Analytical Geometry and Calculus II

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Review of Theorems for Limits

Let a and c be any number, if $F = \lim_{x\to a} f(x)$ and $G = \lim_{x\to a} g(x)$ then

1. $\lim_{x\to a}(f(x) + g(x)) = F + G$
2. $\lim_{x\to a}(f(x) - g(x)) = F - G$
3. $\lim_{x\to a}(c * f(x)) = c * F$
4. $\lim_{x\to a}(\frac{f(x)}{g(x)}) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ except when $G = 0$
5. $\lim_{x\to a}(f(x))^c = F^c$

Limits Involving Infinity while $c \neq 0$

These are templates, where x is either taken to ∞ or 0.

1. $c * (\pm\infty) = \pm\infty$ Example: $\lim_{x\to\infty} 5x = \infty$
2. $\frac{c}{\pm\infty} = 0$ Example: $\lim_{x\to\infty} \frac{5}{x} = 0$
3. $\frac{c}{0} = \pm\infty$ Example: $\lim_{x\to 0} \frac{5}{0} = \infty$
4. $\frac{\pm\infty}{c} = \pm\infty$

Limits of Rational Functions

Theorem: Given a rational function $\frac{f(x)}{g(x)}$, the following is be true. Let d represent the degree of $f(x)$ and e represent the degree of $g(x)$.

1. If $d > e$ then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \pm\infty$
2. If $d < e$ then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$
3. If $d = e$ then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{a}{b}$ where a is the leading term of $f(x)$ and b is the leading term of $g(x)$

Example: To evalute $lim_{x\to\infty} \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$

1. Reduce all terms by x of the leading coefficient's degree.
2. Then take the limit of each term. $\frac{10-3x^{-1}+8x^{-3}}{\sqrt{25+x^{-2}+2x^{-6}}}$
3. Simplifiy. $\frac{10-0+0}{\sqrt{25+0+0}}$
4. $\frac{10}{\sqrt{25}} = \frac{10}{5} = 2$

Inverse Trig Functions

Definition: $y = sin^{-1}x$ is the value of y such that $x = siny$.
Domain: $-1 \leq x \leq 1$
Range: $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$

Definition: $y = cos^{-1}x$ is the value of y such that $x = cosy$.
Domain: $-1 \leq x \leq 1$
Range: $0 \leq y \leq \pi$

Other Trig Functions

- $y = tan^{-1}x \rightarrow x = tany$
  Range: $\frac{-\pi}{2} < y < \frac{\pi}{2}$
- $y = cot^{-1}x \rightarrow x = coty$
  Range: $0 < y < \pi$
- $y = sec^{-1}x \rightarrow x = secy$
  Range: $0 \leq y \leq \pi, y \neq \frac{\pi}{2}$
- $y = csc^{-1}x \rightarrow x = cscy$
  Range: $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$

Inverse Trig Identities

- $sin(sin^{-1}x) = x$
- $cos(cos^{-1}x) = x$
- $sin^{-1}(sinx) = x$ only if x is in range of sin^{-1}
- $cos^{-1}(cosx) = x$ only if x is in range of cos^{-1}

Example: $sin^{-1}(sin\pi) = sin^{-1}(0) = 0 \neq \pi$

Example: $cos(sin^{-1}x)$

1. Let $y = sin^{-1}x$ so that $x = siny$ and $cos(sin^{-1}x) = cosy$
2. Recall that $sin = \frac{opposite}{hypotenuse}$

3. Let $hypotenuse = 1$ and $opposite = b$ where b has yet to be determined.

4. Recall that $cosy = \frac{adjacent}{hypotenuse}$

5. $\frac{adjacent}{hypotenuse} = \frac{b}{1}$, and therefor $cosy = b$

6. Use the Pythagorean Theorem to solve: $x^2 + b^2 = 1^2$

7. $b^2 = 1 - x^2$

8. $b = \sqrt{1 - x^2}$

9. Therefor $cos(sin^{-1}x) = \sqrt{1 - x^2}$

Derivatives of Inverse Trig Functions

- $\frac{d}{dx}(sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(tan^{-1}x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(cot^{-1}x) = \frac{-1}{1-x^2}$
- $\frac{d}{dx}(sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$
- $\frac{d}{dx}(csc^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}$

Antiderivatives Involving Inverse Trig Functions

- $\int \frac{dx}{\sqrt{a^2-x^2}} = sin^{-1}\frac{x}{a} + C$
- $\int \frac{dx}{a^2+x^2} = \frac{1}{a}tan^{-1}\frac{x}{a} + C$
- $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}sec^{-1}\frac{x}{a} + C$

L'Hopital's Rule

L'Hopital's Rule let's us evaluate impossible limits.

Theorem: Suppose $f(x)$ and $g(x)$ are differentiable on an open interval I containing a where $g'(x) \neq 0$ on I when $x \neq a$. If

1. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$
2. $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$. This is also true as $x \to \pm\infty, x \to a^+, x \to a^-$.

Basic Approaches to Integration

- Subtle Substitution
  $\int \frac{dx}{x^{-1}+1}$
  $= \int \frac{x}{x+1} dx$ and suppose $u = x+1, x = u-1, du = dx$
  $= \int \frac{u-1}{u} du$
  $= \int du - \int \frac{1}{u} du = u - \ln|u| + C$
  $= x + 1 - ln|x| + C$

- Splitting Fractions
  $\int \frac{2-3x}{\sqrt{1-x^2}} dx$
  $= \int \frac{2}{\sqrt{1-x^2}} dx - \frac{3x}{\sqrt{1-x^2}} dx$
  $= \int 2\sin^{-1} x - \frac{3x}{\sqrt{1-x^2}} dx$
  etc...

- Completing the Square
  $\int \frac{dx}{\sqrt{27-6x-x^2}}$
  $= \int \frac{dx}{\sqrt{-(x^2+6x-27)}}$
  $= \int \frac{dx}{\sqrt{-((x+3)^2-36)}}$
  $= \int \frac{dx}{\sqrt{36-(x+3)^2}}$
  $= \sin^{-1}(\frac{x+3}{6}) + C$

- Multiplying by 1 (Using Conjugates)
  $\int \frac{dx}{1+\sin x}$ the conjugate of $1+\sin x$ is $1-\sin x$
  $= \int \frac{dx(1-\sin x)}{(1+\sin x)(1-\sin x)}$
  $= \int \frac{1-\sin x}{1-\sin^2 x} dx$
  $= \int \frac{1-\sin x}{\cos^2 x} dx$
  $= \int \frac{1}{\cos^2 x} dx - \int \frac{\sin x}{\cos^2 x} dx$
  $= \int \sec^2 x dx - \int \tan x \sec x dx$
  $= \tan x - \sec x + C$

Integration by Parts

Integration by Parts, or IBP, is used when integrating products of functions. It's not perfect, and can get messy if used incorrectly. The basic form is

$\int u dv = uv - \int v du$

Solve by substituting u and dv, then using the right hand form.

Example: $\int te^t dt$

1. Let $u = t$ and $dv = e^t dt$ so that $du = dt$ and $v = e^t$
2. $= te^t - \int e^t dt$
3. $= te^t - e^t + C$

Trigometetric Integrals

Products of sin and cos

- If the power of sin or cos split off 1 factor and use $\sin^2 x + \cos^2 x = 1$
  Example: $\int \cos^3 x dx \rightarrow \int \cos^2 x \cos x \rightarrow \int (1 - \sin^2 x) \cos x dx$
- If the power of sin or cos is even, use a half-angle identity.
  $\cos^2 x = \frac{1+\cos^2 x}{2}$ and $\sin^2 x = \frac{1-\sin^2 x}{2}$
  Example: $\int \sin^2 x dx = \int \frac{1-\cos^2 x}{2} dx$
  $= \int \frac{1}{2} dx - \int \frac{1}{2} \cos^2 x dx$
  $= \frac{x}{2} - \frac{1}{4} \sin 2x + C$

Products of of powers of sin and cos

- If the power of sinx or cosx is odd, split off a factor and rewrite the resulting power in terms of the opposite, then use U-Substituion.

Powers of tan, sec, cot and csc

- $\int \sec^2 x = \tan x + C$
- $\int \tan^2 x = \int \sec^2 x + 1 = \tan x + x + C$
- $\int \csc^2 x = -\cot x + C$
- $\int \cot^2 x = \int \csc^2 x + 1 = -\cot x + x + C$

Products of Powers of tan and sec

- If the power of sec is even, split $\sec^2$, rewrite in terms of tanx in terms of sec then use U-Sub on $\tan x$.
  Example: $\int \sec^2 x \tan^{1/2} x dx$
  Let $u = \tan x$ and $du = \sec^2 x$
  $= \int u^{1/2} du$
  $= \frac{2}{3} u^{3/2} + C$
- If the power of tan is odd, split off secxtanx, rewrite remaining even power of tanx in terms of secx then use U-Sub on $\sec x$.
  Example: $10 \int \tan^9 x \sec^2 x dx = 10 \int \tan^8 x \sec x \sec x \tan x dx$
  $= 10 \int (\sec^2 x - 1)^4 \sec x \sec x \tan x dx$
  $u = \sec x$ and $du = \sec x \tan x dx$
  $= 10 \int (u^2 - 1)^4 u du$