v \in S$, then the set is called *orthonormal* or an *orthonormal system*. A family $(v_i)_{i\in I}$ of vectors in $\mathfrak{H}$ is called *orthogonal* if $\langle v_i, v_j\rangle = 0$ for all $i, j \in I$ with $i \neq j$ and *orthonormal* if in addition $\|v_i\| = 1$ for all $i \in I$.

12.3.2 Obviously, the set of orthonormal subsets of a Hilbert space is ordered by set-theoretic inclusion. Therefore, the following definition makes sense.

12.3.3 Definition A maximal orthonormal set in a Hilbert space $\mathfrak{H}$ is called an *orthonormal basis* or a *Hilbert basis* of $\mathfrak{H}$.

12.3.4 Proposition *Every Hilbert space $\mathfrak{H}$ has an orthonormal basis.*

Proof. Wothout loss of generality we can assume that $\mathfrak{H} \neq \{0\}$, because $\varnothing$ is a Hilbert basis for $\{0\}$. Let $\mathcal{O}$ denote the set of orthonormal subsets of $\mathfrak{H}$. As mentioned before, $\mathcal{O}$ is ordered by set-theoretic inclusion. Let $\mathcal{C} \subset \mathcal{O}$ be a non-empty chain. Put $U = \bigcup_{S \in \mathcal{C}} S$. Then U is an upper bound of $\mathcal{C}$. So by Zorn's lemma $\mathcal{O}$ has a maximal element. $\square$

12.3.5 Remark (a) By slight abuse of language we sometimes call an orthonormal family $(b_i)_{i \in I}$ in a Hilbert space $\mathfrak{H}$ an *orthonormal basis* or a *Hilbert basis* of $\mathfrak{H}$ if the set $\{b_i \mid i \in I\}$ is an orthornormal basis.

(b) If on an orthonormal basis $B \subset \mathfrak{H}$ a total order relation is given, one calls B an *ordered Hilbert basis* of $\mathfrak{H}$. Likewise, an orthonormal basis of the form $(b_i)_{i \in I}$ is called *orderd* if the index set I carries a total order.

12.3.6 Proposition (Pythagorean theorem for orthogonal families) *An orthogonal family $(v_i)_{i \in I}$ in a Hilbert space $\mathfrak{H}$ is summable if and only if the family of norms $(\|v_i\|)_{i \in I}$ is square summable. In this case one has*

$$\left\| \sum_{i \in I} v_i \right\|^2 = \sum_{i \in I} \|v_i\|^2 .$$

Proof. Assume that $(\|v_i\|)_{i \in I}$ is square summable or in other words that the net of partial sums $\left(\sum_{i \in J} \|v_i\|^2\right)_{J \in \mathcal{F}(I)}$ converges to some $s \in \mathbb{R}$. For $\varepsilon > 0$ choose a finite $J_\varepsilon \subset I$ such that for all finite J with $J_\varepsilon \subset J \subset I$ the relation

$$\left| s - \sum_{i \in J} \|v_i\|^2 \right| < \frac{\varepsilon^2}{2}$$

holds true. For finite $K \subset I$ with $K \cap J_\varepsilon = \varnothing$ one then obtains by the pythagorean theorem for finite orthogonal families, Eq. (12.1.2),

$$\left\| \sum_{i \in K} v_i \right\|^2 = \sum_{i \in K} \|v_i\|^2 \leqslant \left| s - \sum_{i \in K \cup J_\varepsilon} \|v_i\|^2 \right| + \left| s - \sum_{i \in J_\varepsilon} \|v_i\|^2 \right| < \varepsilon^2 .$$

Hence $\left(\sum_{i \in J} v_i\right)_{J \in \mathcal{F}(I)}$ is a Cauchy net in $\mathfrak{H}$, so convergent.

Now let $(v_i)_{i \in I}$ be summable to $v \in \mathfrak{H}$. Then there exists a $J_1 \in \mathcal{F}(I)$ such that for all finite $J \subset I$ containing J_1

$$\left\| v - \sum_{i \in J} v_i \right\| \leqslant 1 .$$

This implies by the pythagorean theorem for finite orthogonal families

$$\sum_{i \in J} \|v_i\|^2 = \left\| \sum_{i \in J} v_i \right\|^2 \leqslant \left(\left\| v - \sum_{i \in J} v_i \right\| + \|v\| \right)^2 \leqslant (1 + \|v\|)^2 .$$

Hence the net of partial sums $\left(\sum_{i\in J}\|v_i\|^2\right)_{J\in\mathcal{F}(I)}$ is bounded, so convergent since each term $\|v_i\|^2$ is $\geqslant 0$.

By continuity of the inner product and pairwise orthogonality of the v_i we finally obtain in the convergent case

$$\left\|\sum_{i\in I} v_i\right\|^2 = \left\langle\sum_{i\in I} v_i, \sum_{j\in I} v_j\right\rangle = \sum_{i\in I}\left\langle v_i, \sum_{j\in I} v_j\right\rangle = \sum_{i\in I}\sum_{j\in I}\langle v_i, v_j\rangle = \sum_{i\in I}\|v_i\|^2 \ .$$

□

12.3.7 Proposition *Let $(v_i)_{i\in I}$ be an orthonormal family in a Hilbert space $\mathfrak{H}$. Then for every $v\in\mathfrak{H}$ the family $(\langle v, v_i\rangle)_{i\in I}$ is square summable and* Bessel's inequality *holds true that is*

$$\sum_{i\in I}|\langle v, v_i\rangle|^2 \leqslant \|v\|^2 \ .$$

Proof.

12.3.8 Theorem *Let B be an orthonormal system in a Hilbert space $\mathfrak{H}$. Then the following are equivalent:*

(1) *The orthonormal system B is maximal, i.e. a Hilbert basis.*

(2) *The orthonormal system B is* total *that is for all $v\in H$ such that $\langle v, b\rangle = 0$ for all $b\in B$ the equality $v = 0$ holds true.*

(3) *For every $b\in B$ let $\mathfrak{H}_b = \{rb\in\mathfrak{H} \mid r\in\mathbb{K}\}$. Then the canonical map*

$$\iota : \widehat{\bigoplus_{b\in B}}\mathfrak{H}_b \to \mathfrak{H},\ (v_b)_{b\in B}\mapsto \sum_{b\in B} v_b$$

is an isometric isomorphism.

(4) *The closed linear span $\overline{\mathrm{Span}}B$ coincides with $\mathfrak{H}$.*

(5) *For all $v\in\mathfrak{H}$, one has the* Fourier expansion

$$v = \sum_{b\in B}\langle v, b\rangle b \ .$$

(6) *For all $v, w\in\mathfrak{H}$, one has*

$$\langle v, w\rangle = \sum_{b\in B}\langle v, b\rangle\langle b, w\rangle \ .$$

(7) *For all $v\in\mathfrak{H}$,* Parseval's identity *holds true that is*

$$\|v\|^2 = \sum_{b\in B}|\langle v, b\rangle|^2 \ .$$

Proof. (1) $\Rightarrow$ (2): If $v \neq 0$, then $\frac{v}{\|v\|}$ is a unit vector orthogonal to each v_i. Hence $\{v\} \cup B$ is an orthonormal system which is strictly larger than B, contradicting (1).

(2) $\Rightarrow$ (3). First note that by the pythagorean theorem for infinite families, Proposition 12.3.6, the canonical map $\iota : \widehat{\bigoplus}_{b\in B} H_b \to H$ is well-defined and an isometry. Hence ι is injective. It remains to show that ι is surjective. To this end observe that $\operatorname{im}\iota$ is closed in $\mathfrak{H}$ since ι is an isometry (the image is complete). If ι is not surjective, then $\operatorname{im}\iota^{\perp}$ is not the zero vector space. Choose $v \in \operatorname{im}\iota^{\perp}\backslash\{0\}$. Then v is orthogonal to each element of B, but $v \neq 0$. This contradicts (2), so $\operatorname{im}\iota = \mathfrak{H}$.

(3) $\Rightarrow$ (5): We can represent any $v \in \mathfrak{H}$ in the form $v = \iota\left((v_b)_{b\in B}\right) = \sum_{b\in B} v_b$ with $(v_b)_{b\in B} \in \widehat{\bigoplus}_{b\in B} H_b$. Write $v_b = r_b\, b$ for every $b \in B$, where $r_b \in \mathbb{K}$ is uniquely determined by v_b. Then compute using continuity of the inner product

$$\langle v, b\rangle = \langle \sum_{c\in B} v_c, b\rangle = \sum_{c\in B} r_c \langle c, b\rangle = r_b \ .$$

Therefore,

$$v = \sum_{b\in B} r_b\, b = \sum_{b\in B} \langle v, b\rangle b \ .$$

(5) $\Rightarrow$ (6): Fourier expansion of $v, w \in H$ gives $v = \sum\limits_{b\in B} \langle v, b\rangle b$ and $w = \sum\limits_{b\in B} \langle w, b\rangle b$. Then, by continuity of the inner product,

$$\langle v, w\rangle = \sum_{b\in B} \langle v, b\rangle\langle b, w\rangle \ .$$

(5) $\Rightarrow$ (4): Let $v \in \mathfrak{H}$. Then $\sum\limits_{b\in J} \langle v, b\rangle b \in \operatorname{Span}(B)$ for all finite $J \subset B$. But by Fourier expansion v is the limit of the net $\left(\sum\limits_{b\in J} \langle v, b\rangle b\right)_{J\in\mathcal{F}(B)}$, so v lies in the closure $\overline{\operatorname{Span}}(B)$.

(4) $\Rightarrow$ (2): Assume that $\langle v, b\rangle = 0$ for all $b \in B$. By (4), v can be written as a limit $v = \lim\limits_{n\to\infty} v_n$, where $v_n \in \operatorname{Span}(B)$ for all $n \in \mathbb{N}$. Then $\langle v, v_n\rangle = 0$ for all $n \in \mathbb{N}$ by assumption. By continuity of the inner product this implies

$$\|v\|^2 = \lim_{n\to\infty} \langle v, v_n\rangle = 0 \ ,$$

so $v = 0$.

(6) $\Rightarrow$ (7): Put $v = w$. Then, by assumption,

$$\|v\|^2 = \langle v, v\rangle = \sum_{b\in B} \langle v, b\rangle\langle b, v\rangle = \sum_{b\in B} |\langle v, b\rangle|^2 \ .$$

(7) $\Rightarrow$ (1): Assume (7) and that (1) is not true. Then there exists $v \in H$ with $\|v\| = 1$ and $\langle v, b\rangle = 0$ for all $b \in B$. But then

$$\|v\|^2 = \sum_{b\in B} |\langle v, b\rangle|^2 = 0,$$

which is a contradiction. $\square$

12.4. The monoidal structure of the category of Hilbert spaces

12.4.1 Let $\mathbb{K}$ be the field of real or complex numbers. Hilbert spaces over $\mathbb{K}$ together with bounded $\mathbb{K}$-linear maps between them form a category denoted by $\mathbb{K}$-Hilb or just Hilb if no confusion can arise. This can be seen immediately by observing that the identity map $\mathbb{1}_{\mathfrak{H}}$ on a Hilbert space is a bounded linear operator and that the composition $B \circ A : \mathfrak{H}_1 \to \mathfrak{H}_3$ of two bounded linear operators between Hilbert spaces $A : \mathfrak{H}_1 \to \mathfrak{H}_2$ and $B : \mathfrak{H}_2 \to \mathfrak{H}_3$ is again a bounded linear operator. We want to endow the category Hilb with a bifunctor $\widehat{\otimes} : \mathsf{Hilb} \times \mathsf{Hilb} \to \mathsf{Hilb}$ so that it becomes a monoidal category. The (bi)functor $\widehat{\otimes}$ will be called the *Hilbert tensor product.*

Unless mentioned differently, Hilbert spaces, vector spaces and the algebraic tensor product $\otimes$ in this section are assumed to be taken over the ground field $\mathbb{K}$.

12.4.2 Proposition *Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be two Hilbert spaces. Then there exists a unique inner product $\langle \cdot, \cdot \rangle : (\mathfrak{H}_1 \otimes \mathfrak{H}_2) \times (\mathfrak{H}_1 \otimes \mathfrak{H}_2) \to \mathbb{K}$ on the algebraic tensor product $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ such that*

$$\langle v_1 \otimes v_2, w_1 \otimes w_2 \rangle = \langle v_1, w_1 \rangle \cdot \langle v_2, w_2 \rangle \quad \textit{for all } v_1, w_1 \in \mathfrak{H}_1,\ v_2, w_2 \in \mathfrak{H}_2 \ . \tag{12.4.1}$$

Proof. Let us first provide some preliminary constructions. Recall that for every pair of vector spaces V_1 and V_2 the bilinear map

$$\begin{aligned} \tau : \mathrm{Hom}(\mathrm{V}_1, \mathbb{K}) \times \mathrm{Hom}(\mathrm{V}_2, \mathbb{K}) &\to \mathrm{Hom}(\mathrm{V}_1 \otimes \mathrm{V}_2, \mathbb{K}), \\ (\lambda_1, \lambda_2) &\mapsto \big(\mathrm{V}_1 \otimes \mathrm{V}_2 \to \mathbb{K},\ v_1 \otimes v_2 \mapsto \lambda_1(v_1) \cdot \lambda_2(v_2)\big) \end{aligned}$$

induces a linear map

$$\widehat{\tau} : \mathrm{Hom}(\mathrm{V}_1, \mathbb{K}) \otimes \mathrm{Hom}(\mathrm{V}_2, \mathbb{K}) \to \mathrm{Hom}(\mathrm{V}_1 \otimes \mathrm{V}_2, \mathbb{K})$$

by the universal property of the tensor product. This map is an isomorphism. To see this choose a basis $(v_{1i})_{i \in I}$ of V_1 and a basis $(v_{2j})_{j \in J}$ of V_2. Let $(v'_{1i})_{i \in I}$ and $(v'_{2j})_{j \in J}$ denote the respective dual bases of V'_1 and V'_2. Then $\left(v'_{1i} \otimes v'_{2j}\right)_{(i,j) \in I \times J}$ is a basis of $\mathrm{Hom}(\mathrm{V}_1, \mathbb{K}) \otimes \mathrm{Hom}(\mathrm{V}_2, \mathbb{K})$ which under $\widehat{\tau}$ is mapped bijectively to the basis $((v_{1i} \otimes v_{2j})')_{(i,j) \in I \times J}$ of $\mathrm{Hom}(\mathrm{V}_1 \otimes \mathrm{V}_2, \mathbb{K})$ dual to the basis $(v_{1i} \otimes v_{2j})_{(i,j) \in I \times J}$ of $\mathrm{V}_1 \otimes \mathrm{V}_2$. Hence $\widehat{\tau}$ is a linear isomorphism as claimed, and we can identify the tensor product $\lambda_1 \otimes \lambda_2$ of two linear functionals $\lambda_i : \mathrm{V}_i \to \mathbb{K}$, $i = 1, 2$ with its image in $\mathrm{Hom}(\mathrm{V}_1 \otimes \mathrm{V}_2, \mathbb{K})$.

Now observe that for two conjugate-linear maps $\mu_1 : \mathrm{V}_1 \to \mathbb{K}$ and $\mu_2 : \mathrm{V}_2 \to \mathbb{K}$ the map $\tau^*(\mu_1, \mu_2) = \overline{\overline{\mu_1} \otimes \overline{\mu_2}} : \mathrm{V}_1 \otimes \mathrm{V}_2 \to \mathbb{K}$ is conjugate-linear and satisfies

$$\tau^*(\mu_1, \mu_2)\,(v_1 \otimes v_2) = \mu_1(v_1) \cdot \mu_2(v_2) \quad \text{for all } v_1 \in \mathrm{V}_1,\ v_2 \in \mathrm{V}_2 \ . \tag{12.4.2}$$

One obtains a map

$$\tau^* : \mathrm{Hom}^*(\mathrm{V}_1, \mathbb{K}) \times \mathrm{Hom}^*(\mathrm{V}_2, \mathbb{K}) \to \mathrm{Hom}^*(\mathrm{V}_1 \otimes \mathrm{V}_2, \mathbb{K}) \ ,$$

where here the symbol $\mathrm{Hom}^*(\mathrm{V}, \mathbb{K})$ denotes the space of all conjugate linear functionals on a vector space V. Since τ^* is biadditive and since $\tau^*(z\mu_1, \mu_2) = \tau^*(\mu_1, z\mu_2)$ for all $\mu_1 \in \mathrm{Hom}^*(\mathrm{V}_1, \mathbb{K})$, $\mu_2 \in \mathrm{Hom}^*(\mathrm{V}_2, \mathbb{K})$, and $z \in \mathbb{K}$, the map τ^* factors through a linear map

$$\widehat{\tau^*} : \mathrm{Hom}^*(\mathrm{V}_1, \mathbb{K}) \otimes \mathrm{Hom}^*(\mathrm{V}_2, \mathbb{K}) \to \mathrm{Hom}^*(\mathrm{V}_1 \otimes \mathrm{V}_2, \mathbb{K}) \ .$$

Using the above bases $(v_{1i})_{i\in I}$ and $(v_{2j})_{j\in J}$ of V_1 and V_2 respectively, one observes that $\widehat{\tau^*}$ is an isomorphism since it maps the basis $\left(\overline{v'_{1i}} \otimes \overline{v'_{2j}}\right)_{(i,j)\in I\times J}$ of $\mathrm{Hom}^*(\mathrm{V}_1, \mathbb{K}) \otimes \mathrm{Hom}^*(\mathrm{V}_2, \mathbb{K})$ bijectively to the basis $\left(\overline{(v_{1i} \otimes v_{2j})'}\right)_{(i,j)\in I\times J}$ of the space $\mathrm{Hom}^*(\mathrm{V}_1 \otimes \mathrm{V}_2, \mathbb{K})$. So $\widehat{\tau^*}$ is also a linear isomorphism, which allows us to identify the tensor product $\mu_1 \otimes \mu_2$ of two conjugate linear functionals $\mu_i : \mathrm{V}_i \to \mathbb{K}$, $i = 1, 2$ with its image in $\mathrm{Hom}^*(\mathrm{V}_1 \otimes \mathrm{V}_2, \mathbb{K})$.

After these preliminary considerations we consider the map

$$\mathfrak{H}_1 \times \mathfrak{H}_2 \to \mathrm{Hom}^*(\mathfrak{H}_1 \otimes \mathfrak{H}_2, \mathbb{K}),\ (v_1, v_2) \mapsto \overline{v_1^\flat} \otimes \overline{v_2^\flat} = \tau^*\left(\overline{v_1^\flat}, \overline{v_2^\flat}\right) = \widehat{\tau^*}\left(\overline{v_1^\flat} \otimes \overline{v_2^\flat}\right) ,$$

which is well-defined and bilinear since the musical isomorphisms $^\flat : \mathfrak{H}_l \to \mathfrak{H}'_l$, $v \mapsto \langle -, v\rangle$, $l = 1, 2$, are conjugate-linear and since τ^* is bilinear. Hence it factors through a linear map

$$\beta : \mathfrak{H}_1 \otimes \mathfrak{H}_2 \to \mathrm{Hom}^*(\mathfrak{H}_1 \otimes \mathfrak{H}_2, \mathbb{K})$$

such that

$$\beta(v_1 \otimes v_2)(w_1 \otimes w_2) = \langle v_1, w_1\rangle \cdot \langle v_2, w_2\rangle \quad \text{for all } v_1, w_1 \in \mathfrak{H}_1,\ v_2, w_2 \in \mathfrak{H}_2 . \tag{12.4.3}$$

Now put

$$\langle \cdot, \cdot\rangle : (\mathfrak{H}_1 \otimes \mathfrak{H}_2) \times (\mathfrak{H}_1 \otimes \mathfrak{H}_2) \to \mathbb{K},\ (v, w) \mapsto \beta(v)(w) .$$

Then $\langle \cdot, \cdot\rangle$ is sesquilinear by construction, and (12.4.1) holds true by (12.4.3).

Let us show that $\langle \cdot, \cdot\rangle$ is positive definite. Let $v = \sum_{k=1}^n v_{1k} \otimes v_{2k} \in \mathfrak{H}_1 \otimes \mathfrak{H}_2$. Choose an orthonormal basis $e_1, \ldots, e_m$ of the linear subspace spanned by $v_{21}, \ldots, v_{2n}$. Expand $v_{2k} = \sum_{i=1}^m c_{ki} e_i$ with $c_{k1}, \ldots, c_{km} \in \mathbb{K}$. Then

$$v = \sum_{k=1}^n v_{1k} \otimes v_{2k} = \sum_{k=1}^n \sum_{i=1}^m v_{1k} \otimes (c_{ki} e_i) = \sum_{i=1}^m \left(\sum_{k=1}^n c_{ki} v_{1k}\right) \otimes e_i = \sum_{i=1}^m w_{1i} \otimes e_i , \tag{12.4.4}$$

where $w_{1i} = \sum_{k=1}^n c_{ki} v_{1k}$. Hence

$$\langle v, v\rangle = \langle \sum_{i=1}^m w_{1i} \otimes e_i, \sum_{j=1}^m w_{1j} \otimes e_j\rangle = \sum_{i=1}^m \sum_{j=1}^m \langle w_{1i}, w_{1j}\rangle\langle e_i, e_j\rangle = \sum_{i=1}^m \|w_{1i}\|^2 \geqslant 0 . \tag{12.4.5}$$

Moreover, if $\langle v, v\rangle = 0$, then $w_{1i} = 0$ for $i = 1, \ldots, m$, which implies $v = \sum_{i=1}^m w_{1i} \otimes e_i = 0$. So $\langle \cdot, \cdot\rangle$ is an inner product on $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ satisfying (12.4.1). It is uniquely determined by this condition since the vectors $v_1 \otimes v_2$ with $v_1 \in \mathfrak{H}_1$ and $v_2 \in \mathfrak{H}_2$ span $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. □

12.4.3 Definition Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be Hilbert spaces. The Hilbert completion of the algebraic tensor product $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ equipped with the unique inner product $\langle \cdot, \cdot\rangle$ fulfilling (12.4.1) will be denoted $\mathfrak{H}_1 \widehat{\otimes} \mathfrak{H}_2$, its inner product again by $\langle \cdot, \cdot\rangle$. One calls the Hilbert space $\left(\mathfrak{H}_1 \widehat{\otimes} \mathfrak{H}_2, \langle \cdot, \cdot\rangle\right)$ the *Hilbert tensor product* of $\mathfrak{H}_1$ and $\mathfrak{H}_2$ or just the *tensor product* of $\mathfrak{H}_1$ and $\mathfrak{H}_2$ if no confusion can arise.

12.4.4 Proposition *Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be Hilbert spaces.*

(i) *If $A_i \subset \mathfrak{H}_i$ for $i = 1, 2$ are total in the ambient Hilbert space, then the set of simple vectors $a_1 \otimes a_2$ with $a_1 \mathfrak{H}_1$ and $a_2 \mathfrak{H}_2$ is total in the Hilbert tensor product $\mathfrak{H}_1 \hat{\otimes} \mathfrak{H}_2$.*

(ii) *If $(e_i)_{i \in I}$ and $(f_j)_{j \in J}$ are orthonormal bases of $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively, then $(e_i \otimes f_j)_{(i,j) \in I \times J}$ is an orthonormal basis of the Hilbert tensor product $\mathfrak{H}_1 \hat{\otimes} \mathfrak{H}_2$.*

Proof. ad (i). Recall that a subset $A \subset \mathfrak{H}$ or a family $A = (a_j)_{j \in J}$ of elements of a Hilbert space $\mathfrak{H}$ is called *total* in $\mathfrak{H}$ if the linear span of A is dense in $\mathfrak{H}$. By density of the algebraic tensor product $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ in the Hilbert tensor product $\mathfrak{H}_1 \hat{\otimes} \mathfrak{H}_2$, the set of simple tensors $v_1 \otimes v_2$ with $v_i \in \mathfrak{H}_i$ for $i = 1, 2$ is total in $\mathfrak{H}_1 \hat{\otimes} \mathfrak{H}_2$. Hence it suffices to find for such v_i and all $\varepsilon > 0$ vectors $w_i \in \operatorname{Span} A_i$ for $i = 1, 2$ such that

$$\|v_1 \otimes v_2 - w_1 \otimes w_2\| < \frac{\varepsilon}{2} .$$

By totality of A_i in $\mathfrak{H}_i$ there exist $w_i \in \operatorname{Span} A_i$ such that

$$\|v_1 - w_1\| < \min\left\{1, \frac{\varepsilon}{2(\|v_2\| + 1)}\right\} \quad \text{and} \quad \|v_2 - w_2\| < \frac{\varepsilon}{2(\|v_1\| + 1)} .$$

Then

$$\|v_1 \otimes v_2 - w_1 \otimes w_2\| \leqslant \|v_1 - w_1\| \, \|v_2\| + \|v_2 - w_2\| \|w_1\| < \varepsilon .$$

ad (ii). The family $(e_i \otimes f_j)_{(i,j) \in I \times J}$ is orthonormal by definition of the inner product on $\mathfrak{H}_1 \hat{\otimes} \mathfrak{H}_2$. It is total by (i) and therefore a Hilbert basis. □

12.4.5 Proposition *Assigning to each pair of Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ the Hilbert tensor product $\mathfrak{H}_1 \hat{\otimes} \mathfrak{H}_2$ and to each pair of bounded linear operators $A_1 : \mathfrak{H}_1 \to \mathfrak{H}_3$ and $A_2 : \mathfrak{H}_2 \to \mathfrak{H}_4$ between Hilbert spaces the unique bounded extension $A_1 \hat{\otimes} A_2 : \mathfrak{H}_1 \hat{\otimes} \mathfrak{H}_2 \to \mathfrak{H}_3 \hat{\otimes} \mathfrak{H}_4$ of the operator $A_1 \otimes A_2 : \mathfrak{H}_1 \otimes \mathfrak{H}_2 \to \mathfrak{H}_3 \hat{\otimes} \mathfrak{H}_4$, $v_1 \otimes v_2 \mapsto A_1(v_1) \otimes A_2(v_2)$ comprises a (covariant) bifunctor*

$$\hat{\otimes} : \mathsf{Hilb} \times \mathsf{Hilb} \to \mathsf{Hilb} .$$

Moreover, $\hat{\otimes}$ is isometric in the sense that

$$\|v_1 \otimes v_2\| = \|v_1\| \, \|v_2\| \quad \textit{for all } v_1 \in \mathfrak{H}_1,\ v_2 \in \mathfrak{H}_1 \textit{ and} \tag{12.4.6}$$

$$\|A_1 \hat{\otimes} A_2\| = \|A_1\| \, \|A_2\| \quad \textit{for all } A_1 \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_3),\ A_2 \in \mathfrak{B}(\mathfrak{H}_2, \mathfrak{H}_4) . \tag{12.4.7}$$

Proof. We first show that $A_1 \otimes A_2$ is a bounded operator. To this end observe that $A_1 \otimes A_2$ can be written as the composition of the two operators $A_1 \otimes \mathbb{1}_{\mathfrak{H}_2}$ and $\mathbb{1}_{\mathfrak{H}_3} \otimes A_2$. Hence it suffices to show that each of these linear maps is bounded. Let $v = \sum_{k=1}^n v_{1k} \otimes v_{2k} \in \mathfrak{H}_1 \otimes \mathfrak{H}_2$ be of norm 1. As in the proof of Proposition 12.4.2 expand $v_{2k} = \sum_{i=1}^m c_{ki} e_i$, $k = 1, \ldots, n$, where $e_1, \ldots, e_m$ is an orthonormal basis of $\operatorname{Span}\{v_{21}, \ldots, v_{2n}\}$ and $c_{k1}, \ldots, c_{km} \in \mathbb{K}$. Equations (12.4.4) and (12.4.5) then entail that

$$v = \sum_{i=1}^m w_{1i} \otimes e_i \quad \text{and} \quad 1 = \langle v, v \rangle = \sum_{i=1}^m \|w_{1i}\|^2$$

for $w_{1i} = \sum_{k=1}^n c_{ki} v_{1k}$, $i = 1, \ldots, m$. Hence

$$\|(A_1 \otimes \mathbb{1}_{\mathfrak{H}_2}) v\|^2 = \left\| \sum_{i=1}^m A_1(w_{1i}) \otimes e_i \right\|^2 = \sum_{i=1}^m \|A_1(w_{1i})\|^2 \leqslant \|A_1\|^2 \sum_{i=1}^m \|w_{1i}\|^2 = \|A_1\|^2 ,$$

so $A_1 \otimes \mathbb{1}_{\mathfrak{H}_2}$ is bounded with norm $\leqslant \|A_1\|$. By symmetry, $\mathbb{1}_{\mathfrak{H}_3} \otimes A_2$ is bounded with norm $\leqslant \|A_2\|$. Hence $A_1 \otimes A_2 = (\mathbb{1}_{\mathfrak{H}_3} \otimes A_2) \circ (A_1 \otimes \mathbb{1}_{\mathfrak{H}_2})$ is bounded and

$$\|A_1 \otimes A_2\| \leqslant \|A_1\| \, \|A_2\| \ .$$

Let us show the converse inequality. For given $\varepsilon > 0$ there exist unit vectors $v_i \in \mathfrak{H}_i$, $i = 1, 2$ such that $\|A_i v_i\| \geqslant \|A_i\| - \frac{\varepsilon}{2(\|A_1\|+\|A_2\|+1)}$. Then

$$\|(A_1 \hat{\otimes} A_2)(v_1 \otimes v_2)\| = \|A_1 v_1\| \, \|A_2 v_2\| \geqslant \|A_1\| \, \|A_2\| - \varepsilon \ .$$

This implies

$$\|A_1 \otimes A_2\| \geqslant \|A_1\| \, \|A_2\|$$

and (12.4.7) follows. Equality (12.4.6) is clear by construction of the Hilbert tensor product.

Next observe that $\mathbb{1}_{\mathfrak{H}_1} \hat{\otimes} \mathbb{1}_{\mathfrak{H}_2} = \mathbb{1}_{\mathfrak{H}_1 \hat{\otimes} \mathfrak{H}_2}$ by definition. Given Hilbert spaces $\mathfrak{H}_1, \ldots, \mathfrak{H}_6$ and bounded linear operators $A_i : \mathfrak{H}_i \to \mathfrak{H}_{i+2}$ and $B_i : \mathfrak{H}_{i+2} \to \mathfrak{H}_{i+4}$ for $i = 1, 2$, the composition $(B_1 \otimes B_2) \circ (A_1 \otimes A_2)$ coincides with $(B_1 \circ A_1) \otimes (B_2 \circ A_2)$ by functoriality of the algebraic tensor product. By continuity of the operators $A_1 \hat{\otimes} A_2$ and $B_1 \hat{\otimes} B_2$ and by density of $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ in $\mathfrak{H}_1 \hat{\otimes} \mathfrak{H}_2$ the equality

$$(B_1 \hat{\otimes} B_2) \circ (A_1 \hat{\otimes} A_2) = (B_1 \circ A_1) \hat{\otimes} (B_2 \circ A_2)$$

follows. Hence $\hat{\otimes}$ is a bifunctor as claimed. $\square$

12.4.6 Proposition *For every Hilbert space $\mathfrak{H}$ one has two natural isomorphisms*

$$\hat{u}_{\mathfrak{H}} : \mathbb{K} \hat{\otimes} \mathfrak{H} \to \mathfrak{H}, \ z \otimes v \to zv \quad \textit{and} \quad {}_{\mathfrak{H}}\hat{u} : \mathfrak{H} \hat{\otimes} \mathbb{K} \to \mathfrak{H}, \ v \otimes z \to zv$$

called the left *and the* right unit*, respectively. Furthermore, for every triple of Hilbert spaces $\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}_3$ there is a natural isomorphism, called* associator

$$\hat{a}_{\mathfrak{H}_1,\mathfrak{H}_2,\mathfrak{H}_3} : (\mathfrak{H}_1 \hat{\otimes} \mathfrak{H}_2) \hat{\otimes} \mathfrak{H}_3 \to \mathfrak{H}_1 \hat{\otimes} (\mathfrak{H}_2 \hat{\otimes} \mathfrak{H}_3), \ (v_1 \otimes v_2) \otimes v_3 \mapsto v_1 \otimes (v_2 \otimes v_3) \ .$$

These data fulfill the so-called coherence conditions *that is the* pentagon diagram

$$\begin{tikzcd}
 & ((\mathfrak{H}_1 \hat{\otimes} \mathfrak{H}_2) \hat{\otimes} \mathfrak{H}_3) \hat{\otimes} \mathfrak{H}_4 \arrow[dl, "\hat{a}_{\mathfrak{H}_1,\mathfrak{H}_2,\mathfrak{H}_3} \hat{\otimes} \mathbb{1}_{\mathfrak{H}_4}"'] \arrow[dr, "\hat{a}_{\mathfrak{H}_1 \hat{\otimes} \mathfrak{H}_2,\mathfrak{H}_3,\mathfrak{H}_4}"] & \\
(\mathfrak{H}_1 \hat{\otimes} (\mathfrak{H}_2 \hat{\otimes} \mathfrak{H}_3)) \hat{\otimes} \mathfrak{H}_4 \arrow[d, "\hat{a}_{\mathfrak{H}_1,\mathfrak{H}_2 \hat{\otimes} \mathfrak{H}_3,\mathfrak{H}_4}"'] & & (\mathfrak{H}_1 \hat{\otimes} \mathfrak{H}_2) \hat{\otimes} (\mathfrak{H}_3 \hat{\otimes} \mathfrak{H}_4) \arrow[d, "\hat{a}_{\mathfrak{H}_1,\mathfrak{H}_2,\mathfrak{H}_3 \hat{\otimes} \mathfrak{H}_4}"] \\
\mathfrak{H}_1 \hat{\otimes} ((\mathfrak{H}_2 \hat{\otimes} \mathfrak{H}_3) \hat{\otimes} \mathfrak{H}_4) \arrow[rr, "\mathbb{1}_{\mathfrak{H}_1} \hat{\otimes} \hat{a}_{\mathfrak{H}_2,\mathfrak{H}_3,\mathfrak{H}_4}"'] & & \mathfrak{H}_1 \hat{\otimes} (\mathfrak{H}_2 \hat{\otimes} (\mathfrak{H}_3 \hat{\otimes} \mathfrak{H}_4))
\end{tikzcd}$$

and the triangle diagram

$$\begin{tikzcd}
(\mathfrak{H}_1 \widehat{\otimes} \mathbb{K}) \widehat{\otimes} \mathfrak{H}_2 \arrow[rr, "\widehat{a}_{\mathfrak{H}_1,\mathbb{K},\mathfrak{H}_3}"] \arrow[dr, "{}_{\mathfrak{H}_1}\widehat{u} \widehat{\otimes} \mathbb{1}_{\mathfrak{H}_2}"'] & & \mathfrak{H}_1 \widehat{\otimes} (\mathbb{K} \widehat{\otimes} \mathfrak{H}_2) \arrow[dl, "\mathbb{1}_{\mathfrak{H}_1} \widehat{\otimes} \widehat{u}_{\mathfrak{H}_2}"] \\
& \mathfrak{H}_1 \widehat{\otimes} \mathfrak{H}_2 &
\end{tikzcd}$$

commute for all Hilbert spaces $\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}_3, \mathfrak{H}_4$. *In other words, the category* Hilb *endowed with the Hilbert tensor product* $\widehat{\otimes}$ *is a monoidal category.*

Proof. The category of $\mathbb{K}$-vector spaces with the usual tensor product as tensor functor is monoidal. Denote the corresponding unit isomorphisms and associator by ${}_{-}u$, u_{-}, and $a_{-,-,-}$, respectively. Then observe that by construction $\mathbb{K} \widehat{\otimes} \mathfrak{H} = \mathbb{K} \otimes \mathfrak{H}$ and $\mathfrak{H} \widehat{\otimes} \mathbb{K} = \mathfrak{H} \otimes \mathbb{K}$ for every Hilbert space $\mathfrak{H}$. In particular this means that putting $\widehat{u}_{\mathfrak{H}} = u_{\mathfrak{H}}$ and ${}_{\mathfrak{H}}\widehat{u} = {}_{\mathfrak{H}}u$ gives the desired units. Next recall that $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ is dense in $\mathfrak{H}_1 \widehat{\otimes} \mathfrak{H}_2$ which by Proposition 12.4.4 implies density of $(\mathfrak{H}_1 \otimes \mathfrak{H}_2) \otimes \mathfrak{H}_3$ and $\mathfrak{H}_1 \otimes (\mathfrak{H}_2 \otimes \mathfrak{H}_3)$ in $(\mathfrak{H}_1 \widehat{\otimes} \mathfrak{H}_2) \widehat{\otimes} \mathfrak{H}_3$ and $\mathfrak{H}_1 \widehat{\otimes} (\mathfrak{H}_2 \widehat{\otimes} \mathfrak{H}_3)$, respectively. Similarly one argues that $\mathfrak{H}_1 \otimes (\mathfrak{H}_2 \otimes (\mathfrak{H}_3 \otimes \mathfrak{H}_4))$ is dense in $\mathfrak{H}_1 \widehat{\otimes} (\mathfrak{H}_2 \widehat{\otimes} (\mathfrak{H}_3 \widehat{\otimes} \mathfrak{H}_4))$, and so on. Since the associator map $a_{\mathfrak{H}_1,\mathfrak{H}_2,\mathfrak{H}_3} : (\mathfrak{H}_1 \otimes \mathfrak{H}_2) \otimes \mathfrak{H}_3 \to \mathfrak{H}_1 \otimes (\mathfrak{H}_2 \otimes \mathfrak{H}_3)$ is bounded, it extends in a unique way to a linear bounded map $\widehat{a}_{\mathfrak{H}_1,\mathfrak{H}_2,\mathfrak{H}_3} : (\mathfrak{H}_1 \widehat{\otimes} \mathfrak{H}_2) \widehat{\otimes} \mathfrak{H}_3 \to \mathfrak{H}_1 \widehat{\otimes} (\mathfrak{H}_2 \widehat{\otimes} \mathfrak{H}_3)$. Using density, continuity, and commutativity of the pentagon and triangle diagrams for the tensor product functor one concludes that the coherence conditions for $\widehat{\otimes}$ with the unit and associator maps ${}_{-}\widehat{u}$, $\widehat{u}_{-}$, and $\widehat{a}_{-,-,-}$ are satisfied. □

12.5. Adjoints of bounded operators

12.5.1 Throughout this section, $\mathfrak{H}$ stands for a Hilbert space over the field $\mathbb{K}$ of real or complex numbers. Let $A \in \mathfrak{B}(\mathfrak{H})$ that is let $A : \mathfrak{H} \to \mathfrak{H}$ be linear and bounded. Then the map

$$b_A : \mathfrak{H} \times \mathfrak{H} \to \mathbb{K},\ (v, w) \mapsto \langle v, Aw \rangle$$

is sesquilinear and bounded with norm

$$\|b_A\| = \sup \left\{ |b_A(v, w)| \;\middle|\; v, w \in \mathfrak{H} \;\&\; \|w\| = \|v\| = 1 \right\} = \|A\| \ .$$

By Corollary 12.2.8 to the Riesz representation theorem there exists a unique element $A^* \in \mathfrak{B}(\mathfrak{H})$ such that

$$b_A(v, w) = \langle A^* v, w \rangle \quad \text{for all } v, w \in \mathfrak{H} \ .$$

This operator satisfies $\|A^*\| = \|b_A\| = \|A\|$.

12.5.2 Definition The unique operator $A^* \in \mathfrak{B}(\mathfrak{H})$ associated to some $A \in \mathfrak{B}(\mathfrak{H})$ such that

$$\langle v, Aw \rangle = \langle A^* v, w \rangle \quad \text{for all } v, w \in \mathfrak{H}$$

is called the *adjoint* of A.

12.5.3 Proposition *If* $A \in \mathfrak{B}(\mathfrak{H})$, *then so is its adjoint* $A^* \in \mathfrak{B}(\mathfrak{H})$.

Proof. First we show that A^* is linear. Given $v, w, w' \in H$, we compute

$$\begin{aligned}\langle v, A^*(w+w')\rangle &= \mu_{A,w+w'}(v) = \langle Av, w+w'\rangle = \langle Av, w\rangle + \langle Av, w'\rangle \\ &= \mu_{A,w}(v) + \mu_{A,w'}(v) = \langle v, A^*w\rangle + \langle v, A^*w'\rangle = \langle v, A^*w + A^*w'\rangle\end{aligned}$$

Since this is true for all $v \in H$, this implies that $A^*(w+w') = A^*w'$. Furthermore, given $\lambda \in \mathbb{K}$, we have

$$\begin{aligned}\langle v, A^*(\lambda w)\rangle &= \mu_{A,\lambda w}(v) = \langle Av, \lambda w\rangle = \bar{\lambda}\langle Av, w\rangle \\ &= \bar{\lambda}\mu_{A,w}(v) = \bar{\lambda}\langle v, A^*w\rangle = \langle v, \lambda A^*w\rangle.\end{aligned}$$

Again, since this is true for all $v \in H$, we know $A^*(\lambda w) = \lambda A^*w$. This proves that A^* is linear.

It remains to show that A^* is bounded. We know

$$\|A^*\| = \sup_{\|v\|=\|w\|=1} |\langle v, A^*w\rangle| = \sup_{\|v\|=\|w\|=1} |\langle w, Av\rangle| = \|A\| < \infty,$$

which is what we wanted to show. Note that $\|A^*\| = \|A\|$. □

We leave it as an exercise to show that

$$\|A\| = \sup_{\|v\|=\|w\|=1} |\langle v, Aw\rangle|$$

for all $A \in \mathfrak{L}(\mathfrak{H})$, as was used in the above proof.

12.5.4 Definition An operator $A \in \mathfrak{L}(\mathfrak{H})$ is called *self-adjoint* if $A = A^*$, *unitary* if $A^* = A^{-1}$, and *normal* if $[A, A^*] = AA^* - A^*A = 0$.

We note that self-adjoint and unitary operators are always normal, but normal operators do not have to be self-adjoint or unitary. In the remainder of these notes, we gather several results on self-adjoint and normal operators.

12.5.5 Lemma *An operator $A \in \mathfrak{L}(\mathfrak{H})$ is self-adjoint if and only if $\langle Av, v\rangle \in \mathbb{R}$ for all $v \in H$.*

Proof. $\Rightarrow$) If A is self-adjoint, then

$$\langle Av, v\rangle = \mu_{A,v}(v) = \langle v, A^*v\rangle = \langle v, Av\rangle = \overline{\langle Av, v\rangle},$$

which implies that $\langle Av, v\rangle \in \mathbb{R}$.

$\Leftarrow$) Suppose that $\langle Av, v\rangle \in \mathbb{R}$ for all $v \in H$. We know

$$\langle A(v+w), v+w\rangle = \langle Av, v\rangle + \langle Av, w\rangle + \langle Aw, v\rangle + \langle Aw, w\rangle. \tag{$*$}$$

By assumption, $\langle A(v+w), v+w\rangle$, $\langle Av, v\rangle$, and $\langle Aw, w\rangle$ are all real. This implies that $\langle Av, w\rangle + \langle Aw, v\rangle$ is real as well, so

$$\mathfrak{Im}\langle Av, w\rangle = -\mathfrak{Im}\langle Aw, v\rangle = \mathfrak{Im}\langle v, Aw\rangle.$$

Since this holds for all $w \in H$, it holds for iw as well. Thus,

$$\mathfrak{Re}\langle Av, w\rangle = \mathfrak{Im}\langle Av, -iw\rangle = \mathfrak{Im}\langle v, A(-iw)\rangle = \mathfrak{Im}\, i\langle v, Aw\rangle = \mathfrak{Re}\langle v, Aw\rangle.$$

Combining the above two lines yields $\langle Av, w\rangle = \langle v, Aw\rangle$ for all $v, w \in H$. Since the adjoint satisfies $\langle Av, w\rangle = \langle v, A^*w\rangle$, this implies that $A = A^*$. □

12.5.6 Proposition *If $A \in \mathfrak{L}(\mathfrak{H})$ and $\langle Av, v\rangle = 0$ for all $v \in H$, then $A = 0$.*

Proof. Since $\langle Av, v\rangle = 0$ for all $v \in H$, equation $(*)$ from Lemma 4 reduces to

$$\langle Av, w\rangle = -\langle Aw, v\rangle = -\langle w, Av\rangle = -\overline{\langle Av, w\rangle} \quad \text{for all } v, w \in H \ ,$$

i.e. $\langle Av, w\rangle$ has no real part for all $v, w \in H$. But then fixing v and setting $w = Av$ implies $\|Av\|^2 = 0$ for all $v \in H$, so $A = 0$. □

12.5.7 Proposition *If $A \in \mathfrak{L}(\mathfrak{H})$ is self-adjoint, then*

$$\|A\| = \sup_{\|v\|=1} |\langle Av, v\rangle|.$$

Proof. We know

$$\|A\| = \sup_{\|v\|=\|w\|=1} |\langle Av, w\rangle|,$$

so we clearly have

$$\sup_{\|v\|=1} |\langle Av, v\rangle| \leqslant \|A\| \, .$$

□

12.5.8 Proposition *If $A \in \mathfrak{L}(\mathfrak{H})$, then A^*A is self-adjoint and $\|A^*A\| = \|A\|^2$.*

Proof. For arbitrary $v \in H$, we have

$$\langle A^*Av, v\rangle = \langle Av, Av\rangle = \|Av\|^2 \in \mathbb{R},$$

so A^*A is self-adjoint by Lemma 4. By Proposition 6,

$$\|A^*A\| = \sup_{\|v\|=1} |\langle A^*Av, v\rangle| = \sup_{\|v\|=1} \|Av\|^2 = \|A\|^2 \, .$$

□

12.5.9 Proposition *If $A \in \mathfrak{L}(\mathfrak{H})$, then there exist $B, C \in \mathfrak{L}(\mathfrak{H})$ self-adjoint such that $A = B + iC$. Furthermore, A is normal if and only if $[B, C] = 0$.*

Proof. We define

$$B = \frac{1}{2}(A + A^*) \quad \text{and} \quad C = \frac{i}{2}(A^* - A).$$

Clearly $A = B + iC$. Note also that $A^* = B - iC$. Furthermore, for all $v \in H$

$$\langle Bv, v\rangle = \frac{1}{2}\langle Av, v\rangle + \frac{1}{2}\langle A^*v, v\rangle = \frac{1}{2}\langle Av, v\rangle + \frac{1}{2}\overline{\langle Av, v\rangle} \in \mathbb{R}$$

and

$$\langle Cv, v\rangle = \frac{i}{2}\langle A^*v, v\rangle - \frac{i}{2}\langle Av, v\rangle = \frac{i}{2}\overline{\langle Av, v\rangle} - \frac{i}{2}\langle Av, v\rangle \in \mathbb{R}$$

This implies that B and C are self-adjoint by Lemma 4.

Finally, we compute

$$[A, A^*] = [B + iC, B - iC] = -i[B, C] + i[C, B] = -2i[B, C],$$

Clearly A is normal if and only if $[B, C] = 0$. □

12.5.10 Proposition *If A is normal, then $\|Av\| = \|A^*v\|$ for all $v \in H$.*

Proof. Using the fact that $A^*A = AA^*$, we compute

$$\|Av\|^2 = \langle Av, Av\rangle = \langle v, A^*Av\rangle = \langle v, AA^*v\rangle = \langle A^*v, A^*v\rangle = \|A^*v\|^2 .$$

Taking a square root yields the desired result. □

12.6. Projection-valued measures and spectral integrals

12.6.1 In this section $\mathfrak{H}$ will always denote a fixed complex Hilbert space.

12.6.2 Definition By a *projection-valued measure* or a *spectral measure* on a measurable space $(\Omega, \mathcal{A})$ one understands a map $E : \mathcal{A} \to \mathfrak{B}(\mathfrak{H})$ having the following properties:

(SM0) For each $\Delta \in \mathcal{A}$ the operator $E(\Delta)$ is an orthogonal projection that is $E(\Delta)^2 = E(\Delta)$ and $E(\Delta)^* = E(\Delta)$.

(SM1) $E(\Omega) = \mathrm{id}_{\mathfrak{H}}$.

(SM2) For every sequence $(\Delta_n)_{n\in\mathbb{N}}$ of pairwise disjoint elements of $\mathcal{A}$ one has

$$E\left(\bigcup_{n\in\mathbb{N}} \Delta_n\right) = \text{s-}\sum_{n=0}^{\infty} E(\Delta_n) ,$$

where convergence is with respect to the strong operator toplogy.

12.6.3 Remark Recall that *convergence* of a sequence of operators $(A_n)_{n\in\mathbb{N}} \subset \mathfrak{B}(\mathfrak{H})$ in the *strong operator topology* to some A means that for every $v \in \mathfrak{H}$ the sequence $(A_n v)_{n\in\mathbb{N}}$ converges in $\mathfrak{H}$ to Av. One denotes this by $A = \text{s-}\lim\limits_{n\to\infty} A_n$. Likewise, $B = \text{s-}\sum\limits_{n=0}^{\infty} A_n$ means that the sequence of partial sums $\left(\sum\limits_{k=0}^{n} A_n\right)_{n\in\mathbb{N}}$ converges in the strong operator topology to some $B \in \mathfrak{B}(\mathfrak{H})$.

12.6.4 Proposition *A spectral measure $E : \mathcal{A} \to \mathfrak{B}(\mathfrak{H})$ has the following properties in addition to the defining axioms:*

(SM1') $E(\emptyset) = 0$.

(SM2') (Finite additivity) *One has for all disjoint $\Delta_1, \Delta_2 \in \mathcal{A}$*

$$E(\Delta_1 \cup \Delta_2) = E(\Delta_1) + E(\Delta_2) .$$

(SM3) *One has for all $\Delta_1, \Delta_2 \in \mathcal{A}$*

$$E(\Delta_1 \cap \Delta_2) = E(\Delta_1) \cdot E(\Delta_2) .$$

Proof. *ad* (SM1').

ad (SM2').

ad (SM3). □

12.7. Spectral theory of bounded operators

12.7.1 We now apply the foundations of Hilbert space theory built in the previous sections to spectral theory. For the moment we will sacrifice generality and work only with bounded linear operators. The spectral theory of unbounded linear operators will be treated later.

Let us a recall that a linear map $A : \mathfrak{H}_1 \to \mathfrak{H}_2$ between Hilbert spaces is continuous if and only if it is bounded, i.e. has finite operator norm, and that $\mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ is a Banach space with the operator norm. For the rest of this section, $\mathfrak{H}$, $\mathfrak{H}_1$, $\mathfrak{H}_2, \ldots$ will always denote complex Hilbert spaces and A, B bounded linear operators. We will also now fix the base field to be complex, i.e. $\mathbb{K} = \mathbb{C}$. Last we agree on writing $I_{\mathfrak{H}}$ or just I for the identity operator on a Hilbert space $\mathfrak{H}$.

Spectrum and Resolvent

12.7.2 Definition Let $A : \mathfrak{H} \to \mathfrak{H}$ be a bounded linear operator. A complex number λ is then called an *eigenvalue* of A if there exists a nonzero $v \in H$ such that $Av = \lambda v$. For every $\lambda \in \mathbb{C}$ one defines the λ-*eigenspace* of A as

$$\mathrm{Eig}_\lambda(A) = \left\{ v \in H \,\middle|\, Av = \lambda v \right\} \subset \mathfrak{H},$$

which is clearly a linear subspace of $\mathfrak{H}$.

12.7.3 By definition it is immediately clear that

$$\mathrm{Eig}_\lambda(A) = \ker(A - \lambda),$$

where the λ on the right stands for the operator λI. In other words this means that $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if $A - \lambda$ is not injective.

12.7.4 Definition Let $A \in \mathfrak{B}(\mathfrak{H})$. We make the following definitions.

(i) A *regular value* of A is a complex number λ such that $A - \lambda$ is invertible.

(ii) The set of all regular values is the *resolvent* of A, denoted $\varrho(A)$.

(iii) A *spectral value* of A is a complex number λ such that $A - \lambda$ is not invertible.

(iv) The set of all spectral values is the *spectrum* of A, denoted $\sigma(A)$.

(v) The *point* or *eigenspectrum* of A is the set

$$\sigma_{\mathrm{p}}(A) = \left\{ \lambda \in \mathbb{C} \,\middle|\, \ker(A - \lambda) \neq \{0\} \right\}.$$

(vi) An *approximate eigenvalue* of A is a complex number λ for which there exists a sequence of unit vectors $(v_n)_{n \in \mathbb{N}} \subset \mathfrak{H}$ such that

$$\lim_{n \to \infty} (A - \lambda) v_n = 0.$$

The set $\sigma_{\mathrm{ap}}(A)$ is the set of all approximate eigenvalues.

12.7.5 Evidently, $\sigma(A) = \mathbb{C}\backslash\varrho(A)$ and $\sigma_{\mathrm{p}}(A) \subset \sigma_{\mathrm{ap}}(A) \subset \sigma(A)$, and these may all be strict inclusions. Note that $A - \lambda$ is bounded for any $\lambda \in \mathbb{C}$, so the open mapping theorem **??** implies that $(A-\lambda)^{-1} \in \mathfrak{B}(\mathfrak{H})$ when $\lambda \in \varrho(A)$. We call the map

$$R_\bullet(A) : \varrho(A) \to \mathfrak{B}(\mathfrak{H}), \quad R_\lambda(A) = (A-\lambda)^{-1}$$

the *resolvent* of A, not to be confused with the resolvent set $\varrho(A)$. To keep the notation clean, we often briefly write R_λ for $R_\lambda(A)$ and leave implicit that R_λ depends on A.

First, we prove some topological properties of the spectrum and resolvent. Recall the following lemma, which generalizes the geometric series.

12.7.6 Lemma (Carl Neumann) *Let $A \in \mathfrak{B}(\mathfrak{H})$. If $\|A\| < 1$, then $I - A$ is invertible,*

$$(I-A)^{-1} = \sum_{n=0}^{\infty} A^n,$$

and

$$\left\|(I-A)^{-1}\right\| \leqslant \frac{1}{1-\|A\|}.$$

Proof. Since $\|A\| < 1$ and $\|A^n\| \leqslant \|A\|^n$ by submultiplicativity of the operator norm, we know $\sum_{n=0}^{\infty} \|A^n\| < \infty$. This implies that the family $(A^n)_{n\in\mathbb{N}}$ is absolutely summable, so $\sum_{n=0}^{\infty} A^n$ exists. Furthermore, for every $N \in \mathbb{N}$ we have

$$(I-A)\sum_{n=0}^{N} A^n = \left(\sum_{n=0}^{N} A^n\right)(I-A) = \sum_{n=0}^{N} A^n - \sum_{n=1}^{N+1} A^n = I - A^{N+1},$$

which implies that

$$\lim_{N\to\infty}(I-A)\sum_{n=0}^{N} A^n = \lim_{N\to\infty}\left(\sum_{n=0}^{N} A^n\right)(I-A) = I.$$

By continuity of multiplication in $\mathfrak{B}(\mathfrak{H})$ one gets

$$(I-A)\sum_{n=0}^{\infty} A^n = \left(\sum_{n=0}^{\infty} A^n\right)(I-A) = I,$$

which proves that $I - A$ is invertible and $(I-A)^{-1} = \sum_{n=0}^{\infty} A^n$.

Finally, one concludes by the triangle inequality and submultiplicativity of the operator norm

$$\left\|(I-A)^{-1}\right\| \leqslant \sum_{n=0}^{\infty} \|A^n\| \leqslant \sum_{n=0}^{\infty} \|A\|^n = \frac{1}{1-\|A\|}.$$ □

12.7.7 Proposition *Let $A \in \mathfrak{B}(\mathfrak{H})$.*

(i) *For any $\lambda \in \varrho(A)$, one has*

$$B_{\|R_\lambda\|^{-1}}(\lambda) \subset \varrho(A) \ .$$

Hence, $\varrho(A) \subset \mathbb{C}$ is open.

(ii) *The spectrum $\sigma(A)$ is compact and*

$$\sigma(A) \subset \bar{B}_{\|A\|}(0) \ .$$

(iii) *If the complex number λ satisfies $|\lambda| > \|A\|$, then $\lambda \in \varrho(A)$ and*

$$R_\lambda = -\frac{1}{\lambda} - \sum_{n=1}^{\infty} \lambda^{-n-1} A^n \ ,$$

where convergence is with respect to the operator norm.

Proof. ad (i). Fix $\lambda \in \varrho(A)$ and set $r = \|R_\lambda\|^{-1}$. Let $\mu \in B_r(\lambda)$. Then

$$\|(\mu - \lambda)R_\lambda\| = |\mu - \lambda| \, \|R_\lambda\| < 1.$$

Thus, by Lemma 12.7.6, one knows that $I - (\mu - \lambda)R_\lambda$ is invertible. Since $A - \lambda$ is invertible, the composition

$$(A - \lambda)\big(I - (\mu - \lambda)R_\lambda\big) = A - \mu$$

is invertible, which proves that $\mu \in \varrho(A)$. Hence $\varrho(A)$ is open.

ad (ii). Since $\varrho(A)$ is open, the complement $\sigma(A) = \mathbb{C}\backslash\varrho(A)$ is closed. Furthermore, if $|\lambda| > \|A\|$, then $\|\lambda^{-1}A\| < 1$, so $I - \lambda^{-1}A$ and hence $A - \lambda$ are invertible by Lemma 12.7.6. This implies that $\lambda \in \varrho(A)$, so $\sigma(A) \subset \bar{B}_{\|A\|}(0)$. Since $\sigma(A)$ is closed and bounded, it is compact.

ad (iii). If $|\lambda| > \|A\|$, then $I - \lambda^{-1}A$ is invertible by Lemma 12.7.6 and

$$(I - \lambda^{-1}A)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n} A^n.$$

Since $-\lambda(A - \lambda)^{-1} = (I - \lambda^{-1}A)^{-1}$, one obtains

$$R_\lambda = -\frac{1}{\lambda} \sum_{n=0}^{\infty} \lambda^{-n} A^n = -\frac{1}{\lambda} - \sum_{n=1}^{\infty} \lambda^{-n-1} A^n,$$

as desired. □

Next, we prove some algebraic properties of the resolvent. Hereby, $[A, B] = AB - BA$ denotes the commutator of two operators, as usual.

12.7.8 Proposition *Let $A, B \in \mathfrak{B}(\mathfrak{H})$. Then the following holds true.*

(i) *The resolvent commutes with the operator which means that*

$$[A, R_\lambda(A)] = 0 \quad \textit{for all } \lambda \in \varrho(A) \ .$$

(ii) *The values of the resolvent commute with each other that is*

$$[R_\lambda(A), R_\mu(A)] = 0 \quad \textit{for all } \lambda, \mu \in \varrho(A) \ .$$

(iii) (First resolvent identity) *For all* $\lambda, \mu \in \varrho(A)$

$$R_\lambda(A) - R_\mu(A) = (\lambda - \mu) R_\lambda(A) R_\mu(A) \ .$$

(iv) (Second resolvent identity) *For all* $\lambda \in \varrho(A) \cap \varrho(B)$

$$R_\lambda(A) - R_\lambda(B) = R_\lambda(A)\,(B - A)\,R_\lambda(B) \ .$$

Proof. *ad (i).* Obviously $[A, A-\lambda] = 0$, so

$$0 = R_\lambda[A, A-\lambda]R_\lambda = R_\lambda A - AR_\lambda,$$

as desired.

ad (iii). We compute

$$\begin{aligned}(R_\lambda - R_\mu)(A-\mu)(A-\lambda) &= (R_\lambda A - \mu R_\lambda)(A-\lambda) - (A-\lambda) \\ &= (A-\mu)R_\lambda(A-\lambda) - (A-\lambda) \\ &= \lambda - \mu,\end{aligned}$$

where we used part (i) to commute R_λ past A in the second step. Now multiplying both sides with $R_\lambda R_\mu$ from the right yields the desired equality.

ad (ii). For $\lambda = \mu$, one obviously has $[A_\lambda, A_\mu] = 0$. For $\lambda \neq \mu$, one concludes from (ii)

$$R_\mu R_\lambda = \frac{R_\mu - R_\lambda}{\mu - \lambda} = \frac{R_\lambda - R_\mu}{\lambda - \mu} = R_\lambda R_\mu,$$

so $[R_\lambda, R_\mu] = 0$ for $\lambda \neq \mu$ as well.

ad (iv). The last equality follows by

$$R_\lambda(A)\,(B-A)\,R_\lambda(B) = R_\lambda(A)\,\big((B-\lambda) - (A-\lambda)\big)\,R_\lambda(B) = R_\lambda(A) - R_\lambda(B) \ . \qquad \square$$

The resolvent $R_\bullet(A)$ also has some nice analytic properties which we are going to prove next.

12.7.9 Proposition *The resolvent* $R_\bullet(A) : \varrho(A) \to \mathfrak{B}(\mathfrak{H})$, $\lambda \mapsto R_\lambda$ *is continuous and complex differentiable with derivative given by*

$$R_\bullet(A)' : \ \varrho(A) \to \mathfrak{B}(\mathfrak{H}), \ \lambda \mapsto \lim_{\mu\to\lambda} \frac{R_\mu - R_\lambda}{\mu - \lambda} = R_\lambda^2$$

Proof. Fix $\lambda \in \varrho(A)$ and $\varepsilon > 0$. Let $0 < |\mu - \lambda| < \delta$, where

$$\delta = \min\left(\frac{\varepsilon}{2\|R_\lambda\|^2}, \frac{1}{2\|R_\lambda\|}\right) .$$

Note that $\mu \in \varrho(A)$ by Proposition 12.7.7. Moreover, $\|(\mu - \lambda)R_\lambda\| < 1$, so $I - (\mu - \lambda)R_\lambda$ is invertible with norm less than $(1 - \|(\mu - \lambda)R_\lambda\|)^{-1}$ by Lemma 12.7.6. Now observe that the first resolvent identity can be rearranged to

$$R_\mu = R_\lambda[I - (\mu - \lambda)R_\lambda]^{-1} .$$

Hence

$$\begin{aligned}
\|R_\mu - R_\lambda\| &\leqslant |\mu - \lambda| \, \|R_\mu\| \, \|R_\lambda\| \\
&\leqslant |\mu - \lambda| \, \|R_\lambda\|^2 \left\|(I - (\mu - \lambda)R_\lambda)^{-1}\right\| \\
&\leqslant \frac{|\mu - \lambda| \, \|R_\lambda\|^2}{1 - \|(\mu - \lambda)R_\lambda\|} \\
&< \frac{\varepsilon/2}{1 - 1/2} = \varepsilon .
\end{aligned}$$

This proves that $\lambda \mapsto R_\lambda$ is continuous.

As for complex differentiability, we simply use the first resolvent identity and continuity to conclude

$$\lim_{\mu \to \lambda} \frac{R_\mu - R_\lambda}{\mu - \lambda} = \lim_{\mu \to \lambda} R_\mu R_\lambda = R_\lambda^2.$$

□

12.7.10 Proposition *Let $A \in \mathfrak{B}(\mathfrak{H})$. Then $\lambda R_\lambda \to -I$ as $|\lambda| \to \infty$. In particular, $R_\lambda \to 0$ as $|\lambda| \to \infty$.*

Proof. Fix $\varepsilon > 0$. For $|\lambda| > \|A\|$, we have by Proposition 12.7.7 (iii)

$$\lambda R_\lambda = -I - \sum_{n=1}^{\infty} \lambda^{-n} A^n.$$

Since

$$\left\| \sum_{n=1}^{\infty} \lambda^{-n} A^n \right\| \leqslant \frac{\|A\|}{|\lambda| - \|A\|},$$

one sees that $\lambda R_\lambda \to -I$ as $|\lambda| \to \infty$. Similarly, for $|\lambda| > \|A\|$ one has

$$\|R_\lambda\| \leqslant \frac{1}{|\lambda|} + \frac{1}{|\lambda|} \sum_{n=1}^{\infty} \left\|\lambda^{-n} A^n\right\| \leqslant \frac{1}{|\lambda|} + \frac{1}{|\lambda|} \frac{\|A\|}{|\lambda| - \|A\|},$$

which shows that $R_\lambda \to 0$ as $|\lambda| \to \infty$. □

12.7.11 Proposition *For all $v, w \in \mathfrak{H}$, the map*

$$\langle R_\bullet(A)v, w\rangle : \varrho(A) \to \mathbb{C},\ \lambda \mapsto \langle R_\lambda v, w\rangle$$

is holomorphic with derivative

$$\langle R_\bullet(A)v, w\rangle' : \varrho(A) \to \mathbb{C},\ \lambda \mapsto \langle R_\lambda^2 v, w\rangle.$$

Proof. Given $\lambda \in \varrho(A)$, we compute

$$\lim_{\mu\to\lambda} \frac{\langle R_\mu v, w\rangle - \langle R_\lambda v, w\rangle}{\mu - \lambda} = \lim_{\mu\to\lambda} \frac{\langle(\mu-\lambda)R_\mu R_\lambda v, w\rangle}{\mu - \lambda} = \lim_{\mu\to\lambda}\langle R_\mu R_\lambda v, w\rangle = \langle R_\lambda^2 v, w\rangle,$$

where we have used the first resolvent identity in the first step and continuity of the inner product in the last. □

12.7.12 Proposition *The spectrum of an operator $A \in \mathfrak{B}(\mathfrak{H})$ is nonempty.*

Proof. Suppose $\sigma(A) = \emptyset$, hence $\varrho(A) = \mathbb{C}$. The map

$$\mathbb{C} \to \mathbb{C},\ \lambda \mapsto \langle R_\lambda v, w\rangle$$

then is entire for every $v, w \in \mathfrak{H}$. Furthermore, one has for $\|v\|, \|w\| \leqslant 1$

$$|\langle R_\lambda v, w\rangle| \leqslant \|R_\lambda\| \, \|v\| \, \|w\| \leqslant \|R_\lambda\| \ .$$

Since $\lambda \mapsto \|R_\lambda\|$ is continuous and $\|R_\lambda\| \to 0$ as $|\lambda| \to \infty$, one sees that $\|R_\lambda\|$ is bounded. Hence $\langle R_\bullet v, w\rangle$ is a bounded entire function, which by Liouville's theorem implies that it is zero for every pair $v, w \in \mathfrak{H}$ with $\|v\| = \|w\| = 1$. This entails that $R_\lambda = 0$ for every $\lambda \in \mathbb{C}$, which is a contradiction to R_λ being invertible. Hence $\sigma(A) \neq \emptyset$. □

12.8. Unbounded linear operators

12.8.1 In this section let V, W always denote Banach spaces over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The symbols $\mathfrak{H}$, $\mathfrak{H}_1$, ... will always stand for Hilbert spaces over $\mathbb{K}$.

12.8.2 Definition By an *unbounded $\mathbb{K}$-linear operator* or shortly by an *unbounded operator* from V to W we understand a linear map $A : \mathrm{Dom}(A) \to \mathrm{W}$ defined on a $\mathbb{K}$-linear subspace $\mathrm{Dom}(A) \subset \mathrm{V}$. As usual, $\mathrm{Dom}(A)$ is called the *domain* of the operator A. The space of unbounded $\mathbb{K}$-linear operators from V to W will be denoted $\mathfrak{L}_\mathbb{K}(V, W)$ or just $\mathfrak{L}(V, W)$.

12.8.3 Remark In this work, the term "unbounded" is meant in the sense of "not necessarily bounded". Sometimes we just say *linear operator* or even only *operator* instead of "unbounded linear operator".

12.8.4 Observe that besides the domain $\mathrm{Dom}(A)$ of an unbounded operator $A \in \mathfrak{L}(\mathrm{V},\mathrm{W})$ the *kernel*

$$\mathrm{Ker}(A) = \big\{ v \in \mathrm{V} \;\big|\; Av = 0 \big\} \subset \mathrm{V} ,$$

the *image*

$$\mathrm{Im}(A) = \big\{ w \in \mathrm{W} \;\big|\; \exists v \in \mathrm{Dom}(A) : w = Av \big\} \subset \mathrm{W} ,$$

and the *graph*

$$\mathrm{Gr}(A) = \big\{ (v,w) \in \mathrm{Dom}(A) \times \mathrm{W} \;\big|\; w = Av \big\} \subset \mathrm{V} \times \mathrm{W}$$

of A are all linear subspaces. We will frequently make use of this.

12.8.5 Definition An unbounded operator $A \in \mathfrak{L}(\mathrm{V},\mathrm{W})$ is called *densely defined* if $\mathrm{Dom}(A)$ is dense in V, and *closed* if the graph $\mathrm{Gr}(A)$ is closed in $\mathrm{V} \times \mathrm{W}$. The operator $A \in \mathfrak{L}(V,W)$ is called *closable* if the closure $\overline{\mathrm{Gr}(A)}$ is the graph of an unbounded operator from V to W.

An operator $A \in \mathfrak{L}(V,W)$ is called an *extension* of $B \in \mathfrak{L}(V,W)$ if $\mathrm{Gr}(B) \subset \mathrm{Gr}(A)$. One writes in this situation $B \subset A$.

13. C^*-Algebras

13.1. Infinite tensor products

13.1.1 Infinite tensor products of Hilbert spaces were introduced by **?**. They were motivated by mathematical physics where one needs to describe quantum systems with infinitely many degrees of freedom, see e.g. Emch (2009); **?**. The original construction of infinite tensor products was generalized to von Neumann and C^*-algebras by **?**, **?**, and others. Meanwhile, the topic has been studied in quite some detail in the operator algebra literature, see e.g. **???**. A purely algebraic or better categorical approach allowing the construction of infinite tensor products of modules over a given commutative ring has been given in (**?**, Sec. III.10). The work **?** is also in that spirit. We will essentially follow **?** and construct the infinite tensor product as a module universal with respect to multilinear maps. First we present the main algebraic construction, then we explain some of the subtleties which distinguish infinite from finite tensor products, and finally we construct infinite Hilbert tensor products and infinite tensor products of C^*-algebras.

13.1.2 Let R be a commutative ring and $(M_i)_{i\in I}$ a possibly infinite family of R-modules. Consider $\prod_{i\in I} M_i$, the product of the family $(M_i)_{i\in I}$ within the category of R-modules. For each $j \in I$ let $\pi_j : \prod_{i\in I} M_i \to M_j$ denote the natural projection onto the j-th factor and $\iota_j : M_j \hookrightarrow \prod_{i\in I} M_i$ the uniquely determined natural embedding such that

$$\pi_j \circ \iota_i = \begin{cases} \mathrm{id}_{M_i} & \text{for } i = j \text{ and} \\ 0 & \text{else.} \end{cases}$$

Given an R-module N one then understands by a *multilinear map* from $\prod_{i\in I} M_i$ to N a map $f : \prod_{i\in I} M_i \to N$ such that for each $j \in I$ and $x \in \prod_{i\in I} M_i$ with $\pi_j(x) = 0$ the map $M_j \to N$, $m \mapsto f(\iota_j(m) + x)$ is linear. The set of multilinear maps from $\prod_{i\in I} M_i$ to N will be denoted by $\mathfrak{Mlin}\big(\prod_{i\in I} M_i, N\big)$. It carries a natural structure of an R-module given by pointwise addition of multilinear maps and pointwise action of a scalar on a multilinear map that is by

$$f + g = \left(\prod_{i\in I} M_i \ni x \mapsto f(x) + g(x) \in N\right) \quad \text{and} \quad rf = \left(\prod_{i\in I} M_i \ni x \mapsto rf(x) \in N\right)$$

for all $f, g \in \mathfrak{Mlin}\big(\prod_{i\in I} M_i, N\big)$ and $r \in R$. Since for $j \in I$ and $x \in \prod_{i\in I} M_i$ with $\pi_j(x) = 0$ the maps $M_j \to N$, $m \mapsto (f + g)(\iota_j(m) + x) = f(\iota_j(m) + x) + g(\iota_j(m) + x)$ and $M_j \to N$, $m \mapsto rf(\iota_j(m) + x)$ are linear by assumption on f and g, the maps $f + g$ and rf are multilinear again, so $\mathfrak{Mlin}\big(\prod_{i\in I} M_i, N\big)$ is an R-module indeed with zero element the constant function mapping to $0 \in N$.

13.1.3 Remarks Before proceeding further let us make several explanations concerning the notation used.

(a) The space of multilinear maps $\mathfrak{Mlin}\big(\prod_{i\in I} M_i, N\big)$ actually depends on the family $(M_i)_{i\in I}$ and the R-module N, so in principle one should write $\mathfrak{Mlin}\big((M_i)_{i\in I}, N\big)$ instead of $\mathfrak{Mlin}\big(\prod_{i\in I} M_i, N\big)$. Nevertheless we stick to the latter notation since it is closer to standard notation for linear maps and since it will not lead to any confusion.

(b) In case the index set I has just two elements i_1, i_2, one calls a multilinear map $\prod_{i\in I} M_i = M_{i_1} \times M_{i_2} \to N$ a *bilinear map*. If the cardinality of I is 3, one sometimes calls a multilinear map $\prod_{i\in I} M_i \to N$ a *trilinear map*.

(c) In the following, when saying that $(I_a)_{a\in A}$ is a partition of the set I we mean that each I_a is a non-empty subset of I, that $I_a \cap I_b = \varnothing$ for $a \neq b$ and that $\bigcup_{a\in A} I_a = I$. The empty family is regarded as a partition of the empty set.

(d) We will frequently use in this section the same symbol for maps with the same "universal" properties despite those maps might be strictly speaking different. For example, π_k will stand for the canonical projections $\prod_{i\in I} M_i \to M_k$ and $\prod_{j\in J} M_j \to M_k$ whenever $k \in J \subset I$. Likewise we use the same notation for the two canonical embeddings $M_k \hookrightarrow \prod_{i\in I} M_i$ and $M_k \hookrightarrow \prod_{j\in J} M_j$ defined in 13.1.2 and denote them both by ι_k.

13.1.4 Lemma (cf. (?, Sec. III.10, Lemma 1 & 2)) *Assume that $(M_i)_{i\in I}$ is a family of R-modules, N an R-module, and $f : \prod_{i\in I} M_i \to N$ a mutilinear map.*

(i) *If $g : N \to N'$ is an R-module map, then $g \circ f : \prod_{i\in I} M_i \to N'$ is multilinear.*

(ii) *Let $J \subset I$ be non-empty, $y = (y_i)_{i\in I\setminus J}$ an element of the product $\prod_{i\in I\setminus J} M_i$, and $\iota_{J,y} : \prod_{j\in J} M_j \to \prod_{i\in I} M_i$ the unique map such that for all $x = (x_j)_{j\in J} \in (M_j)_{j\in J}$ and $k \in I$*

$$\pi_k \circ \iota_{J,y}(x) = \begin{cases} x_k & \text{for } k \in J, \\ y_k & \text{for } k \in I\setminus J. \end{cases}$$

Then the composition $f \circ \iota_{J,x} : \prod_{j\in J} M_j \to N$ is multilinear.

(iii) *Let $(I_a)_{a\in A}$ be a partition of the index set I which is assumed to be non-empty. Let $(N_a)_{a\in A}$ be a family of R-modules, $(g_a)_{a\in A}$ a family of multilinear maps $g_a : \prod_{i\in I_a} M_i \to N_a$, and $h : \prod_{a\in A} N_a \to N$ multilinear. Define $g : \prod_{i\in I} M_i \to \prod_{a\in A} N_a$ as the unique map such that*

$$\pi_b \circ g = g_b \circ \pi_{I_b} \quad \text{for } b \in A,$$

where π_J for $J \subset I$ as on the right side stands for the projection $\pi_J : \prod_{i\in I} M_i \to \prod_{j\in J} M_j$ uniquely determined by $\pi_j \circ \pi_J = \pi_j$ for all $j \in J$. Then the composition $h \circ g : \prod_{i\in I} M_i \to N$ is multilinear.

Proof. *ad (i)*. Let $j \in I$ and $x \in \prod_{i\in I} M_i$ with $\pi_j(x) = 0$. By multilinearity of f and linearity of g, the map $M_j \to N'$, $m \mapsto gf(\iota_j(m) + x)$ then has to be linear, hence $g \circ f$ is multilinear.

ad (ii). Let $j \in J$ and $x \in \prod_{i\in J} M_i$ with $\pi_j(x) = 0$. Then $\pi_j(\iota_{J,y}(x)) = 0$ and $f\iota_{J,y}(\iota_j(m) + x) = f(\iota_j(m) + \iota_{J,y}(x)$ for all $m \in M_j$ by construction of $\iota_{J,y}$. Hence the map $M_j \to N$, $m \mapsto f\iota_{J,y}(\iota_j(m) + x)$ is linear by multilinearity of f. This proves that $f \circ \iota_{J,y}$ is multilinear.

ad (iii). Given $j \in I$ let b be the unique element of A such that $j \in I_b$. Assume that $x \in \prod_{i\in I} M_i$ with $\pi_j(x) = 0$. By construction one has $\pi_j(\pi_{I_b}(x)) = 0$. Now let $y \in \prod_{a\in A} N_a$ such that

$$\pi_a(y) = \begin{cases} 0 & \text{for } a = b, \\ g_a \pi_{I_a}(x) & \text{for } a \neq b. \end{cases}$$

One then obtains for $m \in M_j$

$$\pi_a g(\iota_j(m) + x) = \begin{cases} g_b \pi_{I_b}(\iota_j(m) + x) = g_b(\iota_j(m) + \pi_{I_b}(x)) & \text{for } a = b, \\ g_a \pi_{I_a}(x) = \pi_a(y) & \text{for } a \neq b. \end{cases}$$

Hence

$$hg(\iota_j(m) + x) = h\big(\iota_b\big(g_b(\iota_j(m) + \pi_{I_b}(x)\big) + y\big) \ ,$$

and the map $M_j \to N$, $m \mapsto hg(\iota_j(m) + x)$ is linear as the composition of two linear maps. □

13.1.5 Lemma *Assume to be given a non-empty family of R-modules $(M_i)_{i\in I}$ and a partition $(I_a)_{a\in A}$ of the index set I. Then there exists a natural ismorphism*

$$\kappa_{I,A} : \prod_{i\in I} M_i \to \prod_{a\in A} \prod_{i\in I_a} M_i$$

uniquely determined by the condition that $\pi_a \circ \kappa_{I,A} = \pi_{I_a}$ for all $a \in A$.

Proof. By the universal property of the product the R-module map $\kappa = \kappa_{I,A} : \prod_{i\in I} M_i \to \prod_{a\in A} \prod_{i\in I_a} M_i$ exists and is uniquely determined by the requirement that $\pi_a \circ \kappa_{I,A} = \pi_{I_a}$ for all $a \in A$. Naturality also follows from the universal property of the product. It remains to show that κ is an isomorphism. By construction, $\pi_i(x) = \pi_i \pi_a \kappa(x) = 0$ for all $i \in I$ and $a(i) \in A$ such that $i \in I_{a(i)}$, hence $x = 0$. So κ is injective. It is also surjective. To see this pick $x_a \in \prod_{i\in I_a} M_i$ for each $a \in A$. With $a(i)$ for $i \in I$ defined as before put $x = \big(\pi_i(x_{a(i)})\big)_{i\in I}$. Then, by construction, $\pi_i \pi_a \kappa(x) = \pi_i \pi_a(x) = \pi_i(x) = \pi_i(x_a)$ for all $a \in A$ and $i \in I_a$, hence $\big(\pi_a \kappa(x)\big)_{a\in A} = (x_a)_{a\in A}$ and κ is surjective. □

13.1.6 Proposition (Exponential law for multilinear maps) *Let $(M_i)_{i\in I}$ be a family of R-modules over a commutative ring R, N an R-module, and assume that $J \subset I$ is a non-empty subset such that the complement $K = I \backslash J$ is also non-empty. Then the map*

$$\eta_{I,J} : \mathfrak{Mlin}\left(\prod_{j\in J} M_j, \mathfrak{Mlin}\left(\prod_{k\in K} M_k, N\right)\right) \to \mathfrak{Mlin}\left(\prod_{i\in I} M_i, N\right),$$

$$f \mapsto \left(\prod_{i\in I} M_i \ni (x_i)_{i\in I} \mapsto f\big((x_j)_{j\in J}\big)\,((x_k)_{k\in K}) \in N\right)$$

is an isomorphism which is natural in $(M_i)_{i\in I}$ and N.

Proof. We first show that $\eta = \eta_{I,J}$ is linear. To this end let $f, g \in \mathfrak{Mlin}\Big(\prod_{j\in J} M_j, \mathfrak{Mlin}\,(\prod_{k\in K} M_k, N)\Big)$ and $r \in R$. Then, for all $x = (x_i)_{i\in I} \in \prod_{i\in I} M_i$,

$$\begin{aligned}\big(\eta(f+g)\big)(x) &= \big(f+g\big)\big((x_j)_{j\in J}\big)\,((x_k)_{k\in K}) = \big(f((x_j)_{j\in J}) + g((x_j)_{j\in J})\big)\,((x_k)_{k\in K}) = \\ &= f((x_j)_{j\in J})\,((x_k)_{k\in K}) + g((x_j)_{j\in J})\,((x_k)_{k\in K}) = \big(\eta f\big)(x) + \big(\eta g\big)(x) = \big(\eta f + \eta g\big)(x)\end{aligned}$$

and

$$\big(\eta(rf)\big)(x) = (rf)((x_j)_{j\in J})\,((x_k)_{k\in K}) = \big(rf((x_j)_{j\in J})\big)\,((x_k)_{k\in K}) = r\big(f((x_j)_{j\in J})\,((x_k)_{k\in K})\,\big) = \\ = r\big(\eta f(x)\big) = \big(r(\eta f)\big)(x) \ .$$

Hence η is an R-module map.

Next we show that η is an isomorphism by constructing an inverse. Given $f \in \mathfrak{Mlin}\big(\prod_{i\in I} M_i, N\big)$ we define $f^\sharp : \mathfrak{Mlin}\big(\prod_{j\in J} M_j\big) \to \mathfrak{Mlin}\big(\prod_{k\in K} M_k, N\big)$ by the requirement that

$$f^\sharp(y)(z) = f(x_{y,z}) \quad \text{for all } \ y = (y_j)_{j\in J} \ \text{ and } \ z = (z_k)_{k\in K} \ ,$$

where $x_{y,z}$ is the element of $\prod_{i\in I} M_i$ uniquely determined by

$$\pi_i(x_{y,z}) = \begin{cases} y_i & \text{for } \ i \in J, \\ z_i & \text{for } \ i \in K. \end{cases}$$

One thus obtains an R-module map

$$(-)^\sharp_{I,J} : \mathfrak{Mlin}\left(\prod_{i\in I} M_i, N\right) \to \mathfrak{Mlin}\left(\prod_{j\in J} M_j, \mathfrak{Mlin}\left(\prod_{k\in K} M_k, N\right)\right), \quad f \mapsto f^\sharp$$

which by construction is inverse to $\eta_{I,J}$.

Naturality of $\eta_{I,J}$ in $(M_j)_{j\in J}$ and N is clear by definition. □

13.1.7 Definition Let $(M_i)_{i\in I}$ be a family of R-modules over a commutative ring R. By a *tensor product* of $(M_i)_{i\in I}$ one understands an R-module $\bigotimes_{i\in I} M_i$ together with a multilinear map $\tau : \prod_{i\in I} M_i \to \bigotimes_{i\in I} M_i$ such that the following universal property is fulfilled:

(ITensor) For every R-module N and every multilinear map $f : \prod_{i\in I} M_i \to N$ there exists a unique R-module map $\overline{f} : \bigotimes_{i\in I} M_i \to N$ such that the diagram

$$\begin{tikzcd} \prod_{i\in I} M_i \arrow[r, "f"] \arrow[d, "\tau"'] & N \\ \bigotimes_{i\in I} M_i \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

commutes.

The linear map $\overline{f}$ making the diagram comute will sometimes be called the *linearization* of the multilinear map f.

Given a tensor product $\big(\bigotimes_{i\in I} M_i, \tau\big)$, we will usually denote the image of an element $(x_i)_{i\in I} \in \prod_{i\in I} M_i$ under the map τ by $\otimes_{i\in I} x_i$.

13.1.8 Remarks (a) Strictly speaking, a tensor product of a family $(M_i)_{i\in I}$ of R-modules is a pair $\big(\bigotimes_{i\in I} M_i, \tau\big)$ having the above properties. By slight abuse of language, one usually denotes a tensor product just by its first component, the R-module $\bigotimes_{i\in I} M_i$. When helpful for clarity, the associated map $\tau : \prod_{i\in I} M_i \to \bigotimes_{i\in I} M_i$ will be denoted by $\tau_{(M_i)_{i\in I}}$ or by τ_I.

(b) In the case where the index set I of the family $(M_i)_{i\in I}$ is infinite, one sometimes calls $\bigotimes_{i\in I} M_i$ an *infinite tensor product*.

13.1.9 Theorem *Let $(M_i)_{i\in I}$ be a family of R-modules over a commutative ring R. Then the following holds true.*

(i) *A tensor product $\bigotimes_{i\in I} M_i$ of the family $(M_i)_{i\in I}$ exists and is unique up to isomorphism. If I is the empty set, then $\bigotimes_{i\in I} M_i = R$, if I contains a single element $i_\circ$, then $\bigotimes_{i\in I} M_i = M_{i_\circ}$.*

(ii) *If $(N_i)_{i\in I}$ is a second family of R-modules and $(f_i)_{i\in I}$ a family R-module maps $f_i : M_i \to N_i$, then there exists a unique linear map $\bigotimes_{i\in I} f_i : \bigotimes_{i\in I} M_i \to \bigotimes_{i\in I} N_i$ making the diagram*

$$\begin{tikzcd}
\prod\limits_{i\in I} M_i \arrow[r, "f"] \arrow[d, "\tau"'] & \bigotimes\limits_{i\in I} N_i \\
\bigotimes\limits_{i\in I} M_i \arrow[ur, "\bigotimes\limits_{i\in I} f_i"'] &
\end{tikzcd}$$

commute, where $f : \prod_{i\in I} M_i \to \bigotimes_{i\in I} N_i$ is the multilinear map $(x_i)_{i\in I} \mapsto \otimes_{i\in I} f_i(x_i)$.

(iii) *Let $J \subset I$ be a finite non-empty subset set such that M_j is isomorphic to R for all $j \in J$. Denote for each $j \in J$ by 1_j the image of the unit $1 \in R$ under the isomorphism $R \cong M_j$ and by 1_J the family $(1_j)_{j\in J}$. Moreover, for every family $y = (y_j)_{j\in J}$ let $\iota_{J,y} : \prod_{i\in I\setminus J} M_i \to \prod_{i\in I} M_i$ be the map which associates to $x \in \prod_{i\in I\setminus J} M_i$ the family $(x_i)_{i\in I}$ such that $x_i = \pi_i(x)$ for $i \in I\setminus J$ and $x_i = y_i$ for $i \in J$. Then the linearization $\bar\iota_{J,1_J} : \bigotimes_{i\in I\setminus J} M_i \to \bigotimes_{i\in I} M_i$ of the multilinear map $\tau_I \circ \iota_{J,1_J} : \prod_{i\in I\setminus J} M_i \to \bigotimes_{i\in I} M_i$ is an isomorphism.*

Proof. ad (i). By its universal property, the tensor product of the family $(M_i)_{i\in I}$ is uniquely determined up to isomorphism. Hence it remains to show the existence of the tensor product. To this end consider the free R-module over the set $\prod_{i\in I} M_i$ and denote it by F. Let $\delta : \prod_{i\in I} M_i \hookrightarrow F$ be the canonical injection and U be the submodule of F spanned by the elements

$$\delta\big(\iota_j(ry_j + z_j) + (x_i)_{i\in I}\big) - r\delta\big(\iota_j(y_j) + (x_i)_{i\in I}\big) - \delta\big(\iota_j(z_j) + (x_i)_{i\in I}\big)\ ,$$

where $j \in I$, $y_j, z_j \in M_j$, $r \in R$, and $(x_i)_{i\in I} \in \pi_j^{-1}(0)$. Then put $\bigotimes_{i\in I} M_i = F/U$ and define τ as the composition of the canonical projection $\pi : F \to \bigotimes_{i\in I} M_i$ with $\delta : \prod_{i\in I} M_i \to F$. By construction, τ is multilinear. Assume that N is an R-module and $f : \prod_{i\in I} M_i \to N$ is a multilinear map. By the universal property of free R-modules, f lifts to a unique R-linear map $f' : F \to N$ such that $f = f' \circ \delta$. By multilinearity of f, the map f' vanishes on the submodule U, hence descends to an R-linear $\overline{f} : \bigotimes_{i\in I} M_i \to N$ such that $f' = \overline{f} \circ \pi$. Hence $f = f' \circ \delta = \overline{f} \circ \pi \circ \delta = \overline{f} \circ \tau$. By surjectivity of δ and uniqueness of f', $\overline{f}$ is the unique R-linear map satisfying $f = \overline{f} \circ \tau$. Hence $\big(\bigotimes_{i\in I} M_i, \tau\big)$ is a tensor product of the family $(M_i)_{i\in I}$.

In case $I = \varnothing$, the cartesian product $\prod_{i\in I} M_i$ is final in the category of sets, hence consists of only one element $\star$ only. This means in particular that for an R-module N any map $f : \prod_{i\in I} M_i = \{\star\} \to N$ is multilinear. Put $\bigotimes_{i\in I} M_i = R$ and let $\tau : \{\star\} \to R$ be the map $\star \mapsto 1$. Now let $\overline{f} : R \to N$ be the unique linear map such that $\overline{f}(1) = f(\star)$. Then $f = \overline{f} \circ \tau$ and the pair (R, τ) fulfills the universal property of the tensor product.

If I is a singleton with unique element i_0, then $\prod_{i\in I} M_i = M_{i_0}$ and a map $f : \prod_{i\in I} M_i \to N$ is multilinear if and only if f as a map from $M_{i_\circ}$ to N is linear. This implies that the pair $(M_{i_0}, \mathrm{id}_{M_{i_\circ}})$ then is a tensor product for the family $(M_i)_{i\in I}$.

ad (ii). This is an immediate consequence of the universal property of the tensor product.

ad (iii). We construct an inverse to $\overline{\iota}_{J,1_J} : \bigotimes_{i\in I\setminus J} M_i \to \bigotimes_{i\in I} M_i$. Let $x = (x_i)_{i\in I}$ be an element of $\prod_{i\in I} M_i$ and put

$$\lambda(x) = \left(\prod_{j\in J} x_j\right) \cdot \otimes_{i\in I\setminus J} x_i \left(\prod_{j\in J} x_j\right) \cdot \tau_{I\setminus J}((x_i)_{i\in I\setminus J}) \ .$$

Then $\lambda : \prod_{i\in I} M_i \to \bigotimes_{i\in\setminus J} M_i$ is multilinear by construction, hence factors through a linear map $\overline{\lambda} : \bigotimes_{i\in I} M_i \to \bigotimes_{i\in I\setminus J} M_i$. By definition, $\overline{\lambda}$ is a left inverse of $\overline{\iota}_{J,1_J}$. It is also a right inverse since for all $(x_i)_{i\in I} \in \prod_{i\in I} M_i$ by multilinearity of τ_I

$$\begin{aligned}\overline{\iota}_{J,1_J} \circ \overline{\lambda} \circ \tau_I\left((x_i)_{i\in I}\right) &= \overline{\iota}_{J,1_J}\left(\left(\prod_{j\in J} x_j\right) \cdot \otimes_{i\in I\setminus J} x_i\right) = \left(\prod_{j\in J} x_j\right) \cdot \left(\overline{\iota}_{J,1_J} \circ \tau_{I\setminus J}\left((x_i)_{i\in I\setminus J}\right)\right) = \\ &= \left(\prod_{j\in J} x_j\right) \cdot \left(\tau_I \circ \iota_{J,1_J}\left((x_i)_{i\in I\setminus J}\right)\right) = \tau_I \circ \iota_{J,(x_j)_{j\in J}}\left((x_i)_{i\in I\setminus J}\right) = \tau_I\left((x_i)_{i\in I}\right)\end{aligned}$$

and since by construction of the tensor product the image of τ_I is a generating system for the R-module $\bigotimes_{i\in I} M_i$. □

13.1.10 Lemma *Assume that $(M_i)_{i\in I}$ is a finite family of R-modules such that for every $i \in I$ a generating set S_i of the R-module M_i has been given. Then the set $S = \tau\left(\prod_{i\in I} S_i\right)$ is a generating set of the tensor product $\bigotimes_{i\in I} M_i$.*

Proof. By construction of the tensor product in the proof of Theorem 13.1.9 it is clear that a generating set of $\bigotimes_{i\in I} M_i$ is given by the set of elements of the form $\otimes_{i\in I} x_i$ where $(x_i)_{i\in I} \in \prod_{i\in I} M_i$. Each of the x_i can now be represented in the form

$$x_i = \sum_{k=1}^{n_i} r_{i,k} s_{i,k} \quad \text{with } r_{i,1}, \ldots, r_{i,n_i} \in R, \ \ s_{i,1}, \ldots, s_{i,n_i} \in S_i \ .$$

Hence, by multilinearity of τ and with $I = \{i_1, \ldots, i_d\}$,

$$\otimes_{i\in I} x_i = \tau\left((x_i)_{i\in I}\right) = \sum_{k_{i_1}=1}^{n_{i_1}} \cdots \sum_{k_{i_d}=1}^{n_{i_d}} r_{i_1,k_{i_1}} \cdot \ldots \cdot r_{i_d,k_{i_d}} \cdot \tau\left((s_{i,k_i})_{i\in I}\right) \ ,$$

so $\otimes_{i\in I} x_i$ is a linear combination of elements of S and the claim is proved. □

13.1.11 Lemma *Let $(M_i)_{i\in I}$ be a family of R-modules, $(I_a)_{a\in A}$ a finite partition of the index set I, and N an R-module. For $a \in A$ put $N_a = \bigotimes_{i\in I_a} M_i$ and let $\tau_a : \prod_{i\in I_a} M_i \to N_a$ denote the canonical map. Assume that $f : \prod_{a\in A} \prod_{i\in I_a} M_i \to N$ is a map which is* componentwise multilinear *in the following sense.*

(CM) *Let $b \in A$ and $y = (y_a)_{a\in A} \in \prod_{a\in A}\prod_{i\in I_a} M_i$ a family with $y_b = 0$. If for all $j \in I_b$ and families $x = (x_i)_{i\in I_b} \in \prod_{i\in I_b} M_i$ with $x_j = 0$ the map*

$$M_j \to N, \ \ m \mapsto f(\iota_b(\iota_j(m) + x) + y)$$

is linear, then f factors through $(\tau_a)_{a\in A} : \prod_{a\in A}\prod_{i\in I_a} M_i \to \prod_{a\in A} N_a$. More precisely, there exists a unique multilinear map $\overline{f} : \prod_{a\in A} N_a \to N$ such that

$$f = \overline{f} \circ (\tau_a)_{a\in A} \ .$$

Proof. We prove the claim by induction on the cardinality of A. If A is a singleton, then $\prod_{a\in A}\prod_{i\in I_a} M_i$ canonically coincides with $\prod_{i\in I} M_i$ and $f : \prod_{i\in I_a} M_i \to N$ is multilinear, hence by the universal property of the tensor product there exists a unique linear map $\overline{f} : N_a \to N$ such that $f = \overline{f} \circ \tau_a$.

Now assume that the claim holds whenever the cardinality of the index set A is $\leqslant n$ for some $n \in \mathbb{N}^*$. Assume to be given initial data $(M_i)_{i\in I}$ and N, a partition $(I_a)_{a\in A}$ of A with $|A| = n+1$ and componentwise multilinear map $f : \prod_{a\in A}\prod_{i\in I_a} M_i \to N$. Fix $a \in A$ and put $B = A\backslash\{a\}$. Let $x = (x_i)_{i\in I_a} \in \prod_{i\in I_a} M_i$ and $\widetilde{x}$ be the element of $\prod_{d\in A}\prod_{i\in I_d} M_i$ such that

$$\pi_d(\widetilde{x}) = \begin{cases} x & \text{for } d = a \ , \\ 0 & \text{else} \ . \end{cases}$$

The map

$$f_x : \prod_{b\in B}\prod_{i\in I_b} M_i \to N, \ \ y \mapsto f(\iota_B(y) + \widetilde{x})$$

then is componentwise multilinear. Hence by inductive assumption there exists a unique multilinear map $\overline{f_x} : \prod_{b\in B} N_b \to N$ such that $f_x = \overline{f_x} \circ (\tau_b)_{b\in B}$. By assumption on f the map $\prod_{i\in I_a} M_i \to \mathfrak{Map}\left(\prod_{b\in B}\prod_{i\in I_b} M_i, N\right)$, $x \mapsto f_x$ is multilinear which implies multilinearity of

$$\overline{f_\bullet} : \prod_{i\in I_a} M_i \to \mathfrak{Mlin}\left(\prod_{b\in B} N_b, N\right), \ \ x \mapsto \overline{f_x} \ .$$

Let $F : N_a \to \mathfrak{Mlin}\left(\prod_{b\in B} N_b, N\right)$ be its linearization. Application of the exponential law for multilinear maps, Proposition 13.1.6, now gives a multilinear map $\eta(F) : \prod_{d\in A} N_d \to N$ which we denote by $\overline{f}$. Given a family $(x_d)_{d\in A}$ of families $x_d = (x_i)_{i\in I_d}$ one checks

$$\overline{f}\left(\left(\tau_d(x_d)\right)_{d\in A}\right) = F\left(\tau_a(x_a)\right)\left(\left(\tau_b(x_b)\right)_{b\in B}\right) = \overline{f}_{x_a}\left(\left(\tau_b(x_b)\right)_{b\in B}\right) = f_{x_a}\left((x_b)_{b\in B}\right) = f\left((x_d)_{d\in A}\right) \ .$$

Hence $\overline{f} \circ (\tau_d)_{d\in A} = f$. To finish the induction step it remains to prove uniqueness. So let $\overline{g} : \prod_{d\in A} N_d \to N$ be another multilinear map such that $\overline{g} \circ (\tau_d)_{d\in A} = f$ and consider the induced linear map $\overline{g}^\sharp = \eta^{-1}(\overline{g}) : N_a \mapsto \mathfrak{Mlin}(\prod_{b\in B} N_b, N)$. Then for every $x \in \prod_{i\in I_a} M_i$ the relation

$$\overline{g}^\sharp(\tau_a(x)) \circ (\tau_b)_{b\in B} = f_x = \overline{f}_x \circ (\tau_b)_{b\in B}$$

is satisfied. Hence $\overline{g}^\sharp(\tau(x)) = \overline{f}_x$ for all $x \in \prod_{i\in I_a} M_i$ which entails that $\overline{g}^\sharp$ coincides with F. By Proposition 13.1.6 one obtains $\overline{g} = \overline{f}$. This finishes the induction step and the lemma is proved. □

13.1.12 Proposition *Let $(M_i)_{i\in I}$ be a family of R-modules and $(I_a)_{a\in A}$ a finite partition of the index set I. Then there exists a natural isomorphism*

$$\alpha_{I,A} : \bigotimes_{i\in I} M_i \to \bigotimes_{a\in A} \bigotimes_{i\in I_a} M_i.$$

Proof. Put $N_a = \bigotimes_{i\in I_a} M_i$ for $a \in A$ and let $\tau_a : \prod_{i\in I_a} M_i \to N_a$ be the canonical map to the tensor product. Let $\tau_A : \prod_{a\in A} N_a \to \bigotimes_{a\in A} N_a$ be the canonical map to the tensor product of the modules N_a. Define $\tau_{I,A} : \prod_{i\in I} M_i \to \prod_{a\in A} N_a$ as the unique map so that $\pi_a \circ \tau_{I,A} = \tau_a \circ \pi_{I_a}$ for all $a \in A$. By construction $\tau_{I,A} = (\tau_a)_{a\in A} \circ \kappa_{I,A}$, where $\kappa_{I,A} : \prod_{i\in I} M_i \to \prod_{a\in A}\prod_{i\in I_a} M_i$ is the natural isomorphism from Lemma 13.1.5. The composition $\tau_A \circ \tau_{I,A}$ then is multilinear by Lemma 13.1.4 (iii), hence factors through a linear map $\alpha_{I,A} : \bigotimes_{i\in I} M_i \to \bigotimes_{a\in A} N_a$ that is

$$\tau_A \circ (\tau_a)_{a\in A} \circ \kappa_{I,A} = \alpha_{I,A} \circ \tau_I \ . \tag{13.1.1}$$

Naturality of $\alpha_{I,A}$ in $(M_i)_{i\in I}$ is clear by definition so it remains to construct an inverse to $\alpha_{I,A}$. Consider the composition $\tau_I \circ \kappa^{-1} : \prod_{a\in A}\prod_{i\in I_a} M_i \to \bigotimes_{i\in I} M_i$. Assume that $a \in A$ and $(y_b)_{b\in A\setminus\{a\}} \in \prod_{b\in A\setminus\{a\}}\prod_{i\in I_b} M_i$ have been chosen. Let $y_a \in \prod_{i\in I_a} M_i$ be 0, put $\widetilde{y} = (y_d)_{d\in A} \in \prod_{d\in A}\prod_{i\in I_d} M_i$, and let $y \in \prod_{i\in I} M_i$ be the family such that $\pi_i(y) = \pi_i(y_{a(i)})$ for all $i \in I$, where $a(i)$ denotes the unique element of A such that $i \in I_{a(i)}$. In other words let $y = \kappa^{-1}(\widetilde{y})$. For every $j \in I_a$ and $x = (x_i)_{i\in I_a} \in \prod_{i\in I_a} M_i$ with $\pi_j(x) = 0$ the map

$$M_j \to \bigotimes_{i\in I} M_i, \ \ m \mapsto \tau_I \circ \kappa^{-1}\left(\iota_a(\iota_j(m) + x) + \widetilde{y}\right) = \tau_I\left(\iota_j(m) + \iota_{I_a}(x) + y\right)$$

then is multilinear since τ_I is multilinear and $\pi_j(\iota_{I_a}(x) + y) = \pi_j(x) + \pi_j(y_a) = 0$. Hence $\tau_I \circ \kappa^{-1}$ is componentwise multilinear and therefore, by Lemma 13.1.11, factors through the map $(\tau_a)_{a\in A} : \prod_{a\in A}\prod_{i\in I_a} M_i \to \prod_{a\in A} N_a$ which means that

$$\tau_I \circ \kappa^{-1} = \lambda_{I,A} \circ (\tau_a)_{a\in A} \tag{13.1.2}$$

for some uniquely defined multilinear map $\lambda_{I,A} : \prod_{a\in A} N_a \to \bigotimes_{i\in I} M_i$. Let

$$\overline{\lambda}_{I,A} : \bigotimes_{a\in A} N_a \to \bigotimes_{i\in I} M_i$$

be the linearization of $\lambda_{I,A}$. We claim that $\overline{\lambda_{I,A}}$ is inverse to $\alpha_{I,A}$. By definition of $\overline{\lambda_{I,A}}$ and Eqs. (13.1.1) and (13.1.2) one concludes

$$\overline{\lambda_{I,A}} \circ \alpha_{I,A} \circ \tau_I = \overline{\lambda_{I,A}} \circ \tau_A \circ (\tau_a)_{a\in A} \circ \kappa_{I,A} = \lambda_{I,A} \circ (\tau_a)_{a\in A} \circ \kappa_{I,A} = \tau_I \ .$$

Since the image of τ_I generates $\bigotimes_{i\in I} M_i$ as an R-module, $\overline{\lambda_{I,A}}$ has to be left inverse to $\alpha_{I,A}$. Using Eqs. (13.1.1) and (13.1.2) again compute

$$\alpha_{I,A} \circ \overline{\lambda_{I,A}} \circ \tau_A \circ (\tau_a)_{a\in A} = \alpha_{I,A} \circ \lambda_{I,A} \circ (\tau_a)_{a\in A} = \alpha_{I,A} \circ \tau_A \circ \kappa_{I,A}^{-1} = \tau_A \circ (\tau_a)_{a\in A} \ .$$

Since by Lemma 13.1.10 the image of $\tau_A \circ (\tau_a)_{a\in A}$ generates $\bigotimes_{a\in A}\bigotimes_{i\in I_a} M_i$, the equality

$$\alpha_{I,A} \circ \overline{\lambda_{I,A}} = \mathrm{id}_{\bigotimes_{a\in A}\bigotimes_{i\in I_a} M_i}$$

follows and the proposition is proved. □

13.1.13 Proposition and Definition *Let $(A_i)_{i\in I}$ be a family of R-algebras. Then the tensor product $A = \bigotimes_{i\in I} A_i$ carries in a natural way the structure of an R-algebra where the product map is defined by*

$$\cdot : A \times A \to A, \ \ (\otimes_{i\in I} a_i, \otimes_{i\in I} b_i) \mapsto \otimes_{i\in I}(a_i \cdot b_i) \ .$$

In case each of the algebras A_i is commutative, then A is commutative as well. Likewise, if each A_i is unital and 1_i denotes the unit element of A_i, then A is unital with unit given by $1 = \otimes_{i\in I} 1_i$. One calls A the tensor product algebra *of the family of algebras $(A_i)_{i\in I}$.*

Proof. The map

$$\prod_{(i,k)\in I\times\{1,2\}} A_i \to A, \ \ (a_{i,k})_{(i,k)\in I\times\{1,2\}} \mapsto \otimes_{i\in I}(a_{i,1} \cdot a_{i,2})$$

is multilinear by bilinearity of the product maps on the A_i and multilinearity of τ_I, so factors through a linear map $\mu : A \otimes A \cong \bigotimes_{(i,k)\in I\times\{1,2\}} A_i \to A$. Composition of μ with the canonical bilinear map $A \times A \to A \otimes A$ gives the product map $\cdot : A \times A \to A$ and shows that the product on A is well-defined. By construction, the product map $\cdot$ is bilinear. Given $\otimes_{i\in I} a_i, \otimes_{i\in I} b_i, \otimes_{i\in I} c_i \in A$ one computes

$$\big(\otimes_{i\in I}\, a_i \cdot \otimes_{i\in I} b_i\big) \cdot \otimes_{i\in I} c_i = \otimes_{i\in I}((a_i \cdot b_i) \cdot c_i) = \otimes_{i\in I}(a_i \cdot (b_i \cdot c_i)) = \otimes_{i\in I} a_i \cdot \big(\otimes_{i\in I}\, b_i \cdot \otimes_{i\in I} c_i\big) \ .$$

This entails that the product on A is associative. In the same way one shows that A is commutive respectively unital if each of the A_i is. $\square$

13.1.14 As we have seen, the infinite tensor product construction works well for objects of algebraic categories like R-modules, vector spaces or R-algebras. As soon as a topologies compatible with the algebraic structure come in it becomes difficult and sometimes even impossible to construct or even define

14. Manifolds

14.1. Pro-manifolds

14.2. Hilbert manifolds

14.2.1 In this section we will describe several examples of Hilbert manifolds.

14.2.2 Example Let $\mathfrak{H}$ be a Hilbert space over the field $\mathbb{K}$ of real or complex numbers and $\omega : \mathfrak{H} \to \mathbb{R}$ a continuous nonzero real linear form on $\mathfrak{H}$. Then the *sphere*

$$\mathbb{S}(\mathfrak{H}) = \left\{ v \in \mathfrak{H} \,\middle|\, \|v\| = 1 \right\}$$

is a real analytic Hilbert manifold modelled on the real Hilbert space $\ker \omega$. The sphere has tangent bundle

$$T\mathbb{S}(\mathfrak{H}) = \left\{ (v, w) \in \mathbb{S}(\mathfrak{H}) \times \mathfrak{H} \,\middle|\, \mathfrak{Re}\langle v, w \rangle = 0 \right\} .$$

14.3. The Graßmann manifold of a Banach space

14.3.1 Throughout this section we denote by E a Banach space over the field $\mathbb{K} = \mathbb{R}$ or $= \mathbb{C}$. The main object of study of this section then is the space $\mathbb{G}\mathrm{E}$ of closed $\mathbb{K}$-linear subspaces of E. It is called the *Graßmann manifold* or *Graßmannian* of E. Let us equip $\mathbb{G}\mathrm{E}$ with a natural topology by defining a metric on it. For elements $\mathrm{V}, \mathrm{W} \in \mathbb{G}\mathrm{E}$, the *gap distance* $d_{\mathrm{gap}}(\mathrm{V}, \mathrm{W})$ between V and W is defined as the Haudorff distance of their respective closed unit balls $\overline{\mathbb{B}}_{\mathrm{V}}$ and $\overline{\mathbb{B}}_{\mathrm{W}}$. More precisely that means

$$d_{\mathrm{gap}}(\mathrm{V}, \mathrm{W}) = d_{\mathrm{H}}(\overline{\mathbb{B}}_{\mathrm{V}}, \overline{\mathbb{B}}_{\mathrm{W}}) = \max \left\{ \sup_{v \in \overline{\mathbb{B}}_{\mathrm{V}}} d(v, \overline{\mathbb{B}}_{\mathrm{W}}), \sup_{w \in \overline{\mathbb{B}}_{\mathrm{W}}} d(w, \overline{\mathbb{B}}_{\mathrm{V}}) \right\} , \tag{14.3.1}$$

where, as usual, $d(v, B) = \inf_{w \in B} \|v - w\|$ denotes the *distance* between a point $v \in \mathrm{E}$ and a closed $B \subset \mathrm{E}$.

14.3.2 Lemma *Let*

$$\vec{d}_{\mathrm{gap}}(\mathrm{V}, \mathrm{W}) = \sup_{v \in \overline{\mathbb{B}}_{\mathrm{V}}} d(v, \overline{\mathbb{B}}_{\mathrm{W}})$$

denote the directed *or* one-sided gap *between* $\mathrm{V}, \mathrm{W} \in \mathbb{G}\mathrm{E}$. *Then the following holds true.*

(i) $\vec{d}_{\mathrm{gap}}(0, \mathrm{V}) = \vec{d}_{\mathrm{gap}}(\mathrm{V}, 0) = 1$ *whenever* $\mathrm{V} \neq 0$.

(ii) $\vec{d}_{\mathrm{gap}}(\mathrm{V},\mathrm{W}) = 0$ *if and only if* $\mathrm{V} \subset \mathrm{W}$.

(iii) *For all* $x \in \mathrm{E}$,

$$d(x,\overline{\mathbb{B}}_{\mathrm{V}}) \leqslant d(x,\overline{\mathbb{B}}_{\mathrm{W}}) + \vec{d}_{\mathrm{gap}}(\mathrm{W},\mathrm{V}) \ .$$

Proof. (i) follows immediately by definition and (ii) holds true since $d(v,\overline{\mathbb{B}}_{\mathrm{W}}) = 0$ if and only if $v \in \overline{\mathbb{B}}_{\mathrm{W}}$. It remains to show (iii). To this end let $x \in \mathrm{E}$, $v \in \overline{\mathbb{B}}_{\mathrm{V}}$ and $w \in \overline{\mathbb{B}}_{\mathrm{W}}$. Then, by the triangle inequality for the distance $d : \mathrm{E}\times\mathrm{E} \to \mathbb{R}$, $(x,y) \mapsto \|x-y\|$,

$$d(x,v) \leqslant d(x,w) + d(w,v) \ .$$

This entails, by taking the infimum with respect to $v \in \overline{\mathbb{B}}_{\mathrm{V}}$,

$$d(x,\overline{\mathbb{B}}_{\mathrm{V}}) \leqslant d(x,w) + d(w,\overline{\mathbb{B}}_{\mathrm{V}}) \leqslant d(x,w) + \vec{d}_{\mathrm{gap}}(\mathrm{W},\mathrm{V}) \ .$$

Since $w \in \overline{\mathbb{B}}_{\mathrm{W}}$ was arbitrary, (iii) follows. □

14.3.3 Proposition *The gap distance on the Graßmannian* $\mathbb{G}\mathrm{E}$ *of a Banach space is a metric.*

Proof. By definition, the gap distance is symmetric. By (ii) of Lemma 14.3.2, one has $d_{\mathrm{gap}}(\mathrm{V},\mathrm{W}) = 0$ if and only if $\mathrm{V} = \mathrm{W}$. It remains to show the triangle inequality. Let $\mathrm{V},\mathrm{W},\mathrm{X} \in \mathbb{G}\mathfrak{H}$ and use (iii) in the preceding lemma to verify

$$\begin{aligned}
\vec{d}_{\mathrm{gap}}(\mathrm{X},\mathrm{V}) &= \sup_{x\in\overline{\mathbb{B}}_{\mathrm{X}}} d(x,\overline{\mathbb{B}}_{\mathrm{V}}) \leqslant \sup_{x\in\overline{\mathbb{B}}_{\mathrm{X}}} d(x,\overline{\mathbb{B}}_{\mathrm{W}}) + \vec{d}_{\mathrm{gap}}(\mathrm{W},\mathrm{V}) \leqslant \vec{d}_{\mathrm{gap}}(\mathrm{X},\mathrm{W}) + \vec{d}_{\mathrm{gap}}(\mathrm{W},\mathrm{V}) \ ,\\
\vec{d}_{\mathrm{gap}}(\mathrm{V},\mathrm{X}) &= \sup_{v\in\overline{\mathbb{B}}_{\mathrm{V}}} d(v,\overline{\mathbb{B}}_{\mathrm{X}}) \leqslant \sup_{v\in\overline{\mathbb{B}}_{\mathrm{V}}} d(v,\overline{\mathbb{B}}_{\mathrm{W}}) + \vec{d}_{\mathrm{gap}}(\mathrm{W},\mathrm{X}) \leqslant \vec{d}_{\mathrm{gap}}(\mathrm{V},\mathrm{W}) + \vec{d}_{\mathrm{gap}}(\mathrm{W},\mathrm{X}) \ .
\end{aligned}$$

This entails the triangle inequality for d_{gap}. □

14.3.4 Recall that to every closed linear subspace $\mathrm{V} \subset \mathfrak{H}$ of a Hilbert space $\mathfrak{H}$ there exists a unique orthogonal projection $P_{\mathrm{V}} : \mathfrak{H} \to \mathfrak{H}$ whose image is V. The kernel of the projection P_{V} coincides with the orthogonal complement $\mathrm{V}^{\perp}$. One thus obtains a canonical embedding of $\mathbb{G}\mathfrak{H} \hookrightarrow \mathfrak{B}(\mathfrak{H})$. The restriction of the operator norm distance to $\mathbb{G}\mathfrak{H}$ endows $\mathbb{G}\mathfrak{H}$ with another metric which we denote by δ.

14.3.5 Proposition ((Akhiezer & Glazman, 1993, Sec. 34)) *For every Hilbert space* $\mathfrak{H}$ *the metric*

$$\delta : \mathbb{G}\mathfrak{H} \times \mathbb{G}\mathfrak{H} \to \mathbb{R},\ (\mathrm{V},\mathrm{W}) \mapsto \|P_{\mathrm{V}} - P_{\mathrm{W}}\|$$

coincides with the gap metric $d_{\mathrm{gap}} : \mathbb{G}\mathfrak{H} \times \mathbb{G}\mathfrak{H} \to \mathbb{R}$. *Moreoever, for all* $\mathrm{V},\mathrm{W} \in \mathbb{G}\mathfrak{H}$,

(i) $d_{\mathrm{gap}}(\mathrm{V},\mathrm{W}) \leqslant 1$,

(ii) $\vec{d}_{\mathrm{gap}}(\mathrm{V},\mathrm{W}) = \|(I-P_{\mathrm{W}})P_{\mathrm{V}}\|$, *and*

(iii) $d_{\mathrm{gap}}(\mathrm{V},\mathrm{W}) = \max\left\{\|(I-P_{\mathrm{W}})P_{\mathrm{V}}\|, \|(I-P_{\mathrm{V}})P_{\mathrm{W}}\|\right\}$.

Proof. First note that

$$\|(I - P_{\mathrm{W}})P_{\mathrm{V}}\| = \sup_{v \in \overline{\mathbb{B}}_{\mathrm{V}}} \|v - P_{\mathrm{W}} v\| = \vec{d}_{\mathrm{gap}}(\mathrm{V}, \mathrm{W})$$

since $d(v, \mathrm{W}) = \|v - P_{\mathrm{W}} v\|$ for all $v \in \overline{\mathbb{B}}_{\mathrm{V}}$ by the orthogonal decomposition theorem, 12.2.3. This proves (ii) and (iii). Next observe that

$$P_{\mathrm{V}} - P_{\mathrm{W}} = P_{\mathrm{V}}(I - P_{\mathrm{W}}) - (I - P_{\mathrm{V}})P_{\mathrm{W}} \ .$$

By orthogonality of the images of $P_{\mathrm{V}}(I - P_{\mathrm{W}})$ and $(I - P_{\mathrm{V}})P_{\mathrm{W}}$ this implies for all $x \in \mathfrak{H}$

$$\begin{aligned} \|(P_{\mathrm{V}} - P_{\mathrm{W}})x\|^2 &= \|P_{\mathrm{V}}(I - P_{\mathrm{W}})x\|^2 + \|(I - P_{\mathrm{V}})P_{\mathrm{W}} x\|^2 \leqslant \\ &\leqslant \|(I - P_{\mathrm{W}})x\|^2 + \|P_{\mathrm{W}} x\|^2 = \|x\|^2 \ , \end{aligned} \tag{14.3.2}$$

hence

$$\delta(\mathrm{V}, \mathrm{W}) = \|P_{\mathrm{V}} - P_{\mathrm{W}}\| \leqslant 1 \ . \tag{14.3.3}$$

One also obtains

$$\delta(\mathrm{V}, \mathrm{W}) = \sup_{x \in \overline{\mathbb{B}}_{\mathfrak{H}}} \|(P_{\mathrm{V}} - P_{\mathrm{W}})x\| = \sup_{x \in \overline{\mathbb{B}}_{\mathfrak{H}}} \sqrt{\|P_{\mathrm{V}}(I - P_{\mathrm{W}})x\|^2 + \|(I - P_{\mathrm{V}})P_{\mathrm{W}} x\|^2}. \tag{14.3.4}$$

By restricting x to the closed ball of W this formula entails

$$\delta(\mathrm{V}, \mathrm{W}) \geqslant \sup_{x \in \overline{\mathbb{B}}_{\mathrm{W}}} \|(I - P_{\mathrm{V}})P_{\mathrm{W}} x\| = \sup_{x \in \overline{\mathbb{B}}_{\mathrm{W}}} \|(I - P_{\mathrm{V}})x\| = \vec{d}_{\mathrm{gap}}(\mathrm{V}, \mathrm{W}) \ .$$

By switching V and W in (14.3.3) one gets

$$\delta(\mathrm{V}, \mathrm{W}) \geqslant \sup_{x \in \overline{\mathbb{B}}_{\mathrm{V}}} \|(I - P_{\mathrm{W}})P_{\mathrm{V}} x\| = \sup_{x \in \overline{\mathbb{B}}_{\mathrm{V}}} \|(I - P_{\mathrm{W}})x\| = \vec{d}_{\mathrm{gap}}(\mathrm{W}, \mathrm{V}) \ .$$

Consequently,

$$\delta(\mathrm{V}, \mathrm{W}) \geqslant d_{\mathrm{gap}}(\mathrm{V}, \mathrm{W}) \ . \tag{14.3.5}$$

Let us show that

$$\delta(\mathrm{V}, \mathrm{W}) \leqslant d_{\mathrm{gap}}(\mathrm{V}, \mathrm{W}) \ . \tag{14.3.6}$$

To this end observe that for all $x \in \overline{\mathbb{B}}_{\mathfrak{H}}$ by (ii) and $P_{\mathrm{W}}^2 = P_{\mathrm{W}}$

$$\|(I - P_{\mathrm{V}})P_{\mathrm{W}} x\| \leqslant \vec{d}_{\mathrm{gap}}(\mathrm{W}, \mathrm{V}) \cdot \|P_{\mathrm{W}} x\| \ . \tag{14.3.7}$$

Moreover,

$$\begin{aligned} \|P_{\mathrm{V}}(I - P_{\mathrm{W}})x\|^2 &= \langle P_{\mathrm{V}}(I - P_{\mathrm{W}})x, P_{\mathrm{V}}(I - P_{\mathrm{W}})x \rangle = \langle P_{\mathrm{V}}^2(I - P_{\mathrm{W}})x, (I - P_{\mathrm{W}})^2 x \rangle = \\ &= \langle (I - P_{\mathrm{W}})P_{\mathrm{V}}^2(I - P_{\mathrm{W}})x, (I - P_{\mathrm{W}})x \rangle \leqslant \\ &\leqslant \vec{d}_{\mathrm{gap}}(\mathrm{V}, \mathrm{W}) \, \|P_{\mathrm{V}}(I - P_{\mathrm{W}})x\| \, \|(I - P_{\mathrm{W}})x\| \ , \end{aligned}$$

and therefore

$$\|P_{\mathrm{V}}(I - P_{\mathrm{W}})x\| \leqslant \vec{d}_{\mathrm{gap}}(\mathrm{V}, \mathrm{W}) \, \|(I - P_{\mathrm{W}})x\| \ . \tag{14.3.8}$$

Inserting this estimate and (14.3.7) into the squared right side of (14.3.4) then gives

$$\|P_{\mathrm{V}}(I-P_{\mathrm{W}})x\|^2 + \|(I-P_{\mathrm{V}})P_{\mathrm{W}}x\|^2 \leqslant \vec{d}^{\,2}_{\mathrm{gap}}(\mathrm{W},\mathrm{V})\cdot\|P_{\mathrm{W}}x\|^2 + \vec{d}^{\,2}_{\mathrm{gap}}(\mathrm{V},\mathrm{W})\,\|(I-P_{\mathrm{W}})x\|^2 \leqslant$$
$$\leqslant d^2_{\mathrm{gap}}(\mathrm{W},\mathrm{V})\cdot\left(\|P_{\mathrm{W}}x\|^2 + \|(I-P_{\mathrm{W}})x\|^2\right) = d^2_{\mathrm{gap}}\,\|x\|^2\ .$$

Comparing with the left side of (14.3.4) shows (14.3.6), and the equality of δ and d_{gap} follows. By (14.3.3) the latter also yields (i). □

14.3.6 Remark We will use the symbols d_{gap} and δ interchangeably to denote the gap metric on the Graßmannian of a Banach space .

14.3.7 Theorem *Eqipped with the gap metric the Graßmann manifold of a Banach space is a complete metric space.*

Proof. We present the proof for the underlying Banach space being a Hilbert space $\mathfrak{H}$. Then the claim followsimmediately from the fact that $\mathfrak{B}(\mathfrak{H})$ is complete and that the limit of a Cauchy sequence of orthogonal projections $(P_n)_{n\in\mathbb{N}} \subset \mathfrak{B}(\mathfrak{H})$ is again an orthogonal projection. The general case is more tricky. □

15. Lie groups

15.1. Symmetry groups of bilinear and sesquilinear forms

15.1.1 In this section $\mathbb{K}$ will always stand for the field of real or complex numbers. Before defining their symmetry groups let us recall the notions of bilinear and sesquilinear forms. A *bilinear form* on a $\mathbb{K}$-vector space V is a map $b : \mathrm{V} \times \mathrm{V} \to \mathbb{K}$ having the following properties:

(BF1) The map b is *linear* in its first coordinate which means that

$$b(v_1 + v_2, w) = b(v_1, w) + b(v_2, w) \quad \text{and} \quad b(rv, w) = rb(v, w)$$

for all $v, v_1, v_2, w \in \mathrm{V}$ and $r \in \mathbb{K}$.

(BF2), (SF2) The map b is *linear* in its second coordinate which means that

$$b(v, w_1 + w_2) = b(v, w_1) + b(v, w_2) \quad \text{and} \quad b(v, rw) = rb(v, w)$$

for all $v, w, w_1, w_2 \in \mathrm{V}$ and $r \in \mathbb{K}$.

Bilinear forms with the property that commuting its variables leads to the same or to the negative of the original bilinear form are given a particular name. More precisely, a bilinear map $b : \mathrm{V} \times \mathrm{V} \to \mathbb{K}$ is said to be *symmetric* if

(BF3s) $b(v, w) = b(w, v)$ for all $v, w \in \mathrm{V}$,

and *antisymmetric* or *skew-symmetric* if

(BF3a) $b(v, w) = -b(w, v)$ for all $v, w \in \mathrm{V}$.

A map $b : \mathrm{V} \times \mathrm{V} \to \mathbb{K}$ which satisfies (BF2) and axiom (SF1) below instead of (BF1) is called a *sesquilinear form.*

(SF1) The map b is *conjugate-linear* in its first coordinate which means that

$$b(v_1 + v_2, w) = b(v_1, w) + b(v_2, w) \quad \text{and} \quad b(rv, w) = \overline{r}b(v, w)$$

for all $v, v_1, v_2, w \in \mathrm{V}$ and $r \in \mathbb{K}$.

A sesquilinear form b is called a *hermitian form* if it has the following property:

(SF3c) The map b is *conjugate-symmetric* which means that

$$b(v, w) = \overline{b(w, v)} \quad \text{for all } v, w \in \mathrm{V} \ .$$

If the ground field of the underlying vector space of a bilinear or sesquilinear form b is $\mathbb{C}$, one calls b a *complex* bilinear form respectively a *complex* sesquilinear form. One uses analogous language when the ground field is $\mathbb{R}$. Note that a real sesquilinear form is the same as a real bilinear form.

A bilinear or sesquilinear form b is said to be *weakly-nondegenerate* if it satisfies axiom

(SF4w) The map $^\flat : \mathrm{V} \to \mathrm{V}'$, $v \mapsto v^\flat = b(-, v) = \big(\mathrm{V} \ni w \to b(w, v) \in \mathbb{K}\big)$ from V to its algebraic dual V' is injective .

Note that (SF4w) is equivalent to the requirement that for every $v \in \mathrm{V}$ the map $b(v, -) : \mathrm{V} \to \mathbb{K}$, $w \to b(v, w)$ is the zero map if and only if $v = 0$.

In case the underlying vector space V is normed, there is a stronger version of nondegeneracy for bounded bilinear or sesquilinear forms $b : \mathrm{V} \times \mathrm{V} \to \mathbb{K}$. Namely, one calls such a form *nondegenerate* if it fulfills

(SF4n) The map $^\flat : \mathrm{V} \to \mathrm{V}^*$, $v \mapsto v^\flat = b(v, -) = \big(\mathrm{V} \ni w \to b(v, w) \in \mathbb{K}\big)$ from V to its topological dual V^* is a linear or conjugate-linear topological isomorphism.

Recall that $b(v, v) \in \mathbb{R}$ for every hermitian form b on V and $v \in \mathrm{V}$. In case that such a b satisfies

(SF5s) $b(v, v) \geqslant 0$ for all $v \in \mathrm{V}$,

then one calls the hermitian form b *positive semidefinite*.

Recall from Lemma 12.1.6 that a positive semidefinite hermitian form b on a $\mathbb{K}$-vector space V is weakly-nondegenerate if and only if it is *positive definite* which means that

(SF5p) $b(v, v) > 0$ for all $v \in \mathrm{V}\backslash\{0\}$.

15.1.2 Remark If b is a nondegenerate bilinear or sesquilinear form on a Banach space V, then one sometimes calls the map $^\flat : \mathrm{V} \to \mathrm{V}^*$ from Axiom (SF4n) and its inverse $^\sharp : \mathrm{V}^* \to \mathrm{V}$ the *musical isomorphisms* associated to b.

15.1.3 Examples In addition to the hermitian forms introduced in Examples 12.1.9 let us give a few more examples of bilinear forms which are particularly relevant for mathematics or mathematical physics.

(a) Let p, q be positive integers, $n = p + q$, and $\langle \cdot, \cdot \rangle_{p,q} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ the *pseudo-euclidean* form given by

$$\langle x, y \rangle_{p,q} = \sum_{i=1}^{p} x^i\, y^i - \sum_{j=p+1}^{n} x^j\, y^j \quad \text{for } x = (x^1, \ldots, x^n),\ y = (y^1, \ldots, y^n) \in \mathbb{R}^n .$$

The map $\langle \cdot, \cdot \rangle_{p,q}$ is a nondegenerate bilinear form, but it is not positive semidefinite by definition. The space $\mathbb{R}^n$ together with the pseudo-euclidean form $\langle \cdot, \cdot \rangle_{p,q}$ is sometimes denoted $\mathbb{R}^{p,q}$. For the particular case $(p, q) = (1, d) = (1, n-1)$ one calls $\mathbb{R}^{1,d}$ *Minkowski space* of *space–time dimension* $d + 1$, and $\langle \cdot, \cdot \rangle_{\mathrm{M}} := \langle \cdot, \cdot \rangle_{1,d}$ the corresponding *Minkowski metric*. The components of elements $x, y \in \mathbb{R}^{1,d}$ of Minkowski space are often indexed in the form $x = (x^0, x^1, \ldots, x^d) = (x^\mu)_{\mu=0}^d$ and $y = (y^0, y^1, \ldots, y^d) = (y^\nu)_{\nu=0}^d$. In this notation, the Minkowski metric is given by

$$\langle x, y\rangle_{\mathrm{M}} = x^0\, y^0 - \sum_{i=1}^{d} x^i\, y^i \ .$$

Moreover, one associates to x and y the *space-vectors*

$$\vec{x} = (x^1, \ldots, x^d) = (x^i)_{i=1}^d \quad \text{and} \quad \vec{y} = (y^1, \ldots, y^d) = (y^j)_{j=1}^d \ .$$

When labels run through all space-time indices they are usually denoted in the mathematical physics literature by lower-case Greek letters, when they run only through space indices, they are denoted by lower-case Roman letters. We will follow these conventions.

(b) Next consider $\mathbb{K}^{2n}$ with $n \in \mathbb{N}_{>0}$ and define

$$\omega : \mathbb{K}^{2n} \times \mathbb{K}^{2n} \to \mathbb{K}, \quad (v, w) \mapsto \sum_{i=1}^{n} (v_i\, w_{n+i} - w_i\, v_{n+i}) \ .$$

Then ω is a nondegenerate antisymmetric bilinear form. We call it the *standard symplectic form* on $\mathbb{K}^{2n}$. More generally, a nondegenerate antisymmetric bilinear form $\omega : \mathrm{E} \times \mathrm{E} \to \mathbb{K}$ on a Banach space E over $\mathbb{K}$ is called a *symplectic form*. If $\omega : \mathrm{E} \times \mathrm{E} \to \mathbb{K}$ is only weakly-nondegenerate (but still antisymmetric), then one says that ω is a weakly-symplectic form.

If V is a Banach space and $\mathrm{E} = \mathrm{V} \oplus \mathrm{V}^*$, then

$$\omega : \mathrm{E} \times \mathrm{E} \to \mathbb{K}, \ \big((v, \alpha), (w, \beta)\big) \mapsto \beta(v) - \alpha(w)$$

is a weakly-symplectic form on E which is symplectic if and only if V is reflexive that is if and only if the canonical embedding $\mathrm{V} \hookrightarrow \mathrm{V}^{**}$ is an isomorphism.

Proof. Antisymmetry is clear by definition. □

15.1.4 Next consider a Banach space E over $\mathbb{K}$ with norm $\|\cdot\|$ and the space $\mathfrak{B}(\mathrm{E})$ of bounded $\mathbb{K}$-linear operators on E. Recall that $\mathfrak{B}(\mathrm{E})$ carries the following natural topologies:

(i) the *norm topology* or *uniform operator topology* $\mathcal{T}_{\mathrm{n}}$ defined by the *operator norm*

$$\| - \| : \mathfrak{B}(\mathrm{E}) \to \mathbb{R}_{\geqslant 0}, \ A \mapsto \|A\| := \sup\big\{ \|Av\| \in \mathbb{R}_{\geqslant 0} \mid v \in \mathrm{E} \,\&\, \|v\| \leqslant 1 \big\} \ ,$$

(ii) the *compact-open topology* $\mathcal{T}_{\mathrm{co}}$ defined by the seminorms

$$p_K : \mathfrak{B}(\mathrm{E}) \to \mathbb{R}_{\geqslant 0}, \ A \mapsto p_K(A) := \sup\big\{ \|Av\| \in \mathbb{R}_{\geqslant 0} \mid v \in K \big\} \ ,$$

where K runs through the nonempty compact subsets of E,

(iii) the *strong operator topology* $\mathcal{T}_{\mathrm{s}}$ defined by the seminorms

$$p_v : \mathfrak{B}(\mathrm{E}) \to \mathbb{R}_{\geqslant 0}, \ A \mapsto p_v(A) := \|Av\| \ ,$$

where v runs through the nonzero elements of E,

(iv) the *weak operator topology* $\mathcal{T}_\mathrm{w}$ defined by the seminorms

$$p_{\lambda,v} : \mathfrak{B}(\mathrm{E}) \to \mathbb{R}_{\geqslant 0},\ A \mapsto p_{\lambda,v}(A) := \lambda(Av)\ ,$$

where λ runs through the nonzero bounded linear functionals $\mathrm{E} \to \mathbb{K}$ and v through the nonzero elements of E.

These four operator topologies are comparable. More precisely one has

$$\mathcal{T}_\mathrm{w} \subset \mathcal{T}_\mathrm{s} \subset \mathcal{T}_\mathrm{co} \subset \mathcal{T}_\mathrm{n}\ .$$

In case E is finite dimensional, the topologies coincide, if E is infinite dimensional, then the inclusions are proper.

To denote which topology $\mathfrak{B}(\mathrm{E})$ is endowed with we write $\mathfrak{B}(\mathrm{E})_\mathrm{n}$, $\mathfrak{B}(\mathrm{E})_\mathrm{co}$, $\mathfrak{B}(\mathrm{E})_\mathrm{s}$ and $\mathfrak{B}(\mathrm{E})_\mathrm{w}$, respectively.

15.1.5 Proposition and Definition *Let* E *be a Banach space over* $\mathbb{K}$*, and* $\mathsf{GL}(\mathrm{E}) \subset \mathfrak{B}(\mathrm{E})_\mathrm{n}$ *the space of bounded invertible* $\mathbb{K}$*-linear operators on* E *endowed with the norm topology. Then the following holds true.*

(a) *The space* $\mathsf{GL}(\mathrm{E})$ *is open in* $\mathfrak{B}(\mathrm{E})_\mathrm{n}$.

(b) $\mathsf{GL}(\mathrm{E})$ *together with the operator product and the identity map* id_E *is a group.*

(c) *The group* $\mathsf{G} := \mathsf{GL}(\mathrm{E})$ *endowed with the norm topology is a* topological group *which means that it has the following properties:*

(TopGr1) *The multiplication map* $\cdot : \mathsf{G} \times \mathsf{G} \to \mathsf{G}$ *is continuous.*

(TopGr2) *The inversion map* $i : \mathsf{G} \to \mathsf{G}$ *is continuous.*

Proof. ad (b). By the open mapping theorem the inverse of a bounded invertible operator is bounded as well, hence $g^{-1} \in \mathsf{GL}(\mathrm{E})$ for all $g \in \mathsf{GL}(\mathrm{E})$. Obviously $\mathrm{id}_\mathrm{E} \in \mathsf{GL}(\mathrm{E})$, so $\mathsf{GL}(\mathrm{E})$ is a group indeed.

ad (a). Let $g \in \mathsf{GL}(\mathrm{E})$. Then $\|g^{-1}\| > 0$, since $1 = \|v\| \leqslant \|g^{-1}\|\,\|gv\|$ for every unit vector $v \in \mathrm{E}$. Let $0 < r < \frac{1}{\|g^{-1}\|}$. For $A \in \mathfrak{B}(\mathrm{E})$ with $\|A\| < r$ the series $\sum_{k\in\mathbb{N}} (-1)^k \big(g^{-1}A\big)^k$ then is dominated by the converging geometric series $\sum_{k\in\mathbb{N}} r^k$, hence converges too. Compute

$$\big(\mathrm{id}_\mathrm{E} + g^{-1}A\big)\left(\sum_{k=0}^\infty (-1)^k \big(g^{-1}A\big)^k\right) = \sum_{k=0}^\infty (-1)^k \big(g^{-1}A\big)^k - \sum_{k=1}^\infty (-1)^k \big(g^{-1}A\big)^k = \mathrm{id}_\mathrm{E}$$

and analogously

$$\left(\sum_{k=0}^\infty (-1)^k \big(g^{-1}A\big)^k\right)\big(\mathrm{id}_\mathrm{E} + g^{-1}A\big) = \mathrm{id}_\mathrm{E}\ .$$

Therefore $\mathrm{id}_\mathrm{E} + g^{-1}A$ is invertible with bounded inverse $\sum_{k=0}^\infty (-1)^k \big(g^{-1}A\big)^k$. Hence the operator $g + A = g\big(\mathrm{id}_\mathrm{E} + g^{-1}A\big)$ is invertible as well and the open ball of radius r around g is contained in $\mathsf{GL}(\mathrm{E})$. Thus $\mathsf{GL}(\mathrm{E})$ is open in $\mathfrak{B}(\mathrm{E})$.

ad (c). To verify continuity of multiplication recall that $\|AB\| \leqslant \|A\|\,\|B\|$ for all $A, B \in \mathfrak{B}(\mathrm{E})$. Then

$$\|AB - A'B'\| = \|(AB - A'B) + (A'B - A'B')\| \leqslant \|A - A'\|\,\|B\| + \|A'\|\,\|B - B'\| \ .$$

Hence multiplication is locally Lipschitz continuous, therefore continuous.

To prove continuity of inversion let $g \in \mathsf{GL}(\mathrm{E})$ and choose $0 < r < \frac{1}{\|g^{-1}\|}$. Then $g + A \in \mathsf{GL}(\mathrm{E})$ for all $A \in \mathfrak{B}(\mathrm{E})$ with $\|A\| < r$ by the preceding considerations. Moreover,

$$\begin{aligned}\left\|(g+A)^{-1} - g^{-1}\right\| &= \left\|\big((\mathrm{id}_{\mathrm{E}} + g^{-1}A)^{-1} - \mathrm{id}_{\mathrm{E}}\big)g^{-1}\right\| \leqslant \\ &\leqslant \left\|g^{-1}\right\| \left\|\sum_{k=1}^{\infty}(-1)^k\big(g^{-1}A\big)^k\right\| \leqslant \left\|g^{-1}\right\| \sum_{k=1}^{\infty}\left\|g^{-1}A\right\|^k \leqslant \frac{\left\|g^{-1}\right\|^2}{1 - r\|g^{-1}\|}\|A\| \ .\end{aligned}$$

Hence inversion is locally Lipschitz continuous, so in particular continuous. □

Unless mentioned differently, we assume from now on that $\mathsf{GL}(\mathrm{E})$ carries the norm topology. Sometimes we will write $\mathsf{GL}(\mathrm{E})_{\mathrm{n}}$ to emphasize this.

15.1.6 Assume that $b : \mathrm{E} \times \mathrm{E} \to \mathbb{K}$ is a bounded bilinear or sesquilinear form on a Banach space E over $\mathbb{K}$. Consider the group $\mathsf{GL}(\mathrm{E})$ and define $\mathsf{G}(\mathrm{E}, b)$ as the set of all $g \in \mathsf{GL}(\mathrm{E})$ such that

$$b(gv, gw) = b(v, w) \quad \text{for all } v, w \in \mathrm{E} \ .$$

15.1.7 Proposition *Under the assumptions stated* $\mathsf{G}(\mathrm{E}, b)$ *is a closed subgroup of* $\mathsf{GL}(\mathrm{E})_{\mathrm{n}}$.

Proof. If $g, h \in \mathsf{G}(\mathrm{E}, b)$, then their operator product gh lies in $\mathsf{G}(\mathrm{E}, b)$ as well since

$$b(ghv, ghw) = b(hv, hw) = b(v, w) \quad \text{for all } v, w \in \mathrm{E} \ .$$

Moreover, id_{E} leaves b invariant, so is in $\mathsf{G}(\mathrm{E}, b)$, too. Hence $\mathsf{G}(\mathrm{E}, b)$ is a subgroup of $\mathsf{GL}(\mathrm{E})_{\mathrm{n}}$.

Now assume that $g \in \mathsf{GL}(\mathrm{E})_{\mathrm{n}} \backslash \mathsf{G}(\mathrm{E}, b)$. Then there are $v, w \in \mathrm{E}$ such that

$$b(gv, gw) \neq b(v, w) \quad \text{and} \quad \|v\| = \|w\| = 1 \ .$$

Put $\delta = |b(gv, gw) - b(v, w)|$ and let $C = \sup\big\{|b(x, y)| \bigm| x, y \in \mathrm{E} \ \& \ \|x\| = \|y\| = 1\big\}$. Then one has for all $h \in \mathfrak{B}(\mathrm{E})$

$$\begin{aligned}|b(hv, hw) - b(v, w)| &= \left|\big(b(hv, hw) - b(gv, gw)\big) - \big(b(v, w) - b(gv, gw)\big)\right| \geqslant \\ &\geqslant \big|\delta - |b(hv, hw) - b(gv, gw)|\big| \geqslant \\ &\geqslant \delta - |b(hv, hw) - b(gv, hw)| - |b(gv, hw) - b(gv, gw)| \geqslant \\ &\geqslant \delta - C\,\|h - g\|\,(\|h\| + \|g\|) \ .\end{aligned}$$

Hence, if $\|h - g\| < \varepsilon$ with $\varepsilon = \min\left\{1, \frac{1}{2\|g^{-1}\|}, \frac{\delta}{2(C+1)(2\|g\|+1)}\right\}$, then $h \in \mathsf{GL}(\mathrm{E})$ and

$$|b(hv, hw) - b(v, w)| \geqslant \delta - C\,(2\|g\| + 1)\,\varepsilon \geqslant \frac{1}{2}\delta \ .$$

So $\mathsf{GL}(\mathrm{E})_{\mathrm{n}} \backslash \mathsf{G}(\mathrm{E}, b)$ is open and the claim is proved. □

15.1.8 Examples (a) For a Hilbert space $\mathfrak{H}$, the group $\mathsf{G}(\mathfrak{H}, \langle\cdot,\cdot\rangle)$ is called the *unitary group* of $\mathfrak{H}$ and denoted $\mathsf{U}(\mathfrak{H})$. If the underlying ground field is $\mathbb{R}$, one often writes $\mathsf{O}(\mathfrak{H})$ for $\mathsf{G}(\mathfrak{H}, \langle\cdot,\cdot\rangle)$ and calls it the *orthogonal group* of the real Hilbert space $\mathfrak{H}$. In the finite dimensional case, $\mathsf{U}(n)$ stands for $\mathsf{U}(\mathbb{C}^n)$ and $\mathsf{O}(n)$ for $\mathsf{O}(\mathbb{R}^n)$.

(b) Given two positive integers p, q consider the pseudo-euclidean metric $\langle\cdot,\cdot\rangle_{p,q}$ on $\mathbb{R}^{p,q} \cong \mathbb{R}^{p+q}$, see Example 15.1.3 (a). The invariant group $\mathsf{G}\big(\mathbb{R}^n, \langle\cdot,\cdot\rangle_{p,q}\big)$ then is called a *pseudo-orthogonal group* and is denoted $\mathsf{O}(p,q)$.

(c) Let E be a Banach space over $\mathbb{K}$ with a symplectic form ω. The group $\mathsf{Sp}(\mathrm{E},\omega) := \mathsf{G}\big(\mathrm{E},\omega)$ then is the *symplectic group* associated to (E,ω). If E is $\mathbb{K}^{2n}$ and ω its canonical symplectic form, then one writes $\mathsf{Sp}(2n,\mathbb{K})$ for the associated symplectic group.

15.1.9 It has been claimed wrongly at several places in the mathematical literature, notably in Simms (1968)[Proof of Thm. 1] and Atiyah & Segal (2004)[p. 321] that the unitary group with the strong operator topology respectively with the compact-open topology is not a topological group. The correct(ed) statement appeared in Schottenloher (1995)[III.3.2 Satz], Neeb (1997)[Prop. II.1], and Schottenloher (2008)[Prop.3.11], whose presentation we will essentially follow here.

15.1.10 Proposition *If $\mathfrak{H}$ is a Hilbert space, then $\mathsf{U}(\mathfrak{H})_{\mathrm{s}}$, the unitary group $\mathsf{U}(\mathfrak{H})$ endowed with the strong operator topology, is a complete topological group. Moreover, the compact-open topology, the strong operator topology, and weak operator topology all coincide on $\mathsf{U}(\mathfrak{H})$. Finally, if $\mathfrak{H}$ is separable, then $\mathsf{U}(\mathfrak{H})_{\mathrm{s}}$ is completely metrizable.*

Proof. For $v \in \mathfrak{H}$ and $V \in \mathsf{U}(\mathfrak{H})$ let $p_{v,V} : \mathsf{U}(\mathfrak{H}) \to \mathbb{R}_{\geqslant 0}$ be defined by

$$U \mapsto p_{v,V}(U) = \|(U - V)v\| \ .$$

A subbasis of the strong operator topology on $\mathsf{U}(\mathfrak{H})$ then is given by the sets

$$\big\{U \in \mathsf{U}(\mathfrak{H}) \bigm| p_{v,V}(U) < \varepsilon\big\}, \quad \text{where } v \in \mathfrak{H},\ V \in \mathsf{U}(\mathfrak{H}), \text{ and } \varepsilon > 0 \ . \qquad \square$$

15.2. The Lie group SO(3) and its universal cover SU(2)

15.2.1 Recall that the *orthogonal group* in real dimension 3 is given by

$$\mathsf{O}(3) = \big\{g \in \mathsf{GL}(3,\mathbb{R}) \bigm| \forall \vec{x}, \vec{y} \in \mathbb{R}^3 : \langle g\vec{x}, g\vec{y}\rangle = \langle \vec{x}, \vec{y}\rangle\big\} \ .$$

The *special orthogonal group* in dimension 3 is the subgroup

$$\mathsf{SO}(3) = \big\{g \in \mathsf{O}(3) \bigm| \det g = 1\big\} \ .$$

Let us show that both are Lie groups. Consider the map

$$f : \mathsf{GL}(3,\mathbb{R}) \to \mathfrak{Sym}(3,\mathbb{R}), \quad g \mapsto g^{\mathrm{t}} g \ ,$$

where g^{t} is the transpose of g and $\mathfrak{Sym}(n,\mathbb{R})$ denotes the space of real symmetric $n \times n$ matrices. Note that $\dim \mathfrak{Sym}(n,\mathbb{R}) = \frac{n(n+1)}{2}$ and that f is well-defined since $(g^{\mathrm{t}} g)^{\mathrm{t}} = g^{\mathrm{t}} g$. We show that

f is a submersion. To this end check first that for every $g \in \mathsf{GL}(3,\mathbb{R})$ the tangent map of f at g is

$$T_g f : \mathfrak{gl}(3,\mathbb{R}) \to \mathfrak{Sym}(3,\mathbb{R}), \quad A \mapsto A^{\mathrm{t}} g + g^{\mathrm{t}} A \ .$$

For given $S \in \mathfrak{Sym}(3,\mathbb{R})$ put $A = \frac{1}{2}\left(g^{\mathrm{t}}\right)^{-1} S$ and compute

$$T_g f(A) = \frac{1}{2}\left(S^{\mathrm{t}} + S\right) = S \ .$$

Hence f is a submersion, and $\mathsf{O}(3) = f^{-1}(I_3)$ is a submanifold of $\mathsf{GL}(3,\mathbb{R})$ of dimension $\dim_{\mathbb{R}} \mathsf{GL}(3,\mathbb{R}) - \dim_{\mathbb{R}} \mathfrak{Sym}(3,\mathbb{R}) = 9 - 6 = 3$. Because the group multiplication and inverse on $\mathsf{GL}(3,\mathbb{R})$ are smooth, their restriction to $\mathsf{O}(3)$ is so, too, and $\mathsf{O}(3)$ is a Lie group. Since $(\det g)^2 = \det g \det g^{\mathrm{t}} = 1$ for all $g \in \mathsf{O}(3)$, the subgroup $\mathsf{SO}(3) = \mathsf{O}(3) \cap \det^{-1}(\mathbb{R}_{>0})$ is open in $\mathsf{O}(3)$, and $\mathsf{O}(3)$ is the disjoint union of $\mathsf{SO}(3)$ and $-\mathsf{SO}(3)$. Moreover, $\mathsf{SO}(3)$ becomes a Lie group.

The Lie algebra $\mathfrak{o}(3)$ of $\mathsf{O}(3)$ coincides with the Lie algebra $\mathfrak{so}(3)$ of $\mathsf{SO}(3)$ and can be determined via the submersion f, too. More precisely

$$\mathfrak{o}(3) = \mathfrak{so}(3) = \ker T_{\mathbb{1}} f = \left\{ A \in \mathfrak{gl}(3,\mathbb{R}) \;\middle|\; A^{\mathrm{t}} + A = 0 \right\} ,$$

and $\mathfrak{so}(3)$ is the space of all skew-symmetric real 3×3 matrices. Note that $\operatorname{tr} A = 0$ for every element $A \in \mathfrak{so}(3)$.

15.2.2 Theorem *The Lie algebras* $\left(\mathbb{R}^3, \times\right)$ *and* $\mathfrak{so}(3)$ *are isomorphic. An isomorphism is given by the map*

$$M : \mathbb{R}^3 \to \mathfrak{so}(3), \quad \vec{x} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \mapsto M_{\vec{x}} = \begin{pmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & -x^1 \\ -x^2 & x^1 & 0 \end{pmatrix}$$

Denoting by $\vec{e}_1, \vec{e}_2, \vec{e}_3$ *the standard basis of* $\mathbb{R}^3$*, the elements*

$$J_x = J_1 = M_{\vec{e}_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$J_y = J_2 = M_{\vec{e}_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$J_z = J_3 = M_{\vec{e}_3} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

form a basis of the Lie algebra $\mathfrak{so}(3)$*. These elements are sometimes called the* (standard) infinitesimal generators of rotations.

Proof. By definition M is linear. Moreover, the images $M_{\vec{e}_k}$, $k = 1, 2, 3$, are linearly independent, so by dimension reasons the map M is a linear isomorphism. It remains to show that M preserves the Lie brackets. To this end compute for $\vec{x}, \vec{y} \in \mathbb{R}^3$

$$\vec{x} \times \vec{y} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \times \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix} = \begin{pmatrix} x^2 y^3 - x^3 y^2 \\ x^3 y^1 - x^1 y^3 \\ x^1 y^2 - x^2 y^1 \end{pmatrix} ,$$

and then

$$M_{\vec{x}} \cdot M_{\vec{y}} = \begin{pmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & -x^1 \\ -x^2 & x^1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -y^3 & y^2 \\ y^3 & 0 & -y^1 \\ -y^2 & y^1 & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} -x^3\,y^3 - x^2\,y^2 & x^2\,y^1 & x^3\,y^1 \\ x^1\,y^2 & -x^3\,y^3 - x^1\,y^1 & x^3\,y^2 \\ x^1\,y^3 & x^2\,y^3 & -x^2\,y^2 - x^1\,y^1 \end{pmatrix} .$$

Forming the commutator gives

$$\big[M_{\vec{x}}, M_{\vec{y}}\big] = \begin{pmatrix} 0 & x^2\,y^1 - x^1\,y^2 & x^3\,y^1 - x^1\,y^3 \\ x^1\,y^2 - x^2\,y^1 & 0 & x^3\,y^2 - x^2\,y^3 \\ x^1\,y^3 - x^3\,y^1 & x^2\,y^3 - x^3\,y^2 & 0 \end{pmatrix} = M_{\vec{x}\times\vec{y}} .$$

Hence M preserves Lie brackets and the claim is proved. □

15.2.3 Now let us consider the *special unitary group*

$$\mathsf{SU}(2) = \big\{g \in \mathsf{GL}(2,\mathbb{C}) \;\big|\; \forall v, w \in \mathbb{C}^2 : \langle gv, gw\rangle = \langle v, w\rangle \;\&\; \det g = 1\big\} .$$

To verify that $\mathsf{SU}(2)$ is a Lie group let f be the map

$$f : \mathsf{GL}(2,\mathbb{C}) \to \mathfrak{Herm}(2), \quad g \mapsto g^* g ,$$

where $\mathfrak{Herm}(n)$ denotes the space of hermitian $n \times n$ matrices. The tangent map of f at $g \in \mathsf{GL}(2,\mathbb{C})$ is given by

$$T_g f : \mathfrak{gl}(2,\mathbb{C}) \to \mathfrak{Herm}(2), \quad A \mapsto A^* g + g^* A .$$

For given $H \in \mathfrak{Herm}(2)$ let $A = \frac{1}{2}(g^*)^{-1}H$. Then

$$T_g f(A) = \frac{1}{2}\,(H^* + H) = H ,$$

which entails that f is a submersion. Hence $\mathsf{U}(2) = f^{-1}(I_2)$ is a real Lie group of dimension $\dim_{\mathbb{R}} \mathsf{GL}(2,\mathbb{C}) - \dim_{\mathbb{R}} \mathfrak{Herm}(2) = 8 - 4 = 4$. Recall that $\mathsf{U}(2)$ is the *unitary group* in dimension 2. The Lie algebra of $\mathsf{U}(2)$ is given by $\mathfrak{u}(2) = \ker T_{\mathbb{1}} f$, the space of all skew-hermitian 2×2 matrices. The determinant function $\det : \mathsf{U}(2) \to \mathbb{S}^1$ is a smooth group homomorphism and a submersion. The latter is true because for all $A \in \mathfrak{u}(2)$

$$T_{\mathbb{1}} \det(A) = \frac{\partial}{\partial t}\Big|_{t=0} \det(\exp(tA)) = \frac{\partial}{\partial t}\Big|_{t=0} e^{t\,\mathrm{tr}\,A} = \mathrm{tr}\,A ,$$

and because the matrix $A = \mathsf{i}\mathbb{1} \in \mathfrak{gl}(2,\mathbb{C})$ is skew-hermitian and its trace $\mathrm{tr}\,A = 2\mathsf{i}$ spans $\mathfrak{lie}(\mathbb{S}^1) = \mathbb{R}\mathsf{i}$. Therefore, $\mathsf{SU}(2)$ is a real Lie group of dimension $\dim_{\mathbb{R}} \mathsf{U}(2) - \dim_{\mathbb{R}} \mathbb{R}\mathsf{i} = 3$ and with Lie algebra $\mathfrak{su}(2)$ given by the skew-hermitian matrices of trace 0.

15.2.4 Proposition *The Lie group* $\mathsf{SU}(2)$ *is homeomorphic to* $\mathbb{S}^3$*, so in particular compact and simply connected. A homeomorphism is given by*

$$\Psi : \mathbb{S}^3 \mapsto \mathsf{SU}(2),\ (x^0, x^1, x^2, x^3) \mapsto \begin{pmatrix} x^0 + x^1\mathsf{i} & x^2 + x^3\mathsf{i} \\ -x^2 + x^3\mathsf{i} & x^0 - x^1\mathsf{i} \end{pmatrix} .$$

Proof. One has for $(x^0, x^1, x^2, x^3) \in \mathbb{S}^3$

$$\begin{pmatrix} x^0 + x^1\mathsf{i} & x^2 + x^3\mathsf{i} \\ -x^2 + x^3\mathsf{i} & x^0 - x^1\mathsf{i} \end{pmatrix} \cdot \begin{pmatrix} x^0 - x^1\mathsf{i} & -x^2 - x^3\mathsf{i} \\ x^2 - x^3\mathsf{i} & x^0 + x^1\mathsf{i} \end{pmatrix} = \mathbb{1} \ ,$$

hence the matrix $\Psi(x^0, x^1, x^2, x^3)$ is unitary. So Ψ is well-defined. The map Ψ is obviously continuous and injective. It remains to show that Ψ is surjective, because then, by compactness of the 3-sphere, the map Ψ is a homeomorphism and $\mathsf{SU}(2)$ has to be compact. Let

$$g = \begin{pmatrix} z & u \\ v & w \end{pmatrix}$$

be a unitary matrix with determinant being 1 that is $zw - uv = 1$. By unitarity and the formula for the inverse of a 2×2 matrix one obtains the equality

$$\begin{pmatrix} w & -u \\ -v & z \end{pmatrix} = \begin{pmatrix} \overline{z} & \overline{v} \\ \overline{u} & \overline{w} \end{pmatrix} \ ,$$

hence $w = \overline{z}$ and $v = -\overline{u}$. Inserting this in the equation for the determinant entails that $|z|^2 + |u|^2 = 1$. Now write $z = x^0 + x^1\mathsf{i}$ and $u = x^2 + x^3\mathsf{i}$ with real x^0, x^1, x^2, x^3. Then $(x^0, x^1, x^2, x^3) \in \mathbb{S}^3$ and $g = \Psi(x^0, x^1, x^2, x^3)$, so Ψ is surjective and the proposition is proved. □

15.2.5 Proposition *Consider the space*

$$\mathfrak{Herm}^{\mathrm{tr}0}(2) = \mathsf{i}\,\mathfrak{su}(2) = \left\{ X \in \mathfrak{gl}(2, \mathbb{C}) \ \middle| \ X^* = X \ \& \ \operatorname{tr} X = 0 \right\}$$

of all traceless hermitian 2×2 *matrices. Then* $\mathfrak{Herm}^{\mathrm{tr}0}(2)$ *is a real vector space of dimension* 3 *with a basis given by the* Pauli matrices

$$\sigma_1 = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \sigma_2 = \left(\begin{array}{cc} 0 & -\mathsf{i} \\ \mathsf{i} & 0 \end{array} \right), \quad \sigma_3 = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$$

and with inner product

$$\langle \cdot, \cdot \rangle_{\mathfrak{h}^{\mathrm{tr}0}} : \mathfrak{Herm}^{\mathrm{tr}0}(2) \times \mathfrak{Herm}^{\mathrm{tr}0}(2) \to \mathbb{R}, \quad (X, Y) \mapsto -\frac{1}{2} \left(\det(X + Y) - \det X - \det Y \right)$$

and corresponding norm

$$\| \cdot \|_{\mathfrak{h}^{\mathrm{tr}0}} : \mathfrak{Herm}^{\mathrm{tr}0}(2) \to \mathbb{R}_{\geqslant 0}, \quad X \mapsto \sqrt{-\det(X)} \ .$$

An isometric isomorphism between $\left(\mathbb{R}^3, \langle \cdot, \cdot \rangle\right)$ *and* $\left(\mathfrak{Herm}^{\mathrm{tr}0}(2), \langle \cdot, \cdot \rangle_{\mathfrak{h}^{\mathrm{tr}0}}\right)$ *is given by*

$$\vec{\sigma} : \mathbb{R}^3 \to \mathfrak{Herm}^{\mathrm{tr}0}(2), \quad \vec{x} \mapsto \vec{\sigma} \cdot \vec{x} = \sum_{k=1}^{3} x^k \, \sigma_k \ .$$

Its inverse maps $X \in \mathfrak{Herm}^{\mathrm{tr}0}(2)$ *to the vector* $\vec{x}$ *with components* $x^k = \frac{1}{2} \operatorname{tr}(X \sigma_k)$, *where* $k = 1, 2, 3$.

Proof. Let $X = \begin{pmatrix} a & z \\ w & d \end{pmatrix} \in \mathfrak{Herm}^{\mathrm{tr}0}(2)$. Then $a, d \in \mathbb{R}$ and $w = \overline{z}$, since X is hermitian. The assumption $\operatorname{tr} X = 0$ implies $d = -a$. Hence X is of the form

$$\begin{pmatrix} a & b + c\,\mathsf{i} \\ b - c\,\mathsf{i} & -a \end{pmatrix} = a\sigma_3 + c\sigma_1 + b\sigma_2$$

with $a, c, d \in \mathbb{R}$, and any such matrix is an element of $\mathfrak{Herm}^{\mathrm{tr}0}(2)$. Since the Pauli matrices are obviously linearly independent, they therefore form a basis of $\mathfrak{Herm}^{\mathrm{tr}0}(2)$.

Next compute for $\vec{x} = (x^1, x^2, x^3) \in \mathbb{R}^3$

$$\det(\vec{\sigma} \cdot \vec{x}) = \det \begin{pmatrix} x^3 & x^2 + x^1\,\mathsf{i} \\ x^2 - x^1\,\mathsf{i} & -x^3 \end{pmatrix} = -(x^3)^2 - (x^1)^2 - (x^2)^2 = -\|\vec{x}\|^2 \ . \tag{15.2.1}$$

Hence the map $\mathfrak{Herm}^{\mathrm{tr}0}(2) \to \mathbb{R}_{\geqslant 0}$, $X \mapsto \sqrt{-\det(X)}$ is a norm on $\mathfrak{Herm}^{\mathrm{tr}0}(2)$ which has to fullfill the parallelogram identity since the euclidean norm $\|\cdot\|$ does.

The norm on $\mathfrak{Herm}^{\mathrm{tr}0}(2)$ is therefore induced by an inner product which can be recovered by the polarization identity (12.1.9) that is by

$$\langle X, Y \rangle_{\mathfrak{h}^{\mathrm{tr}0}} = -\frac{1}{2}\left(\det(X+Y) - \det X - \det Y\right) \quad \text{for all } X, Y \in \mathfrak{Herm}^{\mathrm{tr}0}(2) \ .$$

Moreover, $\vec{\sigma}$ preserves norms by (15.2.1), hence is an isometry.

For the remaining part of the claim check first that $(\sigma_k)^2 = \mathbb{1}$ for $k = 1, 2, 3$ and that

$$\sigma_1\,\sigma_2 = \begin{pmatrix} \mathsf{i} & 0 \\ 0 & -\mathsf{i} \end{pmatrix} = \mathsf{i}\sigma_3,\ \sigma_2\,\sigma_3 = \begin{pmatrix} 0 & \mathsf{i} \\ \mathsf{i} & 0 \end{pmatrix} = \mathsf{i}\sigma_1,\ \sigma_3\,\sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathsf{i}\sigma_2 \ . \tag{15.2.2}$$

Since the Pauli matrices are hermitian, forming the hermitian conjugate on both sides of these equations entails

$$\sigma_2\,\sigma_1 = -\mathsf{i}\sigma_3, \quad \sigma_3\,\sigma_2 = -\mathsf{i}\sigma_1, \quad \sigma_1\,\sigma_3 = -\mathsf{i}\sigma_2 \ . \tag{15.2.3}$$

Now compute for $\vec{x} \in \mathbb{R}^3$ and $k = 1, 2, 3$

$$\frac{1}{2}\operatorname{tr}\left((\vec{x} \cdot \vec{\sigma})\,\sigma_k\right) = \frac{1}{2}\operatorname{tr}\left(x_k\,(\sigma_k)^2\right) = x_k \ .$$

The proposition is proved. □

15.2.6 Lemma *For $i, j \in \{1, 2, 3\}$ the Pauli matrices satisfy the following commutation relations:*

$$[\sigma_i, \sigma_j] = 2\mathsf{i} \sum_{k=1}^{3} \varepsilon_{ijk}\,\sigma_k \ ,$$

where for $i, j, k \in \{1, 2, 3\}$ the Levi-Civita symbol ε_{ijk} is defined by

$$\varepsilon_{ijk} = \begin{cases} 1 & \textit{if } (i,j,k) \textit{ is an even permutation of } (1,2,3), \\ -1 & \textit{if } (i,j,k) \textit{ is an odd permutation of } (1,2,3), \textit{ and} \\ 0 & \textit{else.} \end{cases}$$

15.2.7 Remark Recall that a permutation of $(1,2,3)$ is even if and only if it is cyclic.

Proof. The commutation relations follow immediately from equations (15.2.2) and (15.2.3) in the proof of the preceding proposition. $\square$

15.2.8 Theorem *The matrices*

$$\tau_1 = \frac{1}{\mathsf{i}}\sigma_1 = \begin{pmatrix} 0 & -\mathsf{i} \\ -\mathsf{i} & 0 \end{pmatrix}, \quad \tau_2 = \frac{1}{\mathsf{i}}\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau_3 = \frac{1}{\mathsf{i}}\sigma_3 = \begin{pmatrix} -\mathsf{i} & 0 \\ 0 & \mathsf{i} \end{pmatrix}$$

form a basis of the Lie algebra $\mathfrak{su}(2)$ and obey the commutation relations

$$[\tau_i, \tau_j] = 2\tau_k \quad \textit{for every cyclic permutation } (i,j,k) \textit{ of } (1,2,3) \ . \tag{15.2.4}$$

Moreover, the linear map $\Phi : \mathfrak{su}(2) \to \mathbb{R}^3$ uniquely defined by $\tau_k \mapsto 2e_k$ for $k = 1,2,3$ is an isomorphism of Lie algebras, where $\mathbb{R}^3$ carries the Lie algebra structure given by the cross product $\times$. In particular, the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ are isomorphic with an isomorphism given by the composition

$$M \circ \Phi : \mathfrak{su}(2) \to \mathfrak{so}(3), \quad \sum_{k=1}^{3} x^k \tau_k = \begin{pmatrix} -x^3\mathsf{i} & -x^2 - x^1\mathsf{i} \\ x^2 - x^1\mathsf{i} & x^3\mathsf{i} \end{pmatrix} \mapsto 2\begin{pmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & -x^1 \\ -x^2 & x^1 & 0 \end{pmatrix},$$

where $M : \mathbb{R}^3 \to \mathfrak{so}(3)$ is the isomorphism from Theorem 15.2.2.

Proof. Since multiplication by $-\mathsf{i}$ is a real linear isomorphism from $\mathfrak{Herm}^{\mathrm{tr}0}(2)$ to $\mathfrak{su}(2)$ and since the Pauli matrices form a basis of $\mathfrak{Herm}^{\mathrm{tr}0}(2)$, the matrices τ_k, $k = 1,2,3$, form a basis of $\mathfrak{su}(2)$. The commutation relations (15.2.4) are an immediate consequence of the preceding lemma. The Lie bracket is preserved by Φ since

$$(2e_i) \times (2e_j) = 2(2e_k) \quad \text{for every cyclic permutation } (i,j,k) \text{ of } (1,2,3) \ .$$

The rest of the claim now follows by definition of Φ and Theorem 15.2.2. $\square$

15.2.9 Theorem *For every $g \in \mathsf{SU}(2)$ the linear map*

$$\pi_g : \mathbb{R}^3 \to \mathbb{R}^3, \quad \vec{x} \mapsto \vec{\sigma}^{-1}\big(g(\vec{\sigma}\cdot\vec{x})g^*\big)$$

is an orthogonal transformation. Moreover, the map

$$\pi : \mathsf{SU}(2) \to \mathsf{SO}(3), \quad g \mapsto \pi_g$$

is a differentiable surjective group homomorphism with kernel $\{\pm I_2\} \cong \mathbb{Z}/2$. In particular, π is the universal covering map of $\mathsf{SO}(3)$. Finally, the tangent map $T_{\mathbb{1}}\pi : \mathfrak{su}(2) \to \mathfrak{so}(3)$ coincides with the isomorphism $M \circ \Phi$ from Theorem 15.2.8.

Proof. Observe that $\det(gAg^*) = \det A$ and $\mathrm{tr}(gAg^*) = \mathrm{tr}(A)$ for all $g \in \mathsf{SU}(2)$ and $A \in \mathfrak{Herm}^{\mathrm{tr}0}(2)$. This together with the fact that $\vec{\sigma}$ is an isometric isomorphism from $(\mathbb{R}^3, \|\cdot\|)$ to $(\mathfrak{Herm}^{\mathrm{tr}0}(2), \sqrt{-\det(\,\cdot\,)})$ entails that the transformations π_g are orthogonal. $\square$

15.3. The Lorentz group $\mathsf{SO}(1,3)$ and its universal cover $\mathsf{SL}(2,\mathbb{C})$

to do: change signature from $(-,+,+,+)$ back to $(+,-,-,-)$.

15.3.1 Recall from Examples 15.1.3 (a) that the *Minowski inner product* of two elements $x = (x^0, x^1, x^2, x^3) \in \mathbb{R}^4$ and $y = (y^0, y^1, y^2, y^3) \in \mathbb{R}^4$ is defined by $\langle x, y\rangle_{\mathrm{M}} = x^0y^0 - \sum_{k=1}^3 x^k y^k$, and that $\mathbb{R}^4$ endowed with the Minkowski inner product is denoted $\mathbb{R}^{1,3}$. The signature of the Minowski inner product therefore is $(+,-,-,-)$ or in other terms $(1,3)$. As usual we call $\mathbb{R}^{1,3}$ *Minkowski space* of (*space-time*) *dimension* 4.

Recall from Examples 15.1.8 (b) that the *pseudo-orthogonal group* $\mathsf{O}(1,3)$ consists of all $g \in \mathsf{GL}(4,\mathbb{R})$ such that

$$\langle gx, gy\rangle_{\mathrm{M}} = \langle x, y\rangle_{\mathrm{M}} \quad \text{for all } x, y \in \mathbb{R}^4 \ .$$

Following common language in mathematical physics we call $\mathsf{O}(1,3)$ the *Lorentz group* in *space-time dimension* 4. The subgroup

$$\mathsf{SO}(1,3) = \{g \in \mathsf{O}(1,3) \mid \det g = 1\} \subset \mathsf{O}(1,3)$$

is called the *proper Lorentz group*. Let us show that the Lorentz groups $\mathsf{O}(1,3)$ and $\mathsf{SO}(1,3)$ are Lie groups. To this end put

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and observe that $\langle x, y\rangle_{\mathrm{M}} = \langle x, \eta y\rangle$ for all $x, y \in \mathbb{R}^4$, where $\langle -, -\rangle$ denotes the euclidean inner product. Hence a matrix $\Lambda \in \mathsf{GL}(4,\mathbb{R})$ lies in $\mathsf{O}(1,3)$ if and only if

$$\Lambda^{\mathrm{t}}\eta\Lambda - \eta = 0 \ . \tag{15.3.1}$$

Following standard language in mathematical physics we call every such Λ a *Lorentz transformation*. The map $f : \mathsf{GL}(4,\mathbb{R}) \to \mathfrak{Sym}(4,\mathbb{R})$, $\Lambda \mapsto \Lambda^{\mathrm{t}}\eta\Lambda - \eta$ is smooth and has derivative

$$T_\Lambda f : \mathfrak{Mat}(4,\mathbb{R}) \to \mathfrak{Sym}(4,\mathbb{R}),\ A \mapsto A^{\mathrm{t}}\eta\Lambda + \Lambda^{\mathrm{t}}\eta A$$

at $\Lambda \in \mathsf{GL}(4,\mathbb{R})$. The derivative at Λ is surjective since $T_\Lambda f\left(\frac{1}{2}\eta(\Lambda^{\mathrm{t}})^{-1}B\right) = B$ for all $B \in \mathfrak{Sym}(4,\mathbb{R})$. Hence f is a submersion and the preimage $\mathsf{O}(1,3) = f^{-1}(0)$ a Lie subgroup of $\mathsf{GL}(4,\mathbb{R})$. The Lie algebra $\mathfrak{o}(1,3)$ of the Lorentz group then consists of the kernel of $T_{\mathbb{1}} f$ that is of a all matrices $A \in \mathfrak{Mat}(4,\mathbb{R})$ such that

$$A^{\mathrm{t}}\eta + \eta A = 0 \ . \tag{15.3.2}$$

Since $\dim \mathfrak{o}(1,3) = \dim \mathfrak{Mat}(4,\mathbb{R}) - \dim \mathfrak{Sym}(4,\mathbb{R}) = 16 - 10 = 6$, one concludes that the Lorentz group $\mathsf{O}(1,3)$ is a Lie group of (real) dimension 6. By (15.3.1), the determinant of a Lorentz transformation $\Lambda \in \mathsf{O}(1,3)$ fulfills $|\det(\Lambda)| = 1$. Moreover, time reversal

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and parity inversion

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

are both Lorentz transformations with determinant -1. One concludes that $\mathsf{SO}(1,3)$ is a Lie subgroup of the Lorentz group $\mathsf{O}(1,3)$ and that the latter is the disjoint union of $\mathsf{SO}(1,3)$ and $\mathsf{SO}(1,3) \cdot T = \mathsf{SO}(1,3) \cdot P$.

15.3.2 The *special linear group* $\mathsf{SL}(2,\mathbb{C})$ consists of all $g \in \mathsf{GL}(2,\mathbb{C})$ such that $\det g = 1$. It is a complex Lie group by the following argument. Observe that the determinant $\det : \mathsf{GL}(2,\mathbb{C}) \to \mathbb{C}$ is a complex differentiable group homomorphism. Its (complex) tangent map at the identity $\mathbb{1}$ is given by

$$T_{\mathbb{1}} \det : \mathfrak{gl}(2,\mathbb{C}) \to \mathbb{C}, \quad A \mapsto \left.\frac{\partial}{\partial z}\right|_{z=0} \det\exp(zA) = \left.\frac{\partial}{\partial z}\right|_{z=0} e^{z\,\mathrm{tr}(A)} = \mathrm{tr}\, A \ .$$

This entails that $T_{\mathbb{1}} \det(z\mathbb{1}) = 2z$ for each $z \in \mathbb{C}$ hence det is a holomorphic submersion and $\mathsf{SL}(2,\mathbb{C}) = \det^{-1}(1)$ a complex Lie group.

15.3.3 Proposition *The Lie group $\mathsf{SL}(2,\mathbb{C})$ is simply-connected.*

Proof. We first show that $\mathsf{SL}(2)$ is path-connected. So let $g \in \mathsf{SL}(2\mathbb{C})$. Then transform g into Jordan normal form that is choose $S \in \mathsf{GL}(2,\mathbb{C})$ such that

$$SgS^{-1} = \begin{pmatrix} a_1 & e \\ 0 & a_2 \end{pmatrix} ,$$

where $a_1, a_2 \in \mathbb{C}$ with $a_1 a_2 = 1$ and $e \in \{0,1\}$. Then choose a path $\gamma_1 : [0,1] \to \mathbb{C}^\times = \mathbb{C}\backslash\{0\}$ such that $\gamma_1(0) = 1$ and $\gamma_1(1) = a_1$. Let $\gamma_2 : [0,1] \to \mathbb{C}^\times$ be the path which maps t to $\gamma_2(t) = \big(\gamma_1(t)\big)^{-1}$. Now put

$$h(t) = S^{-1} \begin{pmatrix} \gamma_1(t) & te \\ 0 & \gamma_2(t)) \end{pmatrix} .$$

Then $h : [0,1] \to \mathsf{SL}(2,\mathbb{C})$ is a continuous path connecting $h(0) = I_2$ with $h(1) = g$. So $\mathsf{SL}(2,\mathbb{C})$ is path-connected.

Next we prove that $\mathsf{SL}(2,\mathbb{C})$ is simply-connected. To this end recall that the subgroup $\mathsf{SU}(2) \subset \mathsf{SL}(2,\mathbb{C})$ is simply-connected. So to verify that $\pi_1\big(\mathsf{SL}(2,\mathbb{C})\big)$ is trivial it suffices to construct a (strong) deformation retraction from $\mathsf{SL}(2,\mathbb{C})$ onto $\mathsf{SU}(2)$ which means that we have to construct a continuous map $r : \mathsf{SL}(2,\mathbb{C}) \times [0,1] \to \mathsf{SL}(2,\mathbb{C})$ such that

$$r_0 = \mathrm{id}, \quad r_1\big(\mathsf{SL}(2,\mathbb{C})\big) \subset \mathsf{SU}(2), \quad \text{and} \quad r_t|_{\mathsf{SU}(2)} = \mathrm{id}_{\mathsf{SU}(2)} \text{ for all } t \in [0,1] \ .$$

Here, as usual, r_t stands for the map $\mathsf{SL}(2,\mathbb{C}) \to \mathsf{SL}(2,\mathbb{C})$, $g \mapsto r(g,t)$.

Let us agree on the following notation. For every matrix $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathfrak{Mat}(2 \times 2, \mathbb{C})$ we denote by a_i with $i = 1,2$ the column vector $\begin{pmatrix} a_{1i} \\ a_{2i} \end{pmatrix}$ and write $a = (a_1, a_2)$. Vice versa, if

$a_i = \begin{pmatrix} a_{1i} \\ a_{2i} \end{pmatrix} \in \mathbb{C}^2$ with $i = 1, 2$ are two (column) vectors then we denote by (a_1, a_2) the matrix $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. If now $g \in \mathsf{SL}(2, \mathbb{C})$ with column vectors g_1, g_2, then we know that g_1 and g_2 form a basis of $\mathbb{C}^2$. Gram–Schmidt orthonormalization will transform the basis (g_1, g_2) into an orthonormal basis (u_1, u_2):

$$(g_1, g_2) \mapsto (u_1, u_2) = \left(\frac{g_1}{\|g_1\|}, \frac{g_2 - \langle g_2, u_1 \rangle u_1}{\|g_2 - \langle g_2, u_1 \rangle u_1\|} \right) .$$

Therefore, Gram–Schmidt orthonormalization can be understood as a retraction from $\mathsf{SL}(2, \mathbb{C})$ to $\mathsf{SU}(2)$ leaving $\mathsf{SU}(2)$ invariant. So we are almost done, we just need to make the Gram–Schmidt process "continuous" in the sense that it can be deformed to the identity.

To achieve this define the following matrices depending on the parameter $t \in [0, 1]$:

$$p_t(g) = \begin{pmatrix} \frac{1}{\|g_1\|^t} & 0 \\ 0 & 1 \end{pmatrix} , \quad q_t(g) = \begin{pmatrix} 1 & -t\langle g_2, u_1 \rangle \\ 0 & 1 \end{pmatrix} , \quad \widetilde{p}_t(g) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\|g_3\|^t} \end{pmatrix} ,$$

where $u_1 = \frac{g_1}{\|g_1\|}$ and $g_3 = g_2 - \langle g_2, u_1 \rangle u_1$. Each of these matrices lies in $\mathsf{GL}(2, \mathbb{C})$ since their determinant is non-zero. Now we define $r : \mathsf{SL}(2, \mathbb{C}) \times [0, 1] \to \mathsf{SL}(2, \mathbb{C})$ by

$$r(g, t) = g \cdot p_t(g) \cdot q_t(g) \cdot \widetilde{p}_t(g) , \quad \text{where } g \in \mathsf{SL}(2, \mathbb{C}),\ t \in [0, 1] .$$

Then $r_0(g) = g$, $r_1(g) = (u_1, u_2) \in \mathsf{SU}(2)$ (since (u_1, u_2) is an orthonormal basis of $\mathbb{C}^2$), $r(g, t) = g$ if $g \in \mathsf{SU}(2)$, and $r(g, t) \in \mathsf{SL}(2, \mathbb{C})$ for all $g \in \mathsf{SL}(2, \mathbb{C})$, $t \in [0, 1]$. The last property is the only not obvious one and needs to be verified because it guarantees that r is well-defined. The other properties are immediate and just tell that r is a strong deformation retraction of the kind we have been looking for.

We check two identies from which the remaining claim will follow immediately. For every $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in \mathsf{SL}(2, \mathbb{C})$ compute

$$\begin{aligned}
1 &= \det g \cdot \overline{\det g} = (g_{11}\, g_{22} - g_{21}\, g_{12}) \cdot (\overline{g_{11}\, g_{22} - g_{21}\, g_{12}}) = \\
&= \left(|g_{11}|^2 |g_{22}|^2 + |g_{21}|^2 |g_{12}|^2\right) - 2\mathfrak{Re}(g_{11}\, g_{22}\, g_{21}\, g_{12}) = \\
&= \left(|g_{11}|^2 |g_{12}|^2 + |g_{11}|^2 |g_{22}|^2 + |g_{21}|^2 |g_{12}|^2 + |g_{21}|^2 |g_{22}|^2\right) - \\
&\quad - \left(|g_{11}|^2 |g_{12}|^2 + |g_{21}|^2 |g_{22}|^2 + 2\mathfrak{Re}(g_{11}\, g_{22}\, g_{21}\, g_{12})\right) = \|g_1\|^2 \|g_2\|^2 - |\langle g_1, g_2 \rangle|^2 .
\end{aligned}$$

Then

$$\begin{aligned}
\det r(g, t) &= \det g \ \det p_t(g) \ \det q_t(g) \ \det \widetilde{p}_t(g) = \left(\frac{1}{\|g_1\| \, \|g_3\|} \right)^t = \\
&= \left(\frac{1}{\|g_1\|} \cdot \frac{\|g_1\|}{\sqrt{\|g_1\|^2 \|g_2\|^2 - |\langle g_1, g_2 \rangle|^2}} \right)^t = \left(\frac{1}{\sqrt{\|g_1\|^2 \|g_2\|^2 - |\langle g_1, g_2 \rangle|^2}} \right)^t = 1 ,
\end{aligned}$$

which means that $r(g, t)$ is in fact an element of $\mathsf{SL}(2, \mathbb{C})$ for all $g \in \mathsf{SL}(2, \mathbb{C})$ and $t \in [0, 1]$. This finishes the proof. □

15.3.4 Proposition *Consider the space*

$$\mathfrak{Herm}(2) = \left\{X \in \mathfrak{gl}(2,\mathbb{C}) \mid X^* = X\right\}$$

of all hermitian 2×2 matrices. Then $\mathfrak{Herm}(2)$ is a real vector space of dimension 4 with a basis given by the identity matrix plus the Pauli matrices that is by

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -\mathsf{i} \\ \mathsf{i} & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The bilinear form

$$\langle \cdot, \cdot \rangle_{\mathfrak{h}} : \mathfrak{Herm}(2) \times \mathfrak{Herm}(2) \to \mathbb{R}, \quad (X,Y) \mapsto \frac{1}{2}\left(\det(X+Y) - \det X - \det Y\right)$$

is symmetric, non-degenerate, and has signature $(1,3)$. An isometric isomorphism between $\left(\mathbb{R}^{1,3}, \langle \cdot, \cdot \rangle_{\mathrm{M}}\right)$ and $(\mathfrak{Herm}(2), \langle \cdot, \cdot \rangle_{\mathfrak{h}})$ is given by

$$\sigma : \mathbb{R}^{1,3} \to \mathfrak{Herm}(2), \quad x \mapsto \sigma \cdot x = \sum_{k=0}^{3} x^k \sigma_k \ .$$

Its inverse maps $X \in \mathfrak{Herm}(2)$ to the vector x with components $x^k = \frac{1}{2}\operatorname{tr}(X\sigma_k)$, where $k = 0, 1, 2, 3$.

15.3.5 Theorem *For every $g \in \mathsf{SL}(2,\mathbb{C})$ the linear map*

$$\pi_g : \mathbb{R}^4 \to \mathbb{R}^4, \quad x \mapsto \sigma^{-1}\big(g(\sigma \cdot x)g^*\big)$$

is a proper orthochronous Lorentz transformation. Moreover, the map

$$\pi : \mathsf{SL}(2,\mathbb{C}) \to \mathsf{SO}^{\uparrow}(1,3), \quad g \mapsto \pi_g$$

is a differentiable surjective group homomorphism with kernel $\{\pm I_2\} \cong \mathbb{Z}/2$. In particular, the tangent map $T_{\mathbb{1}}\pi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{so}(1,3)$ is an isomorphism and π is the universal covering map of $\mathsf{SO}^{\uparrow}(1,3)$.

Proof. Every element $g \in \mathsf{SL}(2,\mathbb{C})$ induces a linear isomorphism

$$\alpha_g : \mathfrak{Herm}(2) \to \mathfrak{Herm}(2), \quad X \mapsto gXg^*$$

which is isometric since $\det(\alpha_g X) = \det X$ for all $X \in \mathfrak{Herm}(2)$. By Proposition 15.3.4, $\sigma : \mathbb{R}^{1,3} \to \mathfrak{Herm}(2)$ is an isometric isomorphism, hence $\pi_g = \sigma \circ \alpha_g \circ \sigma^{-1}$ leaves the Minkowski metric invariant and therefore is a Lorentz transformation. $\square$

16. Fiber bundles

16.1. Fiber bundles

Fibered manifolds and fibered charts

16.1.1 Definition By a *locpro-fibered manifold* we understand a smooth surjective submersion $\pi : E \to M$ from a locpro-manifold E onto a manifold M. If E is a finite dimensional manifold, one calls a surjective submersion $\pi : E \to M$ just a *fibered manifold.* One usually calls π the *projection*, E the *total space* and M the *base* of the (pro-)fibered manifold. A (locpro-) fibered manifold is often denoted as a triple (E, π, M).

A *morphism of (locpro-) fibered manifolds* $\pi_1 : E_1 \to M_1$ and $\pi_2 : E_2 \to M_2$ consists of a pair (φ, f) of smooth maps $\varphi : E_1 \to E_2$ and $f : M_1 \to M_2$ such that the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\pi_1} & M_1 \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ E_2 & \xrightarrow[\pi_2]{} & M_2 \end{array}$$

commutes. One sometimes also says in this situation that φ is a *morphism of (locpro-) fibered manifolds over the map* $f : M_1 \to M_2$. We in particular make use of this language when the base manifolds M_1 and M_2 coincide and f is the identity map. We then just say that φ is a *morphism of (locpro-) fibered manifolds.*

16.1.2 Obviously, the identity map id_E on the total space of a (locpro-) fibered manifold (E, π, M) is a morphism. Moreover, the composition

$$(\varphi_2, f_2) \circ (\varphi_1, f_1) := (\varphi_2 \circ \varphi_1, f_2 \circ f_1)$$

of morphisms

$$(\varphi_1, f_1) : (E_1, \pi_1, M_1) \to (E_2, \pi_2, M_2) \quad \text{and} \quad (\varphi_2, f_2) : (E_2, \pi_2, M_2) \to (E_3, \pi_3, M_3)$$

is a morphism of (pro-)fibered manifolds from (E_1, π_1, M_1) to (E_2, π_2, M_2), and id_E acts as identity morphism. One concludes that (locpro-)fibered manifolds and their morphisms form a category.

16.1.3 Definition Let (E, π, M) be a fibered manifold. By a *fibered chart* of (E, π, M) or a *chart adapted to* $\pi : E \to M$ one understands a chart (U, ψ) :.

16.1.4 Proposition *Given a locpro-fibered manifold (E, π, M), the fiber $F_p := \pi^{-1}(p)$ over an element $p \in M$ is a locpro-manifold.*

Proof. In the case where E is finite dimensional the claim is an immediate consequence of the submersion theorem So assume that E is infinite dimensional. Since the claim is local, we can assume that there exists a smooth projective representation $(E_i, \eta_{ij}, \eta_i)_{i,j\in\mathbb{N},\, i\leqslant j}$ of E. Since M is finite dimensional and using again that the claim is local we can assume that the the smooth map $\pi : E \to M$ factors in a neighborhood of F_p through some smooth map $\pi_i : E_i \to M$ that means that $\pi = \pi_i \circ \eta_i$. Since π is a smooth surjective submersion, π_i is so, too. Therefore, $F_i := \pi_i^{-1}(p)$ is a submanifold of E_i by the submersion theorem. Now put $\pi_j := \pi_i \circ \eta_{ij}$ for all $j > i$. As a composition of surjective submersions each such π_j is a surjective submersion as well. Hence for every $j > i$ the preimage $F_j := \pi_j^{-1}(p) = \eta_{ij}^{-1}(F_i)$ is a submanifold of E_j. Since $\pi = \pi_i \circ \eta_i = \pi_i \circ \eta_{ij} \circ \eta_j = \pi_j \circ \eta_j$, the fiber F_p coincides with $\eta_j^{-1}(F_j)$ for each $j \geqslant i$. Hence we obtain a smooth projective representation $(F_j, \varphi_{jk}, \varphi_j)_{j,k\in\mathbb{N}\, j\leqslant k}$ of F_p, when defining φ_{jk} as the restriction $\eta_{jk}|_{F_k}$ and φ_j as the restriction $\eta_j|_{F_p}$. So F_p is a pro-manifold and the claim is proved. □

16.1.5 Proposition *A locpro-fibered manifold has local smooth sections that is for every locpro-fibered manifold (E, π, M) and every point $p \in M$ there exists as smooth map $s : U \to E$ defined on an open neighborhood U of p in M such that $\pi \circ s = \mathrm{id}_U$.*

17. Jets

17.1. A combinatorial interlude

Multiindices

17.1.1 Assume that $\mathcal{I}$ is a non-empty set which we call *index set*. By a *multiindex* over $\mathcal{I}$ we then understand an element $\alpha \in \mathbb{N}^{(\mathcal{I})}$ that is a family $\alpha = (\alpha_i)_{i\in\mathcal{I}}$ of natural numbers such that only finitely many α_i are non-zero. The *order* of such a multiindex is defined by $|\alpha| := \sum_{i\in\mathcal{I}} \alpha_i$. For $k \in \mathbb{N}$ and $k_1 \in \mathbb{N}$ and $k_2 \in \mathbb{N} \cup \{\infty\}$ with $k_1 \leqslant k_2$ we denote by $\mathbb{N}_k^{(\mathcal{I})}$ the set of all multiindices over $\mathcal{I}$ of order k and by $\mathbb{N}_{k_1,k_2}^{(\mathcal{I})}$ the set of all multiindices of order less or equal k_2 which have order greater or equal k_1. For reasons of clarity, which will be become obvious below, we sometimes also refer to an element of $\mathbb{N}^{(\mathcal{I})}$ as a *Greek multiindex*.

17.1.2 Example In most cases the index set $\mathcal{I}$ will be of the form $\mathcal{I} = \{1, \ldots, d\}$ or of the form $\mathcal{I} = \{0, \ldots, d-1\}$ for some positive integer d. One then has $\mathbb{N}^{(\mathcal{I})} = \mathbb{N}^{\mathcal{I}} = \mathbb{N}^d$ and multiindices are given by d-tuples of the form $\alpha = (\alpha_1, \ldots, \alpha_d)$ or $\beta = (\beta_0, \ldots, \beta_{d-1})$, respectively.

17.1.3 The space of multiindices $\mathbb{N}^{(\mathcal{I})}$ carries the structure of a module over the semiring $\mathbb{N}$ in the sense of Johnson & Manes (1970) that is $\mathbb{N}^{(\mathcal{I})}$ together with componentwise addition is an abelian monoid, componentwise multiplication with scalars is associative, 0 acts as zero map, 1 acts as identity, and the distributivity laws hold true. Moreover, $\mathbb{N}^{(\mathcal{I})}$ is free over the family of multiindices $(1_i)_{i\in\mathcal{I}}$ defined by

$$1_i(j) := \begin{cases} 1 & \text{for } j = i, \\ 0 & \text{else.} \end{cases}$$

17.1.4 By a *Roman multiindex* of a given *order* $k \in \mathbb{N}_{>0}$ over some index set $\mathcal{I}$ we understand an element I of the cartesian product $\mathcal{I}^k$. For $k = 0$ we define $\mathcal{I}^0$ as the set $\{O\}$, where O is a fixed set not appearing as an element of $\mathcal{I}$. We call O the *Roman multiindex of order* 0 over $\mathcal{I}$. We sometimes write $|I|$ for the order of a Roman multiindex. Note that we denote elements of $\mathcal{I}^k$ by capital Roman letters $I, J, \ldots$ and their components by their respective small Roman letters i_l, j_l, and so on.

For $k \geqslant 1$ the symmetric group S_k acts in a canoncial way on $\mathcal{I}^k$. We denote the orbit space of this action by $\overline{\mathcal{I}^k}$ and the orbit through a Roman multiindex $I \in \mathcal{I}^k$ by $\overline{I}$. In other words $\overline{I}$ is the equivalence class of all Roman multiindices obtained from I by permutation of its components. For $k = 0$ we identify $\overline{\mathcal{I}^0}$ with $\mathcal{I}^0$ and $\overline{O}$ with O.

For Roman multiindices $I = (i_1, \ldots, i_k) \in \mathcal{I}^k$ and $J = (j_1, \ldots, j_l) \in \mathcal{I}^l$ of positive order we denote by $I + J$ the multiindex $(i_1, \ldots, i_k, j_1, \ldots, j_l) \in \mathcal{I}^{k+l}$. Obviously, the equivalence class $\overline{I+J}$ depends only on the equivalence classes $\overline{I}$ and $\overline{J}$, hence the operation $+$ descends to a map

$$+ : \mathcal{I}^k/S_k \times \mathcal{I}^l/S_l \to \mathcal{I}^{k+l}/S_{k+l} \ .$$

It is straightforward to see that this operation is associative and commutative. Next define $I + O = O + I = I$ and $\overline{I} + \overline{O} = \overline{O} + \overline{I} = \overline{I}$ for all Roman multiindices I and set $\overline{\mathcal{I}^\bullet} := \mathcal{I}^\bullet/\sim$, where $\mathcal{I}^\bullet := \bigsqcup_{k\in\mathbb{N}} \mathcal{I}^k$ and $\sim$ is the equivalence relation which defines two Roman multiindices $I \in \mathcal{I}^k$ and $J \in \mathcal{I}^l$ as equivalent if $k = l$ and $\overline{I} = \overline{J}$. Then $\overline{\mathcal{I}^\bullet} = \bigsqcup_{k\in\mathbb{N}} \overline{\mathcal{I}^k}$. Moreover, $\overline{\mathcal{I}^\bullet}$ together with $+$ as binary operation and O as zero element becomes an abelian monoid.

Note that every $i \in \mathcal{I}$ can be regarded as a Roman multiindex of order 1 and that $\overline{i} = i$, so we have the sums $I + i = (i_1, \ldots, i_k, i)$ and $\overline{I} + i = \overline{(i_1, \ldots, i_k, i)}$. Sometimes we also write I, i respectively $\overline{I}, i$ for these sums.

17.1.5 Lemma *Assume that $\mathcal{I}$ is totally ordered by some order relation $\leqslant$. Then every element of $\overline{\mathcal{I}^k}$ of order $k \in \mathbb{N}_{>0}$ has a unique representative $I = (i_1, \ldots, i_k) \in \mathcal{I}^k$ such that $i_1 \leqslant i_2 \leqslant \ldots \leqslant i_k$. We call such a representative an increasing representative or of increasing form.*

Proof. One proves the claim by induction on the order k. For $k = 1$ the claim is obvious. Assume that it holds for some k and let $\overline{J}$ be a Roman multiindex of order $k + 1$. Let j_m be the maximum of the components $j_1, \ldots, j_{k+1}$, and let $\sigma \in S_{k+1}$ be the permutation switching m and $k + 1$ and acting by identity on the rest. By hypothesis there exists a permutation $\tau \in S_k$ such that $j_{\sigma\tau(1)} \leqslant \ldots \leqslant j_{\sigma\tau(k)}$. Put $\tau(k + 1) = k + 1$. Then $\tau \in S_{k+1}$ and $I = (j_{\sigma\tau(1)}, \ldots, j_{\sigma\tau(k+1)})$ is a representative of $\overline{J}$ with the desired properties. This finishes the inductive step and the claim is proved. ☐

17.1.6 Proposition *Let $\mathcal{I}$ be an index set with a total order $\leqslant$ on it and $\kappa : \mathbb{N}^{(\mathcal{I})} \to \mathcal{I}^\bullet$ the map which maps the zero map $0_\mathcal{I} : \mathcal{I} \to \mathbb{N}$ to O and a Greek multiindex α of positive order to the Roman multiindex*

$$\big(\underbrace{i_1, \ldots, i_1}_{\alpha_{i_1} \text{ times}}, \underbrace{i_2, \ldots, i_2}_{\alpha_{i_2} \text{ times}}, \ldots, \underbrace{i_l, \ldots, i_l}_{\alpha_{i_l} \text{ times}} \big) \ ,$$

where $i_1 < \ldots < i_l$ are the (pairwise distinct and ordered) elements $i \in \mathcal{I}$ with non-vanishing component α_i. Then the induced map $\overline{\kappa} : \mathbb{N}^{(\mathcal{I})} \to \overline{\mathcal{I}^\bullet}$, $\alpha \mapsto \overline{\kappa(\alpha)}$ is an isomorphism of monoids and maps the space $\mathbb{N}^{(\mathcal{I})}_k$ of Greek multiindices of order k onto $\overline{\mathcal{I}^k}$. Moreover, if $\mathcal{I}$ is finite, then $\mathbb{N}^{(\mathcal{I})}_k$ and $\overline{\mathcal{I}^k}$ are finite as well and both have cardinality given by

$$|\mathbb{N}^{\mathcal{I}}_k| = |\overline{\mathcal{I}^k}| = \frac{1}{k!} \prod_{l=0}^{k-1} \big(|\mathcal{I}| + l \big) \ .$$

Proof. First we need to show that $\overline{\kappa}$ is a bijection. To this end let us make our notation somewhat more precise and choose for each $\alpha \in \mathbb{N}^{(\mathcal{I})} \backslash \{0_\mathcal{I}\}$ the elements $i^\alpha_1, \ldots, i^\alpha_{l_\alpha} \in \mathcal{I}$ so that $i^\alpha_1 < \ldots < i^\alpha_{l_\alpha}$, $\alpha_{i^\alpha_j} > 0$ for $j = 1, \ldots, l_\alpha$ and $\alpha_i = 0$ for all $i \in \mathcal{I} \backslash \{i^\alpha_1, \ldots, i^\alpha_{l_\alpha}\}$. Then

$$\kappa(\alpha) = \big(\underbrace{i^\alpha_1, \ldots, i^\alpha_1}_{\alpha_{i^\alpha_1} \text{times}}, \underbrace{i^\alpha_2, \ldots, i^\alpha_2}_{\alpha_{i^\alpha_2} \text{times}}, \ldots, \underbrace{i^\alpha_{l_\alpha}, \ldots, i^\alpha_{l_\alpha}}_{\alpha_{i^\alpha_{l_\alpha}} \text{ times}} \big) \ .$$

By construction, $\kappa(\alpha)$ is of increasing form. For each element $\overline{I} \in \overline{\mathcal{J}^\bullet}$ let I be the representative of increasing form. Define $\lambda(\overline{I}) \in \mathbb{N}^{(\mathcal{J})}$ as follows. If $\overline{I} = \overline{O}$, put $\lambda(\overline{I}) = 0_{\mathcal{J}}$. If $\overline{I} \neq \overline{O}$, let $i_1^I < \ldots < i_{l_I}^I$ be the elements of $\mathcal{J}$ which appear in I. Then, for each $j = 1, \ldots, l_I$ define $\alpha_{i_j^I}^I$ to be the number of times the index i_j^I appears in I. For $i \in \mathcal{J}$ not appearing among the i_j^I put $\alpha_i^I = 0$. Then define

$$\lambda(\overline{I}) = \alpha^I = \left(\alpha_i^I\right)_{i \in \mathcal{J}} .$$

So we obtain a map $\lambda : \overline{\mathcal{J}^\bullet} \to \mathbb{N}^{(\mathcal{J})}$. For given $I \neq O$ one has by definition $l_{\alpha^I} = l_I$ and $i_1^{\alpha^I} = i_1^I, \ldots, i_l^{\alpha^I} = i_l^I$ where $l = l_{\alpha^I} = l_I$. Moreover, the index $i = i_j^{\alpha^I}$, $j = 1, \ldots, l$ appears in $\overline{\kappa}\big(\lambda(\overline{I})\big)$ exactly α_i^I times which coincides with the number i appears in I. Hence $\overline{\kappa}\big(\lambda(\overline{I})\big) = \overline{I}$. Now assume $\alpha \in \mathbb{N}^{\mathcal{J}} \backslash \{0_{\mathcal{J}}\}$ to be given and let $I = \kappa(\alpha)$. Then $l_I = l_\alpha$ and $i_1^I = i_1^\alpha, \ldots, i_l^I = i_l^\alpha$ for $l = l_I = l_\alpha$. Now observe that for each of the indices $i = i_j^I$, $j = 1, \ldots, l$ the i-th component of $\lambda\big(\overline{\kappa}(\alpha)\big)$ coincides with α_i. Hence $\lambda\big(\overline{\kappa}(\alpha)\big) = \alpha$, which finishes the proof that $\overline{\kappa}$ is a bijection with inverse λ.

By construction of κ one has $|\kappa(\alpha)| = |\alpha|$ for all Greek multiindices α which entails that for every $k \in \mathbb{N}$ the bijection κ maps $\mathbb{N}_k^{(\mathcal{J})}$ onto $\overline{\mathcal{J}^k}$.

Also by construction it is clear that $\kappa(\alpha+\beta) = \kappa(\alpha)+\kappa(\beta)$ for all $\alpha, \beta \in \mathbb{N}^{(\mathcal{J})}$ and that $\kappa(0_{\mathcal{J}}) = O$. Hence κ is a morphism of monoids.

Now we will prove the formula for the cardinality of $\mathbb{N}_k^{\mathcal{J}}$ by double induction on k and the cardinality of the index set $\mathcal{J}$. Obviously $|\mathbb{N}_0^{\mathcal{J}}| = 1$, so the claim holds for $k = 0$ and all finite index sets. Assume that it holds for some natural k and all finite index sets. Now let $\mathcal{J}$ be an index set of cardinality 1. Then $|\mathbb{N}_{k+1}^{\mathcal{J}}| = 1$ since there is only one natural number with absolute value $k + 1$. Next assume that the claim holds for $k + 1$ and all index sets of cardinality less than d. Let $\mathcal{J}$ be an index set of cardinality d. Order the elements of $\mathcal{J}$ in some way so that $\mathcal{J} = \{i_1, \ldots, i_d\}$ and $i_1 < \ldots < i_d$. The set $\mathbb{N}_{k+1}^{\mathcal{J}}$ is then the disjoint union of the set of all $\alpha \in \mathbb{N}_{k+1}^{\mathcal{J}}$ such that $\alpha_{i_d} = 0$ and the set of all $\alpha \in \mathbb{N}_{k+1}^{\mathcal{J}}$ such that $\alpha_{i_d} \geqslant 1$. The first of these sets has cardinality

$$|\mathbb{N}_{k+1}^{\{i_1,\ldots,i_{d-1}\}}| = \frac{1}{k+1!} \prod_{l=0}^{k} \big(d - 1 + l\big) ,$$

the second has cardinality

$$|\mathbb{N}_k^{\mathcal{J}}| = \frac{1}{k!} \prod_{l=0}^{k-1} \big(d + l\big)$$

since the map

$$\big\{\alpha \in \mathbb{N}_{k+1}^{\mathcal{J}} \bigm| \alpha_{i_d} \geqslant 1\big\} \to \mathbb{N}_k^{\mathcal{J}} : \alpha \mapsto (\alpha_{i_1}, \ldots, \alpha_{i_{d-1}}, \alpha_{i_d} - 1) \in \mathbb{N}_k^{\mathcal{J}}$$

is a bijection. Hence

$$\begin{aligned} |\mathbb{N}_{k+1}^{\mathcal{J}}| &= \frac{1}{k+1!} \prod_{l=0}^{k} \big(d - 1 + l\big) + \frac{1}{k!} \prod_{l=0}^{k-1} \big(d + l\big) = \\ &= \frac{1}{k+1!} (d - 1 + k + 1) \prod_{l=0}^{k-1} \big(d + l\big) = \frac{1}{(k+1)!} \prod_{l=0}^{k} \big(d + l\big) \end{aligned}$$

and the induction step is finished. The claim is proved. □

17.1.7 By a *block* of a Roman multiindex I of positive order we mean a Roman multiindex of the form
$$I_B = (i_{b_1}, \ldots, i_{b_{|B|}}) \ ,$$
where B is a subset of $\{1, \ldots k\}$ and the $b_1, \ldots, b_{|B|} \in \{1, \ldots, k\}$ are the elements of B in increasing order. One can now decompose a multiindex I into blocks as follows. Let $\{B_1, \ldots, B_r\}$ be a partition of $\{1, \ldots, k\}$ which we assume to be lexicographically ordered that means that $b_{11} < b_{21} < \ldots < b_{r1}$, where $B_j = \{b_{j1}, \ldots, b_{j|B_j|}\}$ and $b_{jm} < b_{jn}$ for $j = 1, \ldots, r$ and $1 \leqslant m < n \leqslant |B_j|$. To express that $\{B_1, \ldots, B_r\}$ is a lexicographically ordered partition of $\{1, \ldots, k\}$ by r non-empty sets we write
$$B_1 \sqcup \ldots \sqcup B_r = \{1, \ldots, k\} \quad \& \quad \varnothing < B_1 < \ldots < B_r \ .$$
Now put $I_j := I_{B_j}$ for $j = 1, \ldots, r$. Then the Roman multiindices I and $I_1 + \ldots + I_r$ are equivalent which can be interpreted as I being decomposed into the r blocks $I_1, \ldots, I_r$. More precisely, we call the r-tupel of pairs $\big((I_1, B_1), \ldots, (I_r, B_r)\big)$ a a *decomposition of I into r blocks* and denote the space of such decompositions by $\mathrm{Block}^r(I)$. Note that the cardinality of $\mathrm{Block}^r(I)$ coincides with the Sterling number of the second kind $\left\{ {k \atop r} \right\}$ which gives the number of ways the set $\{1, \ldots, k\}$ can be partitioned into r subsets.

Multipowers and multiderivatives

17.1.8 Let M be a manifold and $x = (x^1, \ldots, x^d) : U \to \mathbb{R}^d$ a local coordinate system. Let $I \in \{1, \ldots, d\}^k$ be a Roman multiindex of positive order k. Then the product
$$x^I := x^{i_1} \cdot \ldots \cdot x^{i_k} \tag{17.1.1}$$
and, for every $f \in \mathcal{C}^\infty(U)$, the higher derivative
$$\frac{\partial^{|I|} f}{\partial x^I} := \frac{\partial^k f}{\partial x^{i_1} \cdot \ldots \cdot \partial x^{i_k}} \tag{17.1.2}$$
are both invariant under permutations of the components of I, hence depend only on the equivalence class $\overline{I}$. We therefore sometimes write $x^{\overline{I}}$ for x^I and $\frac{\partial^{|\overline{I}|} f}{\partial x^{\overline{I}}}$ for $\frac{\partial^{|I|} f}{\partial x^I}$. In order 0 one puts $x^{\overline{O}} := x^O := 1$ and $\frac{\partial^{|\overline{O}|} f}{\partial x^{\overline{O}}} := \frac{\partial^{|O|} f}{\partial x^O} := f$. For a multiindex $\alpha \in \mathbb{N}^d$ one defines as usual
$$x^\alpha := (x^1)^{\alpha_1} \cdot \ldots \cdot (x^d)^{\alpha_d}$$
and
$$\frac{\partial^{|\alpha|} f}{\partial x^\alpha} := \frac{\partial^{|\alpha|} f}{(\partial x^1)^{\alpha_1} \cdot \ldots \cdot (\partial x^d)^{\alpha_d}} := \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{\partial}{\partial x^d}\right)^{\alpha_d} f \ .$$
If now α and I are related by $\overline{I} = \kappa(\alpha)$, then $x^{\overline{I}} = x^\alpha$ and $\frac{\partial^{|\overline{I}|} f}{\partial x^{\overline{I}}} = \frac{\partial^{|\alpha|} f}{\partial x^\alpha}$ by definition of κ and invariance of the product and the higher derivative under permutations of components of the multiindex.

17.1.9 Remark Occasionally we need multipowers and multiderivatives over more general index sets. So let $\mathcal{I}$ be an arbitrary but still finite index set. Assume that the components of a coordinate system $x : U \to \mathbb{R}^{\mathcal{I}}$ are labelled x^i where i runs through the elements of $\mathcal{I}$. For

$I \in \mathcal{I}^k$ equations (17.1.1) and (17.1.2) then can be used again to define multipowers x^I and multiderivatives $\frac{\partial^{|I|} f}{\partial x^I}$. Note that both objects are invariant under permutations of components of I, too, so the corresponding expressions where I is replaced by $\overline{I}$ are also well-defined. Now let $\alpha = (\alpha_i)_{i \in \mathcal{I}} \in \mathbb{N}^{\mathcal{I}}$ be a Greek multiindex and $f \in \mathcal{C}^\infty(U)$. One then defines

$$x^\alpha := \prod_{i \in \mathcal{I}} (x^i)^{\alpha_i}$$

and

$$\frac{\partial^{|\alpha|} f}{\partial x^\alpha} := \prod_{i \in \mathcal{I}} \left(\frac{\partial}{\partial x^i} \right)^{\alpha_i} f \ .$$

One finally checks that when $\overline{\kappa}(\alpha) = \overline{I}$ the equalities $x^\alpha = x^{\overline{I}}$ and $\frac{\partial^{|\alpha|} f}{\partial x^\alpha} = \frac{\partial^{|\overline{I}|} f}{\partial x^{\overline{I}}}$ still hold true in this more general situation.

The formula of Faà-di-Bruno

17.1.10 Theorem (Combinatorial form of Faà-di-Bruno's formula) *Let $\mathcal{I}$ and $\mathcal{J}$ denote finite index sets. Assume that M and N are smooth manifolds and that we are given smooth charts $x : U \hookrightarrow \mathbb{R}^{\mathcal{I}}$ and $y : V \hookrightarrow \mathbb{R}^{\mathcal{J}}$ over open domains $U \subset M$ and $V \subset N$. Assume further that $\varphi : U \to V$ is a smooth map. Denote by $\varphi_j : U \to \mathbb{R}$, $j \in \mathcal{J}$ its components that means that $\varphi = (\varphi^j)_{j \in \mathcal{J}}$. Finally let $I \in \mathcal{I}^k$ be a Roman multiindex of positive order k. Then for every $f \in \mathcal{C}^\infty(V)$ the following equality holds true:*

$$\frac{\partial^{|I|}(f \circ \varphi)}{\partial x^I} = \sum_{r=1}^{k} \sum_{\substack{J \in \mathcal{J}^r \\ J=(j_1,\ldots,j_r)}} \sum_{\substack{B_1 \sqcup \ldots \sqcup B_r = \{1,\ldots,k\} \\ \varnothing < B_1 < \ldots < B_r}} \left(\frac{\partial^{|J|} f}{\partial y^J} \circ \varphi \right) \cdot \frac{\partial^{|I_{B_1}|} \varphi^{j_1}}{\partial x^{I_{B_1}}} \cdot \ldots \cdot \frac{\partial^{|I_{B_r}|} \varphi^{j_r}}{\partial x^{I_{B_r}}} \ . \tag{17.1.3}$$

Proof. We prove the claim by induction on the length of the multiindex I. Assume to be given a Roman multiindex $I \in \mathcal{I}^k$ of length $k = |I| = 1$. Then there exists a unique $i \in \mathcal{I}$ such that $I = (i)$. By the chain rule one computes

$$\frac{\partial^{|I|}(f \circ \varphi)}{\partial x^I} = \frac{\partial (f \circ \varphi)}{\partial x^i} = \sum_{j \in \mathcal{J}} \left(\frac{\partial f}{\partial y^j} \circ \varphi \right) \cdot \frac{\partial \varphi^j}{\partial x^i} = \sum_{\substack{J \in \mathcal{J}^1 \\ J=(j_1)}} \sum_{B_1=\{1\}} \left(\frac{\partial^{|J|} f}{\partial y^J} \circ \varphi \right) \cdot \frac{\partial^{|I_{B_1}|} \varphi^{j_1}}{\partial x^{I_{B_1}}} \ ,$$

hence the claim holds true for $k = 1$.

Now assume that for some $k \geqslant 1$ the claim holds for all Roman multiindices of order $\leqslant k$ over $\mathcal{I}$. Assume that $I = (i_1, \ldots, i_{k+1})$ is a Roman multiindex of order $k+1$ over $\mathcal{I}$. Then $I = I' + i_{k+1}$, where $I' = (i_1, \ldots, i_k)$ is a Roman multiindex of order k. Using the induction hypothesis for I', the product and the chain rule one obtains

$$\begin{aligned}
\frac{\partial^{|I|}(f\circ\varphi)}{\partial x^I} &= \frac{\partial}{\partial x^{i_{k+1}}}\frac{\partial^{|I'|}}{\partial x^{I'}} f\circ\varphi = \\
&= \frac{\partial}{\partial x^{i_{k+1}}} \sum_{r=1}^{k} \sum_{\substack{J\in\mathcal{J}^r \\ J=(j_1,\ldots,j_r)}} \sum_{\substack{B_1\sqcup\ldots\sqcup B_r=\{1,\ldots,k\} \\ \varnothing<B_1<\ldots<B_r}} \left(\frac{\partial^{|J|}f}{\partial y^J}\circ\varphi\right)\cdot\frac{\partial^{|I_{B_1}|}\varphi^{j_1}}{\partial x^{I_{B_1}}}\cdot\ldots\cdot\frac{\partial^{|I_{B_r}|}\varphi^{j_r}}{\partial x^{I_{B_r}}} = \\
&= \sum_{r=1}^{k} \sum_{\substack{J\in\mathcal{J}^r \\ J=(j_1,\ldots,j_r)}} \sum_{j_{r+1}\in\mathcal{J}} \sum_{\substack{B_1\sqcup\ldots\sqcup B_r=\{1,\ldots,k\} \\ \varnothing<B_1<\ldots<B_r}} \left(\frac{\partial^{|J|+1}f}{\partial y^{Jj_{r+1}}}\circ\varphi\right)\cdot\frac{\partial^{|I_{B_1}|}\varphi^{j_1}}{\partial x^{I_{B_1}}}\cdot\ldots\cdot\frac{\partial^{|I_{B_r}|}\varphi^{j_r}}{\partial x^{I_{B_r}}}\frac{\partial\varphi^{j_{r+1}}}{\partial x^{i_{k+1}}} + \\
&+ \sum_{r=1}^{k}\sum_{l=1}^{r} \sum_{\substack{J\in\mathcal{J}^r \\ J=(j_1,\ldots,j_r)}} \sum_{\substack{B_1\sqcup\ldots\sqcup B_r=\{1,\ldots,k\} \\ \varnothing<B_1<\ldots<B_r}} \left(\frac{\partial^{|J|}f}{\partial y^J}\circ\varphi\right)\cdot\frac{\partial^{|I_{B_1}|}\varphi^{j_1}}{\partial x^{I_{B_1}}}\cdot\ldots\cdot\frac{\partial^{|I_{B_l}|+1}\varphi^{j_l}}{\partial x^{I_{B_l}i_{k+1}}}\cdot\ldots\cdot\frac{\partial^{|I_{B_r}|}\varphi^{j_r}}{\partial x^{I_{B_r}}} = \\
&= \sum_{r=2}^{k+1} \sum_{\substack{J\in\mathcal{J}^r \\ J=(j_1,\ldots,j_r)}} \sum_{\substack{B_1\sqcup\ldots\sqcup B_r=\{1,\ldots,k+1\} \\ \varnothing<B_1<\ldots<B_r=\{k+1\}}} \left(\frac{\partial^{|J|}f}{\partial y^J}\circ\varphi\right)\cdot\frac{\partial^{|I_{B_1}|}\varphi^{j_1}}{\partial x^{I_{B_1}}}\cdot\ldots\cdot\frac{\partial^{|I_{B_r}|}\varphi^{j_r}}{\partial x^{I_{B_r}}} + \\
&+ \sum_{r=1}^{k} \sum_{\substack{J\in\mathcal{J}^r \\ J=(j_1,\ldots,j_r)}} \sum_{\substack{B_1\sqcup\ldots\sqcup B_r=\{1,\ldots,k+1\} \\ \varnothing<B_1<\ldots<B_r\neq\{k+1\}}} \left(\frac{\partial^{|J|}f}{\partial y^J}\circ\varphi\right)\cdot\frac{\partial^{|I_{B_1}|}\varphi^{j_1}}{\partial x^{I_{B_1}}}\cdot\ldots\cdot\frac{\partial^{|I_{B_r}|}\varphi^{j_r}}{\partial x^{I_{B_r}}} = \\
&= \sum_{r=1}^{k+1} \sum_{\substack{J\in\mathcal{J}^r \\ J=(j_1,\ldots,j_r)}} \sum_{\substack{B_1\sqcup\ldots\sqcup B_r=\{1,\ldots,k+1\} \\ \varnothing<B_1<\ldots<B_r}} \left(\frac{\partial^{|J|}f}{\partial y^J}\circ\varphi\right)\cdot\frac{\partial^{|I_{B_1}|}\varphi^{j_1}}{\partial x^{I_{B_1}}}\cdot\ldots\cdot\frac{\partial^{|I_{B_r}|}\varphi^{j_r}}{\partial x^{I_{B_r}}} .
\end{aligned}$$

This concludes the induction step and the theorem is proved. □

17.2. Jet bundles

17.2.1 Let us fix in this section a smooth finite dimensional fiber bundle $\pi^E : E \to M$. Denote by F its typical fiber and put $d = \dim M$, $n = \dim F$. The dimension of the total space E then is given by $\dim E = d + n$. Note that for each point $p \in M$ the fiber $F_p = (\pi^E)^{-1}(p)$ is diffeomorphic to F.

Recall that $\Gamma^\infty(\pi^E)$ stands for the *sheaf of smooth local sections* of π^E. Its space of sections over an open $U \subset M$ consists of all smooth $s : U \to E$ such that $\pi^E \circ s = \mathrm{id}_U$ and is denoted by $\Gamma^\infty(U, \pi^E)$. When writing $s \in \Gamma^\infty(\pi^E)$ we mean that s is a smooth local section of E defined over some open subset $U = \operatorname{dom} s \subset M$. If $p \in X$ is a point, then $\Gamma^\infty(p, \pi^E)$ denotes the space of local smooth sections about p that is the space of all smooth sections $s : U \to E$ defined on an open neighborhood $U \subset X$ of p. We will write $\mathfrak{U}_p^\circ$ for the filter basis of all open neighborhoods of p and $\Gamma_p^\infty(\pi^E)$ for the *stalk* of $\Gamma^\infty(\pi^E)$ at p which is defined as the colimit

$$\Gamma_p^\infty(\pi^E) = \operatorname*{colim}_{U\in\mathfrak{U}_p^\circ} \Gamma^\infty(U, \pi^E) = \Gamma^\infty(p, \pi^E)/\sim_p \quad . \tag{17.2.1}$$

Here we have made use of the fact that the colimit can be represented as the quotient of $\Gamma^\infty(p, \pi^E)$ by the equivalence relation $\sim_p$, where equivalence $s_1 \sim_p s_2$ of two smooth sections $s_1 : U_1 \to$

E and $s_2 : U_2 \to E$ over open neighborhoods of p is defined by the existance of an open neighborhood $U \subset U_1 \cap U_2$ of p such that $s_1|_U := s_2|_U$. The equivalence class of a section $s \in \Gamma^\infty(p, \pi^E)$ is denoted $[s]_p$ and is called the *germ* of s at p. So in other words, $\Gamma^\infty_p(\pi^E)$ is the space of all germs of smooth sections at p. To distinguish $\sim_p$ from the later defined m-equivalence we call the relation $\sim_p$ *germ equivalence* at p.

17.2.2 Definition Let $p \in M$ be a point in the base manifold M and $k \in \mathbb{N} \cup \{\infty\}$. Two local smooth sections $s_1 : U_1 \to E$ and $s_2 : U_2 \to E$ defined over open neighborhoods of p are said to be k-*equivalent* at p if $s_1(p) = s_2(p)$ and if for every fibered chart $(x, u) : W \to \mathbb{R}^d \times \mathbb{R}^n$ of π^E with $p \in \pi(W)$

$$\frac{\partial^{|\alpha|}(u^b \circ s_1)}{\partial x^\alpha}(p) = \frac{\partial^{|\alpha|}(u^b \circ s_2)}{\partial x^\alpha}(p) \quad \text{for } b = 1, \ldots, n \text{ and all } \alpha \in \mathbb{N}^d_{0,k} \ . \tag{17.2.2}$$

17.2.3 Proposition and Definition *Let $p \in M$ be a point and $k \in \mathbb{N} \cup \{\infty\}$. Then k-equivalence at p is an equivalence relation on $\Gamma^\infty(p, \pi^E)$. It will be denoted by the symbol $\sim_{k,p}$. The k-equivalence class of a smooth section $s : U \to E$ at p will be written $\mathsf{j}^k_p(s)$. It is called the* k-jet *of s at p. The set of such k-jets at p coincides with the quotient space $\mathsf{J}^k_p(\pi^E) = \Gamma^\infty(p, \pi^E)/\sim_{k,p}$. The union*

$$\mathsf{J}^k(E) = \mathsf{J}^k(\pi^E) = \bigcup_{p \in M} \mathsf{J}^k_p(\pi^E)$$

will be called the space of k-jets *of sections of the bundle $\pi^E : E \to M$. Finally, there is a* projection $\pi^k = \pi^{\mathsf{J}^k(E)} : \mathsf{J}^k(\pi^E) \to M$ *which maps a jet $\mathsf{j}^k_p(s)$, $s \in \Gamma^\infty(p, \pi^E)$ to its* footpoint p.

Proof. The relation of k-equivalence at p is obviously reflexive and symmetric by definition. It is also transitive by transitivity of equality. Hence k-equivalence at p is an equivalence relation indeed. The claim is proved. □

17.2.4 Lemma *The following statements are equivalent for two sections $s_1, s_2 \in \Gamma^\infty(p, \pi^E)$ such that $s_1(p) = s_2(p)$:*

(1) *The local sections s_1 and s_2 are k-equivalent at p.*

(2) *For every fibered chart $(x, u) : W \to \mathbb{R}^d \times \mathbb{R}^n$ of π^E with $p \in \pi(W)$*

$$\frac{\partial^{|I|}(u^b \circ s_1)}{\partial x^I}(p) = \frac{\partial^{|I|}(u^b \circ s_2)}{\partial x^I}(p) \quad \begin{array}{l} \textit{for } b = 1, \ldots, n \textit{ and all} \\ I \in \{1, \ldots, d\}^l, 1 \leqslant l \leqslant k \ . \end{array} \tag{17.2.3}$$

(3) *There exists a fibered chart $(x, u) : W \to \mathbb{R}^d \times \mathbb{R}^n$ of π^E with $p \in \pi(W)$ such that* (17.2.2) *holds true.*

(4) *There exists a fibered chart $(x, u) : W \to \mathbb{R}^d \times \mathbb{R}^n$ of π^E with $p \in \pi(W)$ such that* (17.2.3) *holds true.*

Proof. The claim is an immediate consequence of the formula of Faà-di-Bruno. □

17.2.5 Next we want to define a topology on the jet space $\mathsf{J}^k(\pi^E)$ so that $\pi^k : \mathsf{J}^k(\pi^E) \to M$ becomes a (topological) fiber bundle.

18. Geometric PDEs

18.1. Linear differential operators over commutative rings

18.1.1 In this section, A will always denote a commutative unital algebra over a field of characteristic zero $\Bbbk$. The identity element of A will be denoted by 1. Let M, N be two A-modules. An element $a \in A$ then acts in two natural ways on the space $\mathrm{Hom}_\Bbbk(M,N)$ of $\Bbbk$-linear maps from M to N, namely by

$$a_* : \mathrm{Hom}_\Bbbk(M,N) \to \mathrm{Hom}_\Bbbk(M,N),\ f \mapsto a_* f = af = \big(M \ni m \mapsto af(m) \in N\big) \tag{18.1.1}$$

and

$$a^* : \mathrm{Hom}_\Bbbk(M,N) \to \mathrm{Hom}_\Bbbk(M,N),\ f \mapsto a^* f = fa = \big(M \ni m \mapsto f(am) \in N\big)\ . \tag{18.1.2}$$

18.1.2 Proposition and Definition *The actions a_* and a^* define two A-module structures on* $\mathrm{Hom}_\Bbbk(M,N)$ *which are called the* canonical left *and the* canonical right A-module structures*, respectively. These module structures commute.*

Proof. In the following let $a, b \in A$ and $f, g \in \mathrm{Hom}_\Bbbk(M,N)$. Then one computes for $m \in M$

$$\begin{aligned}
\big((a+b)_* f\big)(m) &= (a+b)\big(f(m)\big) = a\big(f(m)\big) + b\big(f(m)\big) \\
&= \big(a_* f\big)(m) + \big(b_* f\big)(m) = \big(a_* f + b_* f\big)(m)\ , \\
\big(a_*(f+g)\big)(m) &= a\big(f(m)+g(m)\big) = af(m) + ag(m) = \big(a_* f + a_* g\big)(m)\ , \\
\big(a_* b_* f\big)(m) &= a\big(bf(m)\big) = (ab)\big(f(m)\big) = \big((ab)_* f\big)(m)\ , \\
\big(1_* f\big)(m) &= 1 \cdot f(m) = f(m)\ ,
\end{aligned}$$

and

$$\begin{aligned}
\big((a+b)^* f\big)(m) &= f\big((a+b)m\big) = f(am) + f(bm) \\
&= \big(a^* f\big)(m) + \big(b^* f\big)(m) = \big(a^* f + b^* f\big)(m)\ , \\
\big(a^*(f+g)\big)(m) &= f(am) + g(am) = a^* f(m) + a^* g(m) = \big(a^* f + a^* g\big)(m)\ , \\
\big(a^* b^* f\big)(m) &= \big(b^* f\big)(am) = f\big((b(am))\big) = f\big((ab)m\big) = \big((ab)_* f\big)(m)\ , \\
\big(1^* F\big)(m) &= F(1 \cdot m) = F(m)\ .
\end{aligned}$$

This proves the module properties. It remains to show that $a_* b^* f = b^* a_* f$. But that is clear since for all $m \in M$

$$\big(a_* b^* f\big)(m) = a\big((b^* f)(m)\big) = a\big(f(bm)\big) = \big(a_* f\big)(bm) = \big(b^* a_* f\big)(m)\ . \qquad \square$$

18.1.3 Remark By the preceding proposition $\mathrm{Hom}_{\Bbbk}(M,N)$ becomes an A-bimodule which is not symmetric, in general, unless for example $M = N = A$. We regard $\mathrm{Hom}_{\Bbbk}(M,N)$ always as an object in the category of A-bimodules. When we want to consider only the canonical left or the canonical right A-module structure on the space of $\Bbbk$-linear maps from M to N we write ${}_A\mathrm{Hom}_{\Bbbk}(M,N)$ and $\mathrm{Hom}_{\Bbbk,A}(M,N)$, respectively, for the resulting objects in the category of A-modules.

18.1.4 Definition For every $a \in A$ denote by $\mathrm{ad}_a : \mathrm{Hom}_{\Bbbk}(M,N) \to \mathrm{Hom}_{\Bbbk}(M,N)$ the $\Bbbk$-linear map $a_* - a^*$ and call it the *adjoint action* of a.

18.1.5 Lemma *Let M,N,P be A-modules. Then one has for all $f \in \mathrm{Hom}_{\Bbbk}(M,N)$, $g \in \mathrm{Hom}_{\Bbbk}(N,P)$ and all $a,b \in A$*

$$\mathrm{ad}_{ab} f = a_*(\mathrm{ad}_b f) + b^*(\mathrm{ad}_a f) = a^*(\mathrm{ad}_b f) + b_*(\mathrm{ad}_a f) \ , \tag{18.1.3}$$

$$\mathrm{ad}_a(g \circ f) = (\mathrm{ad}_a g) \circ f + g \circ (\mathrm{ad}_a f) \ . \tag{18.1.4}$$

Proof. Compute by observing that the left and right A-module structures commute:

$$\mathrm{ad}_{ab} f = (ab)_* f - (ab)^* f = a_*(b_* f - b^* f) + b^*(a_* f - a^* f) = a_*(\mathrm{ad}_b f) + b^*(\mathrm{ad}_a f) \ .$$

By symmetry in a and b the first claimed equality follows. For the second observe that $(a^* g) \circ f = g \circ (a_* f)$ and compute

$$\begin{aligned}\mathrm{ad}_a(g \circ f) &= a_*(g \circ f) - a^*(g \circ f) = (a_* g - a^* g) \circ f + g \circ (a_* f - a^* f) = \\ &= (\mathrm{ad}_a g) \circ f + g \circ (\mathrm{ad}_a f) \ .\end{aligned}$$

□

18.1.6 Definition For all A-modules M,N the space $\mathcal{D}\mathit{iff}^0(M,N)$ of *linear differential operators* of *order* 0 from M to N is defined as the set of $D \in \mathrm{Hom}_{\Bbbk}(M,N)$ such that

$$\mathrm{ad}_a D = 0 \quad \text{for all } a \in A \ .$$

Recursively, one defines the space $\mathcal{D}\mathit{iff}^k(M,N)$ of *linear differential operators* of *order* $\leqslant k+1$ from M to N as the set of all $D \in \mathrm{Hom}_{\Bbbk}(M,N)$ such that

$$\mathrm{ad}_a D \in \mathcal{D}\mathit{iff}^k(M,N) \quad \text{for all } a \in A \ .$$

The space $\mathrm{Der}_{\Bbbk}(A,N)$ of *derivations* in N is defined as the set of all $D \in \mathrm{Hom}_{\Bbbk}(A,N)$ for which the Leibniz rule holds that is for which

$$D(ab) = aD(b) + bD(a) \quad \text{for all } a,b \in A \ .$$

18.1.7 Remark By definition, $\mathcal{D}\mathit{iff}^0(M,N)$ coincides with the space $\mathrm{Hom}_A(M,N)$ of A-module maps from M to N. By induction on k it becomes clear that $\mathcal{D}\mathit{iff}^k(M,N)$ can be equivalently described as the set of all $D \in \mathrm{Hom}_{\Bbbk}(M,N)$ such that

$$(\mathrm{ad}_{a_0} \circ \ldots \circ \mathrm{ad}_{a_k})\, D = 0 \quad \text{for all } a_0, \ldots, a_k \in A \ .$$

18.1.8 Proposition *Let M, N, P be two A-modules. Then the following holds true for all $k, l \in \mathbb{N}$.*

(i) *The space $\mathcal{D}\mathit{iff}^k(M,N)$ inherits from $\mathrm{Hom}_{\mathbb{k}}(M,N)$ both A-module structures so is an A-subbimodule of $\mathrm{Hom}_{\mathbb{k}}(M,N)$. The two A-module structures coincide on $\mathcal{D}\mathit{iff}^0(M,N)$ but in general not on spaces of differential operators of higher order.*

(ii) *One has a canonical inclusion*

$$\mathcal{D}\mathit{iff}^k(M,N) \subset \mathcal{D}\mathit{iff}^{k+1}(M,N) \ .$$

(iii) *The composition of a differential operator $\Delta \in \mathcal{D}\mathit{iff}^k(N,P)$ with a differential operator $D \in \mathcal{D}\mathit{iff}^l(M,N)$ is a linear differential operator of degree $\leqslant k+l$.*

(iv) *The space of derivations $\mathrm{Der}_{\mathbb{k}}(A,N)$ is an A-submodule of $\mathcal{D}\mathit{iff}^1(M,N)$ with respect to the canonical left A-module structure but in general not an A-submodule of $\mathcal{D}\mathit{iff}^1(M,N)$ with respect to the canonical right A-module structure.*

Proof. *ad (i)*. The claim for $\mathcal{D}\mathit{iff}^0(M,N)$ holds since for every $D \in \mathcal{D}\mathit{iff}^0(M,N)$ and $a \in A$ the operators $a_* D$ and $a^* D$ coincide and are both A-linear again by the following equalities.

$$\begin{aligned}(a^*D)(m) = D(am) = D(am) = a(D(m)) = (a_*D)(m) \quad \text{for all } m \in M \text{ and}\\ (a_*D)(bm) = a(D(bm)) = ab(D(m)) = b(aD(m)) = b(a_*D(m)) \quad \text{for all } b \in A,\ m \in M \ .\end{aligned}$$

Under the assumption that $\mathcal{D}\mathit{iff}^k(M,N)$ inherits the A-bimodule structure from $\mathrm{Hom}_{\mathbb{k}}(M,N)$ one checks for $D \in \mathcal{D}\mathit{iff}^{k+1}(M,N)$

$$\begin{aligned}\mathrm{ad}_b(a_*D) = b_*a_*D - b^*a_*D = a_*(b_*D - b^*D) = a_*(\mathrm{ad}_b D) \in \mathcal{D}\mathit{iff}^k(M,N) \quad \text{and}\\ \mathrm{ad}_b(a^*D) = b_*a^*D - b^*a^*D = a^*(b_*D - b^*D) = a^*(\mathrm{ad}_b D) \in \mathcal{D}\mathit{iff}^k(M,N) \ .\end{aligned}$$

By induction $\mathcal{D}\mathit{iff}^k(M,N)$ therefore is an A-subbimodule of $\mathrm{Hom}_{\mathbb{k}}(M,N)$ for all $k \in \mathbb{N}$. Even though the two A-module structures coincide on $\mathcal{D}\mathit{iff}^0(M,N)$ they do not on spaces of differential operators of order 1 (and higher) as Example 18.1.9 below shows.

ad (ii). This is obvious by definition and an inductive argument.

ad (iii). If $k + l = 0$ the claim is clear since then both Δ and D are A-linear, hence their composition is so, too. Assume that for some natural n the claim holds for all $k, l \in \mathbb{N}$ with $k + l \leqslant n$. Then assume $k + l = n + 1$ and let $\Delta \in \mathcal{D}\mathit{iff}^k(N,P)$ and $D \in \mathcal{D}\mathit{iff}^l(M,N)$. Now compute using Equation (18.1.4)

$$\mathrm{ad}_a(\Delta \circ D) = (\mathrm{ad}_a \Delta) \circ D + \Delta \circ (\mathrm{ad}_a D) \ .$$

By inductive hypothesis the right hand side is a differential operator of order $\leqslant n$, hence $\Delta \circ D \in \mathcal{D}\mathit{iff}^{k+l}(M,P)$.

ad (iv). The space of derivations $\mathrm{Der}_{\mathbb{k}}(A,N)$ is an A-submodule of $\mathcal{D}\mathit{iff}^1(M,N)$ with respect to the canonical left A-module structure. Namely, if $D \in \mathrm{Der}_{\mathbb{k}}(A,N)$ and $a, b, c \in A$, then

$$(a_*D)(bc) = aD(bc) = abD(c) + acD(b) = b(aD(c)) + c(aD(b)) = b(a_*D)(c) + c(a_*D)(b) \ .$$

In general, $\mathrm{Der}_{\mathbb{k}}(A,N)$ is not an A-submodule of $\mathcal{D}\mathit{iff}^1(M,N)$ with respect to the canonical right A-module structure. □

18.1.9 Example Let $A = \Bbbk[X_1, \ldots, X_n]$ be the polynomial ring over $\Bbbk$ in n indeterminates and $\Omega^1_{A/\Bbbk}$ the space of Kähler differentials of A that is the space I/I^2, where I is the kernel of the multiplication map $\mu : A \otimes_\Bbbk A \to A$. The canonical map $d : A \to \Omega^1_{A/\Bbbk}$, $a \mapsto da = 1 \otimes a - a \otimes 1 + I^2$ then is a derivation and $\Omega^1_{A/\Bbbk}$ an A-module which is free over the elements $dX_1, \ldots, dX_n$. If now $a \in A \setminus \Bbbk$, then

$$a^* d(1) = da \neq 0 \ ,$$

so $a^* d$ can not be a derivation. Note that $a_* d$ is a derivation by Proposition 18.1.8 (iv).

18.1.10 By Proposition 18.1.8 one has a (filtered) diagram in the category of A-bimodules

$$\mathcal{D}\!\mathit{iff}^{\,0}(M,N) \hookrightarrow \mathcal{D}\!\mathit{iff}^{\,1}(M,N) \hookrightarrow \ldots \hookrightarrow \mathcal{D}\!\mathit{iff}^{\,k}(M,N) \hookrightarrow \ . \tag{18.1.5}$$

Its colimit exists and coincides with the union of the $\mathcal{D}\!\mathit{iff}^{\,k}(M,N)$, $k \in \mathbb{N}$. We will denote it by $\mathcal{D}\!\mathit{iff}(M,N)$ and call it the A-bimodules of linear differential operators from M to N.

18.1.11 Remark In case we want to consider the spaces $\mathcal{D}\!\mathit{iff}^{\,k}(M,N)$ and $\mathcal{D}\!\mathit{iff}(M,N)$ with their canonical left A-module structure, only, we write ${}_A\mathcal{D}\!\mathit{iff}^{\,k}(M,N)$ and ${}_A\mathcal{D}\!\mathit{iff}(M,N)$, respectively. Analogously, when we regard $\mathcal{D}\!\mathit{iff}^{\,k}(M,N)$ and $\mathcal{D}\!\mathit{iff}(M,N)$ as objects in the category of A-modules with their canonical right A-module structure we denote them by $\mathcal{D}\!\mathit{iff}^{\,k}_A(M,N)$ and $\mathcal{D}\!\mathit{iff}_A(M,N)$, respectively. By $\mathcal{D}\!\mathit{iff}(M)$

18.1.12 Proposition *Assigning to every pair of A-modules (M,N) the A-bimodule $\mathcal{D}\!\mathit{iff}^{\,k}(M,N)$ and to every pair of A-module maps $f : M' \to M$ and $g : N \to N'$ the A-bimodule map $(f^*, g_*) : \mathcal{D}\!\mathit{iff}^{\,k}(M,N) \to \mathcal{D}\!\mathit{iff}^{\,k}(M',N')$, $D \mapsto g \circ D \circ f$ comprises a bifunctor which is contravariant in the first and covariant in the second argument. Analogously, the assignment $(M,N) \to \mathcal{D}\!\mathit{iff}(M,N)$ becomes a bifunctor.*

Proof. By definition, $\big((\mathrm{id}_M)^*, (id_N)_*\big)D = D$ for every $D \in \mathcal{D}\!\mathit{iff}^{\,k}(M,N)$, so

$$\big((\mathrm{id}_M)^*, (id_N)_*\big) = \mathrm{id}_{\mathcal{D}\!\mathit{iff}^{\,k}(M,N)} \ .$$

Let $M_1, M_2, M_3, N_1, N_2, N_3$ denote A-modules and assume to be given A-modules maps $f_1 : M_2 \to M_1$, $f_2 : M_3 \to M_2$, $g_1 : N_1 \to N_2$, and $g_2 : N_2 \to N_3$. Then

$$\begin{aligned} \Big(\big(f_2^*, g_{2*}\big) \circ \big(f_1^*, g_{1*}\big)\Big)D &= \big(f_2^*, g_{2*}\big)\big(g_1 \circ D \circ f_1\big) = (g_2 \circ g_1) \circ D \circ (f_1 \circ f_2) = \\ &= \big((f_1 \circ f_2)^*, (g_2 \circ g_1)_*\big)D \ . \end{aligned}$$

This proves that $\mathcal{D}\!\mathit{iff}^{\,k}(-,-)$ and $\mathcal{D}\!\mathit{iff}(-,-)$ are bifunctors contravariant in the first and covariant in the second argument. □

18.1.13 Theorem *Let N be an A-module. Then the functors $\mathcal{D}\!\mathit{iff}^{\,k}(-,N) : {}_A\mathsf{Mod} \to {}_A\mathsf{Mod}_A$ and $\mathcal{D}\!\mathit{iff}(-,N) : {}_A\mathsf{Mod} \to {}_A\mathsf{Mod}_A$ are representable. Representing objects are given by the A-modules ${}_A\mathcal{D}\!\mathit{iff}^{\,k}(N)$ and ${}_A\mathcal{D}\!\mathit{iff}(N)$, respectively.*

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Bibliography

Akhiezer, N. I. & Glazman, I. M. (1993). *Theory of linear operators in Hilbert space*. Dover Publications, Inc., New York. Translated from the Russian and with a preface by Merlynd Nestell, Reprint of the 1961 and 1963 translations, Two volumes bound as one.

Atiyah, M. & Segal, G. (2004). Twisted K-theory. *Ukr. Mat. Visn.*, *1*(3), 287–330.

Baez, J. (1997). Higher-Dimensional Algebra II. 2-Hilbert Spaces. *Adv. Math.*, *127*, 125–189.

Bargmann, V. (1954). On unitary ray representations of continuous groups. *Ann. of Math. (2)*, *59*, 1–46.

Bargmann, V. (1964). Note on Wigner's theorem on symmetry operations. *J. Mathematical Phys.*, *5*, 862–868.

Cohen, H. (1993). *A Course in Computational Algebraic Number Theory*, volume 138 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin.

Cook, J. M. (1953). The mathematics of second quantization. *Trans. Amer. Math. Soc.*, *74*, 222–245.

Coutinho, S. C. (1995). *A primer of algebraic D-modules*, volume 33 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge.

Emch, G. G. (2009). *Algebraic Methods in Statistical Mechanics and Quantum Field Theory*. Dover Publications, Inc., New York.

Frobenius, F. G. (1878). Über lineare Substitutionen und bilineare Formen. *Journal für die reine und angewandte Mathematik*, *84*, 1–63.

Gehér, G. P. (2014). An elementary proof for the non-bijective version of Wigner's theorem. *Phys. Lett. A*, *378*(30-31), 2054–2057.

Gouvêa, F. Q. (1997). *p-adic numbers* (Second ed.). Universitext. Springer-Verlag, Berlin. An introduction.

Grothendieck, A. (1955). Produits tensoriels topologiques et espaces nucléaires. *Mem. Amer. Math. Soc.*, *No. 16*, 140.

Gustafson, S. J. & Sigal, I. M. (2011). *Mathematical concepts of quantum mechanics* (Second ed.). Universitext. Springer, Heidelberg.

Johnson, J. S. & Manes, E. G. (1970). On modules over a semiring. *J. Algebra*, *15*, 57–67.

Jost, R. (1965). *The general theory of quantized fields*, volume 1960 of *Mark Kac, editor. Lectures in Applied Mathematics (Proceedings of the Summer Seminar, Boulder, Colorado*. American Mathematical Society, Providence, R.I.

Kadison, R. V. & Ringrose, J. R. (1997). *Fundamentals of the theory of operator algebras. Vol. II*, volume 16 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI. Advanced theory, Corrected reprint of the 1986 original.

Kriegl, A. & Michor, P. W. (1997). *The convenient setting of global analysis*, volume 53 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI.

Lang, S. (2002). *Algebra* (third ed.)., volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York.

Lomont, J. S. & Mendelson, P. (1963). The Wigner unitarity-antiunitarity theorem. *Ann. of Math. (2)*, *78*, 548–559.

Montgomery, D. & Zippin, L. (1955). *Topological transformation groups*. Interscience Publishers, New York-London.

Neeb, K.-H. (1997). On a theorem of S. Banach. *J. Lie Theory*, *7*(2), 293–300.

Ostrowski, A. (1916). Über einige Lösungen der Funktionalgleichung $\psi(x) \cdot \psi(x) = \psi(xy)$. *Acta Math.*, *41*(1), 271–284.

Pietsch, A. (1972). *Nuclear locally convex spaces*. Springer-Verlag, New York-Heidelberg. Translated from the second German edition by William H. Ruckle, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 66.

Schottenloher, M. (1995). *Geometrie und Symmetrie in der Physik*. Vieweg Verlagsgesellschaft.

Schottenloher, M. (2008). *A mathematical introduction to conformal field theory* (Second ed.)., volume 759 of *Lecture Notes in Physics*. Springer-Verlag, Berlin.

Simms, D. J. (1968). *Lie groups and quantum mechanics*, volume 59 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York.

Simms, D. J. (1971). A short proof of Bargmann's criterion for the lifting of projective representations of Lie groups. *Rep. Mathematical Phys.*, *2*(4), 283–287.

Streater, R. F. & Wightman, A. S. (2000). *PCT, spin and statistics, and all that*. Princeton Landmarks in Physics. Princeton University Press, Princeton, NJ. Corrected third printing of the 1978 edition.

Uhlhorn, U. (1962). Representation of symmetry transformations in quantum mechanics. *Ark. Fys.*, *23*(30), 307–340.

von Neumann, J. & Wigner, E. (1929). Über das Verhalten von Eigenwerten bei adiabatischen Prozessen. (German) [On the behavior of the eigenvalues of adiabatic processes]. *Physikalische Zeitschrift*, *30*(15), 467–470.

Wightman, A. S. & Gårding, L. (1964). Fields as operator-valued distributions in relativistic quantum theory. *Arkiv f. Fysik, Kungl. Svenska Vetenskapsak*, *28*, 129–189.

Wigner, E. (1944). *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren.* J. W. Edwards, Ann Arbor, Michigan.