MATH 521 - Numerical Analysis of Differential Equations

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Assignment 4 : Heat Equation

Name:

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In this assignment we will solve the follow heat equation (prototype diffusion equation)

$$\begin{aligned} u_t - \Delta u = f, \qquad & x \in \Omega, t \in (0, T], \\ u = 0, \qquad & x \in \Omega, t \in [0, T], \\ u = u_0, \qquad & x \in \Omega, t = 0. \end{aligned}$$

where the solution is a function $u(x,t)$ and T is the final time. Assume that $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, and $f = f(x,t)$ is arbitrarily smooth in space and time. Throughout, you may assume that the solution $u(t,x)$ exists, is unique, and as smooth as needed to carry out your analysis.

We first discretize the problem using a P_1-FEM: Let $\mathcal{T}_h$ be a regular triangulation of the polygonal domain $\Omega \subset \mathbb{R}^2$ and let $V_h := \mathcal{P}_1(\mathcal{T}_h)$ be the corresponding FE space. Then the semi-discrete (continuous in time) formulation is to find $u_h \in C^1([0,T], V_h)$, such that $u_h(t=0) = I_h u_0$ (with I_h the nodal interpolation operator), and

$$(u_{h,t}, v_h) + (\nabla u_h, \nabla v_h) = (f, \eta^{n+1} + \eta^n) \qquad \forall v_h \in V_h, t \in (0, T].$$

with $(\cdot,\cdot) = (\cdot,\cdot)_{L^2(\Omega)}$. The Assignment is primarily concerned with the effect of using different discretization schemes in time.

Throughout this assignment we assume that the family $\mathcal{T}_h$, $h > 0$ is uniformly shape-regular (no need to further mention this) and quasiuniform, i.e. $ch \le h_T \le h$ for all $T \in \mathcal{T}_h$ with c independent of h.

Q1: Crank-Nicholson Scheme [10]

For a system of ODEs, $\dot{u} = F(t,u)$ the CN scheme reads,

$$U^{n+1} = U^n + \Delta t \frac{F(t_n, U^n) + F(t_{n+1}, U^{n+1})}{2}$$

where $t_n = n\Delta t$, Δt is the time-step and U^n the approximation to $u(t_n)$.

(a) Formulate the CN discretization of the semi-discrete FEM.

(b) Under suitable regularity assumptions on the solution u, prove that

$$\max_{n=1,...,N} \|u(t_n) - u_h^n\|_{L^2} \le C(h^2 + \Delta t^2)$$

with $N\Delta t \le T$. How does C depend on N (or on T) in your result?

For part-b there is a part of the proof that is very repetitive (basically the same as in class) and I won't make you go through it again. I'll very briefly sketch this out, and you can then perform the last step.

Solution Q1a

$$\left(\frac{u_h^{n+1} - u_h^n}{\Delta t}, v_h\right) + \frac{1}{2}\big((\nabla u_h^n, \nabla v_h) + (\nabla u_h^{n+1}, \nabla v_h)\big) = \frac{1}{2}\big((f^n, v_h) + (f^{n+1}, v_h)\big) \quad \forall v_h \in V_h$$

Solution Q1b

Sketch of Part 1:

Let $u^n(x) = u(x, t_n)$ and $\tilde{u}_h^n \in V_h$ the Ritz projection of u^n i.e.

$$(\nabla(u^n - \tilde{u}_h^n), \nabla v_h) = 0 \qquad \forall v_h \in V_h$$

We split the error

$$e_h^n = u_h^n - u^n = (u_h^n - \tilde{u}_h^n) + (\tilde{u}_h^n - u^n) = \eta^n + \epsilon^n.$$

Following the class notes almost verbatim we can then prove

$$(\eta^{n+1} - \eta^n, v_h) + \frac{\Delta t}{2}(\nabla \eta^{n+1} + \nabla \eta^n, \nabla v_h) = \frac{\Delta t}{2}(g^n, v_h)$$

where

$$||g^n||_{L^2} \leq C_1(h^2 + \Delta t^2)$$

with C_1 depending on $||\nabla^2 u_t||_{L^\infty(L^2)}$ and $||u_{ttt}||_{L^\infty(L^2)}$, but independent of $h, n, \Delta t$.

Part 2: please complete this.

Hint: $(u - v, u + v)_H = ||u||_H^2 - ||v||_H^2$

Choose $v_h = \eta^{n+1} + \eta^n$, yielding:

$$(\eta^{n+1} - \eta^n, \eta^{n+1} + \eta^n) + \frac{\Delta t}{2}(\nabla \eta^{n+1} + \nabla \eta^n, \nabla \eta^{n+1} + \nabla \eta^n) = \frac{\Delta t}{2}(g^n, \eta^{n+1} + \eta^n)$$

Simplifying the left-hand side using the identity provided in the hint, and the C-S inequality gives:

$$||\eta^{n+1}||^2 - ||\eta^n||^2 + \frac{\Delta t}{2}||\nabla \eta^{n+1}||^2 - ||\nabla \eta^n||^2 \leq \frac{\Delta t}{2}||g^n|| ||\eta^{n+1} + \eta^n||.$$

Above is what I got for Q2, I realize that next steps should be related to the Discrete Gronwall's Lemma, but I do not know how to process from that. Here is the Lemma I searched from Google.

Gronwall's Lemma (Discrete Version)

The recurrence relation above, if summed from $n = 0$ to $n = N - 1$ and applying the discrete version of Gronwall's lemma, suggests that:

$$||\eta^N||^2 \leq \exp\left(C\Delta t \sum_{n=0}^{N-1} 1\right)\left(||\eta^0||^2 + C_1^2(h^2 + \Delta t^2)^2 T\right),$$

where C is a constant depending on the domain and coefficients of the equation, not on N or T. Given that $\eta^0 = 0$ (since $u_h^0 = I_h u_0$ and $u^0 = u_0$), this simplifies to:

$$||\eta^N|| \leq C(h^2 + \Delta t^2),$$

where now the constant may depend on T but not explicitly on N, other than through the product $N\Delta t \leq T$.

Though the detail of the Lemma is not clear to me, but observe from the result, one can conclude: Combining the bounds for η^n and ϵ^n:

$$\max_{n=1,...,N} ||u(t_n) - u_h^n|| \leq \max_{n=1,...,N} (||\eta^n|| + ||\epsilon^n||) \leq C(h^2 + \Delta t^2).$$

Thus, the overall constant C in the error estimate reflects dependencies from the norms of higher derivatives of u and the final time T, but crucially it does not depend explicitly on N. This result ensures that the Crank-Nicolson scheme provides a robust and effective method for the numerical approximation of the heat equation in both space and time.

Q2: Inverse Estimate [5]

Under the assumptions on $\mathcal{T}_h$ stated at the beginning of the assignment, prove that there exists $c^i > 0$ such that

$$||\nabla v_h||_{L^2} \leq c^i h^{-1} ||v_h||_{L^2} \qquad \forall v_h \in \mathcal{P}_1(\mathcal{T}_h).$$

Hint: transform to the reference element.

Solution Q2

Each element T in the triangulation $\mathcal{T}_h$ can be mapped to a reference element $\hat{T}$ by:

$$x = B_T \hat{x} + b,$$

And the area of each element T

$$|T| = |\det(B_T)||\hat{T}|,$$

Thus, the gradients transform as follows:

$$\nabla v_h(x) = B_T^{-1} \nabla \hat{v}_h(\hat{x}).$$

Therefore, the norm of the gradient scales as:

$$\|\nabla v_h\|_{L^2(T)}^2 = \int_T |\nabla v_h|^2 \, dx = \int_{\hat{T}} |B_T^{-1} \nabla \hat{v}_h|^2 |\det(B_T)| \, d\hat{x}.$$

Using the norm of a matrix and its inverse, particularly $\|B_T^{-1}\|$ where the norm is the operator norm (maximum stretching factor), we find:

$$\|B_T^{-1}\| \leq Ch^{-1},$$

assuming that the transformation linearly scales with the diameter h of the elements (since $h_T \leq h$ and is quasiuniform). Thus, we estimate:

$$\|\nabla v_h\|_{L^2(T)}^2 \leq \|B_T^{-1}\|^2 |\det(B_T)| \|\nabla \hat{v}_h\|_{L^2(\hat{T})}^2.$$

Thus

$$\|\nabla v_h\|_{L^2(\Omega)}^2 \leq C^2 h^{-2} \|v_h\|_{L^2(\Omega)}^2.$$

And finally,

$$\|\nabla v_h\|_{L^2(\Omega)} \leq c^i h^{-1} \|v_h\|_{L^2(\Omega)}.$$

Q3: Explicit Euler [15]

For a system of ODEs, $\dot{u} = F(t, u)$ the Explicit Euler (EE) scheme reads,

$$U^{n+1} = U^n + \Delta t F(t_n, U^n),$$

where $t_n = n\Delta t$, Δt is the time-step and U^n the approximation to $u(t_n)$.

(a) Formulate the EE discretization of the semi-discrete FEM.

(b) For $f = 0$ and under a suitable restriction on Δt and h that you should derive, prove the **CONDITIONAL STABILITY** result

$$\|u_h^n\|_{L^2} \leq q_{\Delta t} \|u_h^{n+1}\|_{L^2}$$

where $q_{\Delta t} < 1.$

(NOTE: a full error estimate is a bit more involved, so we only prove this stability estimate instead. This is still surprisingly hard, so I will give you the outline of the proof so you can follow it.)

Solution Q3a

Solution Q3b

Step 1: By testing with $v_h = u_h^{n+1} + u_h^n$ (or otherwise) show that

$$\|u_h^{n+1}\|^2 + \frac{\Delta t}{2} \|\nabla u_h^{n+1}\|^2 = \|u_h^n\|^2 - \frac{\Delta t}{2} \|\nabla u_h^n\|^2 + \frac{\Delta t}{2} \|\nabla u_h^{n+1} - \nabla u_h^n\|^2$$

HINT: $(u, v)_H = \frac{1}{2}\|u\|_H^2 + \frac{1}{2}\|v\|_H^2 - \frac{1}{2}\|u - v\|_H^2$

Step 2: By testing with $v_h = u_h^{n+1} - u_h^n$ and applying the inverse inequality prove that

$$\|u_h^{n+1} - u_h^n\|^2 \leq \Delta t \mu \|\nabla u_h^n\|^2,$$

where you should determine μ in terms of $c^i, h, \Delta t.$

Step 3: Apply the inverse inequality again to the result Step-1, then apply Step-2, then collect your terms. The result should now follow under a restriction on μ that you should specify.

Q4: Implementation of Heat Equation [20]

For this assignment you may choose any code-base you like, our first implementation of P1-FEM, the Ferrite implementation, or any other code in Julia or Python or Matlab. This is a little harder than the previous coding assignments in that I'm giving you a lot less help.

(a) Select one of the three discretizations of the heat equation that we covered: P1-FEM in space and IE, EE or CN in time. Implement this scheme to solve the heat equation on the time interval $t \in [0, 1]$ with $\Omega = (-1, 1)^2$, $f(x, t) = 1$ and $u_0(x) = 0$.

Visualize the solution at the final time $T = 1$ and print out the value of $\max_{x \in \Omega} u(x, t = 1)$. (e.g. put this in the title of the figure)

(b) Design and implement a numerical test that demonstrates the convergence rate we proved numerically.

Solution Q4a

In [ ]:

In [ ]:

In [ ]:

Solution Q4b

In [ ]:

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# to use the method of manufactured solutions the following
# may be useful. (or some variant...)
u_ex = (x, t) -> t * cos(x[1] * t + x[2] * sin(t)) * (x[1]^2 - 1) * (x[2]^2 - 1)
∇²u_ex = (x, t) -> ForwardDiff.hessian( x -> u_ex(x, t), x )
Δu_ex = (x, t) -> tr( ∇²u_ex(x, t) )
∂ₜu_ex = (x, t) -> ForwardDiff.derivative( t -> u_ex(x, t), t )
f_ex = (x, t) -> ∂ₜu_ex(x, t) - Δu_ex(x, t)
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In [ ]: