Foundation of Analytics: Lecture 2

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CONTENT

- Introduction to Statistics: Random Variable
- Empirical View of Random Variable
- Common Probability Distributions
- Random Walk, i.i.d and Central Limit Theorem

Example 1: Roll a dice

There six possible outcome of rolling a dice i.e. "1", "2", "3", ... "6".

- If I roll a dice 60 times, how many times do you get "1"?
- What is the probability of getting "1"? 1/6?

Dice-Rolling: Expectation, Variance etc.

- Unique values: $1, 2, 3, 4, 5, 6$
- min:1; max:6
- expectation: $\frac{\sum x_i}{N}$?
- variance: ?

Random Variable

A **random variable** X can take different values with certain probability. To understand a random variable, we need to consider two things:

- The possible outcome value of an experiment: x
- The probability of an outcome is x: $P(x)$.

Discrete Random Variable

A discrete random variable X

- Can take k possible values x_1, x_2, x_3 ... x_k
- Each with probability of p_1, p_2, p_3 ... p_k. For simplicity, we denote the probabilities using a probability mass function

$$P(x_i) = p_i, i = 1, 2, 3, ...k$$

- The probabilities for all possible values sum up to be 1, i.e. $\sum_{i=1}^{k} p_i = 1$

Histogram/Distribution of Discrete Variables

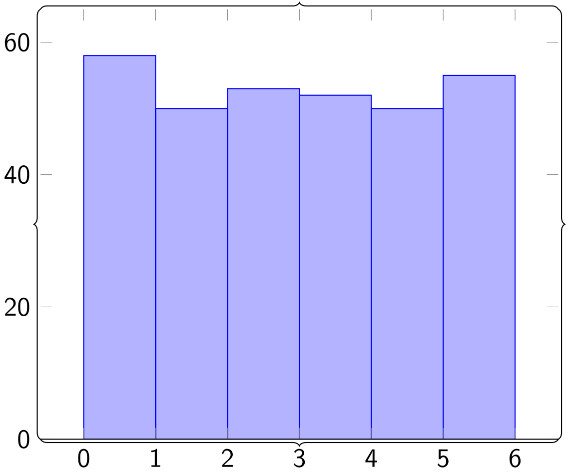
- For data of discrete values e.g. $x^T = [x^1, x^2, x^3 ..., x^n]$,
- Find the unique value of the data
- Count the number of data occured at each discrete value N_i. The total number of data points $N = \sum_i N_i$. The empirical probability mass function can be estimated by

$$P(x_i) = \frac{N_i}{N}$$

Dice-Rolling: Distribution

Given a series of data, $[1, 2, 1, 3, 4, 6, 6, 4, 5, 5, ...]$. What can you tell about the underlying story? Is it from a dice-rolling process?

Count the number of occurence for each value 1, 2, 3, 4, 5, 6



Discrete Random Variable: Function and Expectation

The expectation of a function, $g(X)$ is given by

$$E[g(X)] = \sum_{i=1}^{k} g(x_i)P(x_i)$$

Discrete Random Variable: Expectation, Variance and Moments

In special case when $g(X) = X^n$, we have the n^{th} raw moment of X

$$E[X^n] = \sum_{i=1}^{k} x_i^n P(x_i)$$

The expectation of X is the 1^{st} raw moment of X

$$\mu = E[X]$$

The variance of X is the 2^{nd} moment of X about the expectation

$$\sigma^2 = E[(X - \mu)^2]$$

Bernoulli Distribution

Consider a random variable X that can take value 1 with probability p and 0 with probability $1-p$.

$$P(x) = \begin{cases} p & \text{if } x = 1 \\ 1-p & \text{if } x = 0 \end{cases}$$

The expectation of X is

$$E[X] = p$$

The variance of X is

$$E[(X-\mu)^2] = p(1-p)$$

Joint Distribution and Algebra of Ramdom Variables

If we create a random variable from two random variables:

$$Z = X + Y$$

Distribution: $f(z)$
Expectation:

$$E[Z] = E[X+Y] = E[X] + E[Y] = E[Y] + E[X]$$

Variance:

$$Var[Z] = Var[X+Y] = Var[X] + 2Cov[X,Y] + Var[Y]$$

Covariance is defined as

$$Cov[X,Y] = E\left[(X - E[X])(Y - E[Y])\right]$$

Algebra of Multiple Random Varaibles

Expectation:

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E\left(X_i\right)$$

Variance:

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) + \sum_{i,j} Cov(X_i, X_j)$$

Binomial Distribution

Consider a random event whose outcome is the summation of n independent Bernoulli distribution. The distribution of the corresponding random variable X can be described as

$$P(x) = \binom{n}{k} p^x q^{n-x}, x = 0, 1, 2, ...n$$

The expectation of X is

$$E[X] = np$$

The variance of X is

$$E[(X - \mu)^2] = np(1 - p)$$

Poisson Distribution

Consider a random event, the probability of 1 occurence within a unit time is p. What's the probability distribution of events occurence within time interval of τ (e.g. No. of car accidents occurs in a day in MO). The distribution of the discrete random variable X is

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, ..$$

The expectation of X is

$$E[X] = \lambda$$

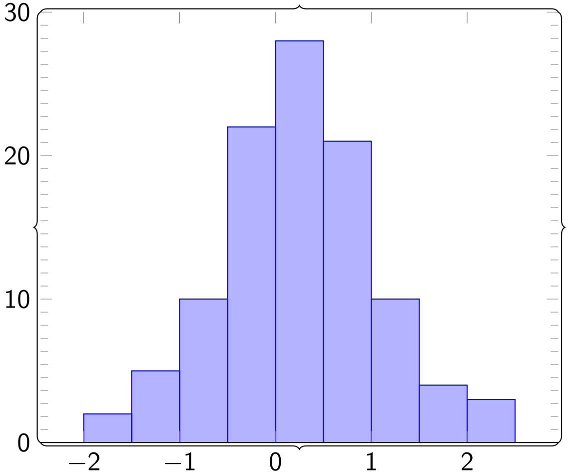
λ is the average number of events per interval, $\lambda = p\tau$
The variance of X is

$$E[(X - \mu)^2] = \lambda$$

Understand the Distribution of Continuous Data

How about real value data, $[-1.407, 0.412, -1.198, 1.552, ...]$?

Count the number of data points that falls into the intervals of $[-2, -1.5), [-1.5, -1.0), ...[0, 0.5), [0.5, 1)...$



Histogram/Empirical Distribution of Continous Variables

- For data of discrete values, count the number of data occured at each discrete value N_i. The total number of data points $N = \sum_i N_i$. The empirical probability mass function is given by

$$P(x_i) = \frac{N_i}{N}$$

- For data of continous values, define k equal-sized-bins (e.g. $[x_i - \Delta x, x_i + \Delta x)$, $i = 1, 2, 3, ...k)$. Count the number of data belong to each bin N_i, the total number of data points $N = \sum_i^k N_i$. The empirical probability distribution is given by

$$P(x_i - \Delta x \leq x < x_i + \Delta x) = \frac{N_i}{N}$$

Continuos Random Variable

A continous random variable X

- Can have a range of values e.g. $(-\infty, +\infty)$, $[0, 1)$, $[0, +\infty)$
- The probability that $a \leq x \leq b$ is defined as

$$P(a \leq x \leq b) = \int_a^b f(x)dx$$

  where $f(x)$ is the probability density function. Note: $f(x)$ is not probability
- The pdf $f(x)$ has to satisfy the following propery

$$P(-\infty \leq x \leq +\infty) = \int_{-\infty}^{+\infty} f(x)dx = 1$$

Continuos Random Variable: Function and Expectation

If we denote a function of a random variable as $g(X)$, the expectation of $g(X)$ is given by

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x)f(x)dx$$

Continuos Random Variable: Expectation, Variance and Moments

In a special case, when $g(X) = X^n$, the expectation of $g(X)$ is called the n^{th} raw moment of X

$$E[X^n] = \int_{-\infty}^{+\infty} x^n f(x) dx$$

The expectation of X is the 1^{st} raw moment of X

$$\mu = E[X]$$

The variance of X is the 2^{nd} moment of X about the expectation

$$\sigma^2 = E[(X - \mu)^2]$$

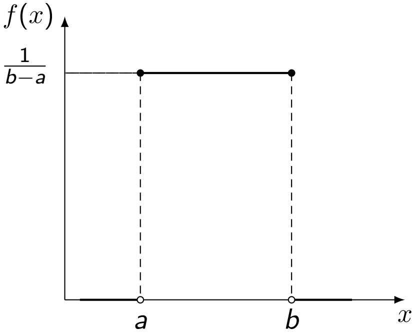
Calculate Expectation using Empirical Probability Distribution

$$
\begin{aligned}
\mu &= \sum_{i=1}^{k} x_i f(x_i - \Delta x \le x < x_i + \Delta x)(2\Delta x) \\
&= \sum_{i=1}^{k} x_i P(x_i - \Delta x \le x < x_i + \Delta x) \\
&= \frac{\sum_{i=1}^{k} x_i N_i}{N}
\end{aligned}
$$

Uniform Distribution

A uniform distribution is given by

$$f(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{if } x > b \end{cases}$$



Gaussian Distribution

A continuous random variable Z is called a standard normal if

$$f(Z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

The probability of $z \le z_0$ is given by

$$P(Z \le z_0) = \int_{-\infty}^{z_0} \frac{1}{\sqrt{2}e^{-z^2/2}} dz$$

Let $X = \mu + \sigma Z$. Then X is a normal distribution with parameters μ and σ^2. Its density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}.$$

The expectation of X: $E[X] = \mu$
The variance of X: $E[(X-\mu)^2] = \sigma^2$

Demo in Python