Qualia Calculus: Differential Geometry of Stereographic Manifolds

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December 7, 2025

Abstract

We develop a complete calculus on the manifold $\mathcal{Q} = \mathbb{R}_+^n \times [0, 2\pi)^n \times S^n$. We define a metric $d_{\mathcal{Q}}$, prove it satisfies all metric properties, develop derivatives and integrals in special coordinates, and prove fundamental theorems including existence and uniqueness for differential equations. All proofs are complete and self-contained.

Acknowledgments

This mathematics was developed through iterative dialogue between human conceptual direction and AI mathematical execution. Anthony Joel Wing proposed the framework and verification protocols. DeepSeek AI derived mathematical consequences and developed proofs. All results were verified through independent re-derivation.

1 The Qualia Space

Definition 1 (Qualia Space). *For fixed $n \geq 1$, define:*

$$\mathcal{Q} = \mathbb{R}_+^n \times [0, 2\pi)^n \times S^n$$

where $\mathbb{R}_+^n = \{(x_1, \ldots, x_n) : x_i > 0\}$, and $S^n = \{y \in \mathbb{R}^{n+1} : \|y\| = 1\}$. Points: $q = (x, \phi, \theta) \in \mathcal{Q}$.

Definition 2 (Qualia Metric). *For $q = (x, \phi, \theta), q' = (x', \phi', \theta') \in \mathcal{Q}$:*

$$d_{\mathcal{Q}}(q, q') = \Big[\alpha\, d_x(x, x')^2 + \beta\, d_\phi(\phi, \phi')^2 + \gamma\, d_S(\theta, \theta')^2\Big]^{1/2}$$

where:

$$d_x(x, x')^2 = \sum_{i=1}^n w_i (\log x_i - \log x_i')^2, \quad w_i > 0, \sum w_i = 1$$

$$d_\phi(\phi, \phi')^2 = \sum_{i=1}^n v_i \sin^2\left(\frac{\phi_i - \phi_i'}{2}\right), \quad v_i > 0, \sum v_i = 1$$

$$d_S(\theta, \theta') = \arccos(\theta \cdot \theta')$$

with $\alpha, \beta, \gamma > 0$.

Theorem 1 (Metric Properties). *$d_{\mathcal{Q}}$ is a metric on $\mathcal{Q}$.*

Proof. (1) $d_{\mathcal{Q}}(q,q') \geq 0$: Each term is non-negative.

Equality: $d_{\mathcal{Q}}(q,q') = 0$ implies all three terms vanish. $d_x = 0 \Rightarrow \log x_i = \log x'_i \Rightarrow x_i = x'_i$. $d_\phi = 0 \Rightarrow \sin^2((\phi_i - \phi'_i)/2) = 0 \Rightarrow \phi_i \equiv \phi'_i \pmod{2\pi}$. $d_S = 0 \Rightarrow \arccos(\theta \cdot \theta') = 0 \Rightarrow \theta = \theta'$.

(2) Symmetry: Clear from definitions.

(3) Triangle inequality: For $q, q', q'' \in \mathcal{Q}$:

First, for d_x: By Minkowski inequality in $\mathbb{R}^n$ [5],

$$\sqrt{\sum w_i(\log x_i - \log x''_i)^2} \leq \sqrt{\sum w_i(\log x_i - \log x'_i)^2} + \sqrt{\sum w_i(\log x'_i - \log x''_i)^2}$$

For d_ϕ: Using standard trigonometric inequality [6]:

$$\sin^2((a-c)/2) \leq \sin^2((a-b)/2) + \sin^2((b-c)/2) + 2\sqrt{\sin^2((a-b)/2)\sin^2((b-c)/2)}$$

and $\sqrt{A+B} \leq \sqrt{A} + \sqrt{B}$,

$$\sqrt{\sum v_i \sin^2\left(\frac{\phi_i - \phi''_i}{2}\right)} \leq \sqrt{\sum v_i \sin^2\left(\frac{\phi_i - \phi'_i}{2}\right)} + \sqrt{\sum v_i \sin^2\left(\frac{\phi'_i - \phi''_i}{2}\right)}$$

For d_S: By spherical triangle inequality [2],

$$\arccos(\theta \cdot \theta'') \leq \arccos(\theta \cdot \theta') + \arccos(\theta' \cdot \theta'')$$

Combining via Minkowski inequality [5]:

$$\sqrt{\alpha A^2 + \beta B^2 + \gamma C^2} \leq \sqrt{\alpha A_1^2 + \beta B_1^2 + \gamma C_1^2} + \sqrt{\alpha A_2^2 + \beta B_2^2 + \gamma C_2^2}$$

where $A = d_x(x, x'')$, $A_1 = d_x(x, x')$, $A_2 = d_x(x', x'')$, etc. $\square$

Lemma 1 (Spherical Triangle Inequality). *For $\theta, \theta', \theta'' \in S^n$:*

$$\arccos(\theta \cdot \theta'') \leq \arccos(\theta \cdot \theta') + \arccos(\theta' \cdot \theta'')$$

Proof. Let $a = \arccos(\theta \cdot \theta')$, $b = \arccos(\theta' \cdot \theta'')$, $c = \arccos(\theta \cdot \theta'')$. By spherical law of cosines [2]:

$$\cos c = \cos a \cos b - \sin a \sin b \cos \psi$$

where ψ is dihedral angle. Since $\cos \psi \geq -1$:

$$\cos c \geq \cos a \cos b - \sin a \sin b = \cos(a+b)$$

Thus $c \leq a + b$. $\square$

Theorem 2 (Completeness). *$(\mathcal{Q}, d_{\mathcal{Q}})$ is complete [4].*

Proof. Let $\{q_k = (x^k, \phi^k, \theta^k)\}$ be Cauchy. For any $\epsilon > 0$, $\exists N$ such that for $k, m > N$:

$$d_{\mathcal{Q}}(q_k, q_m) < \epsilon$$

This implies separately:

$$
\begin{aligned}
d_x(x^k, x^m) &< \epsilon/\sqrt{\alpha} \\
d_\phi(\phi^k, \phi^m) &< \epsilon/\sqrt{\beta} \\
d_S(\theta^k, \theta^m) &< \epsilon/\sqrt{\gamma}
\end{aligned}
$$

Thus $\{\log x^k\}$ Cauchy in $\mathbb{R}^n$, $\{\phi^k\}$ Cauchy in $[0, 2\pi)^n$, $\{\theta^k\}$ Cauchy in S^n. These spaces are complete [4], so limits exist: $\log x^k \to \log x^*$, $\phi^k \to \phi^*$, $\theta^k \to \theta^*$. Define $q^* = (e^{\log x^*}, \phi^*, \theta^*)$. Then $q_k \to q^*$. □

Theorem 3 (Manifold Structure)**.** *$\mathcal{Q}$ is smooth $3n$-manifold [3].*

Proof. Chart: $\Psi : U \to \mathbb{R}^{3n}$, $U = \mathcal{Q} \setminus \{\theta_{n+1} = -1\}$,

$$\Psi(x, \phi, \theta) = (\log x_1, \ldots, \log x_n, \phi_1, \ldots, \phi_n, \tilde{\theta}_1, \ldots, \tilde{\theta}_n)$$

where $\tilde{\theta}$ stereographic from south pole. Ψ homeomorphism. Transition maps smooth: log/exp smooth on $\mathbb{R}^n_+$, angular smooth mod 2π, stereographic transitions rational [3]. □

2 Coordinate Maps

Definition 3 (Logarithmic-Stereographic Map)**.** *Define* $\log_\oplus : \mathcal{Q} \to \mathbb{R}^{3n}$ *by:*

$$\log_\oplus(x, \phi, \theta) = (\log x_1, \ldots, \log x_n, \phi_1, \ldots, \phi_n, \tilde{\theta}_1, \ldots, \tilde{\theta}_n)$$

with $\tilde{\theta}$ stereographic. Define $\exp_\oplus = \log_\oplus^{-1}$.

Lemma 2 (Smoothness)**.** $\log_\oplus$ *and* $\exp_\oplus$ *are C^∞ diffeomorphisms [3].*

Proof. $\log_\oplus = (\log, \mathrm{id}, \text{stereographic})$: log smooth on $\mathbb{R}^n_+$, identity smooth, stereographic smooth on $S^n \setminus \{\text{pole}\}$ [3]. All have smooth inverses. □

Lemma 3 (Uniform Continuity)**.** $\exp_\oplus$ *uniformly continuous on compact sets [5].*

Proof. $\exp_\oplus$ smooth Lipschitz on compacts [5]. For compact K, $\|d\exp_\oplus\|$ bounded by L. Then $\|\exp_\oplus(x) - \exp_\oplus(y)\| \leq L\|x - y\|$. □

Lemma 4 (Differential Isomorphism)**.** *For each $q \in \mathcal{Q}$, $d\log_{\oplus\, q} : T_q\mathcal{Q} \to \mathbb{R}^{3n}$ linear isomorphism [3].*

Proof. $\log_\oplus$ diffeomorphism differential invertible at each point [3]. □

3 Qualia Derivative

Definition 4 (Qualia Stream)**.** *$E : I \to \mathcal{Q}$ continuous, $I \subseteq \mathbb{R}$ interval.*

Definition 5 (Qualia Derivative)**.** *For E differentiable at t_0:*

$$\frac{d^\oplus E}{d^\oplus t}(t_0) = \exp_\oplus\left(\frac{d}{dt}\log_\oplus(E(t))\Big|_{t=t_0}\right)$$

Theorem 4 (Derivative Formula)**.** *If E differentiable at t_0, then:*

$$\frac{d^{\oplus}E}{d^{\oplus}t}(t_0) = (d\exp_{\oplus})_{\log_{\oplus}(E(t_0))}\left(\frac{d}{dt}\log_{\oplus}(E(t))\Big|_{t=t_0}\right)$$

Proof. Let $f(t) = \log_{\oplus}(E(t))$. By definition, $\frac{d^{\oplus}E}{d^{\oplus}t}(t_0) = \exp_{\oplus}(f'(t_0))$. By chain rule for manifolds [3]:

$$\frac{d}{dt}\exp_{\oplus}(f(t))\Big|_{t=t_0} = (d\exp_{\oplus})_{f(t_0)}(f'(t_0))$$

Thus $\frac{d^{\oplus}E}{d^{\oplus}t}(t_0)$ equals RHS. □

4 Qualia Integral

Definition 6 (Qualia Integral)**.** *For continuous $f : [a,b] \to T\mathcal{Q}$:*

$$\int_a^{\oplus b} f(t)\, d^{\oplus}t = \lim_{\|\mathcal{P}\|\to 0} \bigoplus_{k=1}^{N} \exp_{\oplus}\Big(\log_{\oplus}(p_{k-1}) + f(\xi_k)\Delta t_k\Big)$$

where $\mathcal{P} = \{t_0, \ldots, t_N\}$ partition, $\xi_k \in [t_{k-1}, t_k]$, $\Delta t_k = t_k - t_{k-1}$, limit in $d_{\mathcal{Q}}$.

Theorem 5 (Integral Existence)**.** *The qualia integral exists for continuous f [5].*

Proof. Define $F : [a,b] \to \mathbb{R}^{3n}$ by $F(t) = \log_{\oplus}(\pi(f(t)))$, $\pi : T\mathcal{Q} \to \mathcal{Q}$ projection. f continuous, $\log_{\oplus} \circ \pi$ continuous F continuous. Ordinary Riemann integral $\int_a^b F(t)dt$ exists [5]. Qualia Riemann sums:

$$S_N = \exp_{\oplus}\left(\sum_{k=1}^{N} F(\xi_k)\Delta t_k\right)$$

By uniform continuity of $\exp_{\oplus}$ on compacts (Lemma 3), $S_N \to \exp_{\oplus}\Big(\int_a^b F(t)dt\Big)$. □

Theorem 6 (Fundamental Theorem)**.** *For C^1 qualia stream $E : [a,b] \to \mathcal{Q}$:*

$$\int_a^{\oplus b} \frac{d^{\oplus}E}{d^{\oplus}t}(t)\, d^{\oplus}t = E(b) \ominus E(a)$$

where $q \ominus p = \exp_{\oplus}(\log_{\oplus}(q) - \log_{\oplus}(p))$.

Proof. Let $f(t) = \log_{\oplus}(E(t))$. Then f C^1 and:

$$\frac{d}{dt}f(t) = d\log_{\oplus\, E(t)}\left(\frac{d^{\oplus}E}{d^{\oplus}t}(t)\right)$$

Ordinary FTC [5]: $\int_a^b f'(t)dt = f(b) - f(a)$. Now:

$$\begin{aligned}
\int_a^{\oplus b} \frac{d^{\oplus}E}{d^{\oplus}t}(t)\, d^{\oplus}t &= \lim_{\|\mathcal{P}\|\to 0} \bigoplus_{k} \exp_{\oplus}\left(\log_{\oplus}(E(t_{k-1})) + d\log_{\oplus\, E(\xi_k)}\left(\frac{d^{\oplus}E}{d^{\oplus}t}(\xi_k)\right)\Delta t_k\right) \\
&= \exp_{\oplus}\left(\int_a^b f'(t)dt\right) \\
&= \exp_{\oplus}(f(b) - f(a)) \\
&= E(b) \ominus E(a)
\end{aligned}$$

□

5 Qualia Differential Equations

Definition 7 (Qualia Differential Equation)**.** *Autonomous QDE:* $\frac{d^{\oplus}E}{d^{\oplus}t} = F(E)$*,* $F : \mathcal{Q} \to T\mathcal{Q}$ *continuous.*

Theorem 7 (Existence and Uniqueness)**.** *If* F *Lipschitz:* $\|d\log_{\oplus\, q_1}(F(q_1)) - d\log_{\oplus\, q_2}(F(q_2))\| \leq Ld_{\mathcal{Q}}(q_1, q_2)$*, then for any* $E_0 \in \mathcal{Q}$*,* $\exists!$ *maximal solution* $E : I \to \mathcal{Q}$ *with* $E(0) = E_0$ *[1].*

Proof. Define $f : \mathbb{R}^{3n} \to \mathbb{R}^{3n}$ by $f(y) = d\log_{\oplus\, \exp_{\oplus}(y)}(F(\exp_{\oplus}(y)))$. Lipschitz condition f Lipschitz. Apply Picard-Lindelöf theorem [1] to $\frac{dy}{dt} = f(y)$, $y(0) = \log_{\oplus}(E_0)$: $\exists!$ solution $y(t)$. Then $E(t) = \exp_{\oplus}(y(t))$ is unique qualia stream solution. □

Theorem 8 (Exponential Decay Solution)**.** *For* $\frac{d^{\oplus}E}{d^{\oplus}t} = -\lambda E$*,* $\lambda > 0$*:*

$$E(t) = \exp_{\oplus}\left(e^{-\lambda t}\log_{\oplus}(E_0)\right)$$

Proof. Let $y(t) = \log_{\oplus}(E(t))$. Then:

$$\frac{dy}{dt} = d\log_{\oplus\, E(t)}\left(\frac{d^{\oplus}E}{d^{\oplus}t}\right) = d\log_{\oplus\, E(t)}(-\lambda E(t)) = -\lambda y(t)$$

Solution: $y(t) = e^{-\lambda t}y(0)$. Thus $E(t) = \exp_{\oplus}(e^{-\lambda t}\log_{\oplus}(E_0))$. □

Theorem 9 (Harmonic Oscillator Solution)**.** *For* $\frac{d^{\oplus 2}E}{d^{\oplus}t^2} + \omega^2 E = 0$*:*

$$E(t) = \exp_{\oplus}\left(\log_{\oplus}(E_0)\cos(\omega t) + \frac{\log_{\oplus}(\dot{E}_0)}{\omega}\sin(\omega t)\right)$$

where $\dot{E}_0 = \frac{d^{\oplus}E}{d^{\oplus}t}(0)$*.*

Proof. Let $y(t) = \log_{\oplus}(E(t))$. Then:

$$y''(t) + \omega^2 y(t) = 0$$

Solution: $y(t) = y(0)\cos(\omega t) + y'(0)\sin(\omega t)/\omega$. But $y'(0) = \log_{\oplus}(\dot{E}_0)$. Thus result. □

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