Conor Curry - Math 1080 - HW11

April 10, 2017

```
In [1]: Pkg.add("PyPlot")
        using PyPlot
```

```
INFO: Nothing to be done
```

```
In [2]: function conjugate_gradient(A, x, b, , itmax)
            r = b - (A * x)
            d = r
            res_cond =  * norm(b)
            rs = [norm(r)]
            for i in 1:itmax
                r_prev = r
                x_prev = x
                _cond =  * norm(x)
                if dot(d, A*d) != 0
                     = dot(d,r) / dot(d, A*d)
                else
                    return rs, x
                end
                x = x + *d
                r = b - (A * x)
                if norm(r) < res_cond && norm(x - x_prev) < _cond
                    #println("converged")
                    return rs,x
                end
                 = dot(r,r) / dot(r_prev, r_prev)
                d = r + *d
                push!(rs, norm(r))
            end
            return rs, x
        end
```

```
Out[2]: conjugate_gradient (generic function with 1 method)
```

1 Let's run an experiment

1.1 Set up a sample of 500 random 150x150 SPD Matrices. We'll run CG on these.

```
In [3]: dim = 150
        Ms = []
        for i in 1:500
            M = 20.*rand(dim,dim)
            #Make this random matrix SPD
            M = .5(M*M')
            push!(Ms, M)
        end
```

1.1.1 Filter out outlier matrices with absurdly large condition numbers (they make it hard to see trends)

```
In [4]: cond_max = 10e8
        Ms = filter(M->cond(M) < cond_max, Ms)
        @show length(Ms);

length(Ms) = 442
```

1.1.2 ...And then verify that all are SPD.

```
In [5]: if length(filter(M->isposdef(M) && issymmetric(M), Ms)) == length(Ms)
            println("All matrices SPD")
        end

All matrices SPD
```

Setting up variables and empty arrays

```
In [6]: x = zeros(dim)
        b = collect(1:dim)
        k = []
        iters1 = []
        iters2 = [];
```

1.2 All right. Here the experiment begins.

This will run CG twice, for *each* of our sample matrices.

None of these should hit the iteration maximum because of our condition number pruning.

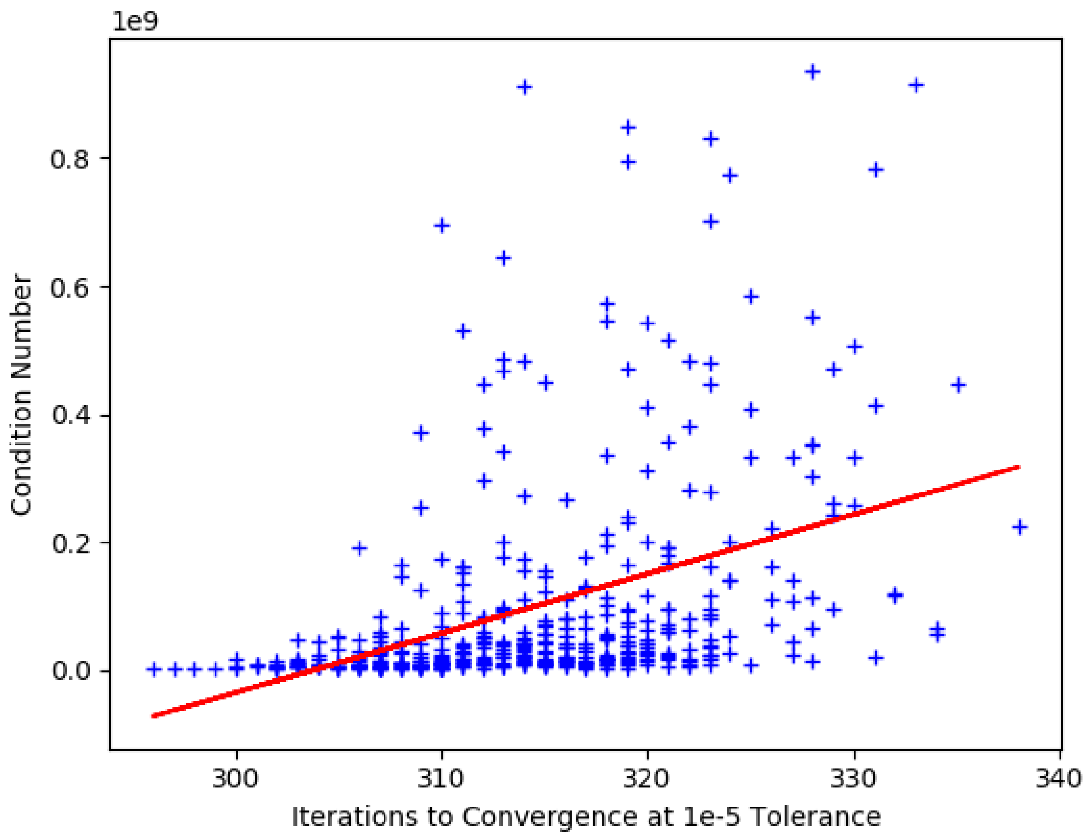
The difference between the runs is in the tolerances. The second run has an extra sig fig in the tolerance.

We'll save the number of iterations here so we can see how CG is doing.

```
In [7]: for M in Ms
            res1, _ = conjugate_gradient(M, x, b, 1e-5 , 10000)
            res2, _ = conjugate_gradient(M, x, b, 1e-6, 10000)
            push!(k, cond(M))
            push!(iters1, length(res1)-1)
            push!(iters2, length(res2)-1)
        end
```

1.3 Looking at the Data

```
In [8]: plot(iters1, k, "b+")
        ylabel("Condition Number")
        xlabel("Iterations to Convergence at 1e-5 Tolerance")
        a, b = linreg(iters1, k)
        plot(iters1, a + b*iters1, "r");
```

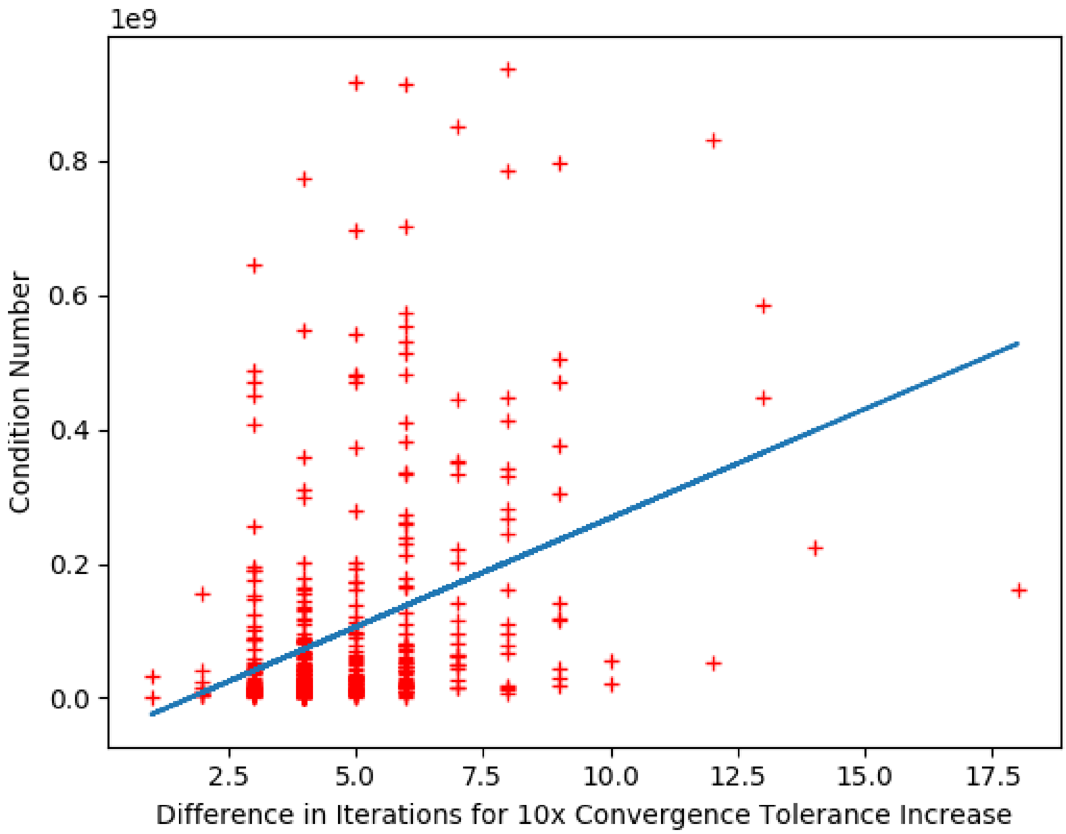


1.3.1 Here we examine only the data from CG with 1e-5 convergence tolerance.

Looking at the number of iterations plotted against the condition number, it is easy to see a clear trend -- an increase in iterations for performing CG on a matrix with a higher condition number.

While the regression performed here is only linear, this trend should theoretically have a quadratic $O(n^2)$ relationship.

```
In [9]: sig_diff = iters2 - iters1
        plot(sig_diff, k, "r+")
        ylabel("Condition Number")
        xlabel("Difference in Iterations for 10x Convergence Tolerance Increase")
        a, b = linreg(sig_diff, k)
        plot(sig_diff, a + b*sig_diff);
```



2 Gram-Schmidt

```
In [10]: function gram_schmidt(basis)
             n = length(basis)
             #this sets ortho[1] = basis[1] as well as sets ortho's dimensions correctly
             ortho = deepcopy(basis)
             for j in 2:n
                 a = zeros(j-1)
                 for i in 1:j-1
                     a[i] = dot(basis[j], ortho[i]) / dot(ortho[i], ortho[i])
```

```
            end
            ortho[j] = basis[j] - sum([a[i]*ortho[i] for i in 1:j-1])
        end
        return ortho
    end;
```

2.0.1 Random non-orthogonal basis to "seed" Gram-Schmidt

The non-zero array is proof that the vectors aren't orthogonal to one another

```
In [11]: basis = [rand(4) for i in 1:4]
         display(basis)
         print([dot(v, w) > eps() ? dot(v,w) : 0 for v in basis for w in basis if v != w])
```

```
4-element Array{Array{Float64,1},1}:
 [0.0608121,0.484729,0.414814,0.219375]
 [0.235343,0.111583,0.472378,0.770868]
 [0.718666,0.00210028,0.649049,0.843008]
 [0.419709,0.538164,0.856795,0.113642]


[0.433458,0.498891,0.666728,0.433458,1.12581,0.65116,0.498891,1.12581,0.954664,0.666728,0.65116,
```

2.0.2 And now orthogonalize the basis with Gram-Schmidt

The same calculation is done as before to produce the (now all zero) array of dot products

```
In [12]: ortho = gram_schmidt(basis)
         display(ortho)
         print([dot(v, w) > eps() ? dot(v,w) : 0 for v in ortho for w in ortho if v != w])
```

```
4-element Array{Array{Float64,1},1}:
 [0.0608121,0.484729,0.414814,0.219375]
 [0.177897,-0.346315,0.0805248,0.563636]
 [0.407797,-0.0484631,0.0872574,-0.170954]
 [-0.074352,-0.10336,0.165349,-0.0636631]


[0,0,0,0,0,0,0,0,0,0,0,0]
```

3 Orthogonalization of Moments

```
In [13]: function ortho_moments(basis, A)
             n = length(basis)
             ortho = deepcopy(basis)
             for j in 1:n-1
                 a = dot(ortho[j], A*A*ortho[j]) / dot(ortho[j], A*ortho[j])
                 if j == 1
                     ortho[2] = A*ortho[1] - *ortho[1]
```

```
            else
                b = dot(ortho[j-1], A*A*ortho[j]) / dot(ortho[j-1], A*ortho[j-1])
                ortho[j+1] = A*ortho[j] - a*ortho[j] - b*ortho[j-1]
            end
        end
        return ortho
    end;
```

In [14]:
```
M = 2.*rand(4,4)
#Make this random matrix SPD
M = .5(M*M')
```

Out[14]:
```
4×4 Array{Float64,2}:
 3.27258  2.39866  3.56423  3.1777
 2.39866  3.10967  2.85869  2.4188
 3.56423  2.85869  4.18123  3.81767
 3.1777   2.4188   3.81767  3.60735
```

In [19]:
```
ortho = ortho_moments(basis, M)
display(ortho)
print([dot(v, M*w) > 200*eps() ? dot(v, M*w) : 0 for v in ortho for w in ortho if v !=
```

```
4-element Array{Array{Float64,1},1}:
 [0.0608121,0.484729,0.414814,0.219375]
 [2.77334,-2.71993,-1.03689,0.984701]
 [-1.66636,-0.317484,1.68427,-0.0177854]
 [0.0853723,0.0474151,-0.334533,0.249874]


[0,0,0,0,0,0,0,0,0,0,0,0]
```