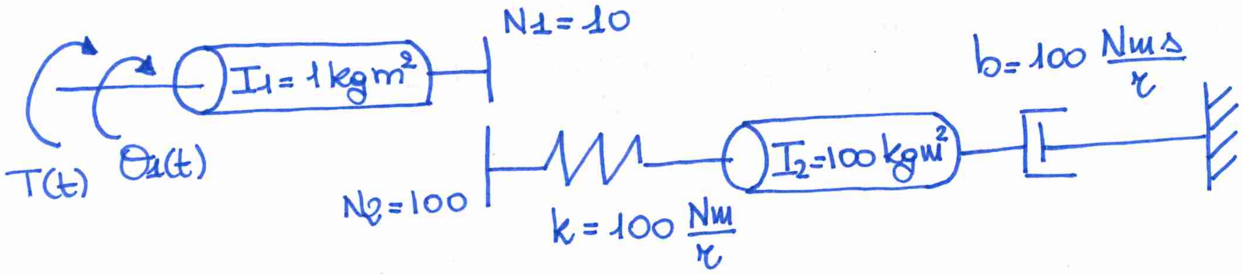
EX. 1

Represent the system in state-space with $\theta_1(t)$ as output



! STATE-SPACE REPRESENTATION:

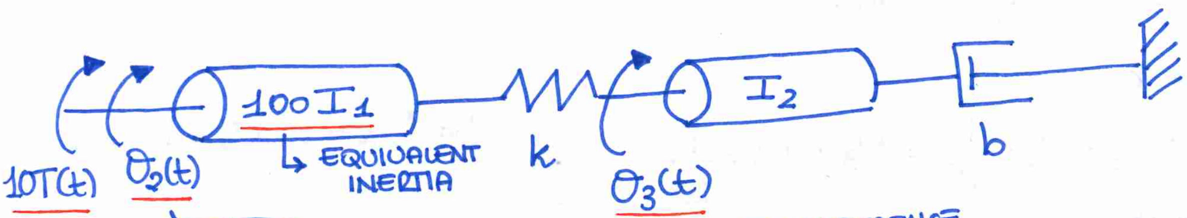
$$\dot{\vec{x}} = A\vec{x} + B\vec{u} \; ; \quad y = C\vec{x} + D\vec{u} \; ;$$

A: system matrix; B: input matrix; C: output matrix; D: feedforward matrix

x = state vector
$\dot{x}$ = derivative of state vec.
y = output vector

SOLUTION:

1) Draw the equivalent system without gears: relate to θ_3 (central reference)



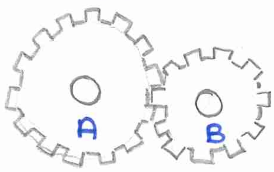
↳ EQUIVALENT INERTIA

↳ EQUIVALENT TORQUE

$\theta_2 = \frac{1}{10}\theta_1$

↳ NEW REFERENCE FRAME

SYSTEM WITH GEARS → EQUIVALENT SYSTEM



N_A, N_B ⇝ # OF TEETHS

ω_A, ω_B ⇝ ANGULAR VELOCITY

$R = \frac{N_B}{N_A} = \frac{\omega_B}{\omega_A}$ GEAR RATIO

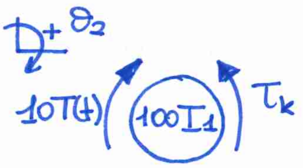
↳ $\frac{N\ destination}{N\ source}$

Equivalent torque: $T_e(t) = \left(\frac{N_2}{N_1}\right)T(t) = 10T(t)$

Equivalent Inertia gear 1: $I_{1e} = \left(\frac{N_2}{N_1}\right)^2 I_1 = 100 I_1$

$$\theta_2(t) = \left(\frac{N_1}{N_2}\right)\theta_1(t) = \frac{1}{10}\theta_1(t)$$

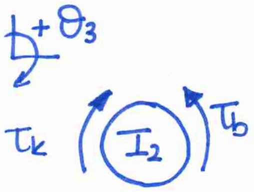
2) Solve the equation of motion:



$$100 I_1 \alpha_2 = 10T(t) - \tau_k(t) = 10T - k(\theta_2 - \theta_3)$$

$$100\ddot{\theta}_2 = 10T - 100(\theta_2 - \theta_3)$$

$$\ddot{\theta}_2 = \theta_3 - \theta_2 + \frac{1}{10}T \qquad \boxed{1}$$



$$I_2 \alpha_3 = \tau_k - \tau_b = k(\theta_2 - \theta_3) - b\dot{\theta}_3$$

$$100\ddot{\theta}_3 = 100(\theta_2 - \theta_3) - 100\dot{\theta}_3$$

$$\ddot{\theta}_3 = \theta_2 - \theta_3 - \dot{\theta}_3 \qquad \boxed{2}$$

3) Define the state variables: output & (n-1) derivatives

$$x_1 = \theta_2 \; ; \quad x_3 = \theta_3$$

$$x_2 = \dot{\theta}_2 \; ; \quad x_4 = \dot{\theta}_3$$

4) Write the state equations:

$$\begin{cases} \dot{x}_1 = \dot{\theta}_2 = x_2 \\ \dot{x}_2 = \ddot{\theta}_2 = \theta_3 - \theta_2 + \dfrac{T}{10} = x_3 - x_1 + \dfrac{T}{10} \\ \dot{x}_3 = \dot{\theta}_3 = x_4 \\ \dot{x}_4 = \ddot{\theta}_3 = \theta_2 - \theta_3 - \dot{\theta}_3 = x_1 - x_3 - x_4 \end{cases} \; ; \qquad y = \theta_1 = 10\theta_2 = 10x_1$$

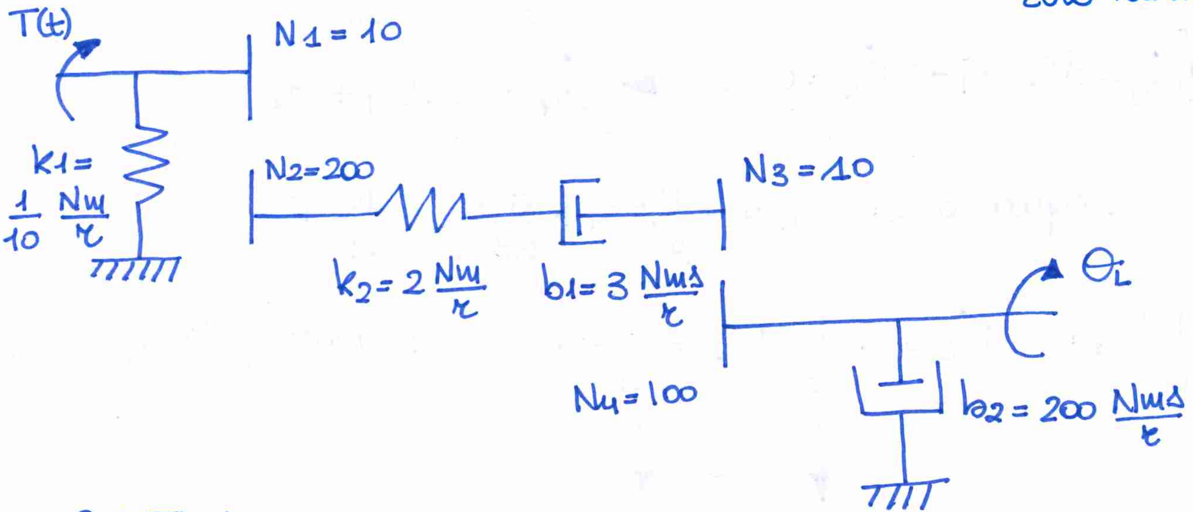
5) Final form: $\dot{\vec{x}} = A\vec{x} + B\vec{u}$; $y = C\vec{x} + D\vec{u}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & +1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/10 \\ 0 \\ 0 \end{bmatrix} T(t)$$

$$y = \begin{bmatrix} 10 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

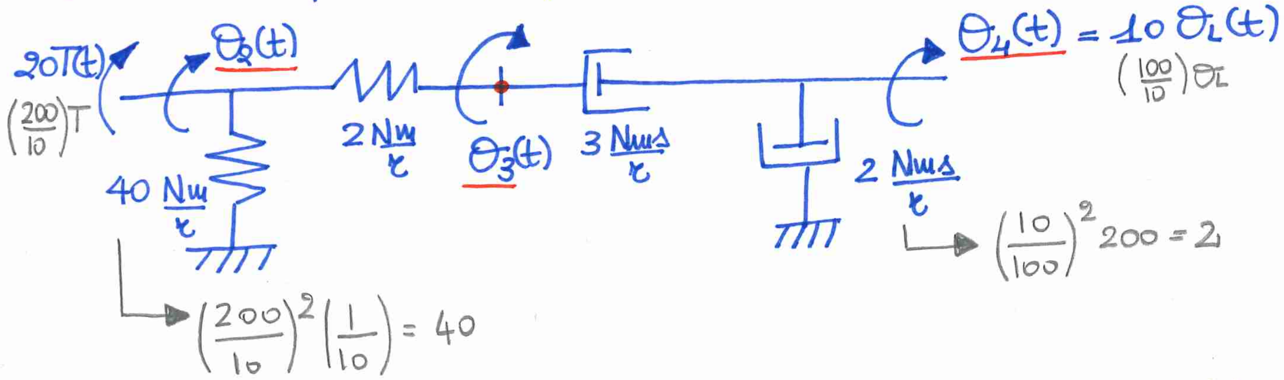
EX. 2

Represent the system in state-space with θ_L as output with zero initial condition.



SOLUTION:

1) Solve the equivalent system : relate to $\theta_3(t) \rightsquigarrow$ central reference



2) Solve the equations of motion.

$\boxed{1}$ $20T = k_1\theta_2 + k_2(\theta_2 - \theta_3) = 42\theta_2 - 2\theta_3$

$k_2(\theta_2 - \theta_3) = b_1(\dot{\theta}_3 - \dot{\theta}_4)$

$\boxed{2}$ $3\dot{\theta}_3 - 3\dot{\theta}_4 - 2\theta_2 + 2\theta_3 = 0$

θ_4 $+$ T_{b1} T_{b2}

$b_1(\dot{\theta}_3 - \dot{\theta}_4) = b_2\dot{\theta}_4$

$\boxed{3}$ $3\dot{\theta}_3 - 5\dot{\theta}_4 = 0 \rightsquigarrow \dot{\theta}_3 = \frac{5}{3}\dot{\theta}_4$

(*)

$\int_0^{t_f} \dot{\theta}_3 = \int_0^{t_f} \frac{5}{3}\dot{\theta}_4 \rightarrow \theta_3 = \frac{5}{3}\theta_4$

Substituting ⊛ into $\boxed{1}$ and $\boxed{2}$:

$$\boxed{1}' \quad 42\,\theta_2 - \frac{10}{3}\theta_4 = 20T \longrightarrow \theta_2 = \frac{5}{63}\theta_4 + \frac{10}{21}T$$

$$\boxed{2}' \quad -2\,\theta_2 + \cancel{\frac{15}{3}}^{5}\dot{\theta}_4 + \frac{10}{3}\theta_4 - 3\,\dot{\theta}_4 = 0 \longrightarrow \dot{\theta}_4 = -\frac{5}{3}\theta_4 + \theta_2$$

merging the 2 equations: $\dot{\theta}_4 = -\frac{5}{3}\theta_4 + \frac{5}{63}\theta_4 + \frac{10}{21}T$

Since $\theta_L = \frac{1}{10}\theta_4$: $\boxed{\dot{\theta}_L = -\frac{10}{63}\theta_L + \frac{10}{21}T}$ 1 DoF system

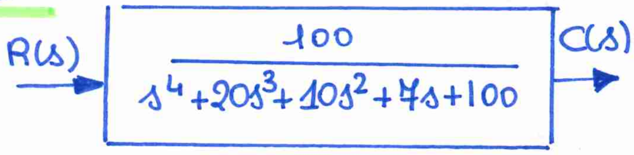
$$A = -\frac{10}{63} \qquad B = \frac{10}{21}$$

$$\vec{\dot{x}} = A\,\vec{x} + B\,\vec{u}$$

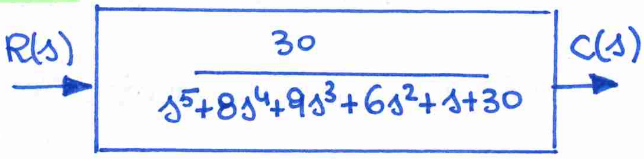
EX. 3

Find the state-space representation for the systems (a) and (b) in phase-variable form.

(a)



(b)



SOLUTION:

(a) ① $G(s) = \frac{C(s)}{R(s)} = \frac{100}{s^4+20s^3+10s^2+7s+100} \Rightarrow 100R(s) = (s^4+20s^3+10s^2+7s+100)C(s)$

! phase-variable choice: select the output and its $(n-1)$ derivatives as the state-variables

Output $\rightsquigarrow$ $C(s)$ (Laplace domain)

② In the time domain: $\ddddot{c} + 20\dddot{c} + 10\ddot{c} + 7\dot{c} + 100c = 100r$

③ Choose the state variables:

$$\begin{aligned} x_1 &= c \\ x_2 &= \dot{c} \\ x_3 &= \ddot{c} \\ x_4 &= \dddot{c} \end{aligned} \longrightarrow \begin{cases} \dot{x}_1 = x_2 = \dot{c} \\ \dot{x}_2 = x_3 = \ddot{c} \\ \dot{x}_3 = x_4 = \dddot{c} \\ \dot{x}_4 = 100r - 20x_4 - 10x_3 - 7x_2 - 100x_1 \end{cases}$$

$$y = c = x_1$$

④ vector-matrix form:

$$\underbrace{\dot{\vec{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -100 & -7 & -10 & -20 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 100 \end{bmatrix} \vec{r}}_{\dot{\vec{x}} = A\vec{x} + B\vec{u}} \; ; \quad \underbrace{y = [1\ 0\ 0\ 0]\,\vec{x}}_{y = C\vec{x} + \underbrace{\cancel{D\vec{u}}}_{=0}}$$

(b)

① $G(s) = \dfrac{C(s)}{R(s)} = \dfrac{30}{s^5+8s^4+9s^3+6s^2+s+30} \Rightarrow 30R(s) = (s^5+8s^4+9s^3+6s^2+s+30)C(s)$

② in the time domain:

$$\overset{.....}{c} + 8\overset{....}{c} + 9\dddot{c} + 6\ddot{c} + \dot{c} + 30c = 30r$$

③ choose the state variable:

$$\begin{aligned} x_1 &= c \\ x_2 &= \dot{c} \\ x_3 &= \ddot{c} \\ x_4 &= \dddot{c} \\ x_5 &= \overset{....}{c} \end{aligned} \longrightarrow \begin{cases} \dot{x}_1 = x_2 = \dot{c} \\ \dot{x}_2 = x_3 = \ddot{c} \\ \dot{x}_3 = x_4 = \dddot{c} \\ \dot{x}_4 = x_5 = \overset{....}{c} \\ \dot{x}_5 = -8\overset{....}{c} - 9\dddot{c} - 6\ddot{c} - \dot{c} - 30c + 30r \end{cases}$$

$$y = c = x_1$$

④ vector-matrix form:

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -30 & -1 & -6 & -9 & -8 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 30 \end{bmatrix} r(t)$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \vec{x}$$

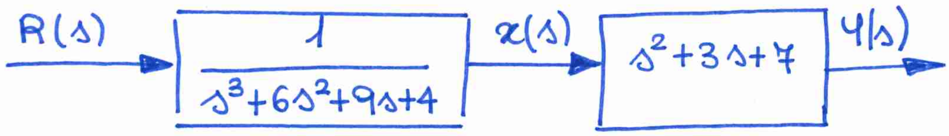
EX. 4

Represent the transfer function in state-space using the vector-matrix form.

$$T(s) = \frac{s^2+3s+7}{(s+1)(s^2+5s+4)}$$

SOLUTION:

It is helpful to split T(s) into two TFs:



- 1º transfer function $\quad \dddot{x} + 6\ddot{x} + 9\dot{x} + 4x = r(t)$

3 state variables: $\quad \begin{matrix} x_1 = x \\ x_2 = \dot{x} \\ x_3 = \ddot{x} \end{matrix} \longrightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -4x_1 - 9x_2 - 6x_3 + r(t) \end{cases}$

- 2º transfer function $\quad y = \ddot{x} + 3\dot{x} + 7x = x_3 + 3x_2 + 7x_1$

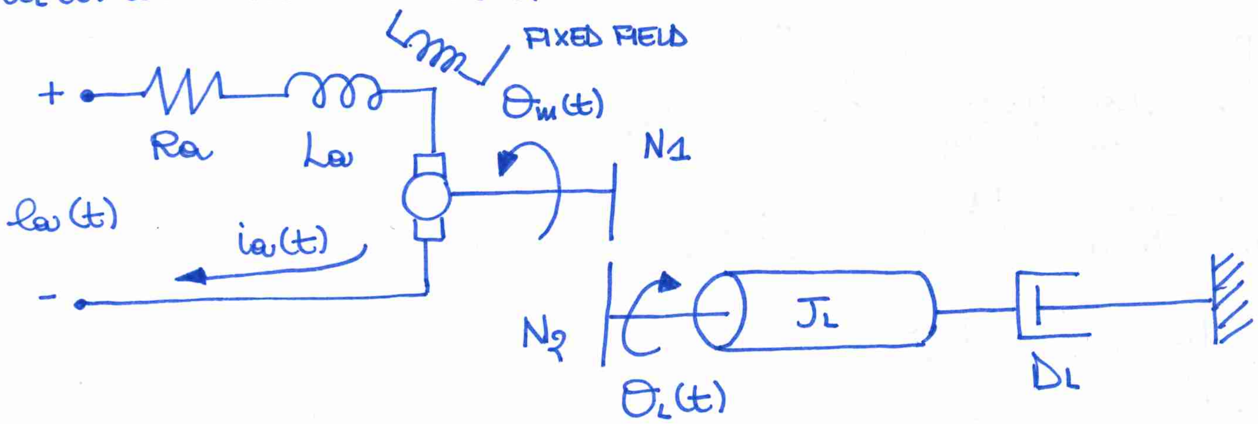
vector-matrix form:

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -9 & -6 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$$y = \begin{bmatrix} 7 & 3 & 1 \end{bmatrix} \vec{x}$$

EX. 5

Represent the system in state-space with $i_a(t)$, $\theta_L(t)$, and $\omega_L(t)$ as state variables.



We know that: $\underbrace{T_m}_{\text{TORQUE AT THE MOTOR}} = \underbrace{K_t \cdot i_a(t)}_{\text{ELECTRICAL}} = \underbrace{J_{eq}\,\dot{\omega}_m + D_{eq}\,\omega_m}_{\text{MECHANICAL}}$

J_{eq} → EQUIVALENT INERTIA OF OUTPUT LOAD

$$\longrightarrow \dot{\omega}_m = -\frac{D_{eq}}{J_{eq}}\omega_m + \frac{K_t}{J_{eq}} i_a(t)$$

From the gear ratio: $\frac{N_2}{N_1} = \frac{\omega_m}{\omega_L} \longrightarrow \omega_m = \frac{N_2}{N_1}\omega_L$

So: $\dot{\omega}_L = -\frac{D_{eq}}{J_{eq}}\omega_L + \frac{N_1}{N_2}\frac{K_t}{J_{eq}} i_a$ 1º STATE EQUATION

$\dot{\theta}_L = \omega_L$ 2º STATE EQUATION

$\dot{i}_a(t) = ?$ 3º STATE EQUATION

Examining the armature loop:

$$e_a(t) = R_a i_a + L_a \dot{i}_a + \underbrace{K_b \omega_m}_{} = R_a i_a + L_a \dot{i}_a + K_b \frac{N_2}{N_1}\omega_L$$

$K_b = \frac{E}{\omega}$ relationship speed-voltage

$$\Rightarrow \dot{i}_a = -\frac{N_2}{N_1}\frac{K_b}{L_a}\omega_L - \frac{R_a}{L_a} i_a + \frac{1}{L_a} e_a$$

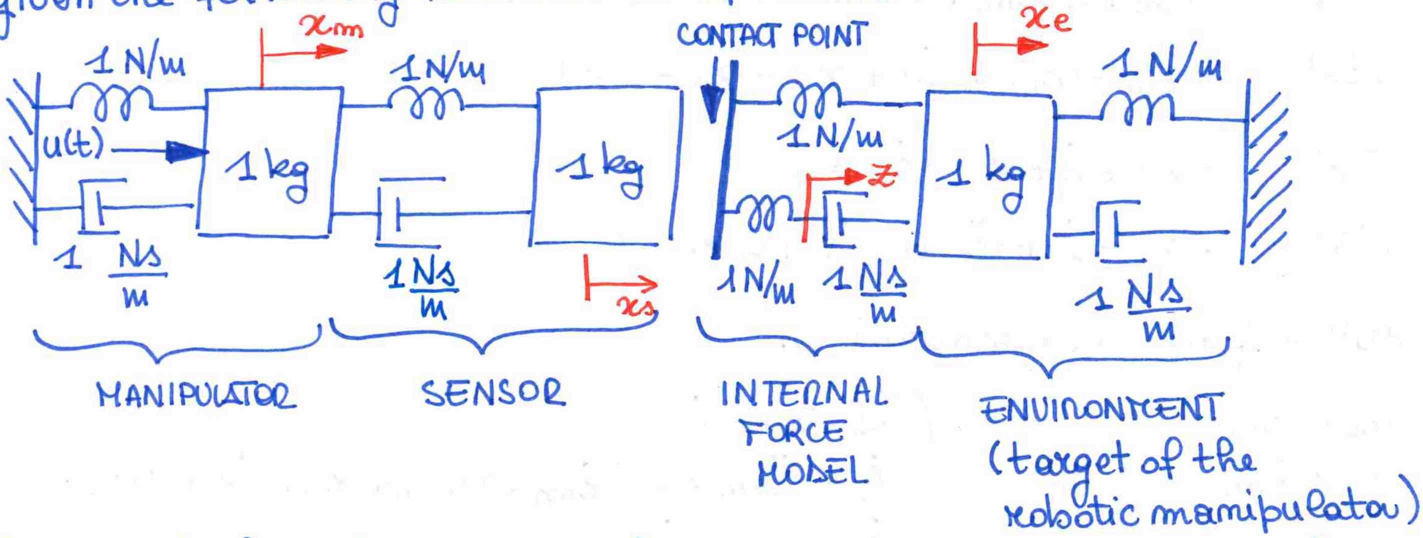
$$\vec{x} = \begin{bmatrix} \vartheta_L \\ \omega_L \\ i_a \end{bmatrix}$$

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{D_{eq}}{J_{eq}} & \frac{N_1}{N_2}\frac{k_b}{J_{eq}} \\ 0 & -\frac{N_2}{N_1}\frac{k_b}{L_a} & -\frac{R_a}{L_a} \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ 1/L_a \end{bmatrix} e_a$$

$$y = \begin{bmatrix} \frac{N_2}{N_1} & 0 & 0 \end{bmatrix} \vec{x}$$

EX. 6

Given the following robotic manipulator:



Represent the 1 DoF manipulator in state space under the following conditions:

a) the manipulator is not in contact with the environm.

b) the manipulator is in contact with the environment.

SOLUTION:

a) NON-CONTACT Equations of motions (AFTER SOLVING THE FREE BODY DIAGRAMS)

$\overrightarrow{x_m}+$ $$\ddot{x}_m + 2\dot{x}_m + 2x_m - x_s - \dot{x}_s = u(t) \longrightarrow \text{INPUT FUNCTION TO THE ROBOTIC MANIPULATOR}$$

$\overrightarrow{x_s}+$ $$-\dot{x}_m - x_m + \ddot{x}_s + \dot{x}_s + x_s = 0$$

State-space representation:

$$\begin{aligned} x_1 &= x_m \\ x_2 &= \dot{x}_m \\ x_3 &= x_s \\ x_4 &= \dot{x}_s \end{aligned} \longrightarrow \begin{cases} \dot{x}_1 = x_2 = \dot{x}_m = x_2 \\ \dot{x}_2 = \ddot{x}_m = u(t) - 2\underset{x_2}{\dot{x}_m} - 2\underset{x_1}{x_m} + \underset{x_3}{x_s} + \underset{x_4}{\dot{x}_s} \\ \dot{x}_3 = x_4 = \dot{x}_s = x_4 \\ \dot{x}_4 = \ddot{x}_s = \underset{x_2}{\dot{x}_m} + \underset{x_1}{x_m} - \underset{x_4}{\dot{x}_s} - \underset{x_3}{x_s} \end{cases}$$

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & -2 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(t) \; ; \quad y = [0 \; 0 \; 1 \; 0]\,\vec{x}$$

b) CONTACT Equations of motions: (AFTER SOLVING THE FBDs)

$\xrightarrow{x_m} +$ $\quad \ddot{x}_m + 2\dot{x}_m + 2x_m - x_s - \dot{x}_s = u(t)$

$\xrightarrow{x_s} +$ $\quad -\dot{x}_m - x_m + \ddot{x}_s + x_s - x_e - z = 0$

$\xrightarrow{z} +$ $\quad -x_s + \dot{z} + z - \dot{x}_e = 0 \quad (*)$

$\xrightarrow{x_e} +$ $\quad -x_s - \dot{z} + \ddot{x}_e + 2\dot{x}_e + 2x_e = 0$

State-space representation:

$$x_1 = x_m \qquad x_2 = \dot{x}_m \qquad x_3 = x_s \qquad x_4 = \dot{x}_s \qquad x_5 = z \qquad x_6 = \dot{z} \qquad x_7 = x_e \qquad x_8 = \dot{x}_e$$

$$\longrightarrow \begin{cases} \dot{x}_1 = x_2 = \dot{x}_m \\ \dot{x}_2 = \ddot{x}_m = -2\overset{x_2}{\dot{x}_m} - 2\overset{x_1}{x_m} + \overset{x_3}{x_s} + \overset{x_4}{\dot{x}_s} + u(t) \\ \dot{x}_3 = x_4 = \dot{x}_s \\ \dot{x}_4 = \ddot{x}_s = \overset{x_2}{\dot{x}_m} + \overset{x_1}{x_m} - \overset{x_3}{x_s} + \overset{x_7}{x_e} + \overset{x_5}{z} \\ \dot{x}_5 = x_6 = \dot{z} \\ \bullet\ \dot{x}_6 = \ddot{z} = \overset{x_4}{\dot{x}_s} - \overset{x_6}{\dot{z}} + \underline{\ddot{x}_e} \quad \text{differentiating } (*) \\ \dot{x}_7 = x_8 = \dot{x}_e \\ \dot{x}_8 = \ddot{x}_e = \overset{x_3}{x_s} + \underset{x_6}{\dot{z}} - 2\underset{x_8}{\dot{x}_e} - 2\underset{x_7}{x_e} \end{cases}$$

From the last equation: $\ddot{x}_e = x_s + \dot{z} - 2\dot{x}_e - 2x_e = x_3 + x_6 - 2x_8 - 2x_7$

substituting in $\bullet$: $\dot{x}_6 = x_4 - \cancel{x_6} + x_3 + \cancel{x_6} - 2x_8 - 2x_7 =$

$= x_4 - 2x_8 - 2x_7 + x_3$

output: x_s; $\rightarrow y = x_s \rightarrow C = [0\ 0\ 1\ 0\ 0\ 0\ 0\ 0]$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -2 & -2 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$