MATH 521 - Numerical Analysis of Differential Equations

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Assignment 2 : Hilbert Spaces, Weak Form of BVPs

Name:

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Background for Q1 [no points]

You will need the following Poincare-type inequalities:

(1) Let Ω be a connected, bounded, domain and $\Gamma_D \subset \partial\Omega$ measurable with surface area $|\Gamma_D| > 0$, then there exists a constant c_P such that

$$\|v\|_{L^2(\Omega)} \le c_P \|\nabla v\|_{L^2(\Omega)} \qquad \forall v \in H^1_{\Gamma_D}(\Omega).$$

(2) Let Ω be a simply connected domain then there exists a constant c_P such that

$$\|v\|_{L^2(\Omega)} \le c_P \|\nabla v\|_{L^2(\Omega)} \qquad \forall v \in H^1(\Omega) \text{ satisfying } (v)_\Omega = 0,$$

where $(v)_\Omega := |\Omega|^{-1} \int_\Omega v \, dx$.

Another way (seemingly stronger but equivalent) to state these results is the following:

(1') Let Ω be a connected, bounded, domain and $\Gamma_D \subset \partial\Omega$ measurable with surface area $|\Gamma_D| > 0$, then there exists a constant c_P such that

$$\|v\|_{L^2(\Omega)} \le c_P \Big(\|v\|^2_{L^2(\Gamma_D)} + \|\nabla v\|^2_{L^2(\Omega)} \Big)^{1/2} \qquad \forall v \in H^1(\Omega).$$

(2) Let Ω be a simply connected domain then there exists a constant c_P such that

$$\|v\|_{L^2(\Omega)} \le c_P \Big(\big|(v)_\Omega\big|^2 + \|\nabla v\|^2_{L^2(\Omega)} \Big)^{1/2} \qquad \forall v \in H^1(\Omega).$$

Note: if you want to prove these results, it is not too difficult. They can both be proven with the same argument. What you need is the compactness of the embedding $L^2(\Omega) \subset H^1(\Omega)$: If $(u_n)_{n\in\mathbb{N}}$ is a sequence that is bounded in $H^1(\Omega)$ then there exists $u \in L^2(\Omega)$ and a subsequence such that $u_{n_j} \to u$ strongly in L^2.

Q1a: [8]

Let Ω be a connected, bounded domain. Are the following spaces H Hilbert spaces when equipped with their stated inner products? If not, then explain what property is missing (no need to justify it at length)

(i) $H = \{v \in C^1(\bar{\Omega}) : v, \nabla v \in L^2(\Omega)\}$ $(u,v)_H = \int_\Omega uv + \nabla u \cdot \nabla v \, dx$

(ii) $H = \{v \in L^2(\Omega) : \text{ weakly differentiable}, \nabla v \in L^2(\Omega)\}$, $(u,v)_H = \int_\Omega \nabla u \cdot \nabla v dx$

(iii) $H = \{v \in L^2(\Omega) : \text{ weakly differentiable}, \nabla v \in L^2(\Omega)\}$,
$(u,v)_H = \int_\Omega uv + \nabla u \cdot \nabla v dx$

(iv) $H = \{v \in L^2(\Omega) : \text{ weakly differentiable}, \nabla v \in L^2(\Omega), v(0) = 0\}$,
$(u,v)_H = \int_\Omega \nabla u \cdot \nabla v dx.$

Solution Q1a

(i) no, it is not complete, it's completion is H^1

(ii) no, $(u,v)_H$ is not an inner product, since $(1,1)_H = 0$ but $1 \neq 0$.

(iii) yes, this is $H^1(\Omega)$.

(iv) yes if $d = 1$, not if $d > 1$ since in that case H isn't even well-defined since point-values are not defined for H^1-functions

Q1b [5+7]

Let Ω be a connected, bounded domain, and $\Gamma_D \subset \partial\Omega$ measurable with surface area $|\Gamma_D| > 0$. Are the following spaces H Hilbert spaces when equipped with their stated inner products? Now please justify your answer in full detail. (except you don't need to show that $(u,v)_H$ is symmetric and bilinear)

(v) $H = \{v \in H^1(\Omega) : (v)_\Omega = 0\}$, $(u,v)_H = \int_\Omega \nabla u \cdot \nabla v dx.$

(vi) $H = H^1(\Omega)$, $(u,v)_H = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\Gamma_D} uv \, dx.$

Solution Q1a

Both are Hilbert spaces. We need to show that $(u,v)_H$ is an inner product (positive) and that H is complete under that inner product. This is equivalent to showing that $\|\cdot\|_H$ is equivalent to $\|\cdot\|_{H_0^1}$.

(v) The upper bound is trivial:

$$\|u\|_H^2 = |u|_1^2 \leq \|u\|_1^2$$

The lower bound is Poincare's inequality (2): since $(u)_\Omega = 0$,

$$\|u\|_1^2 = \|u\|_0^1 + |u|_1^2 \leq (C_p^2+1)|u|_1^2.$$

(vi) Upper bound from the trace inequality

$$\|u\|_H^2 = |u|_1^2 + \|u\|_{L^2(\Gamma_D)}^2 \leq |u|_1^2 + C_{\mathrm{tr}}^2\|u\|_1^2 \leq (1+C_{\mathrm{tr}}^2)\|u\|_1^2.$$

Lower bound from Poincare (1'):

$$\|u\|_1^2 = \|u\|_0^2 + |u|_1^2 \leq c_P^2\big(\|u\|_{L^2(\Gamma_D)}^2 + |u|_1^2\big)^{1/2} + |u|_1^2 \leq (1+c_P^2)\|u\|_H^2.$$

Background to Q2 [no points]

Before starting on Q2, review integration by parts in $\Omega \subset \mathbb{R}^d$. We introduced this as

$$\int_\Omega \partial_i u \cdot v dx = -\int_\Omega u \partial_i v dx + \int_{\partial\Omega} \nu_i u v\, dx.$$

From this expression, please derive the following equivalent formulation: if $g : \Omega \to \mathbb{R}^d, v : \Omega \to \mathbb{R}$ (both weakly differentiable) then

$$\int_\Omega \mathrm{div} g\, v\, dx = -\int_\Omega g \cdot \nabla v\, dx + \int_{\partial\Omega} (\nu \cdot g)\, v\, dx.$$

Q2: Weak forms of 2nd order BVPs [10+10+10]

For the following three problems, derive the weak form and then use the Lax-Milgram theorem to show that the weak forms have unique solutions. Throughout this question,, Ω is a connected domain in $\mathbb{R}^d$, $d > 1$, $p, q \in C(\bar{\Omega})$ with $c_0 \leq p, q \leq c_1$, $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$.

(i) Neumann problem

$$\begin{aligned} -\mathrm{div}\big(p \nabla u\big) + qu &= f, \quad \text{in } \Omega, \\ p\nu \cdot \nabla u &= g, \quad \text{on } \partial\Omega. \end{aligned}$$

(ii) Robin problem

$$\begin{aligned} -\mathrm{div}\big(p \nabla u\big) &= f, \quad \text{in } \Omega, \\ p\nu \cdot \nabla u + u &= g, \quad \text{on } \partial\Omega. \end{aligned}$$

(iii) The classical Neumann problem: in addition to all previous assumptions also assume that $(f)_\Omega = 0$.

$$\begin{aligned} -\Delta u &= f, \quad \text{in } \Omega, \\ \nu \cdot \nabla u &= 0, \quad \text{on } \partial\Omega. \end{aligned}$$

HINT: for (iii) you need to add an additional condition that uniquely determines the solution but doesn't change the problem.

Solution Q2(i)

The weak form for all of these is

$$a(u,v) = \ell(v) \qquad \forall v \in V,$$

where V is a Hilbert space to be defined, and a, ℓ are bounded (and coercive) (bi-)linear forms to be defined on that space.

$$\begin{aligned} V &= H^1(\Omega), \\ a(u,v) &= \int_\Omega p\nabla u \cdot \nabla v + quv\,dx \\ \ell(v) &= \int_\Omega fvdx + \int_{\partial\Omega} gv\,dS. \end{aligned}$$

- V is a hilbert space (cf class) when equipped with the norm $\|u\|_1^2 = \|u\|_0^2 + \|\nabla u\|_0^2$.
- $a(u,u) \geq c_0 \|u\|_1^2$ from assumptions on $p, q \geq c_0$, i.e. a is coercive
- $a(u,v) \leq c_1 \|u\|_1 \|v\|_1$ from assumptions on $p, q \leq c_1$ i.e. a is bounded
- $\ell(v) \leq \|f\|_{L^2(\Omega)} + C_{\mathrm{tr}} \|g\|_{L^2(\partial\Omega)}$ from the trace theorem, i.e. ℓ is bounded.

Lax-Milgram implies that the problem has a unique solution.

Solution Q2(ii)

This is a little more interesting, so we first have to perform a simple calculation:

$$\begin{aligned}\int_\Omega fv\,dx &= \int_\Omega (-\mathrm{div} p\nabla u) v\,dx \\ &= \int_\Omega p\nabla u\cdot\nabla v\,dx - \int_{\partial\Omega} p\nu\cdot\nabla u\, v\,dx \\ &= \int_\Omega p\nabla u\cdot\nabla v\,dx - \int_{\partial\Omega}\Big(gv - uv\Big)\,dx.\end{aligned}$$

This leads to the following weak form:

$$\begin{aligned}V &= H^1(\Omega), \\ a(u,v) &= \int_\Omega p\nabla u\cdot\nabla v\,dx + \int_{\partial\Omega} uv\,dS \\ \ell(v) &= \int_\Omega fvdx + \int_{\partial\Omega} gv\,dS.\end{aligned}$$

- V is a hilbert space (cf class) when equipped with the norm $\|u\|_1^2 = \|u\|_0^2 + \|\nabla u\|_0^2$.
- $a(u,u) \geq c_0|u|_1^2 + \|u\|_{L^2(\partial\Omega)}^2 \geq c_0'\|u\|_1^2$ follows from Poincare (1'); a is coercive.
- $a(u,v) \leq c_1'\|u\|_1\|v\|_1$ follows from $a \leq c_1$ and the trace inequality; a is bounded.
- $\ell(v)$ bounded is the same argument as in (i).

Lax-Milgram implies that the problem has a unique solution.

Solution Q2(iii)

Naively, the weak form becomes

$$
\begin{aligned}
V &= H^1(\Omega), \\
a(u,v) &= \int_\Omega \nabla u \cdot \nabla v \, dx, \\
\ell(v) &= \int_\Omega f v \, dx.
\end{aligned}
$$

The problem is that a is not coercive on V. This is due to the fact that this is a pure Neumann problem, i.e. $1 \in V$ but $a(1,1) = 0$. But rescues us is that $\int f \, dx = 0$, which means that $\ell(1) = 0$ as well. In other words, shifting a possible solution u by a constant $u \to u + c$ we again get a solution. It therefore makes sense that the problem cannot have a unique solution as stated above.

The canonical (but not the only) solution to the problem is to simply pick one solution, e.g. the one for which $(u)_\Omega = 0$. This leads to

$$
\begin{aligned}
V &= \{v \in H^1(\Omega) : (v)_\Omega = 0\}, \\
a(u,v) &= \int_\Omega \nabla u \cdot \nabla v \, dx, \\
\ell(v) &= \int_\Omega f v \, dx.
\end{aligned}
$$

- V is Hilbert
- $a(u,u) = |u|_0^2 \geq \frac{1}{2}(1 + c_P^2)^{1/2} \|u\|_1^2$ by Poincare (2); a is coercive.
- $a(u,v) \leq |u|_0 |v|_0 \leq \|u\|_1 \|v\|_1$; a is bounded.
- $\ell(v) \leq \|f\|_0 \|u\|_0 \leq \|f\|_0 \|u\|_1$ so ℓ is also bounded.

Lax-Milgram implies that this problem has a unique solution.