GRADUATE STUDENT STAT 840 A1

Vsevolod Ladtchenko 20895137

Problem 4

(i)

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\begin{aligned}
\sum_{i=1}^{n}(x_i - \bar{x}_n)^2 &= \sum_{i=1}^{n}(x_i^2 - 2x_i\bar{x}_n + \bar{x}_n^2) \\
&= \sum_{i=1}^{n} x_i^2 - 2\bar{x}_n \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \bar{x}_n^2 \\
&= \sum_{i=1}^{n} x_i^2 - 2\bar{x}_n \sum_{i=1}^{n} x_i + n\bar{x}_n^2 \\
&\texttt{since } n\bar{x}_n = \sum_{i=1}^{n} x_i \\
&= \sum_{i=1}^{n} x_i^2 - 2n\bar{x}_n^2 + n\bar{x}_n^2 \\
&= \sum_{i=1}^{n} x_i^2 - n\bar{x}_n^2
\end{aligned}
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(ii)

$$
\begin{aligned}
\bar{x}_{n+1} &= \frac{1}{n+1}\sum_{i=1}^{n+1} x_i \\
&= \frac{1}{n+1}\left(x_{n+1} + \sum_{i=1}^{n} x_i\right) \\
&= \frac{1}{n+1}\left(x_{n+1} + n\bar{x}_n\right) \\
&= \frac{1}{n+1}x_{n+1} + \frac{n}{n+1}\bar{x}_n \\
&= \frac{1}{n+1}x_{n+1} + \bar{x}_n - \frac{1}{n+1}\bar{x}_n \\
&= \bar{x}_n + \frac{1}{n+1}\left(x_{n+1} - \bar{x}_n\right)
\end{aligned}
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\begin{aligned}
\sum_{i=1}^{n+1}(x_i - \bar{x}_{n+1})^2 &= \sum_{i=1}^{n+1} x_i^2 - (n+1)\bar{x}_{n+1}^2 \\
&\qquad \texttt{by part (i)} \\
&= x_{n+1}^2 + \sum_{i=1}^{n} x_i^2 - (n+1)\bar{x}_{n+1}^2 \\
(n+1)\bar{x}_{n+1}^2 &= (n+1)\left(\bar{x}_n + \frac{1}{n+1}\left(x_{n+1} - \bar{x}_n\right)\right)^2 \\
&\qquad \texttt{from the previous part} \\
&= (n+1)\left(\bar{x}_n^2 + \frac{2\bar{x}_n}{n+1}\left(x_{n+1} - \bar{x}_n\right) + \frac{1}{(n+1)^2}\left(x_{n+1} - \bar{x}_n\right)^2\right) \\
&= (n+1)\bar{x}_n^2 + 2\bar{x}_n\left(x_{n+1} - \bar{x}_n\right) + \frac{1}{n+1}\left(x_{n+1} - \bar{x}_n\right)^2 \\
&= n\bar{x}_n^2 + \bar{x}_n^2 + 2\bar{x}_n\left(x_{n+1} - \bar{x}_n\right) + \frac{1}{n+1}\left(x_{n+1} - \bar{x}_n\right)^2 \\
&= n\bar{x}_n^2 + \epsilon \\
&\qquad \texttt{where } \epsilon = \bar{x}_n^2 + 2\bar{x}_n\left(x_{n+1} - \bar{x}_n\right) + \frac{1}{n+1}\left(x_{n+1} - \bar{x}_n\right)^2 \\
\sum_{i=1}^{n+1}(x_i - \bar{x}_{n+1})^2 &= x_{n+1}^2 + \sum_{i=1}^{n} x_i^2 - n\bar{x}_n^2 - \epsilon \\
&= \sum_{i=1}^{n}(x_i - \bar{x}_n)^2 + x_{n+1}^2 - \epsilon \\
x_{n+1}^2 - \epsilon &= x_{n+1}^2 - \bar{x}_n^2 - 2\bar{x}_n\left(x_{n+1} - \bar{x}_n\right) - \frac{1}{n+1}\left(x_{n+1} - \bar{x}_n\right)^2 \\
&= x_{n+1}^2 - \bar{x}_n^2 - 2\bar{x}_n x_{n+1} + 2\bar{x}_n^2 - \frac{1}{n+1}\left(x_{n+1} - \bar{x}_n\right)^2 \\
&= x_{n+1}^2 - 2\bar{x}_n x_{n+1} + \bar{x}_n^2 - \frac{1}{n+1}\left(x_{n+1} - \bar{x}_n\right)^2 \\
&= \left(x_{n+1} - \bar{x}_n\right)^2 - \frac{1}{n+1}\left(x_{n+1} - \bar{x}_n\right)^2 \\
&= \frac{n}{n+1}\left(x_{n+1} - \bar{x}_n\right)^2 \\
\sum_{i=1}^{n+1}(x_i - \bar{x}_{n+1})^2 &= \sum_{i=1}^{n}(x_i - \bar{x}_n)^2 + \frac{n}{n+1}\left(x_{n+1} - \bar{x}_n\right)^2
\end{aligned}
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