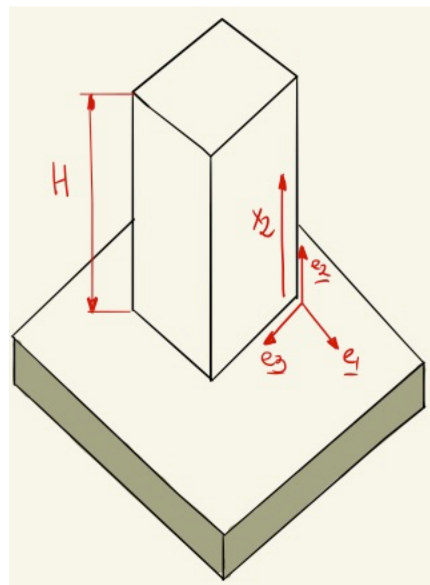
MECH 561 Linear Elasticity

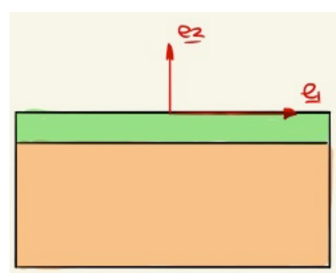
Special Homework Assignment.
Deadline: April 16th at 5.00 pm.

Problem 1: A prismatic concrete column of mass density ρ supports its own weight, as shown in the figure below. (Assume that the solid is subjected to a uniform gravitational body force of magnitude g per unit mass).

- Show that the stress distribution $\sigma_{22} = -\rho g(H - x_2)$ satisfies the equations of static equilibrium $\frac{\partial \sigma_{ij}}{x_i} + \rho b_j = 0$ and also satisfies the boundary conditions $\sigma_{ij} n_i = 0$ on all free boundaries.
- Show that the traction vector acting on a plane with normal $\boldsymbol{n} = \sin\theta\, \boldsymbol{e_1} + \cos\theta\, \boldsymbol{e_2}$ at a height x_2 is given by $\boldsymbol{T} = -\rho g(H - x_2)\cos\theta\, \boldsymbol{e_2}$.
- Deduce that the normal component of traction acting on the plane is $T_n = -\rho g(H - x_2)\cos^2\theta$.
- Show also that the tangential component of traction acting on the plane is $T_t = \rho g(H - x_2)\sin\theta\cos\theta\,(\cos\theta\, \boldsymbol{e_1} - \sin\theta\, \boldsymbol{e_2})$. The easiest way to do this is to note that $\boldsymbol{T} = T_n \boldsymbol{n} + T_t\, \boldsymbol{t}$ and solve for the tangential traction.
- Suppose that the concrete contains a large number of randomly oriented microcracks. A crack which lies at an angle θ to the horizontal will propagate if $|\boldsymbol{T_t}| + \mu T_n > \tau_0$, where μ is the friction coefficient between the faces of the crack and τ_0 is a critical shear stress that is related to the size of the microcracks and the fracture toughness of the concrete, and is therefore a material property.
- Assume that $\mu = 1$. Find the orientation of the microcrack that is most likely to propagate. Hence, find an expression for the maximum possible height of the column. Check your solutions with finite element solvers. Explain differences and similarities.

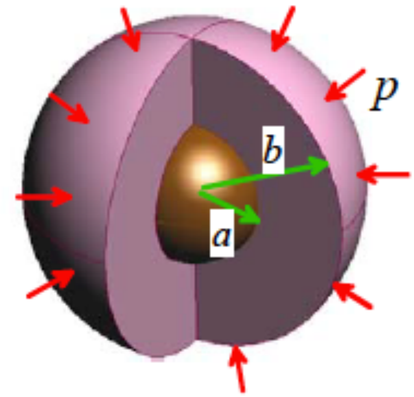


Problem 2: A thin isotropic, linear elastic thin film with Young's modulus E, Poisson's ratio ν and thermal expansion coefficient α is bonded to a stiff substrate. The film is stress free at some initial temperature, and then heated to increase its temperature by T. The substrate prevents the film from stretching in its own plane, so that $\epsilon_{11} = \epsilon_{22} = \epsilon_{12} = 0$, while the surface is traction free, so that the film deforms in a state of plane stress. Calculate the stresses in the film in terms of material properties and temperature, and deduce an expression for the strain energy density in the film.



Problem 3: A baseball can be idealized as a small rubber core with radius a, surrounded by a shell of yarn with outer radius b. As a first approximation, assume that the yarn can be idealized as a

linear elastic solid with Young's modulus E_s and Poisson's ratio ν_s, while the core can be idealized as an incompressible material. Suppose that ball is subjected to a uniform pressure p on its outer surface. Note that, if the core is incompressible, its outer radius cannot change, and therefore the radial displacement u_R at $R = 0$. Calculate the full displacement and stress in the yarn in terms of p and relevant geometric variables and material properties. Check your solutions with finite element solvers. Explain differences and similarities.



Problem 4: Calculate the stresses generated by the Airy function

$$\phi = -\frac{\sigma_0}{2}a^2\log(r) + \frac{\sigma_0}{4}r^2 + \frac{\sigma_0}{4}\left(2a^2 - r^2 - \frac{a^4}{r^2}\right)\cos 2\theta.$$

Verify that the Airy function satisfies the appropriate governing equation. Show that this stress state represents the solution to a large plate containing a circular hole with a radius a at the origin, which is loaded by a tensile stress σ_0 acting parallel to the $\boldsymbol{e_1}$direction. To do this,

- Show that the surface of the hole is traction free, i.e., $\sigma_{rr} = \sigma_{r\theta} = 0$ on $r = a$.
- Show that the stress at $\frac{r}{a} \to \infty$ is $\sigma_{rr} = \sigma_0(1 + \cos 2\theta)\frac{1}{2} = \sigma_0 \cos^2\theta$, and $\sigma_{\theta\theta} = 0$, and $\sigma_{r\theta} = -\sigma_0 \sin 2\theta$.
- Show that the stresses in the point above are equivalent to a stress $\sigma_{11} = \sigma_0$, $\sigma_{22} = \sigma_{12} = 0$.
- Plot a graph showing the variation of the hoop stress $\frac{\sigma_{\theta\theta}}{\sigma_0}$ with θ at $r = a$. (the surface of the hole). What is the value of the maximum stress, and where does it occur?

Problem 5: Using the axisymmetric solution found for a thick-walled cylinder under uniform pressure in its boundaries find the corresponding solution for the two following cases. 1) Infinite medium with a hole under an internal pressure. 2) Infinite medium with a stress-free hole and remote bi-axial stress. Check your solutions with finite element solvers. Explain differences and similarities. *Hint: Using the solution studied in class, take relevant boundary conditions and take limits for the radius and pressures as giving in the examples.*