1 Question 1

1.1 Deformation Gradient (F)

$$F_{ij} = \frac{\partial x_i}{\partial X_j}$$

Note x_i are the spacial coordinates in the deformed configuration and X_j are the material coordinates in the undeformed (reference) configuration.

The displacement vector u_i is defined as following:

$$u_i = x_i - X_i$$

So the deformation vector becomes:

$$F_{ij} = \frac{\partial(X_i + u_i)}{\partial X_j} = \delta_{ij} + \frac{\partial u_i}{\partial X_j}$$

$$\mathbf{F} = \mathbf{I} + \nabla\mathbf{u}$$

1.2 Determinant of F (J)

$$J = \det(\mathbf{F}) = \det(\mathbf{I} + \nabla\mathbf{u})$$

When the deformation and displacements are infinitesimally small, J could be approximated as the determinant of an identity matrix plus a small perturbation,

$$J \approx 1 + \mathrm{Tr}(\nabla\mathbf{u})$$

where $\mathrm{Tr}(\nabla\mathbf{u})$ is the trace of the displacement gradient, equivalent to the divergence of the displacement field, assuming those higher-order terms in the displacement gradient are negligible.

1.3 Infinitesimal Strain Tensor (E)

The Lagrangian strain tensor $\mathbf{E}_L$can be defined in terms of the deformation gradient F as:

$$\mathbf{E}_L = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$$

For infinitesimallysmall deformations,

$$\begin{aligned}
\mathbf{E} &\approx \frac{1}{2}((\mathbf{I} + \nabla\mathbf{u})^T(\mathbf{I} + \nabla\mathbf{u}) - \mathbf{I}) \\
\iff E_{ij} &= \frac{1}{2}((\delta_{ij} + \frac{\partial u_i}{\partial X_j})^T(\delta_{ij} + \frac{\partial u_i}{\partial X_j}) - \delta_{ij}) \\
\iff &= \frac{1}{2}((\delta_{ij}{}^T + \frac{\partial u_i}{\partial X_j}^T)(\delta_{ij} + \frac{\partial u_i}{\partial X_j}) - \delta_{ij}) \\
\iff &= \frac{1}{2}((\delta_{ji} + \frac{\partial u_j}{\partial X_i})(\delta_{ij} + \frac{\partial u_i}{\partial X_j}) - \delta_{ij}) \\
\iff &= \frac{1}{2}(\delta_{ji}\delta_{ij} + \delta_{ji}\frac{\partial u_i}{\partial X_j} + \delta_{ij}\frac{\partial u_j}{\partial X_i} + \frac{\partial u_j}{\partial X_i}\frac{\partial u_i}{\partial X_j} - \delta_{ij})
\end{aligned}$$

Neglecting the higher order term $\frac{\partial u_j}{\partial X_i}\frac{\partial u_i}{\partial X_j}$, and $\delta_{ji} = \delta_{ij}$,

$$\mathbf{E} \approx \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$$

This is known as the infinitesimal strain tensor. The same result could be deduced from the Eulerian Strain tensor approach as well.

2 Question 2