MATH 521 - Numerical Analysis of Differential Equations

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Assignment 2 : Hilbert Spaces, Weak Form of BVPs

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Background for Q1 [no points]

You will need the following Poincare-type inequalities:

(1) Let Ω be a connected, bounded, domain and $\Gamma_D \subset \partial\Omega$ measurable with surface area $|\Gamma_D| > 0$, then there exists a constant c_P such that

$$\|v\|_{L^2(\Omega)} \leq c_P \|\nabla v\|_{L^2(\Omega)} \qquad \forall v \in H^1_{\Gamma_D}(\Omega).$$

(2) Let Ω be a simply connected domain then there exists a constant c_P such that

$$\|v\|_{L^2(\Omega)} \leq c_P \|\nabla v\|_{L^2(\Omega)} \qquad \forall v \in H^1(\Omega) \text{ satisfying } (v)_\Omega = 0,$$

where $(v)_\Omega := |\Omega|^{-1} \int_\Omega v \, dx$.

Another way (seemingly stronger but equivalent) to state these results is the following:

(1') Let Ω be a connected, bounded, domain and $\Gamma_D \subset \partial\Omega$ measurable with surface area $|\Gamma_D| > 0$, then there exists a constant c_P such that

$$\|v\|_{L^2(\Omega)} \leq c_P \Big(\|v\|^2_{L^2(\Gamma_D)} + \|\nabla v\|^2_{L^2(\Omega)} \Big)^{1/2} \qquad \forall v \in H^1(\Omega).$$

(2) Let Ω be a simply connected domain then there exists a constant c_P such that

$$\|v\|_{L^2(\Omega)} \leq c_P \Big(|(v)_\Omega|^2 + \|\nabla v\|^2_{L^2(\Omega)} \Big)^{1/2} \qquad \forall v \in H^1(\Omega).$$

Note: if you want to prove these results, it is not too difficult. They can both be proven with the same argument. What you need is the compactness of the embedding $L^2(\Omega) \subset H^1(\Omega)$: If $(u_n)_{n\in\mathbb{N}}$ is a sequence that is bounded in $H^1(\Omega)$ then there exists $u \in L^2(\Omega)$ and a subsequence such that $u_{n_j} \to u$ strongly in L^2.

Q1a: [8]

Let Ω be a connected, bounded domain. Are the following spaces H Hilbert spaces when equipped with their stated inner products? If not, then explain what property is missing (no need to justify it at length)

(i) $H = \{v \in C^1(\bar{\Omega}) : v, \nabla v \in L^2(\Omega)\}$ $(u, v)_H = \int_\Omega uv + \nabla u \cdot \nabla v \, dx$

(ii) $H = \{v \in L^2(\Omega) : \text{ weakly differentiable}, \nabla v \in L^2(\Omega)\}$, $(u, v)_H = \int_\Omega \nabla u \cdot \nabla v dx$

(iii) $H = \{v \in L^2(\Omega) : \text{ weakly differentiable}, \nabla v \in L^2(\Omega)\}$, $(u, v)_H = \int_\Omega uv + \nabla u \cdot \nabla v dx$

(iv) $H = \{v \in L^2(\Omega) : \text{ weakly differentiable}, \nabla v \in L^2(\Omega), v(0) = 0\}$, $(u, v)_H = \int_\Omega \nabla u \cdot \nabla v dx$.

Solution Q1a

(i) The given space is not a Hilbert space, to make it a Hilbert space, $v \in Clos C^1(\Omega)$.

(ii) The given space is not a Hilbert space, to make it a Hilbert space, $(u, v)_H = 0$ iff $u = v = 0$.

(iii) The given space is a Hilbert space.

(iv) The given space is a Hilbert space.

Q1b [5+7]

Let Ω be a connected, bounded domain, and $\Gamma_D \subset \partial\Omega$ measurable with surface area $|\Gamma_D| > 0$. Are the following spaces H Hilbert spaces when equipped with their stated inner products? Now please justify your answer in full detail. (except you don't need to show that $(u, v)_H$ is symmetric and bilinear)

(v) $H = \{v \in H^1(\Omega) : (v)_\Omega = 0\}$, $(u, v)_H = \int_\Omega \nabla u \cdot \nabla v dx$.

(vi) $H = H^1(\Omega)$, $(u, v)_H = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\Gamma_D} uv \, dx$.

Solution Q1b

(v) Since $(u, v)_H$ is symmetric and bilinear, to determine if the given space was a Hilbert space, we need to show:
1. The given space is complete, since $v \in H^1(\Omega)$, it is cauchy.
2. The inner product is positive definite: $(v, v)_H = \int_\Omega |\nabla v|^2 dx > 0$.
3. And $(v, v)_H = \int_\Omega |\nabla v|^2 dx = 0$ when $\nabla v = 0$, and since $||v||_{L^2(\Omega)} \leq c_P ||\nabla v||_{L^2(\Omega)} \quad \forall v \in H^1(\Omega) \text{ satisfying } (v)_\Omega = 0$, v must be zero. Therefore the inner product satisfying $(v, v)_H = 0$ if and only if $v = 0$
Given the above properties, the given space is indeed a Hilbert space.

(vi) Since $(u, v)_H$ is symmetric and bilinear, to determine if the given space was a Hilbert space, we need to show:
1. The given space is complete, since $H = H^1(\Omega)$.
2. The inner product is positive definite: $(v, v)_H = \int_\Omega |\nabla v|^2 dx + \int_{\Gamma_D} v^2 \, dx > 0$.
3. And $(v, v)_H = \int_\Omega |\nabla v|^2 dx + \int_{\Gamma_D} v^2 \, dx = 0$ when the two terms are equal and opposite, and since $||v||_{L^2(\Omega)} \leq c_P ||\nabla v||_{L^2(\Omega)} \quad \forall v \in H^1_{\Gamma_D}(\Omega)$, v must be zero. Therefore the inner product satisfying $(v, v)_H = 0$ if and only if $v = 0$
Given the above properties, the given space is indeed a Hilbert space.

Background to Q2 [no points]

Before starting on Q2, review integration by parts in $\Omega \subset \mathbb{R}^d$. We introduced this as

$$\int_\Omega \partial_i u \cdot v dx = -\int_\Omega u \partial_i v dx + \int_{\partial\Omega} \nu_i uv \, dx.$$

From this expression, please derive the following equivalent formulation: if $g : \Omega \to \mathbb{R}^d, v : \Omega \to \mathbb{R}$ (both weakly differentiable) then

$$\int_\Omega \operatorname{div} g \, v \, dx = -\int_\Omega g \cdot \nabla v \, dx + \int_{\partial\Omega} (\nu \cdot g) \, v \, dx.$$

Q2: Weak forms of 2nd order BVPs [10+10+10]

For the following three problems, derive the weak form and then use the Lax-Milgram theorem to show that the weak forms have unique solutions. Throughout this question,, Ω is a connected domain in $\mathbb{R}^d$, $d > 1$, $p, q \in C(\bar{\Omega})$ with $c_0 \leq p, q \leq c_1$, $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$.

(i) Neumann problem

$$\begin{aligned} -\operatorname{div}(p\nabla u) + qu = f, &\quad \text{in } \Omega, \\ p\nu \cdot \nabla u = g, &\quad \text{on } \partial\Omega. \end{aligned}$$

(ii) Robin problem

$$\begin{aligned} -\operatorname{div}(p\nabla u) = f, &\quad \text{in } \Omega, \\ p\nu \cdot \nabla u + u = g, &\quad \text{on } \partial\Omega. \end{aligned}$$

(iii) The classical Neumann problem: in addition to all previous assumptions also assume that $(f)_\Omega = 0$.

$$\begin{aligned} -\Delta u = f, &\quad \text{in } \Omega, \\ \nu \cdot \nabla u = 0, &\quad \text{on } \partial\Omega. \end{aligned}$$

HINT for (iii) : you need to introduce an additional condition that uniquely determines the solution but doesn't change the problem.

Solution Q2(i)

The weak form derivation:

$$\int_\Omega -\nabla\cdot(p\nabla u)v\,dx + \int_\Omega quv\,dx = \int_\Omega fv\,dx$$
$$\int_\Omega p\nabla u\cdot\nabla v\,dx - \int_{\partial\Omega}(\nu\cdot p\nabla u)v\,ds + \int_\Omega quv\,dx = \int_\Omega fv\,dx$$
$$\int_\Omega p\nabla u\cdot\nabla v\,dx + \int_\Omega quv\,dx = \int_{\partial\Omega} gv\,ds + \int_\Omega fv\,dx$$

Uniqueness:
Define a bilinear form $a : H^1(\Omega)\times H^1(\Omega)\to \R$, $\quad a(u,v) = \int_\Omega p\nabla u\cdot\nabla v\,dx + \int_\Omega quv\,dx.$
Define a linear functional $l : H^1(\Omega)\to \R$, $\quad l(v) = \int_\Omega fv\,dx + \int_{\partial\Omega} gv\,ds.$

a is continous: For arbitary $u,v\in H^1(\Omega)$ we find the Cauchy-Schwarz inequality:

$$\left|\int_\Omega p\nabla u\cdot\nabla v\,dx\right| \le \left(\int_\Omega p^2|\nabla u|^2\,dx\right)^{1/2}\left(\int_\Omega |\nabla v|^2\,dx\right)^{1/2} \le c_1||\nabla u||_{L^2(\Omega)}||\nabla v||_{L^2(\Omega)}, \quad \text{where} \quad 0\le c_0\le p\le c_1$$
$$\left|\int_\Omega quv\,dx\right| \le \left(\int_\Omega q^2u^2\,dx\right)^{1/2}\left(\int_\Omega v^2\,dx\right)^{1/2} \le c_1||u||_{L^2(\Omega)}||v||_{L^2(\Omega)}, \quad \text{where} \quad 0\le c_0\le q\le c_1$$
$$|a(u,v)| \le c_1||\nabla u||_{L^2(\Omega)}||\nabla v||_{L^2(\Omega)} + c_1||u||_{L^2(\Omega)}||v||_{L^2(\Omega)} \le c_1||u||_{H^1(\Omega)}||v||_{H^1(\Omega)}$$

a is coercive: For arbitary $u\in H^1(\Omega)$ we have by definition of the H^1-norm:

$$a(u,u) = \int_\Omega p|\nabla u|^2\,dx + \int_\Omega qu^2\,dx$$
$$a(u,u) \ge c_0\int_\Omega |\nabla u|^2\,dx + c_0\int_\Omega u^2\,dx$$
$$a(u,u) \ge c_0(||\nabla u||^2_{L^2(\Omega)} + ||u||^2_{L^2(\Omega)}) = c_0||u||^2_{H^1(\Omega)}$$

L is linear and continous: For arbitary $v\in H^1(\Omega)$ we find the Cauchy-Schwarz inequality:

$$\left|\int_\Omega fv\,dx\right| \le ||f||_{L^2(\Omega)}||v||_{L^2(\Omega)} \le ||f||_{L^2(\Omega)}||v||_{H^1(\Omega)}$$
$$\left|\int_{\partial\Omega} gv\,ds\right| \le ||g||_{L^2(\partial\Omega)}||v||_{L^2(\partial\Omega)} \le C_{\text{trace}}||g||_{L^2(\partial\Omega)}||v||_{H^1(\Omega)}$$
$$|L(v)| \le ||f||_{L^2(\Omega)}||v||_{H^1(\Omega)} + C_{\text{trace}}||g||_{L^2(\partial\Omega)}||v||_{H^1(\Omega)}$$
$$C = \max(||f||_{L^2(\Omega)}, C_{\text{trace}}||g||_{L^2(\partial\Omega)})$$
$$|L(v)| \le C||v||_{H^1(\Omega)}$$

Therefore the Lax-Milgram therorem applies and yields the existence of a unique solution to this weak form.

Solution Q2(ii)

The weak form derivation:

$$\int_\Omega -\nabla\cdot(p\nabla u)v\,dx = \int_\Omega fv\,dx$$
$$\int_\Omega p\nabla u\cdot\nabla v\,dx - \int_{\partial\Omega}(\nu\cdot p\nabla u)v\,ds = \int_\Omega fv\,dx$$
$$\int_\Omega p\nabla u\cdot\nabla v\,dx = \int_{\partial\Omega} gv\,ds - \int_{\partial\Omega} uv\,ds + \int_\Omega fv\,dx$$

Uniqueness:
Define a bilinear form $a : H^1(\Omega)\times H^1(\Omega)\to \R$, $\quad a(u,v) = \int_\Omega p\nabla u\cdot\nabla v\,dx + \int_{\partial\Omega} uv\,ds.$
Define a linear functional $l : H^1(\Omega)\to \R$, $\quad l(v) = \int_{\partial\Omega} gv\,ds + \int_\Omega fv\,dx.$

a is continous: For arbitary $u,v\in H^1(\Omega)$ we find the Cauchy-Schwarz inequality:

$$\left|\int_\Omega p\nabla u\cdot\nabla v\,dx\right| \le \left(\int_\Omega p^2|\nabla u|^2\,dx\right)^{1/2}\left(\int_\Omega |\nabla v|^2\,dx\right)^{1/2} \le c_1||\nabla u||_{L^2(\Omega)}||\nabla v||_{L^2(\Omega)}, \quad \text{where} \quad 0\le c_0\le p\le c_1$$
$$||u||_{L^2(\partial\Omega)} \le C_{\text{trace}}||u||_{H^1(\Omega)}$$
$$||v||_{L^2(\partial\Omega)} \le C_{\text{trace}}||v||_{H^1(\Omega)}$$
$$\left|\int_{\partial\Omega} uv\,ds\right| \le ||u||_{L^2(\partial\Omega)}||v||_{L^2(\partial\Omega)} \le C^2_{\text{trace}}||u||_{H^1(\Omega)}||v||_{H^1(\Omega)}$$
$$|a(u,v)| \le (c_1 + C^2_{\text{trace}})||u||_{H^1(\Omega)}||v||_{H^1(\Omega)}$$

a is coercive: For arbitary $u\in H^1(\Omega)$ we have by definition of the H^1-norm:

$$
\begin{gathered}
\int_\Omega p|\nabla u|^2\,dx \geq c_0 \int_\Omega |\nabla u|^2\,dx = c_0\|u\|^2_{H^1(\Omega)} - c_0 \int_\Omega |u|^2\,dx \\
c_0 \int_\Omega |u|^2\,dx \leq \\
0 \leq \int_{\partial\Omega} u^2\,ds \leq C^2_{\text{trace}}\|u\|^2_{H^1(\Omega)} \\
a(u,u) = \int_\Omega p|\nabla u|^2\,dx + \int_{\partial\Omega} u^2\,ds \geq \alpha\|u\|^2_{H^1(\Omega)}
\end{gathered}
$$

L is linear and continous: For arbitary $v \in H^1(\Omega)$ we find the Cauchy-Schwarz inequality:

$$
\begin{gathered}
\left|\int_\Omega fv\,dx\right| \leq \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)} \\
\left|\int_{\partial\Omega} gv\,ds\right| \leq \|g\|_{L^2(\partial\Omega)}\|v\|_{L^2(\partial\Omega)} \leq C_{\text{trace}}\|g\|_{L^2(\partial\Omega)}\|v\|_{H^1(\Omega)} \\
|L(v)| \leq \|f\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)} + C_{\text{trace}}\|g\|_{L^2(\partial\Omega)}\|v\|_{H^1(\Omega)} \\
C = \max(\|f\|_{L^2(\Omega)}, C_{\text{trace}}\|g\|_{L^2(\partial\Omega)}) \\
|L(v)| \leq C\|v\|_{H^1(\Omega)}
\end{gathered}
$$

Therefore the Lax-Milgram therorem applies and yields the existence of a unique solution to this weak form.

Solution Q2(iii)

The weak form derivation:

$$
\begin{aligned}
\int_\Omega -\nabla\cdot(\nabla u)v\,dx &= \int_\Omega fv\,dx \\
\int_\Omega \nabla u\cdot\nabla v\,dx - \int_{\partial\Omega}(\nu\cdot\nabla u)v\,ds &= \int_\Omega fv\,dx \\
\int_\Omega \nabla u\cdot\nabla v\,dx &= \int_\Omega fv\,dx
\end{aligned}
$$

Uniqueness:
Define a bilinear form $a : H^1(\Omega)\times H^1(\Omega) \to$ \R, $\quad a(u,v) = \int_\Omega \nabla u\cdot\nabla v\,dx$.
Define a linear functional $l : H^1(\Omega) \to$ \R, $\quad l(v) = \int_\Omega fv\,dx$.
To ensure uniqueness, we need an additional assumption that $(u)_\Omega := |\Omega|^{-1}\int_\Omega u\,dx = 0$, since if u is a solution, so does u + constant.

a is continous: For arbitary $u, v \in H^1(\Omega)$ we find the Cauchy-Schwarz inequality:

$$
\left|\int_\Omega \nabla u\cdot\nabla v\,dx\right| \leq \left(\int_\Omega 1^2|\nabla u|^2\,dx\right)^{1/2}\left(\int_\Omega |\nabla v|^2\,dx\right)^{1/2} \leq c_1\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} \leq c_1\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}, \quad \text{where} \quad 0 \leq c
$$

a is coercive: For arbitary $u \in H^1(\Omega)$ we have by definition of the H^1-norm:

$$
\begin{gathered}
a(u,u) = \int_\Omega p|\nabla u|^2\,dx \geq c_0\int_\Omega |\nabla u|^2\,dx \\
a(u,u) \geq c_0(\|\nabla u\|^2_{L^2(\Omega)}) = c_0|u|^2_{H^1(\Omega)}
\end{gathered}
$$

L is linear and continous: For arbitary $v \in H^1(\Omega)$ we find the Cauchy-Schwarz inequality:

$$
l(v) = \left|\int_\Omega fv\,dx\right| \leq \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)}
$$

Therefore the Lax-Milgram therorem applies and yields the existence of a unique solution to this weak form.