MATH 521 - Numerical Analysis of Differential Equations

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Assignment 3 : Analysis of FEM

Name:

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As a general comment: The questions may appear long, but they are not. Try to be brief.

Q1: Inhomogeneous Dirichlet Problem [20]

(a) Let $\Omega = (0,1)^2$, $f \in L^2(\Omega)$ and consider the boundary value problem

$$\begin{aligned} -\Delta u &= f, \quad \Omega \\ u &= 0, \quad \partial\Omega. \end{aligned}$$

Derive the weak form of the PDE, then formulate the P_k-finite element method in variational form. (No need to formulate the algebraic equations.)

(b) Now suppose that instead of the homogneous boundary conditions $u = 0$ we solve the inhomogeneous Dirichlet problem

$$\begin{aligned} -\Delta u &= f, \quad \Omega \\ u &= u_D, \quad \partial\Omega. \end{aligned}$$

We assume that $u_D|_{\partial\Omega}$ is the trace of a function $u_D \in H^1(\Omega)$. Reduce the problem to the one in part (a) and use this to formulate a P_k-finite element approximation with trial functions of the form $u_h = u_D + w_h$ where w_h belongs to a suitable finite element space that you should specify.

NOTE: this is an over-simplification. In practice we will use $u_h = I_h u_D + w_h$ which leads to additional variational crimes that we will treat later.

(c) For the problem from part (b) derive an a priori error estimate in the H^1-norm, assuming that $u, u_D \in H^{k+1}$. You may assume boundedness and coercivity and existence/uniqueness without proof, but show the steps for Galerkin orthogonality and Cea's lemma and then deduce the error estimate.

You may state without proof any nodal interpolation error estimate.

Solution Q1a

Multiply the PDE with a test function $v \in C^\infty$ and integrating gives

$$\begin{aligned}\int_\Omega (-\Delta u) v \, dx &= \int_\Omega f v \, dx, && \text{no integrate by parts}\\ \int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} \nu \cdot \nabla u v \, dS &= \int_\Omega f v \, dx, && \text{restrict to } v = 0 \text{ on } \partial\Omega\\ \int_\Omega \nabla u \cdot \nabla v \, dx &= \int_\Omega f v \, dx.\end{aligned}$$

The correct function space is $H_0^1 = H_0^1(\Omega)$, and for $u, v H_0^1$ we define

$$\begin{aligned}a(u, v) &= \int_\Omega \nabla u \cdot \nabla v \, dx,\\ \ell(v) &= \int_\Omega f v \, dx.\end{aligned}$$

Variational form of the PDE: find $u \in H_0^1$ such that $a(u, v) = \ell(v)$ for all $v \in H_0^1$.

To formulate the Pk-FEM, let $\mathcal{T}_h$ be a regular triangulation of Ω and let $V_h := \mathcal{P}_h(\mathcal{T}_h)$ be the space of all $u_h \in C(\bar{\Omega})$ (continuous), piecewise polynomial i.e. $u_h|_T \in P_k$ for all $T \in \mathcal{T}_h$.

Variational form of Pk-FEM: find $u_h \in V_h$ such that $a(u_h, v_h) = \ell(v_h)$ for all $v_h \in V_h$.

Solution Q1b

The idea is to write $u = u_D + w$, then $w|_\partial \Omega = 0$. Going through the derivation of part (a), it was nowhere used that $u|_{\partial\Omega} = 0$. Hence we can write the variational form as follows:

- Find $w \in H_0^1$ such that $a(u_D + w, v) = \ell(v) \quad \forall v \in H_0^1$;

or equivalently

- Find $w \in H_0^1$ such that $a(w, v) = \ell(v) - a(u_D, v) \quad \forall v \in H_0^1$;

The Pk-finite element method can then be formulated as

- Find $w_h \in V_h$ such that $a(u_D + w_h, v_h) = \ell(v_h) \quad \forall v_h \in V_h$; or equivalently,
- Find $w_h \in V_h$ such that $a(w_h, v_h) = \ell(v_h) - a(u_D, v_h) \quad \forall v_h \in V_h$.

Solution Q1c

We define

$$b(v) = \ell(v) - a(u_D, v).$$

Then the PDE and FEM are given by

$$\begin{aligned} a(w, v) &= b(v) \qquad v \in H_0^1, \\ a(w_h, v_h) &= b(v_h) \qquad v_h \in V_h. \end{aligned}$$

Galerkin orthogonality: for any $v_h \in V_h$ we have

$$a(w - w_h, v_h) = b(v_h) - b(v_h) = 0.$$

Cea's Lemma: Let $\tilde{w}_h \in V_h$ be a suitable quasi-best-approximation to w, then

$$\begin{aligned} c_0 \|w - w_h\|_1^2 &\le a(w - w_h, w - w_h) \\ &= a(w - w_h, w - \tilde{w}_h) \\ &\le c_1 \|w - w_h\|_1 \|w - \tilde{w}_h\|_1, \end{aligned}$$

hence

$$\|w - w_h\|_{H^1} \le c \|w - \tilde{w}_h\|_{H^1},$$

where $c = c_1 / c_0$.

Now choose $\tilde{w}_h := I_h w$ the nodal interpolation operator for the space $\mathcal{P}_k(\mathcal{T}_h)$. Since we assumed that $u, u_D \in H^{k+1}$ it follows that also $w = u - u_D \in H^{k+1}$ and hence

$$\|I_h w - w\|_{H^1} \le c h^k \|\nabla^{k+1} w\|_{L^2},$$

where $h = \max_T h_T$ with $h_T = \mathrm{diam}(T), T \in \mathcal{T}_h$; and the constant c depends on $\max_T \kappa_T$ the mesh shape regularity parameter.

We conclude the error estimate for w,

$$\|w - w_h\|_{H^1} \le C h^k \|\nabla^{k+1} w\|_{L^2}.$$

But since $w - w_h = (u - u_D) - (u_h - u_D) = u - u_h$ it follows that also

$$\|u - u_h\|_{H^1} \le C h^k \|\nabla^{k+1} (u - u_D)\|_{L^2} \le C h^k \big(\|\nabla^{k+1} u\|_{L^2} + \|\nabla^{k+1} u_D\|_{L^2} \big).$$

Q2: Advection [10]

Let $\Omega = (0,1)^2$, $f \in L^2(\Omega)$, $b \in \mathbb{R}^2$ constant and consider the boundary value problem

$$
\begin{aligned}
-\Delta u + b \cdot \nabla u &= f, \quad \Omega \\
u &= 0, \quad \partial\Omega.
\end{aligned}
$$

Derive the variational (weak) form of the PDE, then formulate the variational form of the P_k-finite element method. Are the PDE and the FEM well-posed?

Solution Q2

From the boundary condition we already see that we should take test functions $v \in C_c^\infty(\Omega)$. Then we obtain

$$
\begin{aligned}
\int_\Omega (-\Delta u + b \cdot \nabla u) v \, dx &= \int f v \, dx \\
\int_\Omega \nabla u \cdot \nabla v + b \cdot \nabla u \, v \, dx &= \int f v \, dx.
\end{aligned}
$$

No need to integrate by parts the advection term since it is already just one derivative. So we define

$$
a(u,v) = \int_\Omega \nabla u \cdot \nabla v + b \cdot \nabla u \, v \, dx, \qquad \ell(v) = \int_\Omega f v \, dx.
$$

Variational/weak form: Find $u \in H_0^1$ s.t. $a(u,v) = \ell(v) \quad \forall v \in H_0^1$

Let V_h be defined analogously as in Q1a.

Variational form of Pk-FEM: find $u_h \in V_h$ such that $a(u_h, v_h) = \ell(v_h)$ for all $v_h \in V_h$.

As in class and previous exercises, $\ell \in (H_0^1)^*$, and a is easily shown to be bounded. The critical step is to show that a is coercive: we use that $\nabla u \, u = \frac{1}{2} \nabla u^2$

$$
\begin{aligned}
a(u,u) &= \|\nabla u\|_{L^2}^2 + \int_\Omega (b \cdot \nabla u) u \, dx \\
&= \|\nabla u\|_{L^2}^2 + \frac{1}{2} \int_\Omega b \cdot \nabla u^2 \, dx \\
&= \|\nabla u\|_{L^2}^2 + \frac{1}{2} \int_{\partial\Omega} (b \cdot \nu) u^2 \, dx \\
&= \|\nabla u\|_{L^2}^2,
\end{aligned}
$$

where in the last step we used that $u|_{\partial\Omega} = 0$.

Now we can proceed as usual with a Poincare-type inequality to prove that $a(u,u) \geq c_0 \|u\|_{H^1}^2$ for all $u \in H_0^1(\Omega)$.

Q3: Energy [10]

Let H be a Hilbert space and $V_h \subset H$ a finite-dimensional subspace. Let $a : H \times H \to \mathbb{R}$ be a bounded, coercive, symmetric, bilinear form, $\ell \in H^*$, and let

$$J(v) := \frac{1}{2} a(v, v) - \ell(v)$$

be the associated energy functional.

(a) Show that the following two problems are equivalent:

- Find $u \in H$ such that $J(u) \leq J(v)$ for all $v \in H$.
- Find $u \in H$ such that $a(u, v) = \ell(v)$ for all $v \in H$.

(Give full details for the argument that we sketched out in class.)

(b) Conclude as an immediate corollary that the Galerkin projection

- Find $u_h \in V_h$ such that $a(u_h, v_h) = \ell(v_h)$ for all $v_h \in V_h$

can be equivalently written as

- Find $u_h \in V_h$ such that $J(u_h) \leq J(v_h)$ for all $v_h \in V_h$

(c) Prove that the error in energy can be bounded by

$$J(u) \leq J(u_h) \leq J(u) + \frac{1}{2} \|u - u_h\|_a^2,$$

in particular,

$$|J(u) - J(u_h)| \leq \frac{1}{2} \|u - u_h\|_a^2.$$

HINT: You might be tempted to use a duality argument, but it is not needed here.

Solution Q3a and Q3b

If u minimizes J then $J(u+tv) \geq J(u)$ for all $t \in \mathbb{R}, v \in H$. We can expand it,

$$\begin{aligned} J(u+tv) &= \frac{1}{2}a(u+tv, u+tv) - \ell(u+tv) \\ &= \frac{1}{2}a(u,u) + ta(u,v) + \frac{t^2}{2}a(v,v) - \ell(u) - t\ell(v) \\ &= J(u) + t\Big\{a(u,v) - \ell(v)\Big\} + \frac{t^2}{2}a(v,v). \end{aligned}$$

If we define $j(t) = J(u+tv)$ then $t=0$ minimizes j and hence, $j'(0) = 0$ which yields the weak or variational form.

$$(\mathrm{Pw}) \qquad\qquad a(u,v) = \ell(v) \qquad \forall v \in H.$$

Vice-versa, if u satisfies (Pw) and $v \in H$, then

$$J(u+v) = J(u) + \Big\{0\Big\} + \frac{1}{2}a(v,v).$$

Since a is coercive, $a(v,v) \geq 0$ it follows that $J(u+v) \geq J(u)$. Thus, u minimizes J.

Applying the solution of Q3a to $H = V_h$, it follows that the Galerkin project is equivalent to minimizing J over V_h.

Solution Q3c

Since $u_h \in V_h \subset H$ it follows that $J(u) \leq J(u_h)$. This proves the first inequality.

For the second inequality we write $e_h = u_h - u$ and we can use the solution to Q3a to show

$$\begin{aligned} J(u_h) &= J(u+e_h) \\ &= J(u) + \frac{1}{2}a(e_h, e_h). \end{aligned}$$

Finally, the last inequality is obtained by subtracting $J(u_h)$.

Q4: Duality [10]

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain with boundary $\partial\Omega = \Gamma_{\rm D} \cup \Gamma_{\rm N}$ with $|\Gamma_{\rm D}|, |\Gamma_{\rm N}| > 0$, $\mathcal{T}_h$ a regular triangulation of Ω, $f \in L^2(\Omega), g \in L^2(\Gamma_{\rm N})$ and consider the boundary value problem

$$
\begin{aligned}
-\Delta u &= f, \quad && \text{in } \Omega \\
u &= 0, \quad && \text{in } \Gamma_{\rm D}, \\
\nu \cdot \nabla u &= g, \quad && \text{in } \Gamma_{\rm N}.
\end{aligned}
$$

(a) Write down the variational form of the PDE in $H^1_{\Gamma_{\rm D}}$ and the variational form of the Pk-FEM. (no need to give too many details, and no need to prove boundedness and coercivity - you may assume both for step (b).)

(b) Consider the quantity of interest

$$
\Phi(u) = \int_{\Gamma_{\rm N}} u \, dx.
$$

Show that $\Phi \in (H^1_{\Gamma_{\rm D}})^*$.

Let u, u_h solve the variational forms of the PDE and FEM. Prove that

$$
|\Phi(u) - \Phi(u_h)| \leq \|\nabla u - \nabla u_h\|_{L^2} \|\nabla w - \nabla w_h\|_{L^2},
$$

where w is the solution of a dual problem that you should specify and w_h taken from a suitable space is arbitrary.

Solution Q4a

The variational form is to find $u \in H^1_{\Gamma_\mathrm{D}}$ such that

$$a(u,v) = \ell(v) \qquad \forall v \in H^1_{\Gamma_\mathrm{D}}$$

where

$$\begin{aligned} a(u,v) &= \int_\Omega \nabla u \cdot \nabla v \, dx, \\ \ell(v) &= \int_\Omega f v \, dx + \int_{\Gamma_\mathrm{N}} g v \, dx. \end{aligned}$$

The Pk-FEM is defined as follows: Let $\mathcal{T}_h$ be a regular triangulation of Ω and $V_h := \mathcal{P}_k(\mathcal{T}_h) \cap H^1_{\Gamma_\mathrm{D}}$, then the FEM solution is $u_h \in V_h$ such that

$$a(u_h, v_h) = \ell(v_h) \qquad \forall v_h \in V_h.$$

Solution Q4b

The functional $\Phi : H^1_{\Gamma_\mathrm{D}} \to \mathbb{R}$ is linear and bounded due to the trace inequality.

Therefore, the adjoint / dual problem

$$a(v,w) = \Phi(v) \qquad \forall v \in H^1_{\Gamma_\mathrm{D}}.$$

has a unique solution $w \in H^1_{\Gamma_\mathrm{D}}$. With that definition for w we have

$$\Phi(u) - \Phi(u_h) = \Phi(u - u_h) = a(u - u_h, w) = \dots$$

By Galerkin orthogonality,

$$\dots = a(u - u_h, w - w_h) \qquad \forall w_h \in V_h.$$

Hence we can conclude

$$\begin{aligned} |\Phi(u) - \Phi(u_h)| &= |a(u - u_h, w - w_h)| \\ &\le \|u - u_h\|_a \|w - w_h\|_a \\ &= \|\nabla u - \nabla u_h\|_{L^2} \|\nabla w - \nabla w_h\|_{L^2}. \end{aligned}$$