Lecture Note - 02

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Contents

1 Discrete Random Variables

1.1 Probability Distribution/Mass

A discrete random variable X can take on k different values: x_1, x_2, ..., x_k, each value occurs with probability p_1, p_2, ..., p_k. The probability distribution function or probability mass can be denoted as $X : P(x_i) = p_i$

1.2 Momentum

Mean/First Momentum:

$$E(X) = \bar{x} = \sum_{i=1}^{k} x_i p_i$$

n-th Momentum:

$$E(X^n) = \sum_{i=1}^{k} x_i^n p_i$$

Variance

$$Var(X) = \sum_{i=1}^{k} (x_i - \bar{x})^2 p_i$$

Mean of a general function $f(x)$

$$E(f(X)) = \sum_{i=1}^{k} f(x_i) p_i$$

1.3 Bernoulli Distribution

A random variable that follows Bernoulli Distribution can have two possible values 1 or 0, with probability p and $1 - p$, respectively.

$$P(x) = \begin{cases} p, & x = 1 \\ 1 - p, & x = 0 \end{cases} \tag{1}$$

The mean of a Bernoulli random variable can be calculated by definition using the distribution function

$$\bar{x} = 1 \cdot p + 0 \cdot (1 - p) = p$$

A sample Bernoulli Distribution dataset of N samples is a vector composed of 0/1 e.g. $x = [1, 0, 0, 1, ...1]$ and has N elements. Let's assume there are N_1 1s and N_0 0s in the dataset. Numerically, the average of the dataset can be calculated as

$$\bar{x} = \frac{1 + 0 + 0 + 1...1}{N} = \frac{N_0}{N} = \frac{N_0}{N_0 + N_1}$$

Here, $\frac{N_0}{N_0+N_1}$ is also the occurrence probability of 1 in the dataset i.e. $\frac{N_0}{N_0+N_1} \sim p$

1.4 Binomial Distribution

Assume we do 3 experiments and each experiment has an outcome of either 1 or 0. In another word, each experiment's outcome follows Bernoulli Distribution. A sample 3 Bernoulli experiment dataset looks like the following

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ \vdots & & \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

The total number of experiments that have an outcome of 1 can be calculated as the summation of 3 random variables. X can have 4 possible values 0, 1, 2, 3 out of 8 configurations

$$X = \begin{cases} x = 0, [0, 0, 0] \\ x = 1, [1, 0, 0], [0, 1, 0], [0, 0, 1] \\ x = 2, [1, 1, 0], [0, 1, 1], [1, 0, 1] \\ x = 3, [1, 1, 1] \end{cases}$$

The probability distribution of each value is therefore

$$X = \begin{cases} x = 0, p^3 \\ x = 1, 3p(1-p)^2 \\ x = 2, 3p^2(1-p) \\ x = 3, (1-p)^3 \end{cases}$$

In an n-experiments case, the dataset looks like the following

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}}_{n}$$

The value of X could vary between 0 and n. The probability that $X = k$, i.e. k out n experiments have out come 1, can have $\frac{n!}{k!(n-k)!}$ configurations. The probability distribution is therefore

$$P(k) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

The expectation of X can be calculated as

$$\begin{aligned} E(X) &= \sum_{k=0}^{n} kp(k) \\ &= \sum_{k=0}^{n} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\ &= \sum_{k=1}^{n} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}, \text{ the k=0 term is eliminated, the summation starts at k=1} \\ &= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k} \\ &= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k} \\ &= np \sum_{k=0}^{n-1} \frac{(n-1)!}{(k-1)!(n-k-1)!} p^k (1-p)^{n-k-1}, \text{ substitute } k \leftarrow k+1 \text{ and reindex k from 0} \\ &= np \end{aligned}$$

1.5 Poisson Distribution

Probability distribution: consider a random event, the probability of 1 occurrence within a unit time is p. What is the probability distribution of events occurrence within a time interval of τ (e.g. No. of car accidents occurs in a day in MO). Assume the probability of 1 event occurring is p in a unit time period. We then have the following probability distribution within a short interval dt, where $dt \to 0$

$$P = \begin{cases} 1 - pdt, \text{no event} \\ pdt, 1 \text{ event} \\ (pdt)^n \approx 0, \text{n events} \end{cases}$$

The event occurred within a time period of dt follows a Bernoulli distribution

$$\Delta = \begin{cases} 0, 1 - pdt \\ 1, pdt \end{cases}$$

For a period of τ, we can construct a random variable X that is a summation of N random variables $X = \Delta_1 + \Delta_2 + ...\Delta_N$. Each variable on the right hand side follows the Bernoulli distribution described above. In the limit when $dt \to 0$ we need to have $N \to \infty$ to have $Ndt = \tau$. In another word, X is the summation of infinite number of Bernoulli distribution or a binomial distribution when $N \to \infty$

$$X = \Delta_1 + \Delta_2 + ...\Delta_i + ...$$

The distribution of X can be derived by taking the limit of a Binomial distribution.

$$\begin{aligned} P(k) &= \lim_{N\to\infty} \frac{N!}{k!(N-k)!}(pdt)^k(1-pdt)^{N-k} \\ &= \lim_{N\to\infty} \frac{N!}{k!(N-k)!}(\frac{p\tau}{N})^k(1-\frac{p\tau}{N})^{N-k} \\ &= \lim_{N\to\infty} \frac{N!}{k!(N-k)!}(\frac{\lambda}{N})^k(1-\frac{\lambda}{N})^{N-k} \text{ where } \lambda = p\tau \\ &= \lim_{N\to\infty} \frac{N(N-1)...(N-k+1)}{k!}(\frac{\lambda}{N})^k(1-\frac{\lambda}{N})^{N-k} \\ &= \frac{\lambda^k}{k!}e^{-\lambda} \end{aligned}$$

According to Taylor expansion of a exponential function, the summation of $P(k)$ is

$$\sum_{k=0}^{\infty} P(k) = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = e^{-\lambda}e^{\lambda} = 1$$

Mean: there are many ways to calculate the mean of Poisson distribution. Here we use a trick by taking the derivative of the equation above, $e^{-\lambda}\sum\limits_{k=0}^{\infty}\frac{\lambda^k}{k!} = 1$, we then have

$$\frac{d}{d\lambda}e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = 0$$

$$\to -e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} + e^{-\lambda}\sum_{k=0}^{\infty}\frac{k\lambda^{k-1}}{k!} = 0$$

The first term of the equation above is nothing but 1, we therefore have

$$-1 + e^{-\lambda} \sum_{k=0}^{\infty} \frac{k\lambda^{k-1}}{k!} = 0$$

Organize the equation a little we have

$$\frac{1}{\lambda} e^{-\lambda} \sum_{k=0}^{\infty} \frac{k\lambda^{k}}{k!} = 1$$

By definition, $E(X) = e^{-\lambda} \sum\limits_{k=0}^{\infty} \frac{k\lambda^k}{k!}$, we therefore have

$$\frac{1}{\lambda} E(X) = 1 \text{ or } E(X) = \lambda$$

Variance: To calculate the variance we use the same trick of derivative. Here, we take the derivative of the equation $E(k) = \sum\limits_{k=0}^{\infty} \frac{k\lambda^k}{k!} = \lambda$

$$\begin{aligned}
&\frac{d}{d\lambda} E(k) = \frac{d}{d\lambda} \sum_{k=0}^{\infty} \frac{k\lambda^k}{k!} e^{-\lambda} = \frac{d}{d\lambda} \lambda \\
&\rightarrow \sum_{k=0}^{\infty} \frac{k^2\lambda^{k-1}}{k!} e^{-\lambda} - \sum_{k=0}^{\infty} \frac{k\lambda^k}{k!} e^{-\lambda} = 1 \\
&\rightarrow \frac{E(k^2)}{\lambda} = 1 + E(k) \\
&\rightarrow E(k^2) = (1+\lambda)\lambda
\end{aligned}$$

The variance of X by definition is

$$\begin{aligned}
Var(X) &= \sum_{k=0}^{\infty} (k-\lambda)^2 \frac{k\lambda^k}{k!} e^{-\lambda} \\
&= E(k^2) - 2E(k)\lambda + \lambda^2 \\
&= (1+\lambda)\lambda - 2\lambda^2 + \lambda^2 \\
&= \lambda
\end{aligned}$$

2 Continous Random Variable

2.1 Probability Density Function

A continous variable X can take on real value. The probability that X is between $x_i - \Delta x$ and $x_i + \Delta x$ can be described by the following probability function

$$P(x_i - \Delta x < x < x_i + \Delta x) \sim f(x_i)\Delta x$$

where x_i can be any value between $(-\infty, \infty)$. More precisely, we can write the relationship in integral formula

$$P(x_i - \Delta x < x < x_i + \Delta x) = \int_{x_i - \Delta x}^{x_i + \Delta x} f(x)dx$$

Here $f(x)$ is called probability density function and has to satisfy the following constraint

$$\int\limits_{-\infty}^{\infty} f(x)dx = 1$$

2.2 Moments

Mean: the mean of a continous random variable is defined as

$$\bar{x} = E(X) = \int\limits_{-\infty}^{\infty} xf(x)dx$$

Mean of arbitrary function:

$$E(g(x)) = \int\limits_{-\infty}^{\infty} g(x)f(x)dx$$

Variacne: the variance of a continous random variable is defined as

$$Var(X) = \int\limits_{-\infty}^{\infty} (x-\bar{x})^2 f(x)dx$$

3 Gaussian Distribution

The probability density function of a standard Gaussian function is defined as

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$E(X) == \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x e^{-\frac{x^2}{2}} dx = 0$$

$$\begin{aligned}
Var(X) &= \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 e^{-\frac{x^2}{2}} dx \\
&= \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x (e^{-\frac{x^2}{2}})' dx \text{ integrate by parts} \\
&= \frac{1}{\sqrt{2\pi}} x (e^{-\frac{x^2}{2}})|_{-\infty}^{\infty} + \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= 1
\end{aligned}$$

4 Algebra of Multiple Random Variables

Assume the value of a random variable X is the summation of n random variables, X_1, X_2, ... X_n. It can be denoted as

$$X = X_1 + X_2 + ... + X_n$$

The expectation of the random variable can be calculated as

$$E(X) = E(X_1 + X_2 + X_3 ... + X_n) = E(X_1) + E(X_2) + E(X_3) ... + E(X_n)$$

The variance of the random variable is

$$Var(X) = Var(X_1 + X_2 + X_3 ... + X_n) = Var(X_1) + Var(X_2) + Var(X_3) ... + Var(X_n) + \sum_{i \neq j} Cov(X_i, X_j)$$

When X_1, X_2, ... X_n are independent, we have $Cov(X_i, X_j) = 0$, therefore,

$$Var(X) = Var(X_1) + Var(X_2) + Var(X_3) ... + Var(X_n)$$

Consider a binomial distribution that is constructed from n Bernoulli distribution. Since each Bernoulli distribution has mean p and variance $p(1-p)$. The mean of the binomial distribution is the summation of the mean of n Bernoulli distributions i.e. $E(X) = np$. Similarly, the variance follows as $Var(X) = np(1-p)$.