

# Analytical Geometry and Calculus II

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## 1 Review of Theorems for Limits

Let  $a$  and  $c$  be any number, if  $F = \lim_{x \rightarrow a} f(x)$  and  $G = \lim_{x \rightarrow a} g(x)$  then

1.  $\lim_{x \rightarrow a} (f(x) + g(x)) = F + G$
2.  $\lim_{x \rightarrow a} (f(x) - g(x)) = F - G$
3.  $\lim_{x \rightarrow a} (c * f(x)) = c * F$
4.  $\lim_{x \rightarrow a} \left( \frac{f(x)}{g(x)} \right) = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$  except when  $G = 0$
5.  $\lim_{x \rightarrow a} (f(x))^c = F^c$

## 2 Limits Involving Infinity while $c \neq 0$

These are templates, where  $x$  is either taken to  $\infty$  or  $0$ .

1.  $c * (\pm\infty) = \pm\infty$  Example:  $\lim_{x \rightarrow \infty} 5x = \infty$
2.  $\frac{c}{\pm\infty} = 0$  Example:  $\lim_{x \rightarrow \infty} \frac{5}{x} = 0$
3.  $\frac{c}{0} = \pm\infty$  Example:  $\lim_{x \rightarrow 0} \frac{5}{x} = \infty$
4.  $\frac{\pm\infty}{c} = \pm\infty$

## 3 Limits of Rational Functions

Theorem: Given a rational function  $\frac{f(x)}{g(x)}$ , the following is be true. Let  $d$  represent the degree of  $f(x)$  and  $e$  represent the degree of  $g(x)$ .

1. If  $d > e$  then  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \pm\infty$
2. If  $d < e$  then  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0$

3. If  $d = e$  then  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \frac{a}{b}$  where  $a$  is the leading term of  $f(x)$  and  $b$  is the leading term of  $g(x)$

Example: To evaluate  $\lim_{x \rightarrow \infty} \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$

1. Reduce all terms by  $x$  of the leading coefficient's degree.
2. Then take the limit of each term.  $\frac{10 - 3x^{-1} + 8x^{-3}}{\sqrt{25 + x^{-2} + 2x^{-6}}}$
3. Simplify.  $\frac{10 - 0 + 0}{\sqrt{25 + 0 + 0}}$
4.  $\frac{10}{\sqrt{25}} = \frac{10}{5} = 2$

## 4 Inverse Trig Functions

Definition:  $y = \sin^{-1}x$  is the value of  $y$  such that  $x = \sin y$ .

Domain:  $-1 \leq x \leq 1$

Range:  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

Definition:  $y = \cos^{-1}x$  is the value of  $y$  such that  $x = \cos y$ .

Domain:  $-1 \leq x \leq 1$

Range:  $0 \leq y \leq \pi$

### 4.1 Other Trig Functions

- $y = \tan^{-1}x \rightarrow x = \tan y$   
Range:  $-\frac{\pi}{2} < y < \frac{\pi}{2}$
- $y = \cot^{-1}x \rightarrow x = \cot y$   
Range:  $0 < y < \pi$
- $y = \sec^{-1}x \rightarrow x = \sec y$   
Range:  $0 \leq y \leq \pi, y \neq \frac{\pi}{2}$
- $y = \csc^{-1}x \rightarrow x = \csc y$   
Range:  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$

## 5 Inverse Trig Identities

- $\sin(\sin^{-1}x) = x$
- $\cos(\cos^{-1}x) = x$
- $\sin^{-1}(\sin x) = x$  only if  $x$  is in range of  $\sin^{-1}$
- $\cos^{-1}(\cos x) = x$  only if  $x$  is in range of  $\cos^{-1}$

Example:  $\sin^{-1}(\sin \pi) = \sin^{-1}(0) = 0 \neq \pi$

Example:  $\cos(\sin^{-1}x)$

1. Let  $y = \sin^{-1}x$  so that  $x = \sin y$  and  $\cos(\sin^{-1}x) = \cos y$
2. Recall that  $\sin = \frac{\text{opposite}}{\text{hypotenuse}}$
3. Let  $\text{hypotenuse} = 1$  and  $\text{opposite} = b$  where  $b$  has yet to be determined.
4. Recall that  $\cos y = \frac{\text{adjacent}}{\text{hypotenuse}}$
5.  $\frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{1}$ , and therefor  $\cos y = b$
6. Use the Pythagorean Theorem to solve:  $x^2 + b^2 = 1^2$
7.  $b^2 = 1 - x^2$
8.  $b = \sqrt{1 - x^2}$
9. Therefor  $\cos(\sin^{-1}x) = \sqrt{1 - x^2}$

## 6 Derivatives of Inverse Trig Functions

- $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$
- $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$
- $\frac{d}{dx}(\csc^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}$

## 7 Antiderivatives Involving Inverse Trig Functions

- $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + C$
- $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$
- $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} + C$

## 8 L'Hopital's Rule

L'Hopital's Rule let's us evaluate impossible limits.

Theorem: Suppose  $f(x)$  and  $g(x)$  are differentiable on an open interval  $I$  containing  $a$  where  $g'(x) \neq 0$  on  $I$  when  $x \neq a$ . If

1.  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$
2.  $\lim_{x \rightarrow a} f(x) = \pm\infty$  and  $\lim_{x \rightarrow a} g(x) = \pm\infty$

then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ . This is also true as  $x \rightarrow \pm\infty, x \rightarrow a^+, x \rightarrow a^-$ .

## 9 Basic Approaches to Integration

- Subtle Substitution

$$\begin{aligned} & \int \frac{dx}{x^{-1}+1} \\ &= \int \frac{x}{x+1} dx \text{ and suppose } u = x+1, x = u-1, du = dx \\ &= \int \frac{u-1}{u} du \\ &= \int du - \int \frac{1}{u} du = u - \ln|u| + C \\ &= x+1 - \ln|x| + C \end{aligned}$$

- Splitting Fractions

$$\begin{aligned} & \int \frac{2-3x}{\sqrt{1-x^2}} dx \\ &= \int \frac{2}{\sqrt{1-x^2}} dx - \frac{3x}{\sqrt{1-x^2}} dx \\ &= \int 2 \sin^{-1} x - \frac{3x}{\sqrt{1-x^2}} dx \\ & \text{etc...} \end{aligned}$$

- Completing the Square

$$\begin{aligned} & \int \frac{dx}{\sqrt{27-6x-x^2}} \\ &= \int \frac{dx}{\sqrt{-(x^2+6x-27)}} \\ &= \int \frac{dx}{\sqrt{-(x+3)^2-36}} \\ &= \int \frac{dx}{\sqrt{36-(x+3)^2}} \\ &= \sin^{-1} \left( \frac{x+3}{6} \right) + C \end{aligned}$$

- Multiplying by 1 (Using Conjugates)
 
$$\begin{aligned}
 & \int \frac{dx}{1+\sin x} \quad \text{the conjugate of } 1+\sin x \text{ is } 1-\sin x \\
 &= \int \frac{dx(1-\sin x)}{(1+\sin x)(1-\sin x)} \\
 &= \int \frac{1-\sin x}{1-\sin^2 x} dx \\
 &= \int \frac{1-\sin x}{\cos^2 x} dx \\
 &= \int \frac{1}{\cos^2 x} dx - \int \frac{\sin x}{\cos^2 x} dx \\
 &= \int \sec^2 x dx - \int \tan x \sec x dx \\
 &= \tan x - \sec x + C
 \end{aligned}$$

## 10 Integration by Parts

Integration by Parts, or IBP, is used when integrating products of functions. It's not perfect, and can get messy if used incorrectly. The basic form is

$$\int u dv = uv - \int v du$$

Solve by substituting  $u$  and  $dv$ , then using the right hand form.

Example:  $\int te^t dt$

1. Let  $u = t$  and  $dv = e^t dt$  so that  $du = dt$  and  $v = e^t$
2.  $= te^t - \int e^t dt$
3.  $= te^t - e^t + C$

## 11 Trigometetric Integrals

Products of sin and cos

- If the power of sin or cos split off 1 factor and use  $\sin^2 x + \cos^2 x = 1$   
 Example:  $\int \cos^3 x dx \rightarrow \int \cos^2 x \cos x \rightarrow \int (1 - \sin^2 x) \cos x dx$
- If the power of sin or cos is even, use a half-angle identity.  
 $\cos^2 x = \frac{1+\cos^2 x}{2}$  and  $\sin^2 x = \frac{1-\sin^2 x}{2}$   
 Example:  $\int \sin^2 x dx = \int \frac{1-\cos^2 x}{2} dx$   
 $= \int \frac{1}{2} dx - \int \frac{1}{2} \cos^2 x dx$   
 $= \frac{x}{2} - \frac{1}{4} \sin 2x + C$

### 11.1 Products of powers of sin and cos

- If the power of  $\sin x$  or  $\cos x$  is odd, split off a factor and rewrite the resulting power in terms of the opposite, then use U-Substitution.

## 11.2 Powers of tan, sec, cot and csc

- $\int \sec^2 x = \tan x + C$
- $\int \tan^2 x = \int \sec^2 x + 1 = \tan x + x + C$
- $\int \csc^2 x = -\cot x + C$
- $\int \cot^2 x = \int \csc^2 x + 1 = -\cot x + x + C$

## 11.3 Products of Powers of tan and sec

- If the power of sec is even, split  $\sec^2$ , rewrite in terms of tanx in terms of sec then use U-Sub on tan x.  
Example:  $\int \sec^2 x \tan^{1/2} x dx$   
Let  $u = \tan x$  and  $du = \sec^2 x$   
 $= \int u^{1/2} du$   
 $= \frac{2}{3} u^{3/2} + C$
- If the power of tan is odd, split off secxtanx, rewrite remaining even power of tanx in terms of secx then use U-Sub on sec x.  
Example:  $10 \int \tan^9 x \sec^2 x dx = 10 \int \tan^8 x \sec x \sec x \tan x dx$   
 $= 10 \int (\sec^2 x - 1)^4 \sec x \sec x \tan x dx$   
 $u = \sec x$  and  $du = \sec x \tan x dx$   
 $= 10 \int (u^2 - 1)^4 u du$

## 12 Integrating with Trig Substitutions

For forms of  $a^2 - x^2$ ,  $a^2 + x^2$  and  $x^2 - a^2$  - because powers do not distribute over sums / differences.

$a^2 - x^2$  by substituting  $x = a \sin \theta$

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## 13 Partial Functions

Integrating rational functions is typically complicated. Often times they can be rewritten as if the function was created as a some of smaller rational functions. Suppose  $\frac{x+2}{x^3-3x^2+2x}$ . First rewrite the denominator so that each  $x$  term can be solved.

$$= \frac{x+2}{x(x-2)(x-1)}$$

We assume that  $= \frac{x+2}{x(x-2)(x-1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-1}$  in order to split the fraction up

Multiply every term by the denominators.  $x + 2 = A(x - 2)(x - 1) + Bx(x - 1) + Cx(x - 2)$

Solve by substituting  $x$  with numbers to get 0 terms.

$$x = 2 \rightarrow 4 = A(0)(1) + 2B(1) + 2C(0) \rightarrow 4 = 2B \rightarrow 2 = B$$

$$x = 1 \rightarrow 3 = A(-1)(0) + B(0) + C(-1) \rightarrow 3 = -C \rightarrow -3 = C$$

$$x = 0 \rightarrow 2 = A(-2)(-1) + 0B(-1) + 0C(-2) \rightarrow 2 = 2A \rightarrow 1 = A$$

Therefor

$$= \frac{x+2}{x(x-2)(x-1)} = \frac{1}{x} + \frac{2}{x-2} + \frac{-3}{x-1}$$

## 14 Partial Fraction Decomposition (PFD) - Irreducible Quadratic Factors

The denominator  $d$  is a root of the quadratic function  $f(x)$  in and only if  $x - d$  is a factor of  $f(x)$ . Therefor  $f(x)$  is irreducible when  $b^2 - 4ac < 0$

Example:  $\int \frac{x^2+x+2}{(x+1)(x^2+1)} dx$

$x+1$  and  $x^2+1$  are irreducible so use PFD

$$\frac{x^2+x+2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$x^2 + x + 2 = A(x^2 + 1) + (Bx + C)(x + 1)$$

$$x = -1 \rightarrow 2 = 2A \rightarrow 1 = A$$

$$x^2 + x + 2 = x^2 + 1 + (Bx + C)(x + 1)$$

$$x = 0 \rightarrow 2 = 1 + C \rightarrow 1 = C$$

$$x^2 + x + 2 = x^2 + 1 + (Bx + 1)(x + 1)$$

$$x + 2 = 1 + (Bx + 1)(x + 1)$$

$$x + 2 = Bx^2 + Bx + x + 2$$

$$0 = Bx^2 + Bx$$

$$0 = B$$

$$\text{Therefor } \int \frac{x^2+x+2}{(x+1)(x^2+1)} = \int \frac{1}{x+1} + \frac{1}{x^2+1} dx$$

## 15 Numerical Integration

There are three methods generally used to approximate definite integrals. All three rely on splitting the interval into smaller sub-regions whose area is more easily found.

For all three rules, use an arbitrary  $n$  value. Over the interval  $[a, b]$ , define  $\delta x = \frac{b-a}{n}$

Midpoint Rule

$$\text{Area} \approx \delta x \left( \sum_{k=1}^n f\left(\frac{x_{k-1} + x_k}{2}\right) \right)$$

Trapezoid Rule

$$\text{Area} \approx \delta x \left[ \frac{f(x)}{2} + \left( \sum_{k=1}^{n-1} f(x_k) \right) + \frac{f(x_n)}{2} \right]$$

Simpson's Rule

$$\text{Area} \approx \frac{\delta x}{3} (f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 4f(x_{n-1}) + f(x_n))$$

Denote usage of these rules - midpoint, trapezoid, and simpson's - with  $n$

subintervals as following.

$$M(n) = \delta x \left( f\left(\frac{x_0+x_1}{2}\right) + \dots + f\left(\frac{x_{n-1}+x_n}{2}\right) \right)$$

$$T(n) = \delta x \left( \frac{1}{2}f(x_0) + f(x_1) + \dots + f(x_{n-1}) + \frac{1}{2}f(x_n) \right)$$

$$S(n) = \frac{\delta x}{3} (f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 4f(x_{n-1}) + f(x_n))$$

## 16 Numerical Integration Shortcuts

$$T(2n) = \frac{T(n)+M(n)}{2}$$

$$S(2n) = \frac{4T(2n)-T(n)}{3}$$

## 17 Errors in Approximating with Numerical Integrations

Let  $E_M$  be the error using the midpoint rule over the interval  $[a, b]$

$$E_M = \frac{k(b-a)}{24} (\delta x)^2$$

Let  $E_T$  be the error using the trapezoid rule over the interval  $[a, b]$

$$E_M = \frac{k(b-a)}{12} (\delta x)^2$$

Let  $E_S$  be the error using Simpson's rule over the interval  $[a, b]$

$$E_M = \frac{K(b-a)}{180} (\delta x)^4$$

## 18 Sequences

A sequence is an infinite list of numbers like 1, 2, 3, 4.... We define sequences by patterns, formulas, or recurrence relations.

For example, 1, 2, 4, 8, 16, 32... has an explicit formula of  $A_n = 2^n$  for  $n \geq 0$

The arbitrary sequence  $(a_1, a_2, \dots)$  is denoted as  $(a_n)_{n=1}$  or just  $a_n$ . Thus

$$(1, 2, 3, 4, \dots) = (n)$$

$$(1, 2, 4, 8, 16, \dots) = (2^{n-1})$$

Recurrence relations can also define a sequence.

$a_1 = 1$  and  $a_n = 2a_{n-1}$  for  $n \geq 2$  gives us 1, 2, 4, 8, ...

Theorem: Let  $f$  be a function with  $f(n) = a_n$  for all positive integer values of  $n$ .

If  $\lim_{n \rightarrow \infty} f(n) = L$  then  $\lim_{n \rightarrow \infty} a_n = L$

If  $\lim_{n \rightarrow \infty} a_n = L$  then  $a_n$  converges to  $L$  and we write  $(a_n) \rightarrow L$

We can then define the following to be true

- If  $a_n \rightarrow A$  and  $b_n \rightarrow B$  then  $a_n \pm b_n \rightarrow A \pm B$
- $ca_n \rightarrow cA$
- $a_n * b_n \rightarrow AB$



- $\frac{a_n}{b_n} \rightarrow \frac{A}{B}$

We also define the following terms as follows

- $a_n$  is increasing if  $a_{n+1} \geq a_n$  for all  $n$
- $a_n$  is decreasing if  $a_{n+1} \leq a_n$  for all  $n$
- $a_n$  is monotonic if  $a_{n+1}$  is either increasing or decreasing
- $a_n$  is bounded if there is a number  $M$  such that  $|a_n| \leq M$  for all  $n$

Theorem: Bounded monotonic sequences converge

Let  $M$  be the smallest bound. If  $(a_n)$  goes in one direction and is bounded by  $M$  then  $(a_n)$  will approach but never reach  $M$  as  $n \rightarrow \infty$

## 19 Geometric Sequences

A Geometric Sequence is any sequence of the form  $(a * r^n)$  where  $a$  and  $r$  are numbers. If  $(a * r^n)$  converges,  $a$  is not needed to evaluate the bound (given by limit laws).

- If  $(r^n) \rightarrow L$  then  $(a * r^n) \rightarrow aL$
- If  $(a * r^n) \rightarrow M$  then  $(\frac{1}{a} * r^n) \rightarrow \frac{M}{a}$

Definition: We say some property  $P(n)$  holds eventually if there is an integer  $k$  such that  $P(n)$  holds true for all  $n \geq k$

Therefor eventually  $a_n \geq k$  if  $(a_n) = (n)$

Theorem: Let  $a \neq 0$  and  $r$  be numbers. If  $|r| < 1$  then  $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$ . If  $|r| \geq 1$  then  $\sum_{k=0}^{\infty} ar^k$  diverges.

## 20 Squeeze Theorem

An (infinite) series is an infinite sum of numbers

$$a_1 + a_2 + a_3 + \dots = \sum_{k=1}^{\infty} a_k$$

$$\sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k = (a_1 + a_2 + a_3 + \dots)$$

Denote  $\sum_{k=1}^n a_k$  by  $S_n$  which is called the  $n$ th partial.

- $S_1 = a_1$
- $S_2 = a_1 + a_2$
- ...
- $S_n = a_1 + a_2 + \dots + a_n$

We define the limit of  $\sum_{k=1}^{\infty} a_k$  to be  $\lim_{n \rightarrow \infty} S_n = \sum_{k=1}^n a_k$   
 If  $\lim_{n \rightarrow \infty} (S_n) = L$  we write  $\sum_{k=1}^{\infty} a_k = L$   
 If  $L$  exists then  $\sum_{k=1}^{\infty} a_k$  converges. If not,  $\sum_{k=1}^{\infty} a_k$  diverges.

If  $(a_n) = (c)$  then  $S_n = nc$  so  $S_n \rightarrow \infty$  which means non-monotonic sequences cannot produce converging series. To have a converging series given a sequence,  $(a_n) \rightarrow 0$  must be true and it must converge quickly.  
 Example:  $\sum_{k=1}^{\infty} \frac{1}{n}$  diverges only because it is not quick enough.