Basic Statistics and Random Process

Dihui Lai

dlai@wustl.edu

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CONTENT

Introduction to Statistics: Random Variable

Example 1: Roll a dice

There six possible outcome of rolling a dice i.e. "1", "2", "3", ... "6".

- If I roll a dice 60 times, how many times do you get "1"?
- What is the probability of getting "1"? 1/6?

Example 2: Life time of a light bulb

A light bulb can go broken while use. The longer it is used, the more likely the bulb will break.

- The value does not need to be an integer, it can be 120 hours, 120.513 hours
- What's the probability of a light bulb breaks at the 100th hour? $1/100$?

Random Variable

A **random variable** X can take different values with certain probability.
To understand a random variable, we need to consider two things:

- The possible outcome value of an experiment: x
- The probability that an outcome is x.

Discrete Random Variable

A discrete random variable X

- Can take k possible values x_1, x_2, x_3 ... x_k
- Each with probability of p_1, p_2, p_3 ... p_k. For simplicity, we denote the probabilities using a probability mass function

$$P(x) = p_x, x = x_1, x_2, x_3, ...$$

- The probabilities for all possible values sum up to be 1 i.e. $\sum_{i=1}^{k} p_i = 1$

Discrete Random Variable: Function and Expected Value

The expected value of a function, $g(X)$ is given by

$$E[g(X)] = \sum_{i=1}^{k} g(x_i)P(x_i)$$

Discrete Random Variable: Mean, Variance and Moments

In special case when $g(X) = X^n$, we have the n^{th} raw moment of X

$$E[X^n] = \sum_{i=1}^{k} x_i^n P(x_i)$$

The mean of X is the 1^{st} raw moment of X

$$\mu = E[X]$$

The variance of X is the 2^{nd} moment of X about the mean

$$\sigma^2 = E[(X - \mu)^2]$$

Continuos Random Variable

A continous random variable X

- Can have a range of values e.g. $(-\infty, +\infty)$, $[0, 1)$, $[0, +\infty)$
- The probability that $a \leq x \leq b$ is defined as

$$P(a \leq x \leq b) = \int_a^b f(x)dx$$

  where $f(x)$ is the probability density function. Note: $f(x)$ is not probability
- The pdf $f(x)$ has to satisfy the following propery

$$P(-\infty \leq x \leq +\infty) = \int_{-\infty}^{+\infty} f(x)dx = 1$$

Continuos Random Variable: Function and Expected Value

If we denote a function of a random variable as g(X), the expected value of $g(X)$ is given by

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x)f(x)dx$$

Continuos Random Variable: Mean, Variance and Moments

In a special case, when $g(X) = X^n$, the expected value of $g(X)$ is called the n^{th} raw moment of X

$$E[X^n] = \int_{-\infty}^{+\infty} x^n f(x) dx$$

The mean of X is the 1^{st} raw moment of X

$$\mu = E[X]$$

The variance of X is the 2^{nd} moment of X about the mean

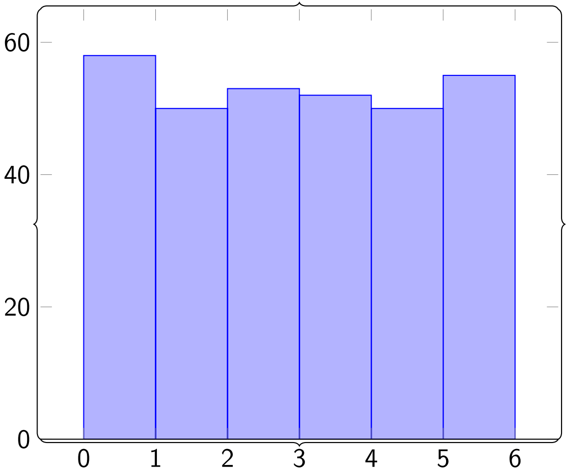
$$\sigma^2 = E[(X - \mu)^2]$$

Empirical View of Random Variable

Understand the Distribution of Discrete Data

Given a series of data, $[1, 2, 1, 3, 4, 6, 6, 4, 5, 5, ...]$. What can you tell about the underlying story? Is it from a dice-rolling process?

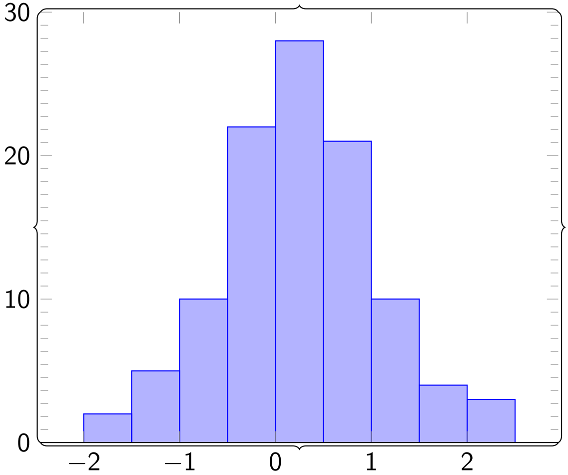
Count the number of occurence for each value 1, 2, 3, 4, 5, 6



Understand the Distribution of Continuous Data

How about real value data, $[-1.407, 0.412, -1.198, 1.552, ...]$?

Count the number of data points that falls into the intervals of $[-2, -1.5), [-1.5, -1.0), ...[0, 0.5), [0.5, 1)...$



Histogram and Empirical Probability Distribution

- For data of discrete values, count the number of data occured at each discrete value N_i. The total number of data points $N = \sum_i N_i$. The empirical probability mass function is given by

$$P(x_i) = \frac{N_i}{N}$$

- For data of continous values, define k equal-sized-bins (e.g. $[x_i - \Delta x, x_i + \Delta x)$, i=1, 2, 3, ...k). Count the number of data belong to each bin n_i, the total number of data points $n = \sum_i^k n_i$. The empirical probability distribution is given by

$$P(x_i - \Delta x \leq x < x_i + \Delta x) = \frac{n_i}{n}$$

Calculate Mean using Empirical Probability Distribution

- Discrete random variable:

$$\mu = \sum_{i=1}^{k} x_i P(x_i) = \frac{\sum_{i=1}^{k} x_i N_i}{N}$$

This last term of the equation is the same as the arithmic mean of the data points

Calculate Mean using Empirical Probability Distribution

- Continuous random variable:

$$\begin{aligned}\mu &= \sum_{i=1}^{k} x_i f(x_i - \Delta x \le x < x_i + \Delta x)(2\Delta x) \\ &= \sum_{i=1}^{k} x_i P(x_i - \Delta x \le x < x_i + \Delta x) \\ &= \frac{\sum_{i=1}^{k} x_i n_i}{n}\end{aligned}$$

This last term of the equation is the approximately arithmic mean of the data points because $x_i n_i \approx \sum_{d \in [x_i - \Delta x, x_i + \Delta x)} d$, $d \in [x_i - \Delta x, x_i + \Delta x)$ denotes the data points belong to the bin $[x_i - \Delta x, x_i + \Delta x)$

Histogram and Empirical Probability Distribution

Demo in Python

Statistical Description of Data

logitude	lattitude	houseAge	medianHouseValue	oceanProx
-122.23	37.88	41	452600.0	NEAR BAY
-122.22	37.86	21	358500.0	NEAR BAY
-122.24	37.85	52	352100.0	NEAR BAY
...	...	...		

Is "medianHouseValue" a random variable?

Common Probability Distribution

Bernoulli Distribution

Consider a random variable X that can take value 1 with probability p and 0 with probability 1-p.

$$P(x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \end{cases}$$

The mean of X is

$$E[X] = p$$

The variance of X is also

$$E[(X - \mu)^2] = p(1 - p)$$

Binomial Distribution

Consider a random event that is composed of n independent experiments, whose outcome could either be succuess (1) or failure (0). The probability of success is p and faliure $1-p$. The corresponding random variable X can be described as

$$P(x) = \binom{n}{k} p^x q^{n-x}, x = 0, 1, 2, ...n$$

The mean of X is

$$E[X] = np$$

The variance of X is also

$$E[(X-\mu)^2] = np(1-p)$$

Poisson Distribution

Consider a random event, the number of occurence within a given interval can be x = 0, 1, 2, 3 ... (e.g. No. of car accidents occurs in a day in MO). The distribution of the discrete random variable X can be described as

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, ..$$

The mean of X is

$$E[X] = \lambda$$

The variance of X is also

$$E[(X-\mu)^2] = \lambda$$

Poisson Distribution

Consider a random event, the number of occurence within a given interval can be x = 0, 1, 2, 3 ... (e.g. No. of car accidents occurs in a day in MO). The distribution of the discrete random variable X can be described as

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, ..$$

The mean of X is

$$E[X] = \lambda$$

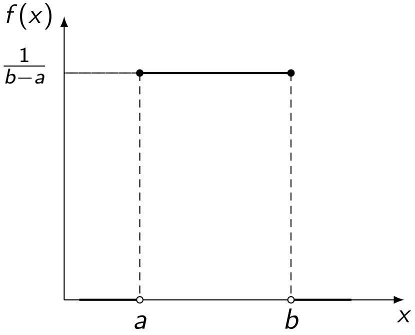
The variance of X is also

$$E[(X - \mu)^2] = \lambda$$

Uniform Distribution

A uniform distribution is given by

$$f(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{if } x > b \end{cases}$$



Gaussian Distribution

A continuous random variable Z is called a standard normal if

$$f(Z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

The probability of $z \leq z_0$ is given by

$$P(Z \leq z_0) = \int_{-\infty}^{z_0} \frac{1}{\sqrt{2}} e^{-z^2/2}\, dz$$

Let $X = \mu + \sigma Z$. Then X is a normal distribution with parameters μ and σ^2. Its density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}.$$

The mean of X: $E[X] = \mu$
The variance of X: $E[(X-\mu)^2] = \sigma^2$

Random Walk, i.i.d, Central Limit Theorem

Random Walk

One-dimension random walk: a random process

- Starting at 0
- The movement at each step could be either $+1$ or -1, of equal probability

Random Walk

Define a random variable B that can take value $+1$ or -1 and have the following random distribution

$$P(b) = \begin{cases} 0.5 & \text{if } b = 1 \\ 0.5 & \text{if } b = -1 \end{cases}$$

The position of a random walk at t^{th} step is a random variable given by

$$Z_t = \sum_{i=1}^{t} B_i$$

Sum of Independent and Identically Distributed (i.i.d)

Suppose we have n independent random variable X_1, X_2, X_3 ... X_n, each have the same probability distribution. We say X_1, X_2, X_3 ... X_n are independent and identically distributed (i.i.d). The sum of i.i.d random variables given by $Z(n) = \sum_{i=1}^{n} X_i$ has the following properties

- $E[Z(n)] = n\mu$;
- $Var[Z(n)] = n\sigma^2$;
- When n is large, distribution of Z(n)/n is close to the normal distribution of mean μ and variance σ^2/n

Here, μ and σ^2 are the mean and variance of X_i, respectively.

Random Walk

Since Z_t is the sum of i.i.d, of mean 0 and variance 1, when t is large, Z_t becomes a normal distribution of mean 0 and variance t (why?)