Vectors and Index Notation

Assistant Prof. Mauricio Ponga

Department of Mechanical Engineering, UBC

MECH 503 - 2021

Vectors and Index notation

- Scalars : α (Greek letter)

  Tempreture
  length

- Vector: $\underline{a}$ (lower case letter) bold

- $\underline{\underline{Q}}$, $\underline{\underline{T}}$ Indicates a 2^{nd} Order tensor or a matrix

$|\underline{a}|$: Magnitude of $\underline{a}$ (Length of the vector)

$|\underline{a}| : 1$ Unit vector

$|\underline{a}| : \underline{0} : |\underline{0}| = 0$ Null vector

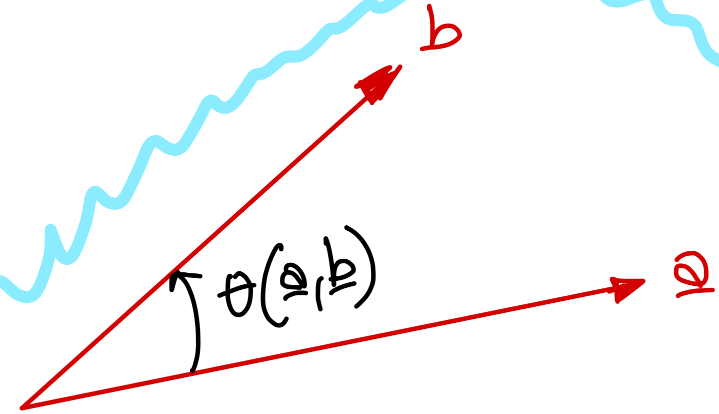
- Operations (Vector)

  Addition $\underline{a} + \underline{b} = \underline{c}$

  Mutiplication by scalar $\alpha \underline{a} = \underline{b}$

  $|\underline{b}| = |\alpha| \cdot |\underline{a}|$

  α : Scalar



Dot product

$\underline{a} \bullet \underline{b} = \alpha$

$\alpha = |\underline{a}||\underline{b}| \cos \theta(\underline{a},\underline{b})$

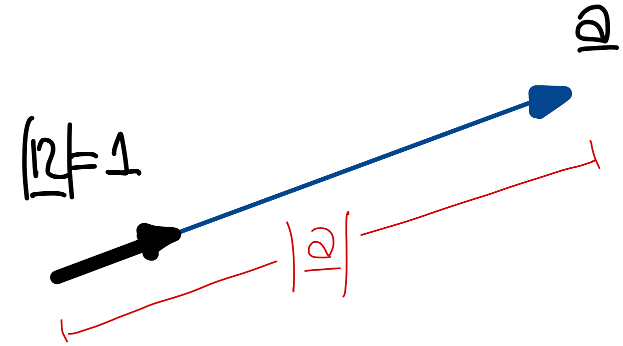
$\alpha = 0 \iff |\underline{a}| = 0$ or $|\underline{b}| = 0$ or

$\theta = \frac{n\pi}{2} \quad n = 1, 2, 3 \dots N$

Vectors and Index notation

- Given an arbitrary vector $\underline{a}$, $|\underline{a}| \neq 0$ then we can define

$$\underline{n} = \frac{\underline{a}}{|\underline{a}|}$$ Unit vector parallel to $\underline{a}$

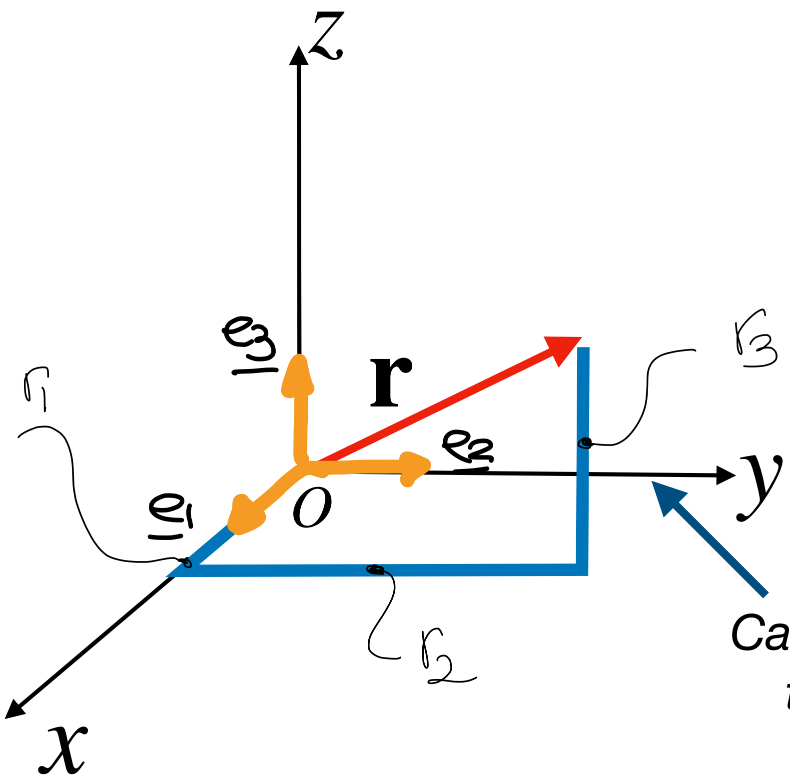


- **Cross product:** $\underline{c} = \underline{a} \times \underline{b}$; $|\underline{c}| = |\underline{a}||\underline{b}| \cdot \sin\theta(\underline{a}, \underline{b})$

$$\left.\begin{array}{l} \bullet\ \underline{a} \times \underline{b} = -\underline{b} \times \underline{a} \\[1em] \bullet\ \underline{a} \cdot (\underline{a} \times \underline{b}) = \underline{b} \cdot (\underline{b} \times \underline{a}) = 0 \end{array}\right\}$$ Proof this at home.

Vectors and Index notation

Positions using vectors



$\{\underline{e}_1, \underline{e}_2, \underline{e}_3\} \rightarrow$ orthonormal basis

$$|\underline{e}_1| = |\underline{e}_2| = |\underline{e}_3| = 1$$

$$\underline{e}_1 \times \underline{e}_2 = \underline{e}_3$$

$$\underline{e}_2 \times \underline{e}_3 = \underline{e}_1$$

$$\underline{e}_3 \times \underline{e}_1 = \underline{e}_2$$

$$\underline{r} = r_1 \underline{e}_1 + r_2 \underline{e}_2 + r_3 \underline{e}_3$$

Components of the vector

$$\underline{r} = \sum_{i=1}^{3} r_i \underline{e}_i = r_i \underline{e}_i$$ Index notation

Dot product

$$\underline{r} \cdot \underline{e}_1 = (r_1 \underline{e}_1 + r_2 \underline{e}_2 + r_3 \underline{e}_3) \cdot \underline{e}_1 =$$

$$\underline{r} \cdot \underline{e}_1 = r_1 \cdot \underline{e}_1 \cdot \underline{e}_1 = r_1$$

$$\underline{r} \cdot \underline{e}_2 = r_2 ; \quad \underline{r} \cdot \underline{e}_3 = r_3$$

Vectors and Index notation

$$\underline{c} = \underline{a} + \underline{b} \Rightarrow c_1 = a_1 + b_1 \; ; \quad c_2 = a_2 + b_2 \quad ; \quad c_3 = a_3 + b_3$$

Dot product:

Index notation for dot product in Cartesian ref. system.

$$\underline{a} \cdot \underline{b} = \sum_{i=1}^{3} a_i b_i = a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 \Rightarrow \text{scalar}$$

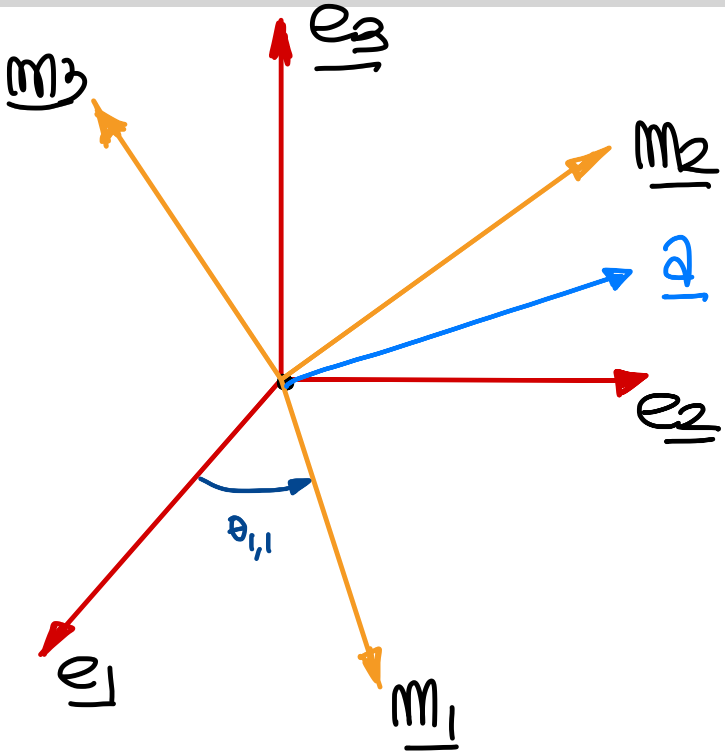
repeated index "i" is called dummy index

Cross product:

$$\underline{c} = \underline{a} \times \underline{b} \Leftrightarrow \begin{cases} c_1 = a_2 b_3 - a_3 b_2 \\ c_2 = a_3 b_1 - a_1 b_3 \\ c_3 = a_1 b_2 - a_2 b_1 \end{cases}$$

$$\begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Vectors and Index notation



$\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ is a Cartesian ref. system

$\{\underline{m}_1, \underline{m}_2, \underline{m}_3\}$ is another Cartesian ref. syst.

Given the vector $\underline{a} = a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_3 \underline{e}_3$

We want to obtain $\underline{a} = \alpha_1 \underline{m}_1 + \alpha_2 m_2 + \alpha_3 \underline{m}_3$

$\alpha_i = ?$ a_i is known.

Doing the dot product between $\underline{a}$ and $\underline{m}_i$

$$\alpha_1 = \underline{a} \cdot \underline{m}_1 = (a_1 \underline{e}_1 + a_2 \underline{e}_3 + a_3 \underline{e}_3) \cdot \underline{m}_1$$

$$\alpha_1 = a_1 \; \underline{m}_1 \cdot \underline{e}_1 + a_2 \; \underline{m}_1 \cdot \underline{e}_2 + a_3 \; \underline{m}_1 \cdot \underline{e}_3$$

Vectors and Index notation

$$\alpha_1 = a_1 \, \underline{m_1} \cdot \underline{e_1} + a_2 \, \underline{m_1} \cdot \underline{e_2} + a_3 \, \underline{m_1} \cdot \underline{e_3}$$

$$\alpha_2 = a_1 \, \underline{m_2} \cdot \underline{e_1} + a_2 \, \underline{m_2} \cdot \underline{e_2} + a_3 \, \underline{m_2} \cdot \underline{e_3}$$

$$\alpha_3 = a_1 \, \underline{m_3} \cdot \underline{e_1} + a_2 \, \underline{m_3} \cdot \underline{e_2} + a_3 \, \underline{m_3} \cdot \underline{e_3}$$

let $\underline{\underline{Q}} = Q_{ij} \quad [3 \times 3] \Rightarrow$

$$\underline{\underline{Q}} = \begin{pmatrix} \underline{m_1} \cdot \underline{e_1} & \underline{m_1} \cdot \underline{e_2} & \underline{m_1} \cdot \underline{e_3} \\ \underline{m_2} \cdot \underline{e_1} & \underline{m_2} \cdot \underline{e_2} & \underline{m_2} \cdot \underline{e_3} \\ \underline{m_3} \cdot \underline{e_1} & \underline{m_3} \cdot \underline{e_2} & \underline{m_3} \cdot \underline{e_3} \end{pmatrix}$$

$$\underline{m_i} \; \underline{e_j} = \overbrace{|\underline{m_i}|}^{1} \, \overbrace{|\underline{e_j}|}^{1} \cos(\underline{m_i}, \underline{e_j})$$

$$\alpha_i = Q_{ij} \, a_j$$

$$\underset{[3 \times 1]}{\underline{\alpha}} = \underset{[3 \times 3]}{\underline{\underline{Q}}} \cdot \underset{[3 \times 1]}{\underline{a}}$$

Vectors and Index notation

$$\underline{\underline{Q}}^{-1} = \underline{\underline{Q}}^{T} \qquad \underline{\underline{Q}} \cdot \underline{\underline{Q}}^{T} = \underline{\underline{Q}}^{T} \cdot \underline{\underline{Q}} = \underline{\underline{I}}$$

Excercise

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Check Matlab script.

- Area between two vectors $\quad A = \frac{1}{2} \left| \underline{a} \times \underline{b} \right|$
- Angle between two vectors $\quad \theta = \cos^{-1}\left(\frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|} \right)$
- Normal to two vectors $\quad \underline{n} = \pm \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$
- Volume enclosed by three vectors

$$V = \left| \underline{c} \cdot (\underline{a} \times \underline{b}) \right|$$

Vectors and Index notation

Summary:

Scalars	0^{th} order Tensor	α (No index)
Vectors	1^{st} order Tensor	$\underline{a} = a_i$ (One index) $i = 1, 2, 3$
Matrix	2^{nd} order Tensor.	$\underline{\underline{Q}} = Q_{ij}$ (Two indices $i = j = 1, 2, 3.$)
	4^{th} order Tensor	$\underline{C} = C_{ijkl}$ $i = j = k = l = 1, 2, 3.$

Vectors and Index notation

Index notation

a) $v_i u_i = \sum_{i=1}^{3} v_i u_i = v_1 u_1 + v_2 u_2 + u_3 v_3$

b) $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33}$

c) $v_{i,i} = v_{1,1} + v_{2,2} + v_{3,3} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$ div.

d) $\delta_{ij,j} =$

e) $\epsilon_{ijk} v_j u_k = \underline{v} \times \underline{u}$

f) $\delta_{mi} \delta_{mj} T_{ij} =$

Vectors and Index notation

Index notation

a) $v_i u_i = \sum_{i=1}^{3} v_i u_i = v_1 u_1 + v_2 u_2 + u_3 v_3$

b) $\delta_{ii} = \sum_{i=1}^{3} \delta_{11} + \delta_{22} + \delta_{33} = 1+1+1 = 3 = \delta_{ij}\delta_{jk}\delta_{ki}$

c) $v_{i,i} = \sum_{i=1}^{3} v_{i,j} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$ = divergence of $\underline{v}$

d) $\delta_{ij,j} = \sum_{j=1}^{3} \delta_{ij,j} = \frac{\partial \delta_{i1}}{\partial x_1} + \frac{\partial \delta_{i2}}{\partial x_2} + \frac{\partial \delta_{i3}}{\partial x_3}$

e) $\underline{\epsilon_{ijk} v_j u_k} = \underline{v} \times \underline{u}$

gives a vector.

$m=1 \quad m=2 \quad m=3$

$\delta_{mi}\left(\delta_{m1} T_{i1} + \delta_{m2} T_{i2} + \delta_{m3} T_{i3}\right)$

f) $\delta_{mi}\delta_{mj} T_{ij}$ = Returns a scalar

$\delta_{mi}\delta_{mj} T_{ij}$

Vectors and Index notation

$m=1 \qquad m=2 \qquad m=3$

$$\delta_{mi}\left(\delta_{m1}T_{i1}+\delta_{m2}T_{i2}+\delta_{m3}T_{i3}\right) = \delta_{1i}\left(\delta_{11}T_{i1}\right)+\delta_{2i}\left(\delta_{22}T_{i2}\right)+\delta_{3i}T_{i3}$$

$$\Rightarrow \delta_{11}\delta_{11}T_{11}+\delta_{22}\delta_{22}T_{22}+\delta_{33}T_{33} = T_{11}+T_{22}+T_{33} =$$

$$\mathrm{Trace}\left(\underline{T}\right)$$

$$Q_{ij}T_{ik}Q_{km} = A_{jm}$$

Matrix $\underline{Q}\,\underline{T}\,\underline{Q}^T$

$$\epsilon_{\underline{ijk}}\,\delta_{\underline{jk}} =$$ Vector.

$\delta_{ij} = e_i \cdot e_j$; $\underline{r} = r_i\,\underline{e_i}$

Vectors and Index notation

$$\delta_{ij}\delta_{jk} = \delta_{ik}$$

$$\delta_{ij}\delta_{jk}\delta_{ki} = \delta_{ik}\delta_{ki} = \delta_{ii} = \sum_{i=1}^{3}\delta_{ii} = 1+1+1 = 3$$