A mathematical model of national-level food system sustainability: supplementary materials

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1 Non-dimensionalisation

To non-dimensionalise the system of equations, we define the dimensionless variables:

$$\tau = \frac{t}{\tilde{t}}, \quad \hat{C} = \frac{C}{\tilde{C}}, \quad \hat{I} = \frac{I}{\tilde{I}}, \quad \hat{D} = \frac{D}{\tilde{D}}, \quad \hat{P} = \frac{P}{\tilde{P}} \tag{1}$$

Re-writing the original system of equations, we have:

$$\frac{d\hat{C}\tilde{C}}{d\tau\tilde{t}} = a\hat{C}\tilde{C}\Big(\frac{\hat{P}\tilde{P}}{b} - 1\Big) - e\hat{C}\tilde{C} \tag{2}$$

$$\frac{d\hat{I}\tilde{I}}{d\tau\tilde{t}} = fg\hat{C}\tilde{C} - w\hat{I}\tilde{I} - \frac{\hat{I}\tilde{I}\hat{D}\tilde{D}}{s\hat{D}\tilde{D} + \hat{I}\tilde{I}} + k(h - fg\hat{C}\tilde{C}) \tag{3}$$

$$\frac{d\hat{D}\tilde{D}}{d\tau\tilde{t}} = m\Big(h\frac{q}{\hat{P}\tilde{P}} - \hat{D}\tilde{D}\Big) \tag{4}$$

$$\frac{d\hat{P}\tilde{P}}{d\tau\tilde{t}} = r\hat{P}\tilde{P}\Big(\frac{s\hat{D}\tilde{D}}{\hat{I}\tilde{I}} - 1\Big) \tag{5}$$

Re-arranging, we have:

$$\frac{d\hat{C}}{d\tau} = a\tilde{t}\hat{C}\Big(\frac{\hat{P}\tilde{P}}{b} - 1\Big) - e\tilde{t}\hat{C} \tag{6}$$

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$$\frac{d\hat{I}}{d\tau} = fg\tilde{t}\tilde{I}^{-1}\tilde{C}\hat{C} - w\tilde{t}\hat{I} - \tilde{t}\frac{\hat{I}\hat{D}\tilde{D}}{s\hat{D}\tilde{D} + \hat{I}\tilde{I}} + \tilde{t}\tilde{I}^{-1}k(h - fg\hat{C}\tilde{C}) \tag{7}$$

$$\frac{d\hat{D}}{d\tau} = m\tilde{t}\tilde{D}^{-1}\Big(h\frac{q}{\hat{P}\tilde{P}} - \hat{D}\tilde{D}\Big) \tag{8}$$

$$\frac{d\hat{P}}{d\tau} = r\tilde{t}\hat{P}\Big(\frac{s\hat{D}\tilde{D}}{\hat{I}\tilde{I}} - 1\Big) \tag{9}$$

We choose the following definitions for the scaling parameters:

$$\tilde{t} = a^{-1}, \quad \tilde{C} = C_0, \quad \tilde{I} = hs, \quad \tilde{D} = h, \quad \tilde{P} = q \tag{10}$$

which results in:

$$\frac{d\hat{C}}{d\tau} = \hat{C}\Big(\frac{q}{b}\hat{P} - 1\Big) - \frac{e}{a}\hat{C} \tag{11}$$

$$\frac{d\hat{I}}{d\tau} = \frac{fgC_0}{ahs}\hat{C} - \frac{w}{a}\hat{I} - \frac{1}{sa}\frac{\hat{I}\hat{D}}{\hat{D} + \hat{I}} + k\Big(\frac{1}{as} - \frac{fgC_0}{ahs}\hat{C}\Big) \tag{12}$$

$$\frac{d\hat{D}}{d\tau} = \frac{m}{a}\Big(\hat{P}^{-1} - \hat{D}\Big) \tag{13}$$

$$\frac{d\hat{P}}{d\tau} = \frac{r}{a}\hat{P}\Big(\frac{\hat{D}}{\hat{I}} - 1\Big) \tag{14}$$

Finally, we rename the dimensionless parameters:

$$\alpha = \frac{q}{b}, \quad \beta = \frac{e}{a}, \quad \delta = \frac{fgC_0}{ahs}, \quad \omega = \frac{w}{a}, \tag{15}$$

$$\gamma = \frac{1}{as}, \quad \kappa = k, \quad \mu = \frac{m}{a}, \quad \rho = \frac{r}{a} \tag{16}$$

and arrive at the full non-dimensionalised system:

$$\frac{d\hat{C}}{d\tau} = \hat{C}\Big(\alpha\hat{P} - 1\Big) - \beta\hat{C} \tag{17}$$

$$\frac{d\hat{I}}{d\tau} = \delta\hat{C} - \omega\hat{I} - \gamma\frac{\hat{I}\hat{D}}{\hat{D} + \hat{I}} + \kappa(\gamma - \delta\hat{C}) \tag{18}$$

$$\frac{d\hat{D}}{d\tau} = \mu\Big(\hat{P}^{-1} - \hat{D}\Big) \tag{19}$$

$$\frac{d\hat{P}}{d\tau} = \rho\hat{P}\Big(\frac{\hat{D}}{\hat{I}} - 1\Big) \tag{20}$$

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2 Stability analysis of the unsustainable domestic production equilibrium

We evaluate the stability of the model equilibria using linear stability analysis (Strogatz 1994). Linear stability analysis is based on a a Taylor series approximation of the four dimensional system, which results in the following matrix representation of the linearised system:

$$
\begin{pmatrix}
\frac{d\zeta_C}{d\tau} \\
\frac{d\zeta_I}{d\tau} \\
\frac{d\zeta_D}{d\tau} \\
\frac{d\zeta_P}{d\tau}
\end{pmatrix}
=
\begin{pmatrix}
\frac{\partial u}{\partial \hat{C}} & \frac{\partial u}{\partial \hat{I}} & \frac{\partial u}{\partial \hat{D}} & \frac{\partial u}{\partial \hat{P}} \\
\frac{\partial v}{\partial \hat{C}} & \frac{\partial v}{\partial \hat{I}} & \frac{\partial v}{\partial \hat{D}} & \frac{\partial v}{\partial \hat{P}} \\
\frac{\partial x}{\partial \hat{C}} & \frac{\partial x}{\partial \hat{I}} & \frac{\partial x}{\partial \hat{D}} & \frac{\partial x}{\partial \hat{P}} \\
\frac{\partial y}{\partial \hat{C}} & \frac{\partial y}{\partial \hat{I}} & \frac{\partial y}{\partial \hat{D}} & \frac{\partial y}{\partial \hat{P}}
\end{pmatrix}_{\hat{C}=\hat{C}^*, \hat{I}=\hat{I}^*, \hat{D}=\hat{D}^*, \hat{P}=\hat{P}^*}
\begin{pmatrix}
\zeta_C \\
\zeta_I \\
\zeta_D \\
\zeta_P
\end{pmatrix}
\quad (21)
$$

where the new variables $\zeta_j = j - j^*$ indicate small displacements of the variables $j = \{\hat{C}, \hat{I}, \hat{D}, \hat{P}\}$ from their equilibriums $j^* = \{\hat{C}^*, \hat{I}^*, \hat{D}^*, \hat{P}^*\}$. The functions u, v, x, y are placeholders for the dimensionless system of equations above. The matrix of partial derivatives is the Jacobian matrix ($\boldsymbol{J}$). The Jacobian equals:

$$
\boldsymbol{J} =
\begin{bmatrix}
\alpha\hat{P} - 1 - \beta & 0 & 0 & \alpha\hat{C} \\
\delta(1-\kappa) & -\omega - \frac{\gamma\hat{D}^2}{(\hat{D}+\hat{I})^2} & -\frac{\gamma\hat{I}^2}{(\hat{D}+\hat{I})^2} & 0 \\
0 & 0 & -\mu & -\mu\hat{P}^{-2} \\
0 & -\rho\hat{P}\hat{D}\hat{I}^{-2} & \rho\hat{P}\hat{I}^{-1} & \rho(\hat{D}\hat{I}^{-1} - 1)
\end{bmatrix}
\quad (22)
$$

which at the unsustainable production equilibrium evaluates to:

$$
\boldsymbol{J}\Big|_{\left\{0, \frac{\kappa\gamma}{\omega+\frac{\gamma}{2}}, \frac{\kappa\gamma}{\omega+\frac{\gamma}{2}}, \frac{\omega+\frac{\gamma}{2}}{\kappa\gamma}\right\}}
=
\begin{pmatrix}
\frac{\alpha(\omega+\frac{\gamma}{2})}{\kappa\gamma} - 1 - \beta & 0 & 0 & 0 \\
\delta(1-\kappa) & -\omega - \frac{\gamma}{4} & -\frac{\gamma}{4} & 0 \\
0 & 0 & -\mu & -\mu(\frac{\kappa\gamma}{\omega+\frac{\gamma}{2}})^2 \\
0 & -\rho(\frac{\omega+\frac{\gamma}{2}}{\kappa\gamma})^2 & \rho(\frac{\omega+\frac{\gamma}{2}}{\kappa\gamma})^2 & 0
\end{pmatrix}
\quad (23)
$$

The Jacobian matrix thus provides us with a linear approximation of the non-linear model at the unsustainable domestic production equilibrium. To ascertain whether small perturbations move away or towards the equilibrium, we calculate the eigenvalues by solving for the roots of the characteristic polynomial $\text{Det}(\boldsymbol{J} - \boldsymbol{I}\lambda)$, equal to:

$$
\left(\frac{\alpha(\omega + \frac{\gamma}{2})}{\kappa\gamma} - 1 - \beta - \lambda\right)\left[-\lambda^3 + \left(-\omega - \frac{\gamma}{4} - \mu\right)\lambda^2 + \left(-\mu(\omega + \rho) - \frac{\gamma}{4}\right)\lambda - \frac{\gamma}{2}\mu\rho\right] = 0 \quad (24)
$$

While the first eigenvalue can be ascertained directly (see the critical ratio in the main text), the roots of the third-degree polynomial in the brackets are more difficult to determine.

Instead, we use the Routh-Hurwitz criterion (Otto and Day 2011) to determine the sign of the remaining roots. Writing the third-degree polynomial as:

$$\lambda^3 + \left(\omega + \frac{\gamma}{4} + \mu\right)\lambda^2 + \left(\mu(\omega + \rho) + \frac{\gamma}{4}\right)\lambda + \frac{\gamma}{2}\mu\rho = 0 \tag{25}$$

$$\lambda^3 + \epsilon_1\lambda^2 + \epsilon_2\lambda + \epsilon_3 = 0 \tag{26}$$

the Routh-Hurwitz criterion state that the eigenvalues will be negative if:

$$\epsilon_1 > 0, \quad \epsilon_3 > 0, \quad \epsilon_1\epsilon_2 > \epsilon_3 \tag{27}$$

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