.3# MATH 521 - Numerical Analysis of Differential Equations

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Assignment 3 : Analysis of FEM

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As a general comment: The questions may appear long, but they are not. Try to be brief.

Q1: Inhomogeneous Dirichlet Problem [20]

(a) Let $\Omega = (0,1)^2$, $f \in L^2(\Omega)$ and consider the boundary value problem

$$\begin{aligned} -\Delta u &= f, \quad \Omega \\ u &= 0, \quad \partial\Omega. \end{aligned}$$

Derive the weak form of the PDE, then formulate the P_k-finite element method in variational form. (No need to formulate the algebraic equations.)

(b) Now suppose that instead of the homogneous boundary conditions $u = 0$ we solve the inhomogeneous Dirichlet problem

$$\begin{aligned} -\Delta u &= f, \quad \Omega \\ u &= u_D, \quad \partial\Omega. \end{aligned}$$

We assume that $u_D|_{\partial\Omega}$ is the trace of a function $u_D \in H^1(\Omega)$. Reduce the problem to the one in part (a) and use this to formulate a P_k-finite element approximation with trial functions of the form $u_h = u_D + w_h$ where w_h belongs to a suitable finite element space that you should specify.

NOTE: this is an over-simplification. In practice we will use $u_h = I_h u_D + w_h$ which leads to additional variational crimes that we will treat later.

(c) For the problem from part (b) derive an a priori error estimate in the H^1-norm, assuming that $u, u_D \in H^{k+1}$. You may assume boundedness and coercivity and existence/uniqueness without proof, but show the steps for Galerkin orthogonality and Cea's lemma and then deduce the error estimate.

You may state without proof any nodal interpolation error estimate.

Solution Q1a

Weak formulation: finding $u \in H_0^1(\Omega)$ such that for all test function $v \in H_0^1(\Omega)$:

$$\begin{aligned} \int_\Omega -\nabla \cdot (\nabla u) v \, dx &= \int_\Omega f v \, dx \\ \int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} (\nu \cdot \nabla u) v \, ds &= \int_\Omega f v \, dx \quad \text{where } u, v \in H_0^1 \\ \int_\Omega \nabla u \cdot \nabla v \, dx &= \int_\Omega f v \, dx \end{aligned}$$

Finite element formulation:
Define $V_h := P_k(T_h) \cap H_0^1$, finding $u_h \in V_h$ such that for all $v_h \in V_h$:

$$\int_\Omega \nabla u_h \cdot \nabla v_h \, dx = \int_\Omega f v_h \, dx, \quad \forall v_h \in V_h$$

Solution Q1b

Reduction to Homogeneous Boundary Conditions:

To handle the inhomogeneous boundary conditions, we introduce a function $w = u - u_D$, $w \in H_0^1(\Omega)$ leading to a problem with homogeneous boundary conditions for w:

$$\begin{aligned} -\Delta w = -\Delta(u - u_D) = -\Delta u + \Delta u_D = f + \Delta u_D, \quad \text{in } \Omega \\ w = 0, \quad \text{in } \partial\Omega \end{aligned}$$

Finite Element Formulation:
The finite element formulation seeks $w_h \in V_h := P_k(T_h) \cap H_0^1(\Omega)$ such that $\forall v_h \in V_h$:

$$\int_\Omega \nabla w_h \cdot \nabla v_h \, dx = \int_\Omega (f + \Delta u_D) v_h \, dx,$$

Solution Q1c

Galerkin Projection:

$$e = w - w_h$$
$$a(e, v_h) = a(w - w_h, v_h) = 0, \quad \forall v_h \in V_h.$$

Cea's lemma:

$$||w - w_h||_{H^1} \leq \frac{c_1}{c_0} \inf_{v_h \in V_h} ||w - v_h||_{H^1}$$

nodal error estimation:

$$||w - I_h w||_{H^1} \leq ch^k ||w||_{H^{k+1}}$$

By knowing u_D exact and $u_h = u_D + w_h, u = u_D + w$, we can deduce the error estimate:

$$||u - u_h||_{H^1} = ||(u_D + w) - (u_D + w_h)||_{H^1} = ||w - w_h||_{H^1} \leq Ch^k ||w||_{H^{k+1}}$$

Q2: Advection [10]

Let $\Omega = (0,1)^2$, $f \in L^2(\Omega)$, $b \in \mathbb{R}^2$ constant and consider the boundary value problem

$$\begin{aligned} -\Delta u + b \cdot \nabla u &= f, \quad \Omega \\ u &= 0, \quad \partial\Omega. \end{aligned}$$

Derive the variational (weak) form of the PDE, then formulate the variational form of the P_k-finite element method. Are the PDE and the FEM well-posed?

Solution Q2

variational (weak) form of the PDE: finding $u \in H_0^1(\Omega)$ such that for all test function $v \in H_0^1(\Omega)$:

$$\int_\Omega -\Delta u v + (b \cdot \nabla u) v \, dx = \int_\Omega f v \, dx,$$
$$\int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega (b \cdot \nabla u) v \, dx = \int_\Omega f v \, dx.$$

variational form of the P_k-finite element method: Define $V_h := P_k(T_h) \cap H_0^1$, finding $u_h \in V_h$ such that for all $v_h \in V_h$:

$$\int_\Omega \nabla u_h \cdot \nabla v_h \, dx + \int_\Omega (b \cdot \nabla u_h) v_h \, dx = \int_\Omega f v_h \, dx.$$

For the given boundary condition and if with a smooth enough data f, the PDE should be Well poseded. But not necessary for FEM which depends on advection speed b, when b is very large, that may break the coercivity and continuity.

Q3: Energy [10]

Let H be a Hilbert space and $V_h \subset H$ a finite-dimensional subspace. Let $a : H \times H \to \mathbb{R}$ be a bounded, coercive, symmetric, bilinear form, $\ell \in H^*$, and let

$$J(v) := \frac{1}{2} a(v, v) - \ell(v)$$

be the associated energy functional.

(a) Show that the following two problems are equivalent:

- Find $u \in H$ such that $J(u) \leq J(v)$ for all $v \in H$.
- Find $u \in H$ such that $a(u, v) = \ell(v)$ for all $v \in H$. (Give full details for the argument that we sketched out in class.)

(b) Conclude as an immediate corollary that the Galerkin projection

- Find $u_h \in V_h$ such that $a(u_h, v_h) = \ell(v_h)$ for all $v_h \in V_h$

can be equivalently written as

- Find $u_h \in V_h$ such that $J(u_h) \leq J(v_h)$ for all $v_h \in V_h$

(c) Prove that the error in energy can be bounded by

$$J(u) \leq J(u_h) \leq J(u) + \frac{1}{2}||u - u_h||_a^2,$$

in particular,

$$|J(u) - J(u_h)| \leq \frac{1}{2}||u - u_h||_a^2.$$

HINT: You might be tempted to use a duality argument, but it is not needed here.

Solution Q3a and Q3b

To find $u \in H$ such that $J(u) \leq J(v)$ for all $v \in H$ is finding a u that is a ture minimizer of $J(\cdot)$.
Which can be written as a varitational problem:
Define a function $v_\lambda := u + \lambda v$, where λ is a scalar, such that:

$$\begin{aligned}
J(v_\lambda) &:= \frac{1}{2}a(u + \lambda v, u + \lambda v) - \ell(u + \lambda v) \\
&= \frac{1}{2}[a(u, u) + 2\lambda a(u, v) + \lambda^2 a(v, v)] - \ell(u) - \lambda \ell(v) \\
\frac{dJ(v_\lambda)}{d\lambda} &= a(u, v) + \lambda a(v, v) - \ell(v) \\
a(u, v) &= \ell(v), \quad \text{when } \lambda = 0
\end{aligned}$$

Therefore the function u is indeed a ture minimizer of $J(\cdot)$, and two statements are equaivlant.
This statement is also ture for Galerkin project, as we subsititute u as u_h and v as v_h

Solution Q3c

$$\begin{aligned}
J(u_h) &:= \frac{1}{2}a(u_h, u_h) - \ell(u_h) \\
&= \frac{1}{2}a(u_h - u + u, u_h - u + u) - \ell(u_h - u + u) \\
&= \frac{1}{2}[a(u_h - u, u_h - u) + 2a(u_h - u, u) + a(u, u)] - \ell(u_h - u) - \ell(u) \\
&= \frac{1}{2}[a(u_h - u, u_h - u) + 2a(u_h - u, u)] - \ell(u_h - u) + J(u)
\end{aligned}$$

From Q3a, we showed that $a(u, v) = \ell(v)$ for all $v \in H$, which also include $v = u_h - u$, therefore:

$$\begin{aligned}
J(u_h) - J(u) &= \frac{1}{2}[a(u_h - u, u_h - u) + 2a(u_h - u, u)] - \ell(u_h - u) \\
&= \frac{1}{2}a(u_h - u, u_h - u) + a(v, u) - \ell(v) \\
&= \frac{1}{2}a(u_h - u, u_h - u)
\end{aligned}$$

Since $a(\cdot, \cdot)$ is positive semi-definite, and energy norm is defined as $||u - u_h||_a = \sqrt{a(u_h - u, u_h - u)}$:

$$\begin{aligned}
J(u) \leq J(u_h) \leq J(u) + \frac{1}{2}||u - u_h||_a^2, \\
|J(u) - J(u_h)| \leq \frac{1}{2}||u - u_h||_a^2.
\end{aligned}$$

Q4: Duality [10]

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain with boundary $\partial\Omega = \Gamma_\mathrm{D} \cup \Gamma_\mathrm{N}$ with $|\Gamma_\mathrm{D}|, |\Gamma_\mathrm{N}| > 0$, $\mathcal{T}_h$ a regular triangulation of Ω, $f \in L^2(\Omega), g \in L^2(\Gamma_\mathrm{N})$ and consider the boundary value problem

$$\begin{aligned}
-\Delta u &= f, \quad \text{in } \Omega \\
u &= 0, \quad \text{in } \Gamma_\mathrm{D}, \\
\nu \cdot \nabla u &= g, \quad \text{in } \Gamma_\mathrm{N}.
\end{aligned}$$

(a) Write down the variational form of the PDE in $H^1_{\Gamma_\mathrm{D}}$ and the variational form of the Pk-FEM. (no need to give too many details, and no need to prove boundedness and coercivity - you may assume both for step (b).)

(b) Consider the quantity of interest

$$\Phi(u) = \int_{\Gamma_\mathrm{N}} u\,dx.$$

Show that $\Phi \in (H^1_{\Gamma_\mathrm{D}})^*$.

Let u, u_h solve the variational forms of the PDE and FEM. Prove that

$$|\Phi(u) - \Phi(u_h)| \le \|\nabla u - \nabla u_h\|_{L^2} \|\nabla w - \nabla w_h\|_{L^2},$$

where w is the solution of a dual problem that you should specify and w_h taken from a suitable space is arbitrary.

Solution Q4a

Variational form of the PDE in $H^1_{\Gamma_\mathrm{D}}$: Finding $u \in H^1_0(\Omega)$ such that for all test function $v \in H^1_{\Gamma_\mathrm{D}}(\Omega)$:

$$\int_\Omega \nabla u \cdot \nabla v\,dx - \int_{\Gamma_\mathrm{N}} gv\,ds = \int_\Omega fv\,dx.$$

variational form of the Pk-FEM: Define $V_h := P_k(T_h) \cap H^1_{\Gamma_D}$ and $S_h := P_k(T_h) \cap H^1_0$, finding $u_h \in S_h$ such that for all $v_h \in V_h$:

$$\int_\Omega \nabla u_h \cdot \nabla v_h\,dx - \int_{\Gamma_\mathrm{N}} gv_h\,ds = \int_\Omega fv_h\,dx.$$

Solution Q4b

To show that Φ is a dual space of $H^1_{\Gamma_\mathrm{D}}$, we need to show that Φ is a linear and continous functional:
Linearity:

$$\Phi(u + \lambda v) = \int_{\Gamma_\mathrm{N}} (u + \lambda v)\,dx = \int_{\Gamma_\mathrm{N}} u\,dx + \lambda \int_{\Gamma_\mathrm{N}} v\,dx = \Phi(u) + \lambda\Phi(v).$$

Continuity:
By Trace Theorem: $\|u\|_{L^2(\Gamma_n)} \le C\|u\|_{H^1(\Omega)}$, and by Cauchy-Schwarz inequality, for $\Phi(u)$, we can show:

$$\begin{aligned}|\Phi(u)| = \left|\int_{\Gamma_\mathrm{N}} u\,dx\right| &\le \left(\int_{\Gamma_\mathrm{N}} u^2\,dx\right)^{1/2} \left(\int_{\Gamma_\mathrm{N}} 1^2\,dx\right)^{1/2} \\ &= \|u\|_{L^2(\Gamma_\mathrm{N})} \cdot |\Gamma_\mathrm{N}|^{1/2} \\ &\le C\|u\|_{H^1(\Omega)} \cdot |\Gamma_\mathrm{N}|^{1/2}\end{aligned}$$

Therefore, $\Phi \in (H^1_{\Gamma_\mathrm{D}})^*$.
For w as the solution of a dual problem : $a(u, w) = \Phi(u)$, then we have:

$$\begin{aligned}\Phi(u) - \Phi(u_h) &= a(u, w) - a(u_h, w) = a(u - u_h, w) \\ &= a(u - u_h, w - w_h) + a(u - u_h, w_h)\end{aligned}$$

w_h taken from a suitable space so that we can let $a(u - u_h, w_h) = 0$, and by Cauchy-Schwarz inequality:

$$|\Phi(u) - \Phi(u_h)| = |a(u - u_h, w - w_h)| \le \|\nabla u - \nabla u_h\|_{L^2} \|\nabla w - \nabla w_h\|_{L^2}$$