Airy’s potential function

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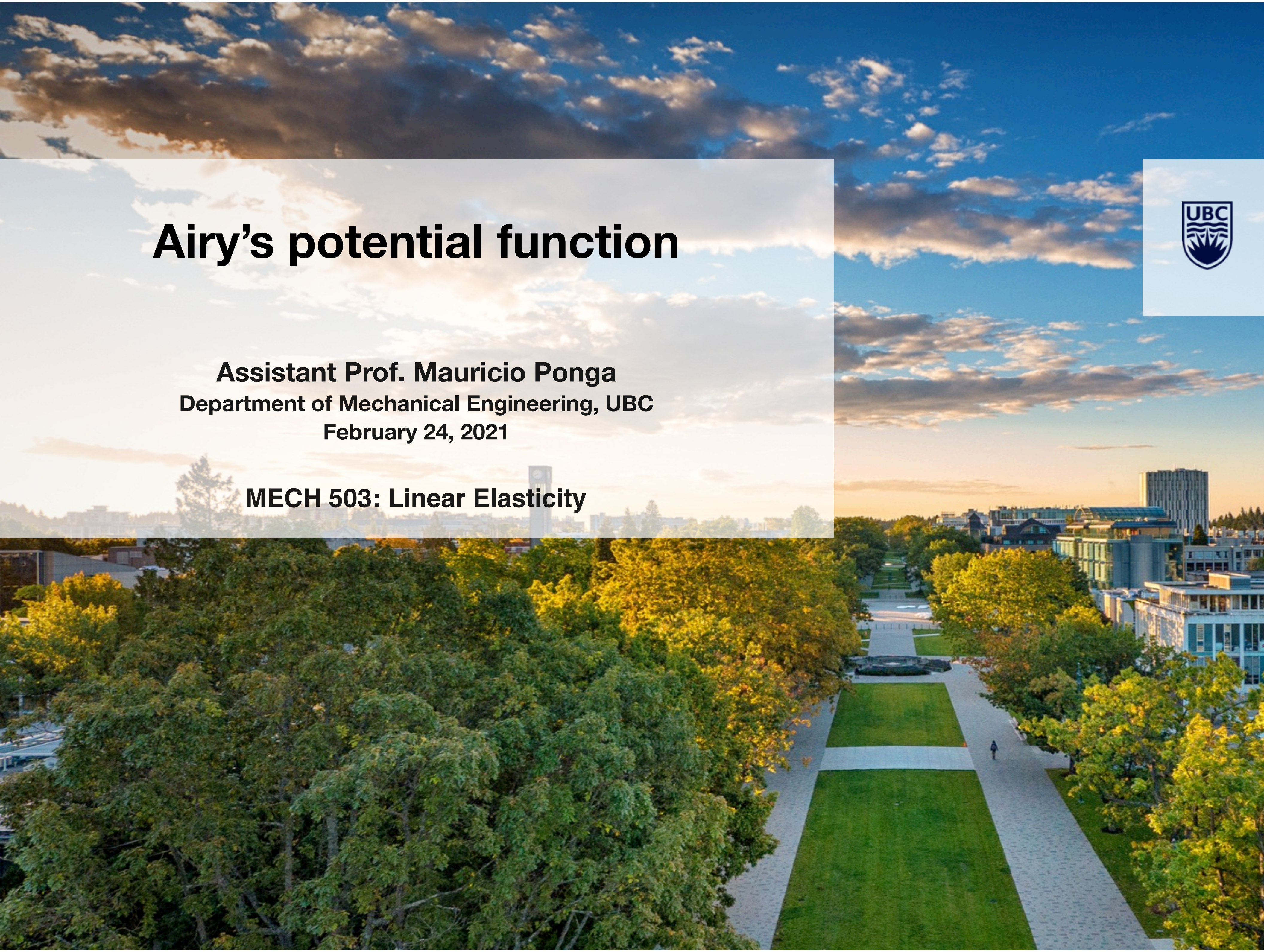
MECH 503: Linear Elasticity



Airy's potential function

$$\begin{cases} \underline{e}_1 = x \\ \underline{e}_2 = y \end{cases}$$

$\sigma_{11} = \sigma_x$ $\quad \sigma_{12} = \sigma_{xy}$ $\quad \varepsilon_{11} = \varepsilon_x$ $\quad \varepsilon_{12} = \varepsilon_{xy}$

$\sigma_{22} = \sigma_y$ $\quad \varepsilon_{22} = \varepsilon_y$ $\quad \gamma_{xy} = 2\varepsilon_{xy}$

$\sigma_{xy} = G\,\gamma_{xy}$

Body forces are given by potential functions

$\underline{b} = (0, \rho_0 g)$ $\quad \Omega = \rho_0 g y$

Compatibility equations (in 2D)

$$\boxed{\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}} \quad (1)$$

Equilibrium Equations

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} = 0 & (2) \\[2ex] \dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \rho_0 g = 0 & (3) \end{cases}$$

subject to some boundary conditions

Hooke's law in plane stress

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

Let $\phi \equiv \phi(x,y)$: Airy's potential function satisfies the following bi-harmonic equation

$$\nabla^4 \phi \equiv \frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = c(\nu)\rho_0\left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right)$$

$$c(\nu) = \begin{cases} \dfrac{1-\nu}{1-2\nu} & \text{Plane strain} \\[2ex] 1/(1-\nu) & \text{Plane stress} \end{cases}$$

The stresses are related to the Airy's potential as follows:

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} - \Omega \quad (A) \qquad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} - \Omega \quad (B) \qquad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \quad (C)$$

Ω : Is a potential function that defines the body forces.

$$\Omega = \rho_0 . g\, y$$

- <u>Proof:</u> If the stresses are defined as in (A), (B), and (C), then we can use equilibrium equations

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} = 0 & (2) \Rightarrow \dfrac{\partial}{\partial x}\left(\dfrac{\partial^2 \phi}{\partial y^2} - \Omega\right) + \dfrac{\partial}{\partial y}\left(-\dfrac{\partial^2 \phi}{\partial x \partial y}\right) = 0 \\[2ex] \dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \rho_0 g = 0 & (3) \Rightarrow \dfrac{\partial}{\partial x}\left(-\dfrac{\partial^2 \phi}{\partial x \partial y}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{\partial^2 \phi}{\partial x^2} - \Omega\right) + \rho_0 g = 0 \end{cases}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2\phi}{\partial y^2}-\Omega\right)+\frac{\partial}{\partial y}\left(-\frac{\partial^2\phi}{\partial x\partial y}\right)=0 \checkmark \Rightarrow \frac{\partial^3\phi}{\partial x\partial y^2}-\underbrace{\cancel{\frac{\partial\Omega}{\partial x}}}_{0}-\frac{\partial^3\phi}{\partial x\partial y^2}=0\Rightarrow 0=0$$

$$\frac{\partial}{\partial x}\left(-\frac{\partial^3\phi}{\partial x\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\partial^2\phi}{\partial x^2}-\Omega\right)+\rho_0 g=0\Rightarrow$$

$$-\frac{\partial^3\phi}{\partial x^2\partial y}+\frac{\partial^3\phi}{\partial x^2\partial y}-\overbrace{\frac{\partial\Omega}{\partial y}}^{\rho_0 g}+\rho_0 g=0$$

Recall $\Omega=\rho_0 g\, y$

$$\frac{\partial\Omega}{\partial y}=\rho_0 g$$

$$0=0 \checkmark$$

- Airy's potential function and stresses satisfy equilibrium

- Show that the bi-harmonic equation satisfies compatibility

$$\boxed{\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y}} \quad (1) \iff \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0$$

Recall Hooke's law

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu \sigma_y) \qquad \varepsilon_y = \frac{1}{E}(\sigma_y - \nu \sigma_x) \qquad \gamma_{xy} = \frac{\overbrace{2(1+\nu)}^{1/G}}{E}\sigma_{xy}$$

Using (1) and replacing the strains by the stresses we find:

$$\frac{1}{\cancel{E}}\frac{\partial^2}{\partial y^2}\Big((\sigma_x - \nu \sigma_y)\Big) + \frac{1}{\cancel{E}}\frac{\partial^2}{\partial x^2}\Big((\sigma_y - \nu \sigma_x)\Big) = \frac{2(1+\nu)}{\cancel{E}}\frac{\partial^2}{\partial x \partial y}\Big(\sigma_{xy}\Big)$$

$$\frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} - \nu\left(\frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_x}{\partial x^2}\right) = 2(1+\nu)\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} \quad (*)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y}\right) = 0 \Rightarrow \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = 0$$

$$\frac{\partial}{\partial y}\left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \rho_o g\right) = 0 \Rightarrow \frac{\partial^2 \sigma_{xy}}{\partial y \partial x} + \frac{\partial^2 \sigma_y}{\partial y^2} + \cancel{\frac{\partial (\rho_o g)}{\partial y}} = 0$$

$$\left\{ \frac{\partial^2 \sigma_x}{\partial x^2} = -\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} \; ; \quad \frac{\partial^2 \sigma_y}{\partial y^2} = -\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} \right.$$

If we add them up

$$\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} = -2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} \quad (**)$$

Let's bring (*) equation f

$$\frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} - \nu\left(\frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_x}{\partial x^2}\right) = 2(1+\nu)\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} \quad (*)$$

$$\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} = -2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} \quad (**)$$

Let's bring (*) equation

$$\frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} \cancel{- \nu \left(\frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_x}{\partial x^2}\right)} + (1+\cancel{\nu})\left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}\right) = 0$$

$$\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} = 0$$

$$\boxed{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0}$$

Compatibility eq in linear elastic isotropic mat.

Replacing σ_x, and σ_y by (A) and (B) Then

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} - \Omega \quad (A) \qquad \sigma_y = \frac{\partial^2\phi}{\partial x^2} - \Omega \quad (B)$$

$$\Omega = \rho_0 g y$$

$$\boxed{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0}$$

$$\frac{\partial^2 \Omega}{\partial x^2} = 0, \qquad \frac{\partial \Omega}{\partial y^2} = 0,$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2\phi}{\partial y^2} - \Omega + \frac{\partial^2\phi}{\partial x^2} - \Omega\right) = 0$$

$$\frac{\partial^4\phi}{\partial x^2 \partial y^2} + \frac{\partial^4\phi}{\partial x^4} + \frac{\partial^4\phi}{\partial y^4} + \frac{\partial^4\phi}{\partial x^2 \partial y^2} = 0$$

$$\boxed{\nabla^4\phi \equiv \frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2 \partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0}$$

As stated!

- Example of solutions of the equilibrium equations using Airy's method:

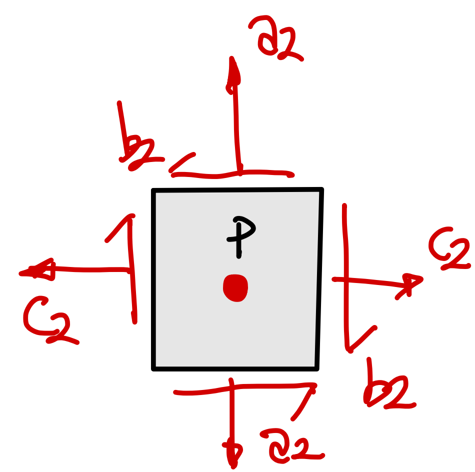
$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \qquad \phi = \phi(x,y)$$

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} - \Omega \qquad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} - \Omega \qquad \sigma_{xy} = -\frac{\partial \phi}{\partial x \partial y}$$

$$\phi_2(x,y) = \frac{a_2}{2}x^2 + b_2\, xy + \frac{c_2}{2} y^2 \Rightarrow$$ Satisfies the bi-harmonic equation

$$\frac{\partial^4 \phi_2}{\partial x^4} = 0 \;;\; \frac{\partial^4 \phi_2}{\partial y^4} = 0 \;;\; \frac{\partial^4 \phi_2}{\partial x^2 \partial y^2} = 0 \;\checkmark$$

$$\sigma_x = c_2 \;;\; \sigma_y = a_2 \;;\; \sigma_{xy} = -b_2 \qquad \underline{\underline{\sigma}} = \begin{bmatrix} c_2 & b_2 \\ b_2 & a_2 \end{bmatrix}$$



- Example #2:

$$\phi_3(x,y) = \frac{a_3}{6}x^3 + \frac{b_3}{2}x^2 y + \frac{c_3}{2}\cdot x y^2 + \frac{d_3}{6}y^3$$

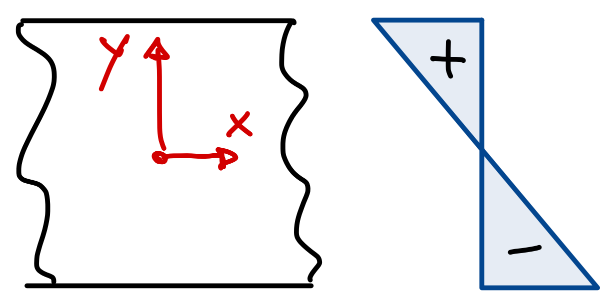
$$\boxed{\nabla^4\phi = \frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0}$$

$$\frac{\partial^4\phi_3}{\partial x^4} = 0 \; ; \quad \frac{\partial^4\phi_3}{\partial y^2} = 0 \; ; \quad \frac{\partial^4\phi}{\partial x^2\partial y^2} = 0 \; \checkmark$$

$$\sigma_x = \frac{\partial^2\phi_3}{\partial y^2} = c_3 x + d_3 y \qquad \sigma_y = \frac{\partial^2\phi_3}{\partial x^2} = a_3 x + b_3 y \; ;$$

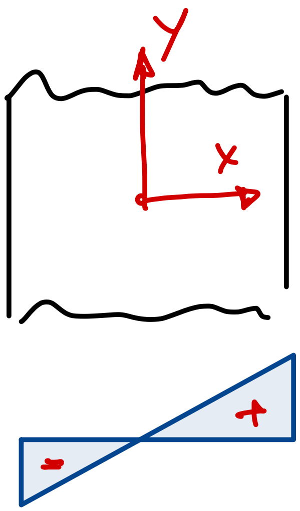
$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -b_3 x - c_3 y.$$

Pure bending



* If $d_3 \neq 0$ $\quad a_3 = b_3 = c_3 = 0$

* If $a_3 \neq 0$ and $b_3 = c_3 = d_3 = 0$



$$\sigma_x = \overset{d_3}{\frac{M \cdot y}{I_{xx}}} \qquad \sigma_y = \overset{a_3}{\frac{M \cdot x}{I_{yy}}}$$

Pure bending

$$M_{(x,z)} = ?$$

?

* IF $b_3 \neq 0$ or $c_3 \neq 0$ $(a_3 = c_3 = d_3 = 0)$

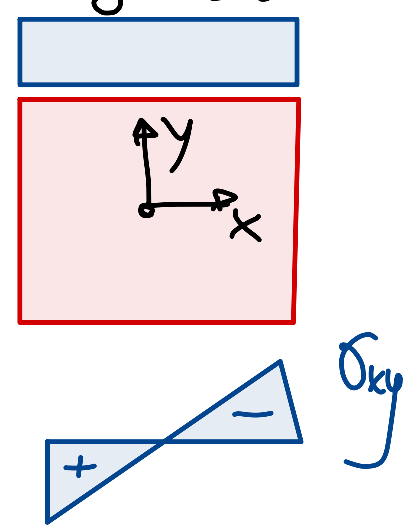
$$\sigma_y = b_3 \cdot y$$

(Linear function in y)

$$\sigma_{xy} = -b_3 x$$

(Linear function in x)

$\sigma_y = b_3 \cdot y_0$



• Example 3 :

$$\phi_4(x,y) = \frac{a_4}{12}x^4 + \frac{b_4}{6}x^3y + \frac{c_4}{2}x^2y^2 + \frac{d_4}{6}xy^3 + \frac{e_4}{12}y^4.$$

$$\nabla^4\phi = \frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0$$

$$\frac{\partial^4\phi}{\partial x^4} = 2a_4 \qquad \frac{\partial^4\phi}{\partial y^4} = 2e_4 \qquad \frac{\partial^4\phi}{\partial x^2\partial y^2} = 4c_4$$

$$\cancel{2}a_4 + \overset{2}{\cancel{4}}c_4 + \cancel{2}e_4 = 0$$

$$\boxed{e_4 = -(a_4 + 2c_4)}$$

$$\phi_4(x,y) = \frac{a_4}{12}x^4 + \frac{b_4}{6}x^3y + \frac{c_4}{2}x^2y^2 + \frac{d_4}{6}xy^3 + \frac{e_4}{12}y^4.$$

$$\sigma_x = \frac{\partial^2\phi_4}{\partial y^2} - \cancelto{0}{\Omega} = c_4x^2 + d_4\,xy - \overbrace{(a_4+2c_4)\,y^2}^{e_4}$$

$$\sigma_y = \frac{\partial^2\phi}{\partial x^2} - \cancelto{0}{\Omega} = a_4x^2 + b_4\,xy + c_4y^2$$

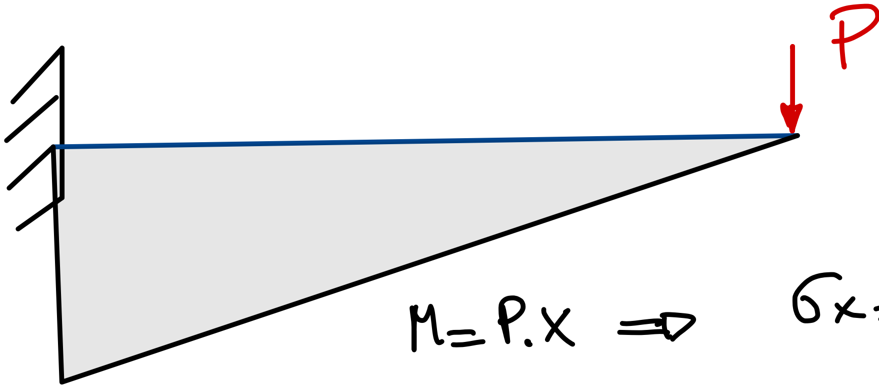
$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -\frac{b_4\,x^2}{2} - 2c_4\,xy - \frac{d_4\,y^2}{2}$$

Coefficients can be arbitrary except for e_4.

If $d_4 \neq 0$ (but others are zero)

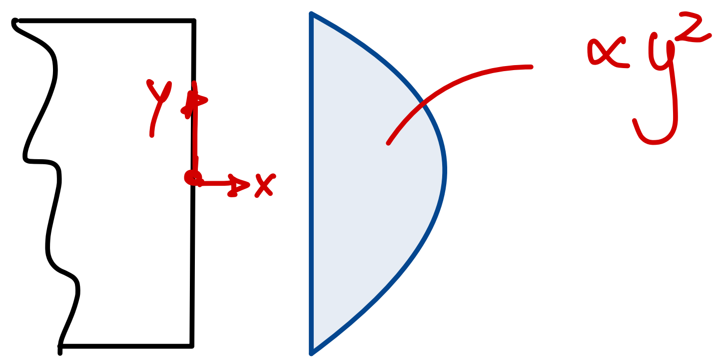
$$\sigma_x = d_4\,xy \quad ; \quad \sigma_{xy} = -\frac{d_4y^2}{2}$$

$$\sigma_x = d_4\, x y \quad ; \quad \sigma_{xy} = -\frac{d_4\, y^2}{2}$$



$$M = P.x \Rightarrow \sigma_x = \frac{M(x).\,y}{I_{xx}} = \frac{P x y}{I_{xx}} = d_4.\,x.\,y$$

$$\sigma_{xy} = -\frac{d_4}{2}.\,y^2$$



- For the following Airy's potential:

$$\phi(x,y) = \frac{a}{2}x^2 + \frac{b}{2}x^2 y + \frac{c}{6}y^3 + \frac{d}{6}x^2 y^3 + \frac{e}{6}y^5$$

Check under what conditions ϕ satisfies the b-h eq.?

Find out the stresses

Identify to what B.Cs the solution correspond to.

$$\boxed{d = -\frac{3}{2}e.}$$

- Generalized solutions using arbitrary polynomial functions

$$\nabla^4\phi = \frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0$$

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} - \Omega \; ; \quad \sigma_y = \frac{\partial^2\phi}{\partial x^2} - \Omega \; ; \quad \sigma_{xy} = -\frac{\partial^2\phi}{\partial x \partial y}.$$

Proposed solution is

$$\phi(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}\, x^m y^n$$

m and n are integers

A_{mn} are constant coeff. To be determine

- If $m+n \leq 1 \Rightarrow$ There will not be resulting stresses.
- If $m+n \leq 3 \Rightarrow$ The function satisfies the biharmonic eq. automatically

$$\frac{\partial^4 \phi}{\partial x^4} = ?$$

$$\frac{\partial \phi}{\partial x} = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} m A_{mn} x^{m-1} y^n$$

$$\frac{\partial^4 \phi}{\partial x^2 \partial y^2}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \sum_{m=2}^{\infty} \sum_{n=0}^{\infty} m(m-1) A_{mn} x^{m-2} y^n$$

$$\frac{\partial^3 \phi}{\partial x^3} = \sum_{m=3}^{\infty} \sum_{n=0}^{\infty} m(m-1)(m-2) A_{mn} x^{m-3} y^n$$

$$\boxed{\frac{\partial^4 \phi}{\partial x^4} = \sum_{m=4}^{\infty} \sum_{n=0}^{\infty} m(m-1)(m-2)(m-3) A_{mn} x^{m-4} y^n}$$

$$\frac{\partial^4 \phi}{\partial y^4} = \sum_{m=0}^{\infty} \sum_{n=4}^{\infty} n(n-1)(n-2)(n-3) A_{mn} x^m y^{n-4}$$

$$\frac{\partial^4 \phi}{\partial x^2 \partial y^2} = \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} m(m-1)n(n-1) A_{mn} x^{m-2} y^{n-2}$$

$$\sum_{m=4}^{\infty}\sum_{n=0}^{\infty} m(m-1)(m-2)(m-3)\, A_{mn}\, x^{m-4} y^{n} + \cdots$$

$$2\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} m(m-1)\, n(n-1)\, A_{mn}\, x^{m-2} y^{n-2} + \cdots$$

$$\sum_{m=0}^{\infty}\sum_{n=4}^{\infty} n(n-1)(n-2)(n-3)\, A_{mn}\, x^{m} y^{n-4} = 0$$

This is the B.H equation

Rearranging and grouping the terms:

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\Big[(m+2)(m+1)m(m-1)\, A_{m+2,n-2} + 2m(m-1)\, n(n-1)\, A_{mn} + \cdots + (n+2)(n+1)\, n(n-1)\, A_{m-2,n+2}\Big]\, x^{m-2} y^{n-2} = 0$$

$$\boxed{\left[(m+2)(m+1)m(m-1)A_{m+2,n-2} + 2m(m-1)n(n-1)A_{mn} + (n+2)(n+1)n(n-1)A_{m-2,n+2}\right] = 0}$$

This is a characteristic equation in m, n that has to be satisfied for each pair (m, n)

- Example:

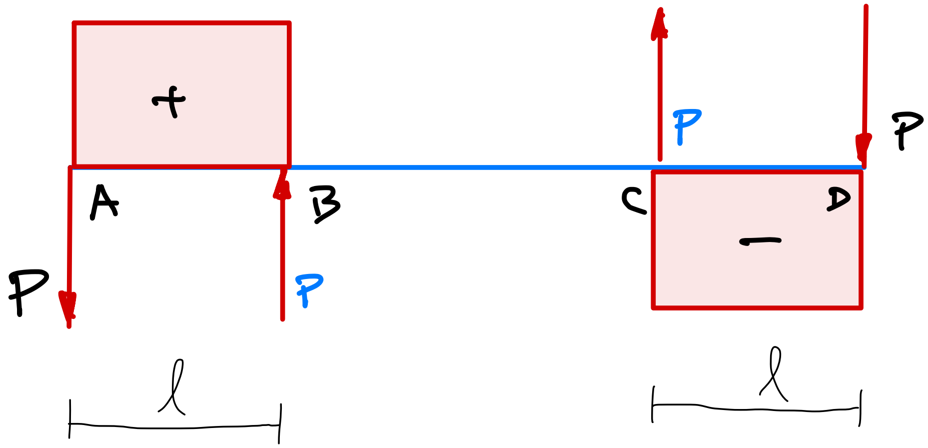
$$\phi(x,y) = A_{40}x^4 + A_{22}x^2y^2 + A_{04}\cdot y^4$$

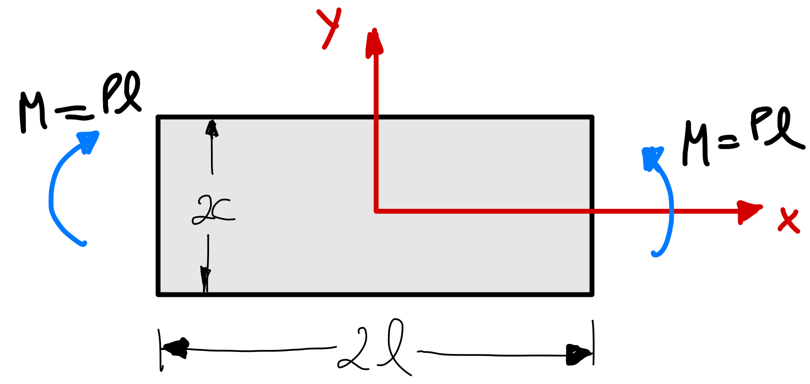
$$\boxed{3A_{40} + A_{22} + 3A_{04} = 0}$$

Characteristic equation $\Rightarrow$ B.H.

- Examples: Pure bending of a Beam

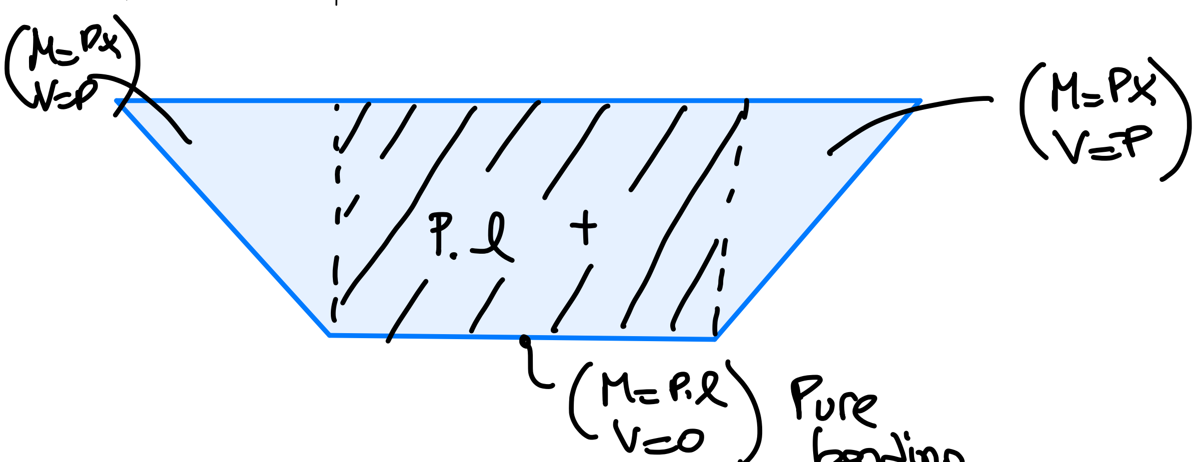
Shear diagram





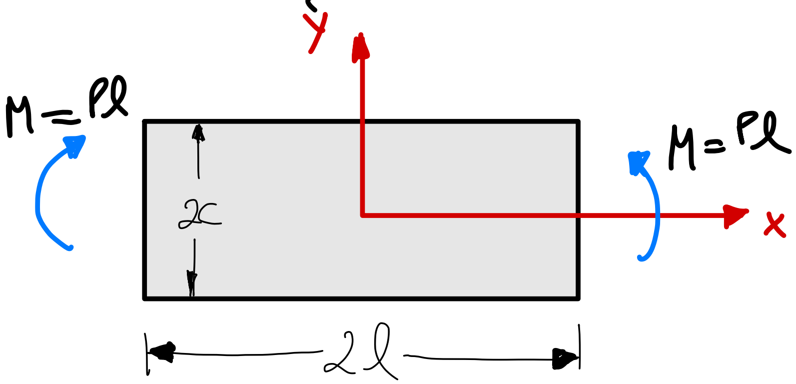
Pure bending condition

Bending Moment



* Boundary conditions for pure bending

$$\sigma_y = (x, \pm c) = 0; \quad \tau_{xy}(x, \pm c) = \tau_{xy}(\pm l, y) = 0$$



Pure bending condition

$$\begin{cases} \sum F_x = 0 & (1) \\ \sum M = 0 \Rightarrow M_{INT} = M_{EXT} = -M & (2) \end{cases}$$

$$dA = b.\,dy \qquad \underline{\underline{b=1}} \Rightarrow dA = dy$$

Using (1)

$$\int_{-c}^{+c} \sigma_x(\pm l, y)\,.\,dy = 0 \qquad (3)$$

$$\boxed{\sigma_x(y) = -\frac{M.y}{I} = -\frac{3}{2}\frac{My}{c^3}}$$

Using (2)

$$\int_{-c}^{c} \sigma_x(\pm l, y)\,.\,y\,.\,dy = -M \qquad (4)$$

$$I = \int_A y^2 dA$$

$$I = \frac{b(2c)^3}{12} = \frac{2}{3}c^3$$

$$\phi(x,y) = \phi(y) = A_{03}\,y^3$$

(Trivially satisfies B.H eq.)

$$\boxed{\sigma_x = \frac{\partial^2 \phi}{\partial y^2} - \Omega = 6A_{03}.\,y}$$

$$\boxed{\sigma_y = \sigma_{xy} = 0}$$

Using (1) $\displaystyle\int_{-c}^{+c} \sigma_x(\pm l, y)\cdot dy = 0$ (3) $\qquad \sigma_x = 6A_{03}\cdot y$

$$6A_{03}\int_{-c}^{+c} y\,dy = \frac{6A_{03}}{2}\, y^2 \Big|_{-c}^{+c} = 3A_{03}\left(c^2 - c^2\right) = 0$$ ✓

Using (2) $\displaystyle 6A_{03}\int_{-c}^{c} y^2\cdot dy = -M$ (4) $\qquad \sigma_x = 6A_{03}\cdot y$

$$\frac{6A_{03}}{3}\, y^3 \Big|_{-c}^{+c} = 2A_{03}\cdot\big[\underbrace{c^3 - (-c^3)}_{2c^3}\big] = 4A_{03}\cdot c^3 = -M$$

$$\boxed{A_{03} = -\frac{M}{4c^3}} \Rightarrow \sigma_x = \frac{-6\cdot M}{4c^3}\, y \Rightarrow \boxed{\sigma_x = \frac{-3My}{2c^3}}$$

Using elasticity (As MoM)

• Displacements: Using Hooke's Law

$$\sigma_x = -\frac{3}{2}\frac{M}{c^3}\cdot y$$

$$\sigma_y = 0$$

$$\frac{\partial u}{\partial x} = \underbrace{\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\cancel{\sigma_y}\right)}\qquad \frac{\partial v}{\partial y} = \underbrace{\varepsilon_y = \frac{1}{E}\left(\cancel{\sigma_y} - \nu\sigma_x\right)};$$

$$(u, v) \Rightarrow \quad \frac{\partial u}{\partial x} = \frac{1}{E}\left(-\frac{3}{2}\frac{M}{c^3}\cdot y\right) \Rightarrow u(x,y) = \int -\frac{3M}{2Ec^3}\cdot y \, dx$$

$$\boxed{u(x,y) = -\frac{3M}{2Ec^3}xy + f(y)}$$

$$\frac{\partial v}{\partial y} = -\frac{\nu}{E}\sigma_x = \frac{3}{2}\frac{M\nu}{Ec^3}y \Rightarrow v(x,y) = \int \frac{3}{2}\frac{M\nu}{Ec^3}\cdot y \, dx =$$

$$\boxed{v(x,y) = \frac{3}{2}\frac{M\nu}{Ec^3}\cdot\frac{y^2}{2} + g(x)} \quad \Rightarrow \quad \boxed{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0}$$

$f(y) = -w_0 y + u_0$

w_0, u_0, v_0 to be determined

$g(x) = \frac{3M}{4EC^3} \cdot x^2 + w_0 x + v_0$

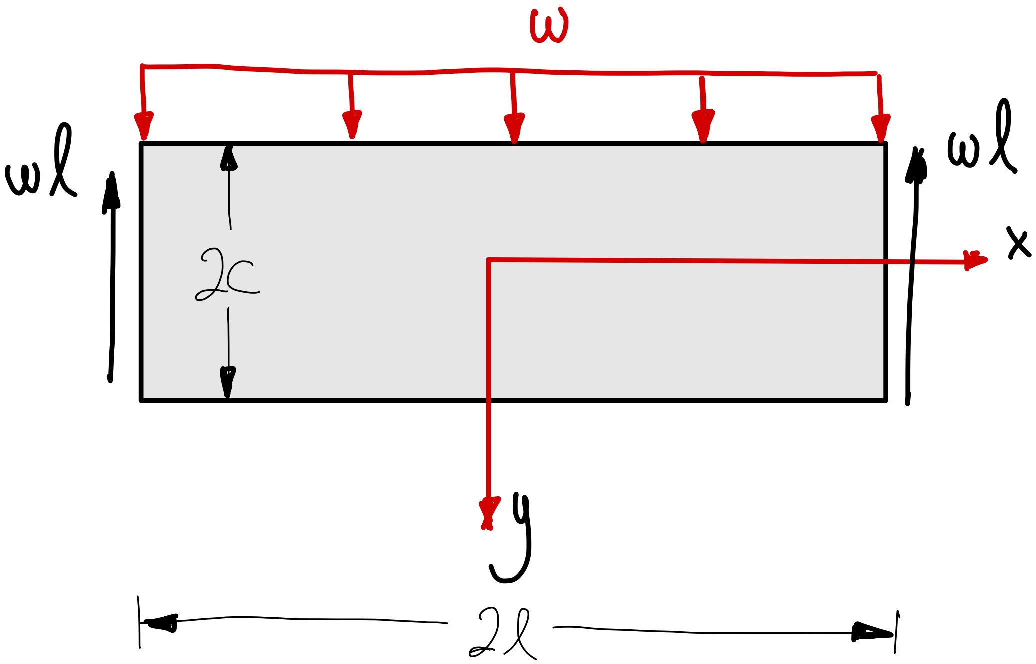
$w_0 = u_0 = 0$; $v_0 = -\frac{3M\ell^2}{4EC^3}$

$$\boxed{g(x) = \frac{3M}{4EC^3}\left(x^2 - \ell^2\right)}$$

In summary

$$\boxed{u = -\frac{Mxy}{EI} \; ; \quad v = \frac{M}{2EI}\left[\nu y^2 + x^2 - \ell^2\right]}$$

• Bending of a beam by uniform transverse loading:



Boundary Conditions:

$$\sigma_{xy}(x, \pm c) = 0 \quad (1)$$

$$\sigma_y(x, c) = 0 \quad (2)$$

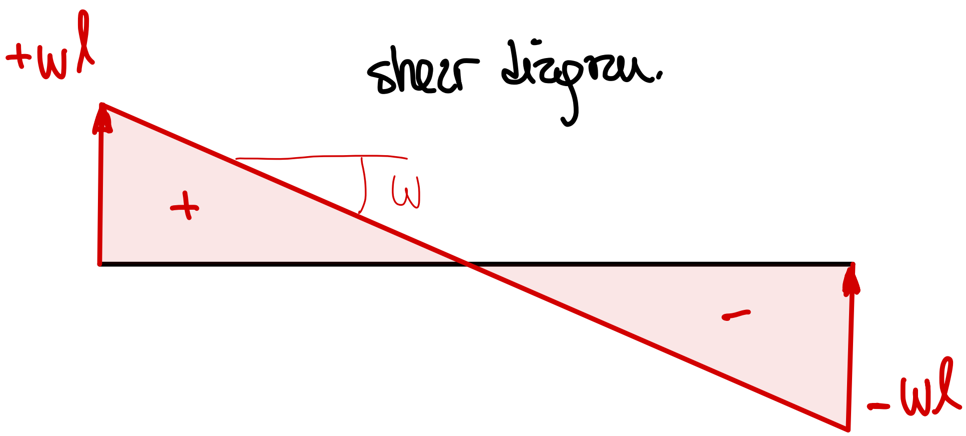
$$\sigma_y(x, -c) = -w \quad (3)$$

$$\int_{-c}^{c} \sigma_x(\pm l, y)\, dy = 0 \quad (4)$$

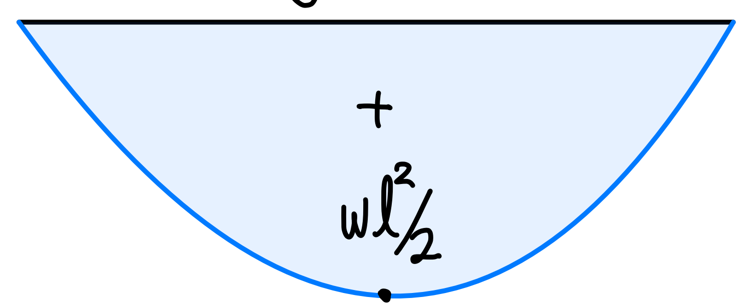
$$\int_{-c}^{c} \sigma_x(\pm l, y)\, y\, dy = 0 \quad (5)$$

$$\int_{-c}^{c} \sigma_{xy}(\pm l, y)\, dy = \pm wl \quad (6)$$

shear diagram.



Bending moment



- Proposed solution: $\phi(x,y) = A_{20}x^2 + A_{21}x^2y + A_{03}y^3 + A_{23}x^2y^3 - \underbrace{\frac{A_{23}}{5}}_{A_{05}}y^5$

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = 6A_{03}y + 6A_{23}x^2 \cdot y - 4A_{23}y^3 = 6A_{03} + 6A_{23}\left(x^2y - \frac{2}{3}y^3\right)$$

$$\sigma_y = \frac{\partial^2\phi}{\partial x^2} = 2A_{20} + 2A_{21}y + 2A_{23}y^3$$

4 unknown coefficients

$A_{20}, A_{21}, A_{03}, A_{23}$

$$\sigma_{xy} = \frac{-\partial^2\phi}{\partial x\partial y} = -2A_{21}x - 6A_{23}xy^2$$

c^2

$\sigma_{xy}(x, \pm c) = 0$ (1) $\Rightarrow$ $\sigma_{xy}(x, \pm c) = -2A_{21}x - 6A_{23} \cdot x \cdot (\pm c)^2 = 0 \Leftrightarrow$

$-\cancel{2}A_{21} = \cancel{6}_{3}A_{23}c^2 \Rightarrow A_{21} = -\frac{A_{23}c^2}{3}$

$\sigma_y(x, c) = 0$ (2) $\Rightarrow$

$\sigma_y(x, -c) = -w$ (3) $\Rightarrow$

$\sigma_y(x, c) = 0 = 2A_{20} + 2A_{21}c + 2A_{23} \cdot c^3$

$\sigma_y(x, -c) = -w = 2A_{20} - 2A_{21}c - 2A_{23} \cdot c^3$

c^2

$$\sigma_{xy}(x, \pm c) = -2A_{21}x - 6A_{23} \cdot x \cdot (\pm c)^2 = 0$$

$$\sigma_y(x, c) = 0 = 2A_{20} + 2A_{21}c + 2A_{23} \cdot c^3$$

$$\sigma_y(x, -c) = -w = 2A_{20} - 2A_{21}c - 2A_{23} \cdot c^3$$

$$\begin{bmatrix} 0 & -2 & -6c^2 \\ 2 & 2c & 2c^3 \\ 2 & -2c & -2c^3 \end{bmatrix} \begin{bmatrix} A_{20} \\ A_{21} \\ A_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -w \end{bmatrix} \Rightarrow \boxed{\begin{aligned} A_{20} &= -w/4 \\ A_{21} &= \frac{3w}{8c} \\ A_{23} &= \frac{-w}{8c^3} \end{aligned}}$$

$$\int_{-c}^{c} \sigma_x(\pm l, y)\, y\, dy = 0 \Rightarrow \boxed{A_{03} = \frac{w}{8c}\left(\frac{l^2}{c^2} - \frac{2}{5}\right)}$$

- **Solution:**

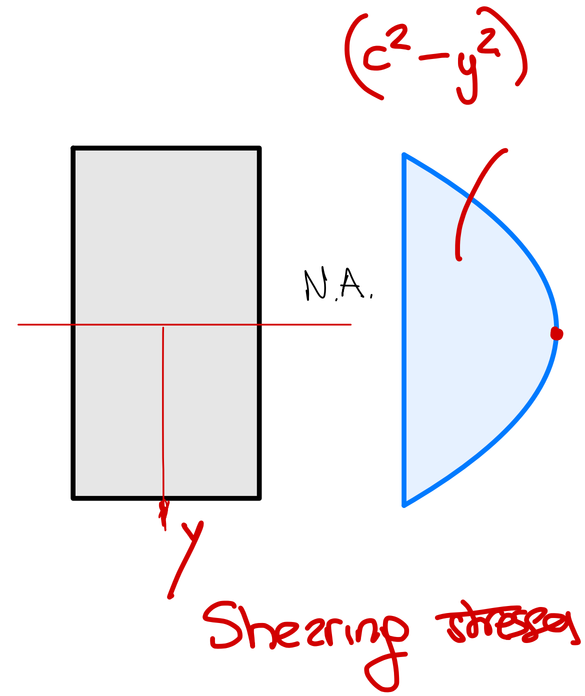
$$I = \frac{2c^3}{3}, \quad b = 1$$

$$\sigma_x = \frac{w}{2I}(l^2 - x^2)y + \frac{w}{I}\left(\frac{y^3}{3} - \frac{c^2 y}{5}\right)$$

$$\sigma_y = -\frac{w}{2I}\left(\frac{y^3}{3} - c^2 y + \frac{2}{3}c^3\right)$$

$$\sigma_{xy} = -\frac{w}{2I}x(c^2 - y^2)$$

Airy's method



- **Mechanics of materials**

$$\sigma_x = \frac{My}{I} = \frac{w}{2I}(l^2 - x^2)y \qquad \sigma_y = 0$$

$$\sigma_{xy} = \frac{VQ}{It} = -\frac{w}{2I}x(c^2 - y^2)$$

• Cartesian coordinate solution using Fourier methods:

Pickett (1944)
Timoshenko (1970)
Goodier
Little (1973)

$$\phi(x,y) = X(x)\, Y(y)$$

X, Y: General functions

Proposed solution

$X(x) = e^{\alpha x}$; $Y = e^{\beta y}$ α, β are coefficients to determine

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

$X'''' = \alpha^4 e^{\alpha x}$; $Y'''' = \beta^4 \cdot e^{\beta y}$, $\frac{\partial^4 \phi}{\partial x^4} = X'''' \cdot Y = \alpha^4 \cdot e^{\alpha x} e^{\beta y}$

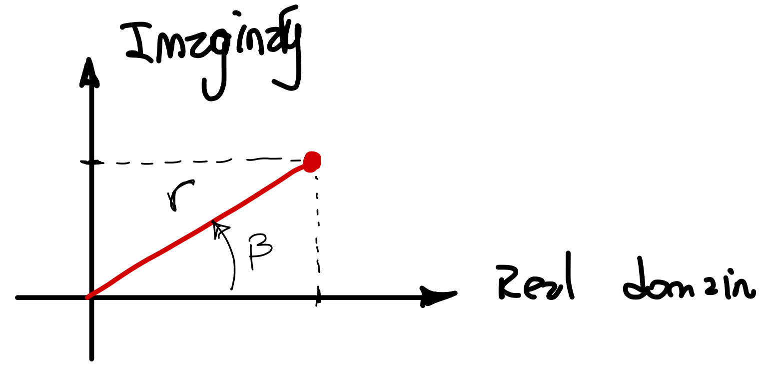
$\frac{\partial^4 \phi}{\partial y^4} = \beta^4 \cdot e^{\alpha x} e^{\beta y}$; $\frac{\partial^2 \phi}{\partial x^2} = X'' Y \Rightarrow \frac{\partial^4 \phi}{\partial x^2 \partial y^2} = X'' Y''$

$X'' = \alpha^2 \cdot e^{\alpha x}$; $Y = \beta^2 e^{\beta y} \Rightarrow \frac{\partial^4 \phi}{\partial x^2 \partial y^2} = \alpha^2 \cdot \beta^2 e^{\alpha x} \cdot e^{\beta x}$

Using the b-h. equation

$$\left(\alpha^4 + 2\alpha^2\beta^2 + \beta^4\right) e^{\alpha x} e^{\beta x} = 0 \iff \alpha^4 + 2\alpha^2\beta^4 + \beta^4 = 0$$

$$\left(\alpha^2 + \beta^2\right)^2 = 0 \Rightarrow \boxed{\alpha = \pm i\beta} \quad \text{where} \quad \boxed{i = \sqrt{-1}}$$



$$e^{i\beta} = \Big(\underbrace{\cos\beta}_{\text{Real part}} + \underbrace{i\sin\beta}_{\text{Imaginary}}\Big)$$

Zero root cases $(\alpha = \beta = 0) \Rightarrow$ General solution

$$\phi_{\beta=0} = C_0 + C_1 x + C_2 x^2 + C_3 x^3 ;$$

$$\phi_{\alpha=0} = C_4 y + C_5 y^2 + C_6 y^3 + C_7 xy + C_8 x^2 y + C_9 x y^2$$

- General case:

$$\phi(x,y) = e^{i\beta x}\left[Ae^{\beta y} + Be^{-\beta y} + Cye^{\beta y} + Dye^{-\beta y}\right] + \ldots$$

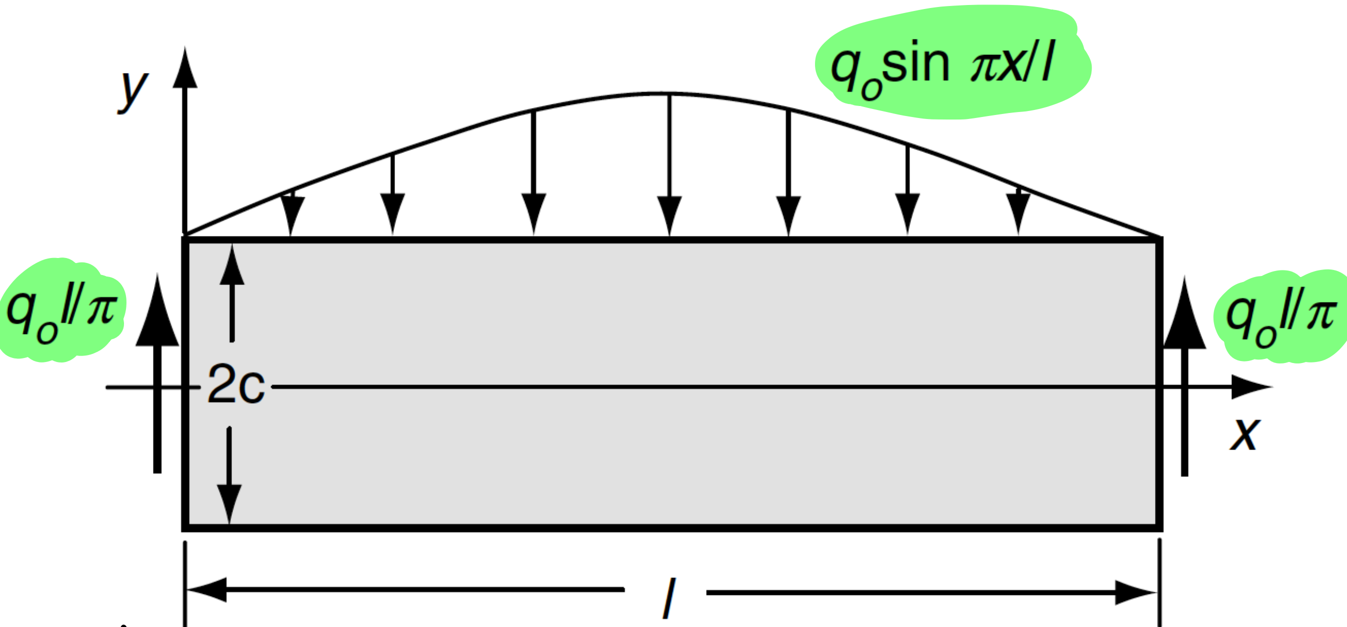
$$e^{-i\beta x}\left[A'e^{\beta y} + B'e^{-\beta y} + C'ye^{\beta y} + D'ye^{-\beta y}\right]$$

$\sinh(x) = -i\sin(x)$; $\cosh(x) = \cos(ix)$.

Realizing the solution is real.

$$\begin{aligned}\phi = {} & \sin\beta x[(A + C\beta y)\sinh\beta y + (B + D\beta y)\cosh\beta y] \\ & + \cos\beta x[(A' + C'\beta y)\sinh\beta y + (B' + D'\beta y)\cosh\beta y] \\ & + \sin\alpha y[(E + G\alpha x)\sinh\alpha x + (F + H\alpha x)\cosh\alpha x] \\ & + \cos\alpha y[(E' + G'\alpha x)\sinh\alpha x + (F' + H'\alpha x)\cosh\alpha x] \\ & + \phi_{\alpha=0} + \phi_{\beta=0}\end{aligned}$$

Beam subject to transverse Sinusoidal loading



- Boundary conditions:

$$\sigma_x(0,y) = \sigma_x(l,y) = 0 \quad (1)$$

$$\sigma_{xy}(x,\pm c) = 0 \quad (2)$$

$$\sigma_y(x,-c) = 0 \quad (3)$$

$$\sigma_y(x,+c) = -q_o \sin\left(\frac{\pi x}{l}\right)$$

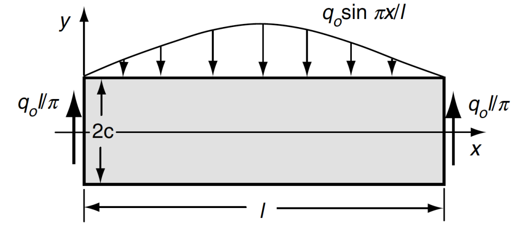
- Reaction forces:

$$\int_{-c}^{+c} \sigma_{xy}(0,y)\,dy = -\frac{q_o l}{\pi}$$

$$\int_{-c}^{c} \sigma_{xy}(l,y)\,dy = -\frac{q_o l}{\pi}.$$

$$\phi = \sin \beta x[(A + C\beta y)\sinh \beta y + (B + D\beta y)\cosh \beta y]$$

Beam subject to transverse Sinusoidal loading



$$\sigma_x = \beta^2 \sin \beta x[A \sinh \beta y + C(\beta y \sinh \beta y + 2 \cosh \beta y) + B \cosh \beta y + D(\beta y \cosh \beta y + 2 \sinh \beta y)]$$

$$\sigma_y = -\beta^2 \sin \beta x[(A + C\beta y) \sinh \beta y + (B + D\beta y) \cosh \beta y]$$

$$\tau_{xy} = -\beta^2 \cos \beta x[A \cosh \beta y + C(\beta y \cosh \beta y + \sinh \beta y) + B \sinh \beta y + D(\beta y \sinh \beta y + \cosh \beta y)]$$

Use σ_{xy} condition $\sigma_{xy}(x, \pm c) = 0$

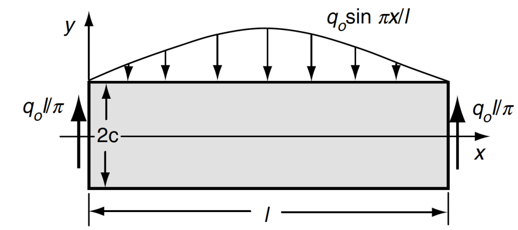
The condition is equivalent to the two following equations

$$\begin{cases} A \cdot \cosh \beta c + D(\beta c \sinh \beta c + \cosh \beta c) = 0 \\ B \sinh \beta c + C(\beta c \cdot \cosh \beta c + \sinh \beta c) = 0 \end{cases}$$

$\boxed{A = -D(\beta c \tanh(\beta c) + 1);}$ $\boxed{B = -C(\beta c \coth(\beta c) + 1)}$

Beam subject to transverse Sinusoidal loading

Equations (3) and (4) can be used to find D and C respectively.



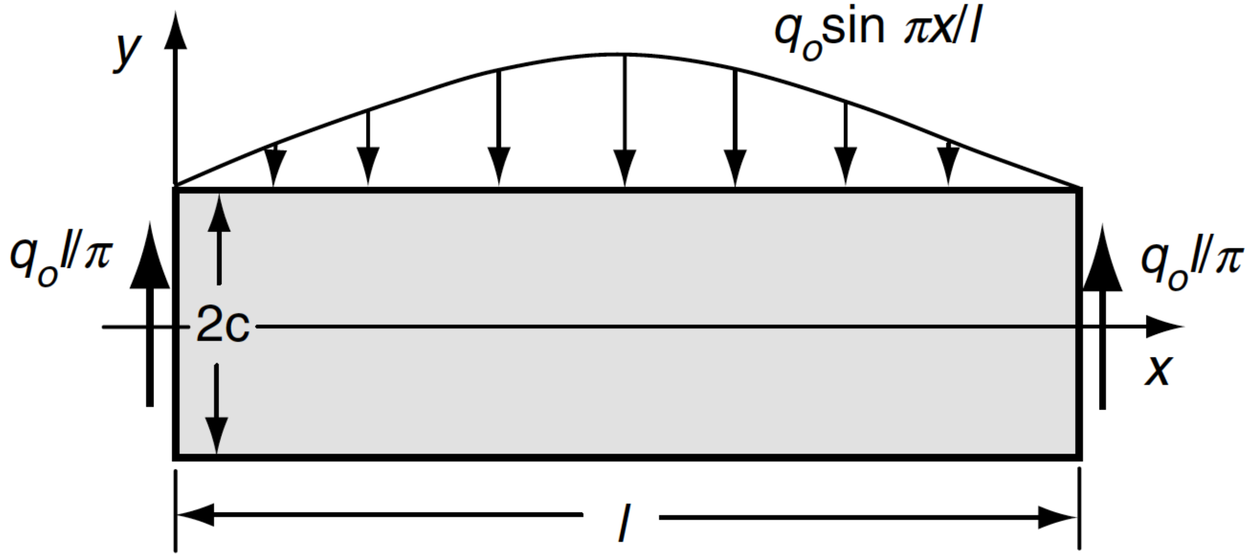
This will lead to the following stresses

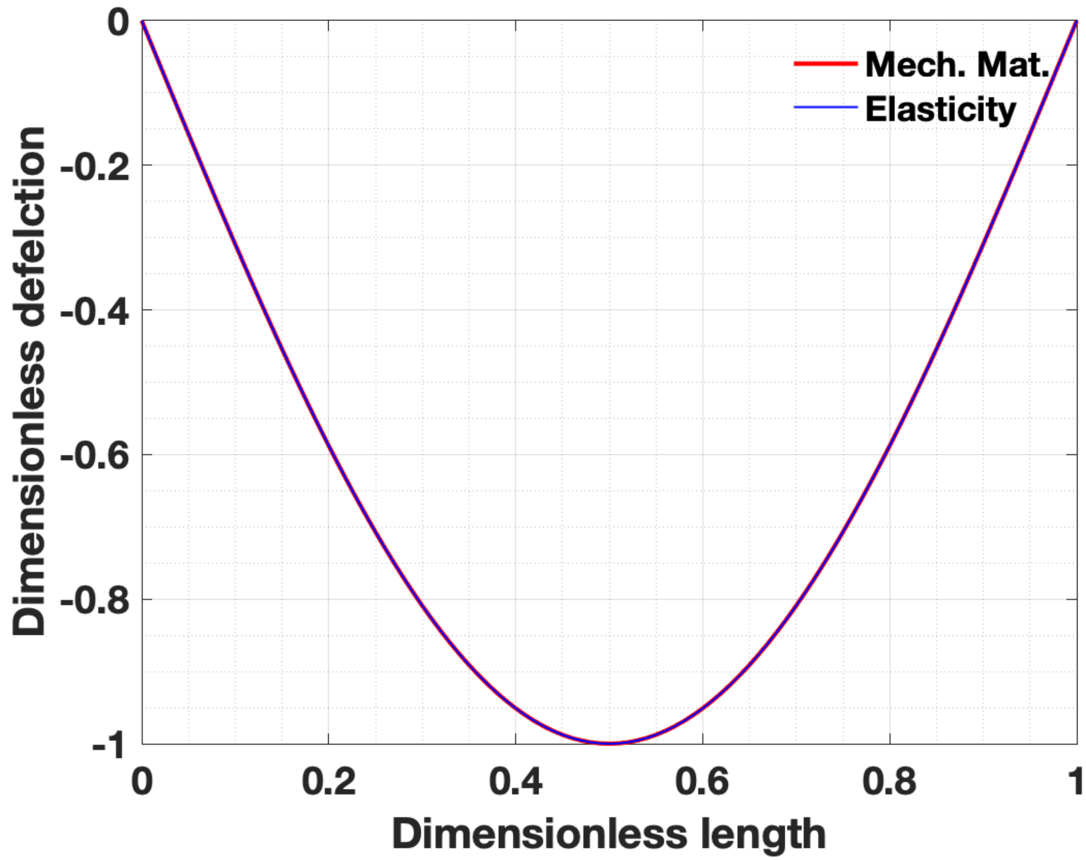
$$\sigma_y = -\beta^2 \sin \beta x \{ D[\beta y \cosh \beta y - (\beta c \tanh \beta c + 1) \sinh \beta y] + C[\beta y \sinh \beta y - (\beta c \coth \beta c + 1) \cosh \beta y] \}$$

The vertical displacement at the mid plane is

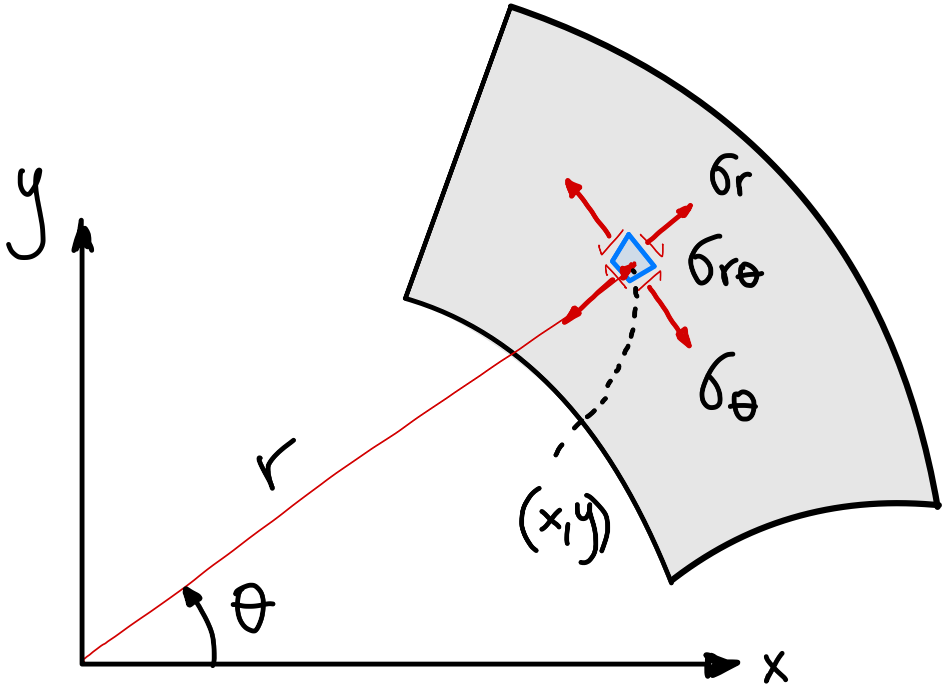
$$v(x,0) = \frac{-3 q_o l^4}{2c^3 \pi^4 E} \sin\left(\frac{\pi x}{l}\right) \left[1 + \frac{1+\nu \pi c}{2l} \tanh\left(\frac{\pi c}{l}\right) \right]$$

Beam subject to transverse Sinusoidal loading





General Solutions in Polar Coordinates - Michel Sol



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad 0 \le \theta \le 2\pi$$

$$\nabla^4 \phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right)^2 \theta = 0$$

σ_r: Radial stress (Normal)

σ_θ: Hoop stress (Normal)

$\sigma_{r\theta}$: Shearing stress.

- **Equilibrium equations in polar coordinates:**

$$\begin{cases} \dfrac{\partial \sigma_r}{\partial r} + \dfrac{1}{r}\dfrac{\partial \sigma_{r\theta}}{\partial \theta} + \dfrac{\sigma_r - \sigma_\theta}{r} = 0 & (1) \\[2ex] \dfrac{1}{r}\dfrac{\partial \sigma_\theta}{\partial \theta} + \dfrac{\partial \sigma_{r\theta}}{\partial r} + 2\dfrac{\sigma_{r\theta}}{r} = 0 & (2) \end{cases}$$

- Relation between stresses and Airy's potential is

$$
\begin{cases}
\sigma_r = \dfrac{1}{r}\dfrac{\partial \phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial r^2}; \\[2ex]
\sigma_\theta = \dfrac{\partial^2 \phi}{\partial r^2}; \\[2ex]
\sigma_{r\theta} = -\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta}\right) = \dfrac{1}{r^2}\dfrac{\partial \phi}{\partial \theta} - \dfrac{1}{r}\dfrac{\partial^2 \phi}{\partial r \partial \theta}
\end{cases}
$$

General Solutions in Polar Coordinates

$$\phi(r,\theta) = f(r)\, e^{b\theta}$$

$f(r)$: Radial function

$e^{b\theta}$: Exponential function that depends on θ

b: Coefficient

$$e^{i\beta} = \cos\beta + i\sin\beta \qquad i = \sqrt{-1}.$$

* Solution has to be subjected to b.c.s.

- Using the bi-harmonic equation:

$$f'''' + \frac{2}{r} f''' + \frac{1-2b^2}{r^2} f'' + \frac{1-2b^2}{r^3} f' + \frac{b^2(4+b^2)}{r^4} f = 0$$

All terms with $e^{b\theta}$ can be cancelled.

General Solutions in Polar Coordinates

Let's introduce a change of variables $r = e^{\xi}$

$$f(r(\xi)) \Rightarrow \frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial \xi}$$

$$\boxed{f'''' - 4f''' + (4+2b^2)f'' - 4b^2 f' + b^2(4+b^2)f = 0}$$

The B-H equation becomes an O.D.E with constant coefficients.

Solution of O.D.E is $\quad f = e^{\lambda \xi}$

General Solutions in Polar Coordinates

Taking derivatives w.r.t ξ we obtain the following characteristic eq.

$$(a^2+b^2)(a^2-4a+4+b^2)=0$$

$a = \pm ib$ where $i=\sqrt{-1}$

Solutions to the characteristic equation

$a = 2 \pm ib$ $n=$ integer

$b = in$

General Solutions in Polar Coordinates

General Solutions in Polar Coordinates

Taking derivatives w.r.t ξ we obtain the following characteristic eq.

$$(a^2+b^2)(a^2-4a+4+b^2)=0$$

$a = \pm ib$ where $i=\sqrt{-1}$

$a = 2 \pm ib$ $n=$ integer

Solutions to the characteristic equation

$b = in$

$$\begin{aligned}
\phi = & \boxed{a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 \log r} \\
& + (a_4 + a_5 \log r + a_6 r^2 + a_7 r^2 \log r)\theta \\
& + \left(a_{11} r + a_{12} r \log r + \frac{a_{13}}{r} + a_{14} r^3 + a_{15} r\theta + a_{16} r\theta \log r\right) \cos\theta \\
& + \left(b_{11} r + b_{12} r \log r + \frac{b_{13}}{r} + b_{14} r^3 + b_{15} r\theta + b_{16} r\theta \log r\right) \sin\theta \\
& + \sum_{n=2}^{\infty} (a_{n1} r^n + a_{n2} r^{2+n} + a_{n3} r^{-n} + a_{n4} r^{2-n}) \cos n\theta \\
& + \sum_{n=2}^{\infty} (b_{n1} r^n + b_{n2} r^{2+n} + b_{n3} r^{-n} + b_{n4} r^{2-n}) \sin n\theta
\end{aligned}$$

Axisymmetric solution

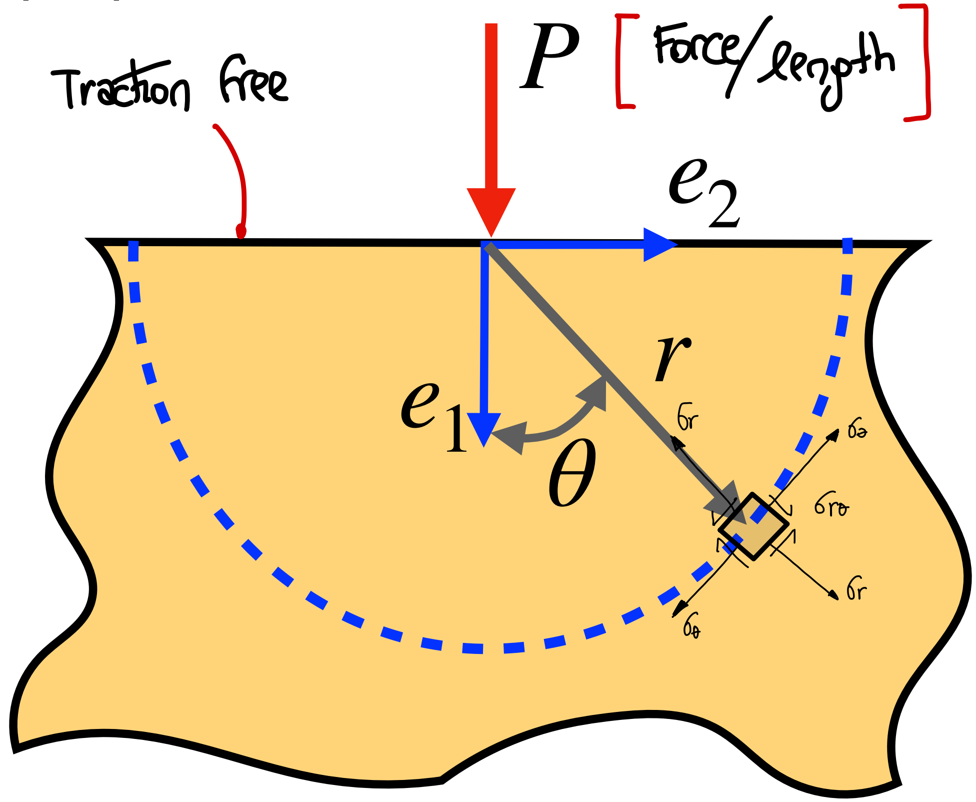
General Solutions in Polar Coordinates - Example

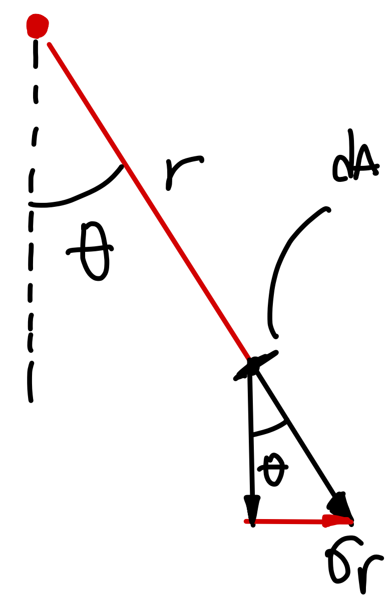
2D Line load acting perpendicular to the surface of an infinite solid

$\sigma_r = ?$

$\sigma_\theta = ?$

$\sigma_{r\theta} = ?$





$$\underbrace{dP}_{\text{vertical component}} = \sigma_r \cdot \cos\theta \cdot \underbrace{dA}_{\underbrace{dl = r\,d\theta}_{\text{length of the arc}}}$$

- Biharmonic equation:

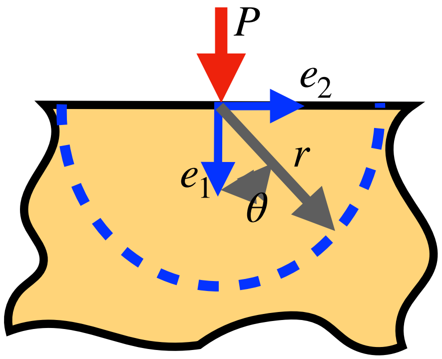
$$\nabla^4 \phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right)^2 \theta = 0$$

- Boundary conditions:

$$\boxed{-P = \int_{-\pi/2}^{+\pi/2} \sigma_r \cdot r \cos\theta \, d\theta}$$

General Solutions in Polar Coordinates - Example

2D Line load acting perpendicular to the surface of an infinite solid



$$\phi(r,\theta) = -\frac{P}{\pi}\, r\theta \,.\, \sin\theta \quad \text{(Proposed solution)}$$

Stress function relations

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2\phi}{\partial\theta^2}\,; \qquad \sigma_\theta = \frac{\partial^2\phi}{\partial r^2}\,; \qquad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\,\partial\theta}$$

It is easy to see that $\sigma_\theta = \sigma_{r\theta} = 0$

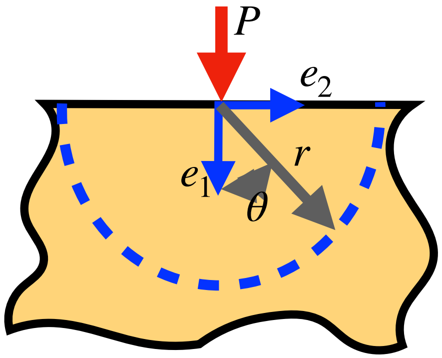
$$\boxed{\sigma_r = -\frac{2P}{\pi}\,\frac{\cos\theta}{r}}$$

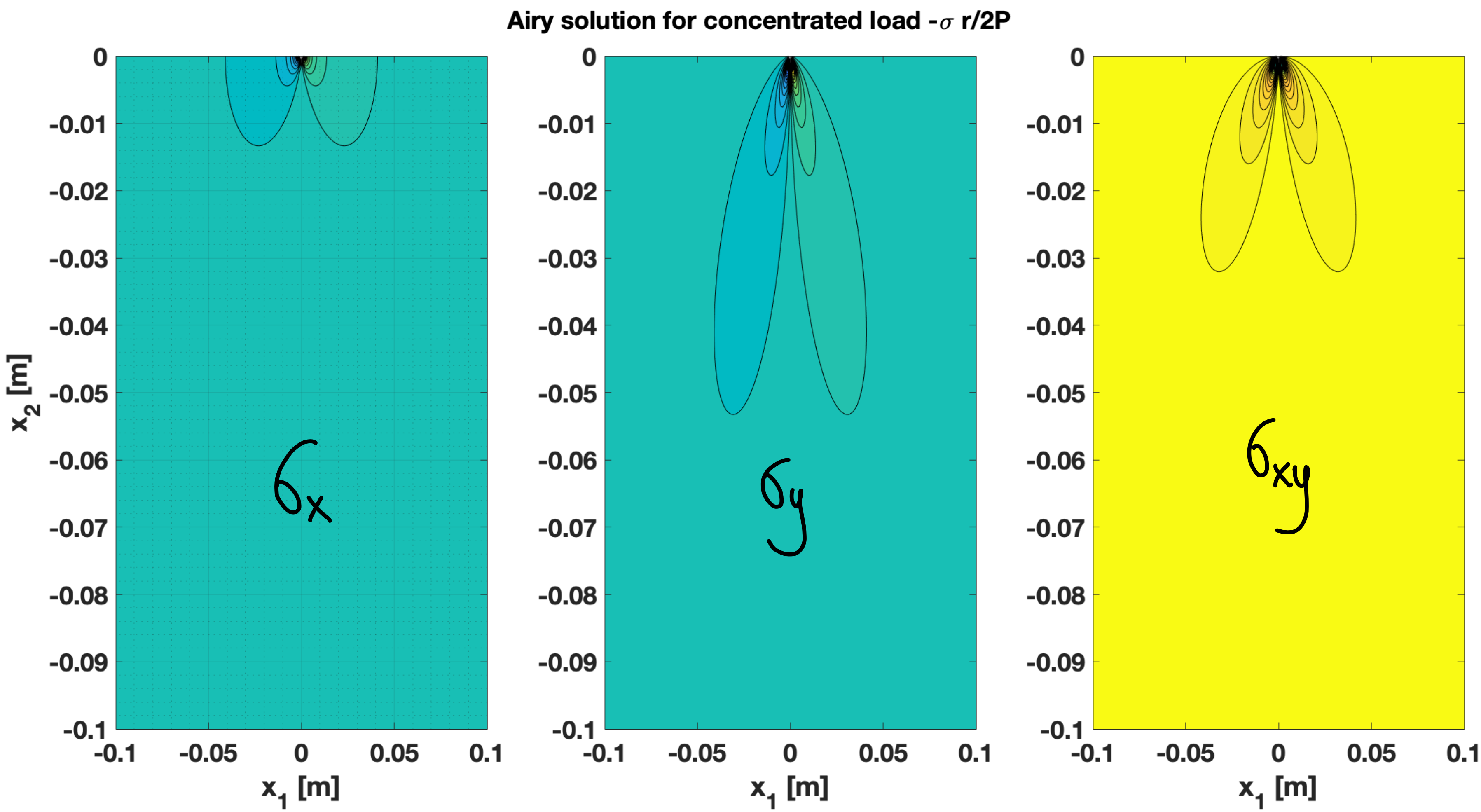
Let's analyze the b.c.s.

$$-P = \int_{-\pi/2}^{+\pi/2} \sigma_r \cdot r\cos\theta \, d\theta = -\frac{2P}{\pi}\int_{-\pi/2}^{+\pi/2} \cancel{\frac{r}{r}}\cos^2\theta \, d\theta = -\frac{2P}{\pi}\left[\frac{1}{2}\left(\theta + \frac{1}{2}\sin\theta\right)\right]_{-\pi/2}^{\pi/2} = -\frac{2P}{\pi}\,\frac{\pi}{2} = -P$$

General Solutions in Polar Coordinates - Example

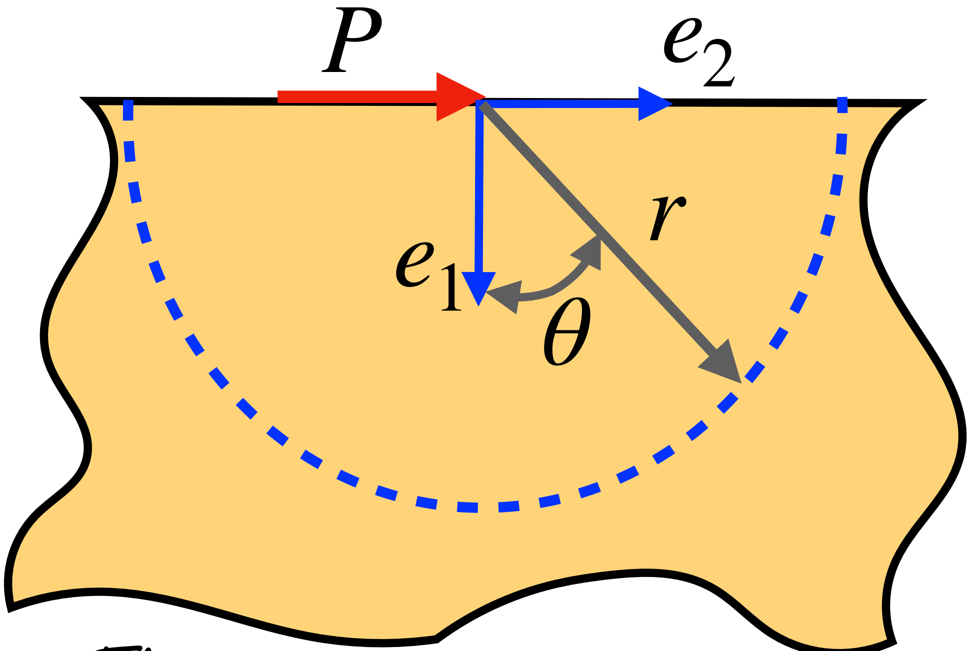
2D Line load acting perpendicular to the surface of an infinite solid





General Solutions in Polar Coordinates - Activity

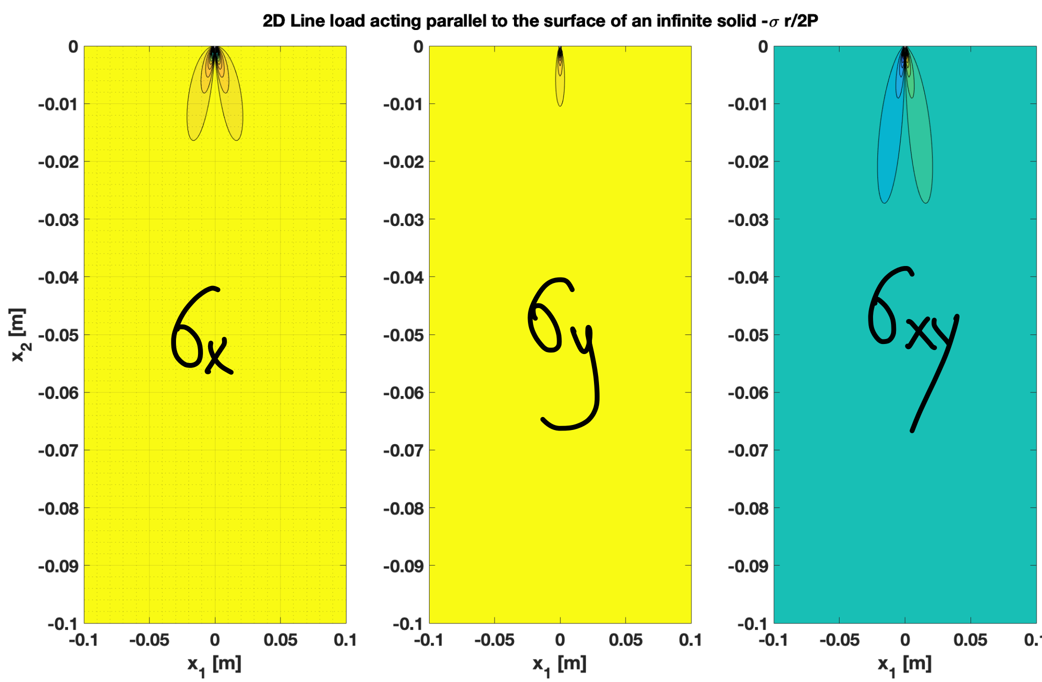
2D Line load acting parallel to the surface of an infinite solid



$$\phi(r,\theta) = -\frac{P}{\pi} r\theta \cos\theta$$

Find the stresses and check that the solution satisfy boundary conditions!

$$-P = \int_{-\pi/2}^{\pi/2} \sigma_r \cdot r\sin\theta \cdot d\theta$$



Axisymmetric solution

$$\phi(r) = a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 \log r$$

Do this in matlab

Using the relations

$$\begin{cases} \sigma_r = \dfrac{1}{r}\cdot\dfrac{\partial \phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2} \\ \sigma_\theta = \dfrac{\partial^2 \phi}{\partial r^2} \qquad \sigma_{r\theta} = \dfrac{1}{r^2}\dfrac{\partial \phi}{\partial \theta} - \dfrac{1}{r}\dfrac{\partial^2 \phi}{\partial r \partial \theta} \end{cases} \Rightarrow \begin{cases} \sigma_r = 2a_3 \log r + \dfrac{a_1}{r^2} + a_3 + 2a_2 \\ \sigma_\theta = 2a_3 \log r - \dfrac{a_1}{r^2} + 3a_3 + 2a_2 \\ \sigma_{r\theta} = 0 \end{cases}$$

Now, we can compute the displacements by

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \qquad \varepsilon_{\theta\theta} = \frac{u_r}{r} + \underbrace{\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}}_{\text{Check this one.}} \qquad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

Axisymmetric solution

Using Hooke's Law

$$\begin{bmatrix}\varepsilon_{rr}\\ \varepsilon_{\theta\theta}\\ \varepsilon_{\theta r}\end{bmatrix} = \frac{1}{E}\begin{bmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{Bmatrix}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \sigma_{\theta r}\end{Bmatrix} \Rightarrow \begin{cases}\sigma_r = 2a_3\log r + \frac{a_1}{r^2} + a_3 + 2a_2\\ \sigma_\theta = 2a_3 \log r - \frac{a_1}{r^2} + 3a_3 + 2a_2\\ \sigma_{r\theta} = 0\end{cases}$$

Then $\varepsilon_{rr} = \frac{1}{E}\left(\sigma_{rr} - \nu\,\sigma_{\theta\theta}\right) = \frac{\partial u}{\partial r}$

$$\varepsilon_{rr} = \frac{1}{E}\left[2a_3 \log r + \frac{a_1}{r^2} + a_3 + 2a_2 - \nu\left(2a_3 \log r - \frac{a_1}{r^2} + 3a_3 + 2a_2\right)\right] = \frac{\partial u_r}{\partial r}$$

$$\boxed{u_r = \int \frac{\partial u}{\partial r}\,dr}$$

$$u_r = \frac{1}{E}\left[-\frac{(1+\nu)a_1}{r} + 2(1-\nu)a_3\, r\log r - (1+\nu)a_3 r + 2a_2(1-\nu)r\right] + A\sin\theta + B\cos\theta$$

$$u_\theta = \frac{4r\theta}{E}a_3 + A\cos\theta - B\sin\theta + Cr$$

Axisymmetric solution

$$u_\theta = \frac{4r\theta}{E} a_3 + A\cos\theta - B\sin\theta + Cr$$

where A, B, C are constants that represent a rigid body motion

If the body has discontinuities, the compatibility equations cannot guarantee single-valued displacements. In these cases we integrate directly the equilibrium equation.

$$\boxed{\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{1}{r^2}u_r = 0}$$

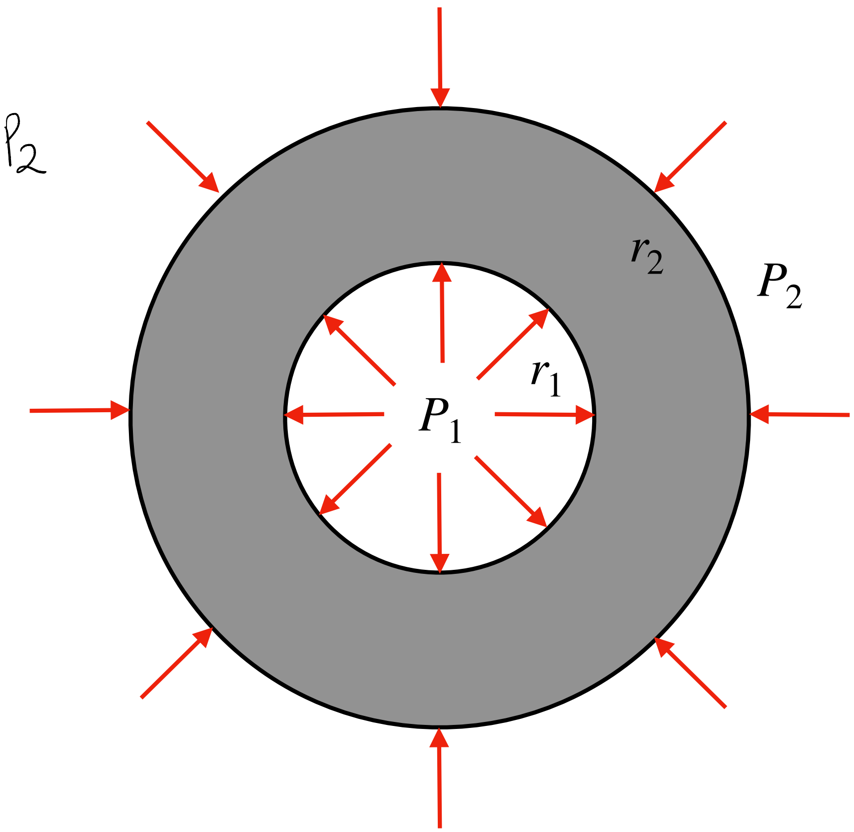
which has solution $u_r = C_1 r + \frac{C_2}{r}$ with C_1, C_2 constants to be determined.

Axisymmetric solution - Example

Thick-walled cylinder under uniform boundary pressure

Boundary conditions

$\sigma_r(r=r_1) = -P_1$; $\sigma_r(r=r_2) = -P_2$



Using $u_r = C_1 r + \frac{C_2}{r}$ we find the solution as

$$\sigma_r = \frac{A}{r^2} + B \qquad \sigma_\theta = -\frac{A}{r^2} + B$$

Axisymmetric solution

This leads to

$$A = \frac{r_1^2 r_2^2 (p_2 - p_1)}{r_2^2 - r_1^2} \qquad B = \frac{r_1^2 p_1 - r_2^2 p_2}{r_2^2 - r_1^2}$$

Then

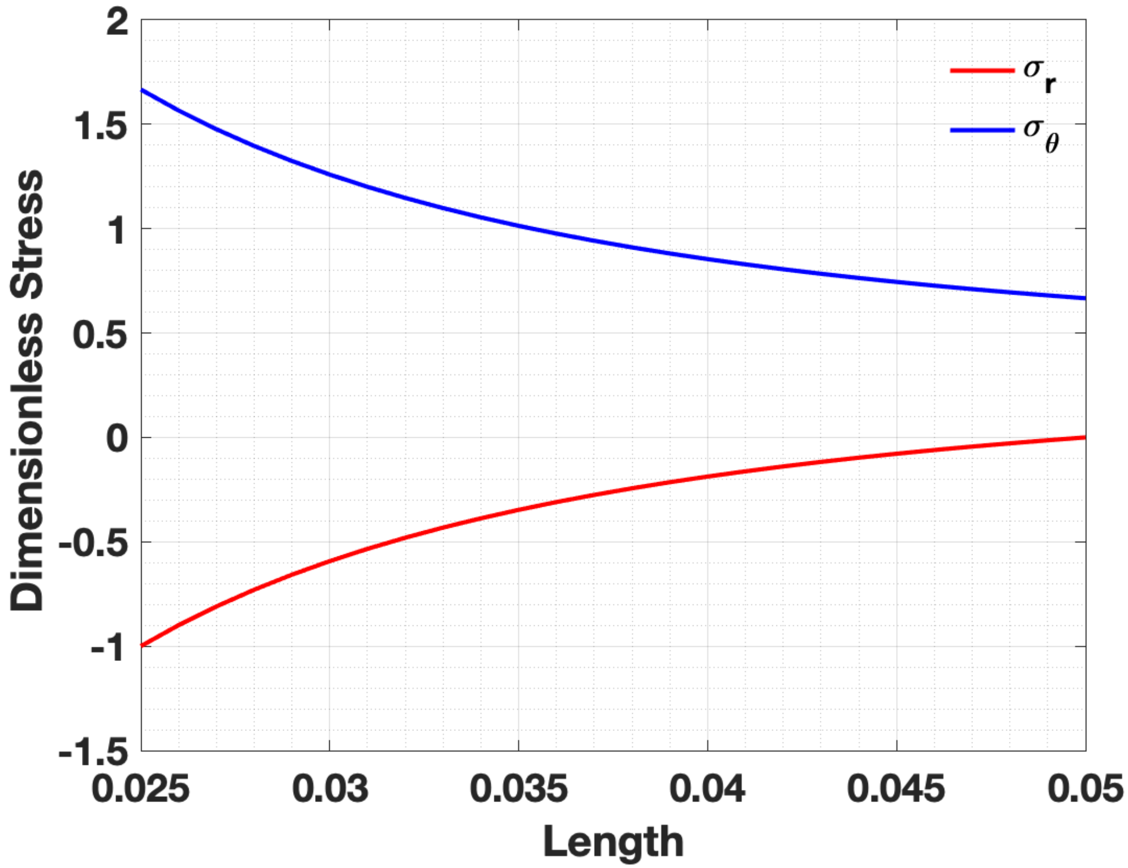
$$\sigma_r = \frac{r_1^2 r_2^2 (p_2 - p_1)}{r_2^2 - r_1^2} \frac{1}{r^2} + \frac{r_1^2 p_1 - r_2^2 p_2}{r_2^2 - r_1^2}$$

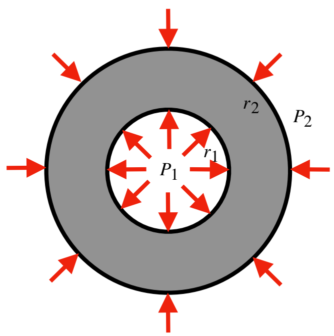
$$\sigma_\theta = \frac{-r_1^2 r_2^2 (p_2 - p_1)}{r_2^2 - r_1^2} \frac{1}{r^2} + \frac{r_1^2 p_1 - r_2 p_2}{r_2^2 - r_1^2}$$

In plane strain theory $\sigma_z = \nu(\sigma_r + \sigma_\theta) = 2\nu \dfrac{r_1^2 p_1 - r_2^2 p_2}{r_2^2 - r_1^2}$

$$u_r = \frac{1+\nu}{E} r \left[(1-2\nu) B - \frac{A}{r^2} \right]$$

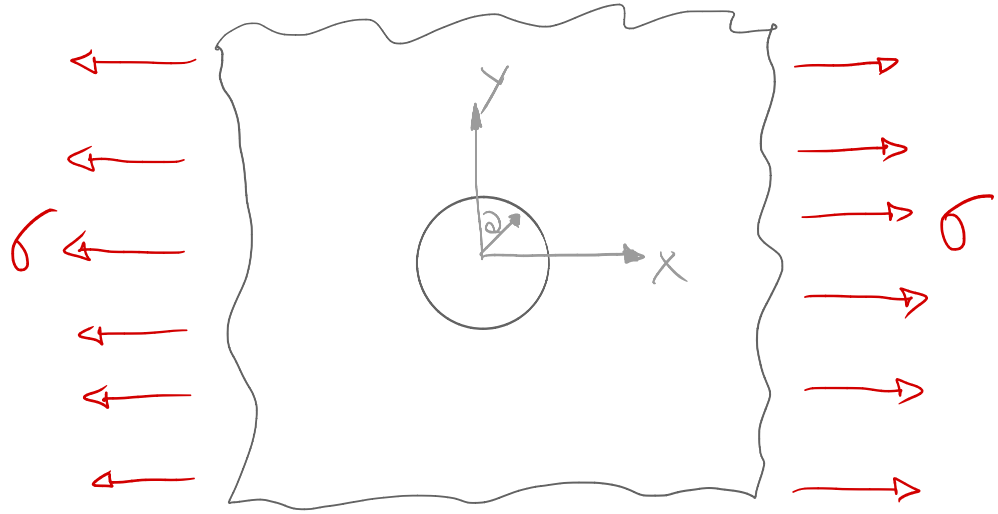
Axisymmetric solution





Axisymmetric solution

Infinite medium with a stress-free hole under uniform far-field tension



The solution for a uniform medium is easy to compute

$$\phi = \frac{1}{2}\sigma y^2 = \frac{\sigma}{2} r^2 \sin^2\theta = \frac{\sigma}{4} r^2 (1-\cos 2\theta)$$

$$\begin{cases} y = r \cdot \sin\theta \\ x = r \cos\theta \end{cases}$$

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = \sigma \checkmark$$

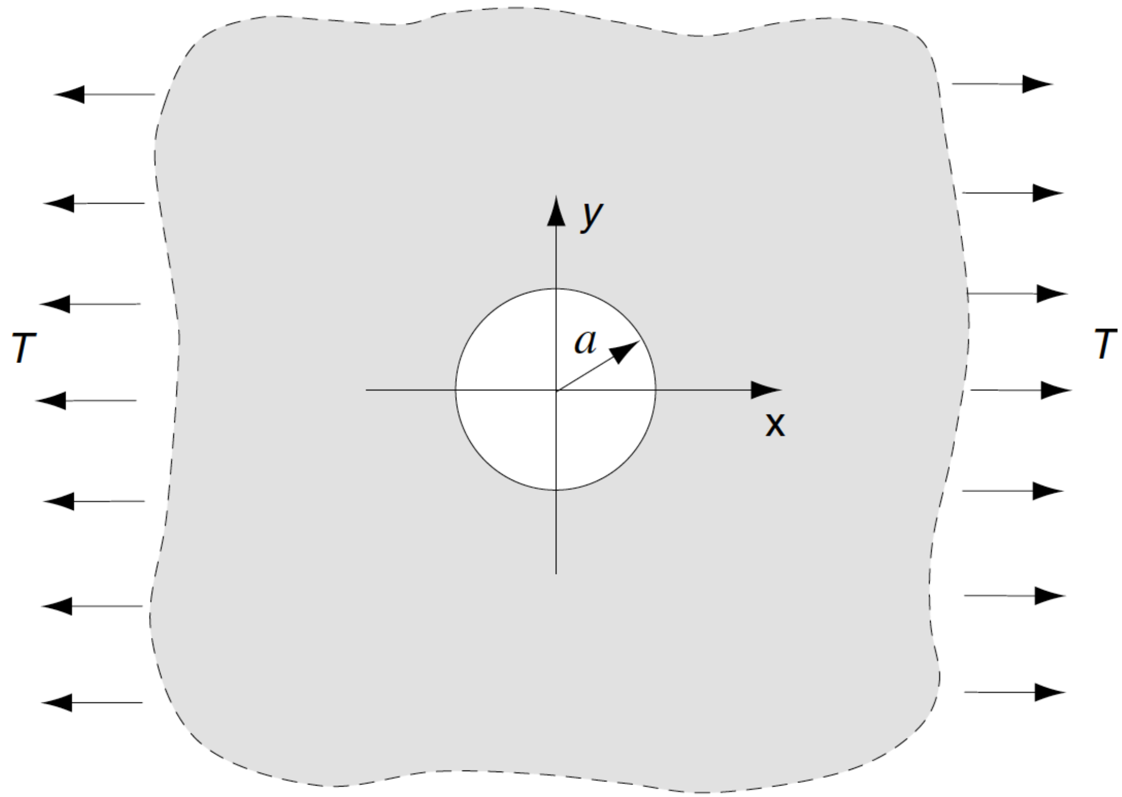
Checks B.Cs.

The presence of the hole make a disturbance near it and should decay to zero far away from the hole. Consider the following function

$$\phi = a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 \log r + \left(a_{21} r^2 + a_{22} r^4 + \frac{a_{23}}{r^2} + a_{24}\right) \cos 2\theta$$

The stresses corresponding to this Airy's function are

Infinite medium with a stress-free hole under uniform far-field tension



B.Cs.

$\sigma_x(r\to\infty)=\sigma$

$\sigma_y(r\to\infty)=0$

$\sigma_{xy}(r\to\infty)=0$

$\sigma_{ij}\to 0$ as $r\to\infty$

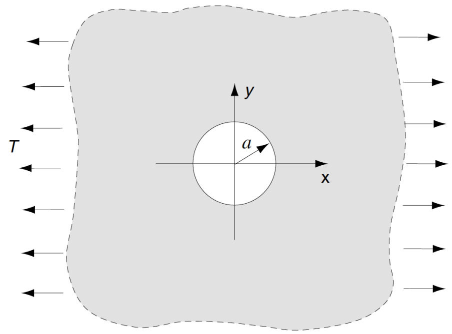
$a_3=a_{22}=0$

$$\sigma_r = \cancel{a_3}(1+2\log r)+2a_2+\frac{a_1}{r^2}-\left(2a_{21}+\frac{6a_{23}}{r^4}+\frac{4a_{24}}{r^2}\right)\cos 2\theta$$

$$\sigma_\theta = \cancel{a_3}(3+2\log r)+2a_2-\frac{a_1}{r^2}+\left(2a_{21}+12\cancel{a_{22}}r^4+\frac{6a_{23}}{r^4}\right)\cos 2\theta$$

$$\sigma_{r\theta} = \left(2a_{21}+6\cancel{a_{22}}r^2-\frac{6a_{23}}{r^4}-\frac{2a_{24}}{r^2}\right)\sin 2\theta$$

Infinite medium with a stress-free hole under uniform far-field tension



Using B.C.s we get the following relations for coefficients

$$2a_2 + \frac{a_1}{a^2} = 0$$

$$2a_{21} + \frac{6a_{23}}{a^4} + \frac{4a_{24}}{a^2} = 0 \qquad 2a_{21} - \frac{6a_{23}}{a^4} - \frac{2a_{24}}{a^2} = 0$$

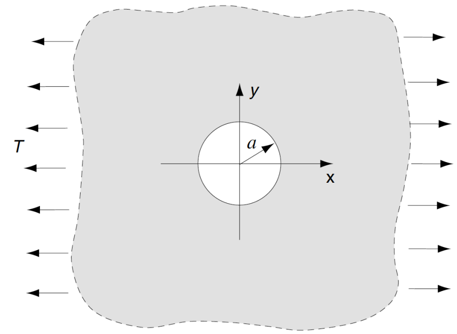
$$2a_{21} = -\sigma/2 \qquad 2a_2 = \sigma/2$$

Solution

$$a_1 = -\frac{a^2\sigma}{2}; \quad a_2 = \frac{\sigma}{4} \quad a_{21} = -\frac{\sigma}{4} \quad a_{23} = \frac{-a^4\sigma}{4} \quad a_{24} = \frac{a^2\sigma}{2}$$

Giving The following stress field.

Infinite medium with a stress-free hole under uniform far-field tension



$$\sigma_r = \frac{\sigma}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{\sigma}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta$$

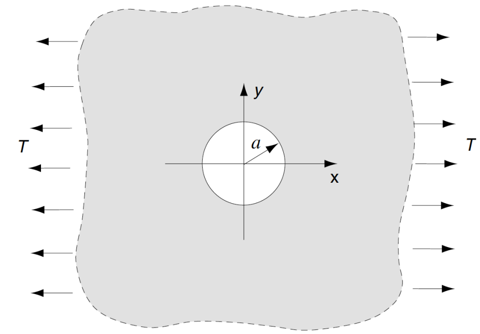
$$\sigma_\theta = \frac{\sigma}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta$$

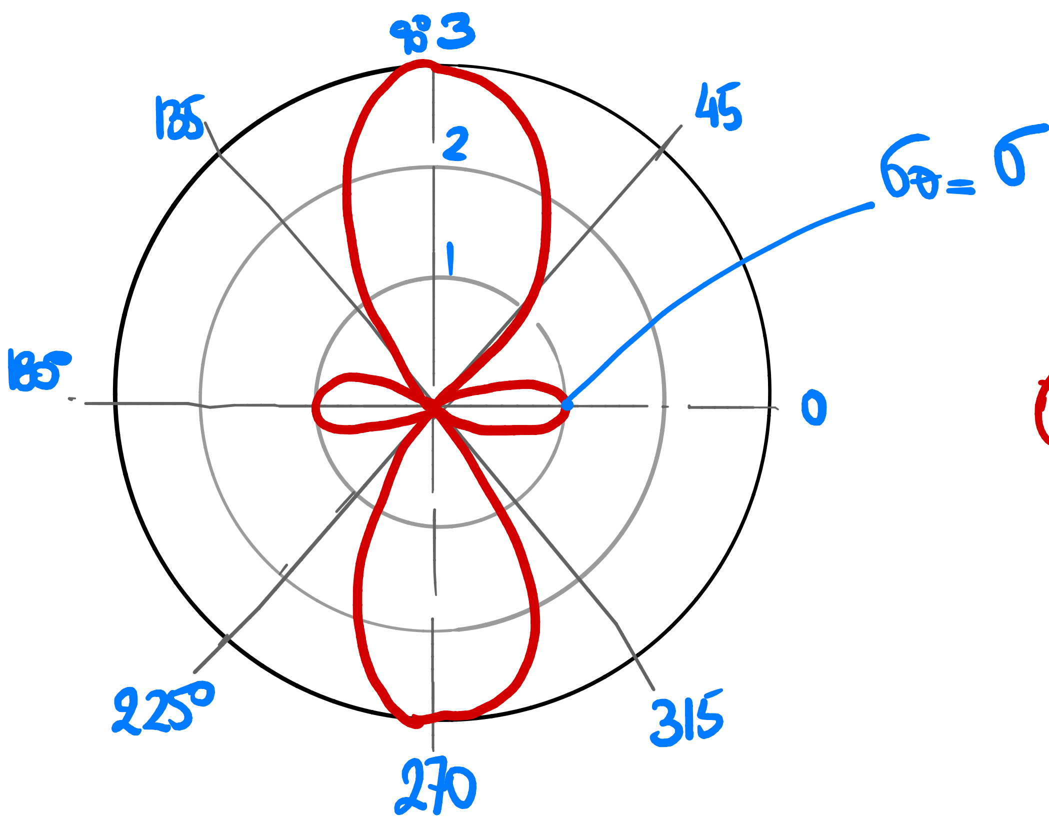
$$\sigma_{r\theta} = -\frac{\sigma}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta$$

The variation of the hoop stress around the boundary of the hole is

$$\sigma_\theta(r=a, \theta) = T(1 - 2\cos 2\theta)$$

Infinite medium with a stress-free hole under uniform far-field tension





$$\sigma_{max} = \sigma_\theta\left(a, \pm \pi/2\right) = 3\sigma$$

- Alternatively:

$$\phi(r,\theta) = -\frac{\sigma}{2} a^2 \log(r) + \frac{\sigma}{4} r^2 + \frac{\sigma}{4}\left(2a^2 - r^2 - \frac{a^4}{r^2}\right)\cos 2\theta$$

You can show that satisfy the B.C.s for the problem.