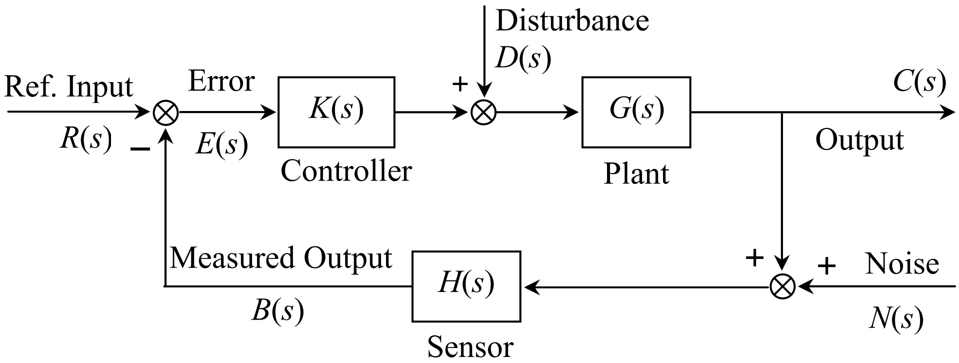
Reduction of Multiple Subsystems Examples

1. The following block diagram represents a generic closed-loop linear system consisting of a reference input $R(s)$, a disturbance $D(s)$, and sensor noise $N(s)$ and output $C(s)$.



Evaluate the transfer functions relating the output $C(s)$ to each of the inputs $R(s)$, $D(s)$ and $N(s)$. *Hint : since the system is linear use the property of superposition*. Assume that $H(s) = 1$ (*i.e.* unity feedback) and $K(s) = K$ (*i.e.* the controller is a simple amplifier). Assume also that there is no noise or disturbance (*i.e.* $N(s) = 0$ and $D(s) = 0$). Derive a simple expression for the error $E(s) = R(s) - B(s)$, in terms of $R(s)$, $G(s)$ and K.

Solution : By using the *principle of superposition* of linear systems, we can compute the transfer functions from each input to the respective output individually. We can then obtain the overall input output function by adding the individual contribution.

Let's start with the transfer function from $R(s)$ to $C(s)$. Disregard the other two inputs, *i.e.* set $D(s) = 0$ and $N(s) = 0$. Therefore,

$$\begin{aligned} C_R(s) &= G(s)K(s)E(s) = G(s)K(s)[R(s) - B(s)] \\ &= G(s)K(s)R(s) - G(s)K(s)H(s)C_R(s) \end{aligned}$$

Re-arranging :

$$C_R(s) = \frac{K(s)G(s)}{1 + K(s)G(s)H(s)} R(s)$$

Similarly, considering only the disturbance input at a time, we get :

$$C_D(s) = \frac{G(s)}{1 + K(s)G(s)H(s)} D(s)$$

and for noise :

$$\begin{aligned} C_N(s) &= -K(s)G(s)H(s)\left[C_N(s) + N(s)\right] \\ &= \frac{-K(s)G(s)H(s)}{1 + K(s)G(s)H(s)} N(s) \end{aligned}$$

As the final result, the overall input output function is

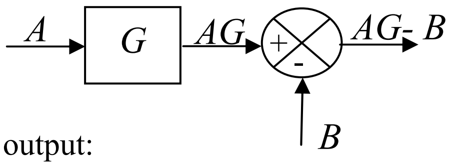
$$C(s) = C_R(s) + C_D(s) + C_N(s)$$
$$= \frac{1}{1+K(s)G(s)H(s)}\left[K(s)G(s)R(s) + G(s)D(s) - K(s)G(s)H(s)N(s)\right]$$

If there is no noise and disturbance, and $K(s) = K$, and $H(s) = 1$, the error signal is given by

$$E(s) = R(s) - B(s) = R(s) - C(s)$$
$$= R(s) - \frac{1}{1+KG(s)}\left[KG(s)R(s)\right] = \frac{1}{1+KG(s)}R(s)$$

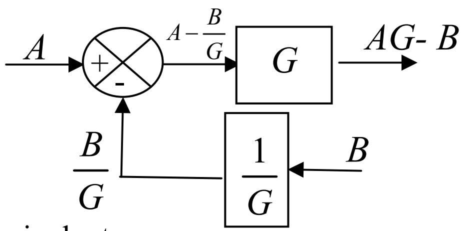
2. To show block diagram equivalence, consider inputs and outputs in different cases and show that they remain the same.

(a) Consider system:



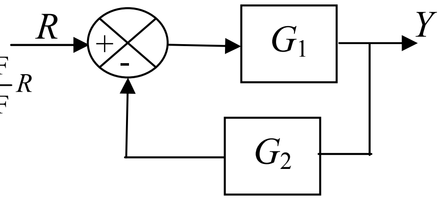
It has two inputs and one output:
Input 1- A
Input 2- B
and Output 1: AG-B

For system:
Input 1- A
Input 2- B
and Output 1: AG-B



Hence the block diagrams are equivalent.

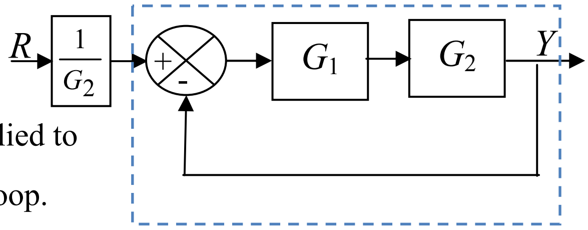
(b) For system:



$$Y = \frac{\text{Forward Loop TF}}{1+\text{Open Loop TF}} R$$

$$= \frac{G_1}{1+G_1G_2} R$$

For system:

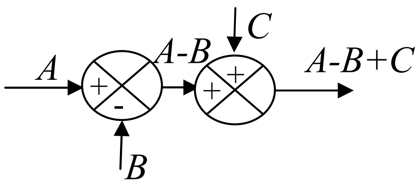


Input $\frac{R}{G_2}$ is being applied to the shaded feedback loop.

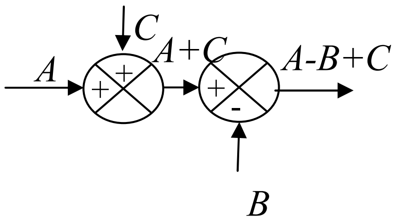
Therefore, $Y = \underbrace{\frac{\text{Forward Loop TF}}{1+\text{Open Loop TF}}}_{\text{For shaded feedback loop}} \underbrace{\frac{R}{G_2}}_{\text{Input to shaded feedback loop}}$

$$\Rightarrow Y = \frac{G_1G_2}{1+G_1G_2}\frac{R}{G_2} = \frac{G_1}{1+G_1G_2} R$$. Hence, given systems are equivalent.

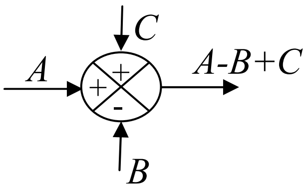
(c) For system 1:



For system 2:

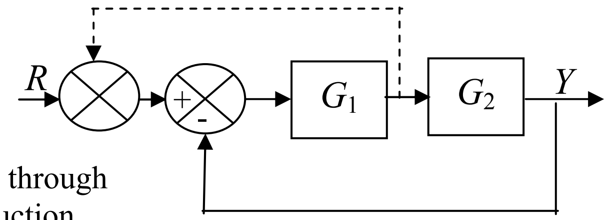


For system 3:

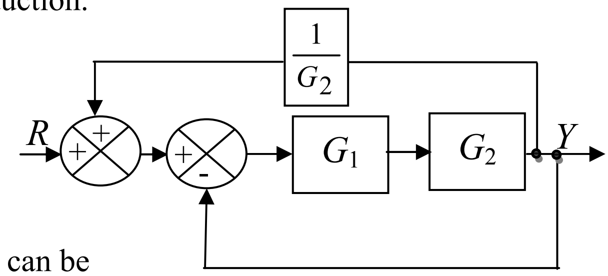


As inputs and output are the same for each of the three systems, therefore, the given systems are equivalent.

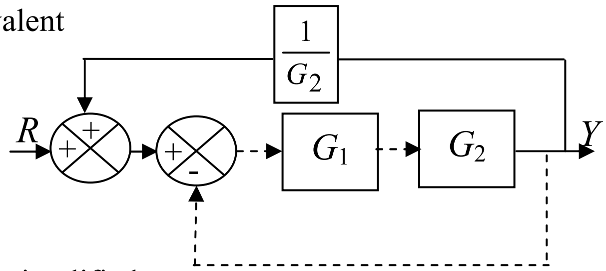
(d) Consider system:



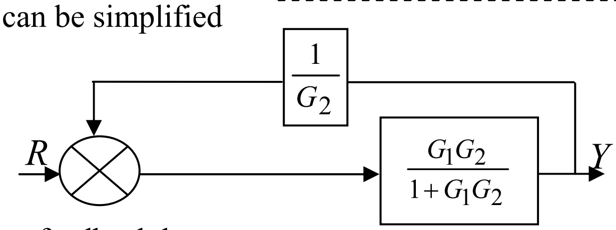
We create another equivalent system, through block diagram reduction.



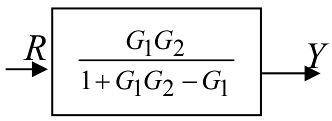
The branch points can be swapped to have an equivalent block diagram as



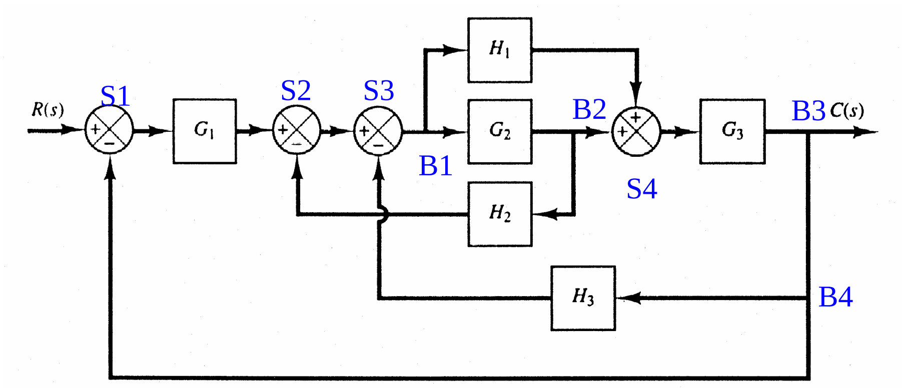
The feedback loop can be simplified



Simplifying the outer feedback loop (notice that it is positive feedback)

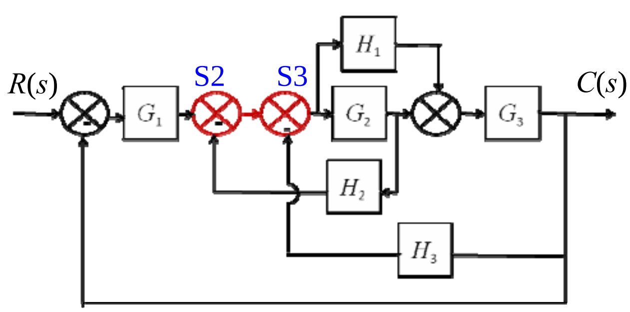


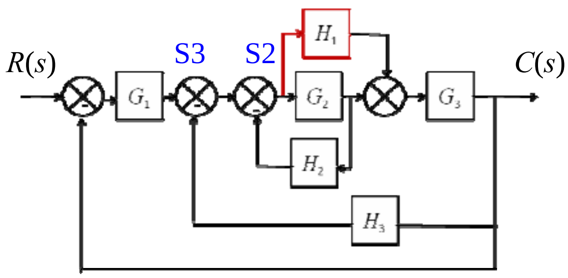
3. The branch points and summation junctions have be labelled in the figure below for ease of understanding



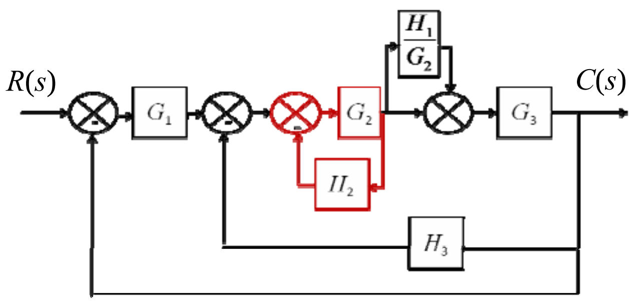
The following step by step approach can be used for simplifying the block diagram

<u>Step 1.</u> Try to simplify the block diagram by starting from the centre, such that the one of the loop can be solved independently. Observe that innermost loop containing H_2 can be simplified if the order of summation junctions S2 and S3 is reversed. This does not affect the inputs and output to the subsystem containing these transfer functions

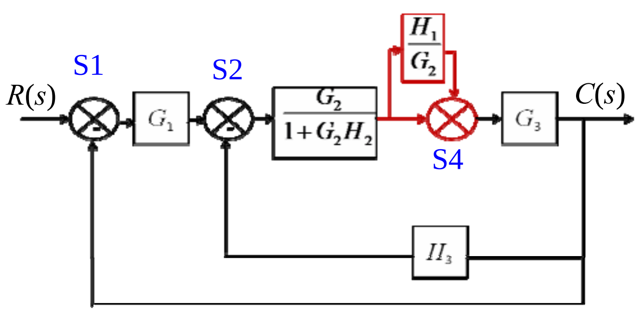




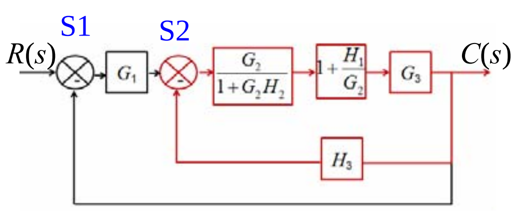
Step 2 Branch point B1 can be moved ahead of transfer function G_2 which allows the innermost loop containing G2 and H2 to be similar to a feedback loop



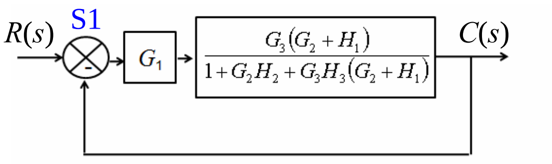
Step 3 Use formula for feedback loop for loop containing G2 and H2



Step 4 The term H1/G1 is a feedforward term which can be combined with unity transfer function.

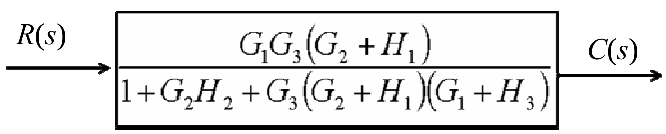


Step 5 Solve the inner feedback loop



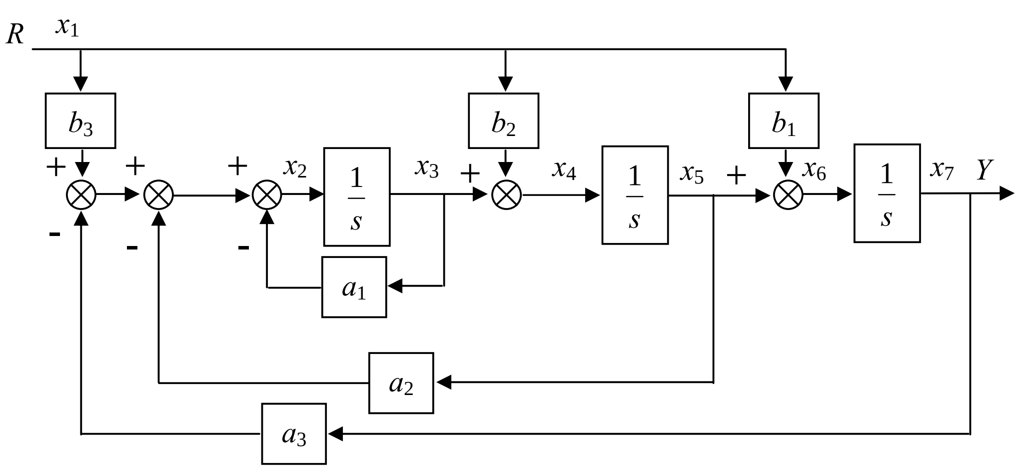
where, $$\frac{\dfrac{G_2}{1+G_2H_2}\left(1+\dfrac{H_1}{G_2}\right)G_3}{1+\dfrac{G_2}{1+G_2H_2}\left(1+\dfrac{H_1}{G_2}\right)G_3H_3}=\frac{G_3(G_2+H_1)}{1+G_2H_2+G_3H_3(G_2+H_1)}$$

<u>Step 6</u> Solve the outer feedback loop

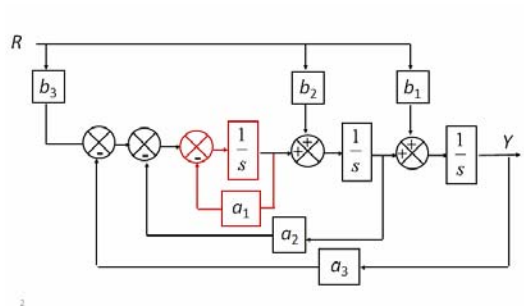


where, $$\frac{\dfrac{G_1G_3(G_2+H_1)}{1+G_2H_2+G_3H_3(G_2+H_1)}}{1+\dfrac{G_1G_3(G_2+H_1)}{1+G_2H_2+G_3H_3(G_2+H_1)}}=\frac{G_1G_3(G_2+H_1)}{1+G_2H_2+G_3H_3(G_2+H_1)+G_1G_3(G_2+H_1)}$$
$$=\frac{G_1G_3(G_2+H_1)}{1+G_2H_2+G_3(G_2+H_1)(G_1+H_3)}$$

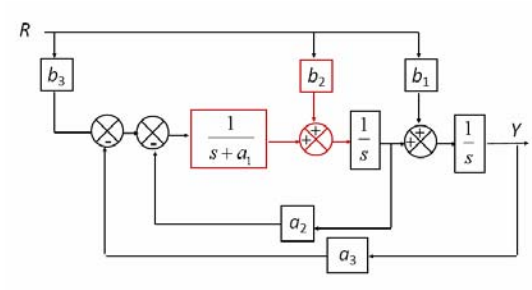
4. From the block diagram shown below, we define the nodes from x_1 to x_{10}.



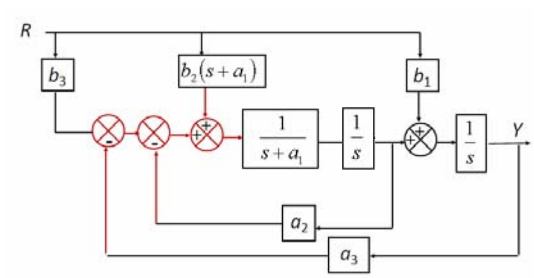
Solution :



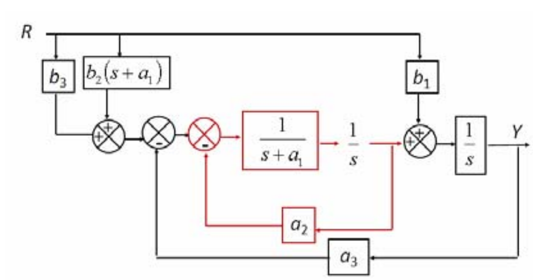
Step 1



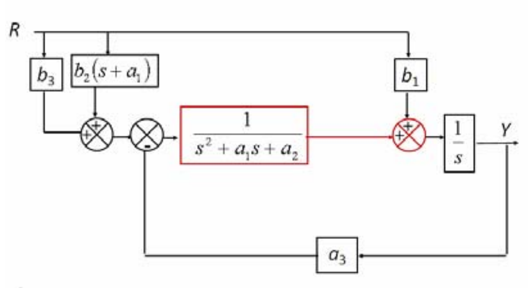
Step 2



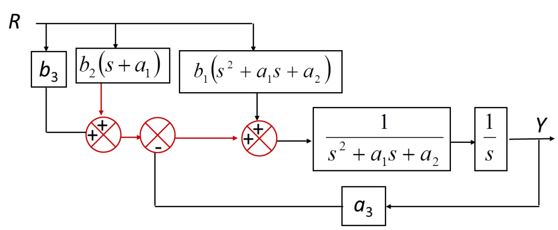
Step 3



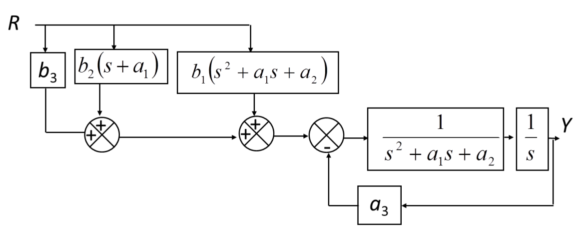
Step 4



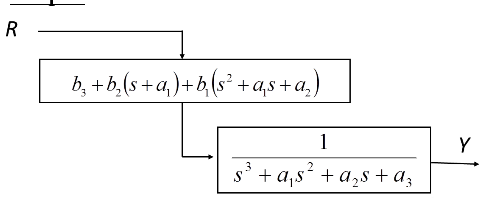
Step 5



Step 6



Step 7



Step 8