MECH 570C

Fluid-Structure Interaction

Module 4: Finite Element Method for Continuum Mechanics (Part 2)

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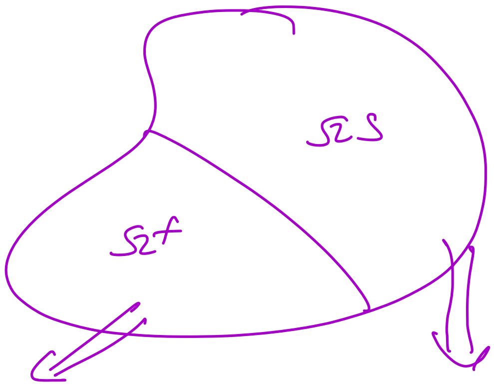
Agenda:

0. Proposal for Course project

   → Pitch

1. Coding Project #1

2.



Elastostatic

$\nabla\cdot\sigma^{s}+\rho\underset{\sim}{b}=0$

Galerkin (FEM) formula

Navier-Stokes

→ Convection-dominated

→ Pressure-velocity coupling

3. FSI coupling, applications

4. FSI physics

Review:

❑ Difficulties in variational formulation

- ► Convection-diffusion-reaction
- ► Incompressibility constraint and pressure velocity coupling

❑ Variational methods

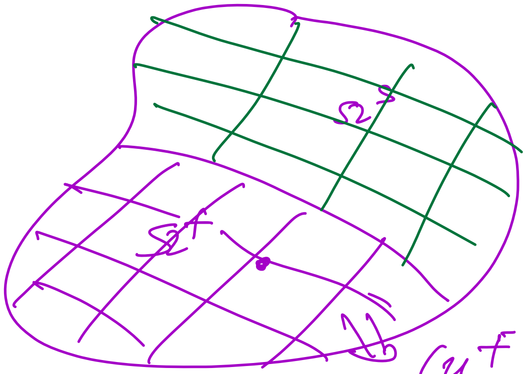
- ► Streamline upwind Petrov-Galerkin (SUPG)
- ► Galerkin/Least-Squares (GLS)
- ► Positivity Preserving Variational (PPV) method

❑ Application:

- ► Incompressible Navier-Stokes equations

Assume :

- incompressible
- viscous
- isothermal



(u^f, p^f)

Navier-Stokes Equations :

$$\rho^f \frac{\partial u^f}{\partial t} + \rho^f u^f \cdot \nabla u^f = \underbrace{\nabla \cdot \overset{-pI + \mu\, \mathrm{sym}(\nabla u)}{\underset{\sim}{\sigma}^f}} + \rho \underset{\sim}{g}$$

Strong (S) form

$$\Rightarrow \begin{cases} \cancelto{0}{\rho^f \frac{\partial \underline{u}^f}{\partial t}} + \underbrace{\cancelto{0}{\rho^f \underline{u}^f \cdot \nabla \underline{u}^f}} = -\nabla p + \mu^f \nabla^2 \underset{\sim}{u}^f + \rho \underset{\sim}{g} \\ \underbrace{\nabla \cdot \underset{\sim}{u}^f = 0} \Rightarrow \mathrm{div}(u^f) = 0 \end{cases}$$

Two difficulties Solving NS Eqns :

- → Convection (nonlinear) $\quad \nabla \cdot (\underset{\sim}{u} \otimes \underset{\sim}{u}) \Rightarrow \nabla \cdot \left(\frac{\underset{\sim}{u}^2}{2}\right)$
- → Incompressibility Constraint !

Incompressibility or Pressure-velocity Coupling problem:

Steady flow, No Convection (Negligible inertia)

$u^f = (u, v, w)$

$$\rightarrow \underset{\sim}{\mu \nabla^2} \boxed{\underset{\sim}{u}}^f - \underset{\sim}{\nabla} \boxed{p} + \underset{\sim}{\rho g} = 0 \quad — ①$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{u}^f = 0 \quad — ②$$

Stokes Problem: $(\underset{\sim}{u}^f, p)$

Four equations, four unknowns

Stokes Problem:

Strong Form

$$\begin{cases} -\mu^f \nabla^2 \underset{\sim}{u}^f + \nabla p = \rho^f g \\ \nabla \cdot u^f = 0 \\ u^f = u^f_D \qquad \Gamma^f_D \\ -\nabla u^f \cdot \underset{\sim}{n} + pn = h^f \qquad \Gamma^f_N \end{cases}$$

$$\begin{tikzcd}[column sep=small] (\underset{\sim}{u}^f, \arrow[d] & p) \arrow[d] & \in S_{u^f} \times S_p \\ (\underset{\sim}{\psi}^f, & q) & : \end{tikzcd}$$

weak statement

$$①^* \rightarrow \int_{\Omega^f} \nabla\underset{\sim}{\psi}^f : (\mu^f \nabla u^f - pI)\, d\Omega = \int \underset{\sim}{\psi}^f \cdot \rho^f \underset{\sim}{g}\, d\Omega$$

$$②^* \int_{\Omega^f} q\,(\nabla \cdot \underset{\sim}{u}^f)\, d\Omega = 0$$

In compact form : (Bilinear form)

$$a(\psi^f, \underset{\sim}{u}^f) + b(\psi^f, p) = (\psi^f, \rho^f g)$$

$$b(v^f, q) = 0$$

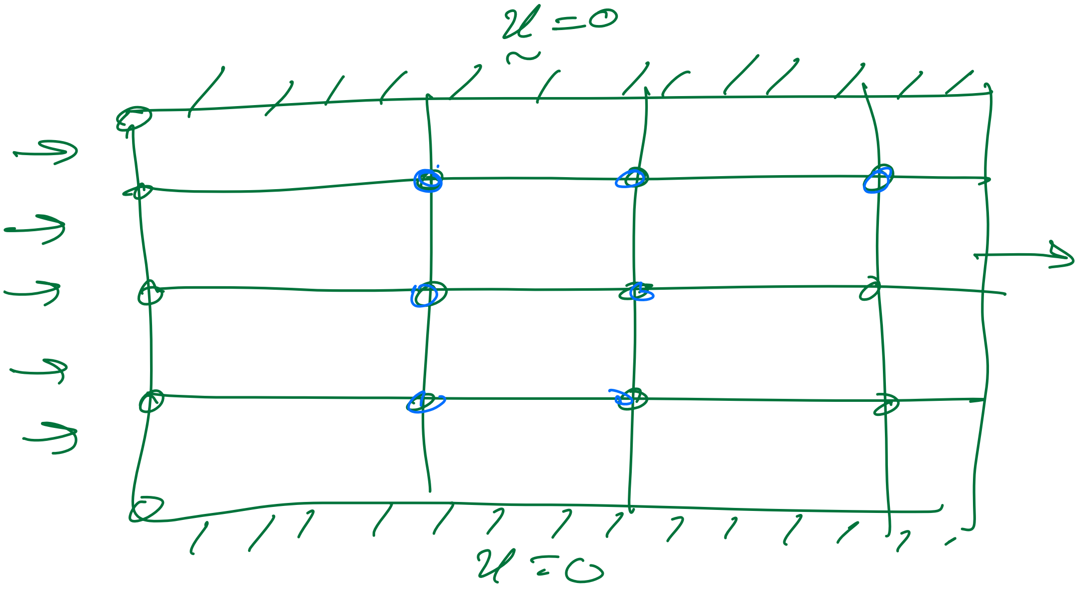
$$a(\psi^f, u^f) = \int_{\Omega} \nabla\psi^f : \mu^f \nabla u^f\, d\Omega$$

$$\Downarrow$$

$$u^f_{h,j} = \sum_{i=1}^{nen} N_i^u \, u^f_{i,j}$$

$$\phi_h = \sum_{i=1}^{nen} N_i^\phi \, \phi_i$$

$$\rightarrow \underbrace{\begin{bmatrix} \boxed{A} & \boxed{B} \\ B^T & \boxed{0} \end{bmatrix}}_{L.H.S\ (Block\ Matrix)} \begin{Bmatrix} u^f \\ \phi \end{Bmatrix} = \begin{Bmatrix} b_{mj} \\ 0 \end{Bmatrix}$$



Mass mat $\leftarrow \int N_i N_j \, d\Omega$

$$A[U^f] = \frac{M}{A\,\Delta t} + K[U^f]$$

$$K[U^f] = \mu\,[A] + \underset{\substack{\uparrow\\ \text{stabilization}\\ \text{matrix}}}{C}[U]$$

$$\underset{\sim}{A} = [a_{ij}] \Rightarrow a_{ij} = \int_\Omega \nabla N_i^u : \nabla N_j^u \, d\Omega$$

$$\underset{\sim}{B} = [b_{kj}], \quad b_{kj} = -\int_\Omega \underset{\sim}{N}_k^p \, \nabla \cdot N_j^u \, d\Omega$$

$$\underset{\sim}{M} = [m_{ij}], \quad m_{ij} = \int_\Omega N_i^u N_j^u \, d\Omega$$

Galerkin vs. Petrov Galerkin

$$\underbrace{\mathcal{L}\,\underset{\sim}{u} = f} \Rightarrow \underset{\sim}{R} = \mathcal{L}\underset{\sim}{u} - f = 0$$

$\uparrow$ Differential operator

$$u^h(x) = \sum N_i u_i$$

$\uparrow$ Shape function (interpolating/ Trial)

Galerkin weak form:

$$(*) \quad \int_{\Omega} \underset{\sim}{\psi}^{*} \underbrace{(\mathcal{L}u^h - f)}$$

$\uparrow$ Test/Weighting function

$\underset{\sim}{R} \rightarrow$ Residual

$$\Rightarrow \quad \psi^{*} = N$$

Petrov-Galerkin: $\psi^{*} \neq N$

$\uparrow$

What should ψ^{*} for NS Eq^{n}s to deal w/ incompressibility & convective terms!

A Zoo of techniques to

Construct ψ^*

Upwinding

Artificial dissipation (Stabilization)

$$\psi^* = \underline{N} \boxed{+ \tau \underline{\mathcal{L}}\, \underline{N}}$$

Scaling parameter

$$\rightarrow \quad \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

$$\underline{\mathcal{L}} = \left(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x}\right)$$

$$\frac{\partial u}{\partial t} + \rho u \cdot \nabla \underset{\sim}{u} + \nabla p = \mu \nabla^2 \underset{\sim}{u} + \cdots$$

$$\mathcal{L} = \left(\frac{\partial}{\partial t} + \rho u \cdot \nabla - \mu \nabla^2\right) \underset{\sim}{u}$$

Stabilized form:

$$\int_{\Omega} \psi^{*} \left(\mathcal{L} \underline{u}^{h} - f \right) d\Omega = 0$$

$$\uparrow$$

$$\psi^{*} = N + \tau \mathcal{L} N$$

$$u^{h} = \sum N_i u_i$$

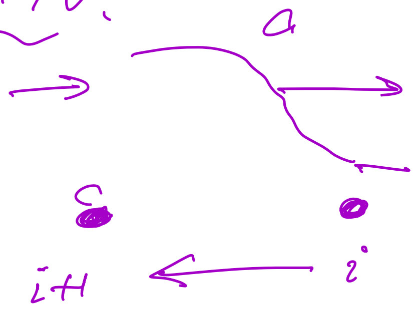
$$\downarrow$$

$$\underbrace{\int_{\Omega} N \left(\mathcal{L} u^{h} - f \right) d\Omega}_{\text{Galerkin}}$$

$$\underbrace{+ \sum_{e=1}^{nel} \int_{\Omega_e} \underline{\tau \mathcal{L} N \left(\mathcal{L} u^{h} - f \right)} \, d\Omega}_{\text{Stabilization term}} = 0$$

Upwinding:

(biasing)



$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

i-1

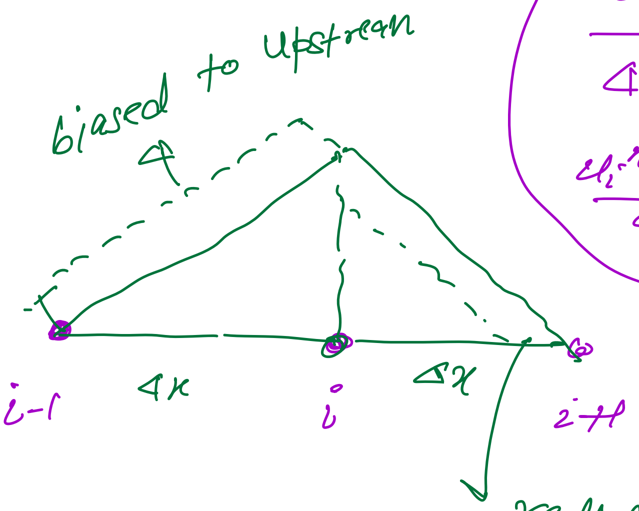
$$\frac{u_i^{n+1}-u_i^n}{\Delta t} + a\frac{u_{i+1}^n-u_{i-1}^n}{2\Delta x} = 0$$

$$\Downarrow$$

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} + a\frac{u_i^n-u_{i-1}^n}{\Delta x}$$

$a > 0$

Left



reduce downstream effects

1-D Convection-diffusion Eqn:

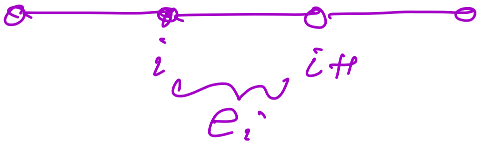
$$c\frac{\partial u}{\partial x} - k\frac{\partial^2 u}{\partial x^2} = f$$

Weak Form:

$$\int_\Omega \left(\psi c\frac{\partial u}{\partial x} + \frac{d\psi}{dx}k\frac{du}{dx}\right)d\Omega$$

$$\mathcal{L} = c\frac{\partial}{\partial x} - k\frac{\partial^2}{\partial x^2}$$

Matrix Form :



Convection term $a^e_{\{i,j\}} = \int_{\Omega^e} \underbrace{N_i} \, u \, \frac{dN_j}{dx} \, d\Omega$

Diffusion term $d^e_{i,j} = \int_{\Omega^e} \frac{dN_i}{dx} \, k \, \frac{dN_j}{dx} \, d\Omega$

Petro Galerkin $\psi^* = N_i + \underset{\uparrow}{\tau} \mathcal{L} u^h$

Peclet number ↘

$$P_e = \frac{ch}{2k} > 2$$

unstable (Galerkin)

Layout

- ❑ Before proceeding with the variational formulation of the fluid-structure coupled system, let us look at the convection-diffusion-reaction (CDR) equation which forms a canonical equation for any continuum transport system (e.g., momentum equation, turbulence transport and heat transfer).

- ❑ The present module discusses the variational formulation and finite element technique applied to the CDR equation and reviews various types of stabilization methods.

The CDR equation

The CDR equation is given as:

$$\frac{\partial \varphi}{\partial t} + \boldsymbol{v} \cdot \nabla \varphi - \nabla \cdot (\boldsymbol{k} \nabla \varphi) + \boldsymbol{s} \varphi = \boldsymbol{f}$$

on a $\boldsymbol{d}$ -dimensional domain $\Omega(t) \subset \mathbb{R}^d$, where φ is the unknown transport variable, $\boldsymbol{v}$, $\boldsymbol{k}$, $\boldsymbol{s}$ and $\boldsymbol{f}$ are the convection velocity, diffusivity tensor ($\boldsymbol{k} = \boldsymbol{k I}$ for isotropic diffusion, $\boldsymbol{k}$ and $\boldsymbol{I}$ being the diffusion coefficient and identity tensor respectively) reaction coefficient and source respectively.

Function Spaces

We review some of the mathematical preliminaries and definitions which are helpful in formulating the weak formulation in a systematic and formal way. Consider a spatial domain $\Omega \subset \mathbb{R}^{n_{sd}}$, where $n_{sd} = 1, 2$ or 3 based on the spatial dimensions. Let Γ denote the boundary to the domain. A mapping function from the domain $\bar{\Omega} = \Omega \cup \Gamma$ to $\mathbb{R}$, $f : \bar{\Omega} \to \mathbb{R}$ is said to be of class $C^m(\Omega)$ if all the derivatives

of the function up to the order m exist and are continuous functions. While solving most of higher-order differential equations, one encounters a boundary where the first derivative becomes discontinuous leading to undefined higher derivatives. Therefore, in the variational formulation, we employ the integral form of the differential equations to reduce the burden of evaluating those higher derivatives. The topic of function spaces gives a mathematical preliminary to such space of functions which obey certain restrictions which can be helpful for circumventing the issue of undefined higher derivatives.

Function Spaces

One particular case of Lebesgue space is the $L^2(\Omega)$ space which consists of functions that are square integrable over the domain. In such a case, the norm is $\|f\|_{L^2(\Omega)} = (f, f)^{1/2}$, where the inner product is defined as

$$(f, g) = \int_{\Omega} fg d\Omega$$

Therefore, $L^2(\Omega)$ is equipped with an inner product with a norm that makes it a complete metric space, a type of Hilbert space.

Sobolev space: Functions in the Sobolev space are such that they belong to L^p space and its derivatives up to a certain order α also belong to L^p, i.e.,

$$W^{k,p}(\Omega) = \left\{ f \in L^p(\Omega) \middle| \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_{n_{sd}}^{\alpha_{n_d}}} \in L^p(\Omega) \forall \middle| \alpha \,| \leq k \right\}$$

where k is a non-negative integer, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{n_{sd}}) \in \mathbb{N}^{n_{sd}}$ and $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_{n_{sd}}$
Note that $W^{0,2}(\Omega) = L^2(\Omega)$. If we consider $p = 2$, the Sobolev space becomes a Hilbert space, i.e., $H^k(\Omega) = W^{k,2}(\Omega)$. For $k = 1$, the Hilbert space is defined as

$$H^1(\Omega) = \left\{ f \in L^2(\Omega) \mid \frac{\partial f}{\partial x_i} \in L^2(\Omega), i = 1, 2, \dots, n_{sd} \right\}$$

with the inner product and the norm respectively,

$$(f, g)_1 = \int_{\Omega} \left(fg + \sum_{i=1}^{n_{\text{sd}}} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i} \right) d\Omega, \quad \|f\|_1 = \sqrt{(f, f)_1}$$

The CDR System: Strong Differential Form

Consider a d -dimensional spatial domain $\Omega(t) \subset \mathbb{R}^d$ with the Dirichlet and Neumann boundaries denoted by $\Gamma_D^{\varphi}(t)$ and $\Gamma_N^{\varphi}(t)$ respectively. The strong form of the CDR equation (with φ as the scalar variable) along with the boundary conditions can be written as

$$\begin{aligned} \frac{\partial \varphi}{\partial t} + v \cdot \nabla \varphi - \nabla \cdot (k \nabla \varphi) + s\varphi = f, \qquad & \text{on } \Omega(t) \times [0,T] \\ \varphi = \varphi_D, \qquad & \text{on } \Gamma_D^{\varphi}(t) \times [0,T] \\ k \nabla \varphi \cdot n^{\varphi} = \varphi_N, \qquad & \text{on } \Gamma_N^{\varphi}(t) \times [0,T] \\ \varphi = \varphi_0, \qquad & \text{on } \Omega(0) \end{aligned}$$

The Convection-Diffusion Stability Issue

Consider the scalar advection-diffusion problem in the domain Ω with boundary Γ

$$\lambda \cdot \nabla\varphi - \nabla \cdot (\kappa \nabla \varphi) = f \quad \text{in } \Omega$$

and boundary conditions

$$\varphi = \bar{\varphi} \quad \text{on } \Gamma_{d\varphi}, \quad \kappa \nabla \varphi \cdot \mathbf{n} = h_{\varphi}$$

$$\text{on } \Gamma_{n\varphi}$$

The relative importance of advection with respect to diffusion is expressed by the Peclet number $Pe = UL/\kappa$ where U and L are some typical speed and length scal respectively.

Stability Problem

- ❑ For advection-dominated problems (high Pe numbers), solutions develop boundary and interior layers in which the transported variable varies rapidly.

Analysis (1)

One-dimensional problem: Consider the 1D problem over the domain $\Omega = [0, 1]$

$$\lambda \frac{d\varphi}{dx} - \kappa \frac{d^2\varphi}{dx^2} = 0$$

with Dirichlet boundary conditions

$$\varphi(x = 0) = \varphi_{\text{in}} \quad \text{and} \quad \varphi(x = 1) = \varphi_{\text{out}}$$

For a constant advection speed λ, it is easy to find the exact solution

$$\varphi(x) = \varphi_{\text{in}} + (\varphi_{\text{out}} - \varphi_{\text{in}}) \frac{e^{(Pe)\,x} - 1}{e^{Pe} - 1}$$

where $Pe = \lambda/\kappa$. The variation of the solution with Peclet number is illustrated in Figure.

Analysis (2)

The Galerkin finite element discretisation is

$$\int_0^1 N_i \lambda \frac{d\varphi^h}{dx} dx + \int_0^1 \frac{dN_i}{dx} \kappa \frac{d\varphi^h}{dx} dx = 0$$

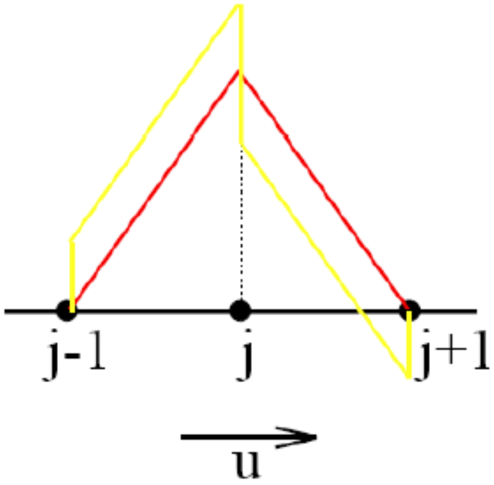
For an interior node i surrounded by two elements L and R of length h, we have

$$\left.\frac{d\varphi^h}{dx}\right)_L = \frac{\varphi_i - \varphi_{i-1}}{h}, \left.\frac{dN_i}{dx}\right)_L = \frac{1}{h} \quad \left.\frac{d\varphi^h}{dx}\right)_R = \frac{\varphi_{i+1} - \varphi_i}{h}, \left.\frac{dN_i}{dx}\right)_R = -\frac{1}{h}$$

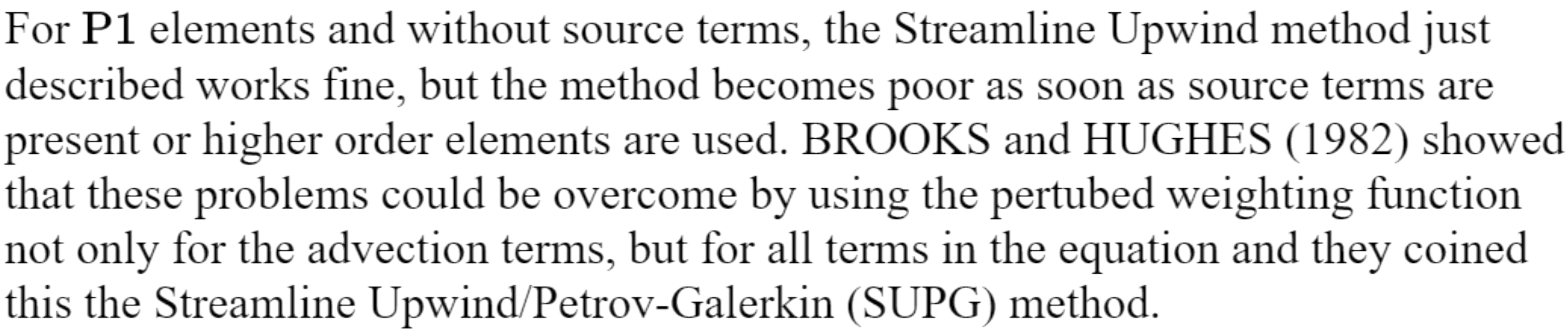
Analysis (3)

Analysis (4)

In the finite element framework, upwinding can be achieved by adding a perturbation $\tau\lambda\frac{dN_i}{dx}$ to the Galerkin weighting function. Indeed, as $dN_i/dx = \pm 1$ on the left and right elements respectively, the term $\tau\lambda\frac{dN_i}{dx}$ will always introduce a positive/negative contribution to the upwind/downwind elements.



SUPG



For **P1** elements and without source terms, the Streamline Upwind method just described works fine, but the method becomes poor as soon as source terms are present or higher order elements are used. BROOKS and HUGHES (1982) showed that these problems could be overcome by using the pertubed weighting function not only for the advection terms, but for all terms in the equation and they coined this the Streamline Upwind/Petrov-Galerkin (SUPG) method.

SUPG

Key Points

Stabilization Parameter

In one dimension, the expression

$$\tau_e = \zeta\left(Pe^h\right)\frac{h}{2\lambda}$$

was shown to provide the same discretisation as the hybrid difference schelme. The formal extension to several dimensions is straightforward as long as the element dimension h_e and the blending function $\zeta\left(Pe^h\right)$ are specified.

A better understanding of the meaning of the stability parameter has recently emerged, based on the relationship of stabilised methods with subgrid scale models and Green's functions.

The definition of the element size is extremely important since it controls directly the amount of diffusion introduced. A critical review of definitions has been made by J.-C. CARETTE.

Generalization: GLS method

The least-square finite element formulation consists in minimising $\int_\Omega r^2(\varphi^h)d\Omega$. Therefore the associated set of weighting functions is $w_i = \partial r(\varphi^h)/\partial\varphi_i$. For the present advection-diffusion equation,

$$r(\varphi^h) = \lambda\cdot\nabla\varphi^h - \nabla\cdot(\kappa\nabla\varphi^h) - f \Rightarrow \frac{\partial r(\varphi^h)}{\partial\varphi_i} = \lambda\cdot\nabla N_i - \nabla\cdot(k\nabla N_i)$$

which, for P1 elements, simplifies to $\partial r(\varphi^h)/\partial\varphi_i = \lambda\cdot\nabla N_i$, so that the stabilisation term in the SUPG formulation can be viewed as a least-square term.

This prompted the development of a generalisation of the SUPG method, i.e. the Galerkin/Least-square (GLS) which differs from the SUPG method by the expression of the stabilisation term

$$\mathrm{ST}_{\mathrm{GLS}} = \sum_e \left(\tau_e\left(\lambda\cdot\nabla N^h - \nabla\cdot(\kappa\nabla N^h\right), r(\varphi^h)\right)_\Omega$$

As pointed out previously, the two methods are identical for P1 elements and differ only for higher order elements.

Unsteady Semi-Discrete Variational Form (1)

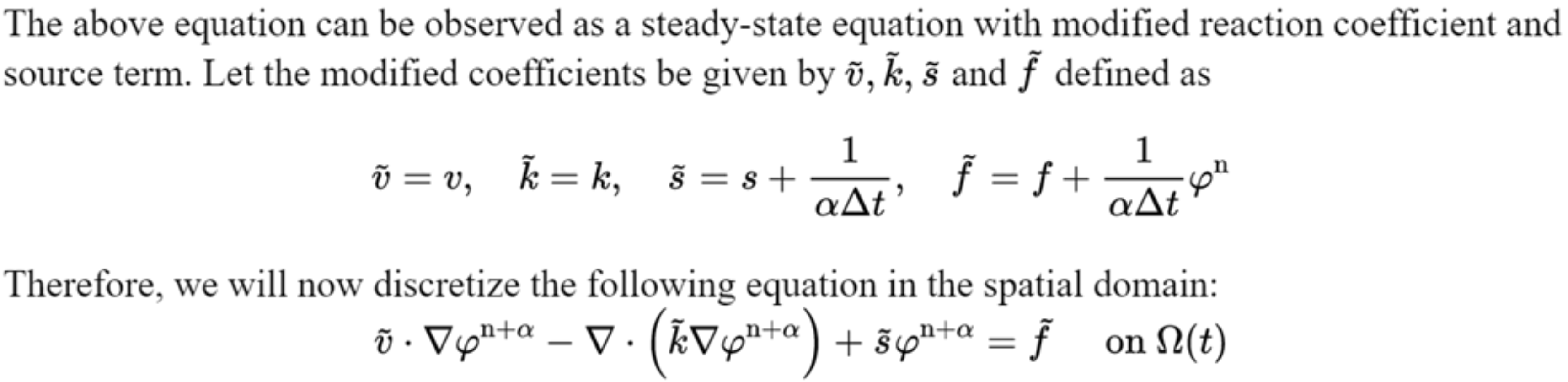
We utilize the generalized- α method to discretize the equation in time. This technique allows user-defined high-frequency damping via controlling a parameter called the spectral radius ρ_∞, which is helpful for coarser discretization in space and time. The following expressions are employed for the temporal discretization:

$$\begin{aligned}\varphi^{\mathrm{n+1}} &= \varphi^{\mathrm{n}} + \Delta t \partial_t \varphi^{\mathrm{n}} + \gamma \Delta t \left(\partial_t \varphi^{\mathrm{n+1}} - \partial_t \varphi^{\mathrm{n}}\right) \\ \partial_t \varphi^{\mathrm{n}+\alpha_{\mathrm{m}}} &= \partial_t \varphi^{\mathrm{n}} + \alpha_{\mathrm{m}} \left(\partial_t \varphi^{\mathrm{n+1}} - \partial_t \varphi^{\mathrm{n}}\right) \\ \varphi^{\mathrm{n}+\alpha} &= \varphi^{\mathrm{n}} + \alpha \left(\varphi^{\mathrm{n+1}} - \varphi^{\mathrm{n}}\right)\end{aligned}$$

where γ, α and α_{m} are the generalized $-\alpha$ parameters given by

$$\alpha_{\mathrm{m}} = \frac{1}{2}\left(\frac{3-\rho_\infty}{1+\rho_\infty}\right), \quad \alpha = \frac{1}{1+\rho_\infty}, \quad \gamma = \frac{1}{2} + \alpha_{\mathrm{m}} - \alpha$$

Semi-Discrete Variational Form (2)



The above equation can be observed as a steady-state equation with modified reaction coefficient and source term. Let the modified coefficients be given by $\tilde{v}$, $\tilde{k}$, $\tilde{s}$ and $\tilde{f}$ defined as

$$\tilde{v} = v, \quad \tilde{k} = k, \quad \tilde{s} = s + \frac{1}{\alpha \Delta t}, \quad \tilde{f} = f + \frac{1}{\alpha \Delta t} \varphi^{\mathrm{n}}$$

Therefore, we will now discretize the following equation in the spatial domain:

$$\tilde{v} \cdot \nabla \varphi^{\mathrm{n}+\alpha} - \nabla \cdot \left(\tilde{k} \nabla \varphi^{\mathrm{n}+\alpha} \right) + \tilde{s} \varphi^{\mathrm{n}+\alpha} = \tilde{f} \quad \text{on } \Omega(t)$$

Spatial Discretization (1)

Therefore, we will now discretize the following equation in the spatial domain:

$$\tilde{v} \cdot \nabla \varphi^{\mathrm{n}+\alpha} - \nabla \cdot \left(\tilde{k} \nabla \varphi^{\mathrm{n}+\alpha} \right) + \tilde{s} \varphi^{\mathrm{n}+\alpha} = \tilde{f} \qquad \text{on } \Omega(t)$$

The domain $\Omega(t)$ is discretized into $\mathrm{n_{el}}$ number of elements such that $\Omega(t) = \cup_{\mathrm{e}=1}^{\mathrm{n_{el}}} \Omega^{\mathrm{e}}$ and $\emptyset = \bigcap_{\mathrm{e}=1}^{\mathrm{n_{el}}} \Omega^{\mathrm{e}}$. The space of trial solution and test function, $\mathscr{S}_{\varphi}^{\mathrm{h}}$ and $\mathscr{V}_{\varphi}^{\mathrm{h}}$ respectively for the variational formulation are defined as

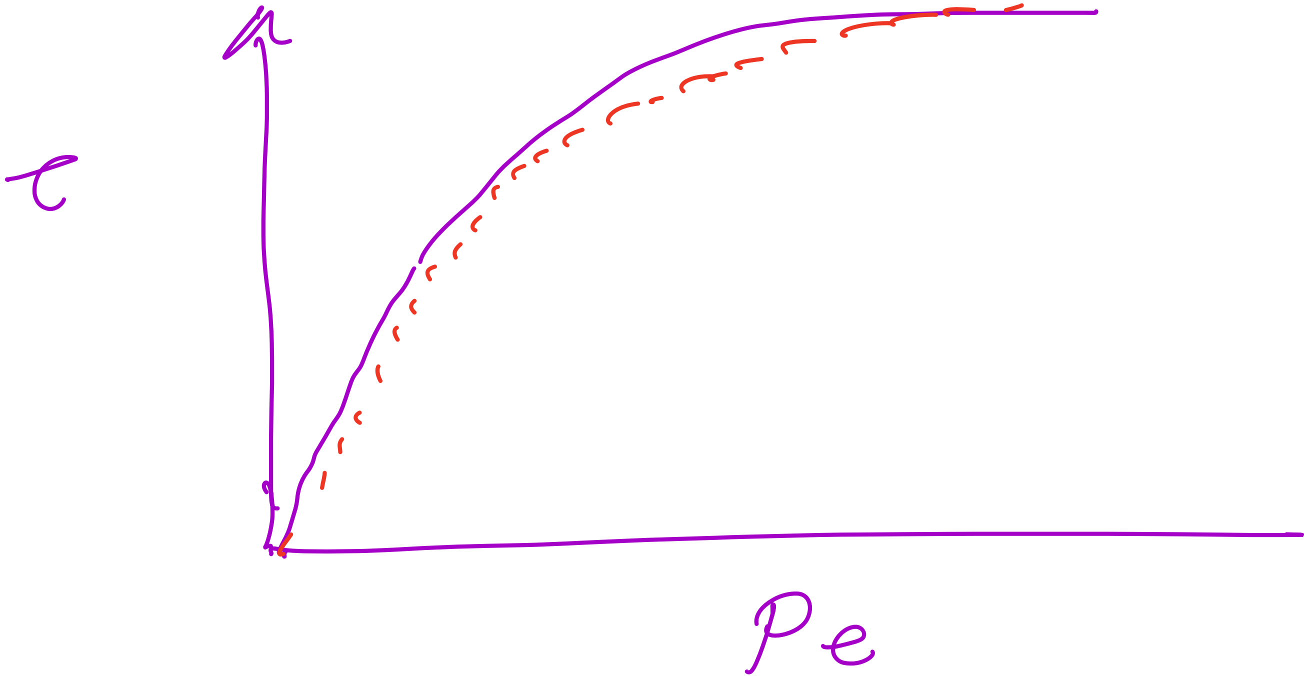
$$\mathscr{S}_{\varphi}^{\mathrm{h}} = \left\{ \varphi_{\mathrm{h}} \mid \varphi_{\mathrm{h}} \in H^1(\Omega(t)), \varphi_{\mathrm{h}} = \varphi_D \text{ on } \Gamma_D^{\varphi}(t) \right\}$$

$$\mathscr{V}_{\varphi}^{\mathrm{h}} = \left\{ w_{\mathrm{h}} \mid w_{\mathrm{h}} \in H^1(\Omega(t)), w_{\mathrm{h}} = 0 \text{ on } \Gamma_D^{\varphi}(t) \right\}$$

Spatial Discretization (2)

$$Pe = \frac{ch}{2K}$$

$$\tau = \frac{h}{2c}\left[\underline{\underline{\coth}}(Pe) - \frac{1}{Pe}\right]$$



$$\tau = \frac{h}{2c} \, f(Pe)$$

$$= \frac{h}{2c}\left(1 + \frac{1}{Pe^2}\right)^{-1/2}$$

$\Swarrow$

Harmonic function

$$\boxed{\tau_m} = \left[\underline{\left(\frac{2c}{h}\right)^2} + \left(\frac{4k}{h2}\right)^2\right]^{-1/2}$$

$\Uparrow$

No more Cotangent

Spatial Discretization (3)

As a result of spurious global oscillations and instability for convection- and reaction-dominated regimes in the Galerkin finite element method, various stabilization techniques have been proposed in the literature, the most widely used of which are SUPG and GLS methods. The stability is introduced through perturbing the test or weighting function so that the effect of upwinding is achieved. The standard variational formulation for such methods is: find $\varphi_{\mathrm{h}}(x, t^{\mathrm{n}+\alpha}) \in \mathscr{S}_{\varphi}^{\mathrm{h}}$ such that $\forall w_{\mathrm{h}} \in \mathscr{V}_{\varphi}^{\mathrm{h}}$

$$\begin{aligned}&\int_{\Omega(t)} \left(w_{\mathrm{h}} (\tilde{v} \cdot \nabla \varphi_{\mathrm{h}}) + \nabla w_{\mathrm{h}} \cdot \left(\tilde{k} \nabla \varphi_{\mathrm{h}} \right) + w_{\mathrm{h}} \tilde{s} \varphi_{\mathrm{h}} \right) d\Omega \\ &+ \sum_{\mathrm{e}=1}^{\mathrm{n_{el}}} \int_{\Omega^{\mathrm{e}}} \mathscr{L}^{m} w_{\mathrm{h}} \tau \left(\tilde{\mathscr{L}} \varphi_{\mathrm{h}} - \tilde{f} \right) d\Omega = \int_{\Omega(t)} w_{\mathrm{h}} \tilde{f} \, d\Omega + \int_{\Gamma_{N}^{\varphi}} w_{\mathrm{h}} \varphi_{N} d\Gamma\end{aligned}$$

where $\mathscr{L}^{m}$ is the operator on the weighting function given in Table 4.1 and the expression for the stabilization parameter τ is

$$\boxed{\tau = \left[\left(\frac{1}{\alpha \Delta t} \right)^{2} + 9 \left(\frac{4 \tilde{k}}{h^{2}} \right)^{2} + \left(\frac{2 |\tilde{v}|}{h} \right)^{2} + \tilde{s}^{2} \right]^{-1/2}}$$

where h is the characteristic element length and $|\tilde{v}|$ is the magnitude of the convection velocity. The formula for τ has been extensively studied in the literature with several variations, and can be estimated through error analysis. The generality of the expression is a topic of discussion later. The residual of the CDR equation is defined as

$$\mathscr{R}(\varphi_{\mathrm{h}}) = \tilde{v} \cdot \nabla \varphi_{\mathrm{h}} - \nabla \cdot \left(\tilde{k} \nabla \varphi_{\mathrm{h}} \right) + \tilde{s} \varphi_{\mathrm{h}} - \tilde{f} = \tilde{\mathscr{L}} \varphi_{\mathrm{h}} - \tilde{f}$$

where $\tilde{\mathscr{L}}$ is the differential operator corresponding to the differential equation.

Spatial Discretization (4)

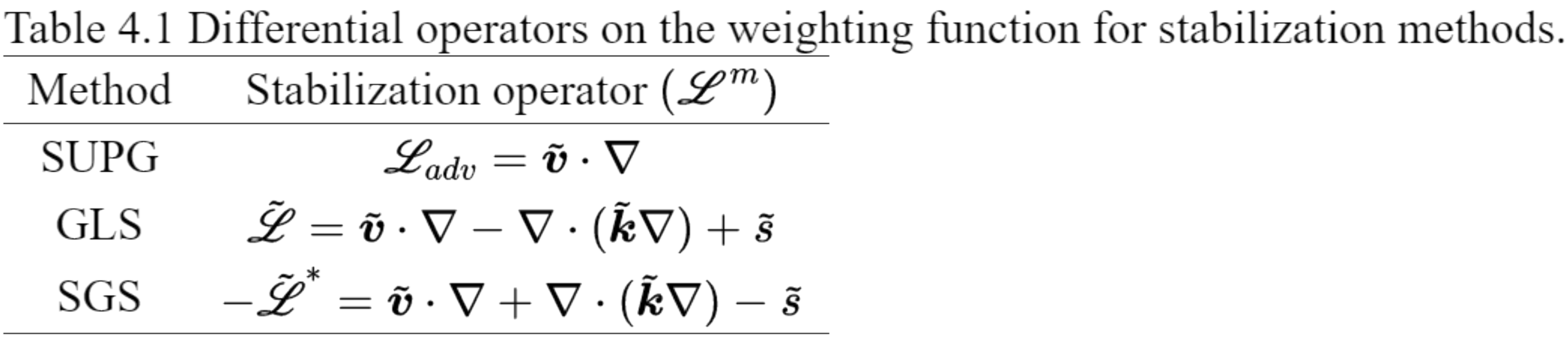


Table 4.1 Differential operators on the weighting function for stabilization methods.

Method	Stabilization operator ($\mathscr{L}^m$)
SUPG	$\mathscr{L}_{adv} = \tilde{\boldsymbol{v}} \cdot \nabla$
GLS	$\tilde{\mathscr{L}} = \tilde{\boldsymbol{v}} \cdot \nabla - \nabla \cdot (\tilde{\boldsymbol{k}} \nabla) + \tilde{\boldsymbol{s}}$
SGS	$-\tilde{\mathscr{L}}^* = \tilde{\boldsymbol{v}} \cdot \nabla + \nabla \cdot (\tilde{\boldsymbol{k}} \nabla) - \tilde{\boldsymbol{s}}$

We begin by analyzing the linear stabilization methods (SUPG, GLS and SGS) with respect to the effect of the sign of the reaction coefficient ($\tilde{\boldsymbol{s}}$), owing to the destruction or production effects. Fourier analysis of the discretized methods (GLS and SGS) showed that the SGS method performs well when $\tilde{\boldsymbol{s}}$ is negative, but, it loses accuracy when $\tilde{\boldsymbol{s}} >> 0$ due to excessive dissipation. On the other hand, the GLS method is not as diffusive as SGS when $\tilde{\boldsymbol{s}} \geq 0$, but it suffers from phase error when $\tilde{\boldsymbol{s}} < 0$. Thus, we select a combination of these methods, so that the formulation is benefited in both production and destruction regimes. Note that the effect of the diffusion term is assumed negligible in the differential operator owing to the use of linear and multilinear finite elements.

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