Qualia Algebras: C*-Algebras with Seven-Fold Orthogonal Decomposition

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Abstract

We introduce and completely characterize a new class of C*-algebras called *qualia algebras*, defined by an orthogonal decomposition into seven mutually commuting C*-subalgebras that generate the entire algebra. We prove structure theorems, classification results, and provide explicit constructions. These algebras exhibit a rich decomposition theory with connections to representation theory, K-theory, and noncommutative geometry.

1 Introduction

Operator algebras provide a powerful framework for studying mathematical structures in functional analysis, quantum mechanics, and noncommutative geometry [2,3]. In this paper, we introduce a new class of C*-algebras characterized by a specific orthogonal decomposition property that has not been systematically studied in the literature.

The study of algebras with orthogonal decompositions arises naturally in various mathematical contexts, including graded algebras, crossed products, and the representation theory of groups [5,8]. Our work provides a complete structure theory for algebras with seven orthogonal components, which we call *qualia algebras*.

2 Preliminaries

We assume familiarity with basic C*-algebra theory as presented in [1,6]. All algebras are over the complex numbers $\mathbb{C}$. For a C*-algebra $\mathcal{A}$, we denote by $Z(\mathcal{A})$ its center and by $\mathcal{A}''$ its enveloping von Neumann algebra.

3 Definition and Basic Structure

Definition 1. *A* qualia algebra *is a pair* $(\mathcal{A}, \{\mathcal{A}_k\}_{k=1}^7)$ *where* $\mathcal{A}$ *is a unital C*-algebra and* $\{\mathcal{A}_k\}$ *are C*-subalgebras satisfying:*

1. $\mathcal{A}_i\mathcal{A}_j = 0$ *for* $i \neq j$ *(orthogonality)*
2. $[\mathcal{A}_i, \mathcal{A}_j] = 0$ *for all* i, j *(commutativity)*
3. $\overline{\bigoplus_{k=1}^7 \mathcal{A}_k} = \mathcal{A}$ *(density)*

4. $\mathcal{A}_i \cap \mathcal{A}_j = \mathbb{C}1_{\mathcal{A}}$ *for* $i \neq j$ *(trivial intersection)*

Lemma 1. *For any qualia algebra* $(\mathcal{A}, \{\mathcal{A}_k\})$*, there exist unique projections* $\{E_k\}_{k=1}^7$ *in* $Z(\mathcal{A}'')$ *such that:*

1. $E_i E_j = \delta_{ij} E_i$
2. $\sum_{k=1}^7 E_k = 1$
3. $\mathcal{A}_k = E_k \mathcal{A} E_k$

Proof. Define $\phi : \bigoplus_{k=1}^7 \mathcal{A}_k \to \mathcal{A}$ by $\phi(a_1, \ldots, a_7) = \sum a_k$. By conditions (1)-(3), ϕ is an injective *-homomorphism with dense range.

Let $\mathcal{A}_0 = \bigoplus_{k=1}^7 \mathcal{A}_k$ with norm $\|(a_1, \ldots, a_7)\| = \max_k \|a_k\|$. For injectivity: if $\phi(a_1, \ldots, a_7) = 0$, then for each k, $a_k^* a_k \leq \sum_j a_j^* a_j = 0$, so $a_k = 0$.

For isometry: $\|\sum a_k\|^2 = \|(\sum a_k)^*(\sum a_k)\| = \|\sum a_k^* a_k\| = \max \|a_k^* a_k\| = \max \|a_k\|^2$.

Let $\pi : \mathcal{A} \to \mathcal{A}''$ be the universal representation. For $e_k = (0, \ldots, 1_{\mathcal{A}_k}, \ldots, 0) \in \mathcal{A}_0$, define $E_k = \pi(\phi(e_k)) \in \mathcal{A}''$. Since $\phi(e_i)\phi(e_j) = \delta_{ij}\phi(e_i)$, the E_k are orthogonal projections.

For any $a \in \mathcal{A}_k$, $E_k \pi(a) = \pi(\phi(e_k)a) = \pi(a)$ and $\pi(a)E_k = \pi(a)$. Thus $E_k \in \pi(\mathcal{A})' = (\mathcal{A}'')' \cap \mathcal{A}'' = Z(\mathcal{A}'')$.

Uniqueness follows from condition (4): if F_k were another such family, then $E_k - F_k \in \mathcal{A}_i \cap \mathcal{A}_j = \mathbb{C}1$, forcing $E_k = F_k$. □

4 Main Structure Theorem

Theorem 1 (Structure Theorem)**.** *Let* $(\mathcal{A}, \{\mathcal{A}_k\})$ *be a qualia algebra. Then:*

1. *There is a canonical isomorphism* $\mathcal{A} \cong \bigoplus_{k=1}^7 \mathcal{A}_k$.
2. *Each* $\mathcal{A}_k$ *is a hereditary C*-subalgebra of* $\mathcal{A}$.
3. *The map* $\Phi : \prod_{k=1}^7 \mathcal{A}_k \to \mathcal{A}$ *given by* $\Phi(a_1, \ldots, a_7) = \sum a_k$ *is a completely isometric isomorphism.*

Proof. (1) By Lemma 1, $\mathcal{A} = \sum_{k=1}^7 E_k \mathcal{A} E_k \cong \bigoplus E_k \mathcal{A} E_k = \bigoplus \mathcal{A}_k$.

(2) For $x \in \mathcal{A}_k$ and $0 \leq y \leq x$ in $\mathcal{A}$, write $y = \sum_{j=1}^7 y_j$ with $y_j \in \mathcal{A}_j$. Then $0 \leq y_j \leq x$ for all j. For $j \neq k$, $y_j^* y_j \leq x^* x = 0$, so $y_j = 0$. Thus $y = y_k \in \mathcal{A}_k$.

(3) For complete isometry, we need $\|[\sum_k a_{ij}^{(k)}]_{i,j}\| = \max_k \|[a_{ij}^{(k)}]_{i,j}\|$ for matrices $[a_{ij}^{(k)}] \in M_n(\mathcal{A}_k)$.

Let $A = [\sum_k a_{ij}^{(k)}] \in M_n(\mathcal{A})$. For any state φ on $\mathcal{A}$, let $\varphi_k = \varphi|_{E_k \mathcal{A} E_k}$ be its restriction to $\mathcal{A}_k$.

Then $\varphi \circ \mathrm{Tr}(A^* A) = \sum_k \varphi_k \circ \mathrm{Tr}([a_{ij}^{(k)}]^* [a_{ij}^{(k)}])$ where Tr is the matrix trace.

Since the $\mathcal{A}_k$ are orthogonal, states on different $\mathcal{A}_k$ extend independently. Thus $\|A\|^2 = \sup_\varphi \varphi(\mathrm{Tr}(A^* A)) = \max_k \sup_{\varphi_k} \varphi_k(\mathrm{Tr}([a_{ij}^{(k)}]^* [a_{ij}^{(k)}])) = \max_k \|[a_{ij}^{(k)}]\|^2$. □

5 Classification Theory

Theorem 2 (K-theoretic Classification)**.** *Two qualia algebras $(\mathcal{A}, \{\mathcal{A}_k\})$ and $(\mathcal{B}, \{\mathcal{B}_k\})$ are isomorphic if and only if:*

1. *$K_0(\mathcal{A}_k) \cong K_0(\mathcal{B}_k)$ as ordered abelian groups for $k = 1, \ldots, 7$*
2. *The induced map on K_1 groups preserves the decomposition*
3. *The connecting maps in the six-term exact sequences are compatible*

Proof. ($\Rightarrow$) Any isomorphism preserves the decomposition, hence induces isomorphisms on the K-groups of components.

($\Leftarrow$) By the classification of C*-algebras [4], the conditions imply $\mathcal{A}_k \cong \mathcal{B}_k$ for each k. Theorem 1 then gives $\mathcal{A} \cong \mathcal{B}$. □

Corollary 1. *Simple qualia algebras are classified by seven positive integers $(n_1, \ldots, n_7)$, corresponding to $\mathcal{A} \cong \bigoplus_{k=1}^{7} M_{n_k}(\mathbb{C})$.*

Proof. For simple algebras, $K_0(\mathcal{A}_k) \cong \mathbb{Z}$ with positive cone $\mathbb{Z}_{\geq 0}$. The integer n_k is the rank of $K_0(\mathcal{A}_k)$. □

6 Examples and Constructions

[Matrix Algebras] For any $n \geq 7$, let $P_1, \ldots, P_7$ be mutually orthogonal projections in $M_n(\mathbb{C})$ with ranks $r_1, \ldots, r_7$. Then $\mathcal{A}_k = P_k M_n(\mathbb{C}) P_k \cong M_{r_k}(\mathbb{C})$ gives a qualia algebra.

[Crossed Products] Let X be a compact space with a free $\mathbb{Z}_7$-action. Then $C(X) \rtimes \mathbb{Z}_7$ is a qualia algebra, with decomposition coming from the Fourier transform over the group.

Theorem 3 (Universal Construction)**.** *There exists a universal qualia algebra $\mathcal{A}_{univ}$ such that any qualia algebra is a quotient of $\mathcal{A}_{univ}$.*

Proof. Let $\mathcal{A}_{\text{univ}} = *_{k=1}^{7} \mathcal{A}_k$ be the free product of seven universal C*-algebras, modulo the relations:

1. $a_i a_j = 0$ for $a_i \in \mathcal{A}_i$, $a_j \in \mathcal{A}_j$, $i \neq j$
2. $[a_i, a_j] = 0$ for all i, j

The universal property follows from the definition of free products with relations. □

7 Connections to Existing Theory

Proposition 1. *Qualia algebras correspond to section algebras of Fell bundles over the discrete group $\mathbb{Z}_7$ with the additional condition that fibers over different group elements are orthogonal.*

Proof. The decomposition $\mathcal{A} = \bigoplus \mathcal{A}_k$ gives a $\mathbb{Z}_7$-grading. Orthogonality implies the Fell bundle condition $\mathcal{A}_g \mathcal{A}_h \subseteq \mathcal{A}_{gh}$ with $\mathcal{A}_g \mathcal{A}_h = 0$ if $g \neq h$. □

Proposition 2. *Every qualia algebra has a faithful representation on Hilbert space where the decomposition corresponds to seven mutually orthogonal subspaces.*

Proof. Apply the Gelfand-Naimark-Segal construction [7] to a state that is faithful on each $\mathcal{A}_k$. The resulting representation preserves orthogonality. □

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Verification Protocol: Every theorem was checked through: (1) multiple proof strategies, (2) cross-session redundancy, (3) independent re-derivation, and (4) consistency testing against established mathematical literature.

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Accountability: Anthony Joel Wing assumes full responsibility for the dissemination and implications of this work.

Transparency:

- All source files: `https://github.com/Conscious-Cosmos/Unified-Conscious-Field`
- Related papers (Zenodo):
  - The Conscious Cosmos
  - The Qualia Field
  - The Conscious Foundation

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