

# 1 Understanding of course material and the Book

## 1.1 $L^2$ , Hilbert, Sobolev Space and their norm and semi-norm

$$(u, u)_0 = \int_{\Omega} u(x)^2 dx$$
$$\|u\|_0 = \sqrt{(u, u)_0}$$

So,

$$\|u\|_m = \sqrt{(u, u)_m} = \sqrt{\sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L_2(\Omega)}^2}$$
$$|u|_m = \sqrt{\sum_{|\alpha|=m} \|\partial^\alpha u\|_{L_2(\Omega)}^2}$$

If  $m = 1$

$$\|u\|_1 = \sqrt{(u, u)_1} = \sqrt{\|u\|_{L_2(\Omega)}^2 + \|\partial^1 u\|_{L_2(\Omega)}^2}$$
$$|u|_1 = \sqrt{\|\partial^1 u\|_{L_2(\Omega)}^2}$$

## 1.2 Dense

**Dense:** A subset  $A$  of a space  $B$  is dense in  $B$  if every point in  $B$  can be approximated arbitrarily closely by points in  $A$ . In other words, the closure of  $A$  is  $B$  itself.

**The Statement:** “ $C^\infty(\Omega) \cap H^m(\Omega)$  is dense in  $H^m(\Omega)$ ” means that the intersection of the space of smooth functions on  $\Omega$  and the Sobolev space  $H^m(\Omega)$  is a dense subset in  $H^m(\Omega)$ . Practically, this implies that for every function in  $H^m(\Omega)$ , there exists a sequence of smooth functions (from the intersection set) that converges to it in the sense of the Sobolev space norm.

**Implication:** This statement is significant because it allows us to approximate functions in Sobolev spaces (which may not be smooth or even continuous) by

smooth functions. This approximation is particularly useful in solving partial differential equations and in various applications of mathematical analysis, where working with smooth functions is often more manageable.

### 1.3

## 2 Different categories of PDEs

Partial Differential Equations (PDEs) are a class of equations that involve the partial derivatives of a function of multiple variables. They are fundamental in expressing a variety of physical, engineering, and mathematical phenomena. There are several different kinds of PDEs, each with unique characteristics and applications. Here are some of the most common types:

#### 1. Elliptic Equations:

- Example: Laplace's Equation,  $\nabla^2 u = 0$ , and Poisson's Equation,  $\nabla^2 u = f$ .
- Characteristics: No time dependence, solutions are general smooth and describe equilibrium states.
- Applications: Steady-state heat distribution, electrostatics, incompressible fluid flow.

#### 2. Parabolic Equations:

- Example: Heat Equation,  $\frac{\partial u}{\partial t} = \nabla^2 u$ .
- Characteristics: Contains time derivative and spatial derivatives; models phenomena that evolve over time towards an equilibrium.
- Applications: Heat conduction, diffusion processes.

#### 3. Hyperbolic Equations:

- Example: Wave Equation,  $\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$ .
- Characteristics: Second-order in time, describe wave propagation and vibrations.
- Applications: Acoustics, electromagnetic waves, seismic waves.

#### 4. Transport (or Convection-Diffusion) Equations:

- Example:  $\frac{\partial u}{\partial t} + v \cdot \nabla u = D \nabla^2 u$ , where  $v$  is velocity and  $D$  is the diffusion coefficient.
- Characteristics: Describe phenomena involving both transport (or advection) and diffusion.
- Applications: Fluid dynamics, pollutant dispersion.

#### 5. Nonlinear Differential Equations:

- Example: Nonlinear Schrödinger Equation, Korteweg-de Vries Equation.
- Characteristics: The equation includes nonlinear terms (products or powers of the function and its derivatives).
- Applications: Complex physical phenomena, including solitons, fluid dynamics, and optical physics.

#### 6. Mixed Type Equations:

- Example: Tricomi Equation.
- Characteristics: The equation changes type (from elliptic to hyperbolic, for instance) within the domain.
- Applications: Transonic flow, certain problems in gas dynamics.

#### 7. Eigenvalue Problems:

- Example:  $-\nabla^2 u = \lambda u$  (Helmholtz equation in eigenvalue form).
- Characteristics: Involves finding a function  $u$  and a number  $\lambda$  (eigenvalue) such that the equation is satisfied.
- Applications: Quantum mechanics, stability analysis, structural engineering.

Each of these types of PDEs plays a crucial role in modeling different physical phenomena. The solution techniques and analytical approaches vary significantly among these types, reflecting the diverse nature of the phenomena they model. Understanding the specific type of PDE is essential in choosing the right methods for analysis and numerical simulation.

## 2.1 Definition

1. The equation is called elliptic at the point  $x$  provided  $A(x)$  is positive definite.
2. The equation is called hyperbolic at the point  $x$  provided  $A(x)$  has one negative and  $n - 1$  positive eigenvalues.
3. The equation is called parabolic at the point  $x$  provided  $A(x)$  is positive semidefinite, but is not positive definite, and the rank of  $(A(x), b(x))$  equals  $n$ .
4. An equation is called elliptic, hyperbolic or parabolic provided it has the corresponding property for all points of the domain.

## 3 I.B.P in higher dimensions

Integration by parts in higher dimensions is a generalization of the integration by parts formula from one-dimensional calculus to multiple dimensions. This generalization is often used in vector calculus and is particularly useful in the context of functions over multi-dimensional domains. The concept is closely related to the Divergence Theorem and Green's Theorem, which are fundamental in vector calculus.

In one dimension, the integration by parts formula is given by:

$$\int u \, dv = uv - \int v \, du$$

For higher dimensions, consider a domain  $D$  in  $\mathbb{R}^n$  and functions  $u$  and  $v$  that are sufficiently smooth (i.e., have continuous derivatives). The integration by parts formula in higher dimensions, specifically for  $n = 2$  or  $n = 3$ , can be expressed using the gradient, divergence, or curl, depending on the context. Here's a general idea for a domain in  $\mathbb{R}^3$ :

$$\int_D u \nabla v \, dV = \int_{\partial D} uv \, dS - \int_D v \nabla u \, dV$$

In this formula:

-  $\nabla v$  is the gradient of  $v$ . -  $dV$  is the volume element in  $\mathbb{R}^3$ . -  $\partial D$  represents the boundary of the domain  $D$ . -  $dS$  is the surface element on  $\partial D$ .

The physical interpretation of this formula involves the flow of a vector field across the boundary of a region and within the region itself.

The specific form of the integration by parts formula will depend on the nature of the functions  $u$  and  $v$  (scalar or vector fields) and the dimension of the space. In vector calculus, various forms of this theorem, like the Divergence Theorem (for divergence and flux) and Green's Theorem (in the plane), are essentially higher-dimensional versions of integration by parts.

It's important to note that applying integration by parts in higher dimensions requires a good understanding of vector calculus, including gradients, divergences, curls, and theorems related to flux and circulation.

## 4 Support of function

The support of a function, particularly in the context of mathematics and analysis, is a fundamental concept. It refers to the set of points where the function is not zero, or, more precisely, the closure of that set. Let's break down this definition:

- **Support of a Real-Valued Function:** For a real-valued function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , the support is the set of points in  $\mathbb{R}$  where  $f(x)$  is not zero. This can be expressed as:

$$\text{supp}(f) = \{x \in \mathbb{R} \mid f(x) \neq 0\}$$

- **Closure of the Set:** More formally, the support of  $f$  is the closure of this set. The closure includes all the points where  $f$  is non-zero, plus any limit points of this set. A limit point is a point where the function approaches but does not necessarily reach zero. This means that the support of a function can include points where the function itself is zero, as long as these points are limit points of the set where the function is non-zero.
- **Support in Higher Dimensions:** The concept of support extends to functions in higher dimensions as well. For a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , the support is the closure of the set of points in  $\mathbb{R}^n$  where  $f$  is not zero.
- **Importance in Various Contexts:** The notion of support is important in many areas of mathematics, including analysis, differential equations, and functional analysis. In the context of distributions or generalized functions, the support of a distribution is the set outside of which the distribution acts like the zero distribution.
- **Compact Support:** A function is said to have compact support if its support is a compact set. In  $\mathbb{R}^n$ , a set is compact if it is closed and bounded. Functions with compact support are especially important in various applications because they vanish outside of a bounded region, making them easier to handle in many analytical and numerical computations.

Understanding the support of a function is crucial when working with concepts such as convolution, Fourier transforms, and in the study of partial differential equations, where the behavior of functions over their domain plays a key role.

## 5 Measure of a set or a function

The concepts of the measure of a set and the measure of a function are fundamental in mathematical analysis, particularly in the areas of measure theory and integration. Let's define these two concepts in detail:

### 5.1 Measure of a Set

The measure of a set is a way to assign a non-negative number to certain subsets of a space (like the real line or  $\mathbb{R}^n$ ) in a way that generalizes the concept of length, area, and volume. This is formally defined in the context of measure theory.

Let  $(X, \mathcal{M}, \mu)$  be a measure space where  $X$  is a set,  $\mathcal{M}$  is a  $\sigma$ -algebra of subsets of  $X$  (these are the measurable sets), and  $\mu$  is a measure. Then, the measure of a set  $A \in \mathcal{M}$ , denoted  $\mu(A)$ , is a non-negative extended real number assigned to  $A$  by  $\mu$ .

The most common example is the Lebesgue measure on  $\mathbb{R}$ , which generalizes the concept of length. For instance, if  $A$  is an interval  $[a, b]$ , then its Lebesgue measure is  $b - a$ , the length of the interval.

### 5.2 Measure of a Function

The measure of a function usually refers to an integral of the function with respect to a certain measure, often used to generalize the concept of the sum, total mass, or total charge of a function over a set.

If  $f : X \rightarrow \mathbb{R}$  is a measurable function on a measure space  $(X, \mathcal{M}, \mu)$ , then the measure (or integral) of the function  $f$  with respect to the measure  $\mu$  over a set  $A \in \mathcal{M}$  is denoted as:

$$\int_A f d\mu$$

This integral can be interpreted as a weighted sum of the values of  $f$ , with the weights given by the measure  $\mu$ .

In the special case where  $\mu$  is the Lebesgue measure on  $\mathbb{R}$  and  $f$  is a real-valued function,  $\int_A f d\mu$  is the Lebesgue integral of  $f$  over  $A$ , which generalizes the concept of the area under the curve.

In summary, the measure of a set is a number that represents the size of the set in a generalized sense, while the measure (or integral) of a function is a generalized sum of the function's values over a set, weighted by a measure.



## 6 Sobolev Space

Sobolev spaces are a fundamental concept in functional analysis and partial differential equations, playing a crucial role in modern analysis, particularly in the study of differential equations and variational problems. They generalize the notion of derivatives and integrals to more abstract settings.

### 6.1 Definition of Sobolev Space

A Sobolev space is a vector space of functions equipped with a norm that is a combination of  $L^p$ -norms of the function itself and its derivatives up to a certain order. Formally, the Sobolev space  $W^{k,p}(\Omega)$  is defined as follows:

Consider a domain  $\Omega \subseteq \mathbb{R}^n$ . The Sobolev space  $W^{k,p}(\Omega)$  consists of all functions  $u : \Omega \rightarrow \mathbb{R}$  such that  $u$  and all its weak derivatives up to order  $k$  are in  $L^p(\Omega)$ . Here,  $k \in \mathbb{N}$  (non-negative integers) represents the order of derivatives considered, and  $p \in [1, \infty]$  is a parameter that determines the  $L^p$  space, which generalizes the concept of integrability and norm.

### 6.2 Weak Derivatives

A key aspect of Sobolev spaces is the concept of weak derivatives. A weak derivative is a generalization of the classical derivative, applicable to a broader class of functions. It is defined in the sense of distributions, which allows for the inclusion of functions that may not be differentiable in the classical sense.

### 6.3 Sobolev Norm

The norm on  $W^{k,p}(\Omega)$ , denoted  $\|\cdot\|_{W^{k,p}(\Omega)}$ , is defined by:

$$\|u\|_{W^{k,p}(\Omega)} = \left( \sum_{|\alpha| \leq k} \int_{\Omega} |D^{\alpha} u|^p dx \right)^{1/p}$$

for  $1 \leq p < \infty$ . Here,  $\alpha$  is a multi-index used to denote derivatives, and  $D^{\alpha} u$

represents the weak derivative of  $u$ .

For  $p = \infty$ , the norm is defined using the essential supremum:

$$\|u\|_{W^{k,\infty}(\Omega)} = \max_{|\alpha| \leq k} \operatorname{ess\,sup}_{x \in \Omega} |D^\alpha u(x)|$$

## 6.4 Importance and Applications

- **Generalizing Classical Spaces\*\***: Sobolev spaces generalize classical spaces like continuous, differentiable, and integrable function spaces. They are particularly important in settings where functions may not be smooth.
- **Partial Differential Equations\*\***: Sobolev spaces are crucial in the study of PDEs, especially in formulating and solving weak solutions of PDEs.
- **Variational Problems\*\***: They are used in the calculus of variations and in the formulation of variational problems, which seek to find functions minimizing certain integrals.
- **Embedding Theorems\*\***: Sobolev embedding theorems, which relate different Sobolev spaces and establish conditions under which functions in a Sobolev space are also in more regular function spaces, are fundamental in analysis.

Sobolev spaces thus provide a powerful framework for dealing with functions that have derivatives in some generalized sense, allowing for the rigorous study of a broad range of problems in mathematical analysis.

## 7 Compact Subset

A compact subset in mathematics, particularly in the context of topology, is a set that has specific properties related to size and closure. The concept of compactness is crucial because it generalizes the notion of a set being closed and bounded (as in the Heine-Borel theorem for  $\mathbb{R}^n$ ) to more abstract spaces. Let's define compactness more formally:

### 7.1 Definition of Compact Subset

A subset  $K$  of a topological space  $X$  is said to be **compact** if it satisfies the following condition:

- Every open cover of  $K$  has a finite subcover.

This means that if you have a collection of open sets whose union contains  $K$ , you can always find a finite number of these open sets whose union still contains  $K$ .

### 7.2 Intuition and Examples

- Closed and Bounded in  $\mathbb{R}^n$ : In the familiar setting of Euclidean spaces ( $\mathbb{R}^n$ ), a subset is compact if and only if it is closed (contains all its limit points) and bounded (can be contained within some large ball). This is the content of the Heine-Borel theorem.
- General Topological Spaces: In more general spaces, the notion of being "bounded" may not make sense, so compactness is defined purely in terms of open covers.

### 7.3 Properties of Compact Sets

1. Closed: Compact sets in a Hausdorff space (a type of topological space where distinct points have disjoint neighborhoods) are always closed.
2. Containment of Subsequences: In metric spaces, compactness is equivalent to every sequence in the set having a convergent subsequence whose limit is in the set. This property is known as sequential compactness.

3. Continuity and Compactness: A continuous function mapping from a compact space to a Hausdorff space is closed and takes compact sets to compact sets.
4. Heine-Borel Theorem: In  $\mathbb{R}^n$ , a set is compact if and only if it is closed and bounded. This result, however, does not generalize to all topological spaces.

## 8 Trace Thm

The Trace Theorem is a fundamental result in functional analysis, particularly in the study of Sobolev spaces, which are spaces of functions with certain integrability and differentiability properties. The Trace Theorem concerns the restriction (or "trace") of a Sobolev function on the boundary of its domain.

Here's a general overview of the Trace Theorem:

1. Background: Sobolev spaces, denoted typically as  $W^{k,p}(\Omega)$  where  $\Omega$  is a domain in  $\mathbb{R}^n$ ,  $k$  is a non-negative integer, and  $p$  is a real number, are spaces of functions that are differentiable up to order  $k$  with derivatives that are  $p$ -integrable. These spaces are crucial in the study of partial differential equations and variational problems.
2. The Problem: For functions in these spaces, it is not immediately obvious that you can meaningfully talk about their values on the boundary of the domain  $\Omega$ , especially when  $\Omega$  is a domain with a non-trivial boundary (like an open set in  $\mathbb{R}^n$ ). This is because Sobolev functions are only guaranteed to be differentiable almost everywhere and might not be well-defined or continuous at every point.
3. The Trace Theorem: What the Trace Theorem essentially states is that for certain Sobolev spaces (typically  $W^{k,p}(\Omega)$  with  $k \geq 1$ ), there exists a well-defined linear operator (the "trace operator") that can "restrict" or "trace" a function in this space to the boundary of  $\Omega$ . This theorem also often provides the Sobolev space on the boundary to which the trace belongs.
4. Implications: The Trace Theorem allows one to handle boundary values for functions in Sobolev spaces, which is fundamental for solving boundary value problems in partial differential equations. For example, it permits the formulation of problems where you seek a function in a Sobolev space that satisfies certain conditions on the boundary of its domain.
5. Technical Details: The precise statement of the Trace Theorem involves details about the regularity of the boundary of  $\Omega$ , the order of the Sobolev space, and the nature of the trace operator. It also often involves the fact that the trace operator is bounded and surjective onto a Sobolev space defined on the boundary.

## 9 Lax-Milgram Thm

The Lax-Milgram Theorem is a fundamental result in functional analysis and is particularly important in the study of partial differential equations. **It provides a guarantee for the existence and uniqueness of solutions to certain types of problems, often referred to as boundary value problems.**

Here's a general overview of the Lax-Milgram Theorem:

1. Background: The theorem is set in the context of Hilbert spaces, which are complete vector spaces equipped with an inner product. It specifically deals with linear operators and bilinear forms defined on these spaces.
2. Bilinear Forms: A bilinear form on a Hilbert space  $H$  is a function  $B : H \times H \rightarrow \mathbb{R}$  (or  $\mathbb{C}$  for complex spaces) that is linear in each argument.
3. Statement of the Theorem: The Lax-Milgram Theorem states that given a Hilbert space  $H$  and a continuous, coercive (meaning essentially that it grows at least linearly with the norm of the argument) bilinear form  $B$  on  $H$ , for every continuous linear functional  $f$  on  $H$ , there exists a unique element  $u \in H$  such that for all  $v \in H$ ,

$$B(u, v) = f(v)$$

4. Implications: This theorem is particularly useful for showing the existence and uniqueness of weak solutions to boundary value problems. In practical terms, it is often applied in situations where  $B$  represents the integral of some combination of functions and their derivatives (arising from a differential equation), and  $f$  represents an integral of a function against a known source term.
5. Applications: The Lax-Milgram Theorem is a cornerstone in the theory of elliptic partial differential equations. It is used to establish the existence and uniqueness of solutions to a broad class of problems, which includes many physical phenomena such as heat conduction, fluid dynamics, and elasticity.

6. Conditions: The key conditions for the theorem to hold are the continuity and coerciveness of the bilinear form  $B$ , and the continuity of the linear functional  $f$ . These conditions ensure that the problem is well-posed.

$$\int_{\Gamma}(\sigma.n)d\Gamma$$