CHBE 552 Problem Set 1

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Question 1

Part a

$$\begin{aligned}
\mathbf{F_1} &= x_1^2 + x_1x_2 + x_1x_3 + x_1x_4 + x_1 \\
&+ x_2^2 + x_2x_3 + x_2x_4 + x_2 + x_3^2 + x_3x_4 \\
&+ x_3 + x_4^2 + x_4 + 1 \\
\mathbf{H_{F_1}} &= \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \\
\mathbf{F_2} &= 8x_1^2 + 4x_1x_2 + 5x_2^2 \\
\mathbf{H_{F_2}} &= \begin{bmatrix} 16 & 4 \\ 4 & 10 \end{bmatrix}
\end{aligned}$$

By implementating Newton's method, the corresponding minimums are found to be:

$$\begin{aligned}
\mathbf{x_{F1}} &= \begin{bmatrix} -0.2 \\ -0.2 \\ -0.2 \\ -0.2 \end{bmatrix} \text{ or } \begin{bmatrix} -0.2 \\ -0.2 \\ -0.2 \\ -0.2 \end{bmatrix} \\
\mathbf{x_{F2}} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

Note that for $\mathbf{F_1}$, two initial guesses are provided the value of $\mathbf{x_{F1}}$ shown above are derived from thoes two initial guesses and they are actually the same.

Part b

$$\mathbf{F} = 4\left(x_1 - 5\right)^2 + \left(x_2 - 6\right)^2$$
$$\nabla\mathbf{F} = [8x_1 - 40,\ 2x_2 - 12]$$

By implementating Fletcher Reeves's method, the minimum is found to be:

$$\mathbf{x} = \begin{bmatrix} 5.11 \\ 2.48 \end{bmatrix}$$

Part c

$$\mathbf{F} = 2x_1^2 + 2x_1x_2 + x_1 + x_2^2 - x_2$$
$$\nabla\mathbf{F} = [4x_1 + 2x_2 + 1,\ 2x_1 + 2x_2 - 1]$$

By implementating DFP method, the minimum is found to be:

$$\mathbf{x} = \begin{bmatrix} -0.65 \\ 0.98 \end{bmatrix}$$

Question 2

Part a

$$
\begin{aligned}
F &= x_1^2 + 10x_1 + x_2^2 + 20x_2 + 25 \\
L &= \lambda\left(x_1 + x_2\right) + x_1^2 + 10x_1 + x_2^2 + 20x_2 + 25 \\
\partial F_{x_1} &= \lambda + 2x_1 + 10 \\
\partial F_{x_2} &= \lambda + 2x_2 + 20 \\
\partial F_{\lambda} &= x_1 + x_2
\end{aligned}
$$

By setting those partial derivative equations to zero, the optimum values are found to be:

$$
\begin{aligned}
x_{1,opt} &= \frac{5}{2} \\
x_{2,opt} &= -\frac{5}{2} \\
\lambda_{opt} &= -15
\end{aligned}
$$

At this optimum point the original function is evaluated to be:

$$
F_{opt} = \frac{25}{2}
$$

For a sensitivity test the constraint condition is changed to $x_1 + x_2 = 0.01$, then the previous steps are repeated:

$$
\begin{aligned}
L_{new} &= \lambda\left(x_1 + x_2 - 0.01\right) + x_1^2 + 10x_1 + x_2^2 + 20x_2 + 25 \\
\partial F_{x_1} &= \lambda + 2x_1 + 10 \\
\partial F_{x_2} &= \lambda + 2x_2 + 20 \\
\partial F_{\lambda} &= x_1 + x_2 - 0.01 \\
x_{1,opt} &= 2.505 \\
x_{2,opt} &= -2.495 \\
\lambda_{opt} &= -15.01 \\
F_{new,opt} &= 12.65005
\end{aligned}
$$

Thus, the increment of the function value is computed to be

$$
\Delta F = 0.15
$$

Part b

$$
\begin{aligned}
F &= -\pi x_1^2 x_2 \\
L &= \lambda \left(2\pi x_1^2 + 2\pi x_1 x_2 - 24\pi\right) - \pi x_1^2 x_2 \\
\partial F_{x_1} &= \lambda \left(4\pi x_1 + 2\pi x_2\right) - 2\pi x_1 x_2 \\
\partial F_{x_2} &= 2\pi\lambda x_1 - \pi x_1^2 \\
\partial F_{\lambda} &= 2\pi x_1^2 + 2\pi x_1 x_2 - 24\pi \\
\text{Solution} &= [(2,\ 4)]
\end{aligned}
$$

Question 3

Question 4

$$\begin{aligned}
F &= y^2 + (x-2)^2 + (z-1)^2 \\
L &= \lambda\left(-4x^2 - 2y^2 + z^2\right) + y^2 + (x-2)^2 + (z-1)^2 \\
\partial F_x &= -8\lambda x + 2x - 4 \\
\partial F_y &= -4\lambda y + 2y \\
\partial F_z &= 2\lambda z + 2z - 2 \\
\partial F_\lambda &= -4x^2 - 2y^2 + z^2 \\
\mathbf{x} &= \left(\frac{4}{5},\ 0,\ \frac{8}{5}\right)
\end{aligned}$$

With $\lambda = -\frac{3}{8}$

Question 5

Part a

$$\begin{aligned}
F &= x_1^2 - 14x_1 + x_2^2 - 6x_2 - 7 \\
L &= \lambda_1 \left(x_1 + x_2 - 2\right) + \lambda_2 \left(x_1 + 2x_2 - 3\right) \\
&\quad + x_1^2 - 14x_1 + x_2^2 - 6x_2 - 7 \\
\partial F_x &= \lambda_1 + \lambda_2 + 2x_1 - 14 \\
\partial F_y &= \lambda_1 + 2\lambda_2 + 2x_2 - 6 \\
\partial F_{\lambda_1} &= x_1 + x_2 - 2 \\
\partial F_{\lambda_2} &= x_1 + 2x_2 - 3
\end{aligned}$$

Determined by KTC, the only optimal point can be found to be:

$$\mathbf{x} = \{x_1 : 3,\ x_2 : -1\}$$

With

$$\lambda_{1,2} = \{\lambda_1 : 8,\ \lambda_2 : 0\}$$

Note here the Lagrangian Multiplier for the second inequality constraint is determined to be zero, which indicates that it is inactive constraint. And the minimum objective function value is computed to be -33.

For the sufficient condition, the Hessian matrix evaluated at the optimal point is determined to be:

$$H^* = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

The corresponding eigen values are:

$$\lambda_{1,2} = \{2, 2\}$$

Since both eigen values are positve, the objective function is considered convex at the optimal point, thus, the optimal point is indeed a true global minimum.

Part b

$$
\begin{aligned}
F &= x_1^2 + 2\left(x_2 + 1\right)^2 \\
L &= \lambda_1\left(-x_1 + x_2 - 2\right) + \lambda_2\left(-x_1 - x_2 - 1\right) + x_1^2 + 2\left(x_2 + 1\right)^2 \\
\partial F_x &= -\lambda_1 - \lambda_2 + 2x_1 \\
\partial F_y &= \lambda_1 - \lambda_2 + 4x_2 + 4 \\
\partial F_{\lambda_1} &= -x_1 + x_2 - 2 \\
\partial F_{\lambda_2} &= -x_1 - x_2 - 1 \\
\mathbf{x} &= \left\{x_1 : -\frac{3}{2},\ x_2 : \frac{1}{2}\right\}
\end{aligned}
$$

With

$$\lambda_{1,2} = \left\{\lambda_1 : -\frac{9}{2},\ \lambda_2 : \frac{3}{2}\right\}$$

And the minimum objective function value is computed to be $\frac{27}{4}$.

For the sufficient condition, the Hessian matrix evaluated at the optimal point is determined to be:

$$H^* = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

The corresponding eigen values are:

$$\lambda_{1,2} = \{2, 4\}$$

Since both eigen values are positve, the objective function is considered convex at the optimal point, thus, the optimal point is indeed a true global minimum.

Question 6

$$
\begin{aligned}
F &= x_1^2 + x_1x_2 - 4x_1 + 1.5x_2^2 - 7x_2 - \log(x_1) - \log(x_2) + 9 \\
L_a &= x_1^2 + x_1x_2 - 4x_1 + 1.5x_2^2 - 7x_2 - \log(x_1) - \log(x_2) + 9 \\
L_b &= \lambda_1(-x_1x_2 + 4) + x_1^2 + x_1x_2 - 4x_1 + 1.5x_2^2 - 7x_2 - \log(x_1) - \log(x_2) + 9 \\
L_c &= \lambda_1(-x_1x_2 + 4) + \lambda_2 \cdot (2x_1 - x_2) + x_1^2 + x_1x_2 - 4x_1 + 1.5x_2^2 - 7x_2 - \log(x_1) - \log(x_2) +
\end{aligned}
$$

$$
\begin{aligned}
\partial L_{a,x_1} &= 2x_1 + x_2 - 4 - \frac{1}{x_1} \\
\partial L_{a,x_2} &= x_1 + 3.0x_2 - 7 - \frac{1}{x_2} \\
\partial L_{b,x_1} &= -\lambda_1x_2 + 2x_1 + x_2 - 4 - \frac{1}{x_1} \\
\partial L_{b,x_2} &= -\lambda_1x_1 + x_1 + 3.0x_2 - 7 - \frac{1}{x_2} \\
\partial L_{b,\lambda_1} &= -x_1x_2 + 4 \\
\partial L_{c,x_1} &= -\lambda_1x_2 + 2\lambda_2 + 2x_1 + x_2 - 4 - \frac{1}{x_1} \\
\partial L_{c,x_2} &= -\lambda_1x_1 - \lambda_2 + x_1 + 3.0x_2 - 7 - \frac{1}{x_2} \\
\partial L_{c,\lambda_1} &= -x_1x_2 + 4 \\
\partial L_{c,\lambda_2} &= 2x_1 - x_2
\end{aligned}
$$

Note that since the partial derivative equations are not linear anymore, the following results are computed numerically. For part a:

$$\mathbf{x_a} = \begin{bmatrix} 1.34754858228762 \\ 2.04699110826639 \end{bmatrix}$$

For part b:

$$\mathbf{x_b} = \begin{bmatrix} 1.79811994048488 \\ 2.22454571018291 \end{bmatrix}$$

With

$$\lambda_1 = \begin{bmatrix} 0.568497719699798 \end{bmatrix}$$

For part c:

$$\mathbf{x_c} = \begin{bmatrix} 1.4142135623731 \\ 2.82842712474619 \end{bmatrix}$$

With

$$\lambda_{1,2} = \begin{bmatrix} 1.06801948466054 \\ 1.03553390593274 \end{bmatrix}$$

And the minimum objective function value is computed to be:

$$\begin{aligned} F_a &= -0.874224083186354 \\ F_b &= -0.494453348930655 \\ F_c &= 0.157861516164399 \end{aligned}$$

For the sufficient condition, the Hessian matrixs for each condition evaluated at the optimal point are determined to be:

$$\begin{aligned} H_a^* &= \begin{bmatrix} 2.55069500468907 & 1 \\ 1 & 3.23865365370371 \end{bmatrix} \\ H_b^* &= \begin{bmatrix} 2.30928772604332 & 1 \\ 1 & 3.20207720752309 \end{bmatrix} \\ H_c^* &= \begin{bmatrix} 2.5 & 1 \\ 1 & 3.125 \end{bmatrix} \end{aligned}$$

The corresponding eigen values are:

$$\begin{aligned} \lambda_{a,1,2} &= \{1.8371669884708, 3.95218166992198\} \\ \lambda_{b,1,2} &= \{1.66057139270888, 3.85079354085753\} \\ \lambda_{c,1,2} &= \{1.76480908660999, 3.86019091339001\} \end{aligned}$$

Since all eigen values for each case are positve, the objective function is considered convex at the optimal points, thus, the optimal points are indeed true global minimum.