MECH 503 Homework 1

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1 Question 1

(a)

$$\nu_i u_i = \sum_{i=1}^{n} \nu_i u_i = \mathbf{v} \cdot \mathbf{u}$$

(b)

$$\delta_{ii} = \sum_{i=1}^{n} \delta_{ii} = n = 3$$

(c)

$$\nu_{i,i} = \sum_{i=1}^{n} \frac{\partial \nu_i}{\partial x_i} = \nabla \cdot \mathbf{v}$$

(d)

$$\delta_{ij,j} = \sum_{j=1}^{n} \frac{\partial \delta_{ij}}{\partial x_j} = 0$$

(e)

$$\epsilon_{ijk} v_j u_k = \sum_{j=1}^{n} \sum_{k=1}^{n} \epsilon_{ijk} v_j u_k = \mathbf{v} \times \mathbf{u}$$

(f)

$$\delta_{mi} \delta_{mj} T_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{mi} \delta_{mj} T_{ij} = T_{mm} = \mathrm{Trace}(\mathbf{T})$$

(g)

$$Q_{ij}T_{ik}Q_{km} = \sum_{j=1}^{n}\sum_{k=1}^{n} Q_{ij}T_{ik}Q_{km} = \mathbf{Q}^T\mathbf{T}\mathbf{Q}$$

(h)

$$\epsilon_{ijk}\sigma_{jk} = \sum_{j=1}^{n}\sum_{k=1}^{n} \epsilon_{ijk}\sigma_{jk} = 0$$

2 Question 2

(a)

$$(u \times v) \cdot w = \varepsilon_{ijk} u_j v_k w_i$$

(b)

$$\det T = \varepsilon_{ijk} T_{1i} T_{2j} T_{3k}$$

(c)

$$(\nabla \times v)_i = \varepsilon_{ijk} \partial_j v_k = \varepsilon_{ijk} v_{k,j}$$

(d)

$$\nabla \cdot v = \partial_i v_i = v_{i,i}$$

(e)

$$(\nabla \times T)_{ij} = \varepsilon_{jkl} \partial_k T_{il} = T_{il,k}$$

(f)

$$(\nabla \times (\nabla T))_i = \varepsilon_{ijk} \partial_j (\partial_k T)$$

(g)

$$(\nabla \cdot T)_i = \partial_j T_{ij} = T_{ij,j}$$

3 Question 3

Define a second order tensor C such that

$$\begin{aligned}
\mathbf{C} &= \mathbf{A} \cdot \mathbf{B} \\
\iff C_{ij} &= A_{ik} \cdot B_{kj} \\
\det\ \mathbf{C} &= \epsilon_{ijk} C_{1i} C_{2j} C_{3k} \\
&= \epsilon_{ijk} (A_{1l} B_{li})(A_{2m} B_{mj})(A_{3n} B_{nk}) \\
&= \epsilon_{ijk} (A_{1l} A_{2m} A_{3n})(B_{li} B_{mj} B_{nk})
\end{aligned}$$

Now consider the determinant of A and B separately

$$\begin{aligned}
\det\ \mathbf{A} &= \epsilon_{lmn} A_{1l} A_{2m} A_{3n} \\
\det\ \mathbf{B} &= \epsilon_{ijk} B_{li} B_{mj} B_{nk} \\
\det\ \mathbf{A} \cdot \det\ \mathbf{B} &= (\epsilon_{lmn} A_{1l} A_{2m} A_{3n})(\epsilon_{ijk} B_{li} B_{mj} B_{nk})
\end{aligned}$$

Thus, one can conclude the following

$$\det\ (\mathbf{A}\ \cdot\ \mathbf{B}) = \det\ \mathbf{A} \cdot \det\ \mathbf{B}$$

4 Question 4

$$
\begin{aligned}
(QQ^T)_{ij} &= Q_{ik}Q^T_{kj} \\
&= Q_{ik}Q_{jk} \\
&= \delta_{ij} \\
\implies QQ^T &= I \\
(Q^TQ)_{ij} &= Q^T_{ik}Q_{kj} \\
&= Q_{ki}Q_{kj} \\
&= \delta_{ji} \\
&= \delta_{ij} \\
\implies Q^TQ &= I \\
Q_{ij}Q^{-1}_{jk} &= Q_{ij}Q^T_{kj} = \delta_{ik} \\
\implies Q^{-1}_{jk} &= Q^T_{kj}
\end{aligned}
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5 Question 5

Figure 1: Undeformed and Deformed Cylinder Visualization