A double counting argument on the hypercube graph

Constantinos Kourouzides

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Abstract

In this note we count in 2 different ways the sum of the edges of all j-dimensional subcubes of Q_m for all j. The resulting equation leads to a new interpretation of the elements in each row of Pascal's triangle.

Consider how many times a particular edge in Q_m contributes to the sum of all edges of all j-dimensional subcubes of Q_m for all j. Let us fix any edge $uv \in E(Q_m)$. We propose the following equation:

$$\sum_{j=1}^{m} \underbrace{\binom{m}{j} 2^{m-j}}_{\#\text{ of } j\text{-dimensional subcubes in } Q_m} \cdot \underbrace{j\, 2^{j-1}}_{\#\text{ of edges in } Q_j} = m\, 2^{2m-2}$$

$$= m\, 2^{m-1}\Big(1 + \sum_{j\geq 2} \#\text{ of } j\text{-dimensional subcubes that contain the edge } uv\Big).$$

The first line of the above equation computes the required sum in a straightforward manner, by using the well-known fact that Q_m contains $\binom{m}{j} 2^{m-j}$, j-dimensional subcubes. The following line uses the fact that any edge in Q_m belongs to the same number of subcubes of any dimension. This "symmetric argument" is the cornerstone of our result.

The task now is to determine the number of subcubes that saturate uv, for different values of j. If so, this will yield the equation:

$$2^{m-1} = 1 + \sum_{j\geq 2} \#\, Q_j \supset uv \mid Q_j \subseteq Q_m.$$

For fixed m, the terms in the RHS of the above equation correspond, and are in the same order as those elements that constitute the (m-1)'th row of Pascal's triangle. That is to say, for a fixed edge, the number of j-dimensional subcubes in Q_m that saturate it, is given by the binomial coefficient $\binom{m-1}{j-1}$. Indeed, if we fix an edge in Q_m, by selecting any $j-1$ from the remaining $m-1$ coordinates, we specify a j-dimensional subcube, which contains that edge, induced by a total of j coordinates which we set to vary. $\square$

For example, if we fix an edge in Q_4, we know that there is one edge that contains it, 3 Q_2's, 3 Q_3's and the entire graph Q_4.