1 Question 1

1.1 Deformation Gradient (F)

$$F_{ij} = \frac{\partial x_i}{\partial X_j}$$

Note x_i are the spatial coordinates in the deformed configuration and X_j are the material coordinates in the undeformed (reference) configuration.

The displacement vector u_i is defined as following:

$$u_i = x_i - X_i$$

So the deformation vector becomes:

$$F_{ij} = \frac{\partial(X_i + u_i)}{\partial X_j} = \delta_{ij} + \frac{\partial u_i}{\partial X_j}$$

$$\mathbf{F} = \mathbf{I} + \nabla \mathbf{u}$$

1.2 Determinant of F (J)

$$J = \det(\mathbf{F}) = \det(\mathbf{I} + \nabla \mathbf{u})$$

When the deformation and displacements are infinitesimally small, J could be approximated as the determinant of an identity matrix plus a small perturbation,

$$J \approx 1 + \mathrm{Tr}(\nabla \mathbf{u})$$

where $\mathrm{Tr}(\nabla \mathbf{u})$ is the trace of the displacement gradient, equivalent to the divergence of the displacement field, assuming those higher-order terms in the displacement gradient are negligible.

1.3 Infinitesimal Strain Tensor (E)

The Lagrangian strain tensor $\mathbf{E}_L$ can be defined in terms of the deformation gradient $\mathbf{F}$ as:

$$\mathbf{E}_L = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$$

For infinitesimally small deformations,

$$\mathbf{E} = \frac{1}{2}((\mathbf{I} + \nabla\mathbf{u})^T(\mathbf{I} + \nabla\mathbf{u}) - \mathbf{I})$$

$$\Longleftrightarrow E_{ij} = \frac{1}{2}((\delta_{ij} + \frac{\partial u_i}{\partial X_j})^T(\delta_{ij} + \frac{\partial u_i}{\partial X_j}) - \delta_{ij})$$

$$\Longleftrightarrow = \frac{1}{2}((\delta_{ij}{}^T + \frac{\partial u_i}{\partial X_j}^T)(\delta_{ij} + \frac{\partial u_i}{\partial X_j}) - \delta_{ij})$$

$$\Longleftrightarrow = \frac{1}{2}((\delta_{ji} + \frac{\partial u_j}{\partial X_i})(\delta_{ij} + \frac{\partial u_i}{\partial X_j}) - \delta_{ij})$$

$$\Longleftrightarrow = \frac{1}{2}(\delta_{ji}\delta_{ij} + \delta_{ji}\frac{\partial u_i}{\partial X_j} + \delta_{ij}\frac{\partial u_j}{\partial X_i} + \frac{\partial u_j}{\partial X_i}\frac{\partial u_i}{\partial X_j} - \delta_{ij})$$

Neglecting the higher order term $\frac{\partial u_j}{\partial X_i}\frac{\partial u_i}{\partial X_j}$, and $\delta_{ji} = \delta_{ij}$,

$$\mathbf{E} \approx \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$$

This is known as the infinitesimal strain tensor. The same result could be deduced from the Eulerian Strain tensor approach as well.

2 Question 2

2.1 Part a

$$\nabla\phi = \mathbf{F} = \begin{bmatrix} \cos(\omega t) & \sin(\omega t) & 0 \\ -\sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 + \alpha t \end{bmatrix}$$

This deformation gradient could be treated by two separate parts, one is the rotation matrix and other one is the strech matrix. With the property of rotation matrix: $\mathbf{Q}^{-1} = \mathbf{Q}^T$

$$\nabla\psi = \mathbf{F}^{-1} = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & \frac{1}{1+\alpha t} \end{bmatrix}$$

Thus, the inverse motion $\mathbf{X} = \psi(\mathbf{x}, t)$ is given by:

$$X_1 = \cos(\omega t)x_1 - \sin(\omega t)x_2$$
$$X_2 = \sin(\omega t)x_1 + \cos(\omega t)x_2$$
$$X_3 = \frac{1}{1 + \alpha t}x_3$$

2.2 Part b

The spatial velocity field $\mathbf{v}(\mathbf{x}, t)$ is given by:

$$v_1 = -\omega \sin(\omega t)x_1 - \omega \cos(\omega t)x_2$$
$$v_2 = \omega \cos(\omega t)x_1 + \omega \sin(\omega t)x_2$$
$$v_3 = -\frac{\alpha}{(1 + \alpha t)^2}x_3$$

2.3 Part c

3 Question 3

By the definition of total/material derivative and partial derivative:

$$D_t f(t, x) = \frac{\partial f(t, x)}{\partial t} + \frac{\partial f(t, x)}{\partial x} \frac{\partial x}{\partial t}$$
$$= \partial_t f(t, x) + \nabla f(t, x) \cdot v$$

This total time derivative measures the rate of change of a function of x and t following the motion of the material. The first term indicates the rate of change of f at a point, and the second term (also called convection term) indicates the effect of that point moving with the material.

4   Question 4

The main conflicts come up when coupling the fluid and the solid part of
a problem since they use different descriptions. In the Eulerian framework,
the motion of the fluid is described from a spatial point of view, focusing
on specific points in space through which the fluid flows. Variables such
as velocity, pressure, and density are expressed as functions of fixed spatial
positions and time. This approach is natural for fluids, as it allows for the
easy handling of the flow's continuous deformation and movement without
needing to track individual fluid particles. However, the Eulerian description
cannot handle moving boundaries. Conversely, the Lagrangian description
focuses on individual particles or points within a solid material, tracking
their positions over time as the material deforms. This method expresses
variables like displacement, strain, and stress as functions of the material's
initial positions and time. The Lagrangian perspective is also natural to solid
mechanics, where the deformation of the material and the paths of specific
material points are of primary interest. Nevertheless, it is challenging to
capture large distortions and topological changes in the domain by Lagrangian
description. In the coupled fluid-structure interaction problem, the deformed
structure tends to change the configuration of the fluid domain too. Thus, we
need to resolve the fluid dynamic equations on a moving (or, time dependent)
domain, where requires ALE.

5 Question 5