MECH 570C-FSI: Fundamentals of variational formulation-A

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Contents

In this chapter, we introduce the concept of variational principle and its relationship with the weak form of any differential equation. We then write the formulation such that it leads to the finite element framework.

1 Review of calculus of variations

We first review some of the basic principles of calculus of variations. In conventional calculus, a *function* $f(x)$ is defined as a mapping from $\mathbb{R}$ to $\mathbb{R}$, $\mathbb{R}$ being the set of real numbers. Here, we define a *functional* $\mathscr{F}[f]$ as a mapping from a function $f(x)$ to $\mathbb{R}$. Therefore, the functional will have a different value based on the choice of the function, or, it is a function of a function.

1.1 Shortest distance between two points

We will demonstrate this concept by taking an example of distance between two points. Consider two points $A(x_1, y_1)$ and $B(x_2, y_2)$ in a two-dimensional Cartesian plane XY. Let a curve $f(x)$ joint these points such that $f(x_1) = y_1$ and $f(x_2) = y_2$. The distance between the points along the curve $f(x)$ can be obtained by dividing the curve into N intervals and assuming that the curve is a straight line in each interval. The length of the ith segment δl_i can be written as

$$\delta l_i = \sqrt{\delta x^2 + \delta y^2} = \delta x \sqrt{1 + \left(\frac{\delta y}{\delta x}\bigg|_i \right)^2}, \tag{1}$$

where δx and δy are the projected length of the segment on the X and Y axis respectively. Therefore, the distance along the curve is

$$\mathscr{F}[f] = \sum_{i=1}^{N} \delta l_i = \sum_{i=1}^{N} \delta x \sqrt{1 + \left(\frac{\delta y}{\delta x}\bigg|_i \right)^2}. \tag{2}$$

In the limit $N \to \infty$,

$$\mathscr{F}[f] = \int_{x_1}^{x_2} \sqrt{1 + (f'(x))^2} dx, \tag{3}$$

where $f'(x)$ is the derivative of $f(x)$ which equals the slope in the given limit. We observe that $\mathscr{F}[f]$ is a functional which gives the total distance between the points A and B and depends on the selection of the curve $f(x)$.

Now, let us move a step further by evaluating the minimum distance between the points. Intuitively, we know that this distance should be along a straight line connecting the points. In the conventional calculus, we find the extremum of a function by taking its derivative and equating it to zero. This point of extremum is also known as a stationary point. Further analysis by calculating the second derivative is required to evaluate whether the function has a maximum, a minimum or point of inflection.

For a functional, we proceed in a similar manner. Suppose we want to find out the shortest distance between the points A and B where the functional for distance along a curve $f(x)$ is given by Eq. (3). Let us assume that $f_m(x)$ is the desired curve when the distance is shortest and $g(x)$ be any smooth function which satisfies the conditions, i.e., $g(x_1) = 0$ and $g(x_2) = 0$. Now, the function $f_m(x)$ can be perturbed as, $\tilde{f}_m(x) = f_m(x) + \varepsilon g(x)$, where ε is a real number. Note that this perturbation does not change the boundary conditions imposed on $f_m(x)$, i.e., $\tilde{f}_m(x_1) = y_1$ and $\tilde{f}_m(x_2) = y_2$. The length along this perturbed curve is given as

$$\mathscr{F}[f_m + \varepsilon g] = \int_{x_1}^{x_2} \sqrt{1 + (\tilde{f}'_m(x))^2} dx \tag{4}$$

$$= \int_{x_1}^{x_2} \sqrt{1 + (f'_m(x) + \varepsilon g'(x))^2} dx. \tag{5}$$

Note that for fixed functions $f_m(x)$ and $g(x)$, $\mathscr{F}[f_m + \varepsilon g]$ gives a different value for the distance for each ε. Therefore, we can say that $\mathscr{F}[f_m + \varepsilon g]$ is a function mapping $\mathbb{R}$ to $\mathbb{R}$. Now, to find the minimum value of the distance, we proceed by differentiating $\mathscr{F}[f_m + \varepsilon g]$ with respect to ε and equating it to zero at $\varepsilon = 0$ since $f_m(x)$ is our desired minimal distance curve.

$$\frac{d}{d\varepsilon} \mathscr{F}[f_m + \varepsilon g]\bigg|_{\varepsilon=0} = 0. \tag{6}$$

After some algebra, we get,

$$\int_{x_1}^{x_2} \left(\frac{f'_m(x)}{\sqrt{1 + (f'_m(x))^2}} \right)' g(x) = 0. \tag{7}$$

Since $g(x)$ is chosen to be an arbitrary function (apart from the conditions $g(x_1) = 0$ and $g(x_2) = 0$), the above integral will vanish only if

$$\left(\frac{f'_m(x)}{\sqrt{1 + (f'_m(x))^2}} \right)' = 0 \tag{8}$$

$$\implies \frac{f'_m(x)}{\sqrt{1 + (f'_m(x))^2}} = a, \tag{9}$$

where a is some constant. We observe that the left-hand side of the above equation will be constant if $f'_m(x) = m$, where m is a function of a. Therefore, $f_m(x) = mx + c$ which is the equation of a straight line and m and c can be evaluated based on the boundary conditions for $f_m(x)$. Equation (8) is the differential equation representing the shortest distance curve and Eq. (7) is related to its weak form, which will be discussed later in this chapter.

Summarizing, we have seen the variational principle work for a simple problem where we calculated the shortest distance curve between two points and showed that it comes out to be a straight line.

1.2 Euler-Lagrange equation

To generalize the methodology taken in the previous section, we consider a functional

$$\mathscr{F}[y] = \int_{x_1}^{x_2} F(x, y, y') dx, \qquad y(x_1) = y_1, y(x_2) = y_2. \tag{10}$$

Following a similar procedure, we finally get what is called as *Euler-Lagrange equation*:

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0, \qquad y(x_1) = y_1, y(x_2) = y_2. \tag{11}$$

2 Function spaces

Next, we review some of the mathematical preliminaries and definitions which will be helpful in formulating the weak formulation in a systematic and formal way. Consider a spatial domain $\Omega \subset \mathbb{R}^{\mathrm{n_{sd}}}$, where $\mathrm{n_{sd}} = 1, 2$ or 3 based on the spatial dimensions. Let Γ denote the boundary to the domain. A mapping function from the domain $\bar{\Omega} = \Omega \cup \Gamma$ to $\mathbb{R}$, $f : \bar{\Omega} \rightarrow \mathbb{R}$ is said to be of class $C^m(\Omega)$ if all the derivatives of the function up to the order m exist and are continuous functions. While solving most of higher-order differential equations, one encounters a boundary where the first derivative becomes discontinuous leading to undefined higher derivatives. Therefore, in the variational formulation, we employ the integral form of the differential equations to reduce the burden of evaluating those higher derivatives. The topic of function spaces gives a mathematical preliminary to such space of functions which obey certain restrictions which can be helpful for circumventing the issue of undefined higher derivatives.

Lebesgue space: A function in the $\mathscr{L}^p(\Omega)$ space is considered to be Lebesgue integrable over the domain Ω to the power $p \in [1, \infty)$, i.e.,

$$\mathscr{L}^p(\Omega) = \left\{ f \middle| \int_\Omega |f(x)|^p d\Omega < \infty \right\}, \tag{12}$$

and is equipped with the norm

$$||f||_{\mathscr{L}^p(\Omega)} = \left(\int_\Omega |f|^p d\Omega \right)^{1/p}. \tag{13}$$

One particular case of Lebesgue space is the $\mathscr{L}^2(\Omega)$ space which consists of functions that are square integrable over the domain. In such a case, the norm is $||f||_{\mathscr{L}^2(\Omega)} = (f, f)^{1/2}$, where the inner product is defined as

$$(f, g) = \int_\Omega fg d\Omega. \tag{14}$$

Therefore, $\mathscr{L}^2(\Omega)$ is equipped with an inner product with a norm that makes it a complete metric space, a type of **Hilbert space**.

Sobolev space: Functions in the Sobolev space are such that they belong to $\mathscr{L}^p$ space and its derivatives up to a certain order α also belong to $\mathscr{L}^p$, i.e.,

$$\mathscr{W}^{k,p}(\Omega) = \left\{ f \in \mathscr{L}^p(\Omega) \middle| \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_{\mathrm{n_{sd}}}^{\alpha_{\mathrm{n_{sd}}}}} \in \mathscr{L}^p(\Omega) \ \ \forall |\alpha| \leq k \right\}, \tag{15}$$

where k is a non-negative integer, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_{\mathrm{n_{sd}}}) \in \mathbb{N}^{\mathrm{n_{sd}}}$ and $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_{\mathrm{n_{sd}}}$.

Note that $\mathscr{W}^{0,2}(\Omega) = \mathscr{L}^2(\Omega)$. If we consider $p = 2$, the Sobolev space becomes a Hilbert space, i.e., $\mathscr{H}^k(\Omega) = \mathscr{W}^{k,2}(\Omega)$. For $k = 1$, the Hilbert space is defined as

$$\mathscr{H}^1(\Omega) = \left\{ f \in \mathscr{L}^2(\Omega) \middle| \frac{\partial f}{\partial x_i} \in \mathscr{L}^2(\Omega), \ i = 1, 2, ..., \mathrm{n_{sd}} \right\}, \tag{16}$$

with the inner product and the norm respectively,

$$(f, g)_1 = \int_\Omega \left(fg + \sum_{i=1}^{\mathrm{n_{sd}}} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i} \right) d\Omega, \qquad ||f||_1 = \sqrt{(f, f)_1}. \tag{17}$$

The above function spaces can be easily extended to vector spaces. For any two vectors, $\boldsymbol{u}, \boldsymbol{v} : \Omega \rightarrow \mathbb{R}^n$, the Hilbert space $\mathscr{H}^k(\Omega)$ has the norm as,

$$||\boldsymbol{u}||_k = \left(\sum_{i=1}^{n} ||u_i||_k^2 \right)^{1/2}. \tag{18}$$

3 Different types of partial differential equations

A second-order partial differential equation can be written in the form

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu = g, \tag{19}$$

where a, b, c, d, e, f and g can be dependent on x and y. Then, based on the value of $b^2 - 4ac$, the partial differential equations can be classified as:

1. Elliptic PDEs: $b^2 - 4ac < 0$, Examples are Poisson and Laplace equations.

$$\nabla^2 u = f. \tag{20}$$

2. Hyperbolic PDEs: $b^2 - 4ac > 0$, Examples are wave equation, convection equation.

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0. \tag{21}$$

3. Parabolic PDEs: $b^2 - 4ac = 0$, Examples include heat conduction equation.

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0. \tag{22}$$

The variational principle can be applied to all the types of PDEs leading to the finite element discretization. To begin with, first we consider the simplest form of Elliptic PDEs such as Poisson equation.

4 Strong and weak forms of Poisson equation

To write the variational or weak form of a differential equation, let us consider a Poisson equation on a domain Ω with boundary $\Gamma = \Gamma_D \cup \Gamma_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$ given by,

$$-\nabla^2 u = f, \qquad \text{in } \Omega, \tag{23}$$

$$u = u_D, \qquad \text{on } \Gamma_D, \tag{24}$$

$$\frac{\partial u}{\partial n} = \mathbf{n} \cdot \nabla u = h, \qquad \text{on } \Gamma_N, \tag{25}$$

where u is the scalar variable we want to solve for, f is a source term, $\mathbf{n}$ is the unit normal to the Neumann boundary Γ_N, u_D is the Dirichlet condition on the Dirichlet boundary Γ_D and h is the Neumann condition on Γ_N. Equations (23)-(25) represent the strong form of the problem. Note that to obtain the solution u in the strong form, u should belong to the class $C^2(\Omega)$, i.e., second-order derivatives should exist.

Now, to write the weak form (or variational form) of the equation, we follow the weighted residual procedure by multiplying the equation with some weighting function w and then integrating over the domain Ω,

$$-\int_\Omega w \nabla^2 u d\Omega = \int_\Omega w f d\Omega. \tag{26}$$

As we have written the equation in the integral form, the second-order derivatives need not be continuous, but they need to be square integrable, i.e., $u \in \mathscr{H}^2(\Omega)$. With the help of Green's identity and Gauss divergence theorem, we get

$$-\int_\Omega w \nabla^2 u d\Omega = -\int_\Omega \big(\nabla \cdot (w \nabla u) - \nabla w \cdot \nabla u\big) d\Omega, \tag{27}$$

$$= \int_\Omega \nabla w \cdot \nabla u d\Omega - \int_\Gamma w (\mathbf{n} \cdot \nabla u) d\Gamma. \tag{28}$$

Notice the change in the requirements of the continuity for w and u. $w \in \mathscr{H}^1(\Omega)$ as well as $u \in \mathscr{H}^1(\Omega)$. Thus, this operation has reduced the requirements of continuity on the solution (which was not the case in the strong form). Moreover, we observe that the Neumann condition is derived naturally in the weighted residual form (which is why it is also called *natural boundary condition*). The Eq. (26) thus reads,

$$\int_\Omega \nabla w \cdot \nabla u d\Omega = \int_\Omega w f d\Omega + \int_{\Gamma_N} w h d\Gamma. \tag{29}$$

To summarize until this point, we have written the weak form of the Poisson equation which naturally gives the Neumann boundary condition while reducing the continuity requirements on the weighting function and the solution. The question now is: What about the Dirichlet condition? To answer this, we define define two function spaces, viz., test or weighting function space and trial or solution space. We will denote the spaces of test and trial functions by $\mathscr{V}$ and $\mathscr{S}$ respectively,

$$\mathscr{V} = \{w \in \mathscr{H}^1(\Omega) | w = 0 \text{ on } \Gamma_D\}, \tag{30}$$

$$\mathscr{S} = \{u \in \mathscr{H}^1(\Omega) | u = u_D \text{ on } \Gamma_D\}. \tag{31}$$

Note the conditions imposed on these spaces which takes care of the Dirichlet condition on Γ_D. Therefore, the statement of the weak form for the Poisson equation is: find $u \in \mathscr{S}$ such that $\forall w \in \mathscr{V}$,

$$\int_\Omega \nabla w \cdot \nabla u d\Omega = \int_\Omega w f d\Omega + \int_{\Gamma_N} w h d\Gamma. \tag{32}$$

The above equation can also be written in a compact form as,

$$a(w,u) = (w,f) + (w,h)_{\Gamma_N}, \tag{33}$$

where

$$a(w,u) = \int_\Omega \nabla w \cdot \nabla u d\Omega, \qquad (w,f) = \int_\Omega w f d\Omega, \qquad (w,h)_{\Gamma_N} = \int_{\Gamma_N} w h d\Gamma. \tag{34}$$

Remark 1. *Try to compare Eqs. (26) and (7). You will observe that $g(x)$ is the weighting function multiplied to the Euler-Lagrange equation of the energy functional in Eq. (26).*

Although the solution of the strong form is a solution of the weak form of the differential equation, the fact that the solution of weak form $u \in \mathscr{S}$ is unique, is proved by the **Lax-Milgram lemma**, the proof of which can be found in [**?**]. Further examples on the variational formulations for different differential equations can be found in [**?**].

5 Review of finite element methods

5.1 One-dimensional Poisson equation

To begin with, let us consider a one-dimensional example of a Poisson equation, for which the variational form was discussed in the previous section. Consider a one-dimensional domain Ω defined as $x \in [0,1]$ with Dirichlet condition specified as $u_D = 0$ at $x = 0$ and $x = 1$. In one-dimension,

$$-\frac{d^2u}{dx^2} = f, \qquad \text{in } \Omega, \tag{35}$$

$$u = u_D, \qquad \text{on } \Gamma_D, \tag{36}$$

and we assume that there is no Neumann boundary for simplicity. The variational form for this equation can be written as: find $u \in \mathscr{S}$ such that for all $w \in \mathscr{V}$,

$$\int_\Omega \frac{dw}{dx}\frac{du}{dx} d\Omega = \int_\Omega w f d\Omega. \tag{37}$$

Next, we need to select appropriate subsets of test and trial function spaces $\mathscr{V}$ and $\mathscr{S}$ respectively which correspond to the finite element interpolation. Let these subsets be denoted by $\mathscr{V}^h$ and $\mathscr{S}^h$ respectively. These subsets are defined by partitioning the domain Ω into $\mathrm{n_{el}}$ non-intersecting "finite elements" such that $\Omega = \cup_{e=1}^{\mathrm{n_{el}}} \Omega^e$ and $\Omega^e \cap \Omega^f = \emptyset, \forall e \neq f$. Therefore, the discrete form of the function spaces now is defined as

$$\mathscr{V}^h = \{w^h \in \mathscr{H}^1(\Omega) | w^h|_{\Omega^e} \in \mathscr{P}_m(\Omega^e) \forall e \text{ and } w^h = 0 \text{ on } \Gamma_D\}, \tag{38}$$

$$\mathscr{S}^h = \{u^h \in \mathscr{H}^1(\Omega) | u^h|_{\Omega^e} \in \mathscr{P}_m(\Omega^e) \forall e \text{ and } u^h = u_D \text{ on } \Gamma_D\}, \tag{39}$$

where $\mathscr{P}_m(\Omega^e)$ is the finite element interpolating space which can consist of polynomials of degree $\leq m$. Therefore, the statement of the variational finite element discretization for the one-dimensional Poisson equation for each element domain Ω^e can be written as: find $u^h \in \mathscr{S}^h$ such that for all $w^h \in \mathscr{V}^h$,

$$\int_{\Omega^e} \frac{dw^h}{dx}\frac{du^h}{dx} d\Omega = \int_{\Omega^e} w^h f^h d\Omega. \tag{40}$$

Remark 2. *Note that here, we are discretizing the spatial dimension using finite elements. A similar approach can be taken for the discretization of the temporal variable t for transient problems where the interpolating space will be both space and time dependent. This calls for what is known as "space-time finite elements," which is out of the scope of the present work. Therefore, in temporal discretization discussed in this work, we will deal with semi-discrete temporal discretization where we use the conventional θ-type methods (such as Forward Euler, Backward Euler, Trapezoidal) or more general generalized-α types of time integrators.*

Within a finite element, the interpolation of the variable of interest (u here) is written as

$$u^h = \sum_{i=1}^{nen} N_i u_i, \qquad \text{such that } \sum_{i=1}^{nen} N_i = 1, \tag{41}$$

where N_i, u_i are the element shape function (polynomial approximation) and the value of the variable u at the ith node of an element respectively and nen is the number of nodes per element. We use Lagrange polynomials to approximate the shape functions in the finite element method.

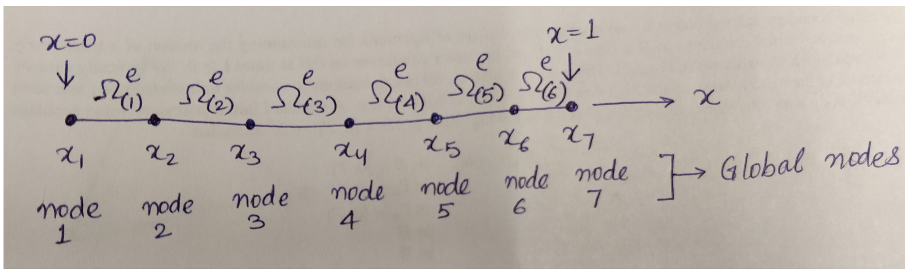


Figure 1: Discretization domain for one-dimensional Poisson equation.

Let us see how to derive the expression for N_i in one-dimension. The one-dimensional domain is shown in Fig. 1. Let us assume that the interpolation is linear within an element, i.e., each shape function can be written as $N_i = ax + b$. Consider the element subdomain $\Omega^e_{(3)}$, u^h in this element can be written as

$$u^h_{(3)} = (ax + b)u_3 + (cx + d)u_4, \tag{42}$$

with the condition that when $x = x_3$, $u^h_{(3)} = u_3$ and when $x = x_4$, $u^h_{(3)} = u_4$. After some simple algebra, we see that

$$u^h_{(3)} = \left(\frac{x_4 - x}{x_4 - x_3}\right)u_3 + \left(\frac{x - x_3}{x_4 - x_3}\right)u_4 = \left(\frac{x_4 - x}{h}\right)u_3 + \left(\frac{x - x_3}{h}\right)u_4, \tag{43}$$

where h is the element size in one-dimension. Therefore, $N_1 = (x_4 - x)/h$ and $N_2 = (x - x_3)/h$ for a one-dimensional linear finite element consisting of two nodes. Also, note that $N_1 + N_2 = 1$. We can write similar expressions for other elements in the discretized domain. However, there is a more effective way for writing these shape functions. For a generalized formulation, we map the element coordinate system to a natural coordinate system of a master element and write the formulation in the natural coordinate system. Let the natural coordinate system be ξ such that for each element $\xi = -1$ at node 1 and $\xi = 1$ at node 2. After the same derivation as above, you will notice that the shape functions now are a function of the natural coordinate ξ as $N_1(\xi) = (1 - \xi)/2$ and $N_2(\xi) = (1 + \xi)/2$, which is independent of the location of the element (which was the case for the previously derived shape functions). This formulation is called isoparametric formulation.

Let us continue our discretization procedure from Eq. (40). We have established that the trial solution u^h is a linear combination of shape functions in a finite element. We have the choice to take the test function as anything else. The most simplistic choice is to take it from the same space as that of trial solution, i.e., $w^h \in \mathscr{V}^h = \text{span}\{N_i\}$. This choice results in what is known as the *Bubnov-Galerkin*, or simply *Galerkin* method. Therefore, Eq. (40) now becomes

$$\int_{\Omega^e} \frac{dN_i}{dx}\frac{dN_j}{dx} u_j d\Omega = \int_{\Omega^e} N_i f^h d\Omega, \tag{44}$$

where Einstein summation notation is employed. Note that in isoparametric formulation, the shape function is a function of the natural coordinate ξ. Therefore, we employ a change of variables,

$$\int_{\Omega^e} \frac{dN_i}{dx}\frac{dN_j}{dx} u_j |J| d\xi = \int_{\Omega^e} N_i(\xi) f^h |J| d\xi, \tag{45}$$

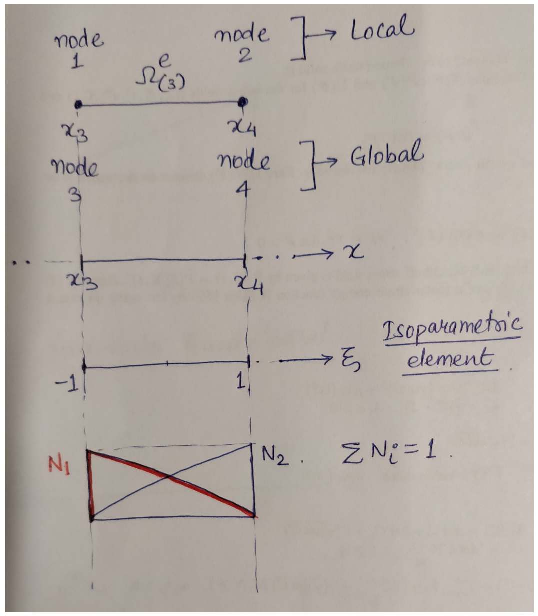


Figure 2: Discretization domain for one-dimensional Poisson equation.

where $|J|$ is the determinant of the Jacobian of the mapping from x to ξ which is derived by the fact that the geometric variables can also be interpolated by the shape functions, i.e., $x = \sum N_i x_i$. In one-dimension, $|J| = h/2$ and $dN_i/dx = (2/h)dN_i(\xi)/d\xi$. Therefore,

$$\int_{\Omega^e} \frac{2}{h}\frac{dN_i(\xi)}{d\xi}\frac{dN_j(\xi)}{d\xi}u_j d\xi = \int_{\Omega^e} \frac{h}{2}N_i(\xi)f^h d\xi, \tag{46}$$

which can be written in the limear matrix form as $\boldsymbol{K}^e\boldsymbol{u}^e = \boldsymbol{f}^e$, where

$$\boldsymbol{K}^e = \frac{2}{h}\int_{\Omega^e}\begin{pmatrix} \frac{dN_1}{d\xi}\frac{dN_1}{d\xi} & \frac{dN_1}{d\xi}\frac{dN_2}{d\xi} \\ \frac{dN_2}{d\xi}\frac{dN_1}{d\xi} & \frac{dN_2}{d\xi}\frac{dN_2}{d\xi} \end{pmatrix} d\xi, \qquad \boldsymbol{f}^e = \frac{h}{2}\int_{\Omega^e}\begin{pmatrix} N_1 f^h \\ N_2 f^h \end{pmatrix} d\xi, \qquad \boldsymbol{u}^e = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{47}$$

where u_1 and u_2 are the unknowns for the particular element e.

Next step is to evaluate the integral expression at the local element level. For this simple problem, it can be done analytically, which gives

$$\boldsymbol{K}^e = \frac{1}{h}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad \boldsymbol{f}^e = h\begin{pmatrix} f_1/3 + f_2/6 \\ f_1/6 + f_2/3 \end{pmatrix}, \qquad \boldsymbol{u}^e = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{48}$$

where we have used the interpolation for $f^h = N_1 f_1 + N_2 f_2$, f_1 and f_2 being the values of the source term at the element nodes respectively.

Let us observe how the discretization stencil at point i looks like by considering two element matrices centered at node i,

$$\left(\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2}\right) = \frac{1}{6}(f_{i-1} + 4f_i + f_{i+1}). \tag{49}$$

Note that the left-hand side stencil is exactly the same as what is obtained from finite difference method (central difference scheme). Therefore, for structured grids, the discretization stencil for the finite element and finite difference methods is the same!

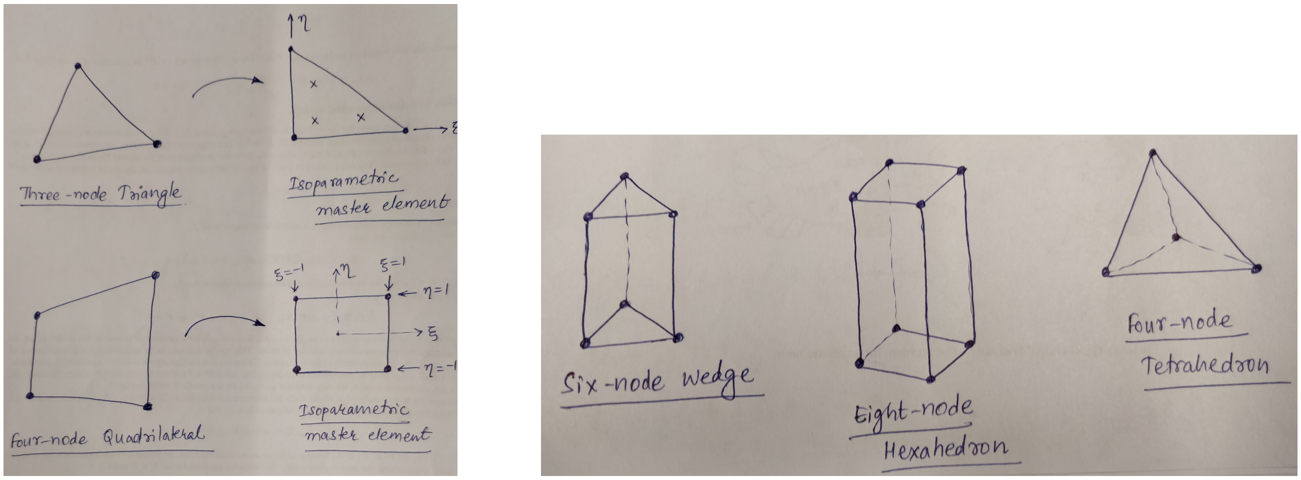
Finally, we assemble the element matrices from all the elements using the global to local node mapping and then solve the global system of equations while applying the Dirichlet boundary conditions to obtain the solution for all the nodes.

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}, \tag{50}$$

where $\boldsymbol{K}$ is the global linear matrix, $\boldsymbol{u}$ is the vector of variables for all the nodes and $\boldsymbol{f}$ is the source vector at each node.

5.2 Higher-dimensional finite elements

In the previous section, we analyzed a simplified one-dimensional example of Poisson equation. In particular, to solve any differential equation in two- or three-dimensions, different types of finite elements are utilized, some of which are shown below with their isoparametric master element.



(a) (b)

Figure 3: (a) Two-dimensional linear finite elements with their master isoparametric element, and (b) three-dimensional linear finite elements.

5.3 Numerical integration: Gauss-Legendre Quadrature Rule

Note that in the previous section, we evaluated the integral analytically. This can be done for simple problems. But as the problem setup becomes large, it is not trivial to do so. Therefore, we opt for numerical integration techniques. A definite integral can be written as an approximation of a weighted sum of the value of the integrand at specific points called as *Gauss points.* The quadrature rule is stated as

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i), \tag{51}$$

where w_i are the weights associated with the n number of quadrature points (or Gauss points) where the integrand $f(x_i)$ is evaluated. This integration is exact for polynomials of degree $2n-1$ or less. The weights and the location of Gauss points for different types of finite elements can be found easily in the literature.

6 Problems

1. Derive the Euler-Lagrange equation for the following functional:

$$\mathscr{F}[y] = 2\pi \int_a^b y\sqrt{1+(y')^2}dx, \tag{52}$$

under the conditions $y(a) = y_1$ and $y(b) = y_2$. Comment on what does it represent.

2. How can one obtain Eq. (8) from Eq. (7)? Show mathematically.

3. Derive and solve the equation obtained for minimizing the following functional on a three-dimensional spatial domain Ω:

$$\mathscr{E}(\phi) = \int_\Omega \left(\frac{\epsilon^2}{2}|\nabla\phi|^2 + F(\phi)\right)d\Omega, \tag{53}$$

where ϵ is a constant and $F(\phi) = (1/4)(\phi^2-1)^2$. Plot the solution obtained in one-dimension.

4. Derive the weak form (Eq. (32)) using variational principle minimizing the following functional:

$$I(u) = \frac{1}{2}\int_{\Omega}(\nabla u)^2 d\Omega - \int_{\Omega} uf d\Omega - \int_{\Gamma_N} uh d\Gamma. \tag{54}$$

5. Derive the isoparametric shape functions for quadratic interpolation in one-dimension.

6. Derive the isoparametric shape functions for the four-node quadrilateral.

References