CHBE 552 Problem Set 1

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Question 1

Part a

$$
\begin{aligned}
\mathbf{F_1} &= x_1^2 + x_1x_2 + x_1x_3 + x_1x_4 + x_1 \\
&+ x_2^2 + x_2x_3 + x_2x_4 + x_2 + x_3^2 + x_3x_4 \\
&+ x_3 + x_4^2 + x_4 + 1 \\
\mathbf{H_{F_1}} &= \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \\
\mathbf{F_2} &= 8x_1^2 + 4x_1x_2 + 5x_2^2 \\
\mathbf{H_{F_2}} &= \begin{bmatrix} 16 & 4 \\ 4 & 10 \end{bmatrix}
\end{aligned}
$$

By implementating Newton's method, the corresponding minimums are found to be:

$$
\begin{aligned}
\mathbf{x_{F1}} &= \begin{bmatrix} -0.2 \\ -0.2 \\ -0.2 \\ -0.2 \end{bmatrix} \text{ or } \begin{bmatrix} -0.2 \\ -0.2 \\ -0.2 \\ -0.2 \end{bmatrix} \\
\mathbf{x_{F2}} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

Note that for $\mathbf{F_1}$, two initial guesses are provided the value of $\mathbf{x_{F1}}$ shown above are derived from thoes two initial guesses and they are actually the same.

Part b

$$\begin{aligned} \mathbf{F} &= 4\left(x_1 - 5\right)^2 + \left(x_2 - 6\right)^2 \\ \nabla \mathbf{F} &= \left[8x_1 - 40,\ 2x_2 - 12\right] \end{aligned}$$

By implementating Fletcher Reeves's method, the minimum is found to be:

$$\mathbf{x} = \begin{bmatrix} 5.11 \\ 2.48 \end{bmatrix}$$

Part c

$$\begin{aligned} \mathbf{F} &= 2x_1^2 + 2x_1x_2 + x_1 + x_2^2 - x_2 \\ \nabla \mathbf{F} &= \left[4x_1 + 2x_2 + 1,\ 2x_1 + 2x_2 - 1\right] \end{aligned}$$

By implementating DFP method, the minimum is found to be:

$$\mathbf{x} = \begin{bmatrix} -0.65 \\ 0.98 \end{bmatrix}$$

Question 2

Part a

$$\begin{aligned} F &= x_1^2 + 10x_1 + x_2^2 + 20x_2 + 25 \\ L &= \lambda\left(x_1 + x_2\right) + x_1^2 + 10x_1 + x_2^2 + 20x_2 + 25 \\ \partial F_{x_1} &= \lambda + 2x_1 + 10 \\ \partial F_{x_2} &= \lambda + 2x_2 + 20 \\ \partial F_{\lambda} &= x_1 + x_2 \end{aligned}$$

By setting those partial derivative equations to zero, the optimum values are found to be:

$$x_{1,opt} = \frac{5}{2}$$
$$x_{2,opt} = -\frac{5}{2}$$
$$\lambda_{opt} = -15$$

At this optimum point the original function is evaluated to be:

$$F_{opt} = \frac{25}{2}$$

For a sensitivity test the constraint condition is changed to $x_1 + x_2 = 0.01$, then the previous steps are repeated:

$$\begin{aligned}
L_{new} &= \lambda\left(x_1 + x_2 - 0.01\right) + x_1^2 + 10x_1 + x_2^2 + 20x_2 + 25 \\
\partial F_{x_1} &= \lambda + 2x_1 + 10 \\
\partial F_{x_2} &= \lambda + 2x_2 + 20 \\
\partial F_{\lambda} &= x_1 + x_2 - 0.01 \\
x_{1,opt} &= 2.505 \\
x_{2,opt} &= -2.495 \\
\lambda_{opt} &= -15.01 \\
F_{new,opt} &= 12.65005
\end{aligned}$$

Thus, the increment of the function value is computed to be

$$\Delta F = 0.15$$

Part b

$$\begin{aligned}
F &= -\pi x_1^2 x_2 \\
L &= \lambda\left(2\pi x_1^2 + 2\pi x_1 x_2 - 24\pi\right) - \pi x_1^2 x_2 \\
\partial F_{x_1} &= \lambda\left(4\pi x_1 + 2\pi x_2\right) - 2\pi x_1 x_2 \\
\partial F_{x_2} &= 2\pi\lambda x_1 - \pi x_1^2 \\
\partial F_{\lambda} &= 2\pi x_1^2 + 2\pi x_1 x_2 - 24\pi \\
\text{Solution} &= [(2,\ 4)]
\end{aligned}$$

Question 3

Question 4

$$
\begin{aligned}
F &= y^2 + (x-2)^2 + (z-1)^2 \\
L &= \lambda\left(-4x^2 - 2y^2 + z^2\right) + y^2 + (x-2)^2 + (z-1)^2 \\
\partial F_x &= -8\lambda x + 2x - 4 \\
\partial F_y &= -4\lambda y + 2y \\
\partial F_z &= 2\lambda z + 2z - 2 \\
\partial F_\lambda &= -4x^2 - 2y^2 + z^2 \\
\mathbf{x} &= \left(\frac{4}{5},\ 0,\ \frac{8}{5}\right)
\end{aligned}
$$

With $\lambda = -\frac{3}{8}$

Question 5

Part a

$$
\begin{aligned}
F &= x_1^2 - 14x_1 + x_2^2 - 6x_2 - 7 \\
L &= \lambda_1\left(x_1 + x_2 - 2\right) + \lambda_2\left(x_1 + 2x_2 - 3\right) + x_1^2 - 14x_1 + x_2^2 - 6x_2 - 7 \\
\partial F_x &= \lambda_1 + \lambda_2 + 2x_1 - 14 \\
\partial F_y &= \lambda_1 + 2\lambda_2 + 2x_2 - 6 \\
\partial F_{\lambda_1} &= x_1 + x_2 - 2 \\
\partial F_{\lambda_2} &= x_1 + 2x_2 - 3 \\
\mathbf{x} &= \{x_1 : 1,\ x_2 : 1\}
\end{aligned}
$$

With

$$\lambda_{1,2} = \{\lambda_1 : 20,\ \lambda_2 : -8\}$$

And the minimum objective function value is computed to be -25.

Part b

$$
\begin{aligned}
F &= x_1^2 + 2\left(x_2 + 1\right)^2 \\
L &= \lambda_1\left(-x_1 + x_2 - 2\right) + \lambda_2\left(-x_1 - x_2 - 1\right) + x_1^2 + 2\left(x_2 + 1\right)^2 \\
\partial F_x &= -\lambda_1 - \lambda_2 + 2x_1 \\
\partial F_y &= \lambda_1 - \lambda_2 + 4x_2 + 4 \\
\partial F_{\lambda_1} &= -x_1 + x_2 - 2 \\
\partial F_{\lambda_2} &= -x_1 - x_2 - 1 \\
\mathbf{x} &= \left\{x_1 : -\frac{3}{2},\ x_2 : \frac{1}{2}\right\}
\end{aligned}
$$

With

$$
\lambda_{1,2} = \left\{\lambda_1 : -\frac{9}{2},\ \lambda_2 : \frac{3}{2}\right\}
$$

And the minimum objective function value is computed to be $\frac{27}{4}$.

Question 6

$$
\begin{aligned}
F &= x_1^2 + x_1 x_2 - 4x_1 + 1.5x_2^2 - 7x_2 - \log(x_1) - \log(x_2) + 9 \\
L_a &= x_1^2 + x_1 x_2 - 4x_1 + 1.5x_2^2 - 7x_2 - \log(x_1) - \log(x_2) + 9 \\
L_b &= \lambda_1(-x_1 x_2 + 4) + x_1^2 + x_1 x_2 - 4x_1 + 1.5x_2^2 - 7x_2 - \log(x_1) - \log(x_2) + 9 \\
L_c &= \lambda_1(-x_1 x_2 + 4) + \lambda_2 \cdot (2x_1 - x_2) + x_1^2 + x_1 x_2 - 4x_1 + 1.5x_2^2 - 7x_2 - \log(x_1) - \log(x_2) +
\end{aligned}
$$

$$
\begin{aligned}
\partial L_{a,x_1} &= 2x_1 + x_2 - 4 - \frac{1}{x_1} \\
\partial L_{a,x_2} &= x_1 + 3.0x_2 - 7 - \frac{1}{x_2} \\
\partial L_{b,x_1} &= -\lambda_1 x_2 + 2x_1 + x_2 - 4 - \frac{1}{x_1} \\
\partial L_{b,x_2} &= -\lambda_1 x_1 + x_1 + 3.0x_2 - 7 - \frac{1}{x_2} \\
\partial L_{b,\lambda_1} &= -x_1 x_2 + 4 \\
\partial L_{c,x_1} &= -\lambda_1 x_2 + 2\lambda_2 + 2x_1 + x_2 - 4 - \frac{1}{x_1} \\
\partial L_{c,x_2} &= -\lambda_1 x_1 - \lambda_2 + x_1 + 3.0x_2 - 7 - \frac{1}{x_2} \\
\partial L_{c,\lambda_1} &= -x_1 x_2 + 4 \\
\partial L_{c,\lambda_2} &= 2x_1 - x_2
\end{aligned}
$$

For part a:

$$\mathbf{x_a} = \begin{bmatrix} 1.34754858228762 \\ 2.04699110826639 \end{bmatrix}$$

For part b:

$$\mathbf{x_b} = \begin{bmatrix} 1.79811994048488 \\ 2.22454571018291 \end{bmatrix}$$

With

$$\lambda_1 = \begin{bmatrix} 0.568497719699798 \end{bmatrix}$$

For part c:

$$\mathbf{x_c} = \begin{bmatrix} 1.4142135623731 \\ 2.82842712474619 \end{bmatrix}$$

With

$$\lambda_{1,2} = \begin{bmatrix} 1.06801948466054 \\ 1.03553390593274 \end{bmatrix}$$

And the minimum objective function value is computed to be:

$$F_a = -0.874224083186354$$
$$F_b = -0.494453348930655$$
$$F_c = 0.157861516164399$$