MECH 550Y-FSI: Part B, Handout 2: Variational formulation for FSI

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In the previous chapter, we studied function spaces and built our fundamentals to write the variational and finite element form for a simple elliptic equation. In this chapter, we look into some challenges in dealing with flow equations, especially the pressure-velocity coupling and techniques which deal with such challenges. We will study about the Petrov-Galerkin stabilization and apply it to the canonical convection-diffusion-reaction equation.

1 Variational form for fluid-structure interaction

As mentioned in the previous chapters, fluid-structure interaction problems involve the coupling of two fields- flow and structure. In the current context, we will consider an incompressible Newtonian fluid with a three-dimensional structure for the variational formulation. However, the variational framework is general in its application and can be applied to any differential equation.

1.1 Review of the partial differential equations

Consider a computational domain $\Omega^{\mathrm{f}}(t) \subset \mathbb{R}^d$ with a piecewise smooth boundary $\Gamma^{\mathrm{f}}(t)$ at time t. The boundary can be decomposed into three components, the Dirichlet boundary $\Gamma^{\mathrm{f}}_D(t)$, the Neumann boundary $\Gamma^{\mathrm{f}}_N(t)$ for the fluid domain and the fluid-structure boundary $\Gamma^{\mathrm{fs}}(t)$. The incompressible flow is modeled by the Navier-Stokes equation

in the ALE reference coordinate system given as

$$\rho^{\mathrm{f}} \left.\frac{\partial \boldsymbol{u}^{\mathrm{f}}}{\partial t}\right|_{\boldsymbol{\chi}} + \rho^{\mathrm{f}}(\boldsymbol{u}^{\mathrm{f}} - \boldsymbol{w}) \cdot \nabla \boldsymbol{u}^{\mathrm{f}} = \nabla \cdot \boldsymbol{\sigma}^{\mathrm{f}} + \boldsymbol{f}^{\mathrm{f}}, \qquad \text{in } \Omega^{\mathrm{f}}(t), \tag{1}$$

$$\nabla \cdot \boldsymbol{u}^{\mathrm{f}} = 0, \qquad \text{in } \Omega^{\mathrm{f}}(t), \tag{2}$$

$$\boldsymbol{u}^{\mathrm{f}} = \boldsymbol{u}^{\mathrm{f}}_D, \qquad \forall \boldsymbol{x}^{\mathrm{f}} \in \Gamma^{\mathrm{f}}_D(t), \tag{3}$$

$$\boldsymbol{\sigma}^{\mathrm{f}} \cdot \mathbf{n}^{\mathrm{f}} = \boldsymbol{h}^{\mathrm{f}}, \qquad \forall \boldsymbol{x}^{\mathrm{f}} \in \Gamma^{\mathrm{f}}_N(t), \tag{4}$$

$$\boldsymbol{u}^{\mathrm{f}} = \boldsymbol{u}^{\mathrm{f}}_0, \qquad \text{on } \Omega^{\mathrm{f}}(0), \tag{5}$$

where ρ^{f} is the fluid density, $\boldsymbol{u}^{\mathrm{f}}$ and $\boldsymbol{w}$ are the fluid and ALE domain velocities respectively and $\boldsymbol{f}^{\mathrm{f}}$ is the body force on the fluid domain. The boundary conditions on the Dirichlet and Neumann boundaries are denoted by $\boldsymbol{u}^{\mathrm{f}}_D$ and $\boldsymbol{h}^{\mathrm{f}}$ respectively with $\mathbf{n}^{\mathrm{f}}$ being the unit normal at $\Gamma^{\mathrm{f}}_N(t)$. The initial condition at $t = 0$ is given by $\boldsymbol{u}^{\mathrm{f}}_0$. The Cauchy stress tensor is represented by $\boldsymbol{\sigma}^{\mathrm{f}}$ for a Newtonian fluid, and is given by

$$\boldsymbol{\sigma}^{\mathrm{f}} = -p\boldsymbol{I} + \boldsymbol{T}^{\mathrm{f}}, \qquad \boldsymbol{T}^{\mathrm{f}} = 2\mu^{\mathrm{f}}\boldsymbol{\epsilon}^{\mathrm{f}}(\boldsymbol{u}^{\mathrm{f}}), \qquad \boldsymbol{\epsilon}^{\mathrm{f}}(\boldsymbol{u}^{\mathrm{f}}) = \frac{1}{2}[\nabla \boldsymbol{u}^{\mathrm{f}} + (\nabla \boldsymbol{u}^{\mathrm{f}})^T], \tag{6}$$

where p, $\boldsymbol{T}^{\mathrm{f}}$, μ^{f} and $\boldsymbol{\epsilon}^{\mathrm{f}}(\boldsymbol{u}^{\mathrm{f}})$ denote the fluid pressure, shear stress tensor, fluid dynamic viscosity and the fluid strain rate tensor respectively.

On the other hand, the structural equations are written in the Lagrangian framework. Consider a structural domain $\Omega^{\mathrm{s}} \subset \mathbb{R}^d$ with a piecewise smooth boundary Γ^{s} at time $t = 0$. Again, the boundary Γ^{s} can be decomposed into Dirichlet boundary Γ^{s}_D, Neumann boundary Γ^{s}_N in the structural domain and the fluid-structure boundary Γ^{fs}. A mapping $\boldsymbol{\varphi}^{\mathrm{s}}(\boldsymbol{x}^{\mathrm{s}}, t)$ maps the domain from its initial configuration Ω^{s} to its deformed configuration $\Omega^{\mathrm{s}}(t)$, where $\boldsymbol{x}^{\mathrm{s}}$ denote the material coordinates. Thus, if $\boldsymbol{\eta}^{\mathrm{s}}(\boldsymbol{x}^{\mathrm{s}}, t)$ denote the displacement field for the structure, $\boldsymbol{\varphi}^{\mathrm{s}}(\boldsymbol{x}^{\mathrm{s}}, t) = \boldsymbol{x}^{\mathrm{s}} + \boldsymbol{\eta}^{\mathrm{s}}(\boldsymbol{x}^{\mathrm{s}}, t)$.

The structural equations can be written as

$$\rho^{\mathrm{s}} \frac{\partial^2 \boldsymbol{\varphi}^{\mathrm{s}}}{\partial t^2} + \nabla \cdot \boldsymbol{\sigma}^{\mathrm{s}} = \boldsymbol{f}^{\mathrm{s}}, \qquad \text{on } \Omega^{\mathrm{s}}, \tag{7}$$

$$\boldsymbol{u}^{\mathrm{s}} = \boldsymbol{u}^{\mathrm{s}}_D, \qquad \forall \boldsymbol{x}^{\mathrm{s}} \in \Gamma^{\mathrm{s}}_D, \tag{8}$$

$$\boldsymbol{\sigma}^{\mathrm{s}} \cdot \mathbf{n}^{\mathrm{s}} = \boldsymbol{h}^{\mathrm{s}}, \qquad \forall \boldsymbol{x}^{\mathrm{s}} \in \Gamma^{\mathrm{s}}_N, \tag{9}$$

$$\boldsymbol{u}^{\mathrm{s}} = \boldsymbol{u}^{\mathrm{s}}_0, \qquad \text{on } \Omega^{\mathrm{s}}, \tag{10}$$

$$\boldsymbol{\varphi}^{\mathrm{s}} = \boldsymbol{\varphi}^{\mathrm{s}}_0, \qquad \text{on } \Omega^{\mathrm{s}}, \tag{11}$$

where ρ^{s} is the structural density, $\boldsymbol{u}^{\mathrm{s}}$ is the velocity of the structural domain and $\boldsymbol{f}^{\mathrm{s}}$ is the body force. The Dirichlet and Neumann conditions on the structural velocity is denoted by $\boldsymbol{u}^{\mathrm{s}}_D$ and $\boldsymbol{u}^{\mathrm{s}}_N$ respectively. The initial condition for the velocity and position are given by $\boldsymbol{u}^{\mathrm{s}}_0$ and $\boldsymbol{\varphi}^{\mathrm{s}}_0$ respectively. The stress tensor $\boldsymbol{\sigma}^{\mathrm{s}}$ is a function of the deformation tensor and is modeled depending on the type of material.

At the fluid-structure interface Γ^{fs},we satisfy the kinematic and dynamic continuity condition for velocity continuity and traction equilibrium. At $t = 0$, let the fluid-structure interface be denoted by $\Gamma^{\mathrm{fs}} = \Gamma^{\mathrm{f}}(0) \cap \Gamma^{\mathrm{s}}$. The interface at time t will then be $\Gamma^{\mathrm{fs}}(t) = \boldsymbol{\varphi}^{\mathrm{s}}(\Gamma^{\mathrm{fs}}, t)$. The required conditions can be mathematically written as

$$\boldsymbol{u}^{\mathrm{f}}(\boldsymbol{\varphi}^{\mathrm{s}}(\boldsymbol{x}^{\mathrm{s}}, t), t) = \boldsymbol{u}^{\mathrm{s}}(\boldsymbol{x}^{\mathrm{s}}, t), \qquad \forall \boldsymbol{x}^{\mathrm{s}} \in \Gamma^{\mathrm{fs}}, \tag{12}$$

$$\int_{\boldsymbol{\varphi}^{\mathrm{s}}(\gamma, t)} \boldsymbol{\sigma}^{\mathrm{f}} \cdot \mathbf{n}^{\mathrm{f}} d\Gamma + \int_{\gamma} \boldsymbol{\sigma}^{\mathrm{s}} \cdot \mathbf{n}^{\mathrm{s}} d\Gamma = 0, \qquad \forall \gamma \subset \Gamma^{\mathrm{fs}}, \tag{13}$$

where $\mathbf{n}^{\mathrm{f}}$ and $\mathbf{n}^{\mathrm{s}}$ denote the unit normals to the deformed fluid element $\boldsymbol{\varphi}^{\mathrm{s}}(\gamma, t)$ and its corresponding undeformed structural element γ in any part of the interface Γ^{fs} respectively.

The motion of the fluid spatial coordinates which are not at Γ^{fs} is modeled as an elastic material in equilibrium by solving the mesh equation given as

$$\nabla \cdot \boldsymbol{\sigma}^{\mathrm{m}} = \mathbf{0}, \qquad \text{on } \Omega^{\mathrm{f}}, \tag{14}$$

$$\boldsymbol{\eta}^{\mathrm{f}} = \boldsymbol{\eta}^{\mathrm{f}}_D, \qquad \forall\ \boldsymbol{x}^{\mathrm{f}} \in \Gamma^{\mathrm{m}}_D, \tag{15}$$

where $\boldsymbol{\sigma}^{\mathrm{m}} = (1 + k_m)[\nabla \boldsymbol{\eta}^{\mathrm{f}} + (\nabla \boldsymbol{\eta}^{\mathrm{f}})^T + (\nabla \cdot \boldsymbol{\eta}^{\mathrm{f}})\boldsymbol{I}]$ is the stress experienced at the fluid spatial coordinates due to the strain induced by the interface deformation, $\boldsymbol{\eta}^{\mathrm{f}}$ represents the displacement motion of the spatial coordinates in the fluid domain and k_m is a local stiffness parameter for the amount of deformation of the spatial coordinates. Here, $\boldsymbol{\eta}^{\mathrm{f}}_D$ is the Dirichlet condition on the fluid mesh displacement at the boundary Γ^{m}_D.

1.2 Semi-discrete formulation for FSI

As the independent variables in the FSI equations are the spatial and temporal coordinates, we need to discretize the domain in both space and time. One way of doing this is to define the trial and test function spaces in the weak form such that they are space and time dependent. Such methods lead to what are called space-time finite element methods. In the present case, we will utilize a simpler approach of discretizing the temporal variable by Taylor series (similar to finite difference methods) first and then write the weak form using spatial dependency of the trial and test function spaces. An example for such discrete time stepping using the generalized-α time integration [1, 2] is described in Section 4.2. While one can use any other time integration technique, we will present the formulation using the generalized-α method for the rest of this chapter.

For the fluid equations, the generalized-α expressions look like:

$$\boldsymbol{u}^{\mathrm{f,n+1}} = \boldsymbol{u}^{\mathrm{f,n}} + \Delta t(\gamma^{\mathrm{f}}\partial_t \boldsymbol{u}^{\mathrm{f,n+1}} + (1-\gamma^{\mathrm{f}})\partial_t \boldsymbol{u}^{\mathrm{f,n}}), \tag{16}$$

$$\partial_t \boldsymbol{u}^{\mathrm{f,n+}\alpha^{\mathrm{f}}_{\mathrm{m}}} = \partial_t \boldsymbol{u}^{\mathrm{f,n}} + \alpha^{\mathrm{f}}_{\mathrm{m}}(\partial_t \boldsymbol{u}^{\mathrm{f,n+1}} - \partial_t \boldsymbol{u}^{\mathrm{f,n}}), \tag{17}$$

$$\boldsymbol{u}^{\mathrm{f,n+}\alpha^{\mathrm{f}}} = \boldsymbol{u}^{\mathrm{f,n}} + \alpha^{\mathrm{f}}(\boldsymbol{u}^{\mathrm{f,n+1}} - \boldsymbol{u}^{\mathrm{f,n}}), \tag{18}$$

and the semi-discrete form for the Navier-Stokes can be written as:

$$\rho^{\mathrm{f}}\partial_t \boldsymbol{u}^{\mathrm{f,n+}\alpha^{\mathrm{f}}_{\mathrm{m}}}\big|_{\boldsymbol{\chi}} + \rho^{\mathrm{f}}(\boldsymbol{u}^{\mathrm{f,n+}\alpha^{\mathrm{f}}} - \boldsymbol{w}^{\mathrm{n+}\alpha^{\mathrm{f}}})\cdot\nabla \boldsymbol{u}^{\mathrm{f,n+}\alpha^{\mathrm{f}}} = \nabla\cdot\boldsymbol{\sigma}^{\mathrm{f,n+}\alpha^{\mathrm{f}}} + \boldsymbol{f}^{\mathrm{n+}\alpha^{\mathrm{f}}}, \qquad \text{in } \Omega^{\mathrm{f}}(t^{\mathrm{n+}\alpha^{\mathrm{f}}}), \tag{19}$$

$$\nabla\cdot\boldsymbol{u}^{\mathrm{f,n+}\alpha^{\mathrm{f}}} = 0, \qquad \text{in } \Omega^{\mathrm{f}}(t^{\mathrm{n+}\alpha^{\mathrm{f}}}). \tag{20}$$

On the other hand, the expressions for the structural equation are

$$\boldsymbol{\varphi}^{\mathrm{s,n+1}} = \boldsymbol{\varphi}^{\mathrm{s,n}} + \Delta t\boldsymbol{u}^{\mathrm{s,n}} + \Delta t^2\bigg(\big(\frac{1}{2}-\beta^{\mathrm{s}}\big)\partial_t \boldsymbol{u}^{\mathrm{s,n}} + \beta^{\mathrm{s}}\partial_t \boldsymbol{u}^{\mathrm{s,n+1}}\bigg), \tag{21}$$

$$\boldsymbol{u}^{\mathrm{s,n+1}} = \boldsymbol{u}^{\mathrm{s,n}} + \Delta t\bigg(\big(1-\gamma^{\mathrm{s}}\big)\partial_t \boldsymbol{u}^{\mathrm{s,n}} + \gamma^{\mathrm{s}}\partial_t \boldsymbol{u}^{\mathrm{s,n+1}}\bigg), \tag{22}$$

$$\partial_t \boldsymbol{u}^{\mathrm{s,n+}\alpha^{\mathrm{s}}_{\mathrm{m}}} = \partial_t \boldsymbol{u}^{\mathrm{s,n}} + \alpha^{\mathrm{s}}_{\mathrm{m}}(\partial_t \boldsymbol{u}^{\mathrm{s,n+1}} - \partial_t \boldsymbol{u}^{\mathrm{s,n}}), \tag{23}$$

$$\boldsymbol{u}^{\mathrm{n+}\alpha^{\mathrm{s}}} = \boldsymbol{u}^{\mathrm{s,n}} + \alpha^{\mathrm{s}}(\boldsymbol{u}^{\mathrm{s,n+1}} - \boldsymbol{u}^{\mathrm{s,n}}), \tag{24}$$

$$\boldsymbol{\varphi}^{\mathrm{n+}\alpha^{\mathrm{s}}} = \boldsymbol{\varphi}^{\mathrm{s,n}} + \alpha^{\mathrm{s}}(\boldsymbol{\varphi}^{\mathrm{s,n+1}} - \boldsymbol{\varphi}^{\mathrm{s,n}}), \tag{25}$$

with the semi-discrete form as

$$\rho^{\mathrm{s}}\partial_t \boldsymbol{u}^{\mathrm{s,n+}\alpha^{\mathrm{s}}_{\mathrm{m}}} + \nabla\cdot\boldsymbol{\sigma}^{\mathrm{s,n+}\alpha^{\mathrm{s}}} = \boldsymbol{f}^{\mathrm{s,n+}\alpha^{\mathrm{s}}}, \qquad \text{on } \Omega^{\mathrm{s}}. \tag{26}$$

The above equations are solved at the time level $\mathrm{n}+\alpha$ and the values at $\mathrm{n}+1$ are updated via a predictor-corrector algorithm (Refer to Section 4.2 for more details).

Although the mesh equation is a steady state equation without the transient term, it can be written at the $n+1$ level and the variables are marched in time as follows:

$$\nabla\cdot\boldsymbol{\sigma}^{\mathrm{m,n+1}} = \boldsymbol{0}, \tag{27}$$

which solves for $\boldsymbol{\eta}^{\mathrm{f,n+1}}$ and the ALE velocity is then given as $\boldsymbol{w}^{\mathrm{n+1}} = (\boldsymbol{\eta}^{\mathrm{f,n+1}} - \boldsymbol{\eta}^{\mathrm{f,n}})/\Delta t$.

1.3 Trial and test function spaces

For writing the weak formulations of the equations discussed above, we need to define the appropriate function spaces to select the trial solution and test or weighting functions which are dependent on the spatial coordinates. For the fluid equation, we define the following spaces

$$\mathscr{V}_{\boldsymbol{\psi}^{\mathrm{f}}} = \{\boldsymbol{\psi}^{\mathrm{f}} \in \mathscr{H}^1(\Omega^{\mathrm{f}}(t^{\mathrm{n+1}}))|\boldsymbol{\psi}^{\mathrm{f}} = \boldsymbol{0} \text{ on } \Gamma^{\mathrm{f}}_D\}, \tag{28}$$

$$\mathscr{V}_q = \{q \in \mathscr{L}^2(\Omega^{\mathrm{f}}(t^{\mathrm{n+1}}))\}, \tag{29}$$

$$\mathscr{S}_{\boldsymbol{u}^{\mathrm{f}}} = \{\boldsymbol{u}^{\mathrm{f}} \in \mathscr{H}^1(\Omega^{\mathrm{f}}(t^{\mathrm{n+1}}))|\boldsymbol{u}^{\mathrm{f}} = \boldsymbol{u}^{\mathrm{f}}_D \text{ on } \Gamma^{\mathrm{f}}_D\}, \tag{30}$$

$$\mathscr{S}_p = \{p \in \mathscr{L}^2(\Omega^{\mathrm{f}}(t^{\mathrm{n+1}}))\}, \tag{31}$$

where $\mathscr{V}_{\boldsymbol{\psi}^{\mathrm{f}}}$ and $\mathscr{V}_q$ denote the test function spaces for the momentum and continuity equations respectively and $\mathscr{S}_{\boldsymbol{u}^{\mathrm{f}}}$ and $\mathscr{S}_p$ denote the spaces from where we select the trial solution for velocity and pressure respectively.

Similarly, for the structural equation, we define the following function spaces

$$\mathscr{V}_{\boldsymbol{\psi}^{\mathrm{s}}} = \{\boldsymbol{\psi}^{\mathrm{s}} \in \mathscr{H}^1(\Omega^{\mathrm{s}}) | \boldsymbol{\psi}^{\mathrm{s}} = \mathbf{0} \text{ on } \Gamma^{\mathrm{s}}_D\}, \tag{32}$$

$$\mathscr{S}_{\boldsymbol{u}^{\mathrm{s}}} = \{\boldsymbol{u}^{\mathrm{s}} \in \mathscr{H}^1(\Omega^{\mathrm{s}}) | \boldsymbol{u}^{\mathrm{s}} = \boldsymbol{u}^{\mathrm{s}}_D \text{ on } \Gamma^{\mathrm{s}}_D\}, \tag{33}$$

where $\mathscr{V}_{\boldsymbol{\psi}^{\mathrm{s}}}$ and $\mathscr{S}_{\boldsymbol{u}^{\mathrm{s}}}$ denote the test function and trial solution spaces for the structural velocity respectively.

For the mesh equation, the following spaces are defined

$$\mathscr{V}_{\boldsymbol{\psi}^{\mathrm{m}}} = \{\boldsymbol{\psi}^{\mathrm{m}} \in \mathscr{H}^1(\Omega^{\mathrm{f}}) | \boldsymbol{\psi}^{\mathrm{m}} = \mathbf{0} \text{ on } \Gamma^{\mathrm{m}}_D\}, \tag{34}$$

$$\mathscr{S}_{\boldsymbol{\eta}^{\mathrm{f}}} = \{\boldsymbol{\eta}^{\mathrm{f}} \in \mathscr{H}^1(\Omega^{\mathrm{f}}) | \boldsymbol{\eta}^{\mathrm{f}} = \boldsymbol{\eta}^{\mathrm{f}}_D \text{ on } \Gamma^{\mathrm{m}}_D\}. \tag{35}$$

Now that we have the appropriate spaces to select the weighting functions and trial solution, we are ready to form the weak form of the flow and structural equations.

1.4 Weak formulation for FSI

In this section, we derive the weak or variational form for the coupled fluid-structure equations. Let us begin with the flow equations. As mentioned earlier, the weak form is written by multiplying the equation by a weighting function and then integrating it over the whole domain. Thus, for the flow equations,

$$\int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} \boldsymbol{\psi}^{\mathrm{f}} \cdot \left(\rho^{\mathrm{f}} \partial_t \boldsymbol{u}^{\mathrm{f,n+\alpha^f_m}}\big|_{\boldsymbol{\chi}} + \rho^{\mathrm{f}} (\boldsymbol{u}^{\mathrm{f,n+\alpha^f}} - \boldsymbol{w}) \cdot \nabla \boldsymbol{u}^{\mathrm{f,n+\alpha^f}} - \nabla \cdot \boldsymbol{\sigma}^{\mathrm{f,n+\alpha^f}} \right) d\Omega = \int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} \boldsymbol{\psi}^{\mathrm{f}} \cdot \boldsymbol{f}^{\mathrm{f,n+\alpha^f}} d\Omega, \tag{36}$$

$$\int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} q(\nabla \cdot \boldsymbol{u}^{\mathrm{f,n+\alpha^f}}) d\Omega = 0, \tag{37}$$

Notice that there are higher order derivatives for velocity in the term $\nabla \cdot \boldsymbol{\sigma}^{\mathrm{f}}$ which makes the requirement for the velocity as $\boldsymbol{u}^{\mathrm{f}} \in \mathscr{H}^2(\Omega^{\mathrm{f}}(t^{\mathrm{n+1}}))$. Therefore, using Green's identity and Gauss' divergence theorem, we relax this requirement.

$$\int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} \boldsymbol{\psi}^{\mathrm{f}} \cdot \left(\rho^{\mathrm{f}} \partial_t \boldsymbol{u}^{\mathrm{f,n+\alpha^f_m}}\big|_{\boldsymbol{\chi}} + \rho^{\mathrm{f}} (\boldsymbol{u}^{\mathrm{f,n+\alpha^f}} - \boldsymbol{w}) \cdot \nabla \boldsymbol{u}^{\mathrm{f,n+\alpha^f}} \right) d\Omega + \int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} \nabla \boldsymbol{\psi}^{\mathrm{f}} : \boldsymbol{\sigma}^{\mathrm{f,n+\alpha^f}} d\Omega$$
$$= \int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} \boldsymbol{\psi}^{\mathrm{f}} \cdot \boldsymbol{f}^{\mathrm{f,n+\alpha^f}} d\Omega + \int_{\Gamma^{\mathrm{f}}} \boldsymbol{\psi}^{\mathrm{f}} \cdot (\mathbf{n}^{\mathrm{f}} \cdot \boldsymbol{\sigma}^{\mathrm{f,n+\alpha^f}}) d\Gamma, \tag{38}$$

$$\int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} q(\nabla \cdot \boldsymbol{u}^{\mathrm{f,n+\alpha^f}}) d\Omega = 0. \tag{39}$$

Now, the continuity requirement on the velocity is $\mathscr{H}^1(\Omega^{\mathrm{f}}(t^{\mathrm{n+1}}))$ and the definitions of the function spaces in Section 1.3 can be employed. The boundary Γ^{f} can be decomposed into Dirichlet, Neumann and fluid-structure boundaries and recalling that $\boldsymbol{\psi}^{\mathrm{f}} = \mathbf{0}$ on Γ^{f}_D,

$$\int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} \boldsymbol{\psi}^{\mathrm{f}} \cdot \left(\rho^{\mathrm{f}} \partial_t \boldsymbol{u}^{\mathrm{f,n+\alpha^f_m}}\big|_{\boldsymbol{\chi}} + \rho^{\mathrm{f}} (\boldsymbol{u}^{\mathrm{f,n+\alpha^f}} - \boldsymbol{w}) \cdot \nabla \boldsymbol{u}^{\mathrm{f,n+\alpha^f}} \right) d\Omega + \int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} \nabla \boldsymbol{\psi}^{\mathrm{f}} : \boldsymbol{\sigma}^{\mathrm{f,n+\alpha^f}} d\Omega$$
$$= \int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} \boldsymbol{\psi}^{\mathrm{f}} \cdot \boldsymbol{f}^{\mathrm{f,n+\alpha^f}} d\Omega + \int_{\Gamma^{\mathrm{f}}_N} \boldsymbol{\psi}^{\mathrm{f}} \cdot \boldsymbol{h}^{\mathrm{f,n+\alpha^f}} d\Gamma + \int_{\Gamma^{\mathrm{fs}}} \boldsymbol{\psi}^{\mathrm{f}} \cdot (\mathbf{n}^{\mathrm{f}} \cdot \boldsymbol{\sigma}^{\mathrm{f,n+\alpha^f}}) d\Gamma, \tag{40}$$

$$\int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} q(\nabla \cdot \boldsymbol{u}^{\mathrm{f,n+\alpha^f}}) d\Omega = 0. \tag{41}$$

We can combine the weak form of the momentum and continuity equations to write the variational statement as: find $[\boldsymbol{u}^{\mathrm{f,n+\alpha^f}}, p^{\mathrm{n+1}}] \in \mathscr{S}_{\boldsymbol{u}^{\mathrm{f}}} \times \mathscr{S}_p$ such that $\forall [\boldsymbol{\psi}^{\mathrm{f}}, q] \in \mathscr{V}_{\boldsymbol{\psi}^{\mathrm{f}}} \times \mathscr{V}_q$,

$$\boxed{\begin{aligned} &\int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} \boldsymbol{\psi}^{\mathrm{f}} \cdot \left(\rho^{\mathrm{f}} \partial_t \boldsymbol{u}^{\mathrm{f,n+\alpha^f_m}}\big|_{\boldsymbol{\chi}} + \rho^{\mathrm{f}} (\boldsymbol{u}^{\mathrm{f,n+\alpha^f}} - \boldsymbol{w}) \cdot \nabla \boldsymbol{u}^{\mathrm{f,n+\alpha^f}} \right) d\Omega + \int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} \nabla \boldsymbol{\psi}^{\mathrm{f}} : \boldsymbol{\sigma}^{\mathrm{f,n+\alpha^f}} d\Omega \\ &+ \int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} q(\nabla \cdot \boldsymbol{u}^{\mathrm{f,n+\alpha^f}}) d\Omega = \int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} \boldsymbol{\psi}^{\mathrm{f}} \cdot \boldsymbol{f}^{\mathrm{f,n+\alpha^f}} d\Omega + \int_{\Gamma^{\mathrm{f}}_N} \boldsymbol{\psi}^{\mathrm{f}} \cdot \boldsymbol{h}^{\mathrm{f,n+\alpha^f}} d\Gamma + \int_{\Gamma^{\mathrm{fs}}} \boldsymbol{\psi}^{\mathrm{f}} \cdot (\mathbf{n}^{\mathrm{f}} \cdot \boldsymbol{\sigma}^{\mathrm{f,n+\alpha^f}}) d\Gamma. \end{aligned}} \tag{42}$$

Proceeding in a similar way for the structural equation, we can write the variational statement as: find $\boldsymbol{u}^{\mathrm{s,n+\alpha^s}} \in \mathscr{S}_{\boldsymbol{u}^{\mathrm{s}}}$ such that $\forall \boldsymbol{\psi}^{\mathrm{s}} \in \mathscr{V}_{\boldsymbol{\psi}^{\mathrm{s}}}$,

$$\begin{aligned}&\int_{\Omega^{\mathrm{s}}} \boldsymbol{\psi}^{\mathrm{s}} \cdot \left(\rho^{\mathrm{s}} \partial_t \boldsymbol{u}^{\mathrm{s,n+\alpha^s_m}}\right) d\Omega + \int_{\Omega^{\mathrm{s}}} \nabla \boldsymbol{\psi}^{\mathrm{s}} : \boldsymbol{\sigma}^{\mathrm{s,n+\alpha^s}} d\Omega \\ &= \int_{\Omega^{\mathrm{s}}} \boldsymbol{\psi}^{\mathrm{s}} \cdot \boldsymbol{f}^{\mathrm{s,n+\alpha^s}} d\Omega + \int_{\Gamma^{\mathrm{s}}_N} \boldsymbol{\psi}^{\mathrm{s}} \cdot \boldsymbol{h}^{\mathrm{s,n+\alpha^s}} d\Gamma + \int_{\Gamma^{\mathrm{fs}}} \boldsymbol{\psi}^{\mathrm{s}} \cdot (\mathbf{n}^{\mathrm{s}} \cdot \boldsymbol{\sigma}^{\mathrm{s,n+\alpha^s}}) d\Gamma. \end{aligned} \tag{43}$$

Similarly for the mesh equation, we get: find $\boldsymbol{\eta}^{\mathrm{f,n+1}} \in \mathscr{S}_{\boldsymbol{\eta}^{\mathrm{f}}}$ such that $\forall \boldsymbol{\psi}^{\mathrm{m}} \in \mathscr{V}_{\boldsymbol{\psi}^{\mathrm{m}}}$,

$$\int_{\Omega^{\mathrm{f}}} \nabla \boldsymbol{\psi}^{\mathrm{m}} : \boldsymbol{\sigma}^{\mathrm{m,n+1}} d\Omega = 0, \tag{44}$$

where $\boldsymbol{\sigma}^{\mathrm{m,n+1}} = (1 + k_m)[\nabla \boldsymbol{\eta}^{\mathrm{f,n+1}} + (\nabla \boldsymbol{\eta}^{\mathrm{f,n+1}})^T + (\nabla \cdot \boldsymbol{\eta}^{\mathrm{f,n+1}})\boldsymbol{I}]$.

Remark 1. *The velocity continuity condition at the fluid-structure interface is satisfied by moving the fluid domain such that the velocity of the fluid domain at the fluid-structure boundary is the same as the structural velocity at the boundary. After solving for the fluid displacement field $\boldsymbol{\eta}^{\mathrm{f}}$, the spatial coordinates are updated as $\boldsymbol{x}^{\mathrm{f}} = \boldsymbol{\chi} + \boldsymbol{\eta}^{\mathrm{f}}(\boldsymbol{\chi}, t)$ for all $\boldsymbol{x}^{\mathrm{f}} \in \Omega^{\mathrm{f}}(t)$ and $\boldsymbol{\chi} \in \Omega^{\mathrm{f}}(0)$.*

2 Finite element formulation for FSI

The next step is to select finite element spaces $\mathscr{V}^h$ and $\mathscr{S}^h$ from the space of test and trial functions $\mathscr{V}$ and $\mathscr{S}$ respectively. We discretize the domain Ω into non-intersecting finite elements $\Omega = \cup_{e=1}^{\mathrm{n_{el}}} \Omega^e$. The discrete spaces for trial and test functions for the fluid equations are defined as

$$\mathscr{V}^h_{\boldsymbol{\psi}^{\mathrm{f}}} = \{\boldsymbol{\psi}^{\mathrm{f}}_h \in \mathscr{H}^1(\Omega^{\mathrm{f}}(t^{\mathrm{n+1}}))\big|\boldsymbol{\psi}^{\mathrm{f}}_h|_{\Omega^e} \in \mathscr{P}_m(\Omega^e) \forall e \text{ and } \boldsymbol{\psi}^{\mathrm{f}}_h = \mathbf{0} \text{ on } \Gamma^{\mathrm{f}}_D\}, \tag{45}$$

$$\mathscr{V}^h_q = \{q_h \in \mathscr{L}^2(\Omega^{\mathrm{f}}(t^{\mathrm{n+1}}))\big|q_h|_{\Omega^e} \in \mathscr{P}_m(\Omega^e) \forall e\}, \tag{46}$$

$$\mathscr{S}^h_{\boldsymbol{u}^{\mathrm{f}}} = \{\boldsymbol{u}^{\mathrm{f}}_h \in \mathscr{H}^1(\Omega^{\mathrm{f}}(t^{\mathrm{n+1}}))\big|\boldsymbol{u}^{\mathrm{f}}_h|_{\Omega^e} \in \mathscr{P}_m(\Omega^e) \forall e \text{ and } \boldsymbol{u}^{\mathrm{f}}_h = \boldsymbol{u}^{\mathrm{f}}_D \text{ on } \Gamma^{\mathrm{f}}_D\}, \tag{47}$$

$$\mathscr{S}^h_p = \{p_h \in \mathscr{L}^2(\Omega^{\mathrm{f}}(t^{\mathrm{n+1}}))\big|p_h|_{\Omega^e} \in \mathscr{P}_m(\Omega^e) \forall e\}, \tag{48}$$

where $\mathscr{P}_m(\Omega^e)$ is the space of polynomials of degree $\leq m$.

Remark 2. *The interpolating space for the velocity and pressure has to be compatible to satisfy the LBB condition (See Section 4.1 for more details). One can circumvent the LBB condition by using stabilization methods like Streamline Upwing Petrov-Galerkin (SUPG) or Galerkin Least Squares (GLS) methods. In the current formulation, we utilize the GLS formulation.*

The finite element variational statement for the element Ω^e for the fluid equations can thus be written as: find $[\boldsymbol{u}^{\mathrm{f,n+\alpha^f}}_h, p^{\mathrm{n+1}}_h] \in \mathscr{S}^h_{\boldsymbol{u}^{\mathrm{f}}} \times \mathscr{S}^h_p$ such that $\forall [\boldsymbol{\psi}^{\mathrm{f}}_h, q_h] \in \mathscr{V}^h_{\boldsymbol{\psi}^{\mathrm{f}}} \times \mathscr{V}^h_q$,

$$\begin{aligned}&\int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} \boldsymbol{\psi}^{\mathrm{f}}_h \cdot \left(\rho^{\mathrm{f}} \partial_t \boldsymbol{u}^{\mathrm{f,n+\alpha^f_m}}_h\big|_{\boldsymbol{\chi}} + \rho^{\mathrm{f}}(\boldsymbol{u}^{\mathrm{f,n+\alpha^f}}_h - \boldsymbol{w}) \cdot \nabla \boldsymbol{u}^{\mathrm{f,n+\alpha^f}}_h\right) d\Omega + \int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} \nabla \boldsymbol{\psi}^{\mathrm{f}}_h : \boldsymbol{\sigma}^{\mathrm{f,n+\alpha^f}}_h d\Omega \\ &+ \int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} q_h (\nabla \cdot \boldsymbol{u}^{\mathrm{f,n+\alpha^f}}_h) d\Omega = \int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} \boldsymbol{\psi}^{\mathrm{f}}_h \cdot \boldsymbol{f}^{\mathrm{f,n+\alpha^f}}_h d\Omega + \int_{\Gamma^{\mathrm{f}}_N} \boldsymbol{\psi}^{\mathrm{f}}_h \cdot \boldsymbol{h}^{\mathrm{f,n+\alpha^f}} d\Gamma + \int_{\Gamma^{\mathrm{fs}}} \boldsymbol{\psi}^{\mathrm{f}}_h \cdot (\mathbf{n}^{\mathrm{f}} \cdot \boldsymbol{\sigma}^{\mathrm{f,n+\alpha^f}}_h) d\Gamma. \end{aligned} \tag{49}$$

The above equation consists of the Galerkin terms without any stabilization terms for the fluid domain. For it to be stable, one needs compatible polynomial spaces for velocity-pressure coupling. With the help of Petrov-Galerkin

stabilization, we introduce extra stabilization terms as:

$$\begin{aligned}
&\int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} \boldsymbol{\psi}_h^{\mathrm{f}} \cdot \left(\rho^{\mathrm{f}} \partial_t \boldsymbol{u}_h^{\mathrm{f,n+\alpha_m^f}} \big|_{\boldsymbol{\chi}} + \rho^{\mathrm{f}} (\boldsymbol{u}_h^{\mathrm{f,n+\alpha^f}} - \boldsymbol{w}) \cdot \nabla \boldsymbol{u}_h^{\mathrm{f,n+\alpha^f}} \right) d\Omega + \int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} \nabla \boldsymbol{\psi}_h^{\mathrm{f}} : \boldsymbol{\sigma}_h^{\mathrm{f,n+\alpha^f}} d\Omega \\
&+ \sum_{e=1}^{\mathrm{n_{el}}} \int_{\Omega^e} \frac{\tau_m}{\rho^{\mathrm{f}}} \left(\rho^{\mathrm{f}} (\boldsymbol{u}_h^{\mathrm{f,n+\alpha^f}} - \boldsymbol{w}) \cdot \nabla \boldsymbol{\psi}_h^{\mathrm{f}} + \nabla q_h \right) \cdot \boldsymbol{R}_m d\Omega^e \\
&+ \int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} q_h (\nabla \cdot \boldsymbol{u}_h^{\mathrm{f,n+\alpha^f}}) d\Omega + \sum_{e=1}^{\mathrm{n_{el}}} \int_{\Omega^e} \nabla \cdot \boldsymbol{\psi}_h^{\mathrm{f}} \tau_c \rho^{\mathrm{f}} \boldsymbol{R}_c d\Omega^e \\
&= \int_{\Omega^{\mathrm{f}}(t^{\mathrm{n+1}})} \boldsymbol{\psi}_h^{\mathrm{f}} \cdot \boldsymbol{f}_h^{\mathrm{f,n+\alpha^f}} d\Omega + \int_{\Gamma_N^{\mathrm{f}}} \boldsymbol{\psi}_h^{\mathrm{f}} \cdot \boldsymbol{h}^{\mathrm{f,n+\alpha^f}} d\Gamma + \int_{\Gamma^{\mathrm{fs}}} \boldsymbol{\psi}_h^{\mathrm{f}} \cdot (\mathbf{n}^{\mathrm{f}} \cdot \boldsymbol{\sigma}_h^{\mathrm{f,n+\alpha^f}}) d\Gamma,
\end{aligned} \tag{50}$$

where the second line represents the stabilization term for the momentum equation and the second term in the third line depicts the same for the continuity equation. $\boldsymbol{R}_m$ and $\boldsymbol{R}_c$ are the residual of the momentum and continuity equations respectively. The stabilization parameters τ_m and τ_c are the least-squares metrics added to the element-level integrals [3, 4, 5, 6] defined as

$$\tau_m = \left[\left(\frac{2}{\Delta t} \right)^2 + (\boldsymbol{u}_h^{\mathrm{f,n+\alpha^f}} - \boldsymbol{w}) \cdot \boldsymbol{G} (\boldsymbol{u}_h^{\mathrm{f,n+\alpha^f}} - \boldsymbol{w}) + C_I \left(\frac{\mu^{\mathrm{f}}}{\rho^{\mathrm{f}}} \right)^2 \boldsymbol{G} : \boldsymbol{G} \right]^{-1/2}, \qquad \tau_c = \frac{1}{\mathrm{tr}(\boldsymbol{G}) \tau_m}, \tag{51}$$

where C_I is a constant derived from inverse estimates [7], tr() denotes the trace and $\boldsymbol{G}$ is the contravariant metric tensor given by

$$\boldsymbol{G} = \frac{\partial \boldsymbol{\xi}^T}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{x}}, \tag{52}$$

where $\boldsymbol{x}$ and $\boldsymbol{\xi}$ are the physical and parametric coordinates respectively.

Coming to the structural equation, we define the finite spaces as follows:

$$\mathscr{V}_{\boldsymbol{\psi}^{\mathrm{s}}}^h = \{ \boldsymbol{\psi}_h^{\mathrm{s}} \in \mathscr{H}^1(\Omega^{\mathrm{s}}) \big| \boldsymbol{\psi}_h^{\mathrm{s}}|_{\Omega^e} \in \mathscr{P}_m(\Omega^e) \forall e \text{ and } \boldsymbol{\psi}_h^{\mathrm{s}} = \mathbf{0} \text{ on } \Gamma_D^{\mathrm{s}} \}, \tag{53}$$

$$\mathscr{S}_{\boldsymbol{u}^{\mathrm{s}}}^h = \{ \boldsymbol{u}_h^{\mathrm{s}} \in \mathscr{H}^1(\Omega^{\mathrm{s}}) \big| \boldsymbol{u}_h^{\mathrm{s}}|_{\Omega^e} \in \mathscr{P}_m(\Omega^e) \forall e \text{ and } \boldsymbol{u}_h^{\mathrm{s}} = \boldsymbol{u}_D^{\mathrm{s}} \text{ on } \Gamma_D^{\mathrm{s}} \}, \tag{54}$$

and the finite element variational statement is written as: find $\boldsymbol{u}_h^{\mathrm{s,n+\alpha^s}} \in \mathscr{S}_{\boldsymbol{u}^{\mathrm{s}}}^h$ such that $\forall \boldsymbol{\psi}_h^{\mathrm{s}} \in \mathscr{V}_{\boldsymbol{\psi}^{\mathrm{s}}}^h$,

$$\begin{aligned}
&\int_{\Omega^{\mathrm{s}}} \boldsymbol{\psi}_h^{\mathrm{s}} \cdot \left(\rho^{\mathrm{s}} \partial_t \boldsymbol{u}_h^{\mathrm{s,n+\alpha_m^s}} \right) d\Omega + \int_{\Omega^{\mathrm{s}}} \nabla \boldsymbol{\psi}_h^{\mathrm{s}} : \boldsymbol{\sigma}_h^{\mathrm{s,n+\alpha^s}} d\Omega \\
&= \int_{\Omega^{\mathrm{s}}} \boldsymbol{\psi}_h^{\mathrm{s}} \cdot \boldsymbol{f}_h^{\mathrm{s,n+\alpha^s}} d\Omega + \int_{\Gamma_N^{\mathrm{s}}} \boldsymbol{\psi}_h^{\mathrm{s}} \cdot \boldsymbol{h}^{\mathrm{s,n+\alpha^s}} d\Gamma + \int_{\Gamma^{\mathrm{fs}}} \boldsymbol{\psi}_h^{\mathrm{s}} \cdot (\mathbf{n}^{\mathrm{s}} \cdot \boldsymbol{\sigma}_h^{\mathrm{s,n+\alpha^s}}) d\Gamma.
\end{aligned} \tag{55}$$

Similarly for the mesh equation, we define the finite element spaces and the variational statement is written as: find $\boldsymbol{\eta}_h^{\mathrm{f,n+1}} \in \mathscr{S}_{\boldsymbol{\eta}^{\mathrm{f}}}^h$ such that $\forall \boldsymbol{\psi}_h^{\mathrm{m}} \in \mathscr{V}_{\boldsymbol{\psi}^{\mathrm{m}}}^h$,

$$\int_{\Omega^{\mathrm{f}}} \nabla \boldsymbol{\psi}_h^{\mathrm{m}} : \boldsymbol{\sigma}_h^{\mathrm{m,n+1}} d\Omega = 0. \tag{56}$$

2.1 Matrix form of the linear system of equations

After the assembly across all the finite elements and mapping to the global nodes for the FSI equations and utilizing linearization strategies like Newton-Raphson, the finite element formulation can be finally written in the matrix form. For the fluid equations, the matrix form looks like the following

$$\begin{pmatrix} \boldsymbol{K}^{\mathrm{f}} & -\boldsymbol{G} \\ \boldsymbol{G}^T & \boldsymbol{C} \end{pmatrix} \begin{pmatrix} \Delta \underline{\boldsymbol{u}}_h^{\mathrm{f,n+\alpha^f}} \\ \Delta \underline{p}_h^{\mathrm{n+1}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{F}^{\mathrm{f}} \\ \mathbf{0} \end{pmatrix}, \tag{57}$$

where $\boldsymbol{K}^{\mathrm{f}}$ consists of the transient, convection, viscous and convective stabilization terms, $\boldsymbol{G}$ is the gradient operator, $\boldsymbol{G}^T$ is the divergence operator for the continuity equation and $\boldsymbol{C}$ consists of the stabilization terms corresponding to

pressure-pressure coupling. The increments in the velocity and pressure for the Newton-Raphson iterative procedure are given by $\Delta \underline{\boldsymbol{u}}_h^{\mathrm{f},\mathrm{n}+\alpha^{\mathrm{f}}}$ and $\Delta \underline{p}_h^{\mathrm{n}+1}$ respectively.

For the structural and mesh equation, we get the following matrix forms:

$$\left(\boldsymbol{K}^{\mathrm{s}}\right)\left(\Delta \underline{\boldsymbol{u}}_h^{\mathrm{s},\mathrm{n}+\alpha^{\mathrm{s}}}\right) = \left(\boldsymbol{F}^{\mathrm{s}}\right), \tag{58}$$

$$\left(\boldsymbol{K}^{\mathrm{m}}\right)\left(\Delta \underline{\boldsymbol{\eta}}_h^{\mathrm{f},\mathrm{n}+1}\right) = \left(\boldsymbol{0}\right), \tag{59}$$

where $\boldsymbol{K}^{\mathrm{s}}$ and $\boldsymbol{K}^{\mathrm{m}}$ are the linearized matrices for the respective equations.

3 Solution procedure

After the temporal and spatial discretization of the FSI equations, they can be solved by either monolithic or partitioned solution strategies. These strategies are discussed in the current section. In the monolithic strategy, the whole problem is considered as a single field and all the components are advanced simultaneously in time. On the other hand, *partitioning* is the process of spatial separation of the discrete model into the interacting components which are called *partitions*. On the other hand, the temporal discretization can be decomposed into what is called *splitting* within a particular time step size (See Fig. 1). The time splitting can be additive or multiplicative.

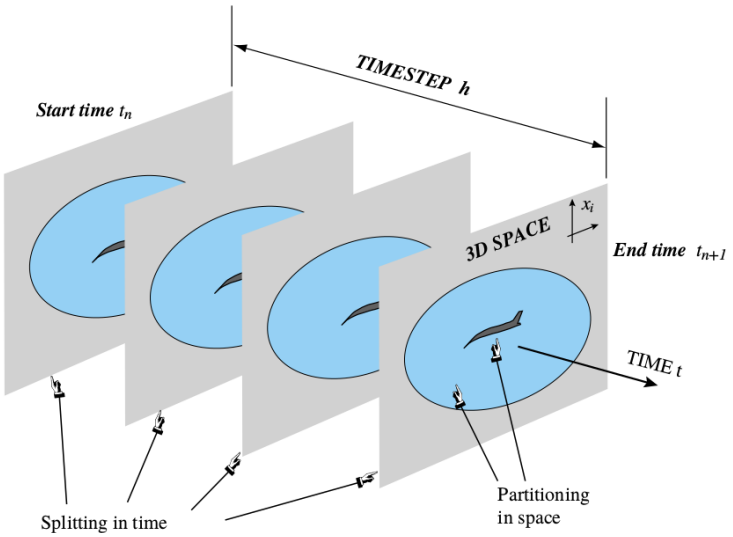

Figure 1: Partition in spatial and splitting in the temporal discretizations.

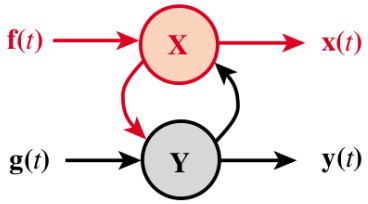

Figure 2: Coupled fields X and Y advancing in time t.

To understand this concept, consider a very simplified problem of two fields X and Y coupled with each other and advancing in time as shown in Fig. 2. Fields X and Y solve for the variables $x(t)$ and $y(t)$ respectively and are coupled as follows:

$$3\dot{x} + 4x - y = f(t), \tag{60}$$

$$\dot{y} + 6y - 2x = g(t), \tag{61}$$

where $f(t)$ and $g(t)$ are the applied forces on the system. Discretizing the system in time using Backward Euler,

$$\begin{pmatrix} 3+4\Delta t & -\Delta t \\ -2\Delta t & 1+6\Delta t \end{pmatrix} \begin{pmatrix} x^{n+1} \\ y^{n+1} \end{pmatrix} = \begin{pmatrix} \Delta t f^{n+1} + 3x^n \\ \Delta t g^{n+1} + y^n \end{pmatrix} \tag{62}$$

whre Δt is the time step size. If one solves Eq. (62) directly in a time step, the technique is called monolithic.

For a partitioned procedure, both the fields are advanced in time separately. It consists of a predictor-corrector algorithm as follows in a particular time step:

1. Predictor step: $y_P^{n+1} = y^n$ or $y_P^{n+1} = y^n + \Delta t \dot{y}^n$
2. Transfer of data: Send the predicted value to the second field
3. Advance the second field: $x^{n+1} = \frac{1}{3+4\Delta t}(\Delta t f^{n+1} + 3x^n + \Delta t y_P^{n+1})$
4. Corrector step: $y^{n+1} = \frac{1}{1+6\Delta t}(\Delta t g^{n+1} + y^n + 2\Delta t x^{n+1})$

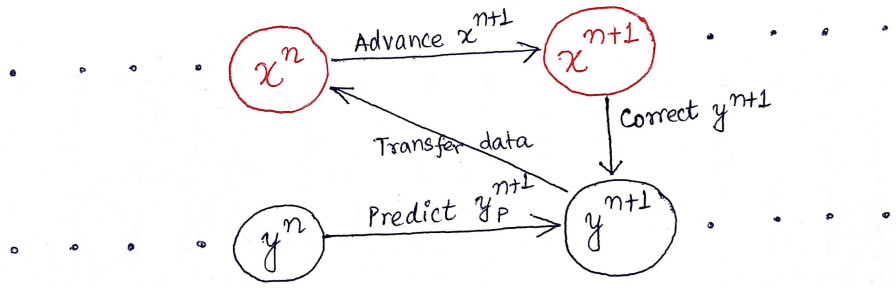

Figure 3: Partitioned staggered technique for coupled fields X and Y advancing in time t.

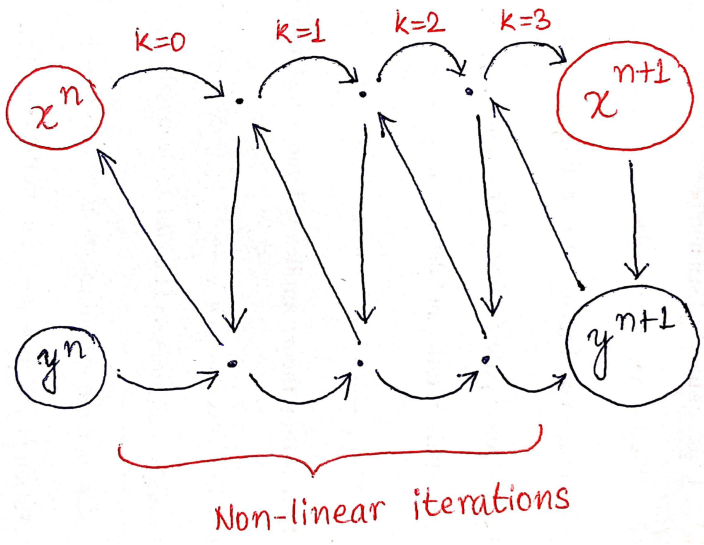

Figure 4: Partitioned staggered technique for coupled fields X and Y advancing in time t.

The above algorithm is called *staggered partitioned* procedure (Fig. 3). For linear problems, it is observed that staggering does not harm stability and accuracy of the problem, given the condition that the predictor is chosen properly. But for more general nonlinear problems, stability could become an issue. On the other hand, accuracy is degraded compared to the same problem solved by a monolithic scheme. This can be resoled by iterating the staggered procedure within a time step so that the scheme now forms a predictor-multicorrector format (Fig. 4). For nonlinear problems, these iterations help in the convergence of the nonlinearities. However, this multiple iterations add to the computational cost and a monolithic scheme could be more advantageous for some problems in that case. Therefore, there is a tradeoff based on the type of problem being solved. Further details can be found in [8, 9].

3.1 Monolithic solution for FSI

In the monolithic technique, the fluid and structural velocities are advanced in time simultaneously and both the fluid and structure are considered as a combined field. The matrix form looks like the following:

$$\begin{pmatrix} \boldsymbol{A}_{\Omega^{\mathrm{s}},\Omega^{\mathrm{s}}} & \boldsymbol{A}_{\Omega^{\mathrm{s}},\Gamma^{\mathrm{f}}} & \boldsymbol{0} \\ \boldsymbol{A}_{\Gamma^{\mathrm{f}},\Omega^{\mathrm{s}}} & \boldsymbol{A}_{\Gamma^{\mathrm{f}},\Gamma^{\mathrm{f}}} & \boldsymbol{A}_{\Gamma^{\mathrm{f}},\Omega^{\mathrm{f}}} \\ \boldsymbol{0} & \boldsymbol{A}_{\Omega^{\mathrm{f}},\Gamma^{\mathrm{f}}} & \boldsymbol{A}_{\Omega^{\mathrm{f}},\Omega^{\mathrm{f}}} \end{pmatrix} \begin{pmatrix} \Delta\underline{\boldsymbol{u}}_{\Omega^{\mathrm{s}}} \\ \Delta\underline{\boldsymbol{y}}_{\Gamma^{\mathrm{f}}} \\ \Delta\underline{\boldsymbol{y}}_{\Omega^{\mathrm{f}}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{R}_1 \\ \boldsymbol{R}_2 \\ \boldsymbol{R}_3 \end{pmatrix}, \tag{63}$$

where $\underline{\boldsymbol{u}}_{\Omega^{\mathrm{s}}}$ denotes the structural velocity, $\underline{\boldsymbol{y}}_{\Gamma^{\mathrm{f}}}$ and $\underline{\boldsymbol{y}}_{\Omega^{\mathrm{f}}}$ are the fluid velocity and pressure variables for the interface and fluid interior nodes respectively.

3.2 Partitioned solution for FSI

In the partitioned procedure for FSI problems, we solve the fluid and structural fields separately and transfer the interface data between the fields. As the equations involved are nonlinear in nature, we opt for the predictor-multicorrector partitioned staggered procedure presented above. The matrices involved in the solution are given in Eqs. (57), (58) and (59) for the fluid, structure and mesh equations. The steps involved in a particular nonlinear iteration of the algorithm are given below:

1. Solve Eq. (58) for the predictor structural velocity based on the fluid forces evaluated at the previous time step t^{n}
2. Solve Eq. (59) for the displacements of the fluid nodes and calculate mesh velocity field
3. Transfer the structural velocity at the fluid-structure interface to the fluid solver by maintaining the velocity continuity condition at the interface, i.e., $\boldsymbol{u}_h^{\mathrm{n}+\alpha^{\mathrm{f}}} = \boldsymbol{w}_h^{\mathrm{n}+\alpha^{\mathrm{f}}}$
4. Solve Eq. (57) for getting the updated velocity and pressure and evaluate the fluid forces
5. Transfer the corrected fluid forces to the structural solver to satisfy the dynamic equilibrium at the fluid-structure interface

The above steps form a nonlinear iteration. The fluid-structure coupling then advances in time at the end of the nonlinear iteration steps once the convergence criteria has been achieved.

4 Appendix

4.1 The pressure-velocity coupling

Consider the Stokes equations for incompressible Newtonian flows in a domain $\Omega^{\mathrm{f}} \subset \mathbb{R}^3$ with a boundary Γ consisting of the Dirichlet and Neumann boundaries Γ_D and Γ_N respectively:

$$-\mu^{\mathrm{f}}\nabla^2\boldsymbol{u}^{\mathrm{f}} + \nabla p = \boldsymbol{f}^{\mathrm{f}}, \qquad \text{in } \Omega^{\mathrm{f}}, \tag{64}$$

$$\nabla \cdot \boldsymbol{u}^{\mathrm{f}} = 0, \qquad \text{in } \Omega^{\mathrm{f}}, \tag{65}$$

$$\boldsymbol{u}^{\mathrm{f}} = \boldsymbol{u}^{\mathrm{f}}_D, \qquad \forall \boldsymbol{x}^{\mathrm{f}} \in \Gamma^{\mathrm{f}}_D, \tag{66}$$

$$-\nabla\boldsymbol{u}^{\mathrm{f}} \cdot \mathbf{n} + p\mathbf{n} = \boldsymbol{h}^{\mathrm{f}}, \qquad \forall \boldsymbol{x}^{\mathrm{f}} \in \Gamma^{\mathrm{f}}_N, \tag{67}$$

where $\boldsymbol{u}^{\mathrm{f}}$ and p are the fluid velocity and pressure respectively, $\boldsymbol{u}^{\mathrm{f}}_D$ and $\boldsymbol{h}^{\mathrm{f}}$ denote the Dirichlet and Neumann conditions respectively.

For writing the weak form of the Stokes equation, we define the following spaces:

$$\mathscr{V}_{\boldsymbol{\psi}^{\mathrm{f}}} = \{\boldsymbol{\psi}^{\mathrm{f}} \in \mathscr{H}^1(\Omega^{\mathrm{f}}) | \boldsymbol{\psi}^{\mathrm{f}} = \boldsymbol{0} \text{ on } \Gamma^{\mathrm{f}}_D\}, \tag{68}$$

$$\mathscr{V}_q = \{q \in \mathscr{L}^2(\Omega^{\mathrm{f}})\}, \tag{69}$$

$$\mathscr{S}_{\boldsymbol{u}^{\mathrm{f}}} = \{\boldsymbol{u}^{\mathrm{f}} \in \mathscr{H}^1(\Omega^{\mathrm{f}}) | \boldsymbol{u}^{\mathrm{f}} = \boldsymbol{u}^{\mathrm{f}}_D \text{ on } \Gamma^{\mathrm{f}}_D\}, \tag{70}$$

$$\mathscr{S}_p = \{p \in \mathscr{L}^2(\Omega^{\mathrm{f}})\}, \tag{71}$$

where $\mathscr{V}_{\boldsymbol{\psi}^{\mathrm{f}}}$ and $\mathscr{V}_q$ denote the test function spaces for the momentum and continuity equations respectively and $\mathscr{S}_{\boldsymbol{u}^{\mathrm{f}}}$ and $\mathscr{S}_p$ denote the spaces from where we select the trial solution for velocity and pressure respectively. We take the

dot product of the momentum equation with the weighting function $\boldsymbol{\psi}^{\mathrm{f}}$ and that of the continuity equation with q and integrate them over the whole domain to write the variational or weak form:

$$\int_{\Omega^{\mathrm{f}}} \boldsymbol{\psi}^{\mathrm{f}} \cdot \Big(-\mu^{\mathrm{f}} \nabla^2 \boldsymbol{u}^{\mathrm{f}} + \nabla p \Big) d\Omega = \int_{\Omega^{\mathrm{f}}} \boldsymbol{\psi}^{\mathrm{f}} \cdot \boldsymbol{f}^{\mathrm{f}} d\Omega, \tag{72}$$

$$\int_{\Omega^{\mathrm{f}}} q(\nabla \cdot \boldsymbol{u}^{\mathrm{f}}) d\Omega = 0, \tag{73}$$

There are higher order derivatives for velocity in the term $\nabla^2 \boldsymbol{u}^{\mathrm{f}}$ which makes the requirement for the velocity as $\boldsymbol{u}^{\mathrm{f}} \in \mathscr{H}^2(\Omega^{\mathrm{f}})$. Therefore, using Green's identity and Gauss' divergence theorem, we relax this requirement.

$$\int_{\Omega^{\mathrm{f}}} \nabla \boldsymbol{\psi}^{\mathrm{f}} : \Big(\mu^{\mathrm{f}} \nabla \boldsymbol{u}^{\mathrm{f}} - p \boldsymbol{I} \Big) d\Omega = \int_{\Omega^{\mathrm{f}}} \boldsymbol{\psi}^{\mathrm{f}} \cdot \boldsymbol{f}^{\mathrm{f}} d\Omega + \int_{\Gamma^{\mathrm{f}}} \boldsymbol{\psi}^{\mathrm{f}} \cdot \boldsymbol{h}^{\mathrm{f}} d\Gamma, \tag{74}$$

$$\int_{\Omega^{\mathrm{f}}} q(\nabla \cdot \boldsymbol{u}^{\mathrm{f}}) d\Omega = 0. \tag{75}$$

Consider a homogeneous Dirichlet and Neumann boundary conditions, i.e., $\boldsymbol{u}^{\mathrm{f}}_D = \boldsymbol{0}$ and $\boldsymbol{h}^{\mathrm{f}} = \boldsymbol{0}$ for simplicity. Therefore, the weak statement can be written as: find $[\boldsymbol{u}^{\mathrm{f}}, p] \in \mathscr{S}_{\boldsymbol{u}^{\mathrm{f}}} \times \mathscr{S}_p$ such that $\forall [\boldsymbol{\psi}^{\mathrm{f}}, q] \in \mathscr{V}_{\boldsymbol{\psi}^{\mathrm{f}}} \times \mathscr{V}_q$,

$$\int_{\Omega^{\mathrm{f}}} \nabla \boldsymbol{\psi}^{\mathrm{f}} : \Big(\mu^{\mathrm{f}} \nabla \boldsymbol{u}^{\mathrm{f}} - p \boldsymbol{I} \Big) d\Omega = \int_{\Omega^{\mathrm{f}}} \boldsymbol{\psi}^{\mathrm{f}} \cdot \boldsymbol{f}^{\mathrm{f}} d\Omega, \tag{76}$$

$$\int_{\Omega^{\mathrm{f}}} q(\nabla \cdot \boldsymbol{u}^{\mathrm{f}}) d\Omega = 0, \tag{77}$$

which in the compact form can be written as

$$a(\boldsymbol{\psi}^{\mathrm{f}}, \boldsymbol{u}^{\mathrm{f}}) + b(\boldsymbol{\psi}^{\mathrm{f}}, p) = (\boldsymbol{\psi}^{\mathrm{f}}, \boldsymbol{f}^{\mathrm{f}}), \tag{78}$$

$$b(\boldsymbol{u}^{\mathrm{f}}, q) = 0, \tag{79}$$

where

$$a(\boldsymbol{\psi}^{\mathrm{f}}, \boldsymbol{u}^{\mathrm{f}}) = \int_{\Omega^{\mathrm{f}}} \nabla \boldsymbol{\psi}^{\mathrm{f}} : \mu^{\mathrm{f}} \nabla \boldsymbol{u}^{\mathrm{f}} d\Omega, \qquad b(\boldsymbol{\psi}^{\mathrm{f}}, p) = \int_{\Omega^{\mathrm{f}}} \nabla \boldsymbol{\psi}^{\mathrm{f}} : (-p \boldsymbol{I}) d\Omega. \tag{80}$$

Remark 3. *Equations (78) and (79) represent the saddle point of the Lagrangian functional*

$$\mathscr{I}(\boldsymbol{\psi}^{\mathrm{f}}, q) = \frac{1}{2} a(\boldsymbol{\psi}^{\mathrm{f}}, \boldsymbol{\psi}^{\mathrm{f}}) + b(\boldsymbol{\psi}^{\mathrm{f}}, q) - (\boldsymbol{\psi}^{\mathrm{f}}, \boldsymbol{f}^{\mathrm{f}}). \tag{81}$$

A saddle point problem is to find a solution $(\boldsymbol{u}^{\mathrm{f}}, p)$ such that the functional is minimized with respect to the first argument and maximized with respect to the second one, i.e., $\mathscr{I}(\boldsymbol{u}^{\mathrm{f}}, q) \leq \mathscr{I}(\boldsymbol{u}^{\mathrm{f}}, p) \leq \mathscr{I}(\boldsymbol{\psi}^{\mathrm{f}}, p)$.

Let us discretize the equation using finite element spaces. The spaces are defined as

$$\mathscr{V}^h_{\boldsymbol{\psi}^{\mathrm{f}}} = \{\boldsymbol{\psi}^{\mathrm{f},h} \in \mathscr{H}^1(\Omega^{\mathrm{f}}) \text{ and } \boldsymbol{\psi}^{\mathrm{f},h} = \boldsymbol{0} \text{ on } \Gamma^{\mathrm{f}}_D\}, \tag{82}$$

$$\mathscr{V}^h_q = \{q^h \in \mathscr{L}^2(\Omega^{\mathrm{f}})\}, \tag{83}$$

$$\mathscr{S}^h_{\boldsymbol{u}^{\mathrm{f}}} = \{\boldsymbol{u}^{\mathrm{f},h} \in \mathscr{H}^1(\Omega^{\mathrm{f}}) \text{ and } \boldsymbol{u}^{\mathrm{f},h} = \boldsymbol{u}^{\mathrm{f}}_D \text{ on } \Gamma^{\mathrm{f}}_D\}, \tag{84}$$

$$\mathscr{S}^h_p = \{p^h \in \mathscr{L}^2(\Omega^{\mathrm{f}})\}. \tag{85}$$

Note that we have not yet defined the polynomial space yet for the test function and solution spaces. Therefore, for each element domain Ω^e, the variational finite element discretization is written as: find $[\boldsymbol{u}^{\mathrm{f},h}, p^h] \in \mathscr{S}^h_{\boldsymbol{u}^{\mathrm{f}}} \times \mathscr{S}^h_p$ such that $\forall [\boldsymbol{\psi}^{\mathrm{f},h}, q^h] \in \mathscr{V}^h_{\boldsymbol{\psi}^{\mathrm{f}}} \times \mathscr{V}^h_q$,

$$\int_{\Omega^e} \nabla \boldsymbol{\psi}^{\mathrm{f},h} : \Big(\mu^{\mathrm{f}} \nabla \boldsymbol{u}^{\mathrm{f},h} - p^h \boldsymbol{I} \Big) d\Omega = \int_{\Omega^e} \boldsymbol{\psi}^{\mathrm{f},h} \cdot \boldsymbol{f}^{\mathrm{f}} d\Omega, \tag{86}$$

$$\int_{\Omega^e} q^h (\nabla \cdot \boldsymbol{u}^{\mathrm{f},h}) d\Omega = 0. \tag{87}$$

Using the polynomial approximation as $u^{\mathrm{f},h}_{(c)} = \sum_{i=1}^{nen} N_i u^{\mathrm{f}}_{i(c)}$ for each velocity component c and $p^h = \sum_{i=1}^{nen} M_i p_i$ and using the Galerkin formulation, the final linear matrix system after assembling the contributions from each element can be written as

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^T & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \underline{\boldsymbol{u}}^{\mathrm{f}} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_m \\ \boldsymbol{0} \end{pmatrix}, \tag{88}$$

where $\underline{\boldsymbol{u}}^{\mathrm{f}}$ and $\underline{p}$ are the unknown vectors for velocity components and pressure respectively at each finite element node, $\boldsymbol{f}_m$ is the right-hand side vector and $\boldsymbol{A}$, $\boldsymbol{B}$ are the block matrices in the left-hand side. Equation (88) is the matrix representation of the saddle point problem. To solve the problem, the left-hand side matrix has to be non-singular, or invertible.

4.1.1 Mixed finite element methods

The polynomial spaces for the velocity and pressure should be compatible with each other based on the requirement of null($\boldsymbol{B}$) = 0. A necessary, but not sufficient condition to ensure this criteria is that $\dim \mathscr{V}^h_q \leq \dim \mathscr{V}^h_{\boldsymbol{\psi}^{\mathrm{f}}}$. The sufficient condition is called Ladyzhenskaya-Babŭska-Brezzi (LBB) or inf-sup condition:

Theorem 1. *There exists a $\beta > 0$ for the bilinear form $b(\boldsymbol{\psi}^{\mathrm{f},h}, q^h)$ such that*

$$\inf_{q^h \in \mathscr{V}^h_q} \sup_{\boldsymbol{\psi}^{\mathrm{f},h} \in \mathscr{V}^h_{\boldsymbol{\psi}^{\mathrm{f}}}} \frac{b(\boldsymbol{\psi}^{\mathrm{f},h}, q^h)}{||\boldsymbol{\psi}^{\mathrm{f},h}||_{\mathscr{V}_{\boldsymbol{\psi}^{\mathrm{f},h}}} ||q^h||_{\mathscr{V}^h_p}} \geq \beta, \tag{89}$$

where β is independent of the mesh size h.

The polynomial spaces for velocity and pressure which satisfy the LBB condition are said to be compatible and LBB stable. For such cases, there exists a unique $\boldsymbol{u}^{\mathrm{f},h} \in \mathscr{S}^h_{\boldsymbol{\psi}^{\mathrm{f},h}}$ and $p^h \in \mathscr{S}^h_q$ for the weak form of the equation.

4.1.2 Stabilization methods

There is a way to circumvent the LBB condition. Till now, we have seen that the only major requirement is for the left-hand side matrix to be non-singular which is not possible due to the zero block matrix. To enforce this positive definiteness property to the global left-hand side matrix, we can modify the weak form of the equation such that we do not get the zero block matrix. These type of methods can be categorized into *Petrov-Galerkin* techniques. In contrast to Galerkin method where we take the weighting function space to be the same as the trial solution space, in the Petrov-Galerkin methods, we perturb the weighting function by certain amount. More details about these methods can be found in [10].

4.2 Generalized-α predictor-corrector method

Here, we discuss the generalized-α [1, 2] predictor-corrector method for the temporal discretization. It enables a user-controlled high frequency damping via a single parameter called the spectral radius ρ_∞, which allows for a coarser discretization in time. We solve the equation at the time interval $\mathrm{n} + \alpha$ while integrating the equation from n to $\mathrm{n}+1$ in time. Suppose the equation can be written as $G(\partial_t u^{\mathrm{n}+\alpha_{\mathrm{m}}}, u^{\mathrm{n}+\alpha}, \phi^{\mathrm{n}+\alpha}) = 0$, where $G(\cdot)$ can be a nonlinear function of the variable ϕ and its first and second-order derivatives, i.e., $u = \partial_t \phi$ and $\partial_t u = \partial^2_{tt}\phi$. The expressions for the variables and their derivatives in the equation can be written as

$$\phi^{\mathrm{n}+1} = \phi^{\mathrm{n}} + \Delta t u^{\mathrm{n}} + \Delta t^2 \big(\beta \partial_t u^{\mathrm{n}+1} + (1/2 - \beta)\partial_t u^{\mathrm{n}}\big), \tag{90}$$

$$u^{\mathrm{n}+1} = u^{\mathrm{n}} + \Delta t \big(\gamma \partial_t u^{\mathrm{n}+1} + (1-\gamma)\partial_t u^{\mathrm{n}}\big), \tag{91}$$

$$\partial_t u^{\mathrm{n}+\alpha_{\mathrm{m}}} = \partial_t u^{\mathrm{n}} + \alpha_{\mathrm{m}} \big(\partial_t u^{\mathrm{n}+1} - \partial_t u^{\mathrm{n}}\big), \tag{92}$$

$$u^{\mathrm{n}+\alpha} = u^{\mathrm{n}} + \alpha(u^{\mathrm{n}+1} - u^{\mathrm{n}}), \tag{93}$$

$$\phi^{\mathrm{n}+\alpha} = \phi^{\mathrm{n}} + \alpha(\phi^{\mathrm{n}+1} - \phi^{\mathrm{n}}), \tag{94}$$

where Δt is the time step size, α_{m}, α, β and γ are defined as

$$\alpha_{\mathrm{m}} = \frac{1}{2}\left(\frac{3-\rho_\infty}{1+\rho_\infty}\right), \qquad \alpha = \frac{1}{1+\rho_\infty}, \qquad \beta = \frac{1}{4}(1 - \alpha_{\mathrm{m}} + \alpha)^2, \qquad \gamma = \frac{1}{2} + \alpha_{\mathrm{m}} - \alpha. \tag{95}$$

Note that the above time integration method simplifies to the following methods under the given conditions:

- Crank-Nicolson method: $\rho_\infty = 1$
- Gear's two-step method: $\rho_\infty = 0$
- Implicit Euler (Backward Euler) method: $\alpha = \alpha_\mathrm{m} = \gamma = 1$

The predictor-corrector algorithm within a time step from n to $\mathrm{n}+1$ consists of the following steps:

1. Predict the variables at $\mathrm{n}+1$.
2. Evaluate the variables at the intermediate time interval $\mathrm{n}+\alpha$.
3. Linearize the nonlinear equation $G(\partial_t u^{\mathrm{n}+\alpha_\mathrm{m}}, u^{\mathrm{n}+\alpha}, \phi^{\mathrm{n}+\alpha}) = 0$ with the help of Newton-Raphson method.
4. Calculate the increments in the variables by solving the linear system of equations.
5. Update the variables at $\mathrm{n}+\alpha$.
6. Correct the variables at $\mathrm{n}+1$ and proceed to the next nonlinear iteration.

References

[1] J. Chung and G. M. Hulbert. A time integration algorithm for structural dynamics with improved numerical dissipation: The generalized-α method. *Journal of Applied Mechanics*, 60(2):371–375, 1993.

[2] K. E. Jansen, C. H. Whitting, and G. M. Hulbert. A generalized-α method for integrating the filtered Navier-Stokes equations with a stabilized finite element method. 190:305–319, 2000.

[3] F. Shakib, T. J. R. Hughes, and Z. Johan. A new finite element formulation for computational fluid dynamics: X. The compressible Euler and Navier-Stokes equations. *Computer Methods in Applied Mechanics and Engineering*, 89:141–219, 1991.

[4] A. N. Brooks and T. J. R. Hughes. Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations. *Computer Methods in Applied Mechanics and Engineering*, 32(1):199–259, 1982.

[5] T. E. Tezduyar, S. Mittal, S. Ray, and R. Shih. Incompressible flow computations with stabilized bilinear and linear equal-order interpolation velocity-pressure elements. *Computer Methods in Applied Mechanics and Engineering*, 95:221–242, 1992.

[6] L. Franca and S. Frey. Stabilized finite element methods: II. The incompressible Navier-Stokes equations. *Computer Methods in Applied Mechanics and Engineering*, 99:209–233, 1992.

[7] I. Harari and T. J. R. Hughes. What are C and h?: Inequalities for the analysis and design of finite element methods. *Computer Methods in Applied Mechanics and Engineering*, 97(2):157–192, 1992.

[8] E. Kuhl, S. Hulshoff, and R. de Borst. An arbitrary Lagrangian Eulerian finite-element approach for fluid-structure interaction phenomena. *International Journal for Numerical Methods in Engineering*, 57(1):117–142, 2003.

[9] Carlos A. Felippa, K.C. Park, and Charbel Farhat. Partitioned analysis of coupled mechanical systems. *Computer Methods in Applied Mechanics and Engineering*, 190(24):3247 – 3270, 2001. Advances in Computational Methods for Fluid-Structure Interaction.

[10] J. Donea and A. Huerta. *Finite Element Methods for Flow Problems*. Wiley, 2003.