Remainder Theorem

We know the property of division which follows in the basic division, i.e.

Dividend = (Divisor × Quotient) + Remainder

This same follows the division of polynomial.

If p(x) and g(x) are two polynomials in which the degree of p(x) ≥ degree of g(x) and g(x) ≠ 0 are given then we can get the q(x) and r(x) so that:

P(x) = g(x) q(x) + r(x),

where r(x) = 0 or degree of r(x) < degree of g(x).

It says that p(x) divided by g(x), gives q(x) as quotient and r(x) as remainder.

Let us understand it with an example

Division of a Polynomial with a Monomial

$$3x^3 + x^2 + x \div x = \frac{3x^3}{x} + \frac{x^2}{x} + \frac{x}{x} = 3x^2 + x + 1$$

We can see that 'x' is common in the above polynomial, so we can write it as

$$3x^3 + x^2 + x = x(3x^2 + x + 1)$$

Hence $3x^2 + x + 1$ and x the factors of $3x^3 + x^2 + x$.

For Example: Divide $3x^2 + x - 1$ by $x + 1$.

(i) Let, $p(x) = 3x^2 + x - 1$ and $g(x) = x + 1$.

(ii) Performing divisions on these polynomials, we get,

$$
\begin{array}{r|l}
 & 3x - 2 \\
\hline
x + 1 & 3x^2 + x - 1 \\
 & \underset{-}{3x^2} + \underset{-}{3x} \\
\hline
 & \quad -2x - 1 \\
 & \quad \underset{+}{-2x} \underset{+}{-2} \\
\hline
 & \qquad\quad 1
\end{array}
$$

(iii) Now, we can re-write p(x) as $3x^2 + x - 1 = (x + 1)(3x - 2) + 1$.

Remainder Theorem:

Statement: Let p(x) be any polynomial of degree greater than or equal to one and let a be any real number. If p(x) is divided by the linear polynomial x – a, then the remainder is p(a).

Proof:

(i) Let p(x) be any polynomial with degree greater than or equal to 1. Suppose that when p(x) is divided by x – a, the quotient is q(x) and the remainder is r(x), i.e., $p(x) = (x - a)\, q(x) + r(x)$

(ii) Since the degree of (x – a) is 1 and the degree of r(x) is less than the degree of (x – a), the degree of r(x) = 0. This means that r(x) is a constant, say r.

(iii) So, for every value of x, r(x) = r.

(iii) Therefore, $p(x) = (x - a)\, q(x) + r$

(iv) In particular, if x = a, this equation gives us

(v) $p(a) = (a - a)\, q(a) + r = r$, which proves the theorem.

In other words, If p(x) and g(x) are two polynomials such that degree of p(x) ≥ degree of g(x) and $g(x) \neq 0$, then there exists two polynomials q(x) and r(x) such that $p(x) = g(x)q(x) + r(x)$, where, q(x) represents the quotient and r(x) represents remainder when p(x) is divided by g(x).