1 Different categories of PDEs

Partial Differential Equations (PDEs) are a class of equations that involve the partial derivatives of a function of multiple variables. They are fundamental in expressing a variety of physical, engineering, and mathematical phenomena. There are several different kinds of PDEs, each with unique characteristics and applications. Here are some of the most common types:

1. Elliptic Equations:
   - Example: Laplace's Equation, $\nabla^2 u = 0$, and Poisson's Equation, $\nabla^2 u = f$.
   - Characteristics: No time dependence, solutions are generally smooth and describe equilibrium states.
   - Applications: Steady-state heat distribution, electrostatics, incompressible fluid flow.
2. Parabolic Equations:
   - Example: Heat Equation, $\frac{\partial u}{\partial t} = \nabla^2 u$.
   - Characteristics: Contains time derivative and spatial derivatives; models phenomena that evolve over time towards an equilibrium.
   - Applications: Heat conduction, diffusion processes.
3. Hyperbolic Equations:
   - Example: Wave Equation, $\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$.
   - Characteristics: Second-order in time, describe wave propagation and vibrations.
   - Applications: Acoustics, electromagnetic waves, seismic waves.
4. Transport (or Convection-Diffusion) Equations:
   - Example: $\frac{\partial u}{\partial t} + v \cdot \nabla u = D \nabla^2 u$, where v is velocity and D is the diffusion coefficient.
   - Characteristics: Describe phenomena involving both transport (or advection) and diffusion.
   - Applications: Fluid dynamics, pollutant dispersion.
5. Nonlinear Differential Equations:

- Example: Nonlinear Schrödinger Equation, Korteweg-de Vries Equation.
- Characteristics: The equation includes nonlinear terms (products or powers of the function and its derivatives).
- Applications: Complex physical phenomena, including solitons, fluid dynamics, and optical physics.

6. Mixed Type Equations:
   - Example: Tricomi Equation.
   - Characteristics: The equation changes type (from elliptic to hyperbolic, for instance) within the domain.
   - Applications: Transonic flow, certain problems in gas dynamics.
7. Eigenvalue Problems:
   - Example: $-\nabla^2 u = \lambda u$ (Helmholtz equation in eigenvalue form).
   - Characteristics: Involves finding a function u and a number λ (eigenvalue) such that the equation is satisfied.
   - Applications: Quantum mechanics, stability analysis, structural engineering.

Each of these types of PDEs plays a crucial role in modeling different physical phenomena. The solution techniques and analytical approaches vary significantly among these types, reflecting the diverse nature of the phenomena they model. Understanding the specific type of PDE is essential in choosing the right methods for analysis and numerical simulation.

1.1 Definition

1. The equation is called elliptic at the point x provided A(x) is positive definite.
2. The equation is called hyperbolic at the point x provided A(x) has one negative and n - 1 positive eigenvalues.
3. The equation is called parabolic at the point x provided A(x) is positive semidefinite, but is not positive definite, and the rank of (A(x), b(x)) equals n.
4. An equation is called elliptic, hyperbolic or parabolic provided it has the corresponding property for all points of the domain.