Deformation gradient

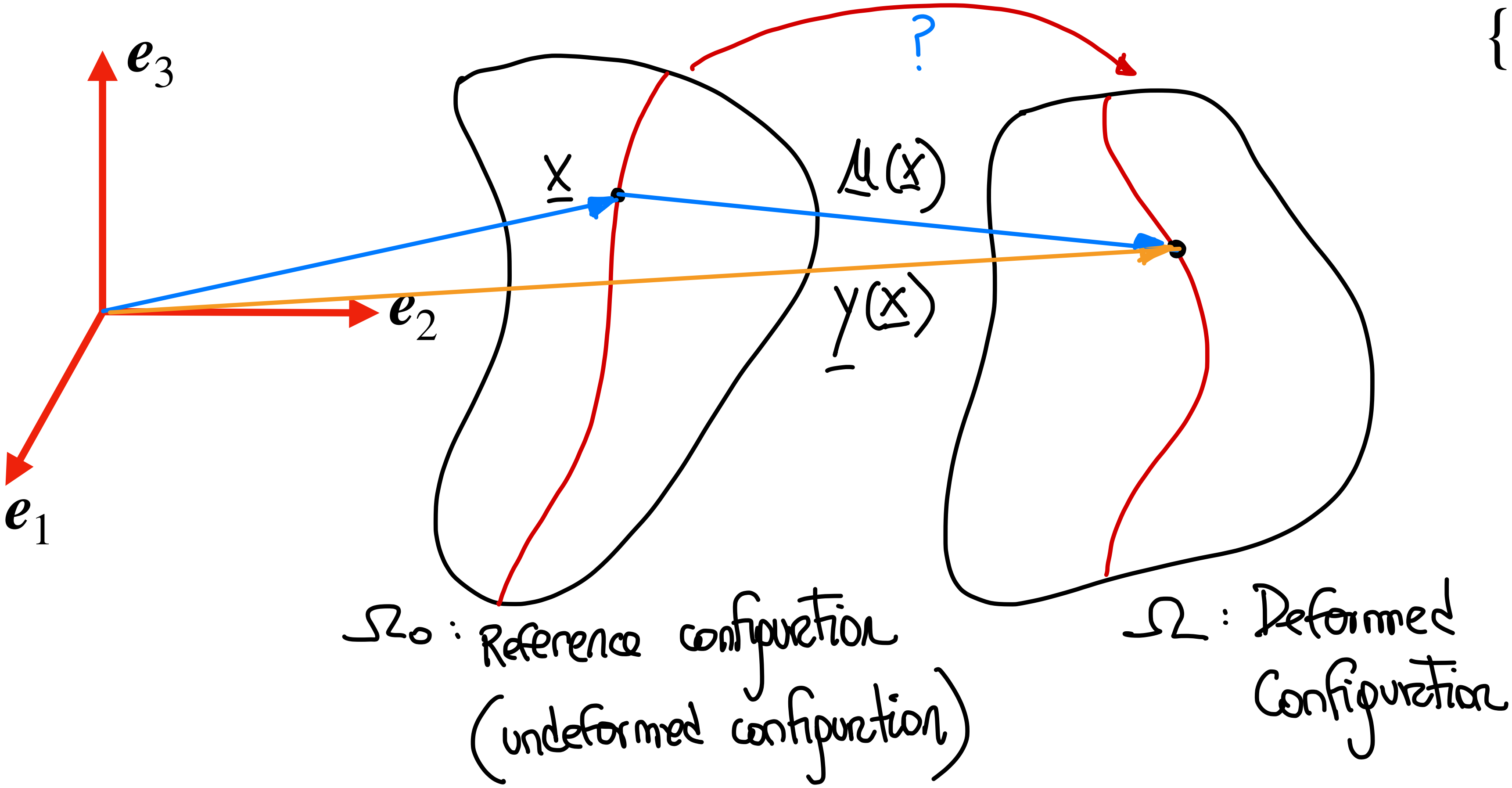
Assistant Prof. Mauricio Ponga

Department of Mechanical Engineering, UBC

MECH 503 - 2021

Deformation gradient

$\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$



$\underline{u}(\underline{x}, t)$ = Displacement vector

$$\boxed{\underline{y} = \underline{x} + \underline{u}(\underline{x}, t)}$$

or

$$\boxed{y_i = x_i + u_i(x_1, x_2, x_3, t) \qquad i = 1, 2, 3.}$$

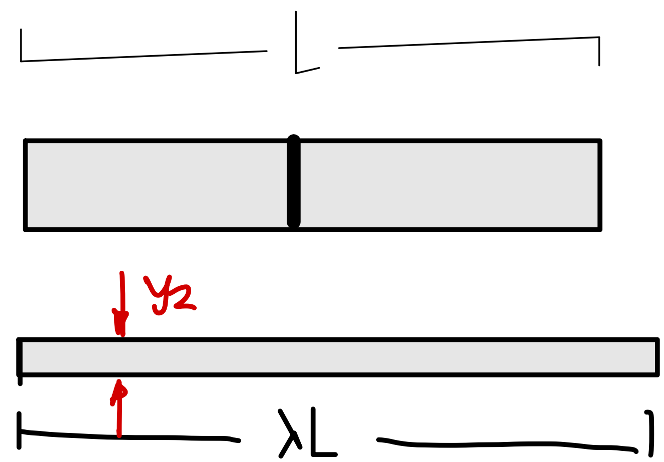
Deformation gradient

Velocity field

$$v_i = \frac{\partial y_i}{\partial t} = \frac{\partial u_i}{\partial t}(x_k, t)\Big|_{x_k = \text{constant}}$$

- Examples of simple deformations

- Volume preserving uniaxial extension



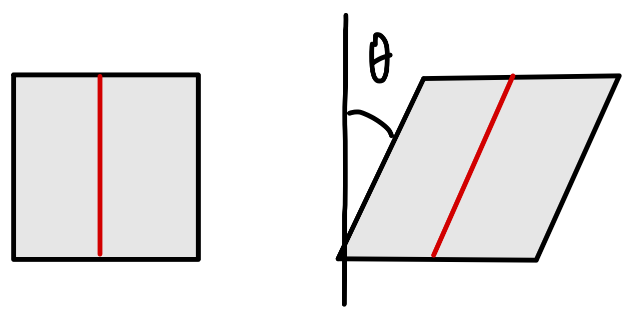
$$y_1 = \lambda x_1$$

$$y_2 = x_2/\sqrt{\lambda}$$

$$y_3 = x_3/\sqrt{3}$$

$$\Omega_0 = \Omega$$

- Simple shear



$$y_1 = x_1 + \tan\theta \, x_2 ; \quad y_2 = x_2 ; \quad y_3 = x_3$$

Deformation gradient

Deformation gradient

- Displacement gradient tensor

$$\underline{u} \otimes \nabla = \frac{\partial u_i}{\partial x_k}$$

Identity Matrix $\rightarrow$ $\underline{\underline{I}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$[\underline{v} \otimes \nabla]_{ij} \equiv \frac{\partial v_i}{\partial x_j}$$

- Deformation gradient tensor

$$\underline{\underline{F}} = \underline{\underline{I}} + \underline{u} \otimes \nabla \quad \text{or} \quad F_{ik} = \delta_{ik} + \frac{\partial u_i}{\partial x_k}$$

Note the following

$$\underline{y} \otimes \nabla = \left(\underline{x} + \underline{u}(\underline{x}, t)\right) \otimes \nabla = \underline{\underline{F}}$$

or

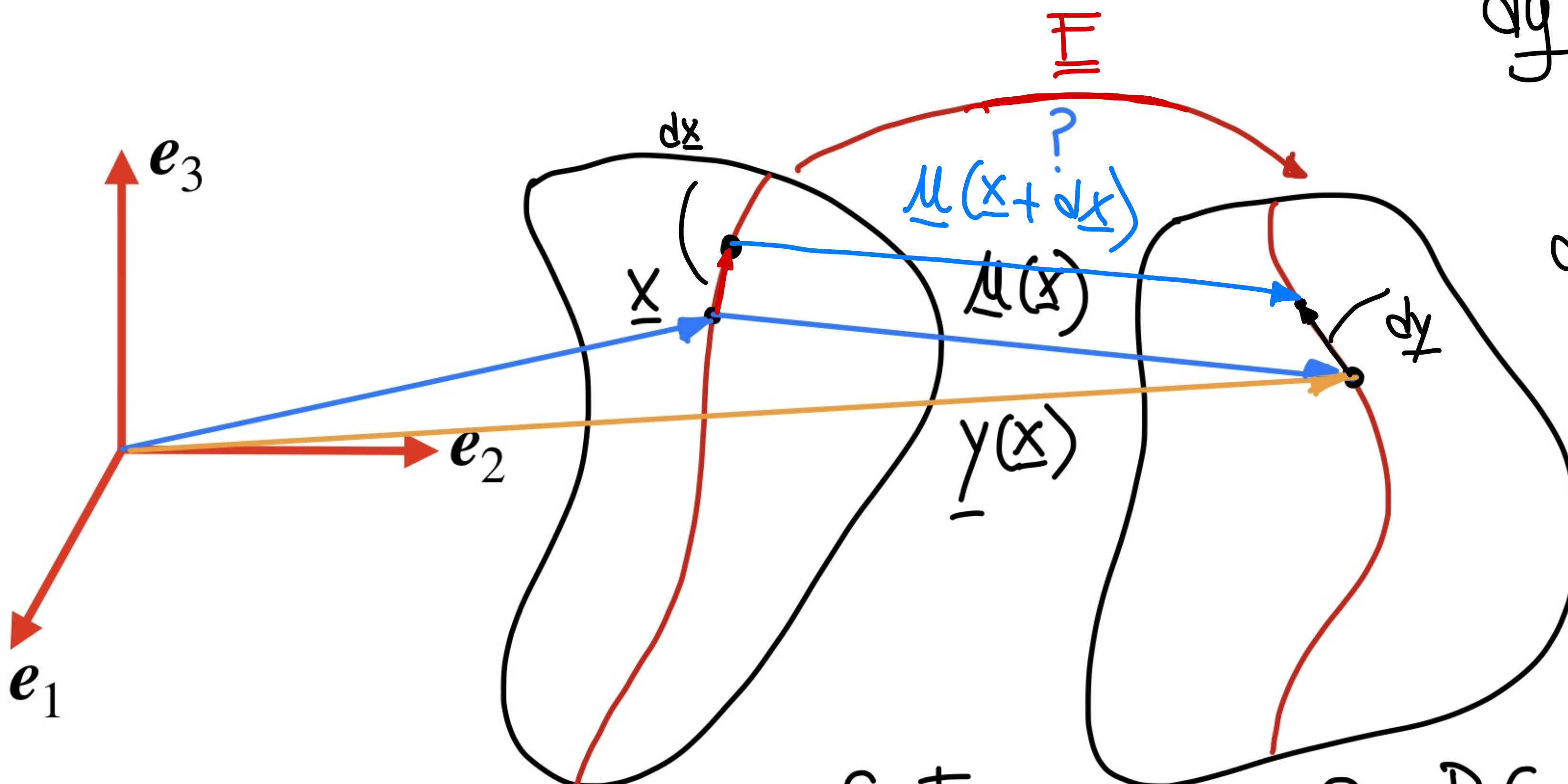
$$\boxed{\frac{\partial y_i}{\partial x_j} = \delta_{ij} + \frac{\partial u_i}{\partial x_j} = F_{ij}}$$

Deformation gradient

$$\underline{dy} = \underline{\underline{F}} \cdot \underline{dx}$$

or

$$dy_i = F_{ij}\, dx_j$$



Ω_0 : Reference configuration (undeformed configuration)

Ω : Deformed Configuration

- Proof :

$$\underline{dy} = \left[\cancel{\underline{x}} + \underline{dx} + \underline{u}(\underline{x} + \underline{dx}) - \left(\cancel{\underline{x}} + \underline{u}(\underline{x})\right)\right]$$

Deformation gradient

$$d\underline{y} = \left[\cancel{\underline{x}} + d\underline{x} + \underline{u}(\underline{x} + d\underline{x}) - (\cancel{\underline{x}} + \underline{u}(\underline{x})) \right]$$

$$dy_i = \cancel{x_i} + dx_i + \underbrace{u_i(x_k + dx_k)} - \cancel{x_i} - u_i(x_k)$$

$$u_i(x_k + dx_k) \simeq u_i(x_k) + \frac{\partial u_i}{\partial x_k} \cdot dx_k + \text{h.o.t}$$

$$dy_i = dx_i + \cancel{u_i(x_k)} + \frac{\partial u_i}{\partial x_k} \cdot dx_k - \cancel{u_i(x_k)}$$

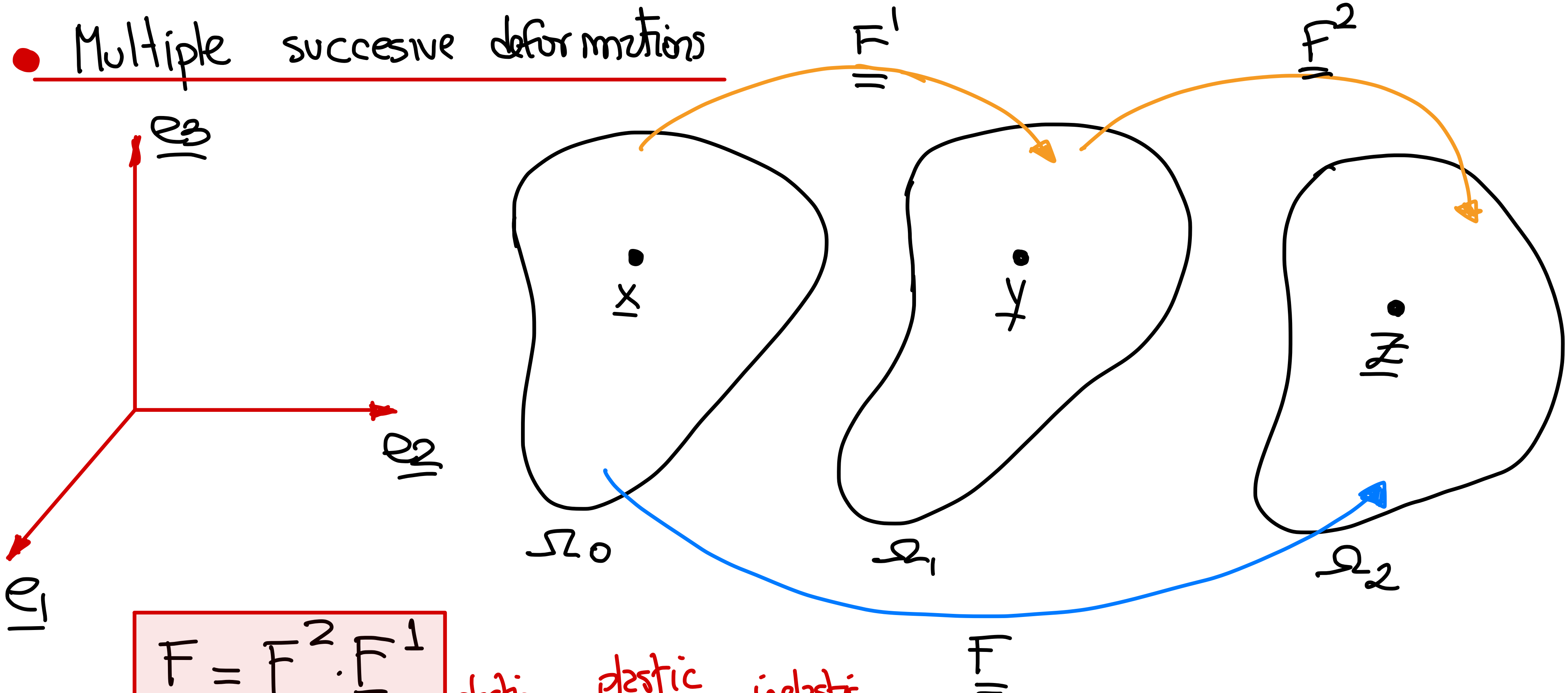
$$dy_i = \left(\delta_{ik} + \frac{\partial u_i}{\partial x_k} \right) dx_k$$

$$\boxed{dy_i = F_{ik}\, dx_k}$$

Deformation gradient

$$dy_i = F_{ij}\,dx_j \quad \Rightarrow \quad dx_i = F^{-1}_{ij}\,dy_j$$

- Multiple succesive deformations



$$\boxed{\underline{\underline{F}} = \underline{\underline{F}}^2 \cdot \underline{\underline{F}}^1}$$

Relevent:

$$\underline{\underline{F}} = \underbrace{\underline{\underline{F}}^e}_{\text{elastic}} \cdot \underbrace{\underline{\underline{F}}^p}_{\text{plastic}} \cdot \underbrace{\underline{\underline{F}}^I}_{\text{inelastic}}$$

Deformation gradient

$$d\underline{y} = \underline{\underline{F}}^1 \, d\underline{x} \qquad d\underline{z} = \underline{\underline{F}}^2 \, d\underline{y}$$

$$dy_i = F^1_{ij} \, dx_j \qquad dz_i = F^2_{ij} \, dy_j = F^2_{ij} \, F^1_{jk} \, . \, dx_k$$

$$dy_j = F^1_{jk} \, dx_k \qquad dz_i = F^2_{ij} \, F^1_{jk} \, dx_k = F_{ij} \, dx_j$$

where $\boxed{F_{ik} = F^2_{ij} \, F^1_{jk}}$

- Jacobien:

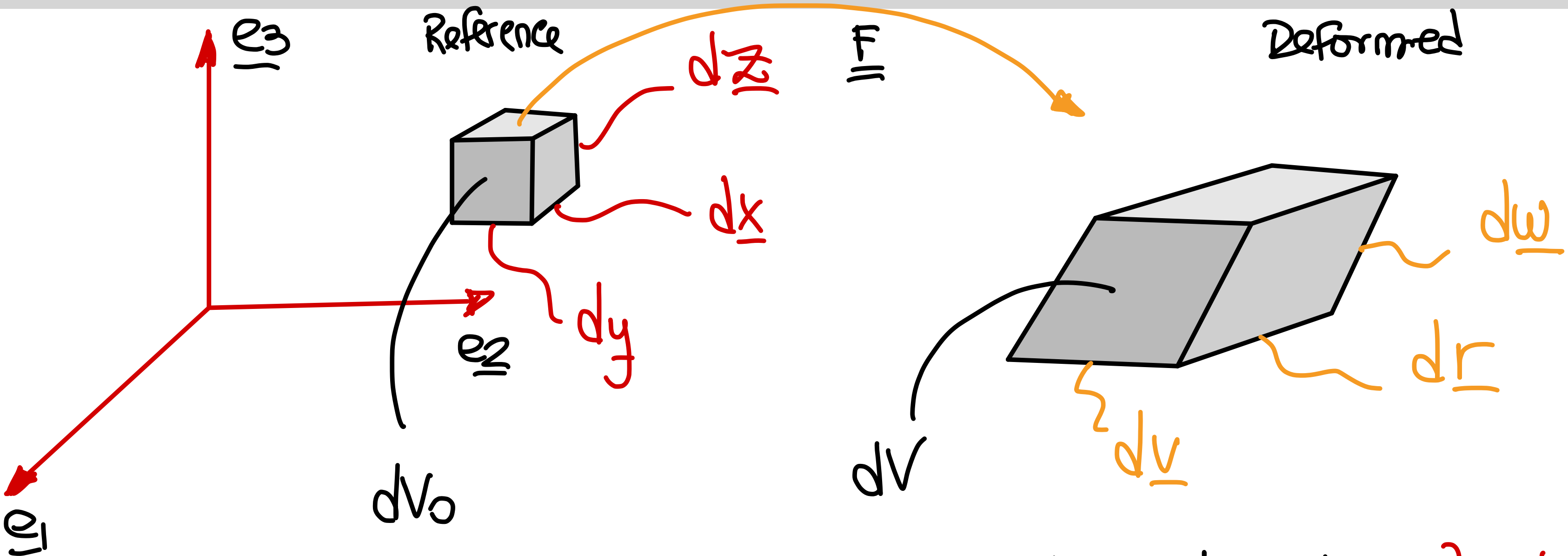
$$J = \det(\underline{\underline{F}}) = \det\left(\delta_{ij} + \frac{\partial u_i}{\partial x_j}\right)$$

$$J = \frac{dV}{dV_0}$$

dV: Volume of an element in the deformed conf.

dV_0: Volume of an element in the reference conf.

Deformation gradient



$$dV_0 = d\underline{z} \cdot \left(d\underline{x} \times d\underline{y} \right) = \epsilon_{ijk}\, dz_i\, dx_j\, dy_k \quad \Big\} \text{ Volume in Reference.}$$

$$\epsilon_{ijk} = \begin{cases} 1 & \\ -1 & \\ 0 & \text{otherwise} \end{cases}$$

$$\underbrace{dV = \epsilon_{ijk}\, dw_i\, dr_j\, dv_k}_{\text{Volume in the deformed configuration.}}$$

Deformation gradient

$$dw_i = F_{il}\, dz_l$$

$$dr_j = F_{jm}\, dx_m$$

$$dv_k = F_{kn}\, dy_n$$

$$dV = \epsilon_{ijk}\, dw_i\, dr_j\, dv_k$$

$$dV = \epsilon_{ijk} \cdot \underbrace{f_{il}\, F_{jm}\, F_{kn}}_{\det(\underline{\underline{F}}) \equiv J}\, dx_m\, dy_n\, dz_l$$

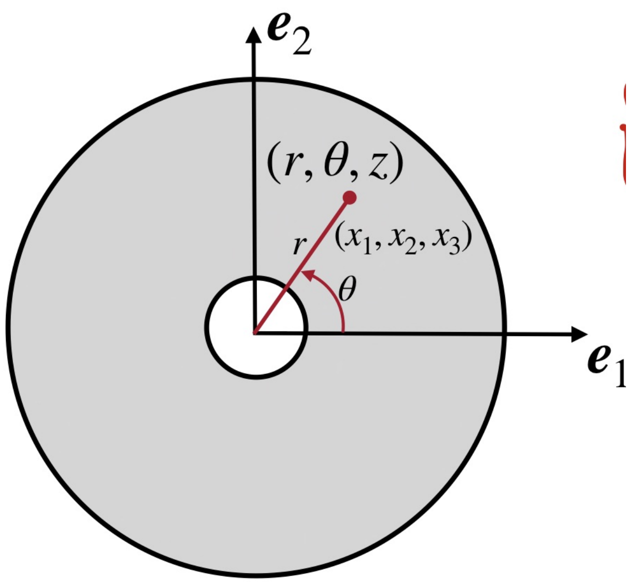
$$dV = J.\ \underbrace{\epsilon_{lmn}\, dx_m\, dy_n\, dz_l}_{\text{volume in the reference.}}$$

$$dV = J \,.\, dV_0$$

$$\boxed{J = \frac{dV}{dV_0}}$$

$$J = \begin{cases} J = 1 : & \text{Density remains the same} \\ J \neq 1 & \begin{cases} \rho > 0 \\ \rho < 1 \end{cases} : \text{Wave problems} \end{cases}$$

Deformation gradient



$$y_1 = f(r)\left[x_1 \cos\phi(x_3) - x_2 \sin\phi(x_3)\right]$$

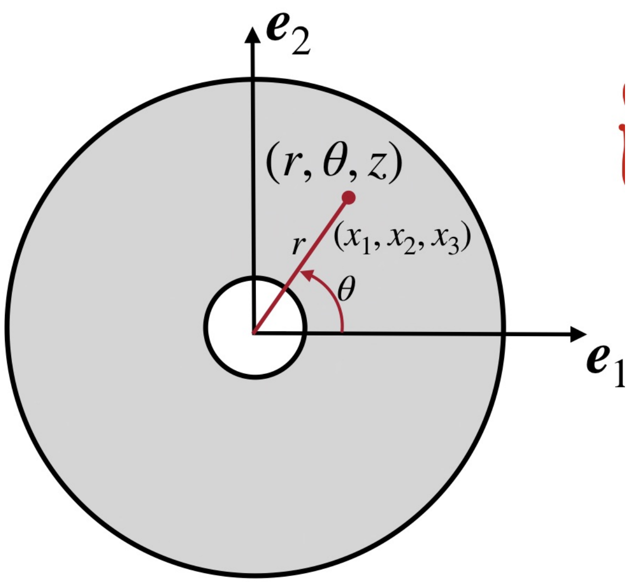
$$y_2 = f(r)\left[x_2 \cos\phi(x_3) + x_1 \sin\phi(x_3)\right]$$

$$y_3 = \lambda x_3 .$$

$$\det(\underline{\underline{F}}) = \det\begin{pmatrix} f(r)\cos\phi(x_3) & -f(r)\sin\phi(x_3) & 0 \\ +f(r)\sin\phi(x_3) & f(r)\cos\phi(x_3) & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$\det(\underline{\underline{F}}) = \lambda f^2(r)\left[\underbrace{\cos^2\phi(x_3) + \sin^2\phi(x_3)}_{1}\right] = \lambda f^2(r)$$

Deformation gradient



$$\begin{cases} y_1 = f(r)\left[x_1 \cos\phi(x_3) - x_2 \sin\phi(x_3)\right] \\ y_2 = f(r)\left[x_2 \cos\phi(x_3) + x_1 \sin\phi(x_3)\right] \\ y_3 = \lambda x_3 . \end{cases}$$

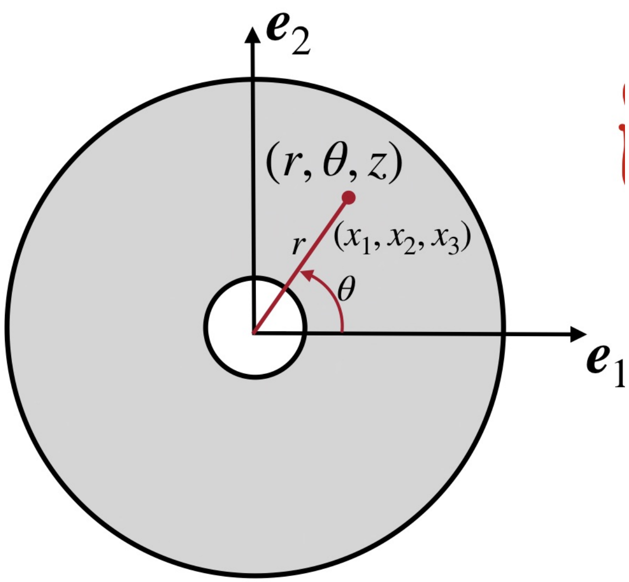
Using $\begin{cases} x_1 = r\cos\theta \\ x_2 = r\sin\theta \end{cases}$

$$\begin{cases} y_1 = r f(r)\left[\overbrace{\cos\theta\cos\phi(x_3) - \sin\theta\sin\phi(x_3)}^{\cos(\theta+\phi(x_3))}\right] \\ y_2 = r f(r)\left[+\underbrace{\cos\theta\sin\phi(x_3) + \sin\theta\cos\phi(x_3)}_{\sin(\theta+\phi(x_3))}\right] \\ y_3 = \lambda x_3 \end{cases}$$

$$y_1^2 + y_2^2 = (r f(r))^2\left[\underbrace{\cos^2(\theta+\phi(x_3)) + \sin^2(\theta+\phi(x_3))}_{1}\right]$$

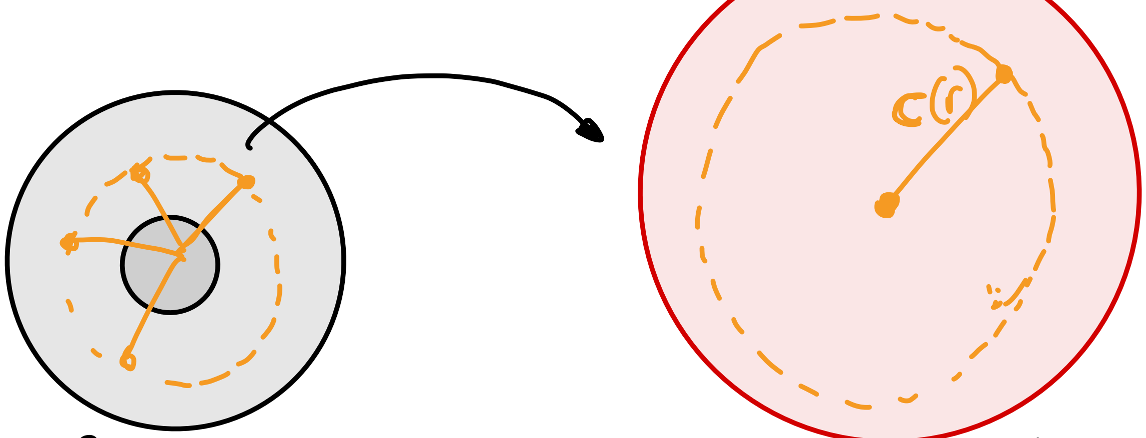
$$\boxed{y_1^2 + y_2^2 = c(r)^2}$$

Deformation gradient

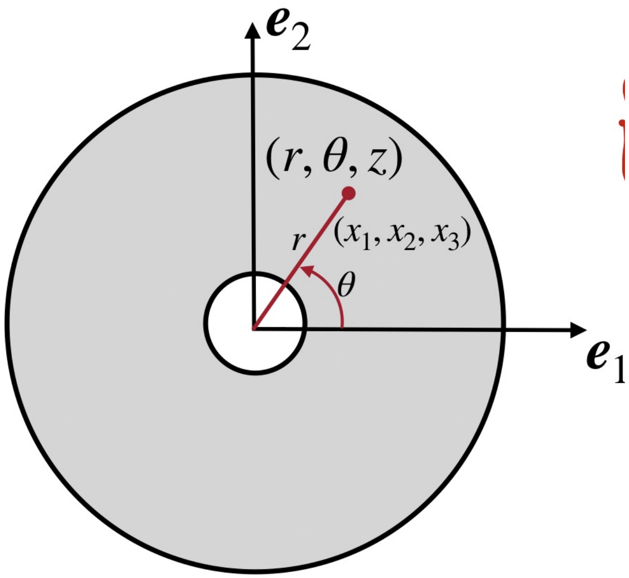


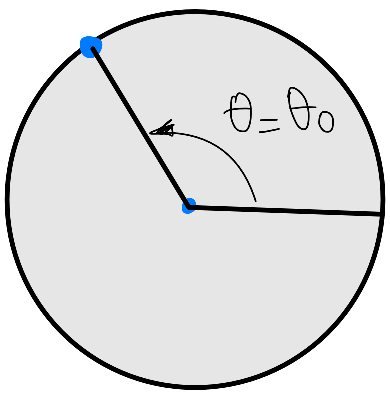
$$
\begin{cases}
y_1 = r f(r) \left[\overbrace{\cos\theta \cos \phi(x_3) - \sin\theta \sin \phi(x_3)}^{\cos(\theta + \phi(x_3))} \right] \\
y_2 = r f(r) \left[\underbrace{+\cos\theta \sin \phi(x_3) + \sin\theta \cos \phi(x_3)}_{\sin(\theta + \phi(x_3))} \right] \\
y_3 = \lambda x_3
\end{cases}
$$

$$
\frac{y_2}{y_1} = \tan(\theta + \phi(x_3))
$$

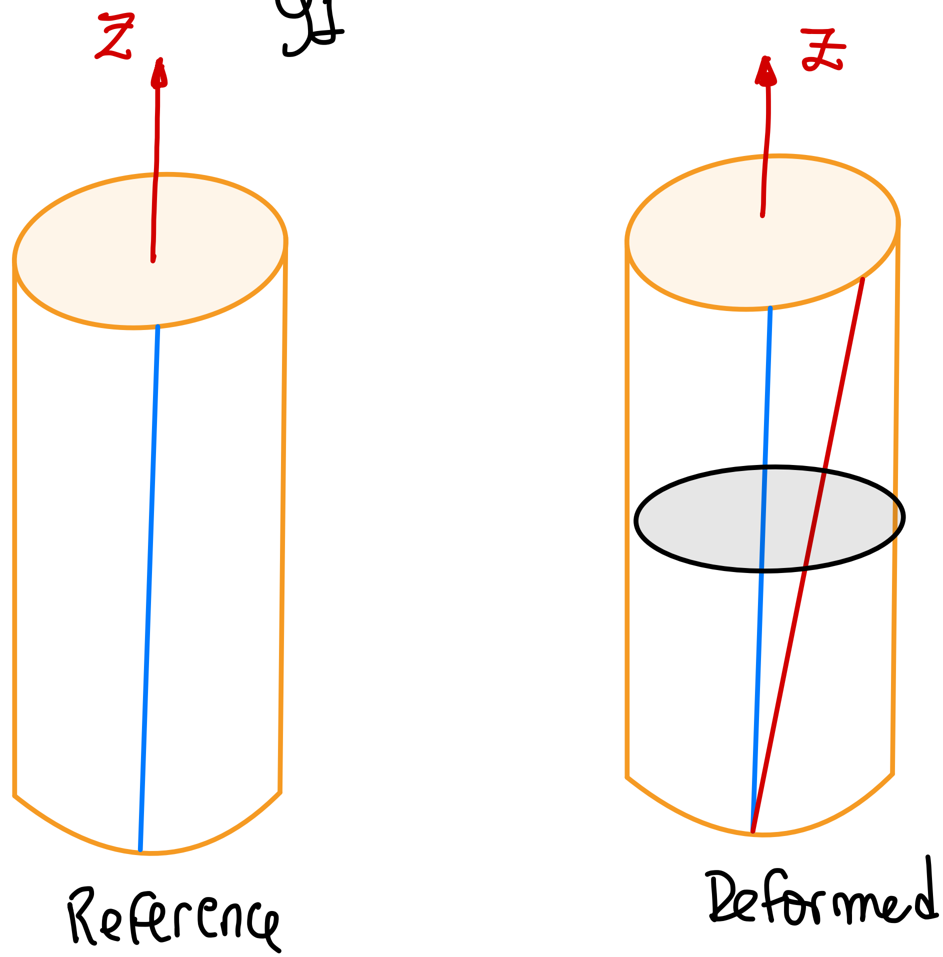


Deformation gradient





$$\frac{y_2}{y_1} = \tan(\theta + \phi(x_3))$$



Try to map this deformation using software.

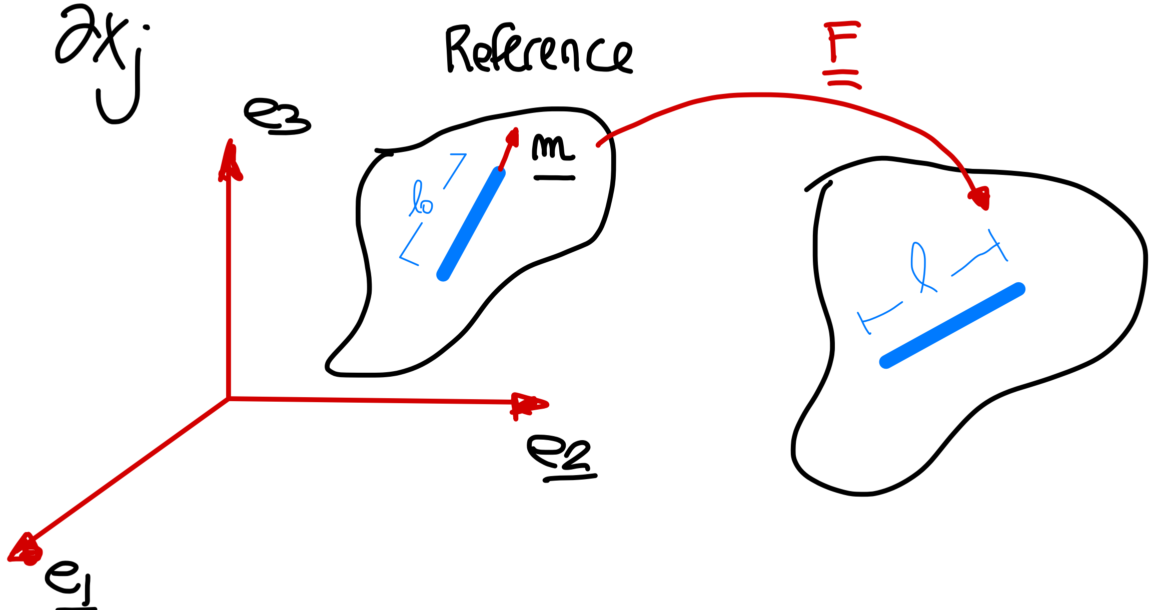
Strain measures

- The Lagrange strain tensor:

$$\underline{\underline{E}} = \frac{1}{2}\left(\underline{\underline{F}}^T \underline{\underline{F}} - \underline{\underline{I}}\right) \quad \text{or} \quad E_{ij} = \frac{1}{2}\left(F_{ki} \cdot F_{kj} - \delta_{ij}\right)$$

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i} \cdot \frac{\partial u_k}{\partial x_j}\right) \qquad i, j, k = 1, 2, 3.$$

$\frac{\partial u_i}{\partial x_j}$: Derivatives of the disp vector.



$$\boxed{\varepsilon_L(m_i) = \frac{l^2 - l_0^2}{2 l_0^2}}$$

$$l = l_0 + \delta l$$

Strain measures

- The Lagrange strain tensor:

$$\varepsilon_L(m_i) = \frac{\ell^2 - \ell_0^2}{2\ell_0^2}$$

$$\varepsilon_L(m_i) = \frac{\ell^2 - \ell_0^2}{2\ell_0^2}$$

$$\ell = \ell_0 + \delta\ell$$

$$\varepsilon_L(m_i) = \frac{(\ell_0 + \delta\ell)^2 - \ell_0^2}{2\ell_0^2} = \frac{(\cancel{\ell_0^2} + 2\ell_0\delta\ell + \delta\ell^2) - \cancel{\ell_0^2}}{2\ell_0^2}$$

$$\boxed{\varepsilon_L(m_i) = \underbrace{\frac{\delta\ell}{\ell_0}}_{\text{Inf. strain}} + \underbrace{\frac{\delta\ell^2}{2\ell_0^2}}_{\text{Non-linear Term.}}}$$

Strain measures

- The Lagrange strain tensor:

$$\boxed{\varepsilon_L(m_i) = \frac{\ell^2 - \ell_0^2}{2\ell_0^2}}$$

$$\varepsilon_L(\underline{m}) = \underline{m} \cdot \underline{\underline{E}} \cdot \underline{m} \quad \text{or}$$

$$\varepsilon_L(m_i) = E_{ij}\, m_i\, m_j \quad \Big\} \text{ scalar}$$

- Proof:

$$d\underline{x} = \ell_0\, \underline{m} \quad \text{or} \quad dx_i = \ell_0\, m_i \qquad dy_i = F_{ij}\, dx_j$$

Recall that $F_{ij} = \left(\delta_{ij} + \frac{\partial u_i}{\partial x_j}\right)$

$$dy_i = \ell_0 F_{ij}\, m_j \qquad \ell^2 = dy_1^2 + dy_2^2 + dy_3^2 = dy_i \cdot dy_i$$

Strain measures

- The Lagrange strain tensor:

$$dy_i = l_0 F_{ij} m_j$$

$$\boxed{\varepsilon_L(m_i) = \frac{l^2 - l_0^2}{2 l_0^2}}$$

$$l = dy_i \cdot dy_i = F_{ij} F_{ik} \cdot l_0^2 \cdot m_j m_k$$

Recall that $F_{ij} = \left(\delta_{ij} + \frac{\partial u_i}{\partial x_j}\right)$

$$l = \left(\delta_{ij} + \frac{\partial u_i}{\partial x_j}\right)\left(\delta_{ik} + \frac{\partial u_i}{\partial x_k}\right) l_0^2 m_j m_k$$

1 : $\delta_{ij} \cdot \delta_{ik} = \delta_{jk}$

2 : $\delta_{ij} \frac{\partial u_i}{\partial x_k} = \frac{\partial u_j}{\partial x_k}$

3 : $\delta_{ik} \cdot \frac{\partial u_i}{\partial x_j} = \frac{\partial u_k}{\partial x_j}$

4 : $\frac{\partial u_i}{\partial x_j} \cdot \frac{\partial u_i}{\partial x_k}$

$$l^2 = \left(\delta_{jk} + \frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} + \frac{\partial u_i}{\partial x_j} \cdot \frac{\partial u_i}{\partial x_k}\right) \cdot l_0^2 m_j m_k$$

Strain measures

- The Lagrange strain tensor:

$$\boxed{\varepsilon_L(m_i) = \frac{l^2 - l_0^2}{2 l_0^2}}$$

$$\boxed{\varepsilon_L(m_i) = \frac{l^2 - l_0^2}{2l_0^2} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\cdot\frac{\partial u_k}{\partial x_j}\right) m_i m_j}$$

$$\left.E_{ij} = \frac{1}{2}\left(F_{ki}\cdot F_{kj} - \delta_{ij}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)\right\}$$

expand E_{ij}

11, 12, 13.

$$E_{11} = \frac{1}{2}\left(F_{11} F_{11} + F_{21} F_{21} + F_{31}\cdot F_{31} - \overbrace{\delta_{11}}^{1}\right)$$

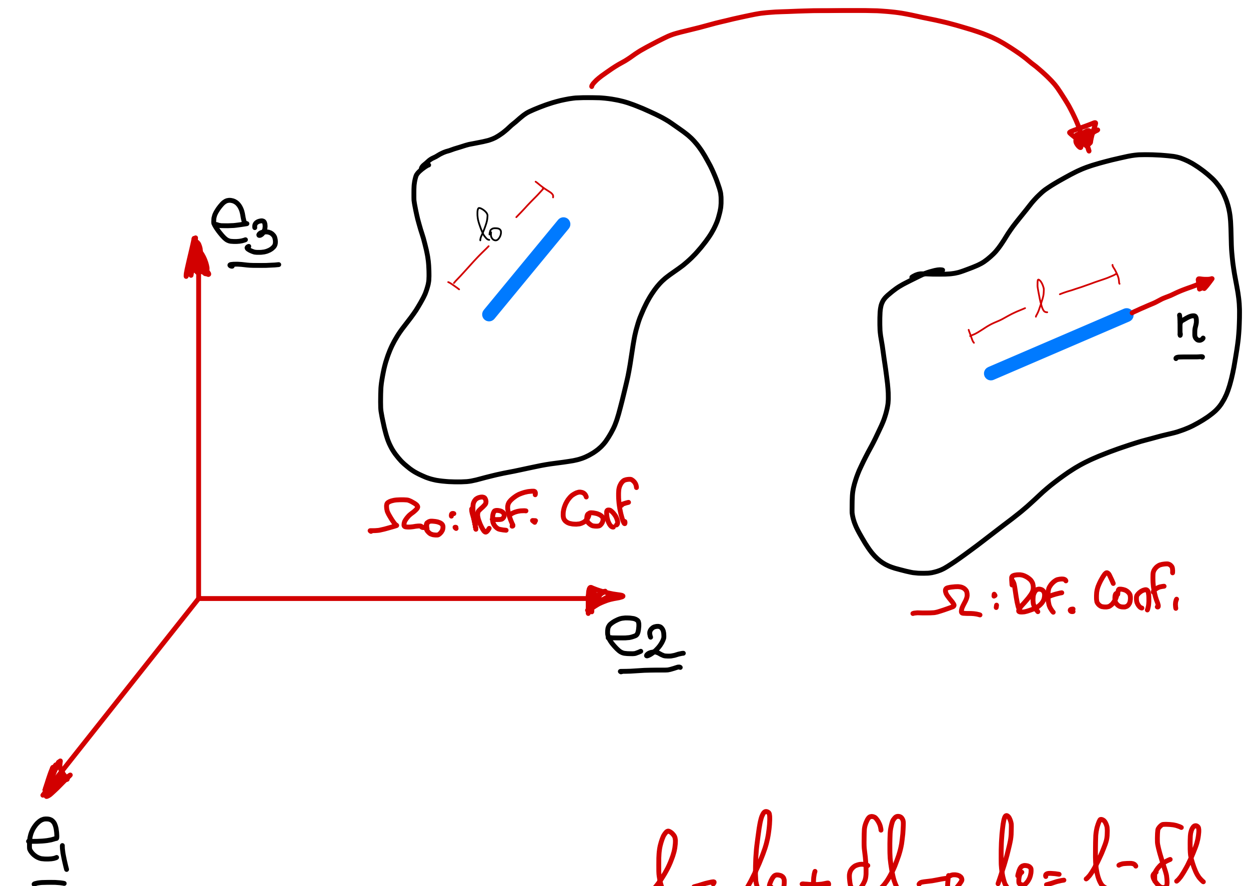
$$E_{12} = \frac{1}{2}\left(F_{11}\cdot F_{12} + F_{21} F_{22} + F_{31} F_{32} - \overbrace{\delta_{12}}^{0}\right)$$

Strain measures

- **The Eulerian strain tensor:**

$$\underline{\underline{E}}^* = \frac{1}{2}\left(\underline{\underline{I}} - \underline{\underline{F}}^{-T}\cdot\underline{\underline{F}}^{-1}\right)$$

$$E^*_{ij} = \frac{1}{2}\left(\delta_{ij} - F^{-1}_{ki}F^{-1}_{kj}\right)$$



$$\boxed{\varepsilon_E(\underline{n}) = \frac{l^2 - l_0^2}{2l^2}}$$

$$\varepsilon_E(\underline{n}) = \underline{n}\cdot\underline{\underline{E}}^*\cdot\underline{n}$$

$$\varepsilon_E(n_i) = E^*_{ij}\cdot n_i\, n_j \Rightarrow$$ Proof exercise for students

$$l = l_0 + \delta l \Rightarrow l_0 = l - \delta l$$

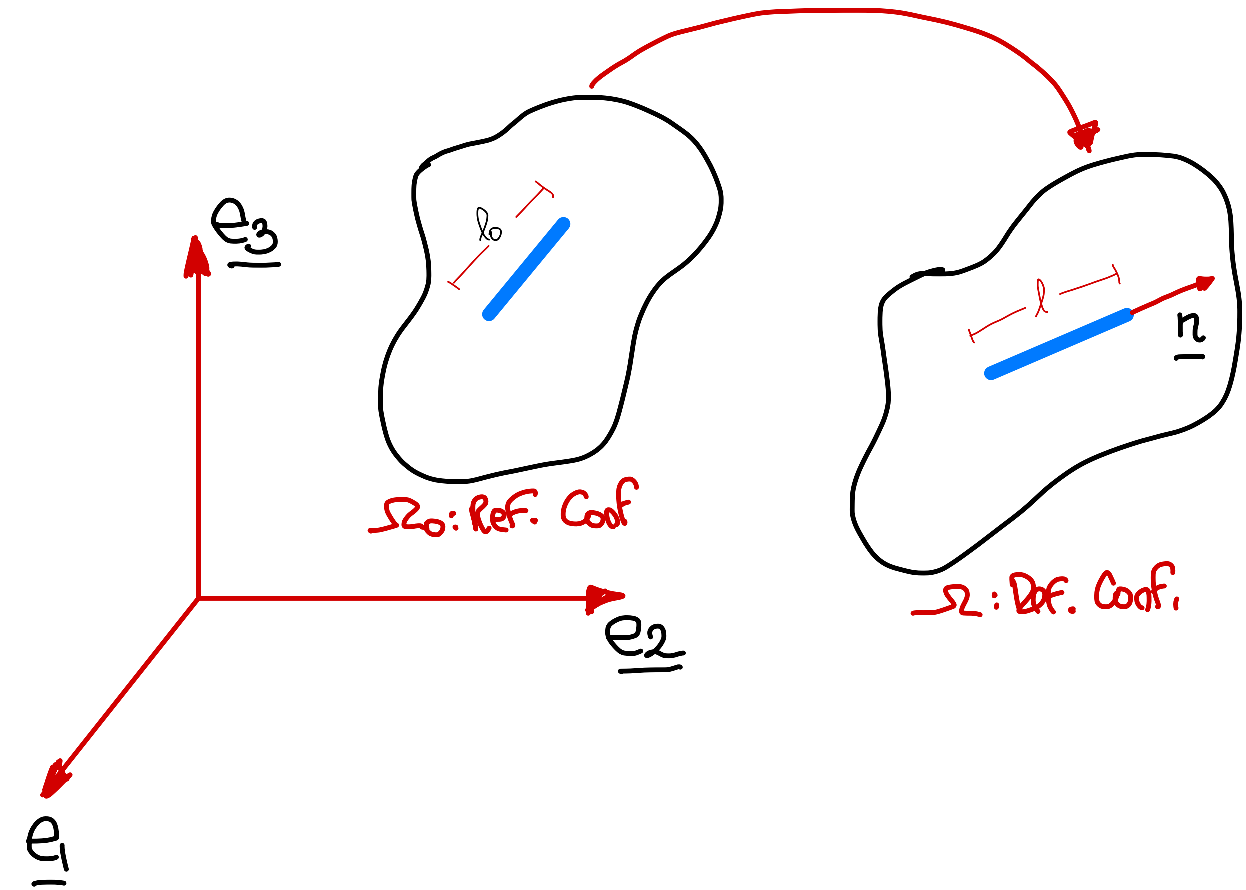
$$\varepsilon_E(\underline{n}) = \frac{l^2 - l_0^2}{2l^2} = \frac{l^2 - (l-\delta l)^2}{2l^2} = \frac{\cancel{l^2} - [\cancel{l^2} - 2l\delta l + \delta l^2]}{2l^2}$$

Strain measures

- The Eulerian strain tensor:

$$\underline{\underline{E}}^* = \frac{1}{2}\left(\underline{\underline{I}} - \underline{\underline{F}}^{-T} \cdot \underline{\underline{F}}^{-1}\right)$$

$$E^*_{ij} = \frac{1}{2}\left(\delta_{ij} - F^{-1}_{ki} F^{-1}_{kj}\right)$$



$$\boxed{\varepsilon_E(\underline{n}) = \frac{\delta l}{l} - \frac{\delta l^2}{2l^2}}$$

$$\varepsilon_E(\underline{n}) = \frac{\cancel{l^2} - [\cancel{l^2} - 2l\delta l + \delta l^2]}{2l^2} = \frac{2l\delta l - \delta l^2}{2l^2} = \overbrace{\frac{\delta l}{l}}^{\text{inf. strain}} - \overbrace{\frac{\delta l^2}{2l^2}}^{\text{non-linear}}$$

Strain measures

- The infinitesimal strain tensor: Only valid when we have small deformation

$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\underline{u}\nabla + (\underline{u}\nabla)^T\right) \quad \text{or} \quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\varepsilon_{ij} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \cdots \\ & \cdots & \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

Approximate strain measure if

$$\frac{\partial u_i}{\partial x_j} \ll 1$$

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i} \cdot \frac{\partial u_k}{\partial x_j}\right) \approx \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \varepsilon_{ij}$$

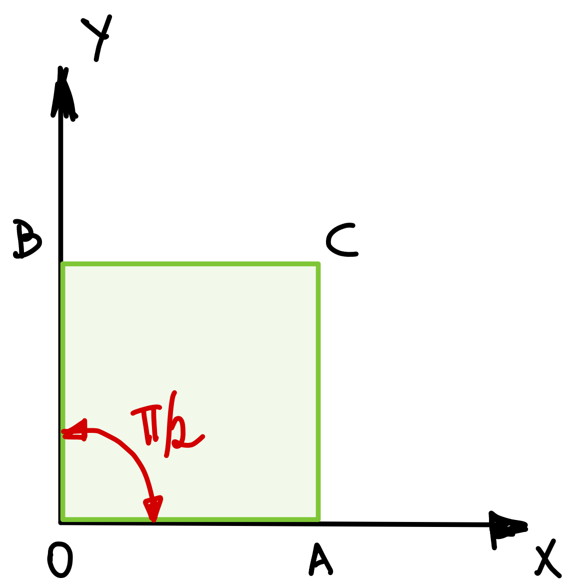
Strain measures

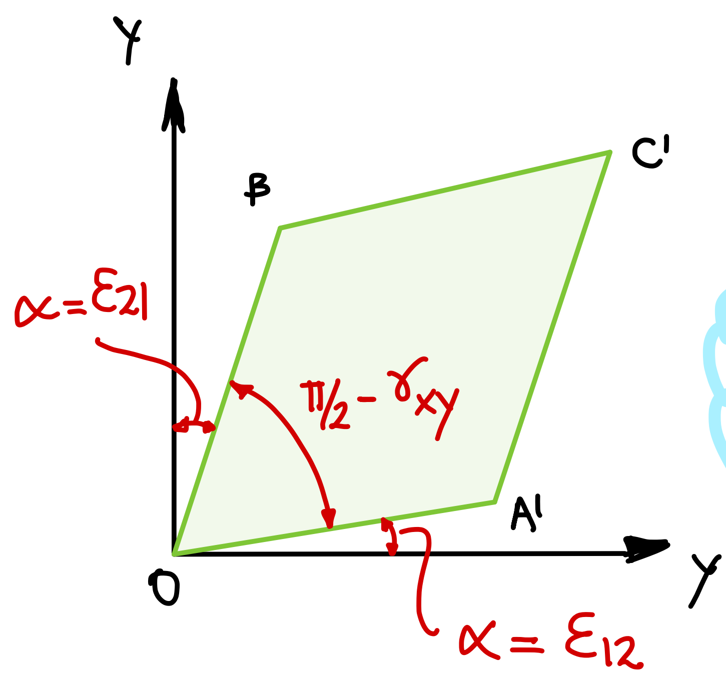
- The infinitesimal strain tensor: Properties of the inf. strain tensor

$$\underline{\underline{\varepsilon}}(\underline{m}) = \frac{l - l_0}{l_0} \simeq \varepsilon_{ij}\, m_i\, m_j$$

Notice that:

$$\mathrm{Trace}\,(\underline{\underline{\varepsilon}}) = \varepsilon_{kk} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{dV - dV_0}{dV_0}$$





$$\underbrace{\varepsilon_{12} = \varepsilon_{21}}_{\text{Inf strain}}$$

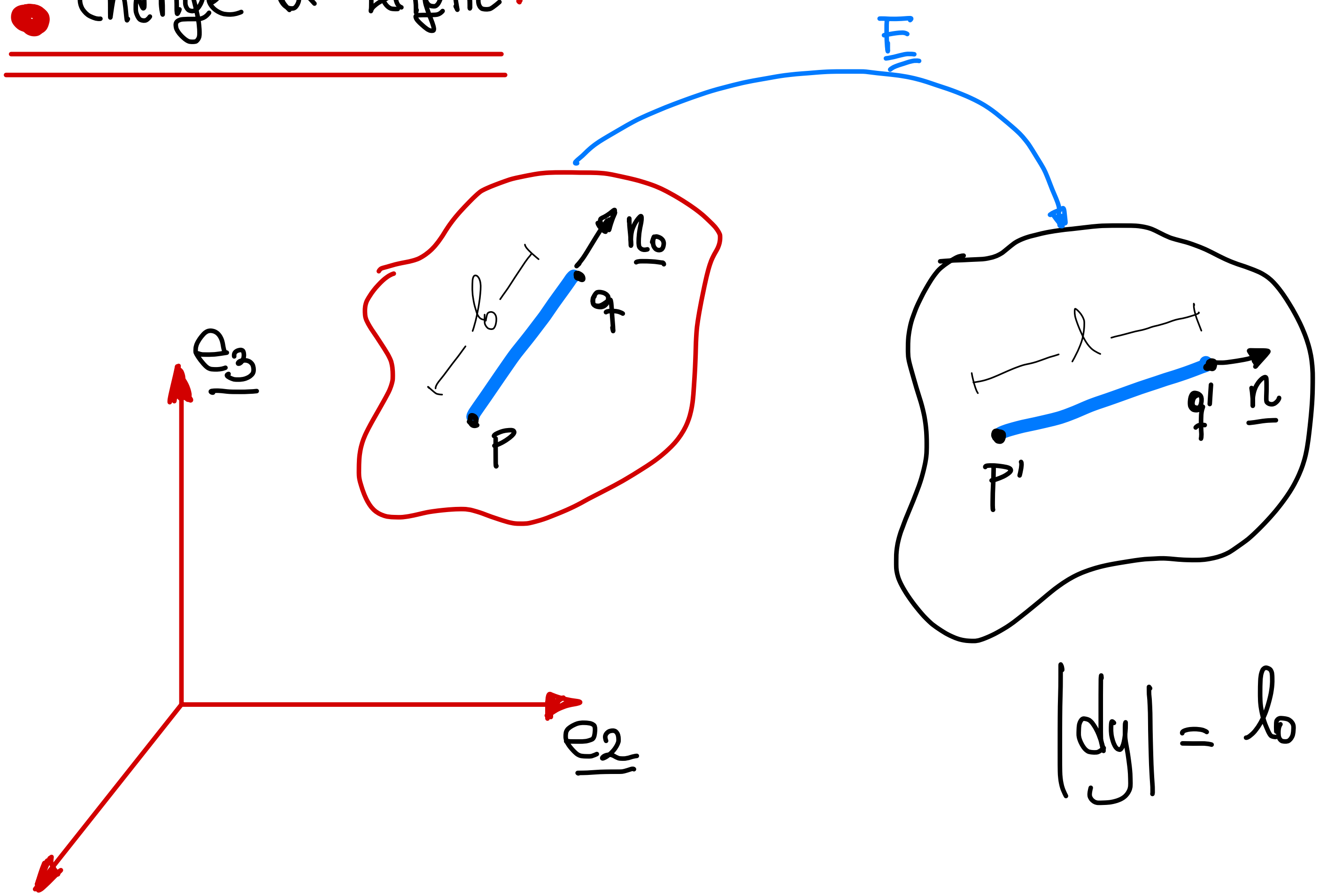
- Engineering

$$\gamma_{xy} = 2\varepsilon_{12}$$

$$\tau_{xy} = G\,\gamma_{xy}$$

Change of length and orientation

- **Change of length:**



$$d\underline{x} = l_0\, \underline{n}_0$$

$$d\underline{y} = l\, \underline{n}$$

$$d\underline{y} = \underline{\underline{F}}\, d\underline{x}$$

$$d\underline{y} = l_0 \underline{\underline{F}} \cdot \underline{n}_0$$

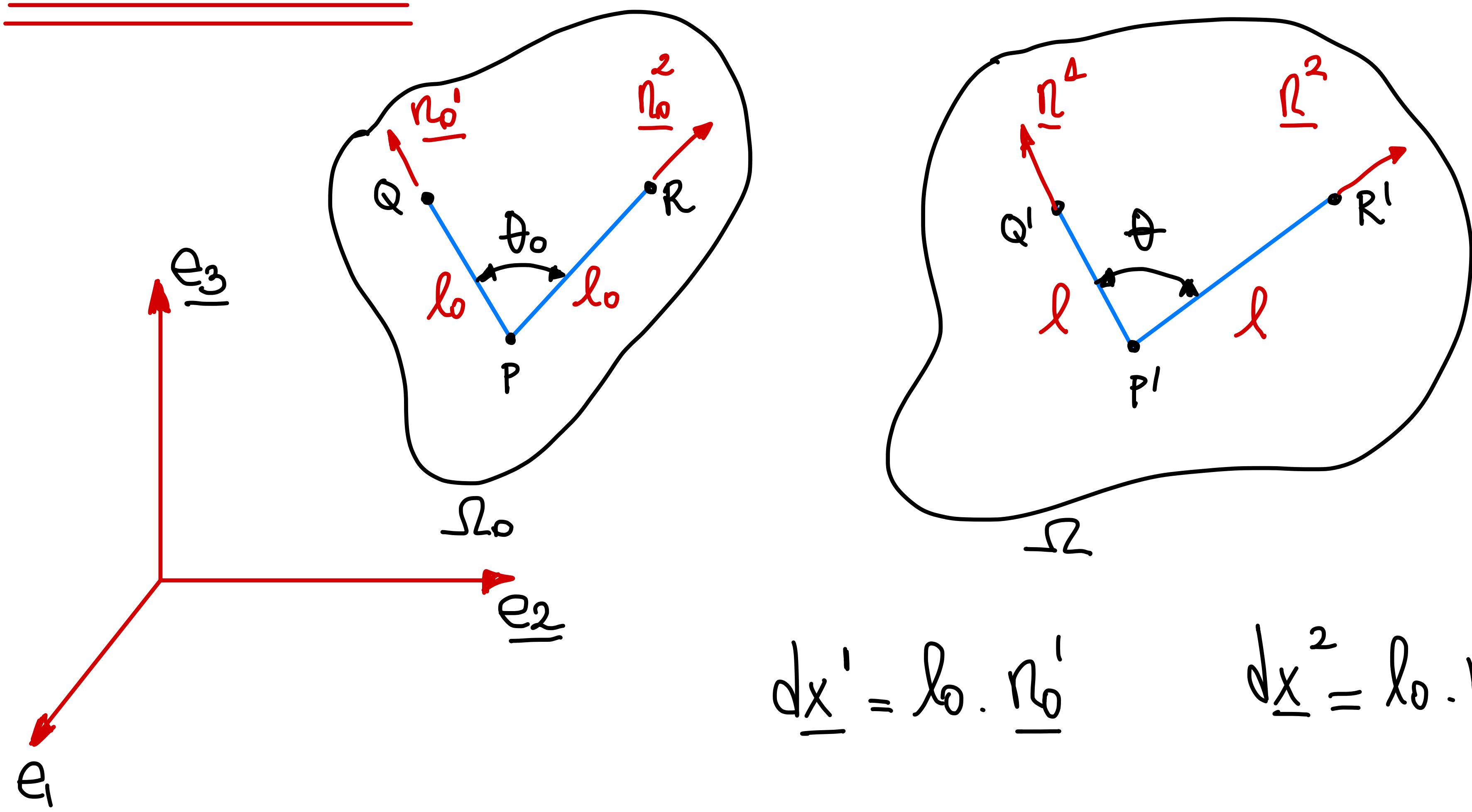
$$|d\underline{y}| = l_0\, |\underline{\underline{F}} \cdot \underline{n}_0|$$

The stretch ratio $\lambda = \dfrac{|d\underline{y}|}{|d\underline{x}|} = \dfrac{\cancel{l_0}}{\cancel{l_0}}\, |\underline{\underline{F}} \cdot \underline{n}_0|$

$$\boxed{\lambda = |\underline{\underline{F}} \cdot \underline{n}_0|}$$

Change of length and orientation

- Change of angle



$$d\underline{x}^1 = l_0 \cdot \underline{n}_0^1 \qquad d\underline{x}^2 = l_0 \cdot \underline{n}_0^2$$

Recall the definition of the dot product

$$\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}| \cdot \cos\theta(\underline{a}, \underline{b})$$

Change of length and orientation

- Change of angle

$$\underbrace{d\underline{x}^1}_{\underline{a}} = l_0 \cdot \underline{n}_0^1 \qquad \underbrace{d\underline{x}^2}_{\underline{b}} = l_0 \cdot \underline{n}_0^2$$

Recall the definition of the dot product

$$\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}| \cdot \cos\theta(\underline{a}, \underline{b})$$

$$\underbrace{d\underline{y}^1}_{\underline{a}} = \underline{\underline{F}} \cdot d\underline{x}^1 \; ; \qquad \underbrace{d\underline{y}^2}_{\underline{b}} = \underline{\underline{F}} \cdot d\underline{x}^2$$

$$d\underline{y}^1 \cdot d\underline{y}^2 = |d\underline{y}^1|\,|d\underline{y}^2| \cdot \cos \overbrace{\theta(d\underline{y}^1, d\underline{y}^2)}^{\theta_y}$$

$$(\underline{\underline{F}} \cdot d\underline{x}^1) \cdot (\underline{\underline{F}} \cdot d\underline{x}^2) = |\underline{\underline{F}} \cdot d\underline{x}^1|\,|\underline{\underline{F}}\, d\underline{x}^2| \cdot \cos\theta_y$$

Change of length and orientation

- Change of angle

$$d\underline{y}^1 \cdot d\underline{y}^2 = |d\underline{y}^1|\,|d\underline{y}^2| \cdot \cos\theta(d\underline{y}^1, d\underline{y}^2)$$

$$(\underline{\underline{F}} \cdot d\underline{x}^1) \cdot (\underline{\underline{F}} \cdot d\underline{x}^2) = |\underline{\underline{F}} \cdot d\underline{x}^1|\,|\underline{\underline{F}}\, d\underline{x}^2| \cdot \cos\theta_y$$

$$\cos\theta_y = \frac{(\underline{\underline{F}} \cdot d\underline{x}^1) \cdot (\underline{\underline{F}}\, d\underline{x}^2)}{|\underline{\underline{F}} \cdot d\underline{x}^1| \cdot |\underline{\underline{F}} \cdot d\underline{x}^2|}$$

$$d\underline{x}^1 = l_0\, \underline{n}_0^1$$

$$d\underline{x}^2 = l_0\, \underline{n}_0^2$$

$$\cos\theta_y = \frac{\cancel{l_0^2}\,(\underline{\underline{F}} \cdot \underline{n}_0^1) \cdot (\underline{\underline{F}} \cdot \underline{n}_0^2)}{\cancel{l_0^2}\, |\underline{\underline{F}} \cdot \underline{n}_0^1|\,|\underline{\underline{F}}\, \underline{n}_0^2|}$$

$$\boxed{\cos\theta_y = \frac{(\underline{\underline{F}} \cdot \underline{n}_0^1) \cdot (\underline{\underline{F}} \cdot \underline{n}_0^2)}{|\underline{\underline{F}} \cdot \underline{n}_0^1|\,|\underline{\underline{F}}\, \underline{n}_0^2|}}$$

Examples

Problem #1:

$r_0 : (x_1, x_2, x_3)$: Is a point in the reference configuration

Consider the following displacement field

$$\mathbf{u} = \begin{pmatrix} x_1^3 + x_1 x_2^2 + x_3^2 \\ x_3^2 - x_2^3 \\ 2x_1x_2x_3 - x_1^3 \end{pmatrix} \times 10^{-4} \; [cm].$$

- Derive the infinitesimal strain tensor.
- Derive the vorticity tensor (defined as $\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$). ✓
- Derive the relative change length (defined as $\frac{\Delta l}{\Delta l_0}$)
- Derive the relative volume change (defined as $\frac{\Delta V}{\Delta V_0}$).
- The deviatoric infinitesimal strain (defined as $e_{ij} = \epsilon_{ij} - \frac{1}{3}\delta_{ij}\epsilon_{kk}$). ✓

$$\varepsilon_{ij} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ - & - & - \\ - & - & - \end{pmatrix}$$

All evaluated at the point $\boldsymbol{x} = \{1, 0, 1\}^T$.

$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\underline{u}\nabla + \underline{u}\nabla^T\right) \Rightarrow \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \varepsilon_{ij}$$

$$\frac{\partial u_i}{\partial x_j}$$

Examples

Problem #1:

Consider the following displacement field

$$\mathbf{u} = \begin{pmatrix} x_1^3 + x_1 x_2^2 + x_3^2 \\ x_3^2 - x_2^3 \\ 2x_1x_2x_3 - x_1^3 \end{pmatrix}.$$

- Derive the infinitesimal strain tensor.
- Derive the vorticity tensor (defined as $\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$).
- Derive the relative change length (defined as $\frac{\Delta l}{\Delta l_0}$)
- Derive the relative volume change (defined as $\frac{\Delta V}{\Delta V_0}$).
- The deviatoric infinitesimal strain (defined as $e_{ij} = \epsilon_{ij} - \frac{1}{3}\delta_{ij}\epsilon_{kk}$).

All evaluated at the point $\boldsymbol{x} = \{1, 0, 1\}^T$.

$$\underline{u}\nabla = \begin{pmatrix} 3x_1^2 + x_2^2 & 2x_1x_2 & 2x_3 \\ 0 & -3x_2^2 & 2x_3 \\ -3x_1^2 + 2x_2x_3 & 2x_1x_3 & 2x_1x_2 \end{pmatrix}$$

$$\underline{u}\nabla(1,0,1) = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 0 & 2 \\ -3 & 2 & 0 \end{pmatrix}$$

$$\underline{u}\nabla^T = \begin{pmatrix} 3 & 0 & -3 \\ 0 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

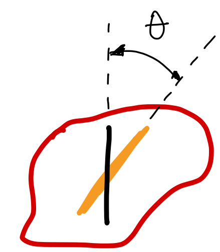
$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\underline{u}\nabla + \underline{u}\nabla^T\right)$$

Examples

$$\underline{u}\nabla(1,0,1) = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 0 & 2 \\ -3 & 2 & 0 \end{pmatrix} \qquad \underline{u}\nabla^T = \begin{pmatrix} 3 & 0 & -3 \\ 0 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} \qquad \underline{\underline{\varepsilon}} = \frac{1}{2}\left(\underline{u}\nabla + \underline{u}\nabla^T\right)$$

$$\underline{\underline{\varepsilon}} = \frac{1}{2}\begin{pmatrix} 6 & 0 & -1 \\ 0 & 0 & 4 \\ -1 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 & -1/2 \\ 0 & 0 & 2 \\ -1/2 & 2 & 0 \end{pmatrix}$$

symmetric by construction.



- Vorticity tensor:

$$w_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \Rightarrow \underline{\underline{w}} = \frac{1}{2}\left(\underline{u}\nabla - \underline{u}\nabla^T\right)$$

$$\underline{\underline{w}} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 5 \\ 0 & 0 & 0 \\ -5 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2.5 \\ 0 & 0 & 0 \\ -2.5 & 0 & 0 \end{pmatrix}$$

Examples

- Relative change of length:

$$\underline{n} = \left\{ \frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}, 0 \right\}$$

$$|\underline{n}| = \left(\frac{1}{3} + \frac{2}{3} \right)^{1/2} = 1 \qquad \text{Unit vector}$$

$$\frac{\Delta l}{l_0} = \underline{n}^T \cdot \underline{\underline{\varepsilon}} \cdot \underline{n}$$

$$\underbrace{[1\times 3]\,[3\times 3]\,[3\times 1]}$$

$$\underbrace{[1\times 3]\,[3\times 1]}$$

$$[1\times 1] = \text{scalar}$$

$$\frac{\Delta l}{l_0} = \left\{ \frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}, 0 \right\}^T \begin{pmatrix} 3 & 0 & -1/2 \\ 0 & 0 & 2 \\ -1/2 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} \\ \sqrt{2/3} \\ 0 \end{pmatrix}$$

$$\frac{\Delta l}{l_0} = \left\{ \frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}, 0 \right\}^T \begin{pmatrix} 3/\sqrt{3} \\ 0 \\ \frac{1}{\sqrt{3}}\left(2\sqrt{2} - \frac{1}{2}\right) \end{pmatrix} = \frac{1}{\sqrt{3}} \cdot \frac{3}{\sqrt{3}} + \cancel{\sqrt{\frac{2}{3}} \cdot 0} + \cancel{0} = 1$$

Examples

- Relative change of length:

$$\frac{\Delta l}{l_0} = 1 \times 10^{-4}.$$

- Relative change in volume:

$$\frac{\Delta V}{V_0} = \mathrm{trace}(\underline{\underline{\varepsilon}}) = \frac{\partial u_k}{\partial x_k} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$$\underline{\underline{\varepsilon}} = \begin{pmatrix} 3 & 0 & -1/2 \\ 0 & 0 & 2 \\ -1/2 & 2 & 0 \end{pmatrix} \qquad \frac{\Delta V}{V_0} = 3 \times 10^{-4}.$$

- Deviatoric part of the $\underline{\underline{\varepsilon}}$:

$$\underline{\underline{e}} = \underline{\underline{\varepsilon}} - \frac{1}{3}\,\mathrm{trace}(\underline{\underline{\varepsilon}})\,\underline{\underline{I}}$$

Examples

- Deviatoric part of the $\underline{\underline{\varepsilon}}$: $\underline{\underline{e}} = \underline{\underline{\varepsilon}} - \frac{1}{3} \operatorname{tra}(\underline{\underline{\varepsilon}}) \underline{\underline{I}}$

$$e_{ij} = \varepsilon_{ij} - \frac{1}{3} \delta_{ij} \varepsilon_{kk}$$

$$\underline{\underline{e}} = \begin{pmatrix} 3 & 0 & -1/2 \\ 0 & 0 & 2 \\ -1/2 & 2 & 0 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 0 & -1/2 \\ 0 & -1 & 0 \\ -1/2 & 2 & -1 \end{pmatrix}$$

Examples

Problem #2:

Consider the following Lagrangian strain tensor

$$\mathbf{E}_L = \begin{pmatrix} 2 & 0.1 & 0 \\ 0.1 & 1.0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

For the fiber given by $\boldsymbol{f} = \{1, 2, 0\}^T$, compute the length in the deformed configuration.

$$E_L = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$$

$$\underline{f} = \{1, 2, 0\}$$

$$\underline{m} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right) \qquad |\underline{m}| = 1$$

$$l_0 = (1^2 + 2^2)^{1/2} = \sqrt{5}$$

$$l_0 = \sqrt{5}$$

$$\varepsilon_L(\underline{m}) = \underline{m}^T \underline{\underline{E_L}} \cdot \underline{m} = E_{Lij}\, m_i\, m_j$$

$$\varepsilon_L(\underline{m}) = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right) \begin{pmatrix} 2 & 0.1 & 0 \\ 0.1 & 1.0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{pmatrix} = 1.28$$

$$\boxed{\begin{aligned} l_0 &= \sqrt{5} \\ l_f &= 8.3066 \end{aligned}}$$

def_grad.m file in Canvas

Contents


```
clear all, close all, clc

% line, square, decomposition 'square';%'decomposition';%'line';%
example = 'square';%

% expansion, compression simple_shear, rigid_rotation, gen_hom_def,
% complex, 'gen_hom_def';%'complex';%'rigid_rotation';%'simple_shear';%'expansion';%
deformation = 'complex';%'compression';
```

Apply a deformation gradient to a line and draw it

```
lambda = 0.001;

x1 =[0; 0; 0];
x2 =[0; 1; 0];
x3 =[1; 1; 0];
x4 =[1; 0; 0];
J =1;
theta = 0.1*pi/180;
n = 100;

f = 1.001; %could be a function of radius.

switch deformation
    case 'compression'
        F  =[1-lambda 0 0; 0 1-lambda 0; 0 0 1-lambda];
    case 'expansion'
        F  =[1+lambda 0 0; 0 1+lambda 0; 0 0 1+lambda];
    case 'simple_shear'
        F  =[1 tan(theta) 0; 0 1 0; 0 0 1];
    case 'rigid_rotation'
        F  =[cos(theta) -sin(theta) 0; sin(theta) cos(theta) 0; 0 0 1];
    case 'gen_hom_def'
        F  =[1.001 0.002 -0.003; 0.003 1.002 -0.001; 0.002 -0.002 1.000];
    case 'complex'
        F = [f*cos(theta) -f*sin(theta) 0; f*sin(theta) f*cos(theta) 0; 0 0 1+lambda];
end

y1 = x1;
y2 = x2;
y3 = x3;
y4 = x4;

switch example
    case 'square'
        for i=1:n

            y1 = F*y1;
```

```
        y2 = F*y2;
        y3 = F*y3;
        y4 = F*y4;

        plot([x1(1) x2(1)],[x1(2) x2(2)],'*r','LineWidth',2.0)
        hold on
        plot([y1(1) y2(1)],[y1(2) y2(2)],'ob-','LineWidth',2.0)
        plot([y2(1) y3(1)],[y2(2) y3(2)],'ob-','LineWidth',2.0)
        plot([y3(1) y4(1)],[y3(2) y4(2)],'ob-','LineWidth',2.0)
        plot([y1(1) y4(1)],[y1(2) y4(2)],'ob-','LineWidth',2.0)

        J= J*det(F);
        text(-1,-1.75,sprintf('Jacobian %f',J),'FontSize',20,'FontWeight','bold')
        text(-1,1.75,deformation,'FontSize',20,'FontWeight','bold')
        set(gca,'FontSize',12,'FontWeight','bold')

        hold off
        axis([-2 2 -2 2])
        axis square
        drawnow

    end

case 'line'

    % plot a line

    for i=1:n

        y1 = F*y1;
        y2 = F*y2;

        plot([x1(1) x2(1)],[x1(2) x2(2)],'*r','LineWidth',2.0)
        hold on
        plot([y1(1) y2(1)],[y1(2) y2(2)],'ob-','LineWidth',2.0)
        hold off
        set(gca,'FontSize',12,'FontWeight','bold')
        text(-1,1.75,deformation,'FontSize',20,'FontWeight','bold')
        axis([-2 2 -2 2])
        axis square
        J= J*det(F);
        text(-1,-1.75,sprintf('Jacobian %f',J),'FontSize',20,'FontWeight','bold')
        drawnow

    end

case 'decomposition'
```

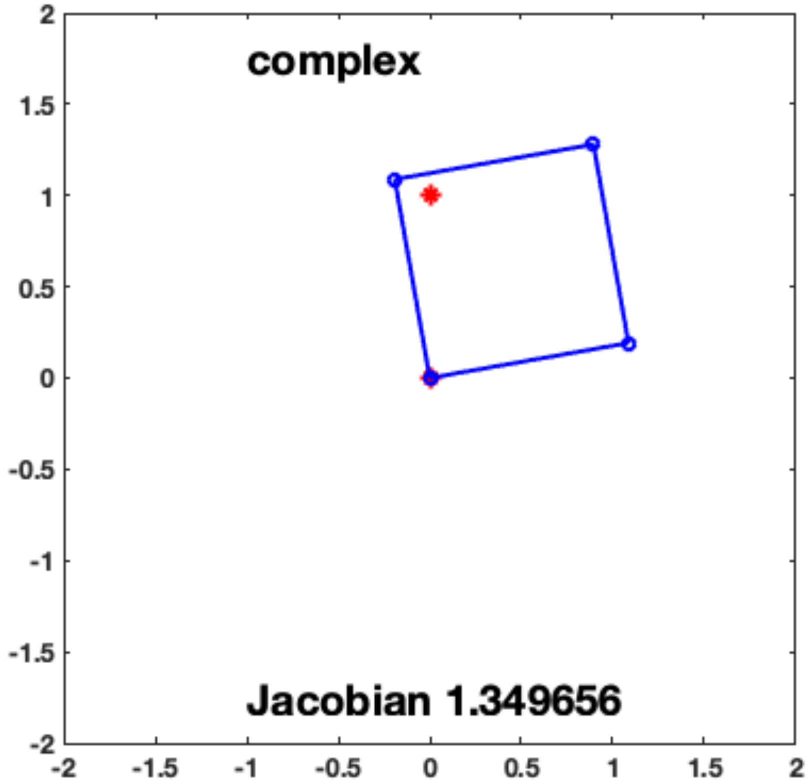
Now show that the deformation gradient can be split in two or more parts

```
    F1 = [cos(theta) -sin(theta) 0; sin(theta) cos(theta) 0; 0 0 1];

    F2 = [1+lambda 0 0; 0 1+lambda 0; 0 0 1+lambda];

    for i=1:n
```

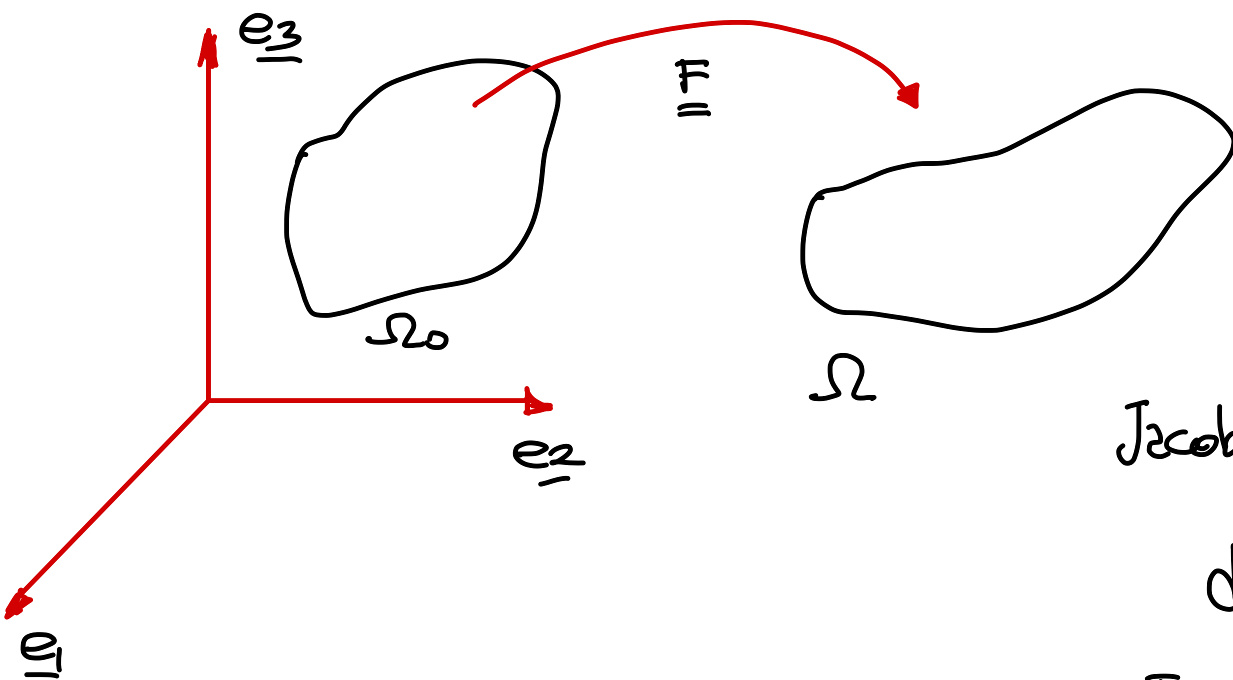
```
            y1 = F2*y1;
            y2 = F2*y2;
            y3 = F2*y3;
            y4 = F2*y4;

            plot([x1(1) x2(1)],[x1(2) x2(2)],'*r','LineWidth',2.0)
            hold on
            plot([y1(1) y2(1)],[y1(2) y2(2)],'ob-','LineWidth',2.0)
            plot([y2(1) y3(1)],[y2(2) y3(2)],'ob-','LineWidth',2.0)
            plot([y3(1) y4(1)],[y3(2) y4(2)],'ob-','LineWidth',2.0)
            plot([y1(1) y4(1)],[y1(2) y4(2)],'ob-','LineWidth',2.0)
            set(gca,'FontSize',12,'FontWeight','bold')
            text(-1,1.75,example,'FontSize',20,'FontWeight','bold')
            J= J*det(F2);
            text(-1,-1.75,sprintf('Jacobian %f',J),'FontSize',20,'FontWeight','bold')
            hold off
            axis([-2 2 -2 2])
            axis square
            drawnow
        end


        for i=1:n

            y1 = F1*y1;
            y2 = F1*y2;
            y3 = F1*y3;
            y4 = F1*y4;

            plot([x1(1) x2(1)],[x1(2) x2(2)],'*r','LineWidth',2.0)
            hold on
            plot([y1(1) y2(1)],[y1(2) y2(2)],'ob-','LineWidth',2.0)
            plot([y2(1) y3(1)],[y2(2) y3(2)],'ob-','LineWidth',2.0)
            plot([y3(1) y4(1)],[y3(2) y4(2)],'ob-','LineWidth',2.0)
            plot([y1(1) y4(1)],[y1(2) y4(2)],'ob-','LineWidth',2.0)
            set(gca,'FontSize',12,'FontWeight','bold')
            text(-1,1.75,example,'FontSize',20,'FontWeight','bold')
            J= J*det(F1);
            text(-1,-1.75,sprintf('Jacobian %f',J),'FontSize',20,'FontWeight','bold')
            hold off
            axis([-2 2 -2 2])
            axis square
            drawnow
        end

%         for i=1:n
%
%             y1 = F2*y1;
%             y2 = F2*y2;
%             y3 = F2*y3;
%             y4 = F2*y4;
%
%             plot([x1(1) x2(1)],[x1(2) x2(2)],'*r','LineWidth',2.0)
%             hold on
%             plot([y1(1) y2(1)],[y1(2) y2(2)],'ob-','LineWidth',2.0)
%             plot([y2(1) y3(1)],[y2(2) y3(2)],'ob-','LineWidth',2.0)
%             plot([y3(1) y4(1)],[y3(2) y4(2)],'ob-','LineWidth',2.0)
```

```
%             plot([y1(1) y4(1)],[y1(2) y4(2)],'ob-','LineWidth',2.0)
%             set(gca,'FontSize',12,'FontWeight','bold')
%             text(-1,1.75,example,'FontSize',20,'FontWeight','bold')
%             J= J*det(F2);
%             text(-1,-1.75,sprintf('Jacobian %f',J),'FontSize',20,'FontWeight','bold')
%             hold off
%             axis([-2 2 -2 2])
%             axis square
%             drawnow
%         end

        % Redefine the points again to plot the total F

        y1 = x1;
        y2 = x2;
        y3 = x3;
        y4 = x4;

        F = F1*F2;
        J = 1;

        for i=1:n

            y1 = F*y1;
            y2 = F*y2;
            y3 = F*y3;
            y4 = F*y4;

            hold on
            plot([y1(1) y2(1)],[y1(2) y2(2)],'sk-','LineWidth',2.0)
            plot([y2(1) y3(1)],[y2(2) y3(2)],'sk-','LineWidth',2.0)
            plot([y3(1) y4(1)],[y3(2) y4(2)],'sk-','LineWidth',2.0)
            plot([y1(1) y4(1)],[y1(2) y4(2)],'sk-','LineWidth',2.0)
            set(gca,'FontSize',12,'FontWeight','bold')
            text(-1,1.75,example,'FontSize',20,'FontWeight','bold')
            %J= J*det(F);
            %text(-1,-1.75,sprintf('Jacobian %f',J))
            hold off
            axis([-2 2 -2 2])
            axis square
            drawnow
        end
```

```
end
```


complex
Jacobian 1.349656
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

Polar decomposition



$$\frac{\partial y_i}{\partial x_j} = F_{ij}$$

(1) $\underline{\underline{F}}$: Nonsingular tensor with

$$\text{Jacobian} = \det(\underline{\underline{F}}) > 0 \qquad J = \frac{dV}{dV_0}$$

$$dV > 0 \quad (\text{positive})$$

(2) $\underline{\underline{F}}^T \cdot \underline{\underline{F}}$: Is positive definite

$$\underline{\underline{F}}^T \cdot \underline{\underline{F}} \cdot \underline{X} \quad \underline{X} > 0$$

$$\underline{\underline{F}} = \underline{\underline{R}} \cdot \underline{\underline{U}}$$

$\underline{\underline{R}}$: Rigid body rotation $\qquad \underline{\underline{R}}^T \cdot \underline{\underline{R}} = \underline{\underline{I}}$

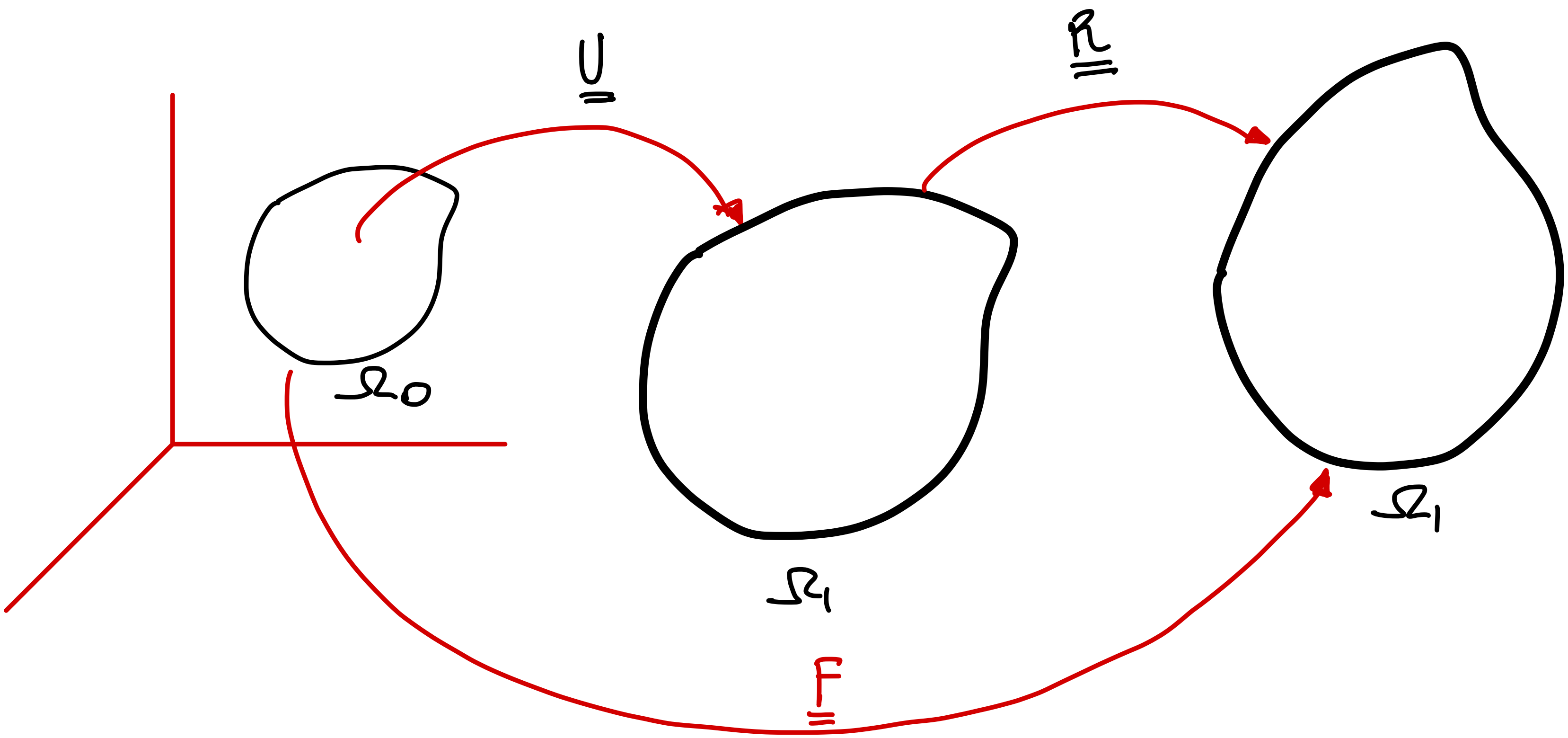
$\underline{\underline{U}}$: Rigth stretch tensor

$$\underline{\underline{F}}^T \cdot \underline{\underline{F}} = (\underline{\underline{R}} \cdot \underline{\underline{U}})^T \cdot \underline{\underline{R}}\,\underline{\underline{U}} \Rightarrow \underline{\underline{F}}^T \underline{\underline{F}} = \underline{\underline{U}}^T \overbrace{\underline{\underline{R}}^T \underline{\underline{R}}}^{\underline{\underline{I}}} \underline{\underline{U}} = \underline{\underline{U}}^2$$

Polar decomposition

$$\underline{\underline{U}} = \left(\underline{\underline{F}}^T \underline{\underline{F}}\right)^{1/2} \qquad \underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} \Rightarrow \underline{\underline{R}} = \underline{\underline{F}} \cdot \underline{\underline{U}}^{-1}$$

$$d\underline{x} \longrightarrow d\underline{y} = \overbrace{\underline{\underline{R}} \cdot \underline{\underline{U}}}^{\underline{\underline{F}}} \cdot d\underline{x} = \underline{\underline{F}}_2 \cdot \underline{\underline{F}}_1 \, d\underline{x}$$



$\underline{\underline{U}}$ is symmetric & positive definite. Then

Polar decomposition

$$\underline{\underline{U}} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}; \qquad \lambda_i > 0 \quad (\text{scalars, greater than } 0)$$

If the components of $d\underline{x}$ in the principal basis are:

$$d\underline{x} = \begin{Bmatrix} dx_1^P \\ dx_2^P \\ dx_3^P \end{Bmatrix} \quad \text{then} \quad d\underline{y} = \underline{\underline{U}} \cdot d\underline{x} = \begin{Bmatrix} \lambda_1 \, dx_1^P \\ \lambda_2 \, dx_2^P \\ \lambda_3 \, dx_3^P \end{Bmatrix}$$

$$\underline{\underline{F}} = \underline{\underline{R}} \cdot \underline{\underline{U}}$$

$\underline{\underline{U}}$: is called the right stretch tensor.

Polar decomposition

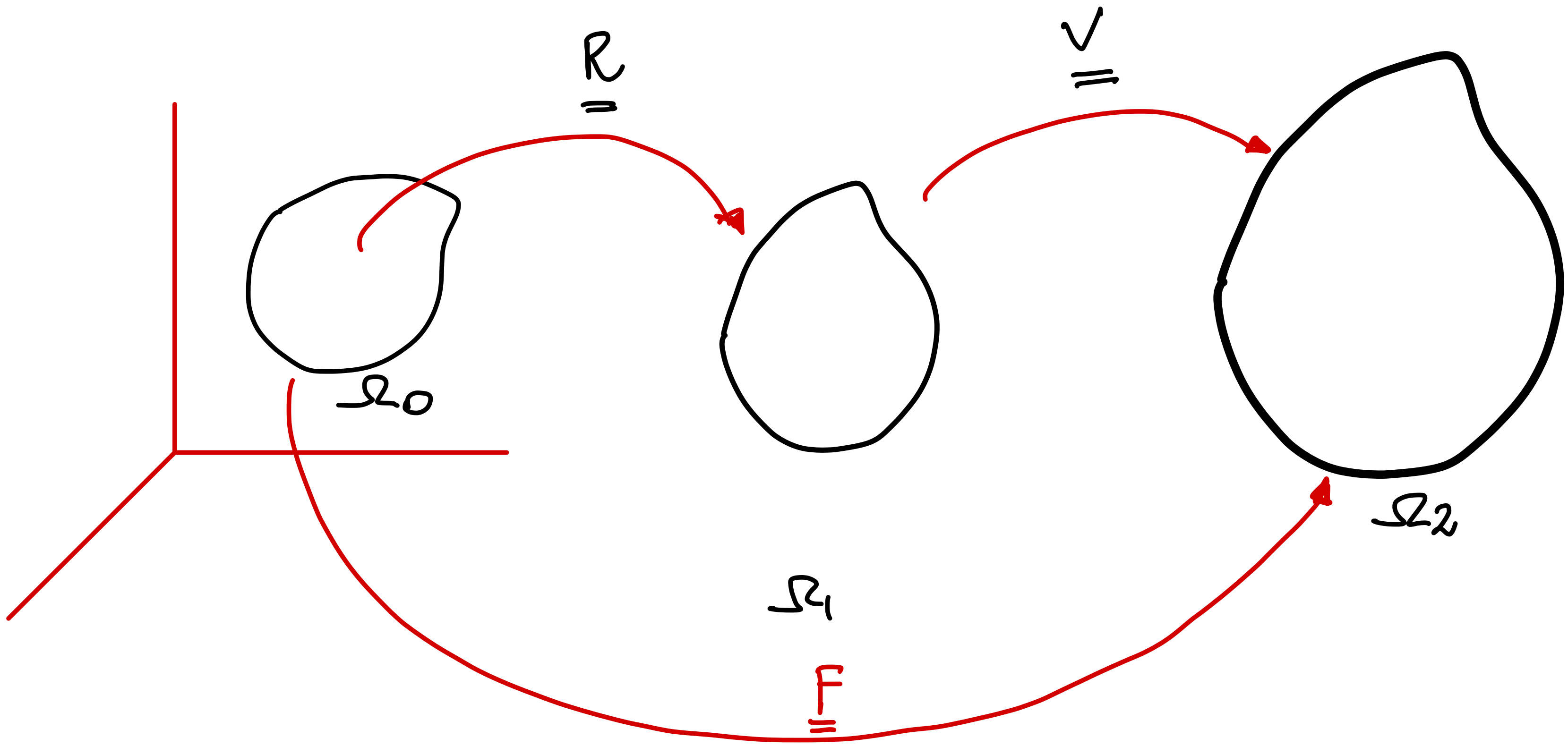
$$\underline{\underline{F}} = \underline{\underline{V}} \cdot \underline{\underline{R}}$$

$\underline{\underline{R}}$: Rigid body rotation

$\underline{\underline{V}}$: Is called the left stretch tensor

$$\underline{\underline{V}}^T \cdot \underline{\underline{V}} = \underline{\underline{V}}^2$$

$$d\underline{x} \longrightarrow d\underline{y} = \underline{\underline{F}} \cdot d\underline{x} = \underline{\underline{V}} \cdot \underline{\underline{R}} \cdot d\underline{x} = \underline{\underline{F}}_2 \, \underline{\underline{F}}_1 \, d\underline{x}$$



Polar decomposition

$$\underline{\underline{F}} = \underline{\underline{V}} \cdot \underline{\underline{R}}$$

$$\underline{\underline{V}} = \left(\underline{\underline{F}} \cdot \underline{\underline{F}}^T\right)^{1/2}$$

$$\underline{\underline{R}} = \underline{\underline{V}}^{-1} \cdot \underline{\underline{F}} = \underline{\underline{F}} \cdot \underline{\underline{U}}^{-1}$$

$$\underline{\underline{V}} = \underline{\underline{F}}\,\underline{\underline{U}}\,\underline{\underline{F}}^{-1}$$

$$\underline{\underline{U}} = \sum_{i=1}^{3} \lambda_i \, \underline{r_i} \otimes \underline{r_i}$$

$$\underline{\underline{V}} = \sum_{i=1}^{3} \lambda_i \, \underline{l_i} \otimes \underline{l_i}$$

r_i : Principal basis

$$\underline{\underline{R}} = \sum_{i=1}^{3} \underline{l_i} \otimes \underline{r_i}$$

$$\underline{\underline{F}} = \sum_{i=1}^{3} \lambda_i \, \underline{l_i} \otimes \underline{r_i}$$

Polar decomposition

$$dl = l_0 \left(\underline{\underline{U}}^2 \cdot \underline{n}_0 \, \underline{n}_0 \right)^{1/2}$$

$$\cos \theta_{ij} = \frac{\underline{\underline{U}}^2 \cdot \underline{n}_0^{(1)} \cdot \underline{n}_0^{(2)}}{\left(\underline{\underline{U}}^2 \, \underline{n}_0^{(1)} \, n_0^{(1)} \right)^{1/2} \left(\underline{\underline{U}}^2 \, \underline{n}_0^{(2)} \, \underline{n}_0^{(2)} \right)^{1/2}}$$

$$dV = dV_0 \, \det(\underline{\underline{U}})$$

$$dA = dA_0 \left(\det \underline{\underline{U}} \right) \left| \underline{\underline{U}}^{-1} \cdot \underline{n}_0 \right|$$

Polar decomposition

Cauchy-Green deformation tensor

$$\underline{\underline{C}} = \underline{\underline{F}}^T \cdot \underline{\underline{F}} = \underline{\underline{U}}^2$$

(Right C.G. def tensor)

$$\underline{\underline{B}} = \underline{\underline{F}} \cdot \underline{\underline{F}}^T = \underline{\underline{V}}^2$$

(Left C.G. def tensor).

$$\underline{\underline{C}} = \sum_{i=1}^{3} \lambda_i^2 \, \underline{r_i} \otimes \underline{r_i}$$

$$\underline{\underline{B}} = \sum_{i=1}^{3} \lambda_i^2 \, \underline{l_i} \otimes \underline{l_i}$$

- <u>Remark</u>

Lagrange strain is $\underline{\underline{E}}_L = \frac{1}{2}\left(\underline{\underline{U}}^2 - \underline{\underline{I}}\right)$

Hencky $= E_H = \ln(\underline{U})$

Generalized strain $\underline{\underline{E}}_G = \frac{1}{m}\left(U^m - I\right)$

Compatibility equations

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i} \cdot \frac{\partial u_k}{\partial x_j} \right) \qquad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$$\begin{array}{ccc} \underline{u} & \rightleftarrows & \varepsilon_{ij} \;\; ? \\ \uparrow & & \uparrow \\ \text{3 components} & & \text{6 pdes} \end{array}$$

$$\left.\begin{array}{l} 6 \text{ equations} \\ 3 \text{ unknowns} \end{array}\right\} \text{OVERDETERMINED SYSTEM}$$

$$\underline{\underline{\varepsilon}} \Rightarrow \begin{cases} \text{It is integrable} \\ \text{Exist a } \underline{u} \text{ from which } \underline{\underline{\varepsilon}} \text{ comes from} \end{cases}$$

- Suppose we have a scalar field

$\phi(x_i, t)$ (Temperature field) ; $\phi\nabla = \dfrac{\partial \phi}{\partial x_i} = V_i \quad i = 1, 2, 3.$

(scalar) (vector)

Compatibility equations

$$\phi \rightleftarrows V_i = \frac{\partial \phi}{\partial x_i}$$

scalar — 3 components, 3 pdes.

System is over determined

$$V_1 = \frac{\partial \phi}{\partial x_1} \; ; \; V_2 = \frac{\partial \phi}{\partial x_2} \; ; \; V_3 = \frac{\partial \phi}{\partial x_3}$$

- Schwartz theorem:

ϕ: Continous function with continous derivatives

$$\boxed{\frac{\partial^2 \phi}{\partial x_i \partial x_j} = \frac{\partial^2 \phi}{\partial x_j \partial x_i} \quad \forall i, j}$$

$$\frac{\partial V_1}{\partial x_2} = \frac{\partial^2 \phi}{\partial x_1 \partial x_2} \; ; \; \frac{\partial V_2}{\partial x_1} = \frac{\partial^2 \phi}{\partial x_2 \partial x_1} \; ; \; \frac{\partial V_3}{\partial x_1} = \frac{\partial \phi}{\partial x_3 \partial x_1}$$

6 equations

$$\frac{\partial V_1}{\partial x_3} = \frac{\partial^2 \phi}{\partial x_1 \partial x_3} \qquad \frac{\partial V_2}{\partial x_3} = \frac{\partial^2 \phi}{\partial x_2 \partial x_3} \qquad \frac{\partial V_3}{\partial x_2} = \frac{\partial \phi}{\partial x_3 \partial x_2}$$

Compatibility equations

$$\frac{\partial v_1}{\partial x_2} = \frac{\partial^2 \phi}{\partial x_1 \partial x_2} = \frac{\partial^2 \phi}{\partial x_2 \partial x_1} = \frac{\partial v_2}{\partial x_1}$$

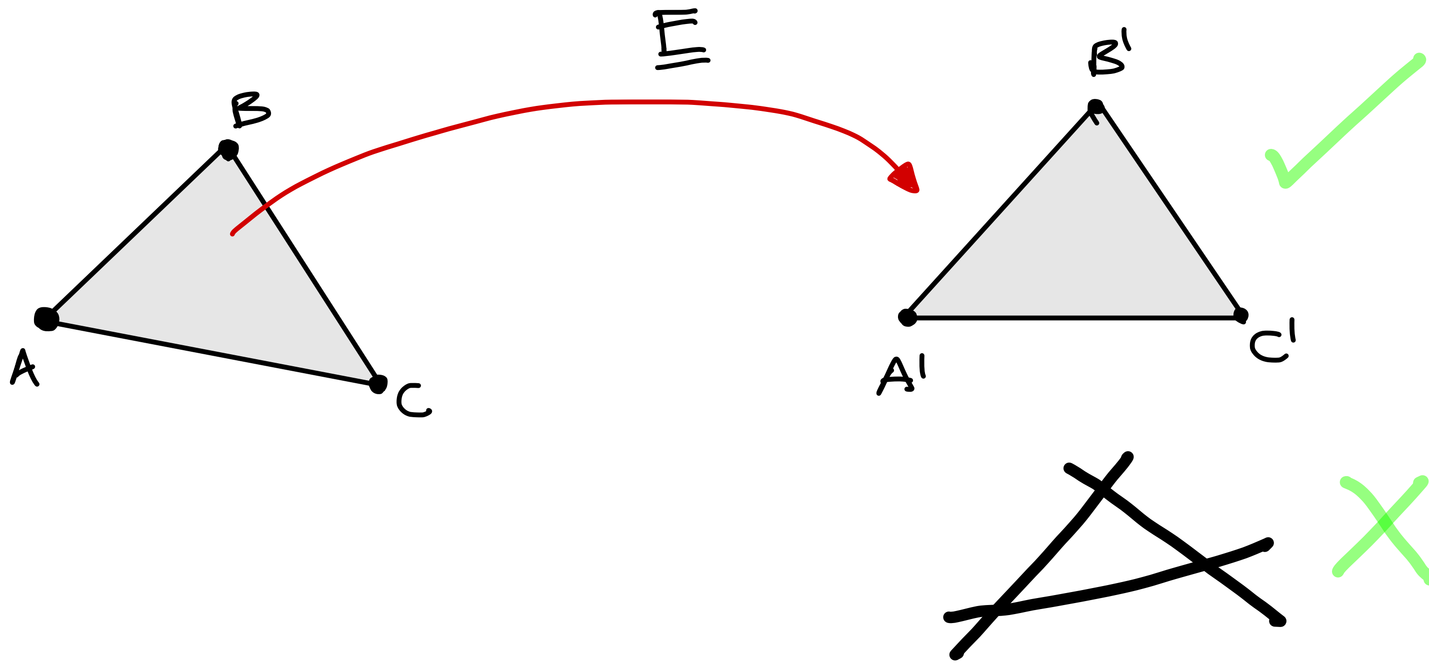
$$\frac{\partial v_1}{\partial x_2} = \frac{\partial v_2}{\partial x_1} \Rightarrow$$

$$\underbrace{\boxed{\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} = 0} \quad \boxed{\frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2} = 0} \quad \boxed{\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} = 0}}_{\text{3 pdes that are dependent on } v_i}$$

$$\boxed{\nabla \times \underline{v} = \underline{0}}$$

If v_i satisfy compatibility, then v_i is a potential vector field

Compatibility equations



Scalar field

$$\phi \Rightarrow v_i = \frac{\partial \phi}{\partial x_i}$$

Inf. Strain

$$u \Rightarrow \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Compatibility equations

In 2D (plane stress)

$$\boxed{\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2}}$$

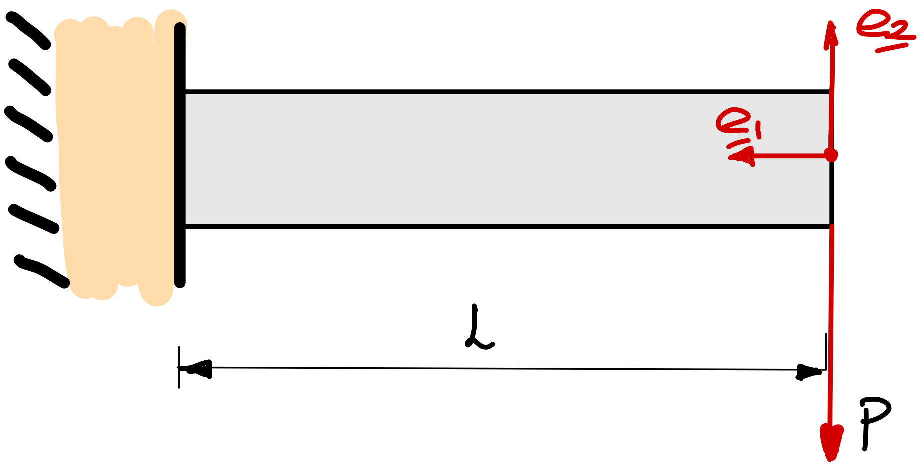
In general 3D case:

$$\boxed{\epsilon_{ipm}\,\epsilon_{jqn}\frac{\partial^2 \varepsilon_{mn}}{\partial x_p \partial x_q} = 0}$$

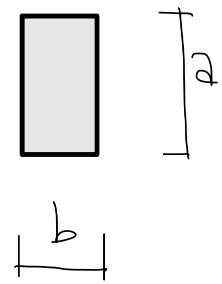
or

$$\boxed{\frac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 \varepsilon_{kl}}{\partial x_i \partial x_j} - \frac{\partial^2 \varepsilon_{ij}}{\partial x_j \partial x_k} - \frac{\partial^2 \varepsilon_{jk}}{\partial x_j \partial x_l} = 0}$$

Compatibility equations



Cross-section



$$C = \frac{3P}{4Ea^3b}$$

$$\varepsilon_{11} = 2C x_1 x_2$$

$$\varepsilon_{22} = -2\nu C x_1 x_2$$

$$\varepsilon_{12} = (1+\nu) C (a^2 - x_2^2)$$

Is the strain field admissible? Yes/No

Find out the displacement field ?

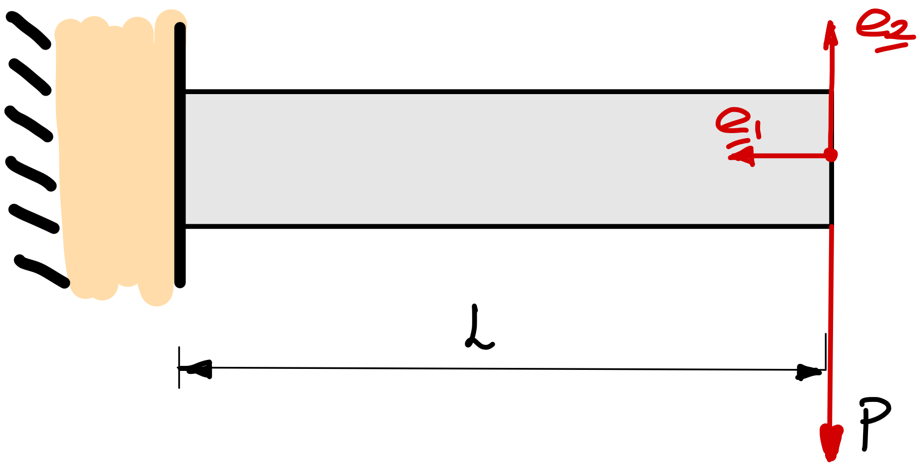
$$u_1 = \int \varepsilon_{11}\, dx_1$$

$$u_2 = \int \varepsilon_{22}\, dx_2$$

Compatibility equation in 2D

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0$$

Compatibility equations



$$\varepsilon_{11} = 2C\,x_1 x_2$$

$$\varepsilon_{22} = -2\nu C\,x_1 x_2$$

$$\varepsilon_{12} = (1+\nu)\,C\,(a^2 - x_2^2)$$

Compatibility equation in 2D

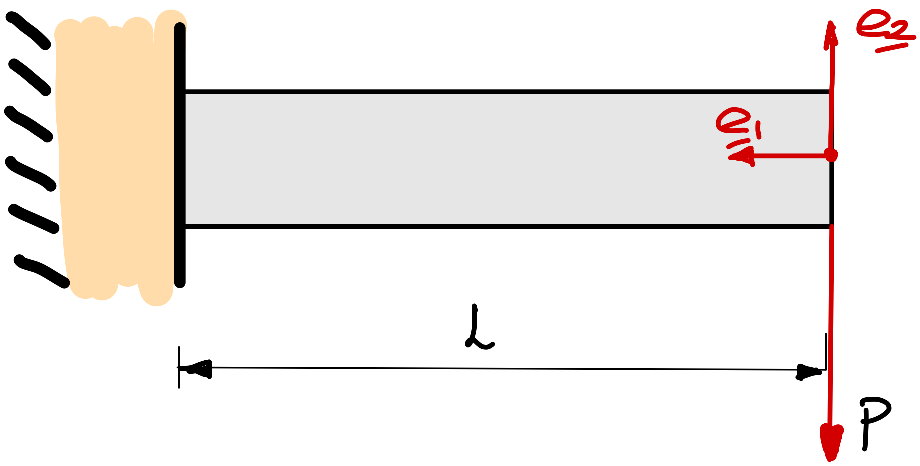
$$\underbrace{\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2}}_{0} + \underbrace{\frac{\partial^2 \varepsilon_{22}}{\partial x_1^2}}_{0} - \underbrace{2\,\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2}}_{0} = 0$$

$$0 + 0 - 0 = 0$$

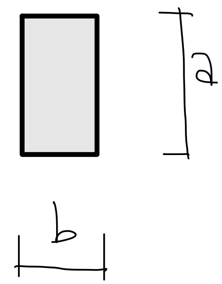
$$\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0$$

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} = \frac{\partial}{\partial x_2^2}\left(2C\,x_1 x_2\right) = 0\,; \qquad \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = \frac{\partial}{\partial x_1^2}\left(-2\nu C\,x_1 x_2\right) = 0$$

Compatibility equations



Cross-section



$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \; ; \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \; ; \quad \varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$$u_1 = \int du_1 = \int \varepsilon_{11}\, dx_1 = \int 2C x_1 x_2 \, dx_1 = 2C x_2 \frac{x_1^2}{2} + f(x_2)$$

$$u_2 = \int du_2 = \int \varepsilon_{22}\, dx_2 = \int -2\nu C x_1 x_2 \, dx_2 = -2\nu C x_1 \frac{x_2^2}{2} + g(x_1)$$

$f(x_2)$ & $g(x_1)$ are functions to be determine.

Compatibility equations

$$u_1 = 2Cx_2\frac{x_1^2}{2} + F(x_2) \qquad u_2 = -2\nu Cx_1\frac{x_2^2}{2} + g(x_1)$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$$(1+\nu)C(a^2-x_2^2) = \frac{1}{2}\left[\left(2C\frac{x_1^2}{2} + \frac{\partial f(x_2)}{\partial x_2}\right) + \left(-2\nu C\frac{x_2^2}{2} + \frac{\partial g(x_1)}{\partial x_1}\right)\right]$$

$$\left[\left(2C\frac{x_1^2}{2} + \frac{\partial f(x_2)}{\partial x_2}\right) + \left(-2\nu C\frac{x_2^2}{2} + \frac{\partial g(x_1)}{\partial x_1}\right) - 2(1+\nu)C(a^2-x_2^2)\right] = 0$$

$$\left[\left(2C\frac{x_1^2}{2} + \frac{\partial g(x_1)}{\partial x_1}\right) + \left(-2\nu C\frac{x_2^2}{2} - 2(1+\nu)C(a^2-x_2^2) + \frac{\partial f(x_2)}{\partial x_2}\right)\right] = 0$$

Compatibility equations

$$\left[\left(2C\frac{x_1^2}{2} + \frac{\partial g(x_1)}{\partial x_1}\right) + \left(-2\nu C\frac{x_2^2}{2} - 2(1+\nu)C\left(a^2 - x_2^2\right) + \frac{\partial f(x_2)}{\partial x_2}\right)\right] = 0$$

$$\alpha \quad - \quad \alpha \quad = 0 \qquad \alpha : \text{Constant.}$$

$$\left(2C\frac{x_1^2}{2} + \frac{\partial g(x_1)}{\partial x_1}\right) = \alpha \Rightarrow \frac{\partial g}{\partial x_1} = \alpha - \cancel{2}C\frac{x_1^2}{\cancel{2}} \Rightarrow g(x_1) = \alpha x_1 - C\frac{x_1^3}{3} + d_1$$

$$\left(-\cancel{2}\nu C\frac{x_2^2}{\cancel{2}} - 2(1+\nu)C\left(a^2 - x_2^2\right) + \frac{\partial f(x_2)}{\partial x_2}\right) = -\alpha$$

$$\frac{\partial f(x_2)}{\partial x_2} = -\alpha + 2(1+\nu)C\left(a^2 - x_2^2\right) + C\nu x_2^2 \Rightarrow \boxed{f(x_2) = \left[2C(1+\nu)a^2 - \alpha\right]x_2 - \frac{C}{3}(2+\nu)x_2^3 + d_2}$$