Assignment 2

WTW 801

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This question uses 1200 randomly generated 3 dimensional points, the first 5 records of this data can be seen in table 1 below.

Table 1: Sample Data

	xdata	ydata	zdata
0	3.366257	4.489859	-0.156650
1	-4.115966	0.381877	-2.615659
2	-0.247866	2.790496	-0.948978
3	5.037184	4.313974	2.301078
4	1.181815	3.453272	0.433186

A principal component analysis was performed on the data, where the first step was to calculate the mean vector with equation 1 shown below

$$\mu_i = \frac{\sum_{i=1}^{N} \boldsymbol{x}_i}{N} \tag{1}$$

the calculations were as follows (rounded to 2 decimals):

$$\begin{aligned} \mu_1 &= \frac{\sum \boldsymbol{x_1}}{N} \\ &= \frac{2172}{1200} \\ &= 1.81 \end{aligned} \qquad \begin{aligned} \mu_2 &= \frac{\sum \boldsymbol{x_2}}{N} \\ &= \frac{3420}{1200} \\ &= 2.85 \end{aligned} \qquad \begin{aligned} \mu_3 &= \frac{\sum \boldsymbol{x_3}}{N} \\ &= \frac{-120}{1200} \\ &= -0.10 \end{aligned}$$

$$\underrightarrow{\therefore \boldsymbol{\mu} = \begin{bmatrix} 1.81 & 2.85 & -0.10 \end{bmatrix}}$$

Next, the covariance matrix ($\boldsymbol{\Sigma}$) was calculated with equation (2) and resulted in the matrix shown below.

$$\boldsymbol{\Sigma} = \frac{\sum_{i=1}^{N} (\boldsymbol{x}_i - \boldsymbol{\mu_i}) \times (\boldsymbol{x}_i - \boldsymbol{\mu_i})^T}{N-1} \tag{2}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 10.38 & 7.21 & 5.18 \\ 7.21 & 6.05 & 4.04 \\ 5.18 & 4.04 & 3.04 \end{bmatrix}$$

The Principal Component Matrix $\mathbf{P}$ is found by determining the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ and the corresponding unit eigenvectors $[\vec{p_1}, \vec{p_2}, \vec{p_3}]$. The eigenvalues are first found by solving for 0 in the following equation.

$$\mathbf{\Sigma} = \begin{bmatrix} 10.38 - \lambda & 7.21 & 5.18 \\ 7.21 & 6.05 - \lambda & 4.04 \\ 5.18 & 4.04 & 3.04 - \lambda \end{bmatrix}$$

Which yields the following principal values:

$$\therefore \lambda_1 = 18.53, \quad \lambda_2 = 0.73, \quad \lambda_3 = 0.21$$

Let $(\mathbf{\Sigma} - \lambda_i \mathbf{I}) = \mathbf{B}$ and solve $\mathbf{B}\boldsymbol{v_i} = \bar{0}$, where $\boldsymbol{v_i}$ is the eigenvector of the respective eigenvalue(λ_i), then convert $\vec{v_i}$ to a unit vector to get the following principal components:

$$\text{The principal component corresponding to the eigenvalue } 18.53 = \begin{bmatrix} 0.74 \\ 0.55 \\ 0.39 \end{bmatrix} \qquad = \boldsymbol{p_1}^T$$

$$\text{The principal component corresponding to the eigenvalue } 0.73 = \begin{bmatrix} 0.66 \\ -0.70 \\ -0.26 \end{bmatrix} \qquad = \boldsymbol{p_2}^T$$

$$\text{The principal component corresponding to the eigenvalue } 0.21 = \begin{bmatrix} -0.13 \\ -0.44 \\ 0.88 \end{bmatrix} \qquad = \boldsymbol{p_3}^T$$

Since principal components should be orthogonal to one another, we compute the dot product of the first and third components, if this equals zero then we know the components are orthogonal.

$$\begin{aligned} \boldsymbol{p_1} \cdot \boldsymbol{p_3} &= (0.74 \times -0.13) + (0.55 \times -0.44) + (0.39 \times 0.88) \\ &= 0.005 \approx 0 \end{aligned}$$

Comparing this with the dot product of xdata and zdata (which are not orthogonal):

$$\boldsymbol{x} \cdot \boldsymbol{z} = 5986.07$$

Code and results for question 1

```
import numpy as np
np.random.seed(2021)
sample = np.random.multivariate_normal(np.array([2,3,0]),
                                       np.array([[10,7,5],
                                       [7,6,4],
                                       [5,4,3]]),
                                       1200).T
xdata = sample[0,:]
ydata = sample[1,:]
zdata = sample[2,:]

mean_vector = sample.mean(axis=1)
Y = sample.T - mean_vector
cov_matrix = np.cov(Y.T)
eigen_values, eigen_vectors = np.linalg.eig(cov_matrix)

print("Mean vector: ", mean_vector)
print("Covariance Matrix: ", cov_matrix)
print("PC1: ", eigen_vectors[0,:])
print("PC2: ", eigen_vectors[1,:])
print("PC3: ", eigen_vectors[2,:])
print("Principal values: ", eigen_values)
print("Dot product of 1st and 3rd component: ",
    round(eigen_vectors[0,:].dot(eigen_vectors[2,:]),5))
print("Dot product of xdata and zdata: ",round(xdata.dot(zdata),5))

results:
Mean vector:  [ 1.81364432  2.85484981 -0.10429726]

Covariance Matrix:
[[10.3836659   7.20769837  5.18187074]
 [ 7.20769837  6.05285977  4.04586087]
 [ 5.18187074  4.04586087  3.04021204]]

PC1:  [ 0.73666019  0.663357   -0.13148863]
PC2:  [ 0.55207529 -0.70219784 -0.44958989]
PC3:  [ 0.39056963 -0.25860335  0.8835042 ]

Principal values:  [18.53270615  0.73384029  0.21019128]

Dot product of 1st and 3rd component:  0.0
Dot product of xdata and zdata:  5986.07325
```

2

Conceptually speaking, the vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are all orthogonal to one another which means that any combination of dot product between any two of these vectors will be 0. Further more, an sum of two vectors will fall in the plane made by those vectors and will subsequently be orthogonal to the remaining vector. Therefore there is no difference between $\mathbf{x}\cdot(\mathbf{y}+\mathbf{z})$ and $\mathbf{x}\cdot\mathbf{y}+\mathbf{x}\cdot\mathbf{z}$ as both of their dot products will be 0. Lastly, the distributive properties of vector dot products shows that these are actually the same sum at different levels of simplification.

$$\begin{aligned} \mathbf{x}\cdot(\mathbf{y}+\mathbf{z}) &= x_1(y_1+z_1)+x_2(y_2+z_2)+\cdots+x_n(y_n+z_n) \\ \therefore \mathbf{x}\cdot(\mathbf{y}+\mathbf{z}) &= \mathbf{x}\cdot\mathbf{y}+\mathbf{x}\cdot\mathbf{z} \end{aligned}$$

3

We can also use the distributive property to prove the following:

$\mathbf{w} = a_1 \cdot \boldsymbol{f_1} + \cdots + a_m\boldsymbol{f_m}$, where $\boldsymbol{f_1}, \cdots, \boldsymbol{f_m}$ are mutually orthogonal, show that $a_i = \frac{\boldsymbol{w}\cdot\boldsymbol{f_i}}{\boldsymbol{f_i}\cdot\boldsymbol{f_i}}$

Given that $\boldsymbol{f_1}, \cdots, \boldsymbol{f_m}$ are mutually orthogonal, we can assume that $\boldsymbol{f_i}\cdot\boldsymbol{f_m} = 0$ for all cases where $i \neq m$, therefore, by multiplying both sides of the equation by $\boldsymbol{f_i}$ we can rewrite the equation as follows:

$$\boldsymbol{w}\cdot\boldsymbol{f_i} = (a_1\cdot\boldsymbol{f_1}+\cdots+a_m\boldsymbol{f_m})\cdot\boldsymbol{f_i}$$

Since $\boldsymbol{f_i}\cdot\boldsymbol{f_m} = 0$ we can simplify to:

$$\boldsymbol{w}\cdot\boldsymbol{f_i} = a_i\boldsymbol{f_i}\cdot\boldsymbol{f_i}$$

Using the distributive property:

$$\frac{\boldsymbol{w}\cdot\boldsymbol{f}_i}{\boldsymbol{f}_i} = a_i\boldsymbol{f_i}$$

$$\therefore \quad a_i = \frac{\boldsymbol{w}\cdot\boldsymbol{f_i}}{\boldsymbol{f_i}\cdot\boldsymbol{f_i}}$$

4

Looking at the equation below, where the set of $\boldsymbol{f}_k$ and $\boldsymbol{g}_i$ are mutually orthogonal and have a mean 0. α_i and $\beta_i k$ are scalars.

$$\boldsymbol{w}_i = \alpha_i + \beta_{i1}\boldsymbol{f}_1 + \cdots + \beta_{ik}\boldsymbol{f}_k + \boldsymbol{g}_i$$

we can assume our set of $\boldsymbol{f}_k$ only contains one vector $\boldsymbol{f}_1$ and write out $\boldsymbol{w}_1, \boldsymbol{w}_2$ and $\boldsymbol{w}_3$ as follows:

$$\begin{aligned}
\boldsymbol{w}_1 &= \alpha_1 + \beta_{11}\boldsymbol{f}_1 + \boldsymbol{g}_1 \\
\boldsymbol{w}_2 &= \alpha_2 + \beta_{21}\boldsymbol{f}_1 + \boldsymbol{g}_2 \\
\boldsymbol{w}_3 &= \alpha_1 + \beta_{31}\boldsymbol{f}_1 + \boldsymbol{g}_3 \\
\vdots &= \qquad \vdots
\end{aligned}$$

Since the set of $\boldsymbol{f}_k$ and $\boldsymbol{g}_i$ have a mean 0 we can write them as zero vectors.

$$\begin{aligned}
\boldsymbol{w}_1 &= \alpha_1 + \beta_{11}\vec{0} + \vec{0} \\
\boldsymbol{w}_2 &= \alpha_2 + \beta_{21}\vec{0} + \vec{0} \\
\boldsymbol{w}_3 &= \alpha_1 + \beta_{31}\vec{0} + \vec{0} \\
\vdots &= \qquad \vdots
\end{aligned}$$

This cancels out the set of $\beta_i \epsilon \{\beta_1, \cdots, \beta_i\}$:

$$\begin{aligned}
\boldsymbol{w}_1 &= \alpha_1 + \cancel{\beta_{11}}\vec{0} \\
\boldsymbol{w}_2 &= \alpha_2 + \cancel{\beta_{21}}\vec{0} \\
\boldsymbol{w}_3 &= \alpha_1 + \cancel{\beta_{31}}\vec{0} \\
\vdots &= \qquad \vdots
\end{aligned}$$

$$\Rightarrow \boldsymbol{w}_i = \alpha_i$$

$$\underrightarrow{\therefore \overline{\boldsymbol{w}} = \overline{\alpha}}$$

5

The factor model in this section made use of 6 securities, namely: ABSA, Standard Bank, Nedbank and Capitec Bank, Johannesburg Stock Exchange Index as well as the global oil price. These prices were all expressed in Rands except oil which was expressed in USD. The oil price **was not** converted into Rands as the model coefficients would adjust to describe the relationship accordingly.

5.1 Factor Model

This model uses 5 years worth of daily data on the securities mentioned above, the first 5 rows of the dataset are shown below in table 2, it was then converted to the daily rates of return shown in 3

Table 2: Financial Data

Date	ABSA	Oil	Capitec	Nedbank	Standard Bank	JSE Index
2016-01-04	9453.48	36.76	48771.43	13231.25	8153.30	49316.61
2016-01-05	9603.62	35.97	48219.30	13314.52	8180.40	49599.72
2016-01-06	9425.68	33.97	47915.62	13259.74	8129.86	49082.29
2016-01-07	9036.42	33.27	47851.21	12862.36	7885.96	48052.78
2016-01-08	9175.44	33.16	47120.56	12559.20	7646.46	48104.68

Table 3: Daily Rates of Return

Date	ABSA	Oil	Capitec	Nedbank	Standard Bank	JSE Index
2016-01-05	0.01588	-0.02149	-0.01132	0.00629	0.00332	0.00574
2016-01-06	-0.01853	-0.05560	-0.00630	-0.00411	-0.00618	-0.01043
2016-01-07	-0.04130	-0.02061	-0.00134	-0.02997	-0.03000	-0.02098
2016-01-08	0.01538	-0.00331	-0.01527	-0.02357	-0.03037	0.00108
2016-01-11	-0.04167	-0.05277	-0.04308	-0.01413	-0.04215	0.00453

A principal component analysis was subsequently conducted to identify $\alpha_i \epsilon \{\alpha_1, \cdots, \alpha_n\}$, $\boldsymbol{\beta_i} \epsilon \{\boldsymbol{\beta_1}, \cdots, \boldsymbol{\beta_k}\}$ and $\boldsymbol{F_i} \epsilon \{\boldsymbol{F}_1, \cdots, \boldsymbol{F}_k\}$. First the mean vector was calculated using equation 1 and yielded the vector $\boldsymbol{\mu}$ (rounded to 5 decimals)[1], this vector corresponds to the set of $\alpha_i \epsilon \{\alpha_1, \cdots, \alpha_n\}$:

$$\underrightarrow{\therefore \boldsymbol{\mu} = \begin{bmatrix} 0.00057 & -0.00156 & 0.00120 & 0.00050 & 0.00060 & 0.00030 \end{bmatrix}}$$

[1]For more accurate results refer to code results

The covariance matrix was calculated using equation 2, and yielded the following matrix:

$$\mathbf{\Sigma} = \begin{bmatrix} 0.000584 & 0.000133 & 0.000327 & 0.000480 & 0.000429 & 0.000146 \\ 0.000133 & 0.008853 & 0.000087 & 0.000147 & 0.000223 & 0.000124 \\ 0.000327 & 0.000087 & 0.000620 & 0.000308 & 0.000313 & 0.000145 \\ 0.000480 & 0.000147 & 0.000308 & 0.000604 & 0.000449 & 0.000156 \\ 0.000429 & 0.000223 & 0.000313 & 0.000449 & 0.000494 & 0.000157 \\ 0.000146 & 0.000124 & 0.000145 & 0.000156 & 0.000157 & 0.000137 \end{bmatrix}$$

The eigenvalues were calculated by solving for $\lambda_i \epsilon \{\lambda_1, \cdots, \lambda_n\}$ in the following:

$$\mathbf{\Sigma} = \begin{bmatrix} 0.000584 - \lambda & 0.000133 & 0.000327 & 0.000480 & 0.000429 & 0.000146 \\ 0.000133 & 0.008853 - \lambda & 0.000087 & 0.000147 & 0.000223 & 0.000124 \\ 0.000327 & 0.000087 & 0.000620 - \lambda & 0.000308 & 0.000313 & 0.000145 \\ 0.000480 & 0.000147 & 0.000308 & 0.000604 - \lambda & 0.000449 & 0.000156 \\ 0.000429 & 0.000223 & 0.000313 & 0.000449 & 0.000494 - \lambda & 0.000157 \\ 0.000146 & 0.000124 & 0.000145 & 0.000156 & 0.000157 & 0.000137 - \lambda \end{bmatrix}$$

$$\therefore \lambda_1 = 0.00887, \quad \lambda_2 = 0.00178, \quad \lambda_3 = 0.00356, \quad \lambda_4 = 0.00012, \quad \lambda_5 = 0.00007, \quad \lambda_6 = 0.0001,$$

Let $(\mathbf{\Sigma} - \lambda_i \mathbf{I}) = \mathbf{B}$ and solve $\mathbf{B}\boldsymbol{v_k} = \bar{0}$, where $\boldsymbol{v_k}$ is the eigenvector of the respective eigenvalue(λ_i) and $\boldsymbol{p_k}$ is the unit vector of $\boldsymbol{v_k}$ which corresponds to the set $\boldsymbol{\beta_i} \epsilon \{\boldsymbol{\beta_1}, \cdots, \boldsymbol{\beta_k}\}$

$$\boldsymbol{p_1}^T = \begin{bmatrix} -0.0196 \\ -0.9989 \\ -0.0135 \\ -0.0213 \\ -0.0296 \\ -0.0156 \end{bmatrix}, \quad \boldsymbol{p_2}^T = \begin{bmatrix} 0.5198 \\ -0.0443 \\ 0.4352 \\ 0.5278 \\ 0.4781 \\ 0.1770 \end{bmatrix}, \quad \boldsymbol{p_3}^T = \begin{bmatrix} -0.2443 \\ 0.0046 \\ 0.8858 \\ -0.3509 \\ -0.1725 \\ 0.0528 \end{bmatrix}, \quad \boldsymbol{p_4}^T = \begin{bmatrix} -0.7964 \\ -0.0077 \\ -0.0228 \\ 0.3587 \\ 0.3754 \\ 0.3092 \end{bmatrix},$$

$$\boldsymbol{p_5}^T = \begin{bmatrix} -0.1416 \\ 0.0025 \\ 0.0775 \\ -0.0710 \\ 0.4801 \\ -0.8593 \end{bmatrix}, \quad \boldsymbol{p_6}^T = \begin{bmatrix} 0.1246 \\ -0.0097 \\ -0.1388 \\ -0.6813 \\ 0.6078 \\ 0.3628 \end{bmatrix}$$

By plotting the scree plot shown in figure 1 , it was decided to only keep 3 principal components as they account for roughly 97% of the original information. The factor loadings of the selected components are displayed in table 4, these represent the contributions of each security to the selected components. The original data (table 3) was then projected along these components to get the 3 factors (table 5) used in the model.

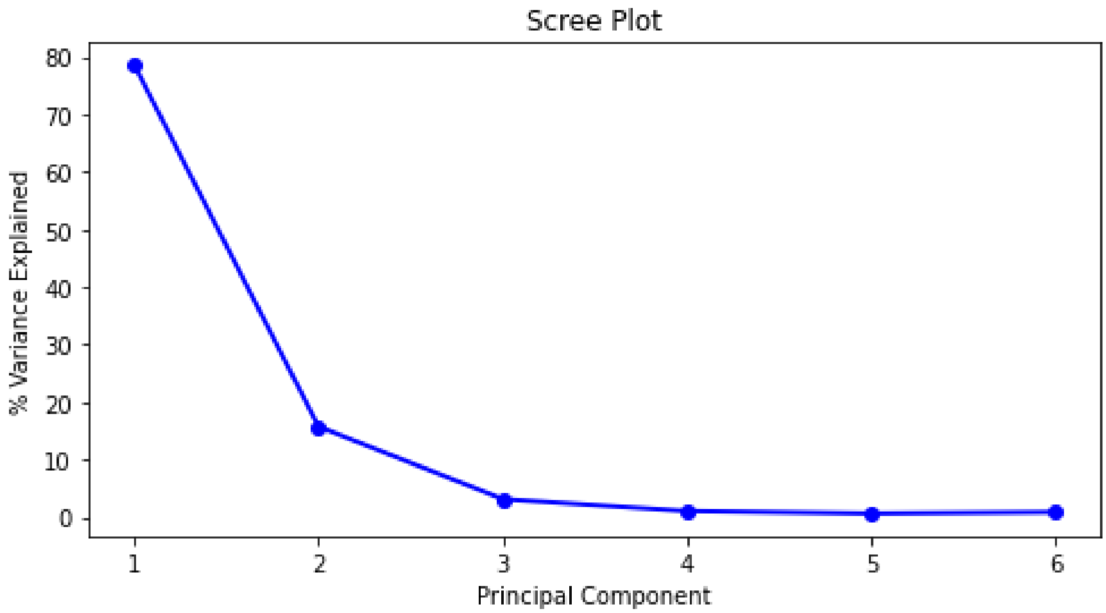

Figure 1

Table 4: Factor Loadings

	FL1	**FL2**	**FL3**
ABSA	-0.0196	0.5198	-0.2443
Oil	-0.9989	-0.0443	0.0046
Capitec	-0.0135	0.4352	0.8858
Nedbank	-0.0213	0.5278	-0.3509
Standard Bank	-0.0296	0.4781	-0.1725
JSE Index	-0.0156	0.1770	0.0528

Table 5: Factors

F1	**F2**	**F3**
0.01949	0.00872	-0.01714
0.05493	-0.01837	0.00000
0.02177	-0.05650	0.02273
0.00310	-0.02676	-0.00439
0.05410	-0.06636	-0.01641
$\vdots$	$\vdots$	$\vdots$

Using these factors, the returns of each stock were estimated with the formula and coefficients shown below, where: $\boldsymbol{\alpha} = \boldsymbol{\mu}$, $\boldsymbol{\beta_1} = \boldsymbol{p_1}$, $\boldsymbol{\beta_2} = \boldsymbol{p_2}$, and $\boldsymbol{\beta_3} = \boldsymbol{p_3}$, which can be seen in table 6.

$$\mathbf{r}_i = \alpha_i + \beta_{i1}\mathbf{f}_1 + \cdots + \beta_{ik}\mathbf{f}_k + \mathbf{e}_i \text{[2]} \tag{3}$$

Table 6: Model Coefficients

$Security_i$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta_1}$	$\boldsymbol{\beta_2}$	$\boldsymbol{\beta_3}$
ABSA	0.00057	-0.01959	0.51980	-0.24430
Oil	-0.00156	-0.99893	-0.04431	0.00456
Capitec	0.00120	-0.01355	0.43522	0.88578
Nedbank	0.00049	-0.02133	0.52778	-0.35092
Standard Bank	0.00060	-0.02960	0.47806	-0.17249
JSE index	0.00029	-0.01562	0.17701	0.05279

5.2 Highest Contributing Financial Assets

Using the factor loadings in table 4 the highest contributing asset will be that with the highest absolute factor loading for the first factor. Arranging the loadings in order of decreasing absolute value the assets level of contributions can be arranged as follows:

1. Oil
2. Standard Bank
3. Nedbank
4. ABSA
5. Capitec
6. JSE Index

[2]Error was omitted as this model was applied on a once off basis

5.3 Variance Explained by the First Principal Component

The amount of variance explained by the first principal component is shown in figure 1. This was calculated using the proportion of each eigenvalue in relation to the sum of all eigenvalues and resulted in an exact value of 78.54% of the original information.

5.4 Interpretation of the First Principal Component

The first principal component can be interpreted by the $\boldsymbol{\beta_1}$ values in table 6. There is a near perfect correlation with the price of oil, and very little correlation with the remaining assets. However, stronger correlation is observed for Capitec, ABSA, Nedbank and Standard Bank when looking at the $\boldsymbol{\beta_2}$ value which represents the second principal component.

Lastly, the overall strength of the model can be assessed by performing a linear regression on each stock using the model shown in 3 with the coefficients in table 6 and observing the R^2 and adjusted R^2. These are shown in table 7 below, where it can be seen that all securities have very high R^2 and adjusted R^2 scores except the JSE Index. This leads us to believe that the model would accurately predict the returns of ABSA, Oil, Capitec, Nedbank and Standard Bank, but it would struggle to predict the returns of the JSE index, this is likely because 4 of the 6 securities are banks, which overrepresents one sector in the JSE Index.

Table 7: R^2 and adjusted R^2 scores

Assets	R2	Adj_R2
ABSA	0.865	0.865
Oil	1.000	1.000
Capitec	0.996	0.996
Nedbank	0.899	0.899
Standard Bank	0.859	0.859
JSE index	0.431	0.430

Code question 5

```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
from sklearn.decomposition import PCA
import yfinance as yf
import statsmodels.api as sm

tickers_list = ['NED.JO', 'SBK.JO', 'CPI.JO', 'ABG.JO', 'CL=F', '^J203.JO']
data = yf.download(tickers_list,'2016-1-1')['Adj Close']
data = data.dropna()

rates = data.pct_change()
rates = rates.iloc[1:,:]

mean_vector = rates.values.T.mean(axis=1)

Y = rates.values - mean_vector
cov = np.cov(Y.T)
eigen_values, eigen_vectors = np.linalg.eig(cov)

x = eigen_values
w = []
z = []
cnt = 1
for i in x:
    w.append(100*(i/sum(x)))
    z.append(cnt)
    cnt+=1
print(sorted(w, key=float, reverse=True))
plt.rcParams["figure.figsize"] = (10,6)
plt.plot(z, w,'o-', linewidth=2, color='blue')
plt.title('Scree Plot')
plt.xlabel('Principal Component')
plt.ylabel('% Variance Explained')
plt.show()

loading = eigen_vectors.T * np.sqrt(eigen_values)
loading = pd.DataFrame(loading)
loading.columns = ['PC1', 'PC2', 'PC3', 'PC4', 'PC5', 'PC6']
loading.index = rates.columns

pc = pd.DataFrame(eigen_vectors)
factors = np.matmul(Y , eigen_vectors)
```

```
factors = pd.DataFrame(factors)
factors = factors.iloc[:,0:3]
factors.columns = ['F1','F2','F3']

X = factors.values
a = sm.add_constant(X)
alph_beta = []
R = []

for i in rates.columns:
    y = rates[i]
    ri = sm.OLS(y, a).fit()
    alph_beta.append(ri.params.values)
    R.append([ri.rsquared, ri.rsquared_adj])

res = pd.DataFrame(np.array(alph_beta), columns = ['alpha', 'beta_1', 'beta_2', 'beta_3'], index = rates.columns)
R = pd.DataFrame(np.array(R), columns = ['R2', 'Adj_R2'], index = rates.columns)

Results:

Covariance Matrix:
[[5.83727756e-04 1.32775471e-04 3.27358410e-04 4.80393124e-04
  4.28827846e-04 1.45578181e-04]
 [1.32775471e-04 8.85308046e-03 8.73307045e-05 1.47142944e-04
  2.23444483e-04 1.23756133e-04]
 [3.27358410e-04 8.73307045e-05 6.19913778e-04 3.08340781e-04
  3.12628918e-04 1.44926721e-04]
 [4.80393124e-04 1.47142944e-04 3.08340781e-04 6.04256346e-04
  4.48944125e-04 1.55984560e-04]
 [4.28827846e-04 2.23444483e-04 3.12628918e-04 4.48944125e-04
  4.94339964e-04 1.56916178e-04]
 [1.45578181e-04 1.23756133e-04 1.44926721e-04 1.55984560e-04
  1.56916178e-04 1.36525306e-04]]

eigenvalues:  [8.86856685e-03 1.77825604e-03 3.55681076e-04 1.19313629e-04
 7.22942503e-05 9.77317671e-05]

eigenvectors:
[[-0.01958793  0.51979562 -0.24430181 -0.7963778  -0.14142495  0.12460671]
 [-0.99892822 -0.04429939  0.00455727 -0.00769481  0.00245305 -0.00969464]
 [-0.013547    0.43522313  0.88578404 -0.02280291  0.07751596 -0.13876324]
 [-0.02133236  0.52778413 -0.35091866  0.35868852 -0.07109736 -0.68127281]
```

```
 [-0.02959896  0.47806031 -0.17248778  0.37550119  0.48010225  0.607726  ]
 [-0.01562183  0.17701174  0.05278774  0.30910292 -0.85932022  0.36285039]]

 Explained Variance:
 [78.53958271413775, 15.748146154896107, 3.14989374877895, 1.056635509753698,
 ↪  0.8655076218287037, 0.6402342506047781]

 Factor Loadings:
               FL1       FL2       FL3       FL4       FL5       FL6
ABG.JO   -0.019571  0.519796 -0.244239 -0.796430 -0.141245  0.124600
CL=F     -0.998929 -0.044277  0.004562 -0.007700  0.002460 -0.009686
CPI.JO   -0.013534  0.435218  0.885779 -0.022734  0.077617 -0.138769
NED.JO   -0.021331  0.527785 -0.350928  0.358657 -0.071247 -0.681268
SBK.JO   -0.029584  0.478067 -0.172558  0.375610  0.479993  0.607720
^J203.JO -0.015620  0.177008  0.052870  0.308876 -0.859389  0.362870

Model Coefficients:
            alpha    beta_1    beta_2    beta_3
ABG.JO   0.000570 -0.019588  0.519796 -0.244302
CL=F    -0.001557 -0.998928 -0.044299  0.004557
CPI.JO   0.001198 -0.013547  0.435223  0.885784
NED.JO   0.000493 -0.021332  0.527784 -0.350919
SBK.JO   0.000603 -0.029599  0.478060 -0.172488
^J203.JO  0.000294 -0.015622  0.177012  0.052788
```