MATH 521 - Numerical Analysis of Differential Equations

Christoph Ortner, 01/2024

Assignment 1 : One Dimension

Name: Jincong Li

Student ID: 60539939

Q1: Implementation of Model Problem [20]

Recall our first boundary value problem that we studied in class,

$$\begin{aligned} -u'' &= f, \quad x \in (0,1), \\ u(0) &= u(1) = 0. \end{aligned}$$

We reformulated this in the weak form: find $u \in H^1_0(0,1)$ such that

$$\int_0^1 u'v' dx = \int_0^1 f v dx \qquad \forall v \in H^1_0(0,1).$$

We then defined the finite element method as follows:

- Specify the nodes for a mesh: $0 = x_0 < x_1 < \cdots < x_N = 1$
- Specify the space $V_h = \{u_h : \text{cts, p.w. affine w.r.t. } (x_j)_j\}$
- Fine $u_h \in V_h$ such that

$$\int_0^1 u_h' v_h' dx = \int_0^1 f v_h dx \qquad \forall v_h \in H^1_0(0,1).$$

Your task: Implement this numerical scheme, using mid-point quadrature (as in class) solve it with $f(x) = 1$, plot both the exact solution and the finite element solution (for $N = 15$).

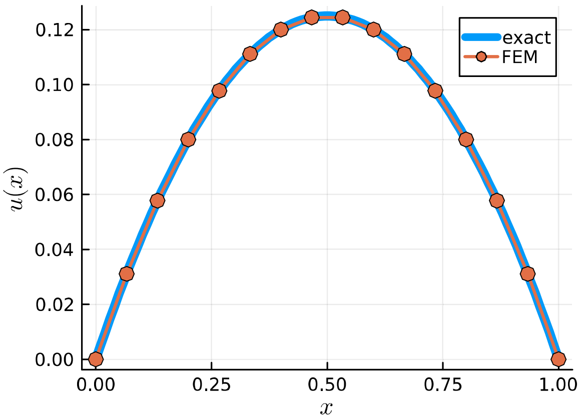
In [ ]:
```
# import Pkg; Pkg.add("Plots")
# import Pkg; Pkg.add("LaTeXStrings")
# using Plots, LaTeXStrings
# import Pkg; Pkg.add("GR")
```

In [ ]:
```
# outline of the implementation
using Plots, LaTeXStrings
function assemble_system(X, f)
    # input
    #   X : list of grid points, e.g. as Vector{Float64}
    #   f : function to evaluate f(x)

    N = length(X) - 1    # number of elements
    A = zeros(N+1, N+1)  # should be sparse, but let's not worry
    F = zeros(N+1)

    for j = 1:N
        # compute the contributions to F and A from the element (xⱼ₋₁, xⱼ)
        # and write them into A, F
        ξⱼ = 0.5 * (X[j]+X[j+1])  # midpoint for quadrature
        hⱼ = X[j+1] - X[j]        # mesh size in current element
        A[j,j] += 1/hⱼ
        A[j+1,j+1] += 1/hⱼ
        A[j,j+1] += -1/hⱼ
        A[j+1,j] += -1/hⱼ
        F[j] += hⱼ * f(ξⱼ)*0.5    # 0.5 is the value of \phi_i(\Xi_i)
        F[j+1] += hⱼ * f(ξⱼ)*0.5
    end

    return A, F
end

# My suggestion is that `assemble_system` returns
# A and F ignoring the boundary condition i.e. for the full
# N+1 DOFs. We can then reduce those to the required size
# for solving only for the free DOFs. (Think about why this works!)
```

```
N = 15
X = range(0, 1, length = N+1)
f = x -> 1.0
A, F = assemble_system(X, f)
U = zeros(N+1)
U[2:N] = A[2:N, 2:N] \ F[2:N];
```

In [ ]:

```
# the postprocessing and visualization should be done in a separate cell
# from the computation.

xp = range(0, 1, 100)
u = xp -> 0.5 * xp .* (1-xp)

plot(; xlabel = L"x", ylabel = L"u(x)", size = (400, 300))
plot!(xp, u, lw=6, label = "exact")
plot!(X, U, lw=2, m=:o, ms=5, label = "FEM")
```



Q2-pre [5]

To solve the following question you will need a little extra piece of information that I hinted at in class but didn't really work out completely: in one dimension, point evaluation is a continuous / bounded operation in the typical Sobolev spaces we encounter. Concretely, the following is true: let $\hat{x} \in (0,1)$ and let $v \in C^1([0,1])$ then

$$|v(\hat{x}) - v(0)| \leq C\|v'\|_{L^2(0,1)}$$

for some suitable constant $C > 0$. Prove this statement.

Solution for Q2-pre:

$$v(\hat{x}) - v(0) = \int_0^{\hat{x}} v'(x)\,dx$$

By applying Cauchy-Schwarz Inequality:

$$|\int_0^{\hat{x}} v'(x)\,dx\,| \leq \sqrt{|\int_0^{\hat{x}} 1^2\,dx|} \cdot \sqrt{|\int_0^{\hat{x}} (v'(x))^2\,dx|}$$

Consider the first term as a positive constant C, and then write the second term in the L^2 norm format:

$$\sqrt{|\int_0^{\hat{x}} 1^2\,dx|} \cdot \sqrt{|\int_0^{\hat{x}} (v'(x))^2\,dx|} = C||v'||_{L^2(0,1)}$$

Therefore, one can conclude

$$|v(\hat{x}) - v(0)| \leq C||v'||_{L^2(0,1)}$$

Q2: Neumann Boundary Condition [15]

Consider the boundary value problem

$$
\begin{aligned}
-u'' &= f, \quad x \in (0,1), \\
u(0) &= 0, \\
u'(1) &= g.
\end{aligned}
$$

where a, f are continuous in $[0,1]$, $a(x) > 0$, $g \in \mathbb{R}$.

(1) Derive the weak form. Prove that it has a unique solution.

HINT: the correct function space this time is not $H_0^1(0,1)$. Remember from class how we chose the test function!

(2) Formulate the corresponding finite element method. Prove that it has a unique solution.

(3) Prove that the FEM solution is the best approximation in a natural norm that you should specify.

Solution (Q2.1) Assume a test function $v(x)$ and by integration by parts:

$$
\begin{aligned}
\int_0^1 u'v' \, dx \; - u'v|_0^1 &= \int_0^1 fv \, dx \\
\int_0^1 u'v' \, dx \; - g \cdot v(1) &= \int_0^1 fv \, dx \\
\int_0^1 u'v' \, dx \; &= \int_0^1 fv \, dx \; + \int_{\Gamma_D} gv \, dx \;\; \forall v \in H^1_{\Gamma_D}
\end{aligned}
$$

One can conclude the first term to be

$$a(u,v)$$

And the second term to be

$$l(v)$$

Thus, the weak form is:

$$a(u,v) = l(v) \; \forall v \in H^1_{\Gamma_D}$$

To prove the existence and uniqueness of the solution, it is necessary to prove the following statements:

Boundness of $l(v)$:

$$
\begin{aligned}
l(v) &= \int_0^1 fv \, dx \; + \int_{\Gamma_D} gv \, dx \\
\text{Considering the first term} & \\
|\int_0^1 fv \, dx \, | &\leq ||f||_{L^2(0,1)} ||v||_{L^2(0,1)} \leq ||f||_{L^2(0,1)} ||v||_1 \\
\text{Then considering the second term} & \\
|\int_{\Gamma_D} gv \, dx \, | &\leq ||g||_{L^2(\Gamma_D)} ||v||_{L^2(\partial\Omega)} \\
\text{By Trace theroem} & \\
&\leq C_{tr} ||g||_{L^2(\Gamma_D)} ||v||_1 \\
\iff |l(v)| \leq (||f||_{L^2(0,1)} + C_{tr} ||g||_{L^2(\Gamma_D)}) ||v||_1 &
\end{aligned}
$$

Thus, the boundedness of $l(v)$ is proved and indicates $l \in (H^1_{\Gamma_D})^*$.

Boundedness of a:

$$
\begin{aligned}
a(u,u) &= \int_\Omega \sum_{ik} a_{ik} \partial_i u \partial_k u + a_0 u^2 \, dx \\
\text{Considering the first term} & \\
\int_\Omega \sum_{ik} a_{ik} \partial_i u \partial_k u \, dx \; &\leq c_1 (|\partial u|)^2 \\
\text{Considering the second term} & \\
\int_\Omega a_0 u^2 \, dx \; &\leq c_1 u^2 \\
\text{Thus,} a(u,u) &\leq c_1 \int_\Omega u^2 + (|\partial u|)^2 \, dx
\end{aligned}
$$

So, the boundedness of a is proved.

Coercivity of a:

$$
\begin{aligned}
a(u,u) &= \int_\Omega \sum_{ik} a_{ik}\partial_i u \partial_k u + a_0 u^2\, dx \text{ assume } a_0 > 0 \\
&\geq c_0 ||\partial u||_{L^2}^2 = c_0 |u|_1^2
\end{aligned}
$$

By Poincaré's inequality,exists a constant $C_p > 0$ s.t.

$$
||u||_{L^2} \leq C_p ||\partial u||_{L^2}\ \forall u \in H^1_{\Gamma_D}
$$

Therefore,

$$
\begin{aligned}
a(u,u) &\geq \min(c_0/2, c_0/(2c_p))(||\partial u||_{L^2}^2 + ||u||_{L^2}^2) \\
&=: \tilde{c_0} ||u||_1^2
\end{aligned}
$$

And by Lax-Milgram Theorem: a is proved to be bilinear, bounded and coercive. l is proved to be linear and bounded There exist unique $u \in H^1_{\Gamma_D}$ such that $a(u,v) = l(v) v \in H^1_{\Gamma_D}$

Solution (Q2.2) The corresponding discretized form is as follows: Suppose N elements in total

$$
\begin{aligned}
&\int_0^1 u_h' v_h'\, dx \ = \int_0^1 f v_h\, dx\ + \int_{\Gamma_D} g v_h\, dx \ \ \forall v \in V_h \\
&\iff V^T A U = V^T F \\
&\iff AU = F \\
&\text{where } A_{ij} = \int \phi_i{}' \phi_j{}'\, dx \ = \sum_{n=1}^{N} \int_{x_{n-1}}^{x_n} \phi_i{}' \phi_j{}'\, dx \\
&F_j = \int f \phi_j\, dx \ \approx h_j f(\xi_j)\phi(\xi_j) + h_{j+1} f(\xi_{j+1})\phi(\xi_{j+1}) \\
&\text{When } j = N,\, F_{j+1} += g
\end{aligned}
$$

I do not know how to express this in the F_j formula above precisely, but it will be clearly stated in the code-wise statement below. In terms of for loop:

$$
\begin{aligned}
&\text{for } j = 1, \ldots, N \\
&\xi_j = 0.5 \cdot (X_j + X_{j+1}) \\
&h_j = X_{j+1} - X_j \\
&F_j += h_j f(\xi_j)\phi(\xi_{j-1}) + h_{j+1} f(\xi_j)\phi(\xi_j) \\
&\text{When } j = N,\, F_{j+1} += g \\
&A_{j-1,j-1} += 1/h_n \\
&A_{j,j} += 1/h_n \\
&A_{j,j-1} += -1/h_n \\
&A_{j-1,j} += -1/h_n \\
&endforloop \\
&U = A/F
\end{aligned}
$$

Following the same steps mentioned in part Q2.1,

$$
\begin{aligned}
C||u-u_h||_{H^1_{\Gamma_D}}^2 \ leq\, & a(u-u_h, u-u_h) \\
&= a(u-u_h, u-u_h) + a(u-u_h, v_h - u_h)
\end{aligned}
$$

By Galerkin orthogonality:

$$
a(u-u_h, v) = 0\ \forall v \in V_h
$$

Note $V_h \subseteq H^1_{\Gamma_D}$.

$$
\begin{aligned}
C_0 ||u-u_h||_{H^1_{\Gamma_D}}^2 &\leq a(u-u_h, u-v_h) \\
&\leq C_1 ||u-u_h||_H ||u-v_h||_H
\end{aligned}
$$

Thus, the coercivity and boundesness of a are proved. By Lax-Milgram Theorem: There exist unique $u_h \in V_h$ such that $a(u_h, v_h) = l(v_h) v \in V_h$

Solution (Q2.3) Suppose u and u_h are the solutions of the variational probelm in $H^1_{\Gamma_D}$ and V_h. Note $V_h \subseteq H^1_{\Gamma_D}$. By definition of u and u_h,

$$
\begin{aligned}
a(u,v) &= l(v)\ \forall v \in H^1_{\Gamma_D} \\
a(u_h,v) &= l(v)\ \forall v \in V_h
\end{aligned}
$$

By Galerkin orthogonality:

$$
a(u-u_h, v) = 0\ \forall v \in V_h
$$

Let $v_h \in V_h$ with $v = v_h - u_h \in V_h$.

$$\begin{aligned}
\alpha ||u-u_h||_m^2 &\leq a(u-u_h, u-u_h) \\
&= a(u-u_h, u-u_h) + a(u-u_h, v_h-u_h) \\
&\leq C||u-u_h||_m^2 ||u-v_h||_m^2 \\
\iff \alpha ||u-u_h||_m &\leq C||u-v_h||_m
\end{aligned}$$

Therefore, in a natural norm sense, u_h is the best approximation of the original problem.

Q3: Implementation of Q2 [10]

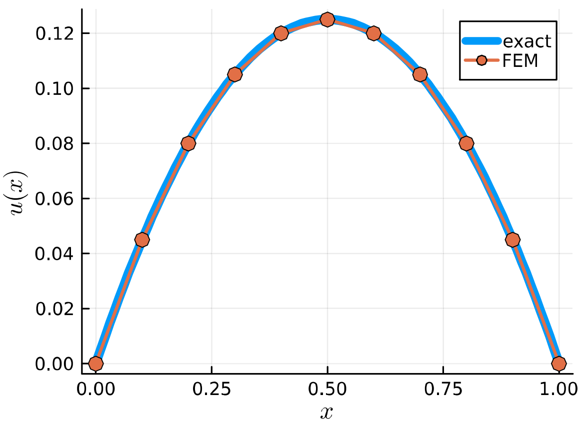
Implement the method you defined in Q2. Copy-paste your code from Q1 and adapt it.

HINT: only a single line needs to be added to the assemble, then the solution script that enforces the boundary condition needs to be adapted suitably.

Use it to solve the BVP from Q2 with $f = 1$ and $g = -1/2$ and $N = 10$. Plot the exact solution and the FEM solution.

In [ ]:
```
# Solution to Q3

# outline of the implementation
using Plots, LaTeXStrings
function assemble_system(X, f)
    # input
    #   X : list of grid points, e.g. as Vector{Float64}
    #   f : function to evaluate f(x)

    N = length(X) - 1    # number of elements
    A = zeros(N+1, N+1)  # should be sparse, but let's not worry
    F = zeros(N+1)

    for j = 1:N
        # compute the contributions to F and A from the element (xⱼ₋₁, xⱼ)
        # and write them into A, F
        ξⱼ = 0.5 * (X[j]+X[j+1])  # midpoint for quadrature
        hⱼ = X[j+1] - X[j]        # mesh size in current element
        A[j,j] += 1/hⱼ
        A[j+1,j+1] += 1/hⱼ
        A[j,j+1] += -1/hⱼ
        A[j+1,j] += -1/hⱼ
        F[j] += hⱼ * f(ξⱼ) * 0.5  # 0.5 is the value of \phi_i(\Xi_i)
        F[j+1] += hⱼ * f(ξⱼ) * 0.5
    end
    F[N+1] += g
    return A, F
end

# My suggestion is that `assemble_system` returns
# A and F ignoring the boundary condition i.e. for the full
# N+1 DOFs. We can then reduce those to the required size
# for solving only for the free DOFs. (Think about why this works!)

N = 10
X = range(0, 1, length = N+1)
f = x -> 1.0
g = -1/2
A, F = assemble_system(X, f)
U = zeros(N+1)
U[2:N+1] = A[2:N+1, 2:N+1] \ F[2:N+1];  # The #N+1 node is included here so that the Newmann BC is taken into acc
```

In [ ]:
```
# postprocessing and visualization

xp = range(0, 1, 100)
u = xp -> 0.5 * xp .* (1-xp)
plot(; xlabel = L"x", ylabel = L"u(x)", size = (400, 300))
plot!(xp, u, lw=6, label = "exact")
plot!(X, U, lw=2, m=:o, ms=5, label = "FEM")
# print(U[N])        # Used to check the last value of FEM solution
```



exact
FEM
0.00
0.02
0.04
0.06
0.08
0.10
0.12
$u(x)$
0.00
0.25
0.50
0.75
1.00
x