Lecture 08

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Multi-class Classification

Multinomial Distribution

Multinomial distribution: there could be c outcome of an experiment, each of probability p_1, p_2, p_3, ..., p_k and $\sum_{k=1}^{c} p_k = 1$. If one perform M experiments, the probability of getting m_1, m_2, m_3, ... m_c of each out come can be described as

$$f(m_1, m_2, m_3, ...m_c) = \frac{M!}{m_1!m_2!...m_c!} p_1^{m_1} p_2^{m_2} ... p_c^{m_c}$$

Multi-class Classification

If a target variable contains more than 2 types of out come, logistic regression can not handle it easily. We need to build a **multi-class classification** model

Assume there can be C possible outcome in your target variable, we can use one-hot encoding and denote the out come of $i^t h$ data point $\vec{y}^i = [y_1^i, y_2^i, ...y_C^i]$, only one element out of C is 1.

Multi-class Classification: Likelihood Function

For each data point i, the likelihood function can be constructed as $L^i = \prod_{k=1}^{C} {p_k^i}^{y_k^i}$ i.e. multinomial distribution of (M=1). For an outcome of class c, we have $y_c = 1$ and all other elements of y are 0. $y_1 = 0$, $y_2 = 0$, ...$y_C = 0$. The log-likelihood of the dat point is $\ell^i = \sum_{k=1}^{C} y_k^i \log(p_k^i)$

The log-likelihood function (a.k.a log-loss) is

$$\ell = \log(L) = \sum_{i=1}^{n} \sum_{k=1}^{C} y_k^i \log(p_k^i)$$

Multi-Class classification: Softmax

To estimate p_k^i, we use softmax function of $\vec{x} \cdot \vec{\beta}_k$

$$p_k^i = \frac{e^{\vec{x}^i \cdot \vec{\beta}_k}}{\sum\limits_{k=1}^{C} e^{\vec{x}^i \cdot \vec{\beta}_k}}$$

Multi-Class classification: Optimization

To find the $\vec{\beta}_k$, we need to solve

$$\frac{\partial \ell}{\partial \beta_{kj}} = 0$$

Python method: *"statsmodels.api.MNLogit"*

Naive Bayes Classifier

Bayes' Theorem

Bayes' Theorem: given two random variables X and Y, the conditional probability of X given Y is expressed as:

$$P(Y \mid X) = \frac{P(X \mid Y)P(Y)}{P(X)}$$

Useful terminologies to interpret the equations: $P(Y)$ is called prior, which is the belif in Y without any other knowledge. $P(Y \mid X)$ is the posterior taking into consideration of X. $P(X \mid Y)$ is the likelihood.

In a discrete case, the probability distribution of X can be calculated as $P(X) = \sum_i P(X \mid Y_i)P(Y_i)$

Baye's Theorem Example

Rain in California

You are planning a trip to california tomorrow. Unfortunately, the weatherman has predicted rain for tomorrow. You know in southern california, it only rains 5 days each year and there is a chance the weather man makes false predictions. You searched on line and find that when it rains, the weatherman correctly forecasts rain of 90% of the time. When it doesn't rain, he incorrectly forecasts rain 10% of the time. What is the probability that it will rain tomorrow.

Baye's Theorem Example

Solution: Denote the event that the weatherman forcast a raining day as F. The probability of rain given weatherman's forcast is

$$P(1 \mid F) = \frac{P(1)P(F \mid 1)}{P(F)}$$

.

Given that we have $P(F \mid 1) = 0.9$, $P(F \mid 0) = 0.1$, $P(1) = \frac{5}{365}$ and $P(0) = 1 - P(1) = \frac{360}{365}$. $P(F) = P(F|1)P(1) + P(F|0)P(0)$, Therefore,

$$P(1 \mid F) = \frac{P(1)P(F \mid 1)}{P(F)} = \frac{\frac{5}{365} \cdot 0.9}{0.1109} = 0.111$$

Joint Probability Distribution and Chain Rule

- The joint probability distrution of two events can be discribed as

$$P(A, B) = P(A \mid B)P(B) = P(B \mid A)P(A)$$

  If A and B are two indpendent events, then we have $P(B) = P(B \mid A)$ and $P(A) = P(A \mid B)$

- **Chain Rule**: Considering n random events X_1, X_2, X_3 ... X_n, their joint probability distribution can be described as

$$\begin{aligned}
&P(X_n, \ldots, X_1) \\
&= P(X_n|X_{n-1}, \ldots, X_1)P(X_{n-1}, \ldots, X_1) \\
&= P(X_n|X_{n-1}, \ldots, X_1)P(X_{n-1}|X_{n-2}, \ldots, X_1)P(X_{n-2}, \ldots, X_1) \\
&= \ldots
\end{aligned}$$

Naive Bayes Classifier

The joint probability distribution of predictor $\vec{x}$ and target variable y can be written as

$$
\begin{aligned}
&p(x_1, x_2, ...x_n, y) \\
&= p(x_1|x_2, x_3, ..., y)p(x_2, x_3, ..., y) \\
&= p(x_1|x_2, x_3, ..., y)p(x_2|x_3, x_4, ..., y)p(x_3, x_4, ..., y) \\
&= p(x_1|x_2, x_3, ..., y)p(x_2|x_3, x_4, ..., y)...p(x_{n-1}|x_n, y)p(x_n|y)p(y)
\end{aligned}
$$

Assuming features are indepedent of each other but only dpendent on the target variable, then we have

$$p(x_1, x_2, ...x_n, y) = p(y)\prod_{i=1}^{N} p(x_i|y)$$

Naive Bayes Classifier

Using Bayes' theorem, we can get the conditional probability distribution of the target variable as

$$p(Y|X) = \frac{p(X, Y)}{p(X)}$$

Therefore, we have

$$p(Y|X) = \frac{p(y) \prod_{i=1}^{N} p(x_i|y)}{\sum_{y} p(y) \prod_{i=1}^{N} p(x_i|y)}$$

The denominator is constant if the features are know. $p(y)$ and $p(x_i \mid y)$ can be calculated from the data. We need to find the y that maximize the $p(Y \mid X)$, i.e.

$$\hat{y} = \operatorname*{argmax}_{y} p(y) \prod_{i=1}^{N} p(x_i|y)$$

Nature Language Process

Word Semantics and Representations

- Homonymous: a word can have multiple definitions e.g. mouse could mean small rodents or it could mean computer devices.
- Synonyms/antonym (words' relations): couch/sofa, vomit/throw up, filbert/hazelnut; long/short, big/little
- Word sentiments
- Can we represent a word using vectors and quantify those measures?

Word Vector Representations

Term-term matrix or word-word matrix: count the number of times a word occurs in a context window around the target word (e.g. ± 7)
sugar, a sliced lemon, a tablespoonful of, **apricot** jam, a pinch each of,

	aardvark	...	computer	data	pinch	result	sugar	..
apricot	0	...	0	0	1	0	1	..
pineapple	0	...	0	0	1	0	1	..
digital	0	...	2	1	0	1	0	..
information	0	...	1	6	0	4	0	..

It can be inferred from the word-word matrxi that apricot and pineapple are more simliar to each other.

Consine Similarity

The similarity of two words could be measured by dot-products of their vector representation

$$\vec{v} \cdot \vec{w} = \sum_{i=1}^{N} v_i w_i$$

The dot-product favors vectors of higher frequency to normalize the similarity without considering word frequency, we use cosine similarity meature

$$cosine(\vec{v}, \vec{w}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|} = \frac{\sum_{i=1}^{N} v_i w_i}{\sqrt{\sum_1^N v_i^2}\sqrt{\sum_1^N w_i^2}}$$

N-gram Language Models

- Models that assign probabilities to sequences of words are called language models or LM.
- An n-gram is a sequence of N words e.g. 2-gram (or bigram) "Good Morning", 3-gram "Turn it on"
- N-gram lanuage models estimate the probability of the last word of an n-gram given the previous words

N-gram Language Models

LM: What is the probability of having a sentence that consists a sequence of words: w_1, w_2, w_3 ... w_N, i.e. $P(w_1, w_2, w_3 ... w_N)$.
Recall the chain rule:

$$P(w_1, w_2, w_3 ... w_N)$$
$$= P(w_1)P(w_2|w_1)P(w_3|w_1, w_2)P(w_4|w_1, w_2, w_3)...P(w_N|w_1, w_2, ...w_{N-1})$$

In the case of bigram, we assume $P(w_N|w_1, ..., w_{N-1}) = P(w_N|w_{N-1})$, since the word is only dependent on the previous word, it is also called Markov assumption. In general case of an n-gram, we assume

$$P(w_N|w_1, w_2, ...w_{N-1}) = P(w_N|w_{N-1}, w_{N-2}, ...w_{N-n+1})$$

MLE Estimation for bigram

In the case of bigram, the MLE estimation can be formulated as

$$P(w_N|w_{N-1}) = \frac{C(w_{N-1}w_N)}{\sum_w C(w_{N-1}w)} = \frac{C(w_{N-1}w_N)}{C(w_{N-1})}$$

Here, C is the count of the words' occurence

Example: MLE Estimation for bigram

Estimate the bigram for the following corpus, here $\langle s \rangle$ and $\langle /s \rangle$ are introduced as the symbols that represents the begining and end of a setence.
$\langle s \rangle$ I am Sam $\langle /s \rangle$
$\langle s \rangle$ Sam I am $\langle /s \rangle$
$\langle s \rangle$ I do not like green eggs and ham $\langle /s \rangle$
We begin buy counting the words occurence and have $C(I) = 3$, $C(Sam) = 2$, $C(\langle /s \rangle) = 3$, $C(\langle s \rangle) = 3 \ldots C(\langle s \rangle I) = 2$, $C(\langle s \rangle Sam) = 1$

So we have $P(I|\langle s \rangle) = \frac{2}{3}$, $P(Sam|\langle s \rangle) = \frac{1}{3}$, $P(do|I) = \frac{1}{3}$, $P(am|I) = \frac{2}{3}$, $P(Sam|am) = \frac{1}{2}$, $P(\langle /s \rangle|Sam) = \frac{1}{2}$

The in-sample probability of $P(\langle s \rangle I\ am\ Sam \langle /s \rangle) = P(I|\langle s \rangle)P(am|I)P(Sam|am)P(\langle /s \rangle|Sam) = 2/3x2/3x1/2x1/2$

Evaluating Language Models

How do we compare two LM?

- A test data/hold out data set can be used to evaluate a LM. Apply the estiamated conditional probability to the test data set and compare the resulting probability.
- Perplexity is used instead of the raw probability.

$$
\begin{aligned}
PP(W) &= P(w_1, w_2, ...w_N)^{-\frac{1}{N}} \\
&= \sqrt[N]{\frac{1}{P(w_1, w_2, ...w_N)}}
\end{aligned}
$$

- Maximize probability is equivalent to minimize perplexity

Smoothing

What do we do with words that appear in a test set with an unseen context for example, $P(John|am) = 0$ because "John" has never appear in training text. We end up getting $P(w_1, w_2, ...w_N) = 0$. One possible solution is smoothing

- Laplace smoothing: increase the bigram count by 1, so what was counted 0 now becomes 1

$$P(w_N|w_{N-1}) = \frac{C(w_{N-1}w_N) + 1}{\sum_w C(w_{N-1}w) + 1} = \frac{C(w_{N-1}w_N) + 1}{C(w_{N-1}) + V}$$

  The denominator is adjusted by the vocabulary size of V

- Add-k smoothing, increase the count by a fraction of k (0.5, 0.8 ...) and we have

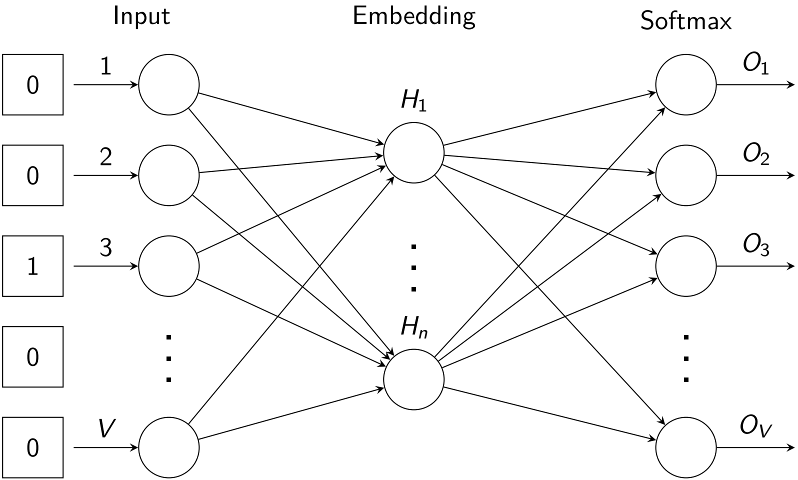
$$P(w_N|w_{N-1}) = \frac{C(w_{N-1}w_N) + k}{\sum_w C(w_{N-1}w) + k} = \frac{C(w_{N-1}w_N) + k}{C(w_{N-1}) + kV}$$

Unknown Words

What do we do if a word in the test data is not in the vocabulary i.e. out of vocabulary (OOV)

- Choose a fixed vocabulary. If a word in the training set is OOV, convert it to $\langle UNK \rangle$. Estimate the probability of $\langle UNK \rangle$ as a regular word.
- replace low frequency word in the training dataset by $\langle UNK \rangle$. Treating $\langle UNK \rangle$ as regular word.

Neural Network Based Language Model: CBOW/Skip-Gram Model



A vocabulary is fed into the neural network using one-hot encoding methods. For a vocabulary of size V, the input vector is of size 1xV

Neural Network Based Language Model: Architecture

- The input variable is a one-hot encoding vector. If the vocabulary is of size V, an input vector is has V components $\vec{x} = [0, 0, 0...1, ...0]$
- The hidden layer has *n* neurons. The input weights matrix W is of size $V \times n$
- The output layer weights W' matrix is of size $n \times V$
- CBOW: take 2*m* words (i.e. w_{c-m}, ...w_{c-1}, w_{c+1}, w_{c+m}) around the center word w_c as input w_c is the target.
- Skip-gram: take the center word w_c as the input and the 2*m* words (i.e. w_{c-m}, ...w_{c-1}, w_{c+1}, w_{c+m}) around it as the target.
  Reference: `https://arxiv.org/pdf/1301.3781.pdf`

Neural Network Based Language Model: Word Embedding

The word representation/embedding can be calculated as

$$w_i = x_i W$$

x_i is the $i^t h$ word in the dictionary, w_i is the $i^t h$ row in the input matrix W