Polynomial Methods in Combinatorics

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What are Polynomial Methods?

A collection of techniques in Combinatorics which use polynomial interpolation and rigidity properties of polynomials to control the size of collections of objects with a certain structure.

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Example (Rigidity and Interpolation)

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- **Rigidity**: If $P \in \mathbb{R}[X_1, \ldots, X_n]$ has degree D and a line ℓ intersects $Z(P)$ in more than D points then $\ell \subset Z(P)$.
- **Interpolation**: We can do parameter-counting arguments using the fact that $\dim \mathbb{R}_{\deg \leq D}[X_1, \ldots, X_n] \sim D^n$.

Theorems proven using polynomial methods

Below lists results that can be proved using the polynomial method:

- **Kakeya Conjecture in Finite Fields**: If $A \subset \mathbb{F}^n$ contains a line in every direction then $|A| \gtrsim |\mathbb{F}|^n$.

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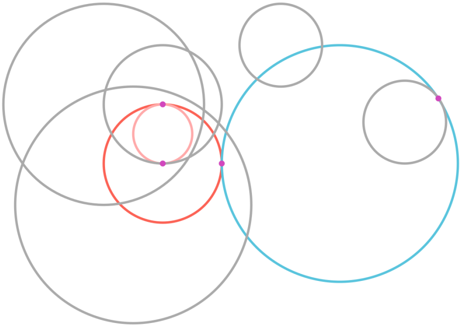
- **Circle Tangencies**: Given a (suitably non-degenerate) collection of N circles in $\mathbb{R}^2$, they determine $\lesssim N^{3/2}$ tangencies.

Circle Tangencies

Theorem

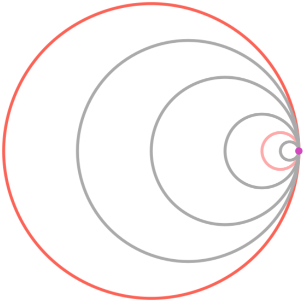
Given a (suitably non-degenerate) collection of N circles in $\mathbb{R}^2$, they determine $\lesssim N^{3/2}$ tangencies[a]*.*

[a] A tangency is a pair of circles (γ, γ') that are tangent.



Circle Tangencies: What's degenerate?

Collection of N circles with $\sim N^2$ tangencies:



Circle Tangencies: Lifting into $\mathbb{R}^3$

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We now present a sketch of a recent proof due to Ellenberg, Solymosi, and Zahl. [1]

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We now present a sketch of a recent proof due to Ellenberg, Solymosi, and Zahl. [1]

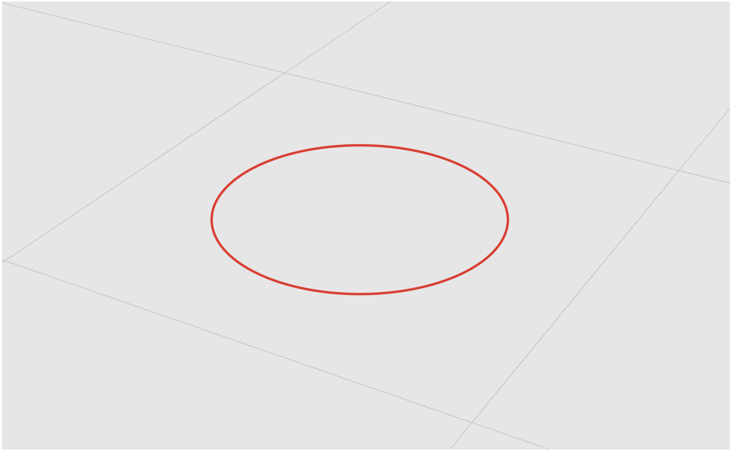
Sketch Proof: Assume:

- There are $\gtrsim N^{3/2}$ tangencies.
- Collection is uniform: each circle tangent to $\gtrsim N^{1/2}$ other circles.

For each circle γ in our collection, we define the curve $\beta(\gamma) \subset \mathbb{R}^3$ as:

$$\beta(\gamma) := \left\{ (x, y, z) \mid (x, y) \in \gamma,\ z = \frac{1}{\text{Slope of tangent at } (x, y)} \right\}$$

Circle Tangencies: Lifting into $\mathbb{R}^3$



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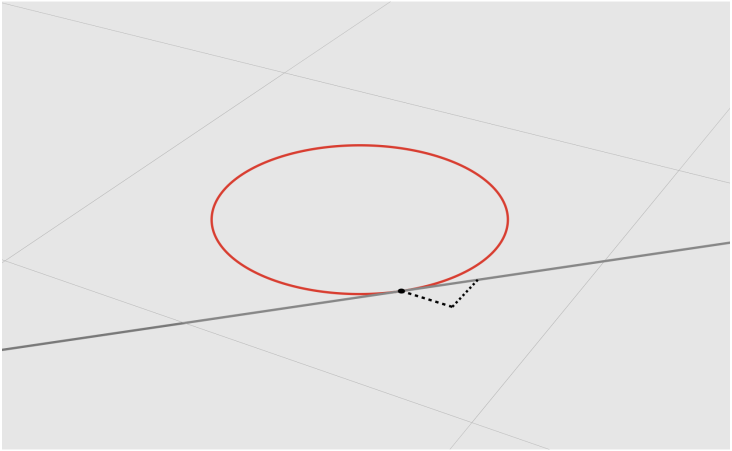

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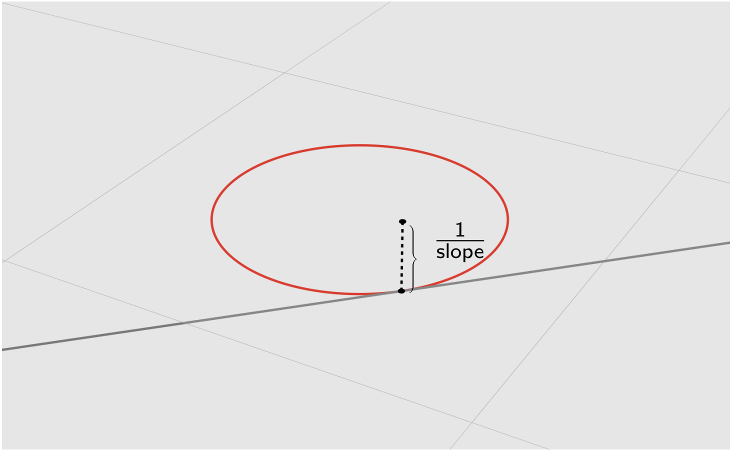

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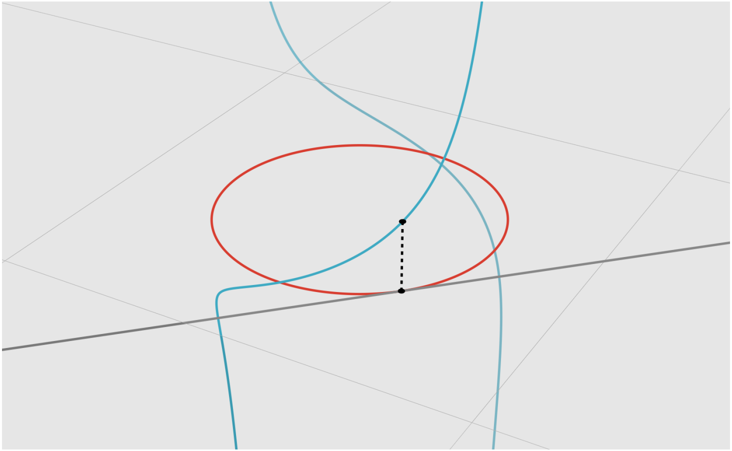
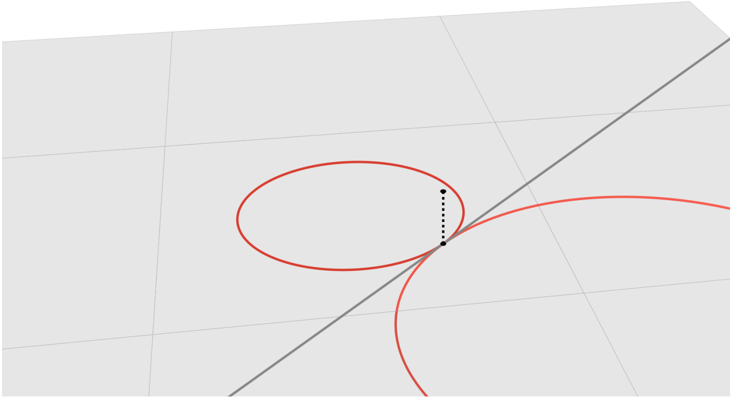


Circle Tangencies: Tangencies to Incidences



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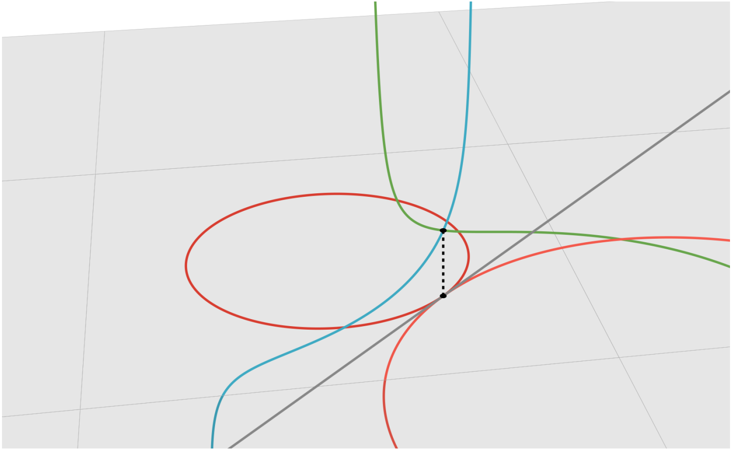

Intersection $\iff$ z co-ords equal $\iff$ circles are tangent.

Tangency problem in $\mathbb{R}^2$ $\iff$ Incidences problem in $\mathbb{R}^3$.

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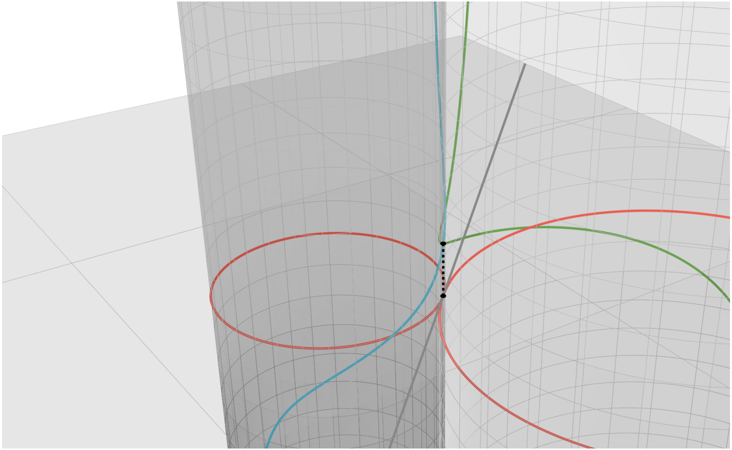

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Circle Tangencies: Tangent Vectors at Incidences

Let us examine the tangent vectors at a point of incidence between two curves:

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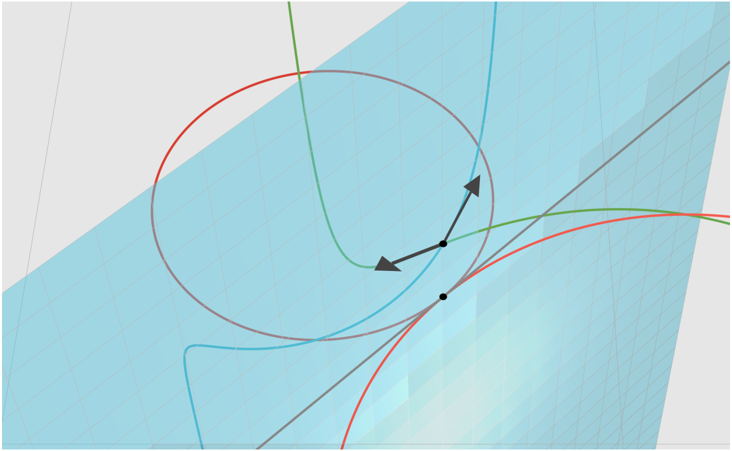
At every point of intersection, the tangent vectors span a vertical plane.



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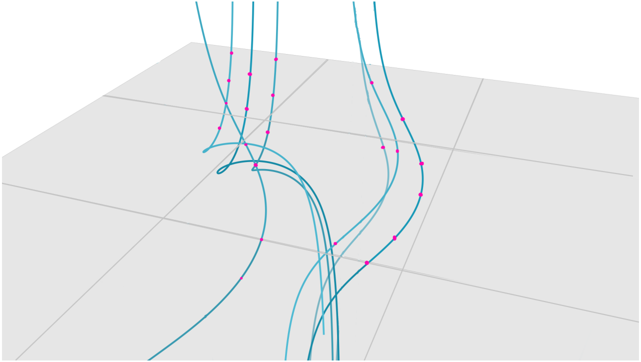
Circle Tangencies: Interpolation

Let P be a polynomial such that all $N^{3/2}$ incidences are contained in $Z(P)$.



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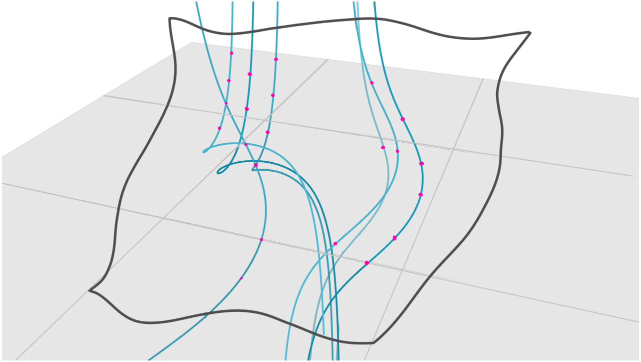
We can find a polynomial P such that all $N^{3/2}$ incidences are contained in $Z(P)$.



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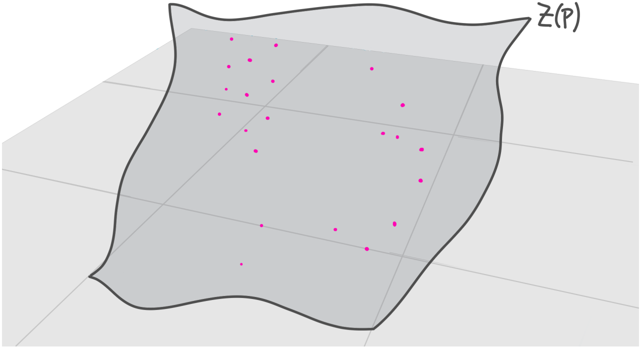

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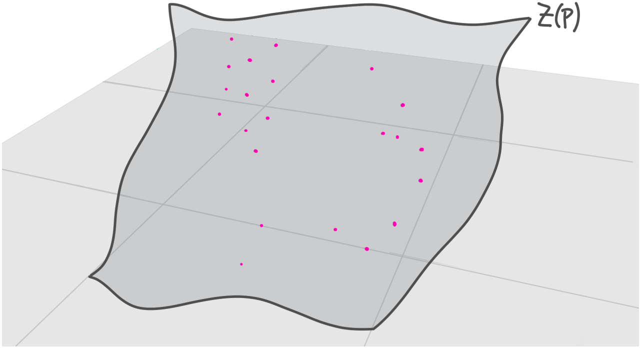

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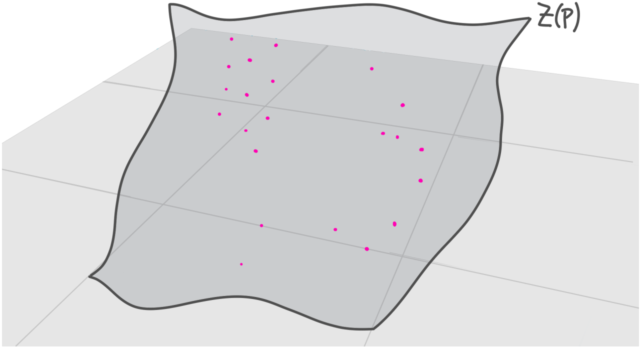

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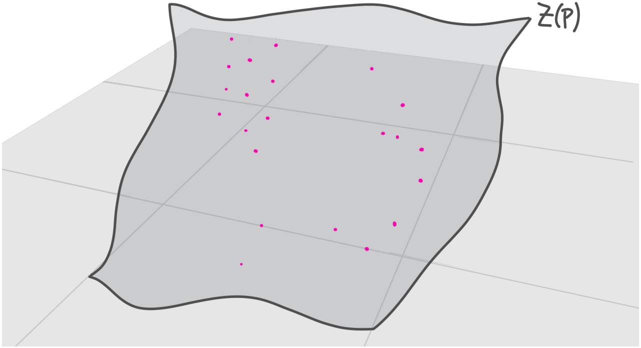
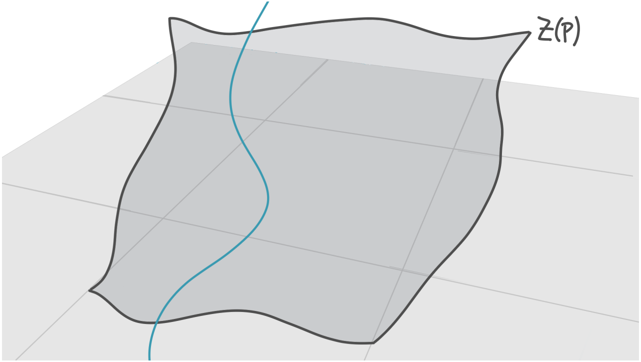


- Recall that $\dim \mathbb{R}_{\deg \leq D}[X, Y, Z] \sim D^3$.
- Each incidence point gives one linear equation for the coefficients.
- $D^3 \sim N^{3/2} \implies D \sim N^{1/2}$.

Circle Tangencies: $\beta(\gamma) \subset Z(P)$



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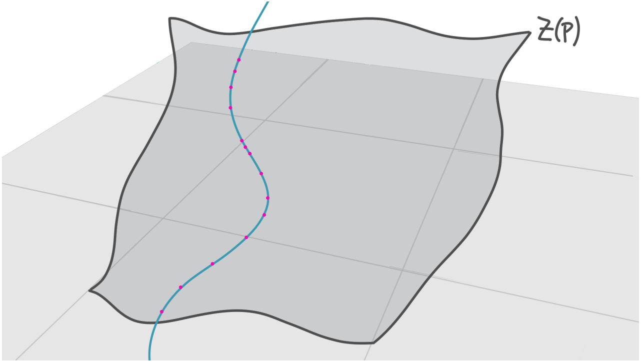

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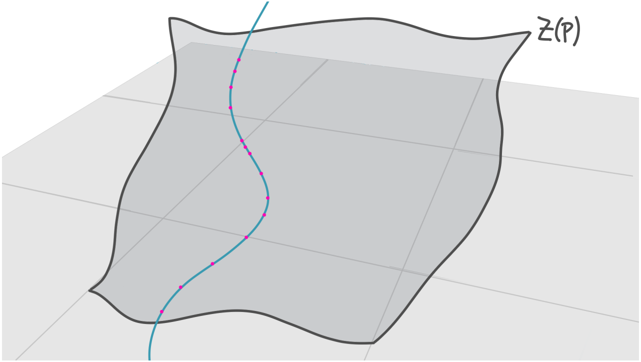

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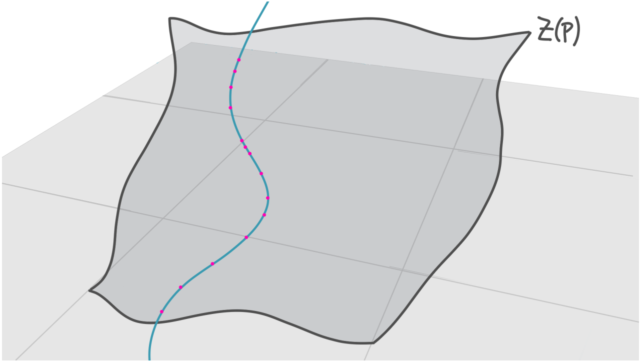

- Each curve $\beta(\gamma)$ intersects $Z(P)$ at $\gtrsim N^{1/2}$ points.
- But $\deg \beta(\gamma) = O(1)$ and $\deg P \sim N^{1/2}$
- $\implies \beta(\gamma) \subset Z(P)$ by Bézout's Theorem! (rigidity)

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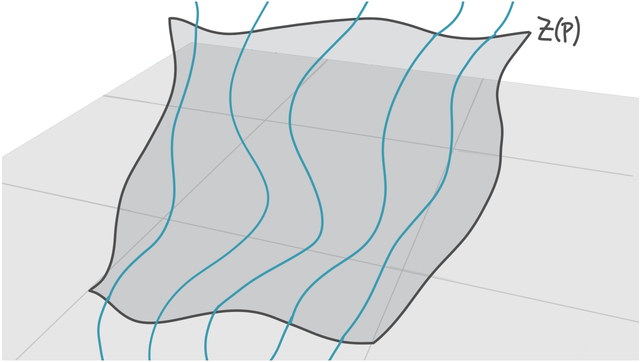

- Each curve $\beta(\gamma)$ contains $\gtrsim N^{1/2}$ points of intersection with $Z(P)$.
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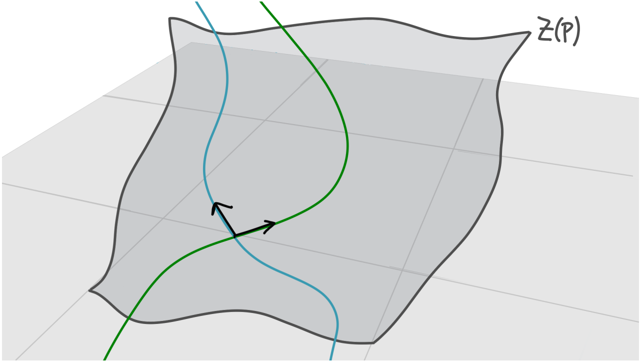
Circle Tangencies: Tangent Vectors



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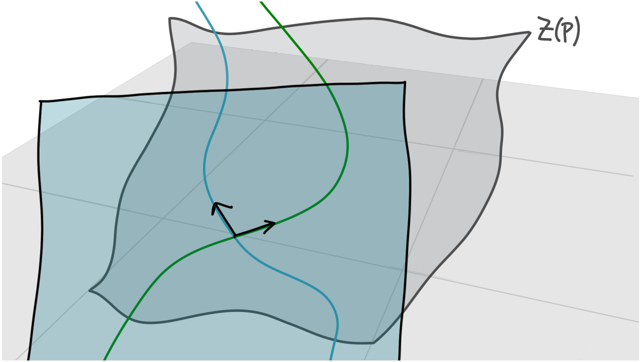

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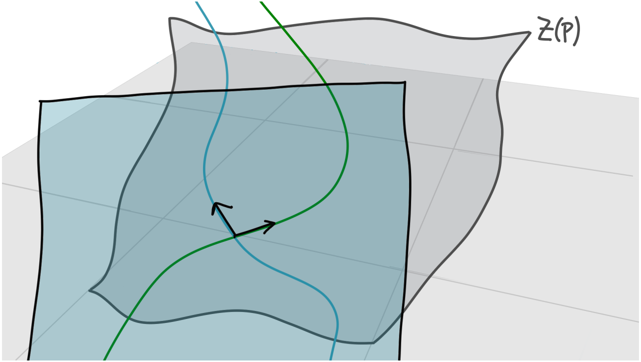

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- $\implies Z(\partial_z P)$ also contains all incidences.

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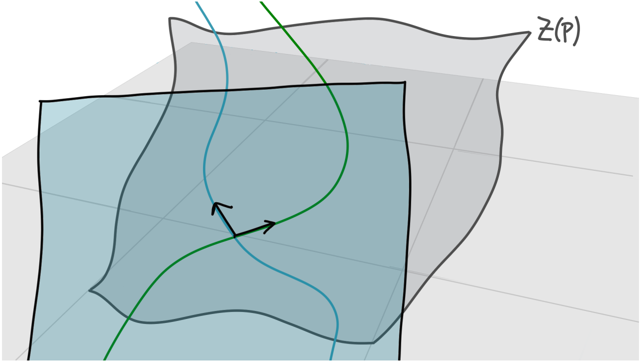
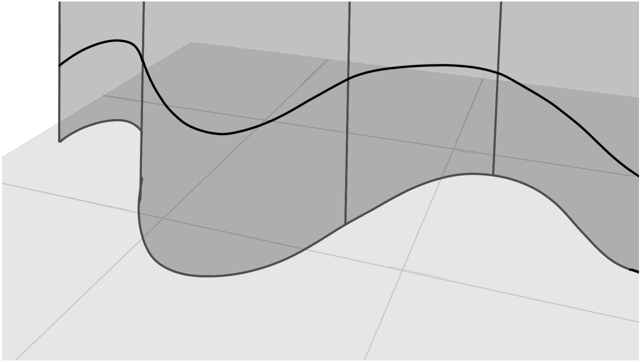


- Before we showed tangent space at incidences is vertical, so $\partial_z P = 0$ on all incidences!
- $\implies Z(\partial_z P)$ also contains all incidences.
- If $\deg P$ minimal $\implies P(X, Y, Z) = Q(X, Y)$.

Circle Tangencies: Contradiction



Recall that $\deg P = \deg Q \sim N^{1/2}$, but $Z(Q)$ contains N circles. Contradiction!

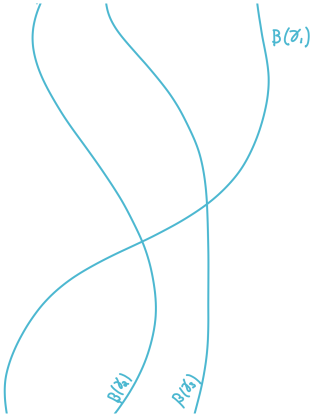
Circle Tangencies: Recap of Argument

Theorem

Given a (suitably non-degenerate) collection of N circles in $\mathbb{R}^2$, they determine $\lesssim N^{3/2}$ tangencies.

1. Assume there are $\gtrsim N^{3/2}$ tangencies.
2. Lift curves into $\mathbb{R}^3$ and change into an incidence problem.
3. Use a low degree polynomial P to interpolate these points. (parameter-counting)
4. Argue that if $Z(P)$ contains $\gtrsim N^{1/2}$ points of $\beta(\gamma)$ then $\beta(\gamma) \subset Z(P)$. (rigidity)
5. Use structure of the objects to argue $P(X, Y, Z) = Q(X, Y)$.
6. Contradiction as degree of Q is $\sim N^{1/2}$ but contains N circles.

Circle Tangencies: New Proof



Polynomial Partitioning: We can find a polynomial P of degree at most D such that $Z(P)$ partitions $\mathbb{R}^3$ into $\sim D^3$ cells such that each cell intersects $\lesssim \frac{N}{D^2}$ curves $\beta(\gamma)$.

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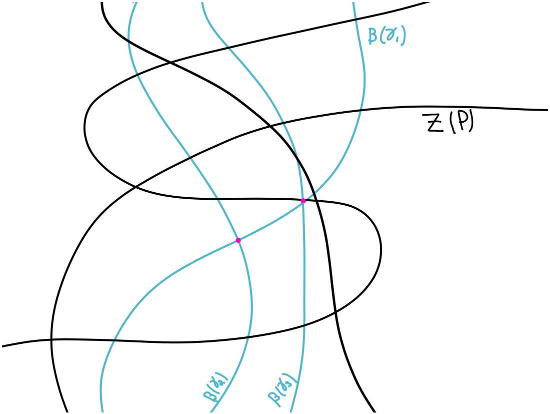

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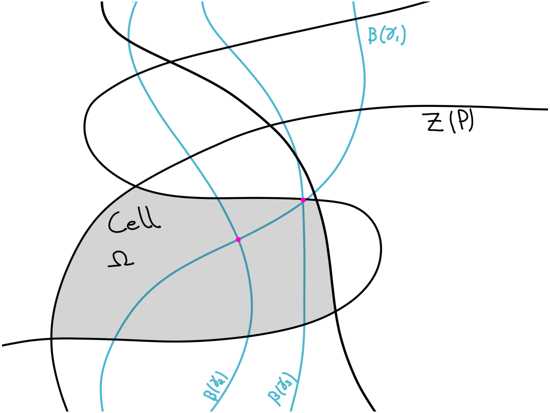

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Set of $\beta(\gamma)$'s can be partitioned:

$$C_1 = \{\beta(\gamma) \not\subset Z(P)\} \text{ and } C_2 = \{\beta(\gamma) \subset Z(P)\}$$

Total incidences of $\beta(\gamma)$'s $= I(C_1, C_1) + 2I(C_1, C_2) + I(C_2, C_2)$

Circle Tangencies: New Proof

$I(C_1, C_1)$ = incidences between curves inside the cells. Each cell contains $\lesssim \frac{N}{D^2}$ curves so:

$$I(C_1, C_1) \leq \sum_{\text{cells}} \left(\frac{N}{D^2}\right)^2$$

$$= D^3(N^2 D^{-4}) = N^2 D^{-1}$$

Circle Tangencies: New Proof

$I(C_1, C_2) =$ incidences between a curve β' in $Z(P)$ and a curve β not in $Z(P)$.

$$I(C_1, C_2) = \sum_{\beta \in C_1} \sum_{\beta' \in C_2} \mathbb{1}[\beta \cap \beta' \neq \emptyset]$$

Each $\beta(\gamma) \in C_1$ can intersect $Z(P)$ at most $\lesssim D$ times. (Bézout)

$$\lesssim \sum_{\beta \in C_1} D \lesssim ND$$

Circle Tangencies: New Proof

$I(C_2, C_2)$ = incidences between $\beta(\gamma)$'s both in $Z(P)$. We need to consider the class of polynomials:

$$\mathcal{P}_0 = \{R \in \mathbb{R}[X, Y, Z] \mid \forall \beta(\gamma) \in C_2,\ \beta(\gamma) \subset Z(R)\}$$

We choose the polynomial of minimal degree in $\mathcal{P}_0$ and label it P_0. We consider $\partial_z P_0$:

Case 1: $Z(\partial_z P_0)$ contains all the curves of C_2. Proceed by a similar argument to the previous proof to yield a bound of $\lesssim D^2$.

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Case 1: $Z(\partial_z P_0)$ contains all the curves of C_2. Proceed by a similar argument to the previous proof to yield a bound of $\lesssim D^2$.

Case 2: $C_2 \backslash Z(\partial_z P_0)$ is non-empty. We partition C_2 as follows:

$$C_1^{(1)} = \{\beta(\gamma) \not\subset Z(\partial_z P_0)\} \text{ and } C_2^{(1)} = \{\beta(\gamma) \subset Z(\partial_z P_0)\}$$

Case 2:

$$I(C_2, C_2) = I(C_1^{(1)}, C_2) + I(C_2^{(1)}, C_2^{(1)})$$

The first term is bounded by $\lesssim |C_1^{(1)}|D$ by Bézout.

The second term continues the recursive process.

Circle Tangencies: New Proof

$$\mathcal{P}_1 = \{R \in \mathbb{R}[X, Y, Z] \mid \forall \beta(\gamma) \in C_2^{(1)},\ \beta(\gamma) \subset Z(R)\}$$

We choose the polynomial of minimal degree and label it P_1. We consider $\partial_z P_1$:

Case 1: $Z(\partial_z P_1)$ contains all the curves of C_2. We pick up a term of $\lesssim D^2$, and our recursion stops.

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Case 2: $C_2 \backslash Z(\partial_z P_1)$ is non-empty. We partition $C_2^{(1)}$ as follows:

$$C_1^{(2)} = \{\beta(\gamma) \not\subset Z(\partial_z P_1)\} \text{ and } C_2^{(2)} = \{\beta(\gamma) \subset Z(\partial_z P_1)\}$$

Again we have

$$I(C_2^{(1)}, C_2^{(1)}) \lesssim |C_1^{(2)}|D + I(C_2^{(2)}, C_2^{(2)}).$$

Circle Tangencies: New Proof

At each step after choosing the minimal degree polynomial P_i in

$$\mathcal{P}_i = \{R \in \mathbb{R}[X, Y, Z] \mid \forall \beta(\gamma) \in C_2^{(i)},\ \beta(\gamma) \subset Z(R)\}.$$

We have either

Case 1: All curves are within $Z(\partial_z P_i)$ and we collect a term of at most D^2. The recursion stops.

Case 2: We partition $C_2^{(i)}$ and bound by $\lesssim |C_1^{(i)}|D + I(C_2^{(i+1)}, C_2^{(i+1)})$.

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Case 2: We partition $C_2^{(i)}$ and bound by $\lesssim |C_1^{(i)}|D + I(C_2^{(i+1)}, C_2^{(i+1)})$.

Hence,

$$I(C_2, C_2) \lesssim D^2 + \sum_i |C_1^{(i)}|D \lesssim D^2 + ND.$$

Circle Tangencies: New Proof

Adding these up we get the number of tangencies to be:

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We optimize D now by setting $N^2 D^{-1} \sim ND \implies D \sim N^{1/2}$.

We achieve:

$$\lesssim N^{3/2}$$

Circle Tangencies: Recap of New Proof

1. Lift curves into $\mathbb{R}^3$ and change into an incidence problem.
2. Use a low degree polynomial P to partition $\mathbb{R}^3$ into D^3 cells, each intersecting $\lesssim ND^{-2}$ curves. (parameter-counting + Borsuk-Ulam)
3. Use a trivial bound in each cell and sum over all cells.
4. Deal with curves contained in the zero set using algebraic tools.
5. Choose D to optimise inequality and achieve $\lesssim N^{3/2}$ bound.

Thank you for your attention.

Any questions?

Jordan S. Ellenberg, Jozsef Solymosi, and Joshua Zahl.
New bounds on curve tangencies and orthogonalities, 2016.