MECH 570C

Fluid-Structure Interaction

Module 4: Finite Element Method for Continuum Mechanics (Part 2)

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MECH 570C FSI

0. Term project Pitch

1. Galerkin vs. Petrov-Galerkin

→ Stabilization Concept

→ Applications to advection-diffusion

& Continuum mechanics

(Navier-Stokes & nonlinear

solids)

2. Linearization

Review: FEA implementation

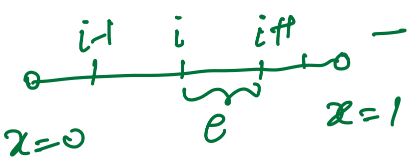
$$\int_0^1$$

→ 1D static diffusion / equilibrium problem

$\boxed{\text{GDE (strong form)}}$

$$\Rightarrow -\frac{\partial^2 u}{\partial x^2} = f \qquad \to R(u) = \mathcal{L}u - f, \quad \mathcal{L} = \frac{\partial^2}{\partial x^2}$$

$$\left.\begin{aligned} &\to u(0) = 0 \\ &-\frac{du}{dx}(1) = \cancel{0}\ 5 \end{aligned}\right\} \text{B.C.'s}$$



$\delta u \rightarrow$ weight function $\quad \psi \in \boxed{H^1[0,1]}$

$\psi(0) = 0$ ↑ Kinematically admissible functions

→ Weak form: $\int \psi\, R(u)\, d\Omega = 0$

$$\Rightarrow \int \psi \left(\frac{\partial^2 u}{\partial x^2} + f\right) d\Omega = 0$$

$$\int_0^1$$

Integration-by-parts

$$\circledast \quad \left.\frac{\partial u}{\partial x}\boxed{\psi}\right|_0^1 - \boxed{\int_0^1} \frac{\partial u}{\partial x}\frac{\partial \psi}{\partial x}\, d\Omega = -\int_0^1 f\,\psi\, d\Omega$$

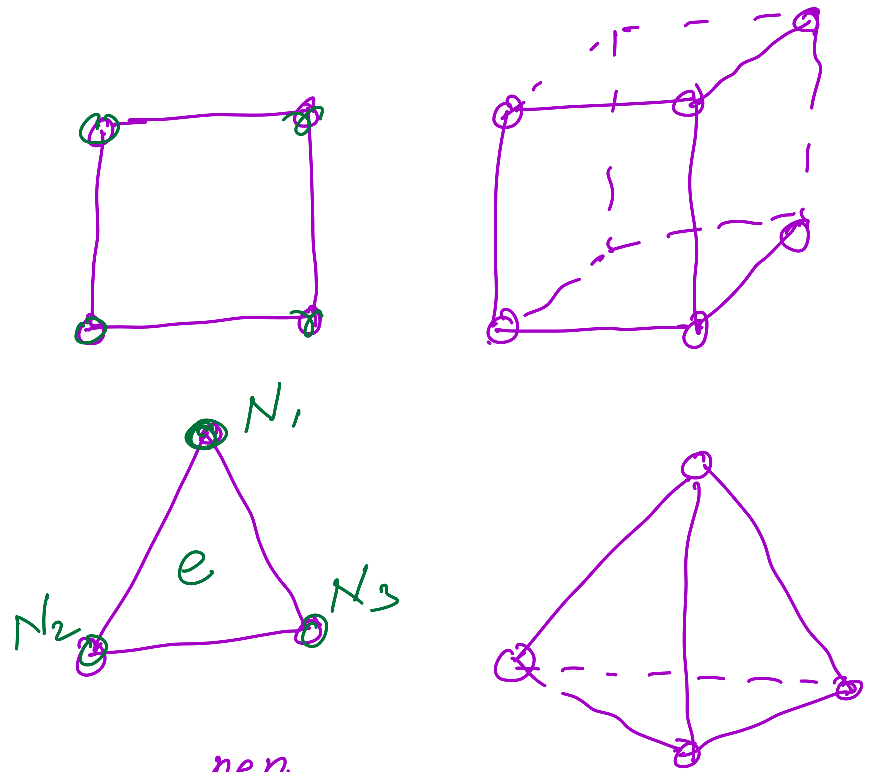
Using above eqn element-by-element:

$$\int_0^1 \square\, dx = \int_0^{0.1} \square\, dx + \int_{0.1}^{0.2} \square\, dx + \cdots \int_{0.9}^{1.0} \square\, dx$$

↑ one element

Types of finite elements

1 D



At each element:

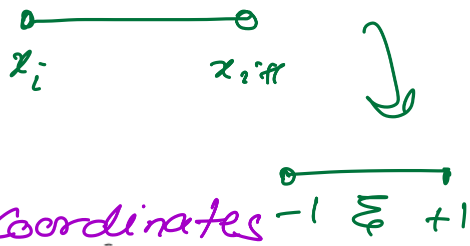
Introduce: $\quad u_e^h = \sum_{i=1}^{nen} u_i \boxed{N_i} \quad$ such that $\quad \sum^{a} N_i = 1$

↑ shape function @ node "a"

Test/weight function

→ $\psi = N$

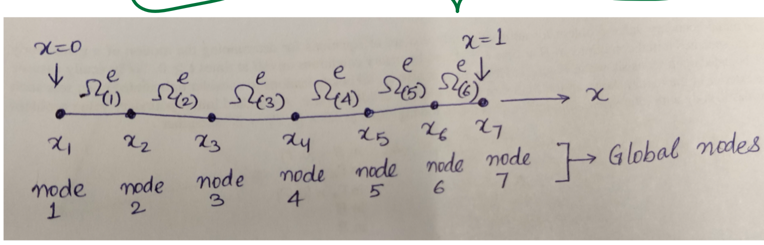
(Galerkin method)



$$\rightarrow \int_{\Omega^e} \frac{dN_i}{dx}\frac{dN_j}{dx}\boxed{u_j}\, d\Omega = \int_{\Omega^e} N_i f^h \, d\Omega,$$

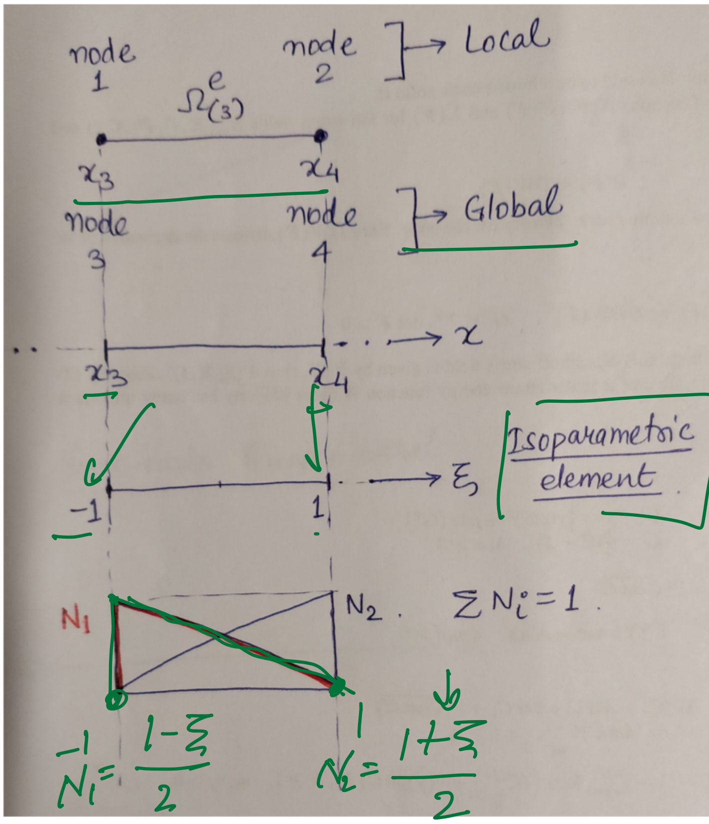
Using transformation to natural coordinates

$$\int_{\Omega^e} \frac{dN_i}{dx}\frac{dN_j}{dx} u_j |J| d\xi = \int_{\Omega^e} N_i(\xi) f^h |J| d\xi,$$



Local element matrix:

$$\boldsymbol{K}^e = \frac{2}{h}\int_{\Omega^e} \begin{pmatrix} \frac{dN_1}{d\xi}\frac{dN_1}{d\xi} & \frac{dN_1}{d\xi}\frac{dN_2}{d\xi} \\ \frac{dN_2}{d\xi}\frac{dN_1}{d\xi} & \frac{dN_2}{d\xi}\frac{dN_2}{d\xi} \end{pmatrix} d\xi, \qquad \boldsymbol{f}^e = \frac{h}{2}\int_{\Omega^e} \begin{pmatrix} N_1 f^h \\ N_2 f^h \end{pmatrix} d\xi, \qquad \boldsymbol{u}^e = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$



$$-\frac{\partial^2 u}{\partial x^2} = f$$

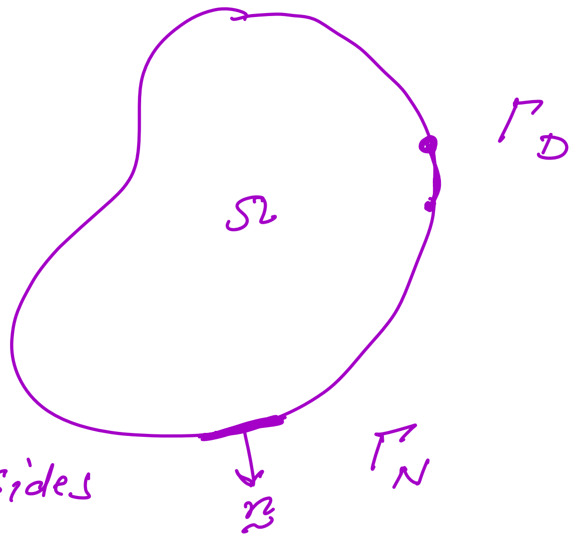
$$\Downarrow$$

$$[K]\{u\} = \{f\}$$

$$\Rightarrow \{u\} = [K]^{-1}\{f\}$$

→ 2D Poisson problem

$$-\nabla^2 u = f$$



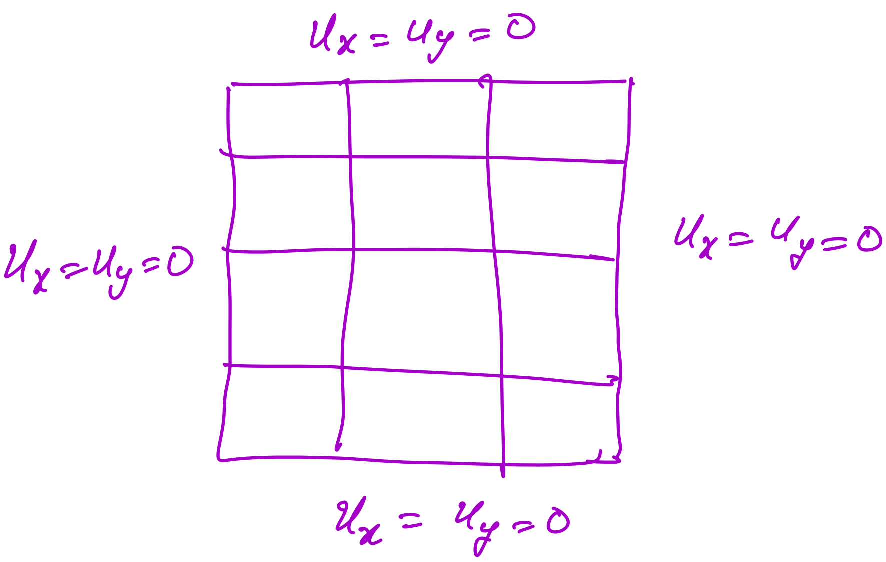
$$u = 0$$

$$\underset{\sim}{n} \cdot \nabla \underset{\sim}{u} = 0$$

Special case:

Ω : $[0, L] \times [0, L]$

Γ_D : $u = 0$ on all four sides



Matlab Code Demo: ✓

Linearization (Increment)

→ Process similar to variation or differentiation

→ first-order Taylor Series

→ Part of Newton-Raphson

Consider $f(\underset{\sim}{x}^{k+1}) = f(\underset{\sim}{x}^{k} + \Delta u^{k})$

$= f(x^{k}) + \left(\dfrac{\partial f}{\partial \underset{\sim}{x}}\right) \cdot \Delta x^{k}$

Continuous form $\rightarrow \underset{\sim}{\mathcal{L}u} - f = R(\underset{\sim}{u})$

$\Downarrow$

Define Residual

$$\rightarrow \underline{R(\underset{\sim}{u})} = \underbrace{a(\underset{\sim}{u}, \underline{\psi})}_{L.H.S} - \underbrace{\ell(\psi)}_{\text{Load vector}}$$

Linearize $\underline{R(\underset{\sim}{u})}$ in the direction of $\underline{\Delta u}$: (increment)

$$L\left[R(\underset{\sim}{u})\right] = \left.\frac{\partial R(u + \epsilon \Delta u)}{\partial \epsilon}\right|_{\epsilon=0}$$

Linearized part of $\underset{\sim}{R}(\underset{\sim}{u})$

$$= \left[\frac{\partial R}{\partial \underset{\sim}{u}}\right]^T \underset{\sim}{\Delta u}$$

$\underline{R(\underset{\sim}{u}) \text{ is nonlinear}}$ but $\underbrace{L[R(\underset{\sim}{u})]}$ is linear w.r.t $\underline{\Delta u}$

$$\underline{\underset{\sim}{R}(\underset{\sim}{u}^{K+1})} \approx \underbrace{\left[\frac{\partial R(u^k)}{\partial \underset{\sim}{u}}\right]^T}_{\text{Tangent matrix } \underline{K}} \underline{\underset{\sim}{\Delta u}}^k + \underbrace{R(u^k)}_{\mathcal{L}u-f} = 0$$

Linearization in Newton-Raphson Method:

$$\rightarrow \underbrace{\left[\frac{\partial R(\underset{\sim}{u}^k)}{\partial \underset{\sim}{u}}\right]^T}_{L.H.S} \underbrace{\Delta u^k}_{\text{update}} = \underbrace{-R(u^k)}_{\text{Residue}}$$

Update: $\underset{\sim}{u}^{\boxed{k+1}} = \underset{\sim}{u}^{\boxed{k}} + \underset{\sim}{\Delta u}^k$

$R(u^k)$ is known, $\left[\frac{\partial R(u^k)}{\partial \underset{\sim}{u}}\right]$?

$$\frac{\partial}{\partial u}\left[R(u)\right] = \underbrace{\frac{\partial}{\partial u}\left[a(u, \psi)\right]}_{L.H.S \text{ matrix}} - \cancel{\frac{\partial}{\partial u} \ell(\psi)} \rightarrow 0$$

Review:

- Difficulties in variational formulation
  - Convection-diffusion-reaction
  - Incompressibility constraint and pressure velocity coupling

- Variational methods
  - Streamline upwind Petrov-Galerkin (SUPG)
  - Galerkin/Least-Squares (GLS)
  - Positivity Preserving Variational (PPV) method

- Application:
  - Incompressible Navier-Stokes equations

Layout

- Before proceeding with the variational formulation of the fluid-structure coupled system, let us look at the convection-diffusion-reaction (CDR) equation which forms a canonical equation for any continuum transport system (e.g., momentum equation, turbulence transport and heat transfer).

- The present module discusses the variational formulation and finite element technique applied to the CDR equation and reviews various types of stabilization methods.

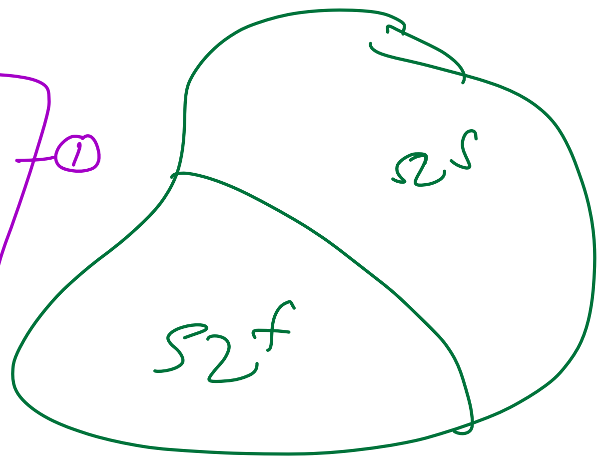
Navier-Stokes Eq^n: $\nabla \cdot \underset{\sim}{\sigma}^f$ ✓

$$\cancel{\rho^f \frac{\partial \underset{\sim}{u}^f}{\partial t}} + \underbrace{\boxed{\cancel{\rho^f \underset{\sim}{u}^f \cdot \nabla \underset{\sim}{u}^f}}}_{\text{Convective}} = \underbrace{-\nabla p}_{\text{pressure}} + \underbrace{\mu \nabla^2 u^f}_{\text{viscous}} + \underbrace{\rho \underset{\sim}{g}}_{\text{body force}}$$

$$\nabla \cdot \underset{\sim}{u}^f = 0$$

viscous-dominated problem

$$\boxed{\begin{aligned} &\mu \nabla^2 u^f - \nabla p + \rho \underset{\sim}{g} = 0 \quad \text{—} (1) \\ &\nabla \cdot \underset{\sim}{u}^f = 0 \quad \text{—} (2) \end{aligned}}$$



$\Downarrow$

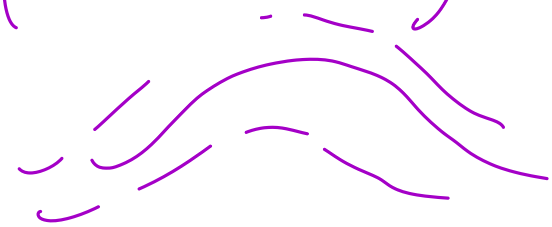
Galerkin finite element has no difficulty!

weak form $\approx$ Galerkin form = Matrix form

$u = T$

Pure diffusion:

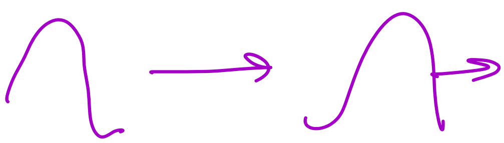
(Parabolic) $\rightarrow$ $$\frac{\partial u}{\partial t} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$



Pure Convection:

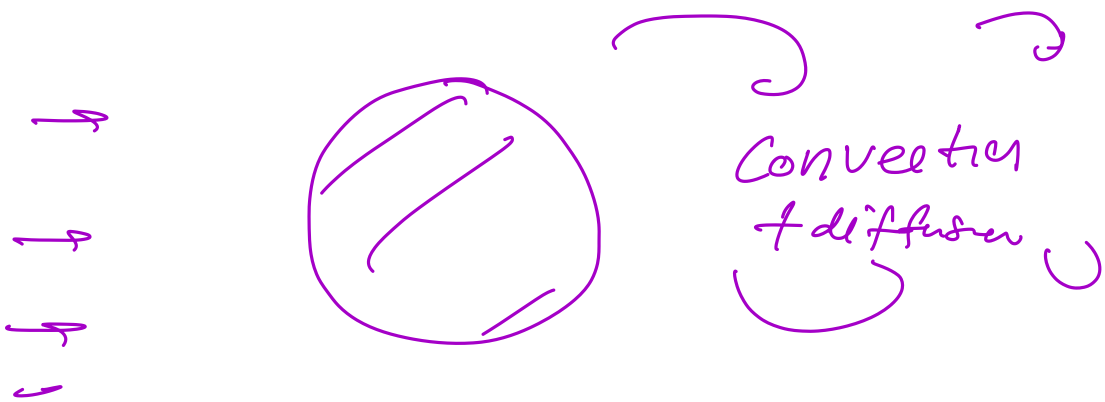
$\rightarrow$ $$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0$$

or $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$



Convection-Diffusion problem:

Example: Momentum balance Laws



interpolation (shape) function

Weight function $\leftarrow \psi = N$

Concept of Petrov - Galerkin

Ex:

1D Convection-diffusion Eqn:

$$\boxed{c}\frac{\partial \boxed{u}}{\partial x} - \boxed{K}\frac{\partial^2 u}{\partial x^2} = \tilde{f}$$

Weak form:
$$\int_{\Omega} \left(\underbrace{\psi\, c \frac{\partial u}{\partial x}}_{\text{convection}} + \underbrace{\frac{d\psi}{dx} K \frac{du}{dx}}_{\text{diffusion}} \right) d\Omega$$

Matrix form: $\psi = N$

$$a^e_{c\{i,j\}} = \int_{\Omega^e} N_i\, u \frac{dN_j}{dx}\, d\Omega$$

Convection matrix

$$a^e_{d\{i,j\}} = \int_{\Omega_e} \frac{dN_i}{dx} \boxed{K} \frac{dN_j}{dx}\, d\Omega$$

Diffusion matrix

Peclet number
$$Pe = \frac{ch}{2K} > 2$$

Consider a general form of PDE

$$\mathcal{L}u = f$$

Differential operator

$$u^h(x) = \sum N_i u_i$$

Galerkin:

$$\int_\Omega \boxed{\psi^*}\left(\mathcal{L}u^h - f\right) d\Omega = 0$$

if $\boxed{\psi^* = N}$

Petrov-Galerkin: $\rightarrow \psi^* \neq N$

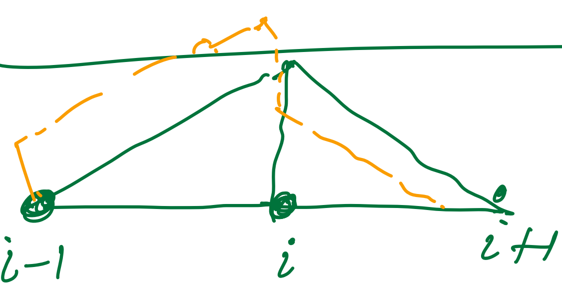
For stabilization.

Weight function

$$\boxed{\psi^* = \psi + \tau \mathcal{L}_{adv} \psi}$$

$\rightarrow$ $c > 0$

Left



Stabilization via product of a perturbation & residual:

Galerkin

$$\int_{\Omega} \underset{\sim}{\psi} \left(\mathcal{L} u^h - f \right) d\Omega \qquad c\frac{\partial}{\partial x}$$

$$+ \sum_{e=1}^{n_{el}} \int_{\Omega_e} \tau \, \mathcal{L}_{adv} \boxed{\underset{\sim}{\psi}} \left(\mathcal{L} u^h - f \right) d\Omega = 0$$

$$\Rightarrow \int_{\Omega} \left(\underset{\sim}{\psi} + \tau \, \mathcal{L}_{adv} \underset{\sim}{\psi} \right) \left(\mathcal{L} u^h - f \right) d\Omega = 0$$

Streamline-Upwind / Petrov-Galerkin (SUPG)

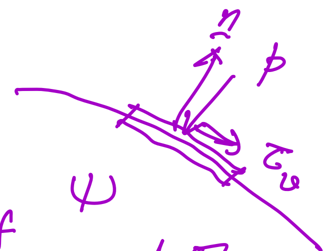
Review:

(I)

$$\int_\Gamma (\underset{\sim}{\sigma}^f \cdot \underset{\sim}{n})\, d\Gamma$$

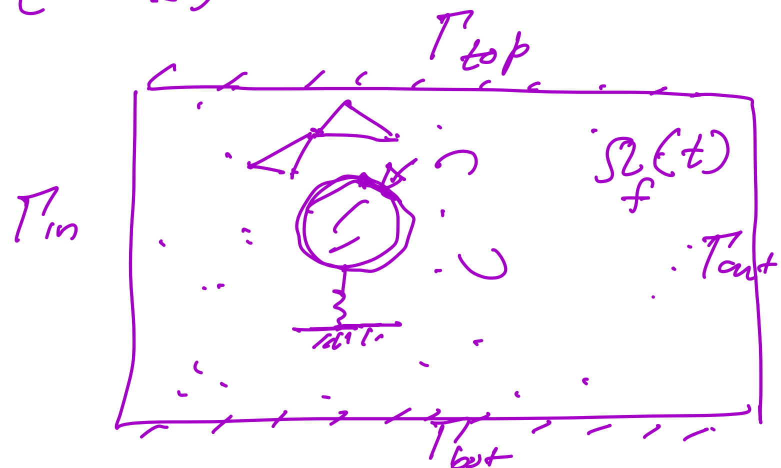
$$\downarrow$$

$$\to \sigma^f = \underbrace{-pI}_{\text{pressure force}} + \underbrace{\mu^f[\nabla u^f + (\nabla u^f)^T]}_{\text{viscous force}}$$



ψ

$$\begin{pmatrix} t_x \\ t_y \end{pmatrix} \qquad \to \int_\Gamma \psi\, (\sigma^f \cdot \underline{n})\, d\Gamma$$



(II) $\to \underline{\nabla \cdot \sigma^m} = 0$

$$\sigma^m = (1+\tau_m)\left\{ \underbrace{[\nabla \eta^f + (\nabla \eta^f)^T]}_{} + \underbrace{(\nabla \cdot \eta^f)\, \underset{\sim}{I}}_{} \right\}$$

Galerkin projection $\rightarrow \int \psi \; \underbrace{\nabla \cdot \underline{\sigma}^{m}} \, d\Omega = 0$

Weight function $\uparrow$ Test function

$\underline{\psi} \rightarrow$ Finite dimensional subspace given by Hilbert space

Find: $\underline{\eta}^{f}$

$\underline{\eta}^{f} = \sum N_i \, \eta_i^{f}$

$\mathcal{V}_{\psi^m} = \{ \psi^m \in H^1(\Omega^f)$

$\quad \psi^m = 0 \in \Gamma_D$

$\uparrow$ Trial / shape function / interpolation

Best possible projection in Hilbert space is given by Galerkin projection

$\nabla \cdot \sigma^{m} = 0$

$\mathcal{L}\underline{u} = \underline{f}$

$\Rightarrow \underline{R} = \mathcal{L}\underline{u} - \underline{f}$

$\rightarrow \int \underset{\sim}{w} \underset{\sim}{R} \, d\Omega = 0$

$\Rightarrow \quad \underset{\sim}{\psi} = \underset{\sim}{N}$

Test function (weight) = shape function

$$\underbrace{\int_{\Omega^f} \psi^m \, \nabla \cdot \sigma^f \, d\Omega} = \int_{\Gamma^f} \psi^m \, \sigma^m \cdot n \, d\Gamma \to 0$$

$$- \int_{\Omega^f} \nabla \psi^m : \sigma^m \, d\Gamma$$

$$= 0$$

$$\Rightarrow \int_{\Omega^f} \nabla \psi^m : \sigma^m \, d\Gamma = 0$$

$$\eta^f = \begin{pmatrix} \eta_i^f(1) \; N_i \\ \eta_i^f(2) \; N_i \end{pmatrix}$$

$$\nabla \eta^f = \begin{pmatrix} \eta_i^f(1) \dfrac{\partial N_i}{\partial x} & (\;) \dfrac{\partial N_i}{\partial y} \\ \eta_i^f(2) & \cdot \quad \cdot \end{pmatrix}_{2 \times 2}$$

$$(\nabla \cdot \eta^f)$$

$$\nabla \psi^m = \begin{pmatrix} \dfrac{\partial N_j}{\partial x} \\[2ex] \dfrac{\partial N_j}{\partial y} \end{pmatrix}$$

$$\int_\Omega \nabla \psi^m : \sigma^m \, d\Omega$$

$$= \int_{\Omega^f} \begin{bmatrix} 3\dfrac{\partial N_i}{\partial x}\dfrac{\partial N_j}{\partial x} + \dfrac{\partial N_i}{\partial y}\dfrac{\partial N_j}{\partial y} \\[2ex] \dfrac{\partial N_i}{\partial y}\dfrac{\partial N_j}{\partial x} + \dfrac{\partial N_i}{\partial x}\dfrac{\partial N_j}{\partial y} \end{bmatrix}$$

$$\dfrac{\partial N_i}{\partial x}\dfrac{\partial N_j}{\partial x} + 3\dfrac{\partial N_i}{\partial y}\dfrac{\partial N_j}{\partial y} \Bigg) \begin{pmatrix} \eta_i^f(1) \\ \eta_i^f(2) \end{pmatrix} d\Omega$$

Summary :

$$\mathcal{L} u = f$$

differential operator

unknown soln

infinite dim

$$\Downarrow$$

$$a(u, \psi) = l(\psi)$$

finite dim

Test function

R.H.S vector

$$u^h = \sum_{i=1}^{N} u_i N_i$$

Galerkin

$N = \psi$

$$\downarrow$$

$$a(u^h, \psi) = l(\psi)$$

$$K_{ij} = A(\underline{N}_i, \psi_i)$$

FEM

$$[K]\{\underline{U}\} = \{F\}$$

$$\rightarrow \frac{\partial^2 u}{\partial x^2} = f$$

Elliptic $\rightarrow$ $\nabla^2 u = \underbrace{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}}$ $= f(x, y)$

(equilibrium

Parabolic $\rightarrow$ $\frac{\partial \underset{\sim}{u}}{\partial t} = \nu \nabla^2 \underset{\sim}{u} + \underset{\sim}{f}$

(Time marching)

Convection effect :

Steady

$$\boxed{c \underbrace{\frac{\partial u}{\partial x}}} - k \frac{\partial^2 u}{\partial x^2} = 0$$

ex

$$u(x) = C_1 e^{(c/k)x} + C_2$$

C_1 & C_2 from Conditions (initial & boundary)

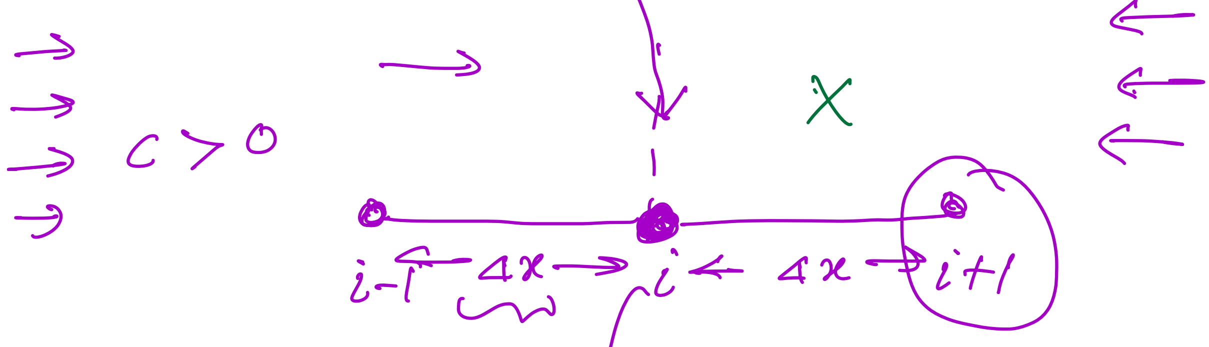
$\rightarrow$ Concept of $\underbrace{\text{upwinding}}$

(Taking care direction of information propagation)

$\underset{\sim}{u} \cdot \nabla \underset{\sim}{u}$

$\boxed{\text{Upwind}}$ Hyperbolic or Convection PDE's

$$\frac{\partial u}{\partial t} + c\left(\frac{\partial u}{\partial x}\right)_i = 0$$



$$\left(\frac{\partial u}{\partial x}\right)_i^{\text{central}} = \frac{u_{i+1} - u_{i-1}}{2\Delta x}$$

Backward difference $\rightarrow$ $$\left(\frac{\partial u}{\partial x}\right)_i^{\text{upwind}} = \frac{u_i - u_{i-1}}{\Delta x}$$

If $c < 0$, forward difference

$$\left(\frac{\partial u}{\partial x}\right)_i^{\text{upwind}} = \frac{u_{i+1} - u_i}{\Delta x}$$

→ Streamline Upwind / Petrov-Galerkin

$\psi \neq N$

Weak Form

$$\int_{\Omega} \psi \left(c \frac{\partial \hat{u}}{\partial x} - k \frac{\partial^2 \hat{u}}{\partial x^2} \right) d\Omega = 0$$

$\psi = N$

$$\hat{u} = \sum N_i(x)\, u_i$$

$$\int_{\Omega} \psi \left(c \frac{\partial \hat{u}}{\partial x} - (k + \tilde{k}) \frac{\partial^2 \hat{u}}{\partial x^2} \right) d\Omega = 0$$

$$\left[\underbrace{\int_{\Omega} c \psi \frac{\partial N}{\partial x} - \hat{k} \psi \frac{\partial^2 N}{\partial x^2} - k \psi \frac{\partial^2 N}{\partial x^2}} \right] \tilde{u} = 0$$

$$* \quad \underbrace{\int_{\Omega} \psi \left(c \frac{\partial \hat{u}}{\partial x} - k \frac{\partial^2 \hat{u}}{\partial x^2} \right)} - \sum_{e=1}^{nel} \int_{\Omega e} \underbrace{\psi \left(\hat{k} \frac{\partial^2 \hat{u}}{\partial x^2} \right)} = 0$$

Numerical diffusion or stabilization

Computational Mechanics of FSI

0. Coding project #2

→ Review τ

1. Stabilization : τ

→ 1D convection-diffusion problem

→ 2D/3D extension

2. Time integration

→ Semi-discrete form

→ Newmark family / α-methods

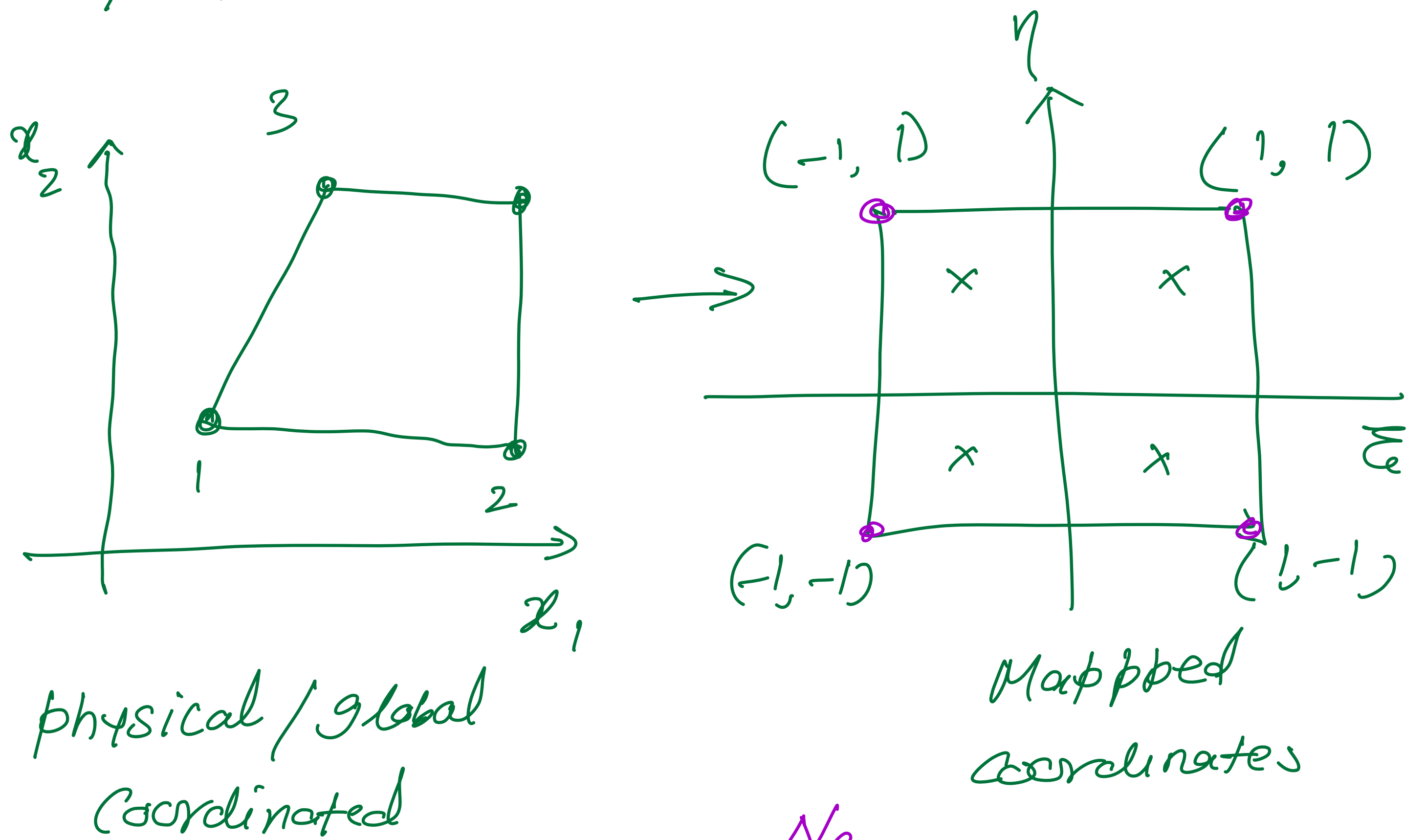
3. FSI Coupling techniques

→ Partitioned vs. monolithic

St. Venant - Kirchoff Solid :

Total Lagrangian update

4-noded linear (P1) elements



Displacement $\rightarrow$ $\underset{\sim}{u}^s = \sum_{i=1}^{N_e} N_i(\underset{\sim}{\xi})\, \underset{\sim}{u}_i^s$

Isoparametric mapping :

nodal coordinates

$$\underset{\sim}{x} = \sum_{i=1}^{N_e} N_i(\underset{\sim}{\xi})\, \underset{\sim}{x}_i$$

$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$ $\qquad$ $N_3 = \frac{1}{4}(1+\xi)(1+\eta)$

$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$ $\qquad$ $N_4 = \frac{1}{4}(1-\xi)(1+\eta)$

Stress - strain relation:

$$\rightarrow \quad \underset{\sim}{S} = f(\underset{\sim}{E})$$

Stress tensor

Strain tensor

Displacement gradient:

$$\frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{X}} = \sum_{i=1}^{N} \frac{\partial N_i(\xi)}{\partial X} u_i$$

Deformation gradient

$$\{\underset{\sim}{F}\} = \{ \underline{F_{11}} \quad \underline{F_{12}} \; \cdots \}$$

$$\underset{\sim}{F} = f(\underset{\sim}{u})$$

Green - Lagrange strain:

$$\{\underset{\sim}{E}\} = \begin{Bmatrix} E_{11} \\ E_{22} \\ 2E_{12} \end{Bmatrix}$$

St. Venant-Kirchoff material

$$\{S\} = [D]\{\underset{\sim}{E}\}$$

$$[D] = \begin{bmatrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

$\underset{\sim}{E}(\underset{\sim}{u})$ is nonlinear, $\bar{E}(\underset{\sim}{u}, \underset{\sim}{\psi})$

$$\underset{\sim}{E}(u, \psi) = \operatorname{sym}\left(\nabla_0 u^T \underset{\sim}{F}\right)$$

$$= \{\underset{\sim}{B}_N\}\{\underset{\sim}{d}\}$$

$$[B]_N = \begin{bmatrix} \underline{F_{11}} \underset{\sim}{N}_{,1} & \cdots & \cdots & - \\ & & & \\ & & & \end{bmatrix}$$

Incremental Strain:

$$\{\underset{\sim}{\Delta E}\} = \{B_N\}\{\Delta d\}$$

Linearization:

$$\iint \underset{\sim}{E} : \underset{\sim}{D} : \Delta E \, d\Omega$$

Summary:

→ SvK Stress-Strain relationship

→ Find linearization terms

→ Use Newton iterations

→ Generalized-α method for time stepping / integration

Stabilization : Pe

1D Steady convection-diffusion problem

$$\underbrace{c\frac{\partial u}{\partial x} - k\frac{\partial^2 u}{\partial x^2}} = 0$$

speed

no-source or reaction ~~effects~~

Galerkin method:

solution

differential operator

$$\mathcal{L}\, u = f \quad \text{in } \Omega$$

$$\mathcal{L} = \left(c\frac{\partial}{\partial x} - k\frac{\partial^2}{\partial x^2}\right)$$

$\Omega = \Omega^f \cup \Omega^s$

Residual form:

$$R = \mathcal{L}u - f = 0$$

Variational/weak form:

$$\int_\Omega (R\,\psi)\, d\Omega = 0$$

test function

Inner product (Hilbert space) → $\langle R, \psi \rangle = 0$

→ $\psi = N + \tau \mathcal{L}_{adv} N$

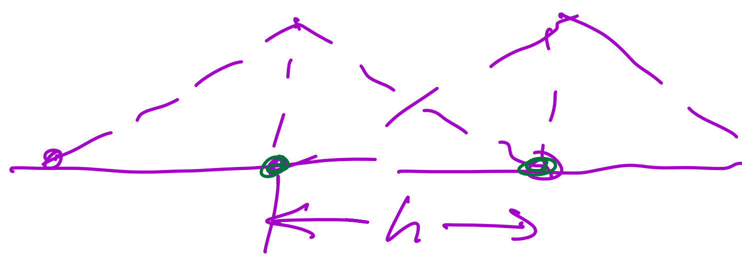
Shape function

$u = \sum N_i \boxed{u_i}$

Galerkin form:

$\psi = \phi = N$

shape function



* Galerkin form of convection-diffusion gives oscillations
* Oscillations depend on $\boxed{Pe = \frac{c\,h}{2K}}$

Upwind & Petrov - Galerkin

Upwind ↑ direction information

$c > 0$

$c < 0$

$\psi \neq N$

ψ: test function

N: shape function

$$\psi = N + \tau \mathcal{L}_{adv} N$$

$$\mathcal{L}_{adv} \to c \frac{\partial}{\partial x}$$

$$\circledast \quad \int_{\Omega} N \underbrace{\left(c \frac{\partial u}{\partial x} - k \frac{\partial^2 u}{\partial x^2} \right)}_{R} d\Omega + \sum_{e=1}^{nel} \int_{\Omega^e} \boxed{\tau}\, c \frac{\partial N}{\partial x} \underbrace{\left(c \frac{\partial u}{\partial x} - k \frac{\partial^2 u}{\partial x^2} \right)}_{R} d\Omega = 0$$

$$\int_{\Omega} \underbrace{\left(N + \tau c \frac{\partial N}{\partial x} \right)}\, R \, d\Omega = 0$$

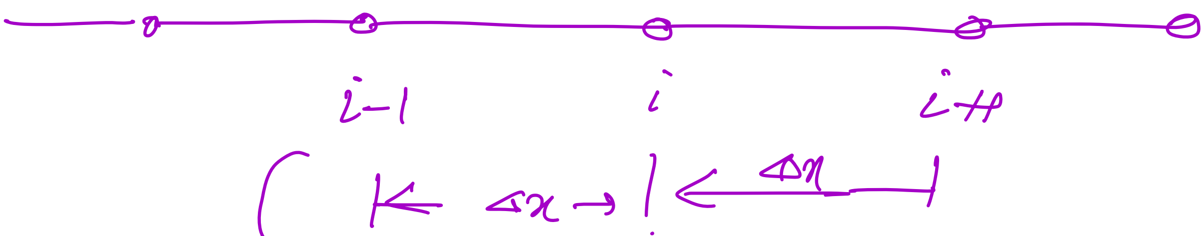
$$\psi = \underbrace{N + \tau c \left(\frac{\partial N}{\partial x} \right)}_{\text{Petrov-Galerkin}} \qquad u = \sum N_i u_i$$

What is τ ? How to set it?

D.E. $\rightarrow c\frac{\partial u}{\partial x} - K\frac{\partial^2 u}{\partial x^2} = 0$

Exact Solution $u(x) = ?$

Exact ? $\rightarrow$ $\boxed{u(x) = C_1 e^{\frac{cx}{K}} + C_2}$



Central difference = Galerkin

$$\left(-\frac{c}{2} - \frac{K}{\Delta x} - \boxed{\tau}\frac{c^2}{\Delta x}\right) u_{i-1}$$

$$+\left(2\frac{K}{\Delta x} + 2\boxed{\tau}\frac{c^2}{\Delta x}\right) u_i$$

$$+\left(\frac{c}{2} - \frac{K}{\Delta x} - \tau\frac{c^2}{\Delta x}\right) u_{i+1} = 0$$

$$u_{i-1} = u(x_{i-1}) = C_1 e^{\frac{c\,x_{i-1}}{k}} + C_2$$

$$u_i = u(x_i) = C_1 e^{\frac{c\,x_i}{k}} + C_2$$

$$u_{i+1} = u(x_{i+1}) = C_1 e^{\frac{c\,x_{i+1}}{k}} + C_2$$

$$\Rightarrow \quad \tau = \frac{-\frac{c}{2}\left(1 + e^{c/k\,\Delta x}\right) - \frac{k}{\Delta x}\left(1 - e^{-c/k\,\Delta x}\right)}{\frac{c^2}{\Delta x}\left(1 - e^{c/k\,\Delta x}\right)}$$

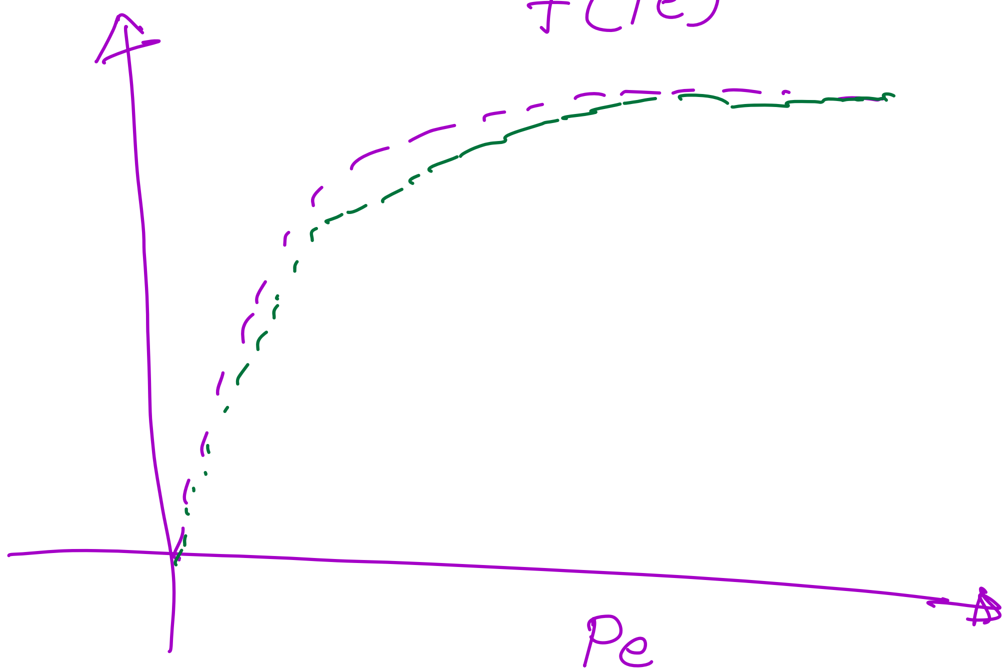
$$\tau = \frac{\Delta x}{2c}\frac{\left(e^{\frac{c\Delta x}{k}} + 1\right)}{\left(e^{\frac{c\Delta x}{k}} - 1\right)} - \frac{k}{c^2}$$

$$\coth(x) = \frac{e^{2x} + 1}{e^{2x} - 1}, \qquad Pe = \frac{c\Delta x}{2k}$$

$$\boxed{\tau = \frac{\Delta x}{2c}\left[\coth(Pe) - \frac{1}{Pe}\right]}$$

$$\boxed{\tau} = \frac{\Delta x}{2c}\left[\underbrace{\coth(Pe) - \frac{1}{Pe}}_{f(Pe)}\right]$$

$f(Pe)$

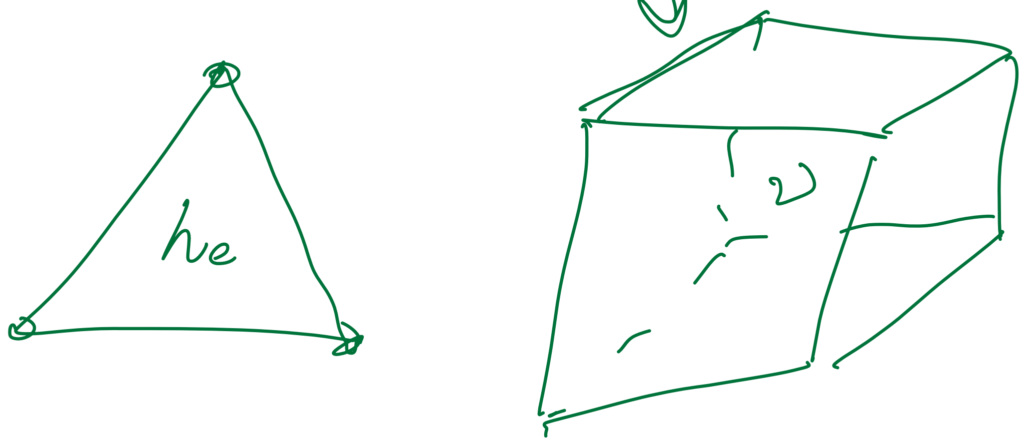


$$\tau = \frac{\Delta x}{2c} f(Pe) = \frac{\Delta x}{2c}\underbrace{\left(1+\frac{1}{Pe^2}\right)^{-1/2}}_{\text{Harmonic}}$$

$$= \left[\left(\frac{2c}{\Delta x}\right)^2 + \left(\frac{4K}{(\Delta x)^2}\right)^2\right]^{-1/2}$$

Stabilization parameter for general time-dependent convection dominated PDE's (Navier-Stokes)

$$\tau_e = \left[\underbrace{\left(\frac{2}{\Delta t}\right)^2}_{\text{inertia}} + \overbrace{\left(\frac{2\|\underset{\sim}{c}\|}{h_e}\right)^2}^{\text{convection}} + \overbrace{\left(\frac{4\kappa}{h_e^2}\right)^2}^{\text{diffusion}} \right]^{-\frac{1}{2}}$$



The CDR equation

The CDR equation is given as:

$$\frac{\partial \varphi}{\partial t} + \boldsymbol{v} \cdot \nabla \varphi - \nabla \cdot (\boldsymbol{k} \nabla \varphi) + s\varphi = f$$

on a d -dimensional domain $\Omega(t) \subset \mathbb{R}^d$, where φ is the unknown transport variable, $\boldsymbol{v}$, $\boldsymbol{k}$, s and f are the convection velocity, diffusivity tensor ($\boldsymbol{k} = k\boldsymbol{I}$ for isotropic diffusion, k and $\boldsymbol{I}$ being the diffusion coefficient and identity tensor respectively) reaction coefficient and source respectively.

The CDR System: Strong Differential Form

Consider a d -dimensional spatial domain $\Omega(t) \subset \mathbb{R}^d$ with the Dirichlet and Neumann boundaries denoted by $\Gamma_D^\varphi(t)$ and $\Gamma_N^\varphi(t)$ respectively. The strong form of the CDR equation (with φ as the scalar variable) along with the boundary conditions can be written as

$$
\begin{aligned}
\frac{\partial \varphi}{\partial t} + v \cdot \nabla \varphi - \nabla \cdot (k \nabla \varphi) + s\varphi = f, &\qquad \text{on } \Omega(t) \times [0, T] \\
\varphi = \varphi_D, &\qquad \text{on } \Gamma_D^\varphi(t) \times [0, T] \\
k \nabla \varphi \cdot n^\varphi = \varphi_N, &\qquad \text{on } \Gamma_N^\varphi(t) \times [0, T] \\
\varphi = \varphi_0, &\qquad \text{on } \Omega(0)
\end{aligned}
$$

The Convection-Diffusion Stability Issue

Consider the scalar advection-diffusion problem in the domain Ω with boundary Γ

$$\lambda \cdot \nabla\varphi - \nabla \cdot (\kappa\nabla\varphi) = f \quad \text{in } \Omega$$

and boundary conditions

$$\varphi = \bar{\varphi} \quad \text{on } \Gamma_{d\varphi}, \quad \kappa\nabla\varphi \cdot \mathbf{n} = h_\varphi$$

$$\text{on } \Gamma_{n\varphi}$$

The relative importance of advection with respect to diffusion is expressed by the Peclet number $Pe = UL/\kappa$ where U and L are some typical speed and length scal respectively.

Stability Problem

- For advection-dominated problems (high Pe numbers), solutions develop boundary and interior layers in which the transported variable varies rapidly.

Analysis (1)

One-dimensional problem: Consider the 1D problem over the domain $\Omega = [0, 1]$

$$\lambda \frac{d\varphi}{dx} - \kappa \frac{d^2\varphi}{dx^2} = 0$$

with Dirichlet boundary conditions

$$\varphi(x = 0) = \varphi_{\text{in}} \quad \text{and} \quad \varphi(x = 1) = \varphi_{\text{out}}$$

For a constant advection speed λ, it is easy to find the exact solution

$$\varphi(x) = \varphi_{\text{in}} + (\varphi_{\text{out}} - \varphi_{\text{in}}) \frac{e^{(Pe)\,x} - 1}{e^{Pe} - 1}$$

where $Pe = \lambda/\kappa$. The variation of the solution with Peclet number is illustrated in Figure.

Analysis (2)

The Galerkin finite element discretisation is

$$\int_0^1 N_i \lambda \frac{d\varphi^h}{dx} dx + \int_0^1 \frac{dN_i}{dx} \kappa \frac{d\varphi^h}{dx} dx = 0$$

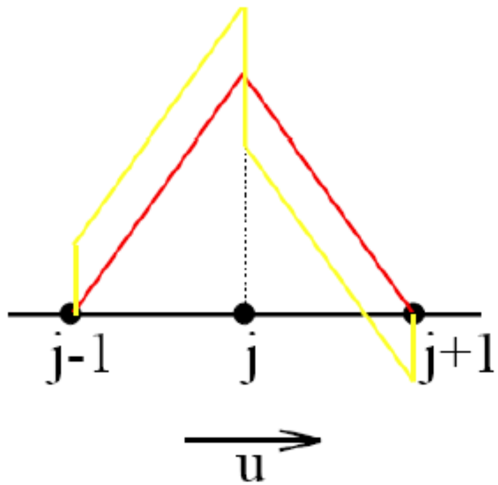
For an interior node i surrounded by two elements L and R of length h, we have

$$\left.\frac{d\varphi^h}{dx}\right)_L = \frac{\varphi_i - \varphi_{i-1}}{h}, \left.\frac{dN_i}{dx}\right)_L = \frac{1}{h} \quad \left.\frac{d\varphi^h}{dx}\right)_R = \frac{\varphi_{i+1} - \varphi_i}{h}, \left.\frac{dN_i}{dx}\right)_R = -\frac{1}{h}$$

Analysis (3)

Analysis (4)

In the finite element framework, upwinding can be achieved by adding a perturbation $\tau\lambda\frac{dN_i}{dx}$ to the Galerkin weighting function. Indeed, as $dN_i/dx = \pm 1$ on the left and right elements respectively, the term $\tau\lambda\frac{dN_i}{dx}$ will always introduce a positive/negative contribution to the upwind/downwind elements.



SUPG

For **P1** elements and without source terms, the Streamline Upwind method just described works fine, but the method becomes poor as soon as source terms are present or higher order elements are used. BROOKS and HUGHES (1982) showed that these problems could be overcome by using the pertubed weighting function not only for the advection terms, but for all terms in the equation and they coined this the Streamline Upwind/Petrov-Galerkin (SUPG) method.

Stabilization Parameter

In one dimension, the expression

$$\tau_e = \zeta\left(Pe^h\right)\frac{h}{2\lambda}$$

was shown to provide the same discretisation as the hybrid difference schelme. The formal extension to several dimensions is straightforward as long as the element dimension h_e and the blending function $\zeta\left(Pe^h\right)$ are specified.

A better understanding of the meaning of the stability parameter has recently emerged, based on the relationship of stabilised methods with subgrid scale models and Green's functions.

The definition of the element size is extremely important since it controls directly the amount of diffusion introduced. A critical review of definitions has been made by J.-C. CARETTE.

Generalization: GLS method

The least-square finite element formulation consists in minimising $\int_\Omega r^2(\varphi^h)\,d\Omega$. Therefore the associated set of weighting functions is $w_i = \partial r(\varphi^h)/\partial\varphi_i$. For the present advection-diffusion equation,

$$r(\varphi^h) = \lambda\cdot\nabla\varphi^h - \nabla\cdot(\kappa\nabla\varphi^h) - f \Rightarrow \frac{\partial r(\varphi^h)}{\partial\varphi_i} = \lambda\cdot\nabla N_i - \nabla\cdot(k\nabla N_i)$$

which, for P1 elements, simplifies to $\partial r(\varphi^h)/\partial\varphi_i = \lambda\cdot\nabla N_i$, so that the stabilisation term in the SUPG formulation can be viewed as a least-square term.
This prompted the development of a generalisation of the SUPG method, i.e. the Galerkin/Least-square (GLS) which differs from the SUPG method by the expression of the stabilisation term

$$\mathrm{ST}_{\mathrm{GLS}} = \sum_e \left(\tau_e(\lambda\cdot\nabla N^h - \nabla\cdot(\kappa\nabla N^h), r(\varphi^h)\right)_\Omega$$

As pointed out previously, the two methods are identical for P1 elements and differ only for higher order elements.

Semi-discrete Time Integration

$$M \frac{\partial u}{\partial t} + K u = f$$

mass matrix — velocity derivative

$$\boxed{u_{n+1} = f(u_n)}$$

$$\frac{1}{2} m |u|^2$$

$$\frac{dy}{dt} = f(t, y)$$

→ Newmark-β method

(Backward Euler & Trapezoidal rule)

→ $\boxed{\text{Generalized-}\alpha \text{ method}}$ $\boxed{\alpha_m}$

Accel → $\dot{u}_{n+1} = \dot{u}_n + (1-\gamma) \Delta t \, \ddot{u}_n + \gamma \Delta t \, \ddot{u}_{n+1}$

Disp → $u_{n+1} = u_n + \Delta t \, \dot{u}_n + \frac{\Delta t^2}{2}\left((1-2\beta)\ddot{u}_n + 2\beta \ddot{u}_{n+1}\right)$

$$M \ddot{u}_{n+1} + K u_{n+1} = F_{n+1}$$

Computational Mechanics of FSI

→ Review: $\underbrace{\text{Stabilization}}$

→ Time integration
  - Trapezoidal
  * - Newmark-family methods

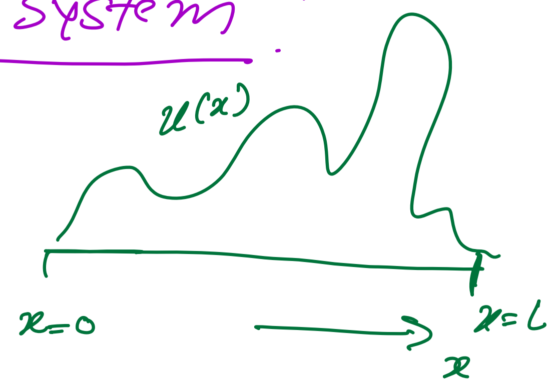
→ Semi-discrete formulation

- fluid system
- solid system

Generic differential system:

$$\frac{d^2}{dx^2}\{u\} = f \rightarrow \mathcal{L}u = f$$

$$\mathcal{L} \equiv \frac{d^2}{dx^2}$$



$$\mathcal{L} = \mathcal{L}_{adv} + \mathcal{L}_{diff}$$

$u(x) = ?$

$$\mathcal{L}_{diff} = -\kappa \frac{d^2}{dx^2}$$

$$\mathcal{L}_{adv} = c\frac{d}{dx}$$

$c > 0$

$$\mathcal{L} = \mathcal{L}_{adv} + \mathcal{L}_{diff}$$

$$\mathcal{L}\, u = f$$

$$\Rightarrow \left(c\frac{d}{dx} - \kappa\frac{d^2}{dx^2}\right) u = f$$

unknown / informatio

Weak form / Variational form:

$$\int \psi \, (\mathcal{L}u - f) \, d\Omega = 0$$

Residual

Test function

$$\mathcal{L}[\boxed{u}] = f$$

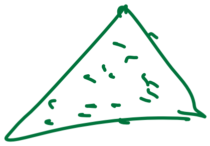
Weighted-residual form

$$\left\{ \begin{aligned} &\psi = N \qquad \text{Galerkin} \\ &\underline{u} = \sum \underset{\sim}{N_i} \, \boxed{u_i} \end{aligned} \right\}$$

$$\sum N_i = 1$$

The finite element method

$$[K]\{u_i\} = \{RHS\}_i$$



$Pe > 1$ Peclet # → $Pe = \dfrac{c\,h}{2K}$

$Pe \le 1$

weak form (Galerkin $\psi = N$)

$$\int_\Omega \underbrace{N(\mathcal{L}u - f)}\, d\Omega = 0$$

$\mathcal{L} = \mathcal{L}_{adv} + \mathcal{L}_{diff}$

$\boxed{\text{Petrov-Galerkin}}$

$$N^* = N + \tau \underbrace{\mathcal{L}_{adv} N}_{\tau c \frac{dN}{dx}}$$

$$\int_\Omega N(\mathcal{L}u - f)\, d\Omega + \sum_{e=1}^{n_{el}} \int_{\Omega_e} \tau \mathcal{L}_{adv} N (\mathcal{L}u - f)\, d\Omega = 0$$

$$\Downarrow$$

$$\int_\Omega (N + \underbrace{\tau \mathcal{L}_{adv} N})(\mathcal{L}u - f)\, d\Omega = 0$$

Petrov-Galerkin Stabilization

→ τ is stabilization parameter which need to be determined

→ $$\tau_e = \frac{h_e}{2c}\left[\underbrace{\coth(Pe_e) - \frac{1}{Pe_e}}\right]$$

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} - k\frac{\partial^2 u}{\partial x^2} = f$$

$$\tau_e = \left[\left(\frac{2}{\Delta t}\right)^2 + \left(\frac{2\|c\|}{h_e}\right)^2 + \left(\frac{4k}{h_e^2}\right)^2\right]^{-1/2}$$

$$Pe = \frac{c\,h_e}{2k}$$

PG Stabilization parameter for convection-diffusion problems!

$$\mathcal{L}_{adv} = \frac{\partial}{\partial t} + c_i \frac{\partial}{\partial x_i}$$

Review: ODE integration

$$\frac{d}{dt}(\text{amount}) = \underbrace{R.H.S}$$

$$u = u(x, t)$$

Semi-discrete:

$$\frac{\partial}{\partial t} + \mathcal{L}u = f$$

$$\left(c\frac{\partial}{\partial x} - k\frac{\partial^2}{\partial x^2}\right)$$

$$\frac{d}{dt}\{u\} = -\text{Res}$$

$$\text{Res} = \underbrace{\mathcal{L}u - f}$$

partial derivatory

$t^n \rightarrow t^{n+1}$

$$\frac{du}{dt} = R(u, t)$$

$$\frac{u^{n+1} - u^n}{\Delta t}$$

Integral form

Taylor Series

$$u^{n+1} = u^n + \int_{t^n}^{t^{n+1}} R(\tau, u)\, d\tau$$

Forward Euler

Backward Euler

Trapezoidal

Forward Euler:

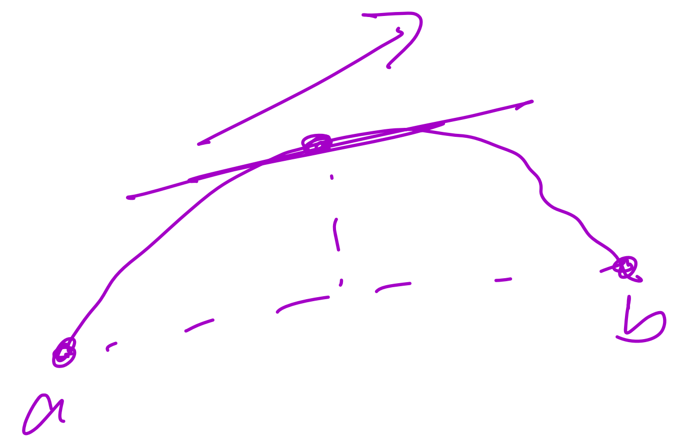
$$u^{n+1} = u^n + \int_{t^n}^{t^{n+1}} R(\tau, u^n)\, d\tau$$

Backward Euler:

$$u^{n+1} = u^n + \int_{t^n}^{t^{n+1}} R(\tau, u^{n+1})\, d\tau$$

Trapezoidal:

$$u^{n+1} = u^n + \int_{t^n}^{t^{n+1}} R\Big(\tau, \underbrace{\frac{u^n + u^{n+1}}{2}}\Big)\, d\tau$$



mid-point rule

Newmark / Generalized α for 2 order system ext

$$M\ddot{\underline{u}} + C\dot{\underline{u}} + \underbrace{Ku}_{f^{int}(\underset{\sim}{u})} = f$$

(I)

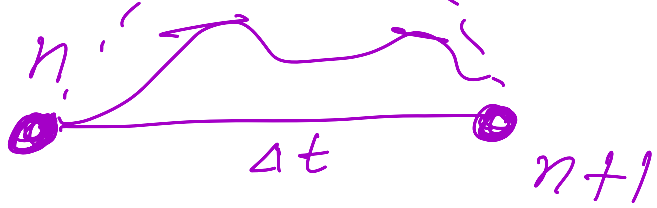
$$\dot{u}_{n+1} = \dot{u}_n + \Delta t\, \ddot{u}_\gamma$$

$\rightarrow 0 \leq \gamma \leq 1$

$$\rightarrow \ddot{u}_\gamma = (1-\gamma)\,\ddot{\underline{u}}_n + \gamma\, \ddot{u}_{n+1}$$

$$\dot{u}_{n+1} = \dot{u}_n + (1-\gamma)\Delta t\, \ddot{u}_n + \gamma \Delta t\, \ddot{u}_{n+1}$$

u : disp, $\dot{u}$: vel $\ddot{u}$: accel



(II)

$$u_{n+1} = \underbrace{u_n}_{\text{disp @ } n} + \Delta t\, \dot{u}_n + \frac{\Delta t^2}{2} \ddot{u}_\beta$$

$$\ddot{u}_\beta = (1-2\beta)\,\ddot{u}_n + 2\beta\, \ddot{u}_{n+1}$$

$$0 \le 2\beta \le 1$$

① $$\dot{u}_{n+1} = \dot{u}_n + (1-\gamma)\Delta t\, \ddot{u}_n + \gamma \Delta t\, \ddot{u}_{n+1}$$

② $$u_{n+1} = u_n + \Delta t\, \dot{u}_n + \frac{\Delta t^2}{2}\left[(1-2\beta)\ddot{u}_n + 2\beta\, \ddot{u}_{n+1}\right]$$

③ $$M\ddot{u}_{n+1} + C\dot{u}_{n+1} + K u_{n+1} = f^{ext}_{n+1}$$

Case : 1

$\gamma = 0.5$, $\beta = 0$

$\Rightarrow$ Explicit central diff

Case 2 : $\gamma = 0.5$, $\beta = 0.25$

Mid-point rule
(Avearage constant acceleration)

Newmark-Beta Method

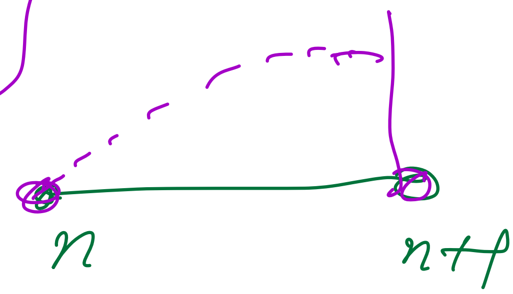
$\downarrow$

β, γ , α

Generalized-α method:

Mass matrix　　　　　　　　　　Stiffness matrix

$$\rightarrow \boxed{M\ddot{u} + Ku} = 0$$



(I) $$u_{n+1} = u_n + \Delta t\left(\gamma a_{n+1} + (1-\gamma) a_n\right)$$

previous　　　　　　　　　　previous

(II) $$M\left\{\alpha_m a_{n+1} + (1-\alpha_m) a_n\right\} + K\left\{\alpha u_{n+1} + (1-\alpha) u_n\right\} = 0$$

α_m : inertia weighting parameter

γ : weighting for accel

Solve for $\{u_{n+1}, a_{n+1}\}$ using previous values $\{u_n, a_n\}$

$$\begin{bmatrix} & & \\ & B & \\ & & \end{bmatrix} \begin{Bmatrix} u_{n+1} \\ a_{n+1} \end{Bmatrix} = \begin{bmatrix} & & \\ & C & \\ & & \end{bmatrix} \begin{Bmatrix} u_n \\ a_n \end{Bmatrix}$$

$$B = \begin{bmatrix} 1 & -\gamma \Delta t \\ \alpha K & \alpha_m M \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & (1-\underline{\gamma})\Delta t \\ (\underline{\alpha}-1)K & (\underline{\alpha_m}-1)M \end{bmatrix}$$

$$A = B^{-1} C$$

$\uparrow$ Amplification matrix

Spectral matrix $\underline{\rho(A) \leq 1}$

$\rightarrow \underline{\alpha = \alpha_m = \gamma = 1}$, (Backward Euler)

$$\rho(A) = \frac{1}{1 + \left(\frac{K\Delta t}{M}\right)} \leq 1$$

Second-order integration :

(Trapezoidal rule)

Tuning

α , α_m γ

$\alpha = \alpha_m = \gamma = \frac{1}{2}$

Neutrally Stable

Can become unstable for high frequency errors !

Damping factor

$\rho_\infty = f(\alpha, \alpha_m, \gamma)$

$0 \leq \rho_\infty \leq 1$ Trapezoidal

$$\alpha = \frac{1}{1+\rho_\infty}$$

$$\alpha_m = \frac{3-\rho_\infty}{2(1+\rho_\infty)}$$

$$\gamma = \frac{1}{1+\rho_\infty}$$

Generalized-α method for variational or energy conserving time integration?

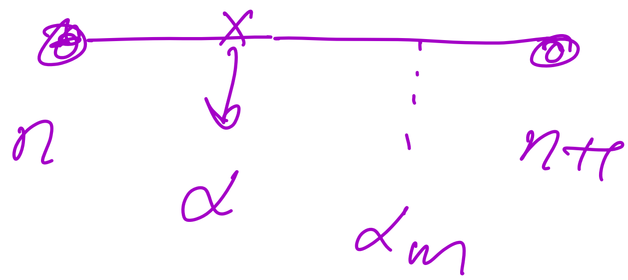
Navier-Stokes system:

$$u^f_{n+1} = u^f_n + \Delta t\left(\underline{\gamma}\, \dot{u}^f_n + (1-\gamma)\dot{u}^f_{n+1}\right)$$

$$\underset{\sim}{u}^f_{n+\underline{\alpha}} = \alpha\, \underset{\sim}{u}^f_{n+1} + (1-\alpha)\, \underset{\sim}{u}^f_n$$

$$a_{n+\alpha_m} = \alpha_m \ \ldots$$

$$t_{n+\alpha} =$$



$$t_n + \alpha \Delta t_{n+1}$$

$$\boxed{\underset{\sim}{M}\ \underset{\sim}{a}_{n+\alpha_m} + \underset{\sim}{N}(u_{n+\alpha}, p_{n+1}) = F(t_{n+\alpha})}$$

mass-matrix

Unsteady Semi-Discrete Variational Form (1)

We utilize the generalized- α method to discretize the equation in time. This technique allows user-defined high-frequency damping via controlling a parameter called the spectral radius ρ_∞, which is helpful for coarser discretization in space and time. The following expressions are employed for the temporal discretization:

$$\begin{aligned}
\varphi^{\mathrm{n}+1} &= \varphi^{\mathrm{n}} + \Delta t \partial_t \varphi^{\mathrm{n}} + \gamma \Delta t \left(\partial_t \varphi^{\mathrm{n}+1} - \partial_t \varphi^{\mathrm{n}}\right) \\
\partial_t \varphi^{\mathrm{n}+\alpha_{\mathrm{m}}} &= \partial_t \varphi^{\mathrm{n}} + \alpha_{\mathrm{m}} \left(\partial_t \varphi^{\mathrm{n}+1} - \partial_t \varphi^{\mathrm{n}}\right) \\
\varphi^{\mathrm{n}+\alpha} &= \varphi^{\mathrm{n}} + \alpha \left(\varphi^{\mathrm{n}+1} - \varphi^{\mathrm{n}}\right)
\end{aligned}$$

where γ, α and α_{m} are the generalized $-\alpha$ parameters given by

$$\alpha_{\mathrm{m}} = \frac{1}{2}\left(\frac{3-\rho_\infty}{1+\rho_\infty}\right), \quad \alpha = \frac{1}{1+\rho_\infty}, \quad \gamma = \frac{1}{2} + \alpha_{\mathrm{m}} - \alpha$$

Semi-Discrete Variational Form (2)

The above equation can be observed as a steady-state equation with modified reaction coefficient and source term. Let the modified coefficients be given by $\tilde{v}$, $\tilde{k}$, $\tilde{s}$ and $\tilde{f}$ defined as

$$\tilde{v} = v, \quad \tilde{k} = k, \quad \tilde{s} = s + \frac{1}{\alpha \Delta t}, \quad \tilde{f} = f + \frac{1}{\alpha \Delta t} \varphi^{\mathrm{n}}$$

Therefore, we will now discretize the following equation in the spatial domain:

$$\tilde{v} \cdot \nabla \varphi^{\mathrm{n}+\alpha} - \nabla \cdot \left(\tilde{k} \nabla \varphi^{\mathrm{n}+\alpha} \right) + \tilde{s} \varphi^{\mathrm{n}+\alpha} = \tilde{f} \quad \text{on } \Omega(t)$$

Spatial Discretization (1)

Therefore, we will now discretize the following equation in the spatial domain:

$$\tilde{v} \cdot \nabla \varphi^{\mathrm{n}+\alpha} - \nabla \cdot \left(\tilde{k} \nabla \varphi^{\mathrm{n}+\alpha} \right) + \tilde{s} \varphi^{\mathrm{n}+\alpha} = \tilde{f} \qquad \text{on } \Omega(t)$$

The domain $\Omega(t)$ is discretized into n_{el} number of elements such that $\Omega(t) = \cup_{\mathrm{e}=1}^{\mathrm{n}_{\mathrm{el}}} \Omega^{\mathrm{e}}$ and $\emptyset = \bigcap_{\mathrm{e}=1}^{\mathrm{n}_{\mathrm{el}}} \Omega^{\mathrm{e}}$. The space of trial solution and test function, $\mathscr{S}_{\varphi}^{\mathrm{h}}$ and $\mathscr{V}_{\varphi}^{\mathrm{h}}$ respectively for the variational formulation are defined as

$$\mathscr{S}_{\varphi}^{\mathrm{h}} = \left\{ \varphi_{\mathrm{h}} \mid \varphi_{\mathrm{h}} \in H^1(\Omega(t)), \varphi_{\mathrm{h}} = \varphi_D \text{ on } \Gamma_D^{\varphi}(t) \right\}$$

$$\mathscr{V}_{\varphi}^{\mathrm{h}} = \left\{ w_{\mathrm{h}} \mid w_{\mathrm{h}} \in H^1(\Omega(t)), w_{\mathrm{h}} = 0 \text{ on } \Gamma_D^{\varphi}(t) \right\}$$

Spatial Discretization (2)

Spatial Discretization (3)

As a result of spurious global oscillations and instability for convection- and reaction-dominated regimes in the Galerkin finite element method, various stabilization techniques have been proposed in the literature, the most widely used of which are SUPG and GLS methods. The stability is introduced through perturbing the test or weighting function so that the effect of upwinding is achieved. The standard variational formulation for such methods is: find $\varphi_{\mathrm{h}}(x, t^{\mathrm{n}+\alpha}) \in \mathscr{S}_{\varphi}^{\mathrm{h}}$ such that $\forall w_{\mathrm{h}} \in \mathscr{V}_{\varphi}^{\mathrm{h}}$

$$\int_{\Omega(t)} \left(w_{\mathrm{h}} (\tilde{v} \cdot \nabla \varphi_{\mathrm{h}}) + \nabla w_{\mathrm{h}} \cdot \left(\tilde{k} \nabla \varphi_{\mathrm{h}} \right) + w_{\mathrm{h}} \tilde{s} \varphi_{\mathrm{h}} \right) d\Omega$$
$$+ \sum_{\mathrm{e}=1}^{\mathrm{n_{el}}} \int_{\Omega^{\mathrm{e}}} \mathscr{L}^{m} w_{\mathrm{h}} \tau \left(\tilde{\mathscr{L}} \varphi_{\mathrm{h}} - \tilde{f} \right) d\Omega = \int_{\Omega(t)} w_{\mathrm{h}} \tilde{f} \, d\Omega + \int_{\Gamma_{N}^{\varphi}} w_{\mathrm{h}} \varphi_{N} d\Gamma$$

where $\mathscr{L}^{m}$ is the operator on the weighting function given in Table 4.1 and the expression for the stabilization parameter τ is

$$\tau = \left[\left(\frac{1}{\alpha \Delta t} \right)^{2} + 9 \left(\frac{4 \tilde{k}}{h^{2}} \right)^{2} + \left(\frac{2 |\tilde{v}|}{h} \right)^{2} + \tilde{s}^{2} \right]^{-1/2}$$

where h is the characteristic element length and $|\tilde{v}|$ is the magnitude of the convection velocity. The formula for τ has been extensively studied in the literature with several variations, and can be estimated through error analysis. The generality of the expression is a topic of discussion later. The residual of the CDR equation is defined as

$$\mathscr{R}(\varphi_{\mathrm{h}}) = \tilde{v} \cdot \nabla \varphi_{\mathrm{h}} - \nabla \cdot \left(\tilde{k} \nabla \varphi_{\mathrm{h}} \right) + \tilde{s} \varphi_{\mathrm{h}} - \tilde{f} = \tilde{\mathscr{L}} \varphi_{\mathrm{h}} - \tilde{f}$$

where $\tilde{\mathscr{L}}$ is the differential operator corresponding to the differential equation.

Spatial Discretization (4)

Table 4.1 Differential operators on the weighting function for stabilization methods.

Method	Stabilization operator ($\mathscr{L}^m$)
SUPG	$\mathscr{L}_{adv} = \tilde{\boldsymbol{v}} \cdot \nabla$
GLS	$\tilde{\mathscr{L}} = \tilde{\boldsymbol{v}} \cdot \nabla - \nabla \cdot (\tilde{\boldsymbol{k}} \nabla) + \tilde{\boldsymbol{s}}$
SGS	$-\tilde{\mathscr{L}}^* = \tilde{\boldsymbol{v}} \cdot \nabla + \nabla \cdot (\tilde{\boldsymbol{k}} \nabla) - \tilde{\boldsymbol{s}}$

We begin by analyzing the linear stabilization methods (SUPG, GLS and SGS) with respect to the effect of the sign of the reaction coefficient ($\tilde{\boldsymbol{s}}$), owing to the destruction or production effects. Fourier analysis of the discretized methods (GLS and SGS) showed that the SGS method performs well when $\tilde{\boldsymbol{s}}$ is negative, but, it loses accuracy when $\tilde{\boldsymbol{s}} >> 0$ due to excessive dissipation. On the other hand, the GLS method is not as diffusive as SGS when $\tilde{\boldsymbol{s}} \geq 0$, but it suffers from phase error when $\tilde{\boldsymbol{s}} < 0$. Thus, we select a combination of these methods, so that the formulation is benefited in both production and destruction regimes. Note that the effect of the diffusion term is assumed negligible in the differential operator owing to the use of linear and multilinear finite elements.

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