THE UNIVERSITY OF BRITISH COLUMBIA
Department of Mechanical Engineering
MECH 570C: Fluid-Structure Interaction (FSI) - Theory and Computation
(Term 2, 2023-2024)

Due Date: Jan 23, 2024

Problem Set 1: Non-dimensional FSI parameters and continuum mechanics aspects

Question 1

Provide an interesting example where fluid-structure interaction takes place in daily life or practical engineering settings. What are characteristic properties and the key non-dimensional parameters in your example? Be thoughtful and creative in your answer.

Question 2

Suppose a body with configuration $B = \{\boldsymbol{x} \in \mathbb{E}^3 | 0 < x_i < 1\}$ is subject to a body force field per unit volume $\boldsymbol{b} = \alpha x_3 \boldsymbol{e}_3$ and a traction field on its bounding surface ∂B given by

$$\boldsymbol{t} = \begin{cases} x_1 x_2 (1 - x_1)(1 - x_2)\, \boldsymbol{e}_3 & \text{on face } x_3 = 0, \\ \boldsymbol{0}, & \text{on all other faces.} \end{cases}$$

Find the value of α (constant) for which the resultant body and surface forces are balanced, that is, $\boldsymbol{r}_b[B] + \boldsymbol{r}_s[\partial B] = \boldsymbol{0}$.

Question 3

Consider a solid body B immersed in a liquid of constant mass density ρ^* and subject to a uniform gravitational force field per unit mass. Suppose that the free surface of the liquid coincides with the plane $x_3 = 0$, and that the downward direction (into the liquid) coincides with e_3. In this case, the liquid exerts a hydrostatic surface force field on the bounding surface of B

$$\boldsymbol{t} = -p\boldsymbol{n}$$

where $\boldsymbol{n}$ is the outward unit normal on the surface of $B, p = \rho^* g x_3$ is the hydrostatic pressure in the liquid, and g is the gravitational acceleration constant.
(a) Use the Divergence Theorem to show that the resultant hydrostatic surface force on B (the

buoyant force) is given by $\boldsymbol{r}_s[\partial B] = -We_3$, where W is the weight of the liquid displaced by B.
(b) Show that the hydrostatic surface force has a zero resultant torque about the center of volume of B, that is, $\boldsymbol{\tau}_s[\partial B] = \mathbf{0}$
Remark: The result in (a) is known as Archimedes' Principle. It states that the buoyant force on an object equals the weight of the displaced liquid. The result in (b) shows that the buoyant force acts at the center of volume of the object.

Question 4

Consider a body $B = \{\boldsymbol{x} \in E^3 | 0 < x_i < 1\}$ with constant mass density $\rho > 0$ subject to a constant body force per unit mass $[\boldsymbol{b}] = (0, 0, -g)^T$. Suppose the Cauchy stress field in B is given by

$$[\boldsymbol{S}] = \begin{pmatrix} x_2 & x_3 & 0 \\ x_3 & x_1 & 0 \\ 0 & 0 & \rho g x_3 \end{pmatrix}$$

(a) Show that $\boldsymbol{S}$ and $\boldsymbol{b}$ satisfy the local equilibrium equations.
(b) Find the traction field on each of the six faces of the bounding surface ∂B.
(c) Find by direct calculation the resultant surface force $\boldsymbol{r}_s[\partial B]$ and the resultant body force $\boldsymbol{r}_b[B]$ and verify that these forces are balanced, that is, $\boldsymbol{r}_s[\partial B] + \boldsymbol{r}_b[\boldsymbol{B}] = \mathbf{0}$. Briefly explain how this result is consistent with part (a).

Question 5

Consider a continuum body B with mass density ρ subject to a body force per unit mass b and a traction h on its bounding surface. Assume the Cauchy stress field $\boldsymbol{S}$ in B is related to a vector field $\boldsymbol{u} : B \to \mathcal{V}$ by the expression

$$\boldsymbol{S} = \mathbf{C}\boldsymbol{E} \quad \text{or} \quad S_{ij} = \mathrm{C}_{ijkl} E_{kl}$$

where $\mathbf{C}$ is a constant fourth-order tensor and $\boldsymbol{E} : B \to \mathcal{V}^2$ is defined by

$$\boldsymbol{E} = \operatorname{sym}(\nabla \boldsymbol{u}) = \frac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T\right)$$

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Here $\boldsymbol{u}$ is the displacement field of the body from an unstressed state and $\mathbf{E}$ is the infinitesimal strain tensor. Moreover, let W denote the strain energy in B, defined by

$$W = \frac{1}{2} \int_B \boldsymbol{E}(\boldsymbol{x}) : \mathbf{C}\boldsymbol{E}(\boldsymbol{x}) dV_{\boldsymbol{x}}$$

Assuming B is in equilibrium show that

$$W = \frac{1}{2}\left(\int_B \rho(\boldsymbol{x})\boldsymbol{b}(\boldsymbol{x}) \cdot \boldsymbol{u}(\boldsymbol{x}) dV_{\boldsymbol{x}} + \int_{\partial B} \boldsymbol{h}(\boldsymbol{x}) \cdot \boldsymbol{u}(\boldsymbol{x}) dA_{\boldsymbol{x}}\right)$$