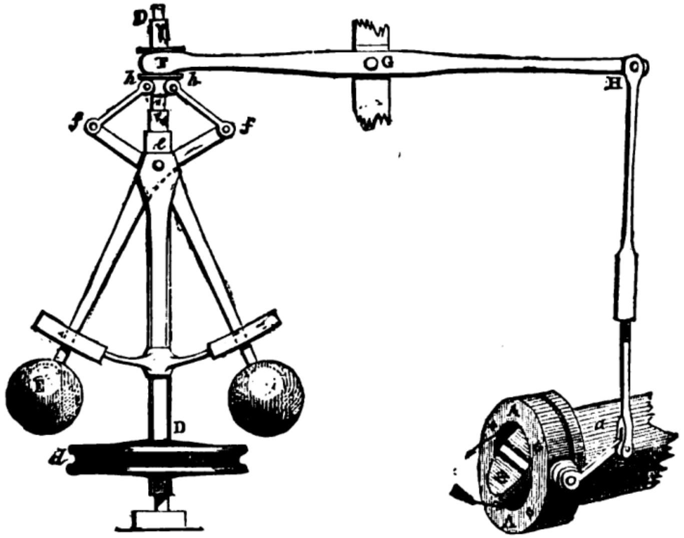

Modelling in the Time Domain

Control System Design

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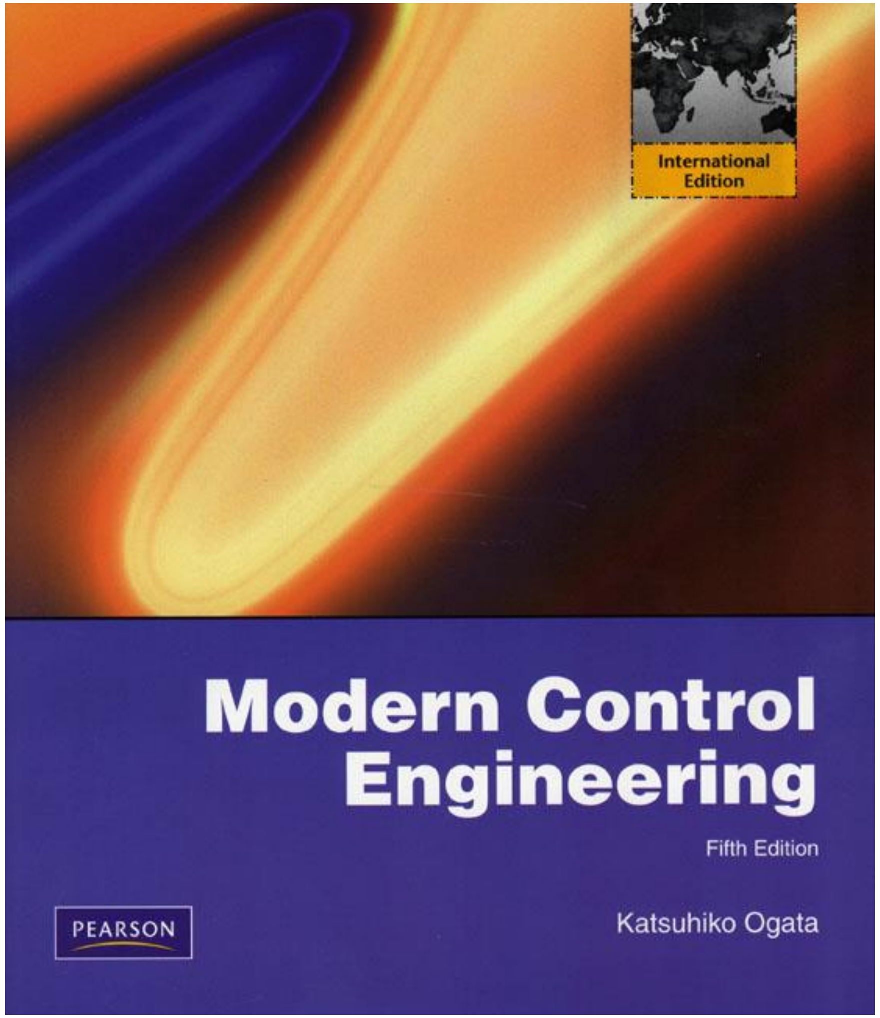
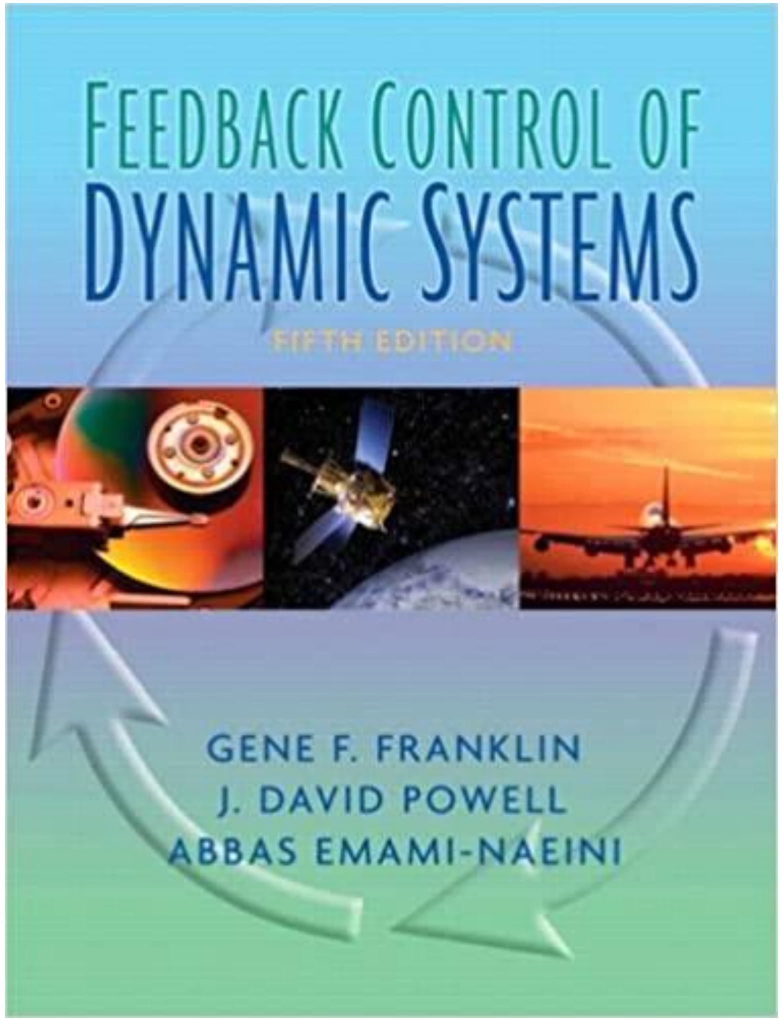
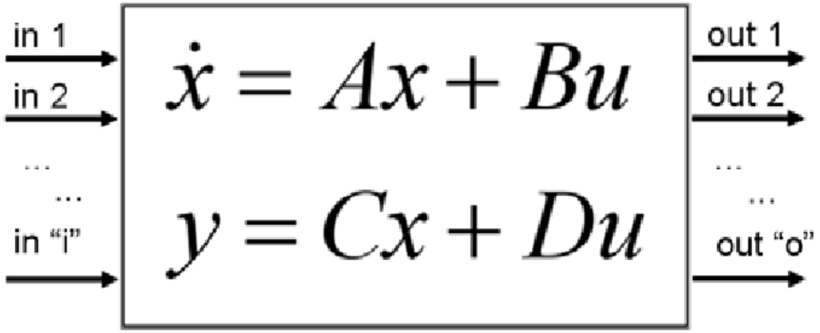
Reading


International Edition
Modern Control Engineering
Fifth Edition
PEARSON
Katsuhiko Ogata


FEEDBACK CONTROL OF DYNAMIC SYSTEMS
FIFTH EDITION
GENE F. FRANKLIN
J. DAVID POWELL
ABBAS EMAMI-NAEINI

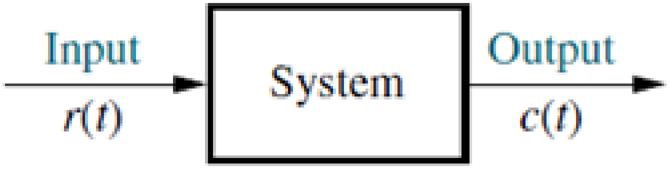
Lesson Learning Outcomes



- Find a mathematical model, called a state-space representation, for a linear, time invariant system
- Model electrical and mechanical systems in state space
- Convert a transfer function to state space
- Convert a state-space representation to a transfer function
- Linearize a state-space representation

Intro

the classical, or ***frequency-domain***, technique from previous class converting a system's differential equation to a transfer function



Advantages

1. thus generating a mathematical model of the system that algebraically relates a representation of the output to a representation of the input.
2. Replacing a differential equation with an algebraic equation not only simplifies the representation of individual subsystems but also simplifies modelling interconnected subsystems.
3. Rapidly provides stability and transient response information.

Disadvantages

1. It can be applied only to linear, time-invariant systems or systems that can be approximated as such.

The state-space approach

Also referred to as the ***modern, or time-domain, approach*** is a unified method for modeling, analyzing, and designing a wide range of systems.

For example, the state-space approach can be used to represent **nonlinear systems** that have backlash, saturation, and dead zone.

We proceed now to establish the state-space approach as an alternate method for representing physical systems.

This section sets the stage for the formal definition of the state-space representation by making some observations about systems and their variables

Approaching the state space representation

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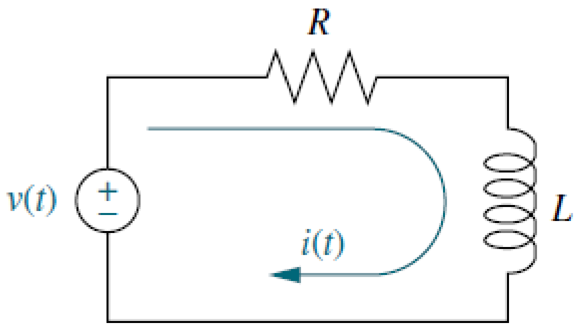
- We select a particular subset of all possible system variables and call the variables in this subset state variables.
- For an nth-order system, **we write n simultaneous, first-order differential equations** in terms of the state variables (state eqns.).
- We algebraically combine the state variables with the system's input and find all of the other system variables for $t \geq t_0$. We call this algebraic equation the **output equation**.
- We call this representation of the system a **state-space representation.**

Example 1



RL network

$$L\frac{di}{dt} + Ri = v(t) \qquad L[sI(s) - i(0)] + RI(s) = V(s)$$

Assuming the input, **v(t)**, to be a unit step, **u(t)**, whose Laplace transform is **V(s) = 1/s**, we solve for **I(s)** and get

$$I(s) = \frac{1}{R}\left(\frac{1}{s} - \frac{1}{s+\frac{R}{L}}\right) + \frac{i(0)}{s+\frac{R}{L}} \qquad i(t) = \frac{1}{R}\left(1 - e^{-(R/L)t}\right) + i(0)e^{-(R/L)t}$$

Example 1

We can now solve for all of the other network variables algebraically in terms of i(t) and the applied voltage, v(t).

$$v_R(t) = Ri(t)$$

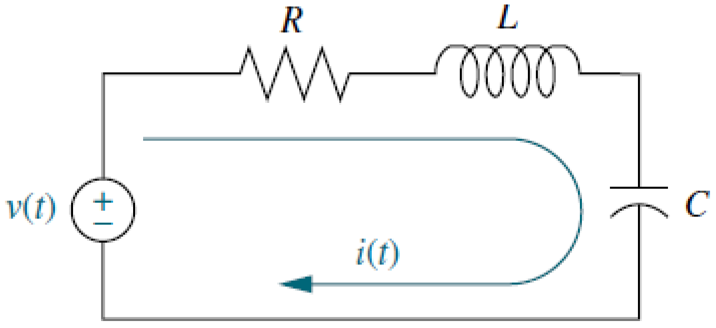
$$v_L(t) = v(t) - Ri(t)$$

$$\frac{di}{dt} = \frac{1}{L}[v(t) - Ri(t)]$$

Hence, the algebraic equations are output equations.

Thus, knowing the state variable, i(t), and the input, v(t), we can find the value, or state, of any network variable at any time, $t \geq t_0$.

Example 2



RLC network

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = v(t)$$

We select ***i(t)*** and ***q(t)***, the charge on the capacitor, as ***the two state variables***.

Converting to charge, using i(t) =dq/dt, we get→ $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = v(t)$$

We can convert the Eq. into two simultaneous, first-order differential equations in terms of i(t) and q(t).

$$\frac{dq}{dt} = i$$

$$\frac{di}{dt} = -\frac{1}{LC}q - \frac{R}{L}i + \frac{1}{L}v(t)$$

STATE EQUATIONS

$v_L(t)$ is a linear combination of the state variables, q(t) and i(t), and the input, v(t).

$$v_L(t) = -\frac{1}{C}q(t) - Ri(t) + v(t)$$

OUTPUT EQUATION

Example 2

- The combined state equations and the output equation form a viable representation of the network, which we call a **state-space representation.**

STATE EQUATIONS

$$\frac{dq}{dt} = i$$

$$\frac{di}{dt} = -\frac{1}{LC}q - \frac{R}{L}i + \frac{1}{L}v(t)$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$\dot{\mathbf{x}} = \begin{bmatrix} dq/dt \\ di/dt \end{bmatrix}; \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} q \\ i \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1/L \end{bmatrix}; \; u = v(t)$$

OUTPUT EQUATION

$$v_L(t) = -\frac{1}{C}q(t) - Ri(t) + v(t)$$

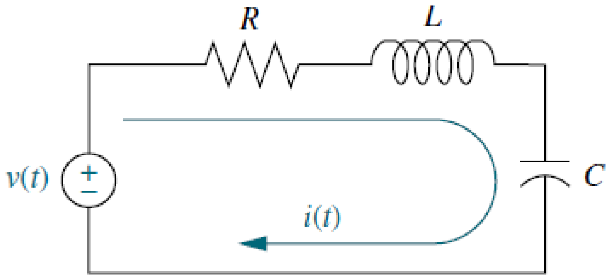
$$y = \mathbf{C}\mathbf{x} + Du$$

$$y = v_L(t); \quad \mathbf{C} = [-1/C \quad -R]; \quad \mathbf{x} = \begin{bmatrix} q \\ i \end{bmatrix}; \quad D = 1; \quad u = v(t)$$

Definition

A state-space representation, therefore, consists of

- (1) the simultaneous, first-order differential equations from which the state variables can be solved and
- (2) the algebraic output equation from which all other system variables can be found.



RLC network

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$\dot{\mathbf{x}} = \begin{bmatrix} dq/dt \\ di/dt \end{bmatrix}; \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} q \\ i \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1/L \end{bmatrix}; \; u = v(t)$$

$$y = \mathbf{C}\mathbf{x} + Du$$

$$y = v_L(t); \quad \mathbf{C} = [-1/C \quad -R]; \quad \mathbf{x} = \begin{bmatrix} q \\ i \end{bmatrix}; \quad D = 1; \quad u = v(t)$$

The General State-Space Representation

System variable. Any variable that responds to an input or initial conditions in a system.

State variables. The **smallest set of linearly independent** system variables such that the values of the members of the set at time $\mathbf{t_0}$ along with known forcing functions completely determine the value of all system variables for all $t \geq t_0$.

State vector. A vector whose elements are the state variables.

State equations. A set of **n** simultaneous, **first-order differential equations** with n variables, where the n variables to be solved are the state variables.

Output equation. The algebraic equation that expresses the output variables of a system **as linear combinations of the state variables and the inputs.**

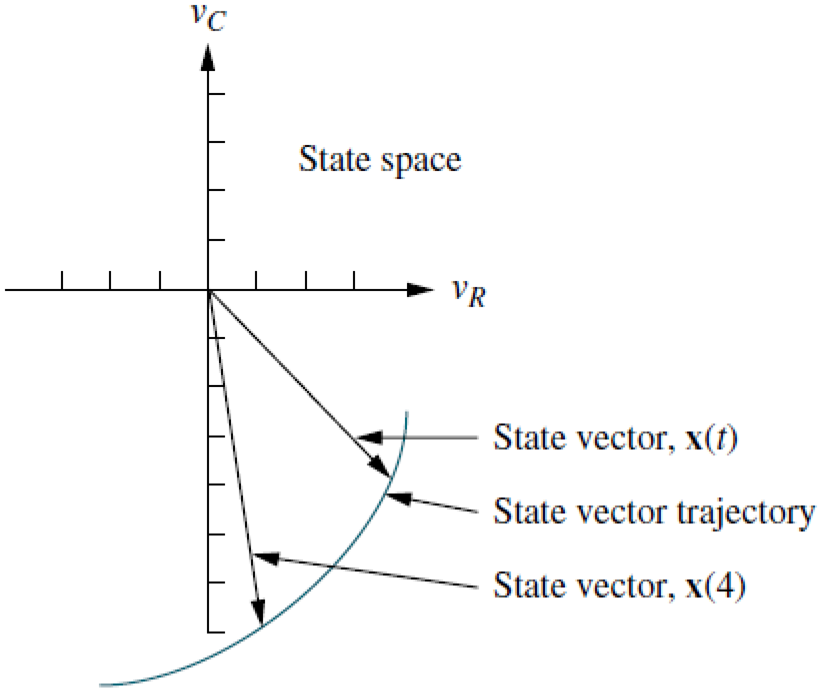
The General State-Space Representation

- **State space**. The n-dimensional space whose axes are the state variables.

Another choice of two state variables can be made, for example, $v_R(t)$ and $v_C(t)$, the resistor and capacitor voltage, respectively

$$\frac{dv_R}{dt} = -\frac{R}{L}v_R - \frac{R}{L}v_C + \frac{R}{L}v(t)$$

$$\frac{dv_C}{dt} = \frac{1}{RC}v_R$$



Graphic representation of state space and a state vector

The General State-Space Representation

- A system is represented in state space by the following equations:

$$\boxed{\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}}$$

$$\boxed{\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}}$$

$\mathbf{x}$ = state vector
$\dot{\mathbf{x}}$ = derivative of the state vector with respect to time
$\mathbf{y}$ = output vector
$\mathbf{u}$ = input or control vector
$\mathbf{A}$ = system matrix
$\mathbf{B}$ = input matrix
$\mathbf{C}$ = output matrix
$\mathbf{D}$ = feedforward matrix

1. **A minimum number of state variables must be selected** as components of the state vector. This minimum number of state variables is sufficient to describe completely the state of the system.
2. The components of the state vector (that is, this minimum number of state variables) **must be linearly independent.**

Representing a Translational Mechanical System

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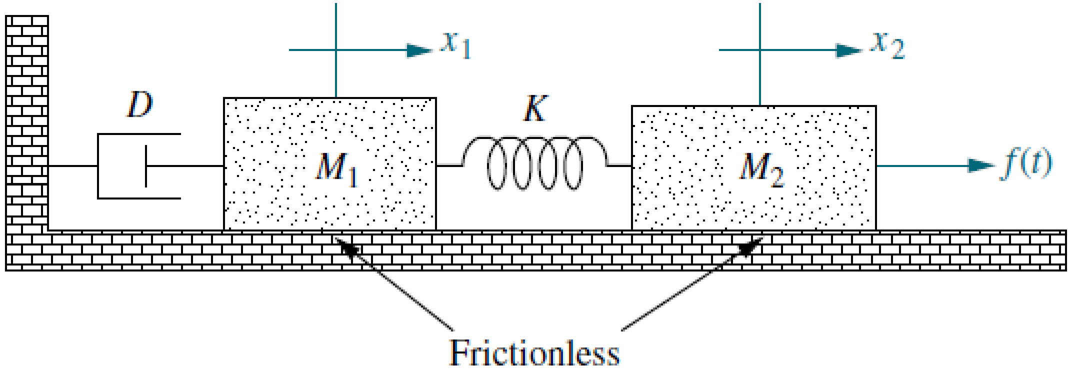
Find the state equations



First write the differential equations for the network

$$M_1 \frac{d^2x_1}{dt^2} + D\frac{dx_1}{dt} + Kx_1 - Kx_2 = 0$$

$$-Kx_1 + M_2\frac{d^2x_2}{dt^2} + Kx_2 = f(t)$$

Let´s choose x_1, v_1, x_2, and v_2 as state variables

$$d^2x_1/dt^2 = dv_1/dt, \text{ and } d^2x_2/dt^2 = dv_2/dt$$

$$\frac{dx_1}{dt} = +v_1$$

$$\frac{dv_1}{dt} = -\frac{K}{M_1}x_1 - \frac{D}{M_1}v_1 + \frac{K}{M_1}x_2$$

$$\frac{dx_2}{dt} = +v_2$$

$$\frac{dv_2}{dt} = +\frac{K}{M_2}x_1 - \frac{K}{M_2}x_2 + \frac{1}{M_2}f(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K/M_1 & -D/M_1 & K/M_1 & 0 \\ 0 & 0 & 0 & 1 \\ K/M_2 & 0 & -K/M_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M_2 \end{bmatrix} f(t)$$

state equations

$$\boxed{\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}}$$

Converting a Transfer Function to State Space

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Using inverse Laplace Transformation we get the Diff Eqn $\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_0 u$

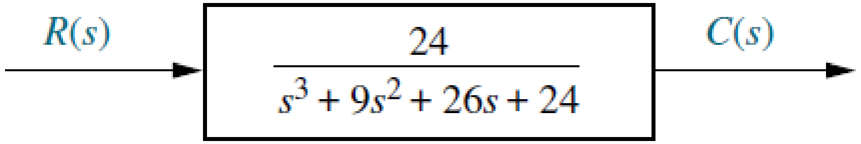
A convenient way to choose state variables is to choose the output, y(t), and its (n-1) derivatives as the state variables. This choice is called the phase-variable choice. Choosing the state variables, x_i, we get:

$$
\begin{aligned}
x_1 &= y \\
x_2 &= \frac{dy}{dt} \\
x_3 &= \frac{d^2y}{dt^2} \\
&\vdots \\
x_n &= \frac{d^{n-1}y}{dt^{n-1}}
\end{aligned}
\qquad
\begin{aligned}
\dot{x}_1 &= \frac{dy}{dt} \\
\dot{x}_2 &= \frac{d^2y}{dt^2} \\
\dot{x}_3 &= \frac{d^3y}{dt^3} \\
&\vdots \\
\dot{x}_n &= \frac{d^n y}{dt^n}
\end{aligned}
$$

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix}
=
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\
\vdots & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\
-a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & \cdots & -a_{n-1}
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}
+
\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u
$$

$$
y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}
$$

Converting a Transfer Function to State Space Example



$$\frac{C(s)}{R(s)} = \frac{24}{(s^3 + 9s^2 + 26s + 24)}$$

Step 1 Find the associated differential equation

$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s) \qquad \dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r$$

Step 2 Select the state variables and differentiate.

$$\begin{aligned} x_1 &= c \\ x_2 &= \dot{c} \\ x_3 &= \ddot{c} \end{aligned}$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -24x_1 - 26x_2 - 9x_3 + 24r \\ y &= c = x_1 \end{aligned}$$

Step 3 In vector-matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Converting a Transfer Function to State Space Example

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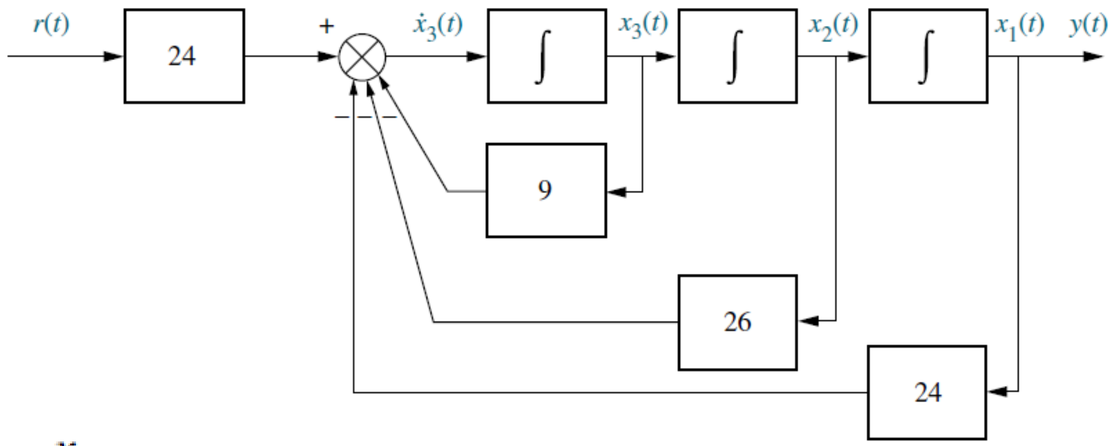
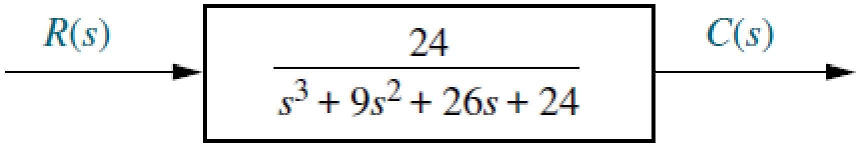
At this point, we can create an equivalent block diagram of the system to help visualize the state variables





$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r$$
$$y = c = x_1$$

$$\frac{C(s)}{R(s)} = \frac{24}{(s^3 + 9s^2 + 26s + 24)}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s)$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Converting a Transfer Function with Polynomial in Numerator

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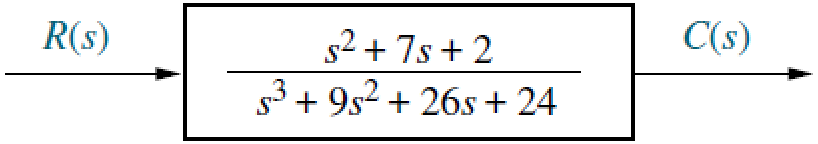
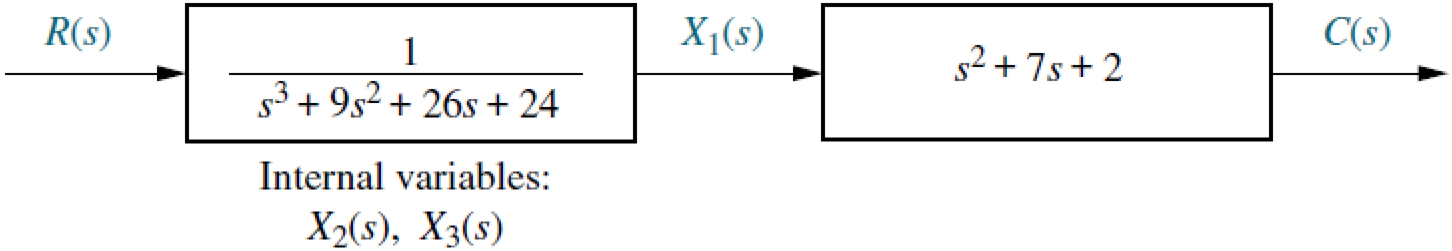
Find the state-space representation of the transfer function shown:



Step 1 Separate the system into two cascaded blocks



Step 2 Find the state equations for the block containing the denominator (same as example before but ***r(t) is not multiplied by 24!!)***

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

STATE EQUATIONS

Converting a Transfer Function with Polynomial in Numerator

Step 3 Introduce the effect of the block with the numerator $C(s) = (b_2 s^2 + b_1 s + b_0) X_1(s) = (s^2 + 7s + 2) X_1(s)$

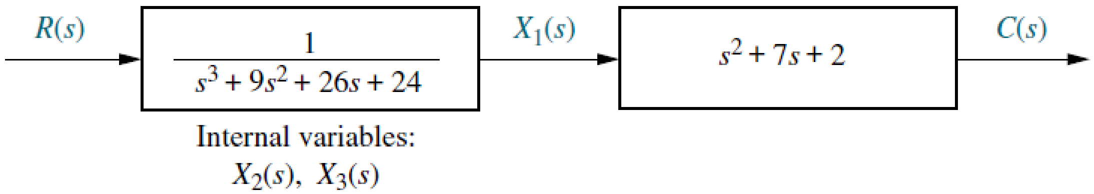
Taking the inverse Laplace transform $c = \ddot{x}_1 + 7\dot{x}_1 + 2x_1$

Choosing the space variables

$$x_1 = x_1$$
$$\dot{x}_1 = x_2$$
$$\ddot{x}_1 = x_3$$

$$y = c(t) = b_2 x_3 + b_1 x_2 + b_0 x_1 = x_3 + 7 x_2 + 2x_1$$

Thus, the last box of Figure block “collects” the states and generates the output equation.



$$y = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

OUTPUT EQUATION

Converting from State Space to a Transfer Function

Now we move in the opposite direction and convert the state-space representation into a transfer function.

Given the state and output equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

take the Laplace transform assuming zero initial conditions

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$
$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

Solving for **X**(s) $(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s)$ $\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s)$$

$$\boxed{T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}}$$

However, if ***U***(s) and ***Y***(s) are scalars, we can find the transfer function

Converting from State Space to a Transfer Function (Example)

Given the system defined by the state space Eq., find the transfer function, T(s) = Y(s)/U(s), where U(s) is the input and Y(s) is the output.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0]\mathbf{x}$$

$$\boxed{\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}}$$

$$\boxed{\mathbf{y} = \mathbf{Cx} + \mathbf{Du}}$$

$$\mathbf{B} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{C} = [1 \quad 0 \quad 0]$$

$$\mathbf{D} = 0$$

SOLUTION: The solution revolves around finding the term $(s\mathbf{I} - \mathbf{A})^{-1}$

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

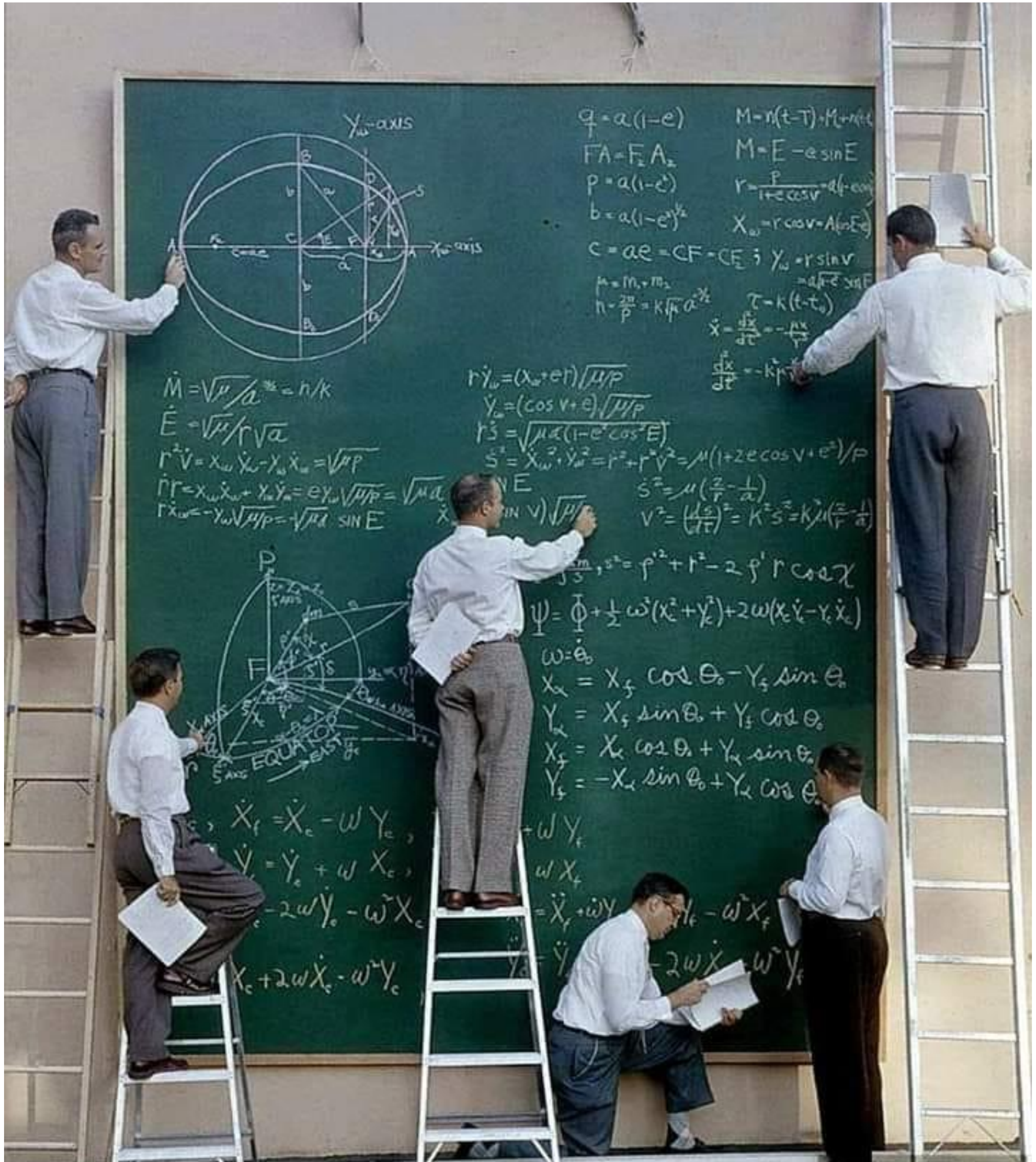
$$\boxed{T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}}$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\operatorname{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})} = \frac{\begin{bmatrix} (s^2+3s+2) & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3 + 3s^2 + 2s + 1}$$

$$\boxed{T(s) = \frac{10(s^2 + 3s + 2)}{s^3 + 3s^2 + 2s + 1}}$$

Few examples:

Paper and pen!



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