MECH 570C

Fluid-Structure Interaction

Module 5: Variational FSI Formulation (Part 1)

Computational Mechanics of FSI

→ Review : Stabilization
↓
Convection - diffusion
+
Incompressibility

→ Time integration :

- Trapezoidal / mid-point
- Newmark - family methods
  (Generalized - α technique)

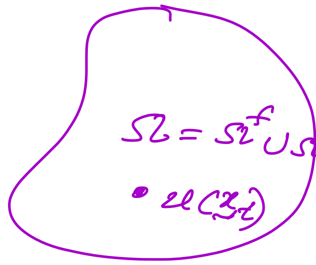
→ Semi - discrete formulation
  → Fluid
  → solid
  Coding project 2

→ Combined FSI system
  → Partitioned
  → Monolithic

Generic differential system:

$$\frac{d^2}{dx^2}\{u\} = f \qquad\qquad \mathcal{L}\,\underset{\sim}{u} = f$$

$\mathcal{L} \equiv \frac{d^2}{dx^2}$ $\qquad$ differential operator



$$\mathcal{L} = \mathcal{L}_{adv} + \mathcal{L}_{diff}$$

$$\mathcal{L}_{adv} = c\frac{d}{dx}$$

$$\mathcal{L}_{diff} = -\kappa\frac{d^2}{dx^2}$$

Consider steady 1D convection - diffusion

$$\underbrace{c\frac{du}{dx}}_{\text{convection}} - \underbrace{\kappa\frac{d^2u}{dx^2}}_{\text{diffusion}}$$

$$\Rightarrow \left(c\frac{d}{dx} - \kappa\frac{d^2}{dx^2}\right) u = f$$

unknown function

Weak form / variational form:

$$\int \psi\,(\mathcal{L}u - f)\,d\Omega = 0$$

Test function $\qquad$ Residual

Weighted-residual form / Galerkin projection

$$\psi = N$$

$$u = \sum N_i u_i$$

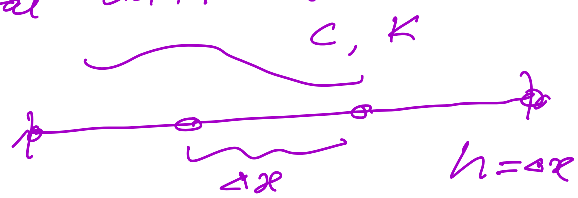
$$\sum N_i = 1$$

The finite element method:

$$[K]\{u_i\} = \{RHS\}_i$$

Galerkin method w/ uniform grid

= Central difference



Peclet # $\quad Pe = \dfrac{c\,h}{2\kappa}$

$Pe > 1$

$Pe \leq 1$

Petrov-Galerkin :

$$\psi = N$$

$$\psi^* = N^* = N + \tau \underbrace{\mathcal{L}_{adv}}_{} N$$

$$= N + \tau \, c \frac{d}{dx} N$$

$$\int_\Omega N (\mathcal{L}u - f) \, d\Omega + \sum_{e=1}^{nel} \tau \mathcal{L}_{adv} N (\mathcal{L}u - f) = 0$$

$$\Downarrow$$

$$\int_\Omega \underbrace{(N + \tau \mathcal{L}_{adv} N)}_{} (\mathcal{L}u - f) \, d\Omega = 0$$

Petrov-Galerkin Stabilized

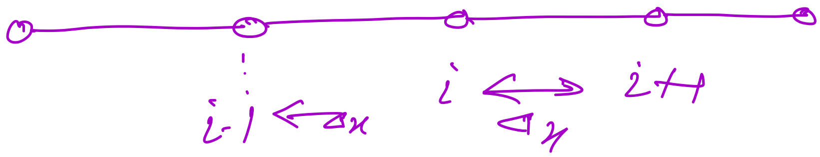
τ is stabilization parameter which needs to be determined from "analytical" solutions!

$$\tau_e = \frac{h_e}{2c} \left[\coth(Pe) - \frac{1}{Pe_e} \right]$$

What is τ ?

$$c\frac{\partial u}{\partial x} - \kappa \frac{\partial^2 u}{\partial x^2} = 0$$

$$u(x) = C_1 e^{cx/\kappa} + C_2$$



$$\tau = \frac{\Delta x}{2c} \underbrace{\frac{\left(e^{c\Delta x/\kappa}+1\right)}{\left(e^{c\Delta x/\kappa}-1\right)}}_{\coth(x)} - \frac{\kappa}{c^2}$$

$$Pe = \frac{c\Delta x}{2\kappa}$$

$$\rightarrow \tau = \frac{\Delta x}{2c}\left[\coth(Pe) - \frac{1}{Pe}\right]$$

$$\Downarrow$$

$$\tau = \left[\left(\frac{2c}{\Delta x}\right)^2 + \left(\frac{4\kappa}{\Delta x^2}\right)^2\right]^{-1/2}$$

Review: $\frac{d}{dt}$ (amount) = R.H.S

$u(\underset{\sim}{x}, t)$

$$\frac{\partial u}{\partial t} + \mathcal{L} u = f$$

partial

$$\Downarrow$$

$$\frac{d}{dt}\{u\} = -Res$$

Total derivative

$$\Downarrow$$

$$Res = \mathcal{L}u - f$$

$$t^n \rightarrow t^{n+1}$$

$$\frac{du}{dt} = \underset{\sim}{R}(u, t)$$

Integral

Taylor

$$u^{n+1} = u^n + \int_{t^n}^{t^{n+1}} R(\tau, u)\, d\tau$$

Forward/backward

Trapezoidal

Newmark

Newmark / Generalized α method
for second-order system

$$\underbrace{M}_{\text{Mass matrix}}\ddot{u} + \underbrace{C}_{\text{damping matrix}}\dot{u} + \underbrace{Ku}_{f^{int}(u)} = f$$

(I)

(*) $$\dot{u}_{n+1} = \dot{u}_n + \Delta t\, \ddot{u}_\gamma$$

$$0 \le \gamma \le 1$$

$$\rightarrow \ddot{u}_\gamma = (1-\gamma)\ddot{u}_n + \gamma\, \ddot{u}_{n+1}$$

$$\rightarrow \dot{u}_{n+1} = \dot{u}_n + (1-\gamma)\Delta t\, \ddot{u}_n + \gamma \Delta t\, \ddot{u}_{n+1}$$

u: disp, $\dot{u}$: vel, $\ddot{u}$: accel

(II)

$$u_{n+1} = u_n + \Delta t\, \dot{u}_n + \frac{\Delta t^2}{2} u_\beta$$

disp @ n

$$\ddot{u}_\beta = (1-2\beta)\,\ddot{u}_n + 2\beta\, \ddot{u}_{n+1}$$

Three steps $\qquad 0 \le 2\beta \le 1$

① $$\dot{u}_{n+1} = \dot{u}_n + (1-\gamma)\Delta t\, \ddot{u}_n + \gamma \Delta t\, \ddot{u}_{n+1}$$

[velocity update

② $$u_{n+1} = u_n + \Delta t\, \dot{u}_n + \frac{\Delta t^2}{2}\left[(1-2\beta)\,\ddot{u}_n + 2\beta\, \ddot{u}_{n+1}\right]$$

③ $$M\ddot{u}_{n+1} + C\dot{u}_{n+1} + K u_{n+1} = f_{n+1}^{\boxed{ext}}$$

Case 1 : $\gamma = 0.5$, $\beta = 0$

$$\dot{u}_{n+1} = \dot{u}_n + \frac{1}{2}\Delta t(\ddot{u}_n + \ddot{u}_{n+1})$$

Case 2 : $\gamma = 0.5$, $\beta = 0.25$

Mid-point rule (average constant accel)

Newmark - Beta method
$\Downarrow$

Variational FSI Formulation

0. Recap

$$\mathcal{L}u = f$$

→ Weak form

→ Linearization (matrix form)

→ Stabilization

→ Time integration → Newmark-β $\frac{\partial^2}{\partial t^2}$; → Generalized-α $\frac{\partial}{\partial t}$

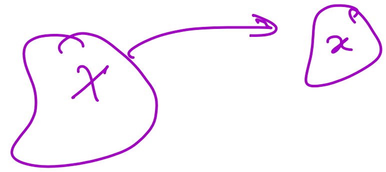
1. Coupled FSI System

→ Fluid part

→ Solid part →

→ Mesh motion (ALE treatment)

Review:



Linear elasticity:

$$\underset{\sim}{\sigma}^s = \lambda \, tr(\underset{\sim}{\epsilon}^s) + 2\mu \, \underset{\sim}{\epsilon}^s$$

Strain:

$$\rightarrow \quad \underset{\sim}{\epsilon}^s = \left(\frac{\partial u_i^s}{\partial x_j} + \frac{\partial u_j^s}{\partial x_i} \right)$$

$$\rho^s \frac{\partial^2 \underset{\sim}{u}^s}{\partial t^2} = \nabla \cdot \sigma^s + f_b^s$$

Nonlinear Elasticity:

material coeff

$$W^{int} = \frac{1}{2} \underset{\sim}{E} : C : \underset{\sim}{E}$$

Green-Lagrange strain

$$S_{ij} = \frac{\partial W^{int}}{\partial E_{ij}}$$

St. Venant Kirchoff Model:

$$\underset{\sim}{S} = \underset{\sim}{C} : \underset{\sim}{E}$$

$$= \lambda \, tr(\underset{\sim}{E}) \, I + 2\mu \, \underset{\sim}{E}$$

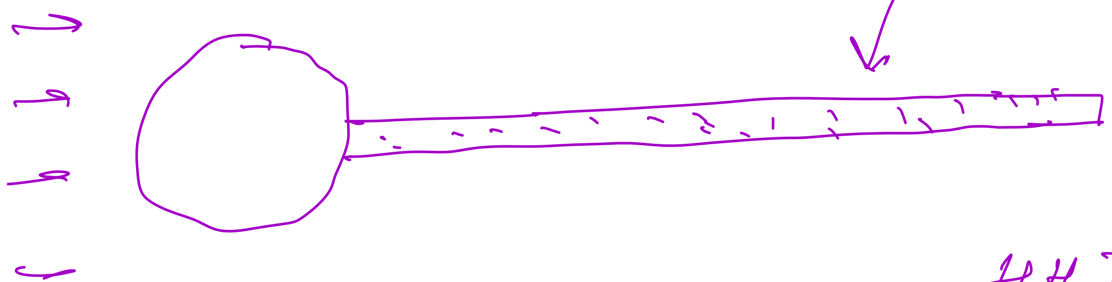
$$\underset{\sim}{E} = \frac{1}{2}\left(\underset{\sim}{F}^T \underset{\sim}{F} - I\right)$$

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \cancel{\frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j}}\right)$$

If ignoring quadratic term or assuming linear strain

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

St. Venant Kirchoff
model for solid !



$$t^n \longrightarrow t^{n+1}$$

HHT-α

Hilber Hughes
Taylor

$$(*) \quad M \ddot{\tilde{u}}^s_{n+\alpha} + f^{int}(u^s_{n+\alpha_m}) = f^{ext}(u_{n+\alpha})$$

$$K(u^s_{n+\alpha})\, u^s_d \longrightarrow$$ Linearized
L.H.S
(tangent matrix)

$$\begin{cases} u_{n+\alpha^s} = (1-\alpha^s)\, \underset{\sim}{u}_n + \alpha^s \underset{\sim}{u}_{n+1} \\ \ddot{u}_{n+\alpha_m^s} = (1-\alpha_m^s)\, \ddot{u}_n + \alpha_m^s \ddot{u}_{n+1} \end{cases}$$

$$\alpha^s = \frac{1}{1+\rho_\infty^s}, \quad \alpha_m^s = \frac{2-\rho_\infty^s}{1+\rho_\infty^s}$$

$$\boxed{\rho_\infty} \in [0, 1]$$

Spectral radius / high frequency damping factor

$$\gamma = \frac{1}{2} + \alpha_m^s - \alpha^s, \quad \beta = \frac{1}{4}(1+\alpha_m^s - \alpha^s)^2$$

Newton – Raphson $n \to n+1$

$$\underset{\sim}{R}_{n+1} = f^{ext}(\underset{\sim}{u}_{n+1}) - \frac{1}{\beta \Delta t^2}[M]\underset{\sim}{u}_{n+1}$$

$\Downarrow$

$$- \underset{\sim}{f}^{int}(\underset{\sim}{u}_{n+1})$$

$$+ M\left[\frac{1}{\beta \Delta t^2} u_n + \frac{1}{\beta \Delta t}\dot{u}_n + \left(\frac{1}{2\beta} - 1\right)\ddot{u}_n\right]$$

Iterative loop $i-1 \to i$

Expand residual about the known solution u^{i-1}_{n+1} (Taylor series)

$$\underset{\sim}{R}^{i}_{n+1} \approx \underset{\sim}{R}^{i-1}_{n+1} + \left.\frac{\partial \underset{\sim}{R}_{n+1}}{\partial u_{n+1}}\right|^{i-1} \cdot \Delta u^{i}_{n+1} = 0$$

Tangent stiffness matrix :

$$\text{L.H.S} \rightarrow K^T \equiv \frac{\partial \underset{\sim}{R}}{\partial \underset{\sim}{u}}$$

$\underset{\sim}{R}$ ← residual vector

$\underset{\sim}{u}$ ← solution vector

old to new

$$\underbrace{\left[K^T \right]^{i-1}_{n+1}} \cdot \underbrace{\Delta u^{i}_{n+1}} = - \underset{\sim}{R}^{i-1}_{n+1} \leftarrow \text{old}$$

If the procedure converges, the residual is gradually (quadratic) reduced to zero

$$\underset{\sim}{u}^{i}_{n+1} = \underset{\sim}{u}^{i-1}_{n+1} + \Delta u^{i}_{n+1}$$

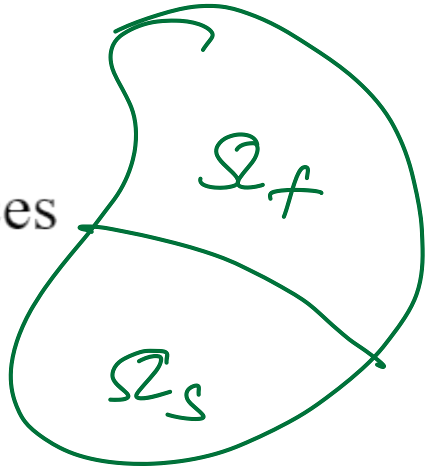
Overview

- In the previous chapter, we studied some of the basics of the finite element variational formulation for a transient convection-diffusion-reaction equation, which forms a canonical form for the nonlinear fluid-structure interaction equations.
- In this chapter, we apply the variational formulation to the separate equations governing the fluid and structural domains
- We present the variational form of the Navier-Stokes equations written in ALE framework for the fluid and the structural dynamics equation written in the Lagrangian coordinate system along with the boundary conditions mentioned

Weak/Variational Form for Fluid-Structure Interaction

- Trial and Test Function Spaces

∇^2



For the fluid equation, we define the following spaces

∇p

$$
\left\{
\begin{aligned}
\mathscr{V}_{\psi^{\mathrm{f}}} &= \{\psi^{\mathrm{f}} \in H^1(\Omega^{\mathrm{f}}(t)) \mid \psi^{\mathrm{f}} = 0 \text{ on } \Gamma^{\mathrm{f}}_D\} \\
\mathscr{V}_q &= \{q \in L^2(\Omega^{\mathrm{f}}(t))\} \\
\mathscr{S}_{\boldsymbol{v}^{\mathrm{f}}} &= \{\boldsymbol{v}^{\mathrm{f}} \in H^1(\Omega^{\mathrm{f}}(t)) \mid \boldsymbol{v}^{\mathrm{f}} = \boldsymbol{v}^{\mathrm{f}}_D \text{ on } \Gamma^{\mathrm{f}}_D\} \\
\mathscr{S}_p &= \{p \in L^2(\Omega^{\mathrm{f}}(t))\}
\end{aligned}
\right.
$$

where $\mathscr{V}_{\psi^{\mathrm{f}}}$ and $\mathscr{V}_q$ denote the test function spaces for the momentum and continuity equations respectively and $\mathscr{S}_{v^{\mathrm{f}}}$ and $\mathscr{S}_p$ denote the spaces from where we select the trial solution for velocity and pressure respectively.

Least square (Banach space) $\subset$ Hilbert $\int (\,)(\,) d\Omega$ $\subset$ Sobolev space

Weak/Variational Form for Fluid-Structure Interaction

❑ Trial and Test Function Spaces

Similarly, for the structural equation, we define the following function spaces

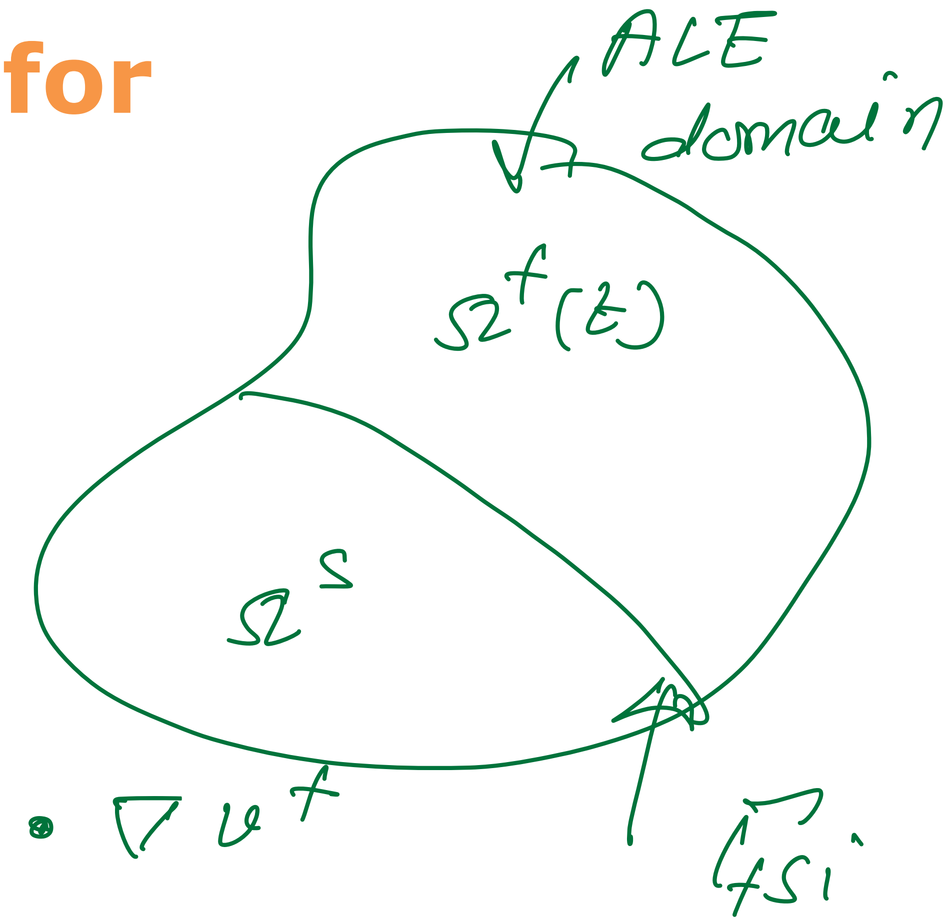
$$\mathscr{V}_{\psi^{\mathrm{s}}} = \left\{ \psi^{\mathrm{s}} \in H^1(\Omega^{\mathrm{s}}) \mid \psi^{\mathrm{s}} = 0 \text{ on } \Gamma_D^{\mathrm{s}} \right\}$$
$$\mathscr{S}_{v^{\mathrm{s}}} = \left\{ v^{\mathrm{s}} \in H^1(\Omega^{\mathrm{s}}) \mid v^{\mathrm{s}} = v_D^{\mathrm{s}} \text{ on } \Gamma_D^{\mathrm{s}} \right\}$$

where $\mathscr{V}_{\psi^{\mathrm{s}}}$ and $\mathscr{S}_{v^{\mathrm{s}}}$ denote the test function and trial solution spaces for the structural velocity respectively. For the mesh equation, the following spaces are defined

$$\mathscr{V}_{\psi^{\mathrm{m}}} = \left\{ \boldsymbol{\psi}^{\mathrm{m}} \in H^1(\Omega^{\mathrm{f}}) \mid \psi^{\mathrm{m}} = 0 \text{ on } \Gamma_D^{\mathrm{m}} \right\}$$
$$\mathscr{S}_{\boldsymbol{u}^{\mathrm{f}}} = \left\{ \boldsymbol{u}^{\mathrm{f}} \in H^1(\Omega^{\mathrm{f}}) \mid \boldsymbol{u}^{\mathrm{f}} = \boldsymbol{u}_D^{\mathrm{f}} \text{ on } \Gamma_D^{\mathrm{m}} \right\}$$

Now that we have the appropriate spaces to select the weighting functions and trial solution, we are ready to form the weak form of the flow and structural equations.

Weak Formulation for FSI



Mom :

mesh velocity

$$\int_{\Omega^f(t)} \Big[\rho^f \frac{\partial v^f}{\partial t}\Big|_{\chi} + \rho^f \big(\underset{\sim}{v}^f - \underset{\sim}{w}\big) \cdot \nabla v^f - \nabla \cdot \sigma^f \Big) \cdot \psi^f \, d\Omega = \int_{\Omega^f(t)} \rho^f b^f \cdot \psi^f \, d\Omega$$

Test function

Weak Formulation for FSI

unknown

$$\left\{ \begin{matrix} \psi^f \longrightarrow v^f \\ q \longrightarrow p \end{matrix} \right\}$$

Continuity:

$$\int_{\Omega^f(t)} (\nabla \cdot v^f)\, q \, d\Omega = 0$$

scalar test function

$$\sigma^f = -pI + \mu\left(\nabla v^f + (\nabla v^f)^T\right)$$

Weak Formulation for FSI

Goal is to find $\left[v_n^f , \phi \right]$

for given $\underbrace{\left[\psi^f , z \right]}_{\Uparrow}$

Weak Formulation for FSI

Proceeding in a similar way for the structural equation, we can write the variational statement as: find $v^s \in \mathscr{S}_{v^s}$ such that $\forall \psi^s \in \mathscr{V}_{\psi^s}$

$$\int_{\Omega^s} \left(\rho^s \frac{\partial v^s}{\partial t}\right) \cdot \psi^s d\Omega + \int_{\Omega^s} \sigma^s : \nabla \psi^s d\Omega$$
$$= \int_{\Omega^s} \rho^s b^s \cdot \psi^s d\Omega + \int_{\Gamma^s_N} \sigma^s_N \cdot \psi^s d\Gamma + \int_{\Gamma} (\sigma^s \cdot n^s) \cdot \psi^s d\Gamma$$

Solid part

Similarly for the mesh equation, we get: find $u^f \in \mathscr{S}_{\boldsymbol{u}}$ f such that $\forall \boldsymbol{\psi}^m \in \mathscr{V}_{\psi^m}$,

$$\int_{\Omega^f} \nabla \psi^m : \sigma^m d\Omega = 0$$
$$\text{where } \sigma^m = (1 + k_m)\left[\nabla u^f + \left(\nabla u^f\right)^T + (\nabla \cdot u^f) I\right]$$

mesh for fluid domain

⇒ Variational/weak form of Coupled FSI system

Semi-Discrete Temporal Discretization

- As the independent variables in the FSI equations are the spatial and temporal coordinates, we need to discretize the domain in both space and time.
- One way of doing this is to define the trial and test function spaces in the weak form such that they are space and time dependent.
- Such methods lead to what are called space-time finite element methods. In the present case, we will utilize a simpler approach of discretizing the temporal variable by Taylor series (similar to finite difference methods) such that the trial and test function spaces have only spatial dependence.

Generalized-α Time Integration

Here, we discuss the generalized- α predictor-corrector method for the temporal discretization. It enables a user-controlled high frequency damping via a single parameter called the spectral radius ρ_∞, which allows for a coarser discretization in time. We solve the equation at the time interval $\mathrm{n}+\alpha$ while integrating the equation from n to $n+1$ in time. Suppose the equation can be written as $G(\partial_t u^{\mathrm{n}+\alpha_\mathrm{m}}, u^{\mathrm{n}+\alpha}, \phi^{\mathrm{n}+\alpha}) = 0$, where $G(\cdot)$ can be a nonlinear function of the variable ϕ and its first and second-order derivatives, i.e., $u = \partial_t \phi$ and $\partial_t u = \partial_{tt}^2 \phi$, respectively. The expressions for the variables and their derivatives in the equation can be written as

Time-Integration Parameters

The predictor-corrector algorithm within a time step from n to $\mathrm{n}+1$ consists of the following steps:
1. Predict the variables at $\mathrm{n}+1$.
2. Evaluate the variables at the intermediate time interval $\mathrm{n}+\alpha$.
3. Linearize the nonlinear equation $G(\partial_t u^{\mathrm{n}+\alpha_{\mathrm{m}}}, u^{\mathrm{n}+\alpha}, \phi^{\mathrm{n}+\alpha}) = 0$ with the help of Newton-Raphson method.
4. Calculate the increments in the variables by solving the linear system of equations.
5. Update the variables at $\mathrm{n}+\alpha$.
6. Correct the variables at $\mathrm{n}+1$ and proceed to the next nonlinear iteration.

Semi-Discrete Temporal Discretization Applied to FSI

Finite Element Space Discretization for FSI

The next step is to select finite element spaces $\mathscr{V}^{\mathrm{h}}$ and $\mathscr{S}^{\mathrm{h}}$ from the space of test and trial functions $\mathscr{V}$ and $\mathscr{S}$ respectively. We discretize the domain Ω into nonintersecting finite elements $\Omega = \cup_{\rho=1}^{\mathrm{n_{el}}} \Omega^e$. The discrete spaces for trial and test functions for the fluid equations are defined as

$$
\begin{aligned}
\mathscr{V}_{\boldsymbol{\psi}^{\mathrm{f}}}^{\mathrm{h}} &= \left\{ \boldsymbol{\psi}_{\mathrm{h}}^{\mathrm{f}} \in H^1\left(\Omega^{\mathrm{f}}\left(t^{\mathrm{n+1}}\right)\right) \middle| \psi_{\mathrm{h}}^{\mathrm{f}}\big|_{\Omega^e} \in \mathbb{P}_m(\Omega^e) \forall e \text{ and } \psi_{\mathrm{h}}^{\mathrm{f}} = 0 \text{ on } \Gamma_D^{\mathrm{f}} \right\} \\
\mathscr{V}_q^{\mathrm{h}} &= \left\{ q_{\mathrm{h}} \in L^2\left(\Omega^{\mathrm{f}}\left(t^{\mathrm{n+1}}\right)\right) | q_{\mathrm{h}}|_{\Omega^e} \in \mathbb{P}_m(\Omega^e) \forall e \right\} \\
\mathscr{S}_{\boldsymbol{v}^{\mathrm{f}}}^{\mathrm{h}} &= \left\{ \boldsymbol{v}_{\mathrm{h}}^{\mathrm{f}} \in H^1\left(\Omega^{\mathrm{f}}\left(t^{\mathrm{n+1}}\right)\right) \middle| v_{\mathrm{h}}^{\mathrm{f}}\big|_{\Omega^e} \in \mathbb{P}_m(\Omega^e) \forall e \text{ and } v_{\mathrm{h}}^{\mathrm{f}} = v_D^{\mathrm{f}} \text{ on } \Gamma_D^{\mathrm{f}} \right\} \\
\mathscr{S}_p^{\mathrm{h}} &= \left\{ p_{\mathrm{h}} \in L^2\left(\Omega^{\mathrm{f}}\left(t^{\mathrm{n+1}}\right)\right) | p_{\mathrm{h}}|_{\Omega^e} \in \mathbb{P}_m(\Omega^e) \forall e \right\}
\end{aligned}
$$

where $\mathbb{P}_m(\Omega^e)$ is the space of polynomials of degree $\leq m$ The finite element variational statement for the element Ω^e for the fluid equations can thus be written as: find $\left[v_{\mathrm{h}}^{\mathrm{f,n}+\alpha^{\mathrm{f}}}, p_{\mathrm{h}}^{\mathrm{n+1}}\right] \in \mathscr{S}_{v^{\mathrm{f}}}^{\mathrm{h}} \times \mathscr{S}_p^{\mathrm{h}}$ such that $\forall \left[\psi_{\mathrm{h}}^{\mathrm{f}}, q_{\mathrm{h}}\right] \in \mathscr{V}_{\psi^{\mathrm{f}}}^{\mathrm{h}} \times$
$\mathscr{V}_q^{\mathrm{h}}$

$$
\begin{aligned}
&\int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} \left(\rho^{\mathrm{f}} \partial_t v_{\mathrm{h}}^{\mathrm{f,n}+\boxed{\alpha_{\mathrm{m}}^{\mathrm{f}}}}\Big|_{\chi} + \rho^{\mathrm{f}} \left(v_{\mathrm{h}}^{\mathrm{f,n}+\alpha^{\mathrm{f}}} - w \right) \cdot \nabla v_{\mathrm{h}}^{\mathrm{f,n}+\alpha^{\mathrm{f}}} \right) \cdot \psi_{\mathrm{h}}^{\mathrm{f}} d\Omega \\
&+ \int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} \sigma_{\mathrm{h}}^{\mathrm{f,n}+\alpha^{\mathrm{f}}} : \nabla \psi_{\mathrm{h}}^{\mathrm{f}} d\Omega + \int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} \left(\nabla \cdot v_{\mathrm{h}}^{\mathrm{f,n}+\alpha^{\mathrm{f}}} \right) q_{\mathrm{h}} d\Omega \\
&= \int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} \rho^{\mathrm{f}} b_{\mathrm{h}}^{\mathrm{f,n}+\alpha^{\mathrm{f}}} \cdot \psi_{\mathrm{h}}^{\mathrm{f}} d\Omega + \int_{\Gamma_N^{\mathrm{f}}} \sigma_N^{\mathrm{f,n}+\boxed{\alpha^{\mathrm{f}}}} \cdot \psi_{\mathrm{h}}^{\mathrm{f}} d\Gamma + \int_{\Gamma(\mathrm{n+1})} \left(\sigma_{\mathrm{h}}^{\mathrm{f,n}+\alpha^{\mathrm{f}}} \cdot n^{\mathrm{f}} \right) \cdot \psi_{\mathrm{h}}^{\mathrm{f}} d\Gamma
\end{aligned}
$$

Finite Element Space Discretization for FSI

Finite Element Space Discretization for FSI

$$+\int_{\Omega^{\mathrm{f}}(t^{\mathrm{n}+1})}\left(\nabla\cdot\boldsymbol{v}_{\mathrm{h}}^{\mathrm{f,n}+\alpha^{\mathrm{f}}}\right)q_{\mathrm{h}}d\Omega+\sum_{e=1}^{\mathrm{nel}}\int_{\Omega^e}\nabla\cdot\psi_{\mathrm{h}}^{\mathrm{f}}\tau_c\boldsymbol{R}_c d\Omega^e$$

$$=\int_{\Omega^{\mathrm{f}}(t^{\mathrm{n}+1})}\rho^{\mathrm{f}}b_{\mathrm{h}}^{\mathrm{f,n}+\alpha^{\mathrm{f}}}\cdot\psi_{\mathrm{h}}^{\mathrm{f}}d\Omega+\int_{\Gamma_N^{\mathrm{f}}}\sigma_N^{\mathrm{f,n}+\alpha^{\mathrm{f}}}\cdot\psi_{\mathrm{h}}^{\mathrm{f}}d\Gamma+\int_{\Gamma(t^{\mathrm{n}+1})}\left(\sigma_{\mathrm{h}}^{\mathrm{f,n}+\alpha^{\mathrm{f}}}\cdot n^{\mathrm{f}}\right)\cdot\psi_{\mathrm{h}}^{\mathrm{f}}d\Gamma$$

where the second line represents the stabilization term for the momentum equation and the second term in the third line depicts the same for the continuity equation. $\boldsymbol{R}_m$ and $\boldsymbol{R}_c$ are the residual of the momentum and continuity equations respectively. The stabilization parameters τ_m and τ_c are the least-squares metrics added to the element-level integrals defined as

$$\tau_m=\left[\left(\frac{2\rho^{\mathrm{f}}}{\Delta t}\right)^2+\left(\rho^{\mathrm{f}}\right)^2\left(v_{\mathrm{h}}^{\mathrm{f,n}+\alpha^{\mathrm{f}}}-w\right)\cdot G\left(v_{\mathrm{h}}^{\mathrm{f,n}+\alpha^{\mathrm{f}}}-w\right)+C_I\left(\mu^{\mathrm{f}}\right)^2 G:G\right]^{-1/2}$$

$$\tau_c=\frac{1}{8\,\mathrm{tr}(G)\tau_m}$$

where C_I is a constant derived from inverse estimates $\mathbf{tr}()$ denotes the trace and G is the contravariant metric tensor given by

$$\boxed{G=\left(\frac{\partial\xi}{\partial x}\right)^T\frac{\partial\xi}{\partial x}}$$

Galerkin Structure and Mesh Motion

Coming to the structural equation, we define the finite spaces as follows:

$$\mathscr{V}_{\psi^{\mathrm{s}}}^{\mathrm{h}} = \left\{\psi_{\mathrm{h}}^{\mathrm{S}} \in H^1(\Omega^{\mathrm{s}}) \middle| \psi_{\mathrm{h}}^{\mathrm{S}}\big|_{\Omega^e} \in \mathbb{P}_m(\Omega^e) \forall e \text{ and } \psi_{\mathrm{h}}^{\mathrm{s}} = 0 \text{ on } \Gamma_D^{\mathrm{s}}\right\}$$

$$\mathscr{S}_{v^{\mathrm{s}}}^{\mathrm{h}} = \left\{v_{\mathrm{h}}^{\mathrm{s}} \in H^1(\Omega^{\mathrm{s}}) \middle| v_{\mathrm{h}}^{\mathrm{S}}\big|_{\Omega^e} \in \mathbb{P}_m(\Omega^e) \forall e \text{ and } v_{\mathrm{h}}^{\mathrm{s}} = v_D^{\mathrm{s}} \text{ on } \Gamma_D^{\mathrm{s}}\right\}$$

and the finite element variational statement is written as: find $v_h^{s,n+\alpha^s} \in \mathscr{S}_{v^s}^h$ such that $\forall \psi_h^s \in \mathscr{V}_{\psi^s}^h$

$$\int_{\Omega^{\mathrm{s}}} \left(\rho^{\mathrm{s}} \partial_t v_{\mathrm{h}}^{\mathrm{s,n}+\alpha_{\mathrm{m}}^{\mathrm{s}}}\right) \cdot \psi_{\mathrm{h}}^{\mathrm{s}} d\Omega + \int_{\Omega^{\mathrm{s}}} \sigma_{\mathrm{h}}^{\mathrm{s,n}+\alpha^{\mathrm{s}}} : \nabla \psi_{\mathrm{h}}^{\mathrm{s}} d\Omega$$

$$= \int_{\Omega^{\mathrm{s}}} \rho^{\mathrm{s}} b_{\mathrm{h}}^{\mathrm{s,n}+\alpha^{\mathrm{s}}} \cdot \psi_{\mathrm{h}}^{\mathrm{s}} d\Omega + \int_{\Gamma_N^{\mathrm{s}}} \sigma_N^{\mathrm{s,n}+\alpha^{\mathrm{s}}} \cdot \psi_{\mathrm{h}}^{\mathrm{s}} d\Gamma + \int_{\Gamma} \left(\sigma_{\mathrm{h}}^{\mathrm{s,n}+\alpha^{\mathrm{s}}} \cdot \boldsymbol{n}^{\mathrm{s}}\right) \cdot \psi_{\mathrm{h}}^{\mathrm{s}} d\Gamma$$

Similarly for the mesh equation, we define the finite element spaces and the variational statement is written as: find $u_{\mathrm{h}}^{\mathrm{f,n+1}} \in \mathscr{S}_{u^{\mathrm{f}}}^{\mathrm{h}}$ such that $\forall \psi_{\mathrm{h}}^{\mathrm{m}} \in \mathscr{V}_{\psi^{\mathrm{m}}}^{\mathrm{h}}$

$$\int_{\Omega^{\mathrm{f}}} \nabla \psi_{\mathrm{h}}^{\mathrm{m}} : \sigma_{\mathrm{h}}^{\mathrm{m,n+1}} d\Omega = 0$$

Matrix Form of the Linear System of Equations

Momentum:

$$M^f \dot{v}^f_{n+1} + K_f^f (v^f_{n+1})\, v_{n+1} - G^f p^{n+1} = f_{n+1}$$

$v^f \cdot \nabla v^f$

∇p

LHS matrix

Gradient

Continuity: $\nabla \cdot v^f_{n+1} = 0$

$$(G^T)\, v_{n+1} = 0$$

$$\begin{bmatrix} \frac{M}{\Delta t} + K^f & -G^f \\ (G^f)^T & C^f \end{bmatrix} \begin{Bmatrix} \Delta \underset{\sim}{v}^f \\ \Delta \phi \end{Bmatrix} = \begin{Bmatrix} R_m^f \\ R_c \end{Bmatrix}$$

$\uparrow$ Stabilization matrix

$\uparrow$ $\nabla \cdot v^f = 0$

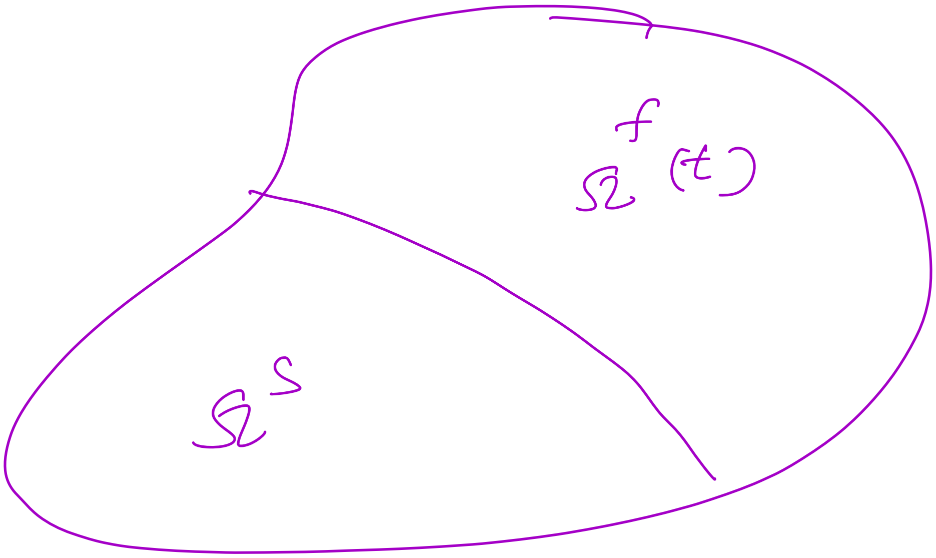
Matrix Form of the Linear System of Equations

Similarly, for the structural and mesh equation, we get the following matrix forms:

$$\left(\boldsymbol{K}^{\mathrm{s}}\right)\left(\Delta \underline{\boldsymbol{v}}_{\mathrm{h}}^{\mathrm{s,n}+\alpha^{\mathrm{s}}}\right) = \left(\boldsymbol{F}^{\mathrm{s}}\right)$$

$$\left(\boldsymbol{K}^{\mathrm{m}}\right)\left(\Delta \underline{\underline{u}}_{\mathrm{h}}^{\mathrm{f,n}+1}\right) = (0)$$

where K^{s} and K^{m} are the linearized matrices for the respective equations.



Unified block (monolithic)

$$\boxed{\Omega^f + \Omega^s}$$

mono- single
lithic: ROCK

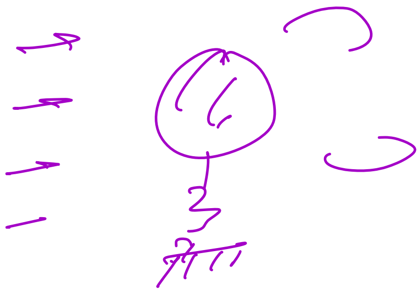
Partitioned/ Split form

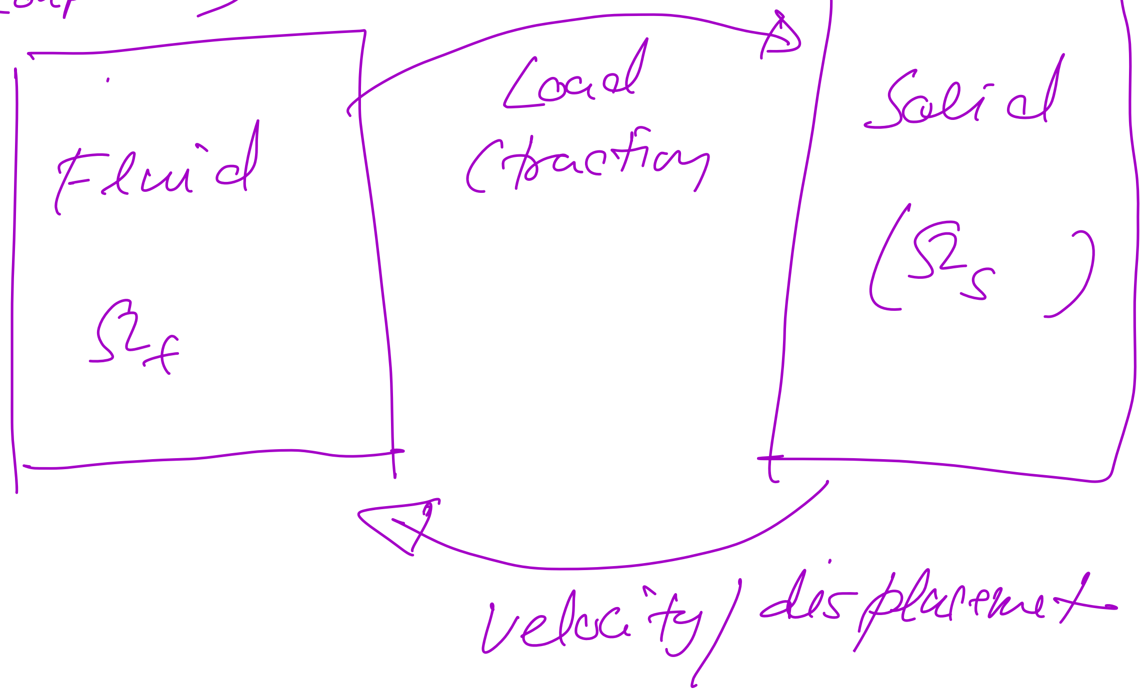
Monolithic:

$$\begin{bmatrix} 10^{16} & - & - \\ - & - & - \\ - & - & 10^{-15} \end{bmatrix} \begin{Bmatrix} \underset{\sim}{v}^f \\ p^f \\ \underset{\sim}{v}^s \end{Bmatrix} = - \begin{Bmatrix} R_m^f \\ R_c^f \\ R_m^s \end{Bmatrix}$$

Partitioned:

(Decoupled)

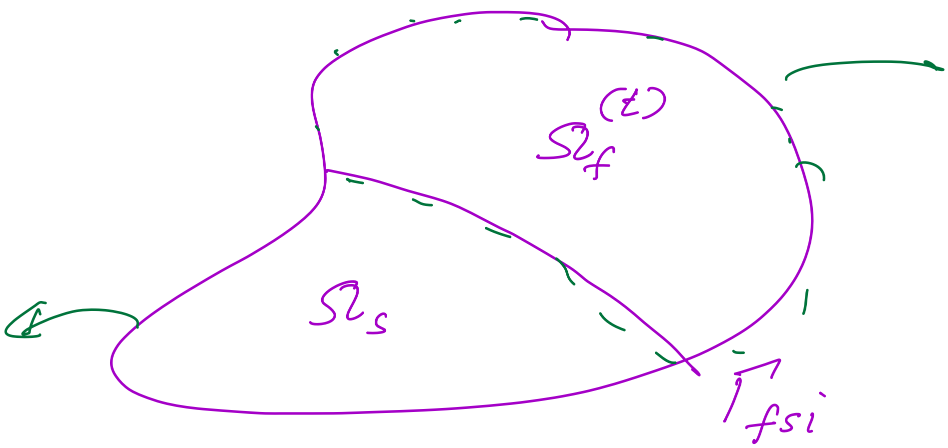




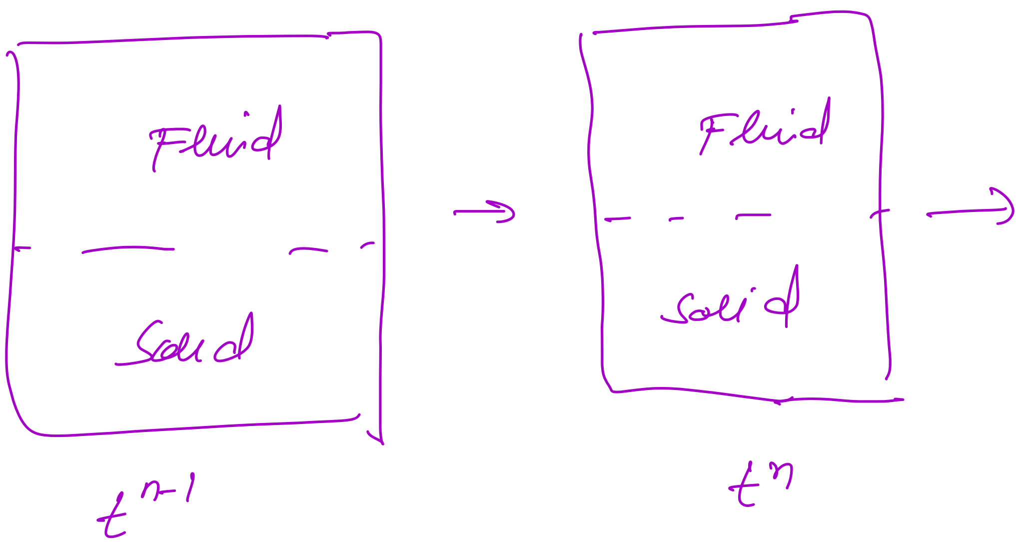
Computational Approaches for Fluid-Structure Interaction

- Coupled FSI system problem
  * Different scales in space & time
  * Solving coupled PDE's // (Hyperbolic, Parabolic, Elliptic)
  * Moving domains of fluid & solid (ALE)
  * Interface Conditions
  * Eulerian-Lagrangian Conflict!

- Monolithic vs. Partitioned



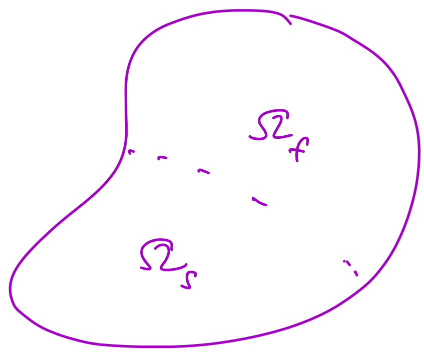
Monolithic:



Mono: Single

Lithic: Rock

$$ma = F$$



Ω^f {
— Momentum for Fluid
— Continuity/mass for fluid] $\mathrm{div}(v^f)=0$

Ω^s { Momentum for solid

Fluid & Solid continuum eq^{n}s are solved simultaneously in a "unified" block solver!

Drawbacks: → Differing / inconsistent (Scales are very broad)

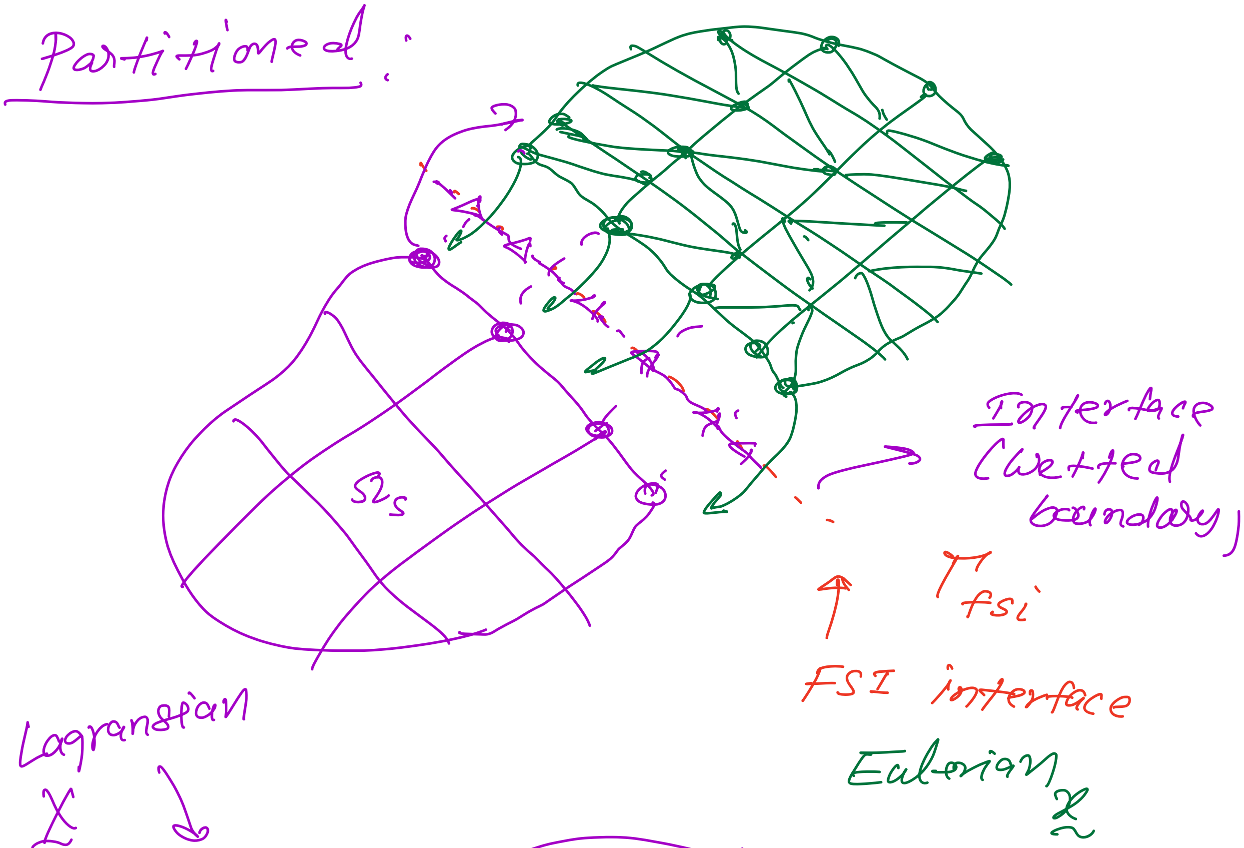
Matrix / Algebra issue

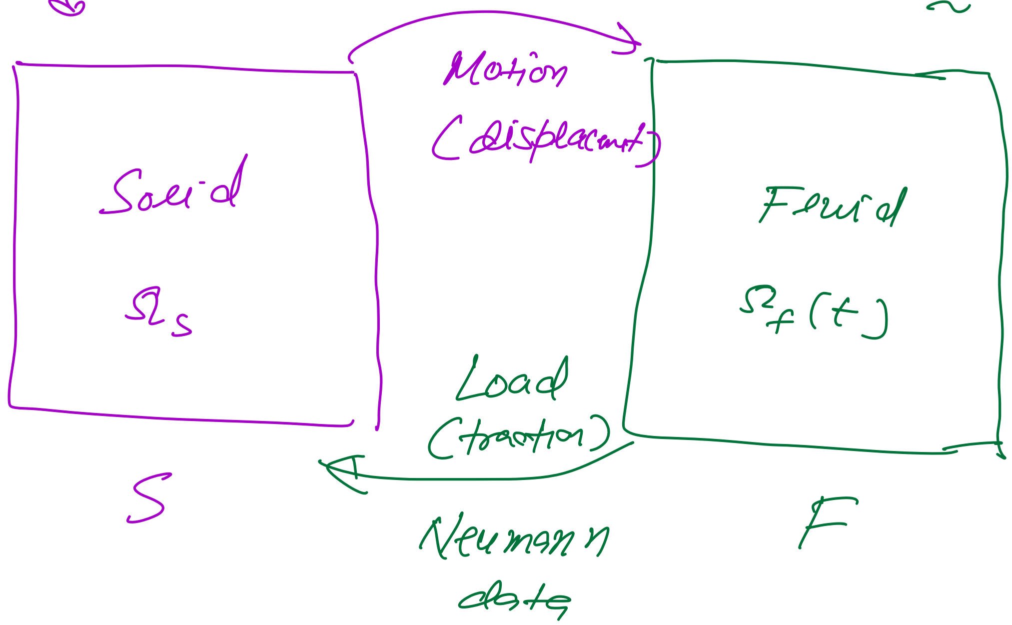
"ill-conditioning" issue!

→ Difficult to build & maintain codes!

→ Linearization! (LHS matrix)

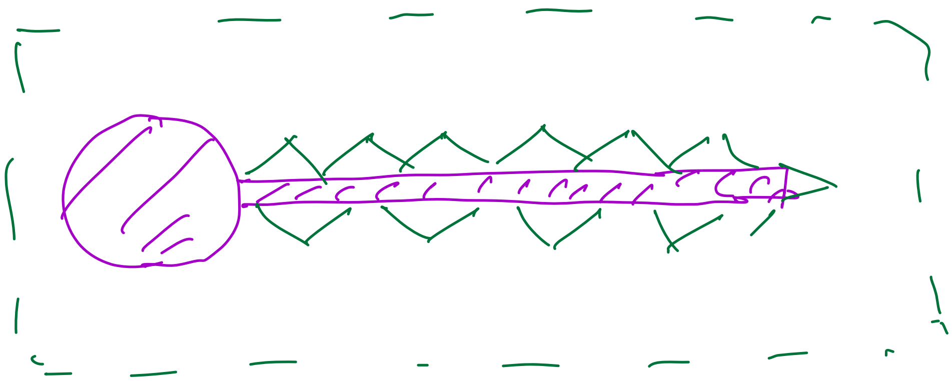
Partitioned:





Partitioned approach decomposes FSI system into two parts

- → Fluid part (F)
- → Solid part (S)



Incompressible NS in <u>ALE</u> form:
(Liquid/Low speed air)

$$\left.\begin{array}{l}\rightarrow \text{Momentum balance}\\ \rightarrow \text{Continuity}\\ \rightarrow \text{Mesh motion}\end{array}\right\}\ \underset{\sim}{x}$$

$$\underset{\sim}{x}^{K} = \underset{\sim}{F}\left(\underset{\sim}{x}^{K-1}, \underset{\sim}{y}\right)$$

Structural effects

Lagrangian Solid: $\underset{\sim}{S}$

$$\underset{\sim}{y}^{K} = \underset{\sim}{S}\left(\underset{\sim}{y}^{K-1}, \underset{\sim}{x}\right)$$

Fluid effect via pressure & viscous shearing @ interface/ wetted boundary

Block Jacobi:

$$\underset{\sim}{x}^{K} = F^{p}(\underset{\sim}{x}^{K-1}, y^{K-1})$$

$$\underset{\sim}{y}^{K} = S^{q}(\underset{\sim}{y}^{K-1}, x^{K-1})$$

At $K=1$, $x^{0}_{K+1} = x_K$, $y^{0}_{K+1} = y_K$

Block Gauss-Seidel:

Fluid part $x^{K} = F^{p}(x^{K-1}, y^{K})$

Solid part $y^{K} = S^{q}(y^{K-1}, \underline{x}^{K})$

$\Rightarrow$ Passing information back-&-forth at each time step with a "lag" is termed "Staggering"!

(Partitioned Staggered approach)

Solution Procedure

After the temporal and spatial discretization of the FSI equations, they can be solved by either monolithic or partitioned solution strategies. These strategies are discussed in the current section. In the monolithic strategy, the whole problem is considered as a single field and all the components are advanced simultaneously in time. On the other hand, partitioning is the process of spatial separation of the discrete model into the interacting components which are called partitions. On the other hand, the temporal discretization can also be decomposed into what is called splitting within a particular time step size (See Fig. 5.1). The time splitting can be additive or multiplicative.

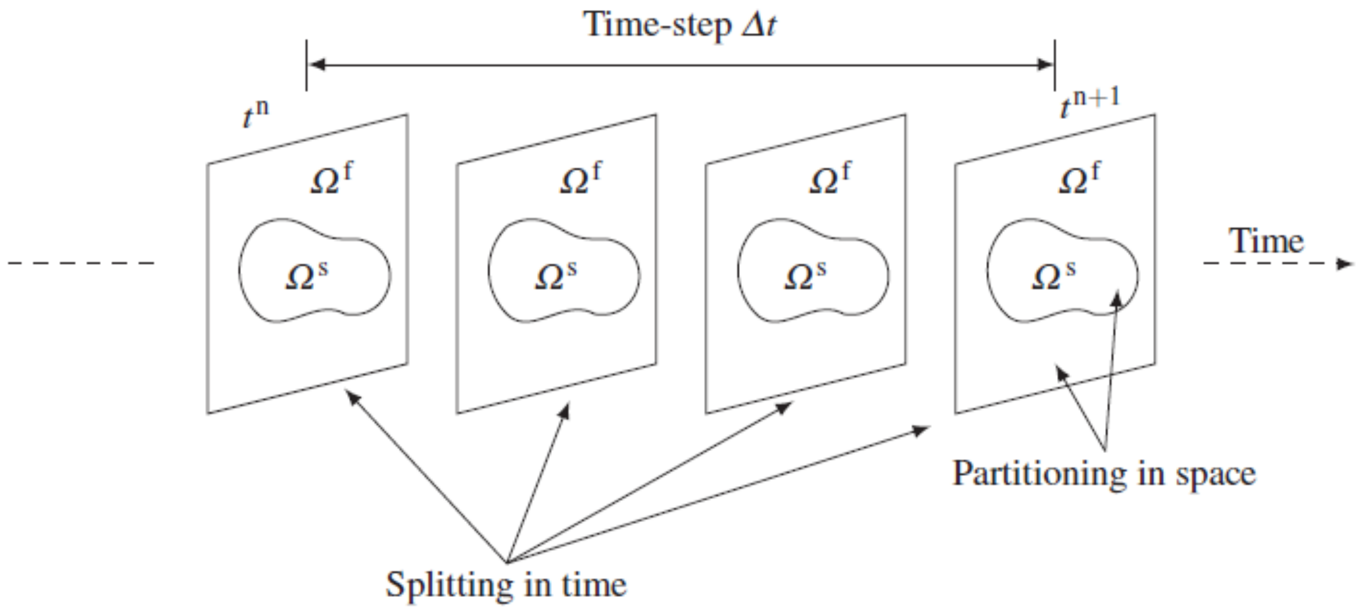
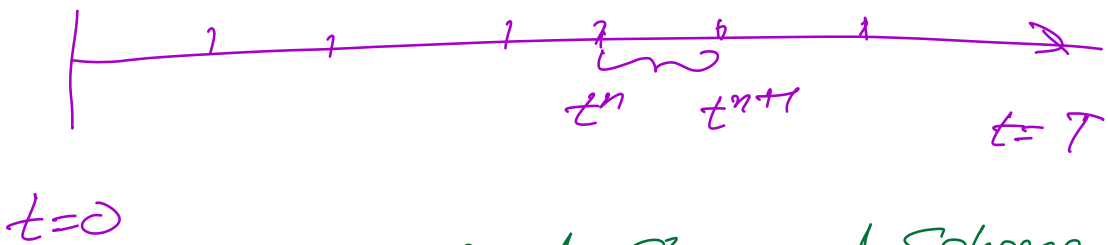
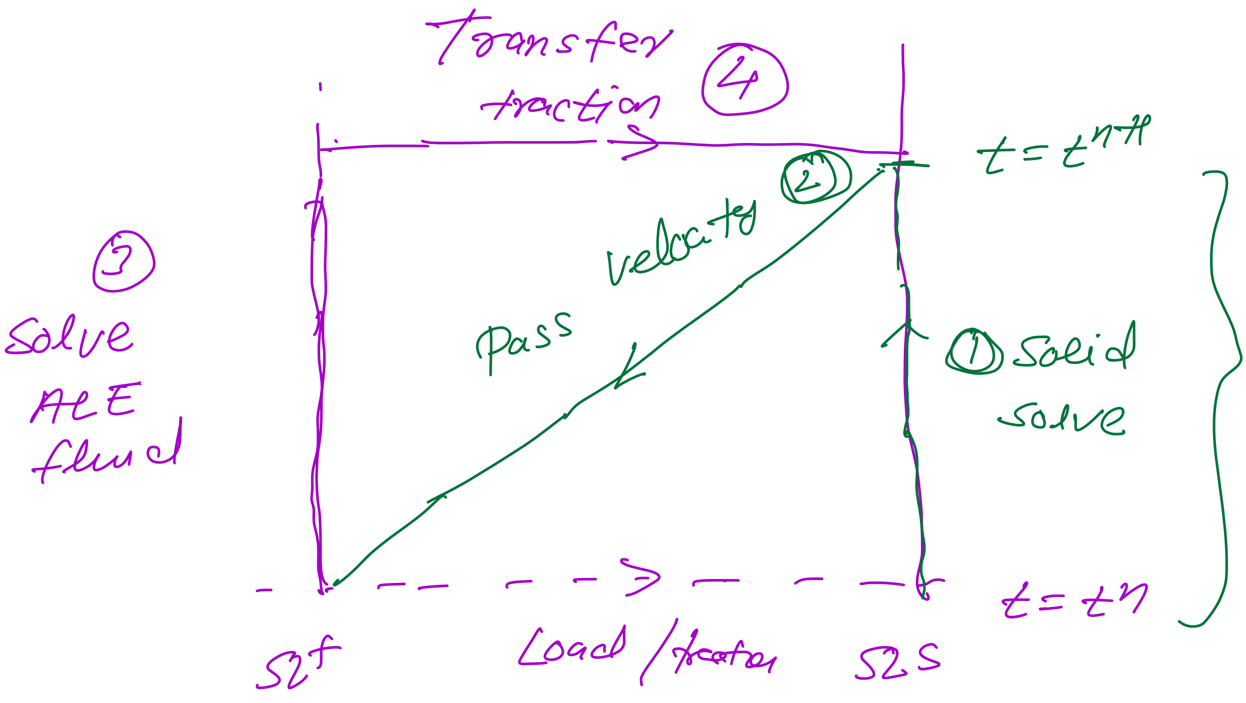


Fig. 5.1 Partition in spatial and splitting in the temporal discretizations.

Sequential Staggered Scheme
(Loosely-coupled method)



Conventional Staggered Scheme
(CSS)



Algorithm: CSS

Start with initial values for fluid, mesh, & solid

At each time step: $t^n \to t^{n+1}$

① Predict velocity from solid

② Transfer velocity to ALE fluid

③ Solve ALE mesh motion & flow system

④ Transfer fluid traction to solid surface

Remark: This is a workhorse technique for aeroelasticity & wind engineering!
(air interaction w/ metals)

	Pros	Cons
Partitioned	(1) Easy to Code & debug (2) specialized models (3) Parallel Coding	(1) Stability & Convergence issues (2) Spurious added mass effects (3) Time lag (staggering) errors
Monolithic	(1) Coupling conditions are implicit (2) Stability can be good for strong FSI couplings { liquid interacting thin solids (3) No issue of time lag	(1) Difficult to Code & debug, maintain (2) Parallel programming is complex (3) Modularity not there

Partitioned Procedure (1)

Time derivative

solid effect

$$X \rightarrow 5\dot{x} + 2x - 3y = f(t) \quad \text{—} \quad (1)$$

$$Y \rightarrow \dot{y} + 8y - 7x = g(t) \quad \text{—} \quad (2)$$

fluid effect

forcing

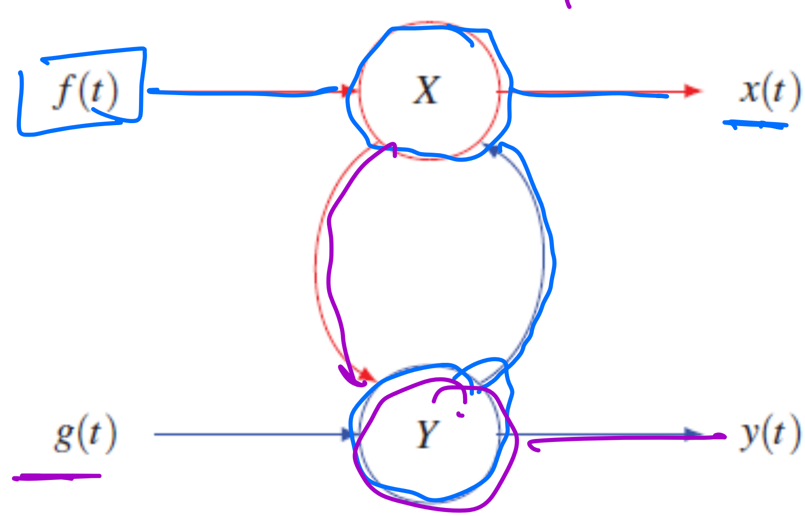


Fig. 5.2 Coupled fields X and Y advancing in time t.

Rewrite in matrix Form :

$$\underbrace{\begin{bmatrix} 5+2\Delta t & -3\Delta t \\ -7\Delta t & 1+8\Delta t \end{bmatrix}}_{\text{Monolithic}} \begin{Bmatrix} x^{n+1} \\ \boxed{y^{n+1}} \end{Bmatrix} = \begin{Bmatrix} \Delta t\, f(t^{n+1}) + 5x^n \\ \Delta t\, g(t^{n+1}) + y^n \end{Bmatrix}$$

→ Let's do partitioning

① Predictor step $y_p^{n+1} = y^n$

or $y_p^{n+1} = y^n + \Delta t\, \dot{y}^n$

② Transfer of data: Sending to Second field (x)

③ Advance Second field

Partitioned Procedure (2)

$$x^{n+1} = \frac{1}{5+2\Delta t}\left[\Delta t f(t^{n+1}) + 5x^{n} + 3\Delta t\, y_p^{n+1}\right]$$

(4) Corrector step:

$$y^{n+1} = \frac{1}{1+8\Delta t}\left[\Delta t g^{n+1} + y^{n} - 7\Delta t\, x^{n+1}\right]$$

For a partitioned procedure, both the fields are advanced in time separately. It consists of a predictor-corrector algorithm as follows in a particular time step:

1. Predictor step: $y_P^{\mathrm{n+1}} = y^{\mathrm{n}}$ or $y_p^{\mathrm{n+1}} = y^{\mathrm{n}} + \Delta t \dot{y}^{\mathrm{n}}$
2. Transfer of data: Send the predicted value to the second field
3. Advance the second field:

$$x^{\mathrm{n+1}} = \frac{1}{5+2\Delta t}\left(\Delta t f\left(t^{\mathrm{n+1}}\right) + 5x^{\mathrm{n}} + 3\Delta t y_P^{\mathrm{n+1}}\right)$$

4. Corrector step:

$$y^{\mathrm{n+1}} = \frac{1}{1+8\Delta t}\left(\Delta\, \mathrm{tg}\left(t^{\mathrm{n+1}}\right) + y^{\mathrm{n}} + 7\Delta t x^{\mathrm{n+1}}\right)$$

Partitioned Procedure (3)

CSS

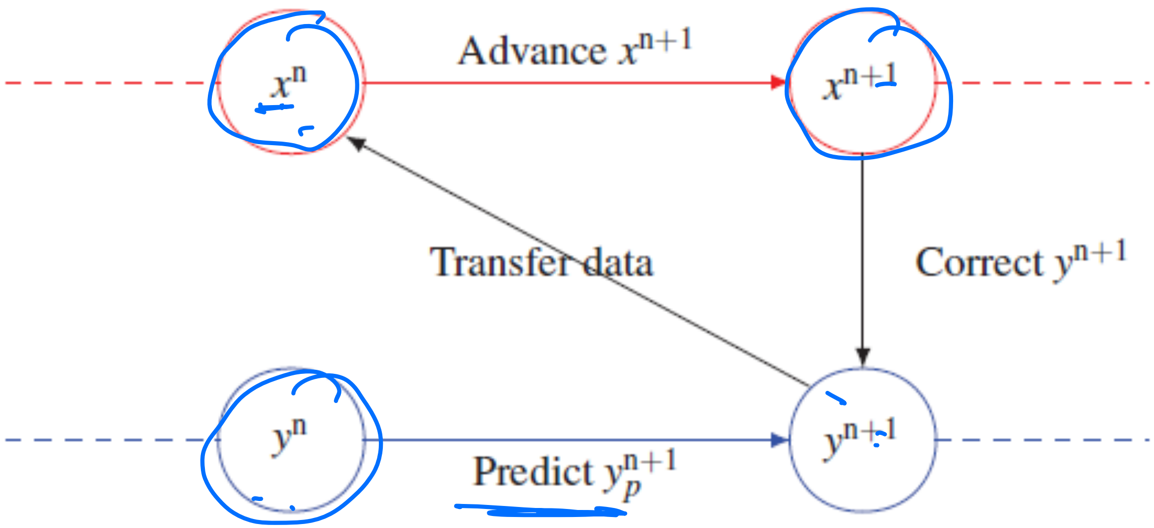


Fig. 5.3 Partitioned staggered technique for coupled fields X and Y advancing in time t.

Partitioned Iterative Procedure (4)

Subiterations

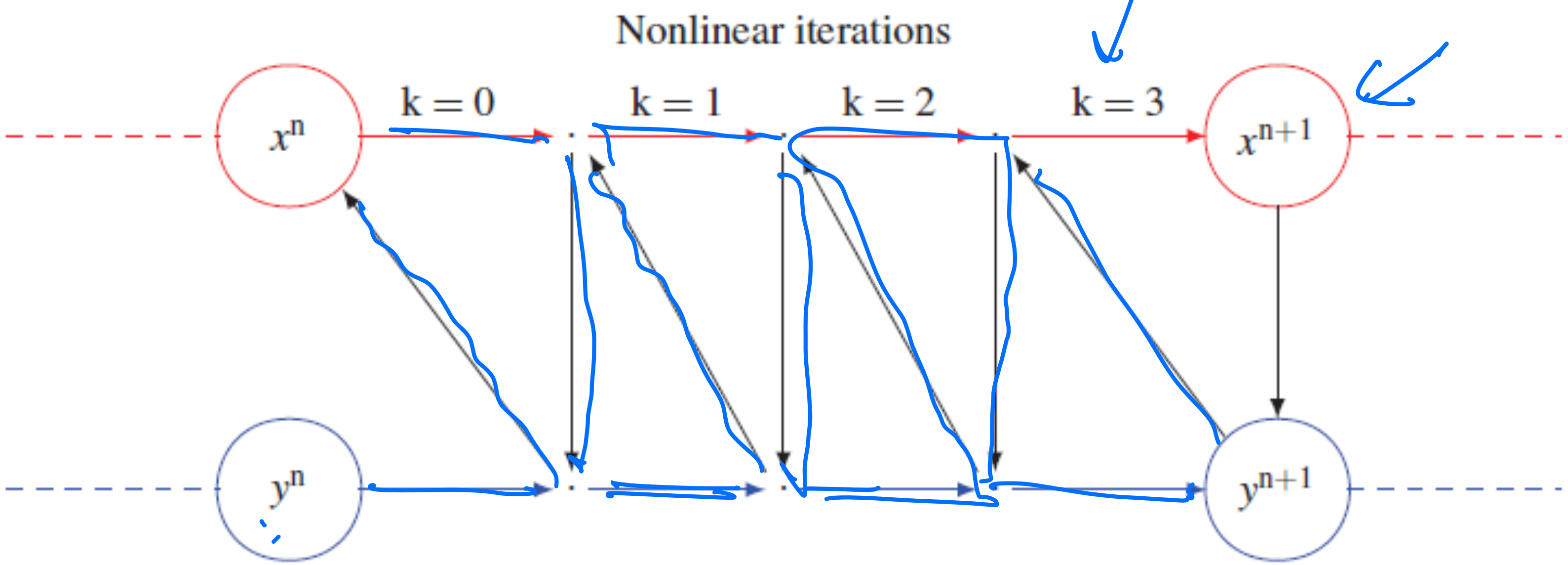


Fig. 5.4 Partitioned iterative staggered technique for coupled fields X and Y advancing in time t.

(Strongly Coupled)

Summary

The above algorithm is called staggered partitioned procedure (Fig. 5.3). For linear problems, it is observed that staggering does not harm stability and accuracy of the problem, given the condition that the predictor is chosen properly. But for more general nonlinear problems, stability could become an issue. On the other hand, accuracy is degraded compared to the same problem solved by a monolithic scheme. This can be resolved by iterating the staggered procedure within a time step so that the scheme now forms a predictor-multicorrector format (Fig. 5.4). For nonlinear problems, these iterations help in the convergence and capture of the nonlinearities. However, these multiple iterations add to the computational cost and a monolithic scheme could be more advantageous for some problems in that case. Therefore, there is a tradeoff based on the type of problem being solved. Next, we briefly look into how these strategies differ for an FSI problem, before going into the details in the upcoming chapters.

Partitioned Solution for FSI

In the partitioned procedure for FSI problems, we solve the fluid and structural fields separately and transfer the interface data between the fields. As the equations involved are nonlinear in nature, we opt for the predictor-multicorrector partitioned staggered procedure presented above.

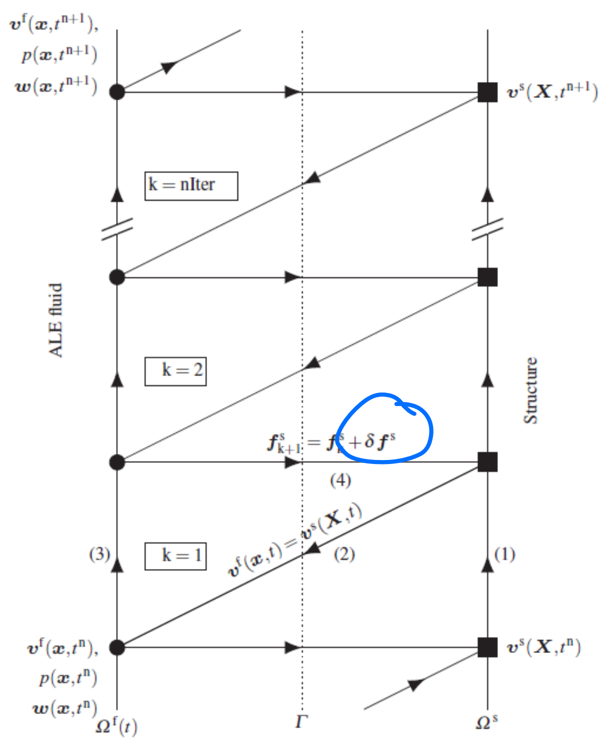


Fig. 5.5 Partitioned iterative staggered technique for coupled fluid-structure interaction system.

Partitioned Solution for FSI

Summary: Partitioned Solution for FSI

- The above steps form a nonlinear iteration. The fluid-structure coupling then advances in time at the end of the nonlinear iteration steps once the convergence criteria has been achieved.
- The various challenges with the transfer of forces along the interface and the satisfaction of the equilibrium conditions have been discussed in detail later.
- To summarize, we began with the fluid-structure continuum equations and expressed the variational finite element form for the stabilized fluid system, structural system and mesh equation.
- We then briefly discussed the two main strategies for solving the coupled FSI system while noting some of the major difference between them.

Monolithic Solution for FSI

- In the monolithic technique, the fluid and structural velocities are advanced in time simultaneously and both the fluid and structure are considered as a combined field.
- Note that when written in a combined field form, the dynamic equilibrium at the fluid-structure interface is satisfied naturally and we don’t need to worry about the transfer of forces along the interface.
- This technique will be discussed in more detail in Chapter 6.

Module 5 (Part 1)

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