GRADUATE STUDENT STAT 840 A2

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Problem 1

Since X has a lognormal distribution, and ϵ has a normal distribution, and Y is a function of (X, ϵ) we can decompose the expectation of Y/X into an expectation over X and an expectation over ϵ.

$$\begin{aligned}
\mathbb{E}\left[\frac{Y}{X}\right] &= \mathbb{E}\left[\frac{\exp(9 + 3\log X + \epsilon)}{X}\right] \\
&= \mathbb{E}_X\left[\mathbb{E}_\epsilon\left[\frac{\exp(9 + 3\log X + \epsilon)}{X}\middle| X = x\right]\right] \\
\mathbb{E}_\epsilon\left[\frac{\exp(9 + 3\log X + \epsilon)}{X}\middle| X = x\right] &= \mathbb{E}_\epsilon\left[\frac{e^9 e^{3\log X} e^\epsilon}{X}\middle| X = x\right] \\
&= \mathbb{E}_\epsilon\left[\frac{e^9 (e^{\log X})^3 e^\epsilon}{X}\middle| X = x\right] \\
&= \mathbb{E}_\epsilon\left[\frac{e^9 X^3 e^\epsilon}{X}\middle| X = x\right] \\
&= \mathbb{E}_\epsilon\left[e^9 X^2 e^\epsilon \middle| X = x\right] \\
&= e^9 X^2 \mathbb{E}_\epsilon\left[e^\epsilon\right] \qquad \texttt{since X constant} \\
&= e^9 X^2 e^{0.5} \qquad \texttt{lnorm(0,1) mean = } e^{0.5} \\
&= e^{9.5} X^2 \\
\mathbb{E}_X\left[e^{9.5} X^2\right] \approx \hat{\theta}_{RB} &= \frac{1}{n}\sum_{i=1}^{n} e^{9.5} X_i^2 \\
\mathbb{E}\left[\frac{Y}{X}\right] \approx \hat{\theta}_{MC} &= \frac{1}{n}\sum_{i=1}^{n} \frac{Y_i}{X_i}
\end{aligned}$$

```r
d_RB = function(x)
{
  return(exp(9.5) * (x^2))
}

d_MC = function(x, e)
{
  return(exp(9) * (x^2) * (exp(e)))
}

get_SE = function(x) # return variance of sample mean
{
  n = length(x)
  mu = sum(x) / n
  s_sq = sum((x - mu)^2) / ((n-1)*n) # notes 2 p 30
```

```r
  return(sqrt(s_sq))
}

get_CI = function(xbar, se, conf=.95)
{
  a = 1 - conf
  L = xbar + qnorm(a/2)*se
  U = xbar + qnorm(1 - a/2)*se
  return(c(L,U))
}



n = 100000000
x = rlnorm(n, 0, 1)
e = rnorm(n, 0, 1)
y = exp(9 + 3*log(x) + e)

# RB
deltas_RB = d_RB(x)
theta_hat_RB = mean(deltas_RB)
SE_RB = get_SE(deltas_RB)
CI_RB = get_CI(theta_hat_RB, SE_RB)

# simple
deltas_MC = d_MC(x, e)
theta_hat_MC = mean(deltas_MC)
SE_MC = get_SE(deltas_MC)
CI_MC = get_CI(theta_hat_MC, SE_MC)

# whenever reporting a MC estimate, report n, SE of estimate, and CI
# Rao Blackwell
round(c(n=n, theta=theta_hat_RB, se=SE_RB, ci=CI_RB), 0)
```

```
##         n     theta        se       ci1       ci2
## 100000000     98732        77     98581     98882
```

```r
# Simple MC
round(c(n=n, theta=theta_hat_MC, se=SE_MC, ci=CI_MC), 0)
```

```
##         n     theta        se       ci1       ci2
## 100000000     98743       121     98506     98981
```

We see that the Rao-Blackwellized estimator enjoys a much smaller variance / standard error, compared to simple Monte-Carlo. The means of the estimates are very similar, agreeing with the theoretical result that both estimators have the same expected value.