A weak Galerkin finite element method for second-order elliptic problems

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A R T I C L E   I N F O

A B S T R A C T

This paper introduces a finite element method by using a weakly defined gradient operator over generalized functions. The use of weak gradients and their approximations results in a new concept called *discrete weak gradients* which is expected to play an important role in numerical methods for partial differential equations. This article intends to provide a general framework for managing differential operators on generalized functions. As a demonstrative example, the discrete weak gradient operator is employed as a building block in the design of numerical schemes for a second order elliptic problem, in which the classical gradient operator is replaced by the discrete weak gradient. The resulting numerical scheme is called a weak Galerkin (WG) finite element method. It can be seen that the weak Galerkin method allows the use of totally discontinuous functions in the finite element procedure. For the second order elliptic problem, an optimal order error estimate in both a discrete H^1 and L^2 norms are established for the corresponding weak Galerkin finite element solutions. A superconvergence is also observed for the weak Galerkin approximation.

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1. Introduction

The goal of this paper is to introduce a numerical approximation technique for partial differential equations based on a new interpretation of differential operators and their approximations. To illustrate the main idea, we consider the Dirichlet problem for second-order elliptic equations which seeks an unknown function $u = u(x)$ satisfying

$$-\nabla \cdot (a\nabla u) + \nabla \cdot (bu) + cu = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = g \quad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is a polygonal or polyhedral domain in $\mathbb{R}^d (d = 2, 3)$, $a = (a_{ij}(x))_{d \times d} \in [L^\infty(\Omega)]^{d^2}$ is a symmetric matrix-valued function, $b = (b_i(x))_{d \times 1}$ is a vector-valued function, and $c = c(x)$ is a scalar function on Ω. Assume that the matrix a satisfies the following property: there exists a constant $\alpha > 0$ such that

$$\alpha \xi^T \xi \le \xi^T a \xi, \quad \forall \xi \in \mathbb{R}^d. \tag{1.3}$$

For simplicity, we shall concentrate on two-dimensional problems only (i.e., $d = 2$). An extension to higher-dimensional problems is straightforward.

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The standard weak form for (1.1) and (1.2) seeks $u \in H^1(\Omega)$ such that $u = g$ on $\partial\Omega$ and

$$(a\nabla u, \nabla v) - (bu, \nabla v) + (cu, v) = (f, v) \quad \forall v \in H_0^1(\Omega), \tag{1.4}$$

where (ϕ, ψ) represents the L^2-inner product of $\phi = \phi(x)$ and $\psi = \psi(x)$—either vector-valued or scalar-valued functions. Here ∇u denotes the gradient of the function $u = u(x)$, and ∇ is known as the gradient operator. In the standard Galerkin method (e.g., see [1,2]), the trial space $H^1(\Omega)$ and the test space $H_0^1(\Omega)$ in (1.4) are each replaced by properly defined subspaces of finite dimensions. The resulting solution in the subspace/subset is called a Galerkin approximation. A key feature in the Galerkin method is that the approximating functions are chosen in a way that the gradient operator ∇ can be successfully applied to them in the classical sense. A typical implication of this property in Galerkin finite element methods is that the approximating functions (both trial and test) are continuous piecewise polynomials over a prescribed finite element partition for the domain, often denoted by $\mathcal{T}_h$. Therefore, a great deal of attention has been paid to a satisfaction of the embedded "continuity" requirement in the research of Galerkin finite element methods in existing literature till recent advances in the development of discontinuous Galerkin methods. But the interpretation of the gradient operator still lies in the classical sense for both "continuous" and "discontinuous" Galerkin finite element methods in current existing literature.

In this paper, we will introduce a weak gradient operator defined on a space of generalized functions. The weak gradient operator will then be employed to discretize the problem (1.4) through the use of a discrete weak gradient operator as building bricks. The corresponding finite element method is called the *weak Galerkin* method. Details can be found in Section 4.

To explain weak gradients, let K be any polygonal domain with interior K^0 and boundary ∂K. A *weak function* on the region K refers to a function $v = \{v_0, v_b\}$ such that $v_0 \in L^2(K)$ and $v_b \in H^{\frac{1}{2}}(\partial K)$. The first component v_0 can be understood as the value of v in the interior of K, and the second component v_b is the value of v on the boundary of K. Note that v_b may not be necessarily related to the trace of v_0 on ∂K should a trace be defined. Denote by $W(K)$ the space of weak functions associated with K; i.e.,

$$W(K) = \left\{ v = \{v_0, v_b\} : \ v_0 \in L^2(K), v_b \in H^{\frac{1}{2}}(\partial K) \right\}. \tag{1.5}$$

Recall that the dual of $L^2(K)$ can be identified with itself by using the standard L^2 inner product as the action of linear functionals. With a similar interpretation, for any $v \in W(K)$, the *weak gradient* of v can be defined as a linear functional $\nabla_w v$ in the dual space of $H(\text{div}, K)$ whose action on each $q \in H(\text{div}, K)$ is given by

$$(\nabla_w v, q) := -\int_K v_0 \nabla \cdot q \, dK + \int_{\partial K} v_b q \cdot \mathbf{n} \, ds, \tag{1.6}$$

where $\mathbf{n}$ is the outward normal direction to ∂K. Observe that for any $v \in W(K)$, the right-hand side of (1.6) defines a bounded linear functional on the normed linear space $H(\text{div}, K)$. Thus, the weak gradient $\nabla_w v$ is well defined. With the weak gradient operator ∇_w being employed in (1.4), the trial and test functions can be allowed to take separate values/definitions on the interior of each element T and its boundary. Consequently, we are left with a greater option in applying the Galerkin formulation to partial differential equations.

Many numerical methods have been developed for the model problem (1.1)–(1.2). The existing methods can be classified into two categories: (1) methods based on the primary variable u, and (2) methods based on the variable u and a flux variable (mixed formulation). The standard Galerkin finite element methods [1–3] and various interior penalty type discontinuous Galerkin methods [4–8] are typical examples of the first category. The standard mixed finite elements [9–16] and various discontinuous Galerkin methods based on both variables [17–20] are representatives of the second category. Due to the enormous amount of publications available in general finite element methods, it is unrealistic to list all the key contributions from the computational mathematics research community in this article. The main intention of the above citation is to draw a connection between existing numerical methods with the one that is to be presented in the rest of the sections.

The weak Galerkin finite element method, as detailed in Section 4, is closely related to the mixed finite element method (see [9–12,14,16]) with a hybridized interpretation of Fraeijs de Veubeke [21,22] and the hybridizable discontinuous Galerkin (HDG) method [18]. The hybridized formulation introduces a new term, known as the Lagrange multiplier, on the boundary of each element. The Lagrange multiplier is known to approximate the original function $u = u(x)$ on the boundary of each element. In fact, the WG schemes introduced in Section 4 are equivalent to some standard mixed finite element methods and the HDG method when $b = 0$, $c = 0$, and $a = \text{const}$ in (1.1). But the WG method is different from these methods when the coefficients are general variable functions. The concept of *weak gradients* shall provide a systematic framework for dealing with discontinuous functions defined on elements and their boundaries in a near classical sense. The same concept is expected to be applied to other types of partial differential equations with new basic differential operators such as *divergence* and *curl*.

2. Preliminaries and notations

We use standard definitions for the Sobolev spaces $H^s(D)$ and their associated inner products $(\cdot, \cdot)_{s,D}$, norms $\| \cdot \|_{s,D}$, and seminorms $| \cdot |_{s,D}$ for $s \geq 0$. For example, for any integer $s \geq 0$, the seminorm $| \cdot |_{s,D}$ is given by

$$|v|_{s,D} = \left(\sum_{|\alpha|=s} \int_D |\partial^\alpha v|^2 dD \right)^{\frac{1}{2}},$$

with the usual notation

$$\alpha = (\alpha_1, \alpha_2), \qquad |\alpha| = \alpha_1 + \alpha_2, \qquad \partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2}.$$

The Sobolev norm $\| \cdot \|_{m,D}$ is given by

$$\|v\|_{m,D} = \left(\sum_{j=0}^m |v|_{j,D}^2 \right)^{\frac{1}{2}}.$$

The space $H^0(D)$ coincides with $L^2(D)$, for which the norm and the inner product are denoted by $\| \cdot \|_D$ and $(\cdot, \cdot)_D$, respectively. When $D = \Omega$, we shall drop the subscript D in the norm and inner product notation. The space $H(\text{div}; \Omega)$ is defined as the set of vector-valued functions on Ω which, together with their divergence, are square integrable; i.e.,

$$H(\text{div}; \Omega) = \left\{ \mathbf{v} : \ \mathbf{v} \in [L^2(\Omega)]^2, \nabla \cdot \mathbf{v} \in L^2(\Omega) \right\}.$$

The norm in $H(\text{div}; \Omega)$ is defined by

$$\|\mathbf{v}\|_{H(\text{div};\Omega)} = \left(\|\mathbf{v}\|^2 + \|\nabla \cdot \mathbf{v}\|^2 \right)^{\frac{1}{2}}.$$

3. A weak gradient operator and its approximation

The goal of this section is to introduce a weak gradient operator defined on a space of generalized functions. The weak gradient operator will then be employed to discretize partial differential equations. To this end, let K be any polygonal domain with interior K^0 and boundary ∂K. A *weak function* on the region K refers to a function $v = \{v_0, v_b\}$ such that $v_0 \in L^2(K)$ and $v_b \in H^{\frac{1}{2}}(\partial K)$. The first component v_0 can be understood as the value of v in the interior of K, and the second component v_b is the value of v on the boundary of K. Note that v_b may not be necessarily related to the trace of v_0 on ∂K should a trace be well defined. Denote by $W(K)$ the space of weak functions associated with K; i.e.,

$$W(K) = \left\{ v = \{v_0, v_b\} : \ v_0 \in L^2(K), v_b \in H^{\frac{1}{2}}(\partial K) \right\}. \tag{3.1}$$

Definition 3.1. The dual of $L^2(K)$ can be identified with itself by using the standard L^2 inner product as the action of linear functionals. With a similar interpretation, for any $v \in W(K)$, the *weak gradient* of v is defined as a linear functional $\nabla_w v$ in the dual space of $H(\text{div}, K)$ whose action on each $q \in H(\text{div}, K)$ is given by

$$(\nabla_w v, q) := -\int_K v_0 \nabla \cdot q \, dK + \int_{\partial K} v_b q \cdot \mathbf{n} \, ds, \tag{3.2}$$

where $\mathbf{n}$ is the outward normal direction to ∂K.

Note that for any $v \in W(K)$, the right-hand side of (3.2) defines a bounded linear functional on the normed linear space $H(\text{div}, K)$. Thus, the weak gradient $\nabla_w v$ is well defined. Moreover, if the components of v are restrictions of a function $u \in H^1(K)$ on K^0 and ∂K, respectively, then we would have

$$-\int_K v_0 \nabla \cdot q \, dK + \int_{\partial K} v_b q \cdot \mathbf{n} \, ds = -\int_K u \nabla \cdot q \, dK + \int_{\partial K} u q \cdot \mathbf{n} \, ds = \int_K \nabla u \cdot q \, dK.$$

It follows that $\nabla_w v = \nabla u$ is the classical gradient of u.

Next, we introduce a discrete weak gradient operator by defining ∇_w in a polynomial subspace of $H(\text{div}, K)$. To this end, for any non-negative integer $r \geq 0$, denote by $P_r(K)$ the set of polynomials on K with degree no more than r. Let $V(K, r) \subset [P_r(K)]^2$ be a subspace of the space of vector-valued polynomials of degree r. A discrete weak gradient operator, denoted by $\nabla_{w,r}$, is defined so that $\nabla_{w,r} v \in V(K, r)$ is the unique solution of the following equation

$$\int_K \nabla_{w,r} v \cdot q \, dK = -\int_K v_0 \nabla \cdot q \, dK + \int_{\partial K} v_b q \cdot \mathbf{n} \, ds, \quad \forall q \in V(K, r). \tag{3.3}$$

It is not hard to see that the discrete weak gradient operator $\nabla_{w,r}$ is a Galerkin-type approximation of the weak gradient operator ∇_w by using the polynomial space $V(K, r)$.

The classical gradient operator $\nabla = (\partial_{x_1}, \partial_{x_2})$ should be applied to functions with a certain smoothness in the design of numerical methods for partial differential equations. For example, in the standard Galerkin finite element method, such a "smoothness" often refers to continuous piecewise polynomials over a prescribed finite element partition. With the weak gradient operator as introduced in this section, derivatives can be taken for functions without any continuity across the boundary of each triangle. Thus, the concept of weak gradient allows the use of generalized functions in approximations.

Analogies of weak gradient can be established for other differential operators such as divergence and curl operators. Details for weak divergence and weak curl operators and their applications in numerical methods will be given in forthcoming papers.

4. A weak Galerkin finite element method

The goal of this section is to demonstrate how discrete weak gradients be used in the design of numerical schemes that approximate the solutions of partial differential equations. For simplicity, we take the second order elliptic equation (1.1) as a model for discussion. With the Dirichlet boundary condition (1.2), the standard weak form seeks $u \in H^1(\Omega)$ such that $u = g$ on $\partial\Omega$ and

$$(a\nabla u, \nabla v) - (bu, \nabla v) + (cu, v) = (f, v) \quad \forall v \in H_0^1(\Omega). \tag{4.1}$$

Let $\mathcal{T}_h$ be a triangular partition of the domain Ω with meshsize h. Assume that the partition $\mathcal{T}_h$ is shape regular so that the routine inverse inequality in the finite element analysis holds true (see [1]). In the general spirit of the Galerkin procedure, we shall design a weak Galerkin method for (4.1) by following two basic principles: (1) *replace $H^1(\Omega)$ by a space of discrete weak functions defined on the finite element partition $\mathcal{T}_h$ and the boundary of triangular elements*; and (2) *replace the classical gradient operator by a discrete weak gradient operator $\nabla_{w,r}$ for weak functions on each triangle T*. Details are to be presented in the rest of this section.

For each $T \in \mathcal{T}_h$, denote by $P_j(T^0)$ the set of polynomials on T^0 with degree no more than j, and $P_\ell(\partial T)$ the set of polynomials on ∂T with degree no more than ℓ (i.e., polynomials of degree ℓ on each line segment of ∂T). A *discrete weak function* $v = \{v_0, v_b\}$ on T refers to a weak function $v = \{v_0, v_b\}$ such that $v_0 \in P_j(T^0)$ and $v_b \in P_\ell(\partial T)$ with $j \geq 0$ and $\ell \geq 0$. Denote this space by $W(T, j, \ell)$, i.e.,

$$W(T, j, \ell) := \left\{ v = \{v_0, v_b\} : \ v_0 \in P_j(T^0), \ v_b \in P_\ell(\partial T) \right\}.$$

The corresponding finite element space would be defined by patching $W(T, j, \ell)$ over all the triangles $T \in \mathcal{T}_h$. In other words, the weak finite element space is given by

$$S_h(j, \ell) := \{v = \{v_0, v_b\} : \ \{v_0, v_b\}|_T \in W(T, j, \ell), \ \forall T \in \mathcal{T}_h\}. \tag{4.2}$$

Denote by $S_h^0(j, \ell)$ the subspace of $S_h(j, \ell)$ with vanishing boundary values on $\partial\Omega$; i.e.,

$$S_h^0(j, \ell) := \{v = \{v_0, v_b\} \in S_h(j, \ell), v_b|_{\partial T \cap \partial\Omega} = 0, \ \forall T \in \mathcal{T}_h\}. \tag{4.3}$$

According to (3.3), for each $v = \{v_0, v_b\} \in S_h(j, \ell)$, the discrete weak gradient of v on each element T is given by the following equation:

$$\int_T \nabla_{w,r} v \cdot q dT = -\int_T v_0 \nabla \cdot q dT + \int_{\partial T} v_b q \cdot \mathbf{n} ds, \quad \forall q \in V(T, r). \tag{4.4}$$

Note that no specific examples of the approximating space $V(T, r)$ have been mentioned, except that $V(T, r)$ is a subspace of the set of vector-valued polynomials of degree no more than r on T.

For any $w, v \in S_h(j, \ell)$, we introduce the following bilinear form

$$a(w, v) = (a\nabla_{w,r} w, \nabla_{w,r} v) - (bw_0, \nabla_{w,r} v) + (cw_0, v_0), \tag{4.5}$$

where

$$(a\nabla_{w,r} w, \nabla_{w,r} v) := \sum_{T \in \mathcal{T}_h} \int_T a\nabla_{w,r} w \cdot \nabla_{w,r} v dT,$$

$$(bw_0, \nabla_{w,r} v) := \sum_{T \in \mathcal{T}_h} \int_T bw_0 \cdot \nabla_{w,r} v dT,$$

$$(cw_0, v_0) := \int_\Omega cw_0 v_0 d\Omega.$$

Weak Galerkin Algorithm 1. *A numerical approximation for* (1.1) *and* (1.2) *can be obtained by seeking* $u_h = \{u_0, u_b\} \in S_h(j, \ell)$ *satisfying* $u_b = Q_b g$ *on* $\partial\Omega$ *and the following equation:*

$$a(u_h, v) = (f, v_0), \quad \forall v = \{v_0, v_b\} \in S_h^0(j, \ell), \tag{4.6}$$

where $Q_b g$ *is an approximation of the boundary value in the polynomial space* $P_\ell(\partial T \cap \partial\Omega)$. *For simplicity,* $Q_b g$ *shall be taken as the standard* L^2 *projection for each boundary segment; other approximations of the boundary value* $u = g$ *can also be employed in* (4.6).

5. Examples of weak Galerkin method with properties

Although the weak Galerkin scheme (4.6) is defined for arbitrary indices j, ℓ, and r, the method can be shown to produce good numerical approximations for the solution of the original partial differential equation only with a certain combination of their values. For one thing, there are at least two prominent properties that the discrete gradient operator $\nabla_{w,r}$ should possess in order for the weak Galerkin method to work well. These two properties are:

P1: For any $v \in S_h(j, \ell)$, if $\nabla_{w,r} v = 0$ on T, then one must have $v \equiv$ constant on T. In other words, $v_0 = v_b =$ constant on T;

P2: Let $u \in H^m(\Omega)$ ($m \geq 1$) be a smooth function on Ω, and $Q_h u$ be a certain interpolation/projection of u in the finite element space $S_h(j, \ell)$. Then, the discrete weak gradient of $Q_h u$ should be a good approximation of ∇u.

The following are two examples of weak Galerkin finite element spaces that fit well into the numerical scheme (4.6).

WG Example 5.1. *In this example, we take* $\ell = j + 1$, $r = j + 1$, *and* $V(T, j + 1) = \left[P_{j+1}(T)\right]^2$, *where* $j \geq 0$ *is an arbitrary integer. Denote by* $S_h(j, j + 1)$ *the corresponding finite element space. More precisely, the finite element space* $S_h(j, j + 1)$ *consists of functions* $v = \{v_0, v_b\}$ *where* v_0 *is a polynomial of degree no more than* j *in* T^0, *and* v_b *is a polynomial of degree no more than* $j + 1$ *on* ∂T. *The space* $V(T, r)$ *used to define the discrete weak gradient operator* $\nabla_{w,r}$ *in* (4.4) *is given as vector-valued polynomials of degree no more than* $j + 1$ *on* T.

WG Example 5.2. *In the second example, we take* $\ell = j$, $r = j + 1$, *and* $V(T, r = j + 1) = \left[P_j(T)\right]^2 + \widehat{P}_j(T)\mathbf{x}$, *where* $\mathbf{x} = (x_1, x_2)^T$ *is a column vector and* $\widehat{P}_j(T)$ *is the set of homogeneous polynomials of order* j *in the variable* $\mathbf{x}$. *Denote by* $S_h(j, j)$ *the corresponding finite element space. Note that the space* $V(T, r)$ *that was used to define a discrete weak gradient is in fact the usual Raviart–Thomas element* [9] *of order* j *for the vector component.*

Let us demonstrate how the two properties **P1** and **P2** are satisfied with the two examples given as above. For simplicity, we shall present results only for Property 5.1. The following result addresses a satisfaction of the property **P1**.

Lemma 5.1. *For any* $v = \{v_0, v_b\} \in W(T, j, j + 1)$, *let* $\nabla_{w,j+1} v$ *be the discrete weak gradient of* v *on* T *as defined in* (4.4) *with* $V(T, r) = \left[P_{j+1}(T)\right]^2$. *Then,* $\nabla_{w,j+1} v = 0$ *holds true on* T *if and only if* $v =$ constant *(i.e.,* $v_0 = v_b =$ constant*).*

Proof. It is trivial to see from (4.4) that if $v =$ constant on T, then the right-hand side of (4.4) would be zero for any $q \in V(T, j + 1)$. Thus, we must have $\nabla_{w,j+1} v = 0$.

Now assume that $\nabla_{w,j+1} v = 0$. It follows from (4.4) that

$$-\int_T v_0 \nabla \cdot q \, dT + \int_{\partial T} v_b q \cdot \mathbf{n} \, ds = 0, \quad \forall q \in V(T, j + 1). \tag{5.1}$$

Let $\bar{v}_0$ be the average of v_0 over T. Using the results of [14], there exists a vector-valued polynomial $q_1 \in V(T, j + 1) = [P_{j+1}(T)]^2$ such that $q_1 \cdot \mathbf{n} = 0$ on ∂T and $\nabla \cdot q_1 = v_0 - \bar{v}_0$. With $q = q_1$ in (5.1), we arrive at $\int_T (v_0 - \bar{v}_0)^2 dT = 0$. It follows that $v_0 = \bar{v}_0$, and (5.1) can be rewritten as

$$\int_{\partial T} (v_b - v_0) q \cdot \mathbf{n} \, ds = 0, \quad \forall q \in V(T, j + 1). \tag{5.2}$$

Now since $v_b - v_0 \in P_{j+1}(\partial T)$, then one may select a $q \in V(T, j + 1) = [P_{j+1}(T)]^2$ such that

$$\int_{\partial T} \phi q \cdot \mathbf{n} \, ds = \int_{\partial T} \phi (v_b - v_0) \, ds, \quad \forall \phi \in P_{j+1}(\partial T),$$

which, together with (5.2) and $\phi = v_b - v_0$ yields

$$\int_{\partial T} (v_b - v_0)^2 \, ds = 0.$$

The last equality implies $v_b = v_0 =$ constant, which completes a proof of the lemma. $\quad\square$

To verify property **P2**, let $u \in H^1(T)$ be a smooth function on T. Denote by $Q_h u = \{Q_0 u, Q_b u\}$ the L^2 projection onto $P_j(T^0) \times P_{j+1}(\partial T)$. In other words, on each element T, the function $Q_0 u$ is defined as the L^2 projection of u in $P_j(T)$ and on ∂T, $Q_b u$ is the L^2 projection in $P_{j+1}(\partial T)$. Furthermore, let R_h be the local L^2 projection onto $V(T, j+1)$. According to the definition of $\nabla_{w,j+1}$, the discrete weak gradient function $\nabla_{w,j+1}(Q_h u)$ is given by the following equation:

$$\int_T \nabla_{w,j+1}(Q_h u) \cdot q \, dT = -\int_T (Q_0 u)\nabla \cdot q \, dT + \int_{\partial T} (Q_b u)q \cdot \mathbf{n} \, ds, \quad \forall q \in V(T, j+1). \tag{5.3}$$

Since Q_0 and Q_b are L^2-projection operators, then the right-hand side of (5.3) is given by

$$-\int_T (Q_0 u)\nabla \cdot q \, dT + \int_{\partial T} (Q_b u)q \cdot \mathbf{n} \, ds = -\int_T u\nabla \cdot q \, dT + \int_{\partial T} uq \cdot \mathbf{n} \, ds$$
$$= \int_T (\nabla u) \cdot q \, dT = \int_T R_h(\nabla u) \cdot q \, dT.$$

Thus, we have derived the following useful identity:

$$\nabla_{w,j+1}(Q_h u) = R_h(\nabla u), \quad \forall u \in H^1(T). \tag{5.4}$$

The above identity clearly indicates that $\nabla_{w,j+1}(Q_h u)$ is an excellent approximation of the classical gradient of u for any $u \in H^1(T)$. Thus, it is reasonable to believe that the weak Galerkin finite element method shall provide a good numerical scheme for the underlying partial differential equations.

6. Mass conservation of weak Galerkin

The second order elliptic equation (1.1) can be rewritten in a conservative form as follows:

$$\nabla \cdot q + cu = f, \quad q = -a\nabla u + bu.$$

Let T be any control volume. Integrating the first equation over T yields the following integral form of mass conservation:

$$\int_{\partial T} q \cdot \mathbf{n} \, ds + \int_T cu \, dT = \int_T f \, dT. \tag{6.1}$$

We claim that the numerical approximation from the weak Galerkin finite element method for (1.1) retains the mass conservation property (6.1) with a numerical flux q_h. To this end, for any given $T \in \mathcal{T}_h$, we chose in (4.6) a test function $v = \{v_0, v_b = 0\}$ so that $v_0 = 1$ on T and $v_0 = 0$ elsewhere. Using the relation (4.5), we arrive at

$$\int_T a\nabla_{w,r} u_h \cdot \nabla_{w,r} v \, dT - \int_T bu_0 \cdot \nabla_{w,r} v \, dT + \int_T cu_0 \, dT = \int_T f \, dT. \tag{6.2}$$

Using the definition (4.4) for $\nabla_{w,r}$, one has

$$\int_T a\nabla_{w,r} u_h \cdot \nabla_{w,r} v \, dT = \int_T R_h(a\nabla_{w,r} u_h) \cdot \nabla_{w,r} v \, dT$$
$$= -\int_T \nabla \cdot R_h(a\nabla_{w,r} u_h) \, dT$$
$$= -\int_{\partial T} R_h(a\nabla_{w,r} u_h) \cdot \mathbf{n} \, ds \tag{6.3}$$

and

$$\int_T bu_0 \cdot \nabla_{w,r} v \, dT = \int_T R_h(bu_0) \cdot \nabla_{w,r} v \, dT$$
$$= -\int_T \nabla \cdot R_h(bu_0) \, dT$$
$$= -\int_{\partial T} R_h(bu_0) \cdot \mathbf{n} \, ds. \tag{6.4}$$

Now substituting (6.4) and (6.3) into (6.2) yields

$$\int_{\partial T} R_h\left(-a\nabla_{w,r} u_h + bu_0\right) \cdot \mathbf{n} \, ds + \int_T cu_0 \, dT = \int_T f \, dT, \tag{6.5}$$

which indicates that the weak Galerkin method conserves mass with a numerical flux given by

$$q_h \cdot \mathbf{n} = R_h\left(-a\nabla_{w,r} u_h + bu_0\right) \cdot \mathbf{n}.$$

The numerical flux $q_h \cdot \mathbf{n}$ can be verified to be continuous across the edge of each element T through a selection of the test function $v = \{v_0, v_b\}$ so that $v_0 \equiv 0$ and v_b are arbitrary.

7. Existence and uniqueness for weak Galerkin approximations

Assume that u_h is a weak Galerkin approximation for the problem (1.1) and (1.2) arising from (4.6) by using the finite element space $S_h(j, j+1)$ or $S_h(j, j)$. The goal of this section is to derive a uniqueness and existence result for u_h. For simplicity, details are only presented for the finite element space $S_h(j, j+1)$; the result can be extended to $S_h(j, j)$ without any difficulty.

First of all, let us derive the following analogy of Gårding's inequality.

Lemma 7.1. *Let $S_h(j, \ell)$ be the weak finite element space defined in (4.2) and $a(\cdot, \cdot)$ be the bilinear form given in (4.5). There exist positive constants K and α_1 satisfying*

$$a(v, v) + K(v_0, v_0) \geq \alpha_1(\|\nabla_{w,r} v\|^2 + \|v_0\|^2), \tag{7.1}$$

for all $v \in S_h(j, \ell)$.

Proof. Let $B_1 = \|b\|_{L^\infty(\Omega)}$ and $B_2 = \|c\|_{L^\infty(\Omega)}$ be the L^∞ norms of the coefficients b and c, respectively. Since

$$|(bv_0, \nabla_{w,r} v)| \leq B_1 \|\nabla_{w,r} v\| \, \|v_0\|,$$
$$|(cv_0, v_0)| \leq B_2 \|v_0\|^2,$$

then it follows from (4.5) that there exists a constant K and α_1 such that

$$\begin{aligned} a(v, v) + K(v_0, v_0) &\geq \alpha \|\nabla_{w,r} v\|^2 - B_1 \|\nabla_{w,r} v\| \, \|v_0\| + (K - B_2)\|v_0\|^2 \\ &\geq \alpha_1(\|\nabla_{w,r} v\|^2 + \|v_0\|^2), \end{aligned}$$

which completes the proof. □

For simplicity of notation, we shall drop the subscript r in the discrete weak gradient operator $\nabla_{w,r}$ from now on. Readers should bear in mind that ∇_w refers to a discrete weak gradient operator defined by using the setups of either Example 5.1 or Example 5.2. In fact, for these two examples, one may also define a projection Π_h for $\mathbf{q} \in H(\text{div}, \Omega)$ such that $\Pi_h \mathbf{q} \in H(\text{div}, \Omega)$, and on each $T \in \mathcal{T}_h$, one has $\Pi_h \mathbf{q} \in V(T, r = j + 1)$ and the following identity (see [13])

$$(\nabla \cdot \mathbf{q}, v_0)_T = (\nabla \cdot \Pi_h \mathbf{q}, v_0)_T, \quad \forall v_0 \in P_j(T^0).$$

The following result is based on the above property of Π_h.

Lemma 7.2. *For any $\mathbf{q} \in H(\text{div}, \Omega)$, we have*

$$\sum_{T \in \mathcal{T}_h} (-\nabla \cdot \mathbf{q}, v_0)_T = \sum_{T \in \mathcal{T}_h} (\Pi_h \mathbf{q}, \nabla_w v)_T, \tag{7.2}$$

for all $v = \{v_0, v_b\} \in S_h^0(j, j+1)$.

Proof. The definition of Π_h and the definition of $\nabla_w v$ imply,

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} (-\nabla \cdot \mathbf{q}, v_0)_T &= \sum_{T \in \mathcal{T}_h} (-\nabla \cdot \Pi_h \mathbf{q}, v_0)_T \\ &= \sum_{T \in \mathcal{T}_h} (\Pi_h \mathbf{q}, \nabla_w v)_T - \sum_{T \in \mathcal{T}_h} \langle v_b, \Pi_h \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T} \\ &= \sum_{T \in \mathcal{T}_h} (\Pi_h \mathbf{q}, \nabla_w v)_T. \end{aligned}$$

Here we have used the fact that $\Pi_h \mathbf{q} \cdot \mathbf{n}$ is continuous across each interior edge and $v_b = 0$ on $\partial \Omega$. This completes the proof. □

Lemma 7.3. *For $u \in H^{1+s}(\Omega)$ with $s > 0$, we have*

$$\|\Pi_h(a\nabla u) - a\nabla_w(Q_h u)\| \leq Ch^s \|u\|_{1+s}, \tag{7.3}$$
$$\|\nabla u - \nabla_w(Q_h u)\| \leq Ch^s \|u\|_{1+s}. \tag{7.4}$$

Proof. Since from (5.4) we have $\nabla_w(Q_h u) = R_h(\nabla u)$, then

$$\|\Pi_h(a\nabla u) - a\nabla_w(Q_h u)\| = \|\Pi_h(a\nabla u) - aR_h(\nabla u)\|.$$

Using the triangle inequality and the definition of Π_h and R_h, we have

$$\begin{aligned} \|\Pi_h(a\nabla u) - aR_h(\nabla u)\| &\leq \|\Pi_h(a\nabla u) - a\nabla u\| + \|a\nabla u - aR_h(\nabla u)\| \\ &\leq Ch^s \|u\|_{1+s}. \end{aligned}$$

The estimate (7.4) can be derived in a similar way. This completes a proof of the lemma. □

We are now in a position to establish a solution uniqueness and existence for the weak Galerkin method (4.6). It suffices to prove that the solution is unique. To this end, let $e \in S_h^0(j, j + 1)$ be a discrete weak function satisfying

$$a(e, v) = 0, \quad \forall v = \{v_0, v_b\} \in S_h^0(j, j + 1). \tag{7.5}$$

The goal is to show that $e \equiv 0$ by using a duality approach similar to what Schatz [23] did for the standard Galerkin finite element methods.

Lemma 7.4. *Let $e = \{e_0, e_b\} \in S_h^0(j, j + 1)$ be a discrete weak function satisfying (7.5). Assume that the dual of (1.1) with homogeneous Dirichlet boundary condition has the H^{1+s} regularity ($s \in (0, 1]$). Then, there exists a constant C such that*

$$\|e_0\| \leq Ch^s \|\nabla_w e\|, \tag{7.6}$$

provided that the meshsize h is sufficient small.

Proof. Consider the following dual problem. Find $w \in H^1(\Omega)$ such that

$$-\nabla \cdot (a\nabla w) - b \cdot \nabla w + cw = e_0 \quad \text{in } \Omega, \tag{7.7}$$
$$w = 0 \quad \text{on } \partial\Omega. \tag{7.8}$$

The assumption of H^{1+s} regularity implies that $w \in H^{1+s}(\Omega)$ and there is a constant C such that

$$\|w\|_{1+s} \leq C\|e_0\|. \tag{7.9}$$

Testing (7.7) against e_0 and then using (7.2) lead to

$$
\begin{aligned}
\|e_0\|^2 &= (-\nabla \cdot (a\nabla w), e_0) - (b \cdot \nabla w, e_0) + (cw, e_0) \\
&= (\Pi_h(a\nabla w), \nabla_w e) - (\nabla w, be_0) + (cw, e_0) \\
&= (\Pi_h(a\nabla w) - a\nabla_w(Q_h w), \nabla_w e) + (a\nabla_w(Q_h w), \nabla_w e) - (\nabla w - \nabla_w(Q_h w), be_0) \\
&\quad - (\nabla_w(Q_h w), be_0) + (cw - c(Q_0 w), e_0) + (Q_0 w, ce_0).
\end{aligned}
$$

The sum of the second, fourth and sixth term on the right hand side of the above equation equals $a(e, Q_h w) = 0$ due to (7.5). Therefore, it follows from Lemma 7.3 that

$$
\begin{aligned}
\|e_0\|^2 &= (\Pi_h(a\nabla w) - a\nabla_w(Q_h w), \nabla_w e) - (\nabla w - \nabla_w(Q_h w), be_0) + (c(w - Q_0 w), e_0) \\
&\leq Ch^s \|w\|_{1+s} (\|\nabla_w e\| + \|e_0\|) + Ch\|w\|_1 \|e_0\|.
\end{aligned}
$$

Using the H^{1+s}-regularity assumption (7.9), we arrive at

$$\|e_0\|^2 \leq Ch^s \|e_0\| (\|\nabla_w e\| + \|e_0\|),$$

which leads to

$$\|e_0\| \leq Ch^s (\|\nabla_w e\| + \|e_0\|).$$

Thus, when h is sufficiently small, one would obtain the desired estimate (7.6). This completes the proof. □

Theorem 7.5. *Assume that the dual of (1.1) with homogeneous Dirichlet boundary condition has H^{1+s}-regularity for some $s \in (0, 1]$. The weak Galerkin finite element method defined in (4.6) has a unique solution in the finite element spaces $S_h(j, j + 1)$ and $S_h(j, j)$ if the meshsize h is sufficiently small.*

Proof. Observe that uniqueness is equivalent to existence for the solution of (4.6) since the number of unknowns is the same as the number of equations. To prove a uniqueness, let $u_h^{(1)}$ and $u_h^{(2)}$ be two solutions of (4.6). By letting $e = u_h^{(1)} - u_h^{(2)}$ we see that (7.5) is satisfied. Now we have from Gårding's inequality (7.1) that

$$a(e, e) + K\|e_0\|^2 \geq \alpha_1 \left(\|\nabla_w e\|^2 + \|e_0\|^2\right).$$

Thus, it follows from the estimate (7.6) of Lemma 7.4 that

$$\alpha_1 \left(\|\nabla_w e\|^2 + \|e_0\|^2\right) \leq KC^2 h^{2s} \|\nabla_w e\|^2$$

for h being sufficiently small. Now choose h small enough so that $KC^2 h^{2s} \leq \frac{\alpha_1}{2}$. Thus,

$$\|\nabla_w e\|^2 + \|e_0\|^2 = 0,$$

which, together with Lemma 5.1, implies that e is a constant and $e_0 = 0$. This shows that $e = 0$ and consequently, $u_h^{(1)} = u_h^{(2)}$. □

8. Error analysis

The goal of this section is to derive some error estimate for the weak Galerkin finite element method (4.6). We shall follow the usual approach in the error analysis: (1) investigating the difference between the weak Galerkin finite element approximation u_h with a certain interpolation/projection of the exact solution through an error equation, and (2) using a duality argument to analyze the error in the L^2 norm.

Let us begin with the derivation of an error equation for the weak Galerkin approximation u_h and the L^2 projection of the exact solution u in the weak Galerkin finite element space $S_h(j, j + 1)$. Recall that the L^2 projection is denoted by $Q_h u \equiv \{Q_0 u, Q_b u\}$, where Q_0 denotes the local L^2 projection onto $P_j(T)$ and Q_b is the local L^2 projection onto $P_{j+1}(\partial T)$ on each triangular element $T \in \mathcal{T}_h$. Let $v = \{v_0, v_b\} \in S_h^0(j, j + 1)$ be any test function. By testing (1.1) against the first component v_0 and using (7.2) we arrive at

$$
\begin{aligned}
(f, v_0) &= \sum_{T \in \mathcal{T}_h} (-\nabla \cdot (a\nabla u), v_0)_T + (\nabla \cdot (bu), v_0) + (cu, v_0) \\
&= (\Pi_h(a\nabla u), \nabla_w v) - (\Pi_h(bu), \nabla_w v) + (cu, v_0).
\end{aligned}
$$

Adding and subtracting the term $a(Q_h u, v) \equiv (a\nabla_w(Q_h u), \nabla_w v) - (b(Q_0 u), \nabla_w v) + (c(Q_0 u), v_0)$ on the right hand side of the above equation and then using (5.4) we obtain

$$
\begin{aligned}
(f, v_0) &= (a\nabla_w(Q_h u), \nabla_w v) - (bQ_0 u, \nabla_w v) + (cQ_0 u, v_0) + (\Pi_h(a\nabla u) - aR_h(\nabla u), \nabla_w v) \\
&\quad - (\Pi_h(bu) - bQ_0 u, \nabla_w v) + (c(u - Q_0 u), v_0),
\end{aligned} \tag{8.1}
$$

which can be rewritten as

$$
a(u_h, v) = a(Q_h u, v) + (\Pi_h(a\nabla u) - aR_h(\nabla u), \nabla_w v) - (\Pi_h(bu) - bQ_0 u, \nabla_w v) + (c(u - Q_0 u), v_0).
$$

It follows that

$$
a(u_h - Q_h u, v) = (\Pi_h(a\nabla u) - aR_h(\nabla u), \nabla_w v) - (\Pi_h(bu) - bQ_0 u, \nabla_w v) + (c(u - Q_0 u), v_0). \tag{8.2}
$$

Eq. (8.2) shall be called the *error equation* for the weak Galerkin finite element method (4.6).

8.1. An estimate in a discrete H^1-norm

We begin with the following lemma which provides an estimate for the difference between the weak Galerkin approximation u_h and the L^2 projection of the exact solution of the original problem.

Lemma 8.1. *Let $u \in H^1(\Omega)$ be the solution of (1.1) and (1.2). Let $u_h \in S_h(j, j + 1)$ be the weak Galerkin approximation of u arising from (4.6). Denote by $e_h := u_h - Q_h u$ the difference between the weak Galerkin approximation and the L^2 projection of the exact solution $u = u(x_1, x_2)$. Then there exists a constant C such that*

$$
\begin{aligned}
\frac{\alpha_1}{2}(\|\nabla_w(e_h)\|^2 + \|e_{h,0}\|^2) \leq C &\left(\|\Pi_h(a\nabla u) - aR_h(\nabla u)\|^2 + \|c(u - Q_0 u)\|^2 \right. \\
&\left. + \|\Pi_h(bu) - bQ_0 u\|^2\right) + K\|u_0 - Q_0 u\|^2.
\end{aligned} \tag{8.3}
$$

Proof. Substituting v in (8.2) by $e_h = \{e_0, e_b\} := u_h - Q_h u = \{u_0 - Q_0 u, u_b - Q_b u\}$ and using the usual Cauchy–Schwarz inequality we arrive at

$$
\begin{aligned}
a(e_h, e_h) &= (\Pi_h(a\nabla u) - aR_h(\nabla u), \nabla_w(u_h - Q_h u)) - (\Pi_h(bu) - bQ_0 u, \nabla_w(u_h - Q_h u)) \\
&\quad + (c(u - Q_0 u), u_0 - Q_0 u) \\
&\leq \|\Pi_h(a\nabla u) - aR_h(\nabla u)\|\|\nabla_w(u_h - Q_h u)\| + \|\Pi_h(bu) - bQ_0 u\|\|\nabla_w(u_h - Q_h u)\| \\
&\quad + \|c(u - Q_0 u)\|\|u_0 - Q_0 u\|.
\end{aligned}
$$

Next, we use Gårding's inequality (7.1) to obtain

$$
\begin{aligned}
\alpha_1(\|\nabla_w e_h\|^2 + \|e_0\|^2) &\leq \|\Pi_h(a\nabla u) - aR_h(\nabla u)\|\|\nabla_w(u_h - Q_h u)\| \\
&\quad + \|\Pi_h(bu) - bQ_0 u\|\|\nabla_w(u_h - Q_h u)\| + \|c(u - Q_0 u)\|\|u_0 - Q_0 u\| + K\|u_0 - Q_0 u\|^2 \\
&\leq \frac{\alpha_1}{2}(\|\nabla_w(u_h - Q_h u)\|^2 + \|u_0 - Q_0 u\|^2) + C\left(\|\Pi_h(a\nabla u) - aR_h(\nabla u)\|^2 \right. \\
&\quad \left. + \|\Pi_h(bu) - bQ_0 u\|^2 + \|c(u - Q_0 u)\|^2\right) + K\|u_0 - Q_0 u\|^2,
\end{aligned}
$$

which implies the desired estimate (8.3). $\quad\square$

8.2. An estimate in L^2 norm

We use the standard duality argument to derive an estimate for the error $u_h - Q_h u$ in the standard L^2 norm over domain Ω.

Lemma 8.2. *Assume that the dual of the problem (1.1) and (1.2) has the H^{1+s} regularity. Let $u \in H^1(\Omega)$ be the solution (1.1) and (1.2), and u_h be a weak Galerkin approximation of u arising from (4.6) by using either the weak finite element space $S_h(j, j+1)$ or $S_h(j, j)$. Let $Q_h u$ be the L^2 projection of u in the corresponding finite element space (recall that it is locally defined). Then, there exists a constant C such that*

$$\|Q_0 u - u_0\| \le Ch^s \left(h\|f - Q_0 f\| + \|\nabla u - R_h(\nabla u)\| + \|a\nabla u - R_h(a\nabla u)\| + \|u - Q_0 u\| \right.$$
$$\left. + \|bu - R_h(bu)\| + \|cu - Q_0(cu)\| + \|\nabla_w(Q_h u - u_h)\| \right),$$

provided that the meshsize h is sufficiently small.

Proof. Consider the dual problem of (1.1) and (1.2) which seeks $w \in H_0^1(\Omega)$ satisfying

$$-\nabla \cdot (a\nabla w) - b \cdot \nabla w + cw = Q_0 u - u_0 \quad \text{in } \Omega. \tag{8.4}$$

The assumed H^{1+s} regularity for the dual problem implies the existence of a constant C such that

$$\|w\|_{1+s} \le C\|Q_0 u - u_0\|. \tag{8.5}$$

Testing (8.4) against $Q_0 u - u_0$ gives

$$\|Q_0 u - u_0\|^2 = (-\nabla \cdot (a\nabla w), Q_0 u - u_0) - (b \cdot \nabla w, Q_0 u - u_0) + (cw, Q_0 u - u_0)$$
$$= \text{I} + \text{II} + \text{III}, \tag{8.6}$$

where I, II, and III are defined to represent corresponding terms. Let us estimate each of these terms one by one.
For the term I, we use the identity (7.2) to obtain

$$\text{I} = (-\nabla \cdot (a\nabla w), Q_0 u - u_0) = (\Pi_h(a\nabla w), \nabla_w(Q_h u - u_h)).$$

Recall that $\nabla_w(Q_h u) = R_h(\nabla u)$ with R_h being a local L^2 projection. Thus,

$$\text{I} = (\Pi_h(a\nabla w), \nabla_w(Q_h u - u_h)) = (\Pi_h(a\nabla w), R_h(\nabla u) - \nabla_w u_h)$$
$$= (\Pi_h(a\nabla w), \nabla u - \nabla_w u_h)$$
$$= (\Pi_h(a\nabla w) - a\nabla w, \nabla u - \nabla_w u_h) + (a\nabla w, \nabla u - \nabla_w u_h). \tag{8.7}$$

The second term in the above equation can be handled as follows. Adding and subtracting two terms $(a\nabla_w Q_h w, \nabla_w u_h)$ and $(a(\nabla w - R_h(\nabla w)), \nabla u)$ and using the fact that $\nabla_w(Q_h u) = R_h(\nabla u)$ and the definition of R_h, we arrive at

$$(a\nabla w, \nabla u - \nabla_w u_h) = (a\nabla w, \nabla u) - (a\nabla w, \nabla_w u_h)$$
$$= (a\nabla w, \nabla u) - (a\nabla_w Q_h w, \nabla_w u_h) - (a(\nabla w - R_h(\nabla w)), \nabla_w u_h)$$
$$= (a\nabla w, \nabla u) - (a\nabla_w(Q_h w), \nabla_w u_h) - (a(\nabla w - R_h(\nabla w)), \nabla_w u_h - \nabla u)$$
$$\quad - (a(\nabla w - R_h(\nabla w)), \nabla u)$$
$$= (a\nabla w, \nabla u) - (a\nabla_w(Q_h w), \nabla_w u_h) - (a(\nabla w - R_h(\nabla w)), \nabla_w u_h - \nabla u)$$
$$\quad - (\nabla w - R_h(\nabla w), a\nabla u - R_h(a\nabla u)). \tag{8.8}$$

Substituting (8.8) into (8.7) yields

$$\text{I} = (\Pi_h(a\nabla w) - a\nabla w, \nabla u - \nabla_w u_h) - (a(\nabla w - R_h(\nabla w)), \nabla_w u_h - \nabla u)$$
$$\quad - (\nabla w - R_h(\nabla w), a\nabla u - R_h(a\nabla u)) + (a\nabla w, \nabla u) - (a\nabla_w(Q_h w), \nabla_w u_h). \tag{8.9}$$

For the term II, we add and subtract $(\nabla_w(Q_h w), b(Q_0 u - u_0))$ from II to obtain

$$\text{II} = -(b \cdot \nabla w, Q_0 u - u_0)$$
$$= -(\nabla w - \nabla_w(Q_h w), b(Q_0 u - u_0)) - (\nabla_w(Q_h w), b(Q_0 u - u_0))$$
$$= -(\nabla w - \nabla_w(Q_h w), b(Q_0 u - u_0)) - (\nabla_w(Q_h w), bQ_0 u) + (\nabla_w(Q_h w), bu_0).$$

In the following, we will deal with the second term on the right hand side of the above equation. To this end, we use (5.4) and the definition of R_h and Q_0 to obtain

$$(\nabla_w(Q_h w), bQ_0 u) = (\nabla_w(Q_h w) - \nabla w, bQ_0 u) + (\nabla w, bQ_0 u)$$
$$= (\nabla_w(Q_h w) - \nabla w, bQ_0 u - bu) + (\nabla_w(Q_h w) - \nabla w, bu)$$
$$\quad + (\nabla w, bQ_0 u - bu) + (\nabla w, bu)$$
$$= (\nabla_w(Q_h w) - \nabla w, bQ_0 u - bu) + (R_h(\nabla w) - \nabla w, bu - R_h(bu))$$
$$\quad + (b \cdot \nabla w - Q_0(b \cdot \nabla w), Q_0 u - u) + (\nabla w, bu).$$

Combining the last two equations above, we arrive at

$$\begin{aligned} II = {} & -(\nabla w - \nabla_w(Q_h w), b(Q_0 u - u_0)) - (\nabla_w(Q_h w) - \nabla w, b Q_0 u - b u) \\ & - (R_h(\nabla w) - \nabla w, b u - R_h(b u)) - (b \cdot \nabla w - Q_0(b \cdot \nabla w), Q_0 u - u) - (\nabla w, b u) + (\nabla_w(Q_h w), b u_0). \end{aligned} \quad (8.10)$$

As to the term III, by adding and subtracting some terms and using the fact that Q_0 is a local L^2 projection, we easily obtain the following

$$\begin{aligned} III = {} & (c w, Q_0 u - u_0) = (c w - c Q_0 w, Q_0 u - u_0) + (c Q_0 w, Q_0 u - u_0) \\ = {} & (c w - c Q_0 w, Q_0 u - u_0) + (c Q_0 w, Q_0 u) - (c Q_0 w, u_0) \\ = {} & (c w - c Q_0 w, Q_0 u - u_0) + (c Q_0 w - c w, Q_0 u) + (c w, Q_0 u - u) + (c w, u) - (c Q_0 w, u_0) \\ = {} & (c w - c Q_0 w, Q_0 u - u_0) + (Q_0 w - w, c Q_0 u - c u) + (Q_0 w - w, c u - Q_0(c u)) \\ & + (c w - Q_0(c w), Q_0 u - u) + (c w, u) - (c Q_0 w, u_0). \end{aligned}$$

Note that the sum of the last two terms in I (see (8.9)), II (see (8.10)), and III (see the last equation above) gives

$$\begin{aligned} & (a \nabla w, \nabla u) - (a \nabla_w(Q_h w), \nabla_w u_h) - (\nabla w, b u) + (\nabla_w(Q_h w), b u_0) + (c w, u) - (c Q_0 w, u_0) \\ & = a(u, w) - a(u_h, Q_h w) \\ & = (f, w) - (f, Q_0 w) \\ & = (f - Q_0 f, w - Q_0 w). \end{aligned}$$

Thus, the sum of I, II, and III can be written as follows:

$$\begin{aligned} \|Q_0 u - u_0\|^2 = {} & (f - Q_0 f, w - Q_0 w) + (\Pi_h(a \nabla w) - a \nabla w, \nabla u - \nabla_w u_h) \\ & - (a(\nabla w - R_h(\nabla w)), \nabla_w u_h - \nabla u) - ((\nabla w - R_h(\nabla w)), a \nabla u - R_h(a \nabla u)) \\ & - (\nabla w - \nabla_w(Q_h w), b(Q_0 u - u_0)) - (\nabla_w(Q_h w) - \nabla w, b Q_0 u - b u) \\ & - (R_h(\nabla w) - \nabla w, b u - R_h(b u)) - (b \cdot \nabla w - Q_0(b \cdot \nabla w), Q_0 u - u) \\ & + (c w - c Q_0 w, Q_0 u - u_0) + (Q_0 w - w, c Q_0 u - c u) \\ & + (Q_0 w - w, c u - Q_0(c u)) + (c w - Q_0(c w), Q_0 u - u). \end{aligned} \quad (8.11)$$

Using the triangle inequality, Lemma 7.3, and (5.4) and (8.5), we can bound the second term on the right hand side in the above equation by

$$\begin{aligned} |(\Pi_h(a \nabla w) - a \nabla w, \nabla u - \nabla_w u_h)| \le {} & |(\Pi_h(a \nabla w) - a \nabla w, \nabla u - \nabla_w(Q_h u))| \\ & + |(\Pi_h(a \nabla w) - a \nabla w, \nabla_w(Q_h u) - \nabla_w u_h)| \\ \le {} & C h^s \left(\|\nabla u - R_h(\nabla u)\| + \|\nabla_w(Q_h u - u_h)\| \right) \|Q_0 u - u_0\|. \end{aligned}$$

The other terms on the right hand side of (8.11) can be estimated in a similar fashion, for which we state the results as follows:

$$|(a(\nabla w - R_h(\nabla w)), \nabla_w u_h - \nabla u)| \le C h^s \left(\|\nabla u - R_h(\nabla u)\| + \|\nabla_w(Q_h u - u_h)\| \right) \|Q_0 u - u_0\|,$$
$$|((\nabla w - R_h(\nabla w)), a \nabla u - R_h(a \nabla u))| \le C h^s \|a \nabla u - R_h(a \nabla u)\| \|Q_0 u - u_0\|,$$
$$|(\nabla w - \nabla_w(Q_h w), b(Q_0 u - u_0))| \le C h^s \|Q_0 u - u_0\|^2,$$
$$|(\nabla_w(Q_h w) - \nabla w, b Q_0 u - b u)| \le C h^s \|u - Q_0 u\| \|Q_0 u - u_0\|,$$
$$|(R_h(\nabla w) - \nabla w, b u - R_h(b u))| \le C h^s \|b u - R_h(b u)\| \|Q_0 u - u_0\|,$$
$$|(b \cdot \nabla w - Q_0(b \cdot \nabla w), Q_0 u - u)| \le C h^s \|u - Q_0 u\| \|Q_0 u - u_0\|,$$
$$|(c w - c Q_0 w, Q_0 u - u_0)| \le C h^s \|Q_0 u - u_0\|^2,$$
$$|(Q_0 w - w, c Q_0 u - c u)| \le C h^s \|u - Q_0 u\| \|Q_0 u - u_0\|,$$
$$|(Q_0 w - w, c u - Q_0(c u))| \le C h^s \|c u - Q_0(c u)\| \|Q_0 u - u_0\|,$$
$$|(c w - Q_0(c w), Q_0 u - u)| \le C h^s \|u - Q_0 u\| \|Q_0 u - u_0\|.$$

Substituting the above estimates into (8.11) yields

$$\begin{aligned} \|Q_0 u - u_0\|^2 \le {} & C h^s \big(h \|f - Q_0 f\| + \|\nabla u - R_h(\nabla u)\| + \|a \nabla u - R_h(a \nabla u)\| + \|u - Q_0 u\| \\ & + \|b u - R_h(b u)\| + \|c u - Q_0(c u)\| + \|\nabla_w(Q_h u - u_h)\| + \|Q_0 u - u_0\| \big) \|Q_0 u - u_0\|. \end{aligned}$$

For sufficiently small meshsize h, we have

$$\begin{aligned} \|Q_0 u - u_0\| \le {} & C h^s \big(h \|f - Q_0 f\| + \|\nabla u - R_h(\nabla u)\| + \|a \nabla u - R_h(a \nabla u)\| + \|u - Q_0 u\| \\ & + \|b u - R_h(b u)\| + \|c u - Q_0(c u)\| + \|\nabla_w(Q_h u - u_h)\| \big), \end{aligned}$$

which completes the proof. $\quad \square$

8.3. Error estimates in H^1 and L^2 norms

With the results established in Lemmas 8.1 and 8.2, we are ready to derive an error estimate for the weak Galerkin approximation u_h. To this end, we may substitute the result of Lemma 8.2 into the estimate shown in Lemma 8.1. If so, for sufficiently small meshsize h, we would obtain the following estimate:

$$\|\nabla_w(u_h - Q_h u)\|^2 + \|u_0 - Q_0 u\|^2 \leq C \left(\|\Pi_h(a\nabla u) - aR_h(\nabla u)\|^2 + \|c(u - Q_0 u)\|^2 + \|\Pi_h(bu) - bQ_0 u\|^2 \right)$$
$$+ Ch^{2s} \left(h^2 \|f - Q_0 f\|^2 + \|\nabla u - R_h(\nabla u)\|^2 + \|a\nabla u - R_h(a\nabla u)\|^2 \right.$$
$$\left. + \|u - Q_0 u\|^2 + \|bu - R_h(bu)\|^2 + \|cu - Q_0(cu)\|^2 \right).$$

A further use of the interpolation error estimate leads to the following error estimate in a discrete H^1 norm.

Theorem 8.3. *In addition to the assumption of Lemma 8.2, assume that the exact solution u is sufficiently smooth such that $u \in H^{m+1}(\Omega)$ with $0 \leq m \leq j + 1$. Then, there exists a constant C such that*

$$\|\nabla_w(u_h - Q_h u)\| + \|u_0 - Q_0 u\| \leq C(h^m \|u\|_{m+1} + h^{1+s} \|f - Q_0 f\|). \tag{8.12}$$

Now substituting the error estimate (8.12) into the estimate of Lemma 8.2, and then using the standard interpolation error estimate we obtain

$$\|u_h - Q_h u\| \leq C \left(h^{1+s} \|f - Q_0 f\| + h^{m+s} \|u\|_{m+1} + h^s (h^m \|u\|_{m+1} + h^{1+s} \|f - Q_0 f\|) \right)$$
$$\leq C \left(h^{1+s} \|f - Q_0 f\| + h^{m+s} \|u\|_{m+1} \right).$$

The result can then be summarized as follows.

Theorem 8.4. *Under the assumption of Theorem 8.3, there exists a constant C such that*

$$\|u_h - Q_h u\| \leq C \left(h^{1+s} \|f - Q_0 f\| + h^{m+s} \|u\|_{m+1} \right), \quad s \in (0, 1], \ m \in (0, j + 1],$$

provided that the meshsize h is sufficiently small.

If the exact solution u of (1.1) and (1.2) has the H^{j+2} regularity, then we have from Theorem 8.4 that

$$\|u_h - Q_h u\| \leq C \left(h^{1+s} h^j \|f\|_j + h^{j+s+1} \|u\|_{j+2} \right)$$
$$\leq Ch^{j+s+1} \left(\|f\|_j + \|u\|_{j+2} \right)$$

for some $0 < s \leq 1$, where s is a regularity index for the dual of (1.1) and (1.2). In the case that the dual has a full H^2 (i.e., $s = 1$) regularity, one would arrive at

$$\|u_h - Q_h u\| \leq Ch^{j+2} \left(\|f\|_j + \|u\|_{j+2} \right). \tag{8.13}$$

Recall that on each triangular element T^0, the finite element functions are of polynomials of order $j \geq 0$. Thus, the error estimate (8.13) in fact reveals a superconvergence for the weak Galerkin finite element approximation arising from (4.6).

Acknowledgments

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