Chapter I
Introduction

In dealing with partial differential equations, it is useful to differentiate between several types. In particular, we classify partial differential equations of second order as *elliptic, hyperbolic*, and *parabolic*. Both the theoretical and numerical treatment differ considerably for the three types. For example, in contrast with the case of ordinary differential equations where either initial or boundary conditions can be specified, here the type of equation determines whether initial, boundary, or initial-boundary conditions should be imposed.

The most important application of the finite element method is to the numerical solution of elliptic partial differential equations. Nevertheless, it is important to understand the differences between the three types of equations. In addition, we present some elementary properties of the various types of equations. Our discussion will show that for differential equations of elliptic type, we need to specify boundary conditions and not initial conditions.

There are two main approaches to the numerical solution of elliptic problems: *finite difference methods* and *variational methods*. The finite element method belongs to the second category. Although finite element methods are particularly effective for problems with complicated geometry, finite difference methods are often employed for simple problems, primarily because they are simpler to use. We include a short and elementary discussion of them in this chapter.

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§ 1. Examples and Classification of PDE's

Examples

We first consider some examples of second order partial differential equations which occur frequently in physics and engineering, and which provide the basic prototypes for elliptic, hyperbolic, and parabolic equations.

1.1 Potential Equation. Let Ω be a domain in $\mathbb{R}^2$. Find a function u on Ω with

$$u_{xx} + u_{yy} = 0. \tag{1.1}$$

This is a differential equation of second order. To determine a unique solution, we must also specify boundary conditions.

One way to get solutions of (1.1) is to identify $\mathbb{R}^2$ with the complex plane. It is known from function theory that if $w(z) = u(z) + iv(z)$ is a holomorphic function on Ω, then its real part u and imaginary part v satisfy the potential equation. Moreover, u and v are infinitely often differentiable in the interior of Ω, and attain their maximum and minimum values on the boundary.

For the case where $\Omega := \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\}$ is a disk, there is a simple formula for the solution. Since $z^k = (re^{i\phi})^k$ is holomorphic, it follows that

$$r^k \cos k\phi, \quad r^k \sin k\phi, \quad \text{for } k = 0, 1, 2, \ldots,$$

satisfy the potential equation. If we expand these functions on the boundary in Fourier series,

$$u(\cos\phi, \sin\phi) = a_0 + \sum_{k=1}^{\infty}(a_k \cos k\phi + b_k \sin k\phi),$$

we can represent the solution in the interior as

$$u(x, y) = a_0 + \sum_{k=1}^{\infty} r^k(a_k \cos k\phi + b_k \sin k\phi). \tag{1.2}$$

The differential operator in (1.1) is the *two-dimensional Laplace operator*. For functions of d variables, it is

$$\Delta u := \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}.$$

The potential equation is a special case of the Poisson equation.

1.2 Poisson Equation. Let Ω be a domain in $\mathbb{R}^d$, $d = 2$ or 3. Here $f : \Omega \to \mathbb{R}$ is a prescribed charge density in Ω, and the solution u of the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega \tag{1.3}$$

describes the potential throughout Ω. As with the potential equation, this type of problem should be posed with boundary conditions.

1.3 The Plateau Problem as a Prototype of a Variational Problem. Suppose we stretch an ideal elastic membrane over a wire frame to create a drum. Suppose the wire frame is described by a closed, rectifiable curve in $\mathbb{R}^3$, and suppose that its parallel projection onto the (x, y)-plane is a curve with no double points. Then the position of the membrane can be described as the graph of a function $u(x, y)$. Because of the elasticity, it must assume a position such that its surface area

$$\int_\Omega \sqrt{1 + u_x^2 + u_y^2}\, dxdy$$

is minimal.

In order to solve this nonlinear variational problem approximately, we introduce a simplification. Since $\sqrt{1+z} = 1 + \frac{z}{2} + \mathcal{O}(z^2)$, for small values of u_x and u_y we can replace the integrand by a quadratic expression. This leads to the problem

$$\frac{1}{2}\int_\Omega (u_x^2 + u_y^2)\, dxdy \to \min! \tag{1.4}$$

The values of u on the boundary $\partial\Omega$ are prescribed by the given curve. We now show that the minimum is characterized by the associated Euler equation

$$\Delta u = 0. \tag{1.5}$$

Since such variational problems will be dealt with in more detail in Chapter II, here we establish (1.5) only on the assumption that a minimal solution u exists in $C^2(\Omega) \cap C^0(\bar{\Omega})$. If a solution belongs to $C^2(\Omega) \cap C^0(\bar{\Omega})$, it is called a *classical solution*. Let

$$D(u, v) := \int_\Omega (u_x v_x + u_y v_y)\, dxdy$$

and $D(v) := D(v, v)$. The quadratic form D satisfies the binomial formula

$$D(u + \alpha v) = D(u) + 2\alpha D(u, v) + \alpha^2 D(v).$$

Let $v \in C^1(\Omega)$ and $v|_{\partial\Omega} = 0$. Since $u + \alpha v$ for $\alpha \in \mathbb{R}$ is an admissible function for the minimum problem (1.4), we have $\frac{\partial}{\partial\alpha} D(u + \alpha v) = 0$ for $\alpha = 0$. Using the above binomial formula, we get $D(u, v) = 0$. Now applying Green's integral formula, we have

$$\begin{aligned} 0 = D(u, v) &= \int_\Omega (u_x v_x + u_y v_y)\, dxdy \\ &= -\int_\Omega v(u_{xx} + u_{yy})\, dxdy + \int_{\partial\Omega} v(u_x dy - u_y\, dx). \end{aligned}$$

The contour integral vanishes because of the boundary condition for v. The first integral vanishes for all $v \in C^1(\Omega)$ if and only if $\Delta u = u_{xx} + u_{yy} = 0$. This proves that (1.5) characterizes the solution of the (linearized) Plateau problem.

1.4 The Wave Equation as a Prototype of a Hyperbolic Differential Equation. The motion of particles in an ideal gas is subject to the following three laws, where as usual, we denote the velocity by v, the density by ρ, and the pressure by p:

1. *Continuity Equation.*

$$\frac{\partial\rho}{\partial t} = -\rho_0 \operatorname{div} v.$$

   Because of conservation of mass, the change in the mass contained in a (partial) volume V must be equal to the flow through its surface, i.e., it must be equal to $\int_{\partial V} \rho v \cdot n dO$. Applying Gauss' integral theorem, we get the above equation. Here ρ is approximated by the fixed density ρ_0.

2. *Newton's Law.*

$$\rho_0 \frac{\partial v}{\partial t} = -\operatorname{grad}\, p.$$

   The gradient in pressure induces a force field which causes the acceleration of the particles.

3. *State Equation.*

$$p = c^2 \rho.$$

   In ideal gases, the pressure is proportional to the density for constant temperature.

Using these three laws, we conclude that

$$\begin{aligned} \frac{\partial^2}{\partial t^2} p = c^2 \frac{\partial^2 \rho}{\partial t^2} &= -c^2 \frac{\partial}{\partial t} \rho_0 \operatorname{div} v = -c^2 \operatorname{div}(\rho_0 \frac{\partial v}{\partial t}) \\ &= c^2 \operatorname{div}\, \operatorname{grad}\, p = c^2 \Delta p. \end{aligned}$$

Other examples of the *wave equation*

$$u_{tt} = c^2 \Delta u$$

arise in two space dimensions for vibrating membranes, and in the one-dimensional case for a vibrating string. In one space dimension, the equation simplifies when c is normalized to 1:

$$u_{tt} = u_{xx}. \tag{1.6}$$

The wave equation leads to a well-posed problem (see Definition 1.8 below) when combined with initial conditions of the form

$$\begin{aligned} u(x,0) &= f(x), \\ u_t(x,0) &= g(x). \end{aligned} \tag{1.7}$$

1.5 Solution of the One-dimensional Wave Equation. To solve the wave equation (1.6)–(1.7), we apply the transformation of variables

$$\begin{aligned} \xi &= x + t, \\ \eta &= x - t. \end{aligned} \tag{1.8}$$

Applying the chain rule $u_x = u_\xi \frac{\partial \xi}{\partial x} + u_\eta \frac{\partial \eta}{\partial x}$, etc., we easily get

$$\begin{aligned} u_x &= u_\xi + u_\eta, & u_{xx} &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \\ u_t &= u_\xi - u_\eta, & u_{tt} &= u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}. \end{aligned} \tag{1.9}$$

Substituting the formulas (1.9) in (1.6) gives

$$4u_{\xi\eta} = 0.$$

The general solution is

$$\begin{aligned} u &= \phi(\xi) + \psi(\eta) \\ &= \phi(x+t) + \psi(x-t), \end{aligned} \tag{1.10}$$

where ϕ and ψ are functions which can be determined from the initial conditions (1.7):

$$\begin{aligned} f(x) &= \phi(x) + \psi(x), \\ g(x) &= \phi'(x) - \psi'(x). \end{aligned}$$

After differentiating the first equation, we have two equations for ϕ' and ψ' which are easily solved:

$$\begin{aligned} \phi' &= \frac{1}{2}(f' + g), & \phi(\xi) &= \frac{1}{2} f(\xi) + \frac{1}{2} \int_{x_0}^{\xi} g(s)\, ds, \\ \psi' &= \frac{1}{2}(f' - g), & \psi(\eta) &= \frac{1}{2} f(\eta) - \frac{1}{2} \int_{x_0}^{\eta} g(s)\, ds. \end{aligned}$$

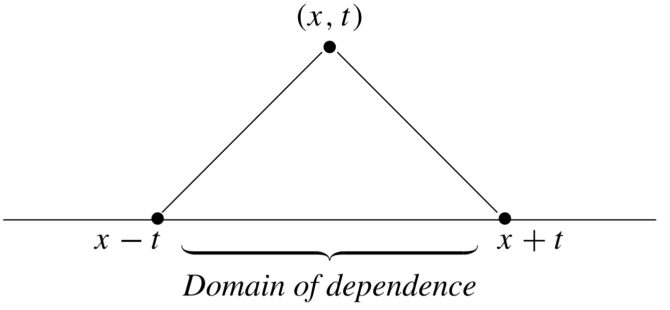

Fig. 1. Domain of dependence for the wave equation

Finally, using (1.10) we get

$$u(x,t) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds. \tag{1.11}$$

We emphasize that the solution $u(x, t)$ depends only on the initial values between the points $x-t$ and $x+t$ (see Fig. 1). [If the constant c is not normalized to be 1, the dependence is on all points between $x - ct$ and $x + ct$]. This corresponds to the fact that in the underlying physical system, any change of data can only propagate with a finite velocity.

The solution u in (1.11) was derived on the assumption that it is twice differentiable. If the initial functions f and g are not differentiable, then neither are ϕ, ψ and u. However, the formula (1.11) remains correct and makes sense even in the nondifferentiable case.

1.6 The Heat Equation as a Prototype of a Parabolic Equation. Let $T(x, t)$ be the distribution of temperature in an object. Then the heat flow is given by

$$F = -\kappa \operatorname{grad} T,$$

where κ is the diffusion constant which depends on the material. Because of conservation of energy, the change in energy in a volume element is the sum of the heat flow through the surface and the amount of heat injection Q. Using the same arguments as for conservation of mass in Example 1.4, we have

$$\begin{aligned}\frac{\partial E}{\partial t} &= -\operatorname{div}\ F + Q \\ &= \operatorname{div} \kappa \operatorname{grad}\ T + Q \\ &= \kappa \Delta T + Q,\end{aligned}$$

where κ is assumed to be constant. Introducing the constant $a = \partial E/\partial T$ for the specific heat (which also depends on the material), we get

$$\frac{\partial T}{\partial t} = \frac{\kappa}{a}\Delta T + \frac{1}{a} Q.$$

For a one-dimensional rod and $Q = 0$, with $u = T$ this simplifies to

$$u_t = \sigma u_{xx}, \tag{1.12}$$

where $\sigma = \kappa/a$. As before, we may assume the normalization $\sigma = 1$ by an appropriate choice of units.

Parabolic problems typically lead to *initial-boundary-value problems*.

We first consider the heat distribution on a rod of finite length ℓ. Then, in addition to the initial values, we also have to specify the temperature or the heat fluxes on the boundaries. For simplicity, we restrict ourselves to the case where the temperature is constant at both ends of the rod as a function of time. Then, without loss of generality, we can assume that

$$\sigma = 1, \quad \ell = \pi \quad \text{and} \quad u(0,t) = u(\pi,t) = 0;$$

cf. Problem 1.10. Suppose the initial values are given by the Fourier series expansion

$$u(x,0) = \sum_{k=1}^{\infty} a_k \sin kx, \qquad 0 < x < \pi.$$

Obviously, the functions $e^{-k^2 t} \sin kx$ satisfy the heat equation $u_t = u_{xx}$, and thus

$$u(x,t) = \sum_{k=1}^{\infty} a_k e^{-k^2 t} \sin kx, \qquad t \geq 0 \tag{1.13}$$

is a solution of the given initial-value problem.

For an infinitely long rod, the boundary conditions drop out. Now we need to know something about the decay of the initial values at infinity, which we ignore here. In this case we can write the solution using Fourier integrals instead of Fourier series. This leads to the representation

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\xi^2/4t} f(x-\xi)\, d\xi, \tag{1.14}$$

where the initial value $f(x) := u(x,0)$ appears explicitly. Note that the solution at a point (x,t) depends on the initial values on the entire domain, and the propagation of the data occurs with infinite speed.

Classification of PDE's

Problems involving ordinary differential equations can be posed with either initial or boundary conditions. This is no longer the case for partial differential equations. Here the question of whether initial or boundary conditions should be applied depends on the *type of the differential equation*.

The general linear partial differential equation of second order in n variables $x = (x_1, \ldots, x_n)$ has the form

$$-\sum_{i,k=1}^{n} a_{ik}(x) u_{x_i x_k} + \sum_{i=1}^{n} b_i(x) u_{x_i} + c(x) u = f(x). \tag{1.15}$$

If the functions a_{ik}, b_i and c do not depend on x, then the partial differential equation has *constant coefficients*. Since $u_{x_i x_k} = u_{x_k x_i}$ for any function which is twice continuously differentiable, without loss of generality we can assume the symmetry $a_{ik}(x) = a_{ki}(x)$. Then the corresponding $n \times n$ matrix

$$A(x) := (a_{ik}(x))$$

is symmetric.

1.7 Definition. (1) The equation (1.15) is called *elliptic at the point x* provided $A(x)$ is positive definite.
(2) The equation (1.15) is called *hyperbolic at the point x* provided $A(x)$ has one negative and $n - 1$ positive eigenvalues.
(3) The equation (1.15) is called *parabolic at the point x* provided $A(x)$ is positive semidefinite, but is not positive definite, and the rank of $(A(x), b(x))$ equals n.
(4) An equation is called *elliptic, hyperbolic* or *parabolic* provided it has the corresponding property for all points of the domain. □

In the elliptic case, the equation (1.15) is usually written in the compact form

$$L u = f, \tag{1.16}$$

where L is an *elliptic differential operator of order 2*. The part with the derivatives of highest order, i.e., $-\sum a_{ik}(x) u_{x_i x_k}$, is called the *principal part* of L. For hyperbolic and parabolic problems there is a special variable which is usually time. Thus, hyperbolic differential equations can often be written in the form

$$u_{tt} + Lu = f, \tag{1.17}$$

while parabolic ones can often be written in the form

$$u_t + Lu = f, \tag{1.18}$$

where L is an elliptic differential operator.

If a differential equation is invariant under *isometric mappings* (i.e., under translation and rotation), then the elliptic operator has the form

$$Lu = -a_0 \Delta u + c_0 u.$$

The above examples all display this invariance.

Well-posed Problems

What happens if we consider a partial differential equation in a framework which is meant for a different type?

To answer this question, we first turn to the wave equation (1.6), and attempt to solve the *boundary-value problem* in the domain

$$\Omega = \{(x, t) \in \mathbb{R}^2;\ a_1 < x + t < a_2,\ b_1 < x - t < b_2\}.$$

Here Ω is a rotated rectangle, and its sides are parallel to the coordinate axes ξ, η defined in (1.8). In view of $u(\xi, \eta) = \phi(\xi) + \psi(\eta)$, the values of u on opposite sides of Ω can differ only by a constant. Thus, the boundary-value problem with general data is not solvable. This also follows for differently shaped domains by similar but somewhat more involved considerations.

Next we study the potential equation (1.1) in the domain $\{(x, y) \in \mathbb{R}^2;\ y \geq 0\}$ as an *initial-value problem*, where y plays the role of time. Let $n > 0$. Assuming

$$\begin{aligned} u(x, 0) &= \frac{1}{n} \sin nx, \\ u_y(x, 0) &= 0, \end{aligned}$$

we clearly get the *formal solution*

$$u(x, y) = \frac{1}{n} \cosh ny\ \sin nx,$$

which grows like e^{ny}. Since n can be arbitrarily large, we draw the following conclusion: there exist *arbitrarily small initial values* for which the corresponding solution at $y = 1$ is *arbitrarily large*. This means that solutions of this problem, when they exist, are not stable with respect to perturbations of the initial values.

Using the same arguments, it is immediately clear from (1.13) that a solution of a parabolic equation is well-behaved for $t > t_0$, but not for $t < t_0$. However, sometimes we want to solve the heat equation in the backwards direction, e.g., in order to find out what initial temperature distribution is needed in order to get a prescribed distribution at a later time $t_1 > 0$. This is a well-known improperly posed problem. By (1.13), we can prescribe at most the low frequency terms of the temperature at time t_1, but by no means the high frequency ones.

Considerations of this type led Hadamard [1932] to consider the solvability of differential equations (and similarly structured problems) together with the stability of the solution.

1.8 Definition. A problem is called *well posed* provided it has a unique solution which depends continuously on the given data. Otherwise it is called *improperly posed*.

In principle, the question of whether a problem is well posed can depend on the choice of the norm used for the corresponding function spaces. For example, from (1.11) we see that problem (1.6)–(1.7) is well posed. The mapping

$$\begin{gathered} C(\mathbb{R}) \times C(\mathbb{R}) \longrightarrow C(\mathbb{R} \times \mathbb{R}_+), \\ f, g \longmapsto u \end{gathered}$$

defined by (1.11) is continuous provided $C(\mathbb{R})$ is endowed with the usual maximum norm, and $C(\mathbb{R} \times \mathbb{R}_+)$ is endowed with the weighted norm

$$\|u\| := \max_{x,t} \{\frac{|u(x,t)|}{1+|t|}\}.$$

The maximum principle to be discussed in the next section is a useful tool for showing that elliptic and parabolic differential equations are well posed.

Problems

1.9 Consider the potential equation in the disk $\Omega := \{(x, y) \in \mathbb{R}^2;\ x^2 + y^2 < 1\}$, with the boundary condition

$$\frac{\partial}{\partial r} u(x) = g(x) \quad \text{for } x \in \partial\Omega$$

on the derivative in the normal direction. Find the solution when g is given by the Fourier series

$$g(\cos\phi, \sin\phi) = \sum_{k=1}^{\infty} (a_k \cos k\phi + b_k \sin k\phi)$$

without a constant term. (The reason for the lack of a constant term will be explained in Ch. II, §3.)

1.10 Consider the heat equation (1.12) for a rod with $\sigma \neq 1$, $\ell \neq \pi$ and $u(0,t) = u(\ell,t) = T_0 \neq 0$. How should the scalars, i.e., the constants in the transformations $t \longmapsto \alpha t$, $x \longmapsto \beta x$, $u \longmapsto u+\gamma$, be chosen so that the problem reduces to the normalized one?

1.11 Solve the heat equation for a rod with the temperature fixed only at the left end. Suppose that at the right end, the rod is isolated, so that the heat flow, and thus $\partial T/\partial x$, vanishes there.

Hint: For k odd, the functions $\phi_k(x) = \sin kx$ satisfy the boundary conditions $\phi_k(0) = 0$, $\varphi'(\frac{\pi}{2}) = 0$.

1.12 Suppose u is a solution of the wave equation, and that at time $t = 0$, u is zero outside of a bounded set. Show that the energy

$$\int_{\mathbb{R}^d} [u_t^2 + c^2 (\operatorname{grad} u)^2]\, dx \tag{1.19}$$

is constant.

Hint: Write the wave equation in the symmetric form

$$\begin{aligned} u_t &= c \operatorname{div} v, \\ v_t &= c \operatorname{grad} u, \end{aligned}$$

and represent the time derivative of the integrand in (1.19) as the divergence of an appropriate expression.

§ 2. The Maximum Principle

An important tool for the analysis of finite difference methods is the discrete analog of the so-called maximum principle. Before turning to the discrete case, we examine a simple continuous version.

In the following, Ω denotes a bounded domain in $\mathbb{R}^d$. Let

$$Lu := -\sum_{i,k=1}^{d} a_{ik}(x)u_{x_i x_k} \tag{2.1}$$

be a linear elliptic differential operator L. This means that the matrix $A = (a_{ik})$ is symmetric and positive definite on Ω. For our purposes we need a quantitative measure of ellipticity.

For convenience, the reader may assume that the coefficients a_{ik} are continuous functions, although the results remain true under less restrictive hypotheses.

2.1 Maximum Principle. *For $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, let*

$$Lu = f \leq 0 \quad \textit{in } \Omega.$$

Then u attains its maximum over $\bar{\Omega}$ on the boundary of Ω. Moreover, if u attains a maximum at an interior point of a connected set Ω, then u must be constant on Ω.

Here we prove the first assertion. For a proof of the second one, see Gilbarg and Trudinger [1983].

(1) We first carry out the proof under the stronger assumption that $f < 0$. Suppose that for some $x_0 \in \Omega$,

$$u(x_0) = \sup_{x\in\Omega} u(x) > \sup_{x\in\partial\Omega} u(x).$$

Applying the linear coordinate transformation $x \longmapsto \xi = Ux$, the differential operator becomes

$$Lu = -\sum_{i,k} (U^t A(x) U)_{ik} u_{\xi_i \xi_k}$$

in the new coordinates. In view of the symmetry, we can find an orthogonal matrix U so that $U^T A(x_0) U$ is diagonal. By the definiteness of $A(x_0)$, we deduce that these diagonal elements are positive. Since x_0 is a maximal point,

$$u_{\xi_i} = 0, \quad u_{\xi_i \xi_i} \leq 0$$

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at $x = x_0$. This means that

$$Lu(x_0) = -\sum_i (U^T A(x_0) U)_{ii} u_{\xi_i \xi_i} \geq 0,$$

in contradiction with $Lu(x_0) = f(x_0) < 0$.

(2) Now suppose that $f(x) \leq 0$ and that there exists $x = \bar{x} \in \Omega$ with $u(\bar{x}) > \sup_{x \in \partial\Omega} u(x)$. The auxiliary function $h(x) := (x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_2)^2 + \cdots + (x_d - \bar{x}_d)^2$ is bounded on $\partial\Omega$. Now if $\delta > 0$ is chosen sufficiently small, then the function

$$w := u + \delta h$$

attains its maximum at a point x_0 in the interior. Since $h_{x_i x_k} = 2\delta_{ik}$, we have

$$\begin{aligned} Lw(x_0) =& Lu(x_0) + \delta Lh(x_0) \\ =& f(x_0) - 2\delta \sum_i a_{ii}(x_0) < 0. \end{aligned}$$

This leads to a contradiction just as in the first part of the proof. □

Examples

The maximum principle has interesting interpretations for the equations (1.1)–(1.3). If the charge density vanishes in a domain Ω, then the potential is determined by the potential equation. Without any charge, the potential in the interior cannot be larger than its maximum on the boundary. The same holds if there are only negative charges.

Next we consider the variational problem 1.3. Let $c := \max_{x \in \partial\Omega} u(x)$. If the solution u does not attain its maximum on the boundary, then

$$w(x) := \min\{u(x), c\}$$

defines an admissible function which is different from u. Now the integral $D(w, w)$ exists in the sense of Lebesgue, and

$$D(w, w) = \int_{\Omega_1} (u_x^2 + u_y^2)\, dxdy < \int_{\Omega} (u_x^2 + u_y^2)\, dxdy,$$

where $\Omega_1 := \{(x, y) \in \Omega;\ u(x) < c\}$. Thus, w leads to a smaller (generalized) surface than u. We can smooth w to get a differentiable function which also provides a smaller surface. This means that the minimal solution must satisfy the maximum principle.

Corollaries

A number of simple consequences of the maximum principle can be easily derived by elementary means, such as taking the difference of two functions, or by replacing u by $-u$.

2.2 Definition. An elliptic operator of the form (2.1) is called *uniformly elliptic* provided there exists a constant $\alpha > 0$ such that

$$\xi' A(x)\xi \geq \alpha \|\xi\|^2 \quad \text{for } \xi \in \mathbb{R}^d, x \in \Omega. \tag{2.2}$$

The largest such constant α is called the *constant of ellipticity*.

2.3 Corollary. Suppose L is a linear elliptic differential operator.

(1) *Minimum Principle.* If $Lu = f \geq 0$ on Ω, then u attains its minimum on the boundary of Ω.

(2) *Comparison Principle.* Suppose $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ and

$$\begin{aligned} Lu &\leq Lv \quad \text{in } \Omega, \\ u &\leq v \quad \text{on } \partial\Omega. \end{aligned}$$

Then $u \leq v$ in Ω.

(3) *Continuous Dependence on the Boundary Data.* The solution of the linear equation $Lu = f$ with Dirichlet boundary conditions depends continuously on the boundary values. Suppose u_1 and u_2 are solutions of the linear equation $Lu = f$ with two different boundary values. Then

$$\sup_{x\in\Omega} |u_1(x) - u_2(x)| = \sup_{z\in\partial\Omega} |u_1(z) - u_2(z)|.$$

(4) *Continuous Dependence on the Right-Hand Side.* Let L be uniformly elliptic in Ω. Then there exists a constant c which depends only on Ω and the ellipticity constant α such that

$$|u(x)| \leq \sup_{z\in\partial\Omega} |u(z)| + c \sup_{z\in\Omega} |Lu(z)| \tag{2.3}$$

for every $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$.

(5) *Elliptic Operators with Helmholtz Terms.* There is a weak form of the maximum principle for the general differential operator

$$Lu := -\sum_{i,k=1}^{d} a_{ik}(x) u_{x_i x_k} + c(x)u \quad \text{with } c(x) \geq 0. \tag{2.4}$$

In particular, $Lu \le 0$ implies

$$\max_{x\in\Omega} u(x) \le \max\{0, \max_{x\in\partial\Omega} u(x)\}. \tag{2.5}$$

Proof. (1) Apply the maximum principle to $v := -u$.

(2) By construction, $Lw = Lv - Lu \ge 0$ and $w \ge 0$ on $\partial\Omega$, where $w := v-u$. It follows from the minimum principle that $\inf w \ge 0$, and thus $w(x) \ge 0$ in Ω.

(3) $Lw = 0$ for $w := u_1 - u_2$. It follows from the maximum principle that $w(x) \le \sup_{z\in\partial\Omega} w(z) \le \sup_{z\in\partial\Omega} |w(z)|$. Similarly, the minimum principle implies $w(x) \ge -\sup_{z\in\partial\Omega} |w(z)|$.

(4) Suppose Ω is contained in a circle of radius R. Since we are free to choose the coordinate system, we may assume without loss of generality that the center of this circle is at the origin. Let

$$w(x) = R^2 - \sum_i x_i^2.$$

Since $w_{x_i x_k} = -2\delta_{ik}$, clearly $Lw \ge 2n\alpha$ and $0 \le w \le R^2$ in Ω, where α is the ellipticity constant appearing in Definition 2.2. Let

$$v(x) := \sup_{z\in\partial\Omega} |u(z)| + w(x) \cdot \frac{1}{2n\alpha} \sup_{z\in\partial\Omega} |Lu(z)|.$$

Then by construction, $Lv \ge |Lu|$ in Ω, and $v \ge |u|$ on $\partial\Omega$. The comparison principle in (2) implies $-v(x) \le u(x) \le +v(x)$ in Ω. Since $w \le R^2$, we get (2.3) with $c = R^2/2n\alpha$.

(5) It suffices to give a proof for $x_0 \in \Omega$ and $u(x_0) = \sup_{z\in\Omega} u(z) > 0$. Then $Lu(x_0) - c(x_0)u(x_0) \le Lu(x_0) \le 0$, and moreover, the principal part $Lu - cu$ defines an elliptic operator. Now the proof proceeds as for Theorem 2.1. □

Problem

2.4 For a uniformly elliptic differential operator of the form (2.4), show that the solution depends continuously on the data.

§ 3. Finite Difference Methods

The finite difference method for the numerical solution of an elliptic partial differential equation involves computing approximate values for the solution at points on a rectangular grid. To compute these values, derivatives are replaced by divided differences. The stability of the method follows from a discrete analog of the maximum principle, which we will call the *discrete maximum principle*. For simplicity, we assume that Ω is a domain in $\mathbb{R}^2$.

Discretization

The first step in the discretization is to put a two-dimensional grid over the domain Ω. For simplicity, we restrict ourselves to a grid with constant mesh size h in both variables; see Fig. 2:

$$\begin{aligned}\Omega_h &:= \{(x, y) \in \Omega;\ x = kh,\ y = \ell h \quad \text{with } k, \ell \in \mathbb{Z}\},\\ \partial\Omega_h &:= \{(x, y) \in \partial\Omega;\ x = kh \ \text{ or } \ y = \ell h \quad \text{with } k, \ell \in \mathbb{Z}\}.\end{aligned}$$

We want to compute approximations to the values of u on Ω_h. These approximate values define a function U on $\Omega_h \cup \partial\Omega_h$. We can think of U as a vector of dimension equal to the number of grid points.

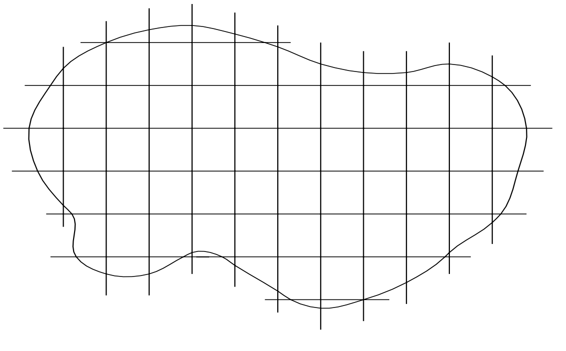

Fig. 2. A grid on a domain Ω

We get an equation at each point $z_i = (x_i, y_i)$ of Ω_h by evaluating the differential equation $Lu = f$, after replacing the derivatives in the representation (2.4) by divided differences. We choose the center of the divided difference to be the grid point of interest, and mark the neighboring points with subscripts indicating their direction relative to the center (see Fig. 3).

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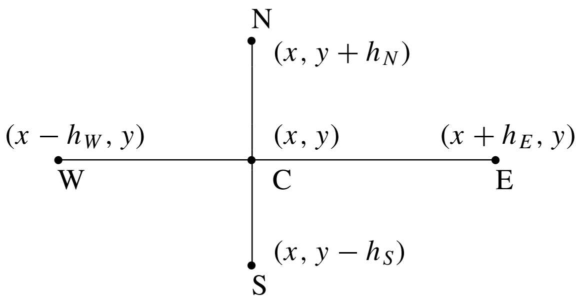

Fig. 3. Coordinates of the neighboring points of C for nonuniform step sizes. The labels of the neighbors refer to the directions east, south, west, and north.

If (x, y) is a point on a square grid whose distance to the boundary is greater than h, we can choose $h_N = h_W = h_S = h_E$ (see Fig. 2). However, for points in the neighborhood of the boundary, we have to choose $h_E \neq h_W$ or $h_N \neq h_S$. Using the Taylor formula, we see that for $u \in C^3(\Omega)$,

$$u_{xx} = \frac{2}{h_E(h_E + h_W)} u_E - \frac{2}{h_E h_W} u_C + \frac{2}{h_W(h_E + h_W)} u_W + \mathcal{O}(h), \tag{3.1}$$

where h is the maximum of h_E and h_W. In the special case where the step sizes are the same, i.e., $h_E = h_W = h$, we get the simpler formula

$$u_{xx} = \frac{1}{h^2}(u_E - 2u_C + u_W) + \mathcal{O}(h^2) \quad \text{for } u \in C^4(\Omega), \tag{3.2}$$

with an error term of second order. Analogous formulas hold for approximating u_{yy} in terms of the values u_C, u_S and u_N. To approximate the mixed derivative u_{xy} by a divided difference, we also need either the values at the NW and SE positions, or those at the NE and SW positions.

Discretization of the Poisson equation $-\Delta u = f$ leads to a system of the form

$$\alpha_C u_C + \alpha_E u_E + \alpha_S u_S + \alpha_W u_W + \alpha_N u_N = h^2 f(x_C) \quad \text{for } x_C \in \Omega_h, \tag{3.3}$$

where for each $z_C \in \Omega_h$, u_C is the associated function value. The variables with a subscript indicating a compass direction are values of u at points which are neighbors of x_C. If the differential equation has constant coefficients and we use a uniform grid, then the coefficients α_* appearing in (3.3) for a point x_C not near the boundary do not depend on C. We can write them in a matrix which we call the *difference star* or *stencil:*

$$\begin{bmatrix} \alpha_{NW} & \alpha_N & \alpha_{NE} \\ \alpha_W & \alpha_C & \alpha_E \\ \alpha_{SW} & \alpha_S & \alpha_{SE} \end{bmatrix}_* . \tag{3.4}$$

For example, for the Laplace operator, (3.2) yields the *standard five-point stencil*

$$\frac{1}{h^2}\begin{bmatrix} & -1 & \\ -1 & +4 & -1 \\ & -1 & \end{bmatrix}_* .$$

To get a higher order discretization error we can use the *nine-point stencil* for $(1/12)[8\Delta u(x, y) + \Delta u(x+h, y) + \Delta u(x-h, y) + \Delta u(x, y+h) + \Delta u(x, y-h)]$.

$$\frac{1}{6h^2}\begin{bmatrix} -1 & -4 & -1 \\ -4 & 20 & -4 \\ -1 & -4 & -1 \end{bmatrix}_*$$

3.1 An Algorithm for the Discretization of the Dirichlet Problem.

1. Choose a step size $h > 0$, and construct Ω_h and $\partial\Omega_h$.
2. Let n and m be the numbers of points in Ω_h and $\partial\Omega_h$, respectively. Number the points of Ω_h from 1 to n. Usually this is done so that the coordinates (x_i, y_i) appear in lexicographical order. Number the boundary points as $n+1$ to $n+m$.
3. Insert the given values at the boundary points:
$$U_i = u(z_i) \quad \text{for } i = n+1, \ldots, n+m.$$
4. For every interior point $z_i \in \Omega_h$, write the difference equation with z_i as center point which gives the discrete analog of $Lu(z_i) = f(z_i)$:
$$\sum_{\ell=C,E,S,W,N} \alpha_\ell U_\ell = f(z_i). \tag{3.5}$$
If a neighboring point z_ℓ belongs to the boundary $\partial\Omega_h$, move the associated term $\alpha_\ell U_\ell$ in (3.5) to the right-hand side.
5. Step 4 leads to a system
$$A_h U = f$$
of n equations in n unknowns U_i. Solve this system and identify the solution U as an approximation to u on the grid Ω_h. (Usually U is called a *numerical solution* of the PDE.)

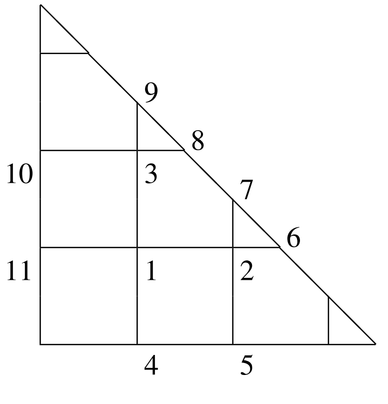

Fig. 4. Grid for Example 3.2

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3.2 Examples. (1) Let Ω be an isosceles right triangle whose nondiagonal sides are of length 7; see Fig. 4. Suppose we want to solve the Laplace equation $\Delta u = 0$ with Dirichlet boundary conditions. For $h = 2$, Ω_h contains three points. We get the following system of equations for U_1, U_2 and U_3:

$$\begin{aligned}
U_1 - \frac{1}{4}U_2 - \frac{1}{4}U_3 &= \frac{1}{4}U_4 + \frac{1}{4}U_{11},\\
-\frac{1}{6}U_1 + U_2 \qquad\qquad &= \frac{1}{6}U_5 + \frac{1}{3}U_6 + \frac{1}{3}U_7,\\
-\frac{1}{6}U_1 \qquad\qquad + U_3 &= \frac{1}{3}U_8 + \frac{1}{3}U_9 + \frac{1}{6}U_{10}.
\end{aligned}$$

(2) Suppose we want to solve the Poisson equation in the unit square:

$$\begin{aligned}
-\Delta u = f \quad & \text{in } \Omega = [0,1]^2,\\
u = 0 \quad & \text{on } \partial\Omega.
\end{aligned}$$

Choose a grid on Ω with mesh size $h = 1/m$. For convenience, we use the double indexing system $U_{ij} \approx u(\frac{i}{m}, \frac{j}{m})$, $1 \le i, j \le m-1$. This leads to the system

$$4U_{i,j} - U_{i-1,j} - U_{i+1,j} - U_{i,j-1} - U_{i,j+1} = f_{i,j}, \quad 1 \le i, j \le m-1, \tag{3.6}$$

where $f_{i,j} = h^2 f(\frac{i}{m}, \frac{j}{m})$. Here terms with indices 0 or m are taken to be zero.

Discrete Maximum Principle

When using the standard five-point stencil (and also in Example 3.2) every value U_i is a weighted average of neighboring values. This clearly implies that no value can be larger than the maximum of its neighbors, and is a special case of the following more general result.

3.3 Star Lemma. *Let $k \ge 1$. Suppose α_i and p_i, $0 \le i \le k$, are such that*

$$\begin{gathered}
\alpha_i < 0 \quad \textit{for } i = 1, 2, \dots, k,\\
\sum_{i=0}^{k} \alpha_i \ge 0, \qquad \sum_{i=0}^{k} \alpha_i p_i \le 0.
\end{gathered}$$

In addition, let $p_0 \ge 0$ or $\sum_{i=0}^{k} \alpha_i = 0$. Then $p_0 \ge \max_{1\le i\le k} p_i$ implies

$$p_0 = p_1 = \cdots = p_k. \tag{3.7}$$

Proof. The hypotheses imply that

$$\sum_{i=1}^{k} \alpha_i (p_i - p_0) = \sum_{i=0}^{k} \alpha_i (p_i - p_0) = \sum_{i=0}^{k} \alpha_i p_i - p_0 \sum_{i=0}^{k} \alpha_i \le 0.$$

Since $\alpha_i < 0$ for $i = 1, \ldots, k$ and $p_i - p_0 \le 0$, all summands appearing in the sums on the left-hand side are nonnegative. Hence, every summand equals 0. Now $\alpha_i \neq 0$ implies (3.7). $\square$

In the following, it is important to note that the discretization can change the topological structure of Ω. If Ω is connected, it does not follow that Ω_h is connected (with an appropriate definition). The situation shown in Fig. 5 leads to a system with a reducible matrix. To guarantee that the matrix is irreducible, we have to use a sufficiently small mesh size.

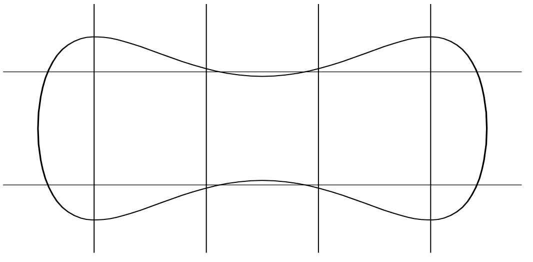

Fig. 5. Connected domain Ω for which Ω_h is not connected

3.4 Definition. Ω_h is said to be *(discretely) connected* provided that between every pair of points in Ω_h, there exists a path of grid lines which remains inside of Ω.

Clearly, using a finite difference method to solve the Poisson equation, we get a system with an irreducible matrix if and only if Ω_h is discretely connected.

We are now in a position to formulate the discrete maximum principle. Note that the hypotheses for the standard five-point stencil for the Laplace operator are satisfied.

3.5 Discrete Maximum Principle. *Let U be a solution of the linear system which arises from the discretization of*

$$Lu = f \quad \text{in } \Omega \quad \text{with } f \le 0$$

using a stencil which satisfies the following three conditions at every grid point in Ω_h:

(i) All of the coefficients except for the one at the center are nonpositive.
(ii) The coefficient in one of the directions is negative, say $\alpha_E < 0$.
(iii) The sum of all of the coefficients is nonnegative.
Then

$$\max_{z_i \in \Omega_h} U_i \le \max_{z_j \in \partial\Omega_h} U_j. \tag{3.8}$$

Furthermore, suppose the maximum over all the grid points is attained in the interior, the coefficients $\alpha_E, \alpha_S, \alpha_W$ and α_N in all four cardinal directions are negative, and Ω_h is discretely connected. Then U is constant.

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Proof. (1) If the maximum value over $\bar{\Omega}_h$ is attained at $z_i \in \Omega_h$, then U_i must have the same values at all of the neighboring points which appear in the stencil of z_i. This follows from the star lemma when U_C is identified with p_0 and $U_E, U_S, \ldots$ are identified with $p_1, p_2, \ldots$.

(2) The assertion now follows using the technique of *marching to the boundary.* Consider all points of Ω_h and $\partial\Omega_h$ which lie on the same horizontal grid line as the point z_i. It follows from (1) by induction that the maximum is attained at all points on this line lying between z_i and the first encountered boundary point.

(3) If Ω_h is connected, by Definition 3.4 we can choose a polygonal path between z_i and any point z_k in Ω_h. Repeating the argument of (2), we get $U_i = U_k$, and thus U is constant. □

The discrete maximum principle implies that the discrete solution U has properties which correspond exactly to those in Corollary 2.3. In particular, we have both the comparison principle and the continuous dependence on f and on the boundary data. In addition, we have

3.6 Corollary. *If the hypotheses of the first part of the discrete maximum principle 3.5 are satisfied, then the system $A_h U = f$ in 3.1(5) has a unique solution.*

Proof. The corresponding homogeneous system $A_h U = 0$ is associated with the discretization of the homogeneous differential equation with zero boundary condition. By 3.5, $\max U_i = \min U_i = 0$. Thus the homogeneous system has only the trivial solution, and the matrix A_h is nonsingular. □

Problem

3.7 Obviously there is always the danger of a misprint in formulas as (3.1). Verify the formula by applying it to the functions 1, x, and x^2 and the points $-h_W$, 0, and h_E. Why is this test sufficient?

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§ 4. A Convergence Theory for Difference Methods

It is relatively easy to establish the convergence of finite difference methods, provided that the solution u of the differential equation is sufficiently smooth up to the boundary, and its second derivatives are bounded. Although these assumptions are quite restrictive, it is useful to carry out the analysis in this framework to provide a first impression of the more general convergence theory. Under weaker assumptions, the analysis is much more complicated; cf. Hackbusch [1986].

Consistency

In the following we shall write L_h for the difference operator (which also specifies the method). Then given $u \in C(\Omega)$, $L_h u$ is a function defined at all points in Ω_h. The symbol A_h will denote the resulting matrix.

4.1 Definition. A finite difference method L_h is called *consistent* with the elliptic equation $Lu = f$ provided

$$Lu - L_h u = o(1) \quad \text{on } \Omega_h \text{ as } h \to 0,$$

for every function $u \in C^2(\bar{\Omega})$. A method has *consistency order m* provided that for every $u \in C^{m+2}(\bar{\Omega})$,

$$Lu - L_h u = \mathcal{O}(h^m) \quad \text{on } \Omega_h \text{ as } h \to 0.$$

The five-point formula (3.1) for the Laplace operator derived by Taylor expansions has order 1 for an arbitrary grid, and order 2 when the four neighbors of the center point are at the same distance from it.

Local and Global Error

The definition of consistency relates to the *local error* $Lu - L_h u$. However, the convergence of a method depends on the *global error*

$$\eta(z_i) := u(z_i) - U_i$$

as z_i runs over Ω_h. The two errors are connected by

4.2 A Difference Equation for the Global Error. Let

$$Lu = f \quad \text{in } \Omega,$$

and suppose

$$A_h U = F$$

is the associated linear system over Ω_h with $F_i = f(z_i)$. In addition, suppose that for the points on the boundary,

$$U(z_j) = u(z_j) \quad \text{for } z_j \in \partial\Omega_h.$$

In view of the linearity of the difference operator, it follows that the global error η satisfies

$$\begin{aligned}(L_h\eta)_i &= (L_h u)(z_i) - (A_h U)_i \\ &= (L_h u)(z_i) - f(z_i) = (L_h u)(z_i) - (Lu)(z_i) \\ &= -r_i,\end{aligned} \tag{4.1}$$

where $r := Lu - L_h u$ is the local error on Ω_h. Thus, η can be interpreted as the solution of the discrete boundary-value problem

$$\begin{aligned} L_h\eta &= -r \quad \text{in } \Omega_h, \\ \eta &= 0 \quad\ \ \text{on } \partial\Omega_h. \end{aligned} \tag{4.2}$$

4.3 Remark. If we eliminate those variables in (4.2) which belong to $\partial\Omega_h$, we get a system of the form

$$A_h\tilde{\eta} = -r.$$

Here $\tilde{\eta}$ is the vector with components $\tilde{\eta}_i = \eta(z_i)$ for $z_i \in \Omega_h$. This shows that convergence is assured provided r tends to 0 and the inverses A_h^{-1} remain bounded as $h \to 0$. This last condition is called *stability*. Thus, *consistency and stability imply convergence.*

In order to illustrate the error calculation by the perturbation method of (4.1), we will turn our attention for a moment to a more formal argument. We investigate the differences between the solutions of the two linear systems of equations

$$\begin{aligned} Ax &= b, \\ (A+F)y &= b, \end{aligned}$$

where F is regarded as a small perturbation. Obviously, $(A+F)(x-y) = Fx$. Thus, the error $x - y = (A+F)^{-1}Fx$ is small provided F is small and $(A+F)^{-1}$ is bounded. – In estimating the global error by the above perturbation calculation, it is important to note that the given elliptic operator and the difference operator operate on different spaces.

We will estimate the size of the solution of (4.2) by considering the difference equation rather than via the norm of the inverse matrices $\|A_h^{-1}\|$.

4.4 Lemma. *Suppose Ω is contained in the disk $B_R(0) := \{(x, y) \in \mathbb{R}^2;\ x^2+y^2 < R^2\}$. Let V be the solution of the equation*

$$\begin{aligned} L_h V &= 1 \quad \text{in } \Omega_h, \\ V &= 0 \quad \text{on } \partial\Omega_h, \end{aligned} \tag{4.3}$$

where L_h is the standard five-point stencil. Then

$$0 \le V(x_i, y_i) \le \frac{1}{4}(R^2 - x_i^2 - y_i^2). \tag{4.4}$$

Proof. Consider the function $w(x, y) := \frac{1}{4}(R^2 - x^2 - y^2)$, and set $W_i = w(x_i, y_i)$. Since w is a polynomial of second degree, the derivatives of higher order which were dropped when forming the difference star vanish. Hence we have $(L_h W)_i = Lw(x_i, y_i) = 1$. Moreover, $W \ge 0$ on $\partial\Omega$. The discrete comparison principle implies that $V \le W$, while the minimum principle implies $V \ge 0$, and (4.4) is proved. □

The essential fact about (4.4) is that it provides a bound which is independent of h. – This lemma can be extended to any elliptic differential equation for which the finite difference approximation is exact for polynomials of degree 2. In this case the factor $\frac{1}{4}$ in (4.4) is replaced by a number which depends on the constant of ellipticity.

4.5 Convergence Theorem. *Suppose the solution of the Poisson equation is a C^2 function, and that the derivatives u_{xx} and u_{yy} are uniformly continuous in Ω. Then the approximations obtained using the five-point stencil converge to the solution. In particular*

$$\max_{z \in \Omega_h} |U_h(z) - u(z)| \to 0 \text{ as } h \to 0. \tag{4.5}$$

Proof. By the Taylor expansion at the point (x_i, y_i),

$$L_h u(x_i, y_i) = -u_{xx}(\xi_i, y_i) - u_{yy}(x_i, \eta_i),$$

where ξ_i and η_i are certain numbers. Because of the uniform continuity, the local discretization error $\max_i |r_i|$ tends to 0. It now follows from (4.2) and Lemma 4.4 that

$$\max |\eta_i| \le \frac{R^2}{4} \max |r_i|, \tag{4.6}$$

which gives the convergence assertion. □

Analogously, using (4.6), we can get $\mathcal{O}(h)$ or $\mathcal{O}(h^2)$ estimates for the global error, provided u is in $C^3(\bar\Omega)$ or in $C^4(\bar\Omega)$, respectively.

Limits of the Convergence Theory

The hypotheses on the derivatives required for the above convergence theorem are often too restrictive.

4.6 Example. Suppose we want to find the solution of the potential equation in the unit disk satisfying the (Dirichlet) boundary condition

$$u(\cos\varphi, \sin\varphi) = \sum_{k=2}^{\infty} \frac{1}{k(k-1)} \cos k\varphi.$$

Since it is absolutely and uniformly convergent, the series represents a continuous function. By (1.2), the solution of the boundary-value problem in polar coordinates is

$$u(x, y) = \sum_{k=2}^{\infty} \frac{r^k}{k(k-1)} \cos k\varphi. \tag{4.7}$$

Now on the x-axis, the second derivative

$$u_{xx}(x, 0) = \sum_{k=2}^{\infty} x^{k-2} = \frac{1}{1-x}$$

is unbounded in a neighborhood of the boundary point $(1, 0)$, and thus Theorem 4.5 is not directly applicable.

A complete convergence theory can be found, e.g., in Hackbusch [1986]. It uses the stability of the differential operator in the sense of the L_2-norm, while here the maximum norm was used (but see Problem 4.8). Since the main topic of this book is the finite element method, we restrict ourselves here to a simple generalization. Using an approximation-theoretical argument, we can extend the convergence theorem at least to a disk with arbitrary continuous boundary values.

By the Weierstrass approximation theorem, every periodic continuous function can be approximated arbitrarily well by a trigonometric polynomial. Thus, for given $\varepsilon > 0$, there exists a trigonometric polynomial

$$v(\cos\varphi, \sin\varphi) = a_0 + \sum_{k=1}^{m} (a_k \cos k\varphi + b_k \sin k\varphi)$$

with $|v - u| < \frac{\varepsilon}{4}$ on $\partial\Omega$. Let

$$v(x, y) = a_0 + \sum_{k=1}^{m} r^k (a_k \cos k\varphi + b_k \sin k\varphi)$$

and let V be the numerical solution obtained by the finite difference method. By the maximum principle and the discrete maximum principle, it follows that

$$|u - v| < \frac{\varepsilon}{4} \quad \text{in } \Omega, \qquad |U - V| < \frac{\varepsilon}{4} \quad \text{in } \Omega_h. \tag{4.8}$$

Note that the second estimate in (4.8) is uniform for all h. Moreover, since the derivatives of v up to order 4 are bounded in Ω, by the convergence theorem, $|V - v| < \frac{\varepsilon}{2}$ in Ω_h for sufficiently small h. Then by the triangle inequality,

$$|u - U| \leq |u - v| + |v - V| + |V - U| < \varepsilon \quad \text{in } \Omega_h. \qquad \square$$

Here we have used an explicit representation for the solution of the Poisson equation on the disk, but we needed only the fact that the boundary values which produce nice solutions are dense. We obtain a generalization if we put this property into an abstract hypothesis.

4.7 Theorem. *Suppose the set of solutions of the Poisson equation whose derivatives u_{xx} and u_{yy} are uniformly continuous in Ω is dense in the set*

$$\{u \in C(\bar{\Omega}); \quad Lu = f\}.$$

Then the numerical solution obtained using the five-point stencil converges, i.e., (4.5) holds.

Problems

4.8 Let L_h be the difference operator obtained from the Laplace operator using (3.1), and let $\Omega_{h,0}$ be the set of (interior) points of Ω_h for which all four neighbors also belong to Ω_h. In order to take into account the fact that the consistency error on the boundary may be larger, in analogy with (4.3) we need to find the solution of

$$\begin{aligned} L_h V &= 1 \text{ in } \Omega_h \backslash \Omega_{h,0}, \\ L_h V &= 0 \text{ in } \Omega_{h,0}, \\ V &= 0 \text{ on } \partial\Omega_h. \end{aligned}$$

Show (for simplicity, on a square) that

$$0 \leq V \leq h^2 \text{ in } \Omega_h.$$

4.9 Consider the eigenvalue problem for the Laplacian on the unit square:

$$\begin{aligned} -\Delta u &= \lambda u \quad \text{in } \Omega = (0,1)^2, \\ u &= 0 \quad\ \ \text{on } \partial\Omega. \end{aligned}$$

Then

$$u_{k\ell}(x, y) = \sin k\pi x \sin \ell\pi y, \quad k, \ell = 1, 2, \ldots, \tag{4.9}$$

are the eigenfunctions with the eigenvalues $(k^2 + \ell^2)\pi^2$. Show that if $h = 1/n$, the restrictions of these functions to the grid are the eigenfunctions of the difference operator corresponding to the five-point stencil. Which eigenvalues are better approximated, the small ones or the large ones?