Smallest Multiple

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Let $S = \{a_1, a_2, \ldots, a_n\}$, be a set of any positive integers. The smallest number m such that m is divisible by every number in set S can be written as:

$$m = \mathrm{LCM}(a_1, a_2, \ldots, a_n)$$

m can be computed by the following operation:

$$m = \mathrm{LCM}(\mathrm{LCM} \ldots \mathrm{LCM}(a_1, a_2), a_3), \ldots, a_n)$$

The lowest common multiple of any two positive integers is shown to be:

$$\mathrm{LCM}(a_1, a_2) = \frac{a_1 \cdot a_2}{\mathrm{GCD}(a_1, a_2)}$$

The greatest common divisor of two positive integers can be calculated quickly using the Euclidean Algorithm. The Euclidean Algorithm is as follows:

1. Given two positive integers a and b, where $a \geq b$.

2. Compute the quotient q and remainder r such that $a = bq + r$.

3. Let $a = b$ and $b = r$.

4. Repeat the process until $b = 0$, at this stage r will equal to the the GCD of the two original numbers.

Complexity Analysis

The Euclidean Algorithm operates in constant space, therefor running the algorithm amy number of times gives an overall space complexity of:

$$O(1) \text{ space}$$

The worst case scenario for the Euclidean Algorithm occurs when the input numbers are consecutive Fibonacci numbers. In this case the number steps is proportional to the number of digits of the smaller number. Given set S is size n and contains maximum element n, the Euclidean Algorithm is run n times and the overall time complexity is:

$$O(n \log(n)) \text{ time}$$