MATH 521 - Numerical Analysis of Differential Equations

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Assignment 3 : Analysis of FEM

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As a general comment: The questions may appear long, but they are not. Try to be brief.

Q1: Inhomogeneous Dirichlet Problem [20]

(a) Let $\Omega = (0,1)^2$, $f \in L^2(\Omega)$ and consider the boundary value problem

$$\begin{aligned} -\Delta u &= f, \quad \Omega \\ u &= 0, \quad \partial\Omega. \end{aligned}$$

Derive the weak form of the PDE, then formulate the P_k-finite element method in variational form. (No need to formulate the algebraic equations.)

(b) Now suppose that instead of the homogneous boundary conditions $u = 0$ we solve the inhomogeneous Dirichlet problem

$$\begin{aligned} -\Delta u &= f, \quad \Omega \\ u &= u_D, \quad \partial\Omega. \end{aligned}$$

We assume that $u_D|_{\partial\Omega}$ is the trace of a function $u_D \in H^1(\Omega)$. Reduce the problem to the one in part (a) and use this to formulate a P_k-finite element approximation with trial functions of the form $u_h = u_D + w_h$ where w_h belongs to a suitable finite element space that you should specify.

NOTE: this is an over-simplification. In practice we will use $u_h = I_h u_D + w_h$ which leads to additional variational crimes that we will treat later.

(c) For the problem from part (b) derive an a priori error estimate in the H^1-norm, assuming that $u, u_D \in H^{k+1}$. You may assume boundedness and coercivity and existence/uniqueness without proof, but show the steps for Galerkin orthogonality and Cea's lemma and then deduce the error estimate.

You may state without proof any nodal interpolation error estimate.

Solution Q1a

The weak form is thus: Find $u \in H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx$$
$$a(u, v) = l(v)$$

FEM: T_h is regular triangularization of Ω

$$V_h := P_h^k(T_h) \cap H_0^1$$

(P_h) Find $u_h \in V_h$ such that for all $v_h \in V_h$

$$a(u_h, v_h) = l(v_h)$$

Solution Q1b

Define $w = u - u_D$, the original problem turns into:

$$\begin{aligned} -\Delta w = -\Delta u + \Delta u_D = f + \Delta u_D, \quad \text{in } \Omega, \\ w = 0, \quad \text{on } \partial\Omega. \end{aligned}$$

The function w should satisfy the homogeneous boundary condition $w = 0$ on $\partial\Omega$, and the modified source term is $f + \Delta u_D$. This allows for solving for w with homogeneous boundary conditions, which simplifies the problem.

FEM: Define T_h is regular triangularization of Ω

$$V_h := P_h^k(T_h) \cap H_0^1$$

Find $w_h \in V_h$ such that

$$\int_\Omega \nabla w_h \cdot \nabla v_h \, dx = \int_\Omega (f + \Delta u_D) v_h \, dx, \quad \forall v_h \in V_h.$$

Solution Q1c

By Galerkin Orthogonality:

$$a(w - w_h, v_h) = 0$$

Since the boundedness and coercivity is assumed, Cea's Lemma gives:

$$\|w - w_h\|_{H_0^1} \leq \frac{c_1}{c_0} \|w - v_h\|_{H_0^1}, \quad \forall v_h \in V_h.$$

By nodal error:

$$\|w - I_h w\|_{H_0^1} \leq c h^k \|w\|_{H^{k+1}}$$

Thus,

$$\|u - u_h\|_{H_0^1} = \|w - w_h\|_{H_0^1} \leq c h^k \|w\|_{H^{k+1}}$$

Q2: Advection [10]

Let $\Omega = (0,1)^2$, $f \in L^2(\Omega)$, $b \in \mathbb{R}^2$ constant and consider the boundary value problem

$$\begin{aligned} -\Delta u + b \cdot \nabla u = f, &\quad \Omega \\ u = 0, &\quad \partial\Omega. \end{aligned}$$

Derive the variational (weak) form of the PDE, then formulate the variational form of the P_k-finite element method. Are the PDE and the FEM well-posed?

Solution Q2

Multiply the equation by v and integrate over Ω:

$$\int_\Omega (-\Delta u + b \cdot \nabla u) v \, d\Omega = \int_\Omega f v \, d\Omega.$$

Apply integration by parts and by $u = 0$ on $\partial\Omega$:

$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega + \int_\Omega (b \cdot \nabla u) v \, d\Omega = \int_\Omega f v \, d\Omega.$$

Thus, the variational form is: Find $u \in H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega + \int_\Omega (b \cdot \nabla u) v \, d\Omega = \int_\Omega f v \, d\Omega.$$
$$a(u, v) = l(v)$$

P_k-FEM: Define T_h is regular triangularization of Ω

$$V_h := P_h^k(T_h) \cap H_0^1$$

Find $u_h \in V_h$ such that for all $v_h \in V_h$,

$$\int_\Omega \nabla u_h \cdot \nabla v_h \, d\Omega - \int_\Omega (b \cdot \nabla u_h) v_h \, d\Omega = \int_\Omega f v_h \, d\Omega.$$

By Lax-Milgram Thm, the weak form of PDE should be able to prove well-posed. But for the strong form of the PDE, it might be well posed if the source term f is smooth enough.

For the FEM form, the behavior of the advection term might break the coercivity and boundedness of a under some circumstances, thus, the FEM might not be well-posed.

Q3: Energy [10]

Let H be a Hilbert space and $V_h \subset H$ a finite-dimensional subspace. Let $a : H \times H \to \mathbb{R}$ be a bounded, coercive, symmetric, bilinear form, $\ell \in H^*$, and let

$$J(v) := \frac{1}{2}a(v,v) - \ell(v)$$

be the associated energy functional.

(a) Show that the following two problems are equivalent:

- Find $u \in H$ such that $J(u) \leq J(v)$ for all $v \in H$.
- Find $u \in H$ such that $a(u,v) = \ell(v)$ for all $v \in H$. (Give full details for the argument that we sketched out in class.)

(b) Conclude as an immediate corollary that the Galerkin projection

- Find $u_h \in V_h$ such that $a(u_h, v_h) = \ell(v_h)$ for all $v_h \in V_h$

can be equivalently written as

- Find $u_h \in V_h$ such that $J(u_h) \leq J(v_h)$ for all $v_h \in V_h$

(c) Prove that the error in energy can be bounded by

$$J(u) \leq J(u_h) \leq J(u) + \frac{1}{2}\|u - u_h\|_a^2,$$

in particular,

$$|J(u) - J(u_h)| \leq \frac{1}{2}\|u - u_h\|_a^2.$$

HINT: You might be tempted to use a duality argument, but it is not needed here.

Solution Q3a

Assume u is a true minimizer of J, meaning $J(u) \leq J(v)$ for all $v \in H$.

To show that u satisfies the variational problem, which indicates thoes two arguments are equivalent:

Consider a function $v \in H$ and a scalar $\lambda \in \mathbb{R}$. Define $v_\lambda = u + \lambda(v - u)$.

The function $J(v_\lambda)$ reaches its minimum at $\lambda = 0$ because u is the minimizer of J. Therefore, $\frac{d}{d\lambda}J(v_\lambda)|_{\lambda=0} = 0$.

Derivative gives:

$$\frac{d}{d\lambda}J(v_\lambda) = \frac{d}{d\lambda}\left(\frac{1}{2}a(v_\lambda, v_\lambda) - \ell(v_\lambda)\right)$$

$$= \frac{d}{d\lambda}\left(\frac{1}{2}a(u + \lambda(v-u), u + \lambda(v-u)) - \ell(u + \lambda(v-u))\right)$$

The linearity of a and l allows separation as follows:

$$= \frac{d}{d\lambda}\left(\frac{1}{2}a(u + \lambda(v-u), u) + \frac{1}{2}a(u + \lambda(v-u), \lambda(v-u)) - \ell(u) - \ell(\lambda(v-u))\right)$$

$$= \frac{d}{d\lambda}\left(\frac{1}{2}a(u,u) + \frac{1}{2}a(u, \lambda(v-u)) + \frac{1}{2}a(\lambda(v-u), u) + \frac{1}{2}a(\lambda(v-u), \lambda(v-u)) - \ell(u) - \ell(\lambda(v-u))\right)$$

$$= \frac{d}{d\lambda}\left(\frac{1}{2}a(u,u) + \lambda a(u, (v-u)) + \lambda^2\frac{1}{2}a((v-u), (v-u)) - \ell(u) - \lambda\ell(v-u)\right)$$

$$= a(u, v-u) + \lambda a(v-u, v-u) - \ell(v-u)$$

$$= a(u,v) - a(u,u) + \lambda a(v-u, v-u) - \ell(v) + \ell(u).$$

When $\lambda = 0$:

$$a(u,v) - a(u,u) - \ell(v) + \ell(u) = 0.$$

Since $a(u,u) - \ell(u)$ is just a constant (not depending on v):

$$a(u,v) = \ell(v), \quad \forall v \in H.$$

Solution Q3b

So, as shown in part a, one can conclude those two statements of Galerkin project are equavalent as well as a corollary.

Solution Q3c

$$J(u_h) = \frac{1}{2}a(u_h,u_h) - \ell(u_h), \quad J(u) = \frac{1}{2}a(u,u) - \ell(u).$$

$$J(u_h) - J(u) = \frac{1}{2}a(u_h,u_h) - \ell(u_h) - \frac{1}{2}a(u,u) + \ell(u).$$

Since a is symmetric and bilinear:

$$a(u_h,u_h) = a(u_h - u + u, u_h - u + u) = a(u_h - u, u_h - u) + 2a(u_h - u, u) + a(u,u).$$

Therefore, the difference $J(u_h) - J(u)$ becomes:

$$J(u_h) - J(u) = \frac{1}{2}(a(u_h - u, u_h - u) + 2a(u_h - u, u)) - \ell(u_h) + \ell(u).$$

Since u satisfies the variational problem, $a(u,v) = \ell(v)$ for any v, including $v = u_h - u$

$$J(u_h) - J(u) = \frac{1}{2}a(u_h - u, u_h - u).$$

Thus,

$$J(u_h) \leq J(u) + \frac{1}{2}\|u - u_h\|_a^2,$$

where $\|u - u_h\|_a = \sqrt{a(u - u_h, u - u_h)}$ is the energy norm of the error.

Therefore,

$$|J(u) - J(u_h)| \leq \frac{1}{2}\|u - u_h\|_a^2.$$

Q4: Duality [10]

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$ with $|\Gamma_D|, |\Gamma_N| > 0$, $\mathcal{T}_h$ a regular triangulation of Ω, $f \in L^2(\Omega), g \in L^2(\Gamma_N)$ and consider the boundary value problem

$$\begin{aligned} -\Delta u &= f, \quad \text{in } \Omega \\ u &= 0, \quad \text{in } \Gamma_D, \\ \nu \cdot \nabla u &= g, \quad \text{in } \Gamma_N. \end{aligned}$$

(a) Write down the variational form of the PDE in $H^1_{\Gamma_D}$ and the variational form of the Pk-FEM. (no need to give too many details, and no need to prove boundedness and coercivity - you may assume both for step (b).)

(b) Consider the quantity of interest

$$\Phi(u) = \int_{\Gamma_N} u \, dx.$$

Show that $\Phi \in (H^1_{\Gamma_D})^*$.

Let u, u_h solve the variational forms of the PDE and FEM. Prove that

$$|\Phi(u) - \Phi(u_h)| \leq \|\nabla u - \nabla u_h\|_{L^2}\|\nabla w - \nabla w_h\|_{L^2},$$

where w is the solution of a dual problem that you should specify and w_h taken from a suitable space is arbitrary.

Solution Q4a

Thus, the variational form is: Find $u \in H^1_{\Gamma_D}(\Omega)$ such that for all $v \in H^1_{\Gamma_D}(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega = \int_\Omega f v \, d\Omega + \int_{\Gamma_N} g v \, ds.$$

P_k-FEM: Define T_h is regular triangularization of Ω and

$$V_h := P_h^k(T_h) \cap H^1_{\Gamma_D}$$

Find $u_h \in V_h$ such that for all $v_h \in V_h$,

$$\int_\Omega \nabla u_h \cdot \nabla v_h \, d\Omega = \int_\Omega f v_h \, d\Omega + \int_{\Gamma_\mathrm{N}} g v_h \, ds.$$

Solution Q4b

The quantity of interest $\Phi(u) = \int_{\Gamma_\mathrm{N}} u \, dx$ can be seen as a linear functional acting on u, since for any scalar a and functions $u, v \in H^1_{\Gamma_\mathrm{D}}(\Omega)$,

- $\Phi(au) = a\Phi(u)$,
- $\Phi(u+v) = \Phi(u) + \Phi(v)$.

Φ is assumed to be bounded.

Thus, $\Phi \in (H^1_{\Gamma_\mathrm{D}}(\Omega))^*$.

The dual problem seeks a function $w \in H^1_{\Gamma_\mathrm{D}}(\Omega)$ such that for all test functions $v \in H^1_{\Gamma_\mathrm{D}}(\Omega)$,

$$\int_\Omega \nabla w \cdot \nabla v \, dx = \int_{\Gamma_\mathrm{N}} v \, dx.$$

By substituting $v = u - u_h$ in the dual problem and using $\Phi(u) - \Phi(u_h)$ as the right-hand side:

$$|\Phi(u) - \Phi(u_h)| = \left| \int_\Omega \nabla w \cdot \nabla (u - u_h) \, dx \right|,$$

Applying the Cauchy-Schwarz inequality yields:

$$|\Phi(u) - \Phi(u_h)| \leq \|\nabla w\|_{L^2(\Omega)} \|\nabla (u - u_h)\|_{L^2(\Omega)}.$$

Given w_h in a suitable space V_h, applying a similar argument for $w - w_h$ with respect to $u - u_h$:

$$|\Phi(u) - \Phi(u_h)| \leq \|\nabla u - \nabla u_h\|_{L^2(\Omega)} \|\nabla w - \nabla w_h\|_{L^2(\Omega)}.$$