THE UNIVERSITY OF BRITISH COLUMBIA
Department of Mechanical Engineering
MECH 570C: Fluid-Structure Interaction (FSI) - Theory and Computation
(Term 2, 2023-2024)

Due Date: Feb 6, 2024

Assignment 2: Eulerian-Lagrangian Modeling

Question 1

In many applications, the deformation and displacements are infinitesimal small and thus Lagrangian and Eulerian frameworks can be identified (based on a first order approximation). Thus there is no need to distinguish between different coordinate systems. Let us justify this point of view in this exercise. Given $\|\nabla u\| \ll 1$, derive the linearized expressions for the deformation gradient $\boldsymbol{F}$, its determinant J and the strain E.

Question 2

Consider the deformation map $\boldsymbol{x} = \varphi(\boldsymbol{X}, t)$ given by

$$
\begin{aligned}
x_1 &= \cos(\omega t) X_1 + \sin(\omega t) X_2 \\
x_2 &= -\sin(\omega t) X_1 + \cos(\omega t) X_2 \\
x_3 &= (1 + \alpha t) X_3
\end{aligned}
$$

Notice that this deformation corresponds to rotation (with rate ω) in the e_1, e_2 -plane together with extension (with rate α) along the e_3 -axis.
(a) Find the components of the inverse motion $X = \psi(x, t)$
(b) Find the components of the spatial velocity field $v(x, t)$
(c) Find the components of the rate of strain and spin tensors $\boldsymbol{L}(x, t)$ and $\boldsymbol{W}(x, t)$. Verify that $\boldsymbol{L}$ is determined by α, whereas $\boldsymbol{W}$ is determined by ω.

Question 3

The material time derivative of an Eulerian field is a key quantity in continuum mechanics and later in fluid-structure interaction modeling. Let $f(t, x) : \mathbb{R}^m \to \mathbb{R}$ and $x \in \mathbb{R}^{m-1}$. Show that:

$$D_t f(t, x) = \partial_t f(t, x) + \nabla f(t, x) \cdot \boldsymbol{v}(t, x)$$

where D_t denotes the total time derivative. Explain this formula in your own words.

Question 4

Explain the Eulerian-Lagrangian conflict in fluid-structure interaction. The transformation $\varphi : \hat{\Omega} \to \Omega$ between different coordinate systems and domains requires to adapt the spatial derivative of a function f. Show that

$$\nabla f = \hat{\nabla} \hat{f} \boldsymbol{F}^{-1}$$

where $\boldsymbol{F}$ denotes the deformation gradient.

Question 5

The divergence of the Piola transformation is a key concept in solid mechanics and later in arbitrary Lagrangian-Eulerian (ALE) for fluid-structure interaction. Let $\hat{\sigma} : \hat{\Omega} \to \mathbb{R}^3$ be the Piola transformation of $\sigma : \Omega \to \mathbb{R}^3$. Show that

$$\hat{\nabla} \cdot \hat{\sigma} = \hat{J} \nabla \cdot \sigma \quad \forall \boldsymbol{x} = \varphi(\hat{\boldsymbol{x}}), \hat{\boldsymbol{x}} \in \hat{\Omega}$$