MATH 521 - Numerical Analysis of Differential Equations

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Assignment 2 : Hilbert Spaces, Weak Form of BVPs

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Background for Q1 [no points]

You will need the following Poincare-type inequalities:

(1) Let Ω be a connected, bounded, domain and $\Gamma_D \subset \partial\Omega$ measurable with surface area $|\Gamma_D| > 0$, then there exists a constant c_P such that

$$\|v\|_{L^2(\Omega)} \leq c_P \|\nabla v\|_{L^2(\Omega)} \qquad \forall v \in H^1_{\Gamma_D}(\Omega).$$

(2) Let Ω be a simply connected domain then there exists a constant c_P such that

$$\|v\|_{L^2(\Omega)} \leq c_P \|\nabla v\|_{L^2(\Omega)} \qquad \forall v \in H^1(\Omega) \text{ satisfying } (v)_\Omega = 0,$$

where $(v)_\Omega := |\Omega|^{-1} \int_\Omega v \, dx$.

Another way (seemingly stronger but equivalent) to state these results is the following:

(1') Let Ω be a connected, bounded, domain and $\Gamma_D \subset \partial\Omega$ measurable with surface area $|\Gamma_D| > 0$, then there exists a constant c_P such that

$$\|v\|_{L^2(\Omega)} \leq c_P \Big(\|v\|^2_{L^2(\Gamma_D)} + \|\nabla v\|^2_{L^2(\Omega)} \Big)^{1/2} \qquad \forall v \in H^1(\Omega).$$

(2) Let Ω be a simply connected domain then there exists a constant c_P such that

$$\|v\|_{L^2(\Omega)} \leq c_P \Big(|(v)_\Omega|^2 + \|\nabla v\|^2_{L^2(\Omega)} \Big)^{1/2} \qquad \forall v \in H^1(\Omega).$$

Note: if you want to prove these results, it is not too difficult. They can both be proven with the same argument. What you need is the compactness of the embedding $L^2(\Omega) \subset H^1(\Omega)$: If $(u_n)_{n\in\mathbb{N}}$ is a sequence that is bounded in $H^1(\Omega)$ then there exists $u \in L^2(\Omega)$ and a subsequence such that $u_{n_j} \to u$ strongly in L^2.

Q1a: [8]

Let Ω be a connected, bounded domain. Are the following spaces H Hilbert spaces when equipped with their stated inner products? If not, then explain what property is missing (no need to justify it at length)

(i) $H = \{v \in C^1(\bar\Omega) : v, \nabla v \in L^2(\Omega)\}$ $(u, v)_H = \int_\Omega uv + \nabla u \cdot \nabla v \, dx$

(ii) $H = \{v \in L^2(\Omega) : \text{ weakly differentiable, } \nabla v \in L^2(\Omega)\}$, $(u, v)_H = \int_\Omega \nabla u \cdot \nabla v dx$

(iii) $H = \{v \in L^2(\Omega) : \text{ weakly differentiable, } \nabla v \in L^2(\Omega)\}$, $(u, v)_H = \int_\Omega uv + \nabla u \cdot \nabla v dx$

(iv) $H = \{v \in L^2(\Omega) : \text{ weakly differentiable, } \nabla v \in L^2(\Omega), v(0) = 0\}$, $(u, v)_H = \int_\Omega \nabla u \cdot \nabla v dx$.

Solution Q1a

(i) The given space is not a Hilbert space. Since $C^1(\bar\Omega)$ is not complete.

(ii) The given space is not a Hilbert space. Since the inner product may be zero when the input functions are constant but not zero.

(iii) The given space is a Hilbert space. Since it is complete and the inner product satifies the required properties.

(iv) The given space is a Hilbert space. Since it is complete and the inner product satifies the required properties.

Q1b [5+7]

Let Ω be a connected, bounded domain, and $\Gamma_D \subset \partial\Omega$ measurable with surface area $|\Gamma_D| > 0$. Are the following spaces H Hilbert spaces when equipped with their stated inner products? Now please justify your answer in full detail. (except you don't need to show that $(u, v)_H$ is symmetric and bilinear)

(v) $H = \{v \in H^1(\Omega) : (v)_\Omega = 0\}$, $(u, v)_H = \int_\Omega \nabla u \cdot \nabla v dx$.

(vi) $H = H^1(\Omega)$, $(u, v)_H = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\Gamma_D} uv \, dx$.

Solution Q1b

(v) Other than the symmetricand bilinear property of the inner product, it is also obvious to show it is positive definite since $(v)_\Omega = 0$, $(v, v)_H = 0$ if and only if $v = 0$. Then the completeness of the given space can be justified by Poincaré inequality as follows:

For any function $v \in H^1(\Omega)$ with a mean value of zero over a connected, bounded domain Ω, the Poincaré inequality asserts that there exists a constant $C > 0$,such that:

$$\|v\|_{L^2(\Omega)} \leq C\|\nabla v\|_{L^2(\Omega)}.$$

Given a Cauchy sequence $\{v_n\}$ in H, for every $\epsilon > 0$, there exists an N such that for all $m, n > N$:

$$\|\nabla(v_n - v_m)\|_{L^2(\Omega)} < \epsilon.$$

Since $\{v_n\}$ is Cauchy in the gradient norm and $H^1(\Omega)$ is complete, it converges to a function $v \in H^1(\Omega)$ in the gradient norm:

$$\lim_{n\to\infty} \|\nabla(v_n - v)\|_{L^2(\Omega)} = 0.$$

The Poincaré inequality implies this convergence also occurs in the L^2 norm for the sequence differences $v_n - v$, as these differences have zero mean. This establishes that v_n converges to v in $H^1(\Omega)$.

Moreover, the mean value of functions is continuous with respect to L^2 convergence. Since each v_n has zero mean, so does the limit function v. Therefore, $v \in H$, demonstrating that the given space is complete. Therefore, it is a Hibert space.

(vi) Since H^1 is complete, the given space is considered to be complete as well.

The positive definiteness of the given space can be proved as follows:

For any non-zero $v \in H^1(\Omega)$, the gradient term $\int_\Omega \nabla v \cdot \nabla v \, dx$ is strictly positive unless v is constant across Ω.

The boundary term $\int_{\Gamma_D} v^2 \, dx$ is non-negative, as it is the integral of the square of v over Γ_D. It contributes to the inner product only when v is non-zero on Γ_D. Since $|\Gamma_D| > 0$, for v not to contribute to this term, it would have to be zero almost everywhere on Γ_D.

Given that v cannot simultaneously be a non-zero constant in Ω and zero on Γ_D (assuming $v \in H^1(\Omega)$), the combined inner product $(u, v)_H$ is positive for any non-zero $v \in H^1(\Omega)$. This satisfies the positive definiteness requirement for an inner product on H. Therefore, it is a Hibert space.

Background to Q2 [no points]

Before starting on Q2, review integration by parts in $\Omega \subset \mathbb{R}^d$. We introduced this as

$$\int_\Omega \partial_i u \cdot v dx = -\int_\Omega u \partial_i v dx + \int_{\partial\Omega} \nu_i uv \, dx.$$

From this expression, please derive the following equivalent formulation: if $g : \Omega \to \mathbb{R}^d, v : \Omega \to \mathbb{R}$ (both weakly differentiable) then

$$\int_\Omega \text{div} g \, v \, dx = -\int_\Omega g \cdot \nabla v \, dx + \int_{\partial\Omega} (\nu \cdot g) \, v \, dx.$$

Answer

Consider $\phi = g\, v$, the divergence of ϕ can be expanded:

$$\text{div}\phi = \text{div}(g v) = \nabla \cdot (g v) = (\nabla v) \cdot g + v(\nabla \cdot g),$$

Divergence theorem gives:

$$\int_\Omega \mathrm{div}\phi \, dx = \int_{\partial\Omega} \nu \cdot (gv) \, dx.$$

Substituting the expanded form of $\mathrm{div}\phi$ and rearranging,

$$\int_\Omega v(\nabla \cdot g) \, dx = \int_{\partial\Omega} (\nu \cdot g) v \, dx - \int_\Omega (\nabla v) \cdot g \, dx.$$

Q2: Weak forms of 2nd order BVPs [10+10+10]

For the following three problems, derive the weak form and then use the Lax-Milgram theorem to show that the weak forms have unique solutions. Throughout this question,, Ω is a connected domain in $\mathbb{R}^d$, $d > 1$, $p, q \in C(\bar{\Omega})$ with $c_0 \leq p, q \leq c_1$, $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$.

(i) Neumann problem

$$\begin{aligned} -\mathrm{div}\big(p\nabla u\big) + qu = f, \quad & \text{in } \Omega, \\ p\nu \cdot \nabla u = g, \quad & \text{on } \partial\Omega. \end{aligned}$$

(ii) Robin problem

$$\begin{aligned} -\mathrm{div}\big(p\nabla u\big) = f, \quad & \text{in } \Omega, \\ p\nu \cdot \nabla u + u = g, \quad & \text{on } \partial\Omega. \end{aligned}$$

(iii) The classical Neumann problem: in addition to all previous assumptions also assume that $(f)_\Omega = 0$.

$$\begin{aligned} -\Delta u = f, \quad & \text{in } \Omega, \\ \nu \cdot \nabla u = 0, \quad & \text{on } \partial\Omega. \end{aligned}$$

HINT for (iii) : you need to introduce an additional condition that uniquely determines the solution but doesn't change the problem.

Solution Q2(i)

Neumann Problem: Weak Form and Uniqueness

Multiply the differential equation by a test function $v \in H^1(\Omega)$ and integrate over Ω:

$$\int_\Omega (-\mathrm{div}(p\nabla u) + qu) v \, dx = \int_\Omega fv \, dx.$$

Applying integration by parts to the divergence term and substituting the boundary condition $p\nu \cdot \nabla u = g$ on $\partial\Omega$,

$$\int_\Omega p\nabla u \cdot \nabla v \, dx + \int_\Omega quv \, dx = \int_\Omega fv \, dx + \int_{\partial\Omega} gv \, ds.$$

Thus, The weak form is: Find $u \in H^1(\Omega)$ such that for all $v \in H^1(\Omega)$,

$$\int_\Omega p\nabla u \cdot \nabla v \, dx + \int_\Omega quv \, dx = \int_\Omega fv \, dx + \int_{\partial\Omega} gv \, ds.$$

To apply the Lax-Milgram theorem,

- The bilinear form $a(u, v) = \int_\Omega p\nabla u \cdot \nabla v \, dx + \int_\Omega quv \, dx,$
- The linear functional $L(v) = \int_\Omega fv \, dx + \int_{\partial\Omega} gv \, ds.$

Boundedness and Coercivity of the Bilinear Form:

- The bilinear form $a(u, u)$ is bounded since $a(u, u) \leq c_1 \int_\Omega (|\nabla u| \cdot |\nabla u| + u^2) \, dx = c_1 ||u||_{H^1}^2$ by C-S inequality.
- The bilinear form $a(u, u)$ is coercive since $a(u, u) \geq c_0 \int_\Omega (|\nabla u| \cdot |\nabla u| + u^2) \, dx = c_0 ||u||_{H^1}^2$.

Boundedness of the Linear Functional: The linear functional $L(v)$ is bounded since $|L(v)| \leq (||f||_{L^2(\Omega)} + C_{tr}||g||_{L^2(\Gamma_N)})||v||_1$ by C-S inequality and Trace theorem.

Given these properties, the Lax-Milgram theorem guarantees the existence and uniqueness of the solution $u \in H^1(\Omega)$ to the weak form of this problem.

Solution Q2(ii)

Robin Problem Weak Form and Solution Uniqueness

Problem Statement

Multiply the differential equation by a test function $v \in H^1(\Omega)$ and integrate over Ω:

$$\int_\Omega (-\mathrm{div}(p\nabla u))v\,dx = \int_\Omega fv\,dx.$$

Applying integration by parts to the divergence term and the divergence theorem,

$$\int_\Omega p\nabla u \cdot \nabla v\,dx - \int_{\partial\Omega} (p\nu \cdot \nabla u)v\,ds = \int_\Omega fv\,dx.$$

Incorporating the Robin boundary condition $p\nu \cdot \nabla u + u = g$ on $\partial\Omega$ leads to the weak form:

$$\int_\Omega p\nabla u \cdot \nabla v\,dx + \int_{\partial\Omega} uv\,ds = \int_\Omega fv\,dx + \int_{\partial\Omega} gv\,ds.$$

Define a bilinear form $a(u, v)$ and a linear functional $L(v)$ as:

$$a(u, v) = \int_\Omega p\nabla u \cdot \nabla v\,dx + \int_{\partial\Omega} uv\,ds, \quad L(v) = \int_\Omega fv\,dx + \int_{\partial\Omega} gv\,ds.$$

To apply the Lax-Milgram theorem,

- The bilinear form $a(u, u) = \int_\Omega p\nabla u \cdot \nabla u\,dx + \int_{\partial\Omega} uv\,ds$.
- The linear functional $L(v) = \int_\Omega fv\,dx + \int_{\partial\Omega} gv\,ds$.

Boundedness and Coercivity of the Bilinear Form:

- The bilinear form $a(u, u)$ is bounded: Considering the first term:

  $$\left|\int_\Omega p\nabla u \cdot \nabla u\,dx\right| \leq c_1 ||\nabla u||_{L^2(\Omega)} ||\nabla v||_{L^2(\Omega)}$$

  Considering the second term:

  $$\left|\int_{\partial\Omega} uv\,ds\right| \leq ||u||_{L^2(\partial\Omega)} ||v||_{L^2(\partial\Omega)} \leq C_{\mathrm{tr}}^2 ||u||_{H^1(\Omega)} ||v||_{H^1(\Omega)}$$
  $$|a(u, v)| \leq (c_1 + C_{\mathrm{tr}}^2) ||u||_{H^1(\Omega)} ||v||_{H^1(\Omega)}$$

- The bilinear form $a(u, u)$ is coercive

  $$\int_\Omega p|\nabla u| \cdot |\nabla u|\,dx \geq c_0 \int_\Omega |\nabla u|^2\,dx = c_0 ||u||^2_{H^1(\Omega)} - c_0 \int_\Omega |u|^2\,dx$$

By Trace theorem,

$$0 \leq \int_{\partial\Omega} u^2\,ds \leq C_{\mathrm{tr}}^2 ||u||^2_{H^1(\Omega)}$$

I actually do not know how to continue this proof. The Trace theorem is applied here since in this problem the uv term appears on the boundary of Ω only, but I do not know how to relate this to proof the coercivity of $a(u, u)$. The same issue exists in the thrid problem as well.

Boundedness of the Linear Functional: The linear functional $L(v)$ is bounded since $|L(v)| \leq (||f||_{L^2(\Omega)} + C_{tr} ||g||_{L^2(\Gamma_N)}) ||v||_1$.

Given these properties, the Lax-Milgram theorem guarantees the existence and uniqueness of the solution $u \in H^1(\Omega)$ to the weak form of this problem.

Solution Q2(iii)

Multiply the PDE by a test function $v \in H^1(\Omega)$ and integrate over Ω:

$$\int_\Omega (-\Delta u)v\,dx = \int_\Omega fv\,dx.$$

Applying integration by parts and considering the Neumann boundary condition leads to:

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx.$$

The weak form is thus: Find $u \in H^1(\Omega)$ such that for all $v \in H^1(\Omega)$,

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx.$$

Given that the Neumann problem specifies the flux across the boundary but not the value of u itself, the solution to this problem is not unique; any constant can be added to a solution to produce another solution.

To ensure uniqueness, one could assume the solution has zero mean over Ω:

$$(u)_{\Omega} = \frac{1}{|\Omega|} \int_{\Omega} u \, dx = 0.$$

This condition uniquely determines the solution by eliminating the constant ambiguity without changing the essence of the problem.

To apply the Lax-Milgram theorem,

The Lax-Milgram theorem requires:

- The bilinear form $a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx$.
- The linear functional $L(v) = \int_{\Omega} f v \, dx$.

Boundedness and Coercivity of the Bilinear Form:

- The bilinear form $a(u, v)$ is bounded since $a(u, v) \leq c_1 \int_{\Omega} (|\nabla u| \cdot |\nabla u|) \, dx = c_1 ||u||_{H^1}^2$ by C-S inequality.
- The bilinear form $a(u, v)$ is coercive, same issue here, I do not know.

Boundedness of the Linear Functional: The linear functional $L(v)$ is bounded since $|L(v)| \leq ||f||_{L^2(\Omega)} ||v||_1$ by C-S inequality.

Given these properties, the Lax-Milgram theorem guarantees the existence and uniqueness of the solution $u \in H^1(\Omega)$ to the weak form of this problem.