Assignment 1

WTW 801

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A Classical Portfolio Optimization

i Time Series Plot

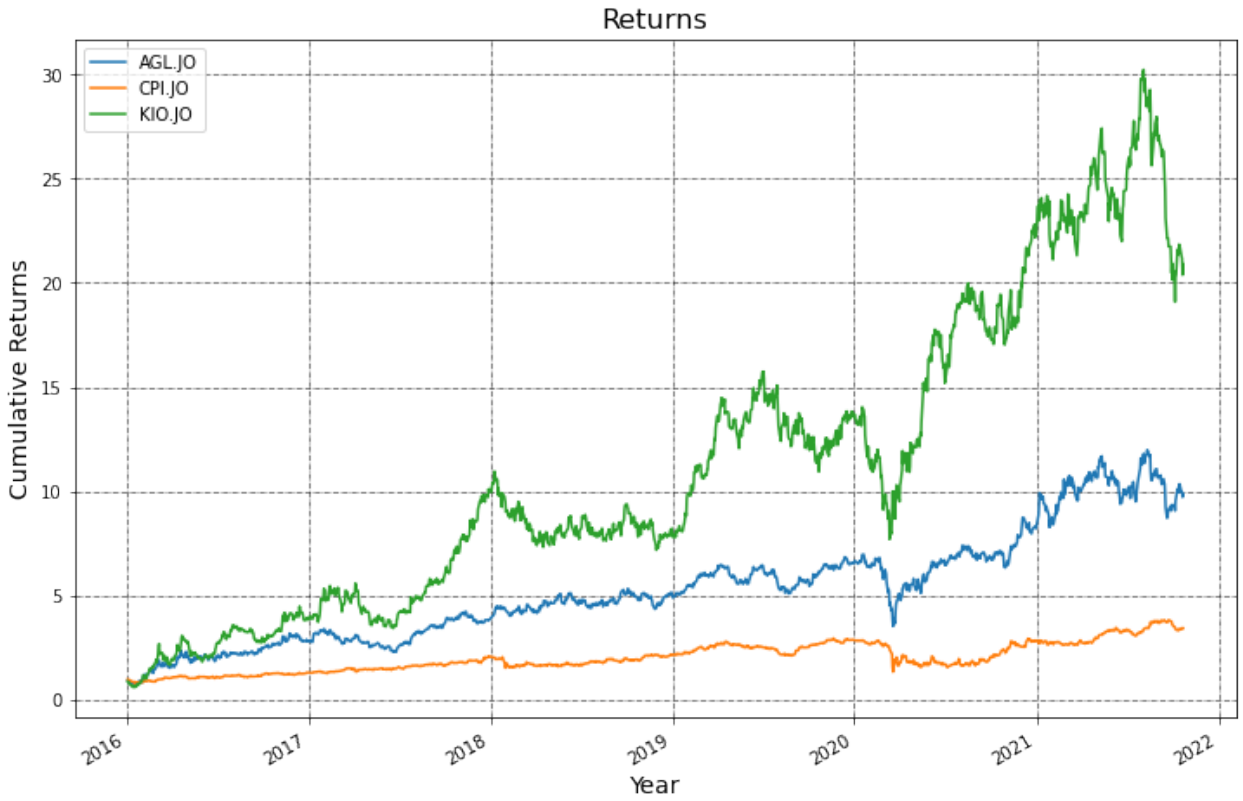

Figure 1

ii Rates of return

Using the equation 1 below, the daily rates of return were calculated and converted into a geometric daily rate of return with equation 2, which can be seen in table 1 on the following page.

$$r_{it} = \frac{I_{i,t} - I_{i,t-1}}{I_{i,t-1}} \tag{1}$$

$$\mu_i = (\prod_{t=1}^{T}(1 + r_{it}))^{\frac{1}{T}} - 1 \tag{2}$$

Table 1: Arithmetic and Geometric Daily Returns

Stocks	**Arithmetic**	**Geometric**
AGL.JO	0.001930	0.001562
BHP.JO	0.000957	0.000739
BTI.JO	-0.000068	-0.000210
CFR.JO	0.000559	0.000418
CPI.JO	0.001134	0.000841
KIO.JO	0.002674	0.002074
MTN.JO	0.000637	0.000238
NPN.JO	0.000719	0.000483
SBK.JO	0.000574	0.000337
SOL.JO	0.000576	-0.000242

The volatility, covariance matrix, and correlation matrix were subsequently calculated as follows:

Table 2: Stock Volatility

Stocks	**Volatility**
AGL.JO	0.027235
BHP.JO	0.020870
BTI.JO	0.016856
CFR.JO	0.016822
CPI.JO	0.024427
KIO.JO	0.034924
MTN.JO	0.028128
NPN.JO	0.021709
SBK.JO	0.021814
SOL.JO	0.040294

Table 3: Covariance Matrix

Stocks	AGL.JO	BHP.JO	BTI.JO	CFR.JO	CPI.JO	KIO.JO	MTN.JO	NPN.JO	SBK.JO	SOL.JO
AGL.JO	0.000742	0.000458	0.000059	0.000167	0.000183	0.000528	0.000221	0.000165	0.000184	0.000421
BHP.JO	0.000458	0.000436	0.000071	0.000147	0.000103	0.000379	0.000151	0.000136	0.000109	0.000325
BTI.JO	0.000059	0.000071	0.000284	0.000091	0.000005	0.000037	0.000057	0.000039	-0.000006	0.000090
CFR.JO	0.000167	0.000147	0.000091	0.000283	0.000059	0.000107	0.000062	0.000086	0.000043	0.000186
CPI.JO	0.000183	0.000103	0.000005	0.000059	0.000597	0.000141	0.000179	0.000113	0.000301	0.000198
KIO.JO	0.000528	0.000379	0.000037	0.000107	0.000141	0.001220	0.000174	0.000150	0.000153	0.000341
MTN.JO	0.000221	0.000151	0.000057	0.000062	0.000179	0.000174	0.000791	0.000105	0.000282	0.000404
NPN.JO	0.000165	0.000136	0.000039	0.000086	0.000113	0.000150	0.000105	0.000471	0.000111	0.000135
SBK.JO	0.000184	0.000109	-0.000006	0.000043	0.000301	0.000153	0.000282	0.000111	0.000476	0.000306
SOL.JO	0.000421	0.000325	0.000090	0.000186	0.000198	0.000341	0.000404	0.000135	0.000306	0.001624

Table 4: Correlation Matrix

Stocks	AGL.JO	BHP.JO	BTI.JO	CFR.JO	CPI.JO	KIO.JO	MTN.JO	NPN.JO	SBK.JO	SOL.JO
AGL.JO	1.000000	0.806573	0.128018	0.364364	0.274641	0.555202	0.289043	0.279662	0.310087	0.383369
BHP.JO	0.806573	1.000000	0.201050	0.417922	0.201883	0.520202	0.257342	0.299635	0.239507	0.386810
BTI.JO	0.128018	0.201050	1.000000	0.319317	0.011013	0.062913	0.120556	0.107810	-0.017439	0.132015
CFR.JO	0.364364	0.417922	0.319317	1.000000	0.143247	0.182380	0.131901	0.236253	0.116567	0.274370
CPI.JO	0.274641	0.201883	0.011013	0.143247	1.000000	0.165655	0.260245	0.212583	0.564489	0.201465
KIO.JO	0.555202	0.520202	0.062913	0.182380	0.165655	1.000000	0.177236	0.198464	0.200243	0.242201
MTN.JO	0.289043	0.257342	0.120556	0.131901	0.260245	0.177236	1.000000	0.171796	0.459246	0.356261
NPN.JO	0.279662	0.299635	0.107810	0.236253	0.212583	0.198464	0.171796	1.000000	0.234390	0.154683
SBK.JO	0.310087	0.239507	-0.017439	0.116567	0.564489	0.200243	0.459246	0.234390	1.000000	0.347688
SOL.JO	0.383369	0.386810	0.132015	0.274370	0.201465	0.242201	0.356261	0.154683	0.347688	1.000000

iii Setting up the Portfolio Optimisation Problem

To generate the optimal portfolio for an expected return R, the following formula was used, where:

- $\mathbf{x}$ represents the proportion of starting capital to be allocated to each stock
- Σ represents the covariance matrix of the stock selection
- A and b ensure that the allocation does not exceed 100%
- C and d ensure that no stock receives a negative allocation.

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \frac{1}{2}\mathbf{x}^T\Sigma\mathbf{x} \\ \text{subject to} \quad & \mu^T\mathbf{x} >= R, \\ & A\mathbf{x} = b, \\ & C\mathbf{x} >= d \end{aligned} \tag{3}$$

This quadratic program was then used iteratively to generate the efficient frontier and composition of efficient portfolios plot seen in figures 2 and 3 on the following page.

iv Efficient Frontier and the Composition of Efficient Portfolios

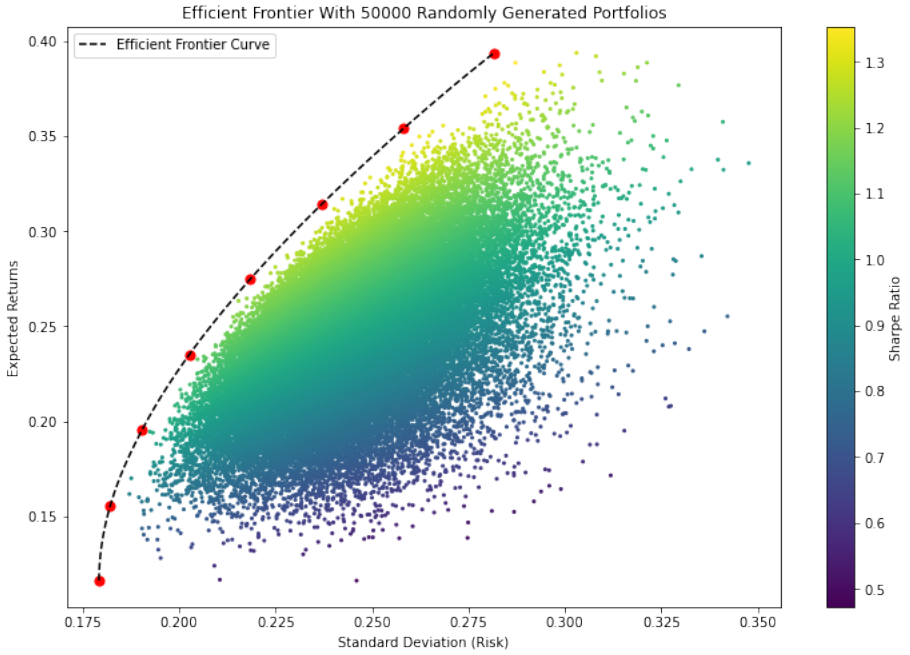

Figure 2

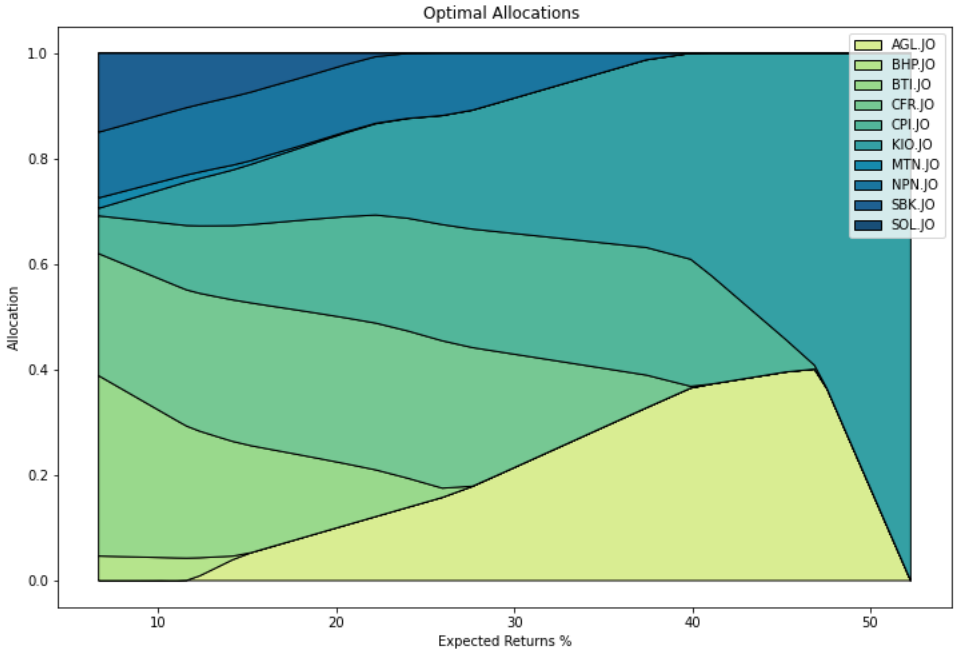

Figure 3

B Large-Scale Portfolio Optimization

i

Optimal portfolios can be significantly different if initial parameters are changed. Markowits recommends using β_i, which is estimated through linear regression to determine u_i and σ_{ij} of a stock. The sum of β_i produces the β of the portfolio. An example of linear constraints restricting β between 0.9 and 1.1 is shown below.

$$\sum_{i=1}^{n} x_i \beta_i > 0.9 \tag{4}$$

$$\sum_{i=1}^{n} x_i \beta_i < 1.1 \tag{5}$$

Where:

- n represents the total maximum number of stocks allowed in the portfolio
- x_i represents the proportion of capital allocated to $stock_i$
- β_i represents the β of $stock_i$

ii

If we want to evenly partition stocks by capitalization: large, medium and small, we can add the following constraints:

$$\sum_{i \epsilon k} x_i <= m_k \tag{6}$$

$$m_l = m_m = m_s \tag{7}$$

Where:

- k represents the level of capitalisation, large, medium, small
- m represents the maximum size of each x_i

iii

This question uses the quadratic program shown in (8) to determine the optimal portfolio that satisfies the median return R. The average daily returns of each stock are shown below:

Table 5: Daily Returns

Stocks	**Daily Return**
AGL.JO	0.001930
BHP.JO	0.000957
BTI.JO	-0.000068
CFR.JO	0.000559
CPI.JO	0.001134
KIO.JO	0.002674
MTN.JO	0.000637
NPN.JO	0.000719
SBK.JO	0.000574
SOL.JO	0.000576

The median of the daily returns column in table 5 is calculated as 0.0006778, this number is used to represent R in the quadratic program shown below, which suggests the allocation shown in table 6:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \frac{1}{2}\mathbf{x}^T\Sigma\mathbf{x} \\ \text{subject to} \quad & \mu^T\mathbf{x} >= R, \\ & A\mathbf{x} = b, \\ & C\mathbf{x} >= d \end{aligned} \tag{8}$$

Table 6: Median Portfolio

Stocks	**Allocation**
AGL.JO	0.04979
BHP.JO	0.10062
BTI.JO	0.15430
CFR.JO	0.12157
CPI.JO	0.09148
KIO.JO	0.01108
MTN.JO	0.11726
NPN.JO	0.11319
SBK.JO	0.12061
SOL.JO	0.12009

iv

Since the median portfolio is based off of historical data, it may not be robust to market fluctuations, to test this we will multiply each of our daily returns by a random number in the range of [0.95,1.05], this will show what could have happened if the stocks had performed either 5% better or 5% worse.

Table 7: Median Portfolio with fluctuations

Stocks	**Median**	**Fluctuation #1**
AGL.JO	0.04979	0.05003
BHP.JO	0.10062	0.10062
BTI.JO	0.15430	0.15404
CFR.JO	0.12157	0.12146
CPI.JO	0.09148	0.09152
KIO.JO	0.01108	0.01151
MTN.JO	0.11726	0.11718
NPN.JO	0.11319	0.11313
SBK.JO	0.12061	0.12051
SOL.JO	0.12009	0.11999

This yielded the results shown in table 8, it was observed that the fluctuated portfolio achieved slightly higher return but also slightly higher risk.

Table 8: Portfolio Results

Portfolio	**Annualised Risk**	**Annualised Return**
Median	12.488137	17.08179
Fluctuation #1	12.492148	17.116919

Three more fluctuations were made and subsequently averaged to generate a more robust portfolio that can handle market fluctuations. This can be seen in table 9, and the performance of each portfolio can be seen in table 10.

By comparing the mean portfolio to the median portfolio we see that the mean portfolio is less risky, but also less profitable with a lower return. The weightings only change slightly across all fluctuations.

Table 9: Fluctuations

Stocks	**Median**	**F #1**	**F #2**	**F #3**	**F #4**	**Mean**
AGL.JO	0.04979	0.05003	0.04969	0.05075	0.05214	0.050480
BHP.JO	0.10062	0.10062	0.10063	0.10061	0.10059	0.100614
BTI.JO	0.15430	0.15404	0.15441	0.15327	0.15177	0.153558
CFR.JO	0.12157	0.12146	0.12161	0.12116	0.12056	0.121272
CPI.JO	0.09148	0.09152	0.09146	0.09165	0.09188	0.091598
KIO.JO	0.01108	0.01151	0.01090	0.01278	0.01524	0.012302
MTN.JO	0.11726	0.11718	0.11730	0.11693	0.11645	0.117024
NPN.JO	0.11319	0.11313	0.11322	0.11294	0.11258	0.113012
SBK.JO	0.12061	0.12051	0.12065	0.12022	0.11965	0.120328
SOL.JO	0.12009	0.11999	0.12013	0.11970	0.11913	0.119808

Table 10: Portfolio Results

Portfolio	**Annualised Risk**	**Annualised Return**
Median	12.488137	17.081789
F #1	12.46475	16.869253
F #2	12.426315	16.484873
F #3	12.466748	16.887958
F #4	12.415614	16.368006
Mean	12.452313	16.738376