Fourier and Wavelet Transforms effect on Signal Processing

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Abstract

The Fourier transform can provide the entire frequency content of a signal but will erase all the time information. While this might be sufficient for several use cases, it won't be sufficient for all signal processing problems. In order to preserve time information in the frequency domain, the Fourier transform must be accompanied with a Gabor transform or wavelet. These transforms are studied with two separate signals. The first signal is studied in depth to understand the capability of the Gabor transform, and observe how similar wavelets behave. The other signal is two recordings of a common nursery rhyme, where the musical score is produced based on the frequency data. The two recordings were done on different instruments, and the difference in the instruments sound properties can be seen in the spectograms of the signal.

1 Introduction and Overview

The fast Fourier transform captures all data and transforms it into a pure frequency domain. This can be very useful information but does not provide the complete picture of the image. In the process of transforming the data we lose all the details of the time domain. With the use of the Gabor transform and wavelets, time information can be kept and added to the frequency data from the Fourier transform.

The first part of this analysis examines different ways of processing an audio file of Handel's Messiah. The Gabor transform was the base of examining the spectograms of the audio. A visual interpolation of the spectogram was used to find the "best" resolution. The window width, and the sampling rate of the Gabor transform were adjusted to obtain the image that contained the best resolution without sacrificing necessary frequencies. With an understanding of the effects of the Gabor transform on this signal, two other wavelets are tested. The Mexican Hat, and Shannon wavelets are analyzed and compared to the Gabor transform with similar parameters.

In the second part of the analysis, two different audio files of the same music but played on different instruments are compared. Each file is a recording of Mary Had a Little Lamb, where one is played on the piano and the other is played on the recorder. Spectograms are made of each file and the frequency components of each can be directly compared.

2 Theoretical Background

2.1 Gabor Transform

The Fourier transform can translate a signal of time into its frequency components. The Fourier transform is defined by:

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} f(x)dx \tag{1}$$

Since the Fourier transform looses the time domain of the signal, a windowed Fourier analysis with the Gabor transform is used to capture the time component. The Gabor transform:

$$g(\tau) = e^{-\omega*(t-\tau)^2} \tag{2}$$

Here ω is the window width of the Gabor transform. The Gabor transform is a Gaussian function which, when applied to a signal, cancels out everything in the signal outside the window. This window is slid across the signal by varying τ and for each sample, the Fourier transform is applied and the data can be stacked across all windows to give both time and frequency resolution.

2.2 Wavelet

The Gabor transform only allows for the adjustment of the window width, and sampling rate. These parameters are sometimes not sufficient for every data set. Another transform, called a wavelet transform, can be constructed to have special properties for the type of data being analyzed. Any type of transformation can be used as long as it satisfies the conditions of the Continuous Wavelet Transform (CWT):

$$C_\Psi = \int_{-\infty}^{\infty} \frac{|\hat{\Psi}(\omega)|^2}{|\omega|} d\omega < \infty \tag{3}$$

In this analysis two wavelets are used to supplement the analysis initially done by the Gabor transform. The first wavelet is called the Mexican Hat Wavelet:

$$\Psi(t) = (1 - t^2)e^{-t^2/2} \tag{4}$$

The other wavelet used is the Shannon wavelet which is just a value of 1 for the window width and 0 everywhere else. These wavelets change the frequencies chosen during the windowed Fourier transform and can help change the ratio of frequency vs time resolution in the Fourier domain.

2.3 Spectogram

A spectogram is a way to visual the frequencies of a signal over time. They are typically shown as a type of heat map where the "hotter" signals are the frequencies that are found within the signal. They are plotted such that the y-axis is the Frequency in Hz and the x-axis is the time axis. The spectogram of each transformation of the signals is used to visually compare the transformations. A visual observation of the spectogram is also how the window and sampling parameters are adjusted. Changing one of these parameters usually adjusts the clarity and the amount of signals present so a visually analysis is sufficient for parameter determination.

3 Algorithm Implementation and Development

There were many parts of this analysis but each part all followed a basic structure. The data would be imported, then transformed with a sliding windowed approach to get the frequency and time data. This data was then combined and plotted into a spectogram to be visually inspected. The loop in Algorithm 1 is across the whole time domain, and the number of points sampled is the sampling rate.

Algorithm 1: Transformations

```
Import sound data and initialize parameters
for j = 1 : L do
  Create sliding Gabor transform or wavelet
  Apply transform to signal
  Take FFT of the transformed signal
  Store FFT data for plotting
end for
Plot the spectogram of the data
```

This algorithm was then repeated with different transform window widths, and different sampling rates to obtain the most visually appeasing spectogram. For each wavelet, transformation, and signal looked at in this analysis this same algorithm was used.

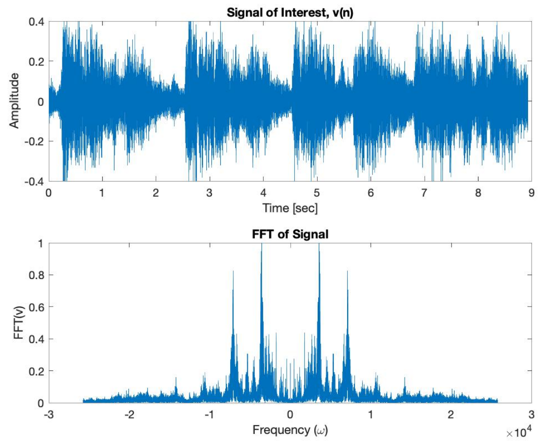

Figure 1: Graph of raw signal and FFT of signal

4 Computational Results

4.1 Part I

The first part we are looking at a snippet of Handel's Messiah. In Figure 1 the raw signal is plotted, along with the FFT of the signal to show how a few frequencies stand out. In the FFT, there are two very distinct frequency peaks, since the FFT is symmetric this corresponds to two unique frequency signatures. These frequencies corresponds to the men's and women's voices in the sample data.

To extract some more information from the signal, the Gabor transform along with the windowed Fourier transform can provide us with some information about the frequencies over time. The first adjusted parameter is the Gabor window width. Figure 2 shows three different window widths tested. With the larger the window width parameter, narrower signal, we get better resolution of the frequencies but we also loose some of the non-core frequency components. From these spectograms we can see the two major frequencies reflected across the origin. We are also able to visualize some of the time component information. These spectograms are starting to show the intensity of the frequencies over time, which corresponds to the larger amplitudes seen in the raw signal in Figure 1. In each of these spectograms the sampling rate was held constant at 80.

The other parameter we can vary alongside window width is the sampling rate. From a visual perspective, the smallest signal width (window width of 70), showed the clearest picture so this value was used for varying the sampling rate. Figure 3 shows how the higher the sampling rate, the clearer the picture that is given. There is point where we start seeing diminishing returns by high computational times with a higher sampling rate. The sampling rate of 80 seemed to have the best middle ground of computational efficiency and resolution. Varying the sampling rate does seem to have less of an effect than window width on the clarity of the spectograms.

To get a better understanding of how other wavelets can affect the signal, the Mexican Hat wavelet and Shannon wavelet transform are used. The Mexican Hat wavelet is the second derivative of the Gabor transform, which is the second derivative of a Gaussian function. The Shannon wavelet is a step function, where the window has an amplitude of 1 and everything else is 0. For both wavelets the sampling rate was kept at 80 and the window width of the Mexican Hat wavelet was set to 70. The Shannon wavelet window width does not easily compare to the width of the other wavelets. A visual effort was applied to try and match the window width to match the Mexican Hat wavelet. Figure 4 shows how both wavelets fared rather well in creating a readable spectogram. The Mexican Hat wavelet was very similar to the Gabor filter, but the Shannon filter provided a slightly clearer picture of the signal compared to the other methods.

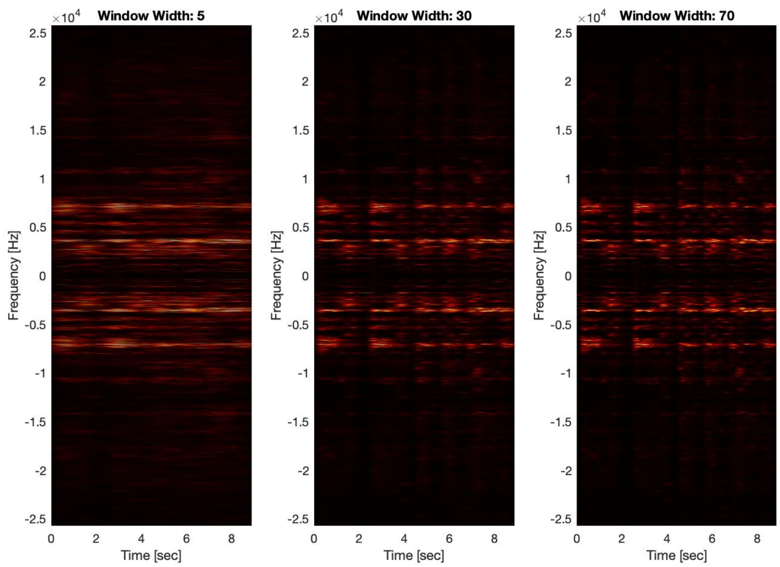

Figure 2: Three different window widths of the Gabor transform with a constant sampling rate

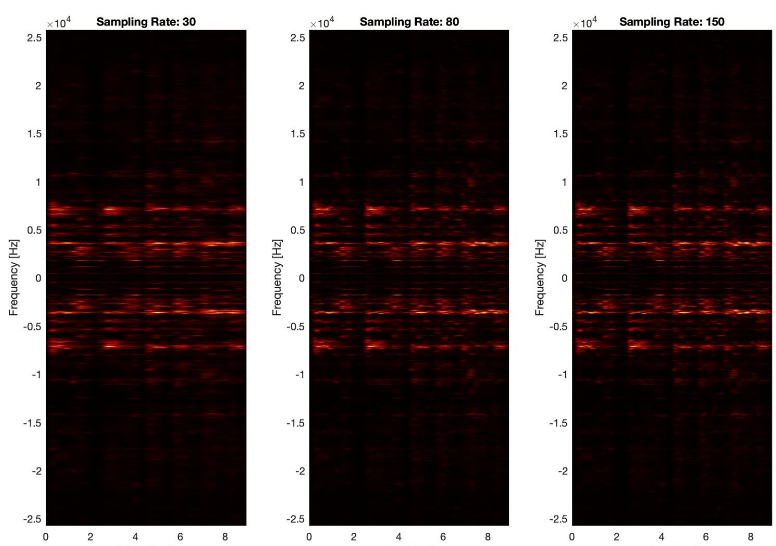

Figure 3: Different sampling rates for a constant window width (70)

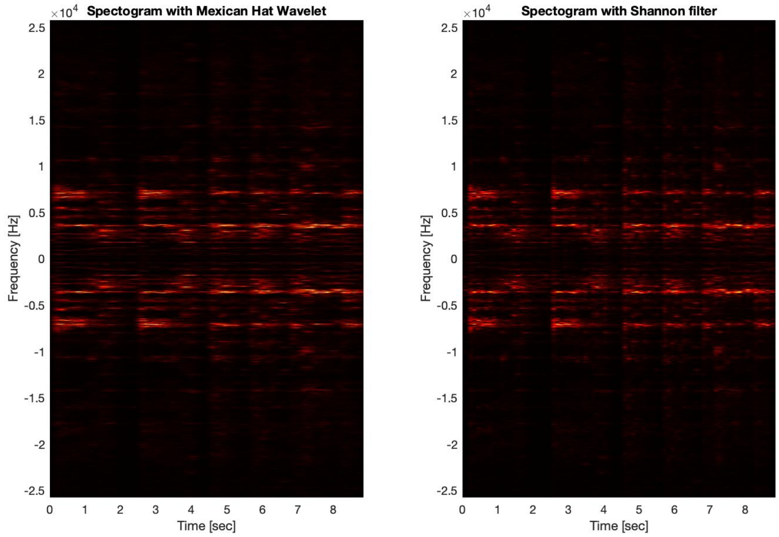

Figure 4: Mexican Hat Wavelet vs Shannon Wavelet

4.2 Part II

The second part of this analysis compares two recordings of the nursery rhyme Mary Had a Little Lamb. The first recording is done on the piano, while the other is done on the recorder. Due to the nature of these instruments there is a fair amount of variability between the signals recorded. Figure 5 compares the two recordings and their corresponding Fourier transforms. Both signals show a similar pattern in their amplitudes which is expected since they follow the same musical score. The transforms show a main frequency emerging, though at different frequency values for the piano compared to the recorder. There are some lesser frequencies shown in the FFT but not much else can be said from just this figure.

In a similar fashion to above, the Gabor transform combined with the windowed Fourier transform will be used to get the time and frequency components out of the raw signal. The spectogram shown in Figure 6, shows the main frequency banded at 2000HZ with the other frequencies creating the rest of the musical scale. Compare this to the spectogram of the recorder, Figure 7 and the musical scale is nearly identical but the absolute frequencies vary greatly. This frequency difference is due to the instruments themselves. Its also interesting to note the range of frequencies compared to the two instruments. The piano is within a 500Hz band but the recorder is spread across a 1500Hz frequency band.

In creating these two spectograms, the overtones had to be removed and the contrast of values had to be exaggerated. To remove the overtones the log of each spectogram was taken, and a scalar value of 1 was added to the spectogram values before the log was taken. This scalar value was used to increase the contrast between signals, and make the important frequencies stand out. For a full comparison, Figure 8 shows the positive half of both spectograms. From this perspective the difference in the frequency values is very clear.

5 Summary and Conclusions

The first part of this analysis was a comparison of a few different techniques to better understand a sample audio recording. All parts of the process used the windowed Fourier transform, but each method used either a different wavelet transform, or varied the sampling rate and window width. By observation, it was noticed that a narrow window width was required to extract the time and frequency data clearly for all techniques. Too narrow of a window width didn't have any increasing effect in resolution so a medium was found to be best. The same was found for sampling rate, too high of a sampling rate and the computational time was too large to justify the modest increase in resolution. The other wavelets provided a similar view to the Gabor transform, but the Shannon wavelet provided a clearer resolution for a broader wavelength. Each method

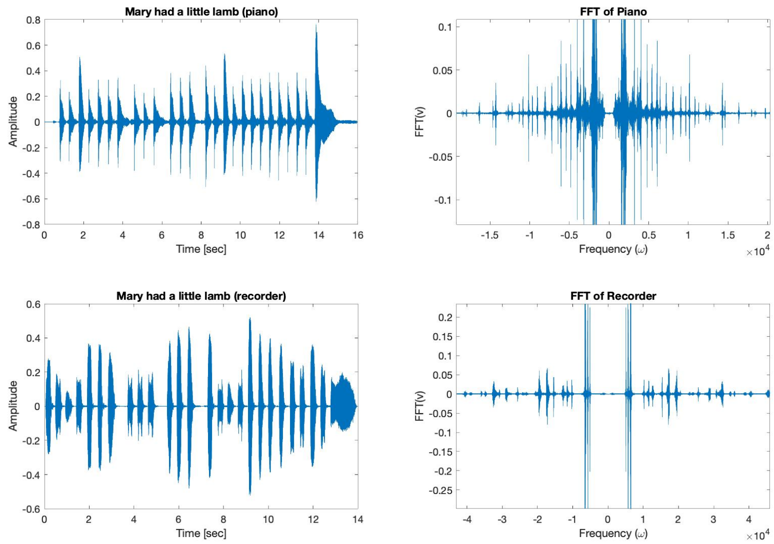

Figure 5: Piano and recorder recording with their respective FFT's

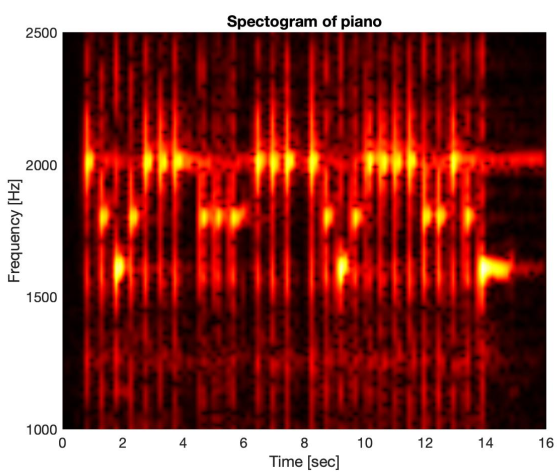

Figure 6: Spectogram of piano recording, zoomed in on important frequencies

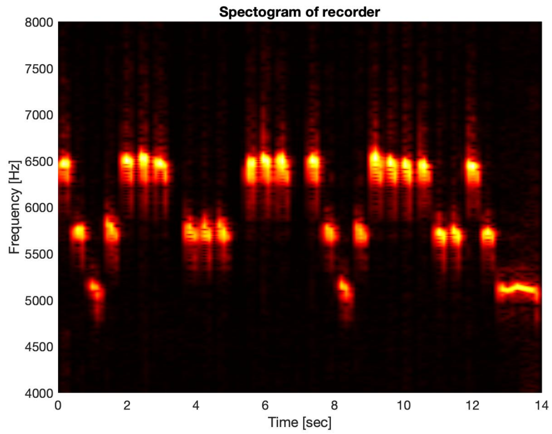

Figure 7: Spectogram of recorder, zoomed in on relevant frequencies

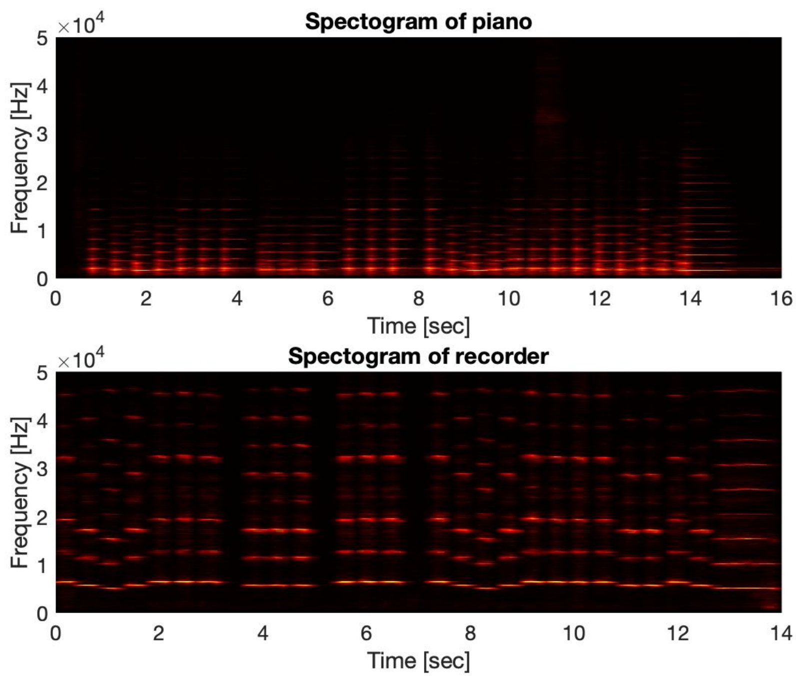

Figure 8: Showing the positive values of the full spectogram for the piano and recorder

created a spectogram that showed the clear frequency bands for both the male and female singers. They also showed that the frequency didn't change during the recording but instead showed the varying intensities corresponding to the pauses in the singing.

In the second part, the piano and recorder audio recordings of Mary Had a Little Lamb was compared. After removing the overtones, the musical score was able to be produced from each recording. The musical score was nearly identical for both instruments, with the main difference being the absolute values of the frequencies. The spectograms were clear enough to show clear step changes in frequencies corresponding to the musical steps in the nursery rhyme.

Appendix A MATLAB Functions

- `fft(V)` creates the two dimensional Fourier Transform of `V` using the fast Fourier transform algorithm. The output is the same size of as the input function and assumed the same dimension of the input function.
- `fftshift(V)` rearranges Fourier transformed data by shifting the zero-frequency component to the center. This is done by alternating half space intervals over the number of dimensions in the input argument.
- `pcolor(x,y,data)` creates the spectograms used in this analysis. The `x, y` values are the x-axis and y-axis. The data is the transformed data, which is a matrix of size `n x m`. `N` is the sampling rate, and `m` is the length of the signal. This "plotting" command will plot the data against the x and y axis while assigning each data point a color value on a defined gradient from the smallest to the highest signal.
- `colormap('hot')` defines the sliding color scale used by the `pcolor` command. The 'hot' command is a color gradient from black, the coldest, to white, the hottest.

Appendix B MATLAB Code

```
% Connor Schleicher AMATH 582 HW 2

% initialize workspace
clear all; close all; clc;

% initialize recording from HW prompt
load handel
v = y'/2;
figure(1)
subplot(2,1,1)
plot((1:length(v))/Fs,v);
xlabel('Time [sec]');
ylabel('Amplitude');
title('Signal of Interest, v(n)');

% initialize fourier modes
L = length(v)/Fs;
n = length(v);
t2=linspace(0,L,n+1); t=t2(1:n);
k=(2*pi/L)*[0:n/2 -n/2:-1];
ks=fftshift(k);


subplot(2,1,2)
```

```
vt = fft(v);
plot(ks,abs(fftshift(vt))/max(abs(vt)));
ylabel('FFT(v)'), xlabel('Frequency (\omega)')
title('FFT of Signal');

% p8 = audioplayer(v,Fs);
% playblocking(p8);

% initialize fourier modes
L = length(v)/Fs;
n = length(v);
t2=linspace(0,L,n+1); t=t2(1:n);
k=(2*pi/L)*[0:n/2 -n/2:-1];
ks=fftshift(k);


%% Spectogram of Handel with Gabor Transform Varying the Window Width

slidew = [5, 30, 70];
slidet = 0:1/L:L;
Specto = [];
%Specto = zeros(length(slidew)*length(slidet),length(v));
for i = 1:length(slidew)
    figure(2)
    for j = 1:length(slidet)
        g = exp(-slidew(i)*(t-slidet(j)).^2); % Gabor transform
        vg = g.*v; % apply the Gabor transform
        vgt = fft(vg); % take the Fourier transform
        Specto = [Specto; abs(fftshift(vgt))]; %storing data for plotting
        subplot(3,1,1), plot(t,v,'k',t,g,'r'), title('Gabor Filtering and signal'), legend(
        xlabel('Time [sec]'), ylabel('Amplitude')
        subplot(3,1,2), plot(t,vg,'k'), title('Gabor Filter of Signal')
        xlabel('Time [sec]'), ylabel('Amplitude')
        subplot(3,1,3), plot(ks, abs(fftshift(vgt))/max(abs(vgt))), title('Transformation o
        xlabel('Frequency (\omega)'), ylabel('FFT')
        drawnow
        %pause(0.05)
    end
figure(3)
subplot(1,3,i)
pcolor(slidet,ks,Specto((i-1)*length(slidet) + 1:i*length(slidet),:).'),
titletext = 'Window Width: ' + string(slidew(i));
title(titletext), xlabel('Time [sec]'), ylabel('Frequency [Hz]')
%set(gca,'Ylim',[0 1500],'Fontsize',[14])
shading interp
colormap(hot)
end

%% Spectogram of Handel with Gabor Transform Varying the Sampling Rate
slidew = 70;
sampling_rate = [30, 80, 150];
Specto = [];
%Specto = zeros(length(slidew)*length(slidet),length(v));
for i = 1:length(sampling_rate)
```

```
    slidet = linspace(0,L,sampling_rate(i));
    figure(3)
    Specto = [];
    for j = 1:length(slidet)
        g = exp(-slidew*(t-slidet(j)).^2); % Gabor transform
        vg = g.*v; % apply the Gabor transform
        vgt = fft(vg); % take the Fourier transform
        Specto = [Specto; abs(fftshift(vgt))]; %storing data for plotting
        subplot(3,1,1), plot(t,v,'k',t,g,'r'), title('Gabor Filtering and signal'), legend(
        xlabel('Time [sec]'), ylabel('Amplitude')
        subplot(3,1,2), plot(t,vg,'k'), title('Gabor Filter of Signal')
        xlabel('Time [sec]'), ylabel('Amplitude')
        subplot(3,1,3), plot(ks, abs(fftshift(vgt))/max(abs(vgt))), title('Transformation o
        xlabel('Frequency (\omega)'), ylabel('FFT')
        drawnow
        %pause(0.05)
    end

    figure(4)
    subplot(1,3,i)
    pcolor(slidet,ks,Specto.'),
    titletext = 'Sampling Rate: ' + string(sampling_rate(i));
    title(titletext), xlabel('Time [sec]'), ylabel('Frequency [Hz]')
    %set(gca,'Ylim',[0 1500],'Fontsize',[14])
    shading interp
    colormap(hot)
end

%% Spectogram of Handel with mexican hat wavelet

width = 70;
slidet = 0:1/L:L;
SpectoMH = zeros(length(slidet),length(v));
%Specto = zeros(length(slidew)*length(slidet),length(v));
figure(5)
for j = 1:length(slidet)
    mh = (1-width*t.^2).*exp(-width*(t-slidet(j)).^2); % mexican hat wavelet
    mh = mh/max(abs(mh)); % scale to 1
    vmh = mh.*v; % apply the Gabor transform
    vmht = fft(vmh); % take the Fourier transform
    SpectoMH(j,:) = abs(fftshift(vmht)); %storing data for plotting
    subplot(3,1,1), plot(t,v,'k',t,mh,'r'), title('Gabor Filtering and signal'), legend('v'
    xlabel('Time [sec]'), ylabel('Amplitude'), ylim([-1 1])
    subplot(3,1,2), plot(t,vmh,'k'), title('Gabor Filter of Signal')
    xlabel('Time [sec]'), ylabel('Amplitude')
    subplot(3,1,3), plot(ks, abs(fftshift(vmht))/max(abs(vmht))), title('Transformation of 
    xlabel('Frequency (\omega)'), ylabel('FFT')
    drawnow
    %pause(0.05)
end

%% Spectogram of Handel with Shannon wavelet

width = 700;
```

```
slidet = 0:1/L:L;
SpectoS = zeros(length(slidet),length(v));
figure(6)
for j = 1:length(slidet)
    k = j-1;
    num_steps = floor(length(v)/length(slidet));
    s = zeros(1,length(t));
    s((k*num_steps)+1:k*num_steps+width) = 1;
    vs = s.*v; % apply the Gabor transform
    vst = fft(vs); % take the Fourier transform
    SpectoS(j,:) = abs(fftshift(vst)); %storing data for plotting
    subplot(3,1,1), plot(t,v,'k',t,s,'r'), title('Gabor Filtering and signal'), legend('v',
    xlabel('Time [sec]'), ylabel('Amplitude')
    subplot(3,1,2), plot(t,vs,'k'), title('Gabor Filter of Signal')
    xlabel('Time [sec]'), ylabel('Amplitude')
    subplot(3,1,3), plot(ks, abs(fftshift(vst))/max(abs(vst))), title('Transformation of Si
    xlabel('Frequency (\omega)'), ylabel('FFT')
    drawnow
    %pause(0.05)
end

%%
figure(6)
subplot(1,2,1)
pcolor(slidet,ks,SpectoMH.'),
title('Spectogram with Mexican Hat Wavelet')
xlabel('Time [sec]'), ylabel('Frequency [Hz]')
shading interp
colormap(hot)

subplot(1,2,2)
pcolor(slidet,ks,SpectoS.'),
title('Spectogram with Shannon filter')
xlabel('Time [sec]'), ylabel('Frequency [Hz]')
shading interp
colormap(hot)

%% Frequency graphs of piano and recorder
figure()
subplot(2,2,1)
tr_piano=16; % record time in seconds
y_p=audioread('music1.wav'); Fs_p=length(y_p)/tr_piano;
plot((1:length(y_p))/Fs_p,y_p);
xlabel('Time [sec]'); ylabel('Amplitude');
title('Mary had a little lamb (piano)'); drawnow
%p8 = audioplayer(y,Fs); playblocking(p8);

subplot(2,2,3)
tr_rec=14; % record time in seconds
y_r=audioread('music2.wav'); Fs_r=length(y_r)/tr_rec;
plot((1:length(y_r))/Fs_r,y_r);
xlabel('Time [sec]'); ylabel('Amplitude');
title('Mary had a little lamb (recorder)');
%p8 = audioplayer(y,Fs); playblocking(p8);
```

```
S_p = y_p'/2;
L_p = length(S_p)/Fs_p;
n_p = length(S_p);
t2_p=linspace(0,L_p,n_p+1); t_p=t2_p(1:n_p);
k_p=(2*pi/L_p)*[0:n_p/2-1 -n_p/2:-1];
ks_p=fftshift(k_p);
St_p = fft(S_p);

S_r = y_r'/2;
L_r = length(S_r)/Fs_r;
n_r = length(S_r);
t2_r = linspace(0,L_r,n_r+1); t_r = t2_r(1:n_r);
k_r = (2*pi/L_r)*[0:n_r/2-1 -n_r/2:-1];
ks_r = fftshift(k_r);
St_r = fft(S_r);

subplot(2,2,2)
plot(ks_p,fftshift(St_p)/max(abs(St_p)))
ylabel('FFT(v)'), xlabel('Frequency (\omega)')
title('FFT of Piano')%, xlim([-0.5 0.5])

subplot(2,2,4)
plot(ks_r,fftshift(St_r)/max(abs(St_r)))
ylabel('FFT(v)'), xlabel('Frequency (\omega)')
title('FFT of Recorder')%, xlim([-0.5 0.5])

%% Spectorgram of Piano

numstep = 100;
width = 150;

slidet_p = linspace(0,tr_piano,numstep);
Specto_p = zeros(length(slidet_p),length(y_p));
figure()
for p = 1:length(slidet_p)
    g_p = exp(-width*(t_p-slidet_p(p)).^2); % Gabor transform
    Sg_p = g_p.*S_p; % apply the Gabor transform
    Sgt_p = fft(Sg_p); % take the Fourier transform
    Specto_p(p,:) = abs(fftshift(Sgt_p)); %storing data for plotting
    subplot(3,1,1), plot(t_p,S_p,'k',t_p,g_p,'r'), title('Gabor Filtering and signal'), lege
    xlabel('Time [sec]'), ylabel('Amplitude')
    subplot(3,1,2), plot(t_p,Sg_p,'k'), title('Gabor Filter of Signal')
    xlabel('Time [sec]'), ylabel('Amplitude')
    subplot(3,1,3), plot(ks_p, abs(fftshift(Sgt_p))/max(abs(Sgt_p))), title('Transformation
    xlabel('Frequency (\omega)'), ylabel('FFT')
    drawnow
end

figure()
pcolor(slidet_p,ks_p,log(Specto_p.'+1)), shading interp
xlabel('Time [sec]'), ylabel('Frequency [Hz]'),title('Spectogram of piano')
set(gca,'Ylim',[1000 2500],'Fontsize',[14])
colormap(hot)
```

```
%% Spectogram of Recorder

numstep = 100;
width = 150;

slidet_r = linspace(0,tr_rec,numstep);
Specto_r = zeros(length(slidet_r),length(y_r));
figure()
for r = 1:length(slidet_r)
    g_r = exp(-width*(t_r-slidet_r(r)).^2); % Gabor transform
    Sg_r = g_r.*S_r; % apply the Gabor transform
    Sgt_r = fft(Sg_r); % take the Fourier transform
    Specto_r(r,:) = abs(fftshift(Sgt_r)); %storing data for plotting
    subplot(3,1,1), plot(t_r,S_r,'k',t_r,g_r,'r'), title('Gabor Filtering and signal'), lege
    xlabel('Time [sec]'), ylabel('Amplitude')
    subplot(3,1,2), plot(t_r,Sg_r,'k'), title('Gabor Filter of Signal')
    xlabel('Time [sec]'), ylabel('Amplitude')
    subplot(3,1,3), plot(ks_r, abs(fftshift(Sgt_r))/max(abs(Sgt_r))), title('Transformation
    xlabel('Frequency (\omega)'), ylabel('FFT')
end

figure ()
pcolor(slidet_r,ks_r,log(Specto_r.'+1)), shading interp
xlabel('Time [sec]'), ylabel('Frequency [Hz]'),title('Spectogram of recorder')
set(gca,'Ylim',[4000 8000],'Fontsize',[14])
colormap(hot)

%% Comparison of full spectogram for piano and recorder

figure ()
subplot(2,1,1)
pcolor(slidet_p,ks_p,log(Specto_p.'+1)), shading interp
xlabel('Time [sec]'), ylabel('Frequency [Hz]'),title('Spectogram of piano')
set(gca,'Ylim', [0 (10^5)/2],'Fontsize',[14])
colormap(hot)

subplot(2,1,2)
pcolor(slidet_r,ks_r,log(Specto_r.'+1)), shading interp
xlabel('Time [sec]'), ylabel('Frequency [Hz]'),title('Spectogram of recorder')
set(gca,'Ylim', [0 (10^5)/2],'Fontsize',[14])
colormap(hot)
```