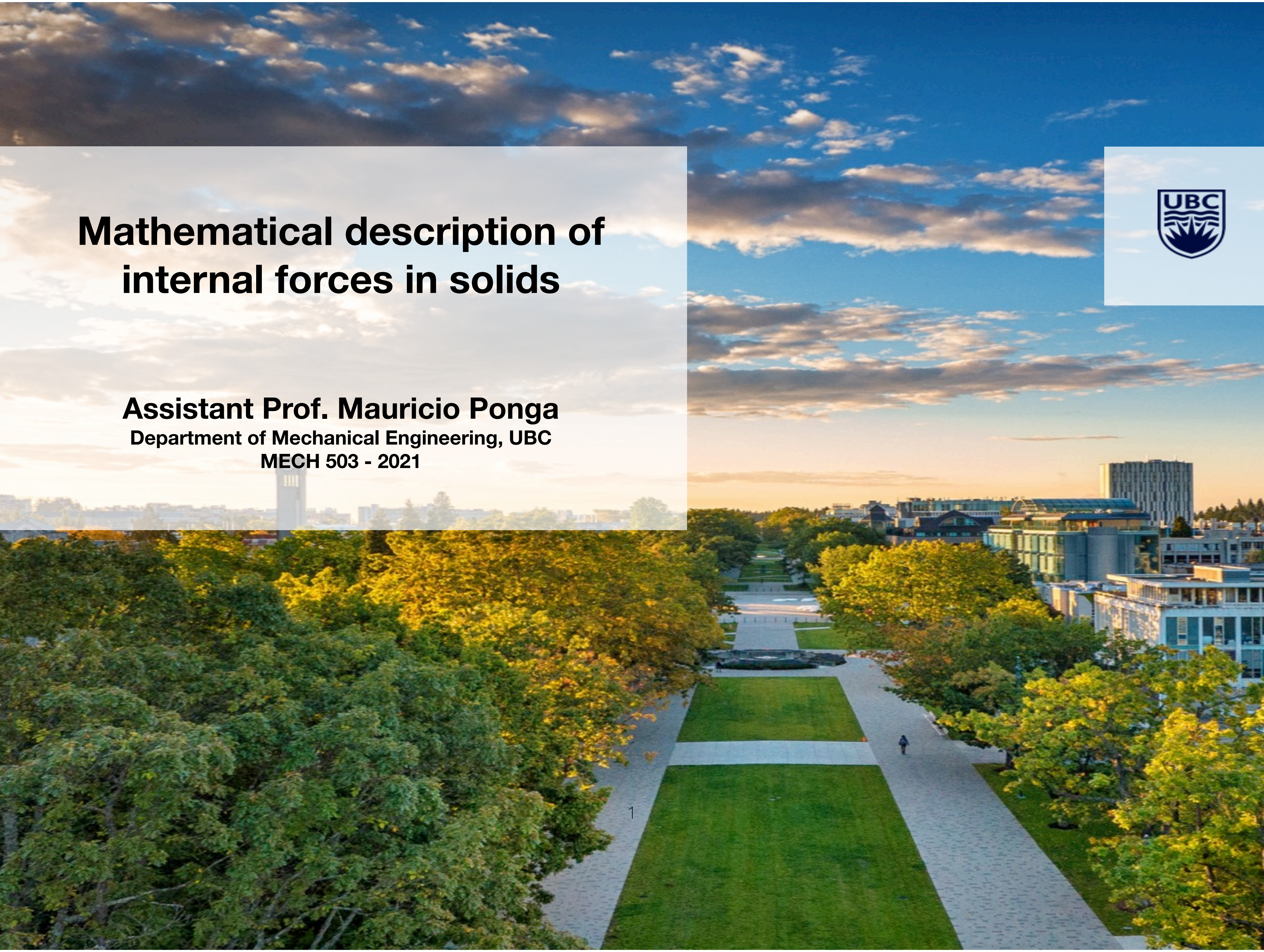
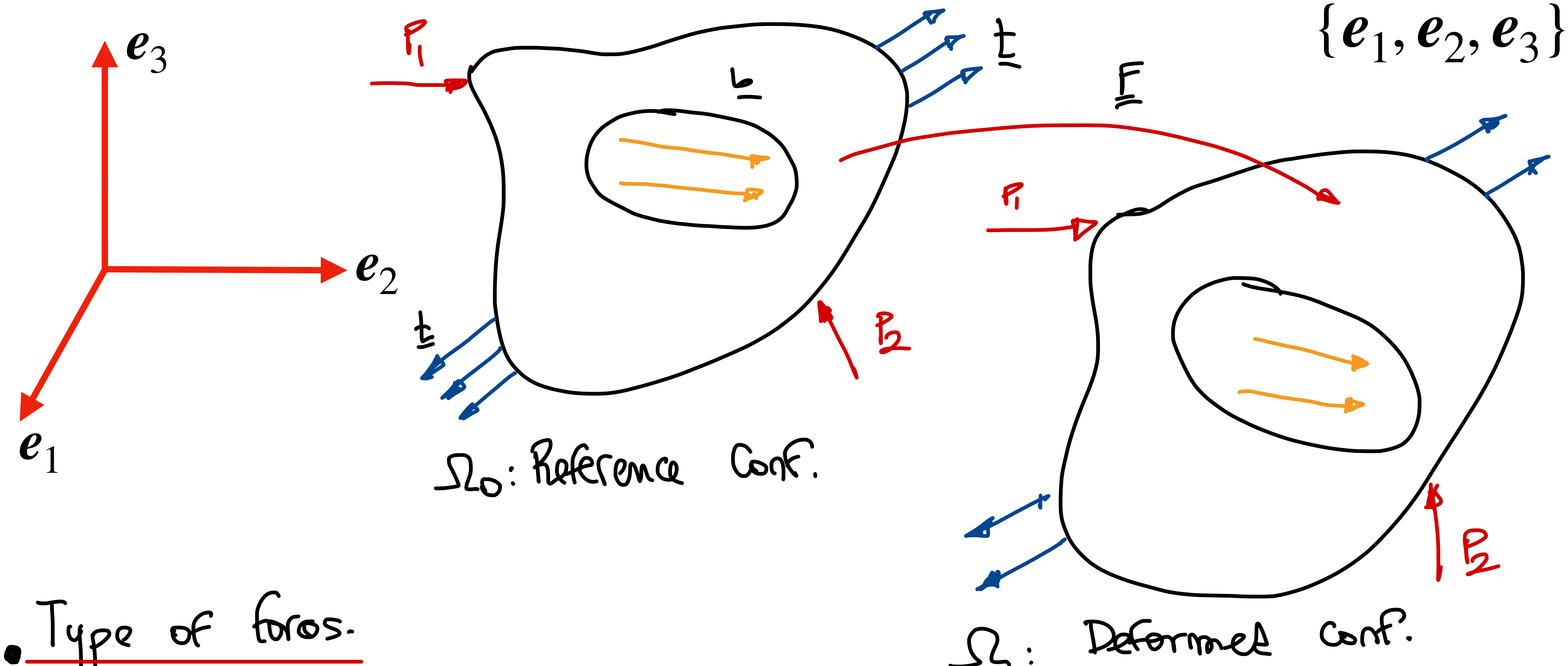


Mathematical description of internal forces in solids

Assistant Prof. Mauricio Ponga
Department of Mechanical Engineering, UBC

MECH 503 - 2021

Mathematical description of forces



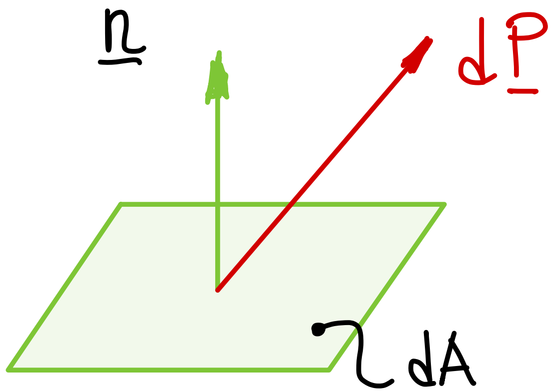
- <u>Type of forces.</u>

1) Surface forces (traction applied on a surface): Lift force in an airfoil.

2) Body forces (Body has mass (ρ)): Gravity, Electrostatic, Mag.,)

Mathematical description of forces

- Surface Traction vector $\underline{t}$



$$\underline{t} = \lim_{dA \to 0} \frac{d\underline{P}}{dA}$$

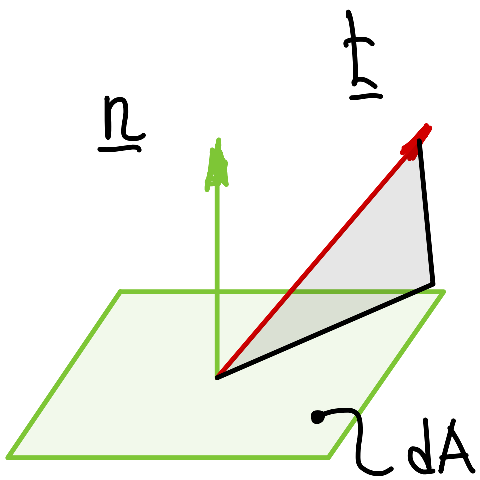
$$\left[\underline{t} = \frac{\text{Force}}{\text{Area}} \right]$$

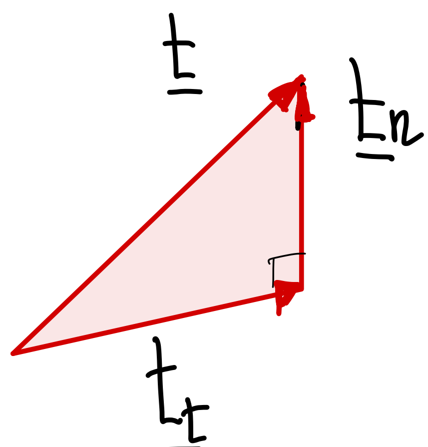
Force intensity vector

$d\underline{P}$ = Inf force vector

$$\underline{P} = \int_A \underline{t} \,.\, dA$$

$$|\underline{n}| = 1$$



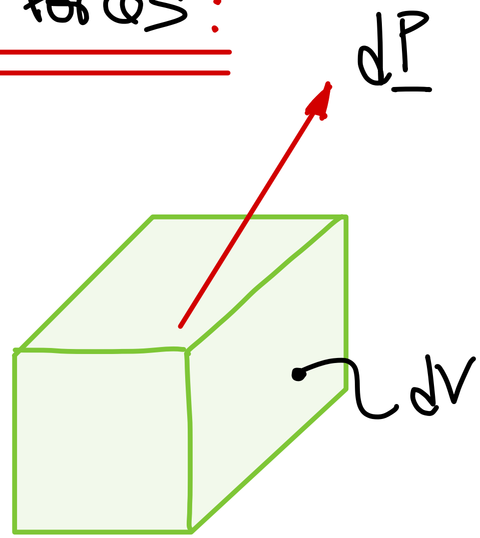


$$\boxed{\underline{t}_n = (\underline{t} \cdot \underline{n}) \,.\, \underline{n}}$$

$$\boxed{\underline{t}_t = \underline{t} - \underline{t}_n}$$

Mathematical description of forces

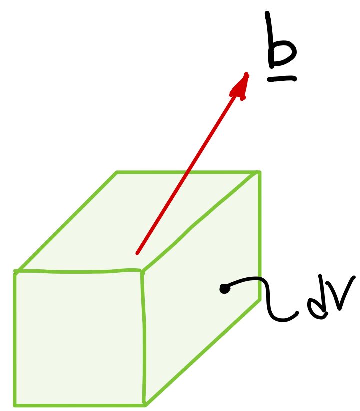
- Body forces:



$$\boxed{\underline{b} = \frac{1}{\rho} \lim_{dV \to 0} \frac{d\underline{P}}{dV}}$$

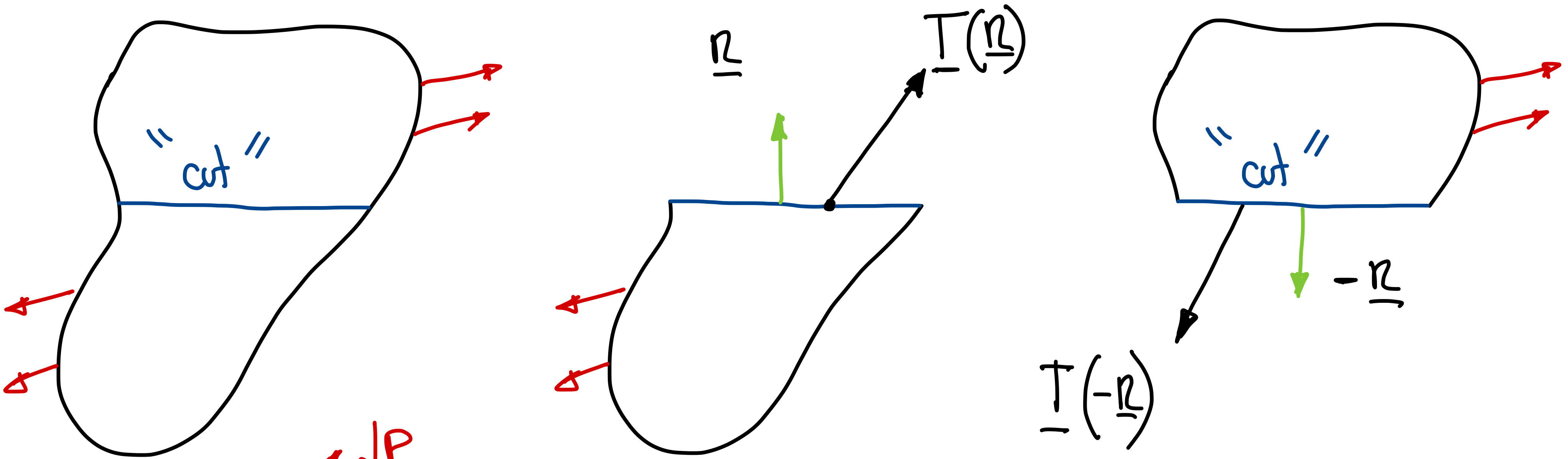
$$\left[\underline{b} = \frac{\cancel{\text{volume}}}{\text{mass}} \cdot \frac{\text{force}}{\cancel{\text{volume}}} = \frac{\text{force}}{\text{mass}} \right]$$

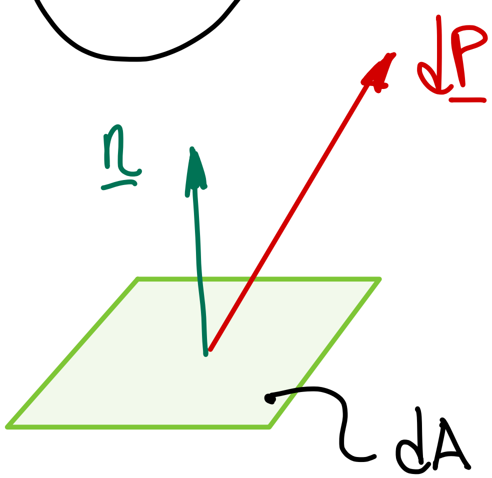
$$\boxed{\underline{P} = \int_V \rho\, \underline{b}\, dV}$$



Mathematical description of forces

- Traction acting on planes within solids:





$$\underline{T}(\underline{n}) = \lim_{dA \to 0} \frac{d\underline{P}^{(\underline{n})}}{dA}$$

$$\left[\underline{T}(\underline{n}) = \frac{\text{Force}}{\text{Area}}\right]$$

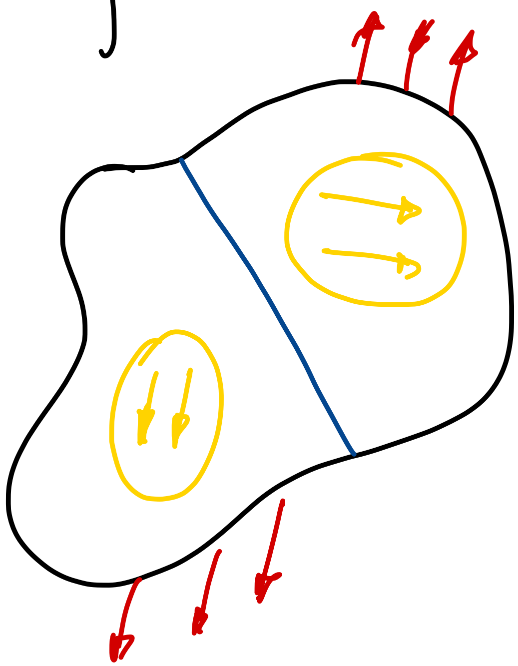
force intensity

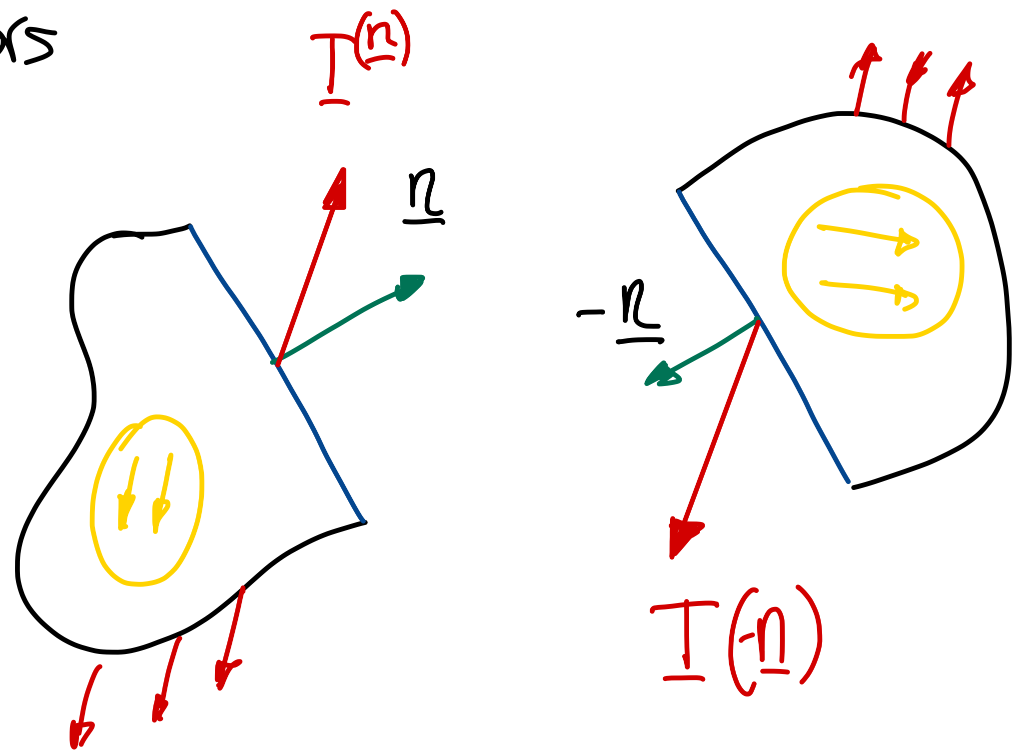
Mathematical description of forces

- Resultant force:

$$\boxed{\underline{P} = \int_A \underline{T}(\underline{n})\, dA + \int_V \rho\, \underline{b}\, dV}$$

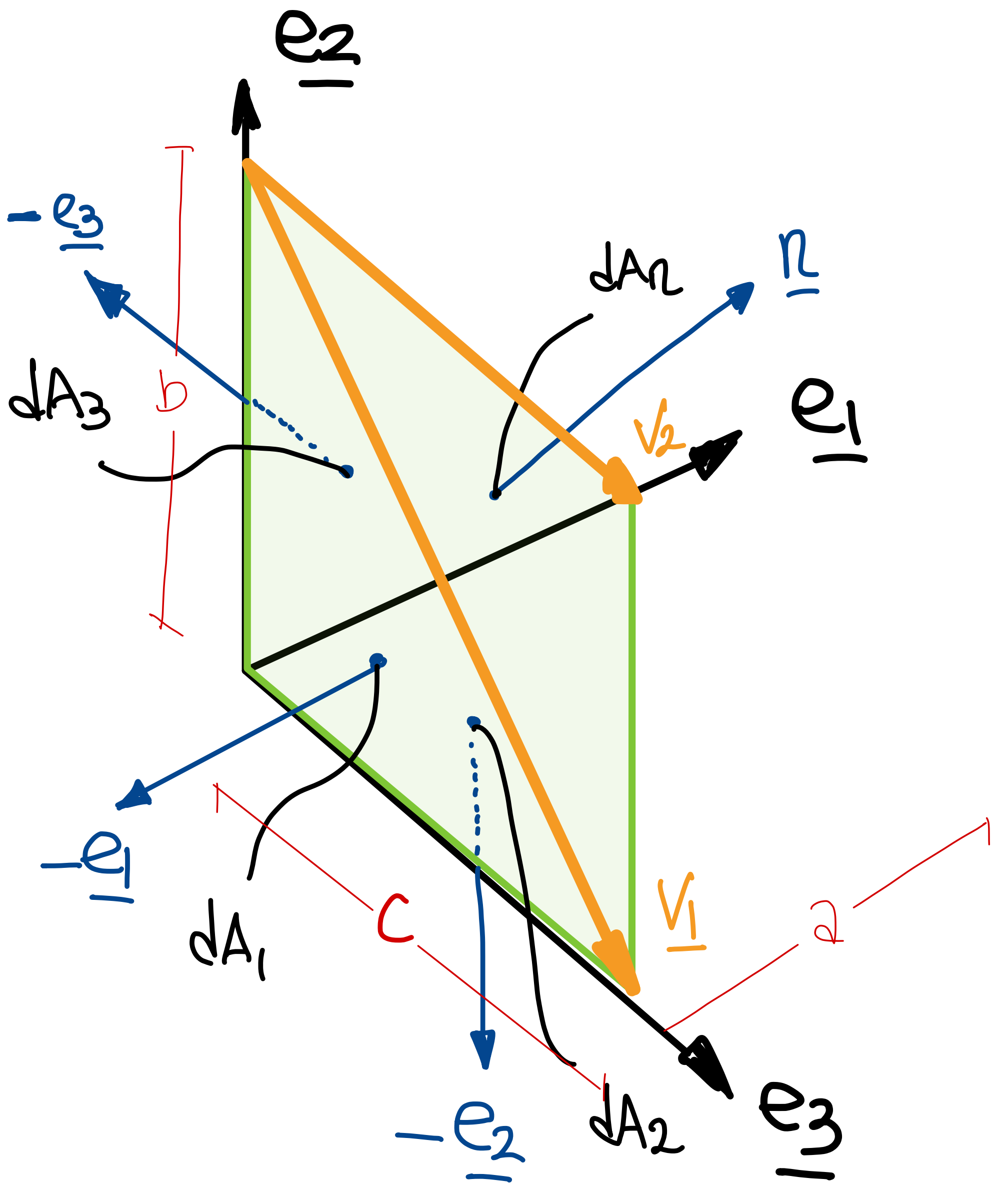
- Analysis of the traction vectors





Newton's third law: $\underline{T}(-\underline{n}) = -\underline{T}(\underline{n})$

Mathematical description of forces



$dA_1,\ dA_2,\ dA_3,\ dA_n$

$dA_1 = \frac{1}{2} bc \qquad dA_2 = \frac{1}{2} ac$

$dA_3 = \frac{1}{2} ab$

$\underline{V}_1 = 0\underline{e}_1 - b\underline{e}_2 + c\underline{e}_3$

$V_2 = a\underline{e}_1 - b\underline{e}_2 + 0\,\underline{e}_3$

$dA_n \cdot \underline{n} = \frac{1}{2}\left(\underline{V}_1 \times \underline{V}_2\right)$

Mathematical description of forces

$dA_1 = \frac{1}{2} bc \qquad dA_2 = \frac{1}{2} ac \qquad dA_3 = \frac{1}{2} ab$

$\underline{V_1} = 0\underline{e_1} - b\underline{e_2} + c\underline{e_3}$

$\underline{V_2} = a\underline{e_1} - b\underline{e_2} + 0\,\underline{e_3}$

$$dA_n \cdot \underline{n} = \frac{1}{2} \begin{vmatrix} \underline{e_1} & \underline{e_2} & \underline{e_3} \\ 0 & -b & c \\ a & -b & 0 \end{vmatrix}$$

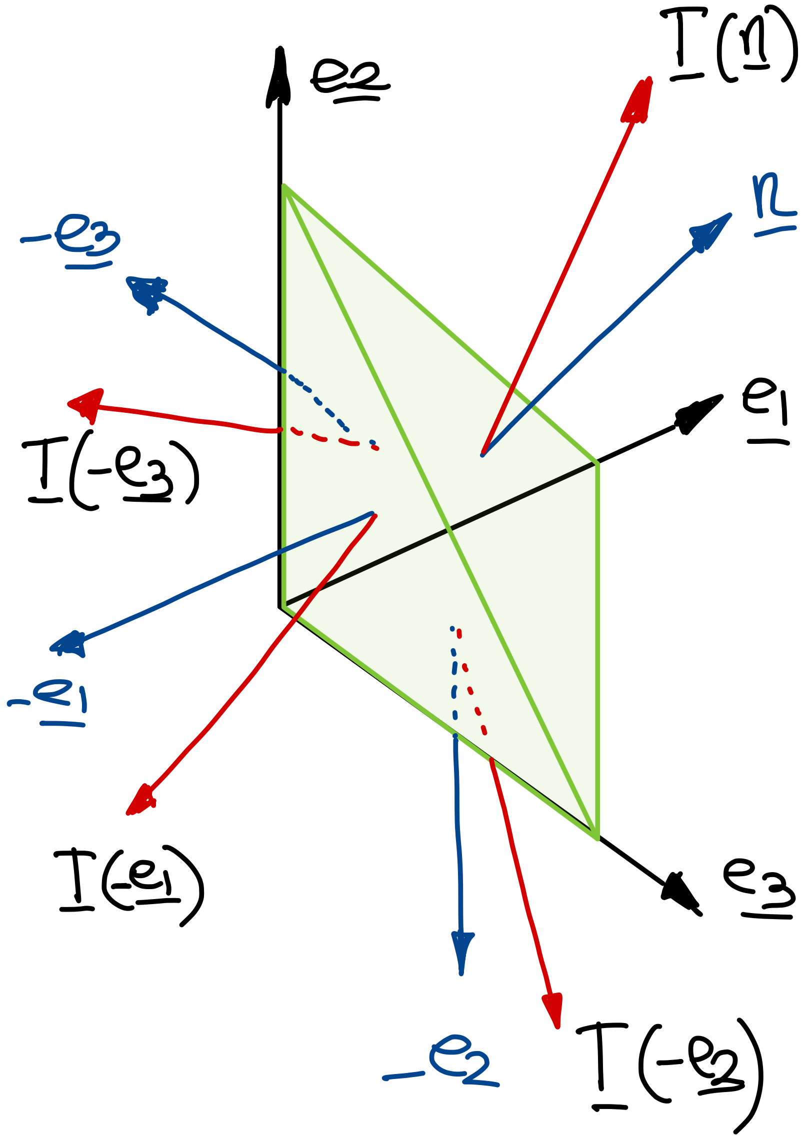
$$dA_n \cdot \underline{n} = \frac{1}{2} \left(\underline{V_1} \times \underline{V_2} \right)$$

$$dA_n \cdot \underline{n} = \frac{1}{2} \left[bc\,\underline{e_1} + ac\,\underline{e_2} + ab\,\underline{e_3} \right]$$

$$\underline{n} = \frac{dA_1}{dA_n}\underline{e_1} + \frac{dA_2}{dA_n}\underline{e_2} + \frac{dA_3}{dA_n}\underline{e_3} = n_1\underline{e_1} + n_2\underline{e_2} + n_3\underline{e_3}$$

$$\boxed{\frac{dA_1}{dA_n} = n_1 \; ; \quad \frac{dA_2}{dA_n} = n_2 \; ; \quad \frac{dA_3}{dA_n} = n_3}$$

Mathematical description of forces



Traction on different planes passing through the same point

$$\underline{T}(\underline{n}) = ? \; ; \; ? \; \underline{T}(\underline{e}_1), \; \underline{T}(\underline{e}_2), \; \underline{T}(\underline{e}_3)$$

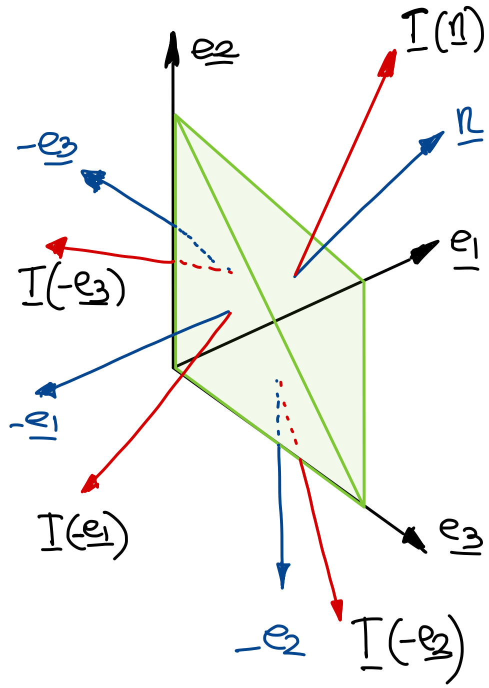
$$\underline{F} = m \cdot \underline{a}$$

$$\underline{T}(\underline{n}) \cdot dA_n + \underline{T}(-\underline{e}_1)\, dA_1 + \underline{T}(-\underline{e}_2)\, dA$$

$$+ \, T(-\underline{e}_3)\, dA_3 + \rho \, \underline{b} \, dV = \rho \cdot dV \cdot \underline{a}$$

$$\underline{T}(-\underline{n}) = -T(\underline{n}) \Rightarrow \underline{T}(-\underline{e}_1) = -T(\underline{e}_1) \; ;$$

Mathematical description of forces



Traction on different planes passing through the same point

$$\underline{T}(\underline{n}) \cdot dA_n - \underline{T}(\underline{e}_1)\, dA_1 - \underline{T}(\underline{e}_2)\, dA - \underline{T}(\underline{e}_3)\, dA_3 + \rho\, \underline{b}\, dV = \rho \cdot dV \cdot \underline{a}$$

$$\lim_{dA_n \to 0} \left(\underline{T}(\underline{n}) - \underline{T}(\underline{e}_1) \overbrace{\frac{dA_1}{dA_n}}^{n_1} - \underline{T}(\underline{e}_2) \overbrace{\frac{dA_2}{dA_n}}^{n_2} - \underline{T}(\underline{e}_3) \overbrace{\frac{dA_3}{dA_n}}^{n_3} + \cancel{\rho\, \underline{b}\, \frac{dV}{dA_n}} \right) = \cancel{\lim_{dA_n \to 0} \rho\, \underline{a}\, \frac{dV}{dA_n}}$$

$$\lim_{dA_n \to 0} \frac{dV}{dA_n} = 0 \checkmark$$

$$\boxed{\underline{T}(\underline{n}) = \underline{T}(\underline{e}_1)\, n_1 + \underline{T}(\underline{e}_2)\, n_2 + \underline{T}(\underline{e}_3)\, n_3}$$

Mathematical description of forces

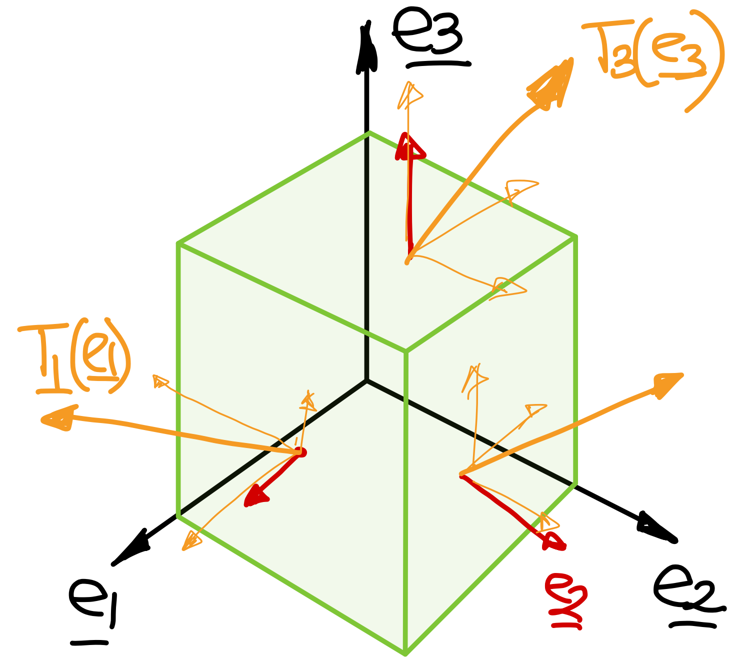
- Cauchy stress tensor:

$i = 1, 2, 3.$

$$T_i(\underline{n}) = T_i(\underline{e_1})\,n_1 + T_i(\underline{e_2})\,n_2 + T_i(\underline{e_3})\,n_3$$

$$\underline{\underline{\sigma}} = \begin{bmatrix} T_1(\underline{e_1}) & T_2(\underline{e_1}) & T_3(\underline{e_1}) \\ T_1(\underline{e_2}) & T_2(\underline{e_2}) & T_3(\underline{e_2}) \\ T_1(\underline{e_3}) & T_2(\underline{e_3}) & T_3(\underline{e_3}) \end{bmatrix}$$

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$



$$\underline{T_1(\underline{e_1})} = T_1(\underline{e_1}) + T_1(\underline{e_2}) + T_1(\underline{e_3})$$

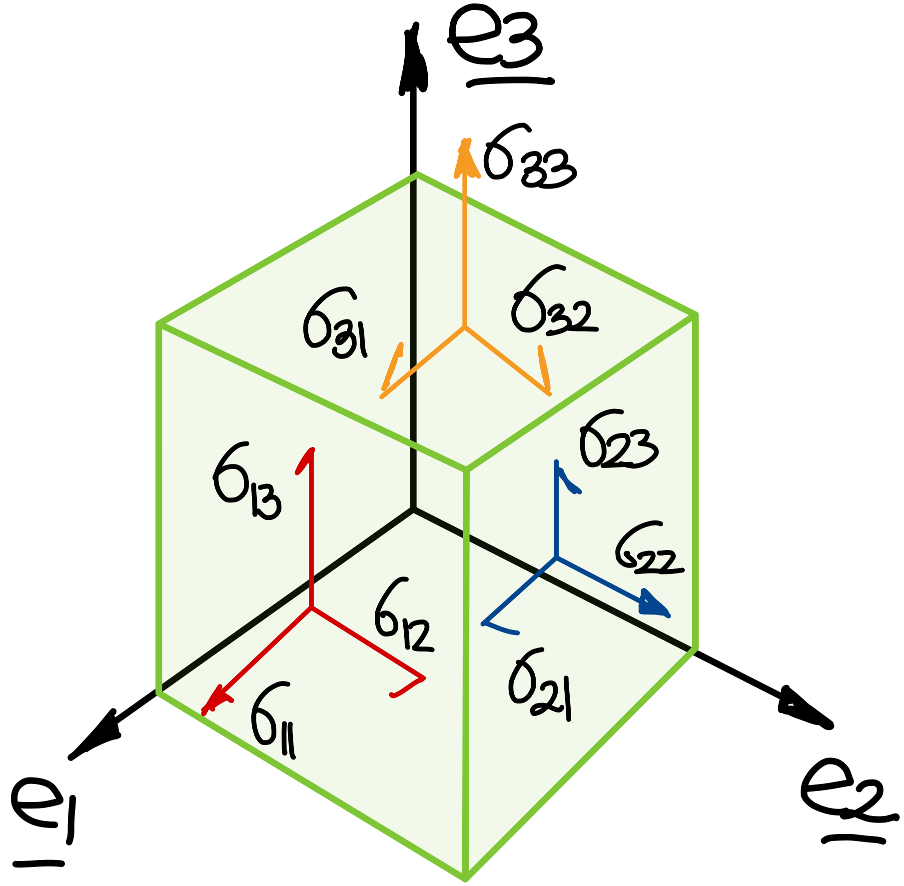
$$\underline{T_2(\underline{e_2})} = T_2(\underline{e_1}) + T_2(\underline{e_2}) + T_2(\underline{e_3})$$

The Cauchy stress tensor allow us to express any traction vector

$$\underline{T}(\underline{n}) = \underline{\underline{\sigma}} \cdot \underline{n} \quad \text{or} \quad T_i(n) = n_j \sigma_{ji}$$

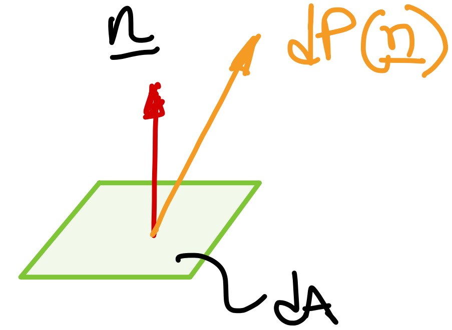
$$T_i(\underline{n}) = T_i(\underline{e_1}) n_1 + T_i(\underline{e_2}) n_2 + T_i(\underline{e_3}) n_3$$

$$T_i(n) = \sigma_{1i} n_1 + \sigma_{2i} n_2 + \sigma_{3i} n_3 \qquad i = 1, 2, 3.$$



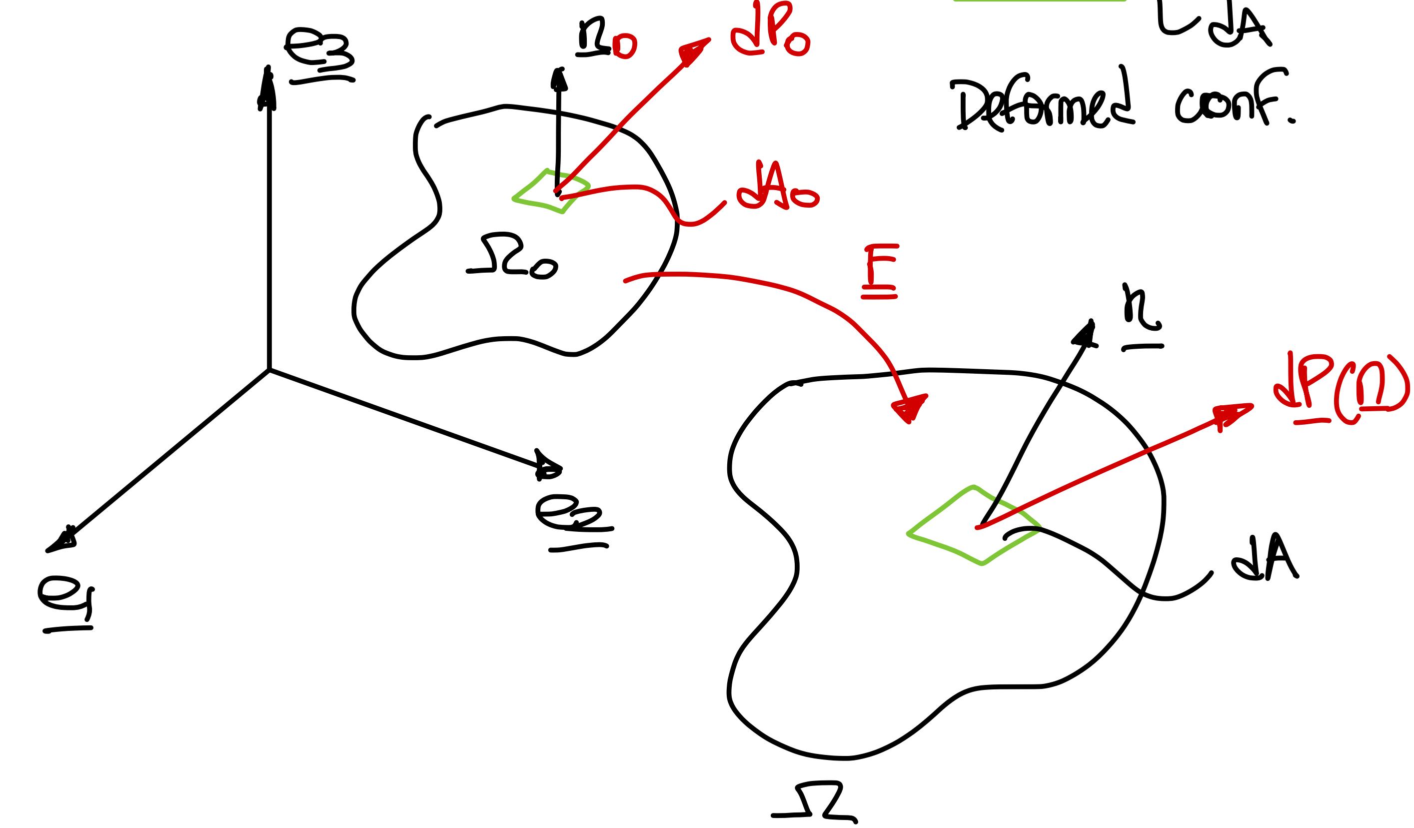
- Other stress measures

- **Cauchy stress tensor**



Deformed conf.

$$\underline{T}(\underline{n}) = \lim_{dA \to 0} \frac{dP(\underline{n})}{dA}$$



$$\underline{\underline{F}} = \underline{\underline{I}} + \underline{u} \otimes \nabla$$

$$F_{ij} = \delta_{ij} + \frac{\partial u_i}{\partial x_j}$$

$$J = \det(\underline{F}) = \frac{dV}{dV_0}$$

- Kirchoff stress: $\underline{\underline{\tau}} = J \underline{\underline{\sigma}}$ or $\tau_{ij} = J \sigma_{ij}$

- Nominal stress (first Piola-Kirchoff stress tensor) $\underline{\underline{S}} = J \underline{F}^{-1} \cdot \underline{\underline{\sigma}}$

  Force deformed / Area is taken in the reference.

  $S_{ij} = J F^{-1}_{ik} \cdot \sigma_{kj}$

- Second Piola-Kirchoff stress (Material) tensor.

$$\underline{\underline{\Sigma}} = J\,\underline{F}^{-1}\,\underline{\underline{\sigma}}\,\underline{\underline{F}}^{-T} \quad \text{or} \quad \Sigma_{ij} = J\,F_{ik}^{-1}\,\sigma_{kl} \cdot F_{jl}^{-1}$$

Force & area in the reference

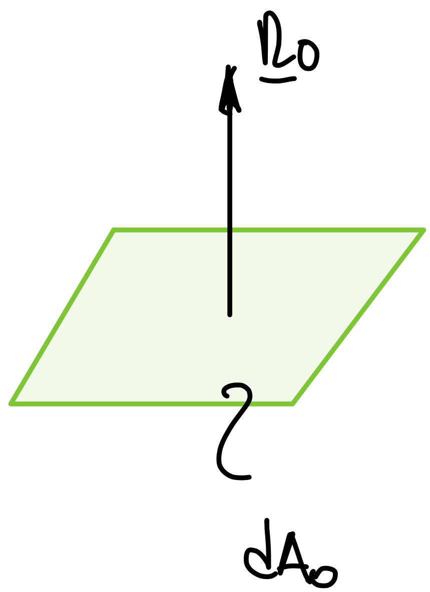
- Inverse relations:

$$\sigma = \frac{\tau}{J} \qquad \underline{\underline{\sigma}} = \frac{1}{J} \cdot \underline{\underline{F}} \cdot \underline{\underline{S}} \qquad \underline{\underline{\sigma}} = \frac{1}{J}\,\underline{F}\,\underline{\underline{\Sigma}}\,\underline{\underline{F}}^{-1}$$

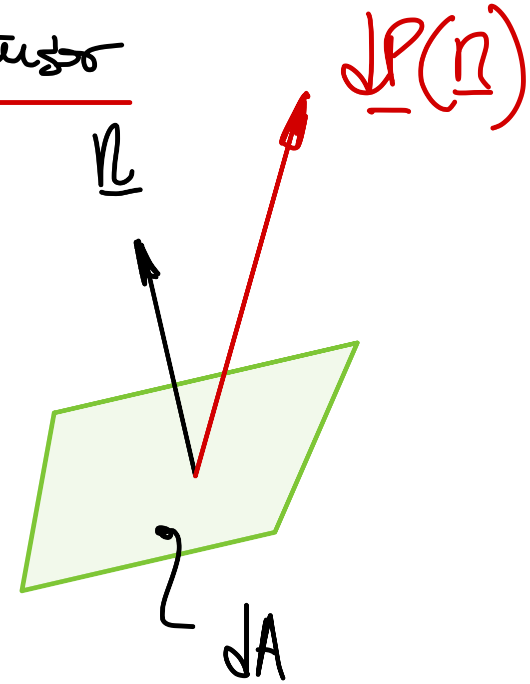
$$\sigma_{ij} = \frac{1}{J} F_{ik} \cdot S_{kj} \qquad \sigma_{ij} = \frac{1}{J} F_{ik}\,\Sigma_{kl}\,F_{jl}^{-1}$$

1st Piola - Kirchhoff stress tensor

$$dP_j(n) = dA_0 n_i^0 S_{ij}$$



Reference Conf.



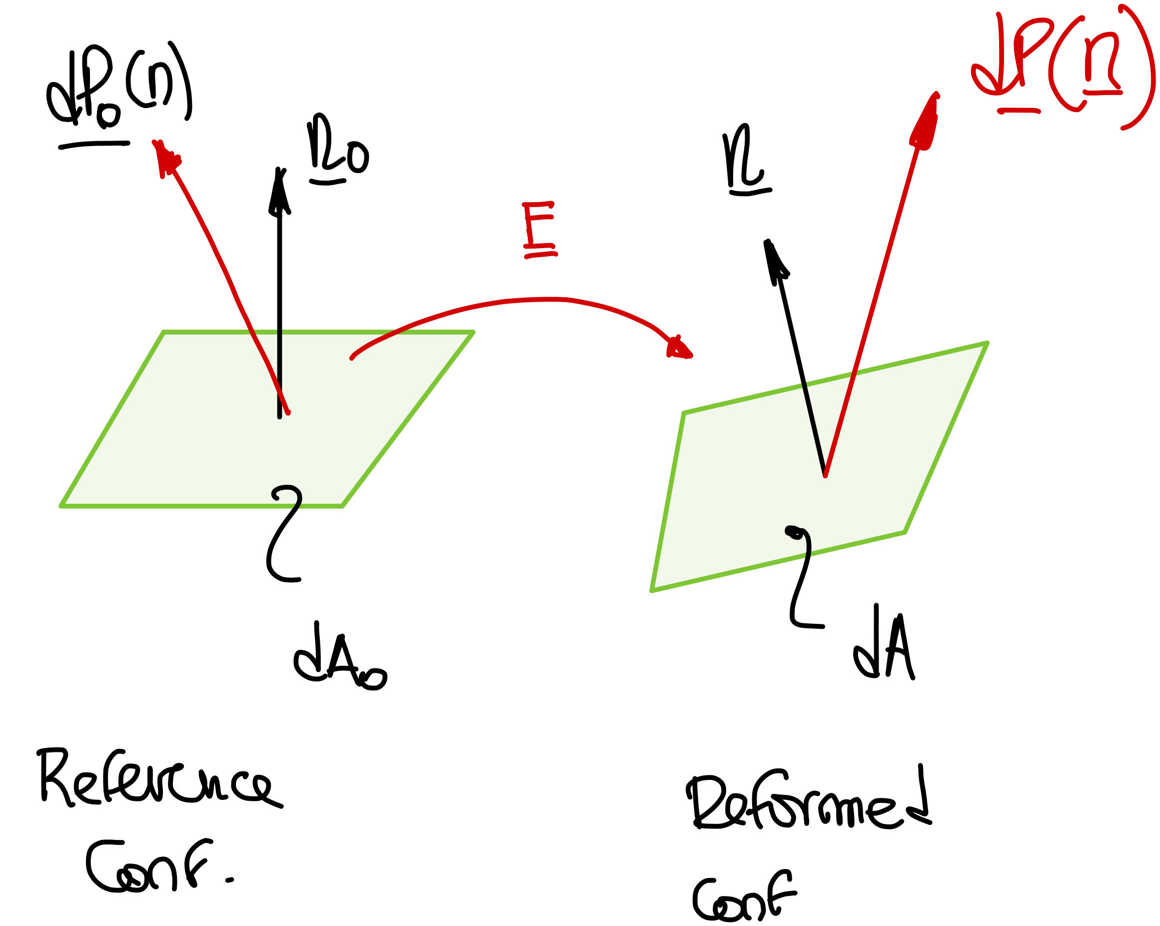
Deformed Conf.

$$dA\,\underline{n} = J\,\underline{F}^T dA_0\,\underline{n_0}$$ (Seen this in class)

$$dP_i(\underline{n}) = dA\, n_j \sigma_{ji} = J F^{-1}_{kj}\, dA_0\, n_k^0\, \sigma_{ji} = dA_0 n_k^0 \overbrace{\left(J F^{-1}_{kj} \cdot \sigma_{ji}\right)}^{S_{ij}}$$

$$dP_i(\underline{n}) = dA_0 \cdot n_k^0\, S_{ij}$$ As stated.

• Material Stress Tensor (2nd P-K): Σ_{ij}



$$\underbrace{d\underline{P}(\underline{n})}_{\text{Int. force Vector in the deformed}} = \underline{\underline{F}}\; \underbrace{d\underline{P}(\underline{n}_0)}_{\text{Internal force vector in the reference}}$$

$$dP_i(\underline{n}) = F_{ij}\, dP_j(\underline{n}_0)$$

$$dP_i(n_0) = dA_0\, n_j^0\, \Sigma_{ij}$$

$$dP_i(\underline{n}) = dA_0\, n_j^0\, S_{ji}$$

$$F_{ik}\, dP_k(\underline{n}_0) = dA_0\, n_j^0\, S_{ji}$$

$$\underbrace{F_{ik}\, F^{-1}_{li}}_{\delta_{kl}}\; dP_k(\underline{n}_0) = dA_0\, n_j^0\, S_{ji}\, F^{-1}_{li}$$

$$dP_\ell(\underline{n_0}) = dA_0 \, n_j^0 \, \underbrace{s_{ji} \, F_{\ell i}^{-1}}_{\Sigma_{j\ell}} \Rightarrow$$

$$\overbrace{\left(J \, F_{kj}^{-1} \cdot \sigma_{ji}\right)}^{S_{ij}} \Rightarrow$$

$$\boxed{dP_\ell(\underline{n_0}) = dA_0 \, n_j^0 \, \Sigma_{j\ell}}$$

- Cauchy stress Tensor $\quad \underline{T}(\underline{n}) = \underline{\underline{\sigma}} \cdot \underline{n}$

- Kirchoff stress Tensor $\quad \underline{\underline{\tau}} = \underline{\underline{\sigma}} / J$

- 1^{st} Piola-Kirchoff stress tensor $\quad \underline{\underline{S}} = J \, \underline{\underline{F}}^{-1} \, \underline{\underline{\sigma}}$

- 2^{nd} Piola-Kirchoff stress Tensor $\quad \underline{\underline{\Sigma}} = J \, \underline{\underline{F}}^{-1} \, \underline{\underline{\sigma}} \, \underline{\underline{F}}^{-T}$

Let $\underline{\dot{w}}$ denotes the work done per unit time made by the stresses. Then,

$$\dot{W} = D_{ij} \sigma_{ji} \, dV = D_{ij} \tau_{ji} \, dV_0 = \dot{F}_{ij} S_{ji} \, dV_0 = \dot{E}_{ij} \Sigma_{ji} \, dV_0$$

$$\left[\frac{\text{Force} \cdot \text{length}}{\text{time}} \right] = \left[\frac{1}{\text{Time}} \; \frac{\text{Force}}{\cancel{\text{length}^2}} \cdot \text{length}^{\cancel{3}} \right]$$

$$\frac{\partial \dot{W}}{\partial D_{ij}} = \sigma_{ij} \, dV \; ; \quad \frac{\partial \dot{W}}{\partial \sigma_{ij}} = D_{ij} \, dV$$

$$F = U - TS$$

Free energy — Internal energy — Abs Temp. — Entropy

$$\frac{\partial F}{\partial T} = -S \qquad \frac{\partial F}{\partial S} = -T$$

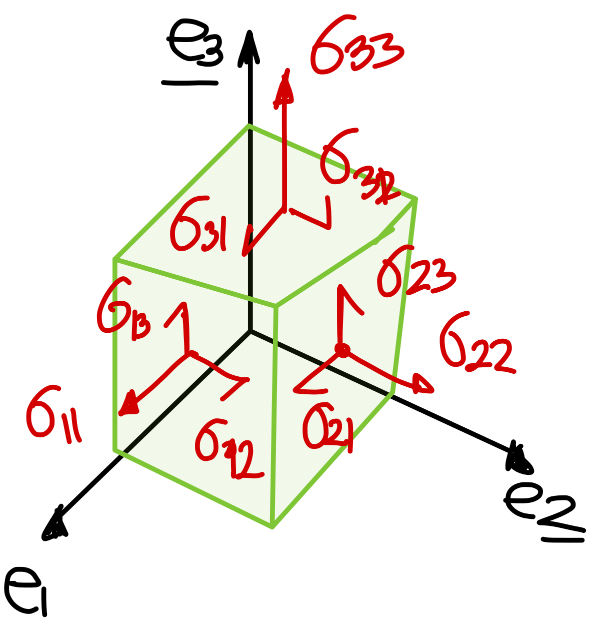
Conjugate variables

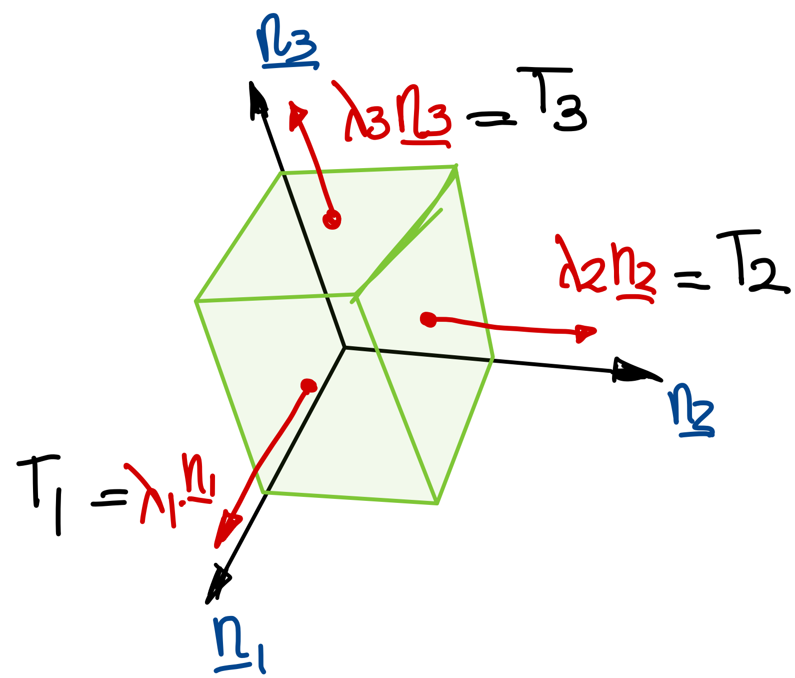
- **Principal stresses and directions:** Given $\underline{\underline{\sigma}}$ find $\{\lambda\}$ and $\{\underline{n}\}$ such that

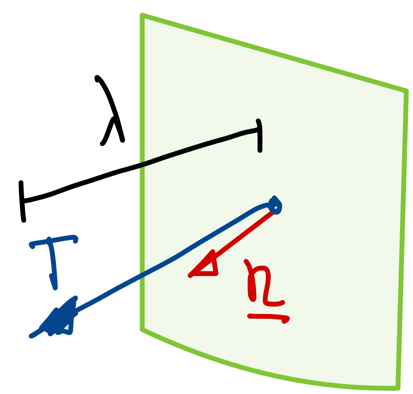
$$\underline{\underline{\sigma}} \cdot \underline{n} = \lambda \, \underline{n}$$

λ: scalar.

$\underline{n}$: vector







$$\underline{\underline{\sigma}} \cdot \underline{n} = \lambda \cdot \underline{n} \Rightarrow \left(\underline{\underline{\sigma}} - \lambda \underline{\underline{I}}\right) \underbrace{\underline{n}}_{\underline{n} \neq \underline{0}} = 0$$

$$\boxed{\det\left(\underline{\underline{\sigma}} - \lambda \underline{\underline{I}}\right) = 0}$$

$$\det\left(\underline{\underline{\sigma}} - \lambda \underline{\underline{I}}\right) = \det\begin{pmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \lambda \end{pmatrix} = 0$$

$$(\sigma_{11}-\lambda)(\sigma_{22}-\lambda)(\sigma_{33}-\lambda) + \sigma_{21}\sigma_{32}\sigma_{13} + \sigma_{31}\sigma_{12}\sigma_{23} - (\sigma_{22}-\lambda)\sigma_{13}\sigma_{31} - (\sigma_{11}-\lambda)\sigma_{23}\sigma_{32} - (\sigma_{33}-\lambda)\sigma_{12}\sigma_{21} = 0$$

: Characteristic eq.

Characteristic eq:

$$\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0$$

Invariants

$$I_1 = \text{trace}(\underline{\underline{\sigma}}) = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{23} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

$$I_2 = \frac{1}{2}\left(\left(\text{trace}(\underline{\underline{\sigma}})\right)^2 - \text{trace}(\underline{\underline{\sigma}}^2) \right)$$

$$I_3 = \det(\underline{\underline{\sigma}})$$

$$I_2 = \sigma_{22} \cdot \sigma_{33} - \sigma_{23}\sigma_{32} + \sigma_{11}\sigma_{33} - \sigma_{13} \cdot \sigma_{31} + \sigma_{11}\sigma_{22} - \sigma_{23}\sigma_{12}$$

$\{\lambda_1, \lambda_2, \lambda_3\}$: Principal stresses of $\underline{\underline{\sigma}}$

$\{\underline{n}_1, \underline{n}_2, \underline{n}_3\}$ Orthonormal vectors

$\underline{n}_1 \perp \underline{n}_2$

$\underline{n}_2 \perp \underline{n}_3$

$n_3 \perp \underline{n}_1$

$|\underline{n}_1| = |\underline{n}_2| = |\underline{n}_3| = 1$

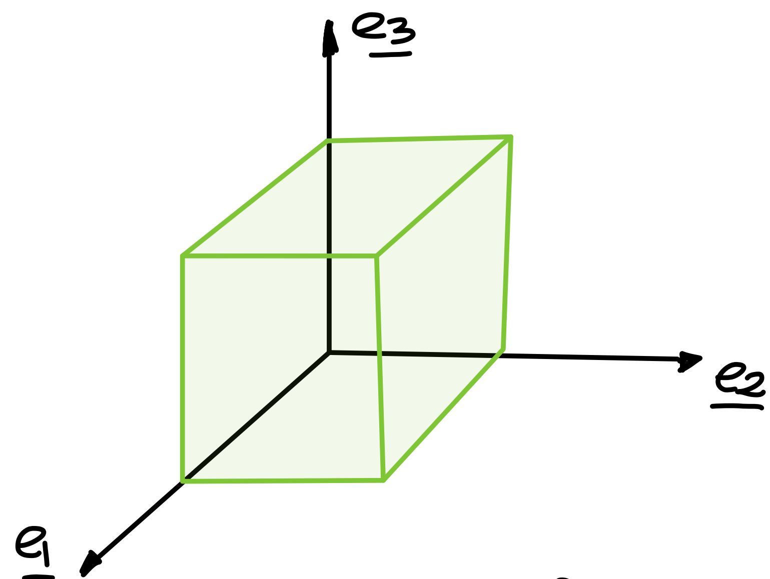
$\{\underline{n}_1, \underline{n}_2, \underline{n}_3\}$: Principal directions of $\underline{\underline{\sigma}}$ at a point

$$\begin{pmatrix} \sigma_{11} - \lambda_1 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \lambda_1 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \lambda_1 \end{pmatrix} \begin{pmatrix} n_1^1 \\ n_1^2 \\ n_1^3 \end{pmatrix} = \mathbf{0}$$

$\underline{n}_1 : (n_1^1, n_1^2, n_1^3)$

$|\underline{n}_1| = 1 = \sqrt{(n_1^1)^2 + (n_1^2)^2 + (n_1^3)^2}$

- Hydrostatic and deviatoric stress, von Mises effective stress



$\underline{\underline{\sigma}} = \sigma_{ij} \quad [3 \times 3]$

$\left.\begin{array}{l} \lambda_1 = \sigma_1 \\ \lambda_2 = \sigma_2 \\ \lambda_3 = \sigma_3 \end{array}\right\}$: Ppl stresses)

$$\underbrace{\sigma_H = \frac{1}{3} \operatorname{trace}(\underline{\underline{\sigma}})}_{\text{Hydrostatic stress}} = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{3}\sigma_{kk} = \overbrace{\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)}$$

Deviatoric

$$\sigma_{ij}' = \sigma_{ij} - \sigma_H \cdot \delta_{ij} = \begin{pmatrix} \sigma_{11} - \sigma_H & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma_H & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma_H \end{pmatrix}$$

Von Mises (equivalent & effective) stress

$$\sigma_e = \left(\frac{3}{2}\sigma_{ij}'\sigma_{ij}'\right)^{1/2}$$

$$\sigma_e = \sqrt{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2\right]}$$

$\sigma_1, \sigma_2, \sigma_3$ are the ppl stresses.

Scalar: Associated with the distortion energy.

- **Problem 1:** Given the following Cauchy stress tensor find the:

a) hydrostatic & deviatoric part of $\underline{\underline{\sigma}}$

b) The invariants I_1, I_2, I_3.

c) The principal stresses $\{\sigma_1, \sigma_2, \sigma_3\}$ and directions

d) and the maximum shearing stresses.

$$\underline{\underline{\sigma}} = \begin{pmatrix} 100 & 200 & -200 \\ 200 & 300 & 400 \\ -200 & 400 & -100 \end{pmatrix} [\text{kPa}]$$

$$\sigma_{ij} = \underbrace{\frac{1}{3}\sigma_{kk}\cdot\delta_{ij}}_{\text{hydro}} + \underbrace{\left(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\cdot\delta_{ij}\right)}_{\text{deviatoric}}$$

$$\boxed{\sigma_H} = \frac{1}{3}\sigma_{kk} = \frac{1}{3}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right) = \frac{1}{3}\left(300\right) = \boxed{100 \text{ kPa}}$$

$$\sigma'_{ij} = \overbrace{\left(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\cdot\delta_{ij}\right)}^{\text{deviatoric}} = \begin{pmatrix} 100-100 & 200 & -200 \\ 200 & 300-100 & 400 \\ -200 & 400 & -100-100 \end{pmatrix}$$

devi T

$$\boxed{\sigma'_{ij} = \begin{pmatrix} 0 & 200 & -200 \\ 200 & 200 & 400 \\ -200 & 400 & -200 \end{pmatrix}}$$

Invariants:

$$\left.\begin{aligned}
I_1 &= \mathrm{trace}(\underline{\underline{\sigma}}) = \sigma_{11} + \sigma_{22} + \sigma_{33} = 300 \text{ kPa} \\[2ex]
I_2 &= \frac{1}{2}\left(\left(\mathrm{trace}(\underline{\underline{\sigma}})\right)^2 - \mathrm{trace}(\underline{\underline{\sigma}}^2)\right) = -32 \times 10^6 \text{ kPa} \\[2ex]
I_3 &= \det(\underline{\underline{\sigma}}) = -280000 \text{ kpa}
\end{aligned}\right\} \text{hydro}$$

$$\left.\begin{aligned}
I_1 &= 0 \\[2ex]
I_2 &= 10^6 \\[2ex]
I_3 &= 30000
\end{aligned}\right\} \text{Deviatoric}$$

$$\det\left(\underline{\underline{\sigma}} - \lambda \cdot \underline{\underline{I}}\right) = 0$$

$$\boxed{\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0}$$

Charecteristic eq.

$\lambda_1 = \sigma_1 = -479$ kPa ; $\lambda_2 = \sigma_2 = 220.59$ kPa ; $\lambda_3 = 558.48$ kPa

$$\underline{n_1} = (0.43, -0.49, 0.75)$$

$$[\lambda, N] = \mathrm{eig}(\underline{\underline{\sigma}})$$

$$\underline{n_2} = (-0.88, -0.076, 0.48)$$

$$\underline{n_3} = (0.43, -0.49, 0.75)$$

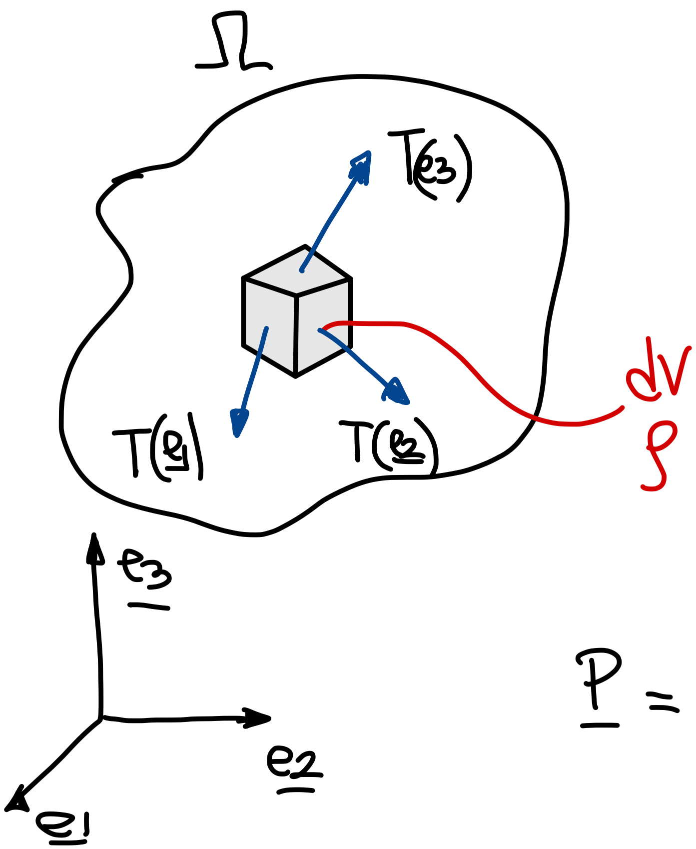
$$\tau_1^{MAX} = \frac{\sigma_3 - \sigma_1}{2} \qquad \tau_2^{MAX} = \frac{\sigma_3 - \sigma_2}{2} \qquad \tau_3^{MAX} = \frac{\sigma_2 - \sigma_1}{2}$$

Equations of motion and equilibrium: Use Newton's second Law to a deformable solid

(Conservation of linear & angular moment)

Linear momentum $\quad \underline{\Delta} = m\,\underline{v}$ $\qquad \underline{v}$: Velocity $\qquad \underline{a} = \dfrac{d\underline{v}}{dt}$: Acceleration

$$\underline{F} = \frac{d}{dt}\left(\underline{\Delta}\right)$$



Cauchy stress tensor

$$\underbrace{\sigma_{jm}}_{\text{Cauchy stress tensor}} \cdot n_m = \underbrace{T_j(\underline{n})}_{\text{Traction vector acting on } \underline{n}}$$

$$\underline{P} = \int_A \underline{T}\, dA + \int_V \rho\, \underline{b}\, dV \quad \text{or} \quad P_i = \int_A T_i\, dA + \int_V \rho\, b_i\, dV$$

Conservation of Linear momentum

$$\Lambda_i = \int_V \rho v_i \, dV \qquad \left[\Lambda_i = \frac{kg}{\cancel{m^3}} \cdot \frac{m}{s} \cdot \cancel{m^3}\right]$$

$$\frac{d\Lambda_i}{dt} = \int_V \rho \overbrace{\frac{dv_i}{dt}}^{a_i} \cdot dV = \int_V \rho a_i \, dV$$

a_i : Components of acceleration vector.

$$\underbrace{\int_A T_i \, dA}_{\text{Area}} + \underbrace{\int_V \rho b_i \, dV}_{\text{Volume}} = \underbrace{\int_V \rho a_i \, dV}_{\text{Volume}}$$

$$\int_A \underbrace{\sigma_{ji} n_j \, dA}_{\text{Area}} + \int_V \underbrace{\rho b_i \, dV}_{\text{Volume}} = \int_V \underbrace{\rho a_i \, dV}_{\text{Volume}}$$

- Divergence theorem

$$\int_A \sigma_{ji} n_j \, dA = \int_V \frac{\partial \sigma_{ji}}{\partial y_j} \, dV$$

$$\int_V \underbrace{\frac{\partial \sigma_{ji}}{\partial y_j} \, dV}_{\text{Volume}} + \int_V \underbrace{\rho b_i \, dV}_{\text{Volume}} = \int_V \underbrace{\rho a_i \, dV}_{\text{Volume}}$$

$$\int_V \left(\frac{\partial \sigma_{ji}}{\partial y_j} + \rho b_i dV - \rho a_i \right) dV = 0 \Leftrightarrow$$

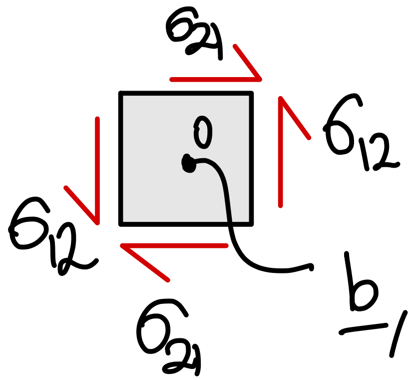
$$\boxed{\frac{\partial \sigma_{ji}}{\partial y_j} + \rho b_i = \rho a_i}$$

Equilibrium equation

$i = j = 1, 2, 3$

$$\boxed{\nabla_y \cdot \underline{\underline{\sigma}} + \rho \underline{b} = \rho \underline{a}}$$

$$\begin{cases} \frac{\partial \sigma_{11}}{\partial y_1} + \frac{\partial \sigma_{21}}{\partial y_2} + \frac{\partial \sigma_{31}}{\partial y_3} + \rho b_1 = \rho a_1 & (1) \quad (i=1, j=1,2,3) \\ \frac{\partial \sigma_{12}}{\partial y_1} + \frac{\partial \sigma_{22}}{\partial y_2} + \frac{\partial \sigma_{32}}{\partial y_3} + \rho b_2 = \rho a_2 & (2) \quad (i=2, j=1,2,3) \\ \frac{\partial \sigma_{13}}{\partial y_1} + \frac{\partial \sigma_{23}}{\partial y_2} + \frac{\partial \sigma_{33}}{\partial y_3} + \rho b_3 = \rho a_3 & (3) \quad (i=3, j=1,2,3) \end{cases}$$



$\underline{b}, \rho, \underline{a}$?

$$\Sigma M_0 = 0$$

$$\sigma_{12} = \sigma_{21}$$

$$\sigma_{ij} = \sigma_{ji} \qquad i, j = 1, 2, 3$$

• Conservation of angular momentum:

$$\int_A \underline{T} \cdot dA + \int_V \rho \underline{b} \, dV = \int_V \rho \underline{a} \, dV$$

$$\int_A \underline{y} \times \underline{T} \, dA + \int_V \underline{y} \times \rho \underline{b} \, dV = \int_V \underline{y} \times \rho \underline{a} \, dV$$

Use definition of $\times$ and index notation

$$\int_A \epsilon_{ijk} y_j T_k \, dA + \int_V \epsilon_{ijk} y_j b_k \rho \, dV = \int_V \epsilon_{ijk} y_j a_k \rho \, dV$$

$$\underbrace{\int_A \epsilon_{ijk}\, y_j\, \sigma_{mk} n_m \, dA}_{\text{Area}} + \underbrace{\int_V \epsilon_{ijk}\, y_j\, b_k\, \rho \, dV}_{\text{Volume}} = \underbrace{\int_V \epsilon_{ijk}\, y_j\, a_k \rho \, dV}_{\text{Volume}}$$

$T_k = \sigma_{mk} n_m$

$\frac{\partial y_j}{\partial y_m} = \delta_{jm}; \quad \frac{\partial \sigma_{mk}}{\partial y_m}$

Using the divergence theorem.

$$\int_A \epsilon_{ijk}\, y_j\, \sigma_{mk}\, n_m \, dA = \int_V \epsilon_{ijk} \frac{\partial}{\partial y_m}\left(y_j\, \sigma_{mk}\right) dV = \int_V \epsilon_{ijk}\left(\delta_{jm}\sigma_{mk} + y_j \frac{\partial \sigma_{mk}}{\partial y_m}\right) dV$$

$$\int_V \epsilon_{ijk}\left(\delta_{jm}\sigma_{mk} + y_j \frac{\partial \sigma_{mk}}{\partial y_m}\right) dV + \int_V \epsilon_{ijk}\, y_j\, b_k\, \rho \, dV = \int_V \epsilon_{ijk}\, y_j\, a_k \rho \, dV$$

$$\int_V \epsilon_{ijk}\left(\delta_{jm}\sigma_{mk}\right)dV + \cancelto{0}{\int_V \epsilon_{ijk}\, y_j \underbrace{\left(\frac{\partial \sigma_{mk}}{\partial y_m} + \rho b_k - \rho a_k\right)}_{\text{Equlibrium equation} = 0} dV}$$

$$\int_V \epsilon_{ijk}\left(\delta_{jm}\sigma_{mk}\right) = 0 \iff \epsilon_{ijk}\left(\delta_{jm}\sigma_{mk}\right) = 0$$

$$\epsilon_{ijk}\,\sigma_{jk} = 0 \Rightarrow \left(\delta_{jm}\delta_{kn} - \delta_{mk}\delta_{nj}\right)\sigma_{jk} = 0$$

$$\sigma_{mn} - \sigma_{nm} = 0$$

Cauchy stress tensor is symmetric & 6 independent components

$$\sigma_{mn} = \sigma_{nm}$$

$$\boxed{\sigma_{ij} = \sigma_{ji}}$$

$$\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix}$$

6 independent components.

- Equlibrium equations in terms of other stress measures

$$\nabla \cdot \underline{\underline{S}} + \rho_0 \underline{b} = \rho_0 \underline{a} \quad \text{or} \quad \frac{\partial S_{ij}}{\partial x_i} + \rho_0 b_j = \rho_0 a_j$$

$$\nabla [\Sigma F^T] + \rho_0 \underline{b} = \rho_0 \underline{a} \quad \text{or} \quad \frac{\partial \Sigma_{ik} F_{jk}}{\partial x_i} + \rho_0 b_j = \rho_0 a_j$$