MECH 570C

Fluid-Structure Interaction

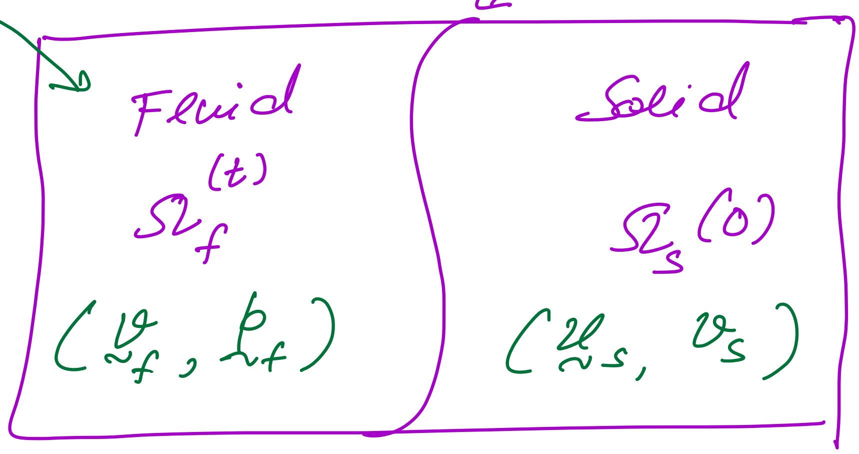
Module 3: Continuum Mechanics of Fluid-Solid Interaction with Moving Domains

MECH 570 C

FSI

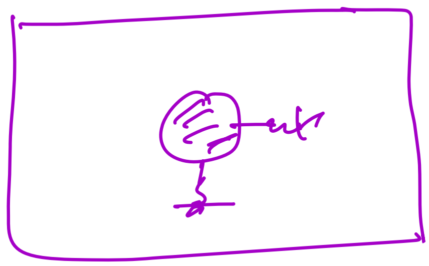
$\underset{\sim}{x}$

Spatial Coordinates



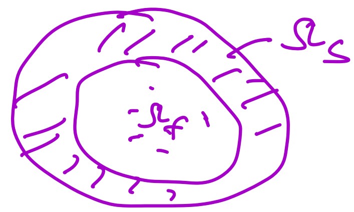
Material frame
$(\underset{\sim}{X})$

→ Setting frame of reference



Eulerian : $\Omega \subset \mathbb{R}^3$

$\underset{\sim}{x}$ (Fixed region space)



Lagrangian : Collection of material points.

Rules :

→ Mass balance $\quad dm = 0$

→ Momentum balance $\quad \frac{d}{dt}(m\underset{\sim}{v}) = \Sigma F$

→ Energy balance

→ Entropy Condition !

Frame/Coordinates	Fluid (liquid + gas)	Solid
Eulerian	✓	Maybe ✓
Lagrangian	Maybe ✓	✓

→ Fluids will be described in Eulerian frame/Coordinates.
(Fixed region space to capture properties)

→ Solids will be represented by Lagrangian (material point/particle-like description)

⇒ At interface: Conflict will occur!

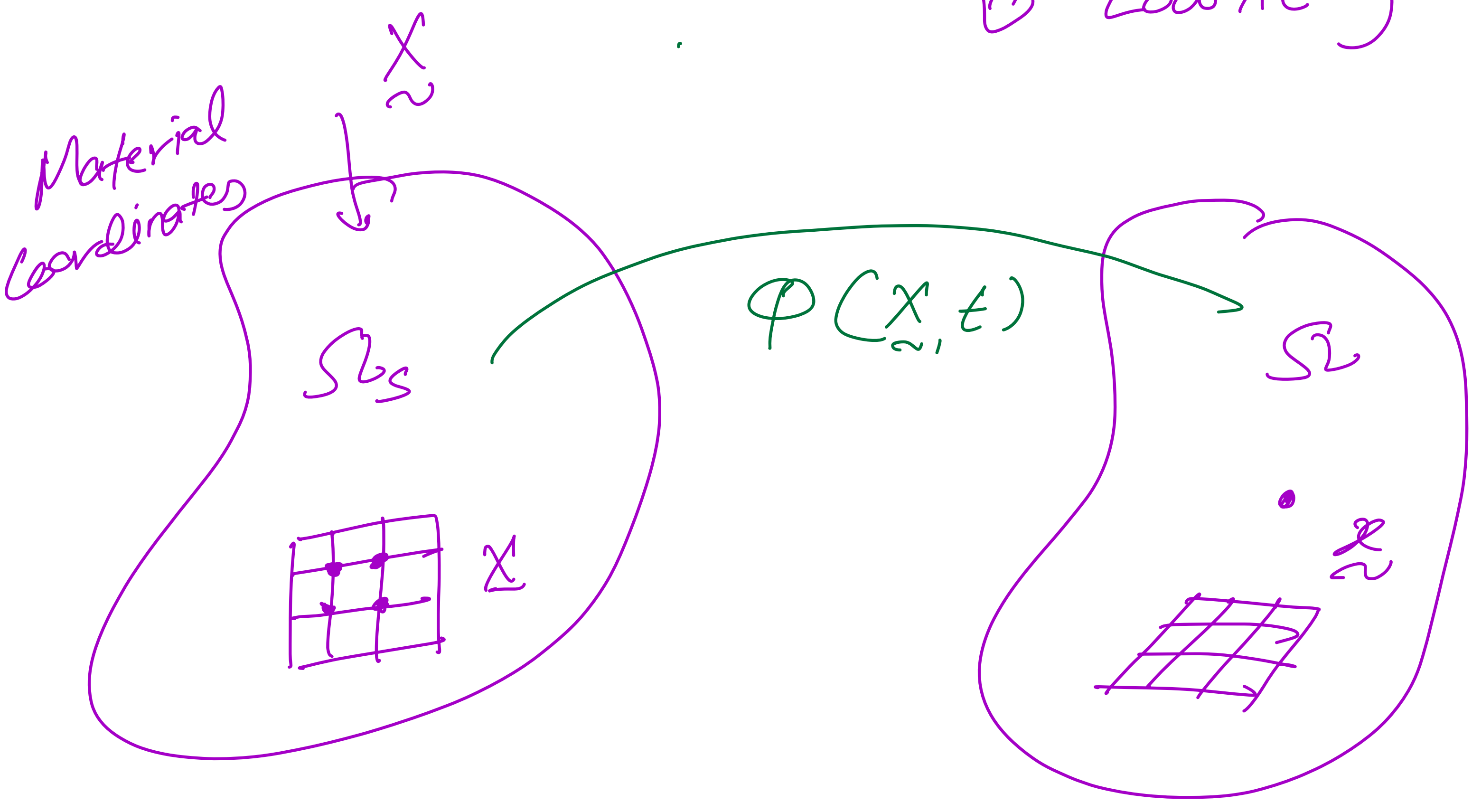
Option A: Fluid E + Solid E

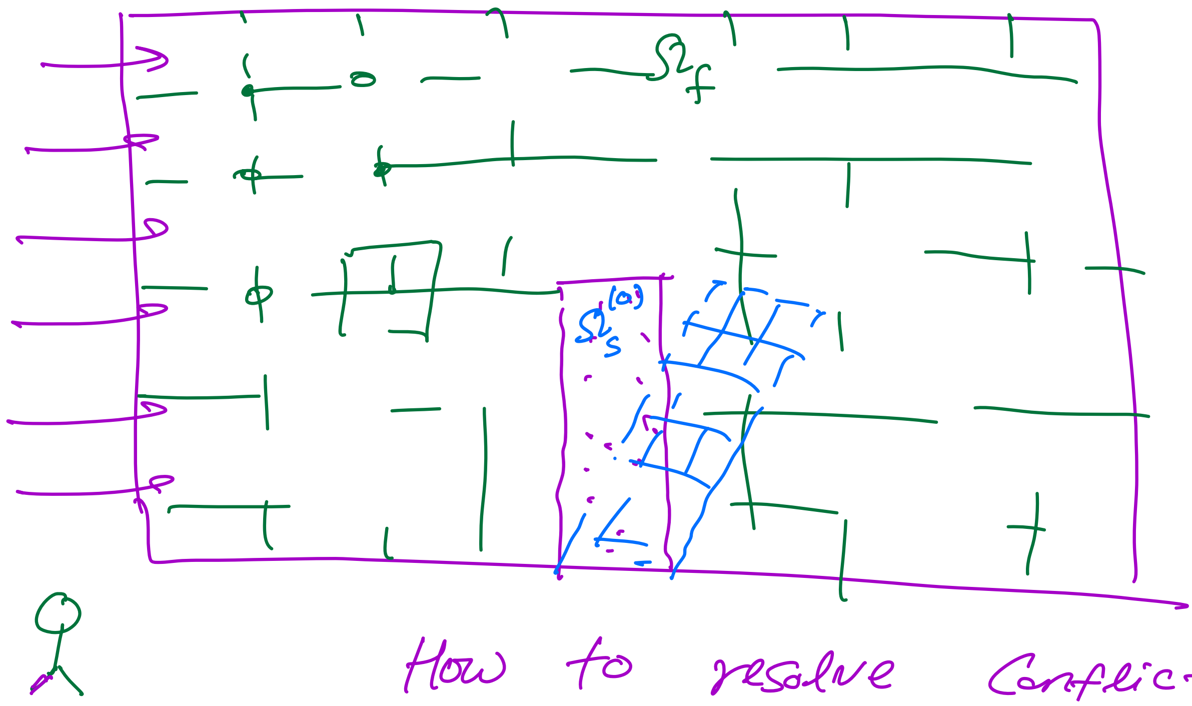
(Fully Eulerian)

Option B: Fluid E + Solid L

Option C: $\underbrace{\text{Fluid L + Solid L}}$

(Fully Lagrangian $\Rightarrow$ Low Re)





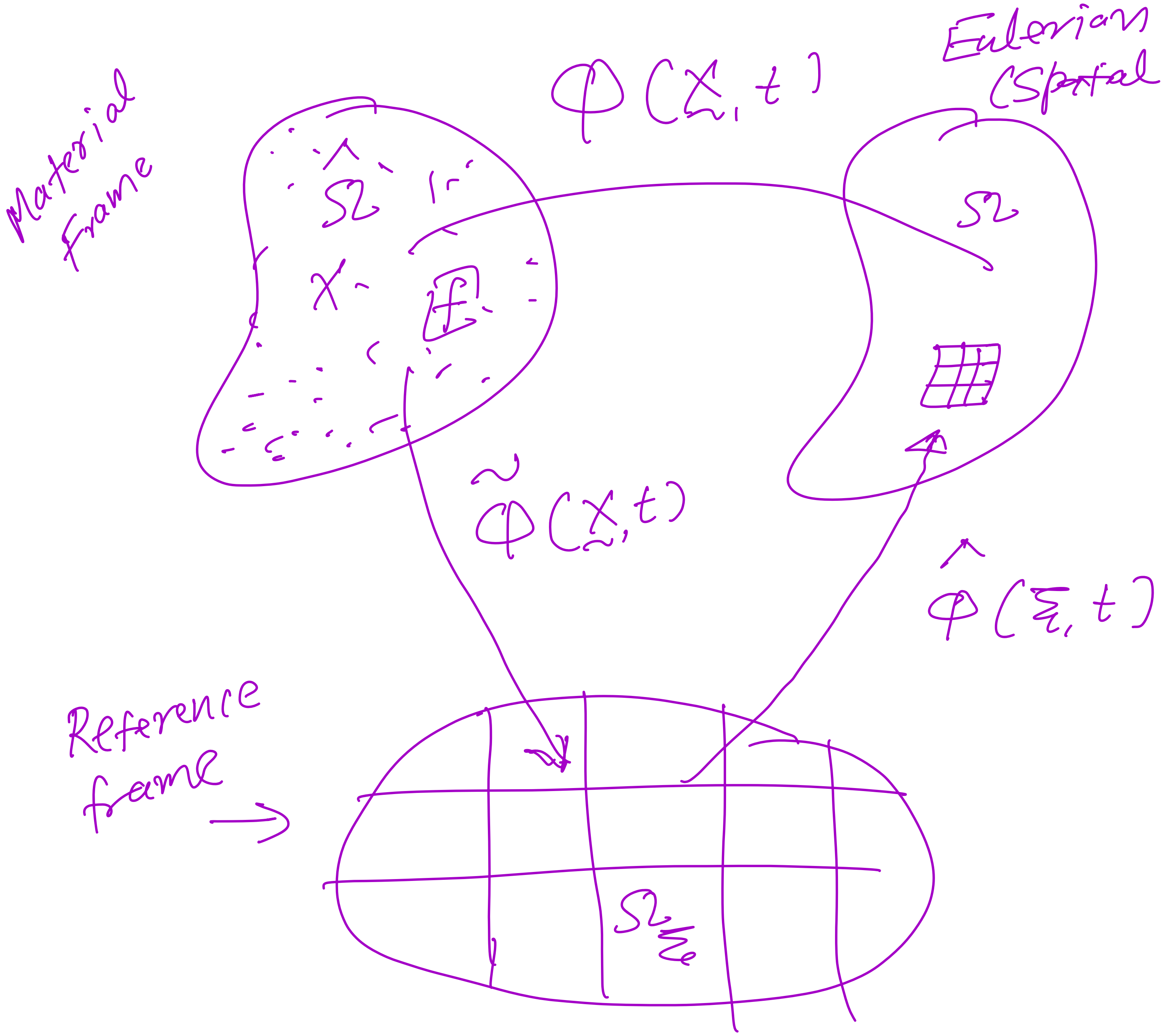
How to resolve conflict

→ in coordinate frames!

→ Introduce new frame
(Reference frame)
for moving fluid domain

→ Adjust / Accomodate Eulerian-
Lagrangian viewpoints

⇒ Arbitrary Lagrangian - Eulerian
frame / viewpoint /
(ALE-frame) coordinates.


Material Frame
Eulerian (Spatial
$\varphi(\underset{\sim}{X}, t)$
Ω
X
Ω
$\tilde{\varphi}(\underset{\sim}{X}, t)$
$\hat{\varphi}(\Xi, t)$
Reference frame
Ω_{Ξ}

Lagrangian to Eulerian Map :

$$\varphi(\cdot, t) : \hat{\Omega} \to \Omega(t)$$

Deformation map $\underset{\sim}{X} \to \underset{\sim}{x} = \varphi(\underset{\sim}{X}, t)$

Displacement $\underset{\sim}{u} = \varphi(\underset{\sim}{X}, t) - \varphi(\underset{\sim}{X}, 0)$

Velocity : $v = \dot{\varphi} = \left.\dfrac{\partial \varphi}{\partial t}\right|_X$, $F = \hat{\nabla} \varphi$

$J = \det(F)$

Consider physical information:
$f(\underset{\sim}{x}, t)$

$$\frac{d}{dt} f(\underset{\sim}{x}, t) = \dot{f}(x, t) = \left.\frac{\partial f}{\partial t}(\underset{\sim}{x}, t)\right|_X$$

$$= \frac{\partial f}{\partial t}\left(\varphi(\underset{\sim}{X}, t), t\right)$$

$$= \left.\frac{\partial f}{\partial t}\right|_x + \frac{\partial f}{\partial X} \cdot \left.\left(\frac{\partial \underset{\sim}{X}}{\partial t}\right)\right|_X$$

$$\boxed{\frac{df}{dt}(\underset{\sim}{x},t) = \left.\frac{\partial f}{\partial t}\right|_{x} + \underset{\sim}{v} \cdot \nabla \underset{\sim}{f}}$$

Referential - to - Eulerian Map:

$$\hat{\varphi} : \Omega_{\xi} \longrightarrow \Omega(t)$$

Referee $\quad\quad$ Eulerian

$$\frac{\partial f}{\partial t}(x,t) = \left.\frac{\partial f}{\partial t}\right|_{x} + v_{\hat{\varphi}} \cdot \nabla f$$

Referential Velocity

Lagrangian - to - Ref map:

Total change in f:

$$(*) \quad \frac{df}{dt} = \frac{d f(\xi, t)}{dt}\Big|_{\xi} + \underbrace{\frac{\partial f(x,t)}{\partial x}}\left(\frac{\partial x}{\partial \xi}\frac{\partial \xi}{\partial t}\right)$$

Spatial Coordinat

$$v^f = \frac{dx}{dt} = \frac{\partial x(\xi, t)}{\partial t} + \left(\frac{\partial x}{\partial \xi}\frac{\partial \xi}{\partial t}\right)$$

$$(**) \quad v^f = \underset{\sim}{\omega} + \underbrace{\frac{\partial x}{\partial \xi}\frac{\partial \xi}{\partial t}}$$

Ref. frame

$$\frac{df}{dt} = \frac{\partial f(\xi, t)}{\partial t} + \frac{\partial f}{\partial x}\cdot(v^f - \omega)$$

$$\Rightarrow \quad \frac{df}{dt} = \frac{\partial f(\xi, t)}{\partial t} + (v^f - \omega)\cdot\nabla f$$

Navier-Stokes Equation in ALE frame will become:

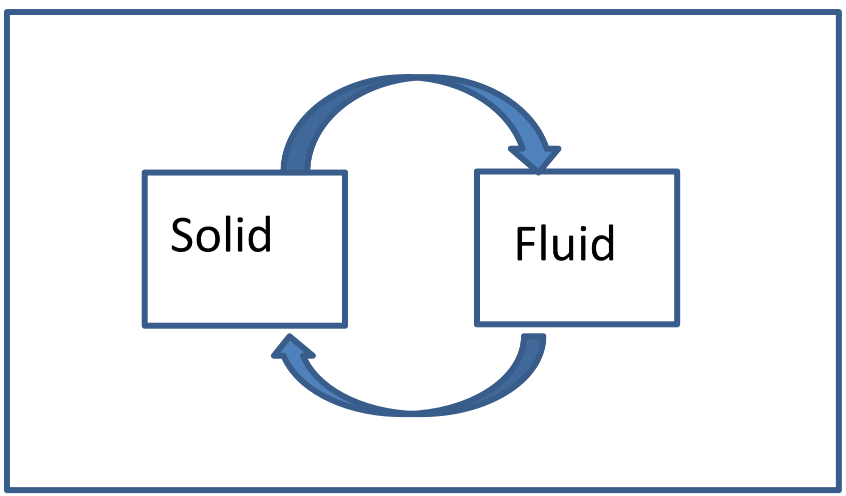
velocity new frame

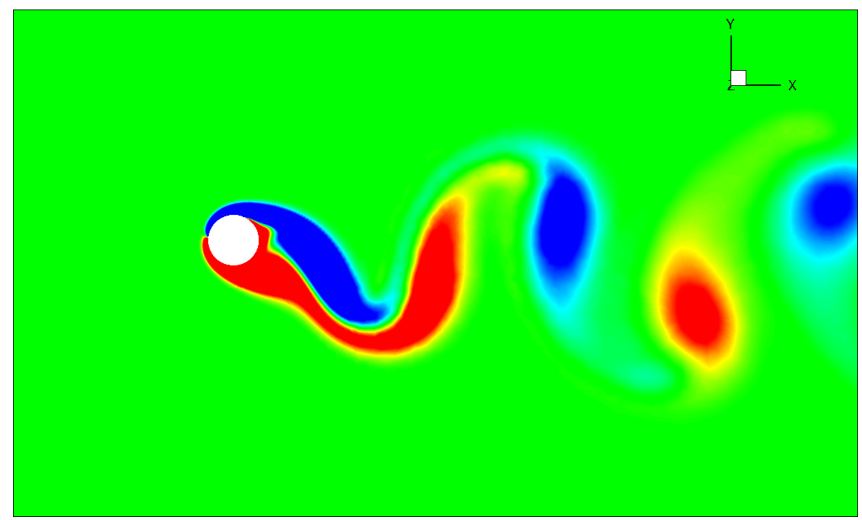
$$\rho^f \frac{\partial v^f}{\partial t}\Big|_{\xi} + \rho^f \underbrace{(\underset{\sim}{v}^f - \underset{\sim}{w})}_{\text{Relative convection velocity}} \cdot \nabla v^f = \underset{\sim}{\nabla \cdot \sigma^f} + \rho^f \underset{\sim}{b}^f$$

What and Why?

- General continuum mechanics and dynamical system

- Ex: Coupled fluid-structure systems
  - ► Feedback between motion and force





Fluid-Structure Interactions

- The main mathematical & computational difficulties:
  - Typically problems with changing domain
  - Interaction happens because forces exerted by the fluid deform the structure, affecting the motion of the fluid

- Conflict of descriptions for fluid in Eulerian coordinates vs. structure (solid) in Lagrangian coordinates

- Interface Conditions
  - Kinematic condition: The velocity of the fluid and the velocity of the solid particles are continuous on the interface.
  - Dynamic condition: The tractions of fluid and solid are continuous on the interface.
  - Geometric condition: Fluid- and solid-domain always match, no holes appear at the interface and the domains do not overlap.

Kinematics of Eulerian and Lagrangian Modeling

- In fluid-structure interaction, we study the kinematical motion and deformation of solid/structures due to the traction applied by fluid flow.
  - It is important to understand the kinematical modeling and coupling of the two fields.

To study the kinematics, there are two common domains in continuum mechanics: the material domain $\Omega_X \subset \mathbb{R}^n_{\text{sd}}$, with n_{sd} spatial dimensions, made up of material particles X and the spatial domain Ω_x consisting of spatial points x. We denote Ω_X or Ω the reference/undeformed configuration while Ω_x or simply $\Omega(t)$ denotes the current/deformed configuration. Furthermore, $\Omega(0)$ is the so-called initial configuration and $\widehat{\Omega} := \Omega(0)$. In the Lagrangian coordinate system, a specific material point and its deformation are observed in time. In contrast, using Eulerian coordinates, we observe a fixed point in space and observe what is happening at this spatial point while time is evolving. While solids/elasticity equations are usually described in the Lagrangian system, fluids (i.e., Navier-Stokes) are preferred in Eulerian coordinates.

Deformation field

Deformation field: A deformation of $\widehat{\Omega}$ is a smooth, one-to-one (i.e., bijective), orientation-preserving mapping

$$\varphi : \widehat{\Omega} \to \Omega(t) \quad \text{with } (\boldsymbol{X}, t) \mapsto (\boldsymbol{x}, t) = (\varphi(\boldsymbol{X}, t), t)$$

This mapping associates each point $X \in \widehat{\Omega}$ (of a reference domain) to a new position $x \in \Omega(t)$ (of the physical domain).

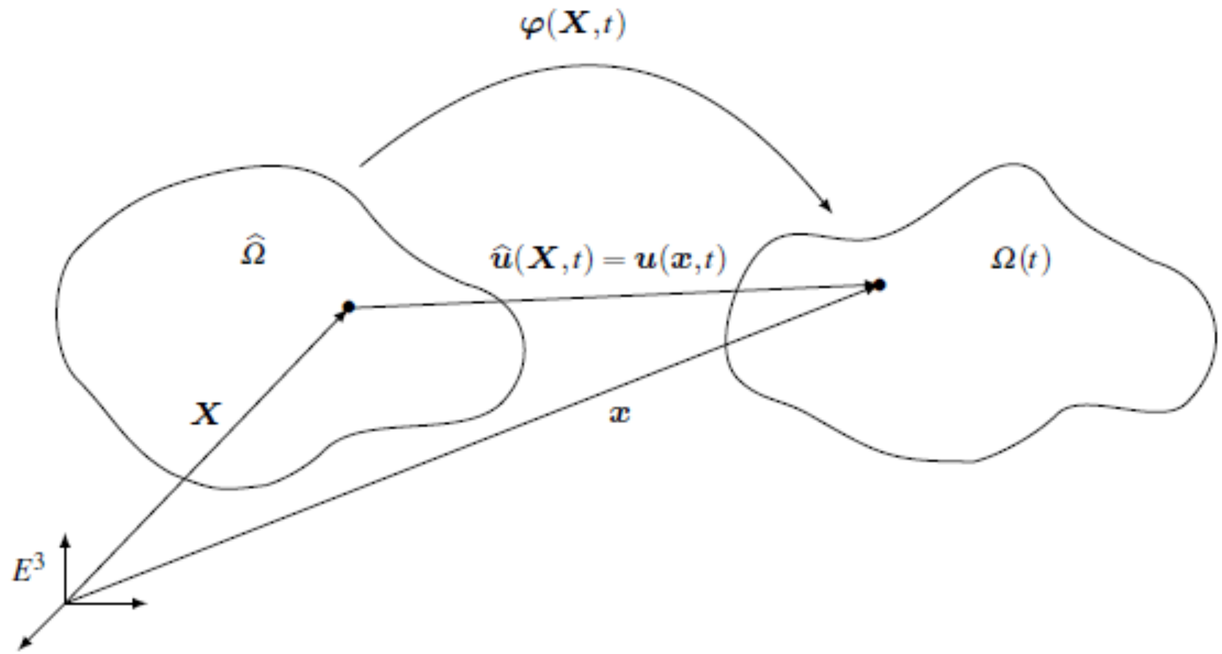


Fig. 3.1 Descriptions of displacements in Eulerian and Lagrangian coordinates. Traversing from the origin via $\widehat{\Omega}$ means $\boldsymbol{X} + \hat{\boldsymbol{u}} = \boldsymbol{x}$ and going from the origin via $\Omega(t)$ leads to $\boldsymbol{x} - \boldsymbol{u} = \boldsymbol{X}$.

Material/Lagrangian and Spatial/Eulerian Descriptions

Material/Lagrangian description:

$$\hat{u} : (X, t) \to \hat{u}(X, t) = x(X, t) - X$$

and it relates a particle's position in the reference configuration X to its corresponding position in the current configuration x at time t.

Spatial/Eulerian description:

$$u(x, t) = x - X(x, t)$$

This is formulated in terms of the current displacement, which results from its original position $\hat{x}$ plus the displacement for that position. We recapitulate that the two displacement descriptions can e transformed through the deformation; namely $x = \varphi(X, t)$

Total Derivatives of Lagrangian and Eulerian Fields

Total derivative of a Lagrangian field:

$$D_t \hat{f}\,(X,t) = \partial_t \hat{f}\,(X,t)$$

The material time derivative measures the rate at which $\hat{f}$ changes in time but following the path line of the particle. This means we measure the rate-change in time of exactly the same particle at all times.

Total derivative of an Eulerian field: The local time derivative of an Eulerian field is defined as

$$\partial_t f(x,t)$$

The current position x is held fixed while measuring the rate at which f changes in time at this fixed point. This means, at each time, f represents a new particle at x. The spatial time derivative is also known to be the local time derivative.

Relation of Deformation Gradient and Displacements

Deformation Gradient:

$$dx = F \cdot dX$$

with $\boldsymbol{F} = \nabla^{\boldsymbol{X}} \boldsymbol{x}$, i.e , $F_{ij} = \frac{\partial x_i}{\partial X_j}$

Deformation Gradient and Displacement:

$$x := x(X,t) = \varphi(X,t) = X + \hat{u}$$

The deformation gradient $\boldsymbol{F}$ can be expressed in terms of $\hat{\boldsymbol{u}}$ as follows:

$$\boldsymbol{F} = \nabla \varphi = \boldsymbol{I} + \widehat{\nabla} \hat{\boldsymbol{u}}$$

where $\widehat{\nabla} = \nabla^{\boldsymbol{X}}$. With the help of the deformation φ, we can represent the deformed configuration as $\Omega(t) = \varphi(\widehat{\Omega})$

Piola Transformation

Piola transformation:

$$\widehat{\boldsymbol{P}}(\boldsymbol{X}) := \widehat{J}(\boldsymbol{X})\boldsymbol{P}(\boldsymbol{x})\boldsymbol{F}^{-T}(\boldsymbol{X}) \quad \text{for} \quad \boldsymbol{x} = \varphi(\boldsymbol{X}, t)$$

In short

$$\widehat{\boldsymbol{P}} := \widehat{JP}\boldsymbol{F}^{-T} \quad \text{for} \quad \boldsymbol{x} = \varphi(\boldsymbol{X}, t)$$

The Piola transformation relates tensor fields between the deformed and reference configurations and is later used to transform the Cauchy stress tensor from Eulerian coordinates to the Piola-Kirchhoff stress tensor into the Lagrangian framework.

Surface Integral Transformation

- Nanson's formula is an important relation that can be used to go from areas in the current configuration to areas in the reference configuration and vice versa. This formula can be given as follows.

$$\boldsymbol{n}(\boldsymbol{x})dA_{\boldsymbol{x}} = \widehat{\boldsymbol{J}\boldsymbol{F}}^{-T}\boldsymbol{N}(\boldsymbol{X})dA_{\boldsymbol{X}}$$

where $\widehat{J} = \det(\boldsymbol{F}(\boldsymbol{X}))$

In view of fluid-structure interaction modeling, $\widehat{J} > 0$ allows to keep the orientation of a mapping φ and consequently φ is locally invertible. More specifically, each point $X \in \widehat{\Omega}$ possesses a neighborhood in which the map is injective. However, this idea can not be extended to the entire domain $\widehat{\Omega}$, which means that local invertibility does not imply general injectivity.

Continuum Mechanics of Moving Domains

The Lagrangian viewpoint consists of following the material particles of the continuum in their motion. The material coordinates, X, allow us to identify the reference configuration, $\widehat{\Omega}$. The motion of the material points relates the material coordinates, X, to the spatial ones, x. It is defined by a mapping φ such that $x := \varphi(X,t)$ Fluid-structure interaction systems often involve the resolution of the fluid dynamic equations on a moving (that is, time dependent) domain. The Lagrangian description allows easy tracking of free surfaces and interfaces between different materials. Its weakness is its inability to follow large distortions and topological changes of the domain without recourse to frequent remeshing operations. Eulerian algorithms are widely used in fluid mechanics. Here, the computational grid is fixed and the fluid moves with respect to the grid. The Eulerian formulation facilitates the treatment of large distortions in the fluid motion and is necessary for high Reynolds number flows. However, the Eulerian formulation cannot handle the moving boundaries due to fluid-structure interaction. Several approaches have been considered in order to deal with such problem. A classical way to overcome the difficulties due to the reconstruction of the mesh at each time step, is the introduction of the arbitrary Lagrangian-Eulerian (ALE) formulation, transporting the equations to a fixed arbitrary reference configuration.

Arbitrary Lagrangian Eulerian

- In the ALE description of motion, neither the material configuration nor the spatial configuration is taken as the reference.

- Thus, a third domain is needed: the referential configuration where reference coordinates are introduced to identify the grid points.

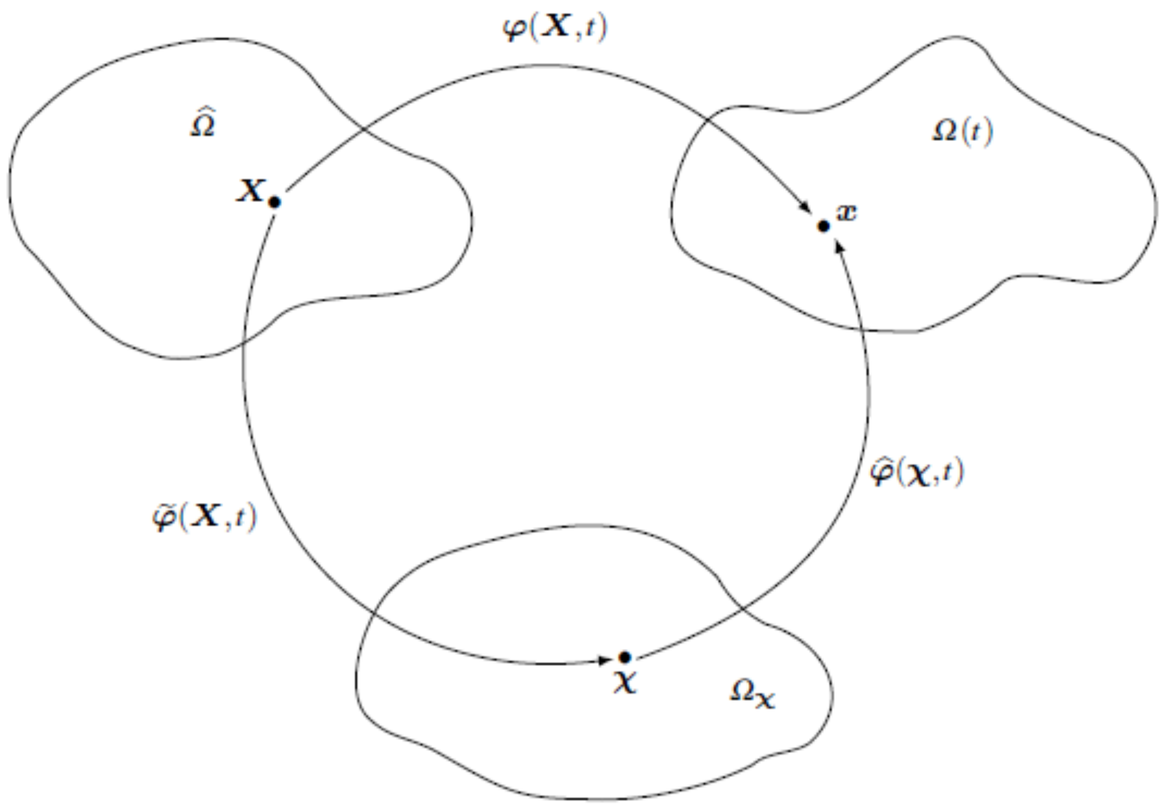


Fig. 3.2 Concept of reference configuration.

In the ALE description of motion, neither the material configuration $\widehat{\Omega}$ nor the spatial configuration $\Omega(t)$ is taken as the reference. Thus, a third domain is needed: the referential configuration Ω_χ where reference coordinates χ are introduced to identify the grid points. The referential domain Ω_χ is mapped into the material and spatial domains by $\widetilde{\varphi}$ and $\widehat{\varphi}$ respectively. The reference configuration moves arbitrarily with respect to the Eulerian and Lagrangian coordinates. The particle motion φ may then be expressed as $\varphi = \tilde{\varphi} \circ \widehat{\varphi}^{-1}$, clearly showing that, the three mappings $\widehat{\varphi}, \tilde{\varphi}$ and φ are not independent. The mapping of $\widehat{\varphi}$ from the referential domain to the spatial domain can be understood as the motion of the grid points in the spatial domain.

Lagrangian-to-Eulerian Map

We first define the Lagrangian-to-Eulerian map φ

$$\begin{aligned} \varphi(\cdot,t) : \widehat{\Omega} \to \Omega(t) = \varphi(\widehat{\Omega},t), \qquad & \forall t \geq 0 \\ X \mapsto x = \varphi(X,t), \qquad & \forall X \in \widehat{\Omega} \end{aligned}$$

where the displacement vector $u = \varphi(X,t) - \varphi(X,0) = \varphi(X,t) - X$. The velocity, deformation gradient and Jacobian are given as:

$$v = \dot{\varphi} = \left.\frac{\partial \varphi}{\partial t}\right|_X, \quad F = \widehat{\nabla}\varphi = \widehat{\nabla}x, \text{ and } \quad J = \det(F)$$

respectively. Let us now represent a scalar physical quantity during this transformation.

$$\begin{aligned} \dot{f}(x,t) &= \left.\frac{\partial f(x,t)}{\partial t}\right|_X \\ &= \left.\frac{\partial f(\varphi(X,t),t)}{\partial t}\right|_X \\ &= \left.\frac{\partial f}{\partial t}\right|_x + \nabla f \cdot \left.\frac{\partial \varphi}{\partial t}\right|_X \\ &= \left.\frac{\partial f}{\partial t}\right|_x + v \cdot \nabla f \end{aligned}$$

Referential-to-Eulerian Map

We next define the Lagrangian-to-Eulerian map $\hat{\varphi}$

$$\begin{aligned} \widehat{\varphi}(\cdot, t) : \Omega_\chi \to \Omega(t) = \widehat{\varphi}(\Omega_\chi, t), & \qquad \forall t \geq 0 \\ \chi \mapsto \boldsymbol{x} = \widehat{\varphi}(\chi, t), & \qquad \forall \chi \in \Omega(t) \end{aligned}$$

where the displacement $\boldsymbol{u}_{\hat{\varphi}} = \widehat{\varphi}(\chi, t) - \hat{\varphi}(\chi, 0) = \hat{\varphi}(\chi, t) - \chi$ and the velocity, deformation gradient and Jacobian are

$$v_{\hat{\varphi}} = \left.\frac{\partial \hat{\varphi}}{\partial t}\right|_{\chi}, \quad \boldsymbol{F}_{\hat{\varphi}} = \nabla^{\boldsymbol{X}} \hat{\varphi} = \nabla^{\chi} \boldsymbol{x}, \text{ and } \quad J_{\hat{\varphi}} = \det\left(\boldsymbol{F}_{\hat{\varphi}}\right)$$

respectively. A scalar physical quantity during this transformation can be represented as

$$\begin{aligned} \left.\frac{\partial f(\boldsymbol{x}, t)}{\partial t}\right|_{\boldsymbol{\chi}} &= \left.\frac{\partial f(\widehat{\varphi}(\boldsymbol{\chi}, t), t)}{\partial t}\right|_{\boldsymbol{\chi}} \\ &= \left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{x}} + \nabla f \cdot \left.\frac{\partial \hat{\varphi}}{\partial t}\right|_{\chi} \\ &= \left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{x}} + \boldsymbol{v}_{\hat{\varphi}} \cdot \nabla f \end{aligned}$$

Lagrangian-to-Referential Map

We next define the Lagrangian-to-Referential map $\widetilde{\varphi}$. The map is defined as: $\widetilde{\varphi} = \hat{\varphi}^{-1} \circ \varphi$

$$\begin{aligned} \widetilde{\varphi}(\cdot,t) : \widehat{\Omega} \to \Omega_\chi = \widetilde{\varphi}(\widehat{\Omega},t), \qquad & \forall t \geq 0 \\ \boldsymbol{X} \mapsto \chi = \widetilde{\varphi}(\boldsymbol{X},t), \qquad & \forall \boldsymbol{X} \in \widehat{\Omega} \end{aligned}$$

The displacement function $u_{\tilde{\varphi}} = \tilde{\varphi}(X,t) - \tilde{\varphi}(X,0) = \tilde{\varphi}(X,t) - X$. The velocity, deformation gradient and Jacobian in this case are given by

$$v_{\tilde{\varphi}} = \left.\frac{\partial \tilde{\varphi}}{\partial t}\right|_X, \quad F_{\tilde{\varphi}} = \widehat{\nabla} \widetilde{\varphi} = \widehat{\nabla} \chi, \text{ and } \quad J_{\tilde{\varphi}} = \det(F_\varphi)$$

respectively. The time derivative of field thus will be:

$$\dot{f}(\chi,t) = \left.\frac{\partial f}{\partial t}\right|_\chi + v_{\tilde{\varphi}} \cdot \nabla^\chi f$$

Using the fundamental kinematic relationship by substituting $f = x = \widetilde{\varphi}(\chi,t)$ in Eq. time derivative is:

$$\begin{aligned} \dot{f}(x,t) &= \left.\frac{\partial f}{\partial t}\right|_x + v \cdot \nabla f \\ &= \left.\frac{\partial f}{\partial t}\right|_\chi - v_{\hat{\varphi}} \cdot \nabla f + v \cdot \nabla f \\ &= \left.\frac{\partial f}{\partial t}\right|_\chi + c \cdot \nabla f \end{aligned}$$

Here c is the particle velocity relative to the mesh as seen from the Eulerian frame and v_φ is the particle velocity as observed from the referential frame.

Concept of ALE for Moving Continua (Deformable Matter)

- Concept of ALE from continuum mechanics
  - ► ALE provides an intermediate state in which the fluid domain is moved according to the solid. This requires a mapping between the deformed state and a reference configuration

The ALE mapping is defined in terms of the fluid mesh displacement $\hat{u}^{\mathrm{f}}$ such that

$$\widehat{\mathscr{A}}(\hat{x},t) : \widehat{\Omega}^{\mathrm{f}} \times I \to \Omega^{\mathrm{f}}(t), \quad \text{with } \widehat{\mathscr{A}}(\hat{x},t) = \hat{x} + \hat{u}^{\mathrm{f}}(\hat{x},t)$$

It is specified through the deformation gradient and its determinant

ALE Mapping

The ALE mapping is defined in terms of the fluid mesh displacement $\hat{u}^{\mathrm{f}}$ such that

$$\widehat{\mathscr{A}}(\hat{x},t) : \widehat{\Omega}^{\mathrm{f}} \times I \to \Omega^{\mathrm{f}}(t), \quad \text{with } \widehat{\mathscr{A}}(\hat{x},t) = \hat{x} + \hat{u}^{\mathrm{f}}(\hat{x},t)$$

It is specified through the deformation gradient and its determinant

$$\widehat{\boldsymbol{F}} := \widehat{\nabla}\widehat{\mathscr{A}} = \hat{\boldsymbol{I}} + \widehat{\nabla}\hat{\boldsymbol{u}}^{\mathrm{f}}, \quad \widehat{J} := \det(\widehat{\boldsymbol{F}})$$

Furthermore, function values in Eulerian and Lagrangian coordinates are identified by

$$\boldsymbol{u}^{\mathrm{f}}(\boldsymbol{x}) =: \hat{\boldsymbol{u}}^{\mathrm{f}}(\hat{\boldsymbol{x}}), \quad \text{with } \boldsymbol{x} = \widehat{\mathscr{A}}(\hat{\boldsymbol{x}},t)$$

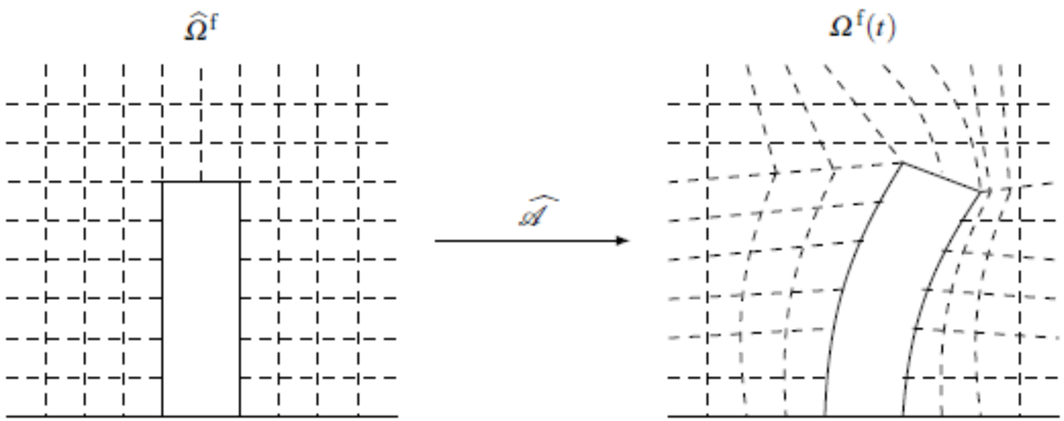


Fig. 3.3 Concept of ALE mapping.

Remarks on Time Derivatives

The derivation of both approaches follows the same rules that we sketch in the following. Let us briefly explain the relations between different time derivatives for different frameworks (such as the Lagrangian, the Eulerian, and the ALE frameworks). In a Lagrangian setting, the total and the partial derivatives coincide:

$$d_t \hat{f}\,(\widehat{\boldsymbol{x}}, t) = \partial_t \hat{f}\,(\widehat{\boldsymbol{x}}, t)$$

In an Eulerian framework, we follow the standard relation between the material time-derivative (the total time derivative) $d_t f$ and the partial time derivative $\partial_t f$:

$$d_t f(x,t) = \partial_i f(x,t) + v \cdot \nabla f$$

where the additional term $v \cdot \nabla f$ is referred to as a transport term.

ALE Time Derivatives

The ALE time derivative is defined as

$$\hat{\partial}_t f(x,t) := \partial_t|_{\hat{f}} f(x,t) = \partial_t f(x,t) + w \cdot \nabla f$$

where the transport term appears due to the motion of the computational domain.

- In a Lagrangian description, we have $w = v$ i.e., the domain is moving with the fluid velocity $\boldsymbol{v}$
- In a fixed Eulerian setting, it holds $w = 0$ i.e., the domain is fixed.
- In ALE, we have $0 \leq w \leq v$. Later we will see that $w = v$ at the fluid-structure interface $\widehat{\Gamma}^{\mathrm{fs}}$ and a bit away, we have $0 < w < v$, while far away $w = 0$ (the mesh is not moving anymore). Thus, in ALE depending on the location in the domain, we use both Eulerian and Lagrangian frameworks with a smooth transition
between them.

MECH 570 C

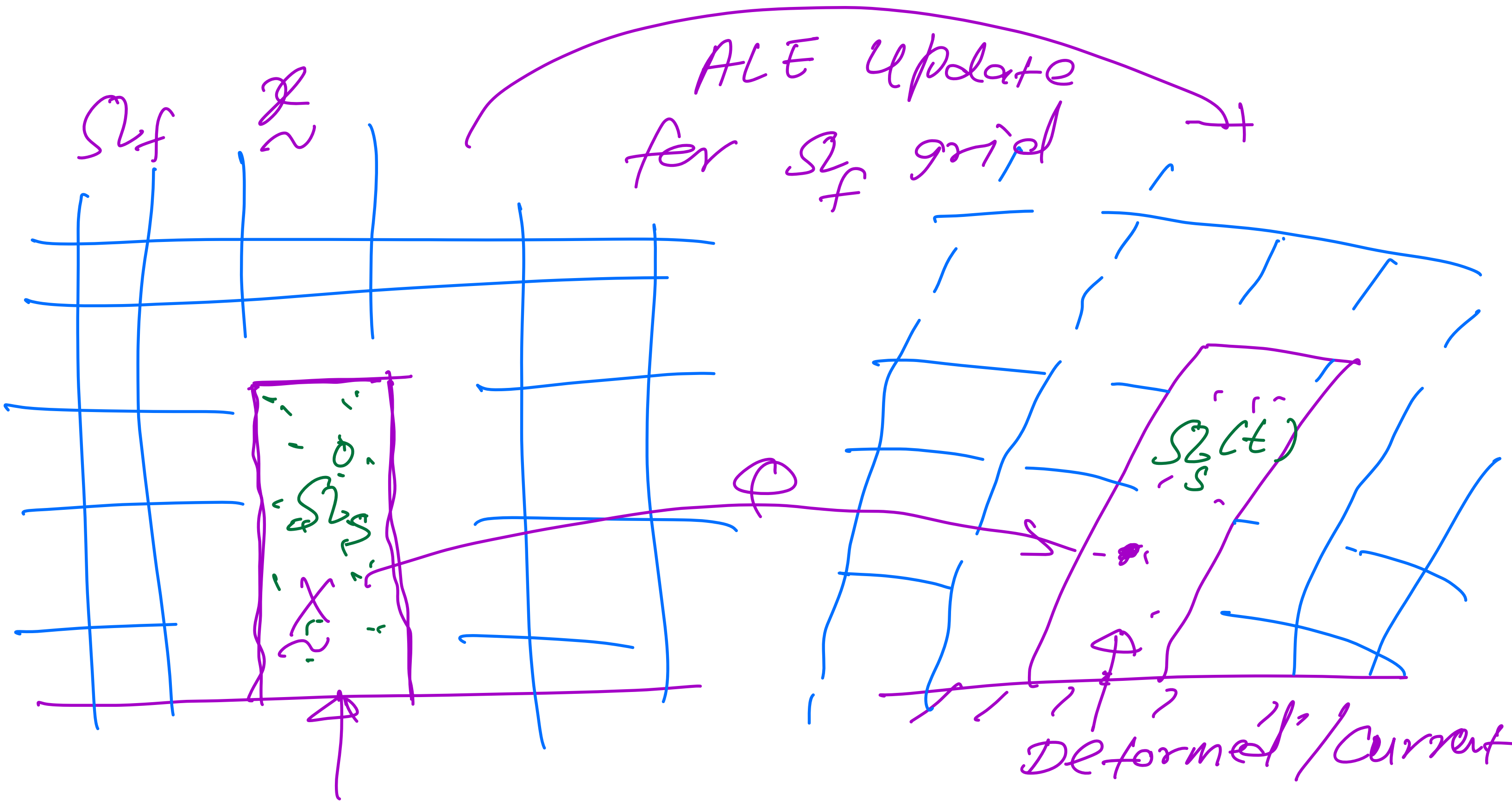
Fluid-Solid Interaction (FSI)

0. Review
   → Concept of ALE Mapping
   → Application to FSI

1. ALE-based Coupled FSI system

2. Interface Conditions

3. Way Forward to Variational principle
   → Basic background
   → Weighted residual / Finite Element Method (FEM)



Reference/Material Coordinate

$$\underset{\sim}{\varphi}(X, t)$$

Cauchy stress

$$\uparrow$$

$$\underset{\sim}{S} \rightarrow [\;\;]$$

$\Downarrow$ Piola transform

$$\mathbb{P}$$

→ ALE-based Navier-Stokes Eqns

→ Incompressible flow

→ Newtonian fluid (Eulerian)

→ Lagrangian solid description

(rigid body, linear/nonlinear elasticity)

→ Additional / Auxiliary reference frame to address the conflict of Eulerian (fluid) vs. Lagrangian (solid)

Fluid $\Omega_f(t)$ $\quad \underset{\sim}{S} \longrightarrow \underset{\sim}{\sigma}^f$

Cauchy stress

Momentum:

$$\rho^f \frac{\partial \underset{\sim}{v}^f}{\partial t} + \rho^f (\underset{\sim}{v}^f \cdot \nabla) \underset{\sim}{v}^f = \underbrace{\nabla \cdot \sigma^f} + \rho^f \underset{\sim}{b}^f$$

$\rightarrow$

$$\Omega^f(t) \times [0, T]$$

Mass / Continuity:

$$\nabla \cdot \underset{\sim}{v}^f = 0$$

$\underset{\sim}{v}^f = \underset{\sim}{v}^f(\underset{\sim}{x}, t)$ for each spatial point $\underset{\sim}{x}$ @ time t,

$\underset{\sim}{\sigma}^f$ is Cauchy Stress tensor

$$\underset{\sim}{\sigma}^f = -p I + \underset{\sim}{\tau}^f, \qquad \underset{\sim}{\tau}^f = 2\mu^f \epsilon^f(v^f)$$

$$\epsilon^f = \frac{1}{2}\left[\nabla v^f + (\nabla v^f)^T\right]$$

This is the Standard Newtonian isothermal Flow system.

Solid Part: $\hat{\Omega} \to \Omega^s$

(Material / Lagrangian frame)

$\underset{\sim}{X} \in \Omega^s$ deforms to new current position via $\varphi^s(\underset{\sim}{X}^s, t)$ at time t

$$v^s = \frac{\partial}{\partial t} \varphi^s(\underset{\sim}{X}^s, t)$$

Momentum balance for Solid:

$$\rho^s \frac{\partial^2 \varphi^s}{\partial t^2} = \nabla \cdot \mathbb{P}^s + \rho^s b^s \quad \text{on } \Omega^s \times [0, T]$$

$\mathbb{P}^s$: First Piola-Kirchhoff Stress tensor

$$\mathbb{P}^s(\varphi^s) = \mu^s\left[\nabla u^s + (\nabla u^s)^T\right] + \lambda^s(\nabla\cdot u^s)\,I$$

$$\dot{\underset{\sim}{u}}^s(\underset{\sim}{X}^s, t) = \varphi^s(\underset{\sim}{X}, t) - \underset{\sim}{X}$$

λ^s & μ^s Lame's parameters

$$\lambda^s = \frac{E^s \nu^s}{(1+\nu^s)(1-2\nu^s)}, \quad \mu^s = \frac{E^s}{2(1+\nu_s)}$$

$$\mathbb{P}^s(\varphi^s) = 2\mu^s \underset{\sim}{F}\,\underset{\sim}{E} + \lambda^s(\mathrm{tr}\,E)\,\underset{\sim}{F}$$

$$\underset{\sim}{F} = \nabla\varphi^s = (I + \nabla \underset{\sim}{u}^s)$$

E : Green-Lagrange Strain

$$E = \frac{1}{2}\left(\underset{\sim}{F}^T \underset{\sim}{F} - I \right)$$

$\Uparrow$ Right Cauchy-Green Strain

$\Rightarrow$ Mapping fluid domain using ALE frame to make Eulerian & Lagrangian frame compatible!

$\underset{\sim}{x}$ $\quad$ $\underset{\sim}{X}$ $\quad$ $\underset{\sim}{\xi}$

chain rule

(*) $$\underset{\sim}{v}^f = \frac{Dx^f}{Dt} = \frac{\partial x^f(\xi, t)}{\underbrace{\partial t}_{\underset{\sim}{\omega}}} + \frac{\partial x}{\partial \xi}\frac{\partial \xi}{\partial t}\bigg|_{\xi}$$

(**) $$\frac{Df}{Dt} = \dot{f} = \frac{\partial f(\xi, t)}{\partial t}\bigg|_{\xi} + \boxed{\frac{\partial f(x,t)}{\partial x}}\frac{\partial x}{\partial \xi}\frac{\partial \xi}{\partial t}\bigg|_{\xi}$$

From $(*)$:

$$\frac{\partial x}{\partial \underset{\sim}{\xi}} \frac{\partial \xi}{\partial t} = \underset{\sim}{v}^f - \underset{\sim}{\omega}$$

$$\frac{D\underset{\sim}{f}}{Dt} = \frac{\partial \underset{\sim}{f}(\underset{\sim}{\xi}, t)}{\partial t} + \underset{\sim}{(\underset{\sim}{v}^f - \underset{\sim}{\omega}) \nabla \underset{\sim}{f}}$$

$$\underset{\sim}{f} \longrightarrow \underset{\sim}{v}^f$$

Transform Navier-Stokes in ALE frame:

$$\rho^f \left.\frac{\partial \underset{\sim}{v}^f}{\partial t}\right|_{\underset{\sim}{\xi}} + \rho^f (\underset{\sim}{v}^f - \underset{\sim}{\omega}) \cdot \nabla v^f = \nabla \cdot \sigma^f + \rho^f b^f$$

ALE mesh velocity

Coupled Fluid-Structure System

We need to consider suitable coordinate systems to solve the governing equations that describe the coupled fluid-structure interactions. In this work, we have considered three different coordinate systems. The Lagrangian or material coordinate system with material points X is considered to represent flexible structure's velocity, $\boldsymbol{v}^{\mathrm{s}}(\boldsymbol{X}, t)$ and displacement, $\boldsymbol{u}^{\mathrm{s}}(\boldsymbol{X}, t)$, at any time t. In this coordinate system, each computational node follows the corresponding material points. The Eulerian or spatial coordinate system has been considered to describe the fluid velocity, $v^{\mathrm{f}}(x, t)$ and pressure, $p(x, t)$, at a spatial point x and time t. Each spatial point in this reference system is fixed in space and fluid flows through the regions formed by these nodes. However, to handle the problem of fluid-structure interactions with moving interface boundaries we introduce a third type of coordinate system for the fluid domain with reference nodes χ, known as ALE or arbitrary Lagrangian-Eulerian. In this coordinate system, the computational nodes can move relative to the spatial coordinate system. The nodes on the fluid-structure interface behave like material points and the nodes inside the fluid domain can be held fixed like spatial points or moved arbitrarily to account for the interface deformation.

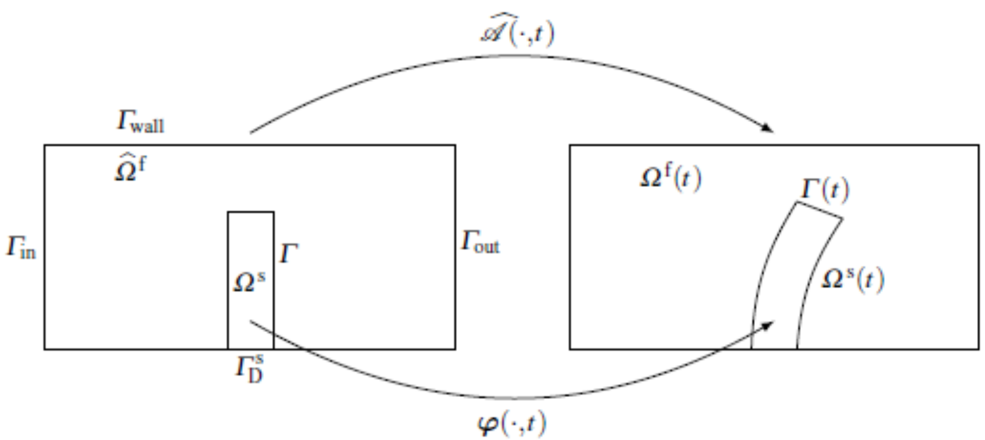


Fig. 3.4 Generic problem in deformable solid.

Coupled FSI Formulation (1)

Fluid problem

Let $\Omega^{\mathrm{f}}(t) \subset \mathbb{R}^d$ be an Eulerian fluid domain at time t, where d is the space dimension. The motion of an incompressible viscous fluid in $\Omega^{\mathrm{f}}(t)$ is governed by the following Navier-Stokes equations:

$$\begin{aligned} \rho^{\mathrm{f}} \frac{\partial v^{\mathrm{f}}}{\partial t} + \rho^{\mathrm{f}} v^{\mathrm{f}} \cdot \nabla v^{\mathrm{f}} &= \nabla \cdot \sigma^{\mathrm{f}} + \rho^{\mathrm{f}} b^{\mathrm{f}}, \qquad && \text{on } \Omega^{\mathrm{f}}(t) \times [0,T] \\ \nabla \cdot v^{\mathrm{f}} &= 0, && \text{on } \Omega^{\mathrm{f}}(t) \times [0,T] \end{aligned}$$

Here $v^{\mathrm{f}} = v^{\mathrm{f}}(x,t)$ represents the fluid velocity defined for each spatial point x at time t, ρ^{f} is the fluid density, $\rho^{\mathrm{f}} b^{\mathrm{f}}$ denotes the body force and σ^{f} is Cauchy stress tensor which is given by

$$\sigma^{\mathrm{f}} = -pI + T^{\mathrm{f}}, \quad T^{\mathrm{f}} = 2\mu^{\mathrm{f}} \epsilon^{\mathrm{f}}(v^{\mathrm{f}}), \quad \epsilon^{\mathrm{f}}(v^{\mathrm{f}}) = \frac{1}{2}\left[\nabla v^{\mathrm{f}} + (\nabla v^{\mathrm{f}})^T\right]$$

where $p = p(x,t)$ is the fluid pressure, I denotes the second-order identity tensor, T represents the shear stress tensor, μ^{f} is the fluid dynamic viscosity and ϵ^{f} is the fluid strain rate tensor.

Coupled FSI Formulation (2)

Solid problem

Let $\Omega^s \subset \mathbb{R}^d$ be the reference Lagrangian domain for a flexible elastic structure. A material point, whose initial position is given by $X \in \Omega^s$, deforms to position $\varphi^S(\boldsymbol{X}, t)$ at time t with the structural momentum equation

$$\rho^s \frac{\partial^2 \varphi^s}{\partial t^2} = \nabla \cdot \sigma^s + \rho^s b^s, \text{ on } \Omega^s \times [0, T]$$

where $\boldsymbol{\sigma}^s$ denotes the first Piola-Kirchhoff stress tensor, $\rho^s \boldsymbol{b}^s$ represents the body force vector acting on the structure, and ρ^s is its mass density. For a linear elastic material,

$$\sigma^s(\varphi^s) = \mu^s \left[\nabla u^s + \left(\nabla \boldsymbol{u}^S \right)^T \right] + \lambda^s \left(\nabla \cdot \boldsymbol{u}^S \right) \boldsymbol{I}$$

where $\boldsymbol{u}^s(\boldsymbol{X}, t) = \varphi^s(\boldsymbol{X}, t) - \boldsymbol{X}$ is the displacement vector, μ^s and λ^s are the Lamé's coefficients of a material satisfying [208]

$$\lambda^s > 0 \quad \text{and} \quad 3\lambda^s + 2\mu^s > 0$$

The relationship between the Lame's coefficients and the flexible structure's elastic properties is given by

$$\lambda^s = \frac{E v^s}{(1 + v^s)(1 - 2v^s)} \quad \text{and} \quad \mu^s = \frac{E}{2(1 + v^s)}$$

where E is Young's modulus and v^s is the Poisson's ratio. For a St. Venant-Kirchhoff (SVK) material.

Coupled FSI Formulation (3)

Solid problem

$$\sigma^{s}(\varphi^{s}) = 2\mu^{s}\boldsymbol{F}\boldsymbol{E} + \lambda^{s}(\mathrm{tr}(\boldsymbol{E}))\boldsymbol{F}$$

where $\mathbf{tr}(\cdot)$ is the tensor trace operator and F is the deformation gradient tensor which is given by

$$\boldsymbol{F} = \nabla\varphi^{\mathrm{S}} = (\boldsymbol{I} + \nabla\boldsymbol{u}^{s})$$

and E represents the Green-Lagrangian strain tensor defined as

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^{T}\boldsymbol{F} - \boldsymbol{I}\right)$$

where the term $\boldsymbol{F}^{T}\boldsymbol{F}$ denotes the right Cauchy-Green deformation tensor. Before we present the Eulerian-Lagrangian coupling, we rewrite the above equation in terms of the structural velocity v^{s}, defined as

$$v^{s}(X,t) = \partial_{t}\varphi^{\mathrm{S}}(X,t)$$

This modification creates the path for the implementation of the velocity continuity (i.e. kinematic matching) along the fluid-structure interface. The structural governing equation can therefore be rewritten as

$$\rho^{s}\frac{\partial v^{s}}{\partial t} = \nabla\cdot\sigma^{s} + \rho^{s}b^{s},\ \text{on}\ \Omega^{s}\times[0,T]$$

Coupled FSI Formulation (4)

ALE formulation

Here, we will reiterate the derivation of the Navier-Stokes equations in the ALE framework. We consider a unique one-to-one mapping function $\hat{\varphi}^{\mathrm{f}}$ defined as

$$\boldsymbol{x} = \widehat{\varphi}^{\mathrm{f}}(\chi, t) \quad \forall t$$

which maps each node χ in the reference coordinate system to its corresponding spatial node $\boldsymbol{x}$. The gradient of the mapping function $\hat{\varphi}^{\mathrm{f}}$ will give us

$$\frac{\partial \widehat{\varphi}^{\mathrm{f}}}{\partial(\chi, t)} = \begin{pmatrix} \frac{\partial \boldsymbol{x}}{\partial \chi} & \boldsymbol{w} \\ 0^{\mathrm{T}} & 1 \end{pmatrix}$$

where $\boldsymbol{w}$ is the mesh velocity, i.e., the velocity with which spatial nodes $\boldsymbol{x}$ are moving with respect to the reference coordinate system and is written as

$$\boldsymbol{w} = \left.\frac{\partial \boldsymbol{x}}{\partial t}\right|_{\chi}$$

Notably, both the material and mesh move with respect to the laboratory frame. We can similarly write the relationships for the inverse map.

Coupled FSI Formulation (5)

ALE formulation

To define the conservation laws in the ALE framework, a relation between time derivatives are needed. The total (or material) time derivatives, which are used in the conservation laws, must be transformed to referential time derivatives. The material derivative of a function $f(\chi, t)$ defined on the ALE reference coordinate system yields

$$\frac{Df}{Dt} = \left.\frac{\partial f(\chi, t)}{\partial t}\right|_{\chi} + \left.\frac{\partial f(x, t)}{\partial x}\frac{\partial x}{\partial \chi}\frac{\partial \chi}{\partial t}\right|_{X}$$

The relation can be further simplified by considering the material derivative of the spatial coordinate x

$$v^{\mathrm{f}} = \frac{Dx}{Dt} = \left.\frac{\partial x(\chi, t)}{\partial t}\right|_{\chi} + \left.\frac{\partial x}{\partial \chi}\frac{\partial \chi}{\partial t}\right|_{X}$$

From the above equations, we get

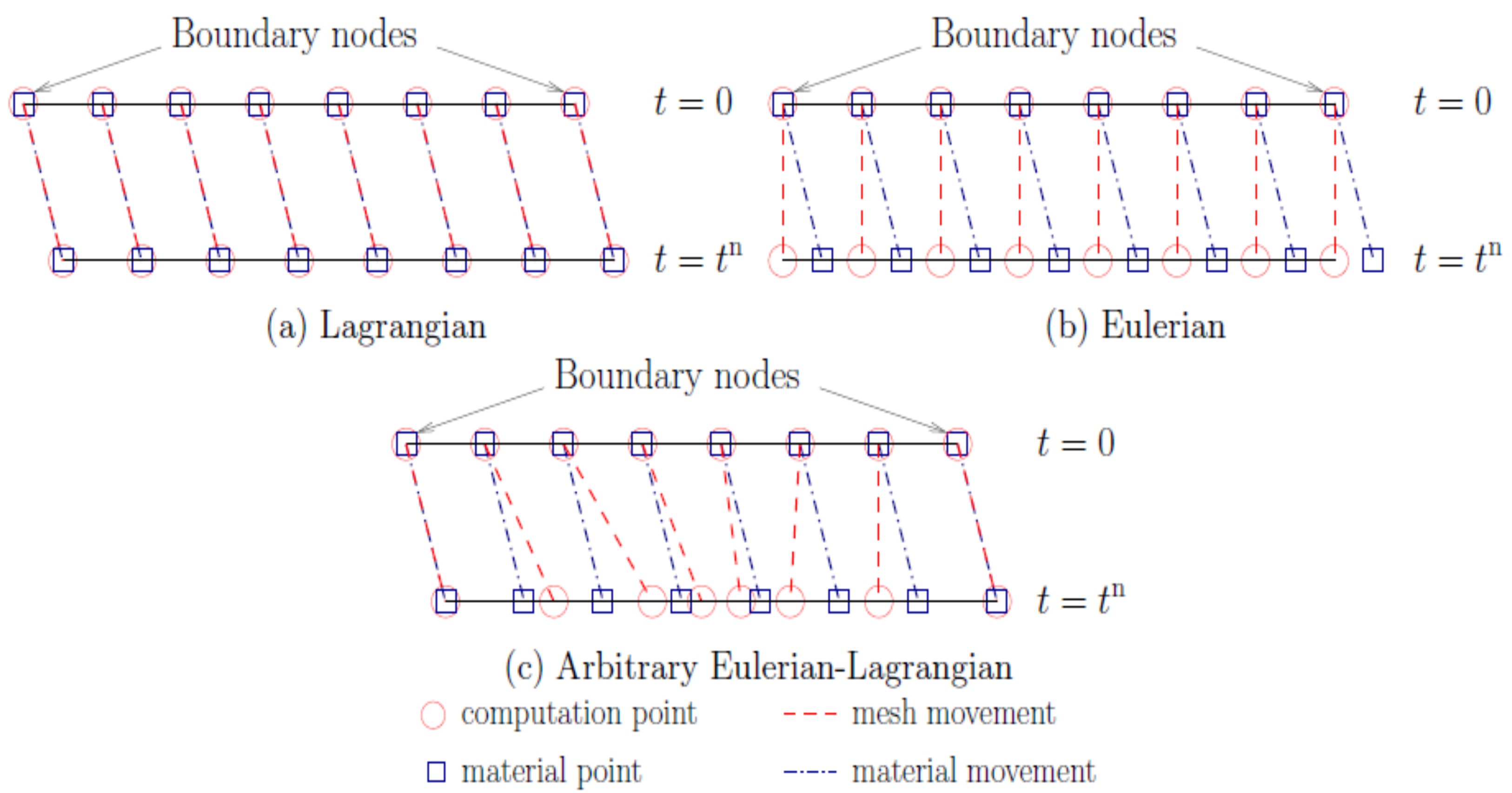
$$\frac{Df}{Dt} = \left.\frac{\partial f(\chi, t)}{\partial t}\right|_{\chi} + \left(v^{\mathrm{f}} - w\right)\frac{\partial f(x, t)}{\partial x}$$

We can obtain the Navier-Stokes equations in the ALE framework as

$$\begin{aligned} \rho^{\mathrm{f}} \left.\frac{\partial v^{\mathrm{f}}}{\partial t}\right|_{\chi} + \rho^{\mathrm{f}}\left(\boldsymbol{v}^{\mathrm{f}} - \boldsymbol{w}\right)\cdot\nabla\boldsymbol{v}^{\mathrm{f}} &= \nabla\cdot\boldsymbol{\sigma}^{\mathrm{f}} + \rho^{\mathrm{f}}\boldsymbol{b}^{\mathrm{f}}, && \text{on } \Omega^{\mathrm{f}}(t)\times[0,T] \\ \nabla\cdot\boldsymbol{v}^{\mathrm{f}} &= 0, && \text{on } \Omega^{\mathrm{f}}(t)\times[0,T] \end{aligned}$$

Arbitrary Lagrangian Eulerian (ALE)

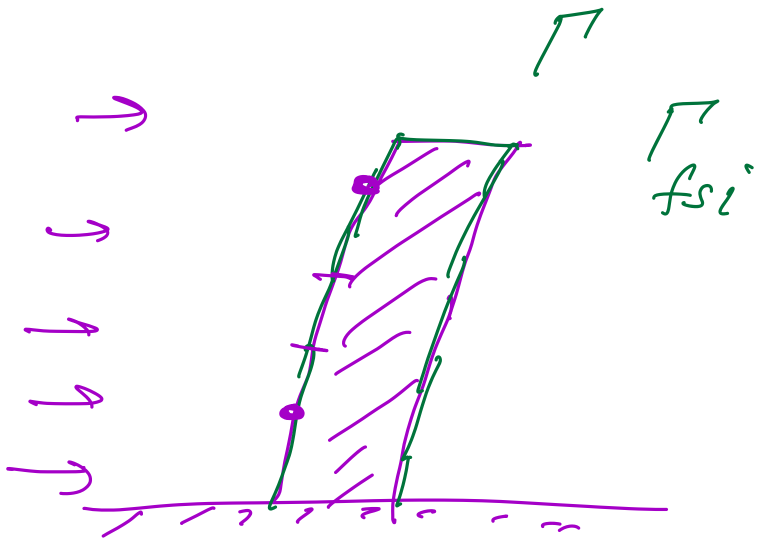
- The nodes of the computational mesh may be moved with the continuum in normal Lagrangian fashion or be held fixed in Eulerian manner



Interface Conditions (Fluid-Solid boundary)

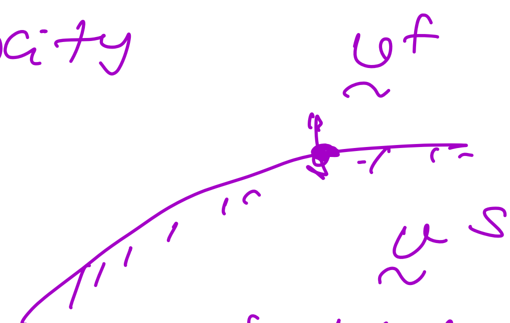
(I) Kinematic Condition :

Matching velocity (no-slip)



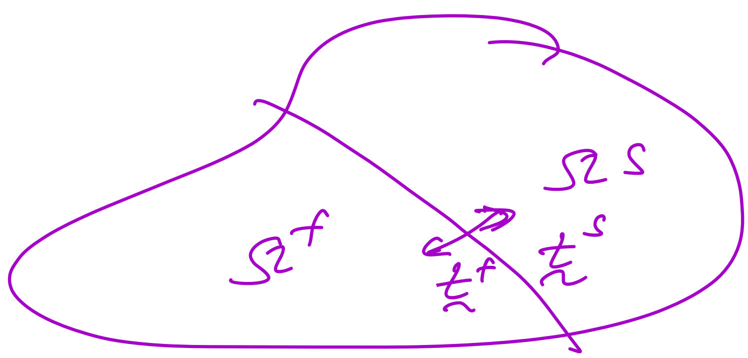
$\Rightarrow$ Continuity of velocity

$$\underset{\sim}{v}^f = \underset{\sim}{v}^s$$



Dirichlet Condition

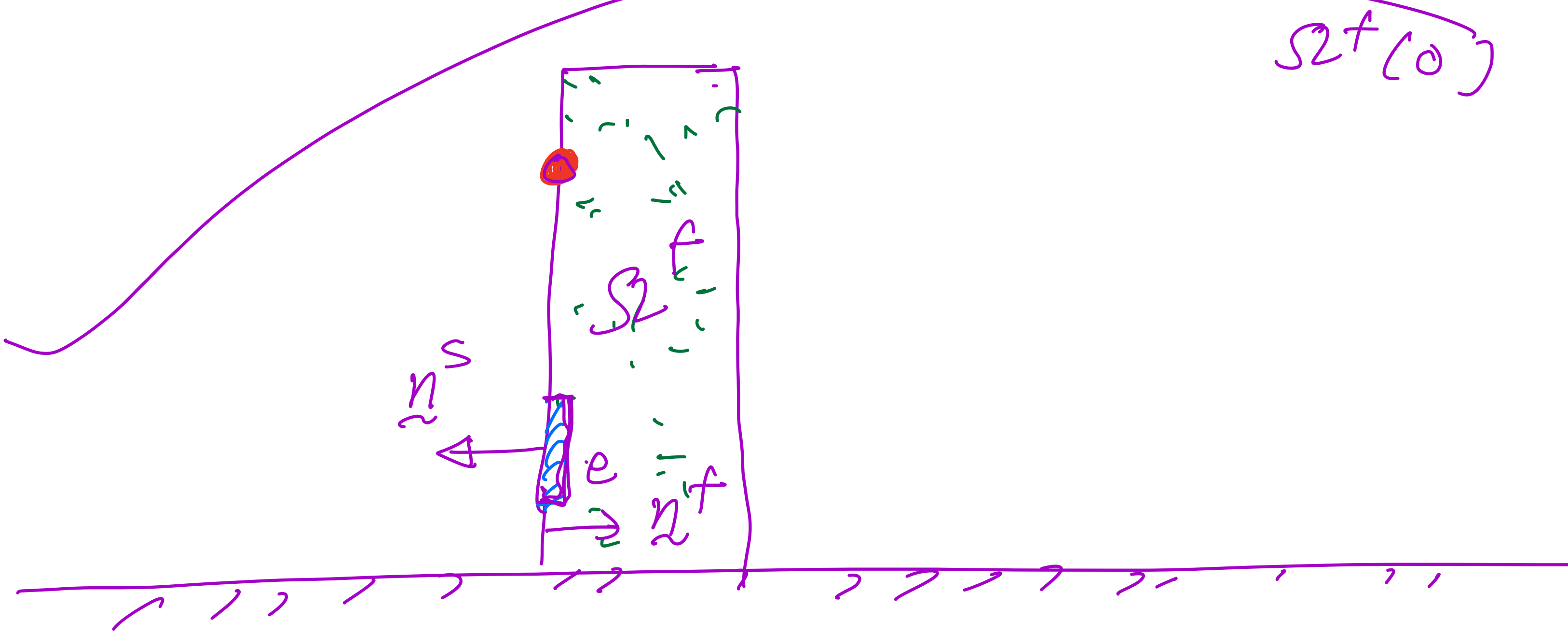
(II) Continuity of traction (Cauchy's Axiom)



(III) Geometry Condition : No holes or gaps allowed

$$\dot{x}^f = \underset{\sim}{v}^s \ @ \ \Gamma_{fsi}$$

At $t=0$



$$\int_e \underset{\sim}{\sigma}^s(\underset{\sim}{X}, t=0) \cdot n^s \, dA_X$$

$$\int \underline{\underset{\sim}{\sigma}^f(\underset{\sim}{x}, t) \cdot \underset{\sim}{n}^f} \, d\Gamma$$

Ref Cordinat

$$\underline{\Phi(e,t)} \quad + \quad \int_e \underset{\sim}{\sigma}^s(\underset{\sim}{X}, t) \cdot \underset{\sim}{n}^s \, d\Gamma$$

$$= 0$$

(Cauchy's traction Law)

Interface Conditions (1)

The interface coupling of the fluid with the structure equations must satisfy three conditions:

- continuity of velocities (no-slip);
- continuity of tractions (dynamic equilibrium);
- continuity of displacements (geometrical coupling of physical solids and fluid mesh motion).

Let us assume that the fluid boundary $\partial\Omega^{\mathrm{f}}(t)$ can be decomposed into three disjoint portions comprising of $\Gamma^{\mathrm{f}}_D(t), \Gamma^{\mathrm{f}}_N(t)$ and $\Gamma(t)$ at any time t, where Γ^{f}_D and Γ^{f}_N represent fluid Dirichlet and Neumann boundaries respectively, Γ represents the interface boundary between the fluid and structural domains at $t = 0$, i.e., $\Gamma = \partial\Omega^{\mathrm{f}}(0) \cap \partial\Omega^{\mathrm{s}}$ and $\Gamma(t)$ is the mapping of Γ from $\Omega^{\mathrm{s}} \to \Omega^{\mathrm{s}}(t)$, i.e., $\Gamma(t) = \varphi^{\mathrm{S}}(\Gamma, t)$. Similarly, we can consider that the solid boundary $\partial\Omega^{\mathrm{s}}$ can be decomposed into $\Gamma^{\mathrm{s}}_D, \Gamma^{\mathrm{s}}_N$ and Γ, where Γ^{s}_D and Γ^{s}_N represent solid Dirichlet and Neumann boundaries respectively.

The fluid and structural governing equations are coupled by the imposition of velocity and traction continuity relations along the fluid-structure interface. Mathematically, these relations can be written as

$$v^{\mathrm{f}}(\varphi^{\mathrm{S}}(\boldsymbol{X}, t), t) = v^{\mathrm{s}}(\boldsymbol{X}, t) \forall \boldsymbol{X} \in \Gamma$$

$$\int_{\varphi^{\mathrm{s}}(\gamma,t)} \boldsymbol{\sigma}^{\mathrm{f}}(\boldsymbol{x}, t) \cdot \boldsymbol{n}^{\mathrm{f}} d\Gamma + \int_{\gamma} \boldsymbol{\sigma}^{\mathrm{S}}(\boldsymbol{X}, t) \cdot \boldsymbol{n}^{\mathrm{s}} d\Gamma = 0 \forall \gamma \subset \Gamma$$

Interface Conditions (2)

where n^{f} and n^{s} are the unit outward normals to the deformed fluid element $\varphi^{\mathrm{s}}(\gamma, t)$ and its corresponding undeformed solid element γ, respectively. Here, γ is any part of fluid-structure interface Γ at $t = 0$. Note that it is equivalent to

$$\left(\det\left(\frac{\partial x}{\partial X}\right)\right)\left(\sigma^{\mathrm{f}}(x(X), t)\right)\left(\frac{\partial x}{\partial X}\right)^{-T} \cdot n^{\mathrm{s}} - \sigma^{\mathrm{s}}(X, t) \cdot n^{\mathrm{s}} = 0$$

F

for any $X \in \Gamma$, where $x(X) =| \varphi^{\mathrm{S}}(X, t)$. Refer to Nanson's formula or the derivation of area transformation.

$$F \quad A \quad (F)^{-T}$$

Application of ALE Formulation for FSI Equations

Given any $t^{\mathrm{n}} \in \mathbb{R}$, when the time t reaches t^{n}, we define a backward in time mapping $\hat{\varphi}^{\mathrm{f,n}}(\cdot, t)$ which maps $\Omega^{\mathrm{f}}(t^{\mathrm{n}})$ to $\Omega^{\mathrm{f}}(t)$ for $t \leq t^{\mathrm{n}}$, and reduces to identity map when $t = t^{\mathrm{n}}$. We can rewrite by the ALE description

$$\int_{\Omega^{\mathrm{f}}(t^{\mathrm{n}})} \rho^{\mathrm{f}} \left(\frac{d}{dt} v^{\mathrm{f}} \left(\widehat{\varphi}^{\mathrm{f,n}}(x,t), t \right) \Big|_{t=t^{\mathrm{n}}} + \left(v^{\mathrm{f}} - \partial_t \widehat{\varphi}^{\mathrm{f,n}}(x, t^{\mathrm{n}}) \right) \cdot \nabla v^{\mathrm{f}} \right) \cdot \psi^{\mathrm{f}} d\Omega$$
$$+ \int_{\Omega^{\mathrm{f}}(t^{\mathrm{n}})} \sigma^{\mathrm{f}} : \nabla \psi^{\mathrm{f}} d\Omega - \int_{\Omega^{\mathrm{f}}(t)} \left(\nabla \cdot v^{\mathrm{f}} \right) q d\Omega$$

$$+ \int_{\Omega^{\mathrm{s}}} \rho^{\mathrm{s}} \partial_t v^{\mathrm{s}}(\boldsymbol{X}, t^{\mathrm{n}}) \cdot \psi^{\mathrm{s}} d\Omega + \int_{\Omega^{\mathrm{s}}} \sigma^{\mathrm{s}} : \nabla \psi^{\mathrm{s}} d\Omega =$$
$$\int_{\Omega^{\mathrm{f}}(t^{\mathrm{n}})} \rho^{\mathrm{f}} b^{\mathrm{f}} \cdot \psi^{\mathrm{f}} d\Omega + \int_{\Sigma_2} \sigma_n^{\mathrm{f}} \cdot \psi^{\mathrm{f}} d\Gamma + \int_{\Omega^{\mathrm{s}}} \rho^{\mathrm{s}} b^{\mathrm{s}} \cdot \psi^{\mathrm{s}} d\Gamma + \int_{\Sigma_4} \sigma_n^{\mathrm{s}} \cdot \psi^{\mathrm{s}} d\Gamma$$

Here $\psi^{\mathrm{f}}, \psi^{\mathrm{s}}$ and q are the weighting or test functions, which will be discussed later in the context of weak/variational formulation.

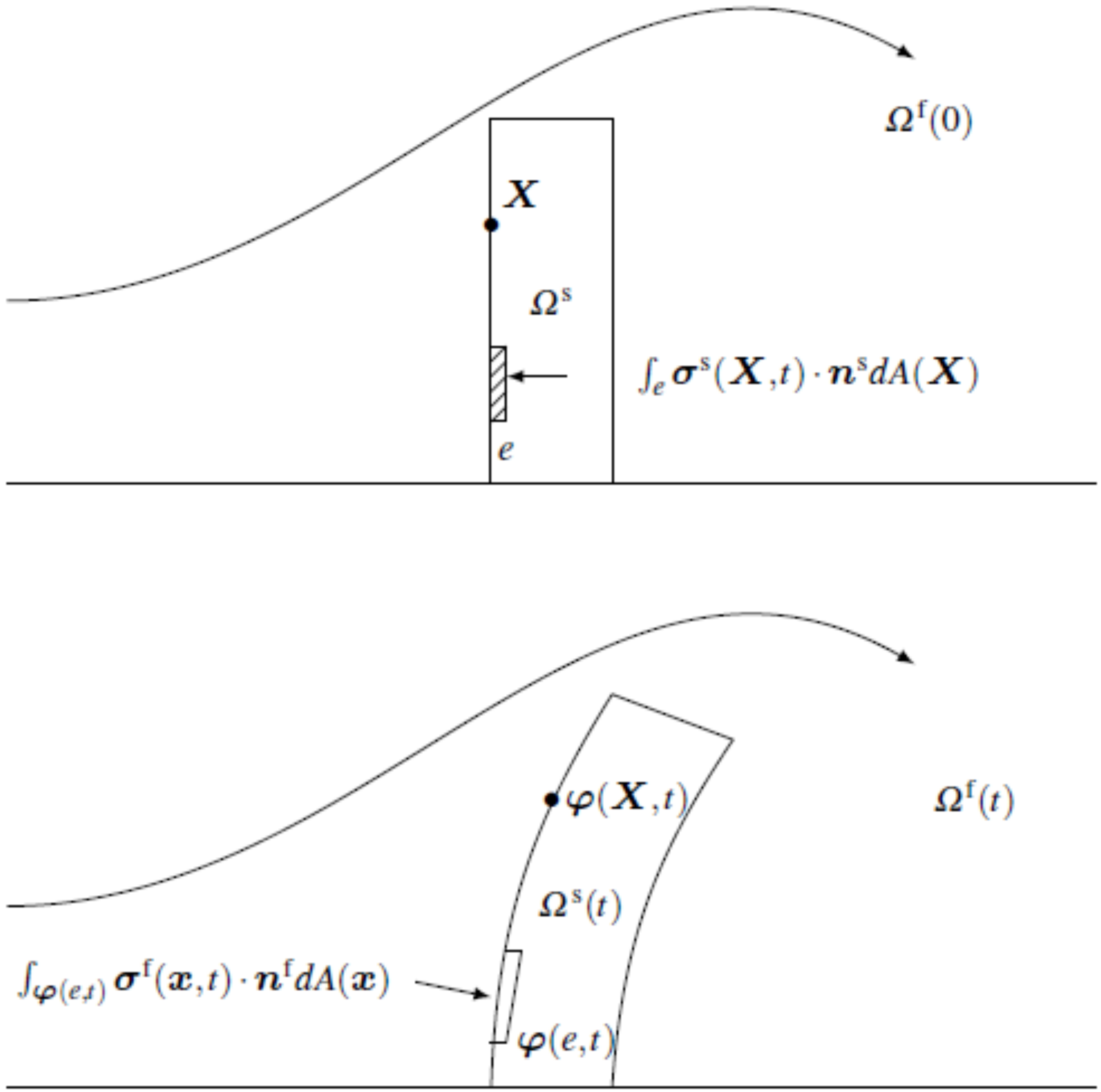
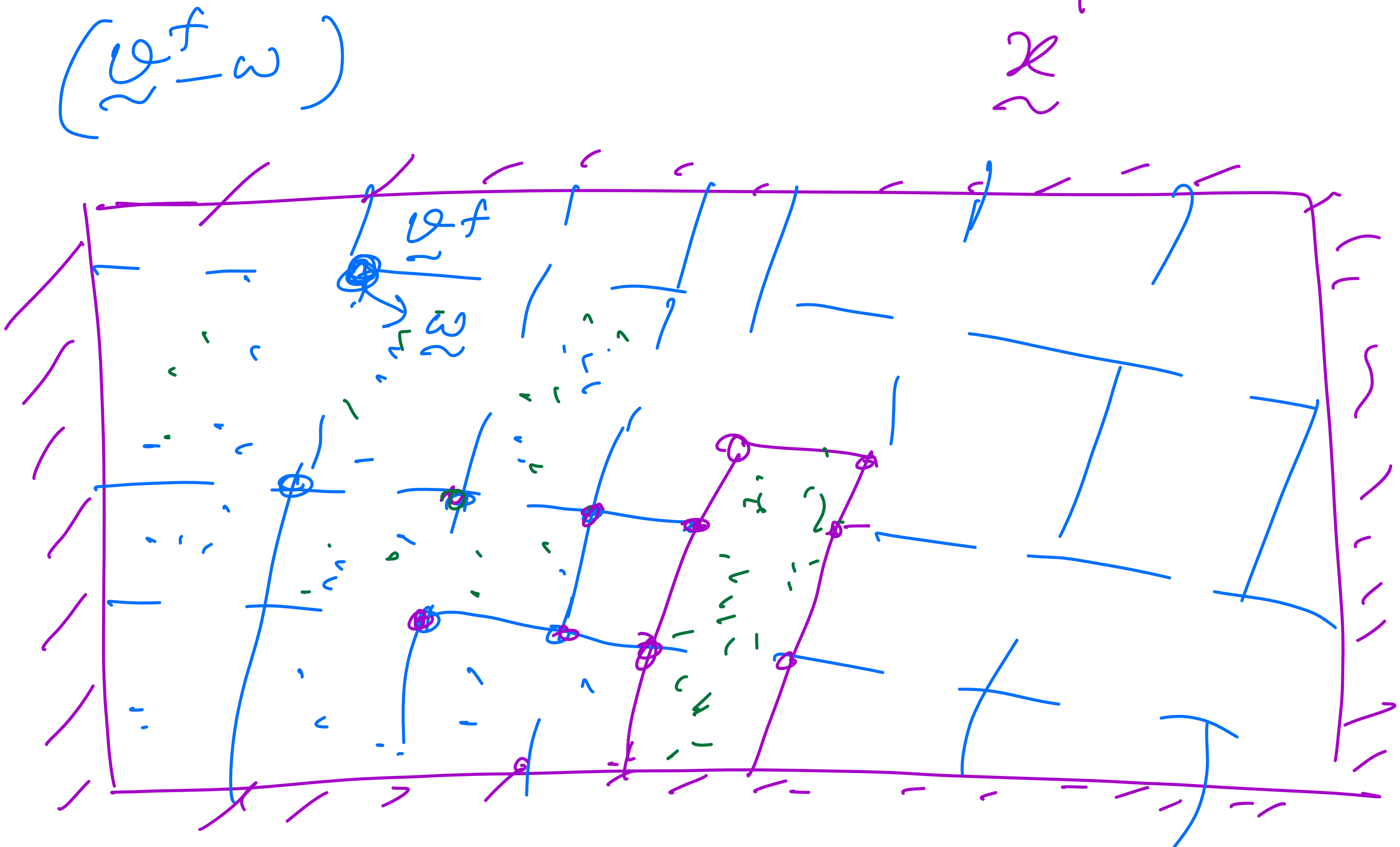


Fig. 3.6 Traction and velocity continuity conditions.

Motion of fluid mesh

(Spatial coordinates $\underset{\sim}{x}^f$)



$$\nabla \cdot \underset{\sim}{\sigma}^m = 0$$

Stiffness

$$\underset{\sim}{\sigma}^m = (1 + k_m)\left[\nabla \underset{\sim}{u}^m + (\nabla u^m)^T\right] + (\nabla \cdot \underset{\sim}{u}^m) I$$

$$\underset{\sim}{u}^m = \varphi(\underset{\sim}{X}^m, t) - \underset{\sim}{X}^m$$

$$\underset{\sim}{x}^f = \underline{X} + \underset{\sim}{u}^m(\underset{\sim}{X}, t)$$

$$\boxed{\dot{\underset{\sim}{x}}_f = \underset{\sim}{v}^s \;@\; \Gamma_{fsi}}$$

The motion of the spatial coordinates on the fluid domain can be simply modeled as an elastic material in equilibrium,

$$\nabla \cdot \sigma^{\mathrm{m}} = 0, \quad \text{with} \quad \sigma^{\mathrm{m}} = (1 + k_{\mathrm{m}})\Big[\Big(\nabla u^{\mathrm{f}} + \big(\nabla u^{\mathrm{f}}\big)^{T}\Big) + \big(\nabla \cdot u^{\mathrm{f}}\big) I\Big]$$

where $\boldsymbol{u}^{\mathrm{f}}$ denotes the ALE mesh nodal displacement satisfying the boundary conditions

$$u^{\mathrm{f}} = \varphi(\boldsymbol{X}, t) - \boldsymbol{X} \quad \text{on } \Gamma$$

$$u^{\mathrm{f}}(\chi, t) = 0 \quad \text{on } \partial\Omega^{\mathrm{f}}(0) \setminus \Gamma$$

In the above equation, $\boldsymbol{k}_{\mathrm{m}}$ represents the local element level mesh stiffness parameter chosen as a function of the element sizes to limit the distortion of the small elements located in the immediate vicinity of the fluid-structure interface. The solution provides us the ALE mesh nodal displacements, which can be used to update the spatial nodes

$$x = \hat{\varphi}^{\mathrm{f}}(\chi, t) = \chi + u^{\mathrm{f}}(\chi, t), \quad \forall x \in \Omega^{\mathrm{f}}(t) \text{ and } \chi \in \Omega^{\mathrm{f}}(0)$$

Module 3

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