

# MATH 521 - Numerical Analysis of Differential Equations

Christoph Ortner, 02/2024

## Assignment 2 : Hilbert Spaces, Weak Form of BVPs

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### Background for Q1 [no points]

You will need the following Poincare-type inequalities:

- (1) Let  $\Omega$  be a connected, bounded, domain and  $\Gamma_D \subset \partial\Omega$  measurable with surface area  $|\Gamma_D| > 0$ , then there exists a constant  $c_P$  such that

$$\|v\|_{L^2(\Omega)} \leq c_P \|\nabla v\|_{L^2(\Omega)} \quad \forall v \in H_{\Gamma_D}^1(\Omega).$$

- (2) Let  $\Omega$  be a simply connected domain then there exists a constant  $c_P$  such that

$$\|v\|_{L^2(\Omega)} \leq c_P \|\nabla v\|_{L^2(\Omega)} \quad \forall v \in H^1(\Omega) \text{ satisfying } (v)_\Omega = 0,$$

where  $(v)_\Omega := |\Omega|^{-1} \int_\Omega v \, dx$ .

Another way (seemingly stronger but equivalent) to state these results is the following:

- (1') Let  $\Omega$  be a connected, bounded, domain and  $\Gamma_D \subset \partial\Omega$  measurable with surface area  $|\Gamma_D| > 0$ , then there exists a constant  $c_P$  such that

$$\|v\|_{L^2(\Omega)} \leq c_P \left( \|v\|_{L^2(\Gamma_D)}^2 + \|\nabla v\|_{L^2(\Omega)}^2 \right)^{1/2} \quad \forall v \in H^1(\Omega).$$

- (2) Let  $\Omega$  be a simply connected domain then there exists a constant  $c_P$  such that

$$\|v\|_{L^2(\Omega)} \leq c_P \left( |(v)_\Omega|^2 + \|\nabla v\|_{L^2(\Omega)}^2 \right)^{1/2} \quad \forall v \in H^1(\Omega).$$

**Note:** if you want to prove these results, it is not too difficult. They can both be proven with the same argument. What you need is the compactness of the embedding  $L^2(\Omega) \subset H^1(\Omega)$ : If  $(u_n)_{n \in \mathbb{N}}$  is a sequence that is bounded in  $H^1(\Omega)$  then there exists  $u \in L^2(\Omega)$  and a subsequence such that  $u_{n_j} \rightarrow u$  strongly in  $L^2$ .

### Q1a: [8]

Let  $\Omega$  be a connected, bounded domain. Are the following spaces  $H$  Hilbert spaces when equipped with their stated inner products? If not, then explain what property is missing (no need to justify it at length)

- (i)  $H = \{v \in C^1(\bar{\Omega}) : v, \nabla v \in L^2(\Omega)\}$   $(u, v)_H = \int_\Omega uv + \nabla u \cdot \nabla v \, dx$
- (ii)  $H = \{v \in L^2(\Omega) : \text{weakly differentiable}, \nabla v \in L^2(\Omega)\}$ ,  $(u, v)_H = \int_\Omega \nabla u \cdot \nabla v \, dx$
- (iii)  $H = \{v \in L^2(\Omega) : \text{weakly differentiable}, \nabla v \in L^2(\Omega)\}$ ,  $(u, v)_H = \int_\Omega uv + \nabla u \cdot \nabla v \, dx$
- (iv)  $H = \{v \in L^2(\Omega) : \text{weakly differentiable}, \nabla v \in L^2(\Omega), v(0) = 0\}$ ,  $(u, v)_H = \int_\Omega \nabla u \cdot \nabla v \, dx$ .

### Solution Q1a

- (i) The given space is not a Hilbert space. Since  $C^1(\bar{\Omega})$  is not complete.
- (ii) The given space is not a Hilbert space. Since the inner product may be zero when the input functions are constant but not zero.
- (iii) The given space is a Hilbert space. Since it is complete and the inner product satisfies the required properties.
- (iv) The given space is a Hilbert space. Since it is complete and the inner product satisfies the required properties.



## Q1b [5+7]

Let  $\Omega$  be a connected, bounded domain, and  $\Gamma_D \subset \partial\Omega$  measurable with surface area  $|\Gamma_D| > 0$ . Are the following spaces  $H$  Hilbert spaces when equipped with their stated inner products? Now please justify your answer in full detail. (except you don't need to show that  $(u, v)_H$  is symmetric and bilinear)

$$(v) H = \{v \in H^1(\Omega) : (v)_\Omega = 0\}, (u, v)_H = \int_\Omega \nabla u \cdot \nabla v dx.$$

$$(vi) H = H^1(\Omega), (u, v)_H = \int_\Omega \nabla u \cdot \nabla v dx + \int_{\Gamma_D} uv dx.$$

### Solution Q1b

(v) Other than the symmetric and bilinear property of the inner product, it is also obvious to show it is positive definite since  $(v)_\Omega = 0$ ,  $(v, v)_H = 0$  if and only if  $v = 0$ . Then the completeness of the given space can be justified by Poincaré inequality as follows:

For any function  $v \in H^1(\Omega)$  with a mean value of zero over a connected, bounded domain  $\Omega$ , the Poincaré inequality asserts that there exists a constant  $C > 0$ , such that:

$$\|v\|_{L^2(\Omega)} \leq C \|\nabla v\|_{L^2(\Omega)}.$$

Given a Cauchy sequence  $\{v_n\}$  in  $H$ , for every  $\epsilon > 0$ , there exists an  $N$  such that for all  $m, n > N$ :

$$\|\nabla(v_n - v_m)\|_{L^2(\Omega)} < \epsilon.$$

Since  $\{v_n\}$  is Cauchy in the gradient norm and  $H^1(\Omega)$  is complete, it converges to a function  $v \in H^1(\Omega)$  in the gradient norm:

$$\lim_{n \rightarrow \infty} \|\nabla(v_n - v)\|_{L^2(\Omega)} = 0.$$

The Poincaré inequality implies this convergence also occurs in the  $L^2$  norm for the sequence differences  $v_n - v$ , as these differences have zero mean. This establishes that  $v_n$  converges to  $v$  in  $H^1(\Omega)$ .

Moreover, the mean value of functions is continuous with respect to  $L^2$  convergence. Since each  $v_n$  has zero mean, so does the limit function  $v$ . Therefore,  $v \in H$ , demonstrating that the given space is complete. Therefore, it is a Hilbert space.

(vi) Since  $H^1$  is complete, the given space is considered to be complete as well.

The positive definiteness of the given space can be proved as follows:

For any non-zero  $v \in H^1(\Omega)$ , the gradient term  $\int_\Omega \nabla v \cdot \nabla v dx$  is strictly positive unless  $v$  is constant across  $\Omega$ .

The boundary term  $\int_{\Gamma_D} v^2 dx$  is non-negative, as it is the integral of the square of  $v$  over  $\Gamma_D$ . It contributes to the inner product only when  $v$  is non-zero on  $\Gamma_D$ . Since  $|\Gamma_D| > 0$ , for  $v$  not to contribute to this term, it would have to be zero almost everywhere on  $\Gamma_D$ .

Given that  $v$  cannot simultaneously be a non-zero constant in  $\Omega$  and zero on  $\Gamma_D$  (assuming  $v \in H^1(\Omega)$ ), the combined inner product  $(u, v)_H$  is positive for any non-zero  $v \in H^1(\Omega)$ . This satisfies the positive definiteness requirement for an inner product on  $H$ . Therefore, it is a Hilbert space.

## Background to Q2 [no points]

Before starting on Q2, review integration by parts in  $\Omega \subset \mathbb{R}^d$ . We introduced this as

$$\int_\Omega \partial_i u \cdot v dx = - \int_\Omega u \partial_i v dx + \int_{\partial\Omega} \nu_i u v dx.$$

From this expression, please derive the following equivalent formulation: if  $g : \Omega \rightarrow \mathbb{R}^d$ ,  $v : \Omega \rightarrow \mathbb{R}$  (both weakly differentiable) then

$$\int_\Omega \operatorname{div} g v dx = - \int_\Omega g \cdot \nabla v dx + \int_{\partial\Omega} (\nu \cdot g) v dx.$$

## Answer

Consider  $\phi = g v$ , the divergence of  $\phi$  can be expanded:

$$\operatorname{div} \phi = \operatorname{div}(g v) = \nabla \cdot (g v) = (\nabla v) \cdot g + v(\nabla \cdot g),$$

Divergence theorem gives:

$$\int_{\Omega} \operatorname{div} \phi \, dx = \int_{\partial \Omega} \nu \cdot (\phi v) \, dx.$$

Substituting the expanded form of  $\operatorname{div} \phi$  and rearranging,

$$\int_{\Omega} v(\nabla \cdot g) \, dx = \int_{\partial \Omega} (\nu \cdot g)v \, dx - \int_{\Omega} (\nabla v) \cdot g \, dx.$$

## Q2: Weak forms of 2nd order BVPs [10+10+10]

For the following three problems, derive the weak form and then use the Lax-Milgram theorem to show that the weak forms have unique solutions. Throughout this question,  $\Omega$  is a connected domain in  $\mathbb{R}^d$ ,  $d > 1$ ,  $p, q \in C(\bar{\Omega})$  with  $c_0 \leq p, q \leq c_1$ ,  $f \in L^2(\Omega)$ ,  $g \in L^2(\partial\Omega)$ .

(i) Neumann problem

$$\begin{aligned} -\operatorname{div}(p\nabla u) + qu &= f, & \text{in } \Omega, \\ p\nu \cdot \nabla u &= g, & \text{on } \partial\Omega. \end{aligned}$$

(ii) Robin problem

$$\begin{aligned} -\operatorname{div}(p\nabla u) &= f, & \text{in } \Omega, \\ p\nu \cdot \nabla u + u &= g, & \text{on } \partial\Omega. \end{aligned}$$

(iii) The classical Neumann problem: in addition to all previous assumptions also assume that  $(f)_\Omega = 0$ .

$$\begin{aligned} -\Delta u &= f, & \text{in } \Omega, \\ \nu \cdot \nabla u &= 0, & \text{on } \partial\Omega. \end{aligned}$$

*HINT for (iii): you need to introduce an additional condition that uniquely determines the solution but doesn't change the problem.*

### Solution Q2(i)

#### Neumann Problem: Weak Form and Uniqueness

Multiply the differential equation by a test function  $v \in H^1(\Omega)$  and integrate over  $\Omega$ :

$$\int_{\Omega} (-\operatorname{div}(p\nabla u) + qu)v \, dx = \int_{\Omega} fv \, dx.$$

Applying integration by parts to the divergence term and substituting the boundary condition  $p\nu \cdot \nabla u = g$  on  $\partial\Omega$ ,

$$\int_{\Omega} p\nabla u \cdot \nabla v \, dx + \int_{\Omega} quv \, dx = \int_{\Omega} fv \, dx + \int_{\partial\Omega} gv \, ds.$$

Thus, The weak form is: Find  $u \in H^1(\Omega)$  such that for all  $v \in H^1(\Omega)$ ,

$$\int_{\Omega} p\nabla u \cdot \nabla v \, dx + \int_{\Omega} quv \, dx = \int_{\Omega} fv \, dx + \int_{\partial\Omega} gv \, ds.$$

To apply the Lax-Milgram theorem,

- The bilinear form  $a(u, v) = \int_{\Omega} p\nabla u \cdot \nabla v \, dx + \int_{\Omega} quv \, dx$ ,
- The linear functional  $L(v) = \int_{\Omega} fv \, dx + \int_{\partial\Omega} gv \, ds$ .

#### Boundedness and Coercivity of the Bilinear Form:

- The bilinear form  $a(u, u)$  is bounded since  $a(u, u) \leq c_1 \int_{\Omega} (|\nabla u| \cdot |\nabla u| + u^2) \, dx = c_1 \|u\|_{H^1}^2$  by C-S inequality.
- The bilinear form  $a(u, u)$  is coercive since  $a(u, u) \geq c_0 \int_{\Omega} (|\nabla u| \cdot |\nabla u| + u^2) \, dx = c_0 \|u\|_{H^1}^2$ .

**Boundedness of the Linear Functional:** The linear functional  $L(v)$  is bounded since  $|L(v)| \leq (\|f\|_{L^2(\Omega)} + C_{tr}\|g\|_{L^2(\Gamma_N)})\|v\|_1$  by C-S inequality and Trace theorem.

Given these properties, the Lax-Milgram theorem guarantees the existence and uniqueness of the solution  $u \in H^1(\Omega)$  to the weak form of this problem.

## Solution Q2(ii)

# Robin Problem Weak Form and Solution Uniqueness

### Problem Statement

Multiply the differential equation by a test function  $v \in H^1(\Omega)$  and integrate over  $\Omega$ :

$$\int_{\Omega} (-\operatorname{div}(p\nabla u))v \, dx = \int_{\Omega} fv \, dx.$$

Applying integration by parts to the divergence term and the divergence theorem,

$$\int_{\Omega} p\nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} (p\nu \cdot \nabla u)v \, ds = \int_{\Omega} fv \, dx.$$

Incorporating the Robin boundary condition  $p\nu \cdot \nabla u + u = g$  on  $\partial\Omega$  leads to the weak form:

$$\int_{\Omega} p\nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} uv \, ds = \int_{\Omega} fv \, dx + \int_{\partial\Omega} gv \, ds.$$

Define a bilinear form  $a(u, v)$  and a linear functional  $L(v)$  as:

$$a(u, v) = \int_{\Omega} p\nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} uv \, ds, \quad L(v) = \int_{\Omega} fv \, dx + \int_{\partial\Omega} gv \, ds.$$

To apply the Lax-Milgram theorem,

- The bilinear form  $a(u, u) = \int_{\Omega} p\nabla u \cdot \nabla u \, dx + \int_{\partial\Omega} uv \, ds$ .
- The linear functional  $L(v) = \int_{\Omega} fv \, dx + \int_{\partial\Omega} gv \, ds$ .

### Boundedness and Coercivity of the Bilinear Form:

- The bilinear form  $a(u, u)$  is bounded: Considering the first term:

$$\left| \int_{\Omega} p\nabla u \cdot \nabla u \, dx \right| \leq c_1 \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}$$

Considering the second term:

$$\begin{aligned} \left| \int_{\partial\Omega} uv \, ds \right| &\leq \|u\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)} \leq C_{tr}^2 \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \\ |a(u, v)| &\leq (c_1 + C_{tr}^2) \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \end{aligned}$$

- The bilinear form  $a(u, u)$  is coercive

$$\int_{\Omega} p|\nabla u| \cdot |\nabla u| \, dx \geq c_0 \int_{\Omega} |\nabla u|^2 \, dx = c_0 \|u\|_{H^1(\Omega)}^2 - c_0 \int_{\Omega} |u|^2 \, dx$$

By Trace theorem,

$$0 \leq \int_{\partial\Omega} u^2 \, ds \leq C_{tr}^2 \|u\|_{H^1(\Omega)}^2$$

I actually do not know how to continue this proof. The Trace theorem is applied here since in this problem the  $uv$  term appears on the boundary of  $\Omega$  only, but I do not know how to relate this to proof the coercivity of  $a(u, u)$ . The same issue exists in the third problem as well.

**Boundedness of the Linear Functional:** The linear functional  $L(v)$  is bounded since  $|L(v)| \leq (\|f\|_{L^2(\Omega)} + C_{tr}\|g\|_{L^2(\Gamma_N)})\|v\|_1$ .

Given these properties, the Lax-Milgram theorem guarantees the existence and uniqueness of the solution  $u \in H^1(\Omega)$  to the weak form of this problem.

## Solution Q2(iii)

Multiply the PDE by a test function  $v \in H^1(\Omega)$  and integrate over  $\Omega$ :

$$\int_{\Omega} (-\Delta u)v \, dx = \int_{\Omega} fv \, dx.$$

Applying integration by parts and considering the Neumann boundary condition leads to:

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} fv \, dx.$$

The weak form is thus: Find  $u \in H^1(\Omega)$  such that for all  $v \in H^1(\Omega)$ ,

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} fv \, dx.$$

Given that the Neumann problem specifies the flux across the boundary but not the value of  $u$  itself, the solution to this problem is not unique; any constant can be added to a solution to produce another solution.

To ensure uniqueness, one could assume the solution has zero mean over  $\Omega$ :

$$(u)_{\Omega} = \frac{1}{|\Omega|} \int_{\Omega} u \, dx = 0.$$

This condition uniquely determines the solution by eliminating the constant ambiguity without changing the essence of the problem.

To apply the Lax-Milgram theorem,

The Lax-Milgram theorem requires:

- The bilinear form  $a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx$ .
- The linear functional  $L(v) = \int_{\Omega} fv \, dx$ .

**Boundedness and Coercivity of the Bilinear Form:**

- The bilinear form  $a(u, v)$  is bounded since  $a(u, v) \leq c_1 \int_{\Omega} (|\nabla u| \cdot |\nabla v|) \, dx = c_1 \|u\|_{H^1}^2$  by C-S inequality.
- The bilinear form  $a(u, v)$  is coercive, same issue here, I do not know.

**Boundedness of the Linear Functional:** The linear functional  $L(v)$  is bounded since  $|L(v)| \leq \|f\|_{L^2(\Omega)} \|v\|_1$  by C-S inequality.

Given these properties, the Lax-Milgram theorem guarantees the existence and uniqueness of the solution  $u \in H^1(\Omega)$  to the weak form of this problem.