# Moduli of Elasticity

The maximum value of the stress within which the body regains its original size and shape is called elastic limit.

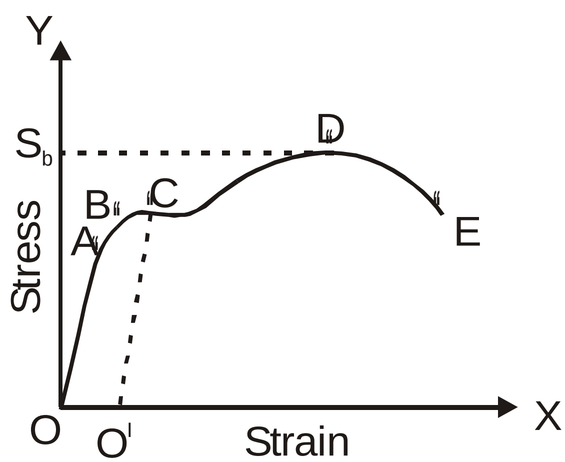
1. **Hooke's law :** Within the elastic limit of a body, stress is directly proportional to strain.

   $$\frac{\text{stress}}{\text{strain}} = E = \text{constant}$$

   Unit of E : CGS : $\frac{\text{dyne}}{\text{cm}^2}$, SI : $\frac{\text{newton}}{\text{m}^2}$ or pascal

   i) Within the proportionality limit stress-strain graph is a straight line passing through the origin.
   ii) A spring balance works on the principle of Hooke's law.
   iii) Modulus of elasticity does not depend upon the dimensions of the body but is a property of the material of the body.
   iv) Within the proportionality limit, the load extension graph is a straight line passing through the origin.

2. **Behaviour of a wire under the action of a load** :




A = Proportionality limit
B = Elastic limit
C = Yielding point
D = Breaking point
$S_b$ = Ultimate tensile strength

a) Stress is proportional to strain upto a limit, which is called proportionality limit. A is the limit of proportionality. Upto this limit, Hooke's law is obeyed.
b) The smallest value of stress which produces a permanent change in the body is called elastic limit.
c) If the wire is loaded beyond the elastic limit, a stage is reached where the wire begins to flow with no increase in the load and this point is called yield point.
d) Beyond the yield point, if the load is increased further the extension increases rapidly and the wire becomes narrower and finally breaks. The point at which the wire breaks is called breaking point.
e) Maximum stress required to break the wire is called ultimate tensile strength.
f) The capacity of a material to withstand large stresses without permanent set is called resilience.
g) The wire regains its original length if the elastic limit is not exceeded.
h) The wire does not obey Hooke's law between the proportionality limit and elastic limit. But wire regains its original length when the load is removed.
i) A permanent set (OP) is produced in the wire beyond elastic limit.
j) The stress required to reach the breaking point is called breaking stress.
k) If the gap between elastic limit and breaking point (BD) of a metal is large, it is called a ductile metal.
l) If the wire breaks soon after exceeding limit, the metal is said to be brittle. (If the gap BD is small).

3. **Types of moduli of elasticity :** There are three moduli of elasticity.

1) Young modulus 'Y'
2) Rigidity modulus 'n'
3) Bulk modulus 'K'

1) **Young's modulus :** Young's modulus is the ratio of longitudinal stress to longitudinal strain within the elastic limit of a body.

$$Y = \frac{\text{longitudinal stress}}{\text{longitudinal strain}} = \frac{\frac{F}{A}}{\frac{e}{l}} = \frac{F}{A} \times \frac{l}{e}$$

When a mass "M" is attached to the lower end,

$Y = \frac{Fl}{Ae}$ but $F = Mg$, $A = \pi r^2$

$$\therefore Y = \frac{Mg}{\pi r^2} \cdot \frac{l}{e}, \quad Y = \left(\frac{M}{e}\right) \frac{g.l}{\pi r^2}$$

i) The stress required to double the length of a wire (or to produce 100% longitudinal strain) is equal to Young's modulus of the wire.

ii) Y of a perfectly elastic material is infinite and that of a perfectly inelastic material is zero.

2) **Rigidity modulus** : Rigidity modulus is the ratio between shearing stress and shearing strain within the elastic limit of a body.

$$\eta = \frac{\text{shearing stress}}{\text{shearing strain}} = \frac{\frac{F}{A}}{\tan\theta} = \frac{F}{A\theta}$$

(for small values of θ, tanθ= θ) (or) $n = \frac{F}{A}.\frac{l}{\Delta l}$.

i) If η is low for a wire, it can be twisted very easily.

ii) Since phosphor-bronze has very low rigidity modulus, it is used as a suspension fiber in moving coil galvanometers.

Rigidity (shear) modulus is used to calculate the strain produced in a rod under twisting stress. It is also used to calculate the restoring torque when a wire or a cylinder is twisted. Torque C produced per unit twist of a wire of length l and radius r is given by $C = \frac{\pi\eta r^4}{2l}$ where □ is rigidity modulus.

3) **Bulk modulus** : Bulk modulus is the ratio between volume stress and volume strain within the elastic limit of a body.

$$K = \frac{\text{volume stress}}{\text{volume strain}} = \frac{\frac{F}{A}}{-\left(\frac{\Delta v}{v}\right)}.$$

(–sign indicates the decrease in volume)

$$K = \frac{Pv}{\Delta v}$$

i) If a block of coefficient of cubical expansion **γ** is heated such that the rise in temperature is θ , the pressure to be applied on it to prevent its expansion=K**γ**θ where K is its bulk modulus.

ii) When a rubber ball of volume 'V' bulk modulus 'K' is taken to a depth 'h' in water, decrease in its volume is $\Delta V = \frac{hdgV}{K}$ (d=density of material).

All modulii of elasticity Y, n, K have same units and dimensions -> $[M^1L^{\square 1}T^{\square 2}] \rightarrow N.m^{\square 2}$

i) Solids possess Y, n and K.

ii) Liquids and gases possess only K.

Bulk modulus of gases is very low, while that of liquids and solids is very high.

Isothermal bulk modulus of a gas=pressure of the gas (P)

iii) Adiabatic bulk modulus of a gas = $\gamma$P where $\gamma$= ratio of two specific heats.

4) **Compressibility :** The reciprocal of bulk modulus is called compressibility $C = \frac{1}{K}$ For incompressible substances C = 0, $K = \infty$

4. Poisson’s ratio ($\sigma$) $= \frac{\text{lateral contraction strain}}{\text{longitudinal elongation strain}}$

$$\frac{\text{transverse strain}}{\textit{longitudinal}\ \text{strain}} = \frac{-\frac{\Delta r}{r}}{\frac{\Delta l}{l}}$$

i) Poisson’s ratio has no unit and has no dimensions.

ii) Theoretical limits of $\sigma$ = –1 to 0.5.

iii) Practical limit of $\sigma$ = 0 to 0.5

iv) If □ = 0.5 the substance is perfectly incompressible.

5. Relation among elastic constants Y, $\gamma$, K, $\sigma$ :

i) $\frac{9}{Y} = \frac{1}{K} + \frac{3}{\eta}$ ii) $Y = 2\eta(1+\sigma)$

iii) Y = 3K(1-2$\sigma$) iv) $\sigma = \frac{3K - 2\eta}{2(\eta + 3K)}$

6. Elastic hysteresis is the result of elastic after effect. There is a lag between stress and strain. The lag is known as elastic hysteresis.

7. **Applications of ‘Y’ :**

i) A long wire suspended vertically can elongate due to its own weight.

ii) Elongation of a wire due to its own weight $e = \frac{l^2 dg}{2Y}$; l is length of the wire, d is density of the wire, Y=Young’s modulus of the material of the wire, g=acceleration due to gravity

A very long wire suspended vertically can break due to its own weight.

iii) Maximum length of the wire that can be hung vertically without breaking=s/dg where s is breaking stress.

iv) **Breaking stress :**

a) The breaking stress of a wire is the maximum stress the material can withstand.

b) Breaking stress= $\frac{\text{breaking force}}{\text{initial area of cross - section}}$

v) Breaking force =

Breaking stress x area of cross-section

vi) **Breaking force :**

1) is independent of length of the wire
2) depends on the area of cross-section and nature of material of the wire.
3) breaking force $\alpha$ area of cross-section.
4) If we cut a cable that can support a maximum load of W into two equal parts, then each part can support a maximum load of W.

8. **Elastic Fatigue :**

a) The state of temporary loss of elastic nature due to continuous strain is called elastic fatigue.

b) Due to elastic fatigue :

i) a wire can be broken within the elastic limit

ii) a wire can be cut into pieces without using instruments

iii) railway tracks and bridges are declared unsafe after long use

iv) spring balances show wrong readings after long use.

9. **Strain energy :**

strain energy of a stretched wire =

$E = \frac{Fxe}{2} = \frac{\text{Force x elongation}}{2}$

Potential energy stored per unit volume in a strained body is called strain energy density.

Potential energy stored in a wire due to twisting= $\frac{1}{2}\tau\theta$.

25. **Strain energy density :**

$= \frac{\text{work}}{\text{volume}} = \frac{1}{2} \text{ x stress x strain}$

$= \frac{1}{2} \text{ x y x (strain)}^2 = \frac{1}{2} \text{ x } \frac{(\text{stress})^2}{\text{y}}$

If 'K' is the force constant, energy stored for extension 'e' is given by E = $\frac{1}{2}Ke^2$

26. **Laws of elongation :**

i) e $\alpha$ l; elongation is proportional to length of wire

ii) e $\alpha$ F; elongation is proportional to force applied

iii) $e \propto 1/A$ or $1/r^2$ elongation is inversely proportional to area of cross-section or square of the radius.

iv) elongation is inversely proportional to Young's modulus.

v) For two wires made of same material, $\frac{e_1}{e_2} = \frac{l_1}{l_2} \cdot \frac{F_1}{F_2} \cdot \frac{A_2}{A_1}$

vi) For two wires made of same material, when same force is applied on them $\frac{e_1}{e_2} = \frac{l_1}{l_2} \cdot \frac{r_2^2}{r_1^2}$.

vii) For two wires, made of same material, and of same volume when same force is applied, elongations ratio is given by $\frac{e_1}{e_2} = \frac{l_1^2}{l_2^2} = \frac{A_2^2}{A_1^2} = \frac{r_2^4}{r_1^4}$ (since, $V = A \times l =$ constant $A_1 l_1 = A_2 l_2$ $\frac{l_1}{l_2} = \frac{A_2}{A_1} = \frac{r_2^2}{r_1^2}$)

viii) If $l_1$ and $l_2$ are the length of a wire under tension $T_1$ and $T_2$, the actual length of the wire $= \frac{l_1 T_2 - l_2 T_1}{T_2 - T_1}$

25. **Springs :**

i) For a spring that obeys Hooke's law, equivalent force constant or spring constant is $K = \frac{YA}{l}$.

ii) $K \propto Y$, $K \propto A$, $K \propto 1/l$

iii) If a spring (or a wire) of force constant K is cut into 'n' equal parts, the force constant of each part of the wire is 'nk'.

iv) If a spring (or a wire) of force constant k is cut in the ratio of m:n, $k_m = \frac{(m+n)k}{m}$; $k_n = \frac{(m+n)k}{n}$

v) Potential energy of a stretched spring $= \frac{1}{2}Fx = \frac{1}{2}Kx^2 = \frac{1}{2}\frac{F^2}{K}$

vi) Two springs have force constants $K_1$ and $K_2$

a) When they are stretched by the same force and if their elastic energies are $E_1$ and $E_2$. $\frac{E_1}{E_2} = \frac{K_2}{K_1}$

b) When they are extended by the same length $\frac{E_1}{E_2} = \frac{K_2}{K_1}$

c) When they are extended till their energies are same, $\frac{F_1}{F_2} = \sqrt{\frac{K_1}{K_2}}$

d) The potential energy of a spring increases, whether it is stretched or compressed.

e) Springs in series $K_{eff}=\frac{K_1K_2}{K_1+K_2}$

f) Springs in parallel $K_{eff}=K_1+K_2$

g) The reciprocal of spring constant is called compliance.

vii) When a spiral spring is stretched, strain involved is longitudinal strain. (thickness is small)

viii) When a helical spring is stretched, strain involved is longitudinal and shearing strain. (thickness is large)

ix) When a wire is stretched, modulus of elasticity involved is Young's modulus

x) When a wire is twisted, modulus of elasticity involved is rigidity modulus.

xi) Inter atomic force constant k = Y. r = Young's modulus x (inter atomic distance)

27. **Thermal force :**

i) When a metal bar is fixed between two walls and the temperature is raised, the bar tries to expand and exerts a force on the walls. This force is called thermal force given by $F=YA\alpha\theta$

$\alpha$=co-efficient of linear expansion of the bar

$\theta$= rise in temperature

Y = Young's modulus, A=area of cross-section

Thermal force is independent of the length of bar.

ii) Thermal stress :

Thermal stress = $\frac{\text{thermal force}}{\text{area}}=\frac{YA\alpha\theta}{A}=Y\alpha\theta$

iii) If a load 'M' produces an elongation 'e' in a wire the rise in temperature required to produce the same elongation is $\frac{Mg}{YA\alpha}$ (since $\frac{e}{l}=\alpha\Delta t$)

where A = area of cross-section of the wire and

$\alpha$= coefficient of linear expansion of the material of the wire.