GRADUATE STUDENT STAT 840 A2

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Problem 6

a)

$$
\begin{aligned}
\pi(\theta \mid \mathbb{X}) &\propto f(\mathbb{X} \mid \theta)\pi(\theta) \\
\pi(\alpha, \eta \mid \mathbb{X}) &\propto f(\mathbb{X} \mid \alpha, \eta)\pi(\alpha, \eta) \\
f(x \mid \alpha, \eta) &\propto \alpha\eta x^{\alpha-1} e^{-\eta x^{\alpha}} \\
f(\mathbb{X} \mid \alpha, \eta) &= \prod_{i=1}^{n} f(x_i \mid \alpha, \eta) \\
&\propto \prod_{i=1}^{n} \alpha\eta x_i^{\alpha-1} e^{-\eta x_i^{\alpha}} \\
&\propto \alpha^n \eta^n \prod_{i=1}^{n} x_i^{\alpha-1} e^{-\eta x_i^{\alpha}} \\
\pi(\alpha, \eta) &\propto e^{-\alpha} \eta^{\beta-1} e^{-c\eta} \\
&\propto e^{-\alpha-c\eta} \eta^{\beta-1} \\
\pi(\alpha, \eta \mid \mathbb{X}) &\propto e^{-\alpha-c\eta} \eta^{\beta-1} \alpha^n \eta^n \prod_{i=1}^{n} x_i^{\alpha-1} e^{-\eta x_i^{\alpha}} \\
&\propto e^{-\alpha-c\eta} \eta^{n+\beta-1} \alpha^n \prod_{i=1}^{n} x_i^{\alpha-1} e^{-\eta x_i^{\alpha}}
\end{aligned}
$$

```r
log_post_pi = function(alp, eta, x, c,b)
{
  n = length(x)
  p1 = (-alp - c*eta)
  p2 = (n+b-1)*log(eta)
  p3 = n*log(alp)
  p4 = (alp-1)*log(x) + (-eta * x^alp)
  return(p1 + p2 + p3 + sum(p4))
}

post_pi = function(alp, eta, x, c,b)
{
  n = length(x)
  p1 = exp(-alp - c*eta)
  p2 = eta^(n+b-1)
  p3 = alp^n
  p4 = x^(alp-1) * exp(-eta * x^alp)
  return(p1 * p2 * p3 * prod(p4))
}
```

```
q = function(a2,n2,a1,n1)
{
  p1 = 1 / (a1 * n1)
  p2 = exp(-a2/a1 -n2/n1)
  return(p1*p2)
}
```

b)

$$\alpha(\theta_n, \theta^*) = \min\left\{\frac{\pi(\theta^* \mid \mathbb{X})q(\theta^*, \theta_n)}{\pi(\theta_n \mid \mathbb{X})q(\theta_n, \theta^*)}, 1\right\} \qquad \texttt{notes 4 p 19}$$

$$\rho(\alpha^*, \eta^* \mid \alpha_{(t)}, \eta_{(t)}) = \min\left\{\frac{\pi(\alpha^*, \eta^* \mid \mathbb{X})q(\alpha^*, \eta^* \mid \alpha_{(t)}, \eta_{(t)})}{\pi(\alpha_{(t)}, \eta_{(t)} \mid \mathbb{X})q(\alpha_{(t)}, \eta_{(t)} \mid \alpha^*, \eta^*)}, 1\right\}$$

$$\pi(\alpha^*, \eta^* \mid \mathbb{X}) \propto e^{-\alpha^* - c\eta^*} \eta^{*n+\beta-1} \alpha^{*n} \prod_{i=1}^{n} x_i^{\alpha^*-1} e^{-\eta^* x_i^{\alpha^*}}$$

$$\pi(\alpha_{(t)}, \eta_{(t)} \mid \mathbb{X}) \propto e^{-\alpha_{(t)} - c\eta_{(t)}} \eta_{(t)}^{n+\beta-1} \alpha_{(t)}^{n} \prod_{i=1}^{n} x_i^{\alpha_{(t)}-1} e^{-\eta_{(t)} x_i^{\alpha_{(t)}}}$$

$$\frac{\pi(\alpha^*, \eta^* \mid \mathbb{X})}{\pi(\alpha_{(t)}, \eta_{(t)} \mid \mathbb{X})} = \frac{e^{-\alpha^* - c\eta^*} \eta^{*n+\beta-1} \alpha^{*n} \prod_{i=1}^{n} x_i^{\alpha^*-1} e^{-\eta^* x_i^{\alpha^*}}}{e^{-\alpha_{(t)} - c\eta_{(t)}} \eta_{(t)}^{n+\beta-1} \alpha_{(t)}^{n} \prod_{i=1}^{n} x_i^{\alpha_{(t)}-1} e^{-\eta_{(t)} x_i^{\alpha_{(t)}}}}$$

$$= e^{\alpha_{(t)} + c\eta_{(t)} - \alpha^* - c\eta^*} \frac{\eta^{*n+\beta-1} \alpha^{*n} \prod_{i=1}^{n} x_i^{\alpha^*-1} e^{-\eta^* x_i^{\alpha^*}}}{\eta_{(t)}^{n+\beta-1} \alpha_{(t)}^{n} \prod_{i=1}^{n} x_i^{\alpha_{(t)}-1} e^{-\eta_{(t)} x_i^{\alpha_{(t)}}}}$$

$$= e^{\alpha_{(t)} + c\eta_{(t)} - \alpha^* - c\eta^*} \frac{\eta^{*n+\beta-1} \alpha^{*n}}{\eta_{(t)}^{n+\beta-1} \alpha_{(t)}^{n}} \prod_{i=1}^{n} x_i^{\alpha^* - \alpha_{(t)}} e^{\eta_{(t)} x_i^{\alpha_{(t)}} - \eta^* x_i^{\alpha^*}}$$

$$q(\alpha^*, \eta^* \mid \alpha_{(t)}, \eta_{(t)}) = \frac{1}{\alpha_{(t)}\eta_{(t)}} \exp\left\{-\frac{\alpha^*}{\alpha_{(t)}} - \frac{\eta^*}{\eta_{(t)}}\right\}$$

$$q(\alpha_{(t)}, \eta_{(t)} \mid \alpha^*, \eta^*) = \frac{1}{\alpha^*\eta^*} \exp\left\{-\frac{\alpha_{(t)}}{\alpha^*} - \frac{\eta_{(t)}}{\eta^*}\right\}$$

$$\frac{q(\alpha^*, \eta^* \mid \alpha_{(t)}, \eta_{(t)})}{q(\alpha_{(t)}, \eta_{(t)} \mid \alpha^*, \eta^*)} = \frac{\alpha^*\eta^*}{\alpha_{(t)}\eta_{(t)}} \exp\left\{-\frac{\alpha^*}{\alpha_{(t)}} - \frac{\eta^*}{\eta_{(t)}} + \frac{\alpha_{(t)}}{\alpha^*} + \frac{\eta_{(t)}}{\eta^*}\right\}$$

$$= \frac{\alpha^*\eta^*}{\alpha_{(t)}\eta_{(t)}} e^{-\frac{\alpha^*}{\alpha_{(t)}} - \frac{\eta^*}{\eta_{(t)}} + \frac{\alpha_{(t)}}{\alpha^*} + \frac{\eta_{(t)}}{\eta^*}}$$

$$\rho(\alpha^*, \eta^* \mid \alpha_{(t)}, \eta_{(t)}) = \min\left\{\frac{\alpha^*\eta^*}{\alpha_{(t)}\eta_{(t)}} e^{-\frac{\alpha^*}{\alpha_{(t)}} - \frac{\eta^*}{\eta_{(t)}} + \frac{\alpha_{(t)}}{\alpha^*} + \frac{\eta_{(t)}}{\eta^*} + \alpha_{(t)} + c\eta_{(t)} - \alpha^* - c\eta^*} \frac{\eta^{*n+\beta-1} \alpha^{*n}}{\eta_{(t)}^{n+\beta-1} \alpha_{(t)}^{n}} \prod_{i=1}^{n} x_i^{\alpha^* - \alpha_{(t)}} e^{\eta_{(t)} x_i^{\alpha_{(t)}} - \eta^* x_i^{\alpha^*}}, 1\right\}$$

$$= \min\left\{e^{-\frac{\alpha^*}{\alpha_{(t)}} - \frac{\eta^*}{\eta_{(t)}} + \frac{\alpha_{(t)}}{\alpha^*} + \frac{\eta_{(t)}}{\eta^*} + \alpha_{(t)} + c\eta_{(t)} - \alpha^* - c\eta^*} \frac{\eta^{*n+\beta} \alpha^{*n+1}}{\eta_{(t)}^{n+\beta} \alpha_{(t)}^{n+1}} \prod_{i=1}^{n} x_i^{\alpha^* - \alpha_{(t)}} e^{\eta_{(t)} x_i^{\alpha_{(t)}} - \eta^* x_i^{\alpha^*}}, 1\right\}$$

c)

How to generate the chain:

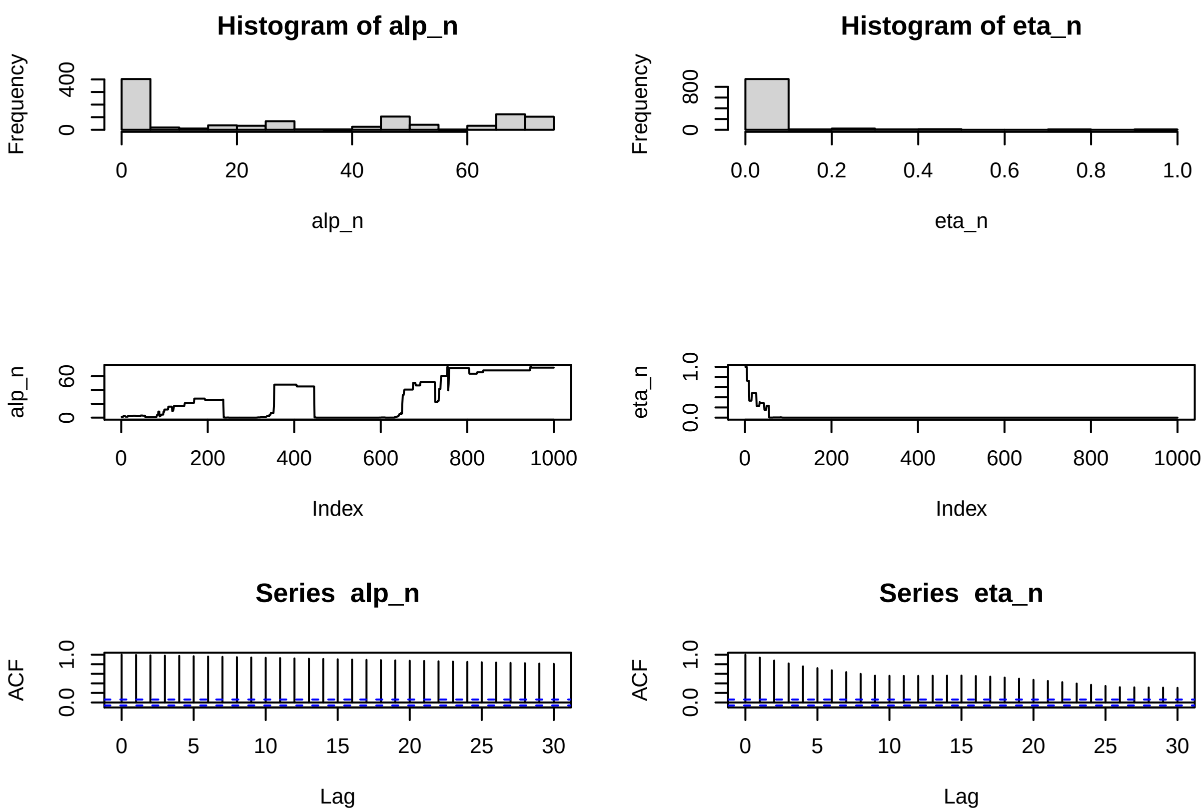
1. initialize $n = 0$ and α_n, η_n
2. sample α^*, η^* from $q(\alpha^*, \eta^* \mid \alpha_n, \eta_n)$, u from $U(0, 1)$
3. if $u \leq \rho(\alpha^*, \eta^* \mid \alpha_n, \eta_n)$ then set $\alpha_{n+1} = \alpha^*$ and $\eta_{n+1} = \eta^*$, else $\alpha_{n+1} = \alpha_n$ and $\eta_{n+1} = \eta_n$
4. set $n = n + 1$ and goto step 2.

```r
p = function(as, at, ns, nt, c, b, n, x)
{
  p1 = exp(-as/at -ns/nt +at/as + nt/ns +at +c*nt -as -c*ns)
  p2 = (ns/nt)^(n+b) * (as/at)^(n+1)
  p3 = prod(x^(as-at) * exp(nt*x^at - ns * x^as))
  return(min(p1*p2*p3, 1))
}

run = function(x,len,c,b)
{
  NN = 1000
  alp_n = rep(NA,NN)
  eta_n = rep(NA,NN)

  # step 1
  alp_n[1] = 1 # initial alpha
  eta_n[1] = 1 # initial eta

  for (i in 2:NN)
  {
    alpha = alp_n[i-1]
    eta = eta_n[i-1]

    # step 2
    alpha_star = rexp(1, 1/alpha)
    eta_star = rexp(1, 1/eta)
    u = runif(1)

    # step 3
    if (u <= p(alpha_star, alpha, eta_star, eta, c, b, len, x))
    {
      alp_n[i] = alpha_star
      eta_n[i] = eta_star
    } else
    {
      alp_n[i] = alpha
      eta_n[i] = eta
    }
  }

  par(mfrow=c(3,2))
  hist(alp_n)
  hist(eta_n)
  plot(alp_n, type='l')
  plot(eta_n, type='l')
  acf(alp_n)
  acf(eta_n)
}

# hyper params
x = c(0.56, 2.26, 1.90, 0.94, 1.40, 1.39, 1.00, 1.45, 2.32, 2.08, 0.89, 1.68)
len = length(x)
```

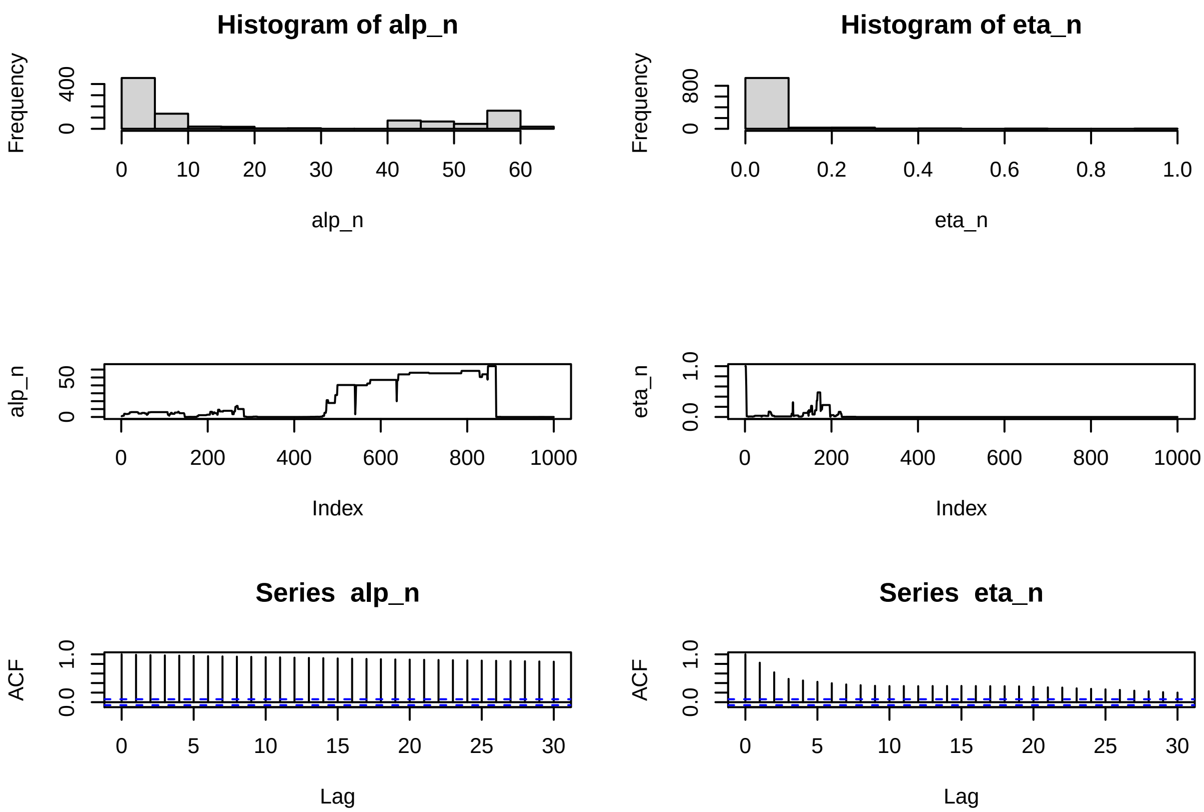
```r
run(x,len,c=0.1,b=1)
```

```
## Error in if (u <= p(alpha_star, alpha, eta_star, eta, c, b, len, x)) {: missing value where TRUE/FALS
```

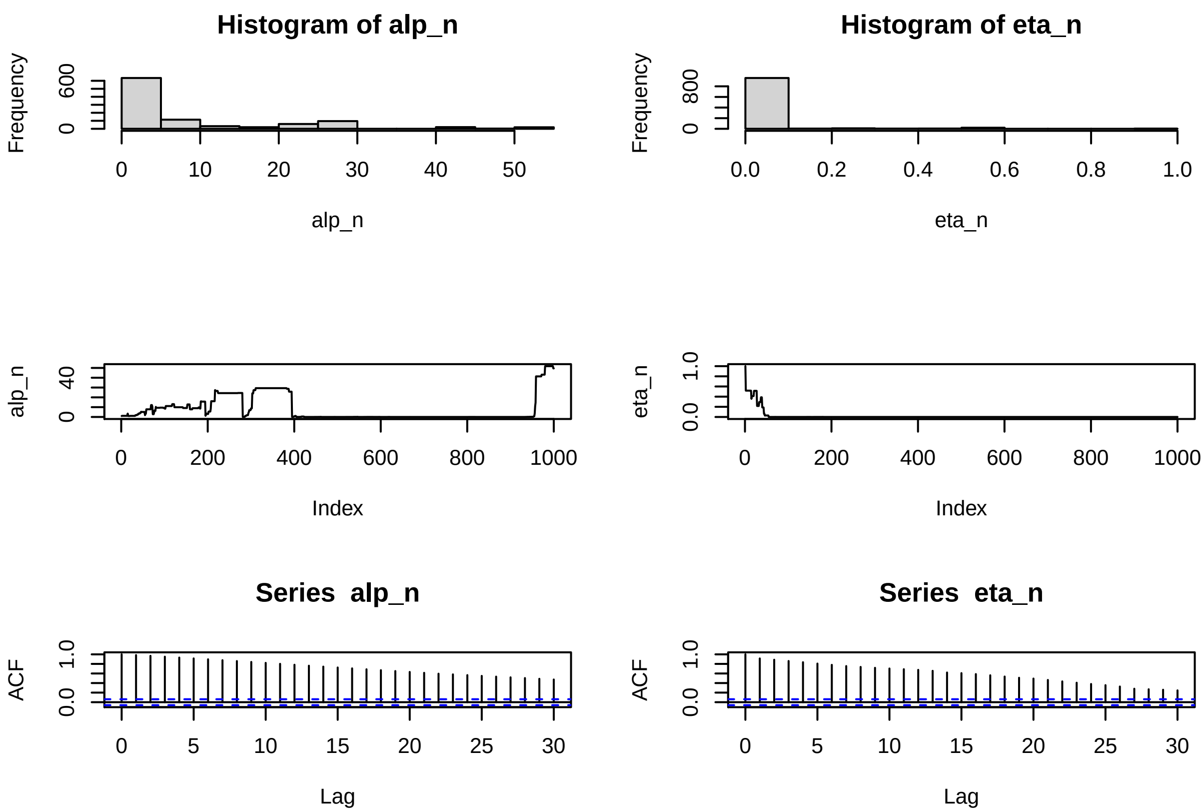
```r
run(x,len,c=4,b=1)
```



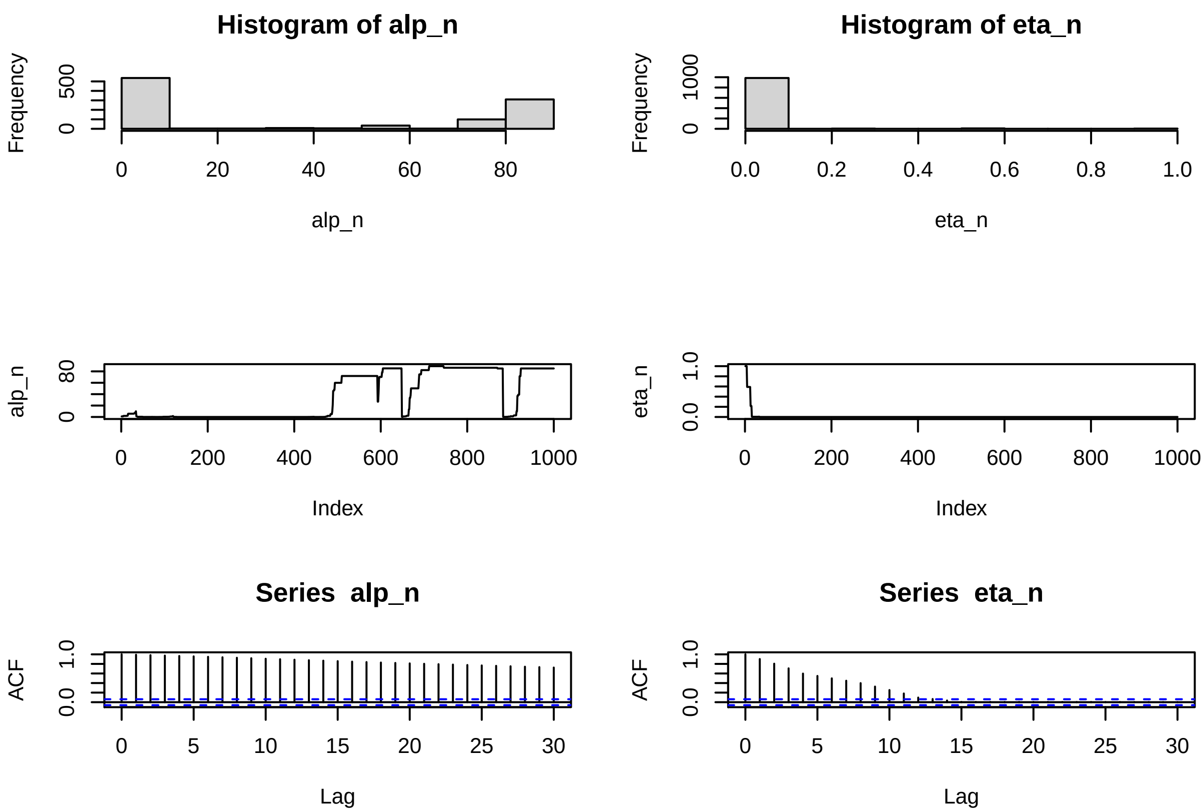

```r
run(x,len,c=4,b=0.1)
```



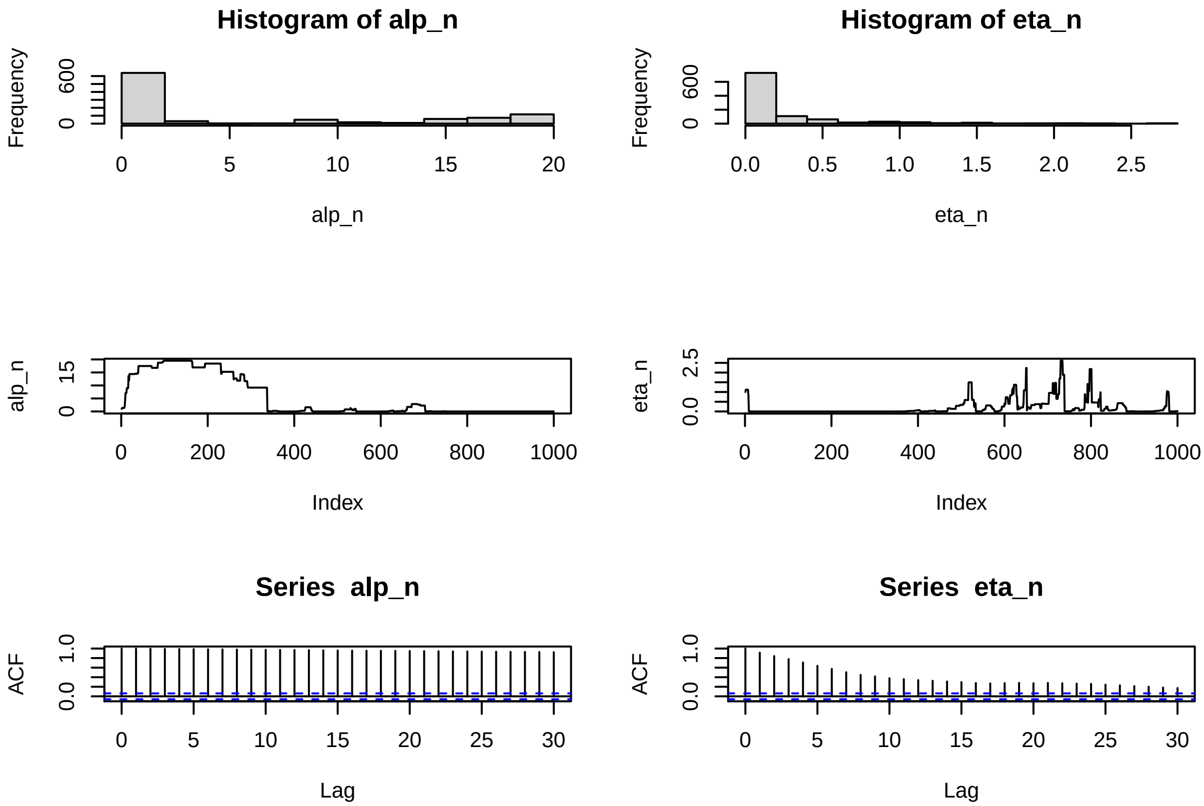

```r
run(x,len,c=10,b=0.1)
```



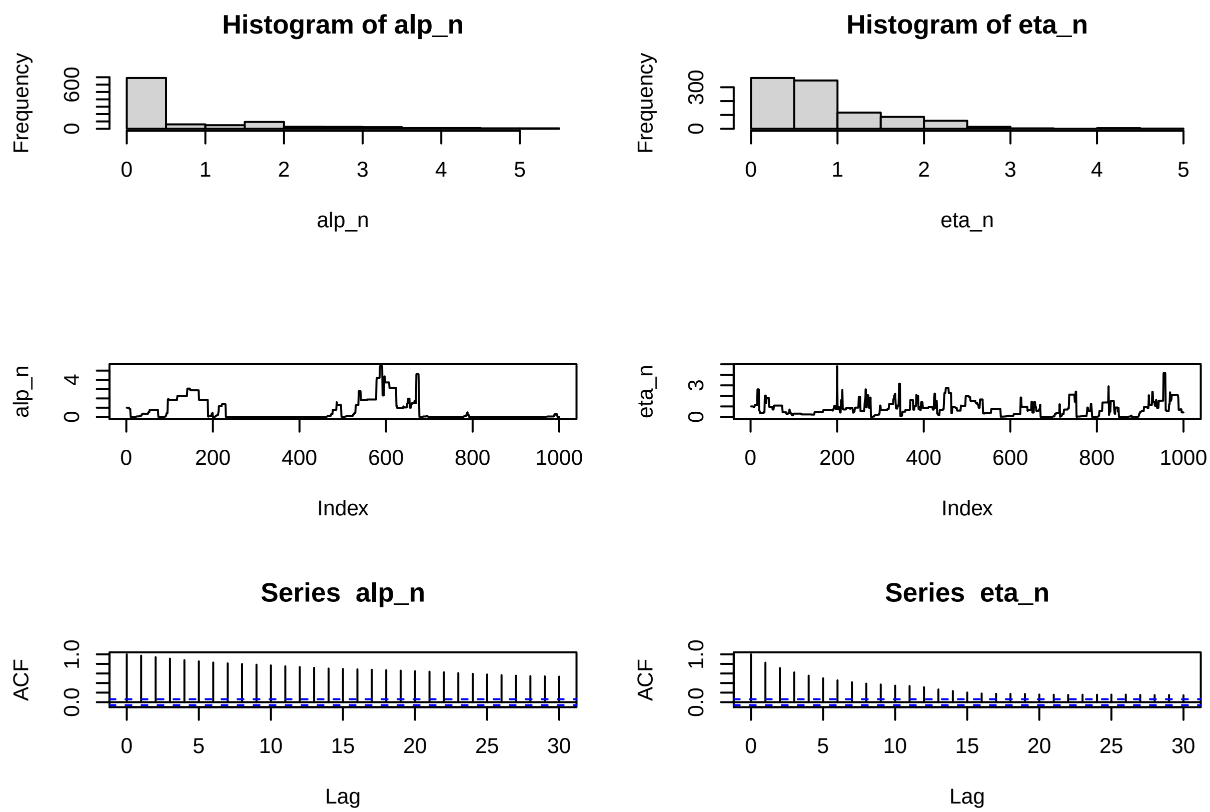

```r
run(x,len,c=0.1,b=0.1)
```




```
run(x,len,c=0.1,b=4)
```




```
run(x,len,c=0.1,b=10)
```



d + e)

```
NN = 1000
alp_n = rep(NA,NN)
eta_n = rep(NA,NN)

# step 1
alp_n[1] = 1 # initial alpha
eta_n[1] = 1 # initial eta

for (i in 2:NN)
{
  alpha = alp_n[i-1]
  eta = eta_n[i-1]

  # step 2
  alpha_star = rexp(1, 1/alpha)
  eta_star = rexp(1, 1/eta)
  u = runif(1)

  # step 3
  if (u <= p(alpha_star, alpha, eta_star, eta, c, b, len, x))
  {
    alp_n[i] = alpha_star
    eta_n[i] = eta_star
```

```r
  } else
  {
    alp_n[i] = alpha
    eta_n[i] = eta
  }
}
```

```
## Error in c * nt: non-numeric argument to binary operator
```

```r
mean(alp_n)
```

```
## [1] NA
```

```r
mean(eta_n)
```

```
## [1] NA
```

```r
quantile(alp_n, probs = c(0.025, 0.975))
```

```
## Error in quantile.default(alp_n, probs = c(0.025, 0.975)): missing values and NaN's not allowed if 'r
```

```r
quantile(eta_n, probs = c(0.025, 0.975))
```

```
## Error in quantile.default(eta_n, probs = c(0.025, 0.975)): missing values and NaN's not allowed if 'r
```

```r
post_pi = function(alp, eta)
{
  n = length(x)
  p1 = exp(-alp - c*eta)
  p2 = eta^(n+b-1)
  p3 = alp^n
  p4 = x^(alp-1) * exp(-eta * x^alp)
  return(p1 * p2 * p3 * prod(p4))
}

max(post_pi(alp_n))
```

```
## Error in post_pi(alp_n): argument "eta" is missing, with no default
```