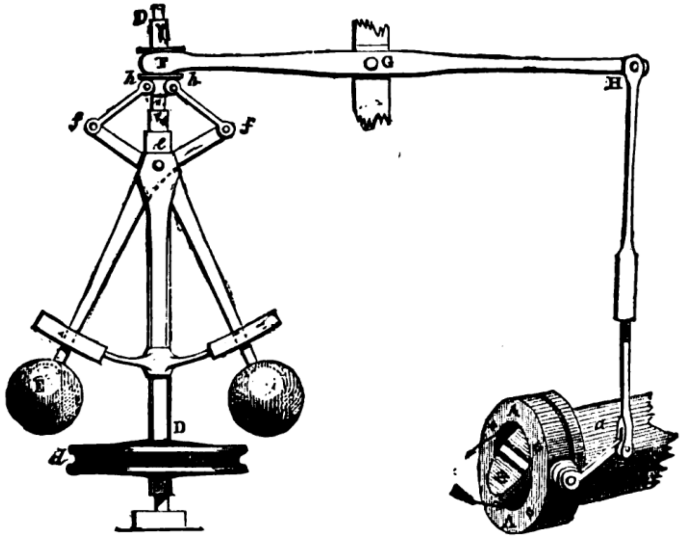

Reduction of Multiple Subsystems

Control System Design

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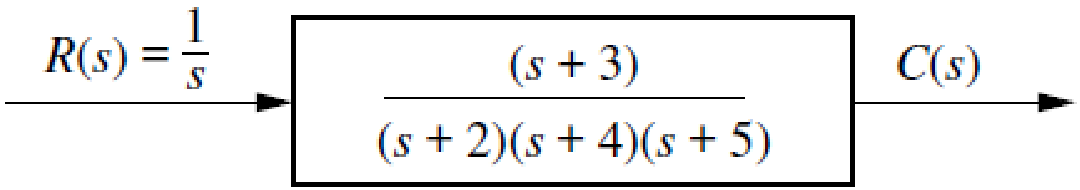
Outlines of the class

- Reduce a block diagram of multiple subsystems to a single block representing the transfer function from input to output
- Analyze and design transient response for a system consisting of multiple subsystems
- Convert block diagrams to signal-flow diagrams
- Find the transfer function of multiple subsystems using Mason's rule
- Represent state equations as signal-flow graphs
- Represent multiple subsystems in state space in cascade, parallel, controller
- canonical, and observer canonical forms
- Perform transformations between similar systems using transformation matrices;
- and diagonalize a system matrix

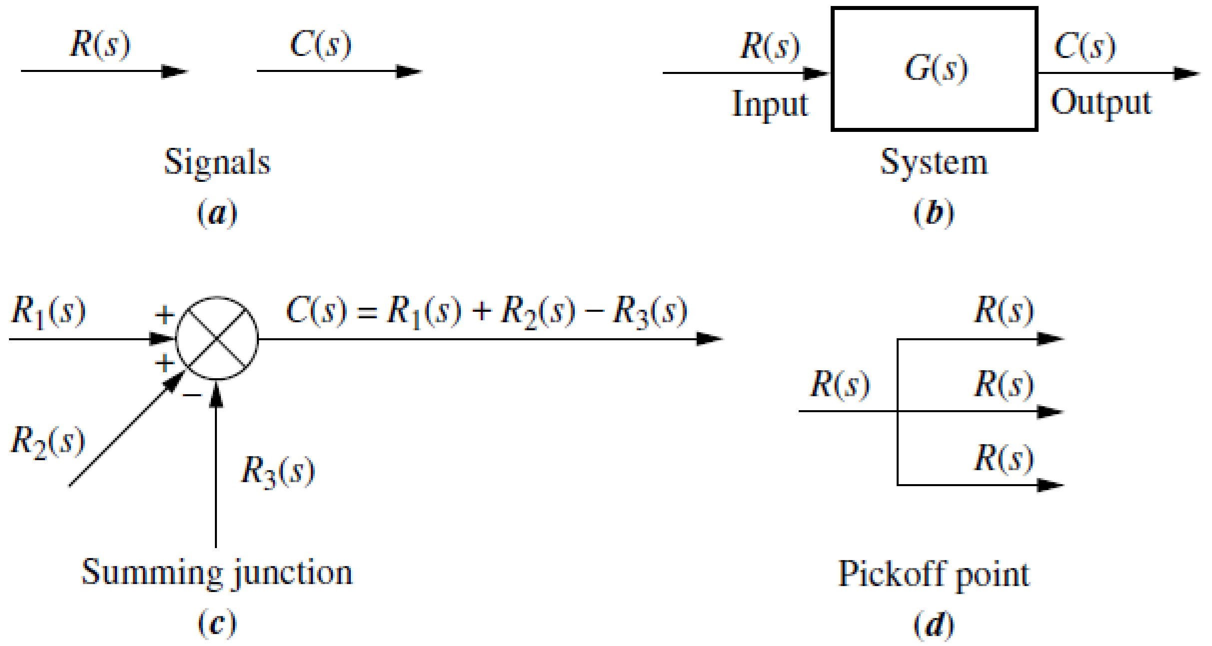
Introduction



- More complicated systems are represented by the interconnection of many subsystems.
- as **block diagrams** and as **signal-flow graphs**.
- We will develop techniques to reduce each representation to **a single transfer function.**
- Block **diagram algebra** will be used to reduce block diagrams and
- **Mason’s rule** to reduce signal-flow graphs.

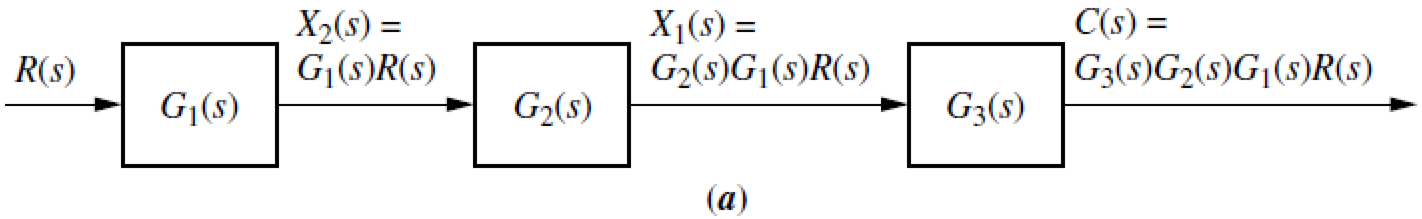
Block Diagrams

- a subsystem is represented as a block with an input, an output, and a transfer function.
- When multiple subsystems are interconnected, a few more schematic elements must be added to the block diagram.
- These new elements are ***summing junctions and pickoff points***.

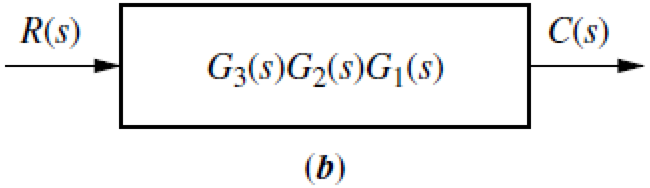


Cascade Form

- We will now examine some common topologies for interconnecting subsystems and derive the single transfer function representation for each of them.



a. Cascaded subsystems; **b.** equivalent transfer function

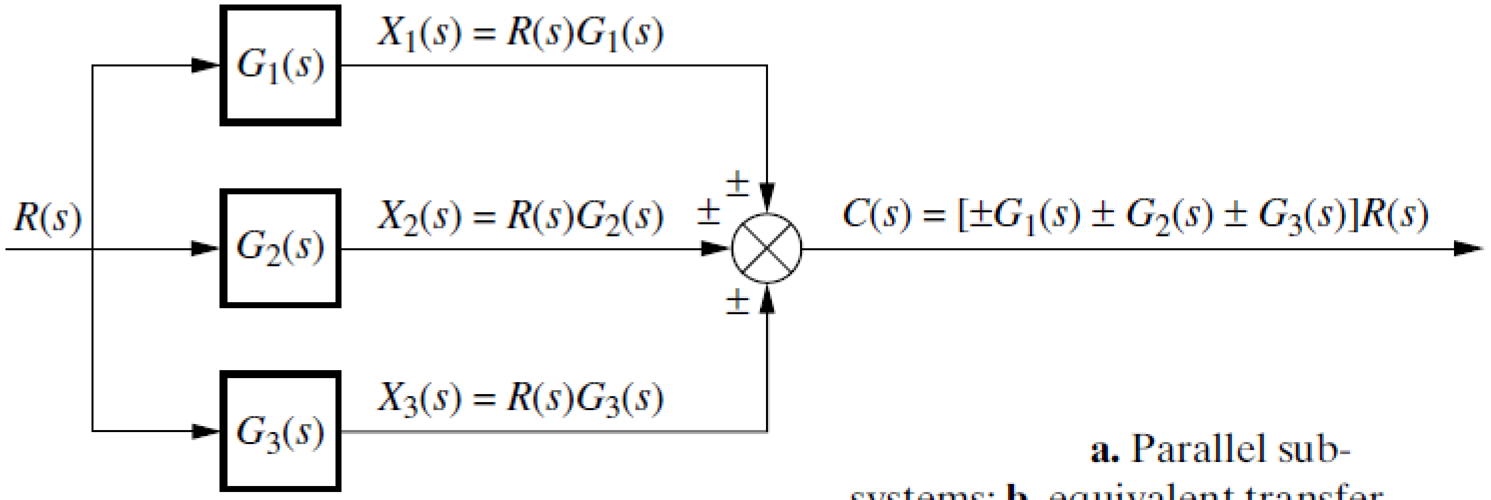


The transfer function is the product of the subsystems’ transfer functions.

$$\boxed{G_e(s) = G_3(s)G_2(s)G_1(s)}$$

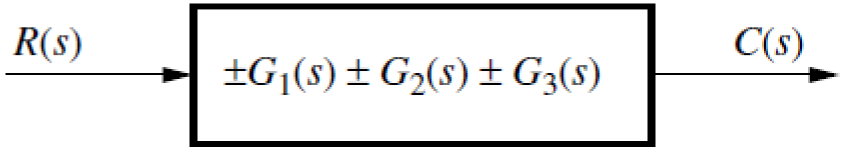
Parallel Form

- Parallel subsystems have a common input and an output formed by the algebraic sum of the outputs from all of the subsystems.



(*a*)

a. Parallel subsystems; **b.** equivalent transfer function

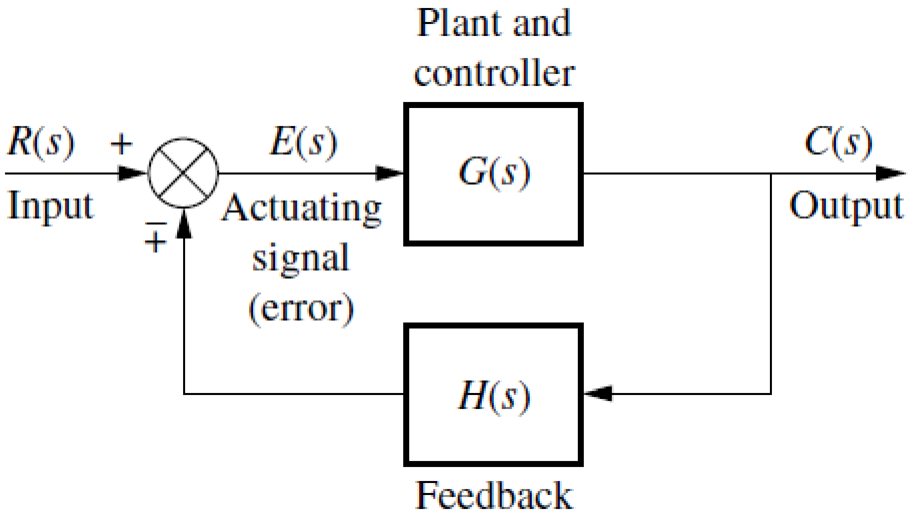


(*b*)

$$\boxed{G_e(s) = \pm G_1(s) \pm G_2(s) \pm G_3(s)}$$

Feedback Form

- The feedback system forms the basis for our study of control systems engineering.
- Let us derive the transfer function that represents the system from its **input** to its **output.**



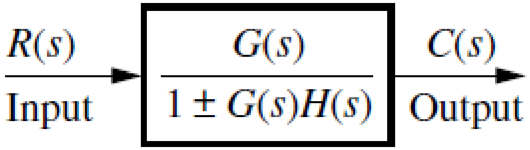
$$E(s) = R(s) \mp C(s)H(s)$$

$$C(s) = E(s)G(s),$$

$$E(s) = \frac{C(s)}{G(s)}$$

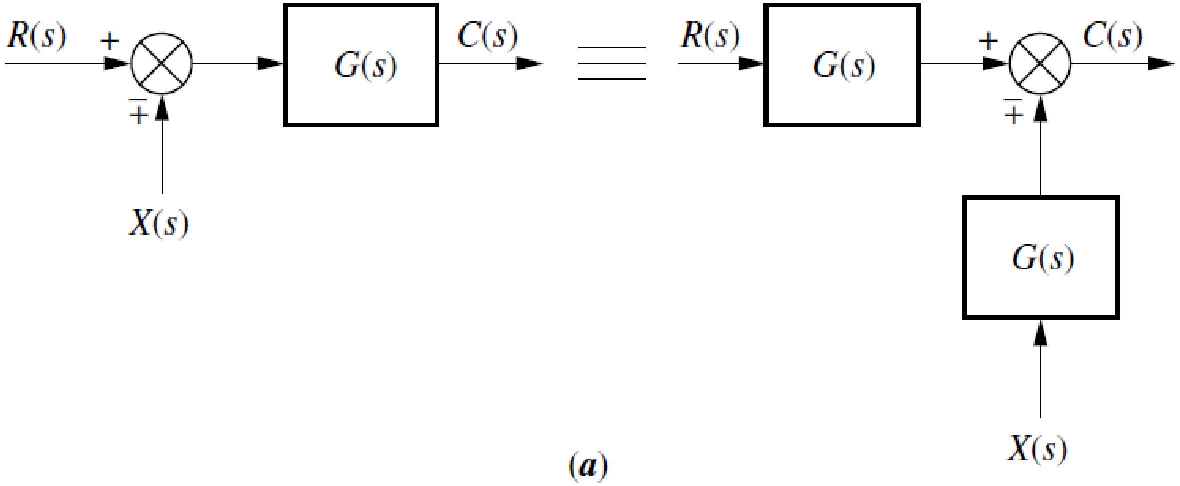
$$\boxed{G_e(s) = \frac{G(s)}{1 \pm G(s)H(s)}}$$

G_e(s) is the equivalent, or **closed-loop, transfer function**



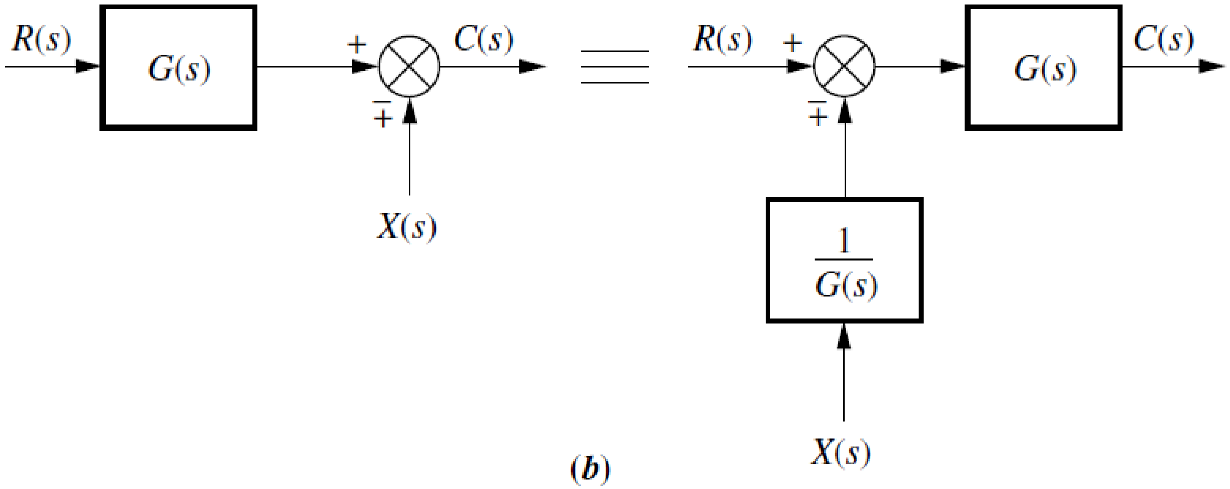
$G(s)H(s)$ is called the **open-loop transfer function**, or loop gain.

Moving Blocks to Create Familiar Forms: equivalence



$$C(s) = R(s)G(s) \mp X(s)G(s).$$

(*a*)

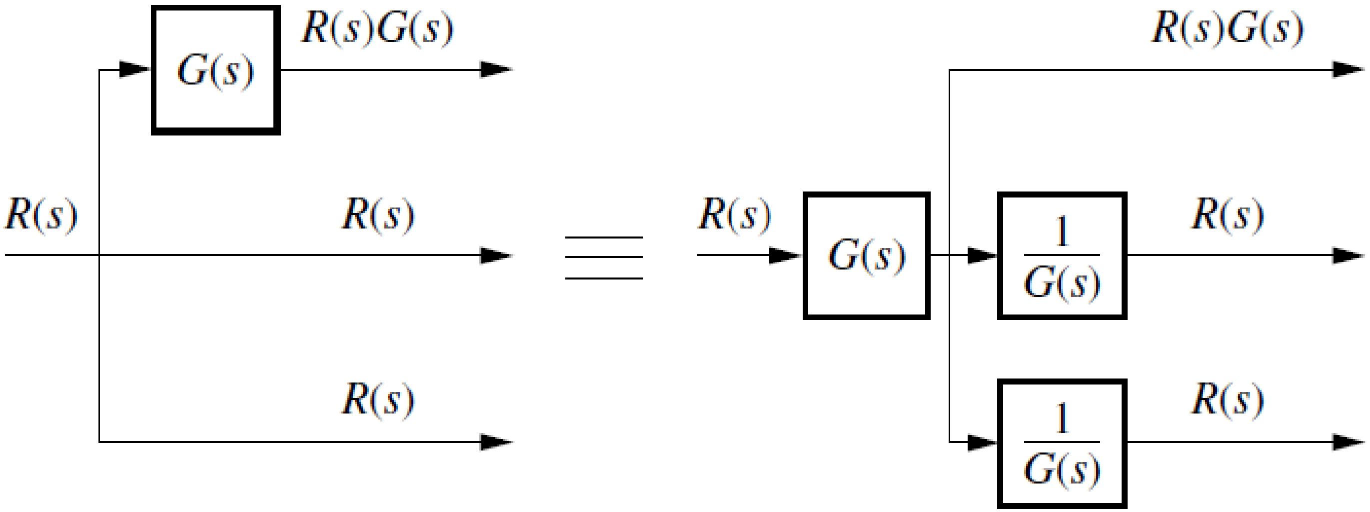


$$C(s) = R(s)G(s) \mp X(s).$$

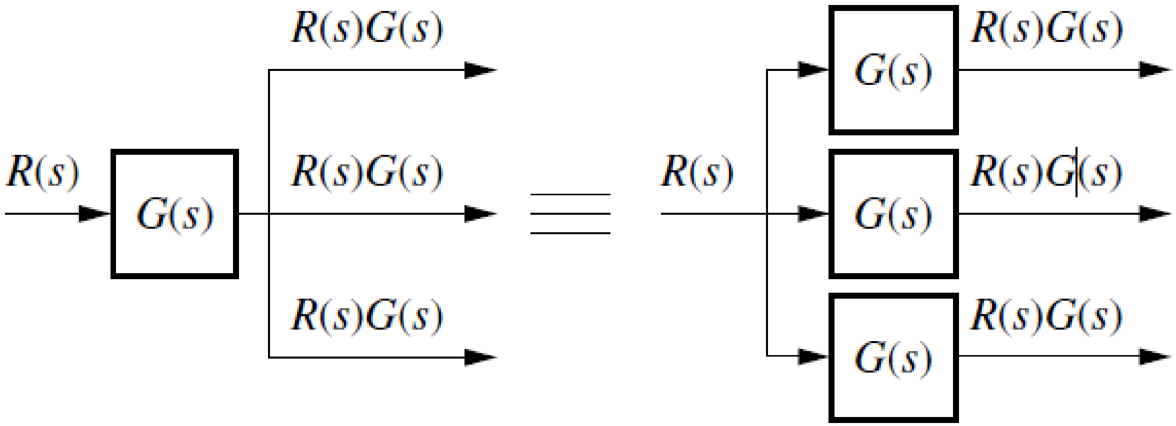
(*b*)

Block diagram algebra for summing junctions—equivalent forms for moving a block **a.** to the left past a summing junction; **b.** to the right past a summing junction

Moving Blocks to Create Familiar Forms: equivalence



(a)

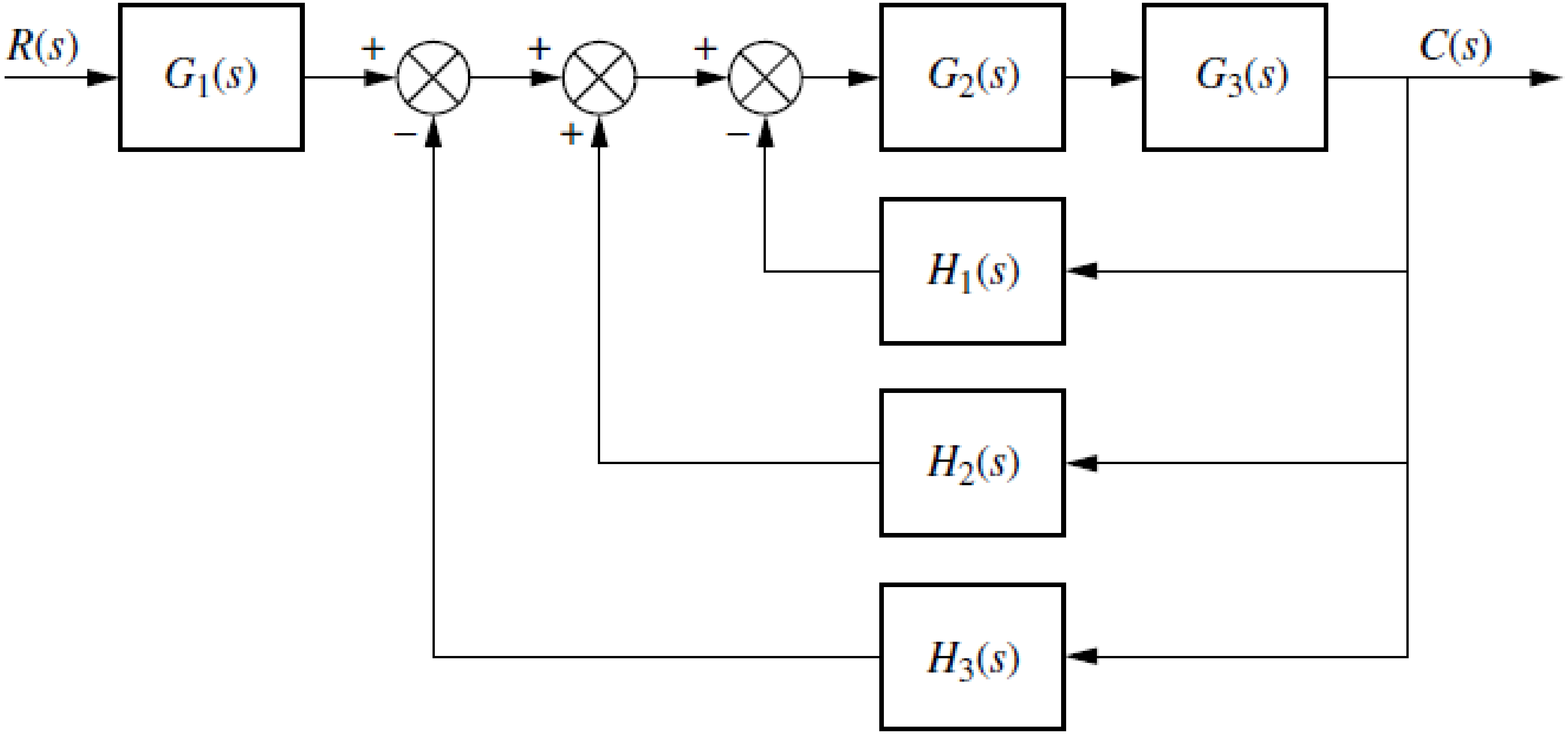


(b)

Block diagram algebra for pickoff points—equivalent forms for moving a block **a.** to the left past a pickoff point; **b.** to the right past a pickoff point

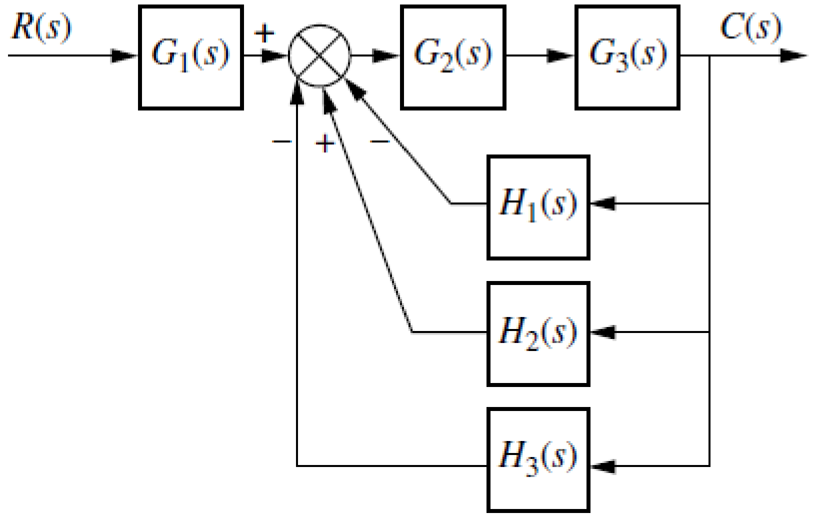
Example 1

PROBLEM: Reduce the block diagram shown in Figure to a single transfer function.



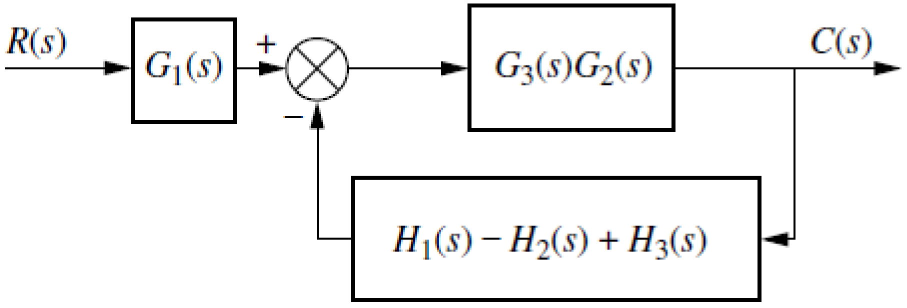
SOLUTION 1:

a. First, the three summing junctions can be collapsed into a single summing junction



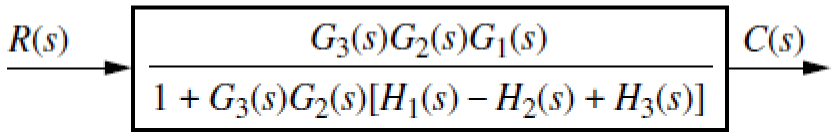
(*a*)

b. Second, recognize that the three feedback functions, $H_1(s)$, $H_2(s)$, and $H_3(s)$, are connected in parallel.
And G_2 and G_3 in cascade.



(*b*)

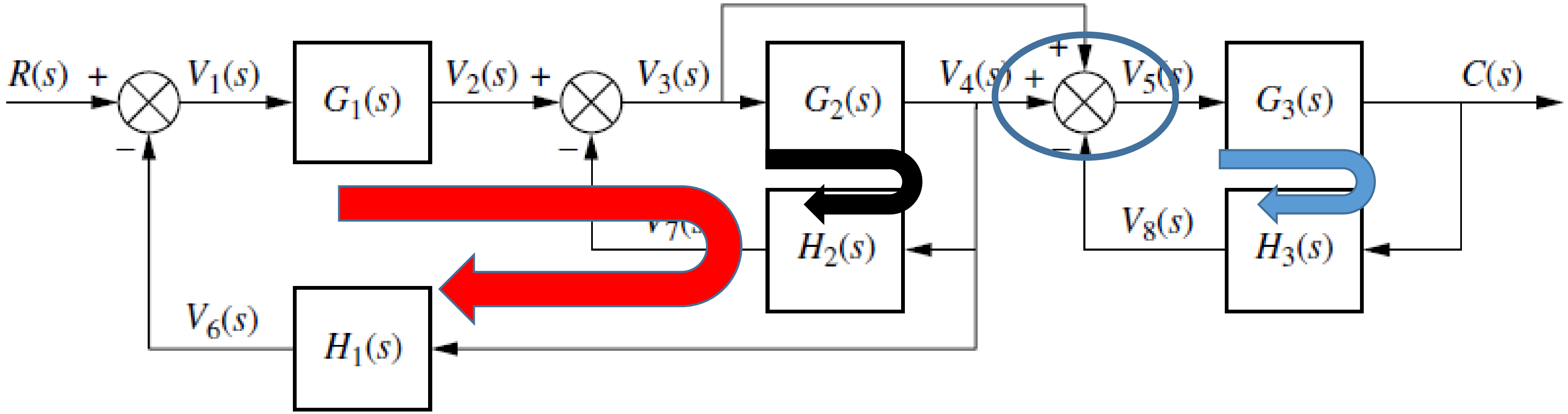
c. And at last we have the canonical form of a feedback system



(*c*)

Example 2

PROBLEM: Reduce the system shown in Figure to a single transfer function.

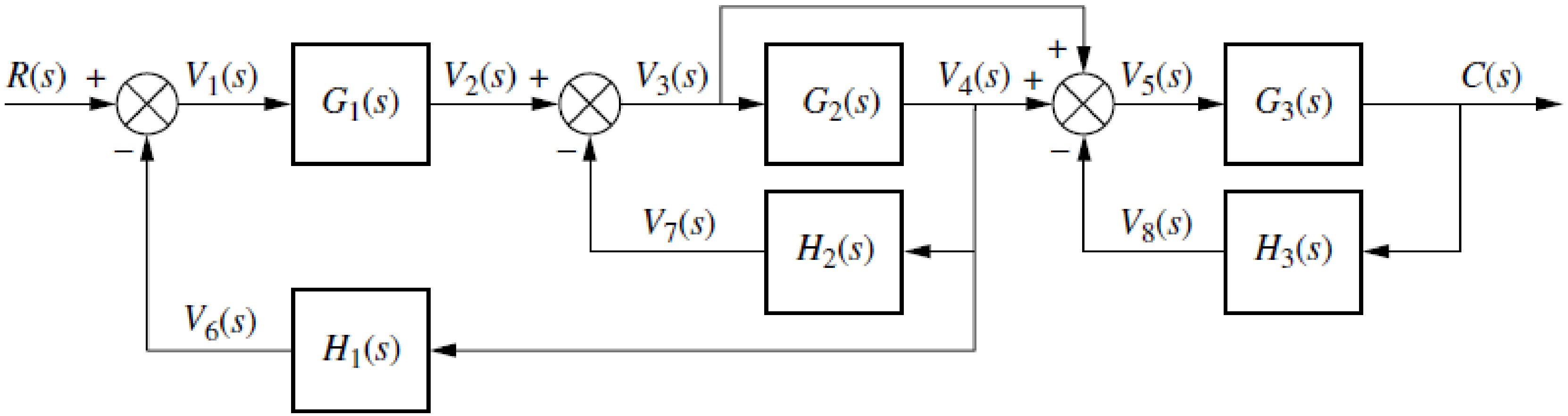


O my **gosh and now**??? It is a mess!

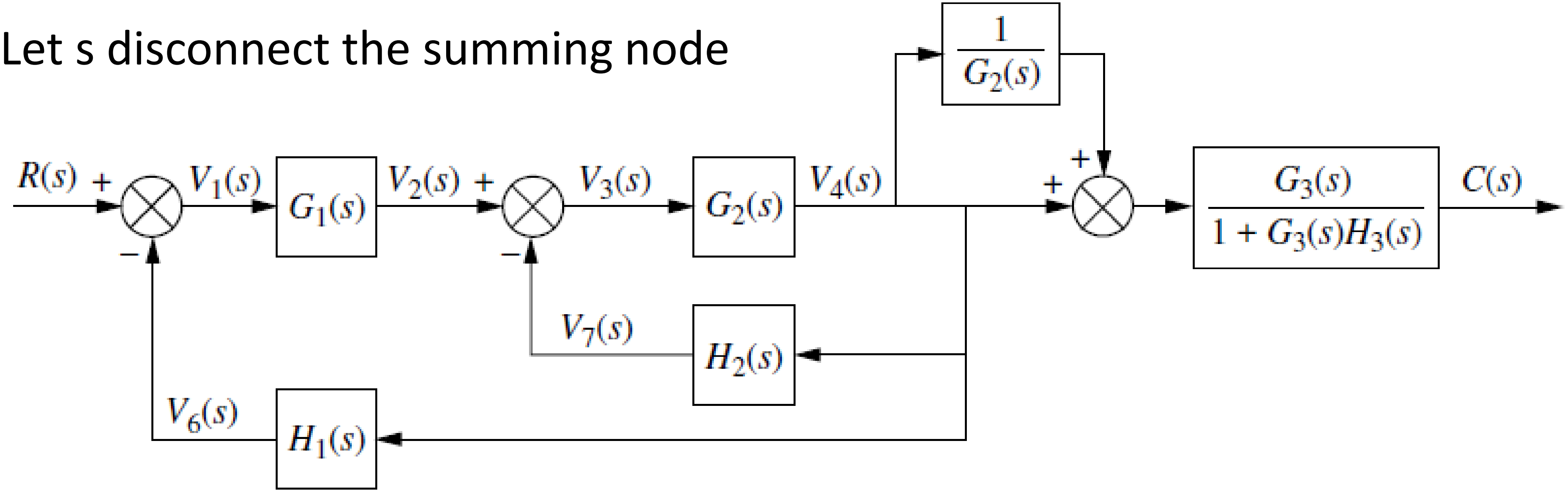
Let s identify the structure that we know:

- I see three feedback loops
- And a summing node which is in the middle and I cannot simplify the three feedback in cascade

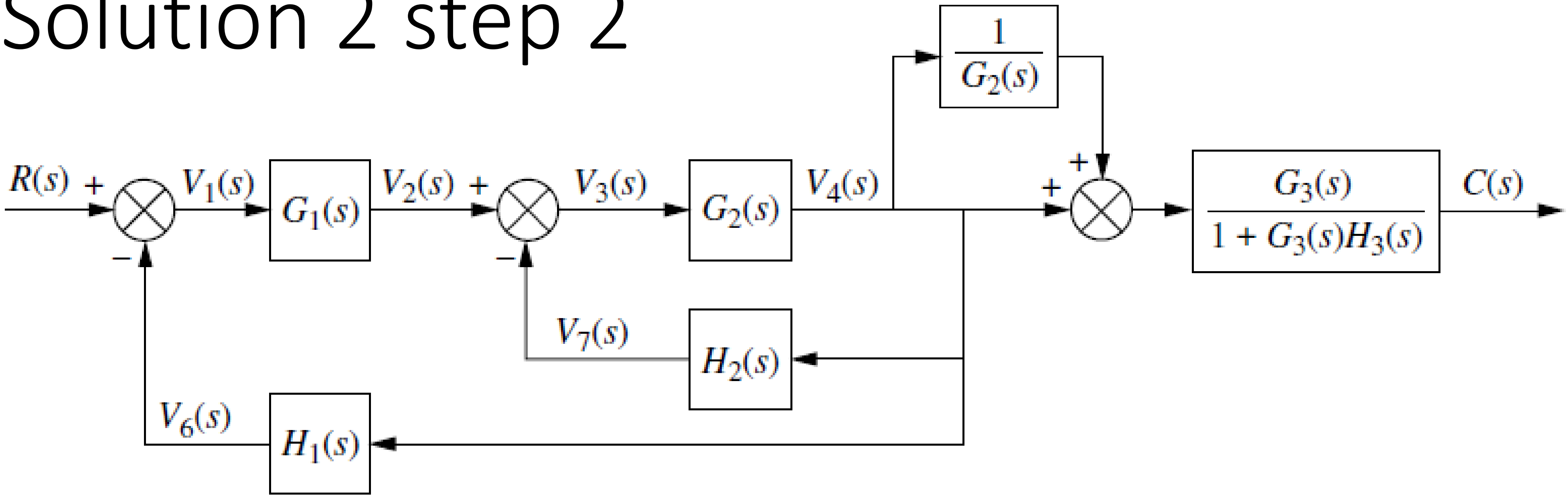
Solution 2 step 1

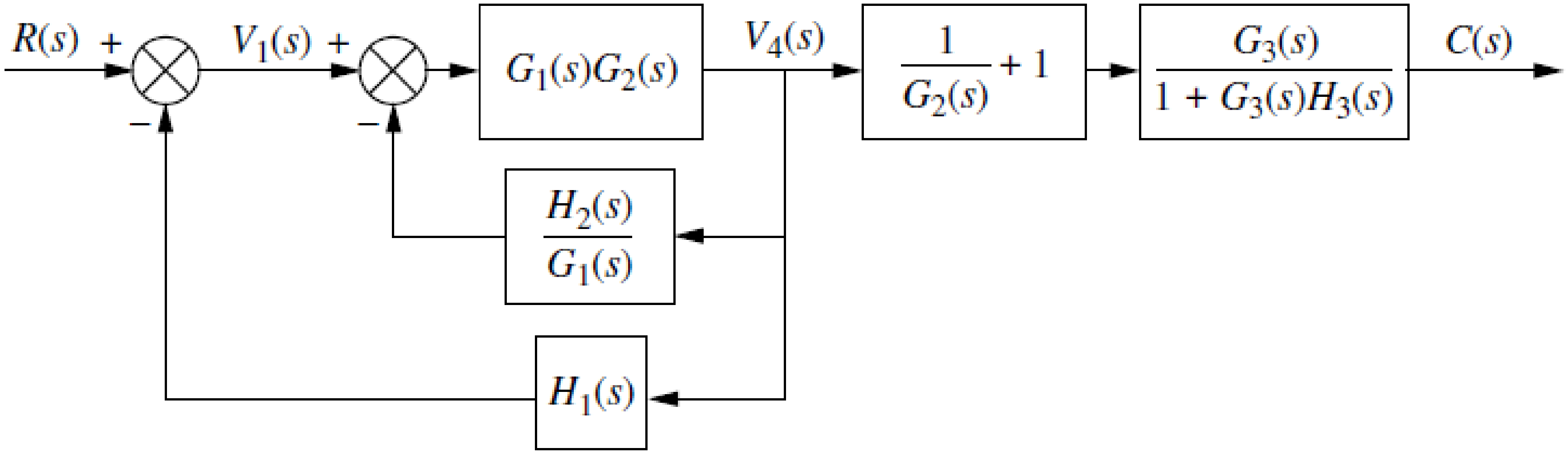


Let s disconnect the summing node

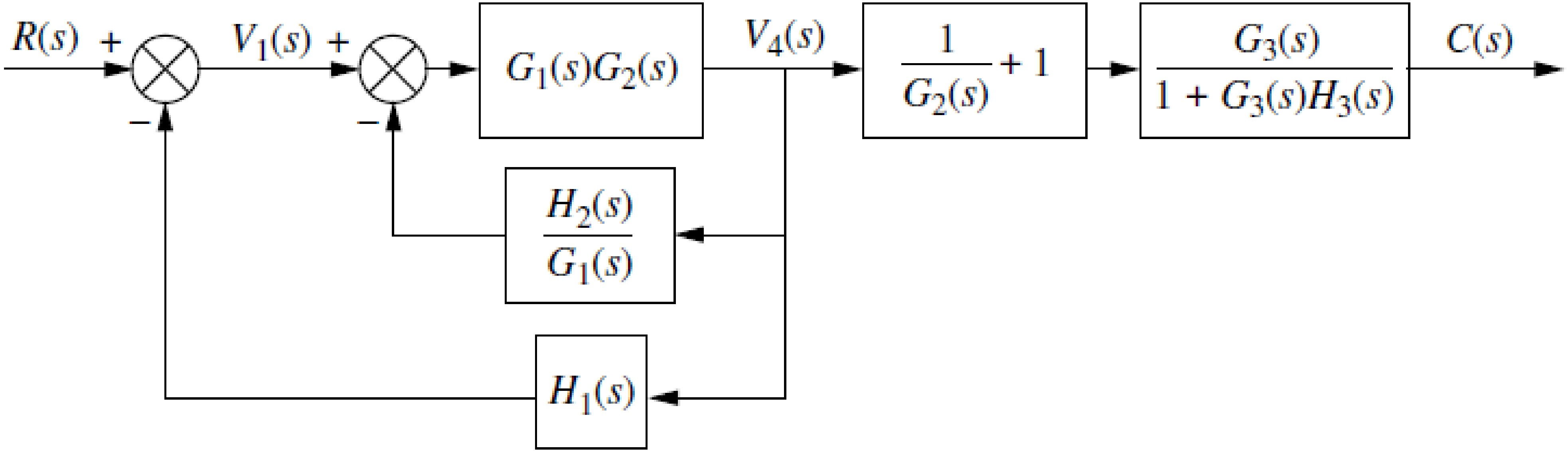


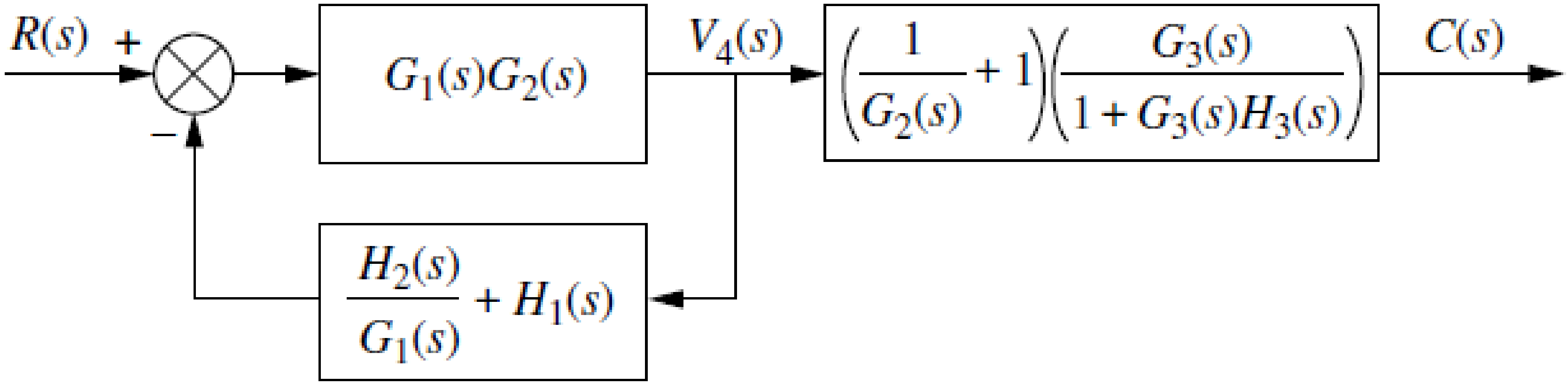
Solution 2 step 2



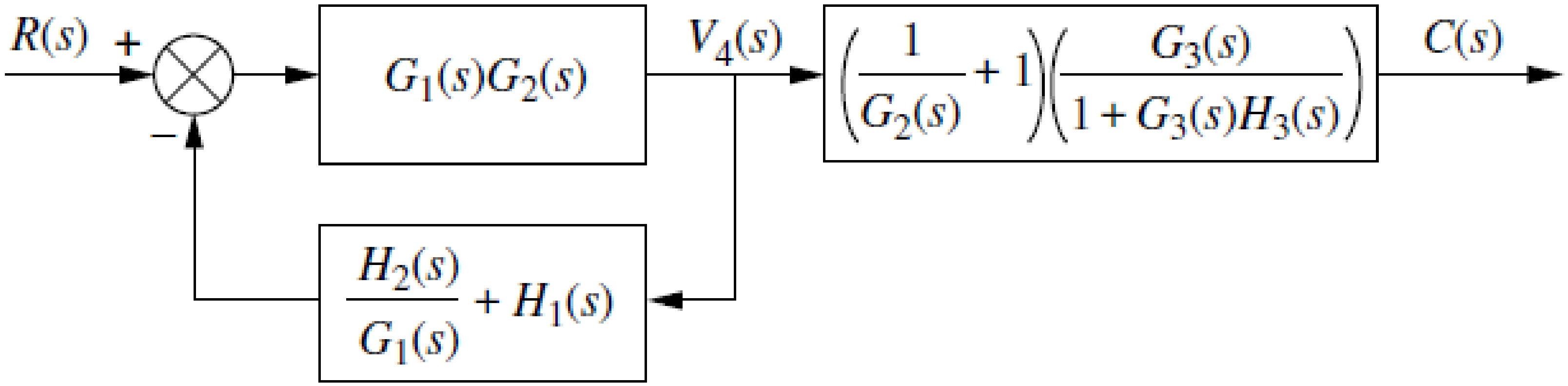


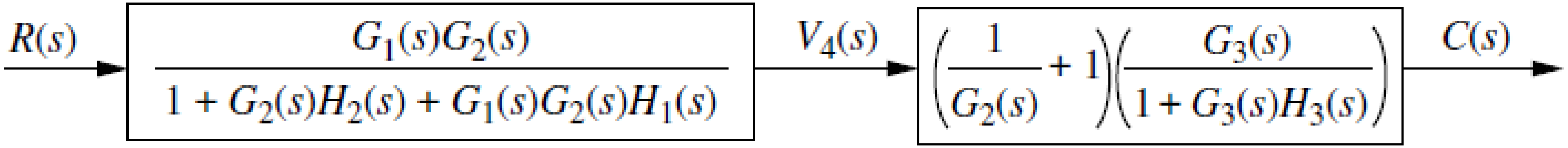
Solution 2 step 3

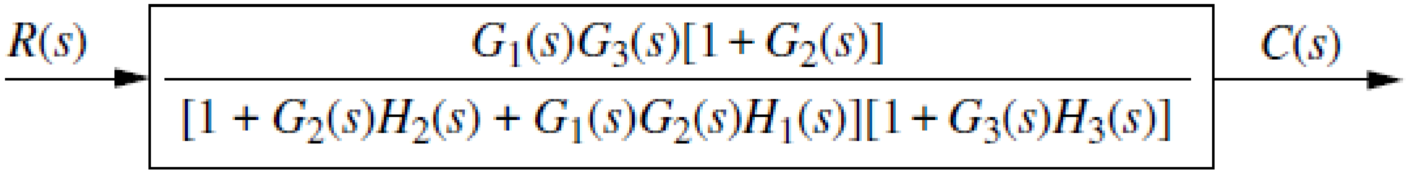




Solution 2 step 3

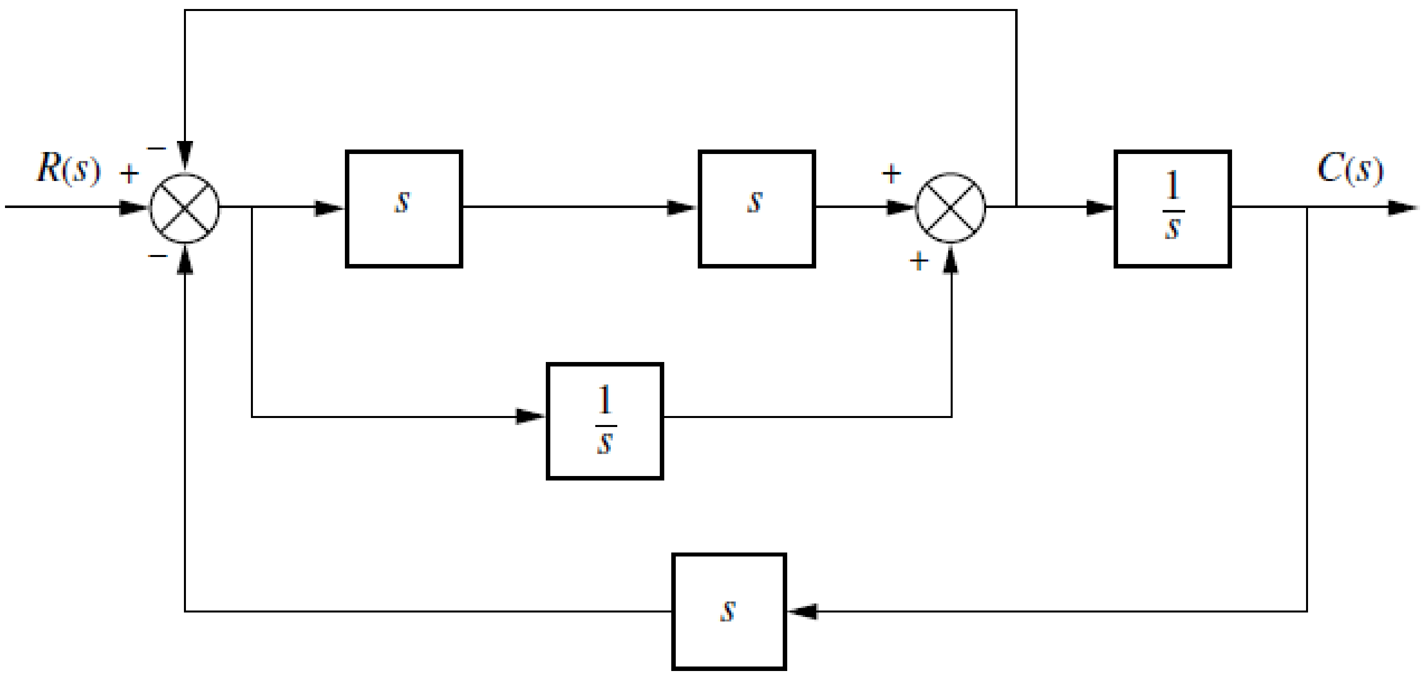






Skill assessment

PROBLEM: Find the equivalent transfer function, $T(s) = C(s)/R(s)$, for the system shown in Figure

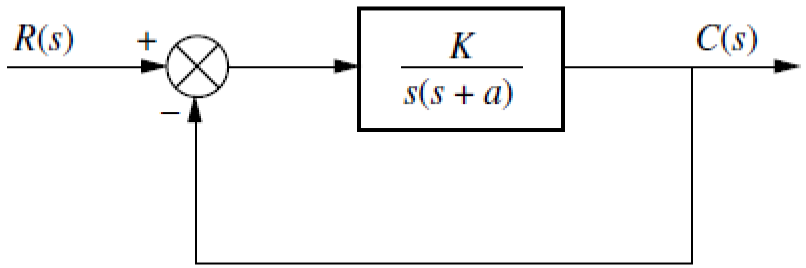


ANSWER:

$$T(s) = \frac{s^3 + 1}{2s^4 + s^2 + 2s}$$

Analysis and Design of Feedback Systems

An immediate application of the previous principles is the analysis and design of feedback systems that reduce to second-order systems.
Percent overshoot, settling time, peak time, and rise time can then be found from the equivalent transfer function.



$$T(s) = \frac{K}{s^2 + as + K}$$

K between 0 and $a^2/4$, the poles of the system are real (**overdamped**)

$$s_{1,2} = -\frac{a}{2} \pm \frac{\sqrt{a^2 - 4K}}{2}$$

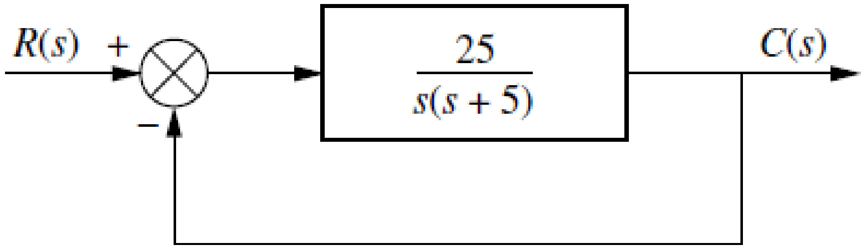
For K above $a^2/4$, the system is **underdamped**, with complex poles located at

$$s_{1,2} = -\frac{a}{2} \pm j\frac{\sqrt{4K - a^2}}{2}$$

Example 3

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Finding Transient Response



PROBLEM: For the system shown in Figure , find the peak time, percent overshoot, and settling time.

SOLUTION: The closed-loop transfer function found is

$$T(s) = \frac{25}{s^2 + 5s + 25}$$

$$\omega_n = \sqrt{25} = 5$$

$$2\zeta\omega_n = 5$$

solving for ζ yields $\zeta = 0.5$

Using the values for ζ and ω_n

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.726 \text{ second}$$

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 = 16.303$$

$$T_s = \frac{4}{\zeta\omega_n} = 1.6 \text{ seconds}$$

Example 4

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

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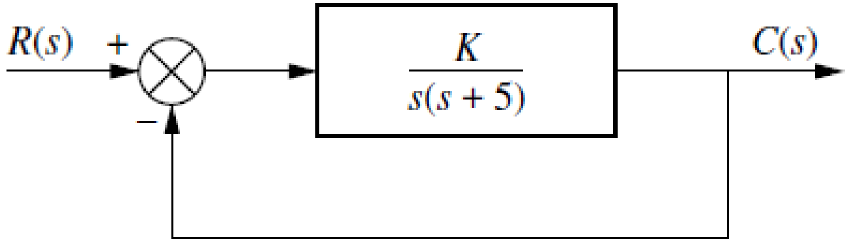
Gain Design for Transient Response

PROBLEM: Design the value of gain, K, for the feedback control system of Figure so that the system will respond with a 10% overshoot.



SOLUTION: The closed-loop transfer function of the system is

$$T(s) = \frac{K}{s^2 + 5s + K}$$

$$2\zeta\omega_n = 5$$

$$\omega_n = \sqrt{K}$$

$$\zeta = \frac{5}{2\sqrt{K}}$$

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$$

A 10% overshoot implies that $\zeta = 0.591$

$$\boxed{K = 17.9}$$

Skill-Assessment Exercise

PROBLEM: For a unity feedback control system with a forward-path transfer function $G(s) = \frac{16}{s(s+a)}$, design the value of a to yield a closed-loop step response that has 5% overshoot.

ANSWER:

$$a = 5.52$$

Use the following MATLAB and Control System Toolbox statements to find ζ, ω_n, $\%OS$, T_s, T_p, and T_r for the closed-loop unity feedback system described in Skill-Assessment Exercise ---> Start with $a = 2$ and try some other values. A step response for the closed-loop system will also be produced.

```
a=2;
numg=16;
deng=poly([0  -a]);
G=tf(numg,deng);
T=feedback(G,1);
[numt,dent]=...
 tfdata(T,'v');
wn=sqrt(dent)3))
z=dent (2)/(2*wn)
Ts=4/(z*wn)
Tp=pi/(wn*...
 sqrt(1-z^2))
pos=exp(-z*pi...
 /sqrt(1-z^2))*100
Tr=(1.76*z^3 - ...
 0.417*z^2 + 1.039*...
 z + 1)/wn
step(T)
```

Signal-Flow Graphs

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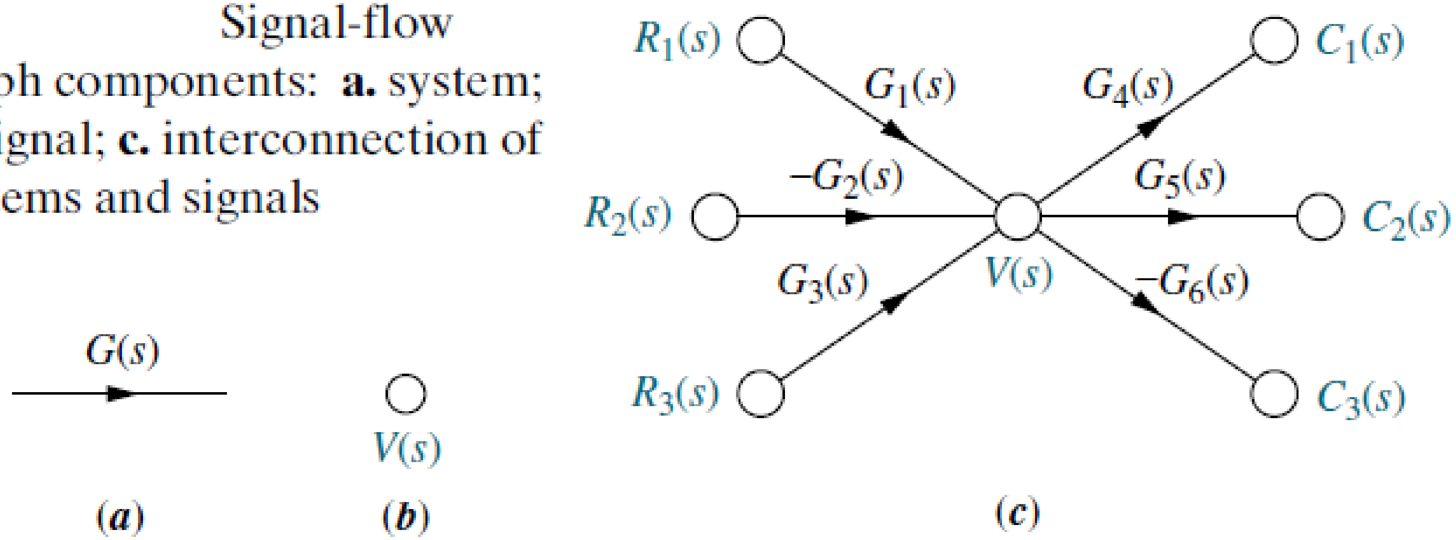
- Signal-flow graphs are an alternative to block diagrams.
- a signal-flow graph consists only of branches, which represent systems, and nodes, which represent signals.
- A system is represented by a line with an arrow showing the direction of signal flow through the system.
- Adjacent to the line we write the transfer function.
- A signal is a node with the signal's name written adjacent to the node.

Signal-flow graph components: **a.** system; **b.** signal; **c.** interconnection of systems and signals



Signal-Flow Graphs

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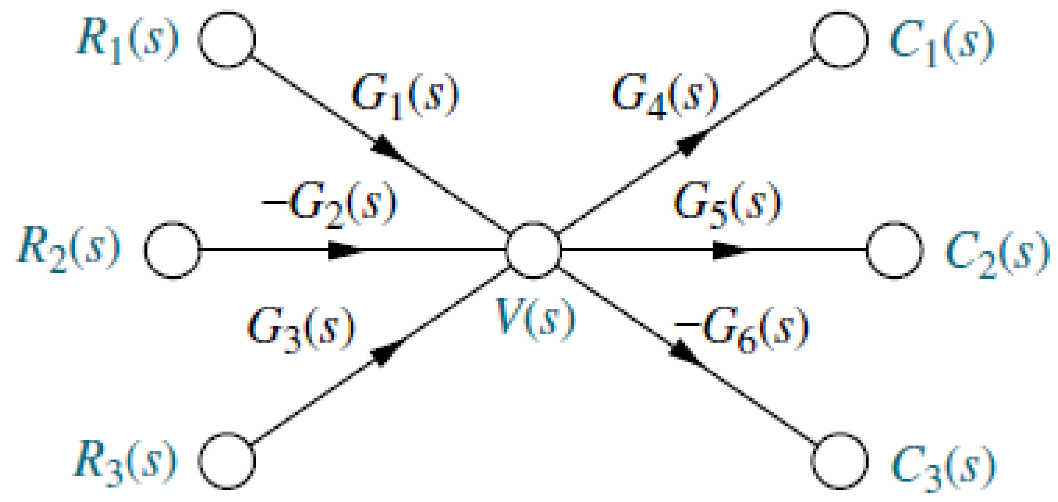


$$V(s) = R_1(s)G_1(s) - R_2(s)G_2(s) + R_3(s)G_3(s).$$

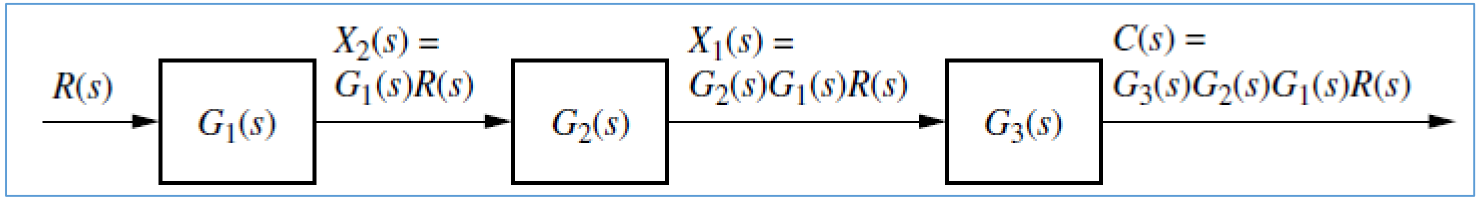
$$C_2(s) = V(s)G_5(s) = R_1(s)G_1(s)G_5(s) - R_2(s)\,G_2(s)G_5(s) + R_3(s)G_3(s)G_5(s).$$

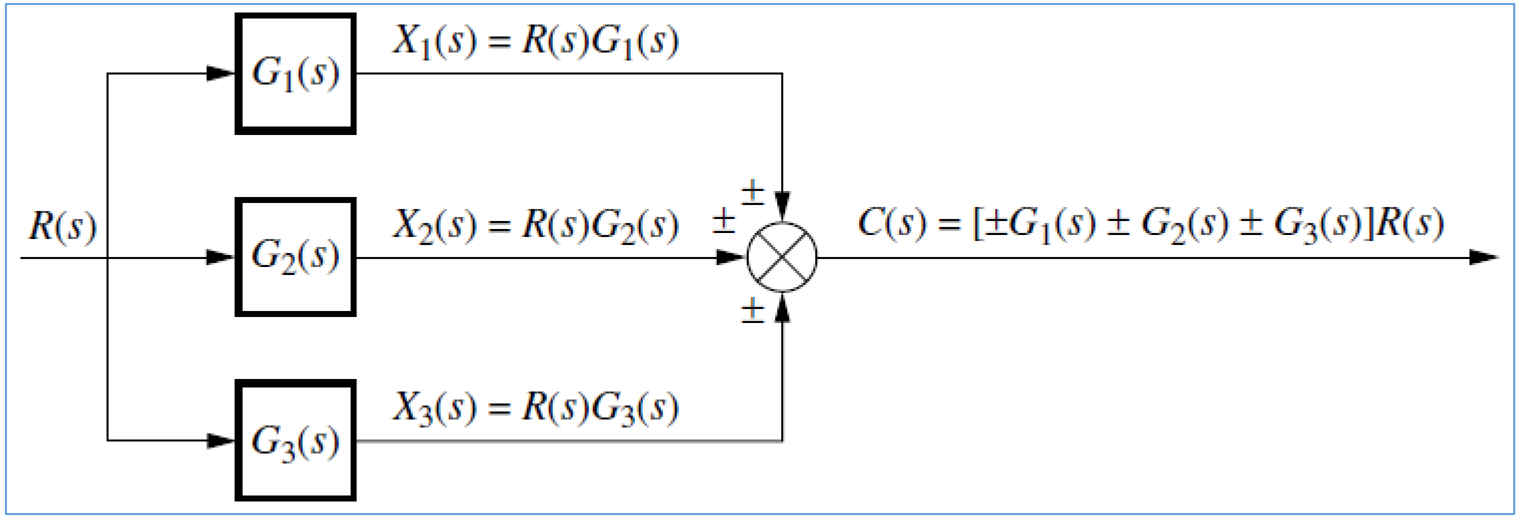
$$C_3(s) = -V(s)G_6(s) = -R_1(s)G_1(s)\,G_6(s) + R_2(s)G_2(s)G_6(s) - R_3(s)G_3(s)G_6(s).$$

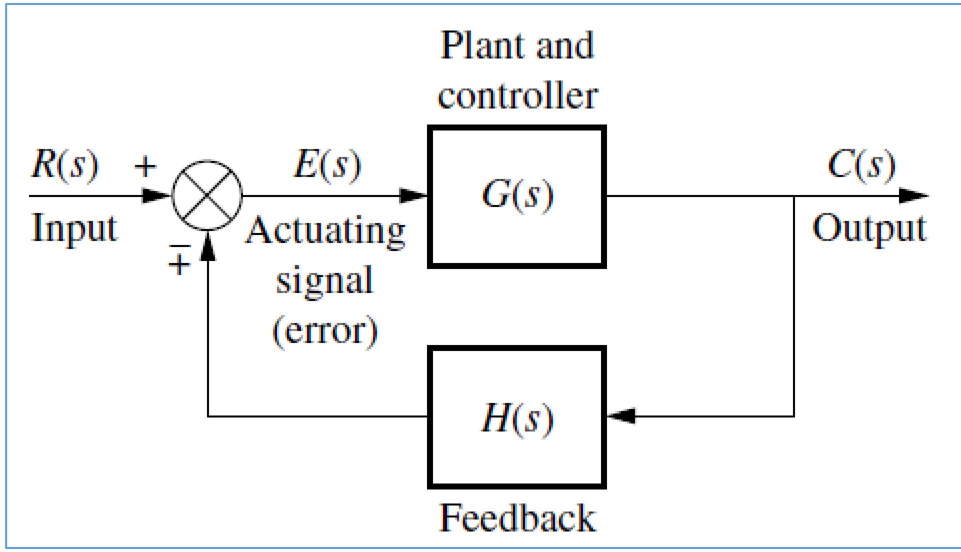
Example 5

Converting Common Block Diagrams to Signal-Flow Graphs

PROBLEM: Convert the cascaded, parallel, and feedback forms of the block diagrams shown in Figures respectively, into signal-flow graphs.

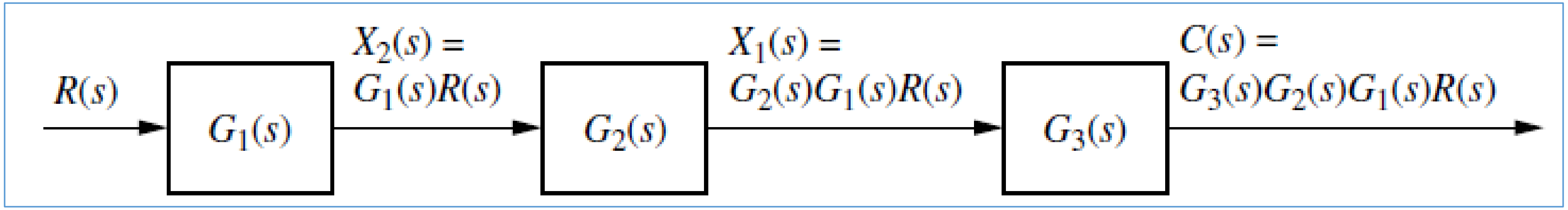


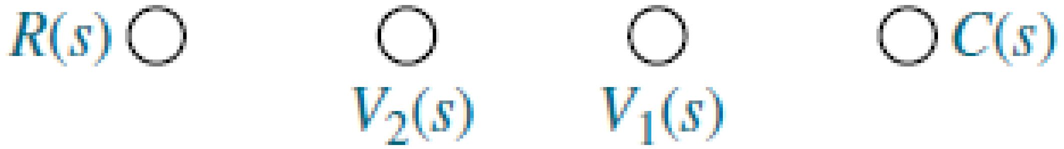


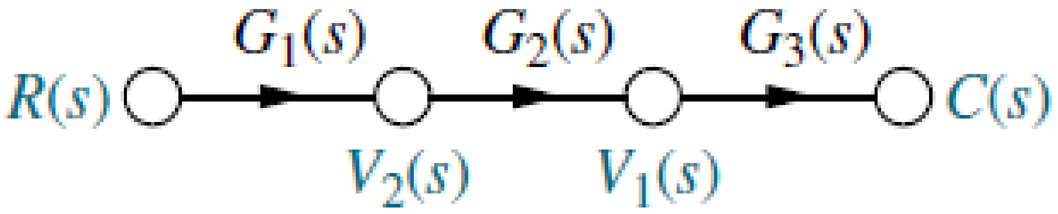


Example 5

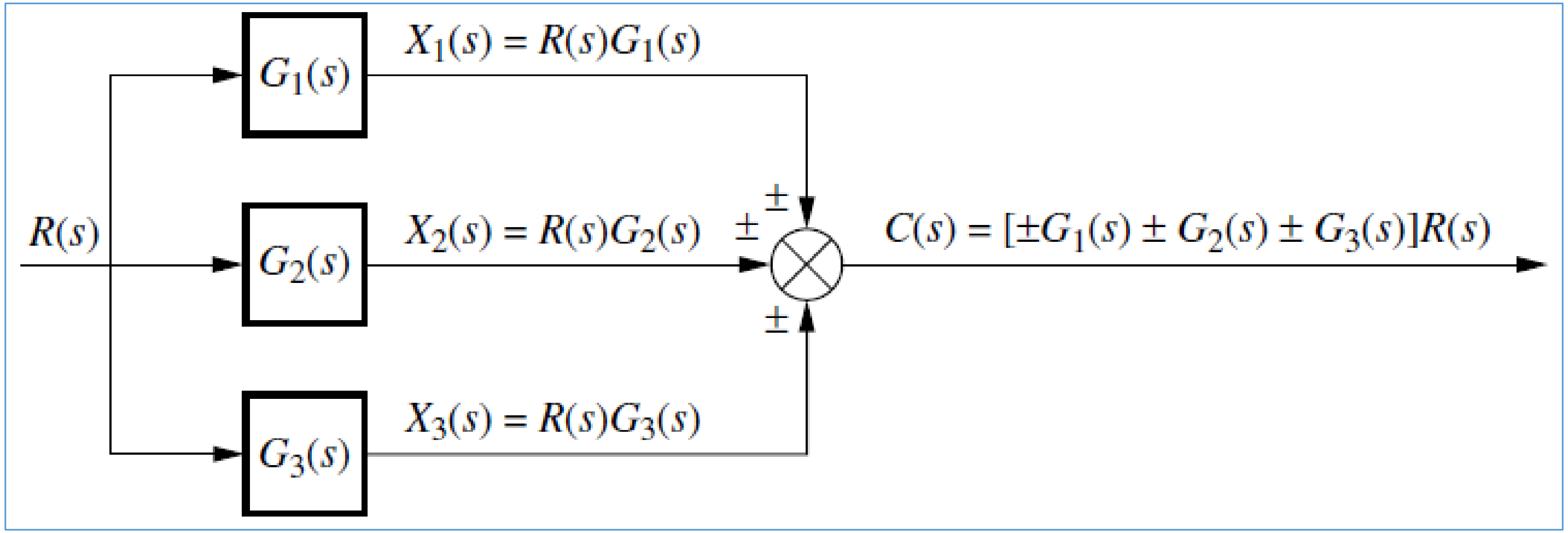
We start by drawing the signal nodes for that system. Next we interconnect the signal nodes with system branches.



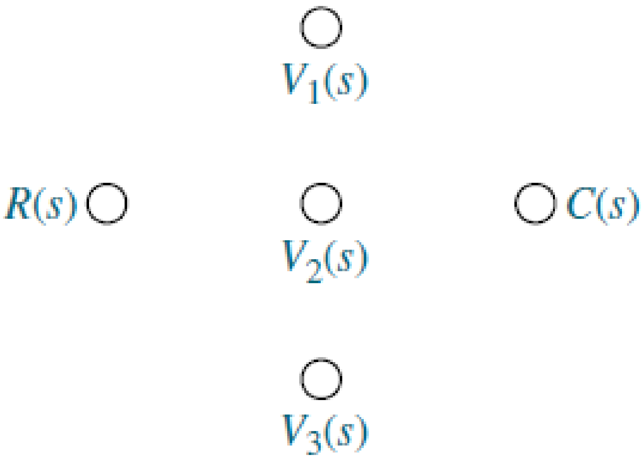




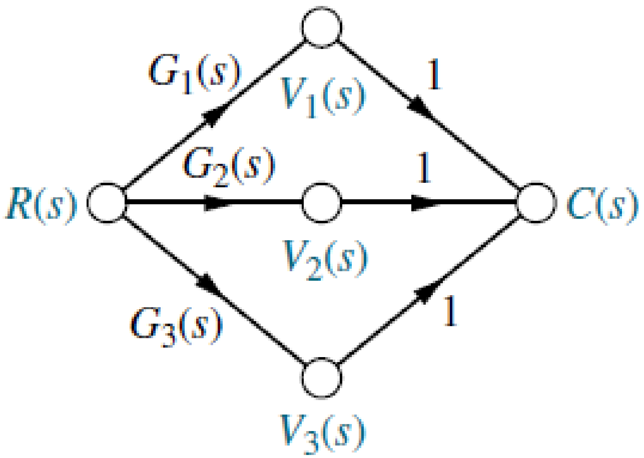
Example 5



$G_1(s)$
$X_1(s) = R(s)G_1(s)$
$R(s)$
$G_2(s)$
$X_2(s) = R(s)G_2(s)$
$\pm$
$\pm$
$\pm$
$C(s) = [\pm G_1(s) \pm G_2(s) \pm G_3(s)]R(s)$
$G_3(s)$
$X_3(s) = R(s)G_3(s)$

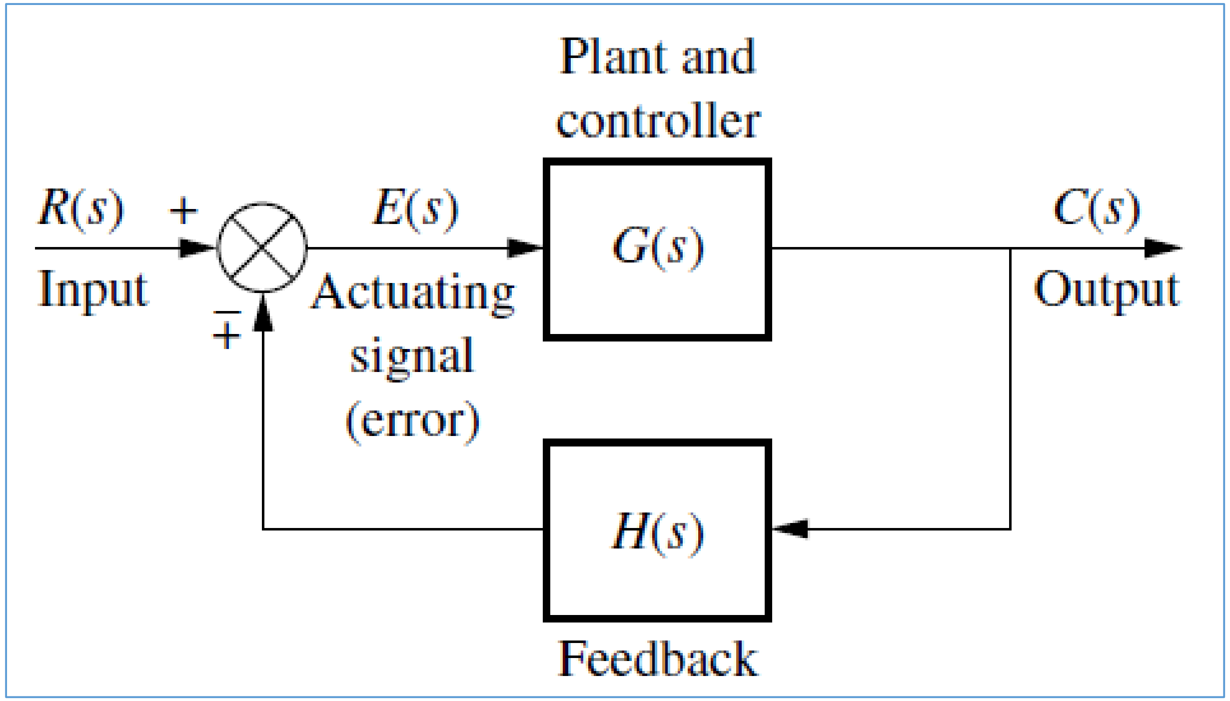


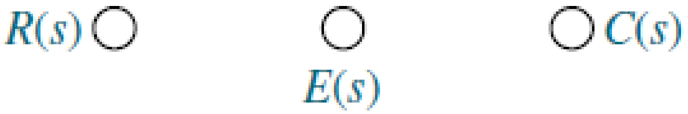
$V_1(s)$
$R(s)$
$V_2(s)$
$C(s)$
$V_3(s)$

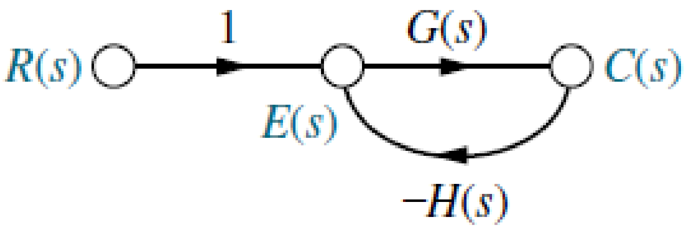


$G_1(s)$
$V_1(s)$
1
$G_2(s)$
1
$R(s)$
$C(s)$
$V_2(s)$
$G_3(s)$
1
$V_3(s)$

Example 5


Plant and
controller
R(s)
+
E(s)
G(s)
C(s)
Input
Actuating
signal
(error)
Output
−
+
H(s)
Feedback

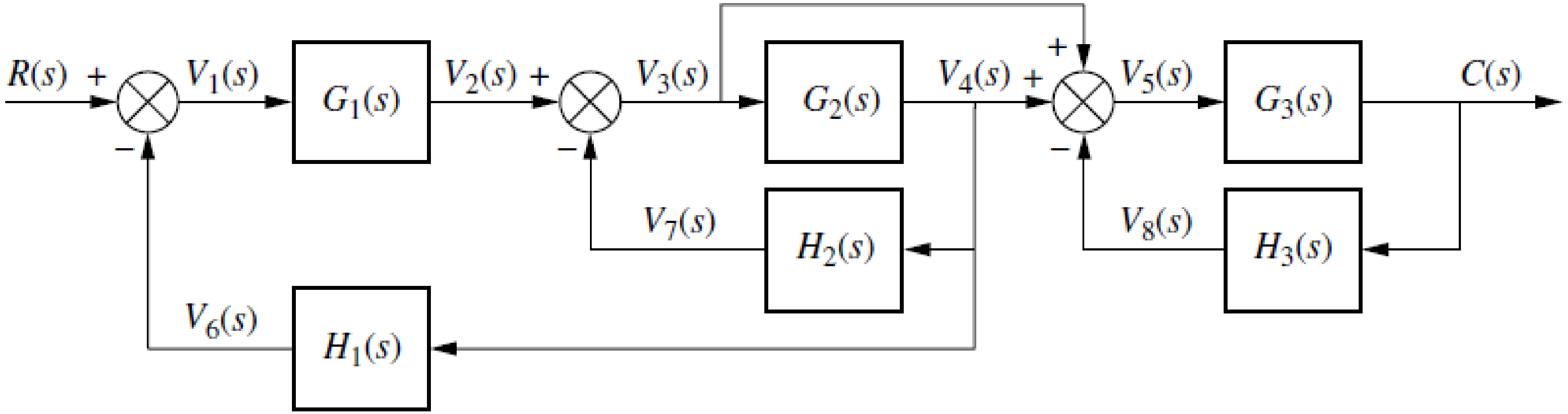

R(s)
E(s)
C(s)


1
G(s)
R(s)
E(s)
C(s)
−H(s)

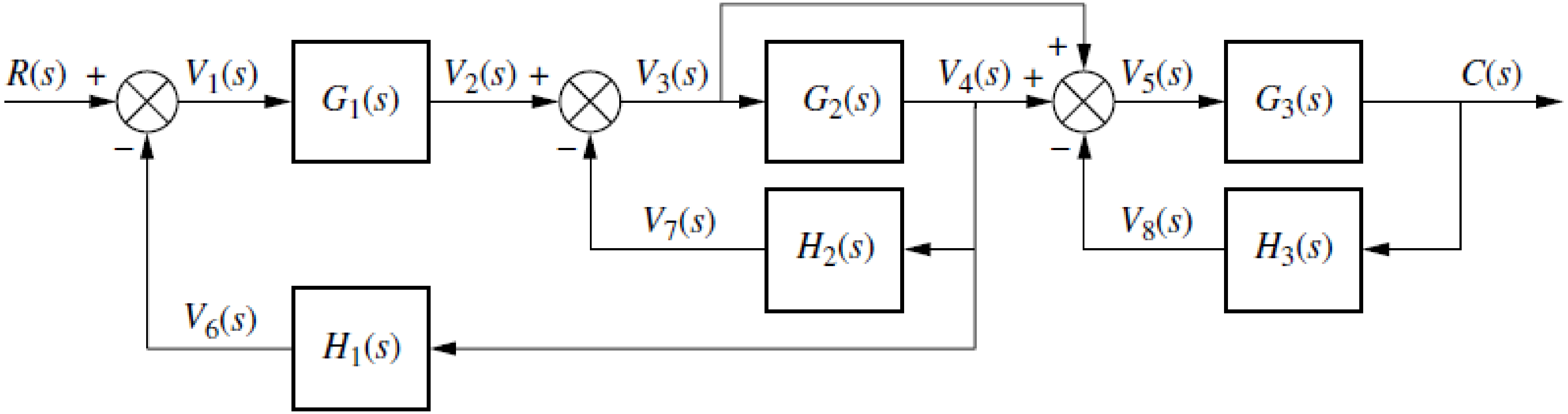
Example 6

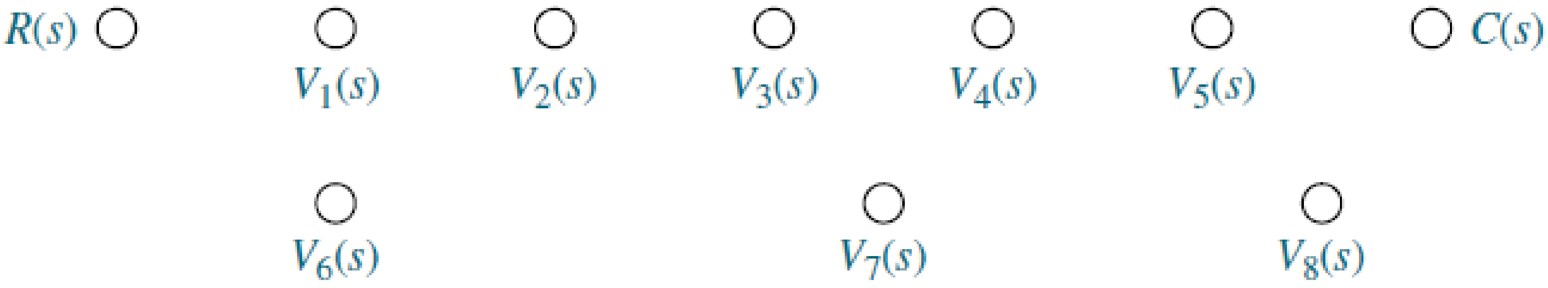
Converting a Block Diagram to a Signal-Flow Graph

PROBLEM: Convert the block diagram of Figure to a signal-flow graph.



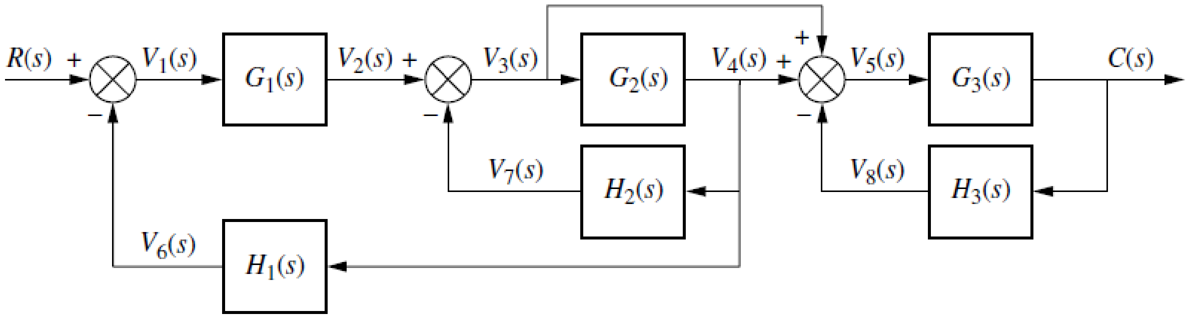
Example 6 (step 1 signals)

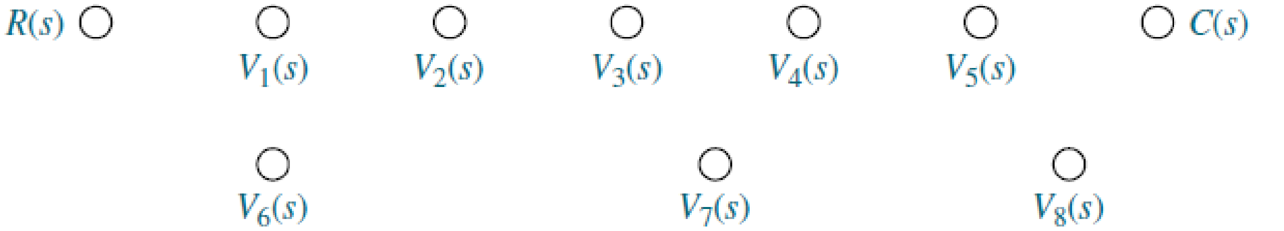


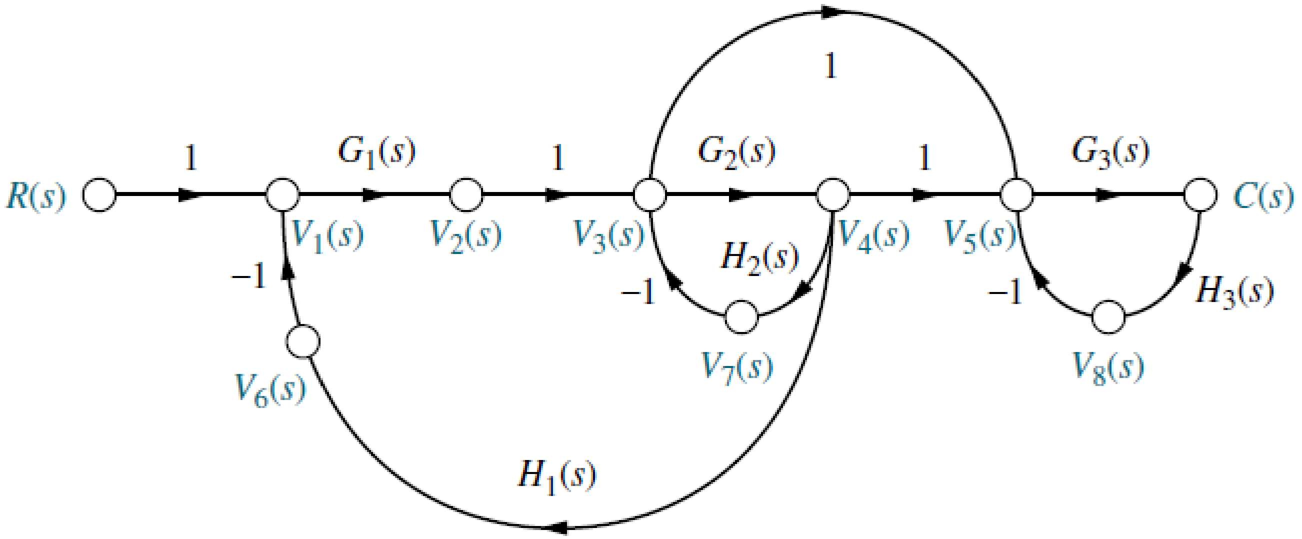


Example 6

(step 2 interconnect the nodes, showing the direction of signal flow and identifying each transfer function)

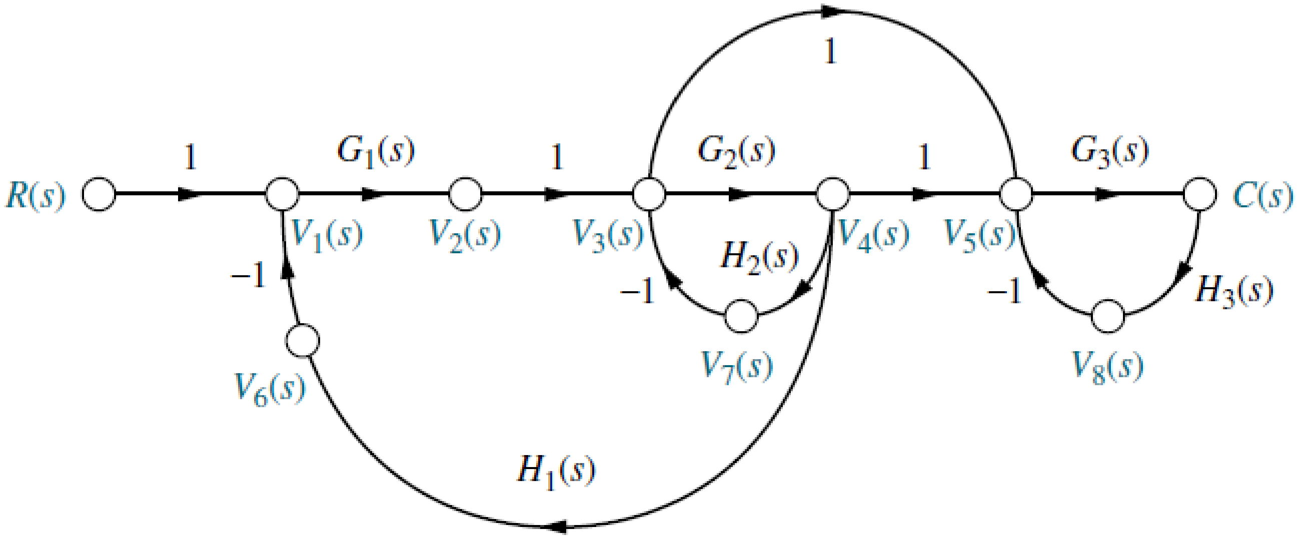


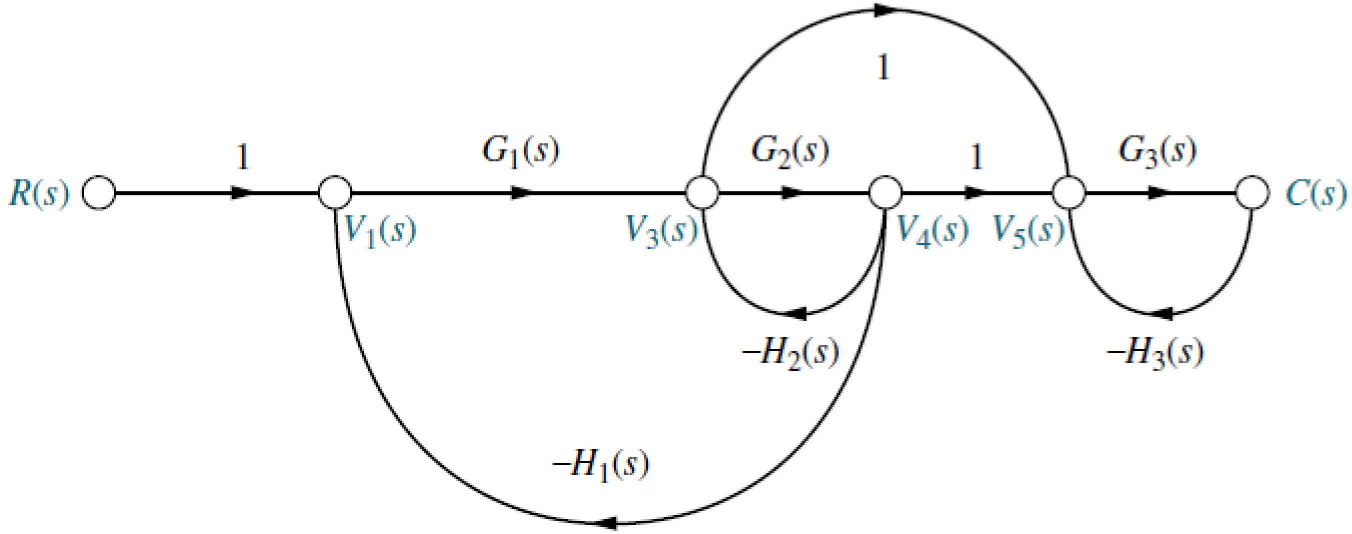




Example 6

Finally, if desired, simplify the signal-flow graph to the one shown below by eliminating signals that have a single flow in and a single flow out **V2(s), V6(s), V7(s), and V8(s)**.





Mason's Rule

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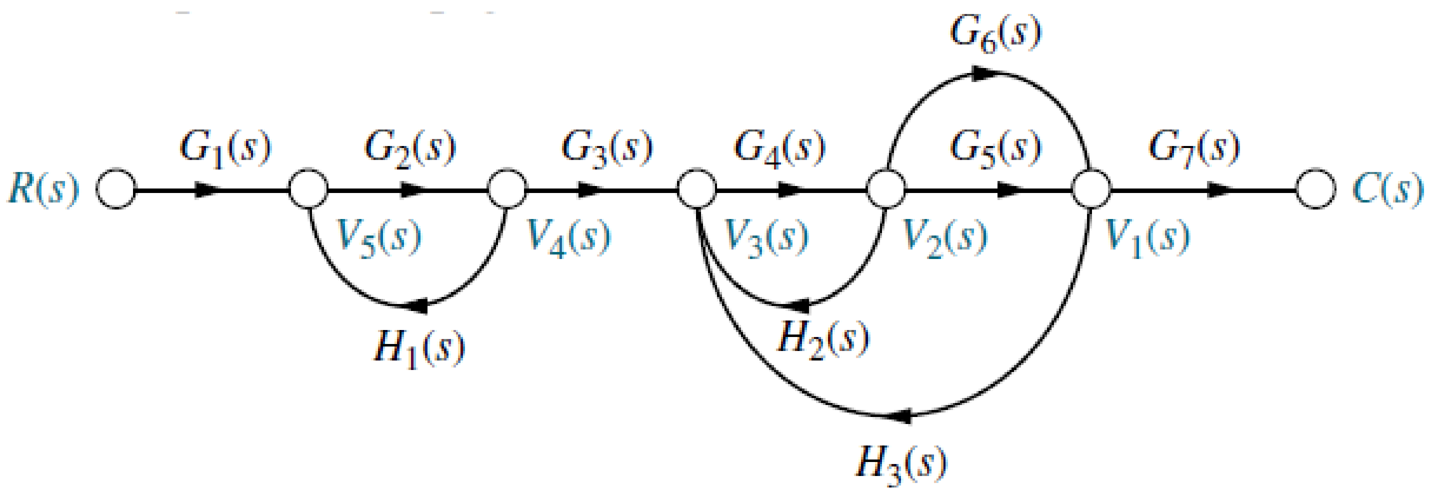
Now we are ready to discuss a technique for **reducing signal-flow graphs** to single transfer functions that relate the output of a system to its input.

Mason's rule for reducing a signal-flow graph to a single transfer function requires the application of **one formula**.

- Mason's formula has several components that must be evaluated.
- First, we must be sure that the definitions of the components are well understood.
- Then we must exert care in evaluating the components.
- To that end, we discuss some basic definitions applicable to signal-flow graphs;

Definitions: Loop Gain

Loop Gain:
The product of branch gains found by traversing a path that starts at a node and ends at the same node, following the direction of the signal flow, without passing through any other node more than once.



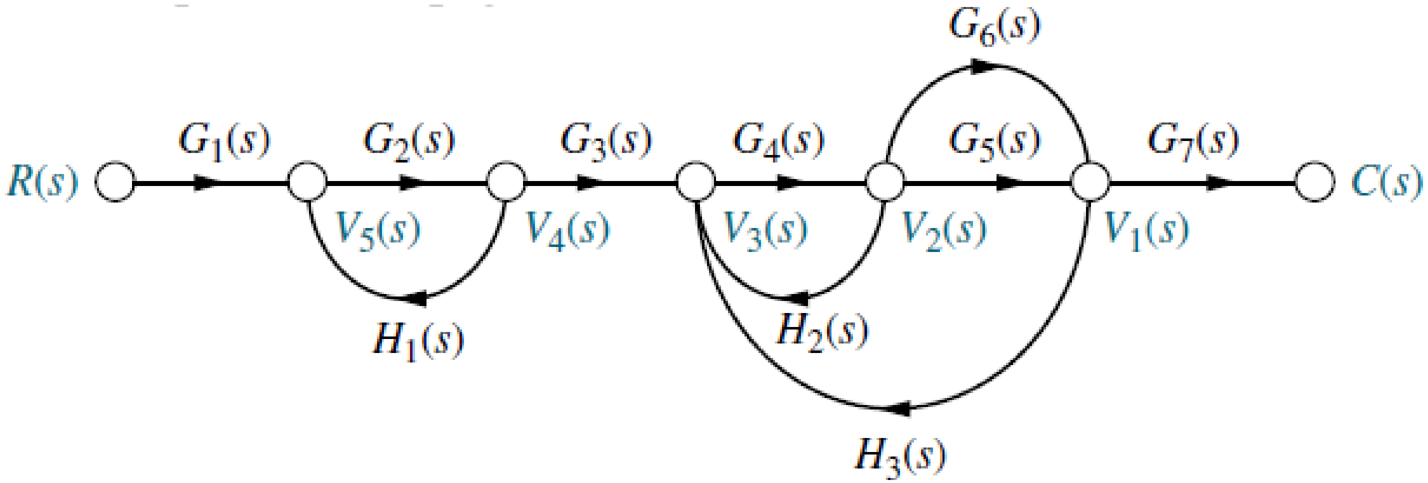
There are 4 loop gains

1. $G_2(s)H_1(s)$
2. $G_4(s)H_2(s)$
3. $G_4(s)G_5(s)H_3(s)$
4. $G_4(s)G_6(s)H_3(s)$

Definitions: Forward path Gain

Forward-path gain. The product of gains found by traversing a path from the input node to the output node of the signal-flow graph in the direction of signal flow

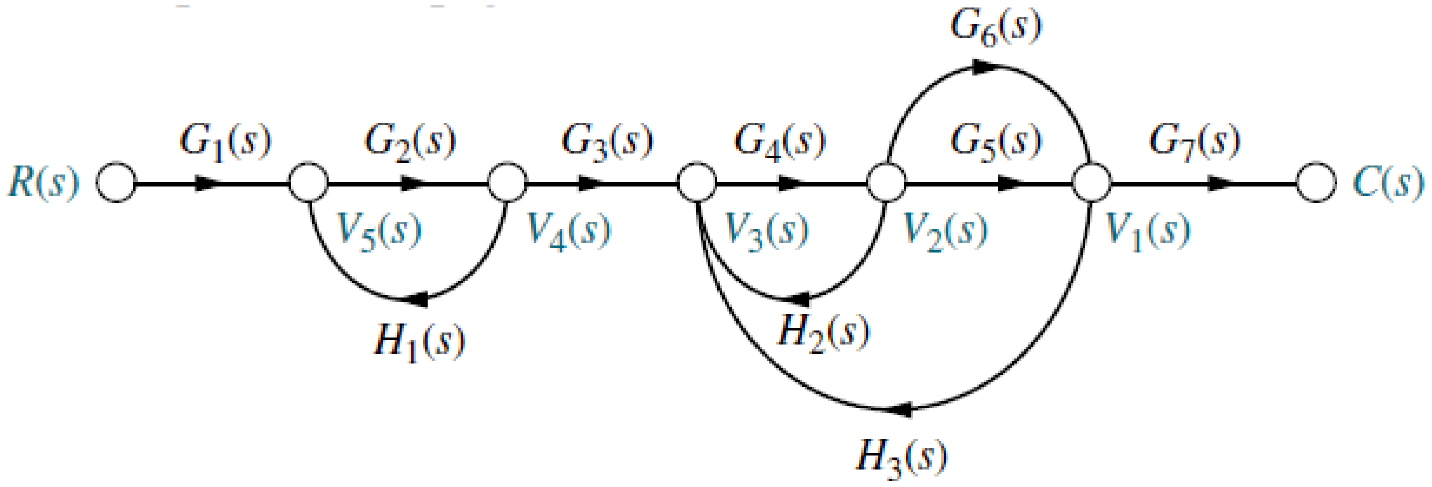
There are two forward-path gains:



1. $G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)G_7(s)$
2. $G_1(s)G_2(s)G_3(s)G_4(s)G_6(s)G_7(s)$

Definitions: Non touching loops

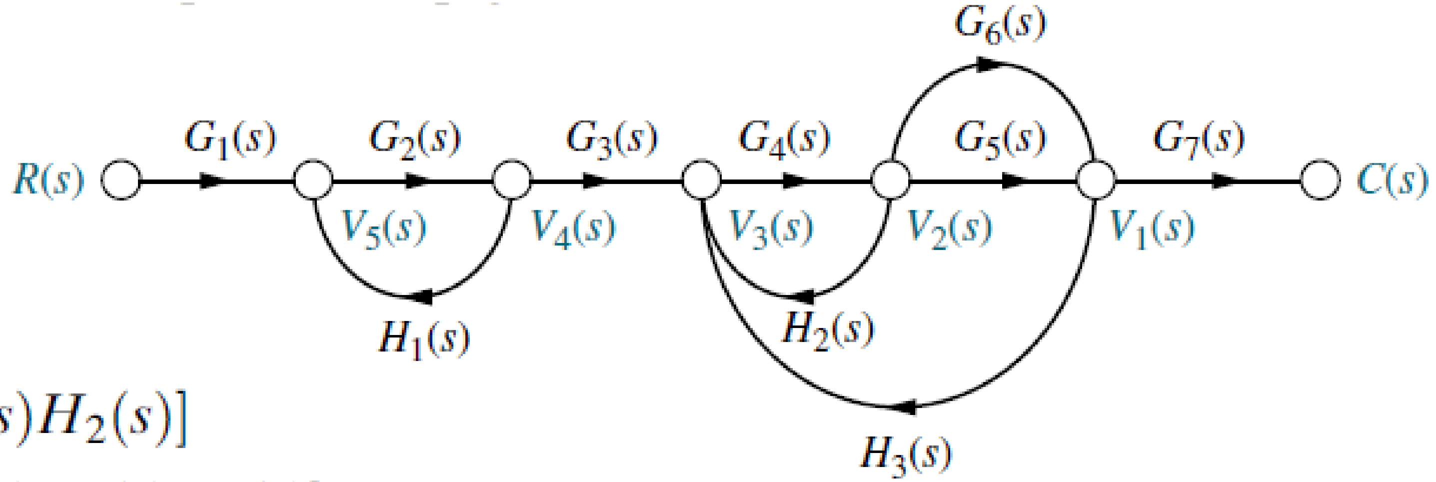
Non touching loops. Loops that do not have any nodes in common.



$G_2(s)H_1(s)$ does not touch the loops $G_4(s)H_2(s)$, $G_4(s)G_5(s)H_3(s)$, and $G_4(s)G_6(s)H_3(s)$.

Definitions: Non touching loop gain

Non touching-loop gain. The product of loop gains from non touching loops taken two, three, four, or more at a time.



1. $[G_2(s)H_1(s)][G_4(s)H_2(s)]$
2. $[G_2(s)H_1(s)][G_4(s)G_5(s)H_3(s)]$
3. $[G_2(s)H_1(s)][G_4(s)G_6(s)H_3(s)]$

The product of loop gains $[G_4(s)G_5(s)H_3(s)][G_4(s)G_6(s)H_3(s)]$
is not a non touching loop gain since these two loops have nodes in common.

We are now ready to state Mason's rule

The transfer function, $C(s)/R(s)$, of a system represented by a signal-flow graph is

$$\boxed{G(s) = \frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta}}$$

where

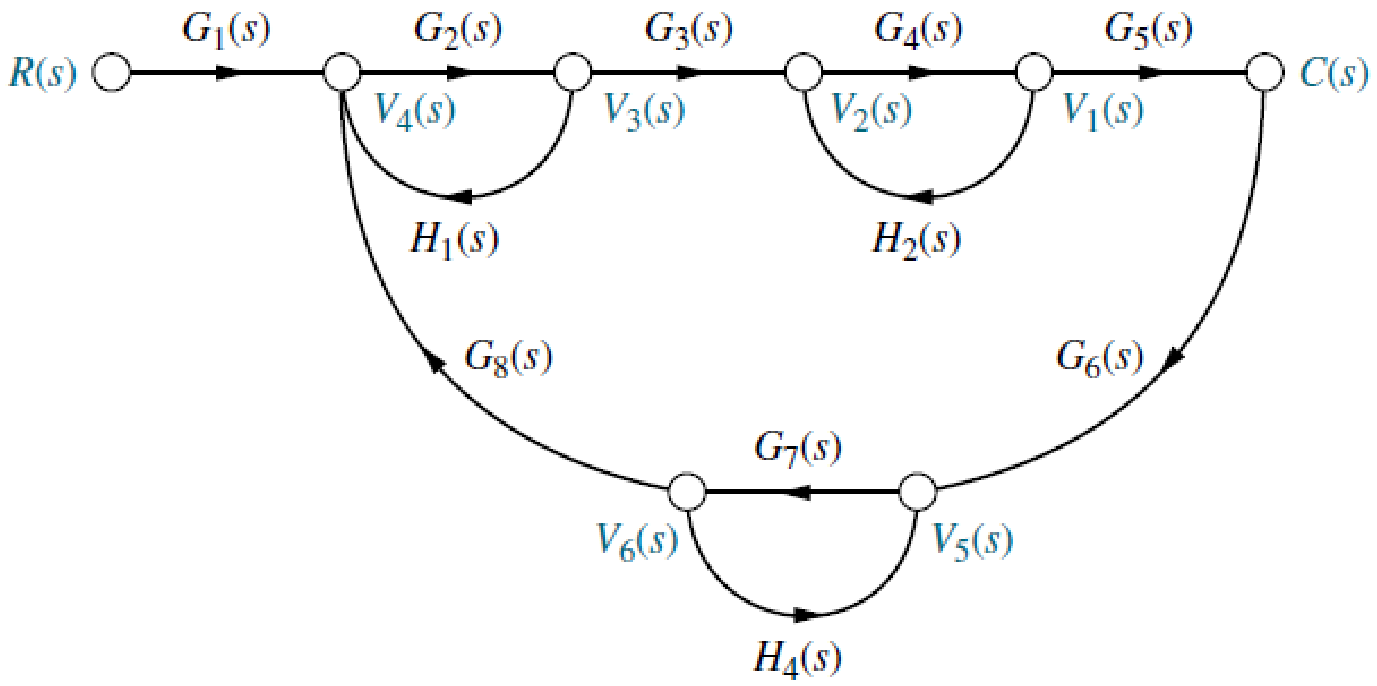
k = number of forward paths
T_k = the kth forward-path gain
$\Delta = 1 - \Sigma$ loop gains $+ \Sigma$ nontouching-loop gains taken two at a time $- \Sigma$ nontouching-loop gains taken three at a time $+ \Sigma$ nontouching-loop gains taken four at a time $- \ldots$
$\Delta_k = \Delta - \Sigma$ loop gain terms in Δ that touch the kth forward path. In other words, Δ_k is formed by eliminating from Δ those loop gains that touch the kth forward path.

Notice the alternating signs for the components of Δ.

Example 7

Transfer Function via Mason's Rule

PROBLEM: Find the transfer function, $C(s)/R(s)$, for the signal-flow graph in Figure



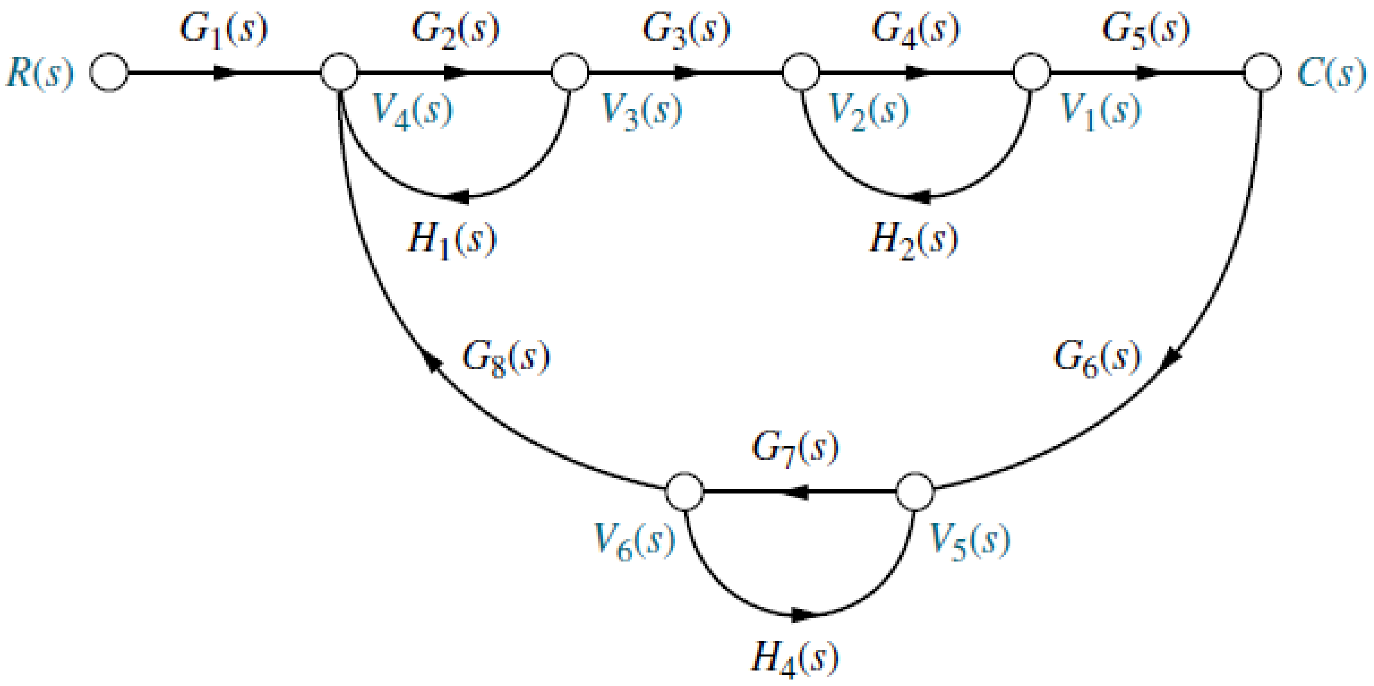
Example 7

SOLUTION:

forward-path gains.

$G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)$

Forward-path gain. The product of gains found by traversing a path from the input node to the output node of the signal-flow graph in the direction of signal flow.

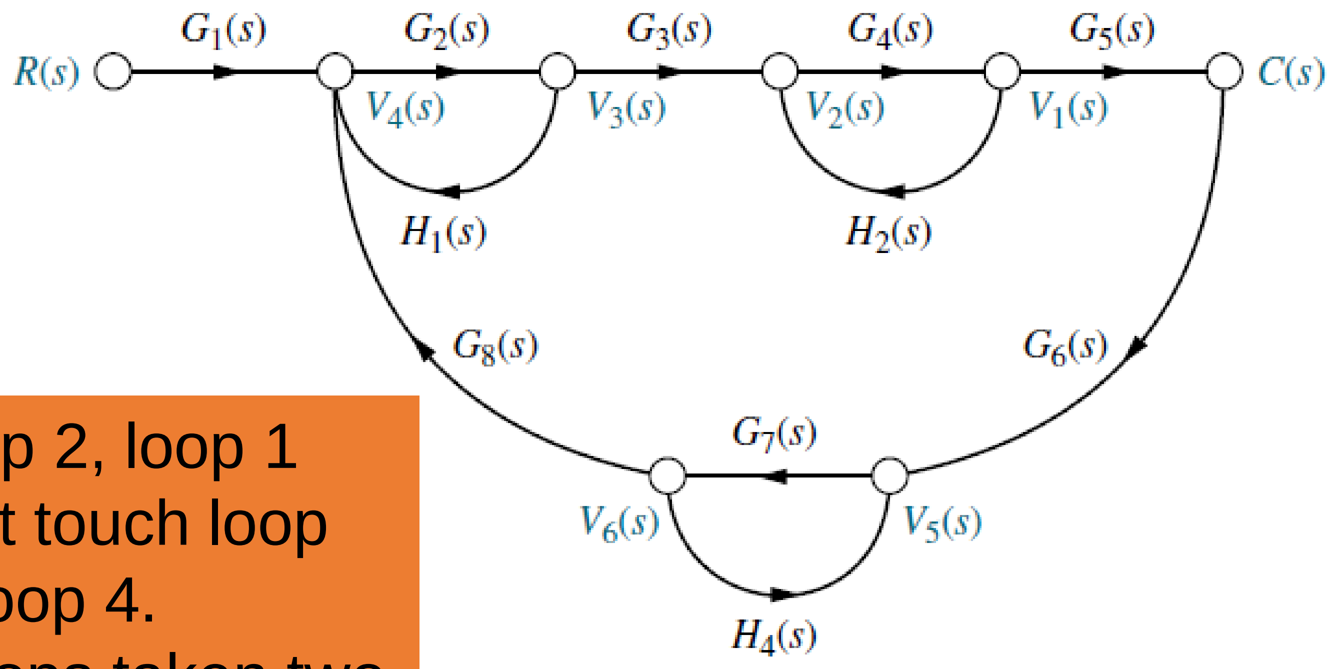


loop gains.

1. $G_2(s)H_1(s)$
2. $G_4(s)H_2(s)$
3. $G_7(s)H_4(s)$
4. $G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)$

Loop Gain:
The product of branch gains found by traversing a path that starts at a node and ends at the same node, following the direction of the signal flow, without passing through any other node more than once.

Example 7



Non touching loops taken **two at a time**

We can see that loop 1 does not touch loop 2, loop 1 does not touch loop 3, and loop 2 does not touch loop 3. Notice that loops 1, 2, and 3 all touch loop 4. Thus, the combinations of non touching loops taken two at a time are as follows:

Loop 1 and loop 2 : $G_2(s)H_1(s)G_4(s)H_2(s)$

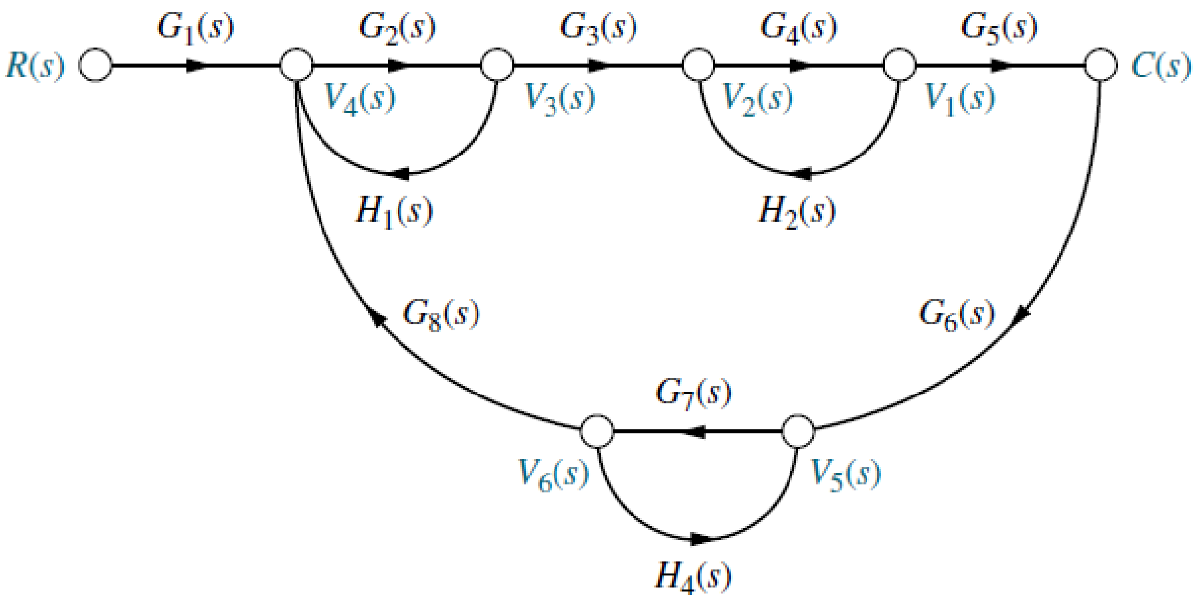
Loop 1 and loop 3 : $G_2(s)H_1(s)G_7(s)H_4(s)$

Loop 2 and loop 3 : $G_4(s)H_2(s)G_7(s)H_4(s)$

Finally, the nontouching loops taken **three at a time** are as follows:
Loops 1; 2; and 3 :

$$G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)$$

Example 7



we form Δ and Δ_k.

$$G(s) = \frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta}$$

$\Delta = 1 - \Sigma$ loop gains $+ \Sigma$ nontouching-loop gains taken two at a time $- \Sigma$ nontouching-loop gains taken three at a time $+ \Sigma$ nontouching-loop gains taken four at a time $- \ldots$

$$\begin{aligned}\Delta = 1 &-[G_2(s)H_1(s) + G_4(s)H_2(s) + G_7(s)H_4(s) \\ &\qquad + G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)] \\ &+[G_2(s)H_1(s)G_4(s)H_2(s) + G_2(s)H_1(s)G_7(s)H_4(s) \\ &\qquad + G_4(s)H_2(s)G_7(s)H_4(s)] \\ &-[G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)]\end{aligned}$$

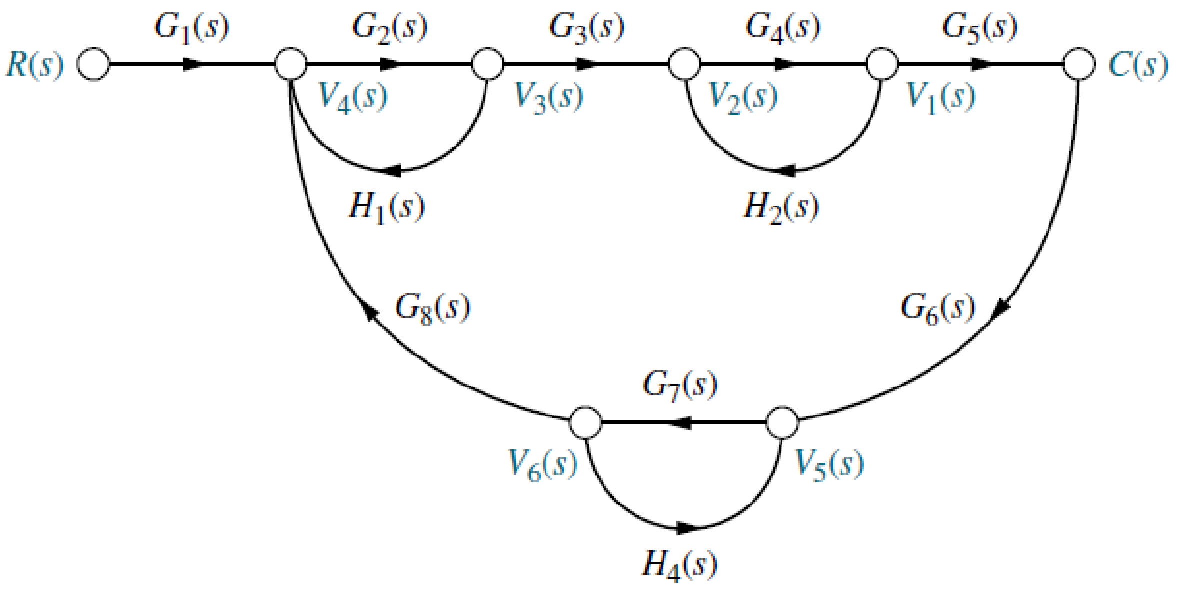
Example 7

$$G(s) = \frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta}$$

$\Delta_k = \Delta - \Sigma$ loop gain terms in Δ that touch the kth forward path. In other words, Δ_k is formed by eliminating from Δ those loop gains that touch the kth forward path.

We form Δ_k by eliminating from Δ the loop gains that touch the kth forward path:

$$\Delta_1 = 1 - G_7(s)H_4(s)$$



Example 7

$$G(s) = \frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta}$$

forward-path gains.

$$G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)$$

$$\Delta_1 = 1 - G_7(s)H_4(s)$$

$$\begin{aligned}\Delta = 1 &- [G_2(s)H_1(s) + G_4(s)H_2(s) + G_7(s)H_4(s) \\ &\qquad + G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)] \\ &+ [G_2(s)H_1(s)G_4(s)H_2(s) + G_2(s)H_1(s)G_7(s)H_4(s) \\ &\qquad + G_4(s)H_2(s)G_7(s)H_4(s)] \\ &- [G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)]\end{aligned}$$

$$G(s) = \frac{T_1\Delta_1}{\Delta} = \frac{[G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)][1 - G_7(s)H_4(s)]}{\Delta}$$

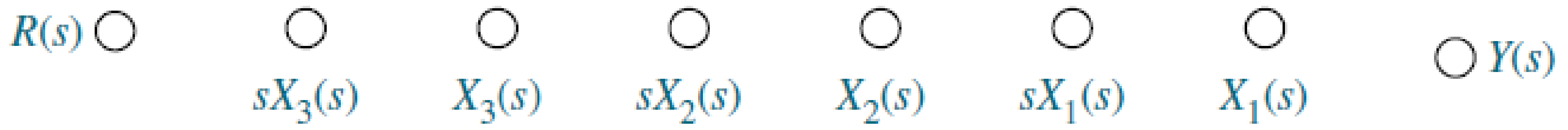
Since there is only one forward path, G(s) consists of only one term, rather than a sum of terms, each coming from a forward path.

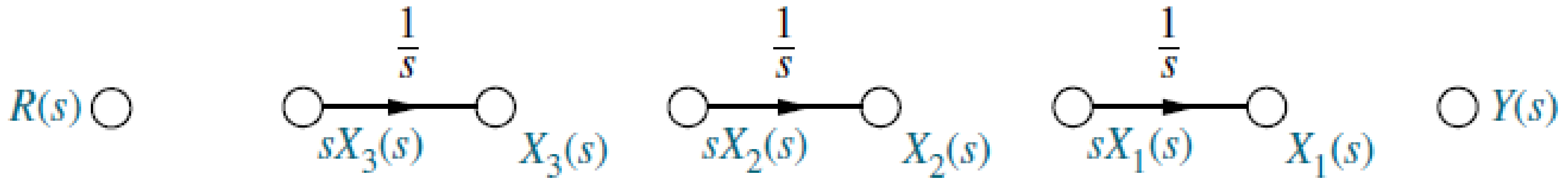
Signal-Flow Graphs of State Equations

we draw signal-flow graphs from state equations.

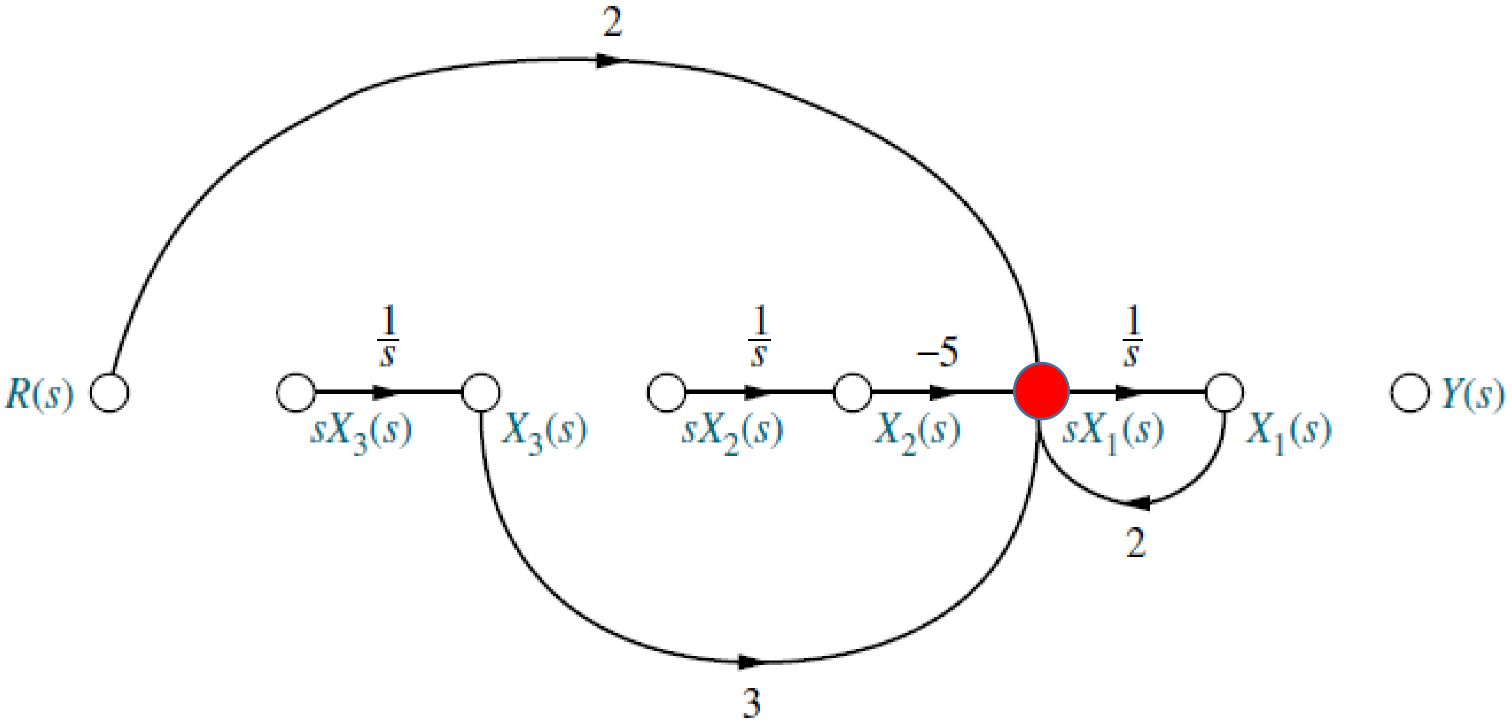
$$\dot{x}_1 = 2x_1 - 5x_2 + 3x_3 + 2r$$
$$\dot{x}_2 = -6x_1 - 2x_2 + 2x_3 + 5r$$
$$\dot{x}_3 = x_1 - 3x_2 - 4x_3 + 7r$$
$$y = -4x_1 + 6x_2 + 9x_3$$

First, identify three nodes to be the three state variables, x_1, x_2, and x_3; also identify three nodes, placed to the left of each respective state variable, to be the derivatives of the state variables:





Signal-Flow Graphs of State Equations
let´s form the state equation **dx₁/dt**



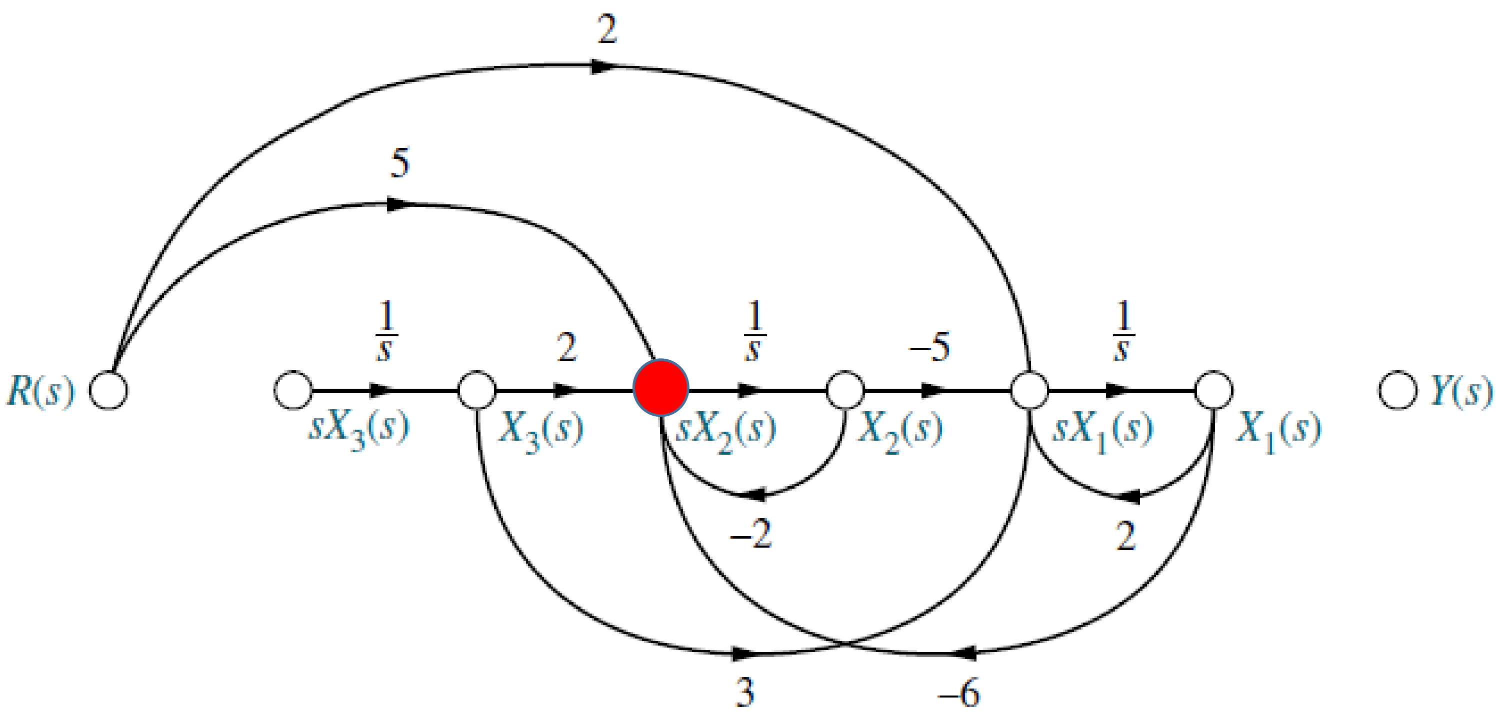
$$\boxed{\dot{x}_1 = 2x_1 - 5x_2 + 3x_3 + 2r}$$

$$\dot{x}_2 = -6x_1 - 2x_2 + 2x_3 + 5r$$

$$\dot{x}_3 = x_1 - 3x_2 - 4x_3 + 7r$$

$$y = -4x_1 + 6x_2 + 9x_3$$

Signal-Flow Graphs of State Equations let´s form the state equation **dx₂/dt**



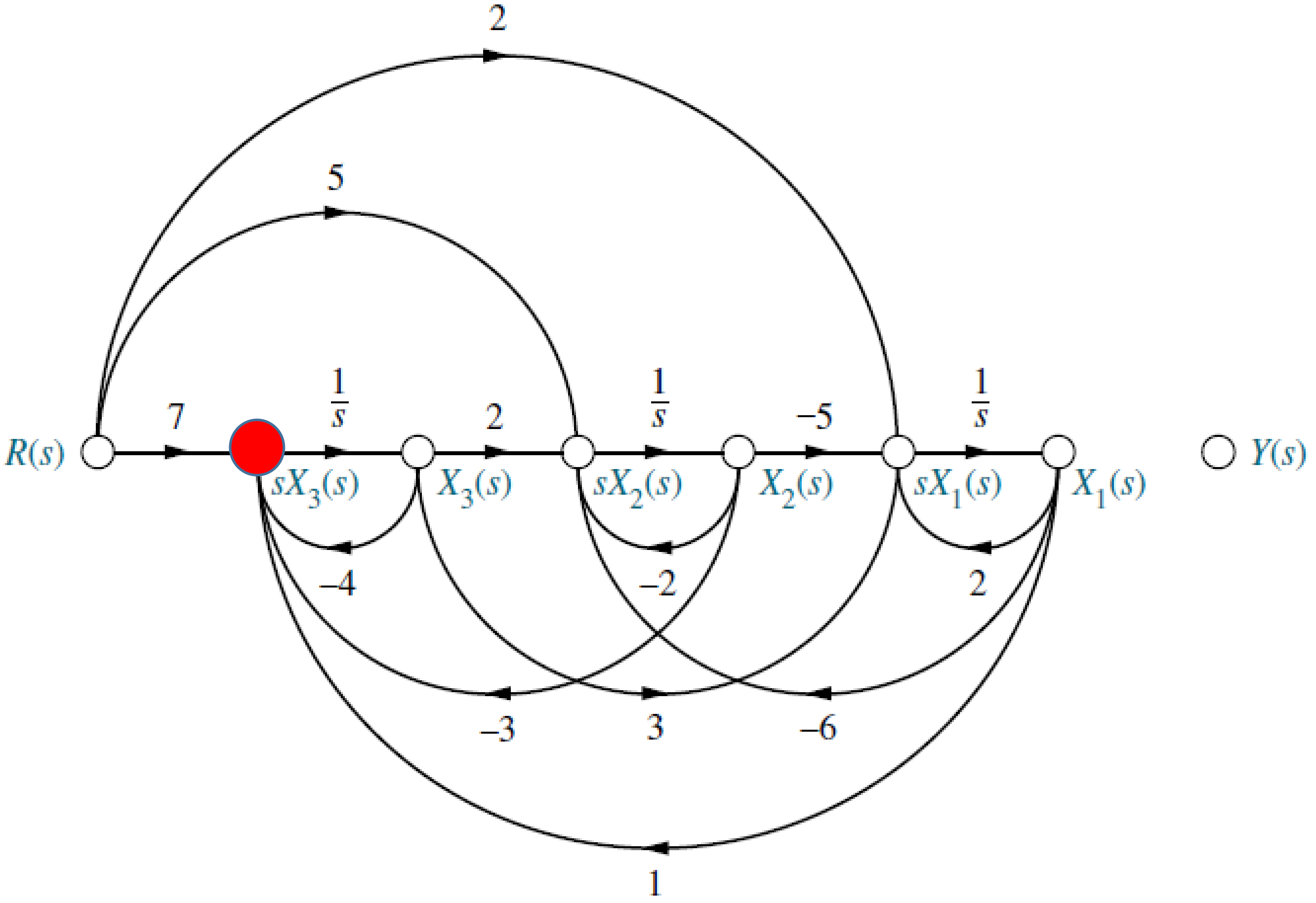
$$\dot{x}_1 = 2x_1 - 5x_2 + 3x_3 + 2r$$

$$\boxed{\dot{x}_2 = -6x_1 - 2x_2 + 2x_3 + 5r}$$

$$\dot{x}_3 = x_1 - 3x_2 - 4x_3 + 7r$$

$$y = -4x_1 + 6x_2 + 9x_3$$

Signal-Flow Graphs of State Equations let´s form the state equation **dx₃/dt**



$$\dot{x}_1 = 2x_1 - 5x_2 + 3x_3 + 2r$$

$$\dot{x}_2 = -6x_1 - 2x_2 + 2x_3 + 5r$$

$$\boxed{\dot{x}_3 = x_1 - 3x_2 - 4x_3 + 7r}$$

$$y = -4x_1 + 6x_2 + 9x_3$$

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Signal-Flow Graphs of State Equations let´s form the output equation **y(t)**

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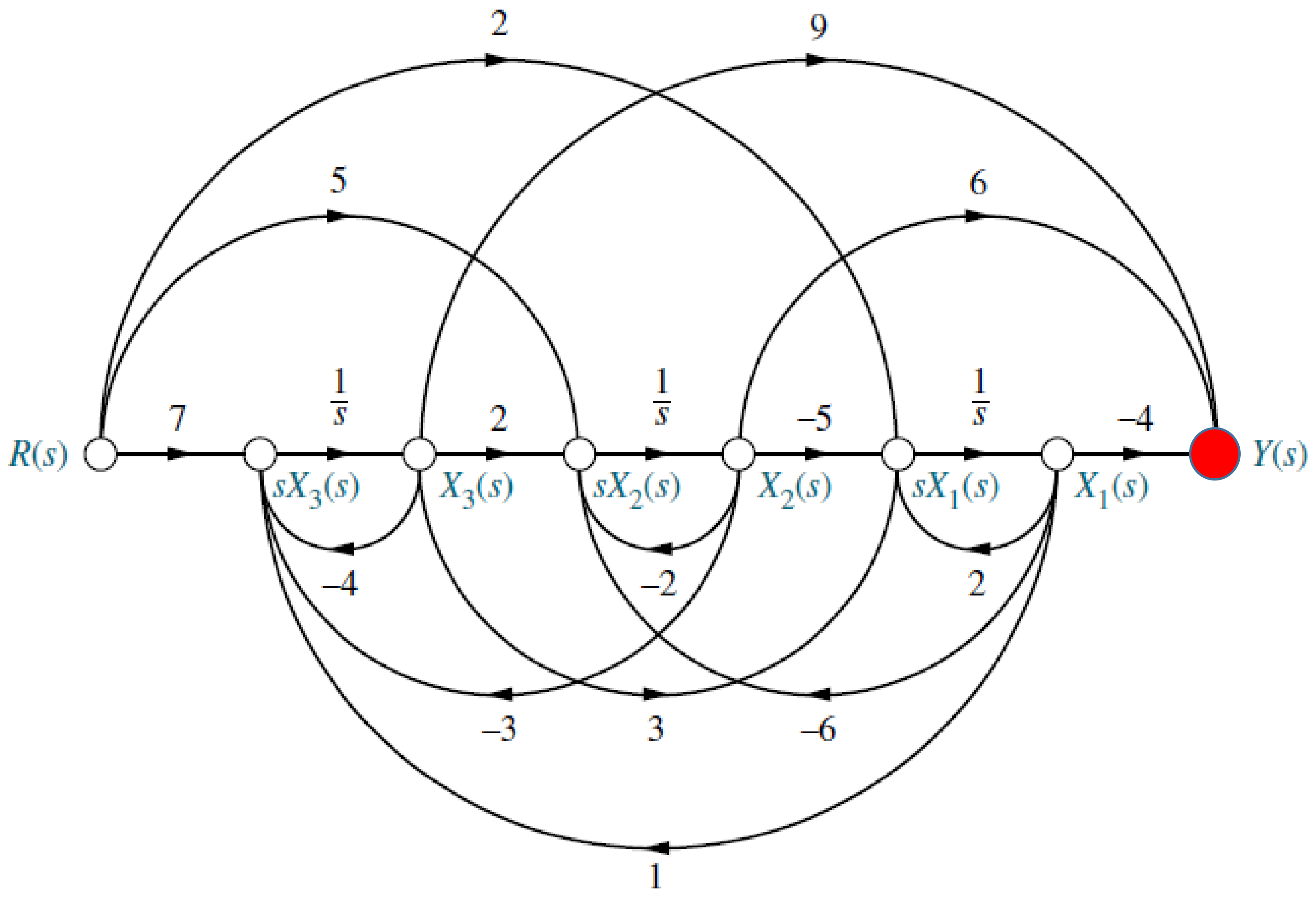


$$\dot{x}_1 = 2x_1 - 5x_2 + 3x_3 + 2r$$

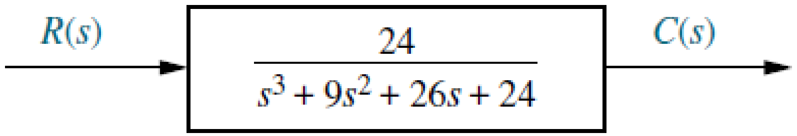
$$\dot{x}_2 = -6x_1 - 2x_2 + 2x_3 + 5r$$

$$\dot{x}_3 = x_1 - 3x_2 - 4x_3 + 7r$$

$$\boxed{y = -4x_1 + 6x_2 + 9x_3}$$

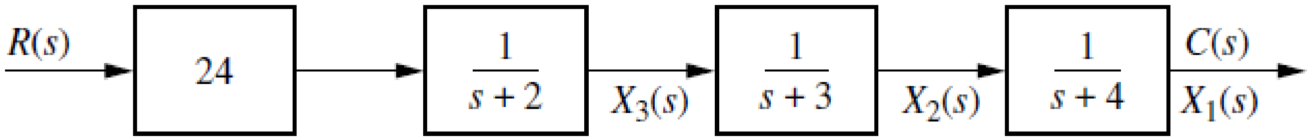
Alternative Representations in State Space

Cascade Form



$$\frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)}$$

It can be represented alternately as



The output of each first-order system block has been labelled as a state variable.

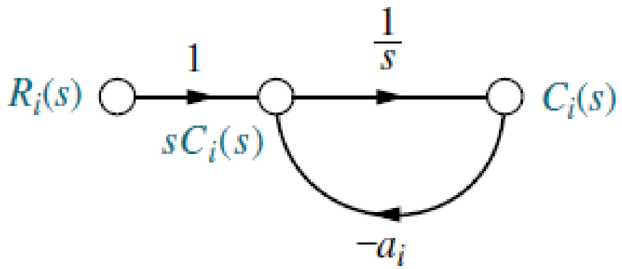
The signal flow for **each first-order** system

$$\frac{C_i(s)}{R_i(s)} = \frac{1}{(s+a_i)} \qquad \frac{dc_i(t)}{dt} + a_i c_i(t) = r_i(t) \qquad \frac{dc_i(t)}{dt} = -a_i c_i(t) + r_i(t)$$

Alternative Representations in State Space

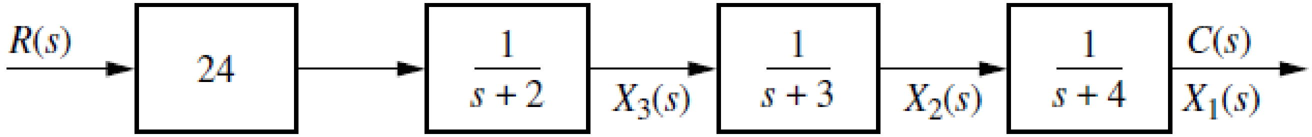
For each first order system we have the signal-flow graph:

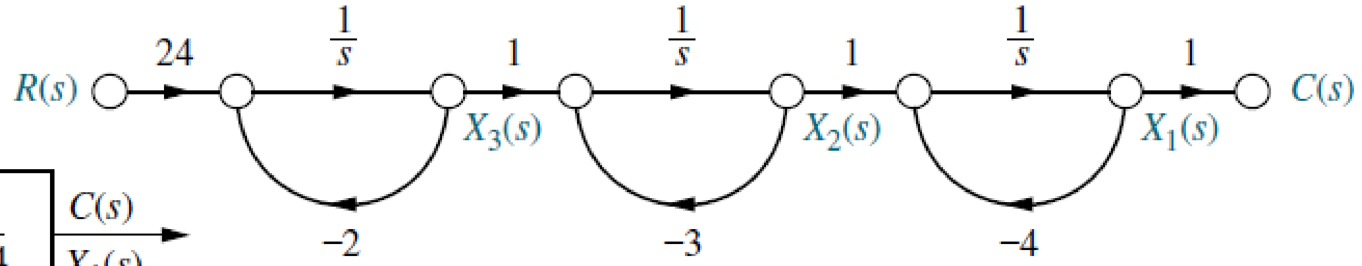
$$\frac{dc_i(t)}{dt} = -a_i c_i(t) + r_i(t)$$



For our system:

$$\frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)}$$





Now write the state equations for the **new representation of the system**.

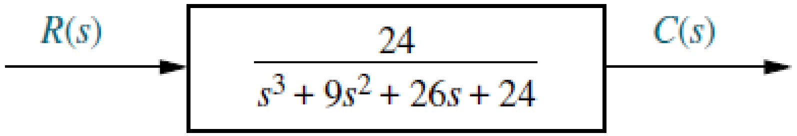
$$\begin{aligned}
\dot{x}_1 &= -4x_1 + x_2 \\
\dot{x}_2 &= -3x_2 + x_3 \\
\dot{x}_3 &= -2x_3 + 24r \\
y &= c(t) = x_1
\end{aligned}$$

$$\boxed{\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r \\
y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}
\end{aligned}}$$

Alternative Representations in State Space

Parallel Form

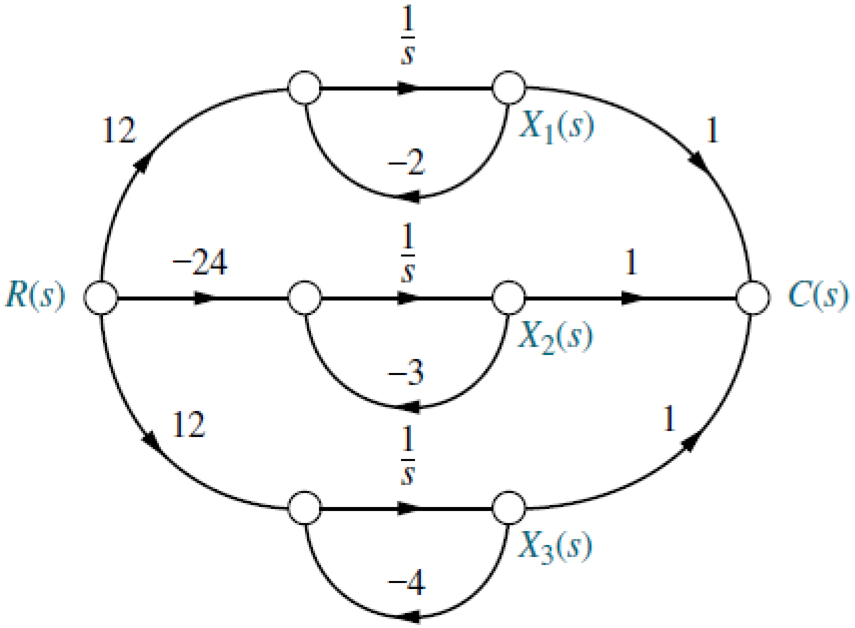
Whereas the **previous** form was arrived at **by cascading the individual first order subsystems**, the **parallel form is derived from a partial-fraction expansion** of the system transfer function.



$$\frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)} = \frac{12}{(s+2)} - \frac{24}{(s+3)} + \frac{12}{(s+4)}$$

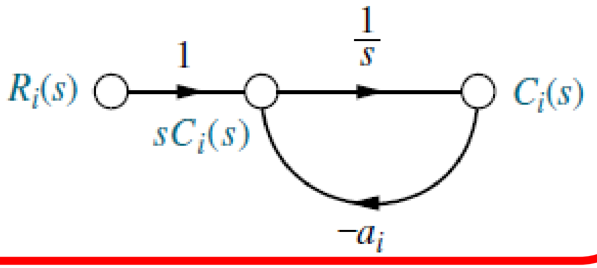
The Equation represents the sum of the individual first-order subsystems. To arrive at a signal-flow graph, first solve for C(s):

$$C(s) = R(s)\frac{12}{(s+2)} - R(s)\frac{24}{(s+3)} + R(s)\frac{12}{(s+4)}$$



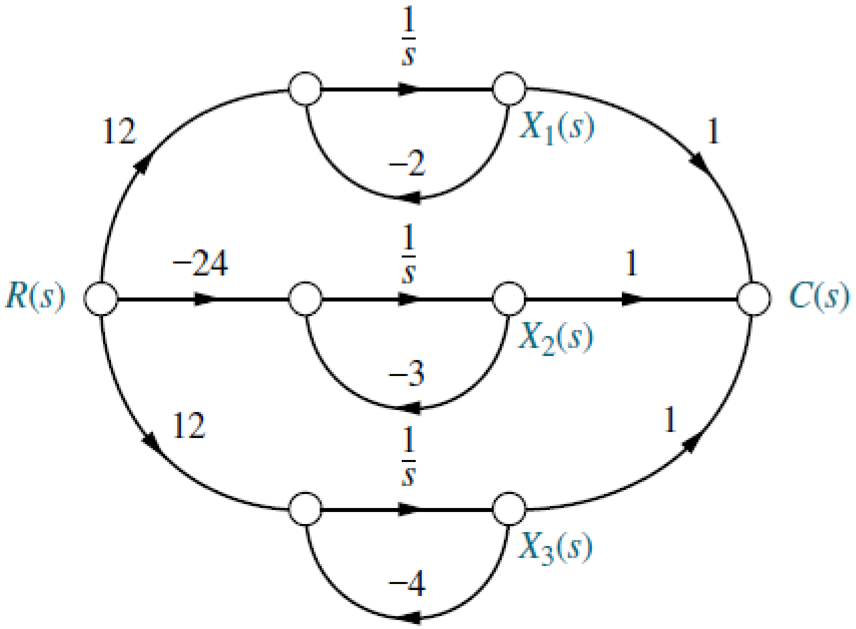
Remember from previous slides 49-50

$$\frac{C_i(s)}{R_i(s)} = \frac{1}{(s+a_i)}$$



Alternative Representations in State Space

Parallel Form



$$
\begin{aligned}
\dot{x}_1 &= -2x_1 && && + 12r \\
\dot{x}_2 &= && -3x_2 && - 24r \\
\dot{x}_3 &= && && -4x_3 + 12r
\end{aligned}
$$

$$y = c(t) = x_1 + x_2 + x_3$$

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 12 \\ -24 \\ 12 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\mathbf{x}$$

What is the advantage of this representation?

Each equation is a first-order differential equation in only one variable.
Thus, we would solve these equations independently.
The equations are said to be **decoupled**.

Example 8 State-Space Representation of Feedback Systems

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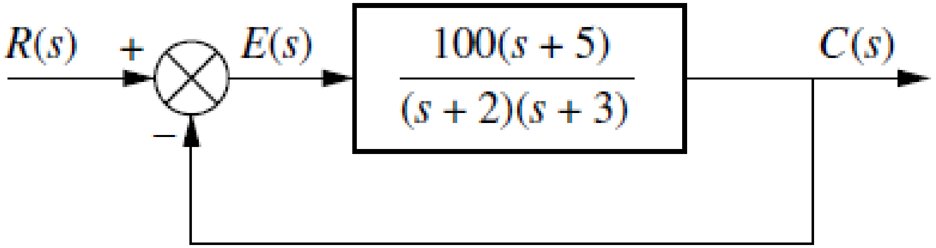
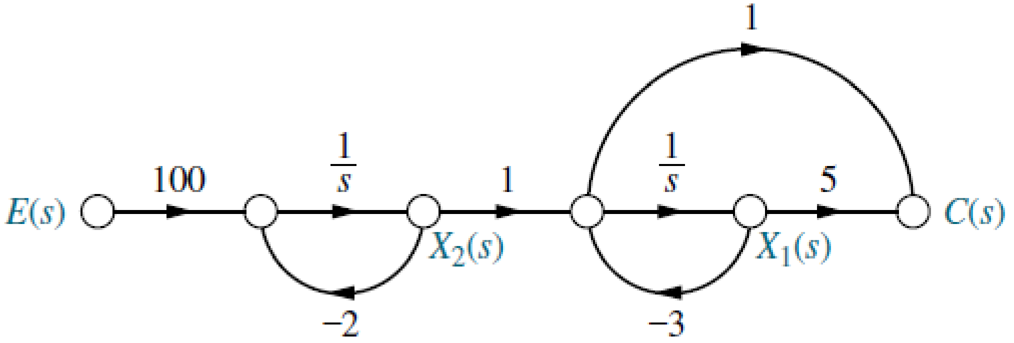
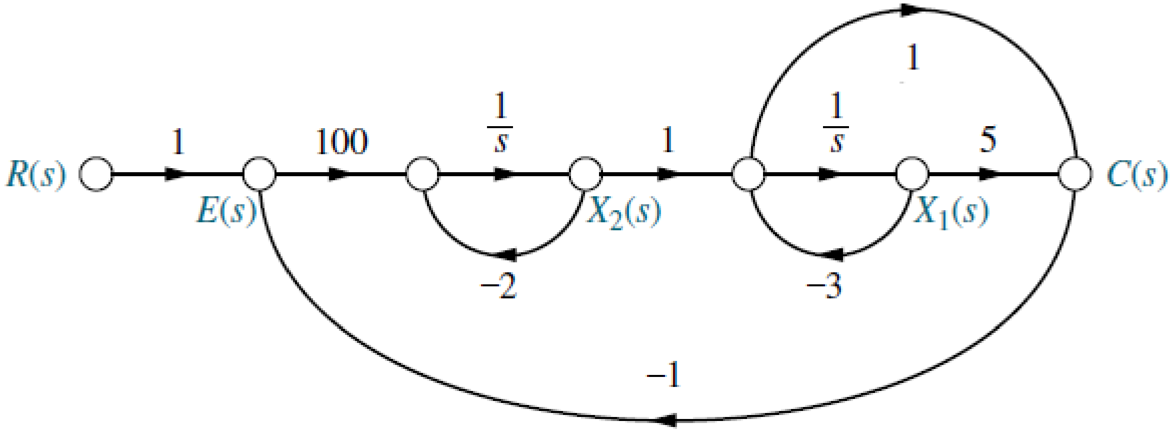
PROBLEM: Represent the feedback control system shown in Figure in state space. Model the forward transfer function in cascade form.



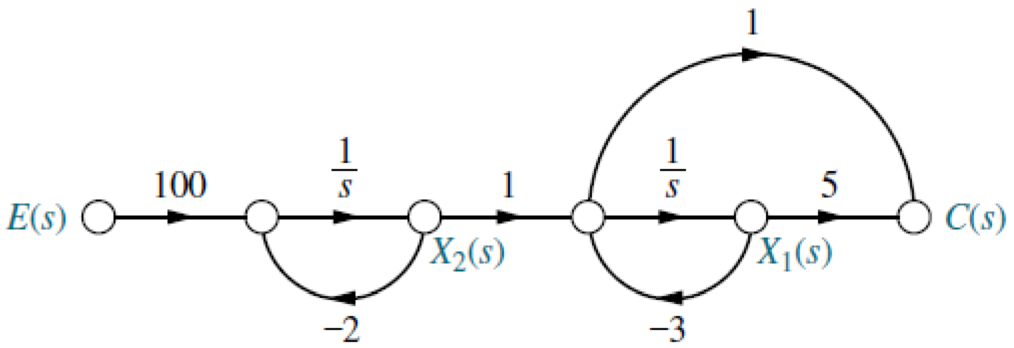
SOLUTION: First we model the **forward transfer function** in cascade form.

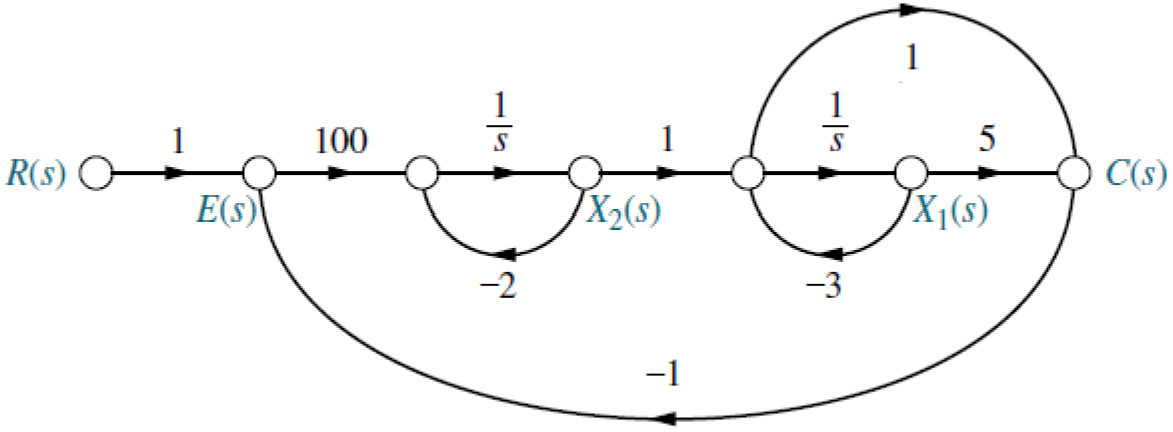


Next add the **feedback** and input paths,



Example 8





$$\dot{x}_1 = -3x_1 + x_2 \qquad \textbf{a}$$

$$\dot{x}_2 = \quad -2x_2 + 100(r - c) \qquad \textbf{b}$$

Substituting equation **c** in **b**

$$c = 5x_1 + (x_2 - 3x_1) = 2x_1 + x_2 \qquad \textbf{c}$$

And we obtain the final state space representation

$$\begin{aligned} \dot{x}_1 &= -3x_1 + x_2 \\ \dot{x}_2 &= -200x_1 - 102x_2 + 100r \\ y = c(t) &= 2x_1 + x_2 \end{aligned}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 1 \\ -200 & -102 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 100 \end{bmatrix} r$$

$$y = \begin{bmatrix} 2 & 1 \end{bmatrix}\mathbf{x}$$

Similarity Transformations

Systems can be represented with **different state variables** even though the transfer function relating the output to the input remains the same.

These systems are called similar systems. Although their state space representations are different, similar systems have the same transfer function and hence the **same poles and eigenvalues.**

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

similar system $\longrightarrow$

$$\mathbf{z} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{z} + \mathbf{P}^{-1}\mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{P}\mathbf{z} + \mathbf{D}\mathbf{u}$$

$$\mathbf{z} = \mathbf{P}^{-1}\mathbf{x}$$

i.e. for a two dimensional space

$$\mathbf{P} = [\mathbf{U}_{z_1}\mathbf{U}_{z_2}] = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mathbf{P}\mathbf{z}$$

Thus, **P** is a transformation matrix whose columns are the coordinates of the basis vectors of the Z_1Z_2 space expressed as linear combinations of the X_1X_2 space.

Example 9

Similarity Transformations on State Equations

Given the system represented in state space by Eqs.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -7 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

$$y = [1 \quad 0 \quad 0]\mathbf{x}$$

transform the system to a new set of state variables, **z**, where the new state variables are related to the original state variables, **x**, as follows:

$$\begin{aligned} z_1 &= 2x_1 \\ z_2 &= 3x_1 + 2x_2 \\ z_3 &= x_1 + 4x_2 + 5x_3 \end{aligned}$$

Example 9

SOLUTION:

$$\mathbf{z} = \mathbf{P}^{-1}\mathbf{x}$$

$$\dot{\mathbf{z}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{z} + \mathbf{P}^{-1}\mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{P}\mathbf{z} + \mathbf{D}\mathbf{u}$$

$$\begin{aligned} z_1 &= 2x_1 \\ z_2 &= 3x_1 + 2x_2 \\ z_3 &= \;\; x_1 + 4x_2 + 5x_3 \end{aligned}$$

$$\mathbf{z} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix} \mathbf{x} = \mathbf{P}^{-1}\mathbf{x}$$

Let´s convert the matrices for the new representation

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -7 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 \\ -0.75 & 0.5 & 0 \\ 0.5 & -0.4 & 0.2 \end{bmatrix} = \begin{bmatrix} -1.5 & 1 & 0 \\ -1.25 & 0.7 & 0.4 \\ -2.5 & 0.4 & -6.2 \end{bmatrix}$$

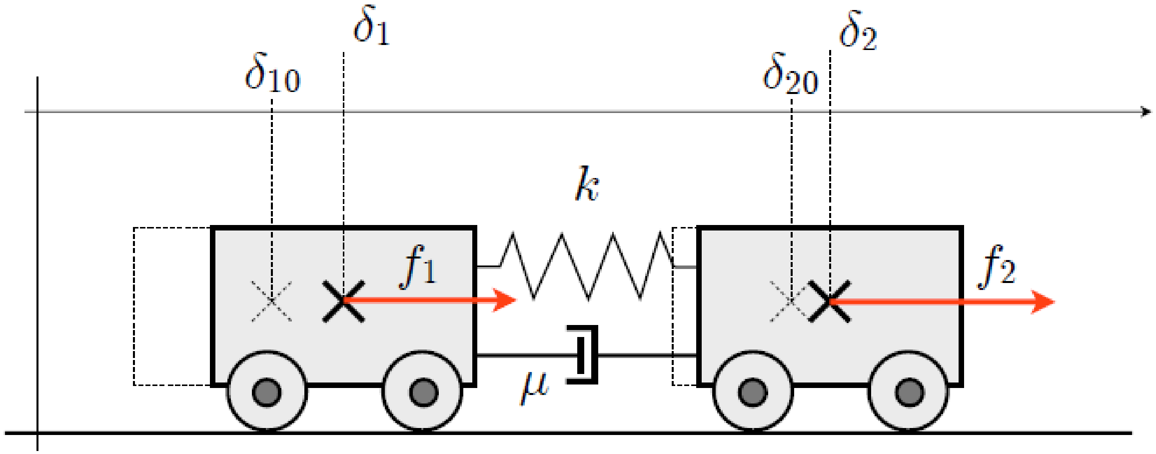
$$\mathbf{P}^{-1}\mathbf{B} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} \qquad \mathbf{C}\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 \\ -0.75 & 0.5 & 0 \\ 0.5 & -0.4 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0 \end{bmatrix}$$

We can hence write the equivalent state space representation

$$\boxed{\begin{aligned} \dot{\mathbf{z}} &= \begin{bmatrix} -1.5 & 1 & 0 \\ -1.25 & 0.7 & 0.4 \\ -2.55 & 0.4 & -6.2 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} u \\ y &= \begin{bmatrix} 0.5 & 0 & 0 \end{bmatrix} \mathbf{z} \end{aligned}}$$

Example 10

Two Degrees of freedom system



$$F_{e1} = -k\left[(\delta_1(t) - \delta_{10}) - (\delta_2(t) + \delta_{20})\right]$$

$$F_{e2} = k\left[(\delta_1(t) - \delta_{10}) - (\delta_2(t) + \delta_{20})\right]$$

elastic forces

$$F_{d1} = -\mu[\dot{\delta}_1(t) - \dot{\delta}_2(t)]$$

$$F_{d2} = \mu[\dot{\delta}_1(t) - \dot{\delta}_2(t)]$$

friction forces

f_1 f_2

external forces

Example 10

equations of motion

$$m_1\ddot{\delta}_1(t) = -k[\delta_1(t) - \delta_{10} - \delta_2(t) + \delta_{20}] - \mu[\dot{\delta}_1(t) - \dot{\delta}_2(t)] + f_1(t)$$

$$m_2\ddot{\delta}_2(t) = k[\delta_1(t) - \delta_{10} - \delta_2(t) + \delta_{20}] + \mu[\dot{\delta}_1(t) - \dot{\delta}_2(t)] + f_2(t)$$

matrix form

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{\delta}_1(t) \\ \ddot{\delta}_2(t) \end{pmatrix} = -\begin{pmatrix} \mu & -\mu \\ -\mu & \mu \end{pmatrix}\begin{pmatrix} \dot{\delta}_1(t) \\ \dot{\delta}_2(t) \end{pmatrix} - \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}\begin{pmatrix} \delta_1(t) - \delta_{10} \\ \delta_2(t) - \delta_{20} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

define

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad F = \begin{pmatrix} \mu & -\mu \\ -\mu & \mu \end{pmatrix}, \quad K = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\delta(t) = \begin{pmatrix} \delta_1(t) - \delta_{10} \\ \delta_2(t) - \delta_{20} \end{pmatrix} \qquad f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$$

$$M\ddot{\delta} + F\dot{\delta} + K\delta = Gf$$

Example 10

$\dot{x} = Ax$ $\quad\quad$ A has rank = 2

$$\dot{x}(t) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m_1 & k/m_1 & -\mu/m_1 & \mu/m_1 \\ k/m_2 & -k/m_2 & \mu/m_2 & -\mu/m_2 \end{pmatrix} x(t)$$

with

$$x(t) = \begin{pmatrix} \text{relative positions} \\ \text{velocities} \end{pmatrix} = \begin{pmatrix} \delta_1(t) - \delta_{10} \\ \delta_2(t) - \delta_{20} \\ \dot{\delta}_1(t) \\ \dot{\delta}_2(t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix}$$

Example 10

Let´s define new coordinates

center of mass

$$\delta_c(t) = \underbrace{\frac{m_1\delta_1(t) + m_2\delta_2(t)}{m_1 + m_2}}_{\substack{\text{center of mass} \\ \text{position}}} - \underbrace{\frac{m_1\delta_{10} + m_2\delta_{20}}{m_1 + m_2}}_{\substack{\text{center of mass} \\ \text{rest position}}}$$

relative displacement

$$\delta_r(t) = \underbrace{\delta_2(t) - \delta_1(t)}_{\substack{\text{relative} \\ \text{position}}} - \underbrace{(\delta_{20} - \delta_{10})}_{\substack{\text{relative} \\ \text{pos. at rest}}}$$

Example 10 The derivatives

$$M = m_1 + m_2$$

$$\begin{aligned}
\delta_c(t) &= \frac{m_1[\delta_1(t) - \delta_{10}] + m_2[\delta_2(t) - \delta_{20}]}{M} = \frac{m_1 x_1(t) + m_2 x_2(t)}{M} \\
\dot{\delta}_c(t) &= \frac{m_1 \dot{x}_1(t) + m_2 \dot{x}_2(t)}{M} = \frac{m_1 x_3(t) + m_2 x_4(t)}{M}
\end{aligned}$$

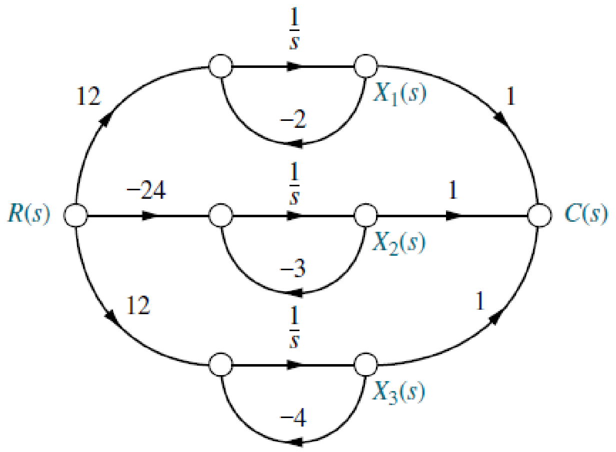
$$\begin{aligned}
\delta_r(t) &= [\delta_2(t) - \delta_{20}] - [\delta_1(t) - \delta_{10}] = x_2(t) - x_1(t) \\
\dot{\delta}_r(t) &= \dot{x}_2(t) - \dot{x}_1(t) = x_4(t) - x_3(t)
\end{aligned}$$

change of coordinates

$$z = \begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{pmatrix} = \begin{pmatrix} \delta_b(t) \\ \delta_s(t) \\ \dot{\delta}_b(t) \\ \dot{\delta}_s(t) \end{pmatrix} = \begin{pmatrix} m_1/M & m_2/M & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & m_1/M & m_2/M \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} = Tx$$

Diagonalizing a System Matrix

- we saw that the **parallel form of a signal-flow graph** can yield a diagonal system matrix.
- A diagonal system matrix has the advantage that each state equation is a function of only **one state variable**
- Hence, each differential equation can be solved independently of the other equations.
- We say that the equations are **decoupled**.



$$\begin{aligned}
\dot{x}_1 &= -2x_1 && + 12r \\
\dot{x}_2 &= \quad -3x_2 && - 24r \\
\dot{x}_3 &= \quad\quad -4x_3 && + 12r \\
y &= c(t) = x_1 + x_2 + x_3
\end{aligned}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 12 \\ -24 \\ 12 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\mathbf{x}$$

Diagonalizing a System Matrix

- Rather than using partial fraction expansion and signal-flow graphs (bye bye Mason), we can decouple a system using matrix transformations.
- If we find the correct matrix, **P**, the transformed system matrix, $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, will be a diagonal matrix
- we are looking for a transformation to another state space that yields a diagonal matrix in that space.

Definitions

Eigenvector. The eigenvectors of the matrix **A** are all vectors, $\mathbf{x}_i \neq \mathbf{0}$, which under the transformation **A** become multiples of themselves; that is

$$\boxed{\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i}$$

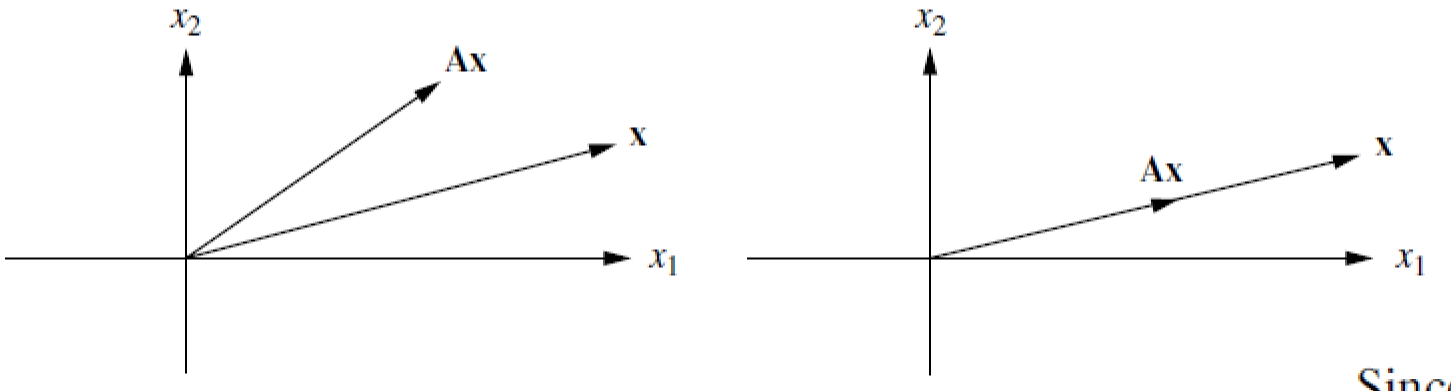
Diagonalizing a System Matrix

Definitions

Eigenvector. The eigenvectors of the matrix **A** are all vectors, $\mathbf{x}_i \neq \mathbf{0}$, which under the transformation **A** become multiples of themselves; that is

$$\boxed{\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i}$$

$$\mathbf{0} = (\lambda_i \mathbf{I} - \mathbf{A})\mathbf{x}_i$$



(*a*) (*b*)

To be an eigenvector, the transformation $\mathbf{A}\mathbf{x}$ must be collinear with $\mathbf{x}$; thus, in (*a*), $\mathbf{x}$ is not an eigenvector; in (*b*), it is

$$\mathbf{x}_i = (\lambda_i \mathbf{I} - \mathbf{A})^{-1}\mathbf{0} = \frac{\operatorname{adj}(\lambda_i \mathbf{I} - \mathbf{A})}{\det(\lambda_i \mathbf{I} - \mathbf{A})}\mathbf{0}$$

Since $\mathbf{x}_i \neq \mathbf{0}$, a nonzero solution exists if

$$\boxed{\det(\lambda_i \mathbf{I} - \mathbf{A}) = 0}$$

Example 11
Diagonalizing a System Matrix

Finding Eigenvectors

Find the eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$$

SOLUTION:

$$\begin{aligned} \det(\lambda\mathbf{I} - \mathbf{A}) &= \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \right| \\ &= \begin{vmatrix} \lambda + 3 & -1 \\ -1 & \lambda + 3 \end{vmatrix} \\ &= \lambda^2 + 6\lambda + 8 \end{aligned}$$

$$\lambda = -2, \text{ and } -4.$$

$$\mathbf{A}\mathbf{x_i} = \lambda\mathbf{x_i}$$

$\lambda = -2$

$$\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \begin{aligned} -3x_1 + x_2 &= -2x_1 \\ x_1 - 3x_2 &= -2x_2 \end{aligned} \qquad \mathbf{x} = \begin{bmatrix} c \\ c \end{bmatrix}$$

$\lambda = -4.$

$$\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -4 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \begin{aligned} -3x_1 + x_2 &= -4x_1 \\ x_1 - 3x_2 &= -4x_2 \end{aligned} \qquad \mathbf{x} = \begin{bmatrix} c \\ -c \end{bmatrix}$$

$$\boxed{\mathbf{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \mathbf{x_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

Example 12 Diagonalizing a System in State Space

PROBLEM: Given the system of Eqs., find the diagonal system that is similar.

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}u$$

$$y = \begin{bmatrix} 2 & 3 \end{bmatrix}\mathbf{x}$$

SOLUTION: First find the eigenvalues and the eigenvectors.
This step was performed in the previous example

$$\mathbf{A} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \qquad \boxed{\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}} \qquad \mathbf{P} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Then we transformed all the other matrices and vectors of the first representation

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$$

$$\mathbf{P}^{-1}\mathbf{B} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$

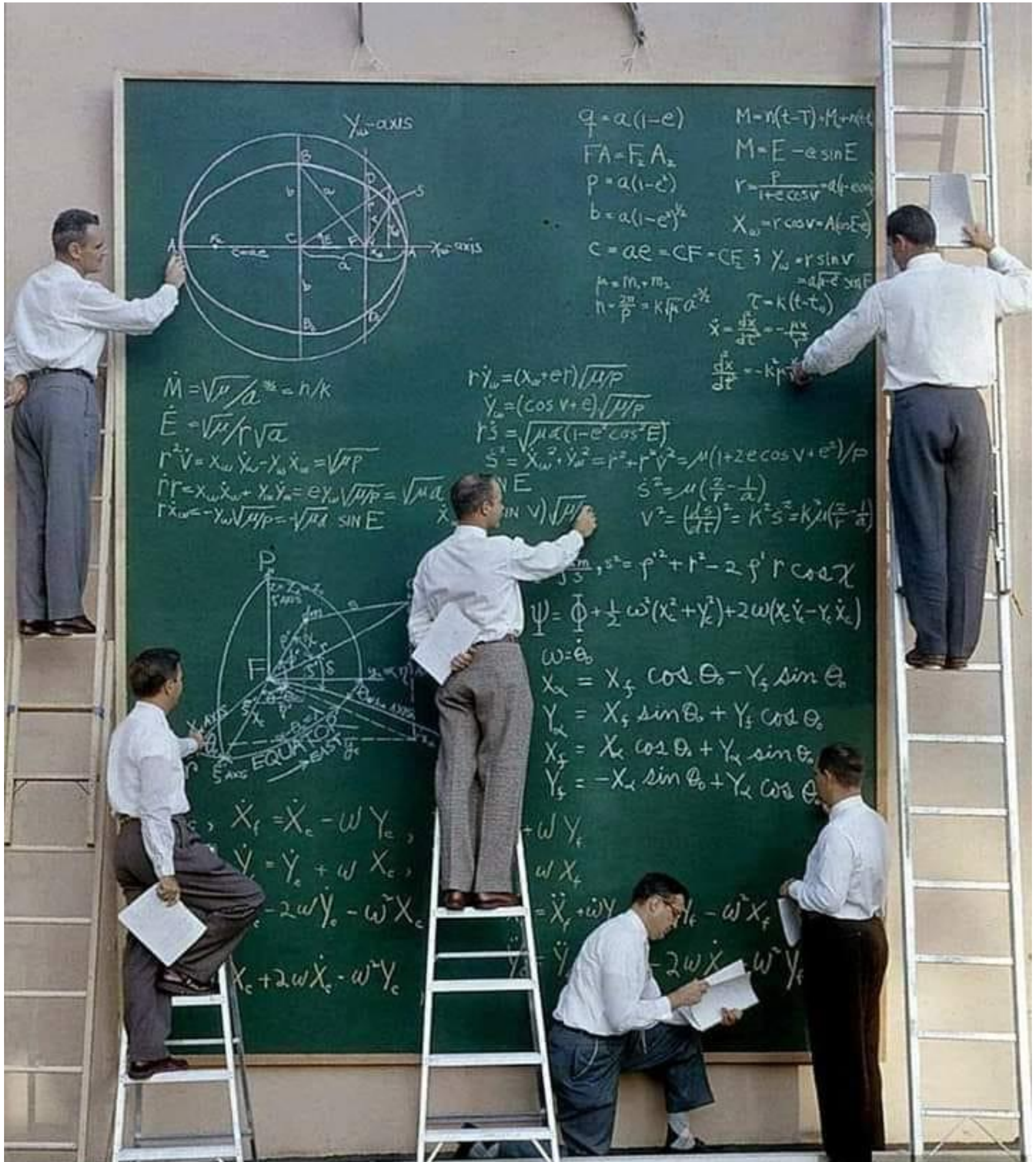
$$\mathbf{C}\mathbf{P} = \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \end{bmatrix}$$

$$\dot{\mathbf{z}} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}u$$

$$y = \begin{bmatrix} 5 & -1 \end{bmatrix}\mathbf{z}$$

Few examples:

Paper and pen!



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