ASEN 5050
SPACEFLIGHT DYNAMICS
Multi-Body Dynamics, Part 1

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Objectives:

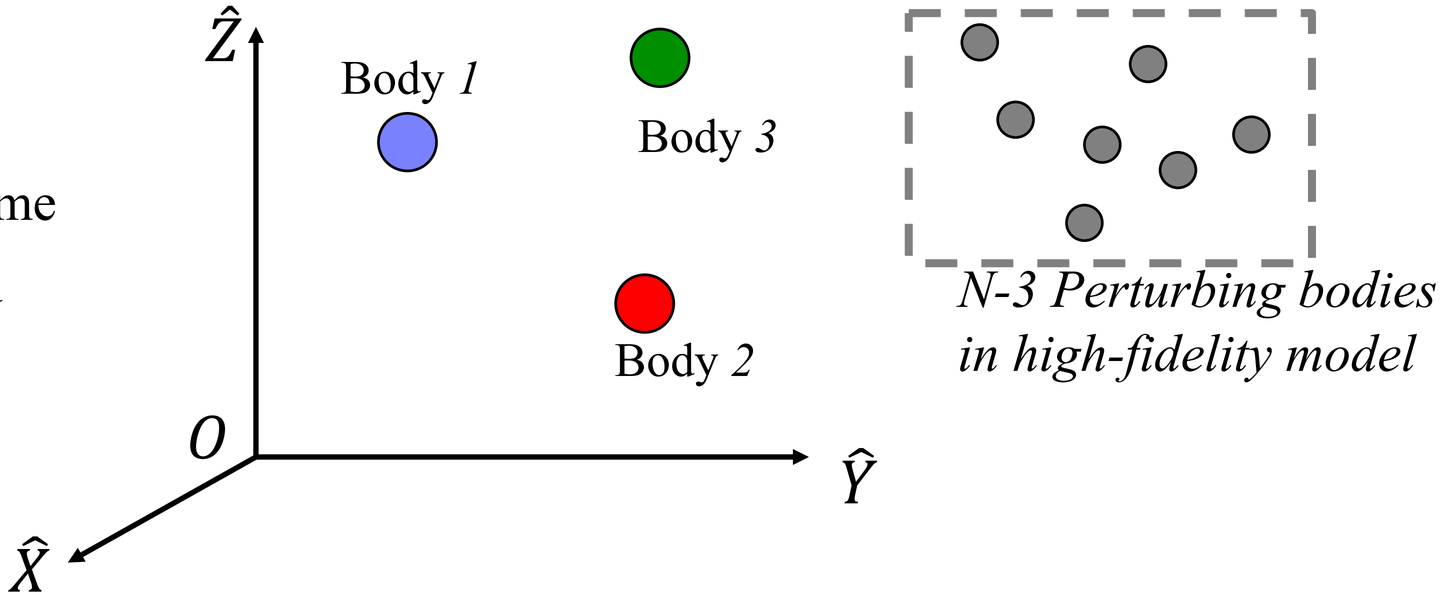
- Define the CR3BP
- Introduce the assumptions and nondimensionalization
- Derive the EOMs and constant of motion
- Introduce types of motion

Multi-Body Problems

Consider the following configuration of bodies, where each body can follow any general path

$\hat{X}\hat{Y}\hat{Z}$: Inertial frame

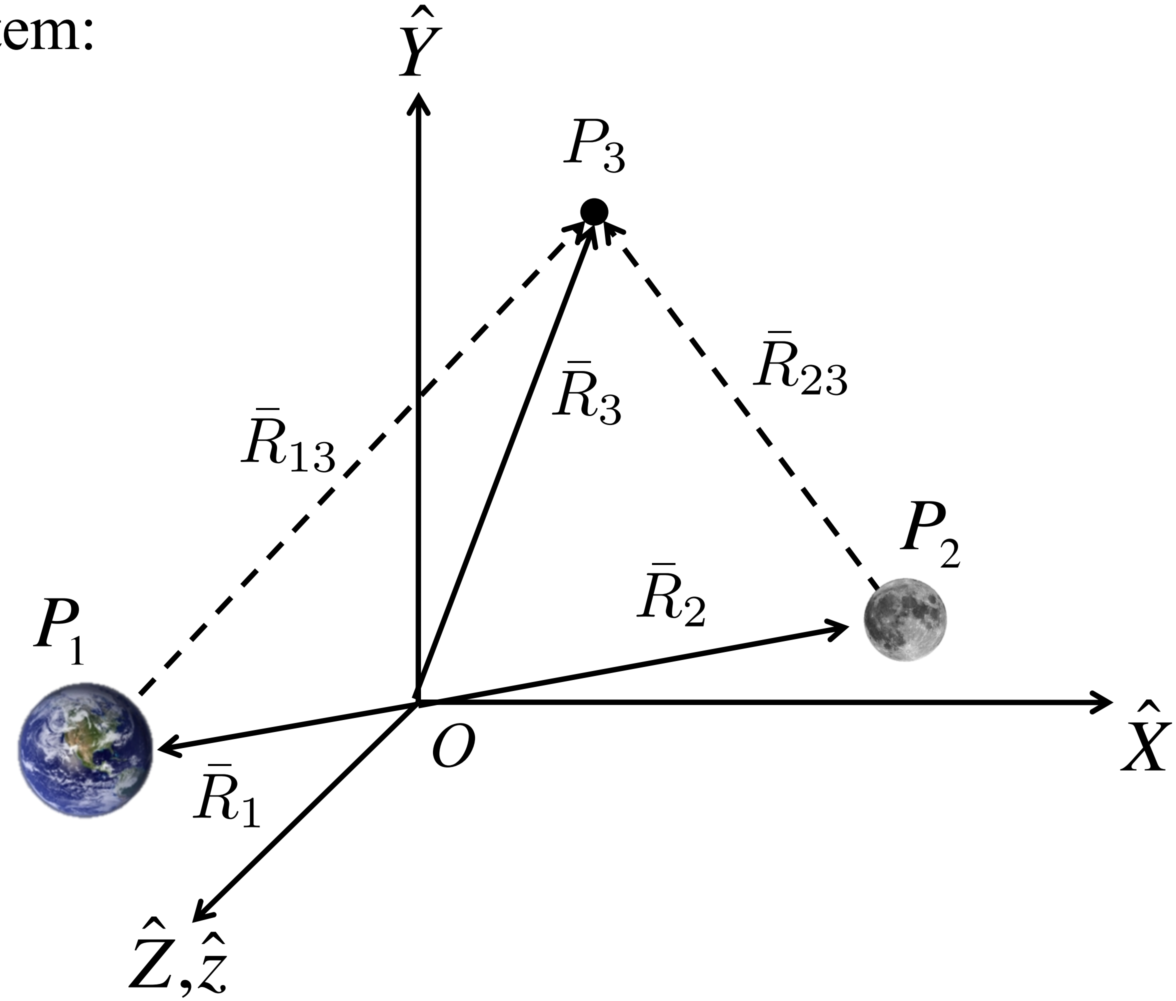
O, Inertially fixed basepoint



Interested in the motion of Body 3 relative to Body 1 and Body 2 (e.g. spacecraft in Earth-Moon system)

Three-Body Problem

Reduce the complexity of this model by assuming only three bodies in the system:



Three-Body Problem

Derive the equations of motion using the potential function for Body 3 due to the gravity of Body 1 and Body 2:

$$\tilde{U}_3 = \frac{\tilde{G}\tilde{M}_1}{\tilde{R}_{13}} + \frac{\tilde{G}\tilde{M}_2}{\tilde{R}_{23}}$$

Where the tilde denotes dimensional quantities

To find the equations of motion (EOMs), note the force per unit mass acting on Body 3 is written as:

$$\bar{\tilde{F}}_3 = \bar{\nabla}\tilde{U}_3 = \bar{\tilde{R}}_3''$$

And assume: $\bar{\tilde{R}}_3'' = \tilde{X}\hat{X} + \tilde{Y}\hat{Y} + \tilde{Z}\hat{Z}$

Three-Body Problem

Taking these derivatives, we find the EOMs are:

$$\frac{\partial \tilde{U}_3}{\partial \tilde{X}} = \tilde{X}_3'' = \frac{\partial}{\partial \tilde{X}} \left(\frac{\tilde{G}\tilde{M}_1}{\tilde{R}_{13}} + \frac{\tilde{G}\tilde{M}_2}{\tilde{R}_{23}} \right) = -\frac{\tilde{G}\tilde{M}_1(\tilde{X} - \tilde{X}_1)}{\tilde{R}_{13}^3} - \frac{\tilde{G}\tilde{M}_2(\tilde{X} - \tilde{X}_2)}{\tilde{R}_{23}^3}$$

$$\frac{\partial \tilde{U}_3}{\partial \tilde{Y}} = \tilde{Y}_3'' = \frac{\partial}{\partial \tilde{Y}} \left(\frac{\tilde{G}\tilde{M}_1}{\tilde{R}_{13}} + \frac{\tilde{G}\tilde{M}_2}{\tilde{R}_{23}} \right) = -\frac{\tilde{G}\tilde{M}_1(\tilde{Y} - \tilde{Y}_1)}{\tilde{R}_{13}^3} - \frac{\tilde{G}\tilde{M}_2(\tilde{Y} - \tilde{Y}_2)}{\tilde{R}_{23}^3}$$

$$\frac{\partial \tilde{U}_3}{\partial \tilde{Z}} = \tilde{Z}_3'' = \frac{\partial}{\partial \tilde{Z}} \left(\frac{\tilde{G}\tilde{M}_1}{\tilde{R}_{13}} + \frac{\tilde{G}\tilde{M}_2}{\tilde{R}_{23}} \right) = -\frac{\tilde{G}\tilde{M}_1(\tilde{Z} - \tilde{Z}_1)}{\tilde{R}_{13}^3} - \frac{\tilde{G}\tilde{M}_2(\tilde{Z} - \tilde{Z}_2)}{\tilde{R}_{23}^3}$$

Simplifying Assumptions

To reduce the complexity of modeling and analyzing the dynamics, introduce the following assumptions:

1. Mass of P_3 << Masses of P_1, P_2
2. P_3 does not influence the paths of P_1 and P_2, so both primaries travel on conics about their mutual barycenter

3. P_1 and P_2 follow circular orbits
4. Model P_1, P_2 and P_3 as point masses

When these assumptions are used, we call the dynamical model the

Nondimensionalization

In practice, time, mass and length quantities can have very different orders of magnitude. Also useful to compare quantities with similar ratios of the mass of P_1 and P_2.

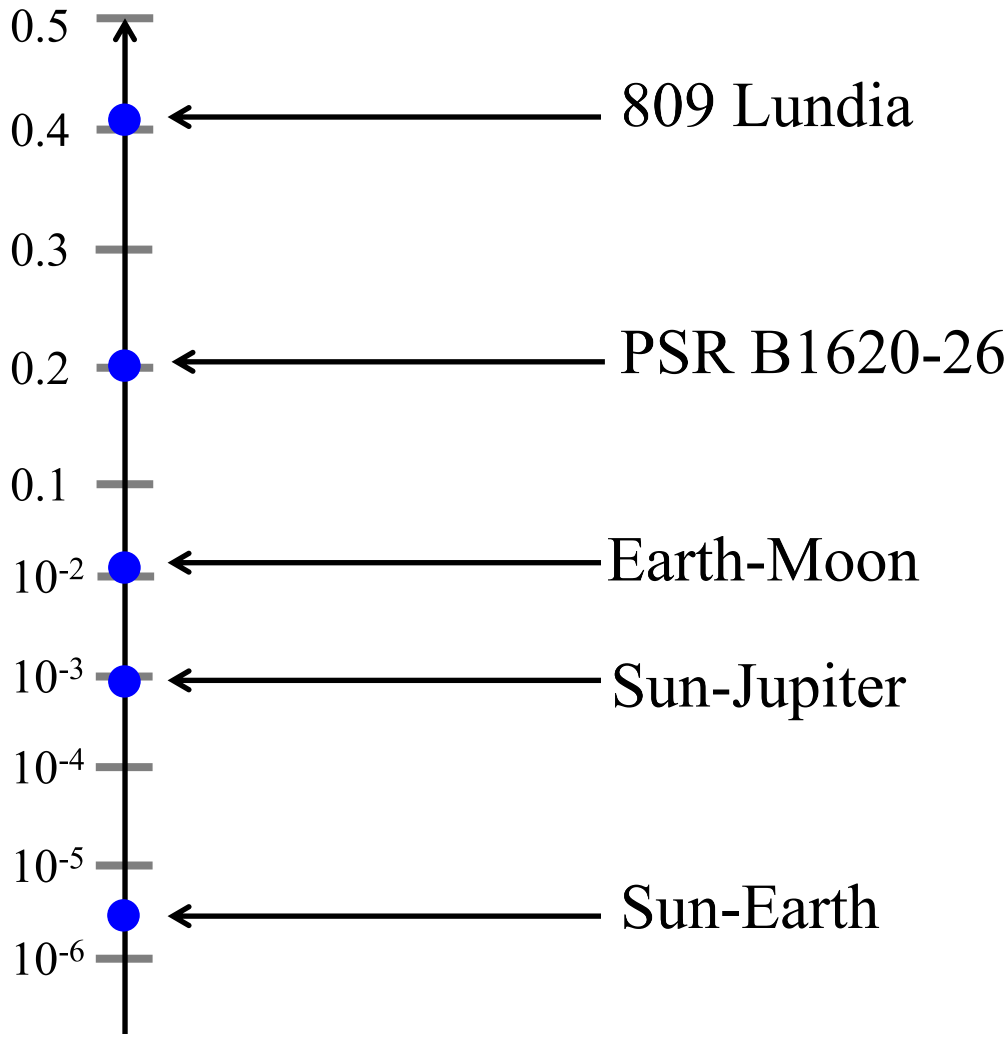
Introduce characteristics quantities m^*, l^*, t^* and nondimensional quantities (no units) have no tilde.

1. Mass: $$m^* = \tilde{M}_1 + \tilde{M}_2$$

$$\mu = M_2 = \frac{\tilde{M}_2}{m^*} \qquad\qquad 1 - \mu = M_1 = \frac{\tilde{M}_1}{m^*}$$

Common Mass Ratios

$$\mu = \frac{\tilde{M}_2}{\tilde{M}_1 + \tilde{M}_2}$$



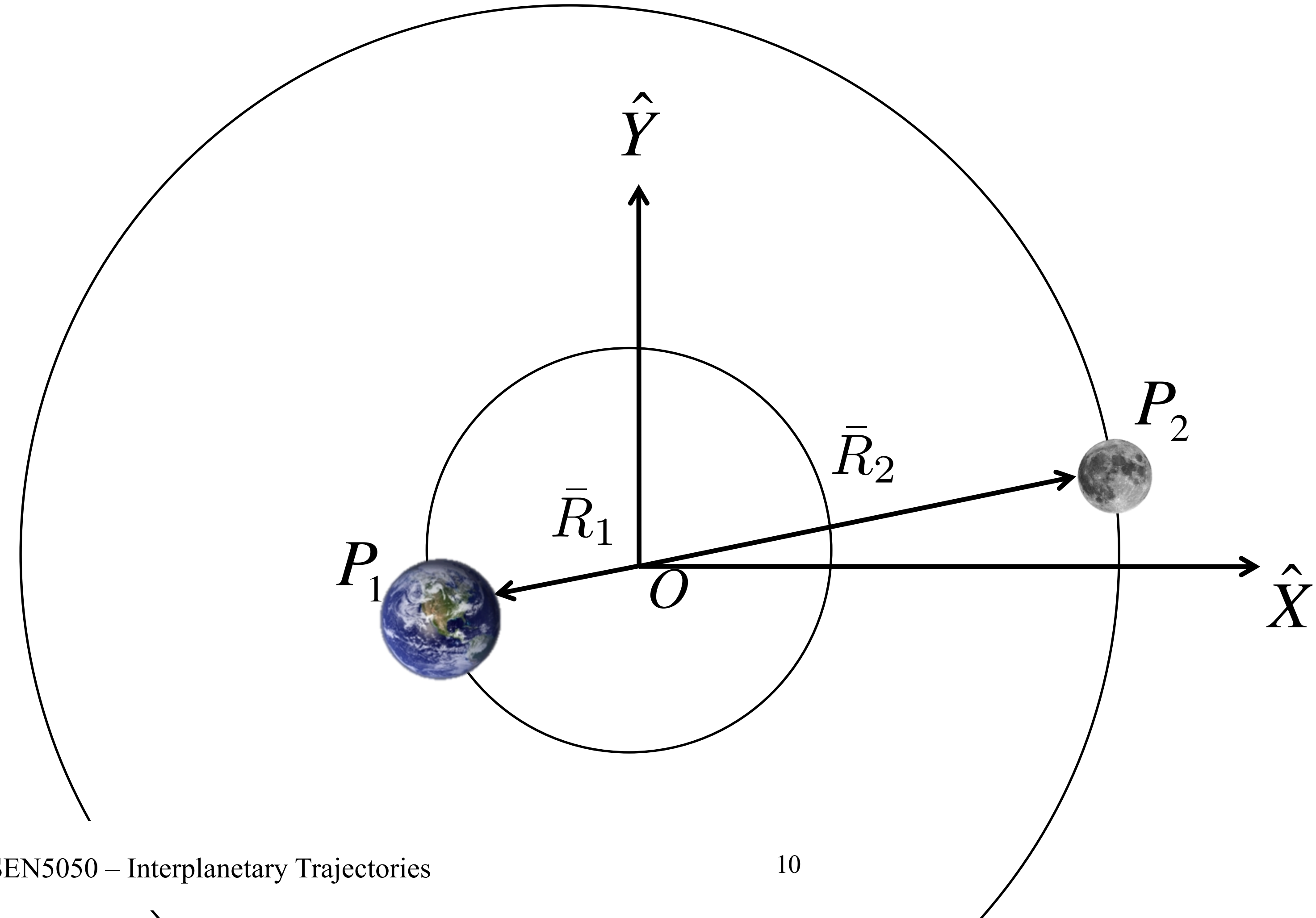
Nondimensionalization

2. Length $$l^* = \tilde{R}_1 + \tilde{R}_2$$

$$R_{13} = \frac{\tilde{R}_{13}}{l^*} \qquad R_{23} = \frac{\tilde{R}_{23}}{l^*}$$

3. Time $$t^* = \left(\frac{(l^*)^3}{\tilde{G}m^*} \right)^{1/2}$$

Path of Primaries



CR3BP EOMs in Inertial Frame

Incorporating the assumptions and nondimensionalization, the equations of motion become:

$$X'' = -\frac{(1-\mu)(X+\mu\cos(t))}{R_{13}^3} - \frac{\mu(X-(1-\mu)\cos(t))}{R_{23}^3}$$

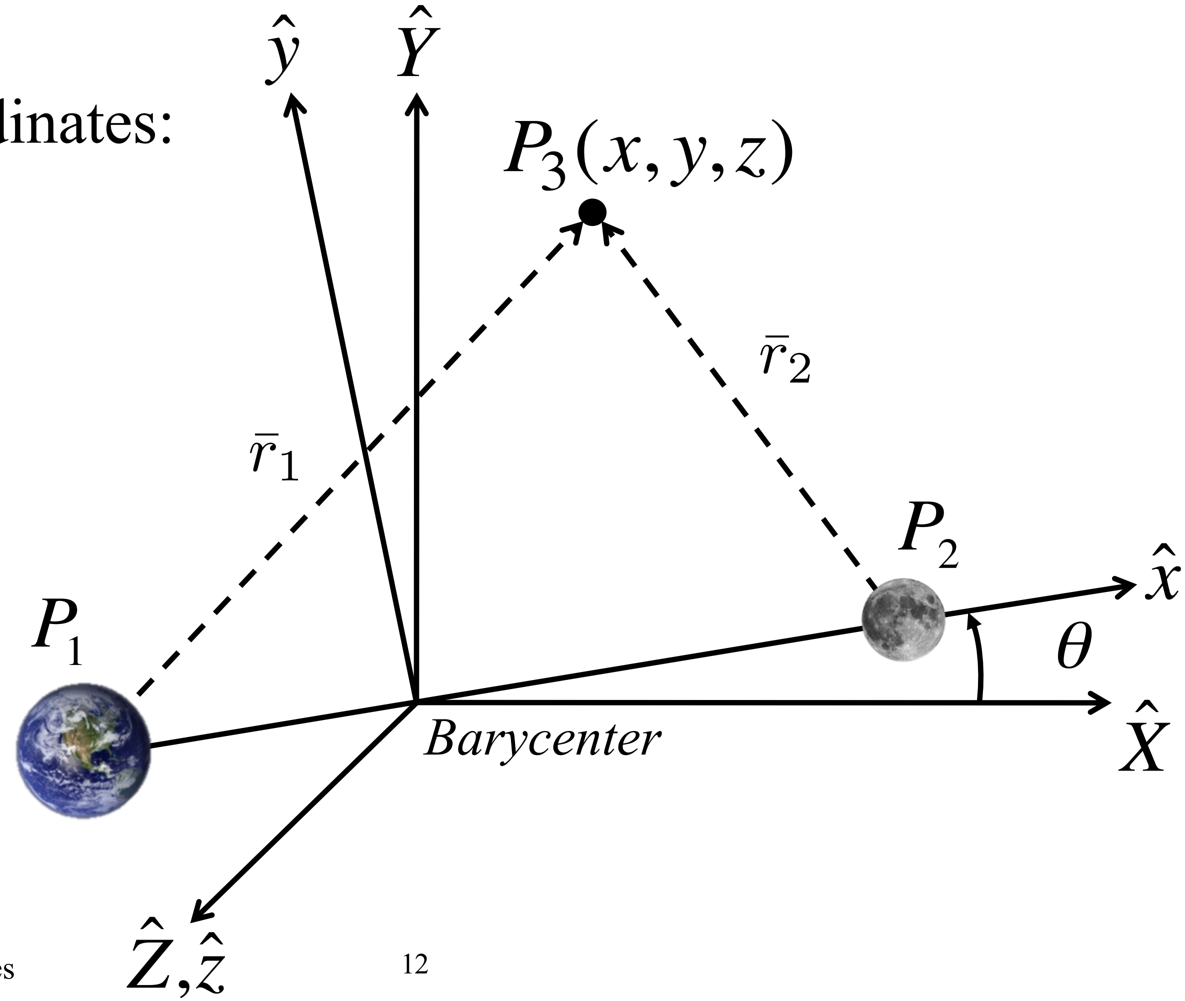
$$Y'' = -\frac{(1-\mu)(Y+\mu\sin(t))}{R_{13}^3} - \frac{\mu(Y-(1-\mu)\sin(t))}{R_{23}^3}$$

$$Z'' = -\frac{(1-\mu)(Z)}{R_{13}^3} - \frac{\mu(Z)}{R_{23}^3}$$

Rotating Frame

Introduce a frame that rotates with the two primaries to remove the time-dependent terms and simplify visualization

Lowercase coordinates: correspond to rotating frame



Acceleration in Rotating Frame

Must rewrite acceleration using terms with a rotating observer

$(\dot{x})$ = derivative with observer in rotating frame

$$\bar{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\bar{v}^I = \bar{r}' = \dot{\bar{r}} + {}^I\bar{\omega}^R \times \bar{r}$$

$$\bar{a}^I = \bar{r}'' = \dot{\bar{v}} + {}^I\bar{\omega}^R \times \bar{v}$$

$${}^I\bar{\omega}^R = n\hat{z} = +1\hat{z}$$

CR3BP EOMs in Rotating Frame

Converting the equations of motion to the rotating frame recovers:

$$\ddot{x} = 2\dot{y} + x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3}$$

$$\ddot{y} = -2\dot{x} + y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3}$$

$$\ddot{z} = -\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3}$$

Where: $r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2}$

$$r_2 = \sqrt{(x-1+\mu)^2 + y^2 + z^2}$$

Definition of Pseudo-Potential Function

Define a "pseudo-potential", which is similar to a potential function and depends only on position variables but incorporates an additional term due to the rotation of the rotating frame

$$\ddot{x} = 2\dot{y} + \boxed{x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3}}$$

$$\ddot{y} = -2\dot{x} + \boxed{y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3}}$$

$$\ddot{z} = \boxed{-\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3}}$$

0 =

Definition of Pseudo-Potential Function

The pseudo-potential function which satisfies these constraints is:

Producing compact equations of motion:

$$\ddot{x} - 2\dot{y} = \frac{\partial U^*}{\partial x}$$

$$\ddot{y} + 2\dot{x} = \frac{\partial U^*}{\partial y}$$

$$\ddot{z} = \frac{\partial U^*}{\partial z}$$

Constant of Motion

From conservation of energy, a constant energy integral exists for these autonomous equations of motion

Dot product of acceleration and velocity vectors:

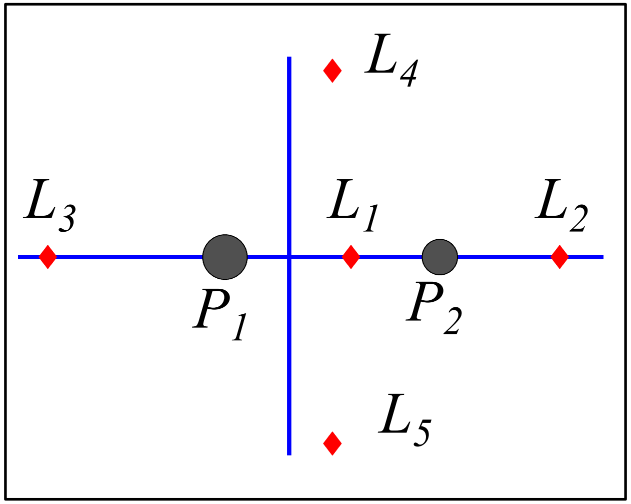
$$\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} = \frac{\partial U^*}{\partial x}\dot{x} + \frac{\partial U^*}{\partial y}\dot{y} + \frac{\partial U^*}{\partial z}\dot{z}$$

Integrating:

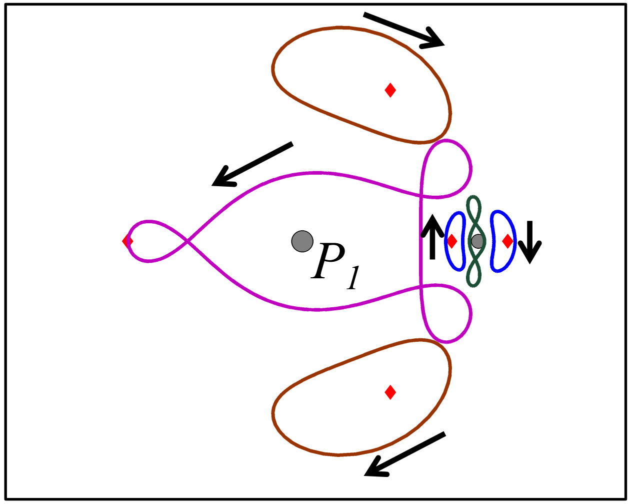
$$\frac{1}{2}v^2 = \int dU^* - \frac{dU^*}{dt} = U^* - \frac{1}{2}C_J$$

Types of Solutions

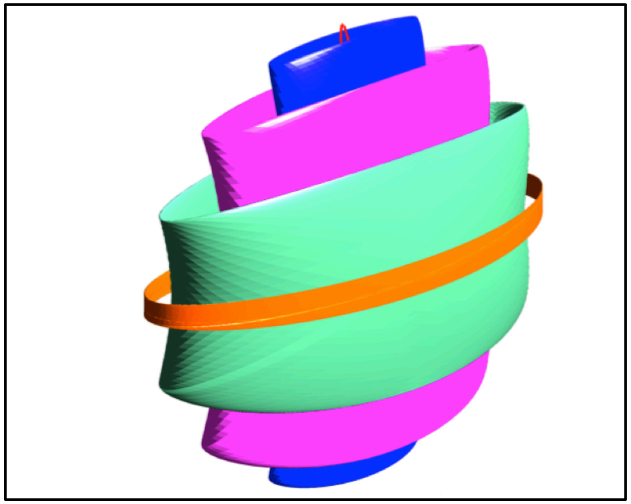
Equilibrium Points



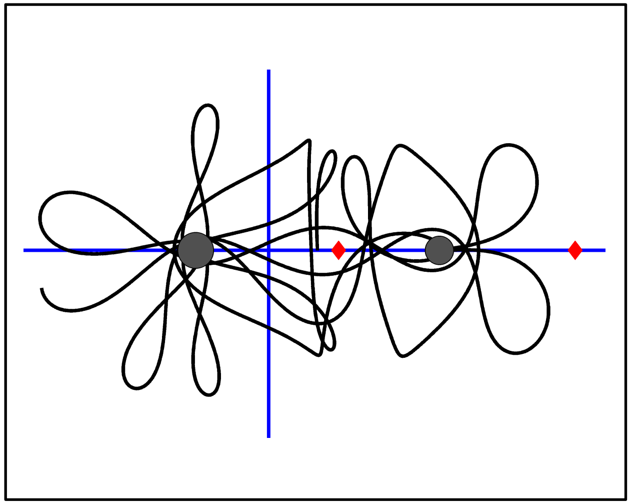
Periodic Orbit



Quasi-Periodic Orbit



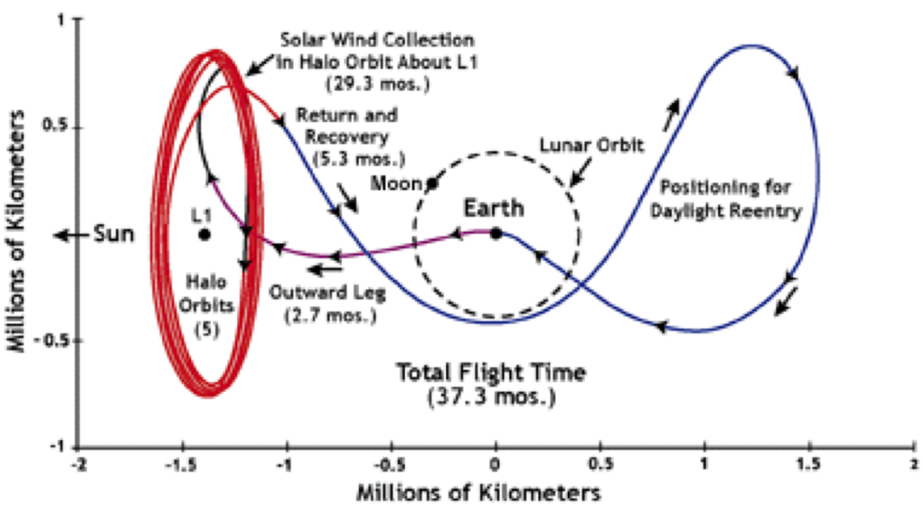
Chaos



Flying the CR3BP

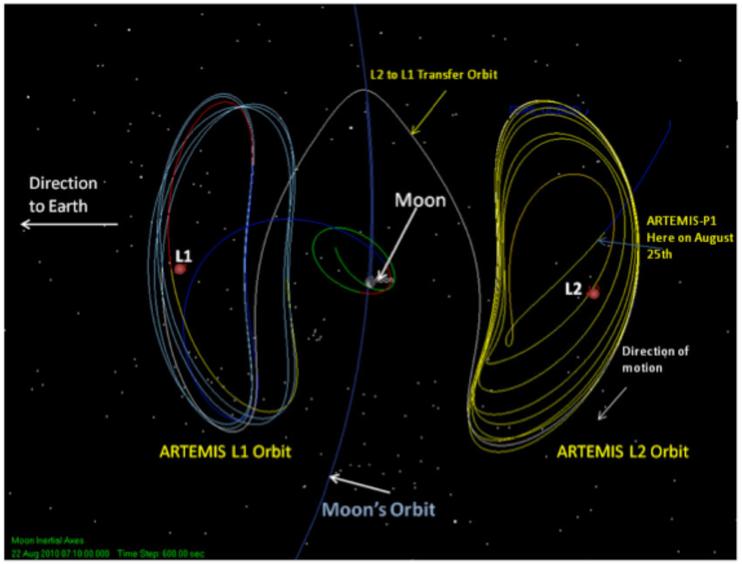
Examples of flown missions:

GENESIS (2001)



Credit: NASA/JPL-Caltech
Purdue University

ARTEMIS (2009, Ext. of THEMIS)

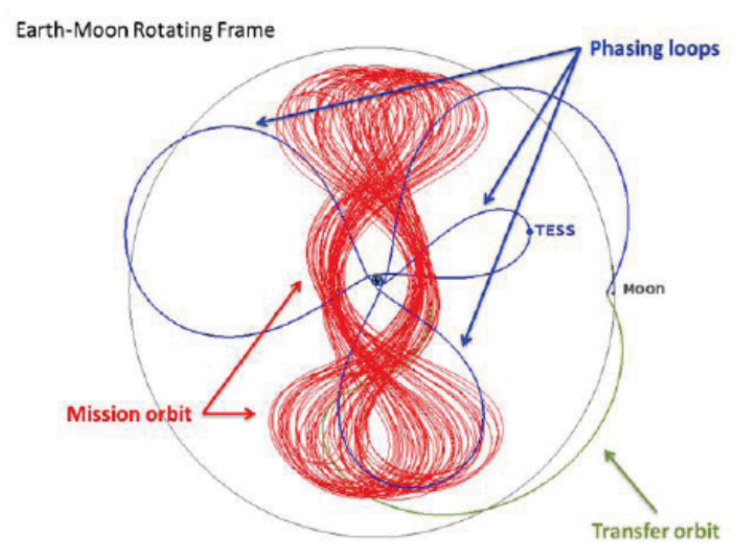


Credit: NASA/GSFC
Purdue University

Flying the CR3BP

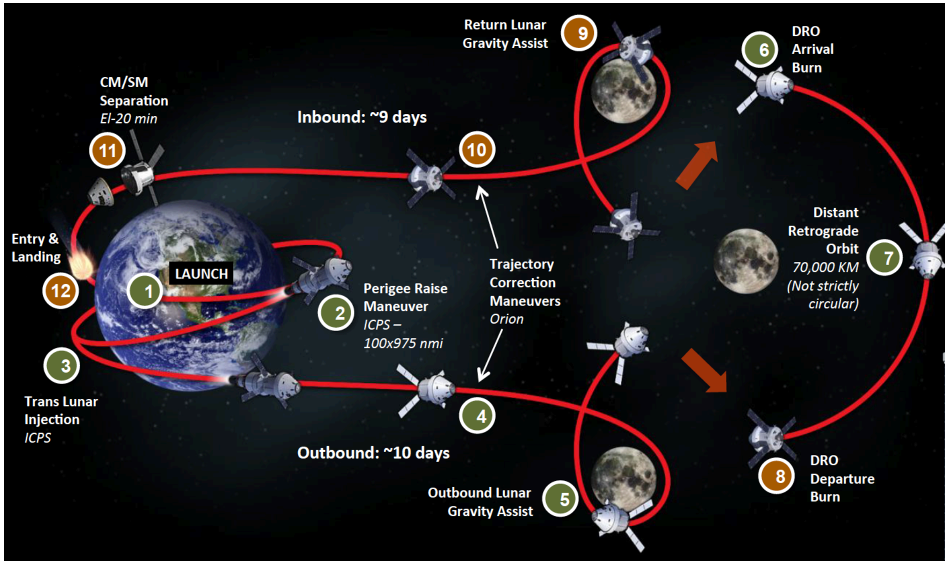
Examples of missions coming soon:

TESS (Est. 2018)



Credit: Dichmann et al, 2014

Exploration Mission-1 (Est. 2019)



Credit: NASA/GSFC