Chapter VI
Finite Elements in Solid Mechanics

Finite element methods are the most widely used tools for computing the deformations and stresses of elastic and inelastic bodies subject to loads. These types of problems involve systems of differential equations with the following special feature: the equations are invariant under translations and orthogonal transformations since the elastic energy of a body does not change under so-called rigid body motions.

Practical problems in structural mechanics often involve small parameters which can appear in both obvious and more subtle ways. For example, for beams, membranes, plates, and shells, the thickness is very small in comparison with the other dimensions. On the other hand, for a cantilever beam, the part of the boundary on which Dirichlet boundary conditions are prescribed is very small. Finally, many materials allow only very small changes in density. These various cases require different variational formulations of the finite element computations. Using an incorrect formulation leads to so-called *locking*. Often, mixed formulations provide a suitable framework for both the computation and a rigorous mathematical analysis.

Most of the characteristic properties appear already in the so-called linear theory, i.e., for small deformations where no genuine nonlinear phenomenon occurs. However, strictly speaking, there is no complete linear elasticity theory, since the above-mentioned invariance under rigid body motions cannot be completely modeled in a linear theory. For this reason, we don't restrict ourselves to the linear theory until later.

§§1 and 2 contain a very compact introduction to elasticity theory. For more details, see Ciarlet [1988], Marsden and Hughes [1983], or Truesdell [1977]. Here we concentrate on those aspects of the theory which we need as background knowledge. In §3 we present several variational formulations for the linear theory, and also include an analysis of locking. Finally, we discuss membranes and plates. In particular, we explore the connection between two widely used plate models.

We limit ourselves to those elements whose construction or analysis is based on different approaches than the elements discussed in Chapters II and III. In particular, we will focus on the stability of the elements.

§ 1. Introduction to Elasticity Theory

Elasticity theory deals with the deformation of bodies under the influence of applied forces, and in particular, with the stresses and strains which result from deformations.

The three-dimensional case provides the foundation for the theory. The essential ingredients are the kinematics, the equilibrium equations, and the material laws.

Kinematics

We assume that we know a *reference configuration* $\bar{\Omega}$ for the body under consideration. Here $\bar{\Omega}$ is the closure of a bounded open set Ω. In general, $\bar{\Omega}$ is just the subset of $\mathbb{R}^3$ where the body is in an unstressed state (natural state). The current state is given by a mapping[14]

$$\phi : \bar{\Omega} \longrightarrow \mathbb{R}^3$$

where $\phi(x)$ represents the position of a point which was located at x in the reference configuration. We write

$$\phi = id + u, \tag{1.1}$$

and call u the *displacement.* Often we will assume that the displacements are small, and will neglect terms of higher order in u.

It is obvious that rigid body motions, i.e., translations and orthogonal transformations, do not alter the stresses in a body. This causes some difficulties since this invariance must be preserved in the finite element results – at least approximately.

In the following, we assume that the mapping ϕ is sufficiently smooth. ϕ represents a *deformation,* provided

$$\det(\nabla\phi) > 0.$$

Here $\nabla\phi$ is the *deformation gradient*, and its matrix representation is

$$\nabla\phi = \begin{bmatrix} \dfrac{\partial\phi_1}{\partial x_1} & \dfrac{\partial\phi_1}{\partial x_2} & \dfrac{\partial\phi_1}{\partial x_3} \\ \dfrac{\partial\phi_2}{\partial x_1} & \dfrac{\partial\phi_2}{\partial x_2} & \dfrac{\partial\phi_2}{\partial x_3} \\ \dfrac{\partial\phi_3}{\partial x_1} & \dfrac{\partial\phi_3}{\partial x_2} & \dfrac{\partial\phi_3}{\partial x_3} \end{bmatrix}. \tag{1.2}$$

[14] As before, we do not use any special notation to distinguish vectors, matrices, or tensors. In general, in this section we use lower case Latin letters for vectors, and capitals for tensors or matrices.

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The word deformation suggests that subdomains with positive volume are mapped into subdomains with positive volume. Deformations are injective mappings locally.

The mapping ϕ induces

$$\phi(x+z) - \phi(x) = \nabla\phi(x)\cdot z + o(z).$$

In terms of the Euclidean distance,

$$\begin{aligned}\|\phi(x+z) - \phi(x)\|^2 &= \|\nabla\phi\cdot z\|^2 + o(\|z\|^2)\\ &= z'\,\nabla\phi^T\,\nabla\phi\, z + o(\|z\|^2).\end{aligned} \tag{1.3}$$

Thus, the matrix

$$C := \nabla\phi^T\,\nabla\phi \tag{1.4}$$

describes the transformation of the length element. It is called the *(right) Cauchy–Green strain tensor.* The deviation

$$E := \frac{1}{2}(C - I)$$

from the identity is called the *strain*, and is one of the most important concepts in the theory. Frequently, we will work with matrix representations of C and E. These matrices are obviously symmetric. Inserting (1.1) into (1.4) gives

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\sum_k \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}. \tag{1.5}$$

In the linear theory we neglect the quadratic terms, leading to the following *symmetric gradient* as an approximation:

$$\varepsilon_{ij} := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{1.6}$$

1.1 Remark. Let Ω be connected. If the strain tensor associated with the deformation $\phi \in C^1(\Omega)$ satisfies the relation

$$C(x) = I \quad \text{for all } x \in \Omega,$$

then ϕ describes a rigid body motion, i.e., $\phi(x) = Qx + b$, where Q is an orthogonal matrix.

Sketch of a proof. Let Γ be a smooth curve in Ω. In view of (1.3) and $C(x) = I$, the rectifiable curves Γ and $\phi(\Gamma)$ always have the same length. This follows directly from the definition of the arc length via an integral. We now use it to establish the desired result.

Since ϕ is locally injective, if Ω is open, then $\phi(\Omega)$ is also open. For every $x_0 \in \Omega$, there exists a convex neighborhood U in Ω such that the convex hull of $\phi(U)$ is contained in $\phi(\Omega)$. The mapping $\phi|_U$ is globally distance preserving, i.e. for all pairs $x, y \in U$,

$$\|\phi(x) - \phi(y)\| = \|x - y\|. \tag{1.7}$$

To see this, let Γ be the line connecting the points x and y. Since $\phi(\Gamma)$ has the same length, $\|\phi(x) - \phi(y)\| \le \|x - y\|$. The equality now follows by examining the preimage of the line connecting $\phi(x)$ and $\phi(y)$.

Because of (1.7), the auxiliary function

$$G(x, y) := \|\phi(y) - \phi(x)\|^2 - \|y - x\|^2$$

vanishes on $U \times U$. G is differentiable with respect to y, and $\frac{1}{2}\frac{\partial G}{\partial y_i}$ satisfies

$$\sum_k \frac{\partial \phi_k}{\partial y_i}(\phi_k(y) - \phi_k(x)) - (y_i - x_i) = 0.$$

This expression is differentiable with respect to x_j, and so

$$-\sum_k \frac{\partial \phi_k}{\partial y_i}\frac{\partial \phi_k}{\partial x_j} + \delta_{ij} = 0,$$

which is just the componentwise version of $\nabla\phi(y)^T\nabla\phi(x) = I$. Multiplying on the left by $\nabla\phi(y)$ and using $C = I$, we immediately get $\nabla\phi(x) = \nabla\phi(y)$. Thus, $\nabla\phi$ is constant on U, and ϕ is a linear transformation.

Now the result follows for the entire domain Ω by a covering argument. $\square$

The Equilibrium Equations

In mechanics we treat the influence of forces axiomatically. Euler and Cauchy both made essential contributions. For details, see Ciarlet [1988].

We assume that the interaction of the body with the outside world is described by two types of applied forces:

(a) applied surface forces (forces distributed over the surface),

(b) applied body forces (forces distributed over the volume).

A typical body force is the force of gravity, while the force caused by a load on a bridge (e.g., a vehicle) is a surface force.

The forces are distinguished by the work they do under deformations.

The body force $f : \Omega \longrightarrow \mathbb{R}^3$ results in a force $f\,dV$ acting on a volume element dV. Surface forces are specified by a function $t : \Omega \times S^2 \longrightarrow \mathbb{R}^3$ where S^2 denotes the unit sphere in $\mathbb{R}^3$: Let V be an arbitrary subdomain of Ω (with a sufficiently smooth boundary), and let dA be an area element on the surface with the unit outward-pointing normal vector n. Then the area element dA contributes $t(x, n)dA$ to the force, which also depends on the direction of n. The vector $t(x, n)$ is called the *Cauchy stress vector.*

The main axiom of mechanics asserts that in an equilibrium state, all forces and all moments add to zero. Here we must take into account both surface forces and body forces.

1.2 Axiom of Static Equilibrium. *(Stress principle of Euler and Cauchy)* Let B be a (deformed) body in equilibrium. Then there exists a vector field t such that in every subdomain V of B, the (volume) forces f and the stresses t satisfy

$$\int_V f(x)dx + \int_{\partial V} t(x, n)ds = 0, \tag{1.8}$$

$$\int_V x \wedge f(x)dx + \int_{\partial V} x \wedge t(x, n)ds = 0. \tag{1.9}$$

Here the symbol $\wedge$ stands for the vector product in $\mathbb{R}^3$.

Once the existence of the Cauchy stress vector is given, its exact dependence on the normal n can be determined. Here and in the sequel, we use the following sets of matrices:

$\mathbb{M}^3$, the set of 3×3 matrices,
$\mathbb{M}^3_+$, the set of matrices in $\mathbb{M}^3$ with positive determinants,
$\mathbb{O}^3$, the set of orthogonal 3×3 matrices,
$\mathbb{O}^3_+ := \mathbb{O}^3 \cap \mathbb{M}^3_+$,
$\mathbb{S}^3$, the set of symmetric 3×3 matrices,
$\mathbb{S}^3_>$, the set of positive definite matrices in $\mathbb{S}^3$.

1.3 Cauchy's Theorem. *Let $t(\cdot, n) \in C^1(B, \mathbb{R}^3)$, $t(x, \cdot) \in C^0(S^2, \mathbb{R}^3)$, and $f \in C(B, \mathbb{R}^3)$ be in equilibrium according to 1.2. Then there exists a symmetric tensor field $T \in C^1(B, \mathbb{S}^3)$ with the following properties:*

$$t(x, n) = T(x)n, \qquad x \in B,\ n \in S^2, \tag{1.10}$$

$$\operatorname{div} T(x) + f(x) = 0, \qquad x \in B, \tag{1.11}$$

$$T(x) = T^T(x), \qquad x \in B. \tag{1.12}$$

The tensor T is called the *Cauchy stress tensor*.

The key assertion of this famous theorem is the representability of the stress vector t in terms of the tensor T. Using the Gauss integral theorem, it follows from (1.8) that

$$\int_V f(x)dx + \int_{\partial V} T(x)n ds = \int_V [f(x) + \operatorname{div} T(x)]dx = 0.$$

This relation also implies the differential equation (1.11). The equilibrium equations (1.9) for the moments imply the symmetry (1.12). $\square$

The Piola Transform

We have formulated the equilibrium equations in terms of the coordinates of the deformed body B (as did Euler). Since these coordinates have to be computed in the first place, it is useful to transform the variables to the reference configuration. To distinguish the expressions, in the following we add a subscript R when referring to the reference configuration. In particular, $x = \phi(x_R)$.

The transformation of the body forces follows directly from the well-known transformation theorem for integrals, where the volume element is given by $dx = \det(\nabla\phi)dx_R$. The forces are proportional to density. Densities are transformed according to conservation of mass: $\rho(x)dx = \rho_R(x_R)dx_R$ which implies $\rho(\phi(x_R)) = \det(\nabla\phi^{-1})\rho_R(x_R)$. Consequently,

$$f(x) = \det(\nabla\phi^{-1}) f_R(x_R). \tag{1.13}$$

The equation (1.13) makes implicit use of the assumption that under the deformation, point masses do not move to positions where we have a different force field. In this case we speak of a *dead load*.

The transformation of stress tensors is more complicated, but can be computed by elementary methods; cf. Ciarlet [1988]. In terms of the reference configuration, we have

$$\operatorname{div}_R T_R + f_R = 0 \tag{1.14}$$

with

$$T_R := \det(\nabla\phi)\, T\, (\nabla\phi)^{-T}. \tag{1.15}$$

Equation (1.14) is the analog of (1.11). However, in contrast to T, the so-called *first Piola–Kirchhoff stress tensor* T_R in (1.15) is not symmetric. To achieve symmetry, we introduce the *second Piola–Kirchhoff stress tensor*

$$\Sigma_R := \det(\nabla\phi)\, (\nabla\phi)^{-1}\, T\, (\nabla\phi)^{-T}. \tag{1.16}$$

Clearly, $\Sigma_R = (\nabla\phi)^{-1} T_R$.

The differences between the three stress tensors can be neglected for small deformation gradients.

Fig. 57. A compression of a body in one direction leads to an expansion in the other directions. The relative size is given by the Poisson ratio ν.

Constitutive Equations

An important problem is to find the deformation of a body and the associated stresses corresponding to given external forces. The equilibrium equation (1.11) (respectively, (1.14)) gives only 3 equations. This does not determine the 6 components of the symmetric stress tensor. The missing equations arise from constitutive equations, which express how the deformations depend on properties of the material as well as the given forces.

1.4 Definition. A material is called *elastic* if there exists a mapping

$$\hat{T} : \mathbb{M}^3_+ \longrightarrow \mathbb{S}^3_+$$

such that for every deformed state,

$$T(x) = \hat{T}(\nabla\phi(x_R)). \tag{1.17}$$

The mapping $\hat{T}$ is called the *response function* for the Cauchy stress, and (1.17) is called the *constitutive equation.*

The constitutive equation implicitly contains the assumption that the stress depends on the displacement in a local way. In view of (1.16), we introduce the response function for the Piola–Kirchhoff stress,

$$\hat{\Sigma}(F) := \det(F)\, F^{-1}\hat{T}(F)\, F^{-T}. \tag{1.18}$$

(Formulas with the variables F will generally be applied with $F := \nabla\phi(x)$.)

For simplicity, we restrict ourselves to *homogeneous* materials, i.e., to materials for which $\hat{T}$ does not depend explicitly on x.

Response functions can be brought into a simpler form on the basis of physical laws. First we make the simple observation that the components $\hat{T}_{ij}$ do not behave like scalar functions. Consider a rectangular parallelepiped whose faces are perpendicular to the coordinate axes. Suppose we press on the surfaces which are perpendicular to the x-axis as in Fig. 57. In addition to a compression in the x-direction, in general the material will react by stretching in the perpendicular directions in order to reduce the change in the volume or density, respectively.

1.5 Axiom of Material Frame-Indifference. The Cauchy stress vector $t(x, n) = T(x)\, n$ is independent of the choice of coordinates, i.e., $Qt(x, n) = t(Qx, Qn)$ for all $Q \in \mathbb{O}^3_+$.

A frame-indifferent material is also called *objective*.

1.6 Theorem. *Suppose the axiom of material frame-indifference holds. Then for every orthogonal transformation* $Q \in \mathbb{O}^3_+$,

$$\hat{T}(QF) = Q\,\hat{T}(F)\,Q^T. \tag{1.19}$$

Moreover, there exists a mapping $\tilde{\Sigma} : \mathbb{S}^3_> \longrightarrow \mathbb{S}^3$ *such that*

$$\hat{\Sigma}(F) = \tilde{\Sigma}(F^T F), \tag{1.20}$$

i.e., $\hat{\Sigma}$ *depends only on* $F^T F$.

Proof. Instead of rotating the coordinate system, we rotate the deformed body:

$$\begin{aligned} x &\longmapsto Qx, \\ \phi &\longmapsto Q\phi, \\ \nabla\phi &\longmapsto Q\nabla\phi, \\ n &\longmapsto Q^{-T} n = Qn, \\ t(x, n) &\longmapsto Qt(x, n). \end{aligned}$$

By Axiom 1.5, $t(Qx, Qn) = Qt(x, n)$, and thus $\hat{T}(QF)Q \cdot n = Q\hat{T}(F) \cdot n$. Replacing Qn by n and using $Q^T Q = I$, we get (1.19).

It follows from (1.18) and (1.19) after some elementary manipulations that

$$\hat{\Sigma}(QF) = \hat{\Sigma}(F) \quad \text{for } Q \in \mathbb{O}^3_+. \tag{1.21}$$

To prove (1.20), we consider the two nonsingular matrices F and G in $\mathbb{M}^3_+$ with $F^T F = G^T G$. Set $Q := FG^{-1}$. Then $Q^T Q = I$ and $\det(Q) > 0$. Now (1.21) implies $\hat{\Sigma}(F) = \hat{\Sigma}(G)$, and so in fact $\hat{\Sigma}$ depends only on the product $F^T F$. □

The axiom of frame-indifference holds for all materials. On the other hand, *isotropy* is purely a material property, which means that no direction in the material is preferred. Layered materials such as wood or crystal are not isotropic. Isotropy implies that the stress vectors do not change if we rotate the nondeformed body, i.e., before the deformation takes place.

1.7 Definition. A material is called *isotropic* provided

$$\hat{T}(F) = \hat{T}(FQ) \quad \text{for all } Q \in \mathbb{O}^3_+. \tag{1.22}$$

The different order of F and Q in (1.19) as compared to (1.22) is important. As in the proof of Theorem 1.6, it can be shown that (1.22) is equivalent to

$$\hat{T}(F) = \bar{T}(FF^T) \tag{1.23}$$

with a suitable function $\bar{T}$.

In view of the transformation properties, the response function depends in an essential way on the invariants of the matrix: every 3×3 matrix $A = (a_{ij})$ is associated with a triple of invariants $\iota_A = (\iota_1(A), \iota_2(A), \iota_3(A))$ defined by the corresponding characteristic polynomial

$$\det(\lambda I - A) = \lambda^3 - \iota_1(A)\,\lambda^2 + \iota_2(A)\,\lambda - \iota_3(A).$$

These principal invariants are closely related to the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of A:

$$\begin{aligned}
\iota_1(A) &:= \sum_i a_{ii} = \mathrm{trace}(A) = \lambda_1 + \lambda_2 + \lambda_3,\\
\iota_2(A) &:= \frac{1}{2}\sum_{ij}(a_{ii}a_{jj} - a_{ij}^2) = \frac{1}{2}[(\mathrm{trace}\ A)^2 - \mathrm{trace}(A^2)]\\
&= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3,\\
\iota_3(A) &:= \det(A) = \lambda_1\lambda_2\lambda_3.
\end{aligned} \tag{1.24}$$

We can now formulate a famous theorem of elasticity theory. We employ the usual notation for diagonal matrices, $D = \mathrm{diag}(d_{11}, d_{22}, \ldots, d_{nn})$.

1.8 Rivlin–Ericksen Theorem [1955]. *A response function $\hat{T} : \mathbb{M}^3_+ \longrightarrow \mathbb{S}^3$ is objective and isotropic if and only if it has the form $\hat{T}(F) = \bar{T}(FF^T)$, and*

$$\begin{aligned}
&\bar{T} : \mathbb{S}^3_> \longrightarrow \mathbb{S}^3\\
&\bar{T}(B) = \beta_0(\iota_B)\, I + \beta_1(\iota_B)\, B + \beta_2(\iota_B)\, B^2.
\end{aligned} \tag{1.25}$$

Here β_0, β_1, and β_2 are functions of the invariants of B.

Proof. By (1.23), $\hat{T}(F) = \bar{T}(FF^T)$ with a suitable function $\bar{T}$. It remains to give the proof for the special form (1.25).

(1) First, let $B = \mathrm{diag}\,(\lambda_1, \lambda_2, \lambda_3)$ be a diagonal matrix, and let $FF^T = B$, e.g., $F = B^{1/2}$. In addition, let $T = (T_{ij}) = \hat{T}(F)$. The matrix $Q := \mathrm{diag}\,(1, -1, -1)$ is orthogonal, and by Theorem 1.6,

$$\hat{T}(QF) = QTQ^T = \begin{pmatrix} T_{11} & -T_{12} & -T_{13}\\ -T_{21} & T_{22} & T_{23}\\ -T_{31} & T_{32} & T_{33}\end{pmatrix}. \tag{1.26}$$

On the other hand, $QF(QF)^T = QBQ^T = B$, and thus by hypothesis, $\hat{T}(QF) = \hat{T}(F) = T$. By (1.26), this can happen only if $T_{12} = T_{13} = 0$. A similar argument with $Q = \operatorname{diag}(-1, -1, +1)$ shows that $T_{23} = 0$. Thus $T(B)$ is diagonal if B is a diagonal matrix.

(2) Suppose again that B is a diagonal matrix. If $B_{ii} = B_{jj}$, then $T_{ii} = T_{jj}$. To verify this we consider the case $B_{11} = B_{22}$, and choose

$$Q = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & -1 \end{pmatrix}.$$

Then $QBQ^T = B$, and analogously to part (1), we deduce that $T_{11} = (QTQ^T)_{11} = T_{22}$.

Thus, we can represent T in the form

$$T = \beta_0 I + \beta_1 B + \beta_2 B^2 \tag{1.27}$$

with suitable coefficients $\beta_0, \beta_1, \beta_2$. Now if we permute the diagonal elements of B, then as we have seen, the elements of T have to be permuted in the same way. This gives the representation (1.27) for the new matrix with the same coefficients β_0, β_1 and β_2 as before. Thus, β_0, β_1 and β_2 are *symmetric* functions of λ_i, and the theorem is proved in the case of a diagonal matrix B.

(3) Suppose $F \in \mathbb{M}^3_+$ and $B = FF^T$ is not diagonal. There exists an orthogonal matrix Q such that $QBQ^{-1} = D$ is a diagonal matrix. Replacing Q by $-Q$ if necessary, we can assume $\det Q > 0$. Note that $\iota_B = \iota_D$. By the above considerations and the material frame-indifference, we deduce that

$$\begin{aligned}
\hat{T}(F) &= Q^{-1}\hat{T}(QF)Q^{-T} \\
&= Q^{-1}\bar{T}(D)Q \\
&= Q^{-1}[\beta_0 I + \beta_1 D + \beta_2 D^2]Q \\
&= \beta_0 I + \beta_1 B + \beta_2 B^2,
\end{aligned}$$

and the proof is complete. □

1.9 Remarks. In the special case where FF^T is a multiple of the unit matrix, $\hat{T}(F)$ is also a multiple of the unit matrix. Then the stress has the character of a pure pressure.

For the transfer of the result on the Cauchy tensor to a corresponding formula for the second Piola–Kirchhoff tensor, we make use of the formula of Cayley–Hamilton: $B^3 - \iota_1(B)B^2 + \iota_2(B)B - \iota_3(B)I = 0$. Eliminating I from (1.25), we get

$$\bar{T}(B) = \tilde{\beta}_1 B + \tilde{\beta}_2 B^2 + \tilde{\beta}_3 B^3$$

with different coefficients. Multiplying on the left by F^{-1} and on the right by F^{-T}, with the notation of Theorem 1.6 we get a reformulation in terms of the Cauchy–Green stress tensor C.

1.10 Corollary. $\Sigma(\nabla\phi) = \tilde{\Sigma}(\nabla\phi^T\nabla\phi)$ *for an isotropic and objective material, where*

$$\tilde{\Sigma}(C) = \gamma_0 I + \gamma_1 C + \gamma_2 C^2, \tag{1.28}$$

and where $\gamma_0, \gamma_1, \gamma_2$ *are functions of the invariants* ι_C.

Linear Material Laws

The stress–strain relationship can be described in terms of two parameters in the neighborhood of a strain-free reference configuration. Setting $C = I + 2E$ in (1.28), $\tilde{\Sigma}(I+2E) = \gamma_0(E)\, I + \gamma_1(E)\, E + \gamma_2(E)\, E^2$, where we have not changed the notation for the functions.

1.11 Theorem. *Suppose that in addition to the hypotheses of Corollary 1.10,* γ_0, γ_1 *and* γ_2 *are differentiable functions of* $\iota_1(E), \iota_2(E)$ *and* $\iota_3(E)$. *Then there exist numbers* π, λ, μ *with*

$$\tilde{\Sigma}(I + 2E) = -\pi I + \lambda\, \mathrm{trace}(E)\, I + 2\mu E + o(E) \quad \text{as } E \to 0.$$

Sketch of a proof. First note that $\tilde{\Sigma}(1+2E) = \gamma_0(E)\, I + \gamma_1\, E + o(E)$. In particular, only the constant term in γ_1 is used. By Remarks 1.9, we know that $\tilde{\Sigma}(I) = -\pi I$ with a suitable $\pi \geq 0$. By (1.24), we deduce that $\iota_2 = \mathcal{O}(E^2)$ and $\iota_3 = \mathcal{O}(E^3)$, and only the constants and the trace remain in the terms of first order in $\gamma_0(E)$. □

Normally, the situation $C = I$ corresponds to an unstressed condition, and $\pi = 0$. The other two constants are called *Lamé constants*. If we ignore the terms of higher order, we are led to the *linear material law of Hooke*:

$$\tilde{\Sigma}(I + 2E) = \lambda\, \mathrm{trace}(E)\, I + 2\mu E. \tag{1.29}$$

A material which satisfies (1.29) in general and not just for small strains is called a *St. Venant–Kirchhoff material.* Note that in the approximation (1.6),

$$\mathrm{trace}(\varepsilon) = \mathrm{div}\, u, \tag{1.30}$$

and thus the Lamé constant λ describes the stresses due to change in density. The other Lamé constant μ is sometimes called the *shear modulus of the material.*

If we use a different set of frequently used parameters, namely *Young's modulus of elasticity* $\mathtt{E}$ and the *Poisson ratio* ν, we have the following relationship:

$$\begin{aligned} \nu &= \frac{\lambda}{2(\lambda+\mu)}, & \mathtt{E} &= \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}, \\ \lambda &= \frac{\mathtt{E}\nu}{(1+\nu)(1-2\nu)}, & \mu &= \frac{\mathtt{E}}{2(1+\nu)}. \end{aligned} \tag{1.31}$$

It follows from physical considerations that $\lambda > 0$, $\mu > 0$, and $\mathtt{E} > 0$, $0 < \nu < \frac{1}{2}$.

The Poisson ratio ν describes the influence of stresses on displacements in the orthogonal directions shown in Fig. 57. For many materials, $\nu \approx 1/3$. On the other hand, for *nearly incompressible materials*, $\lambda \gg \mu$, i.e., ν is very close to $1/2$.

The deformations, stresses, and strains are defined by the kinematics, the equilibrium equations, and the constitutive equations. In principle, only the equilibrium equations (for the Cauchy stress tensor) are linear.

If we assume small deformations, and replace the strain E by the linearization ε, it suffices to work with the so-called *geometrically linear theory*. However, for practical everyday calculation, the complete linear theory where we also assume that the constitutive equations are linear and work with isotropic media is of the greatest importance.

Problem

1.12 Often a *polar factorization* of the deformation gradient

$$F = RU \quad \text{or} \quad F = VR$$

with positive definite Hermitean matrices U, V, and an orthogonal Matrix R is considered. In this way the invariance properties are accentuated. Show that

$$U := (F^T F)^{1/2}, \;\; R := FU^{-1}, \;\text{and}\; V := RUR^T$$

yield the desired factorization and that it is unique. Here F is assumed to be nonsingular.

§ 2. Hyperelastic Materials

By Cauchy's theorem, the equilibrium state of an elastic body is characterized by

$$- \operatorname{div} T(x) = f(x), \quad x \in \Omega, \tag{2.1}$$

and the boundary conditions

$$\begin{aligned} \phi(x) &= \phi_0(x), \quad x \in \Gamma_0, \\ T(x) \cdot n &= g(x), \quad x \in \Gamma_1. \end{aligned} \tag{2.2}$$

Here f is the applied body force and g is the surface traction on the part Γ_1 of the boundary. Γ_0 denotes the part of the boundary on which the displacement is given.

We regard these equations as a boundary-value problem for the deformation ϕ, and write

$$\begin{aligned} - \operatorname{div} \hat{T}(x, \nabla\phi(x)) &= f(x), \quad x \in \Omega, \\ \hat{T}(x, \nabla\phi(x))\, n &= g(x), \quad x \in \Gamma_1, \\ \phi(x) &= \phi(x_0), \quad x \in \Gamma_0. \end{aligned} \tag{2.3}$$

For simplicity, we neglect the dependence of the forces f and g on ϕ, i.e., we consider them to be dead loads; cf. Ciarlet [1988, §2.7].

To be more precise, Ω is the domain occupied by the deformed body, and is also unknown. For simplicity, we identify Ω with the reference configuration, and restrict ourselves to an approximation which makes sense for small deformations.

2.1 Definition. An elastic material is called *hyperelastic* if there exists an energy functional $\hat{W} : \Omega \times \mathbb{M}^3_+ \longrightarrow \mathbb{R}$ such that

$$\hat{T}(x, F) = \frac{\partial \hat{W}}{\partial F}(x, F) \quad \text{for } x \in \Omega,\ F \in \mathbb{M}^3_+ .$$

There is a variational formulation corresponding to the boundary-value problem (2.3) for hyperelastic materials, provided that the vector fields f and g can be written as gradient fields: $f = \operatorname{grad} \mathcal{F}$ and $g = \operatorname{grad} \mathcal{G}$. In this case the solutions of (2.3) are stationary points of the total energy

$$I(\psi) = \int_\Omega [\hat{W}(x, \nabla\psi(x)) - \mathcal{F}(\psi(x))]dx + \int_{\Gamma_1} \mathcal{G}(\psi(x))dx. \tag{2.4}$$

As deformations we admit functions ψ which satisfy Dirichlet boundary conditions on Γ_0 along with the local injectivity condition $\det(\nabla\psi(x)) > 0$. – We introduce appropriate function spaces later.

The expression (2.4) refers to the variational formulation for the displacements. We note that frequently the stresses are also included as variables in the variational problem. Because of the coupling of the kinematics with the constitutive equations, we get a saddle point problem, and thus mixed methods need to be applied.

2.2 Remark. The properties of the material laws discussed in §1 may be rediscovered in analogous properties of the energy functionals. To save space, we present them without proof.

For an objective material, $\hat{W}(x,\cdot)$ is a function of only $C = F^T F$:

$$\hat{W}(x, F) = \tilde{W}(x, F^T F)$$

and

$$\tilde{\Sigma}(x, C) = 2\frac{\partial\tilde{W}(x, C)}{\partial C} \quad \text{for all } C \in \mathbb{S}^3_>.$$

The dependence of C can be made more precise. $\tilde{W}$ depends only on the principal invariants of C, i.e., $\tilde{W}(x, C) = \dot{W}(x, \iota_C)$ for $C \in \mathbb{S}^3_>$. Analogously, for isotropic materials, we have

$$\hat{W}(x, F) = \hat{W}(x, FQ) \quad \text{for all } F \in \mathbb{M}^3_+,\ Q \in \mathbb{O}^3_+.$$

In particular, for small deformations,

$$\tilde{W}(x, C) = \frac{\lambda}{2}\,(\text{trace } E)^2 + \mu\, E : E + o(E^2) \tag{2.5}$$

with $C = I + 2E$. Here, as usual,

$$A : B := \sum_{ij} A_{ij} B_{ij} = \text{trace}(A^T B),$$

for any two matrices A and B.

2.3 Examples. (1) For St. Venant–Kirchhoff materials,

$$\begin{aligned} \hat{W}(x, F) &= \frac{\lambda}{2}\,(\text{trace } F - 3)^2 + \mu\, F : F \\ &= \frac{\lambda}{2}\,(\text{trace } E)^2 + \mu\ \text{trace}\, C. \end{aligned} \tag{2.6}$$

(2) For so-called *neo-Hookean materials*,

$$\tilde{W}(x, C) = \frac{1}{2}\mu[\operatorname{trace}(C - I) + \frac{2}{\beta}\{(\det C)^{-\beta/2} - 1\}], \tag{2.7}$$

where $\beta = \frac{2\nu}{1-2\nu}$.

We note that (2.6) is restricted to strains which are not too large. Indeed, we expect that

$$\hat{W}(x, F) \longrightarrow \infty \ \text{ as } \det F \to 0, \tag{2.8}$$

since $\det F \to 0$ means that the density of the deformed material becomes very large. The condition (2.8) implies that $\hat{W}$ is not a convex function of F. Indeed, the set of matrices

$$B = \{F \in \mathbb{M}^3;\ \det F > 0\} \tag{2.9}$$

is not a convex set; see Problem 2.4. There are many matrices F_0 with $\det F_0 = 0$ which are the convex combination of two matrices F_1 and F_2 with positive determinants. By the continuity of $\hat{W}$ at F_1 and F_2, we would get the boundedness in a neighborhood of F_0 whenever $\hat{W}$ is assumed to be convex.

Problems

2.4 Show that (2.9) does not define a convex set by considering the convex combinations of the matrices

$$\begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & & \\ & -4 & \\ & & 1 \end{pmatrix}.$$

2.5 Consider a St. Venant–Kirchhoff material with the energy function (2.6), and show that there would exist negative energy states if $\mu < 0$ were to hold.

2.6 Consider a neo-Hookean material for small strains, and establish Hooke's law with the same parameters μ and ν.

2.7 Often the energy functional depends on $J := \det F$. Show that the derivate is given by

$$D_F J : \delta F = J \operatorname{trace}(F^{-1}\delta F), \qquad D_C J : \delta C = \frac{1}{2} J \operatorname{trace}(C^{-1}\delta C).$$

Hint. For $F = I$ we have obviously $D_I J\,\delta F = \operatorname{trace}(\delta F)$ and $\det(F + \delta F) = \det F \det(I + F^{-1}\delta F)$.

§ 3. Linear Elasticity Theory

In the linearized equations of elasticity theory we take account only of terms of first order in the displacement u while terms of higher order are neglected. This affects the kinematics in terms of the approximation (1.6), and the constitutive equations in terms of (1.29) or (2.6). Here we restrict ourselves to the isotropic case for two reasons: to keep the discussion more accessible, and because this case is more important in practice. In this framework, we do not have to distinguish between different stress tensors. In order to make this clear, we write

$$\sigma \text{ instead of } \Sigma \quad \text{and} \quad \varepsilon \text{ instead of } E.$$

We begin with a short overview, and then in the framework of three-dimensional elasticity theory consider various formulations of the variational problems, including mixed methods in particular.

In order to make this discussion as independent of the previous sections as possible, we first recall the necessary equations.

The Variational Problem

In the framework of the linear theory, the variational problem is to minimize the energy

$$\Pi := \int_\Omega \Big[\frac{1}{2}\varepsilon : \sigma - f \cdot u\Big] dx + \int_{\Gamma_1} g \cdot u \, dx. \tag{3.1}$$

Here $\varepsilon : \sigma := \sum_{ik} \varepsilon_{ik}\sigma_{ik}$. The variables σ, ε and u in (3.1) are not independent, but instead are coupled by the kinematic equations

$$\begin{aligned} \varepsilon_{ij} &= \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) \\ \text{or} \quad \varepsilon &= \varepsilon(u) =: \nabla^{(s)} u, \end{aligned} \tag{3.2}$$

where $\nabla^{(s)}$ is the symmetric gradient, and the linear constitutive equations

$$\varepsilon = \frac{1+\nu}{\mathtt{E}}\sigma - \frac{\nu}{\mathtt{E}} \operatorname{trace} \sigma \, I. \tag{3.3}$$

In order to establish the connection between (3.1) and (2.4), we first invert (3.3). Since $\operatorname{trace} I = 3$, it follows from (3.3) that $\operatorname{trace} \varepsilon = (1-2\nu)/\mathtt{E} \operatorname{trace} \sigma$, and solving for σ gives

$$\sigma = \frac{\mathtt{E}}{1+\nu}\Big(\varepsilon + \frac{\nu}{1-2\nu} \operatorname{trace} \varepsilon \, I\Big). \tag{3.4}$$

In contrast to (1.29), the constants here are expressed in terms of the modulus of elasticity and the Poisson ratio. Moreover, $\varepsilon : I = \operatorname{trace}\varepsilon$, and hence

$$\frac{1}{2}\sigma : \varepsilon = \frac{1}{2}(\lambda \operatorname{trace}\varepsilon\, I + 2\mu\varepsilon) : \varepsilon = \frac{\lambda}{2}(\operatorname{trace}\varepsilon)^2 + \mu\,\varepsilon : \varepsilon \tag{3.5}$$

coincides with the energy functional in (2.6).

We note that the equation (3.4) is often written componentwise:

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{13}\\ \sigma_{23}\end{bmatrix} = \frac{\mathtt{E}}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & & & \\ \nu & 1-\nu & \nu & & 0 & \\ \nu & \nu & 1-\nu & & & \\ & & & 1-2\nu & & \\ & 0 & & & 1-2\nu & \\ & & & & & 1-2\nu\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{12}\\ \varepsilon_{13}\\ \varepsilon_{23}\end{bmatrix}$$

$$\text{or}\quad \sigma = \mathcal{C}\varepsilon, \tag{3.6}$$

see Problem 4.7.[15] The fact that the matrix $\mathcal{C}$ is positive definite for $0 \le \nu < \frac{1}{2}$ can be seen by applying the Gerschgorin theorem to the *compliance matrix*, i.e. the inverse,

$$\mathcal{C}^{-1} = \frac{1}{\mathtt{E}}\begin{bmatrix}1 & -\nu & -\nu & & & \\ -\nu & 1 & -\nu & & 0 & \\ -\nu & -\nu & 1 & & & \\ & & & 1+\nu & & \\ & 0 & & & 1+\nu & \\ & & & & & 1+\nu\end{bmatrix}. \tag{3.7}$$

Clearly, (3.1), (3.2), and (3.3) lead to a mixed variational formulation. We can now eliminate one or two variables. Thus, there are three distinct formulations in the engineering literature; see Stein and Wunderlich [1973]. Before treating them in detail, we give a short overview.

(1) The displacement formulation.
We eliminate σ with the help of (3.6), and then ε using (3.2):

$$\begin{aligned}\Pi(v) &= \int_\Omega \Big[\frac{1}{2}\nabla^{(s)}v : \mathcal{C}\,\nabla^{(s)}v - f\cdot v\Big]dx + \int_{\Gamma_1} g\cdot v\,dx\\ &= \int_\Omega \big[\mu\,\varepsilon(v) : \varepsilon(v) + \frac{\lambda}{2}(\operatorname{div} v)^2 - f\cdot v\big]dx + \int_{\Gamma_1} g\cdot v\,dx \longrightarrow \min!\end{aligned} \tag{3.8}$$

[15] In engineering references the nondiagonal components of ε are usually normalized so that they differ from (3.2) by a factor of 2. With that notation, called *Voigt notation*, some of our constants will be changed by a factor 2.

Here $\partial\Omega$ is divided into Γ_0 and Γ_1 depending on the boundary conditions as in (2.2). Assuming for simplicity that zero boundary conditions are specified on Γ_0, we have to find the minimum over

$$H^1_\Gamma := \{v \in H^1(\Omega)^3;\ v(x) = 0 \text{ for } x \in \Gamma_0\}.$$

The associated weak formulation is the following: *Find* $u \in H^1_\Gamma$ *with*

$$\int_\Omega \nabla^{(s)} u : \mathcal{C}\, \nabla^{(s)} v \, dx = (f, v)_0 - \int_{\Gamma_1} g \cdot v \, dx \quad \text{for all } v \in H^1_\Gamma.$$

In terms of the L_2-scalar product for matrix-valued functions, we can write these equations in the short form

$$(\nabla^{(s)} u, \mathcal{C}\, \nabla^{(s)} v)_0 = (f, v)_0 - \int_{\Gamma_1} g \cdot v \, dx \quad \text{for all } v \in H^1_\Gamma(\Omega), \tag{3.9}$$

and in particular, for St. Venant–Kirchhoff materials as

$$2\mu(\nabla^{(s)} u, \nabla^{(s)} v)_0 + \lambda(\operatorname{div} u, \operatorname{div} v)_0 = (f, v)_0 - \int_{\Gamma_1} g \cdot v \, dx. \tag{3.10}$$

The associated classical elliptic differential equation is the Lamé differential equation

$$\begin{aligned} -2\mu \operatorname{div} \varepsilon(u) - \lambda \operatorname{grad} \operatorname{div} u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_0, \\ \sigma(u) \cdot n &= g \quad \text{on } \Gamma_1. \end{aligned} \tag{3.11}$$

(2) The mixed method of Hellinger and Reissner

In this method, also called the Hellinger–Reissner principle, the displacement and stresses remain as unknowns, while the strains are eliminated:

$$\begin{aligned} (\mathcal{C}^{-1}\sigma - \nabla^{(s)} u, \tau)_0 &= 0 && \text{for all } \tau \in L_2(\Omega), \\ -(\sigma, \nabla^{(s)} v)_0 &= -(f, v)_0 + \int_{\Gamma_1} g \cdot v \, dx && \text{for all } v \in H^1_\Gamma(\Omega). \end{aligned} \tag{3.12}$$

The equivalence of (3.9) and (3.12) can be seen as follows: Let u be a solution of (3.9). Since $u \in H^1$,

$$\sigma := \mathcal{C}\, \nabla^{(s)} u \in L_2. \tag{3.13}$$

Because of the symmetry of $\mathcal{C}$, (3.9) implies the second equation of (3.12). The first equation of (3.12) is just the weak formulation of (3.13). As soon as we establish that the two variational problems are uniquely solvable, we have the equivalence.

We can write (3.12) as a classical differential equation in the form

$$
\begin{aligned}
\operatorname{div}\sigma &= -f && \text{in } \Omega,\\
\sigma &= \mathcal{C}\nabla^{(s)}u && \text{in } \Omega,\\
u &= 0 && \text{on } \Gamma_0,\\
\sigma\cdot n &= g && \text{on } \Gamma_1.
\end{aligned}
$$

In particular, in view of the second equation, σ is a symmetric tensor.[16]

The equations fit (at least formally) in the general framework of Ch. III in the following canonical form:[17]

$$
\begin{aligned}
X &= L_2(\Omega), \quad M = H^1_\Gamma(\Omega),\\
a(\sigma,\tau) &= (\mathcal{C}^{-1}\sigma,\tau)_0, \quad b(\tau,v) = -(\tau,\nabla^{(s)}v)_0.
\end{aligned}
$$

As for the mixed formulation of the Poisson equation (see Ch. III, §5), there is an alternative: Fix

$$
\begin{aligned}
X &:= H(\operatorname{div},\Omega), \quad M = L_2(\Omega),\\
a(\sigma,\tau) &= (\mathcal{C}^{-1}\sigma,\tau)_0, \quad b(\tau,v) = (\operatorname{div}\sigma,v)_0,
\end{aligned}
\tag{3.14}
$$

where $H(\operatorname{div},\Omega)$ is once again the closure of $C^\infty(\Omega,\mathbb{S}^3)$ w.r.t. the norm (III.5.4),

$$
\|\tau\|_{H(\operatorname{div},\Omega)} := (\|\tau\|_0^2 + \|\operatorname{div}\tau\|_0^2)^{1/2}.
$$

Integrating by parts, we get $b(\tau,v) = (\operatorname{div}\tau,v)_0$. Which formulation makes the most sense depends among other things on the boundary conditions (see below). The connection with the Cauchy equilibrium equations (1.11) is clear from the second version.

---

16 If we give up the linearization in the kinematics, we get the nonlinear system

$$
\begin{aligned}
\partial_j(\sigma_{ij}+\sigma_{kj}\partial_k u_i) &= -f_i && \text{in } \Omega,\\
\sigma &= \mathcal{C}E(u) && \text{in } \Omega,\\
u &= 0 && \text{on } \Gamma_0,\\
(\sigma_{ij}+\sigma_{kj}\partial_k u_i)n_j &= g_i && \text{on } \Gamma_1.
\end{aligned}
$$

Here the sums are to be taken over the double indices using the so-called Einstein convention.

17 For simplicity, we do not write the more precise formulation of $\varepsilon,\sigma\in L_2(\Omega,\mathbb{S}^3)$.

(3) The mixed method of Hu and Washizu (Hu–Washizu principle). Here all three variables remain in the equations:

$$\begin{aligned}
(\mathcal{C}\varepsilon - \sigma, \eta)_0 &= 0 && \text{for all } \eta \in L_2(\Omega),\\
(\varepsilon - \nabla^{(s)} u, \tau)_0 &= 0 && \text{for all } \tau \in L_2(\Omega),\\
-(\sigma, \nabla^{(s)} v)_0 &= -(f, v)_0 + \int_{\Gamma_1} g \cdot v\, dx && \text{for all } v \in H^1_\Gamma(\Omega).
\end{aligned} \tag{3.15}$$

In comparison with (3.12), we have now added the relation $\varepsilon := \mathcal{C}^{-1}\sigma \in L_2(\Omega)$, so that (3.12) and (3.15) are equivalent. To fit this in the general framework, we set

$$X := L_2(\Omega) \times L_2(\Omega), \quad M := H^1_\Gamma(\Omega),$$
$$a(\varepsilon, \sigma, \eta, \tau) = (\mathcal{C}\,\varepsilon, \eta)_0, \quad b(\eta, \tau, v) = (\tau, \nabla^{(s)} v - \varepsilon)_0.$$

We consider all three approaches in more detail below.

The simplest of the three is the displacement formulation. Establishing the validity of the Babuška–Brezzi condition for the mixed methods is considerably more difficult than for the Stokes problem; see below. On the other hand, for applications we are mostly interested in computing the stresses with more accuracy than the displacements. Thus, we look for approaches where the stresses are computed directly rather than via subsequent evaluation of derivatives. We will see more reasons for turning to mixed methods despite their complexity. In fact, we prefer the Hellinger–Reissner rather than the Hu–Washizu principle.

The Hellinger–Reissner principle arose from leaving all components of the stress tensor in the equations. Versions where only some special terms with strains and stresses remain in the equations are also important in practice. This typically leads to mixed methods with penalty terms. As an example, we later discuss a method for nearly incompressible material and plate bending problems.

The Displacement Formulation

It follows from (3.8) that the energy for the displacement method is H^1-elliptic, provided that the quadratic form $\int \varepsilon(v) : \varepsilon(v)\, dx$ has this property. This is the content of a famous inequality. Here we do not restrict the dimension d to be 3.

3.1 Korn's Inequality *(Korn's first inequality). Let Ω be an open bounded set in $\mathbb{R}^d$ with piecewise smooth boundary. Then there exists a number $c = c(\Omega) > 0$ such that*

$$\int_\Omega \varepsilon(v) : \varepsilon(v)\, dx + \|v\|_0^2 \ge c\|v\|_1^2 \quad \textit{for all } v \in H^1(\Omega)^d.$$

For a proof, see Duvaut and Lions [1976], Nitsche [1981], or the end of this §. Its structure is similar to the proof that the divergence satisfies an inf-sup condition as a mapping of $H^1(\Omega)^d$ into $L_2(\Omega)$; cf. Ch. III, §6. A special case in which the inequality can be easily verified is dealt with in Remark 3.5 below.

3.2 Remark. If the strain tensor E of a deformation is trivial, then by Remark 1.1 the deformation is an affine distance-preserving transformation. An analogous assertion holds for the linearized strain tensor ε: *Let $\Omega \subset \mathbb{R}^3$ be open and connected. Then for $v \in H^1(\Omega)$,*

$$\varepsilon(v) = 0,$$

if and only if

$$v(x) = a \wedge x + b \quad \text{with } a, b \in \mathbb{R}^3. \tag{3.16}$$

For the proof, we note that

$$\frac{\partial^2}{\partial x_i \partial x_j} v_k = \frac{\partial}{\partial x_i}\varepsilon_{jk} + \frac{\partial}{\partial x_j}\varepsilon_{ik} - \frac{\partial}{\partial x_k}\varepsilon_{ij} = 0 \tag{3.17}$$

in $H^{-1}(\Omega)$ if $\varepsilon(v) = 0$. From this we conclude that every component v_k must be a linear function. But then a simple computation shows that a displacement of the form $v(x) = Ax+b$ can only be compatible with $\varepsilon(v) = 0$ if A is skew-symmetric. This leads to (3.16).

On the other hand, it is easy to verify that the linear strains for the displacements of the form (3.16) vanish. □

Korn's inequality is simplified for functions which satisfy a zero boundary condition. In the sense of Remark II.1.6, it is only necessary that v vanishes on a part Γ_0 of the boundary, and that Γ_0 possesses a positive $(n-1)$-dimensional measure.

3.3 Korn's Inequality *(Korn's second inequality). Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with piecewise smooth boundary. In addition, suppose $\Gamma_0 \subset \partial\Omega$ has positive two-dimensional measure. Then there exists a positive number $c' = c'(\Omega, \Gamma_0)$ such that*

$$\int_\Omega \varepsilon(v) : \varepsilon(v)dx \ge c'\|v\|_1^2 \quad \text{for all } v \in H^1_\Gamma(\Omega). \tag{3.18}$$

Here $H^1_\Gamma(\Omega)$ is the closure of $\{v \in C^\infty(\Omega)^3;\ v(x) = 0 \text{ for } x \in \Gamma_0\}$ w.r.t. the $\|\cdot\|_1$-norm.

Proof. Suppose that the inequality is false. Then there exists a sequence $(v_n) \in H^1_\Gamma(\Omega)$ with

$$\|\varepsilon(v_n)\|_0^2 := \int \varepsilon(v_n) : \varepsilon(v_n)dx \le \frac{1}{n} \quad \text{and} \quad |v_n|_1 = 1.$$

Because of the hypothesis on Γ_0, Friedrichs' inequality implies $\|v_n\|_1 \le c_1$ for all n and some suitable $c_1 > 0$. Since $H^1(\Omega)$ is compact in $H^0(\Omega)$, there is a subsequence of (v_n) which converges w.r.t. the $\|\cdot\|_0$-norm. With the constant c from Theorem 3.1, we have $c\|v_n - v_m\|_1^2 \le \|\varepsilon(v_n - v_m)\|_0^2 + \|v_n - v_m\|_0^2 \le 2\|\varepsilon(v_n)\|_0^2 + 2\|\varepsilon(v_m)\|^2 + \|v_n - v_m\|_0^2 \le \frac{2}{n} + \frac{2}{m} + \|v_n - v_m\|_0^2$.

The L_2-convergent subsequence is thus a Cauchy sequence in $H^1(\Omega)$, and so converges in the sense of H^1 to some u_0. Hence, $\|\varepsilon(u_0)\| = \lim_{n\to\infty} \|\varepsilon(v_n)\| = 0$, and $|u_0|_1 = \lim_{n\to\infty} |v_n|_1 = 1$. By Remark 3.2, we deduce from $\varepsilon(u_0) = 0$ that u_0 has the form (3.17). In view of the zero boundary condition on Γ_0, it follows that $u_0 = 0$. This is a contradiction to $|u_0|_1 = 1$. □

Korn's inequality asserts that the variational problem (3.8) is elliptic. Thus, the general theory immediately leads to

3.4 Existence Theorem. *Let $\Omega \subset \mathbb{R}^3$ be a domain with piecewise smooth boundary, and suppose Γ_0 has positive two-dimensional measure. Then the variational problem (3.8) of linear elasticity theory has exactly one solution.*

3.5 Remark. In the special case where Dirichlet boundary conditions are prescribed (i.e., $\Gamma_0 = \Gamma$ and $H^1_\Gamma = H^1_0$), the proof of Korn's first inequality is simpler. In this case

$$|v|_{1,\Omega} \le \sqrt{2}\, \|\varepsilon(v)\|_{0,\Omega} \quad \text{for all } v \in H_0^1(\Omega)^3. \tag{3.19}$$

It suffices to show the formula for smooth vector fields. In this case we have

$$2\nabla^{(s)}v : \nabla^{(s)}v - \nabla v : \nabla v = \operatorname{div}[(v\nabla)v - (\operatorname{div} v)v] + (\operatorname{div} v)^2. \tag{3.20}$$

Here $(v\nabla)$ is to be interpreted as $\sum_i v_i \frac{\partial}{\partial x_i}$. The formula (3.20) can be verified, for example, by solving for all terms in the double sum. Since $v = 0$ on $\partial\Omega$, it follows from the Gauss integral theorem that

$$\int_\Omega \operatorname{div}[(v\nabla)v - (\operatorname{div} v)v]dx = \int_{\partial\Omega} [(v\nabla)v - (\operatorname{div} v)v]n ds = 0.$$

Integrating (3.20) over Ω, we have

$$2\|\nabla^{(s)}v\|_0^2 - |v|_1^2 = \int_\Omega (\operatorname{div} v)^2 dx \ge 0,$$

and (3.19) is proved. □

Note that the constant in (3.19) is independent of the domain. If we are given Neumann boundary conditions on a part of the boundary, the constant can easily depend on Ω. We will see the consequences in connection with the locking effect for the cantilever beam shown in Fig. 58 below. — On the other hand, for the pure traction problem, i.e., for $\Gamma_0 = \emptyset$, there is again a compatibility condition; see Problem 3.17.

The Mixed Method of Hellinger and Reissner

The two-field formulation with the displacements and the stresses is usually denoted as the *Hellinger–Reissner principle*, but also as the *Hellinger–Prange–Reissner principle*. The basic idea is contained in Hellinger [1914]; a first proof for the traction problem is due to Prange [1916][18] and for mixed boundary conditions due to Reissner [1950]; see Gurtin [1972], p. 124 and Orava and McLean [1966].

The method has many similarities to the mixed formulation of the Poisson equation in Ch. III, §5. The variational formulation according to (3.12) is

$$\begin{aligned}
(\mathcal{C}^{-1}\sigma, \tau)_0 \; - (\tau, \nabla^{(s)} u)_0 &= 0 && \text{for all } \tau \in L_2(\Omega), \\
-(\sigma, \nabla^{(s)} v)_0 \qquad\qquad &= -(f, v)_0 + \int_{\Gamma_1} g \cdot v \, dx && \text{for all } v \in H^1_\Gamma(\Omega),
\end{aligned} \tag{3.21}$$

which corresponds to the standard displacement formulation. Since $\nu < \frac{1}{2}$, $\mathcal{C}$ is positive definite and the bilinear form $(\mathcal{C}^{-1}\sigma, \tau)_0$ is L_2-elliptic. The following lemma shows that the inf-sup condition follows from Korn's inequality.

3.6 Lemma. *Suppose the hypotheses for Korn's second inequality are satisfied. Then for all $v \in H^1_\Gamma(\Omega)$,*

$$\sup_{\tau \in L_2(\Omega, \mathbb{S}^3)} \frac{(\tau, \nabla^{(s)} v)_0}{\|\tau\|_0} \ge c' \|v\|_1,$$

where c' is the constant in (3.18).

Proof. Given $v \in H^1_\Gamma(\Omega)$, $\tau := \nabla^{(s)} v$ is a symmetric L_2-tensor. Moreover, by (3.18), $\|\tau\|_0 = \|\nabla^{(s)} v\|_0 \ge c' \|v\|_1$. It suffices to consider the case $v \ne 0$:

$$\frac{(\tau, \nabla^{(s)} v)_0}{\|\tau\|_0} = \frac{\|\nabla^{(s)} v\|_0^2}{\|\nabla^{(s)} v\|_0} \ge c' \|v\|_1,$$

which establishes the inf-sup condition. □

The formulation with the spaces as in (3.21) is almost equivalent to the displacement formulation. Specifically, it can be understood as a displacement formulation combined with a softening of the energy. It is suitable for the method of *enhanced assumed strains* by Simo and Rifai [1990]; see Ch. III, §5. As in the discretization of the Poisson equation using the Raviart–Thomas element, generally

[18] Prange's "Habilitationsschrift" (a thesis for an academic degree at a level above the doctorate, which is usually a prerequisite for a professorship in Germany) was unpublished due to the first world war. It was only edited with an introduction by K. Knothe in 1999.

the pairing (3.14) is more appropriate. *Find* $\sigma \in H(\mathrm{div}, \Omega)$ *and* $u \in L_2(\Omega)^3$ *with*

$$\begin{aligned}
(\mathcal{C}^{-1}\sigma, \tau)_0 \; + (\mathrm{div}\, \tau, u)_0 &= 0 && \text{for all } \tau \in H(\mathrm{div}, \Omega),\ \tau n = 0 \text{ on } \Gamma_1,\\
(\mathrm{div}\, \sigma, v)_0 &= -(f, v)_0 && \text{for all } v \in L_2(\Omega)^3,\\
\sigma n &= g && \text{on } \Gamma_1.
\end{aligned} \tag{3.22}$$

We assume that an inhomogeneous boundary condition has been reduced to a homogeneous one in the sense of Ch. II, §2. The equations (3.22) are the Euler equations for the saddle point problem

$$(\mathcal{C}^{-1}\sigma, \sigma)_0 \longrightarrow \min_{\sigma \in H(\mathrm{div}, \Omega)} !$$

with the restriction

$$\mathrm{div}\, \sigma = f$$

and the boundary condition $\sigma n = g$ on Γ_1. This is often called the *dual mixed method*.

Just as in $H_0^1(\Omega)$ where boundary values for the function are prescribed, in the (less regular) space $H(\mathrm{div}, \Omega)$ we can specify the normal components on the boundary. This becomes clear from the jump conditions in Problem II.5.14. Here we assume that the boundary is piecewise smooth.

Although in (3.22) formally we required only that $u \in L_2(\Omega)^3$, in fact the solution satisfies $u \in H_\Gamma^1(\Omega)$. It follows from (3.22) that $\varepsilon(u) = \mathcal{C}^{-1}\sigma \in L_2(\Omega)$. Indeed, suppose $i, j \in \{1, 2, 3\}$ and that only $\tau_{ij} = \tau_{ji}$ are nonzero. In addition, let $\tau_{ij} \in C_0^\infty(\Omega)$. Then writing w instead of τ_{ij}, it follows from (3.22) that

$$\frac{1}{2}\int_\Omega \Big(u_i \frac{\partial w}{\partial x_j} + u_j \frac{\partial w}{\partial x_i}\Big) dx = -\int_\Omega (\mathcal{C}^{-1}\sigma)_{ij} w \, dx.$$

Recalling Definition II.1.1, we see that the symmetric gradient $(\nabla^{(s)} u)_{ij}$ exists in the weak sense, and coincides with $(\mathcal{C}^{-1}\sigma)_{ij} \in L_2(\Omega)$. Now Korn's first inequality implies $u \in H^1(\Omega)^3$. Finally, we apply Green's formula. Because of the symmetry, it follows that for all test functions τ as in (3.22),

$$\begin{aligned}
\int_{\partial\Omega} u \cdot \tau n \, ds &= \int_\Omega \nabla u : \tau \, dx + \int_\Omega u \cdot \mathrm{div}\, \tau \, dx\\
&= \int_\Omega \nabla^{(s)} u : \tau \, dx + \int_\Omega u \cdot \mathrm{div}\, \tau \, dx\\
&= \int_\Omega \mathcal{C}^{-1}\sigma : \tau \, dx + \int_\Omega u \cdot \mathrm{div}\, \tau \, dx = 0.
\end{aligned}$$

Since this holds for all test functions, it follows that $u = 0$ on $\Gamma_0 = \partial\Omega \backslash \Gamma_1$. □

The inf-sup condition and the V-ellipticity follow exactly as in Ch. III, §5. However, we emphasize that they are by no means trivial for the finite element spaces, and it is not easy to find stable pairings of finite element spaces. We explore the consequences for the two-dimensional case in §4.

We emphasize that we obtain different natural boundary conditions for the two formulations: with (3.21) they are $\sigma n = g$ on Γ_1, while with (3.22) we have $u = 0$ on Γ_0.

3.7 Remark. When $\Gamma_0 = \Gamma$, $\Gamma_1 = \emptyset$, i.e., for pure displacement boundary conditions, we need an extra argument for the Hellinger–Reissner principle. In this case the stresses lie in the subspace

$$\hat{H}(\mathrm{div}, \Omega) := \left\{\tau \in H(\mathrm{div}, \Omega);\ \int_\Omega \mathrm{trace}\,\tau\, dx = 0\right\}. \tag{3.23}$$

Indeed, combining (1.30), (3.3), the Gauss integral theorem, and the fact that $u = 0$ on the boundary, we have

$$\begin{aligned}\int_\Omega \mathrm{trace}\,\sigma\, dx &= \frac{\mathtt{E}}{1-2\nu}\int_\Omega \mathrm{trace}\,\varepsilon\, dx = \frac{\mathtt{E}}{1-2\nu}\int_\Omega \mathrm{div}\, u\, dx\\ &= \frac{\mathtt{E}}{1-2\nu}\int_{\partial\Omega} u\cdot n\, ds = 0.\end{aligned}$$

The Mixed Method of Hu and Washizu

In the Hu–Washizu principle, the stresses take the role of the Lagrange multipliers; cf. Hu [1955] and Washizu [1955]. As was pointed out by Felippa [2000], the notation *de Veubeke–Hu–Washizu principle* would be more appropriate since the three-field formulation can already be found in Fraeijs de Veubeke [1951]. Let

$$\begin{aligned}&X := L_2(\Omega)\times H^1_\Gamma(\Omega), \quad M := L_2(\Omega),\\ &a(\varepsilon, u; \eta, v) = (\varepsilon, \mathcal{C}\eta)_0, \quad b(\varepsilon, u; \tau) = -(\varepsilon,\tau)_0 + (\nabla^{(s)}u, \tau)_0.\end{aligned} \tag{3.24}$$

We seek $(\varepsilon, u) \in X$ and $\sigma \in M$ with

$$\begin{aligned}(\varepsilon, \mathcal{C}\eta)_0 \qquad\qquad &- (\eta,\sigma)_0 = 0 && \text{for all } \eta \in L_2(\Omega),\\ &(\nabla^{(s)}v, \sigma)_0 = (f, v)_0 - \int_{\Gamma_1} g\cdot v\, dx && \text{for all } v \in H^1_\Gamma(\Omega),\\ -(\varepsilon,\tau)_0 + (\nabla^{(s)}u, \tau)_0 \qquad &= 0 && \text{for all } \tau \in L_2(\Omega).\end{aligned}$$

By the definiteness of $\mathcal{C}$, there exists $\beta > 0$ such that

$$\begin{aligned}a(\eta, v; \eta, v) = (\eta, \mathcal{C}\eta)_0 &= \frac{1}{2}(\eta, \mathcal{C}\eta)_0 + \frac{1}{2}(\nabla^{(s)}v, \mathcal{C}\nabla^{(s)}v)_0\\ &\ge \frac{\beta}{2}(\|\eta\|_0^2 + \|\nabla^{(s)}v\|_0^2) \ge \beta(\|\eta\|_0^2 + c'\|v\|_1^2)\end{aligned}$$

holds with c' from Korn's inequality on the subspace

$$V = \{(\eta, v) \in X;\ -(\eta, \tau)_0 + (\nabla^{(s)} v, \tau)_0 = 0 \text{ for } \tau \in M\}.$$

Thus the bilinear form a is V-elliptic.

The inf-sup condition is easily verified. We need only evaluate b with $\eta = \tau$ and $v = 0$.

As a second possibility, using the same bilinear form a, we can work with the pairing

$$\begin{aligned} &X := L_2(\Omega) \times L_2(\Omega)^3, \quad M := \{\tau \in H(\mathrm{div}, \Omega);\ \tau n = 0 \text{ on } \Gamma_1\}, \\ &b(\varepsilon, u; \tau) = -(\varepsilon, \tau)_0 - (u, \mathrm{div}\, \tau)_0. \end{aligned} \tag{3.25}$$

The argument is the same as in the second formulation of the Hellinger–Reissner principle.

In regard to the finite element approximation, we should mention one difference as compared with the Stokes problem. The bilinear form a is elliptic on the entire space X only for the first version of the Hellinger–Reissner principle, while in the other cases it is only V-elliptic. The ellipticity on V_h can only be obtained if the space X_h is not too large in comparison with M_h; see Problem III.4.18. On the other hand, since the inf-sup condition requires X_h to be sufficiently large, the finite element spaces X_h and M_h have to fit together.

There is one more reason why it is not easy to provide stable, genuine elements for the Hu–Washizu principle. Here elements are said to be *genuine* if they are not equivalent to some elements for the Hellinger–Reissner theory or for the displacement formulation.

3.8 First Limit Principle of Stolarski and Belytschko [1966]. *Assume that $u_h \in V_h$, $\varepsilon_h \in E_h$, and $\sigma_h \in S_h$ constitute the finite element solution of a problem by the Hu–Washizu method. If the finite element spaces satisfy the relation*

$$S_h \subset \mathcal{C} E_h, \tag{3.26}$$

then (σ_h, u_h) is the finite element solution of the Hellinger–Reissner formulation with the (same) spaces S_h and V_h.

Proof. The arguments in the proof are purely algebraic and apply to the pairings (3.24) and (3.14), or (3.25) and (3.15), respectively. In order to be specific we restrict ourselves to the first case and assume that

$$\begin{aligned} (\varepsilon_h, \mathcal{C}\eta)_0 \quad &- (\eta, \sigma_h)_0 = 0 && \text{for all } \eta \in E_h, \\ &(\nabla^{(s)} v, \sigma_h)_0 = (f, v)_0 - \int_{\Gamma_1} g \cdot v\, dx && \text{for all } v \in V_h, \\ -(\varepsilon_h, \tau)_0 + (\nabla^{(s)} u_h, \tau)_0 \quad &= 0 && \text{for all } \tau \in S_h. \end{aligned} \tag{3.27}$$

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From the first equation and the symmetry of the bilinear forms we conclude that

$$(\varepsilon_h - \mathcal{C}^{-1}\sigma_h, \mathcal{C}\eta)_0 = 0 \quad \text{for all } \eta \in E_h.$$

By the assumption (3.26), we may set $\eta := \varepsilon_h - \mathcal{C}^{-1}\sigma_h$ and obtain

$$(\varepsilon_h - \mathcal{C}^{-1}\sigma_h, \mathcal{C}(\varepsilon_h - \mathcal{C}^{-1}\sigma_h))_0 = 0.$$

Since $\mathcal{C}$ is positive definite, it follows that $\varepsilon_h = \mathcal{C}^{-1}\sigma_h$. Inserting this into the other equations of (3.27), we see that σ_h and u_h are finite element solutions of (3.21). □

Nevertheless, the three-field formulation has advantages for nonlinear problems and is often used as a point of departure; an example is the EAS method by Simo and Rifai [1990].

Nearly Incompressible Material

The mixed methods discussed in this section thus far refer to standard saddle point formulations. There are situations in which saddle point problems with penalty terms are the appropriate tool. We start with a typical example that can serve as a model problem.

Some materials such as rubber are nearly incompressible. It requires a great deal of energy to produce a small change in density. This results in a large difference in the magnitude of the Lamé constants:

$$\lambda \gg \mu.$$

The bilinear form in the displacement formulation (3.10),

$$a(u, v) := \lambda(\operatorname{div} u, \operatorname{div} v)_0 + 2\mu(\varepsilon(u), \varepsilon(v))_0,$$

is indeed H^1-elliptic, since in principle,

$$\alpha \|v\|_1^2 \le a(v, v) \le C\|v\|_1^2 \quad \text{for all } v \in H^1_\Gamma(\Omega), \tag{3.28}$$

where $\alpha \le \mu$ and $C \ge \lambda + 2\mu$. Therefore, C/α is very large. Since by Céa's lemma the ratio C/α enters in the error estimate, we can expect errors which are significantly larger than the approximation error. This phenomenon is frequently observed in finite element computations, and is called *Poisson locking* or *volume locking*. This is a special case of a *locking effect*, and we now examine it in a preparation for a more general discussion of the effect.

One way to overcome locking is a variational formulation involving a *mixed problem with a penalty term.* We start with the displacement formulation (3.10), and write the linear functional in more abstract form as

$$\lambda(\operatorname{div} u, \operatorname{div} v)_0 + 2\mu(\varepsilon(u), \varepsilon(v))_0 = \langle \ell, v\rangle \quad \text{for all } v \in H^1_\Gamma. \tag{3.29}$$

Substituting

$$\lambda \operatorname{div} u = p, \tag{3.30}$$

and using the weak version of (3.30), we are led to the following problem: *Find* $(u, p) \in H^1_\Gamma(\Omega) \times L_2(\Omega)$ *such that*

$$\begin{aligned} 2\mu(\varepsilon(u), \varepsilon(v))_0 \ + (\operatorname{div} v, p)_0 &= \langle \ell, v \rangle \quad \textit{for all } v \in H^1_\Gamma(\Omega), \\ (\operatorname{div} u, q)_0 \ - \frac{1}{\lambda}(p, q)_0 \ &= 0 \qquad\quad \textit{for all } q \in L_2(\Omega). \end{aligned} \tag{3.31}$$

Since the bilinear form $(\varepsilon(u), \varepsilon(v))_0$ is elliptic on H^1_Γ, (3.31) is very similar to the Stokes problem; see Ch. III, §6. As we observed there, in the case where $\Gamma_0 = \partial\Omega$ (more precisely if the two-dimensional measure of Γ_1 vanishes), $\int_\Omega p\, dx = 0$, and $L_2(\Omega)$ should be replaced by $L_2(\Omega)/\mathbb{R}$.

We know from the theory of mixed problems with penalty terms that the stability of (3.31) is the same as for the problem

$$\begin{aligned} 2\mu(\varepsilon(u), \varepsilon(v))_0 \ + (\operatorname{div} v, p)_0 &= \langle \ell, v \rangle, \\ (\operatorname{div} u, q)_0 \qquad\qquad\qquad &= 0, \end{aligned}$$

as $\lambda \to \infty$. The situation here is simple, since the quadratic form $(\varepsilon(v), \varepsilon(v))_0$ is *coercive on the entire space* and not just for divergence-free functions. Moreover, the penalty term is a regular perturbation. Therefore, we can solve (3.31) using the same elements as for the Stokes problem. Since the inverse of the associated operator

$$L : H^1_\Gamma \times L_2 \to (H^1_\Gamma \times L_2)'$$

is bounded independently of the parameter λ, the finite element solution converges *uniformly in* λ.

To be more specific, consider the discretization

$$\begin{aligned} 2\mu(\varepsilon(u_h), \varepsilon(v))_0 \ + (\operatorname{div} v, p_h)_0 &= \langle \ell, v \rangle \quad \text{for all } v \in X_h, \\ (\operatorname{div} u_h, q)_0 \ - \lambda^{-1}(p, q_h)_0 &= 0 \qquad\quad \text{for all } q \in M_h, \end{aligned} \tag*{$(3.31)_h$}$$

with $X_h \subset H^1_\Gamma(\Omega)$, $M_h \subset L_2(\Omega)$. The commonly used Stokes elements have the following approximation property. Given $v \in H^1_\Gamma(\Omega)$ and $q \in L_2(\Omega)$, there exist $P_h v \in X_h$ and $Q_h q \in M_h$ such that

$$\begin{aligned} \|v - P_h v\|_1 &\le ch\|v\|_2\,, \\ \|q - Q_h q\|_0 &\le ch\|q\|_1\,. \end{aligned} \tag{3.32}$$

For convenience, we restrict ourselves to pure displacement boundary conditions. Good approximation is guaranteed by the following regularity result. *If* Ω *is a convex polygonal domain, or if* Ω *has a smooth boundary, then*

$$\|u\|_2 + \lambda\|\operatorname{div} u\|_1 \le c|f\|_0\,; \tag{3.33}$$

see Theorem A.1 in Vogelius [1983]. Following the usual procedure (see, e.g., Theorem III.4.5) we have

$$\begin{aligned} 2\mu(\varepsilon(u_h - P_h u), \varepsilon(v))_0 \ &+ (\operatorname{div} v, p_h - Q_h p)_0 = \langle \ell_u, v\rangle \quad \text{for all } v \in X_h\,, \\ (\operatorname{div}(u_h - P_h u), q)_0 \ &- \lambda^{-1}(p_h - Q_h p, q)_0 = \langle \ell_p, q\rangle \quad \text{for all } q \in M_h\,. \end{aligned}$$

The functionals ℓ_u and ℓ_p can be expressed in terms of $u - P_h u$ and $p - Q_h p$. From the estimates (3.32) and (3.33) we immediately obtain $\|\ell_u\|_{-1} + \|\ell_p\|_0 \le ch\|f\|_0$. Hence,

$$\|u - u_h\|_1 + \|\lambda \operatorname{div} u - p_h\|_0 \le ch\|f\|_0, \tag{3.34}$$

where c is a constant independent of λ. *The finite element method $(3.31)_h$ is robust.*

Nonconforming methods are also very popular for treating nearly incompressible materials. The finite element discretization $(3.31)_h$ of the saddle point problem can also be interpreted as a nonconforming method, and this discretization will serve as a model for analyzing nonconforming methods for the problem with a small parameter.

3.9 Remark. *Let $u_h \in X_h \subset H^1_\Gamma(\Omega)$ and $p_h \in M_h \subset L_2(\Omega)$ be the solution of the finite element discretization $(3.31)_h$. Define a discrete divergence operator by*

$$\begin{aligned} &\operatorname{div}_h : H^1(\Omega) \to M_h \\ &\quad (\operatorname{div}_h v, q)_0 = (\operatorname{div} v, q)_0 \quad \textit{for all } q \in M_h. \end{aligned} \tag{3.35}$$

Then u_h is also the solution of the variational problem

$$2\mu\|\varepsilon(v)\|_0^2 + \lambda\|\operatorname{div}_h v\|_0^2 - \langle \ell, v\rangle \longrightarrow \min_{v\in X_h}! \tag{3.36}$$

Indeed, the solution u_h of (3.36) is characterized by

$$2\mu(\varepsilon(u_h), \varepsilon(v))_0 + \lambda(\operatorname{div}_h u_h, \operatorname{div}_h v)_0 = \langle \ell, v\rangle \qquad \text{for all } v \in X_h.$$

Setting $p_h := \lambda \operatorname{div}_h u_h$ by analogy to (3.30), we have

$$\begin{aligned} 2\mu(\varepsilon(u_h), \varepsilon(v))_0 \ &+ (\operatorname{div}_h v, p_h)_0 = \langle \ell, v\rangle \quad \text{for all } v \in X_h, \\ (\operatorname{div}_h u_h, q)_0 \ &- \lambda^{-1}(p_h, q)_0 \ = 0 \qquad\quad \text{for all } q \in M_h. \end{aligned}$$

Here the operator div_h is met only in inner products with the other factor in M_h. Now, from the definition (3.35) we know that the operator div_h may be replaced here by div. Therefore, u_h together with $p_h := \lambda \operatorname{div}_h u_h$ satisfy $(3.31)_h$. □

We turn to the analysis of (3.36). The inequality $(3.32)_2$ implies the estimate $\|\operatorname{div} v - \operatorname{div}_h v\|_0 \le ch\|\operatorname{div} v\|_1$, but in most of the nonconforming methods there is only a weaker estimate available. Moreover, the discrete divergence does not result from an orthogonal projection in many cases. A basis for an alternative is offered by the following regularity result in Theorem 3.1 of Arnold, Scott, and Vogelius [1989]. *Given u with* $\operatorname{div} u \in H^1(\Omega)$*, there exists $w \in H^2(\Omega) \cap H^1_0(\Omega)$ such that*

$$\operatorname{div} w = \operatorname{div} u \ \text{ and } \ \|w\|_2 \le c\|\operatorname{div} u\|_1. \tag{3.37}$$

3.10 Lemma. *Assume that (3.34) and (3.35) hold. Let the mapping* $\operatorname{div}_h : X_h \to L_2(\Omega)$ *satisfy*

$$\|\operatorname{div} v - \operatorname{div}_h v\|_0 \le ch\|v\|_2 \tag{3.38}$$

and

$$\operatorname{div}_h v = 0 \quad \textit{if} \;\; \operatorname{div} v = 0. \tag{3.39}$$

Then we have

$$\lambda\|\operatorname{div} u - \operatorname{div}_h u\|_0 \le c'h\|f\|_0 \tag{3.40}$$

where u denotes the solution of the variational problem (3.31).

Proof. By (3.34) and (3.37) there exists $w \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $\operatorname{div} w = \operatorname{div} u$ and

$$\|w\|_2 \le c\|\operatorname{div} u\|_1 \le c\lambda^{-1}\|f\|_0.$$

From (3.38) we conclude that

$$\|\operatorname{div} w - \operatorname{div}_h w\|_0 \le c'h\|w\|_2 \le c'h\lambda^{-1}\|f\|_0. \tag{3.41}$$

Since $\operatorname{div}(w - u) = 0$, it follows from (3.39) that $\operatorname{div}_h(w - u) = 0$ and $\operatorname{div} u - \operatorname{div}_h u = \operatorname{div} w - \operatorname{div}_h w$. Combining this with (3.41) we obtain

$$\|\operatorname{div} u - \operatorname{div}_h u\|_0 = \|\operatorname{div} w - \operatorname{div}_h w\|_0 \le c'h\lambda^{-1}\|f\|_0\,,$$

and the proof is complete. □

Now the discretization error for nearly incompressible material is estimated by the lemma of Berger, Scott, and Strang [1972]. The term for the approximation error is the crucial one. Let $P_h : H^2(\Omega) \cap H_0^1(\Omega) \to X_h$ be an interpolation operator. It is sufficient to make provision for

$$\begin{aligned} \|v - P_h v\|_1 + \|\operatorname{div} v - \operatorname{div}_h v\|_0 &\le ch\|v\|_2, \\ \|\operatorname{div}_h v_h\|_0 \le c\|v_h\|_1 &\quad \text{for all } v_h \in X_h. \end{aligned}$$

These inequalities are clear for the model problem in (3.35). Moreover, let the mesh-dependent bilinear form a_h be defined by polarization of the quadratic form in (3.36). It follows from Lemma 3.10 that we have for $w_h \in X_h$

$$\begin{aligned} a_h(u - P_h u, w_h) &= \mu(\varepsilon(u - P_h u), \varepsilon(w_h))_0 + \lambda(\operatorname{div} u - \operatorname{div}_h u, \operatorname{div}_h w_h)_0, \\ &\le ch\,\|u\|_2\|w_h\|_1 + ch\,\|u\|_2\,\|\operatorname{div}_h w_h\|_0, \\ &\le ch\,\|u\|_2\|w_h\|_1. \end{aligned}$$

Finally, the consistency error is given by the term $\lambda(\operatorname{div} u - \operatorname{div}_h u, \operatorname{div}_h w_h)_0$. An estimate of this expression is already included in the formula above, and we have a robust estimate with a constant c that does not depend on λ, namely

$$\|u - u_h\|_1 \le ch\|f\|_0.$$

The following theory describes why uniform convergence cannot be expected with some discretizations. The equation (3.29) is included as a special case if we set $X := H_\Gamma^1(\Omega)$, $a_0(u, v) := 2\mu(\varepsilon(u), \varepsilon(v))_0$, $Bv := \operatorname{div} v$, and $t^2 = 1/\lambda$.

Locking

The concept of *locking effect* is used frequently by engineers to describe the case where a finite element computation produces significantly smaller displacements than it should. In addition to *volume locking*, we also know *shear locking*, *membrane locking*, and *thickness locking* as well as others with no special names. The essential point is that because of a small parameter t, as in (3.28) the quotient C/α grows, and the convergence of the finite element solution to the true solution is *not uniform in* t as $h \to 0$. The papers of Arnold [1981], Babuška and Suri [1992], and Suri, Babuška, and Schwab [1995] have made fundamental contributions to the understanding of locking effects. The following general framework covers nearly incompressible material and applies to the treatment of the Mindlin–Reissner plate in §6.

Let X be a Hilbert space, $a_0 : X \times X \to \mathbb{R}$ a continuous, symmetric, coercive bilinear form with $a_0(v, v) \geq \alpha_0 \|v\|^2$, and $B : X \to L_2(\Omega)$ a continuous linear mapping. Generally, B has a nontrivial kernel and $\dim \ker B = \infty$. In addition, let t be a parameter $0 < t \leq 1$. Given $\ell \in X'$, we seek a solution $u := u_t \in X$ of the equation

$$a_0(u_t, v) + \frac{1}{t^2}(Bu_t, Bv)_{0,\Omega} = \langle \ell, v\rangle \quad \text{for all } v \in X. \tag{3.42}$$

The existence and uniqueness are guaranteed by the coercivity of $a(u, v) := a_0(u, v) + t^{-2}(Bu, Bv)_{0,\Omega}$.

Suppose there exists $u_0 \in X$ with

$$Bu_0 = 0, \quad d := \langle \ell, u_0\rangle > 0. \tag{3.43}$$

After multiplying u_0 by a suitable factor, we can assume that $a_0(u_0, u_0) \leq \langle \ell, u_0\rangle$. In particular, the energy of the minimal solution satisfies

$$\Pi(u_t) \leq \Pi(u_0) = \frac{1}{2}\Big[a_0(u_0, u_0) + \frac{1}{t^2}\|Bu_0\|_{0,\Omega}^2\Big] - \langle \ell, u_0\rangle \leq -\frac{1}{2}d$$

with a bound that is independent of t. Thus, $\langle \ell, u_t\rangle \geq -\Pi(u_t) \geq \frac{1}{2}d$, and so

$$\|u_t\| \geq \|\ell\|^{-1}\frac{1}{2}d \quad \text{ for all } t > 0 \tag{3.44}$$

is bounded below, where $\|\ell\| := \|\ell\|_{X'}$.

We now consider the solution of the variational problem in the finite element space $X_h \subset X$. Looking at the results of Babuška and Suri [1992] or Braess [1998], we recognize that the locking effect occurs when $X_h \cap \ker B = \{0\}$ and[19]

$$\|Bv_h\|_{0,\Omega} \geq C(h)\|v_h\|_X \quad \text{for all } v_h \in X_h. \tag{3.45}$$

[19] To be precise, we have to exclude functions from X_h which are polynomials in Ω or belong to the low-dimensional kernel. This is why on the right-hand side of (3.45) we should replace $\|v_h\|_X$ by a norm of a quotient space as in (3.47).

The coercivity of a on X_h follows from (3.45) with the ellipticity constant $\alpha = \alpha_0 + t^{-2}C(h)^2$. By the stability result II.4.1,

$$\|u_h\| \le \alpha^{-1}\|\ell\| \le t^2\, C(h)^{-2}\,\|\ell\|. \tag{3.46}$$

For a small parameter t this gives a solution which is too small in contrast to (3.44) – and this is what engineers recognize as locking. The convergence cannot be uniform in t as $h \to 0$. On the other hand, a finite element method is called *robust* for a problem with a small parameter t provided that the convergence is uniform in t.

A poor approximation of the kernel as specified by (3.45) is characteristic for locking and will be verified for the Timoshenko beam and linear elements in (3.55) below. Concerning volume locking, for bilinear elements on rectangular grids one has

$$\|\operatorname{div} v_h\|_0 \ge \inf_{\substack{z_h \in (Q_1)^2 \\ \operatorname{div} z_h = 0}} \frac{h}{12\,\mathrm{diam}(\Omega)} |v_h - z_h|_{1,\Omega} \quad \text{for all } v_h \in (Q_1)^2; \tag{3.47}$$

cf. Braess [1996]. The discrete kernel $\{z_h \in (Q_1)^2;\ \operatorname{div} z_h = 0\}$ contains

- rigid body motions,
- linear (global) polynomials with zero divergence,
- deformations in the x-direction which depend only on y,
- deformations in the y-direction which depend only on x.

These special functions do not approximate the functions in the kernel of the divergence operator in the Sobolev space. Thus (3.47) shows Poisson locking of bilinear elements and elucidates that special means as described above are indeed required.

3.11 Remark. From a mathematical standpoint, it would be better to say we have a *poorly conditioned* problem than to call it locking. The key point is the appearance of the large ratio C/α in the constant in Céa's lemma (for example, as in (3.28)). In this case the condition number $\|L\| \cdot \|L^{-1}\|$ of the corresponding isomorphism $L : X \to X'$ is large.

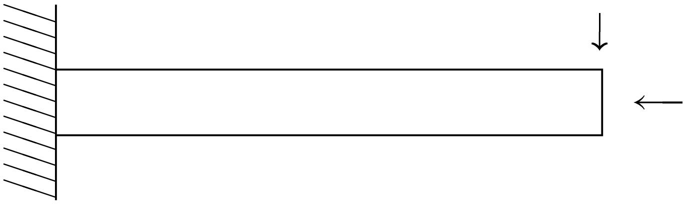

Fig. 58. The poor conditioning is clear for a cantilever beam. A vertical load leads to a significantly larger deformation than an equally large horizontal one. The constant in Korn's inequality is very small since Dirichlet boundary conditions are prescribed on only a small part of the boundary.

There is a qualitative difference between a poor approximation due to using too coarse a grid in the discretization and a poor approximation due to locking. It can be understood in terms of the associated eigenvalue problem; cf. Braess [1988]. As illustrated in Problem I.4.8, the lower eigenvalues are normally approximated well. This observation was also an important argument in the multigrid theory. On the other hand, in the discretization even the lower eigenvalues will clearly be shifted, once we have a tendency toward locking.

3.12 Remark. There are several approaches to reducing the effects of locking:

1. Convert the variational problem (3.42) to a saddle point problem with a penalty term (as explained above for nearly incompressible materials). With $p = t^{-2}Bu$, we get

$$\begin{aligned} a_0(u, v) + (Bv, p)_{0,\Omega} &= \langle \ell, v\rangle \quad \text{for all } v \in X, \\ (Bu, q)_{0,\Omega} - t^2(p, q)_{0,\Omega} &= 0 \qquad\quad \text{for all } q \in L_2(\Omega). \end{aligned} \tag{3.48}$$

2. Use *selective reduced integration*. Then in setting up the system matrix, the term
$$t^{-2}(Bu_h, Bv_h)_{0,\Omega}$$
will be relaxed as in (3.36) so that the constant in (3.45) is also reduced. This makes sense if the process can be understood as one where the approximating functions v_h are replaced in a neighborhood by others for which Bv is small. Strict mathematical proofs usually involve working with equivalent mixed formulations; see §6.
3. Simo and Rifai [1990] expand the space in the spirit of Remark III.5.7 by including nonconforming approximating functions so that sufficiently many functions are available in order to make $\|Bv\|_{0,h}$ small (if an appropriate discrete norm is chosen).

The first two methods listed in 3.12 are partially equivalent; cf. Remark 3.9. The finite element approximation of (3.48),

$$\begin{aligned} a_0(u_h, v) + (Bv, p_h)_{0,\Omega} &= \langle \ell, v\rangle \quad \text{for all } v \in X_h, \\ (Bu_h, q)_{0,\Omega} - t^2(p, q)_{0,\Omega} &= 0 \qquad\quad \text{for all } q \in M_h, \end{aligned} \tag{3.49}$$

is equivalent to the minimization problem

$$\frac{1}{2}a_0(u_h, u_h) + \frac{1}{2}t^{-2}\|R_h Bu_h\|_0 - \langle \ell, u_h\rangle \longrightarrow \min_{u_h \in X_h} ! \tag{3.50}$$

Here $R_h : L_2(\Omega) \to M_h$ is the orthogonal L_2-projector. In practice other projectors are used along with so-called *selected reduced integration.*

Locking of the Timoshenko Beam and Typical Remedies

Shear locking has been observed when computations for the Timoshenko beam are performed with finite elements which are piecewise polynomials of low degree. The locking of P_1 elements is easily verified on the basis of (3.45), and negative as well as positive results can be completely provided.

We will see in §5 that the stored energy of a beam is given by

$$\Pi(\theta, w) := \frac{1}{2}\int_0^b (\theta')^2 dx + \frac{t^{-2}}{2}\int_0^b (w' - \theta)^2 dx, \tag{3.51}$$

if b is the length of the beam and t is the thickness (multiplied by a correction factor). Here, the Lamé constants are abandoned since they enter only as a multiplicative factor in the 1-dimensional case. The rotation θ and the deflection w are in $H_0^1(0, b)$ and in the above setting,

$$B(\theta, w) := w' - \theta. \tag{3.52}$$

Given $g \in H_0^1(0, b)$, we obtain a pair of functions with $B(\theta, w) = 0$ by defining

$$\begin{aligned} \theta(x) &:= g(x) - \frac{6}{b^3}x(b - x)\int_0^b g(\xi)d\xi, \\ w(x) &:= \int_0^x \theta(\xi)d\xi. \end{aligned} \tag{3.53}$$

Hence, the kernel of B is infinite dimensional.

Now assume that the interval $[0, b]$ is divided into subintervals of length h and that $\theta_h, w_h \in \mathcal{M}_{0,0}^1$. Note that

$$\int_\xi^{\xi+h} (\alpha x + \beta)^2 dx \ge \frac{h^3}{6}\alpha^3 = \frac{h^2}{6}\int_\xi^{\xi+h} \alpha^2 dx,$$

whenever $\alpha, \beta \in \mathbb{R}$. From this inequality it follows that on each subinterval of the partition

$$\int_\xi^{\xi+h} (w_h' - \theta_h)^2 dx \ge \frac{h^2}{6}\int_\xi^{\xi+h} (\theta_h')^2 dx.$$

After summing over all subintervals we have

$$\|w_h' - \theta_h\|_0 \ge \frac{h}{3}|\theta_h|_1. \tag{3.54}$$

Friedrichs' inequality and the triangle inequality yield $|w_h|_1 \le \|w_h' - \theta_h\|_0 + \|\theta_h\|_0 \le \|w_h' - \theta_h\|_0 + c|\theta_h|_1$. We estimate $h|\theta_h|_1$ by (3.54) to obtain $h|w_h|_1 \le c\|w_h' - \theta_h\|_0$. Combining the last inequalities we conclude that

$$\|w_h' - \theta_h\|_0 \ge ch(\|\theta_h\|_1 + \|w_h\|_1). \tag{3.55}$$

This proves (3.45) with $C(h) = ch$, and it follows from the preceding investigations that the P_1 element locks.

On the other hand, it is clear that the construction (3.53) of pairs (θ, w) with $B(\theta, w) = 0$ can be approximated by piecewise polynomials of degree 2, and then the locking is avoided.

We consider now the remedies for the low degree elements discussed in Remark 3.12 and start with a mixed method with a penalty term. The weak solution (θ, w) of the variational problem with the internal stored energy (3.51) and a load $(f, w) := \int_0^b f w \, dx$ is given by

$$(\theta', \psi') + t^{-2}(w' - \theta, v' - \psi) = (f, v) \qquad \text{for } \psi, v \in H_0^1 . \tag{3.56}$$

The introduction of the shear term $\gamma := t^{-2}(w' - \theta) \in L_2$ leads to the saddle point formulation

$$\begin{aligned} (\theta', \psi') + (v' - \psi, \gamma) &= (f, v) \quad && \psi, v \in H_0^1 , \\ (w' - \theta, \eta) - t^2(\gamma, \eta) &= 0 && \eta \in L_2 . \end{aligned} \tag{3.57}$$

The ellipticity on the kernel $\{(\psi, v) \in (H_0^1)^2;\ v' - \psi = 0\}$ is obtained by applying Friedrichs' inequality

$$\|\psi'\|_0^2 = \frac{1}{2}|\psi|_1^2 + \frac{1}{2}|\psi|_1^2 \geq \frac{1}{2}|\psi|_1^2 + \frac{c}{2}\|\psi\|_0^2 = \frac{1}{2}|\psi|_1^2 + \frac{c}{2}\|v'\|_0^2 . \tag{3.58}$$

Given $\eta \in L_2$, we define $\rho(x) := x(b - x)$, and an appropriate pair (ψ, v) for verifying the inf-sup condition is given by

$$\begin{aligned} A &:= \frac{\int_0^b \eta(\xi) d\xi}{\int_0^b \rho(\xi) d\xi} , \\ v(x) &:= \int_0^x \eta(\xi) d\xi - A \int_0^x \rho(\xi) d\xi , \qquad \psi(x) := -A\rho(x). \end{aligned} \tag{3.59}$$

By Theorem III.4.11 the saddle point problem is stable.

We consider the discretization of (3.57). The finite element spaces for θ and w are the same as above. Specifically, let $\theta_h,\ w_h \in \mathcal{M}_{0,0}^1$, i.e., they are piecewise linear functions on a partition of the interval $[0, b]$. Now the shear terms are chosen as piecewise constant functions, i.e., $\gamma_h \in \mathcal{M}^0$, and the finite element equations are

$$\begin{aligned} (\theta_h', \psi') + (v' - \psi, \gamma_h) &= (f, v) \quad && \psi, v \in \mathcal{M}_{0,0}^1 , \\ (w' - \theta_h, \eta) - t^2(\gamma_h, \eta) &= 0 && \eta \in \mathcal{M}^0 . \end{aligned} \tag{3.60}$$

The inf-sup condition can obviously be established in the same way as for (3.57), and the ellipticity can be verified also with minor changes since w' is piecewise constant. The mixed method is stable. Thus locking is now eliminated.

In real-life computations the method of *selected reduced integration* is preferred for avoiding locking. Let the partition of the interval be given by $[0, b] = I_1 \cup I_2 \cup \ldots \cup I_M$. The second integral of the energy (3.51) is evaluated by a 1-point quadrature formula,

$$\frac{t^{-2}}{2} \sum_{j=1}^{m} |I_j| (w_h'(\xi_j) - \theta_h(\xi_j))^2,$$

where ξ_j denotes the midpoint of the subinterval I_j. Note that this expression is not larger than the original integral, and the 1-point quadrature formula induces a softening. The nonconforming method looks quite different than the previous one, but it is equivalent.

Indeed, let

$$Q_h : L_2[0, b] \to \mathcal{M}^0$$

be the L_2 projector. Then $(3.60)_2$ yields

$$\gamma_h = t^{-2}(w_h' - \theta_h))),$$

and $(3.60)_1$ may be rewritten as

$$(\theta_h', \psi') + t^{-2}(Q_h(w_h' - \theta_h), v' - \psi) = (f, v). \tag{3.62}$$

Obviously, (3.62) characterizes the minimal solution for the modified energy

$$\frac{1}{2}\int_0^b (\theta_h')^2 dx + \frac{t^{-2}}{2}\int_0^b [Q_h(w' - \theta)^2] dx - \int_0^b f w \, dx.$$

The second term coincides with (3.61), since the mean value of a linear function is attained at the midpoint of an interval.

The elimination of locking by the EAS method is also possible although the nonconforming character is not so obvious. The finite element spaces for θ and w (and the notation) are as above, but the derivative of w_h is enhanced by extra functions $\hat{\varepsilon}_h$. The result is the minimization of

$$\frac{1}{2}\int_0^b (\theta_h')^2 dx + \frac{t^{-2}}{2}\int_0^b (w_h' + \hat{\varepsilon}_h - \theta_h)^2] dx - \int_0^b f w_h \, dx. \tag{3.63}$$

Here, $\hat{\varepsilon}_h$ belongs to the m-dimensional space of noncontinuous functions which are linear polynomials in each subinterval I_j and vanish at the midpoint ξ_j of I_j. Given w_h and θ_h, the total energy (3.63) is minimal for the enhanced derivative $\hat{\varepsilon}_h$ that makes $w_h' + \hat{\varepsilon}_h - \theta_h$ constant in each I_j. Hence, $w_h' - \theta_h = Q_h(w_h' + \hat{\varepsilon}_h - \theta_h)$. The EAS method is here equivalent to selected reduced integration.

Proof of Korn's first inequality. From (3.17), the triangle inequality, and Problem III.3.13 it follows that $\|\operatorname{grad}\frac{\partial v_k}{\partial x_i}\|_{-1} \le 3\,\|\operatorname{grad}\varepsilon(v)\|_{-1} \le 3\,\|\varepsilon(v)\|_0$. The application of Nečas'inequality (III.6.11) to $p = \frac{\partial v_k}{\partial x_i}$ yields

$$\Big\|\frac{\partial v_k}{\partial x_i}\Big\|_0 \le c\Big(\Big\|\frac{\partial v_k}{\partial x_i}\Big\|_{-1} + \|\varepsilon(v)\|_0\Big) \le c\Big(\|v\|_0 + \|\varepsilon(v)\|_0\Big).$$

The summation over all components completes the proof. □

Problems

3.13 Verify that in the case of pure Dirichlet boundary conditions, i.e., $H^1_\Gamma(\Omega) = H^1_0(\Omega)$, the classical solution of (3.10) is in fact a solution of the differential equation (3.11).

3.14 Does $\operatorname{div} u = 0$ imply the relation $\operatorname{div}\sigma = 0$, or does the converse hold? Hint: The connection between $\operatorname{div} u$ and $\operatorname{trace}\varepsilon$ is useful.

3.15 Verify Green's formula

$$\int_\Omega \tau : \varepsilon(v)\,dx = -\int_\Omega v\cdot\operatorname{div}\tau\,dx + \int_{\partial\Omega} v\cdot\tau n\,ds$$

for symmetric tensors $\tau \in H(\operatorname{div},\Omega)$ and $v \in H^1(\Omega)$. Why don't the boundary terms play a role in passing from (3.21) to (3.22)?

3.16 Verify by explicit computation that the expressions $(\operatorname{div} v)^2$ and $\varepsilon(v) : \varepsilon(v)$ are invariant under orthogonal transformations.

3.17 Consider the Hellinger–Reissner principle for a nearly incompressible material. Is the introduction of the variables $p = \lambda \operatorname{div} u$ simpler for the formulation (3.21) or for (3.22)?

3.18 Consider the pure traction problem, i.e., the problem with pure Neumann boundary conditions. Show that the displacement problem (3.9) has only a solution if the compatibility condition

$$-\int_\Omega f\cdot v\,dx + \int_{\partial\Omega} g\cdot v\,dx = 0 \quad \text{for all } v \in \mathrm{RM}$$

holds. Here RM denotes the space of rigid body motions, i.e. the set of all functions v of the form (3.17).
How many compatibility conditions do we encounter for $d = 2$ and $d = 3$, respectively?

§ 4. Membranes

In solving three-dimensional problems, it is often possible to work in two (or even one) dimensions since the length of the domain in one of more space directions is very small. In such cases it is useful to consider the problem for the lower-dimensional continuum, and then discretize afterwards. Some typical examples are bars, beams, membranes, plates, and shells. The simplest two-dimensional example is the membrane. However, this example already shows that the reduction in dimension cannot be accomplished by simply eliminating one coordinate. Moreover, the boundary conditions have some influence on the reduction process. There are two cases depending on the boundary condition.

Plane Stress States

Let $\omega \subset \mathbb{R}^2$ be a domain, and suppose $t > 0$ is a number which is significantly smaller than the diameter of ω. We suppose that there are only external forces operating on the body $\Omega = \omega \times (-\frac{t}{2}, +\frac{t}{2})$, and that their z-components vanish so that they depend only on x and y.[20] If the membrane is thin and a deformation in the z-direction is possible, we have the so-called *plane stress state*, i.e.,

$$\begin{aligned} \sigma_{ij}(x, y, z) &= \sigma_{ij}(x, y), \quad & i, j &= 1, 2, \\ \sigma_{i3} = \sigma_{3i} &= 0, & i &= 1, 2, 3. \end{aligned} \tag{4.1}$$

Then in particular $\varepsilon_{i3} = \varepsilon_{3i} = 0$ for $i = 1, 2$. In order for $\sigma_{33} = 0$, by the constitutive equations (3.6) we have

$$\varepsilon_{33} = -\frac{\nu}{1-\nu}(\varepsilon_{11} + \varepsilon_{22}). \tag{4.2}$$

If we now eliminate the strain ε_{33}, we get the constitutive equations for the plane stress state:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{\mathtt{E}}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix}$$

or

$$\sigma = \frac{\mathtt{E}}{1+\nu}\Big[\varepsilon + \frac{\nu}{1-\nu}(\varepsilon_{11} + \varepsilon_{22}) I\Big]. \tag{4.3}$$

[20] It is easier to visualize the situation if we exchange the y- and z-coordinates. Suppose the middle surface of a thin wall lies in the (x, y)-plane, and that the displacement in the y-direction is very small. Now suppose the wall is subject to a load in the vertical direction so that the external forces operate in a direction parallel to the (middle surface of the) wall.

Finally, the kinematics are compatible with (4.1) and (4.2) provided

$$\begin{aligned} u_i(x, y, z) &= u_i(x, y), \qquad i = 1, 2, \\ u_3(x, y, z) &= z \cdot \varepsilon_{33}(x, y) \end{aligned}$$

and terms of order $\mathcal{O}(z)$ are neglected in the strain term.

Plane Strain States

If boundary conditions are enforced at $z = \pm t/2$ which ensure that the z-component of the displacement vanishes, then we have the *plane strain state*:

$$\begin{aligned} \varepsilon_{ij}(x, y, z) &= \varepsilon_{ij}(x, y), \quad && i, j = 1, 2, \\ \varepsilon_{i3} = \varepsilon_{3i} &= 0, && i \quad = 1, 2, 3. \end{aligned} \tag{4.4}$$

The associated displacements satisfy $u_i(x, y, z) = u_i(x, y)$ for $i = 1, 2$, and $u_3 = 0$. It follows from $\varepsilon_{33} = 0$ along with (3.7) that

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22}), \tag{4.5}$$

and σ_{33} can be eliminated. We obtain the constitutive equations for the plane strain state if we restrict (3.6) to the remaining components:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{\mathtt{E}}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix}. \tag{4.6}$$

Membrane Elements

Both plane elasticity problems lead to a two-dimensional problem with the same structure as the full three-dimensional elasticity problem.

The displacement model thus involves the (two-dimensional and isoparametric versions of the) conforming elements which also play a role for scalar elliptic problems of second order:

(a) bilinear quadrilateral elements,
(b) quadratic triangular elements,
(c) biquadratic quadrilateral elements,
(d) eight-node quadrilateral elements in the serendipity class.

On the other hand, the simplest linear triangular elements are frequently unsatisfactory. For practical problems, there are often preferred directions because of certain geometric relationships. In this case, higher order elements or quadrilateral elements prove to be more flexible.

The PEERS Element

Mixed methods have not been heavily used for problems in plane elasticity, since in the finite element approximation of the Hellinger–Reissner principle (3.22), two stability problems occur simultaneously. The bilinear form $a(\sigma,\tau) := (\mathcal{C}^{-1}\sigma,\tau)_0$ is not elliptic on the entire space $X := H(\mathrm{div},\Omega)$, but only on the kernel V. Here we are using the notation of the general theory in Ch. III, §4.

In order to ensure the ellipticity on V_h, for two decades the best possibility was considered to choose $V_h \subset V$ which assures that the condition in III.4.7 is satisfied. As Brezzi and Fortin [1991, p. 284] have shown via a dimensional argument, the *symmetry of the stress tensor* σ is a major difficulty. For this reason, Arnold, Brezzi, and Douglas [1984] have developed the PEERS element (plane elasticity element with reduced symmetry). It has been studied further by Stenberg [1988], among others. The so-called BDM elements of Brezzi, Douglas, and Marini [1985] are constructed in a similar way. All of these elements are also useful for nearly incompressible materials.

Finite element spaces for stresses with full symmetry have been established by Arnold and Winther [2002]. They can be understood as extensions of the Raviart–Thomas element, and the adaptation of the commuting diagram properties in Ch. III, §5 play an important role. A disadvantage is however that there are 24 local degrees of freedom. There are no satisfactory mixed methods with less degrees of freedom and full symmetry. Often there exist so-called *zero energy modes*, which must be filtered out, since otherwise we get the (hour-glass) instabilities discussed in Ch. III, §7, due to the violation of the inf-sup condition.

In discussing the PEERS elements, for simplicity we restrict ourselves to pure displacement boundary conditions. In this case (3.22) simplifies to

$$\begin{aligned} &(\mathcal{C}^{-1}\sigma,\tau)_0 + (\mathrm{div}\,\tau, u)_0 = 0 && \text{for all } \tau \in H(\mathrm{div},\Omega),\\ &(\mathrm{div}\,\sigma, v)_0 \qquad\qquad\quad = -(f,v)_0 && \text{for all } v \in L_2(\Omega)^2. \end{aligned} \tag{4.7}$$

Since we allow unsymmetric tensors in the following, the *antisymmetric part*

$$as(\tau) := \tau - \tau^T \in L_2(\Omega)^{2\times 2},$$

i.e. $as(\tau)_{ij} = \tau_{ij} - \tau_{ji}$, plays a role. Since $as(\tau)$ is already completely determined by its (2,1)-component, we will refer to this component.

We consider the following saddle point problem:
Find $\sigma \in X := H(\mathrm{div},\Omega)^{2\times 2}$ and $(u,\gamma) \in M := L_2(\Omega)^2 \times L_2(\Omega)$ such that

$$\begin{aligned} &(\mathcal{C}^{-1}\sigma,\tau)_0 + (\mathrm{div}\,\tau,u)_0 + (as(\tau),\gamma) = 0, && \tau \in H(\mathrm{div},\Omega)^{2\times 2},\\ &(\mathrm{div}\,\sigma, v)_0 && = -(f,v)_0\,, \quad v \in L_2(\Omega)^2,\\ &(as(\sigma),\eta) && = 0, \qquad \eta \in L_2(\Omega). \end{aligned} \tag{4.8}$$

Here $(as(\tau),\eta) := \int_\Omega (\tau_{12} - \tau_{21})\eta\, dx$.

Note that the rotations of scalar- and vector-valued functions in $\mathbb{R}^2$ are defined differently:[21]

$$\operatorname{curl} p := \begin{pmatrix} \frac{\partial p}{\partial x_2} \\ -\frac{\partial p}{\partial x_1} \end{pmatrix}, \qquad \operatorname{rot} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}. \tag{4.9}$$

4.1 Lemma. *The saddle point problems (4.7) and (4.8) are equivalent. If (σ, u, γ) is a solution of (4.8), then (σ, u) is a solution of (4.7). Conversely, if (σ, u) is a solution of (4.7), then $(\sigma, u, \gamma = \frac{1}{2} \operatorname{rot} u)$ is a solution of (4.8).*

Proof. (1) Let (σ, u, γ) be a solution of (4.8). The third equation asserts that σ is symmetric. For symmetric τ we have $(as(\tau), \gamma) = 0$, and the first equation in (4.8) reduces to the first one in (4.7). Since the second relation in (4.7) can be read off directly from (4.8), we have shown that (4.7) holds.

(2) Let (σ, u) be a solution of (4.7). By the discussion following (3.21), it follows that $u \in H_0^1(\Omega)^2$, and in the same way as in the derivation of (3.20), we deduce that

$$(\mathcal{C}^{-1}\sigma, \tau)_0 - (\tau, \nabla^{(s)} u)_0 = 0 \tag{4.10}$$

for all *symmetric* fields τ. On the other hand, the symmetry of the expressions $\mathcal{C}^{-1}\sigma$ and $\nabla^{(s)} u$ implies that (4.10) also holds for all skew-symmetric fields. Now the decomposition of ∇u gives

$$\begin{aligned} \nabla^{(s)} u &= \nabla u - \frac{1}{2} as(\nabla u) \\ &= \nabla u - \frac{1}{2}\Big(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\Big) \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \\ &= \nabla u - \frac{1}{2} \operatorname{rot} u \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}. \end{aligned}$$

$\varepsilon(u)$ is the symmetric gradient. The skew-symmetric part is coupled with the rotation. Finally, another application of Green's formula gives

$$\begin{aligned} \int_\Omega \tau : \nabla^{(s)} u \, dx &= \int_\Omega \tau : \nabla u \, dx + \frac{1}{2} \int_\Omega (\tau_{21} - \tau_{12}) \operatorname{rot} u \, dx \\ &= -\int_\Omega \operatorname{div} \tau \, u \, dx + \frac{1}{2} \int_\Omega (\tau_{21} - \tau_{12}) \operatorname{rot} u \, dx. \end{aligned}$$

Combining this with (4.10) leads to the first relation in (4.8). Since the second equation can be taken from (4.7) and the third is obvious because of the symmetry of σ, this establishes (4.8). □

[21] We follow the convention of writing curl p rather than rot p if p is a scalar function. The reader should also be aware that the operators in (4.9) sometimes appear in the literature with different signs.

The equivalence does not immediately imply that the saddle point problem satisfies the hypotheses of Theorem III.4.5. Clearly, the kernel is

$$V = \{\tau \in H(\mathrm{div}, \Omega)^{2\times 2};\ (\mathrm{div}\,\tau, v)_0 = 0 \text{ for } v \in L_2(\Omega)^2,$$
$$(as(\tau), \eta) = 0 \text{ for } \eta \in L_2(\Omega)\}$$

just as for the saddle point problem (4.7). Now the ellipticity of the quadratic form $(\mathcal{C}^{-1}\sigma, \sigma)_0$ can be carried over.

To establish the inf-sup condition, suppose we are given a pair $(v, \eta) \in L_2(\Omega)^2 \times L_2(\Omega)$. Then we construct $\tau \in H(\mathrm{div}, \Omega)^{2\times 2}$ with

$$\begin{aligned} \mathrm{div}\,\tau &= v, \\ as(\tau) &= \eta, \\ \|\tau\|_{H(\mathrm{div},\Omega)} &\le c(\|v\|_0 + \|\eta\|_0), \end{aligned} \tag{4.11}$$

where we write $as(\tau)$ instead of $as(\tau)_{21}$. In view of Remark 3.7, we want

$$\int_\Omega \mathrm{trace}\,\tau\, dx = 0. \tag{4.12}$$

First we determine $\tau_0 \in H(\mathrm{div}, \Omega)^{2\times 2}$ which satisfies just the equation $\mathrm{div}\,\tau_0 = v$. For example, let $\psi \in H_0^1(\Omega)$ be a solution of the Poisson equation $\Delta\psi = v$ and $\tau_0 := \nabla\psi$. Then $\|\psi\|_1 \le c\|v\|_0$ immediately implies $\|\tau_0\|_{H(\mathrm{div},\Omega)} \le \|\tau_0\|_0 + \|\,\mathrm{div}\,\tau_0\|_0 = \|\nabla\psi\|_0 + \|v\|_0 \le c\|v\|_0$.

In order to satisfy the second equation in (4.11), we set

$$\begin{aligned} s &:= \int_\Omega [\eta - as(\tau_0)]dx \Big/ \int_\Omega dx, \\ \beta &:= \eta - as(\tau_0) - s, \end{aligned}$$

and construct $q \in H^1(\Omega)^2$ with

$$\mathrm{div}\, q = \beta$$

via a Neumann problem in the same way as we constructed τ_0. In particular, $\|q\|_1 \le c\|\beta\|_0 \le c(\|\eta\|_0 + \|v\|_0)$. Finally, let

$$\tau := \tau_0 + \begin{pmatrix} \mathrm{curl}\, q_1 \\ \mathrm{curl}\, q_2 \end{pmatrix} + \frac{s}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

To show that τ is a solution of (4.11), we first recall that the divergence of a rotation vanishes. Thus, $\mathrm{div}\,\tau = \mathrm{div}\,\tau_0 = v$. Moreover, by construction of q,

$$\begin{aligned} as(\tau) &= as(\tau_0) + as\begin{pmatrix} \frac{\partial q_1}{\partial y} & -\frac{\partial q_1}{\partial x} \\ \frac{\partial q_2}{\partial y} & -\frac{\partial q_2}{\partial x} \end{pmatrix} + s \\ &= as(\tau_0) + \mathrm{div}\, q + s = \eta. \end{aligned}$$

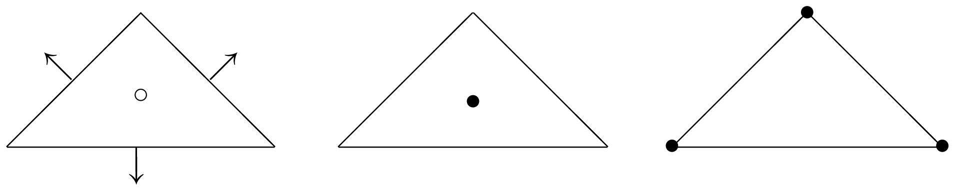

Fig. 59. PEERS element ($\circ$ stands for the rotation of the bubble function)

Since q is a gradient field, $\frac{\partial q_1}{\partial y} - \frac{\partial q_2}{\partial x} = 0$, the constraint (4.12) is easy to verify. Now (4.11) implies that

$$(\operatorname{div}\tau, v)_0 + (as(\tau), \eta) = \|v\|_0^2 + \|\eta\|_0^2$$
$$\geq c\|\tau\|_{H(\operatorname{div},\Omega)}(\|v\|_0 + \|\eta\|_0),$$

and thus the inf-sup condition holds. $\square$

The variational formulation (4.8) is appropriate for nearly incompressible material. In this context it is crucial that the ellipticity constant can be bounded uniformly in the Lamé constant λ. This follows from

$$\|\operatorname{trace}\sigma\| \leq C(\|\sigma_{dev}\| + \|\operatorname{div}\sigma\|).$$

A proof of the inequality can be found in Brezzi and Fortin [1991, p. 161].

4.2 Remark. If the traction is prescribed on a part of the boundary, i.e., Γ_1 is nonempty, then the proof of the inf-sup condition is more complicated. More precisely, the solution of $\operatorname{div}\tau_0 = v$ and $\operatorname{div} q = \beta$ requires more care. We recall that the Neumann boundary condition is not a natural boundary condition for the mixed formulation (4.7) and we have

$$\sigma \in \{\tau \in H(\operatorname{div},\Omega);\ \tau n = 0 \ \text{ on } \Gamma_1\}.$$

We cannot proceed as we have done after (4.12). We first construct τ with $\tau_0 \cdot n = 0$ on Γ_1 by prescribing a Neumann boundary condition there, i.e., $\nabla\psi \cdot n = 0$. Next observe that $\operatorname{rot} q \cdot n = 0$ implies

$$\nabla q \cdot t = 0; \quad \text{thus } q = const \quad \text{on } \Gamma_1.$$

We can find q by solving the Stokes problem $\int(\nabla q)^2 dx \longrightarrow$ min! under the constraints $\operatorname{div} q = \beta$ and $q = 0$ on Γ_1. The inf-sup condition for the elasticity problem (4.8) then follows from the inf-sup condition for the Stokes problem in view of the fact that the auxiliary problem is well defined.

The analysis of (4.8) was performed independently of Korn's inequality. On the other hand, the inf-sup condition for the divergence operator was used. Now it follows from the equivalence described in Lemma 4.1 that the inf-sup condition for the Stokes problem implies Korn's inequality in $\mathbb{R}^2$.

A different proof of this implication was presented by Falk [1991].

To define a simple member of the family of PEERS elements, we employ the usual notation. (See also Fig. 59.) Let $\mathcal{T}$ be a triangulation of Ω, and let

$$\begin{aligned}
\mathcal{M}^k &:= \{v \in L_2(\Omega);\ v|_T \in \mathcal{P}_k \quad \text{for every } T \in \mathcal{T}\},\\
\mathcal{M}_0^k &:= \mathcal{M}^k \cap H^1(\Omega), \qquad \mathcal{M}_{0,0}^k := \mathcal{M}^k \cap H_0^1(\Omega),\\
RT_k &:= \{v \in (\mathcal{M}^{k+1})^2 \cap H(\mathrm{div}, \Omega);\\
&\qquad v|_T = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + p_3 \begin{pmatrix} x \\ y \end{pmatrix},\ p_1, p_2, p_3 \in \mathcal{P}_k\},\\
B_3 &:= \{v \in \mathcal{M}_0^3;\ v(x) = 0 \text{ on every edge of the triangulation}\}.
\end{aligned} \tag{4.13}$$

The PEERS element is the one with the smallest number of local degrees of freedom:

$$\begin{aligned}
\sigma_h &\in X_h := (RT_0)^2 \oplus \mathrm{curl}(B_3)^2,\\
v_h &\in W_h := (\mathcal{M}^0)^2,\\
\gamma_h &\in \Gamma_h := \mathcal{M}_0^1.
\end{aligned}$$

Note that the divergence of the functions in $\mathrm{curl}(B_3)^2$ vanishes, and that the divergence of a piecewise differentiable function is an L_2 function if and only if the normal components are continuous on the inter-element boundaries.

By construction, $\mathrm{div}\,\tau_h \in W_h$. Thus, $\mathrm{div}\,\tau_h = 0$ follows immediately from $(\mathrm{div}\,\tau_h, v_h) = 0$ for all $v_h \in W_h$. Thus, the condition in III.4.7 is satisfied, and the form $(\mathcal{C}^{-1}\sigma_h, \tau_h)$ is elliptic on the kernel. The inf-sup condition can be established in a similar way as for the continuous problem. Since (4.11) must be replaced by finite element approximations, however, here the details are more involved, see Arnold, Brezzi, and Douglas [1984].

The implementation and postprocessing described by Arnold and Brezzi [1985], see Ch. III, §5 is also advantageous for computations with the PEERS element. The postprocessing was also used to estimate the local error by adaptive grid refinement, see Braess, Klaas, Niekamp, Stein, and Wobschal [1995].

Problems

4.3 Show that we get the constitutive equations (4.3) for plane stress states by restricting the relationship $\varepsilon = \mathcal{C}^{-1}\sigma$ to the components with $i = 1, 2$.

What comparable assertion holds for plane strain states?

4.4 For a nearly incompressible material, ν is close to $\frac{1}{2}$. This causes difficulties in the denominators $(1 - 2\nu)$ of (1.31) and (3.6), respectively. On the other hand, in view of (4.3), for the plane stress states, $\nu = \frac{1}{2}$ does not cause any problem. Give an explanation.

4.5 Let Ω be a domain in $\mathbb{R}^2$ with piecewise smooth boundary, and suppose $\psi \in H^1(\Omega)$ satisfies the Neumann boundary condition $\frac{\partial \psi}{\partial n} = 0$ on $\partial\Omega$. What can you say about $\operatorname{rot} \psi$ on the boundary?
Hint: First suppose $\psi \in C^1(\bar{\Omega})$, and consider the tangential component of the rotation on the boundary.

4.6 Does the relation

$$\Delta u = -\operatorname{rot}\operatorname{rot} u + \operatorname{grad}\operatorname{div} u$$

between the derivatives hold for vector fields in $\mathbb{R}^2$ and $\mathbb{R}^3$?

4.7 It would be mathematically cleaner if (3.6) were written in the form

$$\sigma_{ij} = \sum_{k\ell} \mathcal{C}_{ijk\ell}\, \varepsilon_{k\ell}. \tag{4.14}$$

Give a formula for the elasticity tensor – or more precisely for its components $\mathcal{C}_{ijk\ell}$ – which involves only the Lamé constants and the Kronecker symbol so that (4.14) describes the material law $\sigma = 2\mu\varepsilon + \lambda \operatorname{trace} \varepsilon\, I$ for a St. Venant–Kirchhoff material.

Do this also for the plane strain state and the plane stress state.

4.8 Let $\Omega \subset \mathbb{R}^2$, and let $H(\operatorname{rot}, \Omega)$ be the completion of $C^\infty(\Omega)^n$ w.r.t. the norm

$$\|v\|^2_{H(\mathrm{rot})} := \|v\|^2_{0,\Omega} + \|\operatorname{rot} v\|^2_{0,\Omega}.$$

Show that a set $S_h \subset L_2(\Omega)^2$ of piecewise polynomials lies in $H(\mathrm{rot})$ if and only if the component $v \cdot \tau$ in the direction of the tangent is continuous on the edges of the element.

Hint: See Problem II.5.14 for $H(\operatorname{div}, \Omega)$.

4.9 Let u be a solution of the Lamé equations in 2-space, $v \in H^1(\Omega)^2$, and let τ be a symmetrical tensor in $H(\operatorname{div})$ with $\operatorname{div} \tau + f = 0$. Show that

$$\int [\varepsilon(v) - \varepsilon(u)] : [\tau - \mathcal{C}\varepsilon(u)] = 0$$

holds under appropriate boundary conditions. Formulate an estimate of Prager–Synge type.

§ 5. Beams and Plates: The Kirchhoff Plate

A plate is a thin continuum subject to applied forces which – in contrast to the case a membrane – are orthogonal to the middle surface. We distinguish between two cases. The *Kirchhoff plate* leads to a fourth order elliptic problem. Usually it is solved using nonconforming or mixed methods.

The *Mindlin–Reissner plate* (that is also called Reissner–Mindlin plate or Mindlin plate) involves somewhat weaker hypotheses. It is described by a differential equation of second order, and so at first glance its numerical treatment appears to be simpler. However, it turns out that the calculations for the Mindlin–Reissner plate are actually more difficult, and that the problems plaguing the Kirchhoff plate are still present, although concealed. In particular, the Mindlin plate tends to shear locking, and using standard elements leads to poor numerical results.

The analogous reduction of thin membranes, i.e., of membranes with one very small dimension, leads to beams.

After introducing both plate models, we turn our attention first to a discussion of the Kirchhoff plate, and in particular to the *clamped plate*.

The Hypotheses

We consider a thin plate of constant thickness t whose middle surface coincides with the (x, y)-plane. Thus, $\Omega = \omega \times (-\frac{t}{2}, +\frac{t}{2})$ with $\omega \subset \mathbb{R}^2$. We suppose that the plate is subject to external forces which are orthogonal to the middle surface.

5.1 Hypotheses of Mindlin and Reissner.

H1. *Linearity hypothesis.* Segments lying on normals to the middle surface are linearly deformed and their images are segments on straight lines again.

H2. The displacement in the z-direction does not depend on the z-coordinate.

H3. The points on the middle surface are deformed only in the z-direction.

H4. The normal stress σ_{33} vanishes.

Under hypotheses H1–H3 the displacements have the form

$$\begin{aligned} u_i(x, y, z) &= -z\theta_i(x, y), \quad \text{for } i = 1, 2, \\ u_3(x, y, z) &= w(x, y). \end{aligned} \tag{5.1}$$

We call w the *transverse displacement* or *(normal) deflection*, and $\theta = (\theta_1, \theta_2)$ the *rotation*.

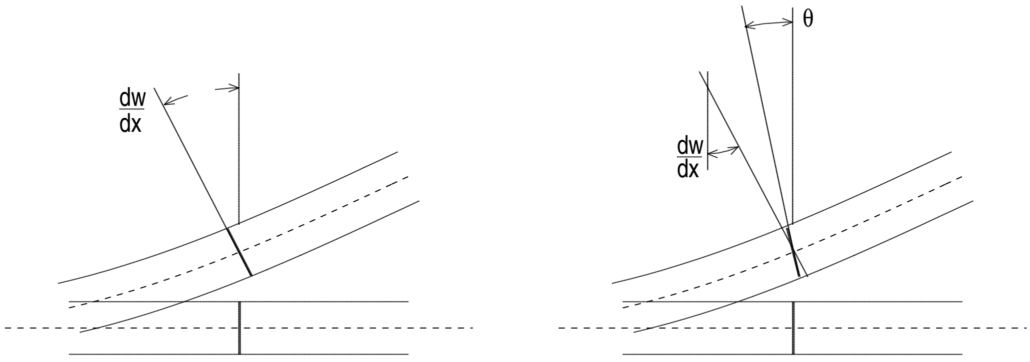

Fig. 60. Beam or section through a plate with and without the normal hypothesis

5.2 The Hypotheses of Kirchhoff–Love. Suppose that in addition to hypotheses H1–H4, we have:

H5. *Normal hypothesis.* The deformations of normal vectors to the middle surface are again orthogonal to the (deformed) middle surface.

This hypothesis is often found under the shorter names *Love's hypothesis* or *Kirchhoff's hypothesis*.

The normal hypothesis implies that the rotations are no longer independent of the deflections (see Fig. 60).

$$\left.\begin{aligned} \theta_i(x,y) &= \frac{\partial}{\partial x_i} w(x,y), \\ u_i(x,y,z) &= -z\frac{\partial w}{\partial x_i}(x,y), \end{aligned}\right\} \quad i = 1, 2. \tag{5.2}$$

We restrict ourselves to body forces which we assume to be independent of z. The associated strains are then

$$(\nabla^{(s)} u)_{ij} = -z(\nabla^{(s)}\theta)_{ij}, \quad (\nabla^{(s)} u)_{i3} = \frac{1}{2}\Big(\frac{\partial}{\partial x_i} w - \theta_i\Big), \quad i, j = 1, 2. \tag{5.3}$$

In view of the hypothesis $\sigma_{33} = 0$, we can make use of the formulas (4.2) and (4.3) for the plane stresses, when we evaluate the bilinear form from the energy functional (3.1):

$$\begin{aligned} \varepsilon : \sigma &= \sum_{i,j=1}^{2} \varepsilon_{ij}\sigma_{ij} + 2\sum_{j=1}^{2} \varepsilon_{3j}\sigma_{3j} \\ &= \frac{\mathrm{E}}{1+\nu}\Big[\sum_{i,j=1}^{2} \varepsilon_{ij}^2 + \frac{\nu}{1-\nu}(\varepsilon_{11}+\varepsilon_{22})^2 + 2\sum_{j=1}^{2} \varepsilon_{3j}^2\Big] \\ &= 2\mu \sum_{\substack{i,j=1 \\ (i,j)\neq(3,3)}}^{3} \varepsilon_{ij}^2 + \lambda\frac{2\mu}{\lambda+2\mu}(\varepsilon_{11}+\varepsilon_{22})^2. \end{aligned}$$

With the model (5.1) and the derivatives (5.3), the integration in (3.1) over the z variable is easily evaluated:[22]

$$\Pi(u) := \Pi(\theta, w) = \frac{t^3}{12} a(\theta, \theta) + \frac{\mu t}{2} \int_\omega |\nabla w - \theta|^2 dx_1 dx_2 - t \int_\omega f w \, dx_1 dx_2 \tag{5.4}$$

with

$$a(\theta, \psi) := \int_\omega \Big[2\mu\, \varepsilon(\theta) : \varepsilon(\psi) + \frac{\lambda}{2} \frac{2\mu}{\lambda + 2\mu} \operatorname{div} \theta \operatorname{div} \psi \Big] dx_1 dx_2. \tag{5.5}$$

The symmetric gradient

$$\varepsilon_{ij}(\theta) := \frac{1}{2} \Big(\frac{\partial \theta_i}{\partial x_j} + \frac{\partial \theta_j}{\partial x_i} \Big), \quad i, j = 1, 2 \tag{5.6}$$

and the divergence are now based on functions of two variables. The first term in (5.4) contains the *bending part* of the energy, and the second term contains the *shear term*. Clearly the latter vanishes in the Kirchhoff model.

The solution of the variational problem does not change if the energy functional is multiplied by a constant. Without altering the notation, we multiply by t^{-3}, replace $t^2 f$ by f in the load, and normalize μ, leading to the (dimensionless) expression

$$\Pi(u) = \frac{1}{2} a(\theta, \theta) + \frac{t^{-2}}{2} \int_\Omega |\nabla w - \theta|^2 dx - \int_\Omega f w \, dx. \tag{5.7}$$

To stay with the usual notation, we have written Ω instead of ω.

Now in the framework of the Kirchhoff model, (5.2) and (5.6) imply

$$\varepsilon_{ij}(\theta) = \partial_{ij} w,$$

and thus (5.5) gives the bilinear form

$$a(\nabla w, \nabla v) = \int_\Omega \Big[\mu \sum_{i,j} \partial_{ij}\, w\, \partial_{ij} v + \lambda' \Delta w \Delta v \Big] dx, \tag{5.8}$$

with a suitable constant λ'. This is a variational problem of fourth order of the same structure as the variational formulation of the biharmonic equation.

[22] Since $D_{33} u = 0$, the model (5.1) is consistent with hypothesis H4 and with (4.2), only if $\operatorname{div} \theta = 0$. The two hypotheses lead to slightly different factors in (5.5). Therefore in computations of plates some so-called *shear-correction factors* may be found, but this has no effect on our analysis.

By Korn's inequality, the bilinear form $a(\nabla w, \nabla v)$ is H^2-elliptic on $H_0^2(\Omega)$. For a conforming treatment, the variational problem

$$\frac{1}{2}a(\nabla v, \nabla v) - (f, v) \longrightarrow \min_{v \in H_0^2(\Omega)} ! \tag{5.9}$$

for the clamped Kirchhoff plate requires C^1 elements, which is computationally expensive. In this respect, the numerical treatment of the Mindlin plate appears at first glance to be simpler since the problem (5.7) is obviously H^1-elliptic for $(w, \theta) \in H_0^1(\Omega) \times H_0^1(\Omega)^2$. However, as we shall see, the Mindlin plate contains a small parameter.

Finally, we would like to mention the so-called *Babuška paradox*; see, e.g., Babuška and Pitkäranta [1990]. If we approximate a domain with a smooth boundary by polygonal domains, then the solutions for Kirchhoff plates on these domains usually do not converge to the solution for the original domain. This holds for the clamped plates as well as for some other boundary conditions.

Note on Beam Models

While plate models refer to elliptic problems in 2-space, the beam models lead to boundary-value problems with ordinary differential equations. The beam with the Kirchhoff hypothesis is called the *Bernoulli beam*, and the beam which corresponds to the Mindlin plate is the *Timoshenko beam*. If we eliminate the Lamé constants, we obtain the energy of the Timoshenko beam by a reduction of (5.7) to the one-dimensional case:

$$\Pi(\theta, w) := \frac{1}{2}\int_0^b (\theta')^2 dx + \frac{t^{-2}}{2}\int_0^b (w' - \theta)^2 dx - \int f w dx.$$

Here $\theta, w \in H_0^1(0, b)$ where b denotes the length of the beam.

We have already considered this model in §3 when illustrating shear locking. It should be emphasized that the computation and the analysis of plates is much more involved and cannot be understood as a simple generalization; cf. Problem 5.13.

Mixed Methods for the Kirchhoff Plate

Nonconforming and mixed methods play an important role in the theory of Kirchhoff plates. We begin with mixed methods since we will also use them for the analysis of nonconforming elements. In the following, $a(\cdot, \cdot)$ always denotes the H^1-elliptic bilinear form on $H_0^1(\Omega)^2$ defined in (5.5).

The minimization of

$$\frac{1}{2}a(\theta,\theta)-(f,w) \tag{5.10}$$

subject to the constraint

$$\nabla w=\theta \quad \text{in } \Omega$$

leads to the following saddle point problem: Find $(w,\theta)\in X$ and $\gamma\in M$ such that

$$\begin{aligned} a(\theta,\psi)+(\nabla v-\psi,\gamma)_0 &= (f,v) && \text{for all } (v,\psi)\in X,\\ (\nabla w-\theta,\eta)_0 &= 0 && \text{for all } \eta\in M. \end{aligned} \tag{5.11}$$

In choosing the spaces X and M for the plate, our first choice would be

$$X:=H_0^2(\Omega)\times H_0^1(\Omega)^2, \quad M:=H^{-1}(\Omega)^2. \tag{5.12}$$

Clearly, the bilinear forms in (5.11) are continuous. In view of Korn's inequality and the constraint $\nabla v=\psi$, we have

$$\begin{aligned} a(\psi,\psi) &\ge c\|\psi\|_1^2\\ &= \frac{c}{2}\|\psi\|_1^2+\frac{c}{2}\|\nabla v\|_1^2\\ &\ge c'(\|\psi\|_1^2+\|v\|_2^2), \end{aligned} \tag{5.13}$$

and a satisfies the ellipticity condition for a saddle point problem. Moreover,

$$\sup_{v,\psi}\frac{(\nabla v-\psi,\eta)_0}{\|v\|_2+\|\psi\|_1}\ge \sup_{\psi\in H_0^1}\frac{-(\psi,\eta)_0}{\|\psi\|_1}=\|\eta\|_{-1},$$

and the inf-sup condition is also satisfied. The solution of (5.11) can now be estimated from the general theory using Theorem III.4.3:

$$\|w\|_2+\|\theta\|_1+\|\gamma\|_{-1}\le c\|f\|_{-2}. \tag{5.14}$$

More importantly, the regularity theory for convex domains Ω yields the sharper result

$$\|w\|_3+\|\theta\|_2+\|\gamma\|_0\le c\|f\|_{-1}, \tag{5.15}$$

see Blum and Rannacher [1980]. Since $H_0^2(\Omega)$ is dense in $H_0^1(\Omega)$, (5.11) even holds for all $v\in H_0^1(\Omega)$ provided the domain is convex and $f\in H^{-1}(\Omega)$.

The following alternative pairing is also important since we do not need C^1 elements for a conforming treatment of the normal deflections.

$$X:=H_0^1(\Omega)\times H_0^1(\Omega)^2, \quad M:=H^{-1}(\text{div},\Omega). \tag{5.16}$$

Here $H^{-1}(\mathrm{div}, \Omega)$ is the completion of $C^\infty(\Omega)^2$ with respect to the norm

$$\|\eta\|_{H^{-1}(\mathrm{div},\Omega)} := (\|\eta\|_{-1}^2 + \|\operatorname{div}\eta\|_{-1}^2)^{\frac{1}{2}}. \tag{5.17}$$

If the domain Ω satisfies the hypotheses of Theorem II.1.3, then $H^{-1}(\mathrm{div}, \Omega)$ can be viewed as the set of H^{-1} functions whose divergences lie in $H^{-1}(\Omega)$.

If the constraint $\nabla v = \psi$ is satisfied, then by (5.13) we have $a(\psi, \psi) \geq c'(\|\psi\|_1^2 + \|v\|_1^2)$, and the ellipticity of a is clear. Moreover,

$$\begin{aligned}
\sup_{v,\psi} \frac{(\nabla v - \psi, \eta)_0}{\|v\|_1 + \|\psi\|_1} &= \frac{1}{2}\sup_{v,\psi} \frac{(\nabla v - \psi, \eta)_0}{\|v\|_1 + \|\psi\|_1} + \frac{1}{2}\sup_{v,\psi} \frac{(\nabla v - \psi, \eta)_0}{\|v\|_1 + \|\psi\|_1} \\
&\geq \frac{1}{2}\sup_{\psi} \frac{(-\psi, \eta)_0}{\|\psi\|_1} + \frac{1}{2}\sup_{v} \frac{-(v, \operatorname{div}\eta)_0}{\|v\|_1} \\
&= \frac{1}{2}\|\eta\|_{-1} + \frac{1}{2}\|\operatorname{div}\eta\|_{-1}.
\end{aligned} \tag{5.18}$$

This establishes the inf-sup condition for the pairing (5.16).

Similarly, we have $|(\nabla v - \psi, \eta)| \leq \|v\|_1 \|\operatorname{div}\ \eta\|_{-1} + \|\psi\|_1 \|\eta\|_{-1}$. Once we establish the continuity, we have all of the hypotheses of Theorem III.4.3, and the general theory implies existence, stability, and

$$\|w\|_1 + \|\theta\|_1 + \|\gamma\|_{H^{-1}(\mathrm{div},\Omega)} \leq c\|f\|_{-1}. \tag{5.19}$$

For convex domains, this estimate is obviously weaker than the regularity result (5.15).

It is not entirely obvious that the two pairings (5.12) and (5.15) lead to the same solution of the variational problem (5.11) for $f \in H^{-1}(\Omega)$. For the solution based on the spaces (5.12), the regularity assertion (5.15) guarantees the inclusion $\gamma \in L_2(\Omega) \subset H^{-1}(\mathrm{div}, \Omega)$, and thus that the solution is also consistent with the other pairing. Conversely, the second equation in (5.11) asserts that w has a weak derivative in H_0^1 which lies in H_0^2. However, this result requires a homogeneous right-hand side in the second equation of (5.11).

DKT Elements

Finite element computations with C^1 elements are circumvented if the normal hypothesis (and C^1 continuity) is satisfied at the nodes of a triangular partition rather than on the entire domain. We call this a *discrete Kirchhoff condition*, and the corresponding element a *discrete Kirchhoff triangle* or for short a DKT element.

Strictly speaking, DKT elements are nonconforming elements for the displacement formulation (5.9). Nevertheless, the connection with mixed methods

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simplifies the analysis since the consistency error can be estimated in terms of the Lagrange multiplier of the mixed formulation.

We consider two examples which require different treatments. Both involve reduced polynomials of degree 3. Given a triangle, let a_1, a_2, a_3 be its vertices, and let a_0 be its center of gravity:

$$\mathcal{P}_{3,\text{red}} := \Big\{ p \in \mathcal{P}_3;\ 6p(a_0) - \sum_{i=1}^{3} [2p(a_i) - \nabla p(a_i) \cdot (a_i - a_0)] = 0 \Big\}.$$

Here the constraint excludes the bubble function, which in other cases is usually appended to polynomials of lower degree. In comparison with standard interpolation using cubic polynomials, the interior point is missing, and only the nine points on the boundary are used.

Instead of using these nine function values, we can also use the function values and first derivatives at the three vertices; cf. Problem II.5.13.

5.3 Example. Following Batoz, Bathe, and Ho [1980], let

$$\begin{aligned} W_h &:= \{w \in H_0^1(\Omega);\ \nabla w \text{ is continuous at all nodes of } \mathcal{T}_h, \\ &\qquad\qquad\qquad\quad w|_T \in \mathcal{P}_{3,\text{red}} \text{ for } T \in \mathcal{T}_h\}, \\ \Theta_h &:= \{\theta \in H_0^1(\Omega)^2;\ \theta|_T \in (\mathcal{P}_2)^2 \text{ and } \theta \cdot n \in \mathcal{P}_1(e) \\ &\qquad\qquad\qquad\quad \text{for every edge } e \in \partial T \text{ and } T \in \mathcal{T}_h\}. \end{aligned} \tag{5.20}$$

Every deflection is associated with a rotation via a discrete Kirchhoff condition:

$$\begin{aligned} w_h \longmapsto \theta_h(w_h): \quad \theta_h(a_i) &= \nabla w_h(a_i) \qquad \text{for the vertices } a_i, \\ \theta_h(a_{ij}) \cdot t &= \nabla w_h(a_{ij}) \cdot t \text{ for the midpoint } a_{ij}. \end{aligned} \tag{5.21}$$

As usual, n denotes a unit normal vector, and t is a tangential vector (to points on the edges). Moreover, $a_{ij} := \frac{1}{2}(a_i + a_j)$. The mapping $W_h \to \Theta_h$ in (5.21) is well defined since by the definition of the space Θ_h, the normal components at the midpoints of the sides are already determined by their values at the vertices:

$$\theta_h(a_{ij}) \cdot n = \frac{1}{2}[\nabla w_h(a_i) + \nabla w_h(a_j)] \cdot n.$$

Thus, $\theta_h(w_h) \in (P_2)^2$ is defined by the interpolation conditions at the six canonical points.

We note that on the edges, the tangential components of ∇w_h are always quadratic polynomials, and by construction coincide with the tangential components of $\theta_h = \theta_h(w_h)$. Thus, θ_h can only vanish in a triangle T if w_h is constant

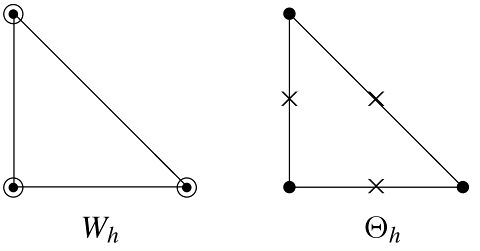


Fig. 61. The DKT element of Batoz, Bathe, and Ho (The tangential component only is fixed at the nodes marked with $\times$.)

on ∂T. Since the restriction of W_h to a triangle has dimension 9, the restriction of $\theta_h(W_h)$ has the same dimension as ∇W_h, i.e., 8.

Note that $\theta_h(W_h)$ is a proper subspace of Θ_h. Since $\dim(\mathcal{P}_2)^2 = 12$, the three kinematic relations $[2\,\theta(a_{ij}) - \theta(a_i) - \theta(a_j)]n = 0$ in (5.20) imply that locally, Θ_h has dimension $12 - 3 = 9$. The fact that the mapping (5.21) only gives a subspace is not a problem for either the implementation or the analysis of the elements.

The associated finite element approximation is the solution of the equation

$$a_h(w_h, v_h) = (f, v_h)_0 \quad \text{for all } v_h \in W_h$$

with the bilinear form

$$a_h(w_h, v_h) := a(\theta_h(w_h), \theta_h(v_h)). \tag{5.22}$$

For the analysis, a_h is defined on $W_h \oplus H^3(\Omega) \cap H_0^1(\Omega)$ by extending the mapping (5.21) to this space in the canonical way. In addition, following Pitkäranta [1988], we make use of the mesh-dependent norms

$$|v|_{s,h} := \Big(\sum_{T\in\mathcal{T}_h} |v|_{s,T}^2\Big)^{\frac{1}{2}}, \qquad \|v\|_{s,h} := \Big(\sum_{T\in\mathcal{T}_h} \|v\|_{s,T}^2\Big)^{\frac{1}{2}}. \tag{5.23}$$

5.4 Remark. The semi-norms

$$\begin{aligned} &\|\nabla v_h\| \text{ and } \|\theta_h(v_h)\|_0, \\ &|\nabla v_h|_{1,h} \text{ and } |\theta_h(v_h)|_1, \\ &\|\nabla v_h\|_{1,h} \text{ and } \|\theta_h(v_h)\|_1, \quad \text{respectively,} \end{aligned} \tag{5.24}$$

are equivalent on W_h, provided that the triangulations are shape-regular.

Proof. In view of the finite dimensionality and the linearity of the mapping $\nabla v_h \mapsto \theta(v_h)$, on the reference triangle we have

$$\begin{aligned} \|\theta_h(v_h)\|_{0,T_{\rm ref}} &\le c\|\nabla v_h\|_{0,T_{\rm ref}}, \\ |\theta_h(v_h)|_{1,T_{\rm ref}} &\le c\|\nabla v_h\|_{1,T_{\rm ref}}. \end{aligned}$$

Now $\theta_h - \nabla v_h = 0$ for $v_h \in \mathcal{P}_1$. By the Bramble–Hilbert lemma, we can estimate $|\theta_h - \nabla v_h|_{1,T_{\text{ref}}}$ by $c|\nabla v_h|_{1,T_{\text{ref}}}$, and thus insert the semi-norm in the second part of (5.24). Since the mapping $\nabla v_h \mapsto \theta(v_h)$ is injective, the converse follows in the same way. Finally, the transformation theorems of Ch. II, §6 imply the estimate for the triangles T of a shape-regular triangulation. The result for the domain Ω follows by summation. □

5.5 Corollary. *The DKT element 5.3 satisfies*

$$a(\theta_h(v_h), \theta_h(v_h)) \geq c \sum_{T \in \mathcal{T}_h} \|v_h\|_{2,T}^2$$

for some constant $c > 0$ independent of h.

Proof. Since $\theta_h \in H_0^1(\Omega)^2$ and $v_h \in H_0^1(\Omega)$, it follows from Korn's inequality, Friedrichs' inequality, and the previous remark that

$$\begin{aligned} a(\theta_h(v_h), \theta_h(v_h)) &\geq c\|\theta_h(v_h)\|_1^2 = c \sum_{T \in \mathcal{T}_h} \|\theta_h(v_h)\|_{1,T}^2 \\ &\geq c\|\nabla v_h\|_{1,h}^2 = c(|\nabla v_h|_{1,h}^2 + |v_h|_{1,\Omega}^2)^2 \\ &\geq c'(|\nabla v_h|_{1,h}^2 + \|v_h\|_{1,\Omega}^2) \geq c'\|v_h\|_{2,h}^2. \end{aligned}$$ □

Since the gradient of linear functions is exactly interpolated, the same argument implies that

$$\|\theta_h(v_h) - \nabla v_h\|_0 \leq c\, h|v_h|_{2,h} \quad \text{for all } v_h \in W_h.$$

We are now in a position to estimate the consistency error via the second lemma of Strang. Since $H^3(\Omega) \subset C^1(\Omega)$, we can directly extend the mapping θ_h in (5.21) to $W_h \oplus H^3(\Omega)$. We also have to take into consideration the fact that, in general, $\theta_h(w) \neq \nabla w$ for the solution w of (5.11). Equality holds if ∇w is linear. Hence, it follows from the Bramble–Hilbert lemma that

$$\|\theta_h(w) - \nabla w\|_0 \leq ch^2\|w\|_3.$$

In the first equation of the mixed formulation (5.11) we now set $v = v_h$ and $\psi = \theta_h(v_h)$. Together with the regularity result (5.15) for the mixed method, this implies

$$\begin{aligned} a_h(w, v_h) - (f, v_h)_0 &= a(\theta_h(w), \theta_h(v_h)) - (f, v_h)_0 \\ &= [a(\theta, \theta_h(v_h)) - (f, v_h)_0] + a(\theta_h(w) - \theta, \theta_h(v_h)) \\ &= (\nabla v_h - \theta_h(v_h), \gamma)_0 + a(\theta_h(w) - \theta, \theta_h(v_h)) \\ &\leq \|\nabla v_h - \theta_h(v_h)\|_0\, \|\gamma\|_0 + ch\|w\|_3\|v_h\|_{2,h} \\ &\leq c\, h\|v_h\|_{2,h}\, \|f\|_{-1}. \end{aligned}$$

Since the approximation error $\inf\{\|w - \psi_h\|_{2,h};\ \psi_h \in W_h\}$ is also of order $\mathcal{O}(h)$, the second lemma of Strang implies

5.6 Theorem. *Let $\mathcal{T}_h$ be a family of shape-regular triangulations. Then the finite element solution using the DKT element 5.3 satisfies*

$$\|w - w_h\|_{2,h} \le ch\|f\|_{-1},$$

with a constant c independent of h.

In Example 5.3 we have used a rather large finite element space for the rotations. The amount of computation can be reduced by using smaller spaces.

5.7 Example *(Zienkiewicz triangle).* Let

$$\begin{aligned} W_h &:= \{w \in H_0^1(\Omega);\ \nabla w \text{ is continuous at all nodes of } \mathcal{T}_h, \\ &\qquad\qquad w|_T \in \mathcal{P}_{3,\mathrm{red}} \text{ for } T \in \mathcal{T}_h\}, \qquad (5.25) \\ \Theta_h &:= \{\theta \in H_0^1(\Omega)^2;\ \theta|_T \in (\mathcal{P}_1)^2 \text{ for } T \in \mathcal{T}_h\} = (\mathcal{M}_{0,0}^1)^2. \end{aligned}$$

Every deflection can be associated with a rotation by means of a discrete Kirchhoff condition:

$$w_h \longmapsto \theta_h(w_h): \quad \theta_h(a_i) = \nabla w_h(a_i) \text{ for the vertices } a_i.$$

It is clear from a dimensionality argument that $\theta_h = 0$ can hold in a triangle, even though $\nabla w_h \neq 0$. Thus, the bilinear form a_h must be defined differently than in (5.22):

$$a_h(w_h, v_h) := \sum_{T \in \mathcal{T}_h} a(\nabla w_h, \nabla v_h)_T.$$

This element is called the *Zienkiewicz triangle*, see Zienkiewicz [1971]. It is interesting to note that both theoretical results and numerical experience show that the convergence of this nonconforming method occurs only for *three-direction meshes*, i.e. for meshes where the grid lines run only in three directions; see Lascaux and Lesaint [1975].

It is possible to dispense with the restriction to three-direction meshes by adding a penalty term, which leads to the quadratic form

$$a_h(v_h, v_h) := a(\theta_h(v_h), \theta_h(v_h)) + \sum_{T \in \mathcal{T}_h} \frac{1}{h_T^2} \int_T |\nabla v_h - \theta_h(v_h)|^2 dx. \qquad (5.26)$$

It is clear from (5.7) that the penalty term was selected on the basis of the theory of the Mindlin–Reissner plate.

For simplicity, we restrict ourselves to a uniform grid. Then the various mesh sizes h_T can be replaced by a global one:

$$a_h(v_h, v_h) := a(\theta_h(v_h), \theta_h(v_h)) + \frac{1}{h^2}\|\nabla v_h - \theta_h(v_h)\|_0^2.$$

In addition to the norms (5.23), the energy norm

$$|||v, \psi|||_2 := \Big(\|\psi\|_1^2 + \frac{1}{h^2}\|\nabla v - \psi\|_0^2\Big)^{\frac{1}{2}}$$

enters into the analysis.

5.8 Lemma. *For all $v_h \in W_h$, $h \le 1$,*

$$c^{-1}\|v_h\|_{2,h} \le |||v_h, \theta_h(v_h)|||_2 \le c\|v_h\|_{2,h}, \tag{5.27}$$

where c is a constant independent of h.

Proof. (1) By the approximation results,

$$\|\nabla v - \theta_h(v)\|_{s,T} \le c\,h^{1-s}|v|_{2,T} \quad \text{for } s = 0, 1.$$

With $s = 0$, we have $\|\nabla v_h - \theta_h(v_h)\|_{0,\Omega} \le c\,h\|v_h\|_{2,h}$. Thus, with $s = 1$,

$$\|\theta_h(v_h)\|_{1,T}^2 \le (\|v_h\|_{1,T} + \|v_h - \theta_h(v_h)\|_{1,T})^2 \le 2\|v_h\|_{1,T}^2 + 2c|v_h|_{2,T}^2.$$

Hence, $\|\theta_h(v_h)\|_{1,\Omega} \le 2(1+c)\|v_h\|_{2,h}$, and the inequality on the right in (5.27) is proved.

(2) To prove the other inequality, we establish the stronger assertion

$$c^{-1}\|v_h\|_{2,h} \le |||v_h, \psi_h|||_2 \quad \text{for all } v_h \in W_h, \psi_h \in \Theta_h. \tag{5.28}$$

First we use Friedrichs' inequality:

$$\begin{aligned}
\frac{1}{c}\|v_h\|_1^2 \le \|\nabla v_h\|_0^2 &\le 2\|\nabla v_h - \psi_h\|_0^2 + 2\|\psi_h\|_0^2 \\
&\le 2h^{-2}\|\nabla v_h - \psi_h\|_0^2 + 2\|\psi_h\|_1^2 \le 2|||v_h, \psi_h|||_2^2.
\end{aligned}$$

Similar estimates along with the usual inverse inequality lead to

$$\begin{aligned}
\sum_T |v_h|_{2,T}^2 &\le \sum_T (|\nabla v_h - \psi_h|_{1,T} + |\psi_h|_{1,T})^2 \\
&\le 2\sum_T (|\nabla v_h - \psi_h|_{1,T}^2 + |\psi_h|_{1,T}^2) \\
&\le 2\sum_T h^{-2}|\nabla v_h - \psi_h|_{0,T}^2 + 2\|\psi_h\|_{1,\Omega}^2 = 2|||v_h, \psi_h|||_2^2,
\end{aligned}$$

and the proof is complete. □

Taking account of Remark III.1.3, we can now directly carry over the method of proof of Theorem 5.6 to establish convergence. For details, see Problem 5.11.

5.9 Theorem. *Let $\mathcal{T}_h$ be a family of shape-regular triangulations. Then the finite element solution using the DKT element 5.7 satisfies*

$$\|w - w_h\|_{2,h} \le c\,h\|f\|_{-1},$$

where c is a constant independent of h.

For the treatment of DKT elements by multigrid methods, see, e.g., Peisker, Rust, and Stein [1990].

Problems

5.10 Generalize Theorem 5.9 to shape-regular grids, and verify (5.27) for a_h with the associated energy norm

$$|||v,\psi|||_2 := \Big(\|\psi\|_1^2 + \sum_{T\in\mathcal{T}_h} \frac{1}{h_T^2}\int_T |\nabla v - \psi|^2 dx\Big)^{\frac{1}{2}}.$$

5.11 To treat the consistency error $a_h(w, v_h) - (f, v_h)_0$ in Theorem 5.9, estimate the contribution of the penalty term

$$\frac{1}{h^2}\int_\Omega (\nabla w - \theta_h(w)) \cdot (\nabla v_h - \theta_h(v_h))\, dx \quad \text{for } v_h \in W_h$$

in terms of $|w|_{3,\Omega}|v_h|_{2,h}$ with the correct power of h.

5.12 Express the dimension of the finite element spaces with DKT elements in terms of the number of triangles, nodes, and edges.

5.13 Show that the mixed method for the Timoshenko beam which corresponds to (5.11) is stable for $X := H_0^1(0, b) \times H_0^1(0, b)$ and $M := L_2(0, b)$. With which Standard Sobolev space does $H^{-1}(\text{div}, (0, b))$ coincide? [Hence, we do not need the space $H^{-1}(\text{div})$ in contrast to the investigation of plates.]

§ 6. The Mindlin–Reissner Plate

The Mindlin–Reissner model for a bending plate involves minimizing (5.7) over $(w, \theta) \in X := H_0^1(\Omega) \times H_0^1(\Omega)^2$. Here the shear term does not vanish since the normal hypothesis is not assumed. The Mindlin plate turns out to be a singular perturbation problem. This is consistent with the observation that the directions of the rotations differ from the normal directions only near the boundary; see Arnold and Falk [1989], Pitkäranta and Suri [1997].

Here t should be thought of as a small parameter. In order to avoid shear locking, we treat the plate as a mixed problem with penalty term. Now we proceed in exactly the same way as we did in going from (3.32) to (3.37) and from (3.42) to (3.48). Introducing the shear term

$$\gamma := t^{-2}(\nabla w - \theta), \tag{6.1}$$

we get the following – at first purely formal – mixed problem with penalty term: Find $(w, \theta) \in X := H_0^1(\Omega) \times H_0^1(\Omega)^2$ and $\gamma \in M := L_2(\Omega)^2$ such that

$$\begin{aligned} a(\theta, \psi) \ &+ (\nabla v - \psi, \gamma)_0 = (f, v) && \text{for all } (v, \psi) \in X, \\ (\nabla w - \theta, \eta)_0 \ &- t^2(\gamma, \eta)_0 \qquad = 0 && \text{for all } \eta \in M. \end{aligned} \tag{6.2}$$

Here the bilinear form a is defined in (5.5). The equation (6.2) and the mixed formulation (5.11) for the Kirchhoff plate differ only by the penalty term $-t^2(\gamma, \eta)_0$. However, we should not overlook the fact that here the variable γ in the equation has a different meaning. In (5.11) it serves as a *Lagrange multiplier*, while by (6.1), here it is to be regarded as a *normed shear term*.

The variational problem cannot be dealt with directly using the general results for the saddle point theory in Ch. III, §4. From the theory of the Kirchhoff plate, we know that $H^{-1}(\text{div}, \Omega)$ is the natural space for the Lagrange multiplier. Theorems III.4.11 and III.4.13 are not applicable since $H^{-1}(\text{div}, \Omega) \not\subset L_2(\Omega)$, and because the bilinear form a is not elliptic on the entire space X.

Since these arguments may appear quite formal to the reader, we present another reason: For $\theta \in H_0^1(\Omega)^2$, $w \in H_0^1(\Omega)$, $\text{rot}\,\theta \in L_2(\Omega)$ and $\text{rot}\,\text{grad}\, w = 0 \in L_2(\Omega)$, and thus

$$\nabla w - \theta \in H_0(\text{rot}, \Omega).$$

The rotation is defined in (4.9). Hence, the proper space is not $L_2(\Omega)$, but rather

$$H_0(\text{rot}, \Omega) := \{\eta \in L_2(\Omega)^2;\ \text{rot}\,\eta \in L_2(\Omega),\ \eta \cdot \tau = 0 \text{ on } \partial\Omega\}. \tag{6.3}$$

Here $\tau = \tau(x)$ is defined (almost everywhere) on $\partial\Omega$ as the direction of the tangent in the counterclockwise direction. We endow the space (6.3) with the norm

$$\|\eta\|_{0,\mathrm{rot}} := (\|\eta\|_0^2 + \|\operatorname{rot}\eta\|_0^2)^{\frac{1}{2}}. \tag{6.4}$$

In terms of the general theory in Ch. III, §4, the bilinear form b for the mixed formulation is given by $b(w,\theta,\eta) := (\nabla w - \theta, \eta)_0$. Now we specialize η to be an element of $L_2(\Omega)^2$ of the form $\eta = \operatorname{curl} p$. Then because of the orthogonality of the rotation and the gradient,

$$b(w,\theta;\eta) = (\nabla w - \theta, \eta)_0 = 0 - (\theta,\eta)_0 \le \|\theta\|_1\,\|\eta\|_{-1},$$

for $w \in H_0^1(\Omega),\ \theta \in H_0^1(\Omega)^2$, and thus

$$\sup_{w,\theta} \frac{b(w,\theta,\eta)}{\|w\|_1 + \|\theta\|_1} \le \|\eta\|_{-1}.$$

In order to ensure that the inf-sup condition holds, we have to endow M with a norm which is weaker than the L_2-norm, that is with the one which is dual to (6.4), i.e., $\|\cdot\|_{H^{-1}(\mathrm{div},\Omega)}$; see below.

As shown in Brezzi and Fortin [1986], the analyis is simplified if we use the Helmholtz decomposition of the shear term into a gradient field and a rotational field. Using the decomposition we get expressions and estimates which involve the usual Sobolev norms.

The Helmholtz Decomposition

In the following we shall see that the space

$$H^{-1}(\mathrm{div},\Omega) := \{\eta \in H^{-1}(\Omega)^2;\ \operatorname{div}\eta \in H^{-1}(\Omega)\}$$

with the graph norm (5.17) is the dual space of $H_0(\mathrm{rot},\Omega)$. Clearly,

$$H_0(\mathrm{rot},\Omega) \subset L_2(\Omega)^2 \subset H^{-1}(\mathrm{div},\Omega).$$

As usual, we identify functions in $L_2(\Omega)/\mathbb{R}$ which differ only by a constant. The norm of an element in this space is just the L_2-norm of the representer which is normalized to have zero integral; see Problem III.6.6.

6.1 Lemma. *Assume that $\Omega \subset \mathbb{R}^2$ is simply connected. Then every function $\eta \in H^{-1}(\mathrm{div}, \Omega)$ is uniquely decomposable in the form*

$$\eta = \nabla\psi + \mathrm{curl}\, p \tag{6.5}$$

with $\psi \in H_0^1(\Omega)$ and $p \in L_2(\Omega)/\mathbb{R}$. Moreover, the norms

$$\|\eta\|_{H^{-1}(\mathrm{div},\Omega)} \quad \textit{and} \quad (\|\psi\|_{1,\Omega}^2 + \|p\|_0^2)^{\frac{1}{2}} \tag{6.6}$$

are equivalent, where p is the representer satisfying $\int_\Omega p\, dx = 0$.

Proof. By hypothesis, $\chi := \mathrm{div}\, \eta \in H^{-1}(\Omega)$. Let $\psi \in H_0^1(\Omega)$ be the solution of the equation $\Delta\psi = \chi$. Then $\mathrm{div}(\eta - \nabla\psi) = \mathrm{div}\, \eta - \Delta\psi = 0$. By classical estimates, every divergence-free function in Ω can be represented as a rotation, i.e., $\eta - \nabla\psi = \mathrm{curl}\, p$ with a suitable $p \in L_2(\Omega)/\mathbb{R}$. This establishes the decomposition.

We also observe that

$$\|\mathrm{div}\, \eta\|_{-1} = \|\Delta\psi\|_{-1} = |\psi|_1, \tag{6.7}$$

and

$$\|\eta\|_{-1} = \|\nabla\psi + \mathrm{curl}\, p\|_{-1} \le \|\nabla\psi\|_{-1} + \|\mathrm{curl}\, p\|_{-1} \le \|\psi\|_0 + \|p\|_0.$$

After summation, it follows that $\|\eta\|_{H^{-1}(\mathrm{div},\Omega)}^2 \le 2\|\psi\|_1^2 + 2\|p\|_0^2$.

In view of (6.7), to complete the proof we need only show that $\|p\|_0 \le c\|\eta\|_{H^{-1}(\mathrm{div},\Omega)}$. Note that $\int_\Omega p\, dx = 0$. It is known from the Stokes problem (see Problem III.6.7) that there exists a function $v \in H_0^1(\Omega)^2$ with

$$\mathrm{div}\, v = p \quad \text{and} \quad \|v\|_1 \le c\|p\|_0. \tag{6.8}$$

Then for $\xi = (\xi_1, \xi_2) := (-v_2, v_1)$, it clearly follows that $\xi \in H_0^1(\Omega)^2$,

$$\mathrm{rot}\, \xi = p \quad \text{and} \quad \|\xi\|_1 \le c\|p\|_0.$$

Moreover, taking account of (6.7), we see that the decomposition (6.5) satisfies

$$\begin{aligned}
\|p\|_0^2 &= (p, \mathrm{rot}\, \xi) = (\mathrm{curl}\, p, \xi) = (\eta - \nabla\psi, \xi) \\
&\le \|\eta\|_{-1}\|\xi\|_1 + |\psi|_1\|\xi\|_0 \\
&\le c(\|\eta\|_{-1} + \|\mathrm{div}\, \eta\|_{-1})\|p\|_0.
\end{aligned}$$

□

Supplement. If η lies in $L_2(\Omega)^2$ and not just in $H^{-1}(\mathrm{div}, \Omega)$, then we even have $p \in H^1(\Omega)/\mathbb{R}$ for the second component of the Helmholtz decomposition (6.5), and $L_2(\Omega) = \nabla H_0^1(\Omega) \oplus \text{curl}(H^1(\Omega)/\mathbb{R})$. Thus, p is a solution of the Neumann problem $(\text{curl}\, p, \text{curl}\, q)_0 = (\text{div}(\eta - \nabla\psi), \text{curl}\, q)_0$ for given $q \in H^1(\Omega)$.

We assert that

$$(H_0(\text{rot}, \Omega))' = H^{-1}(\text{div}, \Omega) \tag{6.9}$$

but we will verify only the inclusion $H^{-1}(\text{div}, \Omega) \subset (H_0(\text{rot}, \Omega))'$, and leave the proof of the converse to Problem 6.11. Let $\gamma \in H_0(\text{rot}, \Omega)$ and $\eta = \nabla\psi + \text{curl}\, p \in H^{-1}(\text{div}, \Omega)$. Using Lemma 6.1, we conclude that

$$\begin{aligned}
(\gamma, \eta)_0 &= (\gamma, \nabla\psi)_0 + (\gamma, \text{curl}\, p)_0 \\
&= (\gamma, \nabla\psi)_0 + (\text{rot}\,\gamma, p)_0 \\
&\le \|\gamma\|_0 \|\psi\|_1 + \|\text{rot}\,\gamma\|_0\, \|p\|_0.
\end{aligned}$$

Now by (6.6) and the Cauchy–Schwarz inequality for $\mathbb{R}^2$, we get

$$(\gamma, \eta)_0 \le c\|\gamma\|_{H_0(\text{rot},\Omega)} \cdot \|\eta\|_{H^{-1}(\text{div},\Omega)}.$$

This shows that the bilinear form $(\gamma, \eta)_0$ can be extended from a dense subset to all of $H_0(\text{rot}, \Omega) \times H^{-1}(\text{div}, \Omega)$, and the inclusion is established.

The Mixed Formulation with the Helmholtz Decomposition

We now return to the variational problem (6.2) for the Mindlin–Reissner plate. Following Brezzi and Fortin [1986], we now assume that the shear term has the form

$$\gamma = \nabla r + \text{curl}\, p \tag{6.10}$$

with $r \in H_0^1(\Omega)$ and $p \in H^1(\Omega)/\mathbb{R}$. We decompose the test function η in the same way as $\eta = \nabla z + \text{curl}\, q$. Also note that the gradients and rotations are L_2-orthogonal; see Problem 6.10. Now we apply Green's formula to the rotation, so that (6.2) leads to the equivalent system

$$\begin{aligned}
(\nabla r, \nabla v)_0 &= (f, v)_0 && \text{for } v \in H_0^1(\Omega), \\
a(\theta, \psi) - (\text{rot}\,\psi, p)_0 &= (\nabla r, \psi)_0 && \text{for } \psi \in H_0^1(\Omega)^2, \\
-(\text{rot}\,\theta, q)_0 - t^2(\text{curl}\, p, \text{curl}\, q)_0 &= 0 && \text{for } q \in H^1(\Omega)/\mathbb{R}, \\
(\nabla w, \nabla z)_0 &= (\theta, \nabla z)_0 + t^2(f, z)_0 && \text{for } z \in H_0^1(\Omega).
\end{aligned} \tag{6.11}$$

The first equation is a Poisson equation which can be solved first. The second and third equations together constitute an equation of Stokes type with penalty term. The fourth equation is also a Poisson equation which is independent of the others, and can be solved afterwards.

We now show that the middle equations

$$\begin{aligned} a(\theta, \psi) - (\operatorname{rot} \psi, p)_0 &= (\nabla r, \psi)_0, \\ -(\operatorname{rot} \theta, q)_0 - t^2 (\operatorname{curl} p, \operatorname{curl} q)_0 &= 0 \end{aligned} \tag{6.12}$$

of (6.11) are indeed of Stokes type. By Korn's inequality, the bilinear form a is H^1-elliptic. Moreover,

$$\operatorname{rot} \psi = \operatorname{div} \psi^\perp \quad \text{with the convention } \ x^\perp := (x_2, -x_1)$$

for any vector in $\mathbb{R}^2$. Clearly, $\|\psi^\perp\|_{s,\Omega} = \|\psi\|_{s,\Omega}$ for $\psi \in H^s(\Omega)$. Thus, (6.12) represents a (generalized) Stokes problem for $\theta^\perp$ with singular penalty term.

6.2 Theorem. *Equations (6.12) and (6.2) describe stable variational problems on $H_0^1(\Omega)^2 \times H^1(\Omega)/\mathbb{R}$ w.r.t. the norm*

$$\|\theta\|_1 + \|p\|_0 + t\, \|\operatorname{curl} p\|_0,$$

and on $H_0^1(\Omega) \times H_0^1(\Omega)^2 \times L_2(\Omega)^2$ w.r.t. the norm

$$\|w\|_1 + \|\theta\|_1 + \|\gamma\|_{H^{-1}(\operatorname{div},\Omega)} + t\, \|\gamma\|_0.$$

The constants in the associated inf-sup conditions are independent of t.

Proof. The first assertion follows immediately from Theorem III.4.13 with $X := H_0^1(\Omega)^2$, $M := L_2(\Omega)/\mathbb{R}$, and $M_c := H(\operatorname{rot}, \Omega)/\mathbb{R}$.

As mentioned above, to prove the second assertion, we cannot directly apply Theorem III.4.13. However, once we check the inf-sup condition for every component of (6.11), it follows for (6.2) with the help of the Helmholtz decompositions of $H^{-1}(\operatorname{div}, \Omega)$ and $L_2(\Omega)^2$, respectively. The argument proceeds in exactly the same way as going from (6.2) to (6.11). □

MITC Elements

The system which arises from the discretization of (6.2) can be transformed in the sense of (3.39) into displacement form (with reduction operators):

$$a(\theta_h, \theta_h) + t^{-2} \|\nabla w_h - R_h \theta_h\|_0^2 - 2(f, w_h)_0 \longrightarrow \min_{w_h, \theta_h}!, \tag{6.13}$$

where the minimization is performed over the spaces W_h and Θ_h, respectively. Here

$$R_h : H^1(\Omega)^2 \longrightarrow \Gamma_h \tag{6.14}$$

is a so-called *reduction operator*, i.e., a linear mapping defined on the finite element space for the shear terms which does not affect the elements in Γ_h.

If possible, the finite element calculations are performed using the displacement model (6.13), since it leads to systems of equations with positive definite matrices and with fewer unknowns. On the other hand, for the convergence analysis, it is still best to use the mixed formulations. However, there is a problem: In general, the functions in Γ_h cannot be represented in the form

$$\gamma_h = \operatorname{grad} r_h + \operatorname{curl} p_h,$$

where r_h and p_h again belong to finite element spaces. Thus, except for a special case treated by Arnold and Falk [1989], a modification of the Helmholtz decomposition is necessary.

The notation for the following finite element spaces is suggested by the variables in (6.11).

6.3 The Axioms of Brezzi, Bathe, and Fortin [1989]. Suppose the spaces

$$W_h \subset H_0^1(\Omega), \quad \Theta_h \subset H_0^1(\Omega)^2, \quad Q_h \subset L_2(\Omega)/\mathbb{R}, \quad \Gamma_h \subset H_0(\mathrm{rot}, \Omega)$$

and the mapping R_h defined in (6.14) have the following properties:

(P_1) $\nabla W_h \subset \Gamma_h$, i.e., the discrete shear term $\gamma_h := t^{-2}(\nabla w_h - R_h\theta_h)$ lies in Γ_h.

(P_2) $\mathrm{rot}\,\Gamma_h \subset Q_h$ – this requirement is consistent with $\gamma_h \in H(\mathrm{rot}, \Omega)$, and thus with $\mathrm{rot}\,\gamma_h \in L_2(\Omega)$.

(P_3) The pair (Θ_h, Q_h) satisfies the inf-sup condition

$$\inf_{q_h \in Q_h} \sup_{\psi_h \in \Theta_h} \frac{(\mathrm{rot}\,\psi_h, q_h)}{\|\psi_h\|_1 \|q_h\|_0} =: \beta > 0,$$

where β is independent of h. The spaces are thus suitable for the Stokes problem.

(P_4) Let P_h be the L_2-projector onto Q_h. Then

$$\mathrm{rot}\, R_h\eta = P_h \,\mathrm{rot}\,\eta \quad \text{for all } \eta \in H_0^1(\Omega)^2,$$

i.e., the following diagram is commutative:

$$\begin{array}{ccc} H_0^1(\Omega)^2 & \xrightarrow{\mathrm{rot}} & L_2(\Omega) \\ R_h \big\downarrow & & \big\downarrow P_h \\ \Gamma_h & \xrightarrow{\mathrm{rot}} & Q_h. \end{array}$$

(P_5) If $\eta_h \in \Gamma_h$ and $\operatorname{rot} \eta_h = 0$, then $\eta_h \in \nabla W_h$. This means that the sequence

$$W_h \xrightarrow{\text{grad}} \Gamma_h \xrightarrow{\text{rot}} Q_h$$

is exact.[23] – This condition corresponds to the fact that rotation-free fields are gradient fields.

We now observe that the *rotation* can be defined as a *weak derivative*, in analogy with Definition II.1.1: A function $u \in L_2(\Omega)$ lies in $H(\operatorname{rot}, \Omega)$ and $v \in L_2(\Omega)^2$ is equal to $\operatorname{curl} u$ in the weak sense provided that

$$\int_\Omega v \cdot \varphi \, dx = \int_\Omega u \operatorname{rot} \varphi \, dx \quad \text{for all } \varphi \in C_0^\infty(\Omega)^2.$$

Similarly, we now define the rotation on the finite element space Q_h in the weak sense; see Peisker and Braess [1992]. The *discrete curl operator* is used indirectly also by Brezzi, Fortin, and Stenberg [1991].

6.4 Definition. The mapping

$$\operatorname{curl}_h : Q_h \longrightarrow \Gamma_h$$

called *discrete curl operator* is defined by

$$(\operatorname{curl}_h q_h, \eta)_0 = (q_h, \operatorname{rot} \eta)_0 \quad \text{for } \eta \in \Gamma_h. \tag{6.15}$$

Since $\Gamma_h \subset H_0(\operatorname{rot}, \Omega)$, the functional $\eta \longmapsto (q_h, \operatorname{rot} \eta)_0$ is well defined and continuous. Thus, $\operatorname{curl}_h q_h$ is uniquely determined by (6.15).

6.5 Theorem. *Suppose properties* (P_1), (P_2) *and* (P_5) *hold. Then*

$$\Gamma_h = \nabla W_h \oplus \operatorname{curl}_h Q_h$$

defines an L_2-orthogonal decomposition (which is called a discrete Helmholtz decomposition).

Proof. (1) It follows directly from Definition 6.1 and (P_1) that $\nabla W_h \oplus \operatorname{curl}_h Q_h \subset \Gamma_h$.

(2) For $q_h \in Q_h$ and $w_h \in W_h$, we have $(\operatorname{curl}_h q_h, \nabla w_h)_0 = (q_h, \operatorname{rot} \nabla w_h)_0 = (q_h, 0)_0 = 0$. Thus, $\operatorname{curl}_h q_h$ and ∇w_h are orthogonal in $L_2(\Omega)$.

23 A sequence of linear mappings $A \xrightarrow{f} B \xrightarrow{g} C$ is called *exact* provided that the image of f coincides with the kernel of g.

(3) Given $\gamma_h \in \Gamma_h$, let η_h be the L_2-projection onto $\mathrm{curl}_h Q_h$. Then η_h is characterized by

$$(\gamma_h - \eta_h, \mathrm{curl}_h q_h)_0 = 0 \quad \text{for all } q_h \in Q_h.$$

By Definition 6.4, $(\mathrm{rot}(\gamma_h - \eta_h), q_h)_0 = 0$ for $q_h \in Q_h$. Since $\mathrm{rot}(\gamma_h - \eta_h) \in Q_h$, it follows that $\mathrm{rot}(\gamma_h - \eta_h) = 0$. By (P_5) we deduce that $\gamma_h - \eta_h \in \nabla W_h$, which implies that $\gamma_h \in \mathrm{curl}_h Q_h \oplus \nabla W_h$. □

We can now follow the same arguments leading from the variational problem (6.7) to the equation (6.11) in exactly the same way as for the finite element version (6.13), see [Peisker and Braess 1992 or Brezzi and Fortin 1991]. This leads to the following problem: Find $(r_h, \theta_h, p_h, w_h) \in W_h \times \Theta_h \times Q_h \times W_h$ such that

$$\begin{aligned}
(\nabla r_h, \nabla v_h)_0 &= (f, v_h)_0 && v_h \in W_h,\\
a(\theta_h, \psi_h) - (p_h, \mathrm{rot}\, \psi_h)_0 &= (\nabla r_h, \psi_h)_0 && \psi_h \in \Theta_h,\\
-(\mathrm{rot}\, \theta_h, q_h)_0 - t^2(\mathrm{curl}_h\, p_h, \mathrm{curl}_h\, q_h)_0 &= 0 && q_h \in Q_h,\\
(\nabla w_h, \nabla z_h)_0 &= (\theta_h, \nabla z_h)_0 + t^2(f, z_h)_0 && z_h \in W_h.
\end{aligned} \tag{6.16}$$

Finite element spaces with properties (P_1)–(P_5) automatically satisfy inf-sup conditions analogous to (6.2) with constants which are independent of t and h.

We compare this also with the discrete version of (6.2): Find $(w_h, \theta_h) \in X_h$ and $\gamma_h \in M_h$ such that

$$\begin{aligned}
a(\theta_h, \psi_h) + (\nabla v_h - \psi_h, \gamma_h)_0 &= (f, v_h) && \text{for all } (\psi_h, v_h) \in X_h,\\
(\nabla w_h - \theta_h, \eta_h)_0 - t^2(\gamma_h, \eta_h)_0 &= 0 && \text{for all } \eta_h \in M_h\,.
\end{aligned} \tag{6.17}$$

6.6 Example. The so-called MITC7 element (element with *mixed interpolated tensorial components*) is a triangular element involving up to seven degrees of freedom per triangle and variable: The shear terms belong to a more complicated space; see Fig. 62:

$$\Gamma_h := \{\eta \in H_0(\mathrm{rot}, \Omega);\ \eta_{|T} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + p_3\begin{pmatrix} y \\ -x \end{pmatrix},\ p_1, p_2, p_3 \in \mathcal{P}_1,\ T \in \mathcal{T}_h\},$$
$$W_h := \mathcal{M}^2_{0,0}, \qquad \Theta_h := \mathcal{M}^2_{0,0} \oplus B_3, \qquad Q_h := \mathcal{M}^1/\mathbb{R}.$$

Here we have made use of the usual notations as in (4.13). Finally, we define the operator R_h by

$$\begin{aligned}
\int_e (\eta - R_h\eta)\tau\, p_1 ds &= 0 \quad \text{for every edge } e \text{ and every } p_1 \in \mathcal{P}_1,\\
\int_T (\eta - R_h\eta) dx &= 0 \quad \text{for every } T \in \mathcal{T}_h.
\end{aligned} \tag{6.18}$$

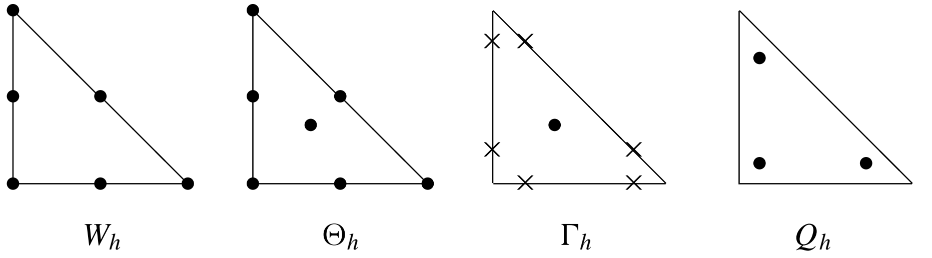


Fig. 62. MITC7 Element (only tangential components are fixed at the points marked with $\times$)

We recall that elements in $H_0(\mathrm{rot}, \Omega)$ have to have continuous tangential components along the edges between the triangles. As with the Raviart–Thomas element, it is easy to check that the *tangential components* of the vector expression

$$\begin{pmatrix} y \\ -x \end{pmatrix}$$

are constant on every edge. Thus, the functions in Γ_h are linear on the edges, and so are determined by the values at two points. In particular, these points can be the sample points for a Gaussian quadrature formula which exactly integrates quadratic polynomials. Thus (in agreement with Fig. 62) the six degrees of freedom for functions in Γ_h are determined by the values on the sides. These six values along with the two components at the midpoint of the triangle determine the eight local degrees of freedom.

Therefore, the restriction operator R_h described by (6.18) can be computed from interpolation at the above six points on the sides along with two integrals over the triangle. Thus, the values of $R_h\eta$ in a triangle depend only on the values of η in the same triangle. This means that the system matrix can be assembled locally, triangle by triangle. This would not have been the case if we had used the L_2-projector in place of R_h.

For numerical results using this element, see Bathe, Brezzi, and Cho [1989] and Bathe, Bucalem, and Brezzi [1991/92].

The Model without a Helmholtz Decomposition

The formulation (6.2) for the Mindlin plate is motivated by the similarity with the mixed formulation of the Kirchhoff plate. Arnold and Brezzi [1993] developed a clever modification which permits a development without using the Helmholtz decomposition. There is a simple treatment in terms of the theory of saddle point problems with penalty term developed in Ch. III, §4; cf. also Braess [1996]. A similar modification can also be found in the treatment of shells by Pitkäranta [1992].

We will partly follow the modification of Chapelle and Stenberg [1998]. The advantage and the disadvantagge of the two models were elucidated by Pitkäranta and Suri [2000].

Let $t, h < 1$. We again start with the minimization of the functional (6.7), but now combine a part of the shear term with the bending part:

$$\Pi(u) = \frac{1}{2}a_p(w,\theta;w,\theta) + \frac{t'^{-2}}{2}\int_\Omega |\nabla w - \theta|^2 dx - \int_\Omega fw dx, \tag{6.19}$$

where

$$\begin{aligned} a_p(w,\theta;v,\phi) &:= a(\theta,\phi) + \frac{1}{h^2+t^2}\int_\Omega (\nabla w - \theta)\cdot(\nabla v - \phi)\,dx, \\ \frac{1}{t^2} &= \frac{1}{h^2+t^2} + \frac{1}{t'^2} \quad \text{or} \quad t'^2 = t^2\frac{h^2}{h^2+t^2}. \end{aligned} \tag{6.20}$$

Thus, we seek $(w,\theta) \in X = H_0^1(\Omega) \times H_0^1(\Omega)^2$ such that

$$a_p(w,\theta;v,\phi) + \frac{1}{t'^2}(\nabla w - \theta, \nabla v - \phi)_0 = (f,w)_0 \quad \text{for all } (v,\phi) \in X. \tag{6.21}$$

By analogy with the derivation of (6.2) from (5.7), with the introduction of (modified) shear terms $\gamma := t'^{-2}(\nabla w - \theta)$, we now get the following mixed problem with penalty term. Find $(w,\theta) \in X$ and $\gamma \in M$ such that

$$\begin{aligned} a_p(w,\theta;v,\phi) + (\nabla v - \phi, \gamma)_0 &= (f,v)_0 && \text{for all } (v,\phi) \in X, \\ (\nabla w - \theta, \eta)_0 - t'^2(\gamma,\eta)_0 &= 0 && \text{for all } \eta \in M. \end{aligned} \tag{6.22}$$

The essential difference compared to (6.2) is the coercivity of the enhanced form a_p.

6.7 Lemma. *There exists a constant* $c := c(\Omega) > 0$ *such that*

$$a_p(w,\theta;w,\theta) \ge c(\|w\|_1^2 + \|\theta\|_1^2) \quad \textit{for all } w \in H_0^1(\Omega),\ \ \theta \in H_0^1(\Omega)^2. \tag{6.23}$$

Proof. By Korn's inequality, $a(\phi,\phi) \ge c_1\|\phi\|_1^2$. In addition,

$$\|\nabla w\|_0^2 \le (\|\nabla w - \theta\|_0 + \|\theta\|_0)^2 \le 2\|\nabla w - \theta\|_0^2 + 2\|\theta\|_1^2.$$

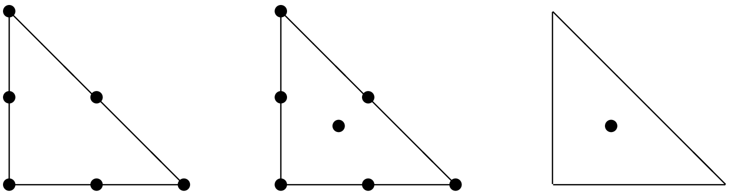
Friedrichs' inequality now implies

$$\|w\|_1^2 \le c_2|w|_1^2 \le 2c_2(\|\nabla w - \theta\|_0^2 + \|\theta\|_1^2),$$

and so

$$\begin{aligned} \|w\|_1^2 + \|\theta\|_1^2 &\le (1+2c_2)\left(\|\nabla w - \theta\|_0^2 + \|\theta\|_1^2\right) \\ &\le (1+2c_2)\left(\|\nabla w - \theta\|_0^2 + c_1^{-1}a(\theta,\theta)\right) \\ &\le (1+2_2)\,(1+c_1^{-1})\,a_p(w,\theta;w,\theta). \end{aligned}$$

This establishes the coercivity with the constant $c := (1+2c_2)^{-1}(1+c_1^{-1})^{-1}$. □



W_h Θ_h Γ_h

Fig. 63. Plate element without the Helmholtz decomposition: $W_h = \mathcal{M}^2_{0,0}$, $\Theta_h = (\mathcal{M}^2_{0,0} \oplus B_3)^2$ and $\Gamma_h = (\mathcal{M}^0)^2$.

The additional term ensures that the coercivity of the quadratic form holds on more than the kernel. This is essential not only for theoretical reasons, but numerical computations have shown that the factor in front of the shear term in (6.20) has to be chosen appropriately and that it should be of the order $O(h^{-2})$. Now Theorem III.4.13 is applicable since the Brezzi condition holds by (5.18).

6.8 Theorem. *The variational formulation (6.21) for the Mindlin–Reissner plate is stable w.r.t. the spaces*

$$X = H^1_0(\Omega) \times H^1_0(\Omega)^2, \quad M := H^{-1}(\mathrm{div}, \Omega), \quad M_c := L_2(\Omega)^2. \tag{6.24}$$

In particular, we have stability w.r.t. the norm

$$\|w\|_1 + \|\theta\|_1 + \|\gamma\|_{H^{-1}(\mathrm{div},\Omega)} + t\|\gamma\|_0. \tag{6.25}$$

Using Fortin's criterion (Lemma III.4.8), it is now easy to show that

$$W_h := \mathcal{M}^2_{0,0}, \quad \Theta_h := (\mathcal{M}^2_{0,0} \oplus B_3)^2, \quad \Gamma_h = (\mathcal{M}^0)^2$$

provides a stable combination of finite element spaces. Here we assume that the domain Ω either is convex or has a smooth boundary.

Once more we make use of the operators $\pi^0_h : H^1_0(\Omega) \to \mathcal{M}^1_{0,0}$ as well $\Pi^1_h : H^1_0(\Omega)^2 \to (\mathcal{M}^1_{0,0} \oplus B_3)^2$ appearing in the proof of Theorem III.7.2. In particular, $\int_T (\Pi_h v - v)_i dx = 0$ for every $T \in \mathcal{T}_h$ and $i = 1, 2$. Since Γ_h contains piecewise constant fields, $\int_\Omega (\Pi_h\theta - \theta) \cdot \gamma_h = 0$ for all $\gamma_h \in \Gamma_h$ and $\theta \in H^1_0(\Omega)^2$.

The transverse displacement can be treated analogously. For every edge e of the triangulation, we define a linear mapping $\pi^2_h : H^1_0(\Omega) \to \mathcal{M}^2_{0,0}$ with

$$\int_e (\pi^2_h v - v) ds = 0.$$

This requires one degree of freedom per edge, which can be the value at the midpoint of the edge. In analogy with Π_h, we set

$$\Pi_h^2 v := \pi_h^0 v + \pi_h^2(v - \pi_h^0 v),$$

so that $\int_e (\Pi_h^2 v - v)ds = 0$ for every edge e of the triangulation. Using Green's formula, we have

$$\int_T \operatorname{grad}(\Pi_h^2 v - v)\cdot \gamma_h dx = \int_{\partial T} (\pi_h^2 v - v)\gamma_h \cdot n ds - \int_T (\pi_h^2 v - v) \operatorname{div} \gamma_h dx = 0.$$

The contour integral vanishes by construction, and the second integral also vanishes since γ_h is constant on T. The boundedness of Π_h^2 follows as for π_h^1, and so the hypotheses of Fortin's criterion are satisfied since $(\operatorname{grad} \Pi_h^2 w - \Pi_h\theta, \gamma_h)_0 = (\operatorname{grad} w - \theta, \gamma_h)_0$ for $\gamma_h \in \mathcal{M}^0$. $\square$

More recently, Chapelle and Stenberg [1998] analyzed the finite element discretization with $\Theta_h := (\mathcal{M}_{0,0}^1 \oplus B_3)^2$ keeping W_h and Γ_h as in Fig. 63. They avoided the $H^{-1}(\mathrm{div})$-norm by using mesh-dependent norms and showed stability with respect to the norm whose square is

$$\|w\|_1^2 + \|\theta\|_1^2 + \frac{1}{h^2+t^2}\|\nabla w - \theta\|_0^2 + (h^2+t^2)\,\|\gamma\|_0^2.$$

In particular, the duality argument of Aubin–Nitsche could be more easily performed with these norms.

Another mesh-dependent norm which less conceals the connection with $H^{-1}(\mathrm{div})$ was introduced by Carstensen and Schöberl [2000].

Problems

6.9 Let $\eta \in L_2(\Omega)^2$. Show that the spaces for the components of the decomposition (6.5) can be exchanged, i.e., that we can choose $\psi \in H^1(\Omega)/\mathbb{R}$ and $p \in H_0^1(\Omega)$. To this end, decompose $\eta^\perp$ according to (6.5), and write the result for $\eta^\perp$ as a decomposition of η.

6.10 Does $\psi \in H^1(\Omega)$, $q \in H^1(\Omega)/\mathbb{R}$ suffice to establish the orthogonality relation

$$(\nabla\psi, \operatorname{curl} q)_0 = 0,$$

or is a zero boundary condition required?

6.11 Show that

$$\| \operatorname{div} \eta \|_{-1} \le \text{ const } \sup_{\gamma} \frac{(\gamma, \eta)_0}{\|\gamma\|_{H(\mathrm{rot},\Omega)}},$$

and thus that $\operatorname{div} \eta \in H^{-1}(\Omega)$ for $\eta \in (H_0(\mathrm{rot}, \Omega))'$. Since $H_0(\mathrm{rot}, \Omega) \supset H_0^1(\Omega)$ implies $(H_0(\mathrm{rot}, \Omega))' \subset H^{-1}(\Omega)$, this completes the proof of (6.9).

6.12 In what sense do the solutions of (5.9) and (6.2) satisfy

$$\operatorname{div} \gamma = f?$$

6.13 Let $t > 0$, and suppose $H^1(\Omega)$ is endowed with the norm

$$|||v||| := (\|v\|_0^2 + t^2 \|v\|_1^2)^{1/2}.$$

The norm of the dual space $|||u|||_{-1} := \sup_v \frac{(u,v)_0}{|||v|||}$ can be estimated easily from above by

$$|||u|||_{-1} \le \min\left\{ \|u\|_0, \ \frac{1}{t} \|u\|_{-1} \right\}.$$

Give an example to show that there is no corresponding estimate from below with a constant independent of t by computing the size of

$$u(x) := \sin x + n \sin n^2 x \in H^0[0, \pi] \quad (1/t \le n \le 2/t)$$

sufficiently exactly in each of these norms.

6.14 The finite element space W_h contains H^1 conforming elements, and thus lies in $C(\Omega)$. Show that $\nabla W_h \subset H_0(\mathrm{rot}, \Omega)$. What property of the rotation is responsible for this?