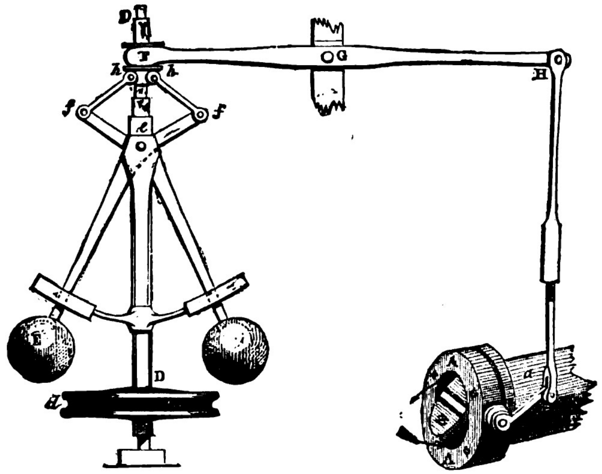


The Time Response
Control System Design

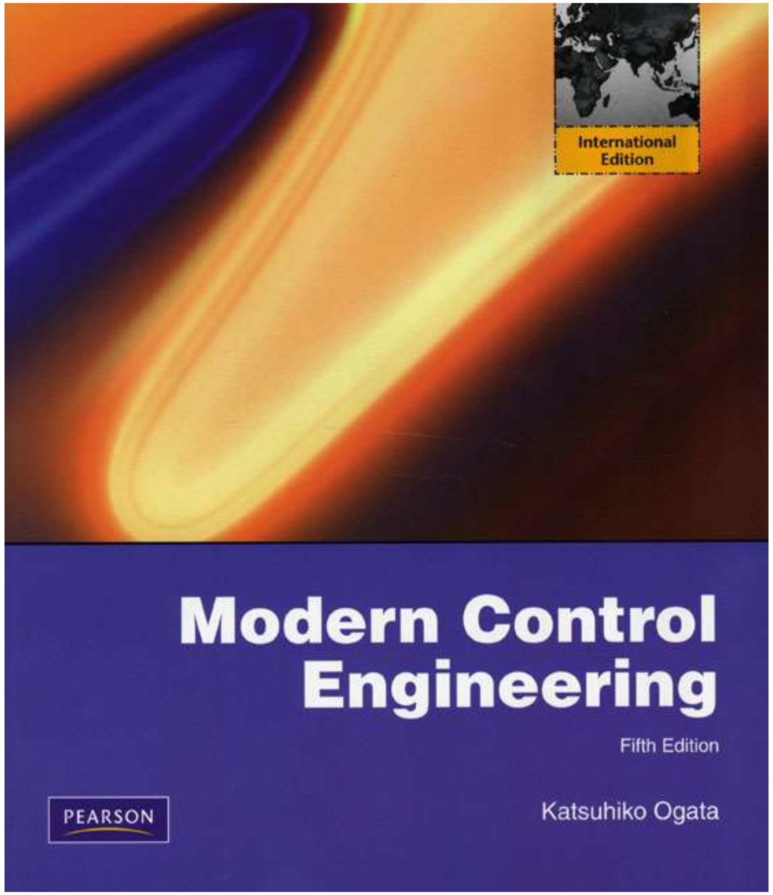
Prof Dr Lorenzo Masia

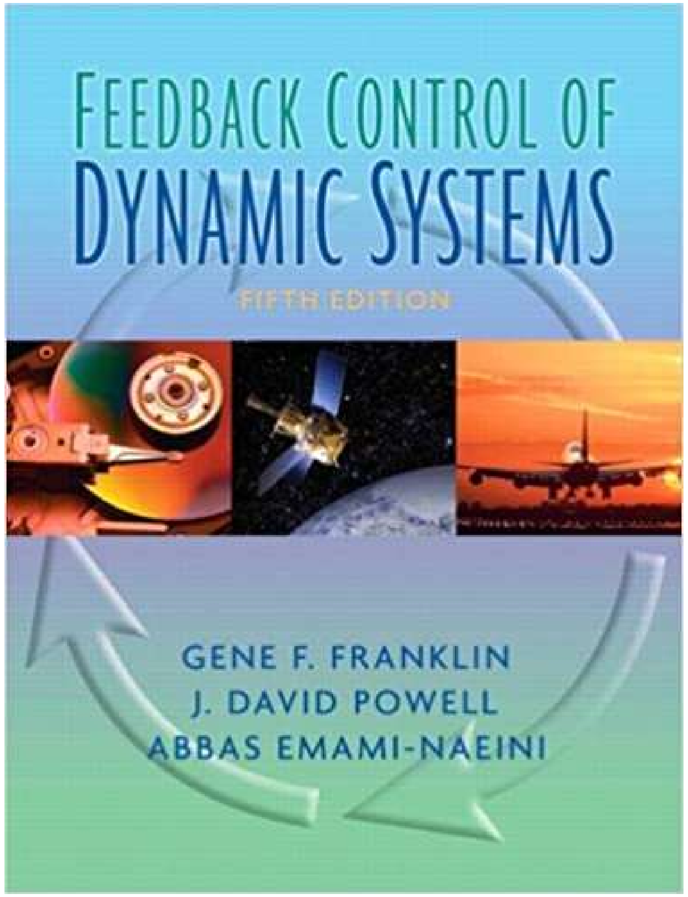
Lorenzo.masia@ziti.uni-Heidelberg.de

www.lorenzomasia.com

UNIVERSITÄT HEIDELBERG
ZUKUNFT SEIT 1386

Reading


International
Edition
Modern Control
Engineering
Fifth Edition
PEARSON
Katsuhiko Ogata


FEEDBACK CONTROL OF
DYNAMIC SYSTEMS
FIFTH EDITION
GENE F. FRANKLIN
J. DAVID POWELL
ABBAS EMAMI-NAEINI

Learning Outcomes

- Use **poles and zeros** of transfer functions to determine the time response of a control system
- Describe quantitatively the **transient response** of **first-order** systems
- Write the general response of **second-order** systems given the pole location
- Find the **damping ratio and natural frequency** of a second-order system
- Find the **settling time, peak time, percent overshoot, and rise time** for an underdamped second-order system
- Approximate higher-order systems and systems with zeros as first- or second-order systems
- Describe the **solution** for the system time response **starting from the state space equations**

Poles, Zeros, and System Response

UNIVERSITÄT HEIDELBERG
ZUKUNFT SEIT 1386

The output response of a system is the sum of two responses:
the **forced response** and the **natural response**

Although many techniques, such as solving a **differential equation** or taking the **inverse Laplace transform**, enable us to evaluate this **output response**, these techniques **are laborious and time-consuming**

The use of **poles and zeros** and their relationship to the time response of a system is such a technique.
Learning this relationship gives us a qualitative "handle" on problems

Poles and Zeros of a Transfer Function

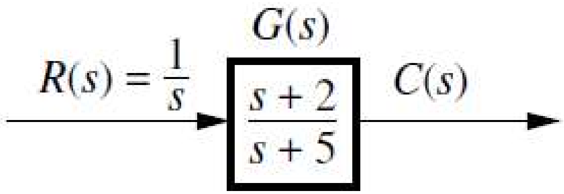
$$R(s) \longrightarrow \boxed{\frac{(b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0)}} \longrightarrow C(s)$$

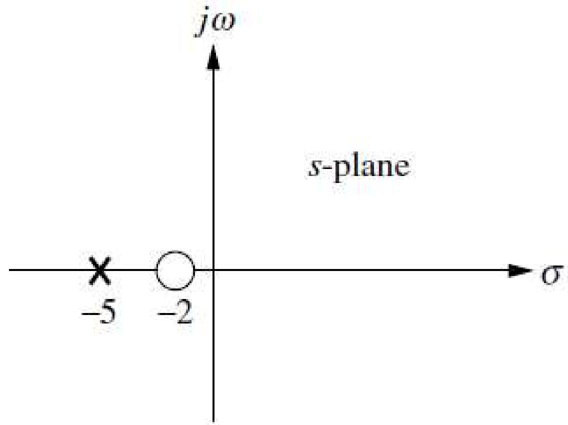
$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_m)\cdots(s+p_n)}$$

The **poles** of a transfer function are
(1) the values of the Laplace transform variable, s, that cause the transfer function to become infinite or
(2) any roots of the denominator of the transfer function that are not common to roots of the numerator.

The **zeros** of a transfer function are
(1)the values of the Laplace transform variable, s, that cause the transfer function to become zero, or
(2) any roots of the numerator of the transfer function that are not common to roots of the denominator.

Poles and Zeros of a First-Order System: An Example





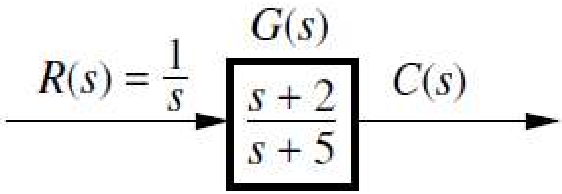
$$C(s) = \frac{(s+2)}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5} = \frac{2/5}{s} + \frac{3/5}{s+5}$$

To show the properties of the poles and zeros, let us find the unit step response of the system:

$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$

Poles and Zeros of a First-Order System: An Example

$$C(s) = \frac{2/5}{s} + \frac{3/5}{s+5}$$

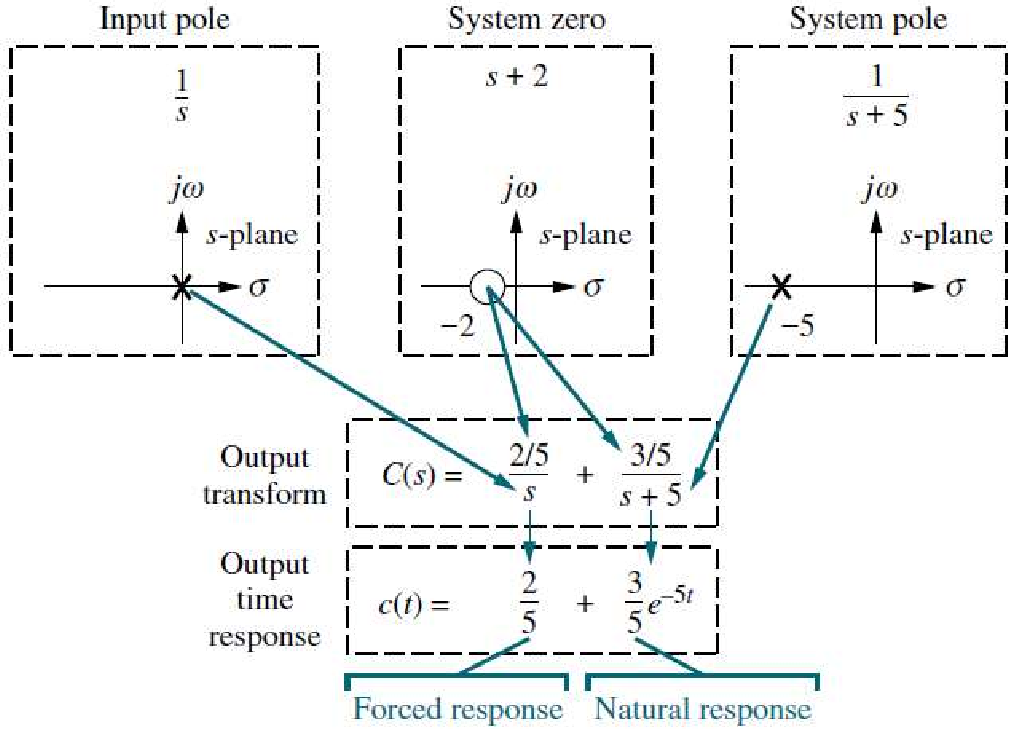


(1) A pole of the input function (1/s) → the forced response

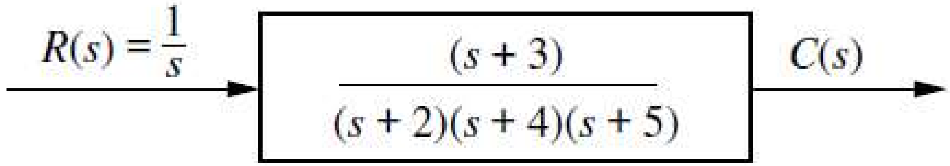
(2) A pole of the transfer function →the natural response (that is, the pole at 5 generated e^{-5t})

(3) A pole on the real axis generates an exponential response e^{-at}, where **a** is the pole location on the real axis

(4) The zeros and poles generate the amplitudes for both the forced and natural responses



Example 4.1



PROBLEM: Given the system in Figure, write the output, c(t), **in general terms (with no calculation for the partial fraction expansion)**. Specify the forced and natural parts of the solution.

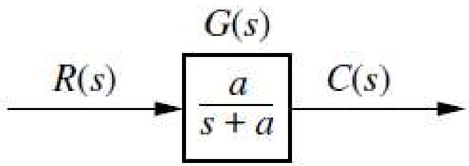
SOLUTION: By inspection, each system pole generates an exponential as part of the natural response. The input's pole generates the forced response.

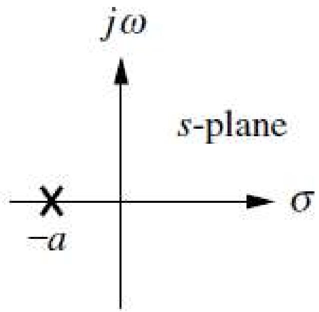
$$C(s) \equiv \underbrace{\frac{K_1}{s}}_{\text{Forced response}} + \underbrace{\frac{K_2}{s+2} + \frac{K_3}{s+4} + \frac{K_4}{s+5}}_{\text{Natural response}}$$

$$c(t) \equiv \underbrace{K_1}_{\text{Forced response}} + \underbrace{K_{2e^{-2t}} + K_{3e^{-4t}} + K_{4e^{-5t}}}_{\text{Natural response}}$$

First-Order Systems

A first-order system without zeros can be described by the transfer function shown:





Taking : R(s) = 1/s, $$C(s) = R(s)G(s) = \frac{a}{s(s+a)}$$ $$\boxed{c(t) = c_f(t) + c_n(t) = 1 - e^{-at}}$$

Let us examine the significance of parameter **a**, the only parameter needed to describe the transient response:

$$e^{-at}\big|_{t=1/a} = e^{-1} = 0.37$$

$$c(t)\big|_{t=1/a} = 1 - e^{-at}\big|_{t=1/a} = 1 - 0.37 = 0.63$$

First-Order Systems: Time Constant

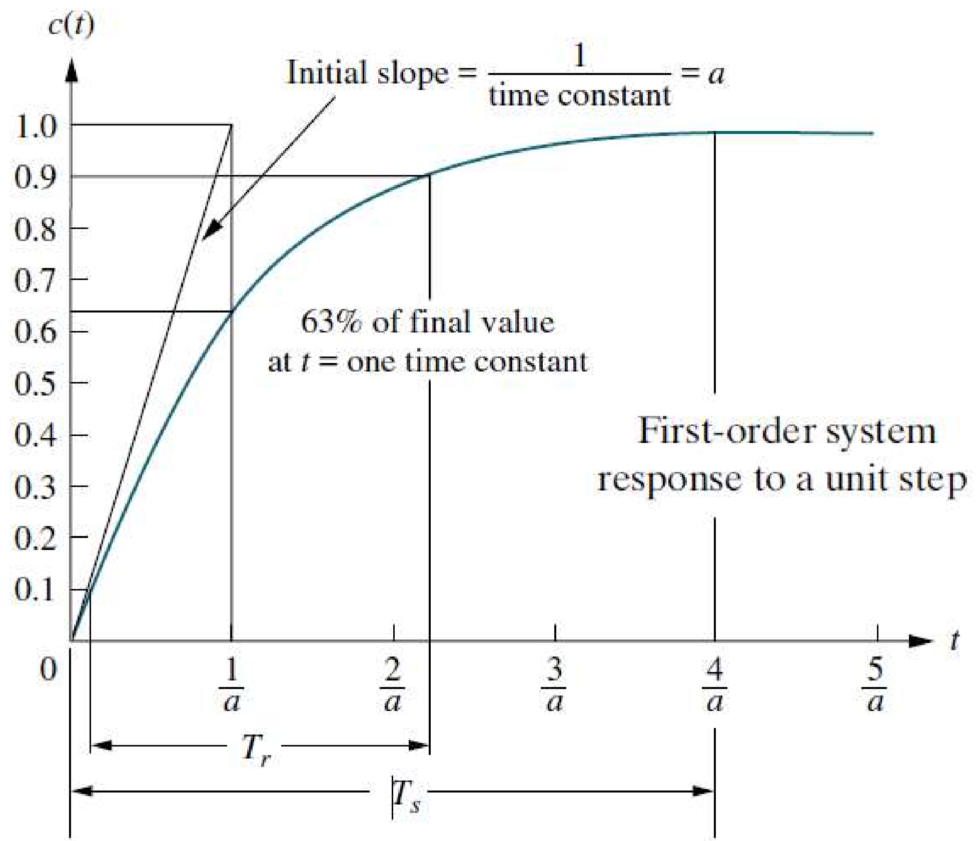
We call ***1/a*** the time constant of the response

The time for e^{-at} to decay to **37%** of its initial value.

Alternately, from:

$$c(t)|_{t=1/a} = 1 - e^{-at}|_{t=1/a} = 1 - 0.37 = 0.63$$

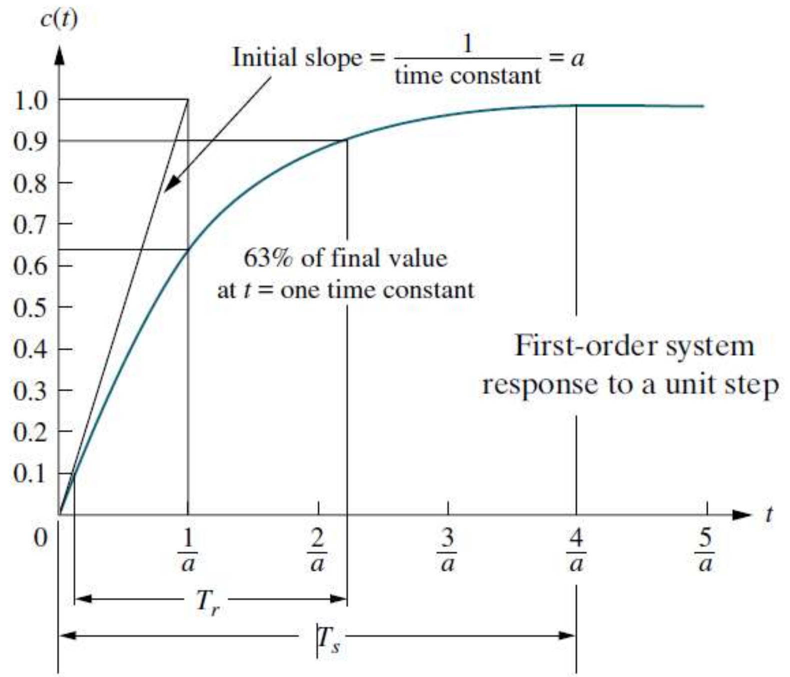
the time constant is the time it takes for the step response to rise to **63%** of its final value



First-Order Systems: Time Rise(T_r)

Rise time is defined as the time for the waveform to go from 0.1 to 0.9 of its final value.

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$



Rise time is found for the difference in time at c(t) = 0,9 and c(t) =0,1.

and solving for time, t, we find the rise time to be:

$$T_r = \frac{2.31}{a} - \frac{0.11}{a} = \frac{2.2}{a}$$

First-Order Systems: Settling Time, Ts

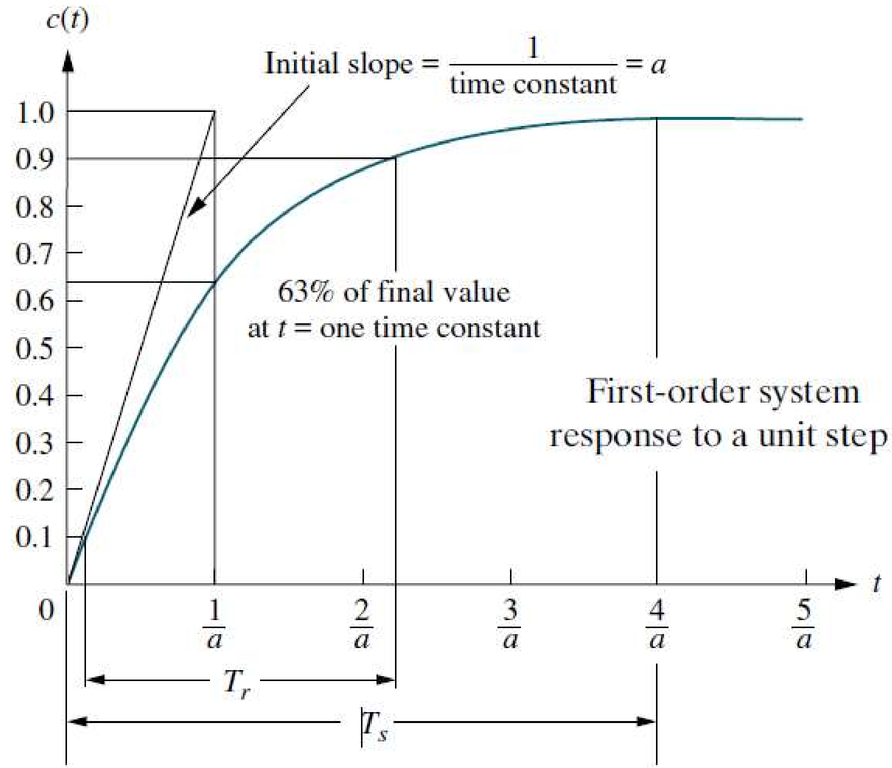
Settling time is defined as the time for the response to reach, and stay within, **2% of its final value**

Letting c(t) = 0,98 in

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

and solving for time, t, we find the settling time to be

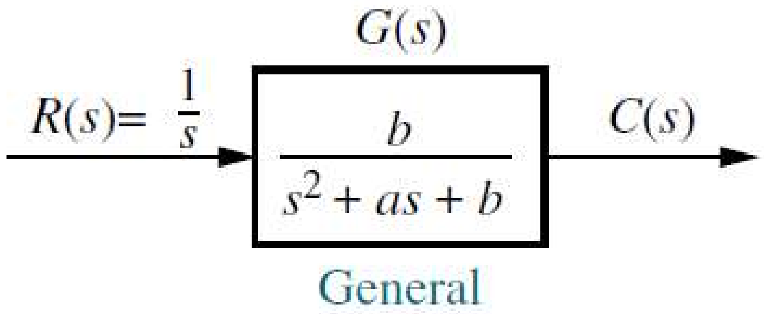
$$T_s = \frac{4}{a}$$



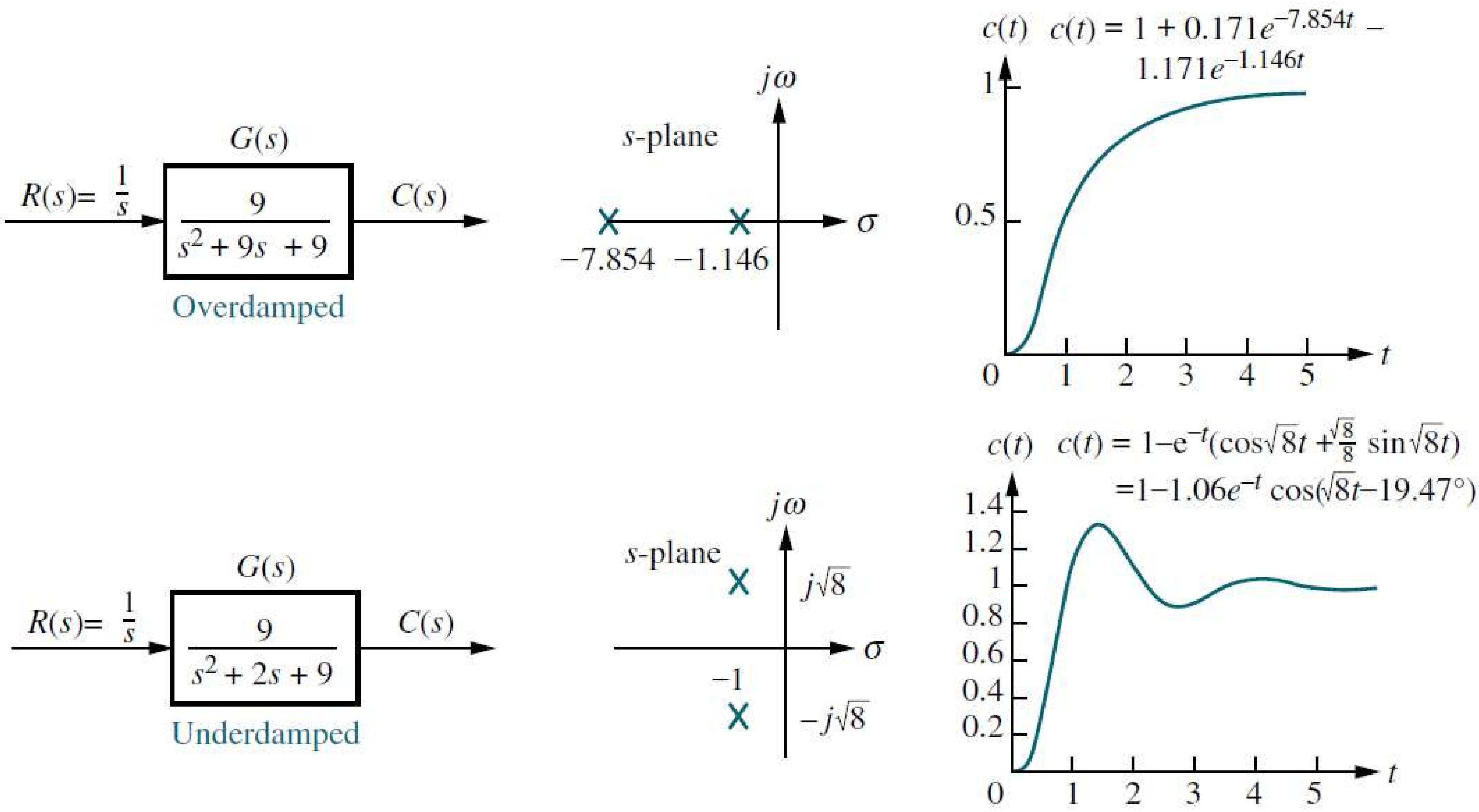
Second-Order Systems: Introduction

Compared to the simplicity of a first-order system, a second-order system exhibits a wide range of responses that must be analyzed and described.

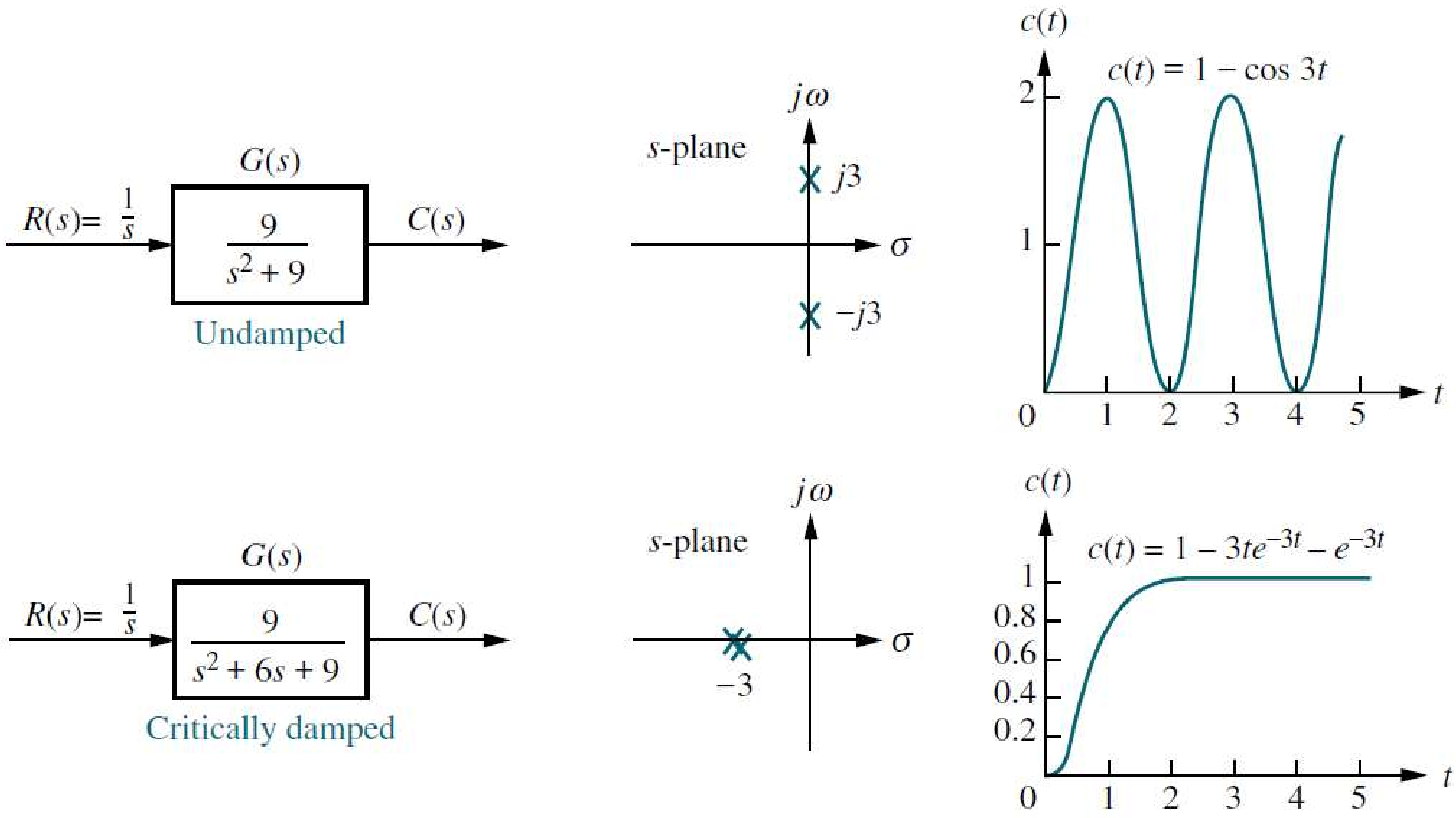
Whereas varying a **first-orde**r system's parameter simply changes the **speed** of the response, changes in the parameters of a **second-order** system can change the **form of the response**.



Second-Order Systems: Introduction



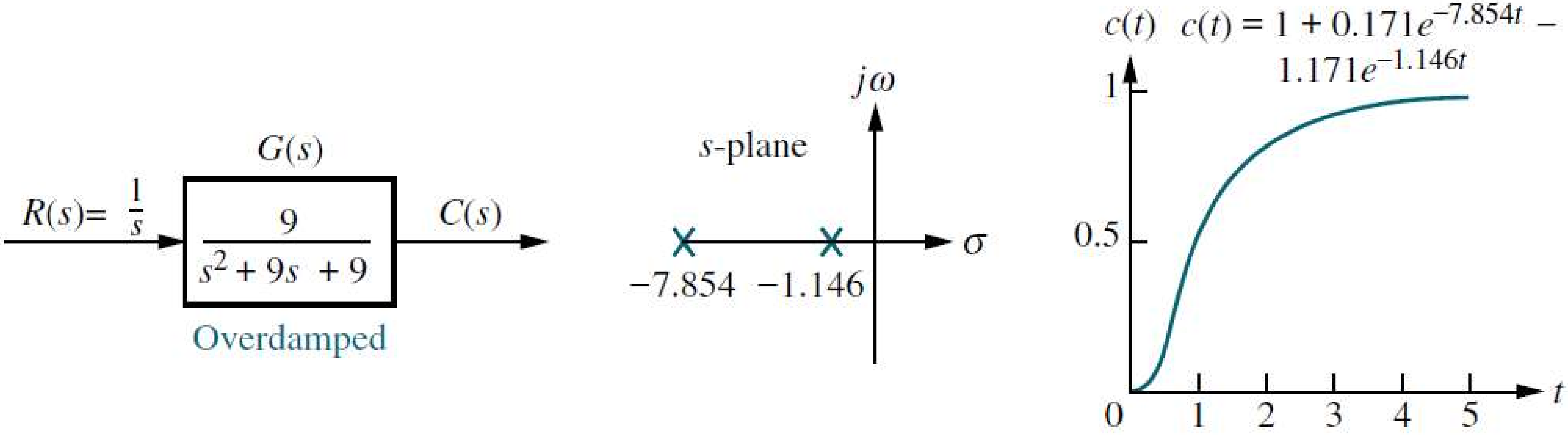
Second-Order Systems: Introduction



Second-Order Systems

Overdamped Response

$$C(s) = \frac{9}{s(s^2+9s+9)} = \frac{9}{s(s+7.854)(s+1.146)}$$

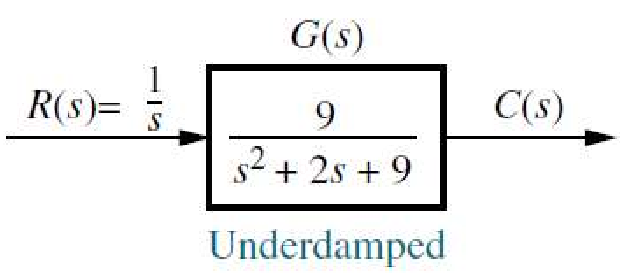


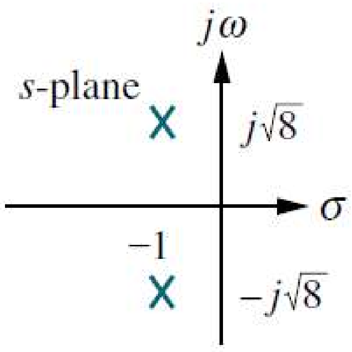
The **input pole at the origin** generates the constant **forced response**; each of the **two system poles** on the real axis generates **an exponential natural response** whose exponent is equal to the pole location.

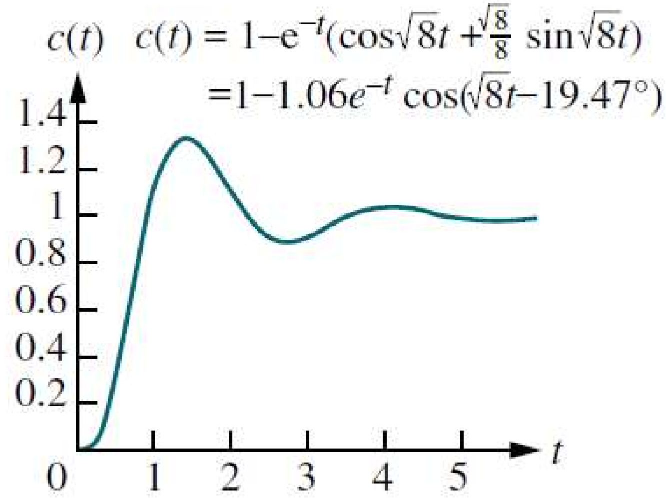
Second-Order Systems

Underdamped Response

$$C(s) = \frac{9}{s(s^2 + 2s + 9)}$$



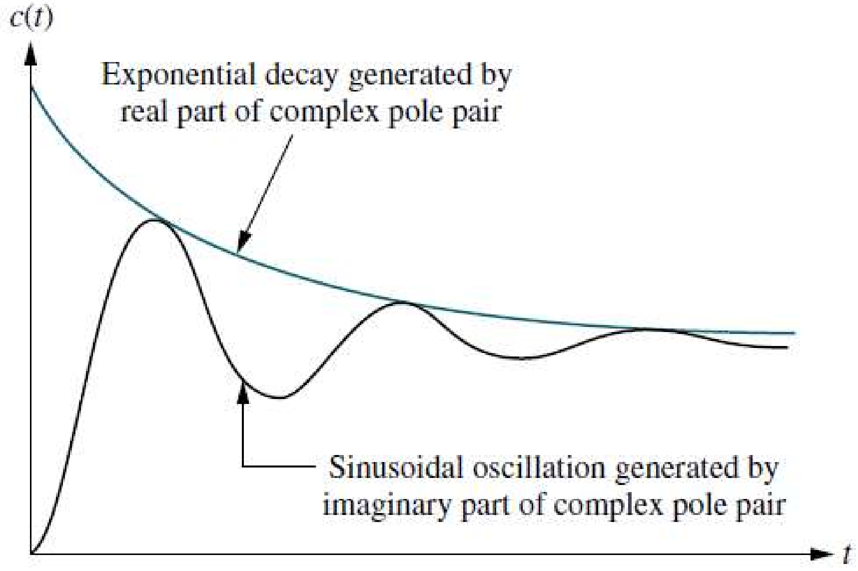




This function has a pole at the origin that comes from the unit step input and **two complex conjugate poles** that come from the system.

Let´s analyse in general what means having complex conjugate poles in terms of response.

Second-Order Systems
Underdamped Response



The **transient** response consists of

- an **exponentially** decaying amplitude generated by the real part of the system pole times
- a **sinusoidal** waveform generated by the imaginary part of the system pole.
- The **time constant** of the exponential decay is equal to the reciprocal of the **real part** of the system pole
- The value of the **imaginary part** is the actual frequency of the sinusoid

This sinusoidal frequency is given the name **damped frequency of oscillation, ω_d.**

Finally, the steady-state response (unit step) was generated by the input pole (1/s) located at the origin.

Example 4.2

Form of Underdamped Response Using Poles

PROBLEM: By inspection, write the form of the step response of the system in Figure

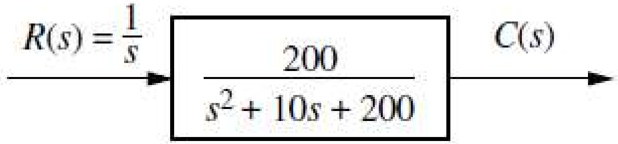


FIGURE System for Example

SOLUTION: First we determine that the form of the forced response is a step. Next we find the form of the natural response. Factoring the denominator of the transfer function in Figure ', we find the poles to be $s = -5 \pm j13.23$. The real part, -5, is the exponential frequency for the damping. It is also the reciprocal of the time constant of the decay of the oscillations. The imaginary part, 13.23, is the radian frequency for the sinusoidal oscillations.

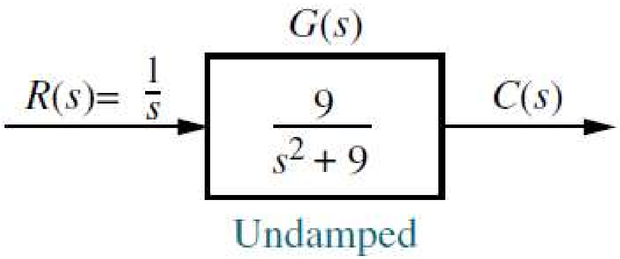
$$c(t) = K_1 + e^{-5t}(K_2 \cos 13.23t + K_3 \sin 13.23t) = K_1 + K_4 e^{-5t}(\cos 13.23t - \phi),$$

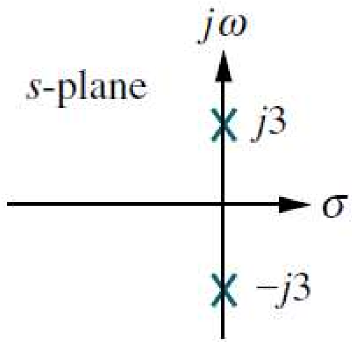
where $\phi = \tan^{-1} K_3/K_2$, $K_4 = \sqrt{K_2^2 + K_3^2}$, and $c(t)$ is a constant plus an exponentially damped sinusoid.

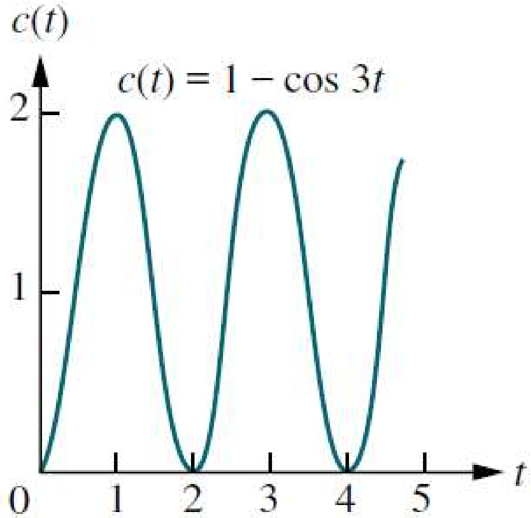
Second-Order Systems

Undamped Response

$$C(s) = \frac{9}{s(s^2+9)}$$







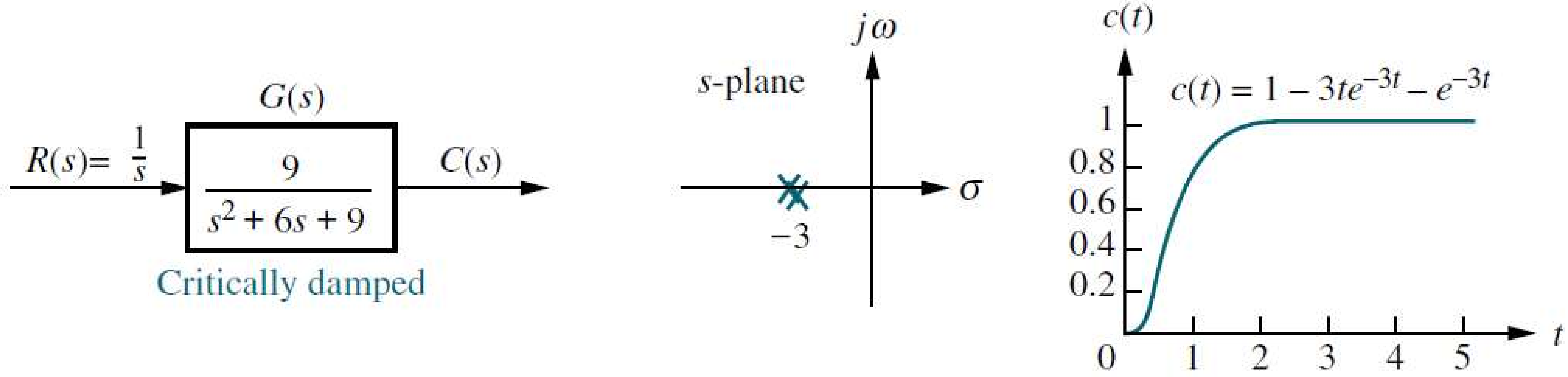
This function has **a pole at the origin** that comes from the unit step input and **two imaginary poles** that come from the system.

The **input pole at the origin** generates the **constant forced response**, and the **two system poles** on the **imaginary axis at ±j3** generate a **sinusoidal natural response** whose frequency is equal to the location of the imaginary poles.

Second-Order Systems

Critically damped Response

$$C(s) = \frac{9}{s(s^2+6s+9)} = \frac{9}{s(s+3)^2}$$



This function has **a pole at the origin** that comes from the unit step input and **two multiple real poles** that come from the system.

The input pole at the origin generates the constant forced response, and the two poles on the real axis at 3 generate a natural response consisting of an exponential and an exponential multiplied by time.

It is called **critically damped**. Critically damped responses are the **fastest possible** without the **overshoot** that is characteristic of the underdamped response.

We now summarize our observations

1. *Overdamped responses*

Poles: Two real at $-\sigma_1, \ -\sigma_2$

Natural response: Two exponentials with time constants equal to the reciprocal of the pole locations, or

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$$

2. *Underdamped responses*

Poles: Two complex at $-\sigma_d \pm j\omega_d$

Natural response: Damped sinusoid with an exponential envelope whose time constant is equal to the reciprocal of the pole's real part. The radian frequency of the sinusoid, the damped frequency of oscillation, is equal to the imaginary part of the poles, or

$$c(t) = Ae^{-\sigma_d t}\cos(\omega_d t - \phi)$$

We now summarize our observations

3. *Undamped responses*

Poles: Two imaginary at $\pm j\omega_1$

Natural response: Undamped sinusoid with radian frequency equal to the imaginary part of the poles, or

$$c(t) = A\cos(\omega_1 t - \phi)$$

4. *Critically damped responses*

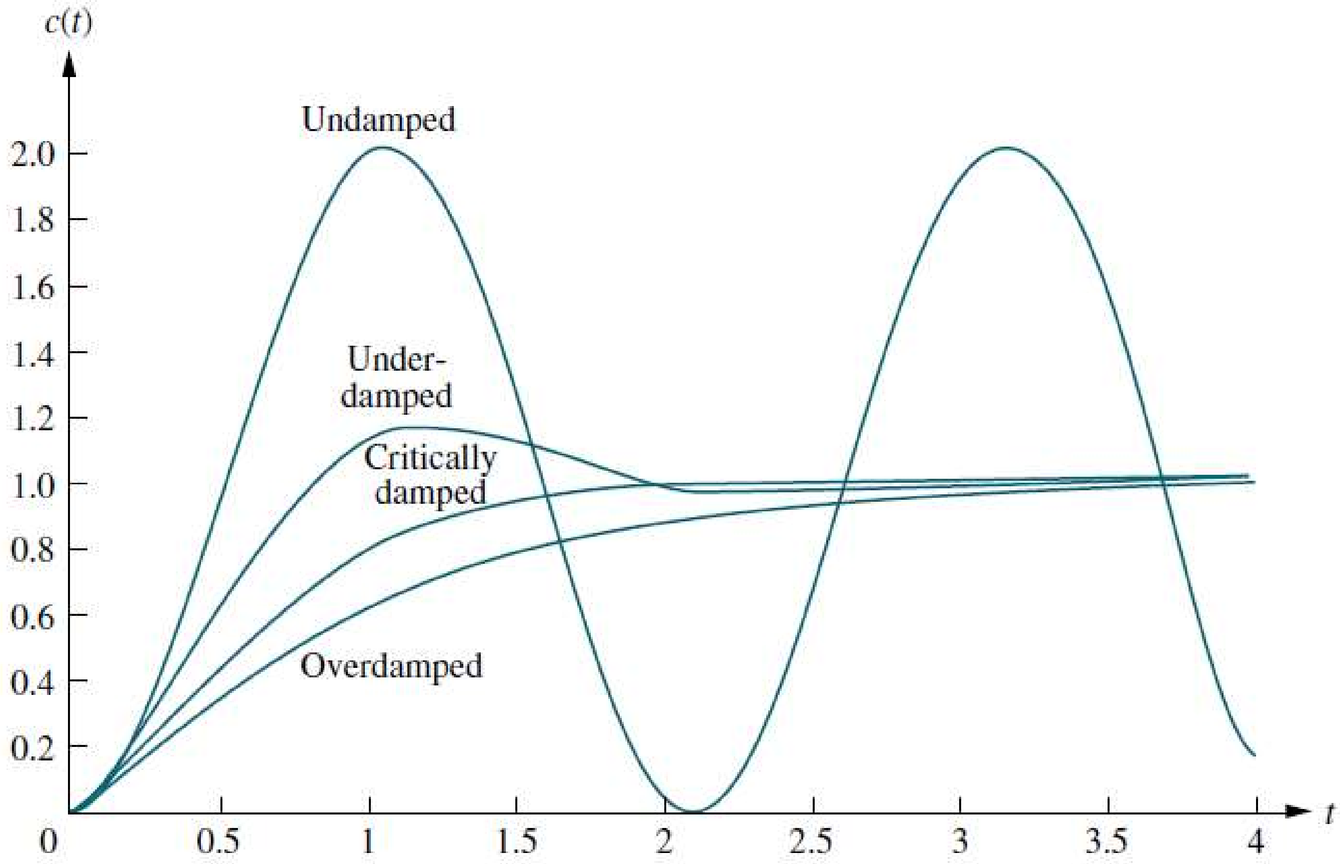
Poles: Two real at $-\sigma_1$

Natural response: One term is an exponential whose time constant is equal to the reciprocal of the pole location. Another term is the product of time, t, and an exponential with time constant equal to the reciprocal of the pole location, or

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_1 t}$$

We now summarize our observations

These responses in the figure below are always referred to a unit step forcing input **R(s)=1/s**



Step responses for second-order system damping cases

Skill-Assessment Exercise

PROBLEM: For each of the following transfer functions, write, by inspection, the general form of the step response:

a. $G(s) = \dfrac{400}{s^2 + 12s + 400}$

b. $G(s) = \dfrac{900}{s^2 + 90s + 900}$

c. $G(s) = \dfrac{225}{s^2 + 30s + 225}$

d. $G(s) = \dfrac{625}{s^2 + 625}$

ANSWERS:

a. $c(t) = A + Be^{-6t}\cos(19.08t + \phi)$

b. $c(t) = A + Be^{-78.54t} + Ce^{-11.46t}$

c. $c(t) = A + Be^{-15t} + Cte^{-15t}$

d. $c(t) = A + B\cos(25t + \phi)$

The General Second-Order System

- we generalize the discussion and establish quantitative specifications defined in such a way that the response of a second-order system can be described to a designer without the need for sketching the response
- we define two physically meaningful specifications for second-order systems
- The two quantities are called natural frequency $\boldsymbol{\omega_n}$ and damping ratio $\boldsymbol{\zeta}$

Natural Frequency , $\boldsymbol{\omega_n}$
Damping Ratio, ζ

The natural frequency of a second-order system is the frequency of oscillation of the system without damping

We define the damping ratio:

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/second)}} = \frac{1}{2\pi}\frac{\text{Natural period (seconds)}}{\text{Exponential time constant}}$$

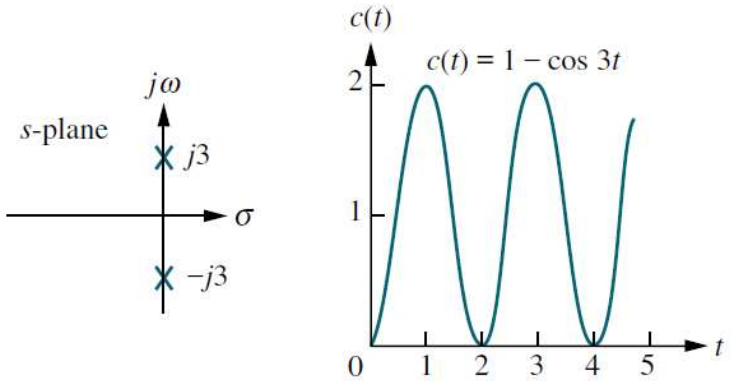
$$G(s) = \frac{b}{s^2 + as + b}$$

For pure imaginary poles with a=0 then we have :

$$G(s) = \frac{b}{s^2 + b}$$

$$\omega_n = \sqrt{b}$$

$$b = \omega_n^2$$



Damping Ratio, ζ

$$G(s) = \frac{b}{s^2 + as + b}$$

Now what is the term ***a*** in Eq?

Assuming an underdamped system, the complex poles (σ ±jω) have σ real part equal to ***–a/2.***

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/second)}} = \frac{|\sigma|}{\omega_n} = \frac{a/2}{\omega_n}$$

$$a = 2\zeta\omega_n$$

Our general second-order transfer function finally looks like this:

$$\boxed{G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

Solving for the poles of the transfer function:

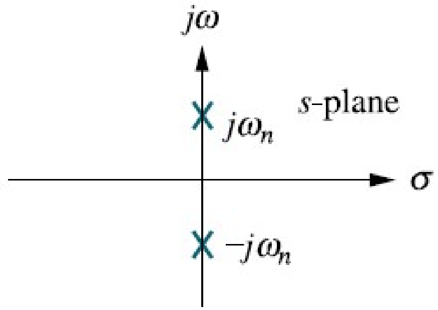
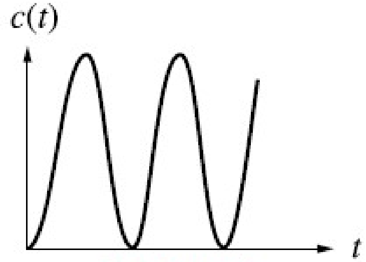
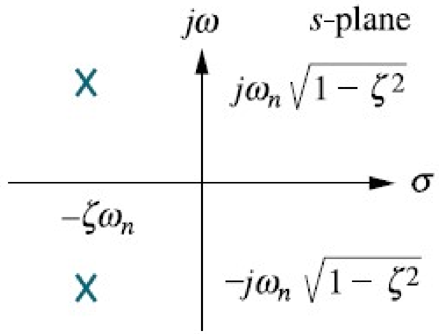
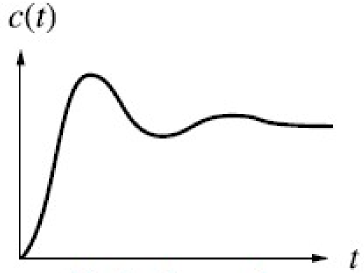
$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Damping Ratio, ζ

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/second)}} = \frac{|\sigma|}{\omega_n} = \frac{a/2}{\omega_n}$$

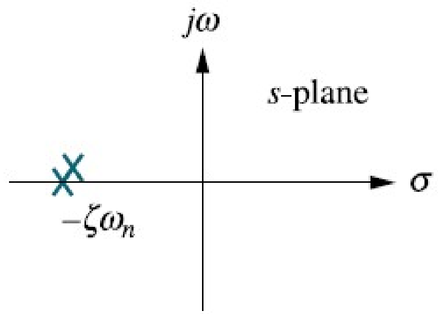
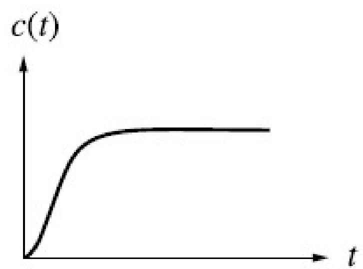
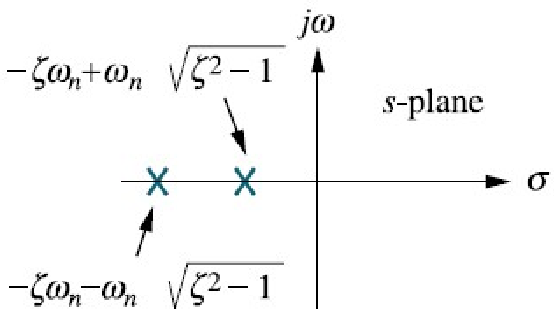
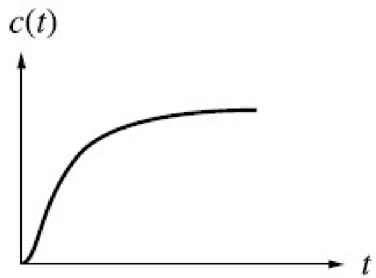
$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

ζ	Poles	Step response
0		 Undamped
$0 < \zeta < 1$		 Underdamped

Damping Ratio, ζ

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

ζ	Poles	Step response
$\zeta = 1$		 Critically damped
$\zeta > 1$		 Overdamped

Example

UNIVERSITÄT HEIDELBERG
ZUKUNFT SEIT 1386

Characterizing Response from the Value of ζ

PROBLEM: For each of the systems shown in Figure , find the value of ζ and report the kind of response expected.

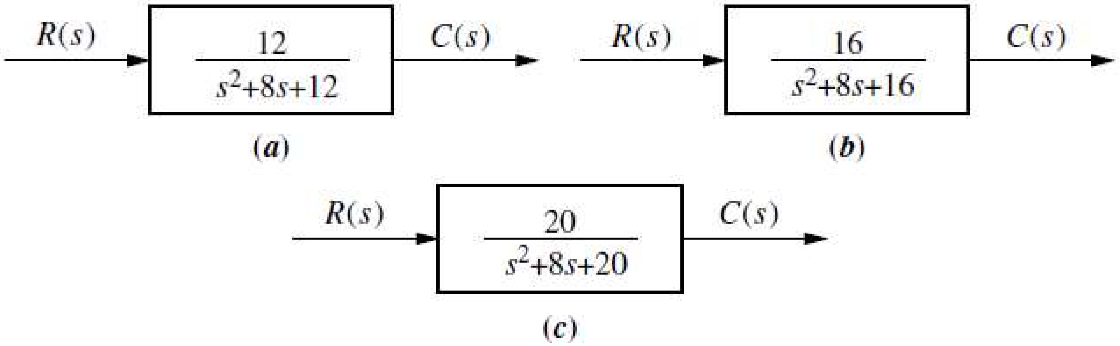


FIGURE Systems for Example

SOLUTION: Since $a = 2\zeta\omega_n$ and $\omega_n = \sqrt{b}$,

$$\zeta = \frac{a}{2\sqrt{b}}$$

Using the values of a and b from each of the systems of Figure , we find $\zeta = 1.155$ for system (a), which is thus overdamped, since $\zeta > 1$; $\zeta = 1$ for system (b), which is thus critically damped; and $\zeta = 0.894$ for system (c), which is thus underdamped, since $\zeta < 1$.

Digressions on: Underdamped Second-Order Systems

UNIVERSITÄT HEIDELBERG
ZUKUNFT SEIT 1386

The underdamped second order system (**$\zeta<1$** and **$s_{1,2}=\sigma \pm j\omega$**), a **common model for physical problems**, displays unique **behaviour** that must be **itemized**; a detailed description of the underdamped response is necessary for both analysis and design.

Let us begin by finding the step response for the general second-order system

$$\boxed{G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Expanding by partial fractions

$$C(s) = \frac{1}{s} - \frac{(s + \zeta\omega_n) + \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

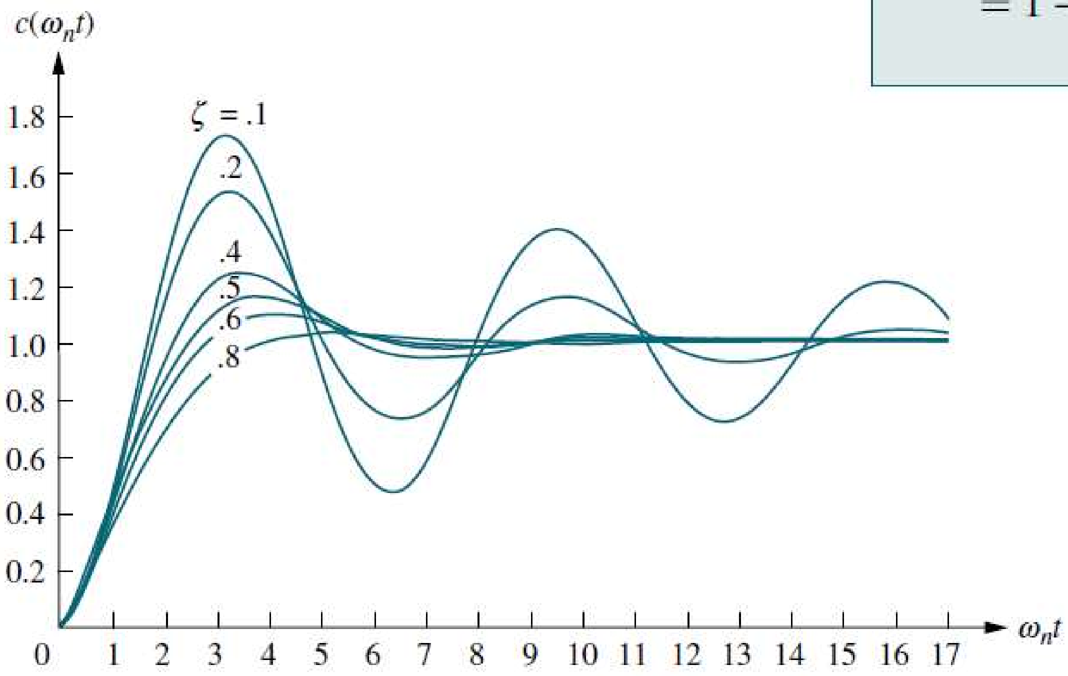
Digressions on: Underdamped Second-Order Systems

Taking the inverse Laplace transform

$$\phi = \tan^{-1}(\zeta/\sqrt{1-\zeta^2}).$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos \omega_n\sqrt{1-\zeta^2}t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin \omega_n\sqrt{1-\zeta^2}t\right)$$

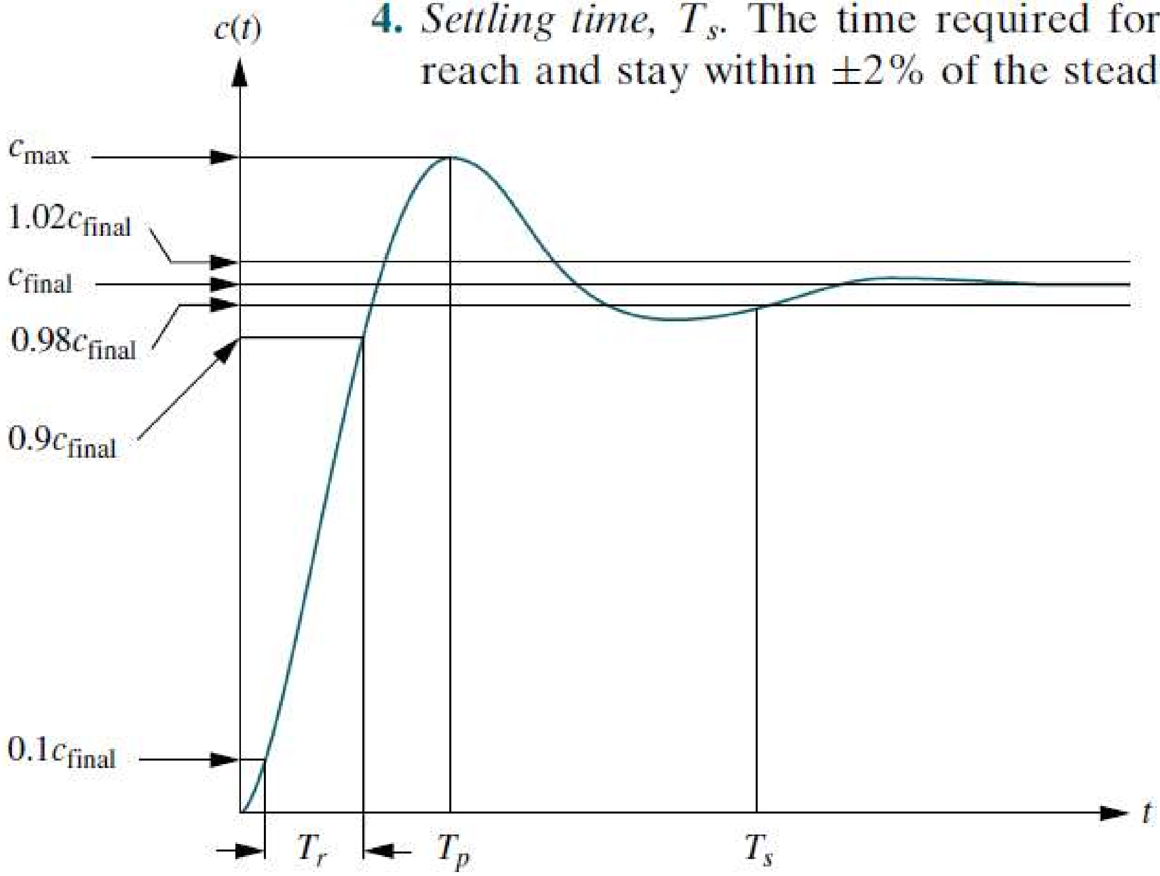
$$= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2}t - \phi)$$



Second-order underdamped responses for damping ratio values

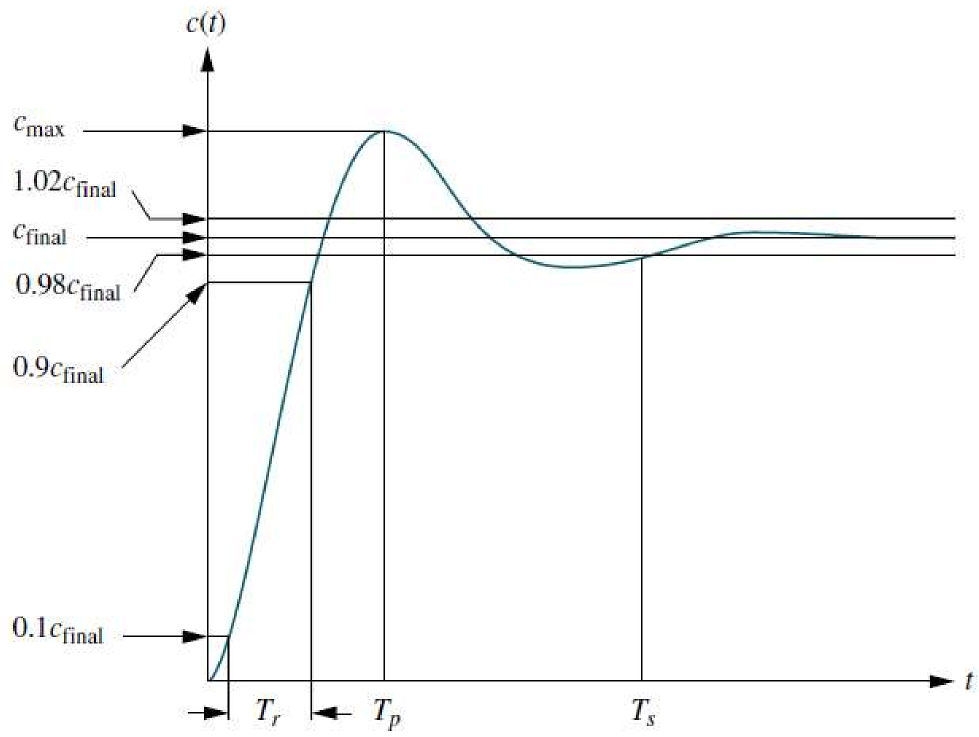
Associated parameters

1. *Rise time, T_r.* The time required for the waveform to go from 0.1 of the final value to 0.9 of the final value.
2. *Peak time, T_P.* The time required to reach the first, or maximum, peak.
3. *Percent overshoot, %OS.* The amount that the waveform overshoots the steady-state, or final, value at the peak time, expressed as a percentage of the steady-state value.
4. *Settling time, T_s.* The time required for the transient's damped oscillations to reach and stay within ±2% of the steady-state value.



Notes on the Second Order System

- All definitions are also valid for systems of order higher than 2,
- <u>analytical expressions for these parameters cannot be found unless the response of the higher-order system can be approximated as a second-order system</u>
- Rise time, peak time, and settling time yield information about the speed of the **transient response**.
- But also the **settling time gives information on the capacity of the system to stabilize on an input**



Evaluation of Tp

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos(\omega_n \sqrt{1-\zeta^2}\, t - \phi)$$

UNIVERSITÄT
HEIDELBERG
ZUKUNFT
SEIT 1386

T_p is found by differentiating c(t) and finding the first zero crossing (peak)

$$\mathscr{L}[\dot{c}(t)] = sC(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\mathscr{L}[\dot{c}(t)] = \frac{\omega_n^2}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} = \frac{\frac{\omega_n}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

$$\dot{c}(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t$$

Setting the derivative to zero

$$\omega_n \sqrt{1-\zeta^2}\, t = n\pi$$

$$t = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Each value of n yields the time for local maxima or minima. Letting n=1 the first peak, which occurs at the peak time, T_p

$$\boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}$$

Evaluation of %OS

The percent overshoot, %OS, is given by $\%OS = \frac{c_{\text{max}} - c_{\text{final}}}{c_{\text{final}}} \times 100$

The term c_{max} is found by evaluating c(t) at the peak time, $c(T_p)$.

For the unit step $c_{\text{final}} = 1$

$$c_{\text{max}} = c(T_p) = 1 - e^{-(\zeta\pi/\sqrt{1-\zeta^2})}\left(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right)$$
$$= 1 + e^{-(\zeta\pi/\sqrt{1-\zeta^2})}$$

Applying the formula $\boxed{\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100}$

Notice that the percent overshoot is a function only of the damping ratio, **ζ**.

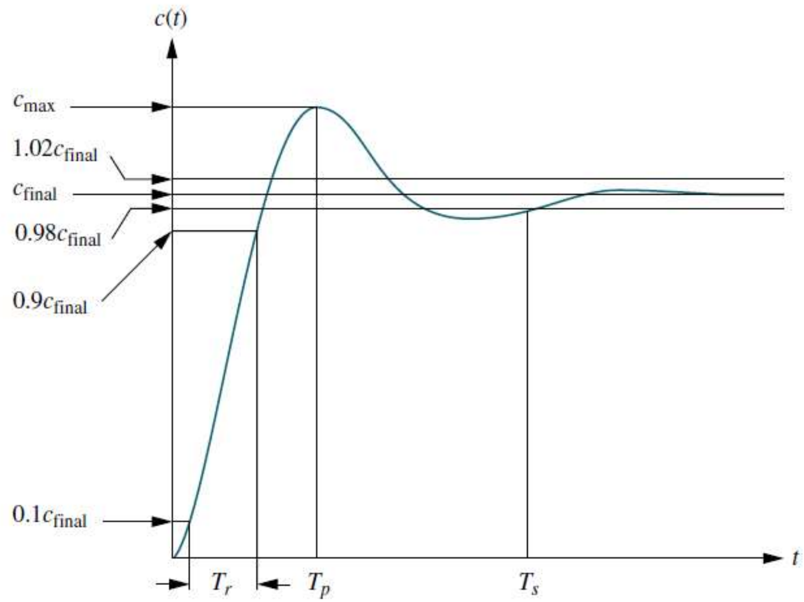
The inverse is given by $\boxed{\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}}$

Evaluation of Ts

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos(\omega_n \sqrt{1-\zeta^2}\, t - \phi)$$

In order to find the settling time, we must find the time for which *c(t)* reaches and stays within 2% of the steady-state value, **C**final.

$$e^{-\zeta\omega_n t} \frac{1}{\sqrt{1-\zeta^2}} = 0.02 \qquad T_s = \frac{-\ln(0.02\sqrt{1-\zeta^2})}{\zeta\omega_n}$$

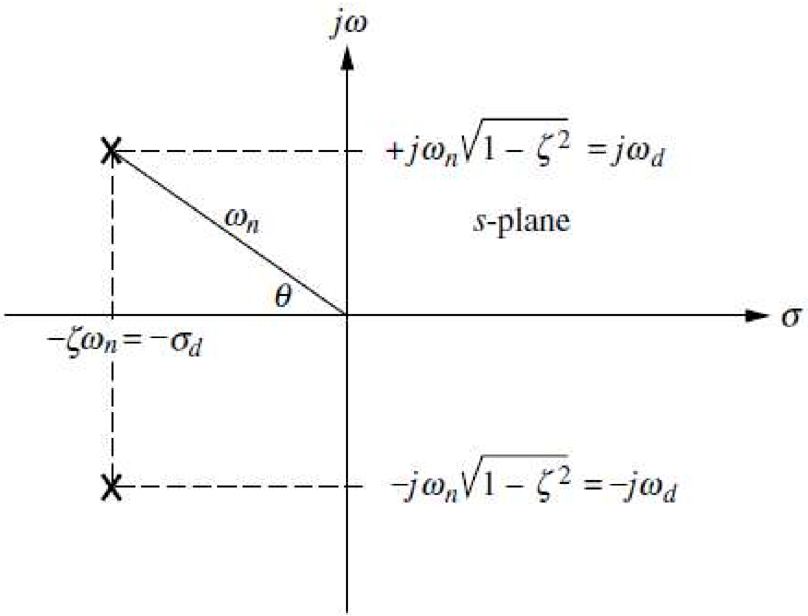


Let us agree on an approximation for the settling time that will be used for all values of **ζ**; let it be:

$$\boxed{T_s = \frac{4}{\zeta\omega_n}}$$

Relating the poles to the response

We now have expressions that relate peak time, percent overshoot, and settling time to the natural frequency and the damping ratio.
Now let us relate these quantities to the location of the poles that generate these characteristics.



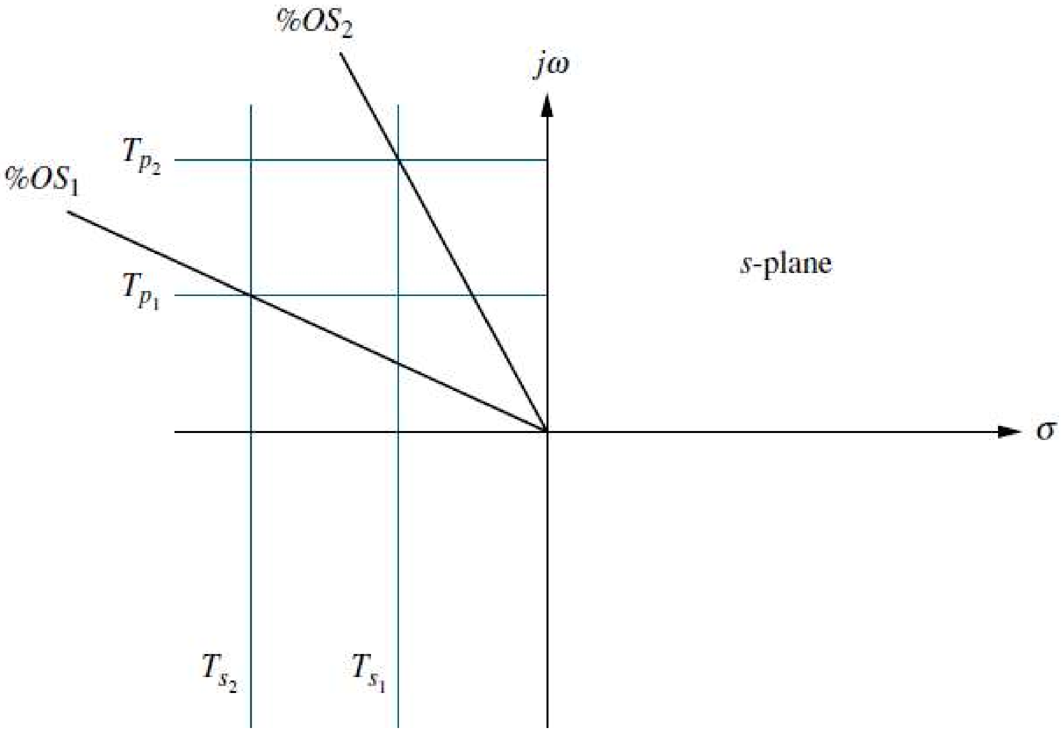
Pole plot for an underdamped second-order system

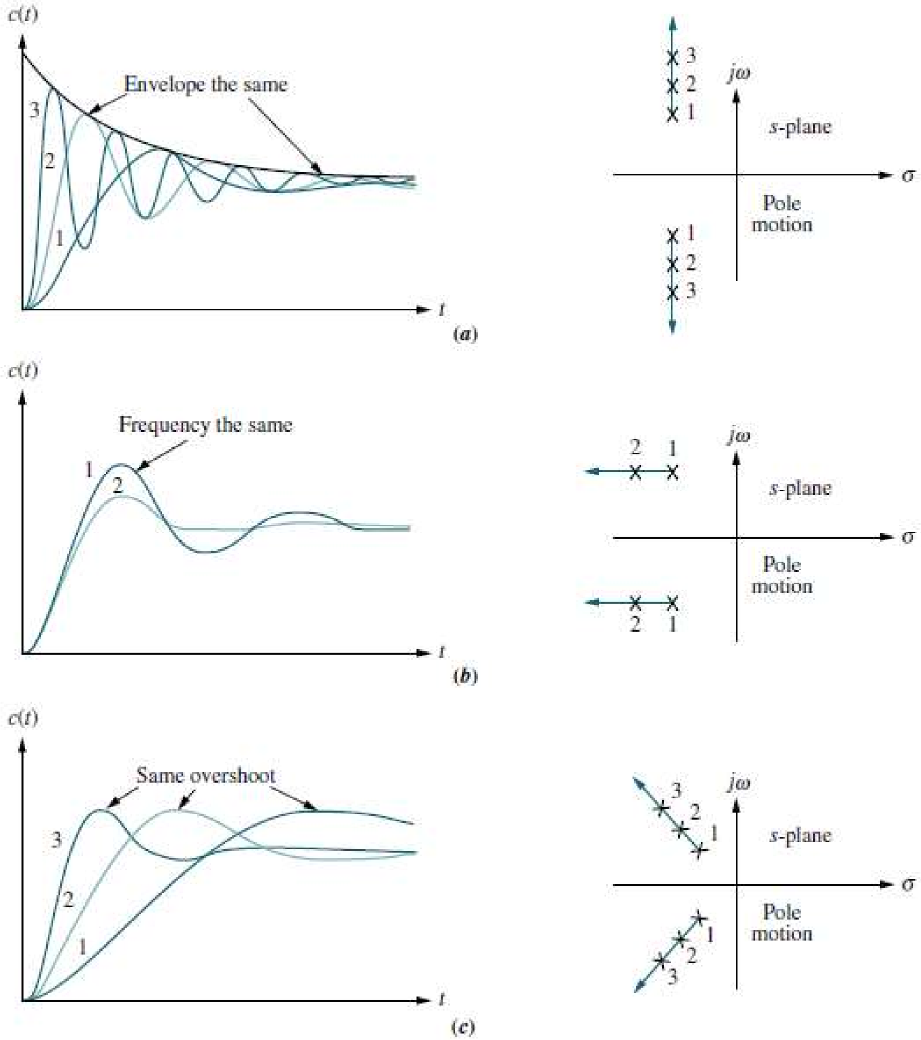
the radial distance from the origin to the pole is the natural frequency, ω_n, and the $\cos(\theta) = \zeta$.

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{\pi}{\sigma_d}$$

Relating the poles to the response





$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100$$

$$T_s = \frac{-\ln(0.02\sqrt{1-\zeta^2})}{\zeta\omega_n}$$

Example

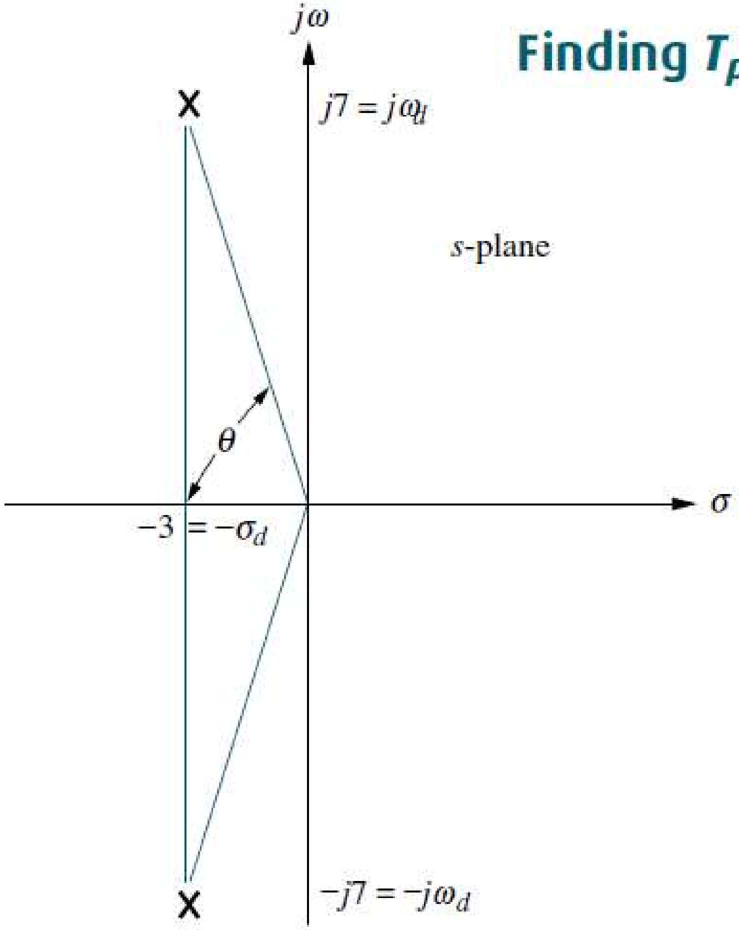


FIGURE Pole plot for Example 4.6

Finding T_p, %OS, and T_s from Pole Location

PROBLEM: Given the pole plot shown in Figure , find ζ, ω_n, T_p, $\%OS$, and T_s.

SOLUTION: The damping ratio is given by $\zeta = \cos\theta = \cos[\arctan(7/3)] = 0.394$. The natural frequency, ω_n, is the radial distance from the origin to the pole, or $\omega_n = \sqrt{7^2 + 3^2} = 7.616$. The peak time is

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{7} = 0.449 \text{ second}$$

The percent overshoot is

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100 = 26\%$$

The approximate settling time is

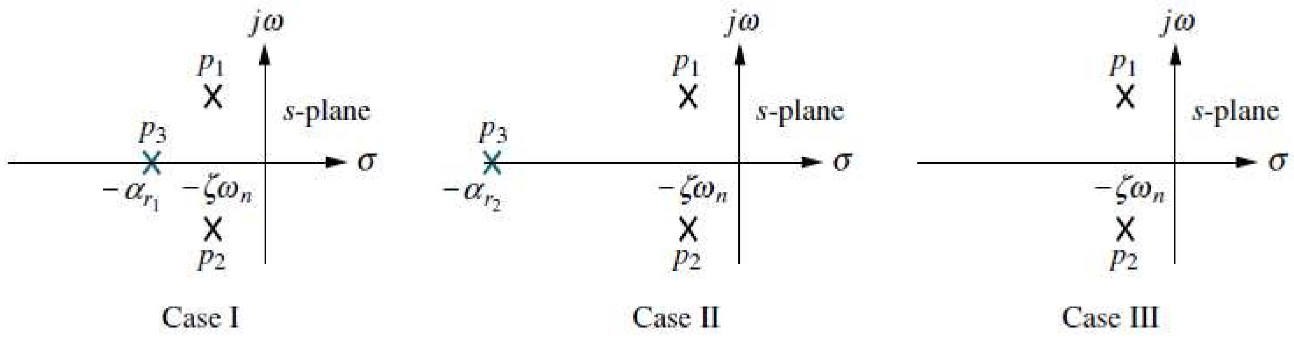
$$T_s = \frac{4}{\sigma_d} = \frac{4}{3} = 1.333 \text{ seconds}$$

System Response with Additional Poles

If a system has more than two poles or has zeros, we cannot use the formulas to calculate the performance specifications that we derived.

However, under certain conditions, a system with more than two poles or with zeros can be approximated to a second order system and **the dominant poles** are the one imposing the dynamics

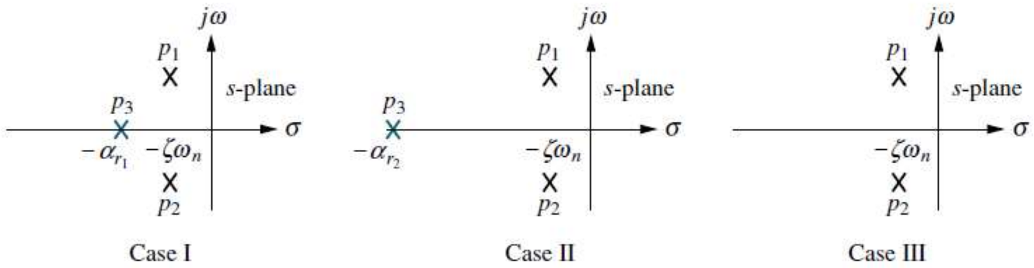
$$C(s) = \frac{A}{s} + \frac{B(s+\zeta\omega_n) + C\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2} + \frac{D}{s+\alpha_r} \qquad c(t) = Au(t) + e^{-\zeta\omega_n t}(B\cos\omega_d t + C\sin\omega_d t) + De^{-\alpha_r t}$$

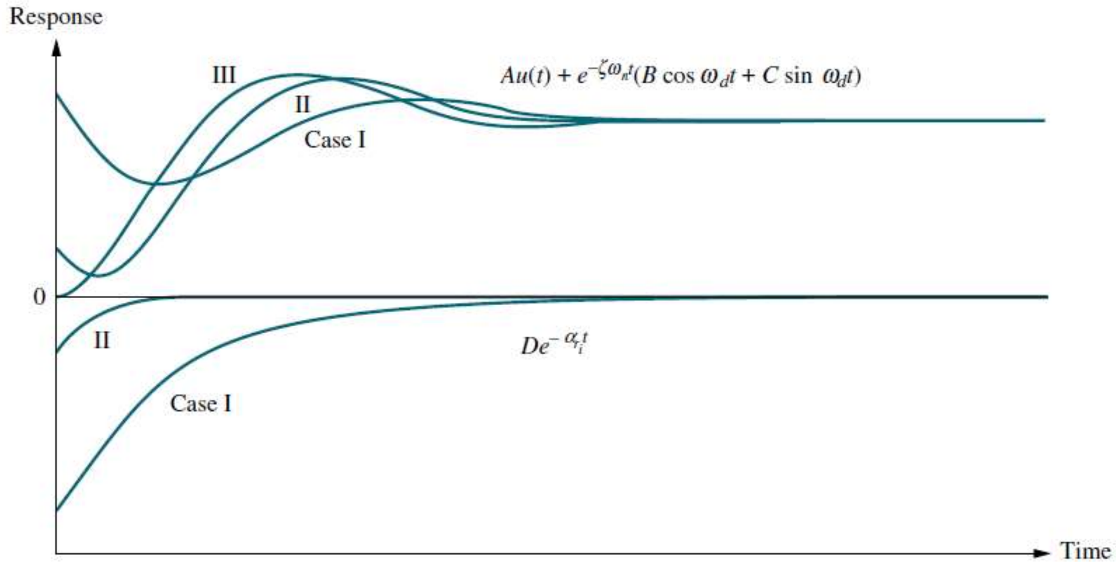


Three possible cases depending where the pole a_r is located

System Response with Additional Poles

$$c(t) = Au(t) + e^{-\zeta\omega_n t}(B\cos\omega_d t + C\sin\omega_d t) + De^{-\alpha_r t}$$





System Response with Additional Poles

example **Comparing Responses of Three-Pole Systems**

$$T_1(s) = \frac{24.542}{s^2 + 4s + 24.542} \qquad T_2(s) = \frac{245.42}{(s+10)(s^2 + 4s + 24.542)} \qquad T_3(s) = \frac{73.626}{(s+3)(s^2 + 4s + 24.542)}$$

The step response, C_i (s), for the transfer function, T_i (S), can be found by multiplying the transfer function by 1/s and then inverse Laplace transform

$$c_1(t) = 1 - 1.09e^{-2t}\cos(4.532t - 23.8^\circ)$$
$$c_2(t) = 1 - 0.29e^{-10t} - 1.189e^{-2t}\cos(4.532t - 53.34^\circ)$$
$$c_3(t) = 1 - 1.14e^{-3t} + 0.707e^{-2t}\cos(4.532t + 78.63^\circ)$$

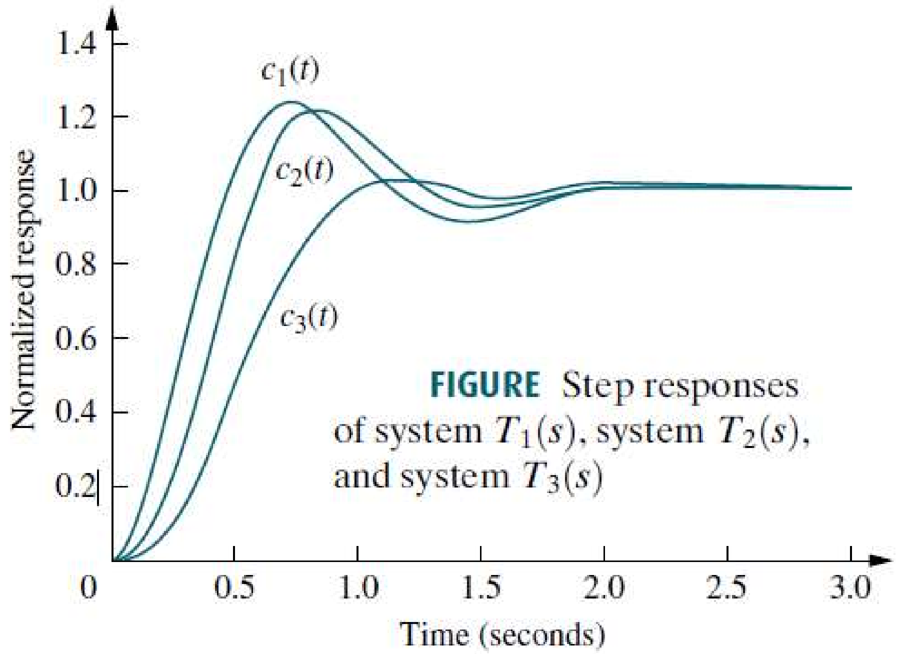


FIGURE Step responses of system $T_1(s)$, system $T_2(s)$, and system $T_3(s)$

System Response With Zeros

Now that we have seen the effect of an additional pole, let us add a zero to the second-order system.

$$T(s) = \frac{(s+a)}{(s+b)(s+c)}$$

In this section, we add a **real-axis zero to a two-pole system**.

- The zero will be added first in **the left half-plane** and then in the right half-plane and its effects noted and analysed.

System Response With Zeros

$$T(s) = \frac{(s+a)}{(s+b)(s+c)} = \frac{A}{s+b} + \frac{B}{s+c}$$

$$= \frac{(-b+a)/(-b+c)}{s+b} + \frac{(-c+a)/(-c+b)}{s+c}$$

If the zero is far from the poles, then ***a*** is large compared to ***b*** and ***c***, and

$$T(s) \approx a\left[\frac{1/(-b+c)}{s+b} + \frac{1/(-c+b)}{s+c}\right] = \frac{a}{(s+b)(s+c)}$$

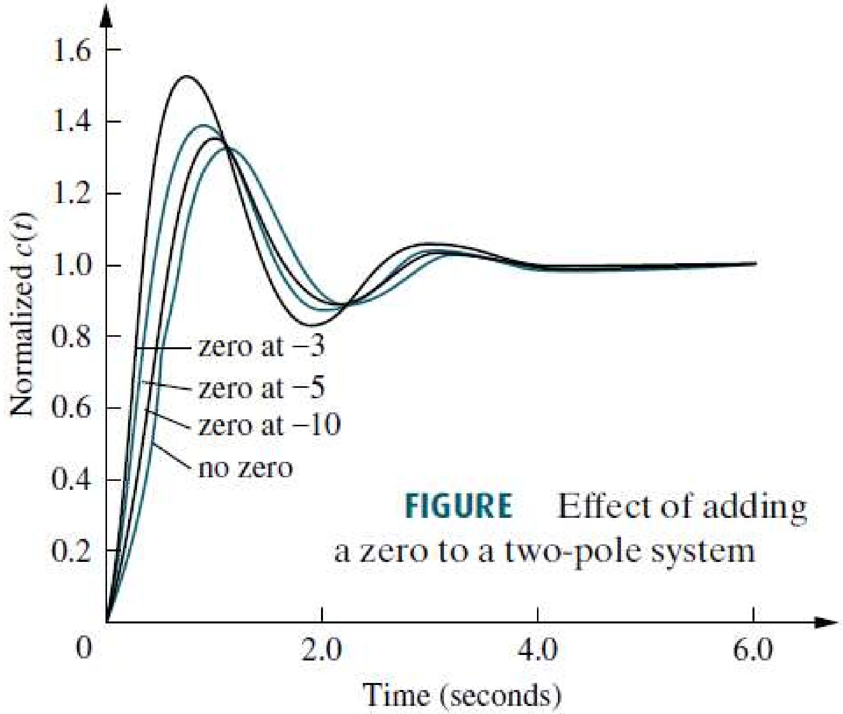


FIGURE Effect of adding a zero to a two-pole system

System Response With Zeros

Another way to look at the effect of a zero, which is more general. Let **C(s)** be the response of a system, **T(s)**. If we add a zero to the transfer function, yielding ***(s+a)T(s)***, the Laplace transform of the new response will be:

$$\boxed{(s+a)C(s) = sC(s) + aC(s)}$$

Thus, the response of a system with a zero consists of two parts: the **derivative of the original response** and a **scaled version of the original response**

- If a, the negative of the zero, is very large, the Laplace transform of the response is approximately ***aC(s)***
- If a is not very large, the response has an additional component consisting of the derivative of the original response.
- As ***a*** becomes smaller, the derivative term contributes more to the response and has a greater effect.

Laplace Transform Solution of State Equations

UNIVERSITÄT HEIDELBERG ZUKUNFT SEIT 1386

Consider the state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)$$

$$\boxed{\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)}$$

$$\boxed{\begin{aligned}\mathbf{X}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \\ &= \frac{\operatorname{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}[\mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)]\end{aligned}}$$

Let s write in form of transfer function output/input:

$$\frac{Y(s)}{U(s)} = \mathbf{C}\left[\frac{\operatorname{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}\right]\mathbf{B} + \mathbf{D} = \frac{\mathbf{C}\operatorname{adj}(s\mathbf{I} - \mathbf{A})\mathbf{B} + \mathbf{D}\det(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}$$

The roots of the denominator of the above Equation are the poles of the system which are the EIGEN VALUES OF THE STATE MATRX **A**.

Example Laplace Transform Solution; Eigenvalues and Poles

PROBLEM: Given the system represented in state space by Eqs.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t}$$

$$y = [1 \quad 1 \quad 0]\mathbf{x}$$

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

a. Solve the preceding state equation and obtain the output for the given exponential input.

b. Find the eigenvalues and the system poles.

Example Laplace Transform Solution; Eigenvalues and Poles

UNIVERSITÄT HEIDELBERG
ZUKUNFT SEIT 1386

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t}$$

$$s\mathbf{I} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix}$$

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 24 & 26 & s+9 \end{bmatrix}$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\begin{bmatrix} (s^2 + 9s + 26) & (s+9) & 1 \\ -24 & s^2 + 9s & s \\ -24s & -(26s + 24) & s^2 \end{bmatrix}}{s^3 + 9s^2 + 26s + 24}$$

Since $\mathbf{U}(s)$ (the Laplace transform for e^{-t}) is $1/(s+1)$, $\mathbf{X}(s)$ can be calculated.

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)]$$

Example

Laplace Transform Solution; Eigenvalues and Poles

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)]$$

using B and x(0)

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t} \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$X_1(s) = \frac{(s^3 + 10s^2 + 37s + 29)}{(s+1)(s+2)(s+3)(s+4)}$$

$$X_2(s) = \frac{(2s^2 - 21s - 24)}{(s+1)(s+2)(s+3)(s+4)}$$

$$X_3(s) = \frac{s(2s^2 - 21s - 24)}{(s+1)(s+2)(s+3)(s+4)}$$

The output equation is found

$$y = [1 \quad 1 \quad 0]\mathbf{x}$$

$$Y(s) = [1 \quad 1 \quad 0]\begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix} = X_1(s) + X_2(s)$$

Example

Laplace Transform Solution; Eigenvalues and Poles

UNIVERSITÄT HEIDELBERG
ZUKUNFT SEIT 1386

$$Y(s) = [1 \quad 1 \quad 0] \begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix} = X_1(s) + X_2(s)$$

$$X_1(s) = \frac{(s^3 + 10s^2 + 37s + 29)}{(s+1)(s+2)(s+3)(s+4)}$$

$$X_2(s) = \frac{(2s^2 - 21s - 24)}{(s+1)(s+2)(s+3)(s+4)}$$

$$Y(s) = \frac{(s^3 + 12s^2 + 16s + 5)}{(s+1)(s+2)(s+3)(s+4)} = \frac{-6.5}{s+2} + \frac{19}{s+3} - \frac{11.5}{s+4}$$

$$\boxed{y(t) = -6.5e^{-2t} + 19e^{-3t} - 11.5e^{-4t}}$$

b. Find the eigenvalues and the system poles.

$$\frac{Y(s)}{U(s)} = \mathbf{C}\left[\frac{\operatorname{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}\right]\mathbf{B} + \mathbf{D} \qquad Y(s) = \frac{-6.5}{s+2} + \frac{19}{s+3} - \frac{11.5}{s+4}$$

Since $\mathbf{U}(s)$ (the Laplace transform for e^{-t}) is $1/(s+1)$

<u>the eigenvalues -2; -3, and -4.</u>

Time Domain Solution of State Equations

we solve the equations directly in the time domain using a method closely allied to the classical solution of differential equations.

The solution in the time domain is given directly by

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \\ &= \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \Phi(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau\end{aligned}$$

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t}$$

is called the state-transition matrix

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \Phi(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau$$

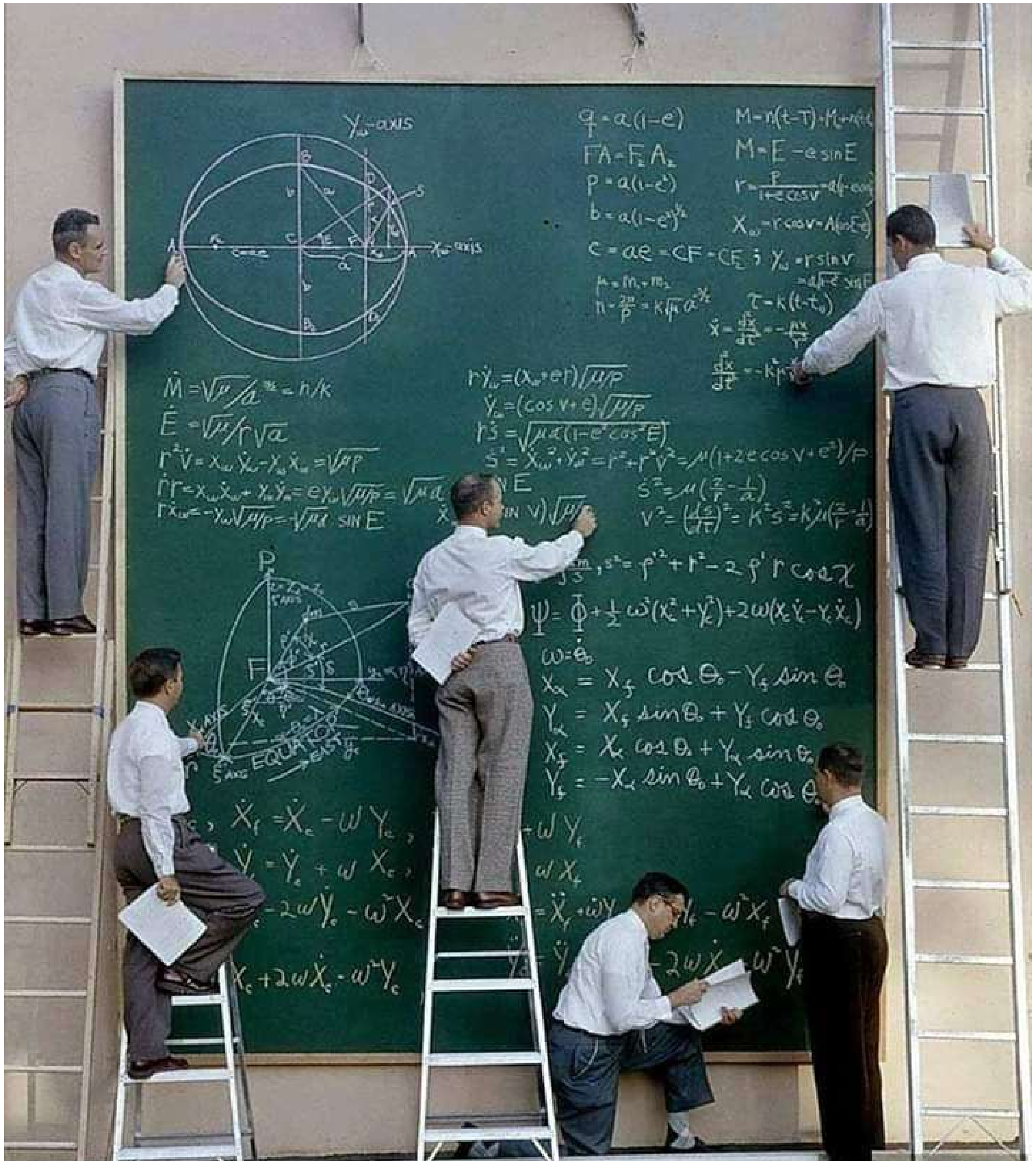
Zero input response

Convolution integral

UNIVERSITÄT
HEIDELBERG
ZUKUNFT
SEIT 1386

Few examples:

Paper and pen!




The End