Polynomial Methods in Combinatorics

Conrad Crowley

Supervisor: Marco Vitturi

March 2022

What are Polynomial Methods?

A collection of techniques in Combinatorics which use polynomial interpolation and rigidity properties of polynomials to control the size of collections of objects with a certain structure.

What are Polynomial Methods?

A collection of techniques in Combinatorics which use polynomial interpolation and rigidity properties of polynomials to control the size of collections of objects with a certain structure.

Example (Rigidity and Interpolation)

- **Rigidity**: If $P \in \mathbb{R}[X_1, \ldots, X_n]$ has degree D and a line ℓ intersects $Z(P)$ in more than D points then $\ell \subset Z(P)$.

What are Polynomial Methods?

A collection of techniques in Combinatorics which use polynomial interpolation and rigidity properties of polynomials to control the size of collections of objects with a certain structure.

Example (Rigidity and Interpolation)

- **Rigidity**: If $P \in \mathbb{R}[X_1, \ldots, X_n]$ has degree D and a line ℓ intersects $Z(P)$ in more than D points then $\ell \subset Z(P)$.
- **Interpolation**: We can do parameter-counting arguments using the fact that $\dim \mathbb{R}_{\deg \leq D}[X_1, \ldots, X_n] \sim D^n$.

Theorems proven using polynomial methods

Below lists results that can be proved using the polynomial method:

- **Kakeya Conjecture in Finite Fields**: If $A \subset \mathbb{F}^n$ contains a line in every direction then $|A| \gtrsim |\mathbb{F}|^n$.

Theorems proven using polynomial methods

Below lists results that can be proved using the polynomial method:

- **Kakeya Conjecture in Finite Fields**: If $A \subset \mathbb{F}^n$ contains a line in every direction then $|A| \gtrsim |\mathbb{F}|^n$.
- **Cauchy-Davenport Theorem**: $|A + B| \geq \min\{p, |A| + |B| - 1\}$ where $A, B \subset \mathbb{Z}_p$ and $A + B := \{a + b \mid a \in A,\ b \in B\}$.

Theorems proven using polynomial methods

Below lists results that can be proved using the polynomial method:

- **Kakeya Conjecture in Finite Fields**: If $A \subset \mathbb{F}^n$ contains a line in every direction then $|A| \gtrsim |\mathbb{F}|^n$.
- **Cauchy-Davenport Theorem**: $|A + B| \geq \min\{p, |A| + |B| - 1\}$ where $A, B \subset \mathbb{Z}_p$ and $A + B := \{a + b \mid a \in A, \ b \in B\}$.
- **Joints Problem**: A collection of N lines in $\mathbb{R}^3$ can form at most $N^{3/2}$ joints. A joint is a point which lies in three non-coplanar lines.

Theorems proven using polynomial methods

Below lists results that can be proved using the polynomial method:

- **Kakeya Conjecture in Finite Fields**: If $A \subset \mathbb{F}^n$ contains a line in every direction then $|A| \gtrsim |\mathbb{F}|^n$.
- **Cauchy-Davenport Theorem**: $|A + B| \geq \min\{p, |A| + |B| - 1\}$ where $A, B \subset \mathbb{Z}_p$ and $A + B := \{a + b \mid a \in A,\ b \in B\}$.
- **Joints Problem**: A collection of N lines in $\mathbb{R}^3$ can form at most $N^{3/2}$ joints. A joint is a point which lies in three non-coplanar lines.
- **Szemerédi-Trotter Theorem**: Given S points and L lines, there are $\lesssim (SL)^{2/3} + S + L$ point-line incidences. (i.e. (p, ℓ) s.t. $p \in \ell$)

Theorems proven using polynomial methods

Below lists results that can be proved using the polynomial method:

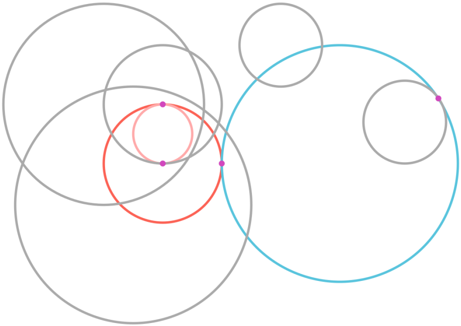
- **Kakeya Conjecture in Finite Fields**: If $A \subset \mathbb{F}^n$ contains a line in every direction then $|A| \gtrsim |\mathbb{F}|^n$.
- **Cauchy-Davenport Theorem**: $|A+B| \geq \min\{p, |A|+|B|-1\}$ where $A, B \subset \mathbb{Z}_p$ and $A+B := \{a+b \mid a \in A,\ b \in B\}$.
- **Joints Problem**: A collection of N lines in $\mathbb{R}^3$ can form at most $N^{3/2}$ joints. A joint is a point which lies in three non-coplanar lines.
- **Szemerédi-Trotter Theorem**: Given S points and L lines, there are $\lesssim (SL)^{2/3} + S + L$ point-line incidences. (i.e. (p, ℓ) s.t. $p \in \ell$)
- **Circle Tangencies**: Given a (suitably non-degenerate) collection of N circles in $\mathbb{R}^2$, they determine $\lesssim N^{3/2}$ tangencies.

Circle Tangencies

Theorem

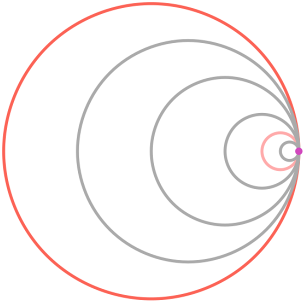
Given a (suitably non-degenerate) collection of N circles in $\mathbb{R}^2$, they determine $\lesssim N^{3/2}$ tangencies[a]*.*

[a] A tangency is a pair of circles (γ, γ') that are tangent.



Circle Tangencies: What's degenerate?

Collection of N circles with $\sim N^2$ tangencies:



Circle Tangencies: Lifting into $\mathbb{R}^3$

Theorem

Given a (suitably non-degenerate) collection of N circles in $\mathbb{R}^2$, they determine $\lesssim N^{3/2}$ tangencies.

We now present a sketch of a recent proof due to Ellenberg, Solymosi, and Zahl. [1]

Circle Tangencies: Lifting into $\mathbb{R}^3$

Theorem

Given a (suitably non-degenerate) collection of N circles in $\mathbb{R}^2$, they determine $\lesssim N^{3/2}$ tangencies.

We now present a sketch of a recent proof due to Ellenberg, Solymosi, and Zahl. [1]

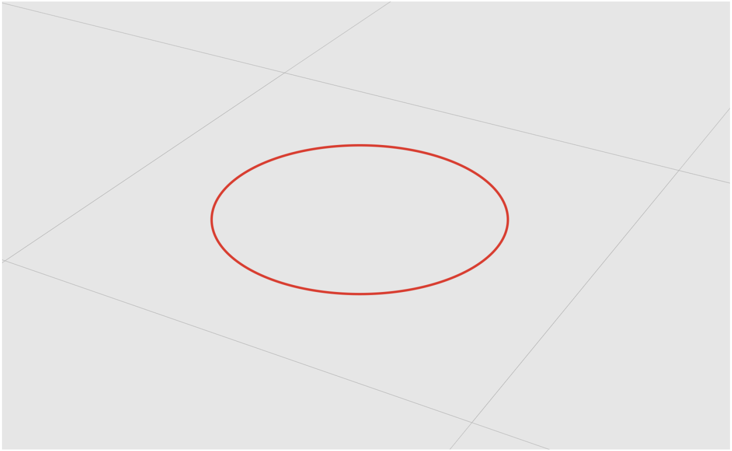
Sketch Proof: Assume:

- There are $\gtrsim N^{3/2}$ tangencies.
- Collection is uniform: each circle tangent to $\gtrsim N^{1/2}$ other circles.

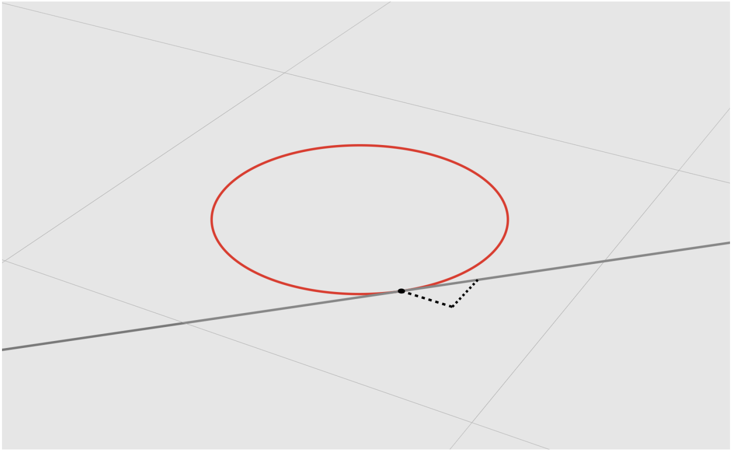
For each circle γ in our collection, we define the curve $\beta(\gamma) \subset \mathbb{R}^3$ as:

$$\beta(\gamma) := \left\{ (x, y, z) \mid (x, y) \in \gamma,\ z = \frac{1}{\text{Slope of tangent at } (x, y)} \right\}$$

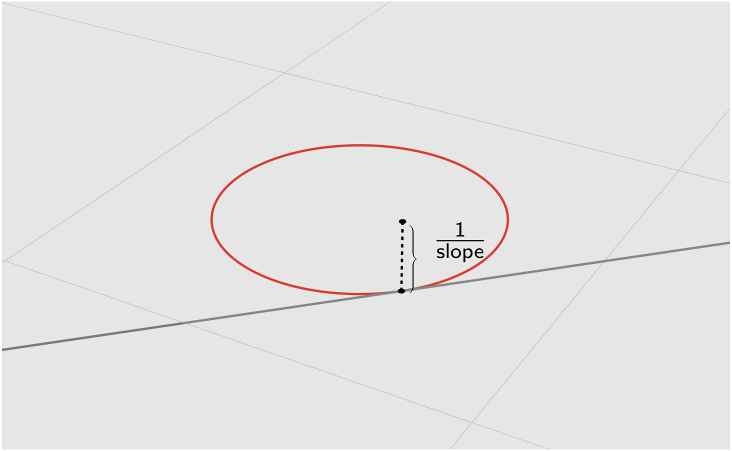
Circle Tangencies: Lifting into $\mathbb{R}^3$



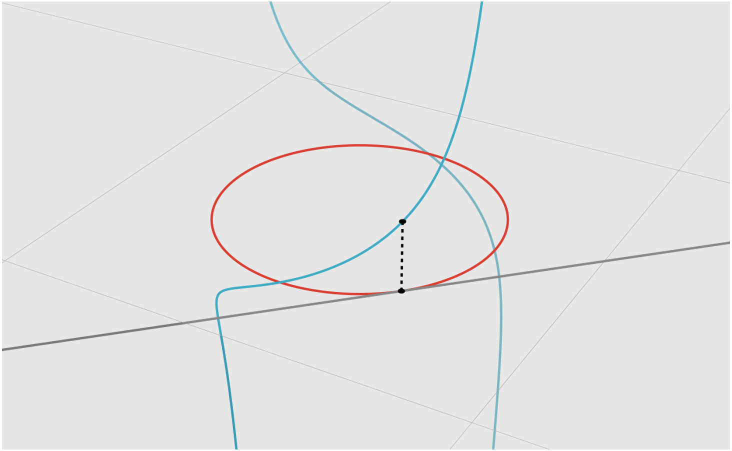
Circle Tangencies: Lifting into $\mathbb{R}^3$



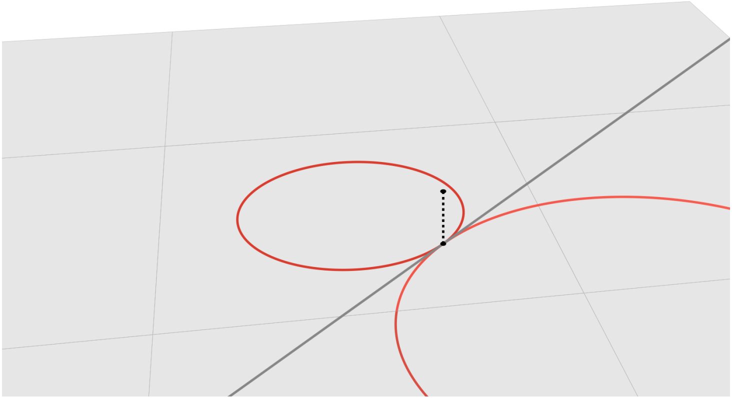
Circle Tangencies: Lifting into $\mathbb{R}^3$



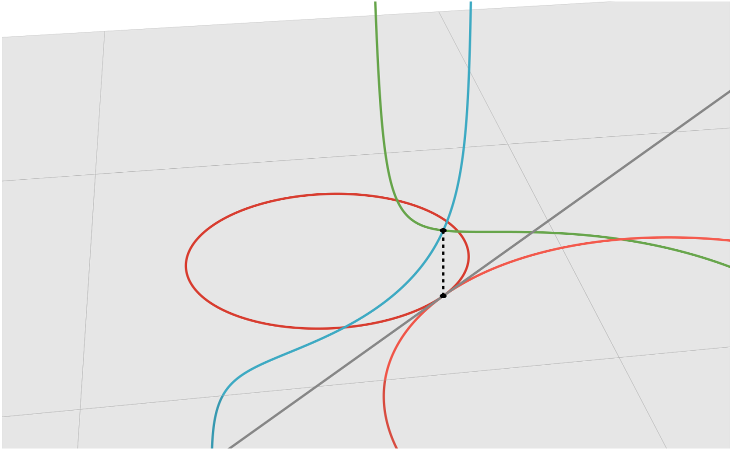
Circle Tangencies: Lifting into $\mathbb{R}^3$



Circle Tangencies: Tangencies to Incidences



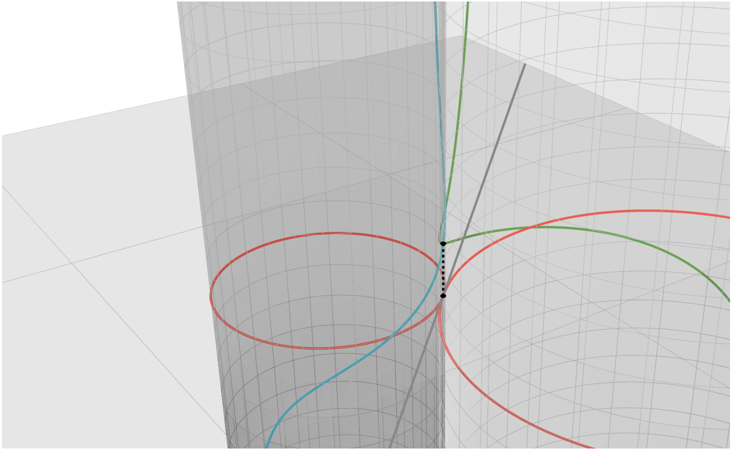
Circle Tangencies: Tangencies to Incidences



Intersection $\implies$ z co-ords equal $\implies$ circles are tangent.

Tangency problem in $\mathbb{R}^2$ $\iff$ Incidences problem in $\mathbb{R}^3$.

Circle Tangencies: Tangencies to Incidences



Intersection $\implies$ z co-ords equal $\implies$ circles are tangent.

Tangency problem in $\mathbb{R}^2$ $\iff$ Incidences problem in $\mathbb{R}^3$.

Circle Tangencies: Tangent Vectors at Incidences

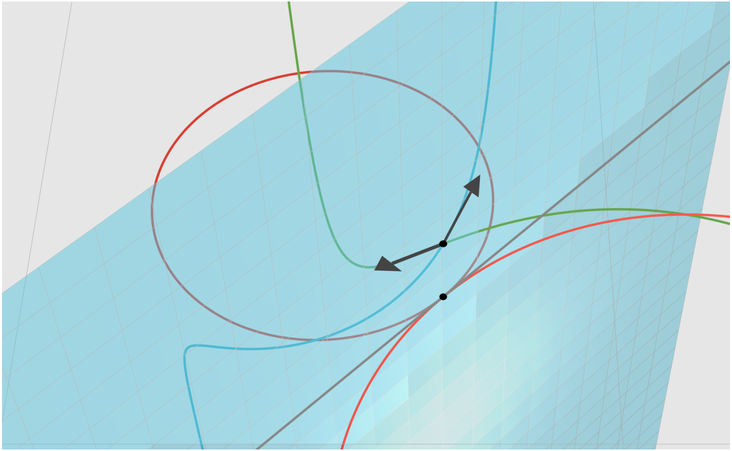
Let us examine the tangent vectors at a point of incidence between two curves:

Circle Tangencies: Tangent Vectors at Incidences

Circle Tangencies: Tangent Vectors at Incidences

At every point of intersection, the tangent vectors span a vertical plane.

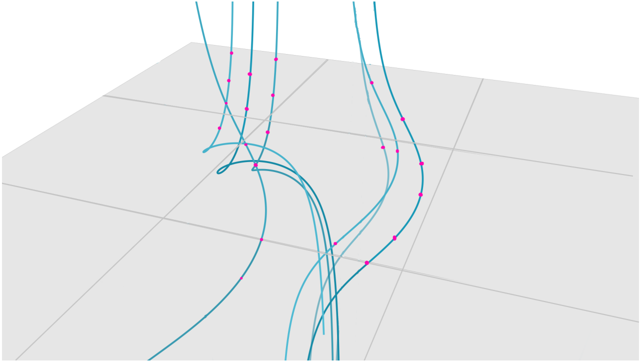
Circle Tangencies: Tangent Vectors at Incidences



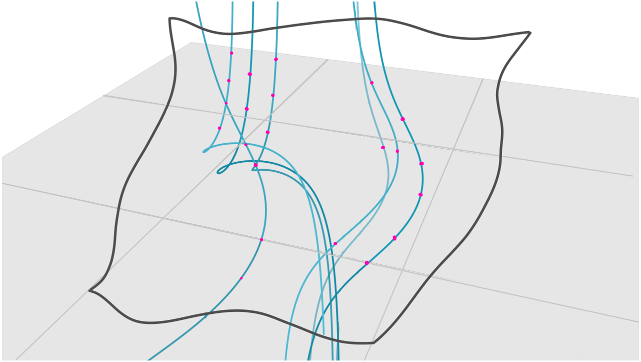
At every point of intersection, the tangent vectors span a vertical plane.

Circle Tangencies: Interpolation

Let P be a polynomial such that all $N^{3/2}$ incidences are contained in $Z(P)$.

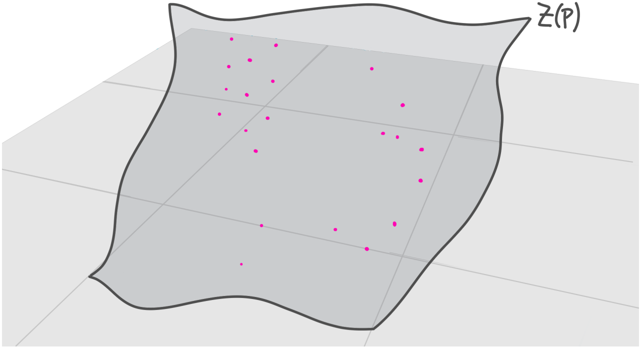


We can find a polynomial P such that all $N^{3/2}$ incidences are contained in $Z(P)$.



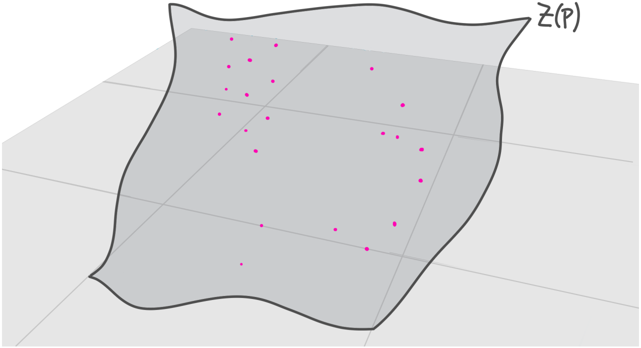
Circle Tangencies: Interpolation

Let P be a polynomial such that all $N^{3/2}$ incidences are contained in $Z(P)$.



Circle Tangencies: Interpolation

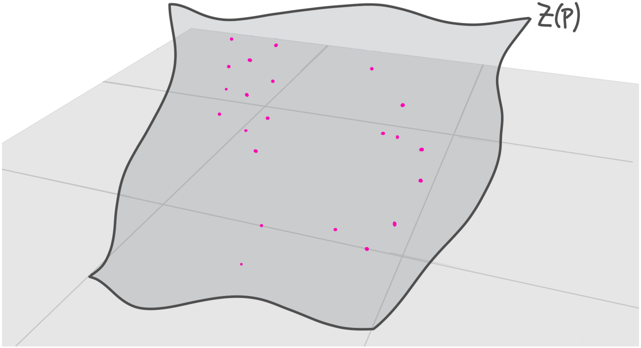
Let P be a polynomial such that all $N^{3/2}$ incidences are contained in $Z(P)$.



- Recall that $\dim \mathbb{R}_{\deg \leq D}[X, Y, Z] \sim D^3$.

Circle Tangencies: Interpolation

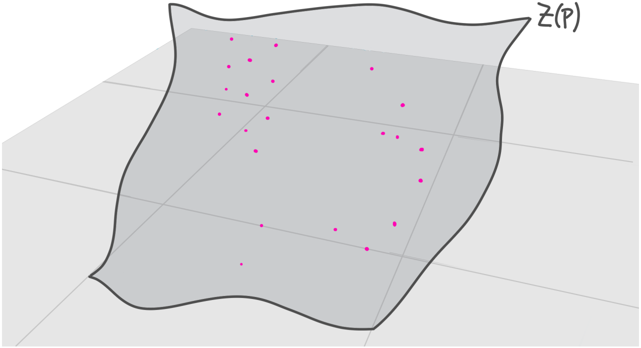
Let P be a polynomial such that all $N^{3/2}$ incidences are contained in $Z(P)$.



- Recall that $\dim \mathbb{R}_{\deg \leq D}[X, Y, Z] \sim D^3$.
- Each incidence point gives one linear equation for the coefficients.

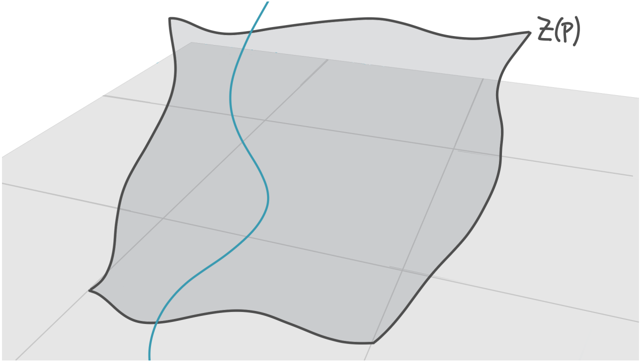
Circle Tangencies: Interpolation

Let P be a polynomial such that all $N^{3/2}$ incidences are contained in $Z(P)$.

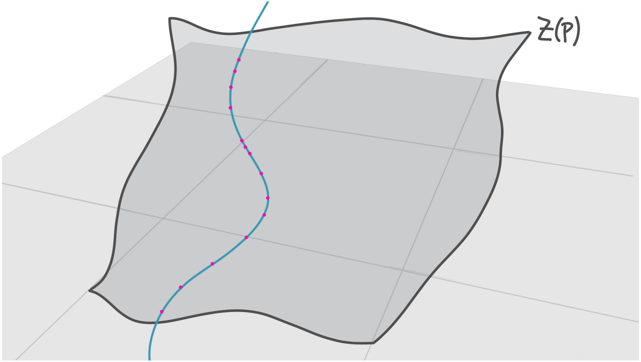


- Recall that $\dim \mathbb{R}_{\deg \leq D}[X, Y, Z] \sim D^3$.
- Each incidence point gives one linear equation for the coefficients.
- $D^3 \sim N^{3/2} \implies D \sim N^{1/2}$.

Circle Tangencies: $\beta(\gamma) \subset Z(P)$

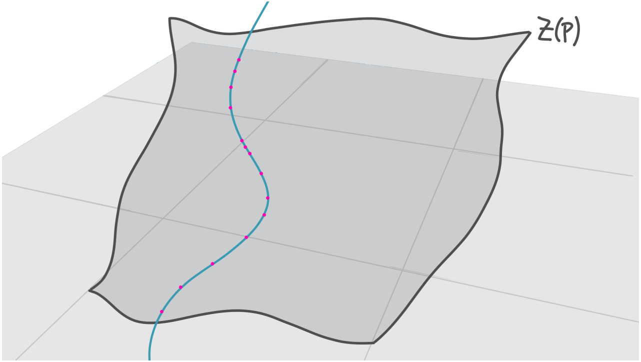


Circle Tangencies: $\beta(\gamma) \subset Z(P)$



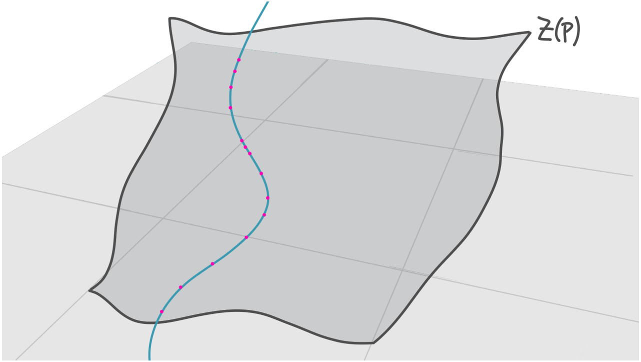
- Each curve $\beta(\gamma)$ intersects $Z(P)$ at $\gtrsim N^{1/2}$ points.

Circle Tangencies: $\beta(\gamma) \subset Z(P)$



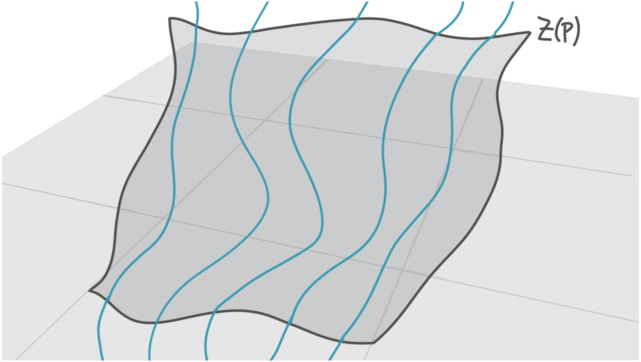
- Each curve $\beta(\gamma)$ intersects $Z(P)$ at $\gtrsim N^{1/2}$ points.
- But $\deg \beta(\gamma) = O(1)$ and $\deg P \sim N^{1/2}$

Circle Tangencies: $\beta(\gamma) \subset Z(P)$



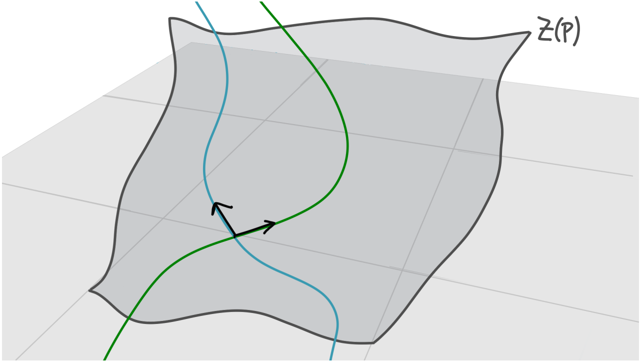
- Each curve $\beta(\gamma)$ intersects $Z(P)$ at $\gtrsim N^{1/2}$ points.
- But $\deg \beta(\gamma) = O(1)$ and $\deg P \sim N^{1/2}$
- $\implies \beta(\gamma) \subset Z(P)$ by Bézout's Theorem! (rigidity)

Circle Tangencies: $\beta(\gamma) \subset Z(P)$

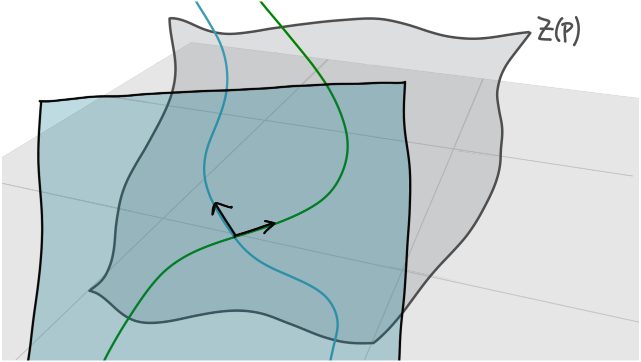


- Each curve $\beta(\gamma)$ contains $\gtrsim N^{1/2}$ points of intersection with $Z(P)$.
- $\deg \beta(\gamma) = O(1)$ and $\deg P \sim N^{1/2}$
- $\implies \beta(\gamma) \subset Z(P)$ by Bézout's Theorem.

Circle Tangencies: Tangent Vectors

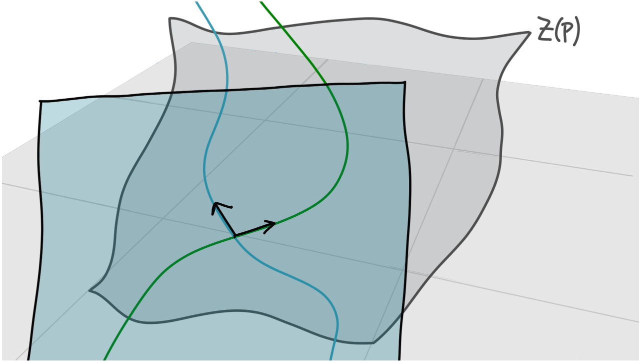


Circle Tangencies: Tangent Vectors



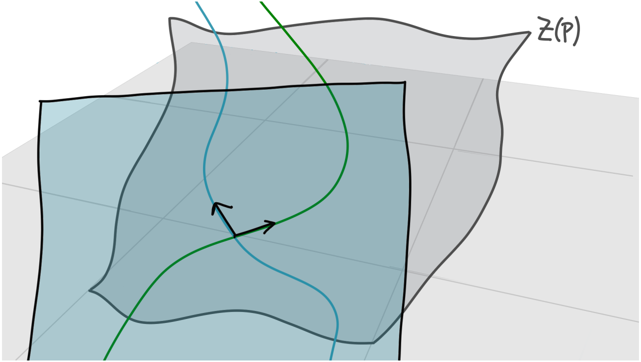
- Before we showed tangent space at incidences is vertical, so $\partial_z P = 0$ on all incidences!

Circle Tangencies: Tangent Vectors



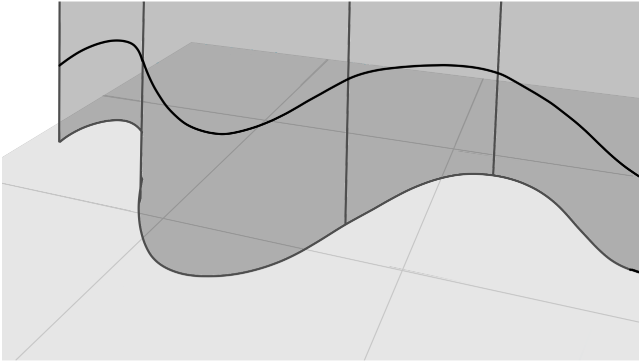
- Before we showed tangent space at incidences is vertical, so $\partial_z P = 0$ on all incidences!
- $\implies Z(\partial_z P)$ also contains all incidences.

Circle Tangencies: Tangent Vectors



- Before we showed tangent space at incidences is vertical, so $\partial_z P = 0$ on all incidences!
- $\implies Z(\partial_z P)$ also contains all incidences.
- If $\deg P$ minimal $\implies P(X, Y, Z) = Q(X, Y)$.

Circle Tangencies: Contradiction



Recall that $\deg P = \deg Q \sim N^{1/2}$, but $Z(Q)$ contains N circles.
Contradiction!

Circle Tangencies: Recap of Argument

Theorem

Given a (suitably non-degenerate) collection of N circles in $\mathbb{R}^2$, they determine $\lesssim N^{3/2}$ tangencies.

1. Assume there are $\gtrsim N^{3/2}$ tangencies.
2. Lift curves into $\mathbb{R}^3$ and change into an incidence problem.
3. Use a low degree polynomial P to interpolate these points. (parameter-counting)
4. Argue that if $Z(P)$ contains $\gtrsim N^{1/2}$ points of $\beta(\gamma)$ then $\beta(\gamma) \subset Z(P)$. (rigidity)
5. Use structure of the objects to argue $P(X, Y, Z) = Q(X, Y)$.
6. Contradiction as degree of Q is $\sim N^{1/2}$ but contains N circles.

Circle Tangencies: New Proof

image to be drawn

Circle Tangencies: New Proof

image to be drawn

Circle Tangencies: New Proof

Polynomial Partitioning:

- We can find a polynomial P of degree D such that $Z(P)$ partitions $\mathbb{R}^3$ into $\sim D^3$ cells. (parameter-counting + Borsuk-Ulam)
- Each cell is intersected by $\lesssim \frac{N}{D^2}$ curves $\beta(\gamma)$.
- $\#$ of incidences inside the cells $\lesssim D^3 \left(\frac{N}{D^2}\right)^2 = N^2 D^{-1}$

Circle Tangencies: New Proof

For the incidences between curves on $Z(P)$ there two cases:

- incidence on $Z(P)$ but $\beta(\gamma) \not\subset Z(P)$. $\implies \lesssim D$ such incidences per curve. (Bézout)
- $\beta(\gamma) \subset Z(P)$. Treat in a similar way to the original proof to achieve $\lesssim D^2$

Circle Tangencies: New Proof

Adding these up we get the number of tangencies to be:

$$\lesssim N^2 D^{-1} + ND + D^2$$

We optimize D now by setting $N^2 D^{-1} \sim ND \implies D \sim N^{1/2}$.

Circle Tangencies: New Proof

Adding these up we get the number of tangencies to be:

$$\lesssim N^2 D^{-1} + ND + D^2$$

We optimize D now by setting $N^2 D^{-1} \sim ND \implies D \sim N^{1/2}$.

We achieve:

$$\lesssim N^{3/2}$$

Thank you for your attention.

Any questions?

Jordan S. Ellenberg, Jozsef Solymosi, and Joshua Zahl.
New bounds on curve tangencies and orthogonalities, 2016.