Analytical Geometry and Calculus II

Conrad A. Mearns

February 26, 2017

Review of Theorems for Limits

Let a and c be any number, if $F = \lim_{x\to a} f(x)$ and $G = \lim_{x\to a} g(x)$ then

1. $\lim_{x\to a}(f(x) + g(x)) = F + G$
2. $\lim_{x\to a}(f(x) - g(x)) = F - G$
3. $\lim_{x\to a}(c * f(x)) = c * F$
4. $\lim_{x\to a}(\frac{f(x)}{g(x)}) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ except when $G = 0$
5. $\lim_{x\to a}(f(x))^c = F^c$

Limits Involving Infinity while $c \neq 0$

These are templates, where x is either taken to ∞ or 0.

1. $c * (\pm\infty) = \pm\infty$ Example: $\lim_{x\to\infty} 5x = \infty$
2. $\frac{c}{\pm\infty} = 0$ Example: $\lim_{x\to\infty} \frac{5}{x} = 0$
3. $\frac{c}{0} = \pm\infty$ Example: $\lim_{x\to 0} \frac{5}{0} = \infty$
4. $\frac{\pm\infty}{c} = \pm\infty$

Limits of Rational Functions

Theorem: Given a rational function $\frac{f(x)}{g(x)}$, the following is be true. Let d represent the degree of $f(x)$ and e represent the degree of $g(x)$.

1. If $d > e$ then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \pm\infty$
2. If $d < e$ then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$
3. If $d = e$ then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{a}{b}$ where a is the leading term of $f(x)$ and b is the leading term of $g(x)$

Example: To evalute $lim_{x\to\infty} \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$

1. Reduce all terms by x of the leading coefficient's degree.
2. Then take the limit of each term. $\frac{10-3x^{-1}+8x^{-3}}{\sqrt{25+x^{-2}+2x^{-6}}}$
3. Simplifiy. $\frac{10-0+0}{\sqrt{25+0+0}}$
4. $\frac{10}{\sqrt{25}} = \frac{10}{5} = 2$

Inverse Trig Functions

Definition: $y = sin^{-1}x$ is the value of y such that $x = siny$.
Domain: $-1 \leq x \leq 1$
Range: $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$

Definition: $y = cos^{-1}x$ is the value of y such that $x = cosy$.
Domain: $-1 \leq x \leq 1$
Range: $0 \leq y \leq \pi$

Other Trig Functions

- $y = tan^{-1}x \to x = tany$
  Range: $\frac{-\pi}{2} < y < \frac{\pi}{2}$
- $y = cot^{-1}x \to x = coty$
  Range: $0 < y < \pi$
- $y = sec^{-1}x \to x = secy$
  Range: $0 \leq y \leq \pi, y \neq \frac{\pi}{2}$
- $y = csc^{-1}x \to x = cscy$
  Range: $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$

Inverse Trig Identities

- $sin(sin^{-1}x) = x$
- $cos(cos^{-1}x) = x$
- $sin^{-1}(sinx) = x$ only if x is in range of sin^{-1}
- $cos^{-1}(cosx) = x$ only if x is in range of cos^{-1}

Example: $sin^{-1}(sin\pi) = sin^{-1}(0) = 0 \neq \pi$

Example: $cos(sin^{-1}x)$

1. Let $y = sin^{-1}x$ so that $x = siny$ and $cos(sin^{-1}x) = cosy$
2. Recall that $sin = \frac{opposite}{hypotenuse}$

3. Let $hypotenuse = 1$ and $opposite = b$ where b has yet to be determined.

4. Recall that $cosy = \frac{adjacent}{hypotenuse}$

5. $\frac{adjacent}{hypotenuse} = \frac{b}{1}$, and therefor $cosy = b$

6. Use the Pythagorean Theorem to solve: $x^2 + b^2 = 1^2$

7. $b^2 = 1 - x^2$

8. $b = \sqrt{1 - x^2}$

9. Therefor $cos(sin^{-1}x) = \sqrt{1 - x^2}$

Derivatives of Inverse Trig Functions

- $\frac{d}{dx}(sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(tan^{-1}x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(cot^{-1}x) = \frac{-1}{1-x^2}$
- $\frac{d}{dx}(sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$
- $\frac{d}{dx}(csc^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}$

Antiderivatives Involving Inverse Trig Functions

- $\int \frac{dx}{\sqrt{a^2-x^2}} = sin^{-1}\frac{x}{a} + C$
- $\int \frac{dx}{a^2+x^2} = \frac{1}{a}tan^{-1}\frac{x}{a} + C$
- $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}sec^{-1}\frac{x}{a} + C$

L'Hopital's Rule

L'Hopital's Rule let's us evaluate impossible limits.

Theorem: Suppose $f(x)$ and $g(x)$ are differentiable on an open interval I containing a where $g'(x) \neq 0$ on I when $x \neq a$. If

1. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

2. $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$. This is also true as $x \to \pm\infty, x \to a^+, x \to a^-$.

Basic Approaches to Integration

- Subtle Substitution
  $\int \frac{dx}{x^{-1}+1}$
  $= \int \frac{x}{x+1} dx$ and suppose $u = x+1, x = u-1, du = dx$
  $= \int \frac{u-1}{u} du$
  $= \int du - \int \frac{1}{u} du = u - \ln|u| + C$
  $= x + 1 - ln|x| + C$

- Splitting Fractions
  $\int \frac{2-3x}{\sqrt{1-x^2}} dx$
  $= \int \frac{2}{\sqrt{1-x^2}} dx - \frac{3x}{\sqrt{1-x^2}} dx$
  $= \int 2\sin^{-1} x - \frac{3x}{\sqrt{1-x^2}} dx$
  etc...

- Completing the Square
  $\int \frac{dx}{\sqrt{27-6x-x^2}}$
  $= \int \frac{dx}{\sqrt{-(x^2+6x-27)}}$
  $= \int \frac{dx}{\sqrt{-((x+3)^2-36)}}$
  $= \int \frac{dx}{\sqrt{36-(x+3)^2}}$
  $= \sin^{-1}(\frac{x+3}{6}) + C$

- Multiplying by 1 (Using Conjugates)
  $\int \frac{dx}{1+\sin x}$ the conjugate of $1+\sin x$ is $1-\sin x$
  $= \int \frac{dx(1-\sin x)}{(1+\sin x)(1-\sin x)}$
  $= \int \frac{1-\sin x}{1-\sin^2 x} dx$
  $= \int \frac{1-\sin x}{\cos^2 x} dx$
  $= \int \frac{1}{\cos^2 x} dx - \int \frac{\sin x}{\cos^2 x} dx$
  $= \int \sec^2 x dx - \int \tan x \sec x dx$
  $= \tan x - \sec x + C$

Integration by Parts

Integration by Parts, or IBP, is used when integrating products of functions. It's not perfect, and can get messy if used incorrectly. The basic form is

$\int udv = uv - \int vdu$

Solve by substituting u and dv, then using the right hand form.

Example: $\int te^t dt$

1. Let $u = t$ and $dv = e^t dt$ so that $du = dt$ and $v = e^t$
2. $= te^t - \int e^t dt$
3. $= te^t - e^t + C$

Trigometetric Integrals

Products of sin and cos

- If the power of sin or cos split off 1 factor and use $\sin^2 x + \cos^2 x = 1$
  Example: $\int \cos^3 x dx \to \int \cos^2 x \cos x \to \int (1 - \sin^2 x) \cos x dx$
- If the power of sin or cos is even, use a half-angle identity.
  $\cos^2 x = \frac{1+\cos^2 x}{2}$ and $\sin^2 x = \frac{1-\sin^2 x}{2}$
  Example: $\int \sin^2 x dx = \int \frac{1-\cos^2 x}{2} dx$
  $= \int \frac{1}{2} dx - \int \frac{1}{2} \cos^2 x dx$
  $= \frac{x}{2} - \frac{1}{4} \sin 2x + C$

Products of of powers of sin and cos

- If the power of sinx or cosx is odd, split off a factor and rewrite the resulting power in terms of the opposite, then use U-Substituion.

Powers of tan, sec, cot and csc

- $\int \sec^2 x = \tan x + C$
- $\int \tan^2 x = \int \sec^2 x + 1 = \tan x + x + C$
- $\int \csc^2 x = -\cot x + C$
- $\int \cot^2 x = \int \csc^2 x + 1 = -\cot x + x + C$

Products of Powers of tan and sec

- If the power of sec is even, split $\sec^2$, rewrite in terms of tanx in terms of sec then use U-Sub on $\tan x$.
  Example: $\int \sec^2 x \tan^{1/2} x dx$
  Let $u = \tan x$ and $du = \sec^2 x$
  $= \int u^{1/2} du$
  $= \frac{2}{3} u^{3/2} + C$

- If the power of tan is odd, split off secxtanx, rewrite remaining even power of tanx in terms of secx then use U-Sub on $\sec x$.
  Example: $10 \int \tan^9 x \sec^2 x dx = 10 \int \tan^8 x \sec x \sec x \tan x dx$
  $= 10 \int (\sec^2 x - 1)^4 \sec x \sec x \tan x dx$
  $u = \sec x$ and $du = \sec x \tan x dx$
  $= 10 \int (u^2 - 1)^4 u du$

Integrating with Trig Substitutions For forms of $a^2 - x^2$, $a^2 + x^2$ and $x^2 - a^2$ - because powers do not distribute ovr sums / differences.
$a^2 - x^2$ by substituting $x = a \sin\theta$

-

Partial Functions

Integrating rational functions is typically complicated. Often times they can be rewritten as if the function was created as a some of smaller rational functions. Suppose $\frac{x+2}{x^3-3x^2+2x}$. First rewrite the denominator so that each x term can be solved.
$= \frac{x+2}{x(x-2)(x-1)}$
We assume that $= \frac{x+2}{x(x-2)(x-1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-1}$ in order to split the fraction up
Multiply every term by the denominators. $x + 2 = A(x-2)(x-1) + Bx(x-1) + Cx(x-2)$
Solve by substituting x with numbers to get 0 terms.
$x = 2 \rightarrow 4 = A(0)(1) + 2B(1) + 2C(0) \rightarrow 4 = 2B \rightarrow 2 = B$
$x = 1 \rightarrow 3 = A(-1)(0) + B(0) + C(-1) \rightarrow 3 = -C \rightarrow -3 = C$
$x = 0 \rightarrow 2 = A(-2)(-1) + 0B(-1) + 0C(-2) \rightarrow 2 = 2A \rightarrow 1 = A$
Therefor
$= \frac{x+2}{x(x-2)(x-1)} = \frac{1}{x} + \frac{2}{x-2} + \frac{-3}{x-1}$

Partial Fraction Decomposition (PFD) - Irreducible Quadratic Factors

The denomonator d is a root of the quadratic function $f(x)$ in and only if $x - d$ is a factor of $f(x)$. Therefor $f(x)$ is irreducible when $b^2 - 4ac < 0$
Example: $\int \frac{x^2+x+2}{(x+1)(x^2+1)} dx$
$x + 1$ and $x^2 + 1$ are irreducible so use PFD
$\frac{x^2+x+2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$
$x^2 + x + 2 = A(x^2+1) + (Bx + C)(x + 1)$
$x = -1 \rightarrow 2 = 2A \rightarrow 1 = A$
$x^2 + x + 2 = x^2 + 1 + (Bx + C)(x + 1)$
$x = 0 \rightarrow 2 = 1 + C \rightarrow 1 = C$
$x^2 + x + 2 = x^2 + 1 + (Bx + 1)(x + 1)$
$x + 2 = 1 + (Bx + 1)(x + 1)$
$x + 2 = Bx^2 + Bx + x + 2$
$0 = Bx^2 + Bx$
$0 = B$
Therefor $\int \frac{x^2+x+2}{(x+1)(x^2+1)} = \int \frac{1}{x+1} + \frac{1}{x^2+1} dx$

Numerical Integration

There are three methods generally used to approximate definite integerals. All three rely on splitting the interval into smaller sub-regions whose area is more easliy found.

For all three rules, use an arbitray n value. Over the interval $[a, b]$, define $\delta x = \frac{b-a}{n}$
Midpoint Rule
$Area \approx \delta x(\sum_{k=1}^{n} f(\frac{x_{k-1}+x_k}{2}))$
Trapezoid Rule
$Area \approx \delta x[\frac{f(x)}{2} + (\sum_{k=1}^{n-1} f(x_k)) + \frac{f(x_n)}{2}]$
Simpson's Rule
$Area \approx \frac{\delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + ... + 4f(x_{n-1}) + f(x_n))$

Denote usage of these rules - midpoint, trapezoid, and simpson's - with n subintervals as following.
$M(n) = \delta x(f(\frac{x_0+x_1}{2}) + ... + f(\frac{x_{n-1}+x_n}{2}))$
$T(n) = \delta x(\frac{1}{2}f(x_0) + f(x_1) + ... + f(x_{n-1}) + \frac{1}{2}f(x_n))$
$S(n) = \frac{\delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + ... + 4f(x_{n-1}) + f(x_n))$

Numerical Integration Shortcuts

$T(2n) = \frac{T(n)+M(n)}{2}$
$S(2n) = \frac{4T(2n)-T(n)}{3}$

Errors in Approximating with Numerical Integration

Let E_M be the error using the midpoint rule over the interval $[a, b]$
$E_M = \frac{k(b-a)}{24}(\delta x)^2$
Let E_T be the error using the trapezoid rule over the interval $[a, b]$
$E_M = \frac{k(b-a)}{12}(\delta x)^2$
Let E_S be the error using Simpson's rule over the interval $[a, b]$
$E_M = \frac{K(b-a)}{180}(\delta x)^4$

1 Sequences

A sequence is an infinite list of numbers like $1, 2, 3, 4....$ We define sequences by patterns, forumlas, or recurrence relations.
For example, $1, 2, 4, 8, 16, 32...$ has an explicit forumula of $A_n = 2^n$ for $n \geq 0$
The arbitrary sequence $(a_1, a_2, ...)$ is denoted as $(a_n)_{n=1}$ or just a_n. Thus
$(1, 2, 3, 4, ...) = (n)$
$(1, 2, 4, 8, 16, ...) = (2^{n-1})$
Recurrence relations can also define a sequence.
$a_1 = 1$ and $a_n = 2a_{n-1}$ for $n \geq 2$ gives us $1, 2, 4, 8, ...$
Theorem: Let f be a function with $f(n) = a_n$ for all positive integer values of n. If $\lim_{n\to\infty} f(n) = L$ then $\lim_{n\to\infty} a_n = L$
If $\lim_{n\to\infty} a_n = L$ then a_n converges to L and we write $(a_n) \to L$
We can then define the following to be true

- If $a_n \to A$ and $b_n \to B$ then $a_n \pm b_n \to A \pm B$

- $ca_n \to cA$
- $a_n * b_n \to AB$
- $\frac{a_n}{b_n} \to \frac{A}{B}$

We also define the following terms as follows

- a_n is increasing if $a_{n+1} \geq a_n$ for all n
- a_n is decreasing if $a_{n+1} \leq a_n$ for all n
- a_n is monotonic if a_{n+1} is either increasing or decreasing
- a_n is bounded if there is a number M such that $|a_n| \leq M$ for all n

Theorem: Bounded monotonic sequences converge
Let M be the smallest bound. If (a_n) goes in one direction and is bounded by M then (a_n) will approach but never reach M as $n \to \infty$

2 Geometric Sequences

A Geometric Sequence is any sequence of the form $(a * r^n)$ where a and r are numbers. If $(a * r^n)$ converges, a is not needed to evaluate the bound (given by limit laws).

- If $(r^n) \to L$ then $(a * r^n) \to aL$
- If $(a * r^n) \to M$ then $(\frac{1}{a} * r^n) \to \frac{M}{a}$

Definition: We say some property $P(n)$ holds eventually if there is an integer k such that $P(n)$ holds true for all $n \geq k$
Therefor eventually $a_n \geq k$ if $(a_n) = (n)$
Theorem: Let $a \neq 0$ nd r be numbers. If $|r| < 1$ then $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$. If $|r| \geq 1$ then $\sum_{k=0}^{\infty} ar^k$ diverges.

3 Squeeze Theorem

An (infinite) series is an infinite sum of numbers
$a_1 + a_2 + a_3 + ... = \sum_{k=1}^{\infty} a_k$
$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = (a_1 + a_2 + a_3 + ...)$
Denote $\sum_{k=1}^{n} a_k$ by S_n which is called the nth partial.

- $S_1 = a_1$
- $S_2 = a_1 + a_2$
- ...
- $S_n = a_1 + a_2 + ... + a_n$

We define the limit of $\sum_{k=1}^{\infty} a_k$ to be $\lim_{n\to\infty} S_n = \sum_{k=1}^{n} a_k$
If $\lim_{n\to\infty}(S_n) = L$ we write $\sum_{k=1}^{\infty} a_k = L$
If L exists then $\sum_{k=1}^{\infty} a_k$ converges. If not, $\sum_{k=1}^{\infty} a_k$ diverges.

If $(a_n) = (c)$ then $S_n = nc$ so $S_n \to \infty$ which means non-monotonic sequences cannot produce converging series. To have a converging series given a sequence, $(a_n) \to 0$ must be true and it must converge quickly.
Example: $\sum_{k=1}^{\infty} \frac{1}{n}$ diverges only because it is not quick enough.