GRADUATE STUDENT STAT 840 A2

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Problem 6

a)

$$
\begin{aligned}
f(x \mid \alpha, \eta) &\propto \alpha\eta x^{\alpha-1} e^{-\eta x^{\alpha}} \\
&\quad 0 < x < \infty \\
f(\mathbf{X} \mid \alpha, \eta) &\propto \prod_{i=1}^{n} \alpha\eta x_i^{\alpha-1} e^{-\eta x_i^{\alpha}} \\
&\propto \alpha^n \eta^n \prod_{i=1}^{n} x_i^{\alpha-1} e^{-\eta x_i^{\alpha}} \\
\log f(\mathbf{X} \mid \alpha, \eta) &=? + n\log(\alpha) + n\log(\eta) + \sum_{i=1}^{n} (\alpha-1)\log(x_i) - \eta x_i^{\alpha} \\
? &= \log(\text{proportionality constant}) \\
\pi(\alpha, \eta) &\propto e^{-\alpha} \eta^{\beta-1} e^{-c\eta} \\
&\propto e^{-\alpha - c\eta} \eta^{\beta-1} \\
\log \pi(\alpha, \eta) &=? - \alpha - c\eta + (\beta-1)\log(\eta) \\
\pi(\alpha, \eta \mid \mathbf{X}) &= f(\mathbf{X} \mid \alpha, \eta)\pi(\alpha, \eta) \\
&\propto e^{-\alpha - c\eta} \eta^{\beta-1} \alpha^n \eta^n \prod_{i=1}^{n} x_i^{\alpha-1} e^{-\eta x_i^{\alpha}} \\
&\propto e^{-\alpha - c\eta} \eta^{n+\beta-1} \alpha^n \prod_{i=1}^{n} x_i^{\alpha-1} e^{-\eta x_i^{\alpha}} \\
\log \pi(\alpha, \eta \mid \mathbf{X}) &=? - \alpha - c\eta + (n+\beta-1)\log(\eta) + n\log(\alpha) + \sum_{i=1}^{n} (\alpha-1)\log(x_i) - \eta x_i^{\alpha}
\end{aligned}
$$

```
log_post_pi = function(alp, eta, c,b,x)
{
  n = length(x)
  p1 = (-alp -c*eta)
  p2 = (n + b - 1)*log(eta)
  p3 = n * log(alp)
  p4 = (alp-1)*log(x) -eta * (x^alp)
  return(p1 + p2 + p3 + sum(p4))
}

post_pi = function(alp, eta, c,b,x)
{
  n = length(x)
```

```
  p1 = exp(-alp -c*eta)
  p2 = eta^(n + b - 1)
  p3 = alp^n
  p4 = (x^(alp-1)) * exp(-eta * (x^alp))
  return(p1 * p2 * p3 * prod(p4))
}
```

b)

$$q(\alpha_*, \eta_* \mid \alpha_{(t)}, \eta_{(t)}) = \frac{1}{\alpha_{(t)}\eta_{(t)}} \exp\left\{-\frac{\alpha_*}{\alpha_{(t)}} - \frac{\eta_*}{\eta_{(t)}}\right\}$$

$$\alpha(\theta_n, \theta_*) = \min\left\{\frac{\pi(\theta_* \mid X)q(\theta_*, \theta_n)}{\pi(\theta_n \mid X)q(\theta_n, \theta_*)}, 1\right\}$$

$$\rho(\alpha_*, \eta_* \mid \alpha_{(t)}, \eta_{(t)}) = \min\left\{\frac{\pi(\alpha_*, \eta_* \mid X)q(\alpha_*, \eta_* \mid \alpha_{(t)}, \eta_{(t)})}{\pi(\alpha_{(t)}, \eta_{(t)} \mid X)q(\alpha_{(t)}, \eta_{(t)} \mid \alpha_*, \eta_*)}, 1\right\}$$

$$\begin{aligned}
\frac{q(\alpha_*, \eta_* \mid \alpha_{(t)}, \eta_{(t)})}{q(\alpha_{(t)}, \eta_{(t)} \mid \alpha_*, \eta_*)} &= \frac{\frac{1}{\alpha_{(t)}\eta_{(t)}} \exp\left\{-\frac{\alpha_*}{\alpha_{(t)}} - \frac{\eta_*}{\eta_{(t)}}\right\}}{\frac{1}{\alpha_*\eta_*} \exp\left\{-\frac{\alpha_{(t)}}{\alpha_*} - \frac{\eta_{(t)}}{\eta_*}\right\}} \\
&= \frac{\alpha_*\eta_*}{\alpha_{(t)}\eta_{(t)}} \exp\left\{-\frac{\alpha_*}{\alpha_{(t)}} - \frac{\eta_*}{\eta_{(t)}}\right\} \exp\left\{\frac{\alpha_{(t)}}{\alpha_*} + \frac{\eta_{(t)}}{\eta_*}\right\} \\
&= \frac{\alpha_*\eta_*}{\alpha_{(t)}\eta_{(t)}} \exp\left\{-\alpha_*/\alpha_{(t)} - \eta_*/\eta_{(t)}\right\} \exp\left\{\alpha_{(t)}/\alpha_* + \eta_{(t)}/\eta_*\right\} \\
&= \frac{\alpha_*\eta_*}{\alpha_{(t)}\eta_{(t)}} e^{-\alpha_*/\alpha_{(t)} - \eta_*/\eta_{(t)} + \alpha_{(t)}/\alpha_* + \eta_{(t)}/\eta_*}
\end{aligned}$$

$$\begin{aligned}
\frac{\pi(\alpha_*, \eta_* \mid X)}{\pi(\alpha_{(t)}, \eta_{(t)} \mid X)} &= \frac{e^{-\alpha_* - c\eta_*} \eta_*^{n+\beta-1} \alpha_*^n \prod_{i=1}^n x_i^{\alpha_*-1} e^{-\eta_* x_i^{\alpha_*}}}{e^{-\alpha_{(t)} - c\eta_{(t)}} \eta_{(t)}^{n+\beta-1} \alpha_{(t)}^n \prod_{i=1}^n x_i^{\alpha_{(t)}-1} e^{-\eta_{(t)} x_i^{\alpha_{(t)}}}} \\
&= e^{\alpha_{(t)} + c\eta_{(t)} - \alpha_* - c\eta_*} \frac{\eta_*^{n+\beta-1} \alpha_*^n}{\eta_{(t)}^{n+\beta-1} \alpha_{(t)}^n} \prod_{i=1}^n x_i^{\alpha_* - \alpha_{(t)}} e^{\eta_{(t)} x_i^{\alpha_{(t)}} - \eta_* x_i^{\alpha_*}}
\end{aligned}$$

$$\begin{aligned}
\rho(\alpha_*, \eta_* \mid \alpha_{(t)}, \eta_{(t)}) &= \min\left\{\frac{\alpha_*\eta_*}{\alpha_{(t)}\eta_{(t)}} e^{-\alpha_*/\alpha_{(t)} - \eta_*/\eta_{(t)} + \alpha_{(t)}/\alpha_* + \eta_{(t)}/\eta_*} e^{\alpha_{(t)} + c\eta_{(t)} - \alpha_* - c\eta_*} \frac{\eta_*^{n+\beta-1} \alpha_*^n}{\eta_{(t)}^{n+\beta-1} \alpha_{(t)}^n} \prod_{i=1}^n x_i^{\alpha_* - \alpha_{(t)}} e^{\eta_{(t)} x_i^{\alpha_{(t)}} - \eta_* x_i^{\alpha_*}}, 1\right\} \\
&= \min\left\{e^{-\alpha_*/\alpha_{(t)} - \eta_*/\eta_{(t)} + \alpha_{(t)}/\alpha_* + \eta_{(t)}/\eta_* + \alpha_{(t)} + c\eta_{(t)} - \alpha_* - c\eta_*} \frac{\eta_*^{n+\beta} \alpha_*^{n+1}}{\eta_{(t)}^{n+\beta} \alpha_{(t)}^{n+1}} \prod_{i=1}^n x_i^{\alpha_* - \alpha_{(t)}} e^{\eta_{(t)} x_i^{\alpha_{(t)}} - \eta_* x_i^{\alpha_*}}, 1\right\}
\end{aligned}$$

```
log_p = function(a2,n2,a1,n1, b,c,x)
{
  n = length(x)
  p1 = -a2/a1 - n2/n1 + a1/a2 + n1/n2 + a1 + c*n1 -a2 -c*n2
  p2 = (n+b)*log(n2) + (n+1)*log(a2) - (n+b)*log(n1) - (n+1)*log(a1)
  p3 = sum((a2-a1)*log(x) + ((n1*(x^a1)) - (n2*(x^a2))))
  return(min(p1+p2+p3,0))
}

p = function(a2,n2,a1,n1, b,c,x)
```

```
{
  n = length(x)
  p1 = exp(-a2/a1 - n2/n1 + a1/a2 + n1/n2 + a1 + c*n1 -a2 -c*n2)
  p2 = ((n2^(n+b)) * (a2^(n+1))) / ((n1^(n+b)) * (a1^(n+1)))
  p3 = prod(((x^(a2-a1)) * exp(((n1*(x^a1)) - (n2*(x^a2))))))
  return(min(p1*p2*p3, 1))
}
```

c)

Explain how to generate the chain:

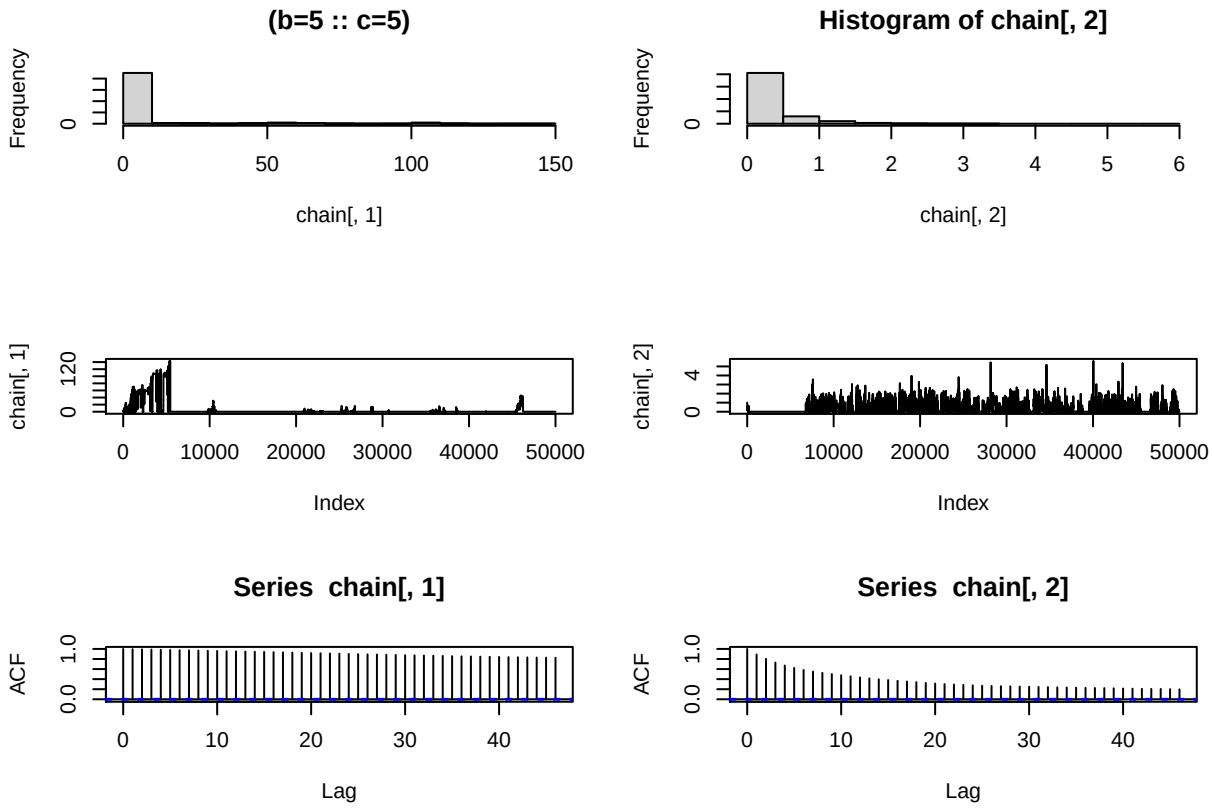
1. Initialize $n = 0$ and α_n, η_n.
2. sample $\alpha_*, \eta_* \sim q(\alpha_*, \eta_* \mid \alpha_{(t)}, \eta_{(t)})$ and $u \sim U(0,1)$
3. if $u \leq \rho(\alpha_*, \eta_* \mid \alpha_{(t)}, \eta_{(t)})$ set $\alpha_{n+1} = \alpha_*, \eta_{n+1} = \eta_*$. else $\alpha_{n+1} = \alpha_n, \eta_{n+1} = \eta_n$
4. set $n = n + 1$ and goto step 2.

```
run = function(b,c,x, NN = 50000, plot_graf=F)
{
  chain = matrix(nrow=NN, ncol=2)
  chain[1,] = c(1,1)

  for (i in 2:NN)
  {
    a1 = chain[i-1,1]
    n1 = chain[i-1,2]

    a2 = rexp(1, 1/a1)
    n2 = rexp(1, 1/n1)

    if (log(runif(1)) <= log_p(a2,n2,a1,n1, b,c,x))
      chain[i,] = c(a2,n2)
    else
      chain[i,] = chain[i-1,]
  }

  if (plot_graf) # plot graphs
  {
    par(mfrow = c(3,2))
    hist(chain[,1], main=paste0("(b=", b, " :: c=", c, ")"))
    hist(chain[,2])
    plot(chain[,1],type='l')
    plot(chain[,2],type='l')
    acf(chain[,1])
    acf(chain[,2])
  }

  return(chain)
}

if (F) # iterate over possible values of b,c to monitor
{
  MM = 15
```

```
  params = rep(1, MM)
  for (i in 2:MM) params[i] = 1.4*params[i-1]
  params
  for (i in params)
  {
    for (j in params)
    {
      tryCatch({
        run(b=i, c=j, x, NN=10000, T)
      }, warning = function(w) {
        print(paste(i,j,e))
      }, error = function(e) {
        print(paste(i,j,e))
      }, finally = {
      })
    }
  }
}
```

d)

```
# posterior mean
b = 5
c = 5
x = c(0.56, 2.26, 1.90, 0.94, 1.40, 1.39, 1.00, 1.45, 2.32, 2.08, 0.89, 1.68)
NN = 50000
chain = run(b,c,x,NN,T)
```




```
c(alpha=mean(chain[,1]),eta=mean(chain[,2]))
```

```
##     alpha       eta
## 6.7172669 0.2350239
```

e)

```
# MAP estimate
MAPs = rep(NA, NN)
for (i in 1:NN) MAPs[i] = post_pi(chain[i,1],chain[i,2],b,c,x)
map_idx = which(MAPs == max(MAPs))[1]
c(chain[map_idx,1],chain[map_idx,2])
```

```
## [1] 2.1861949 0.4185015
```

```
# 95% credible interval
quantile(chain[,1], probs = c(0.025, 0.975))
```

```
##         2.5%        97.5%
## 6.939828e-49 1.031694e+02
```

```
quantile(chain[,2], probs = c(0.025, 0.975))
```

```
##         2.5%        97.5%
## 5.608384e-42 1.357109e+00
```