The Idea of Approximation: From ancient India to Modern World

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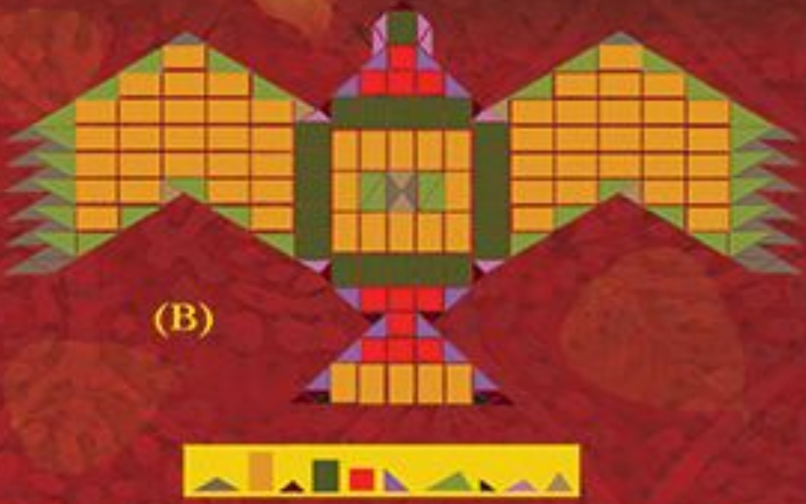
SULBA SUTRA

The Sulba Sutras, one of the ancient texts contain procedures about the building of ceremonial platforms.

(A)



(B)

Sulba Sutra

- Indian mathematics finds its early beginnings in the famous Sulba Sutras of Vaidika literature.
- Written to facilitate accurate construction of various types of sacrificial altars for the Vaidika ritual, these sutras lay down the basic geometrical properties of plane figures like the triangle, rectangle, rhombus and circle.

Vedanga Jyotisha

- Basic categories of the Indian astronomical tradition were similarly established in the various Vedanga Jyotisha texts.
- Vedanga Jyotisha is described as one of the six branches of knowledge called Shad Vedangas. Vedangas are the ancillary subjects of the Vedas and help in understanding the processes associated with the life in Vedic times.
- The purpose of the Jyotisha was to fix suitable times for performing different Yajnas and this involves knowledge of astronomy.

"Vedanga Jyotisha is not astrology"

Indian astronomy

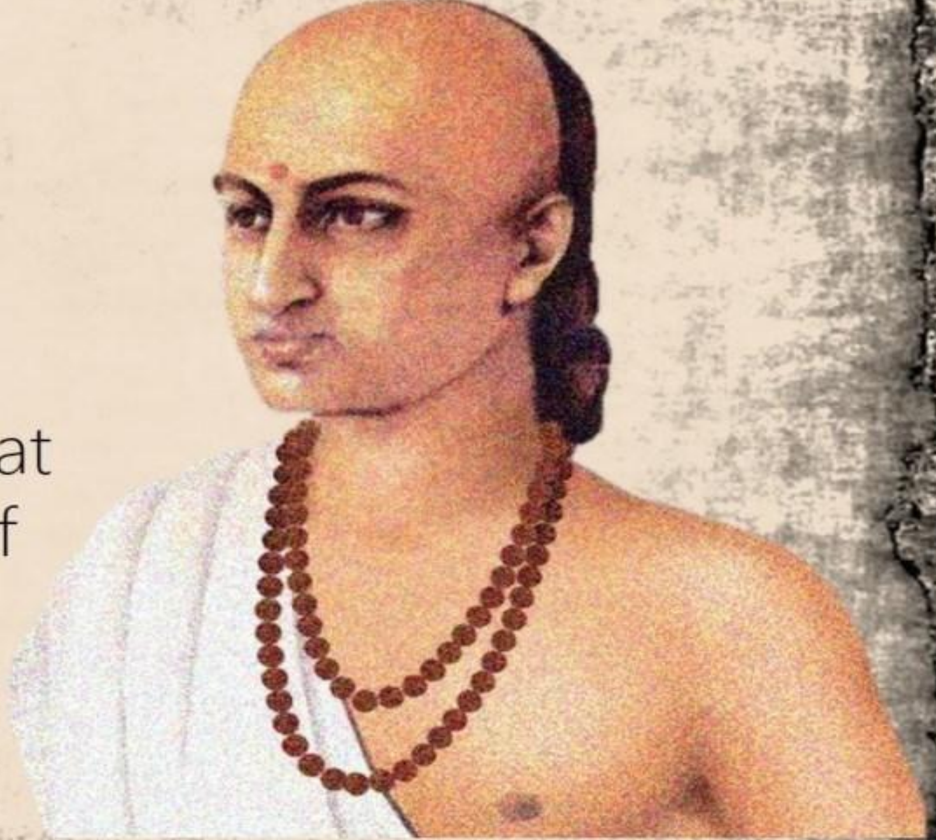
- Rigorous systematisation of India astronomy begins with the Siddhantas, especially the Brahma or Pitamaha Siddhanta and Surya Siddhanta.

- Unfortunately, no authentic original versions of these Sidhanta texts are available.

- The earliest exposition of the Siddhantic tradition is found in the work of Aryabhata (b.476 AD).

π

The great Indian mathematician Aryabhat calculated the value of pi at **3.1416**



Aryabhata (476-529)

Approximation !!!

- One of Aryabhata's contribution in trigonometry was that, he gave methods of calculating their **approximate** numerical values.
- Bhaskara I (c. 600–680) expanded the work of Aryabhata in his books titled *Mahabhaskariya*, *Aryabhatiya-bhashya* and *Laghu-bhaskariya* and gave a rational **approximation** of the sine function.

Some facts

Aryabhata also demonstrated solutions to simultaneous quadratic equations, and produced an **approximation** for the value of π equivalent to 3.1416, correct to four decimal places. He used this to estimate the circumference of the Earth, arriving at a figure of **24,835 miles, only 70 miles off its true value.** But, perhaps even more astonishing, he seems to have been aware that π is an irrational number, and that any calculation with pi can only be an **approximation**, something not proved in Europe until 1761.

A method of Successive approximations

HISTORIA MATHEMATICA **23** (1996), 246–256
ARTICLE NO. 0026

An Example of the Secant Method of Iterative Approximation in a Fifteenth-Century Sanskrit Text

Kim Plofker

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Mathematical approximation by iterative algorithms is well attested in Sanskrit astronomical texts, but its use has not been studied systematically. In his 14th-century supercommentary on Govindasvāmin's commentary on Bhāskara I's *Mahābhāskarīya*, Parameśvara, a student of the renowned Kerala astronomer Mādhava, presents a one-point iterative technique for calculating the Sine of a given angle, as well as a modification of this technique that involves a two-point algorithm essentially identical to the modern secant method. This paper presents a mathematical and historical interpretation of his remarks.

L'approximation mathématique par les procédés d'itération se manifeste fréquemment dans les textes sanscrites astronomiques, mais elle n'est pas beaucoup étudiée. Dans son surcommentaire sur le commentaire de Govindasvāmin sur le *Mahābhāskarīya* de Bhaskara I, Parameśvara, un élève de Mādhava, l'astronome renommé de Kerala, donne un calcul itératif en virgule fixe pour le calcul du Sinus d'un angle donné, aussi bien qu'une modification de cette méthode dans laquelle paraît une technique virtuellement identique à la forme discrète de la méthode de Newton–Raphson. L'article suivant présente une interpretation mathématique et historique de ses rémarques.

Die mathematische Annäherung bei den Iterationsverfahren ist in den sanskritischen astro-

Source: Historia Mathematica 28(2001), 283–295 doi:10.1006/hmat.2001.2331, available online at ttp://www.idealibrary.com on

Approximation algorithm

- Iterative techniques for solving equations have been common in Indian astronomical calculations at least since the use in the Paitāmahasiddhānta (fifth century) of such a technique for correcting planetary longitudes.

- Parameśvara gives a more efficient **approximation** algorithm equivalent to what is now called the “secant method,” in which the root of some function f(x) is approximated by

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}.$$

Value of Pi

- **Π** = 3 Baudhyana Shulba Sutra - (The Oldest **approximate** Value of **Π**)
- **Π** = 3 Mahabharata (Bhishmaparva, XII: 44)
- **Π** to be 18 * (3 – 2 √2) = 3.088 (other Shulba Sutras)
- **Π** to be 28/5= 3.125 (Manava Shulba Sutra)
- **Π** = √10 (Ancient Jaina School)
- **Π** = 62832/20000 = 3.1416. This was astonishingly correct to 4 decimal places Aryabhatta (476 AD)
- Madhava series (also Leibniz series) of **Π**/4

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots=\frac{\pi}{4}.$$

Square root of 2

- Rational **approximation** of root 2 occurs in Baudhayana, Apastamba and Katyayana Sulva Sutras

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34} = 1.4142156\ldots$$

- A commentator by name Rama who lived in the middle of the 15th century A.D., in a place called Naimis .a near modern Lucknow, improved upon this **approximation** and obtained

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{(3)(4)} - \frac{1}{(3)(4)(34)} - \frac{1}{((3)(4)(34)(33)} + \frac{1}{(3)(4)(34)(34)}$$

which gives a better **approximation**, correct up to seven decimal places

Approximation formula

The formula for calculating sin of an angle is given in verses 17 – 19, Chapter VII, Mahabhaskariya of Bhaskara I.

$$\sin x^{\circ} \approx \frac{4x(180-x)}{40500 - x(180-x)}$$

Bhaskara I's sine **approximation** formula can be expressed using the radian measure of angles as follows

$$\sin x \approx \frac{16x(\pi - x)}{5\pi^2 - 4x(\pi - x)}$$

Indian style of Mathematics

- The reason for this spectacular success of the Indian mathematicians lies in the explicitly algorithmic and computational nature of Indian mathematics.

- Indian mathematicians were not trying to discover the ultimate axiomatic truths in mathematics; they were interested in finding methods of solving specific problems that arose in the astronomical and other contexts.

- Therefore, Indian mathematicians were prepared to work with simple algorithms that may give only **approximate** solutions to the problem at hand; and they evolved sophisticated theories of error and recursive procedures to keep the approximations in check.

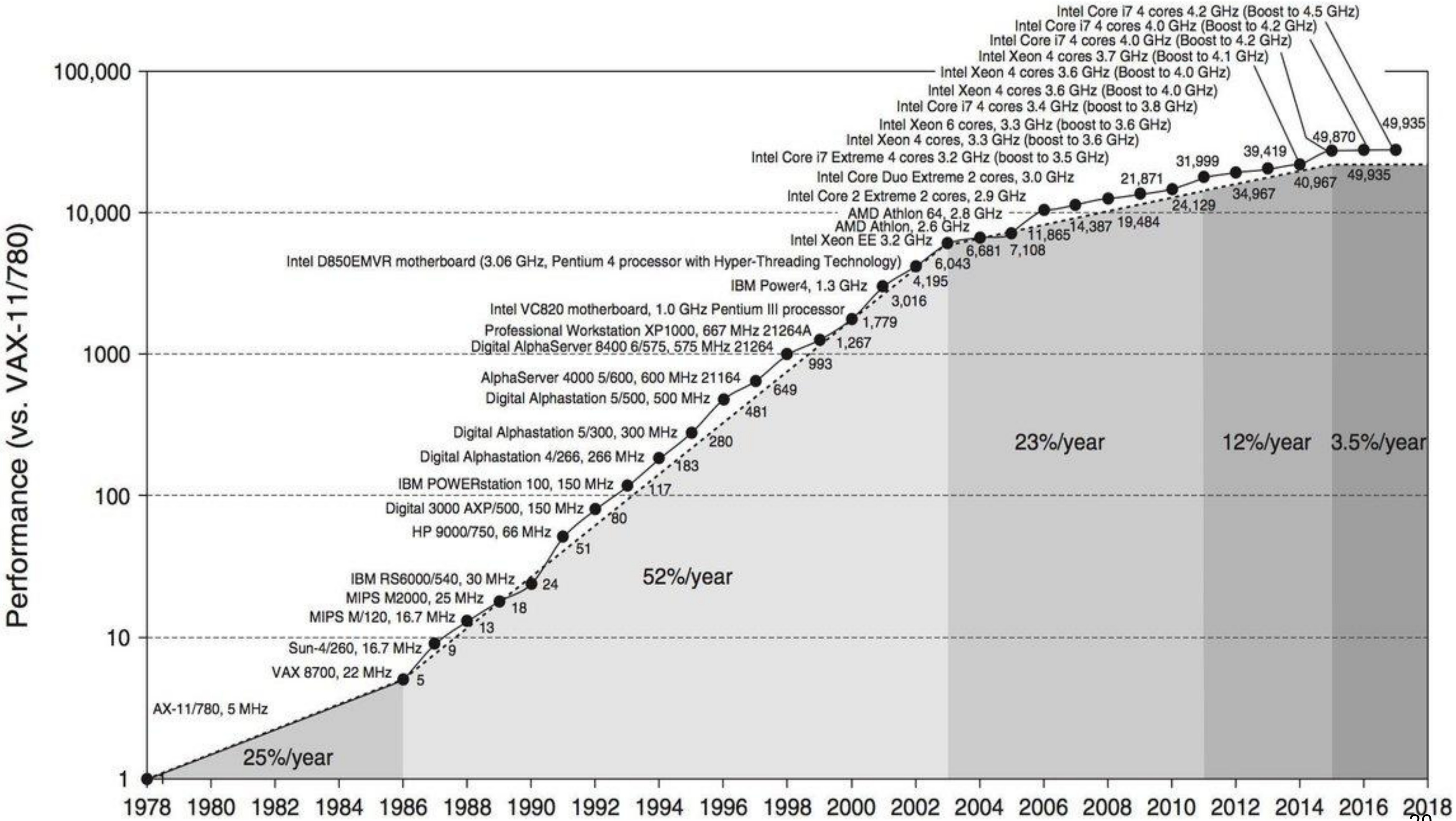
What is so special about approximation?

- Errors
  - But with Faster Calculation
  - And Less Effort
- Can we use approximation for good in today's computing system?
  - A definite YES.
- So why we are talking about approximation Now?
  - Moore’s Law
  - Dennard Scaling

Disclaimer

- The Idea of approximation existed in almost all the civilization in the world.
  - But the Indians exploited it well.
- And the topic I will discuss later in the presentation has nothing to do with India.
  - It is a new emerging paradigm for computing systems. And Research communities across the globe working on it.

Source: https://www.flickr.com/photos/oninnovation/4335512856


Performance (vs. VAX-11/780)
100,000
10,000
1000
100
10
1
1978 1980 1982 1984 1986 1988 1990 1992 1994 1996 1998 2000 2002 2004 2006 2008 2010 2012 2014 2016 2018
25%/year
52%/year
23%/year
12%/year
3.5%/year
AX-11/780, 5 MHz
VAX 8700, 22 MHz
5
Sun-4/260, 16.7 MHz
9
MIPS M/120, 16.7 MHz
13
MIPS M2000, 25 MHz
18
IBM RS6000/540, 30 MHz
24
HP 9000/750, 66 MHz
51
Digital 3000 AXP/500, 150 MHz
80
IBM POWERstation 100, 150 MHz
117
Digital Alphastation 4/266, 266 MHz
183
Digital Alphastation 5/300, 300 MHz
280
Digital Alphastation 5/500, 500 MHz
481
AlphaServer 4000 5/600, 600 MHz 21164
649
Digital AlphaServer 8400 6/575, 575 MHz 21264
993
Professional Workstation XP1000, 667 MHz 21264A
1,267
Intel VC820 motherboard, 1.0 GHz Pentium III processor
1,779
IBM Power4, 1.3 GHz
3,016
Intel D850EMVR motherboard (3.06 GHz, Pentium 4 processor with Hyper-Threading Technology)
4,195
Intel Xeon EE 3.2 GHz
6,043
AMD Athlon, 2.6 GHz
6,681
AMD Athlon 64, 2.8 GHz
7,108
Intel Core 2 Extreme 2 cores, 2.9 GHz
11,865
Intel Core Duo Extreme 2 cores, 3.0 GHz
14,387
Intel Core i7 Extreme 4 cores 3.2 GHz (boost to 3.5 GHz)
19,484
Intel Xeon 4 cores, 3.3 GHz (boost to 3.6 GHz)
21,871
Intel Xeon 6 cores, 3.3 GHz (boost to 3.6 GHz)
24,129
Intel Core i7 4 cores 3.4 GHz (boost to 3.8 GHz)
31,999
Intel Xeon 4 cores 3.6 GHz (Boost to 4.0 GHz)
34,967
Intel Xeon 4 cores 3.6 GHz (Boost to 4.0 GHz)
39,419
Intel Xeon 4 cores 3.7 GHz (Boost to 4.1 GHz)
40,967
Intel Core i7 4 cores 4.0 GHz (Boost to 4.2 GHz)
49,870
Intel Core i7 4 cores 4.0 GHz (Boost to 4.2 GHz)
49,935
Intel Core i7 4 cores 4.2 GHz (Boost to 4.5 GHz)
49,935

INTEL INNOVATION LEADERSHIP

Intel 90nm 65nm 45nm 32nm 22nm 14nm 10nm

Year '03 '04 '05 '06 '07 '08 '09 '10 '11 '12 '13 '14 '15 '16 '17 '18 '19 '20

Others 90nm 65nm 40nm 28nm 20nm 16nm 10nm

Intel leads the industry by at least 3 years in introducing major process innovations

Source: Intel

Approximate Computing

Error: Essential Part of the Design Process

John von Neumann's View on Error (1952):

"Our present treatment of error is unsatisfactory and ad hoc. … Error is viewed (in this work), therefore, not as an extraneous and misdirected or misdirecting accident, but as an essential part of the process under consideration …" [1]

Source : https://ece.uwaterloo.ca/~ssundara/courses/prob_logics.pdf

What is Approximate Computing?

- Relax the accuracy of computation
  - Improve Performance
  - Reduce Energy
  - Reduce Power
  - Reduce Delay

Exact

Approximate

```
[[164, 164, 159, ..., 171, 155, 129],
 [163, 162, 161, ..., 172, 154, 130],
 [163, 160, 162, ..., 168, 157, 129],
 ...,
 [ 44,  40,  49, ..., 104, 101,  99],
 [ 43,  43,  53, ..., 104, 105, 109],
 [ 42,  41,  56, ..., 105, 104, 109]]
```

Exact

Approximate

```
[[162, 162, 162, ..., 170, 155, 128],
 [162, 162, 162, ..., 170, 155, 128],
 [162, 162, 162, ..., 170, 155, 128],
 ...,
 [ 43,  43,  50, ..., 104, 100,  98],
 [ 44,  44,  55, ..., 104, 105, 108],
 [ 44,  44,  55, ..., 104, 105, 108]],
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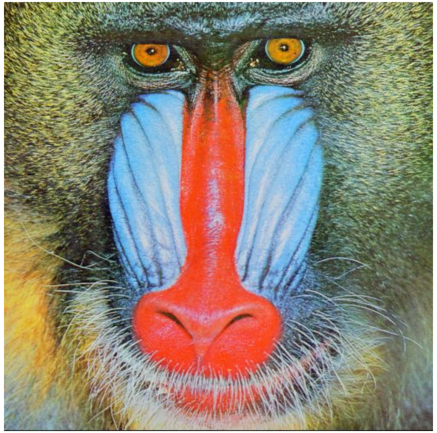
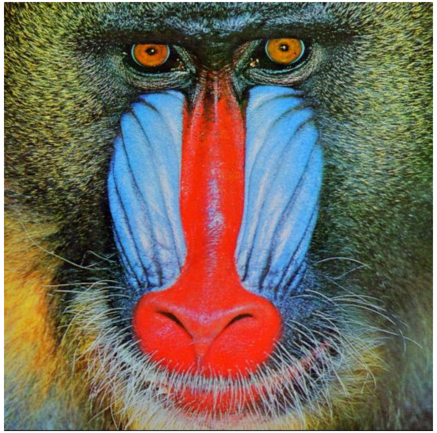
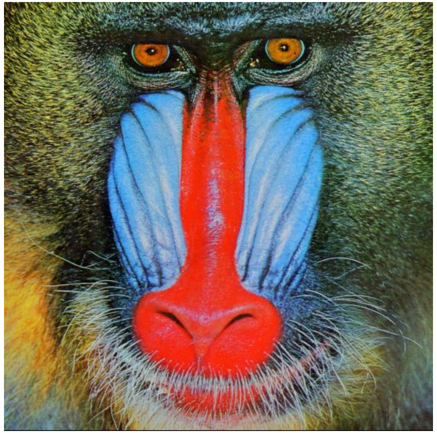
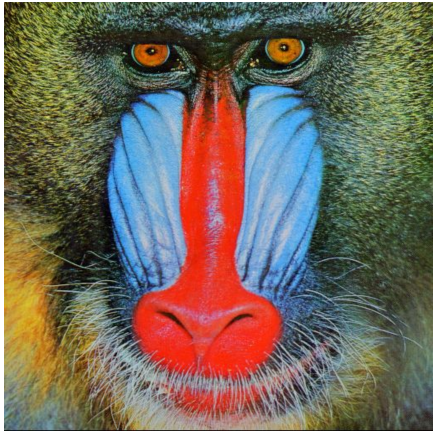
Power Law Transformation Contrast Enhancement

$$EnhancedImage = Image^{\sqrt{2}}$$

$\sqrt{2} = 1.414$ $\sqrt{2} = 1.41$ $\sqrt{2} = 1.4$

What is the advantage of Approximation?

- With reduction in precision of every decimal
  - Hardware Reduces
  - Power Reduces
  - Delay Reduces
  - Performance Improves

Approximation is everywhere.

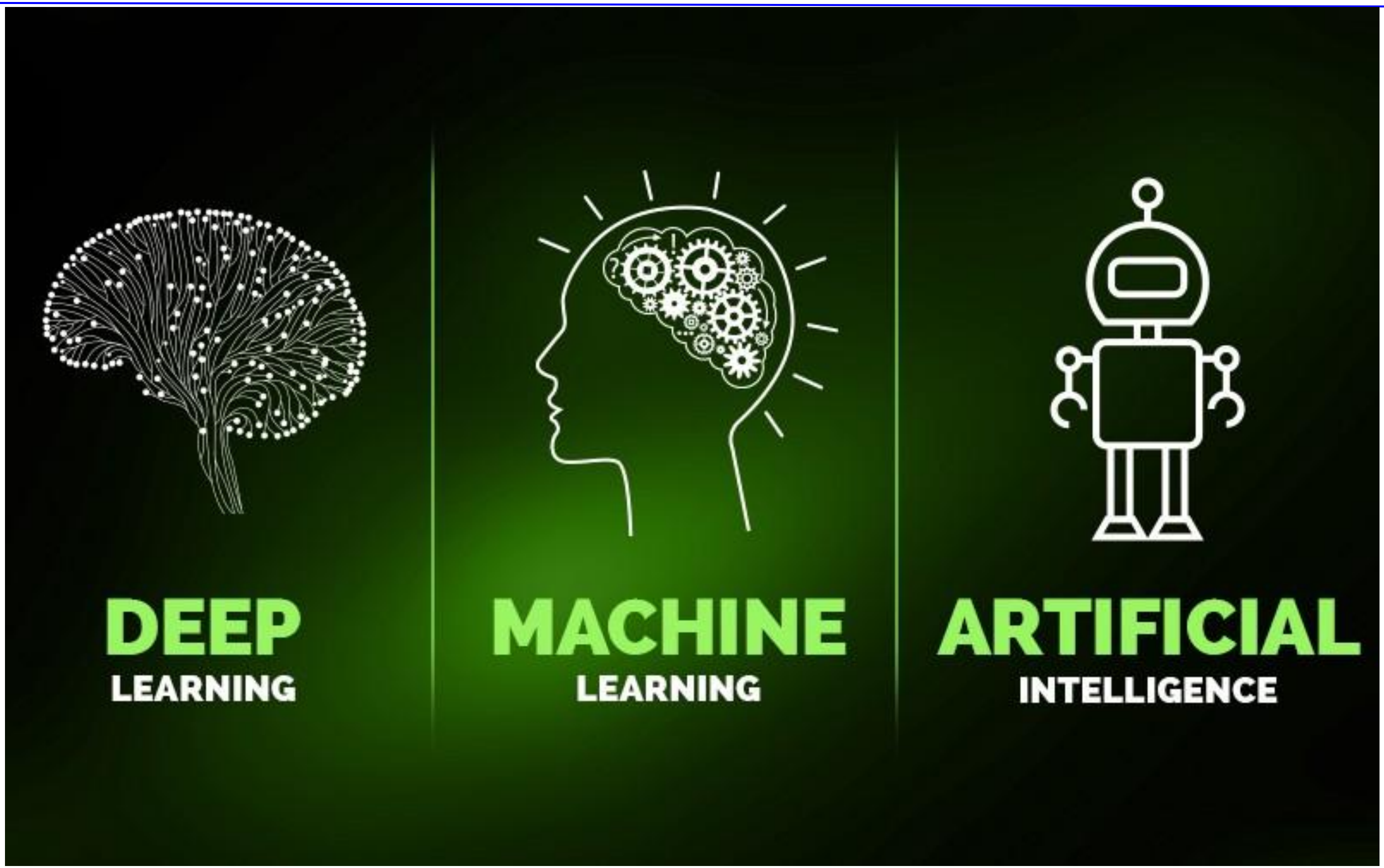
IEEE 754 Single precision Arithmetic.

$$\left(\frac{27/10-e}{\pi-\left(\sqrt{2}+\sqrt{3}\right)}\right)^{67/16}$$

Correct Answer: **302.8827196…**

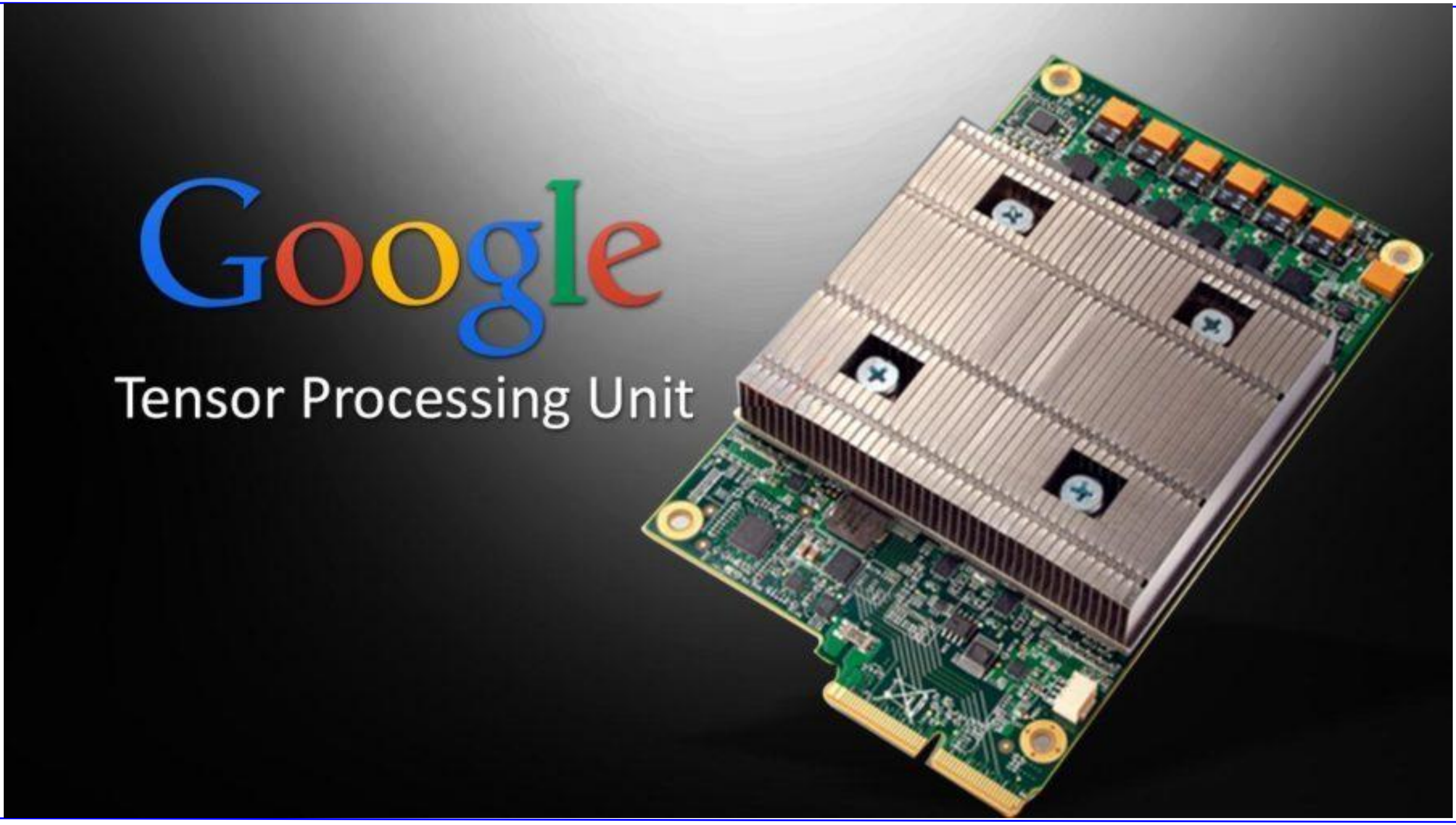
IEEE 754 32 Bit Answer: **302.912...**

Biggest Application



Source: https://becominghuman.ai/artificial-intilligence-machine-learning-deep-learning-df6dd0af500e

Hardware



Source: https://outfresh.com/latest-technology/google-tpu-quantum-computer-milestones-ai/

Should We approximate Everything?

NO

What should and What should not be approximated?

High Precision Application

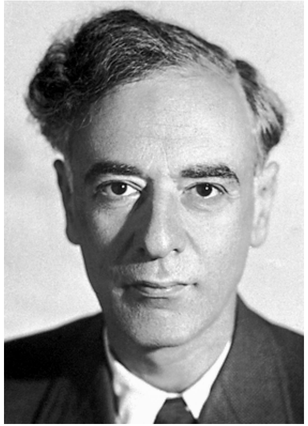
- Aeronautics - Chandrayaan 2
- Health care
- Nuclear Missions

Error Tolerant Application

- Image Processing
- Video Processing
- Machine Learning

Some Quotes

Truth is much too complicated to allow anything but approximations.
John von Neumann



Most important part of doing physics is the knowledge of approximation.
Lev Landau

Although this may seem a paradox, all exact science is based on the idea of approximation. If a man tells you he knows a thing exactly, then you can be safe in inferring that you are speaking to an inexact man.
Bertrand Russell

Thank You