Analytical Geometry and Calculus II

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Review of Theorems for Limits

Let a and c be any number, if $F = \lim_{x\to a} f(x)$ and $G = \lim_{x\to a} g(x)$ then

1. $\lim_{x\to a}(f(x) + g(x)) = F + G$
2. $\lim_{x\to a}(f(x) - g(x)) = F - G$
3. $\lim_{x\to a}(c * f(x)) = c * F$
4. $\lim_{x\to a}(\frac{f(x)}{g(x)}) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ except when $G = 0$
5. $\lim_{x\to a}(f(x))^c = F^c$

Limits Involving Infinity while $c \neq 0$

These are templates, where x is either taken to ∞ or 0.

1. $c * (\pm\infty) = \pm\infty$ Example: $\lim_{x\to\infty} 5x = \infty$
2. $\frac{c}{\pm\infty} = 0$ Example: $\lim_{x\to\infty} \frac{5}{x} = 0$
3. $\frac{c}{0} = \pm\infty$ Example: $\lim_{x\to 0} \frac{5}{0} = \infty$
4. $\frac{\pm\infty}{c} = \pm\infty$

Impossible Limits

- $\frac{0}{0}$
- $\frac{\infty}{\infty}$
- $\infty - \infty$
- $0 * \infty$

Limits of Rational Functions

Theorem: Given a rational function $\frac{f(x)}{g(x)}$, the following is be true. Let d represent the degree of $f(x)$ and e represent the degree of $g(x)$.

1. If $d > e$ then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \pm\infty$
2. If $d < e$ then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$
3. If $d = e$ then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{a}{b}$ where a is the leading term of $f(x)$ and b is the leading term of $g(x)$

Example: To evalute $lim_{x\to\infty} \frac{10x^3-3x^2+8}{\sqrt{25x^6+x^4+2}}$

1. Reduce all terms by x of the leading coefficient's degree.
2. Then take the limit of each term. $\frac{10-3x^{-1}+8x^{-3}}{\sqrt{25+x^{-2}+2x^{-6}}}$
3. Simplifiy. $\frac{10-0+0}{\sqrt{25+0+0}}$
4. $\frac{10}{\sqrt{25}} = \frac{10}{5} = 2$

Inverse Trig Functions

Definition: $y = sin^{-1}x$ is the value of y such that $x = siny$.
Domain: $-1 \leq x \leq 1$
Range: $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$

Definition: $y = cos^{-1}x$ is the value of y such that $x = cosy$.
Domain: $-1 \leq x \leq 1$
Range: $0 \leq y \leq \pi$

Other Trig Functions

- $y = tan^{-1}x \to x = tany$
  Range: $\frac{-\pi}{2} < y < \frac{\pi}{2}$
- $y = cot^{-1}x \to x = coty$
  Range: $0 < y < \pi$
- $y = sec^{-1}x \to x = secy$
  Range: $0 \leq y \leq \pi, y \neq \frac{\pi}{2}$
- $y = csc^{-1}x \to x = cscy$
  Range: $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$

Inverse Trig Identities

- $sin(sin^{-1}x) = x$
- $cos(cos^{-1}x) = x$
- $sin^{-1}(sinx) = x$ only if x is in range of sin^{-1}
- $cos^{-1}(cosx) = x$ only if x is in range of cos^{-1}

Example: $sin^{-1}(sin\pi) = sin^{-1}(0) = 0 \neq \pi$

Example: $cos(sin^{-1}x)$

1. Let $y = sin^{-1}x$ so that $x = siny$ and $cos(sin^{-1}x) = cosy$
2. Recall that $sin = \frac{opposite}{hypotenuse}$
3. Let $hypotenuse = 1$ and $opposite = b$ where b has yet to be determined.
4. Recall that $cosy = \frac{adjacent}{hypotenuse}$
5. $\frac{adjacent}{hypotenuse} = \frac{b}{1}$, and therefor $cosy = b$
6. Use the Pythagorean Theorem to solve: $x^2 + b^2 = 1^2$
7. $b^2 = 1 - x^2$
8. $b = \sqrt{1 - x^2}$
9. Therefor $cos(sin^{-1}x) = \sqrt{1 - x^2}$

Derivatives of Inverse Trig Functions

- $\frac{d}{dx}(sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(tan^{-1}x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(cot^{-1}x) = \frac{-1}{1-x^2}$
- $\frac{d}{dx}(sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$
- $\frac{d}{dx}(csc^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}$

Antiderivatives Involving Inverse Trig Functions

- $\int \frac{dx}{\sqrt{a^2-x^2}} = sin^{-1}\frac{x}{a} + C$
- $\int \frac{dx}{\sqrt{a^2+x^2}} = \frac{1}{a}tan^{-1}\frac{x}{a} + C$

- $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} sec^{-1} \frac{x}{a} + C$

L'Hopital's Rule

L'Hopital's Rule let's us evaluate impossible limits.

Theorem: Suppose $f(x)$ and $g(x)$ are differentiable on an open interval I containing a where $g'(x) \neq 0$ on I when $x \neq a$. If

1. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$
2. $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$. This is also true as $x \to \pm\infty, x \to a^+, x \to a^-$.