Computational Lab 5: Flow Over Finite Wings

Connor Ott - 104690690

University of Colorado - Boulder

Introduction

With the introduction of finite wings, methods different than those used for airfoils must be implement to approximate lift and drag on aerodynamic bodies. Mainly, effects stemming from the presence of the wingtips drastically changes the lift distribution over the wing and must be accounted for accordingly. Prandtl Lifting Line Theory (PLLT) models these changes by superposing vortex filaments of varying strength along the span of a finite wing. This results in a more appropriate approximation of lift. Moreover, the model is extremely robust, and is able to model tapered wings and wings with aerodynamic and geometric twist. The crux of this model is a Fourier expansion of the function for circulation, Γ, which varies along the span of the wing. Solving for this expansion allows for the calculation of the span efficiency factor of the wing, and subsequently the lift and induced drag acting on the wing. This lab explores the accuracy of this expansion, and specifically, it determines the number of terms in the expansion required for different values of relative error in lift and induced drag. Additionally, this lab will delve into the effect of taper ratio and aspect ratio on a finite wing's span efficiency factor.

Theory

The circulation function described in the introduction is determined by generating and then solving a system of equations for the coefficients of each term in the Fourier expansion. Equation 1 is the Fundamental Equation of PLLT, adjusted to replace Γ with a finite sum of sine terms.[1] Note that the indexing in each sum removes any even terms from the series. This ensures a symmetric circulation distribution about the midspan of the wing, which, in turn, ensures a symmetric lift distribution. Henceforth, any reference to N refers to the number of odd terms in the series.

$$\alpha_{geo}(\theta) = \frac{4b(\theta)}{a_0(\theta)c(\theta)} \sum_{n=1}^{N} A_{2n-1} sin((2n-1)\theta) + \alpha_{L=0}(\theta) + \sum_{n=1}^{N} (2n-1) A_{2n-1} \frac{sin((2n-1)\theta)}{sin(\theta)} \tag{1}$$

With the above equation, N equations can be generated through plugging in N unique values for θ between 0 and 2π. With these, the coefficients of each term in the series can be solved for. This operation was carried out in MATLAB using the function *prandtlLLT.m* which generates and solves the system of equations for the coefficients based on wing characteristics. These are . These coefficients can then be used to determine the span efficiency factor, lift coefficient, and induced drag coefficient for the wing in question. The equations for these values can be found below in Equations 2.

$$C_L = A_1 \pi AR \qquad e = \left(1 + \sum_{n=2}^{N} (2n-1) \left[\frac{A_{2n-1}}{A_1}\right]^2\right)^{-1} \qquad C_{D_i} = \frac{C_L^2}{\pi e AR} \tag{2}$$

Results

Figure 1 shows the decrease in error seen by both lift and drag as more odd terms are added to the Fourier expansion of $\Gamma(\theta)$. The lines of best fit are also included to help describe the trends' behaviors mathematically. The tabulated results indicating terms required for 5%, 1%, and 0.1% error are displayed in Table 1. These exact results were gathered for a thick wing with aerodynamic twist and geometric twist, an example of the robustness of PLLT.

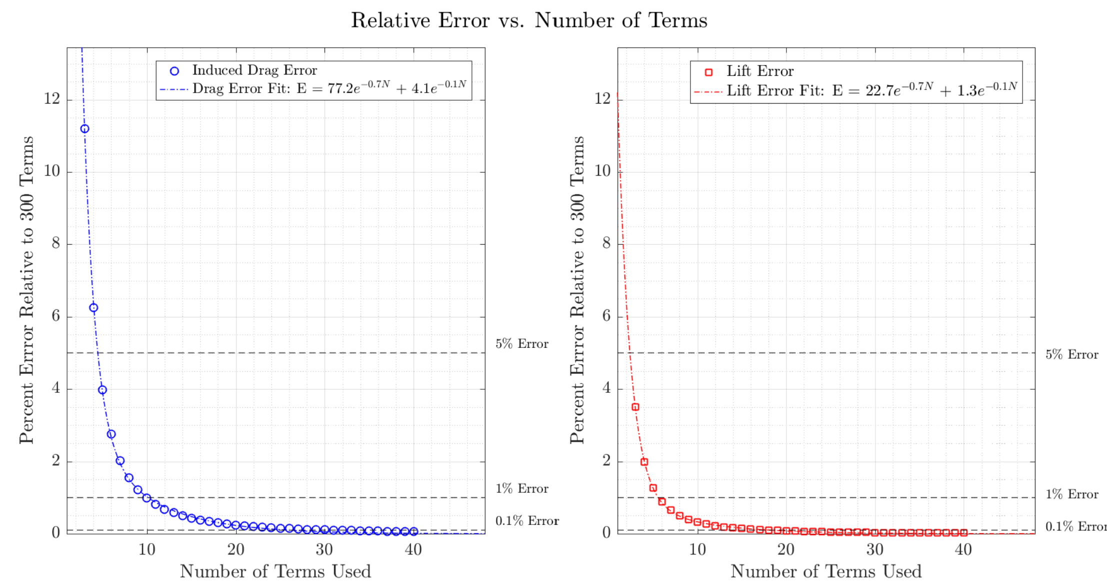

Figure 1: Relative error for lift and drag with varying number of terms in Fourier expansion. Also includes lines of best fit for both trends.

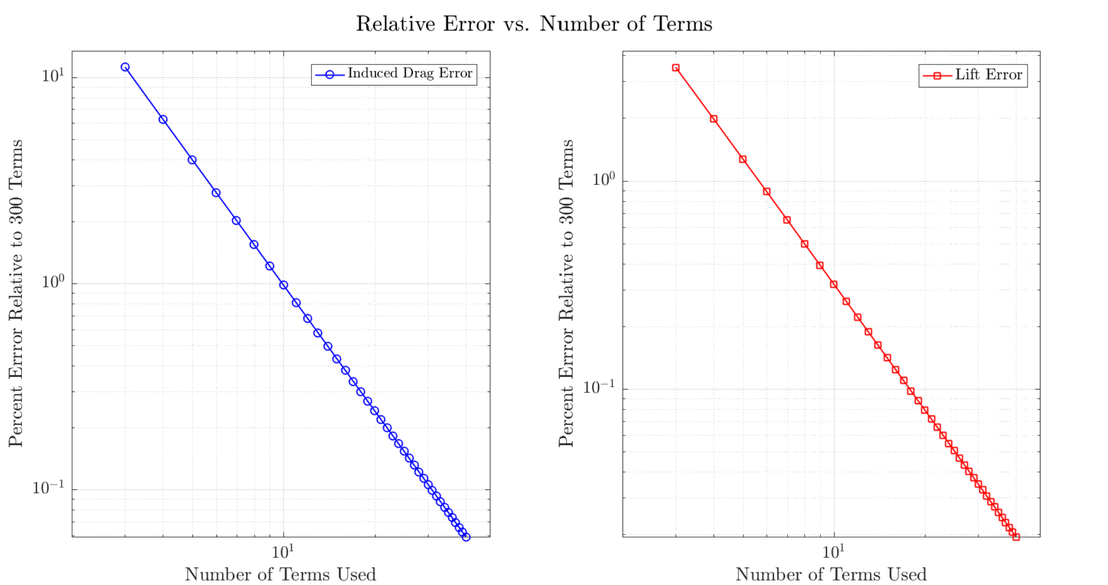

Figure 2: Relative error for lift and drag on a logarithmic scale to emphasize exponential decay of the error with increasing number of terms.

Table 1: Number of odd terms required in Fourier expansion to achieve specific relative error values.

Relative Error	Lift	Induced Drag
5%	3	5
1%	6	10
0.1%	18	31
Exact Value	6033.2 [lbs]	179.4 [lbs]

Figure 3 indicates how span efficiency factor, e, varies with different wing taper ratios and aspect ratios. This plot was gathered by setting wing planform area to a constant and determining wing characteristic for the different aspect ratios and taper ratios seen in the figure.

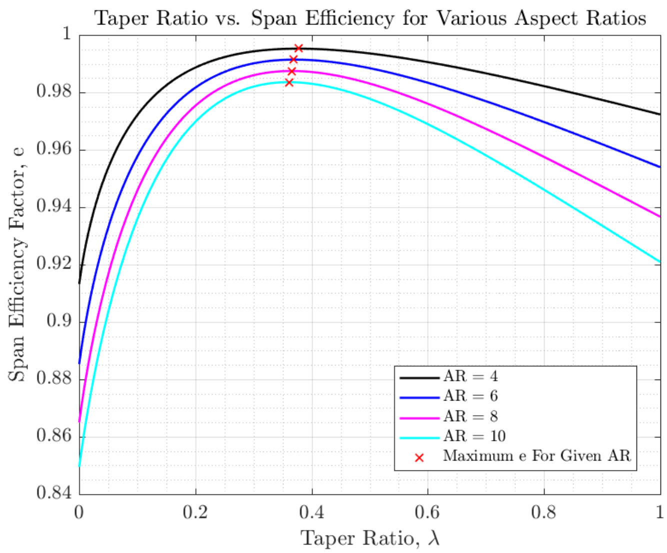

Figure 3: Relationship between taper ratio and span efficiency factor for various aspect ratio wings.

Discussion

Figure 1 shows that PLLT is capable of achieving extremely high accuracy with relatively few terms. The line of best fit plotted over the experimental data implies that the rate of decay of error is exponential in nature. This is corroborated by Figure 2 since the trend appears linear on a logarithmic scale, which is indicative of an exponential relationship. This is great news for scientists and engineers look for a computationally inexpensive and accurate way of approximating finite wing characteristics. It's worth noting that different numbers of terms are required to achieve the same accuracy for lift and drag. This may be due to the fact that induced drag is effectively the component of lift parallel to the free stream, meaning induced drag is a function of lift. Any error in lift calculation will then be propagated through the drag calculation, resulting in error additional to any innate error associated with drag.

Figure 3 immediately illustrates that there is an optimal taper ratio for a wing with a given aspect ratio. This varies slightly between aspect ratios, but is ultimately about 0.3685. The fact that this varies so little between aspect ratios means that a taper ratio of 0.3685 for a given trapezoidal wing will result in a maximum span efficiency factor. However, it's important to realize that near the peak of the curves in Figure 3, slope is fairly shallow. This indicates that a taper ratio in the range of 0.3, to 0.4 will also yield good results, with very little loss in span efficiency. The figure also indicates that, for low taper ratios ($\lambda < 0.15$), taper ratio has a much greater affect on span efficiency. Conversely, at higher taper ratios, aspect ratio takes over. This is evident by the convergence of the aspect ratio curves at lower taper ratios, and divergence of the aspect ratio curves at higher taper ratios.

Figure 3 also indicates that increasing aspect ratio results in decreased span efficiency. This makes it tempting to say that low aspect ratios should be used on planes. This is obviously not the way to go about making a plane since so many have large aspect ratio wings. Looking back at Figure 2, it can be seen that increasing the span efficiency factor and aspect ratio will result in a decrease in induced drag. Note also that 2.5x increase in aspect ratio results in extremely small drops in span efficiency, especially at the optimal taper ratio. The difference being on the order of 1%. This 2.5x increase in AR combined with 1% decrease in e yields a 2.47x decrease in drag. Because the benefit of a high aspect ratio greatly outweighs the loss in span efficiency it causes, the choice between the two becomes clear for the purpose of decreasing drag.

Acknowledgements

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References

[1]Evans, J. Dr., *ASEN 311 Computational Lab #5: Flow Over Finite Wings*, `https://learn.colorado.edu/d2l/le/content/217338/viewContent/3255145/View?ou=217338`