Preface to the German Edition

The method of finite elements is one of the main tools for the numerical treatment of elliptic and parabolic partial differential equations. Because it is based on the variational formulation of the differential equation, it is much more flexible than finite difference methods and finite volume methods, and can thus be applied to more complicated problems. For a long time, the development of finite elements was carried out in parallel by both mathematicians and engineers, without either group acknowledging the other. By the end of the 60's and the beginning of the 70's, the material became sufficiently standardized to allow its presentation to students. This book is the result of a series of such lectures.

In contrast to the situation for ordinary differential equations, for elliptic partial differential equations, frequently no classical solution exists, and we often have to work with a so-called weak solution. This has consequences for both the theory and the numerical treatment. While it is true that classical solutions do exist under approriate regularity hypotheses, for numerical calculations we usually cannot set up our analisis in a framework in which the existence of classical solutions is guaranteed.

One way to get a suitable framework for solving elliptic boundary-value problems using finite elements is to pose them as variational problems. It is our goal in Chapter II to present the simplest possible introduction to this approach. In Sections 1 – 3 we discuss the existence of weak solutions in Sobolev spaces, and explain how the boundary conditions are incorporated into the variational calculation. To give the reader a feeling for the theory, we derive a number of properties of Sobolev spaces, or at least illustrate them. Sections 4 – 8 are devoted to the foundations of finite elements. The most difficult part of this chapter is §6 where approximation theorems are presented. To simplify matters, we first treat the special case of regular grids, which the reader may want to focus on in a first reading.

In Chapter III we come to the part of the theory of finite elements which requires deeper results from functional analysis. These are presented in §3. Among other things, the reader will learn about the famous Ladyshenskaja–Babuška–Brezzi condition, which is of great importance for the proper treatment of problems in fluid mechanics and for mixed methods in structural mechanics. In fact, without this knowledge and relying only on *common sense*, we would very likely find ourselves trying to solve problems in fluid mechanics using elements with an unstable behavior.

It was my aim to present this material with as little reliance on results from real analysis and functional analysis as possible. On the other hand, a certain basic knowledge is extremely useful. In Chapter I we briefly discuss the difference between the different types of partial differential equations. Students confronting the numerical solution of elliptic differential equations for the first time often find the finite difference method more accessible. However, the limits of the method usually become apparent only later. For completeness we present an elementary introduction to finite difference methods in Chapter I.

For fine discretizations, the finite element method leads to very large systems of equations. The operation count for solving them by direct methods grows like n^2. In the last two decades, very efficient solvers have been developed based on multigrid methods and on the method of conjugate gradients. We treat these subjects in detail in Chapters IV and V.

Structural mechanics provides a very important application area for finite elements. Since these kinds of problems usually involve systems of partial differential equations, often the elementary methods of Ch. II do not suffice, and we have to use the extra flexibility which the deeper results of Ch. III allow. I found it necessary to assemble a surprisingly wide set of building blocks in order to present a mathematically rigorous theory for the numerical treatment by finite elements of problems in linear elasticity theory.

Almost every section of the book includes a set of Problems, which are not only excercises in the strict sense, but also serve to further develop various formulae or results from a different viewpoint, or to follow a topic which would have disturbed the flow had it been included in the text itself. It is well-known that in the numerical treatment of partial differential equations, there are many opportunities to go down a false path, even if unintended, particularly if one is thinking in terms of classical solutions. Learning to avoid such pitfalls is one of the goals of this book.

This book is based on lectures regularly presented to students in the fifth through eighth semester at the Ruhr University, Bochum. Chapters I and II and parts of Chapters III and V were presented in one semester, while the method of conjugate gradients was left to another course. Chapter VI is the result of my collaboration with both mathematicians and engineers at the Ruhr University.

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