Polynomial Methods in Combinatorics

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A collection of techniques in Combinatorics which use polynomials and their rigidity properties to argue a . . . .

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- Rigidity: If $P \in \mathbb{R}[X_1, \ldots, X_n]$ has degree D and a line ℓ intersects P in more than D points then $\ell \subset Z(P)$.

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- Interpolation: We can do parameter-counting arguments using the fact that $\dim \mathbb{R}_{\deg \leq D}[X_1, \ldots, X_n] \sim D^n$.

Theorems proven using polynomial methods

We examined proofs of these theorems using the polynomial method:

- Kakeya Conjecture in Finite Fields: If $A \subset \mathbb{F}^n$ contains a line in every direction then $|A| \gtrsim |\mathbb{F}|^n$.

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- Circle Tangencies: Given a (suitably non-degenerate) collection of N circles in $\mathbb{R}^2$, they determine $\lesssim N^{3/2}$ tangencies.

Circle Tangencies

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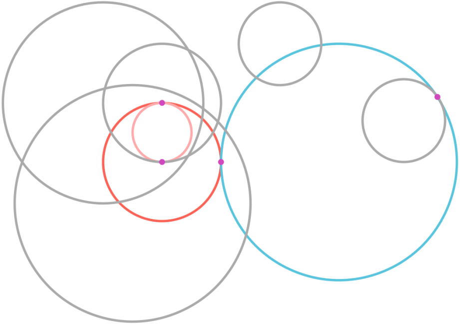
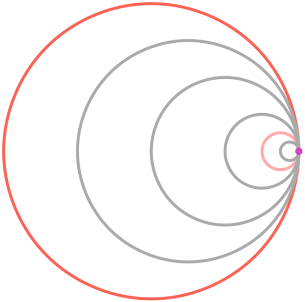


Circle Tangencies: What's degenerate?



Circle Tangencies: Lifting into $\mathbb{R}^3$

Theorem

Given a (suitably non-degenerate) collection of N circles in $\mathbb{R}^2$, they determine $\lesssim N^{3/2}$ tangencies.

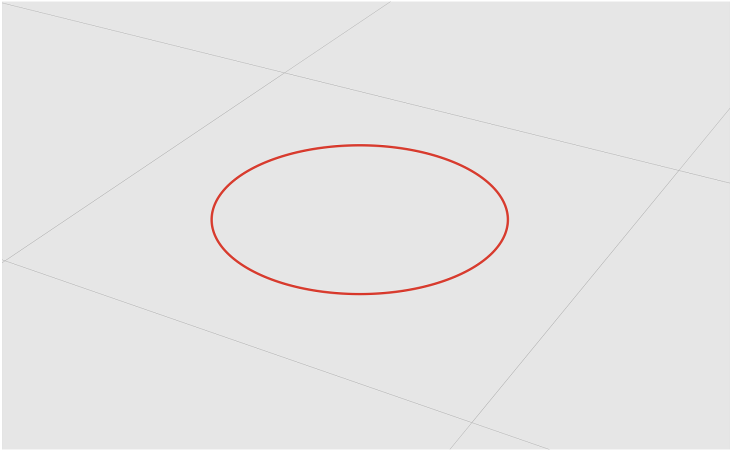
We now present a sketch of a recent proof due to Ellenberg, Solymosi, and Zahl. [1]

Sketch Proof: Assume that there are $\gtrsim N^{3/2}$ tangencies with each circle tangent to $\gtrsim N^{1/2}$ other circles in our collection.

For each circle γ in our collection, we define the curve $\beta(\gamma) \subset \mathbb{R}^3$ as:

$$\beta(\gamma) := \left\{ (x, y, z) \mid (x, y) \in \gamma, \; z = \frac{-1}{\text{Slope of tangent at } (x, y)} \right\}$$

Circle Tangencies: Lifting into $\mathbb{R}^3$



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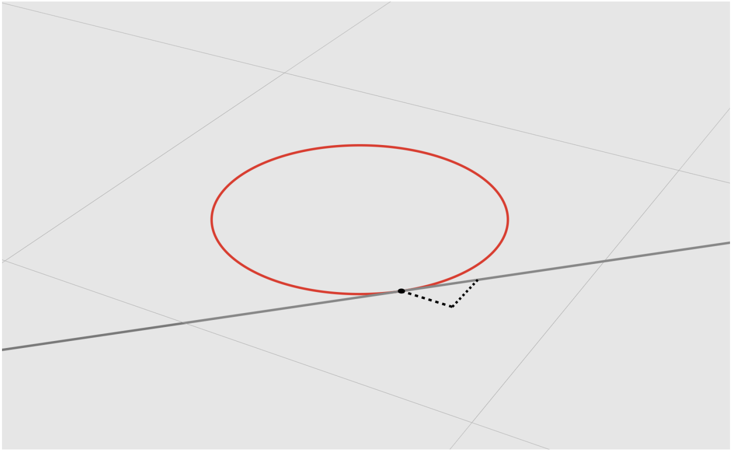

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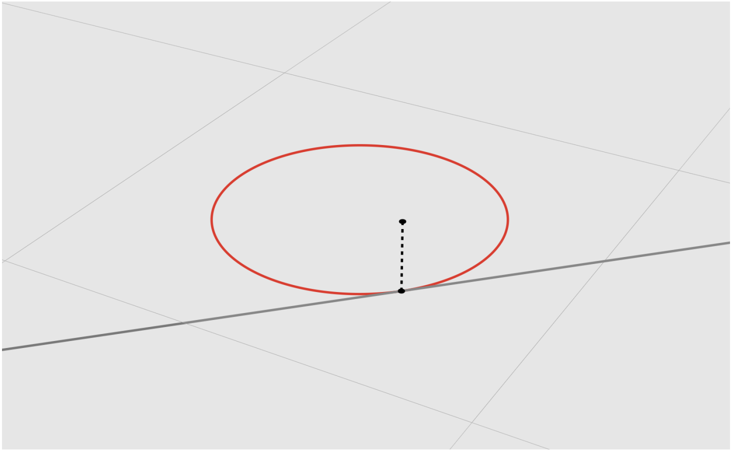

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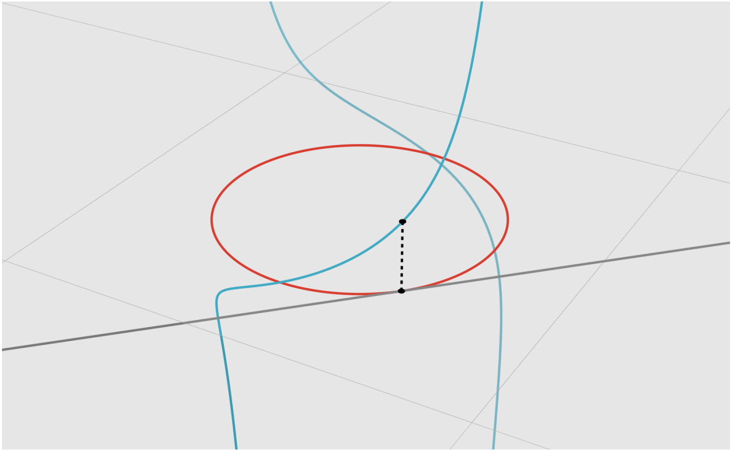
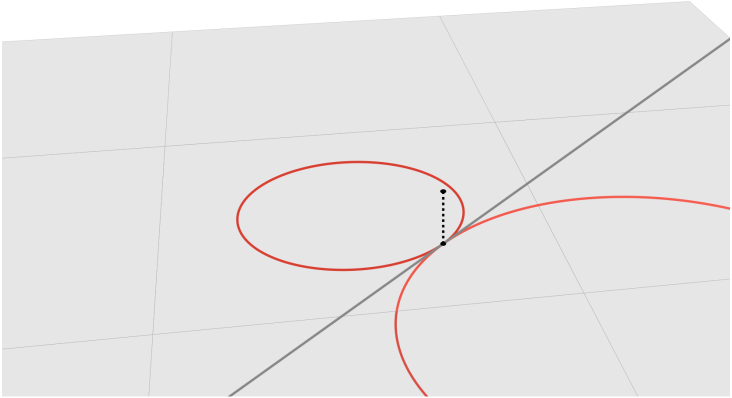


Circle Tangencies: Tangencies to Incidences



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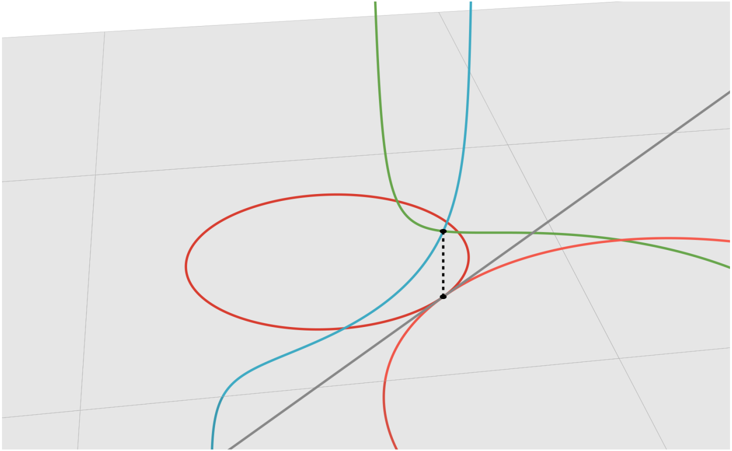

Tangency problem in $\mathbb{R}^2 \rightarrow$ Incidences problem in $\mathbb{R}^3$.

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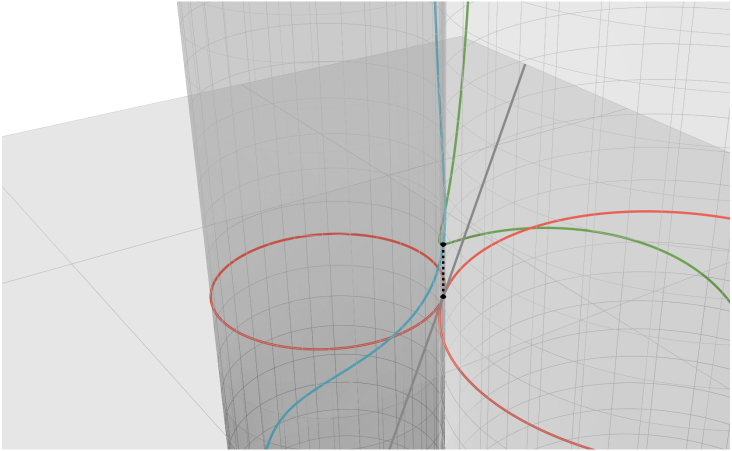

Tangency problem in $\mathbb{R}^2 \to$ Incidences problem in $\mathbb{R}^3$.

Circle Tangencies: Tangent Vectors at Incidences

Let us examine the tangent vectors at a point of incidence between two curves:

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At every point of intersection, the tangent vectors span a vertical plane.

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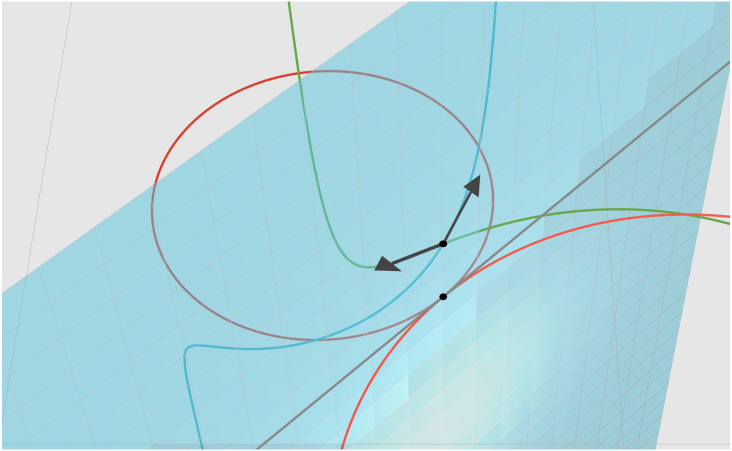

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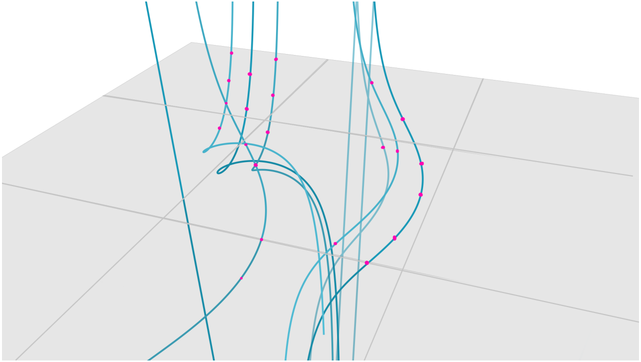
Circle Tangencies: Interpolation

Let P be a polynomial such that all $N^{3/2}$ incidences are contained in $Z(P)$.



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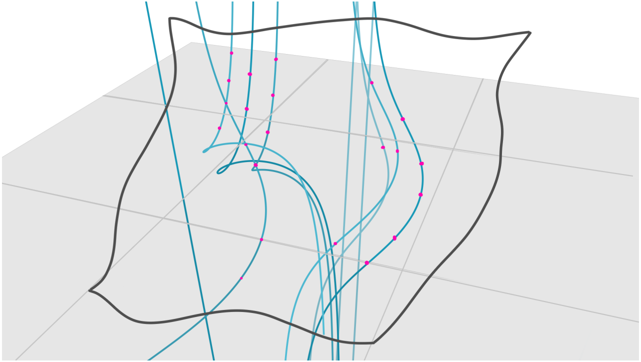

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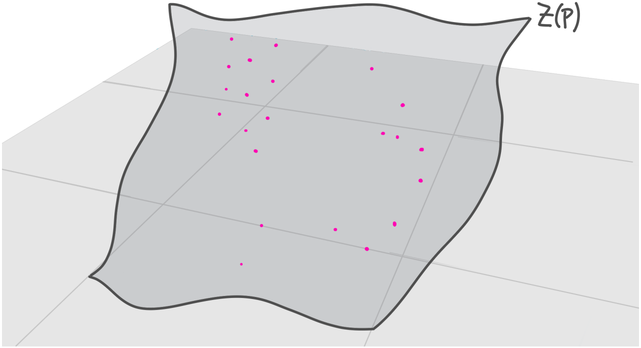

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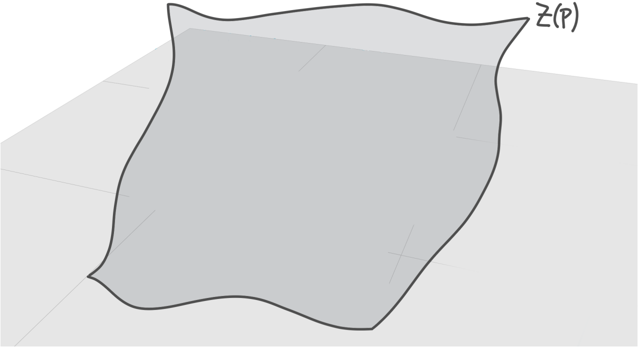

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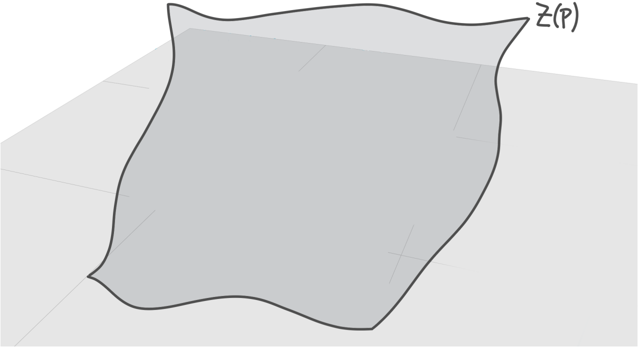

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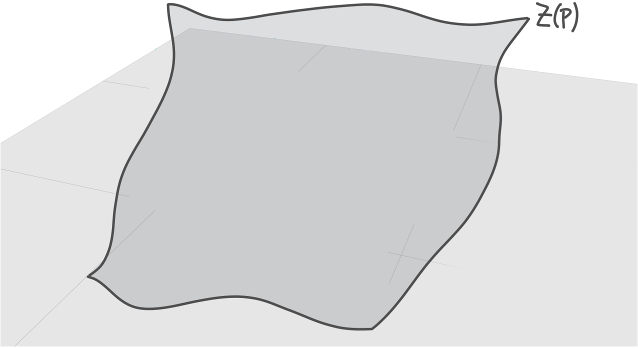

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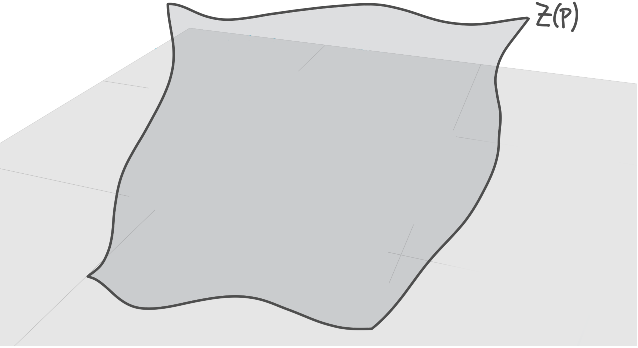

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- Each incidence point gives one linear equation for the coefficients.
- $D^3 \sim N^{3/2} \implies D \sim N^{1/2}$.

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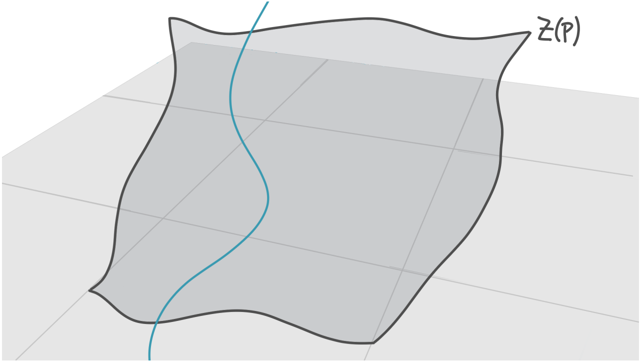

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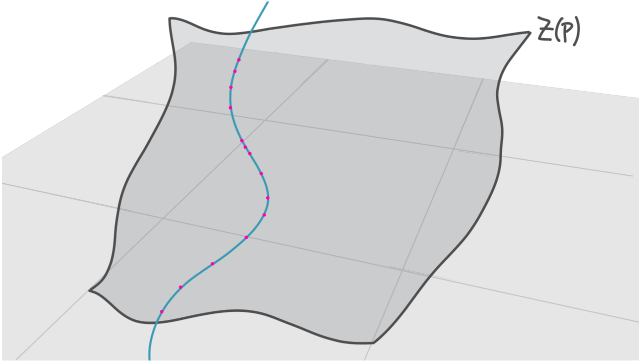

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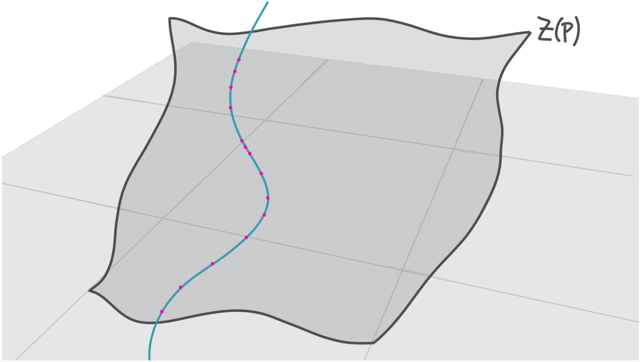

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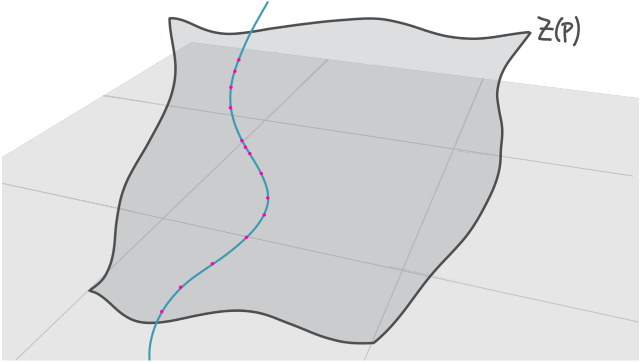

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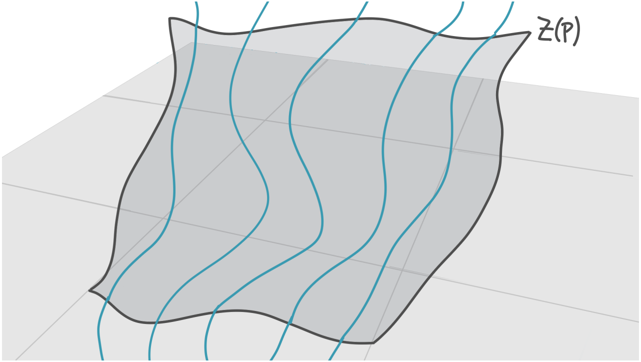

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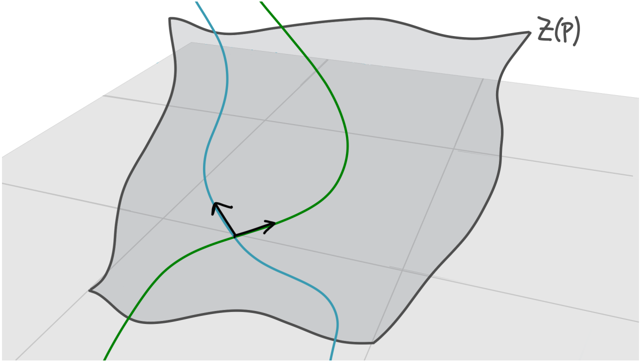

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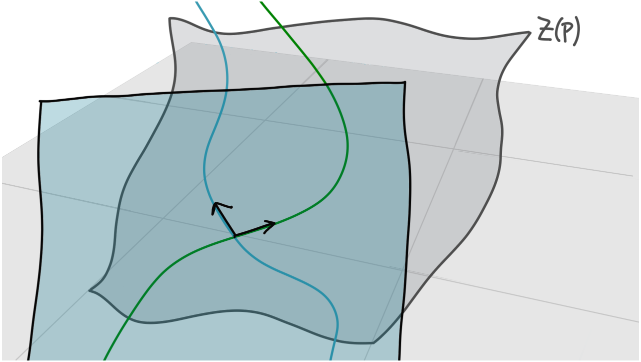

- Tangent vectors span the z-axis at tangencies, so $\partial_z P = 0$ on all incidences!
- $\implies Z(\partial_z P)$ also contains all incidences.
- $\implies P(X, Y, Z) = Q(X, Y)$.

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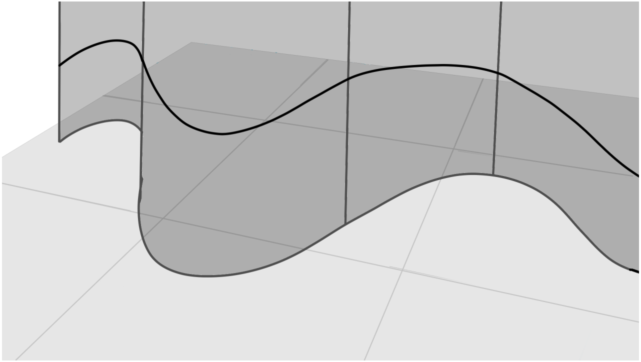

deg $Q \sim N^{1/2}$, but $Z(Q)$ contains N circles. Contradiction!

Circle Tangencies: Recap of Argument

1. Assume there are $\gtrsim N^{3/2}$ tangencies in our collection.
2. Lift curves into $\mathbb{R}^3$ and change into an incidence problem.
3. Use a low degree polynomial P to interpolate these points. (parameter-counting)
4. Argue that if $Z(P)$ contains $\gtrsim N^{1/2}$ points of $\beta(\gamma)$ then $\beta(\gamma) \subset Z(P)$. (rigidity)
5. Use structure of the objects to argue $P(X, Y, Z) = Q(X, Y)$
6. Contradiction as degree of Q is $N^{1/2}$ but contains N circles.

Jordan S. Ellenberg, Jozsef Solymosi, and Joshua Zahl.
New bounds on curve tangencies and orthogonalities, 2016.