

GRADUATE STUDENT STAT 840 A2

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Problem 7

a)

$$\pi(x, y) \propto x^2 \exp(-xy^2 - y^2 + 2y - 4x)$$

where $x > 0, -\infty < y < \infty$

$$\pi(x | y) = \frac{\pi(x, y)}{\pi(y)}$$

$$\propto \pi(x, y)$$

since y given, $\pi(y)$ is constant

$$\propto x^2 \exp(-xy^2 - 4x) \exp(-y^2 + 2y)$$

$$\propto x^2 \exp(-xy^2 - 4x)$$

$$\propto x^2 \exp(-x(y^2 + 4))$$

$$\propto x^2 \exp(-(y^2 + 4)x)$$

$$\propto x^2 e^{-(y^2 + 4)x}$$

$$\propto x^{\alpha-1} e^{-\beta x}$$

$$\sim \text{Gamma}(\alpha = 3, \beta = y^2 + 4)$$

$$\begin{aligned}
\pi(y | x) &= \frac{\pi(x, y)}{\pi(x)} \\
&\propto \pi(x, y) \\
&\quad \text{since } x \text{ given, } \pi(x) \text{ is constant} \\
&\propto x^2 \exp(-xy^2 - y^2 + 2y) \exp(-4x) \\
&\propto \exp(-xy^2 - y^2 + 2y) \\
&\quad \text{since } x > 0 \\
&\propto \exp\left\{y^2(-x-1) + 2y\right\} \\
&\propto \exp\left\{-\frac{1}{2}(-2)\left(y^2(-x-1) + 2y\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(y^2 2(x+1) - 4y\right)\right\} \\
&\quad \text{now complete the square:} \\
&\propto \exp\left\{-\frac{1}{2}\left(y^2 2(x+1) - 4y\right)\right\} \exp\left\{-\frac{1}{2} \frac{2}{x+1}\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(y^2 2(x+1) - 4y\right) - \frac{1}{2} \frac{2}{x+1}\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(y^2 2(x+1) - 4y + \frac{2}{x+1}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(y\sqrt{2}\sqrt{x+1} - \frac{\sqrt{2}}{\sqrt{x+1}}\right)^2\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(\frac{\frac{1}{\sqrt{2}\sqrt{x+1}}}{\frac{1}{\sqrt{2}\sqrt{x+1}}} y\sqrt{2}\sqrt{x+1} - \frac{\frac{1}{\sqrt{2}\sqrt{x+1}}}{\frac{1}{\sqrt{2}\sqrt{x+1}}} \frac{\sqrt{2}}{\sqrt{x+1}}\right)^2\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(\frac{y}{\frac{1}{\sqrt{2}\sqrt{x+1}}} - \frac{\frac{1}{\sqrt{2}\sqrt{x+1}} \frac{\sqrt{2}}{\sqrt{x+1}}}{\frac{1}{\sqrt{2}\sqrt{x+1}}}\right)^2\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(\frac{y - \frac{1}{\sqrt{2}\sqrt{x+1}} \frac{\sqrt{2}}{\sqrt{x+1}}}{\frac{1}{\sqrt{2}\sqrt{x+1}}}\right)^2\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(\frac{y - \frac{1}{x+1}}{\frac{1}{\sqrt{2}\sqrt{x+1}}}\right)^2\right\} \\
&\sim N\left(\mu = \frac{1}{x+1}, \sigma = \frac{1}{\sqrt{2}\sqrt{x+1}}\right)
\end{aligned}$$

b)

Let us rename the variables to x_1 and x_2 . At time n , they will be $x_1^{(n)}$ and $x_2^{(n)}$. The Gibbs sampler steps are:

1. Initialize:

$$\mathbf{x}^{(0)} = (x_1^{(0)}, x_2^{(0)})^T$$
$$n = 0$$

2. Generate:

$$\begin{aligned} x_1^{(n+1)} &\sim \pi(x_1 \mid x_2^{(n)}) &&= \text{Gamma}(\alpha = 3, \beta = (x_2^{(n)})^2 + 4) \\ x_2^{(n+1)} &\sim \pi(x_2 \mid x_1^{(n+1)}) &&= N(\mu = \frac{1}{x_1^{(n+1)} + 1}, \sigma = \frac{1}{\sqrt{2}\sqrt{x_1^{(n+1)} + 1}}) \\ \mathbf{x}^{(n+1)} &= (x_1^{(n+1)}, x_2^{(n+1)})^T \end{aligned}$$

3. Increment n and goto step 2.

According to the notes chapter 4 page 24, the stationary distribution of this Gibbs Sampler will be π .

c)

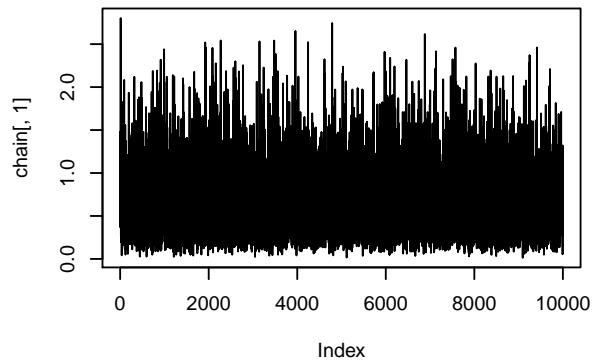
```
n = 10000
chain = matrix(ncol=2, nrow=n)
chain[1,] = c(1,1)

for (i in 2:n)
{
  x2_old = chain[i-1, 2]

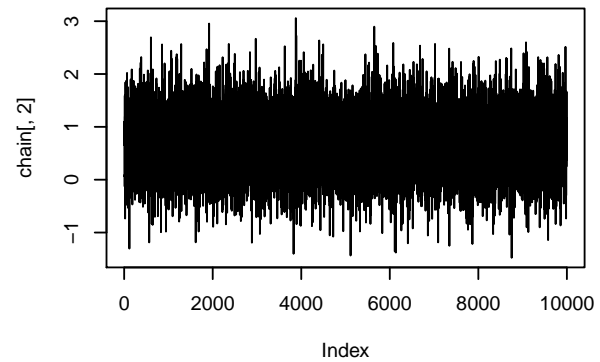
  x1_new = rgamma(n=1, shape=3, rate=(x2_old^2 + 4))
  x2_new = rnorm(n=1, mean=(1/(x1_new + 1)), sd=(1 / (sqrt(2) * sqrt(x1_new + 1))))

  chain[i,] = c(x1_new, x2_new)
}

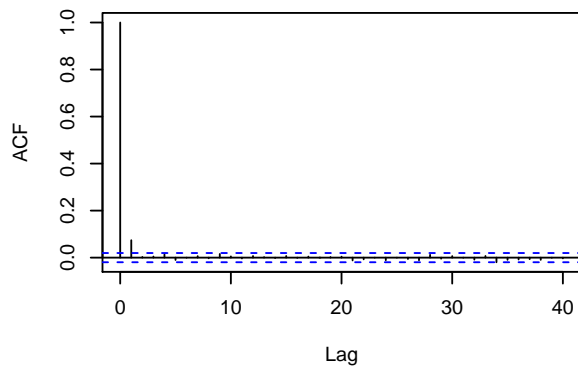
par(mfrow=c(3,2))
# 1. plot chain as time series
plot(chain[,1], type='l')
plot(chain[,2], type='l')
# 2. check that it is stationary using acf(), 1 bar ok
acf(chain[,1])
acf(chain[,2])
# 3. Q7: marginal histograms
hist(chain[,1])
hist(chain[,2])
```



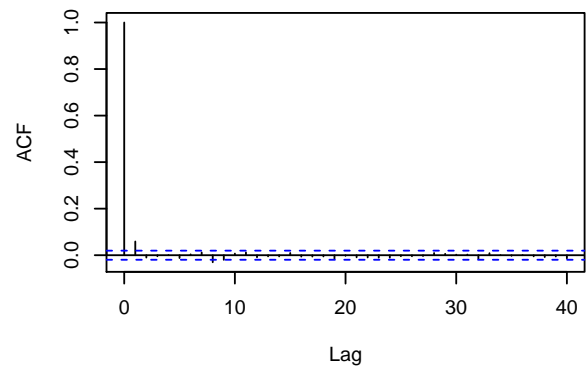
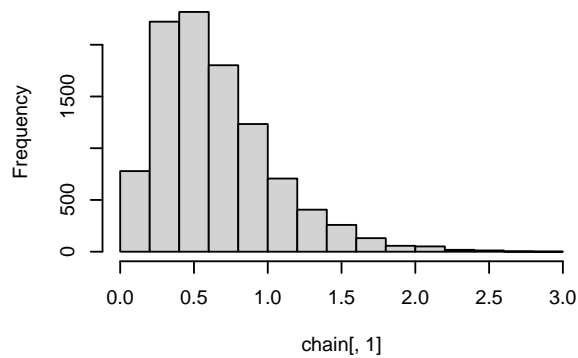
Series chain[, 1]



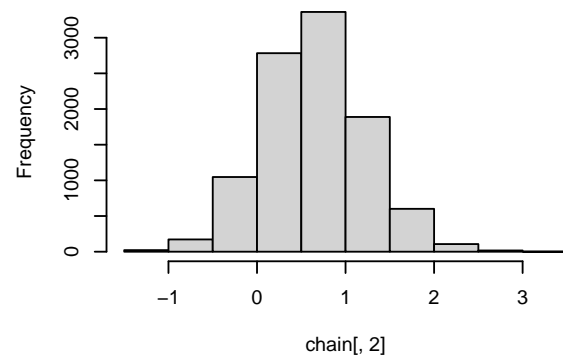
Series chain[, 2]



Histogram of chain[, 1]



Histogram of chain[, 2]



To monitor for convergence, we plot the chain as a time series and look at its autocorrelation. As expected, there is a small autocorrelation at $T = 1$ since this is a Markov Chain. The rest of the correlations are negligible, agreeing with the fact that a Markov Chain contains dependency only on the previous time point.

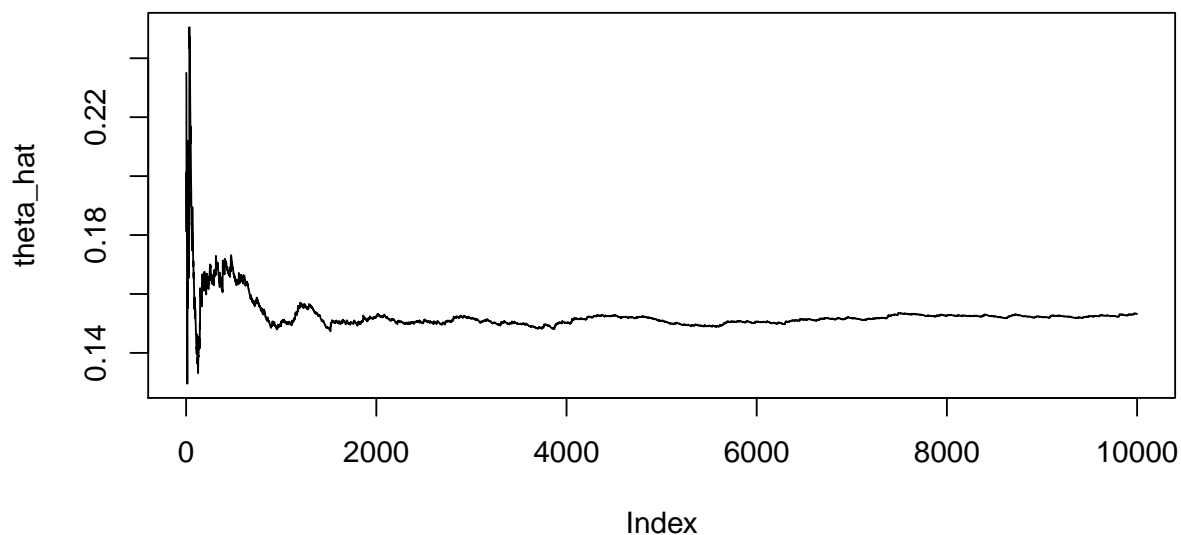
Looking at the marginal histograms, the distributions look smooth. Because the time series of the chain itself are exhibiting consistent behavior, we can assume that the draws from the marginal distributions are also consistent.

d)

```
d = function(chain)
{
  x = chain[,1]
  y = chain[,2]
  return((x^2) * (y^3) * exp(-(x^2)))
}

# monitor theta_hat
theta_hat = rep(NA,n)
for (i in 2:n)
{
  theta_hat[i] = mean(d(chain[1:i,]))
}

# 4. plot estimator as n increases, does it converge?
par(mfrow=c(1,1))
plot(theta_hat, type='l')
```



```
# calculate estimate after burn-in
burn = 1000
chain_b = chain[burn:n,]
mean(d(chain_b))
```

```
## [1] 0.1536044
```

Plotting the estimator at each time point n , we see that it appears to converge around 2000 steps. This is further evidence that the chain has reached its stationary distribution.