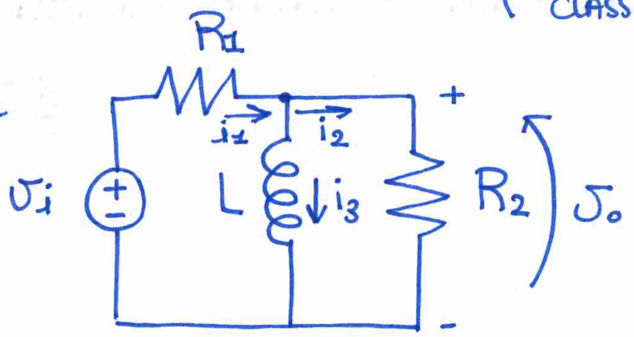
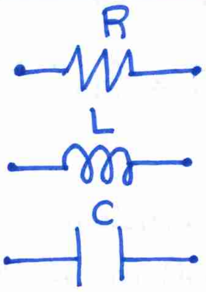
EX. 1 @ × pag. 112 n° 16 (DONE IN CLASS)

Find the transfer function for the system:



$$G(s) = \frac{v_o(s)}{v_i(s)}$$

$R_1 = 1\,\Omega$;
$R_2 = 1\,\Omega$;
$L = 1\,H$



$v_R(t) = R \cdot i(t) \Rightarrow V_R(s) = RI(s)$

$v_L(t) = L \cdot \frac{di(t)}{dt} \Rightarrow V_L(s) = LsI(s)$

$v_C(t) = \frac{1}{C}\int i(t)\,dt \Rightarrow V_C(s) = \frac{1}{C}\frac{I(s)}{s}$

SOLUTION:

Applying KIRCHHOFF CURRENT LAW (KCL): $i_1 = i_2 + i_3$

Solving for the voltage and applying s-domain transformation

$$i_1 = \frac{v_i - v_o}{R_1}$$

$$i_2 = \frac{v_o}{R_2} \qquad \Rightarrow \frac{v_i - v_o}{R_1} = \frac{v_o}{R_2} + \frac{v_o}{Ls}$$

$$i_3 = \frac{v_o}{Ls}$$

$$\Rightarrow v_i - v_o = v_o + \frac{v_o}{Ls} \quad (R_1 = R_2 = 1\,\Omega)$$

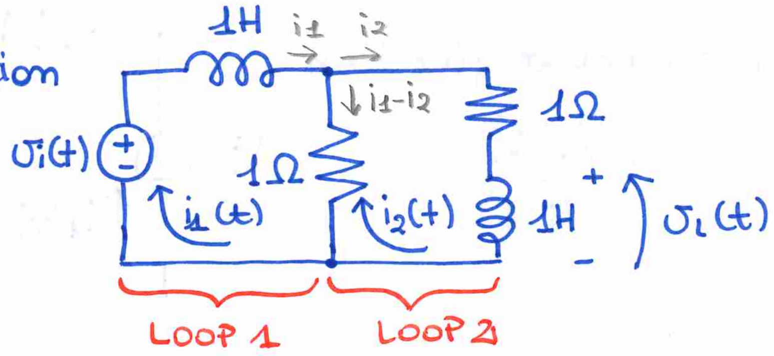
$$\Rightarrow v_i = v_o\left(2 + \frac{1}{s}\right)$$

$$\Rightarrow \boxed{\frac{v_o}{v_i} = \frac{s}{2s+1}}$$

EX. 2 (a) × pag. 112 n° 17 (DONE IN CLASS)

Find the transfer function for the circuit:

$$G(s) = \frac{V_L(s)}{V_i(s)}$$



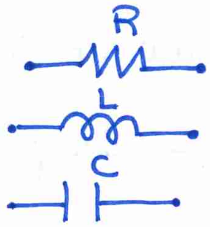
SOLUTION:

$V_R(s) = R\,I(s)$

$V_L(s) = L\,s\,I(s)$

$V_C(s) = \frac{1}{C}\,\frac{I(s)}{s}$

Laplace domain



$v_R = R \cdot i(t)$

$v_L = L\,\frac{di(t)}{dt}$

$v_C = \frac{1}{C}\int i(t)\,dt$

Use Kirchhoff voltage law in s-domain:

LOOP 1: $v_i(s) = L s I_1(s) + R(I_1(s) - I_2(s))$

$= I_1(s)(s+1) - I_2(s)$ (1)

LOOP 2: $R(-I_1(s) + I_2(s)) + R I_2(s) + L s I_2(s) = 0$

$I_2(s)(s+2) - I_1(s) = 0$ (2)

(3) $I_2 \Rightarrow$ current in the inductor : $V_L = L s I_2(s) = s I_2(s)$

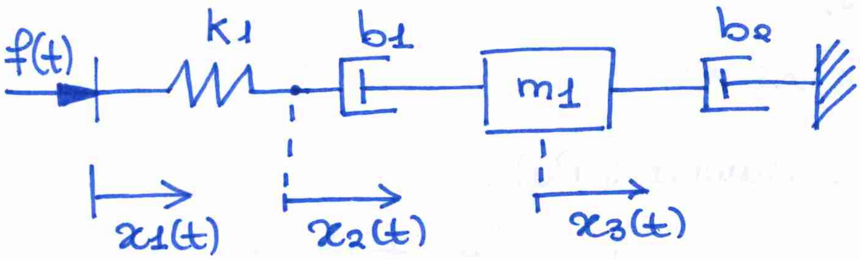
Solving algebrically the system (1)-(2)-(3):

$(s+1)(s+2)I_2(s) - I_2(s) = V_i(s)$

$$\Rightarrow \frac{I_2(s)}{V_i(s)} = \frac{1}{s^2+3s+1} \Rightarrow \boxed{\frac{V_L(s)}{V_i(s)} = \frac{s}{s^2+3s+1}}$$

EX. 3 × pag. 115 n° 25 (TRANSLATIONAL MECHANICAL SYSTEM) (DONE IN CLASS)

Find the transfer function of the mechanical system:

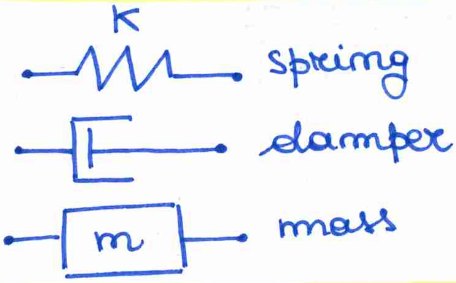


$m_1 = 10\ kg$

$b_1 = 5\ \frac{Ns}{m}$

$b_2 = 2\ \frac{Ns}{m}$

$k_1 = 2\ N/m$



TF: $G(s) = \dfrac{X_2(s)}{F(s)}$

SOLUTION:

1) Define the reference frame $(x_1(t);\ x_2(t);\ x_3(t))$

2) Solve the equations of motions:

x_1 (+):

$$f(t) = f_k(t) = k_1(x_1 - x_2) =$$

$$f(t) = 2(x_1(t) - x_2(t)) \quad (1)$$

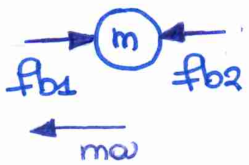
x_2 (+):

$$f_{b1}(t) = b_1 \frac{d}{dt}(x_2(t) - x_3(t))$$

$$+k(x_1 - x_2) - b_1 \frac{d}{dt}(x_2 - x_3) = 0$$

$$+2(x_1 - x_2) - 5 \frac{d}{dt}(x_2 - x_3) = 0 \quad (2)$$

x_3 (+):



$$+f_{b1} - f_{b2} = m_1 a = m_1 \frac{d^2 x_3}{dt^2}$$

$$+b_1 \frac{d}{dt}(x_2 - x_3) - b_2 \frac{d}{dt} x_3 = m_1 \frac{d^2 x_3}{dt^2}$$

$$+5 \frac{d}{dt}(x_2 - x_3) - 2 \frac{d}{dt} x_3 = 10 \frac{d^2 x_3}{dt^2} \quad (3)$$

3) Convert to the Laplace domain

① $2(X_1(s)-X_2(s)) = F(s)$

② $-2X_1(s)+(5s+2)X_2(s)-5sX_3(s)=0$

③ $-5sX_2(s)+(10s^2+7s)X_3=0$

4) Solve for $X_2(s)$ with the Cramer's Rule

$$X_2(s)=\frac{|D_2|}{|D|} \longrightarrow D_2 = \begin{array}{ccc} X_1 & b & X_3 \\ \end{array}$$
$$D_2=\begin{pmatrix} 2 & F(s) & 0 \\ -2 & 0 & -5s \\ 0 & 0 & 10s^2+7s \end{pmatrix}$$

coeff: X_1 ↓ X_2 ↓ X_3 ↓

$$D=\begin{pmatrix} 2 & -2 & 0 \\ -2 & 5s+2 & -5s \\ 0 & -5s & 10s^2+7s \end{pmatrix}$$

$$X_2(s)=\frac{F(20s^2+14s)}{100s^3+20s^2} \Rightarrow X_2(s)=F(s)\,\frac{10s+7}{50s^3+10s^2}$$

$$\Rightarrow \boxed{\frac{X_2(s)}{F(s)}=\frac{10s+7}{50s^3+10s^2}}$$

CRAMER'S RULE: formula for the solution of a system of linear equations: $A\vec{x}=\vec{b}$

⇝ $\boxed{x_i=\frac{\det(A_i)}{\det(A)}}$ with A_i ≡ matrix formed by replacing the i-th column of A with the column vector b

REMEMBER: if A 3x3

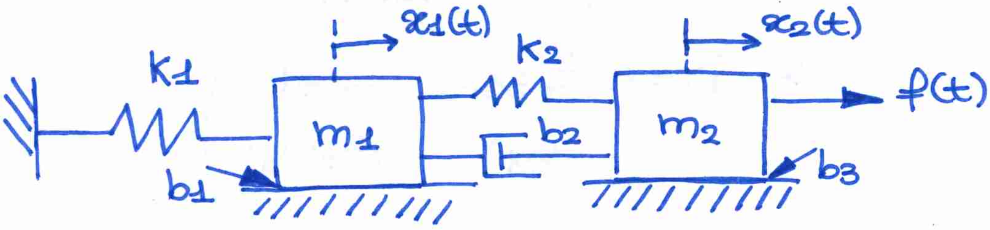
$$\det(A)=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

SARRU'S RULE

$=[①\times②\times③]-[④\times⑤\times⑥]$

EX. 4 × pag. 115 n° 26 (TRANSLATIONAL MECHANICAL SYSTEM) (DONE IN CLASS)

Find the transfer function of the mechanical system:



$k_1 = 1\ N/m$
$k_2 = 1\ N/m$
$b_1 = 2\ \frac{Ns}{m}$
$b_2 = 1\ \frac{Ns}{m}$
$b_3 = 1\ \frac{Ns}{m}$
$m_1 = 1\ kg$
$m_2 = 1\ kg$

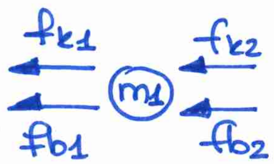
$$G(s) = \frac{X_1(s)}{F(s)}$$

Double-mass system

SOLUTION:

1) Define the reference frame ($x_1(t)$; $x_2(t)$)
2) Solve the equation of motion

$x_1(t)$ + :



$$-f_{k1} - f_{k2} - f_{b1} - f_{b2} = m_1 a_1$$

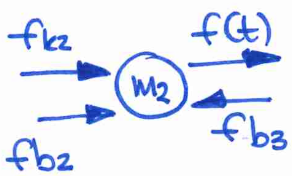
$$-k_1 x_1 - k_2(x_1 - x_2) - b_1 \frac{dx_1}{dt} - b_2 \frac{d(x_1 - x_2)}{dt} = m_1 \frac{d^2 x_1}{dt^2}$$

⇒ Laplace domain

$$s^2 X_1(s) = -X_1(s) - X_1(s) + X_2(s) - 2sX_1(s) - sX_1(s) + sX_2(s)$$

① $X_1(s^2 + 3s + 2) + (-s - 1)X_2 = 0$

$x_2(t)$ + :



$$f(t) + f_{k2} + f_{b2} - f_{b3} = m_2 a_2$$

$$f(t) + k_2(x_1 - x_2) + b_2 \frac{d}{dt}(x_1 - x_2) - b_3 \frac{d}{dt} x_2 = m_2 \frac{d^2 x_2}{dt^2}$$

⇒ Laplace domain

$$m_2 s^2 X_2 - k_2 X_1 + k_2 X_2 - b_2 s X_1 + b_2 s X_2 + b_3 s X_2 = F(s)$$

② $(s^2 + 2s + 1)X_2 + (-s - 1)X_1 = F(s)$

3) Solving for $X_1(s)$ (Cramer's Rule):

$$X_1(s) = \frac{|D_1|}{|D|} \longrightarrow D_1 = \begin{pmatrix} 0 & -s-1 \\ F & s^2+2s+1 \end{pmatrix}$$

$$\longrightarrow D = \begin{pmatrix} s^2+3s+2 & -(s+1) \\ -(s+1) & s^2+2s+1 \end{pmatrix}$$

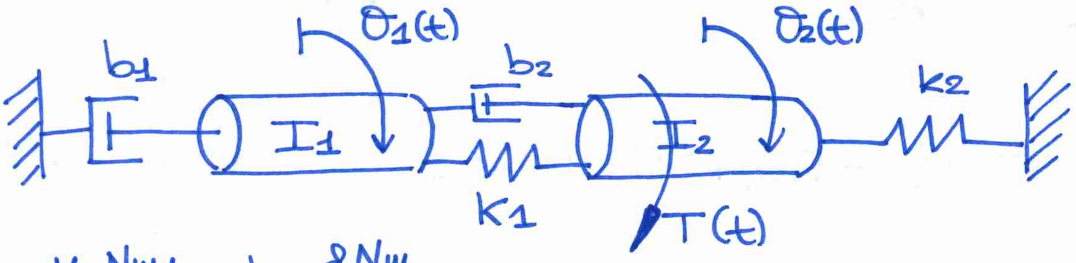
$$|D_1| = F(s+1)$$

$$|D| = (s+2)(s+1)(s+1)(s+1) - (s+1)^2 = (s+2)(s+1)^3 - (s+1)^2$$

$$X_1(s) = \frac{|D_1|}{|D|} = F\frac{s+1}{(s+2)(s+1)^3-(s+1)^2} \Rightarrow \boxed{\frac{X_1(s)}{F(s)} = \frac{1}{s^3+4s^2+5s+1}}$$

$\boxed{\text{EX. 5}}$ × pag. 117 n° 30 (ROTATIONAL MECHANICAL SYSTEM) (DONE IN CLASS)

Find the equations of motion for the following rotational system:



→ APPLIED TORQUE

$b_1 = 7\ \frac{Nms}{rad}$; $k_1 = 8\ \frac{Nm}{rad}$

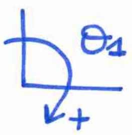
$b_2 = 2\ \frac{Nms}{rad}$; $k_2 = 3\ \frac{Nm}{rad}$

$I_1 = 1\ kgm^2$; $I_2 = 1\ kgm^2$

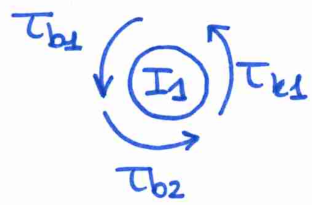
SOLUTION:

1) Define the reference frame $(\theta_1(t); \theta_2(t))$
2) Write the equations of motion

X-section:



$T \equiv$ torque



Assuming a pure rotational system:

$$\sum \tau = I \alpha$$

sum of torques = moment of inertia · angular acceleration

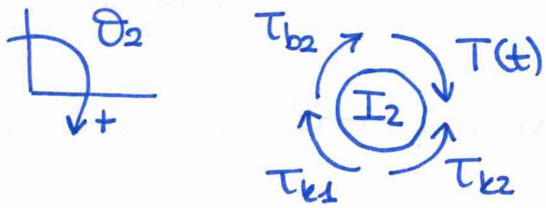
$$I_1 \alpha_1 = I \frac{d^2\theta_1}{dt^2} = -\tau_{b1} - \tau_{b2} - \tau_{k1}$$

$$= -b_2 \frac{d}{dt}(\theta_1 - \theta_2) - b_1 \frac{d\theta_1}{dt} - k_1(\theta_1 - \theta_2)$$

⇒ Laplace domain

$$I s^2 \Theta_1(s) = -b_2 s \Theta_1(s) + b_2 s \Theta_2(s) - b_1 s \Theta_1(s) - k_1 \Theta_1(s) + k_1 \Theta_2(s)$$

(1) $(s^2 + 9s + 8)\Theta_1(s) + (-2s - 8)\Theta_2(s) = 0$



$$I_2 \frac{d^2\theta_2}{dt^2} = T(t) + \tau_{b2} + \tau_{k1} - \tau_{k2}$$

$$= T(t) + b_2 \frac{d}{dt}(\theta_1 - \theta_2) + k_1(\theta_1 - \theta_2) - k_2\theta_2$$

$\Rightarrow$ Laplace domain

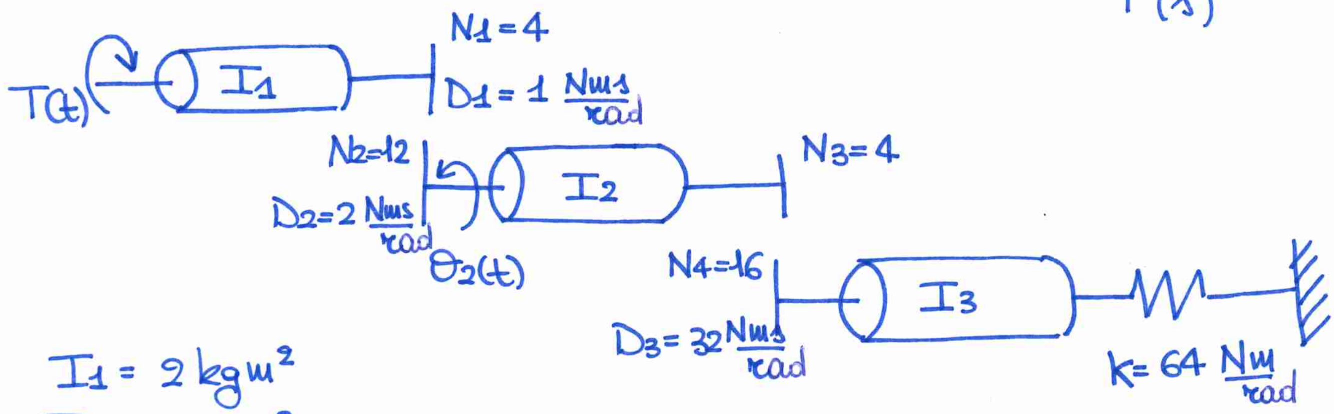
$$s^2\Theta_2(s) = T(s) + 2s\Theta_1(s) - 2s\Theta_2(s) + 8\Theta_1(s) - 8\Theta_2(s) + $$
$$- 3\Theta_2(s)$$

② $$\underline{(s^2 + 2s + 11)\Theta_2(s) + (2s - 8)\Theta_1(s) = T(s)}$$

EX. 6 × pag. 118 n° 33 (ROTATIONAL MECHANICAL SYSTEM WITH GEARS) (DONE IN CLASS)

Find the transfer function of the following rotational systems in line with a gear train:

$$G(s) = \frac{\theta_2(s)}{T(s)}$$



$I_1 = 2\ kg\,m^2$

$I_2 = 1\ kg\,m^2$

$I_3 = 16\ kg\,m^2$

SOLUTION:

1 DoF problem → the torque $T(t)$ is applied to I_1 which is attached to all other bodies through the gear trains

Relate all the mechanical impedances to θ_2 → central reference frame

Effective Inertia: $$I_{eff} = I_2 + I_1\left(\frac{N_2}{N_1}\right)^2 + I_3\left(\frac{N_3}{N_4}\right)^2$$

↳ GEAR RATIO → scales the other contributions with respect to the central ref. frame

$$\left(\frac{\text{\# OF TEETH OF GEAR ON DESTINATION SHAFT}}{\text{\# OF TEETH OF GEAR ON SOURCE SHAFT}}\right)^2$$

Effective Damping: $$D_{eff} = D_2 + D_1\left(\frac{N_2}{N_1}\right)^2 + D_3\left(\frac{N_3}{N_4}\right)^2$$

Effective Stiffness: $$K_{eff} = K\left(\frac{N_3}{N_4}\right)^2$$

$$\Rightarrow T(s)\left(\frac{N_2}{N_1}\right) = \left(I_{eff}s^2 + D_{eff}s + K_{eff}\right)\theta_2(s)$$

$$m\ddot{x} + b\dot{x} + kx$$

substituting the values:

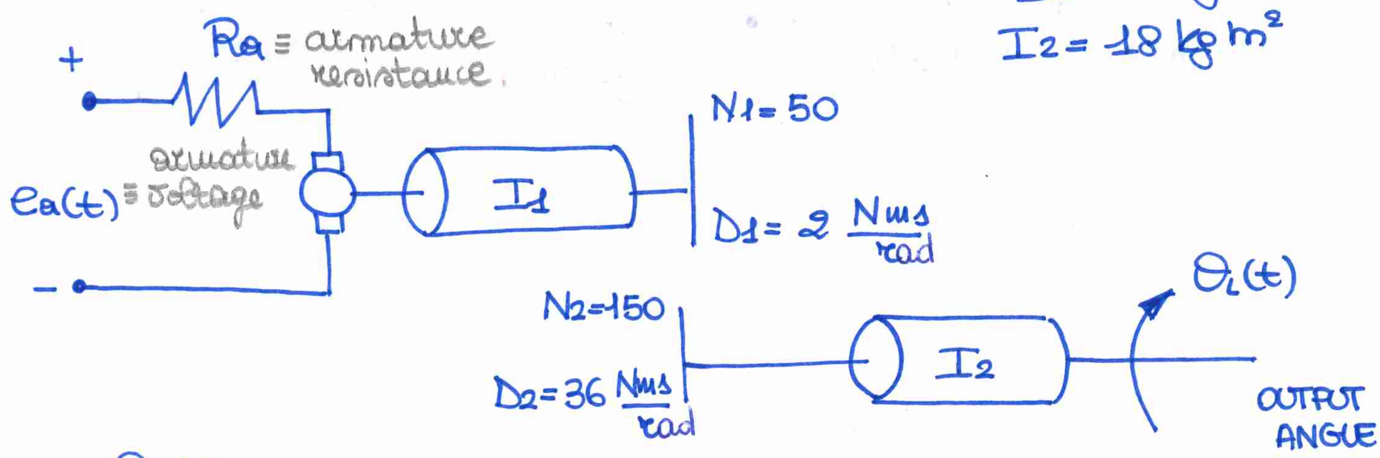
$$\boxed{G(s) = \frac{3}{20s^2 + 13s + 4}}$$

EX. 7 × pag. 121 n° 42 (ELECTROMECHANICAL SYSTEM) (DONE IN CLASS)

Find the transfer function for the system composed by rotational elements, gear trains, and an electromechanical motor.

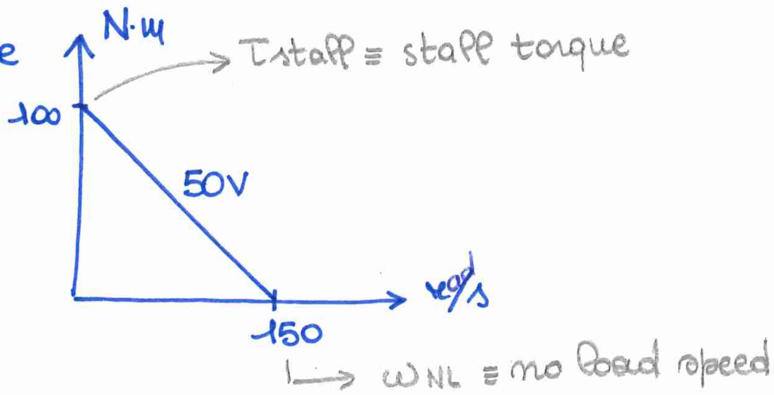
$I_1 = 2 \ kg\,m^2$

$I_2 = 18 \ kg\,m^2$



TF: $\dfrac{\Theta_L(s)}{E_a(s)}$

speed-torque curve of the motor



SOLUTION:

1) Solve motor characteristic

$$\frac{K_t}{R_a} = \frac{T_{stall}}{E_a} = \frac{100}{50} = 2 \ ; \quad K_b = \frac{E_a}{\omega_{NL}} = \frac{50}{150} = \frac{1}{3} \ ; \quad \theta_L(t) = \frac{N_1}{N_2}\theta_m$$

TORQUE CONSTANT $T = K_t \cdot \text{current} \ (V/R)$

MOTOR VELOCITY CONSTANT (back EMF const.) relationship speed-voltage $\left[\frac{V \cdot s}{rad}\right]$

ω_{NL} → NO-LOAD SPEED

the output rotation will be affected by the gear ratio

2) Find inertia, damping as seen by the motor:

$$I_m = I_1 + I_2\left(\frac{N_1}{N_2}\right)^2 = 4$$

$$D_m = D_1 + D_2\left(\frac{N_1}{N_2}\right)^2 = 6$$

EQUIVALENT LOAD AND DAMPING SEEN BY THE ROTOR

$$\frac{\Theta_m(s)}{E_a(s)} = \frac{K_t/R_a I_m}{s\left[s+\frac{1}{I_m}\left(D_m+\frac{K_t K_b}{R_a}\right)\right]}$$

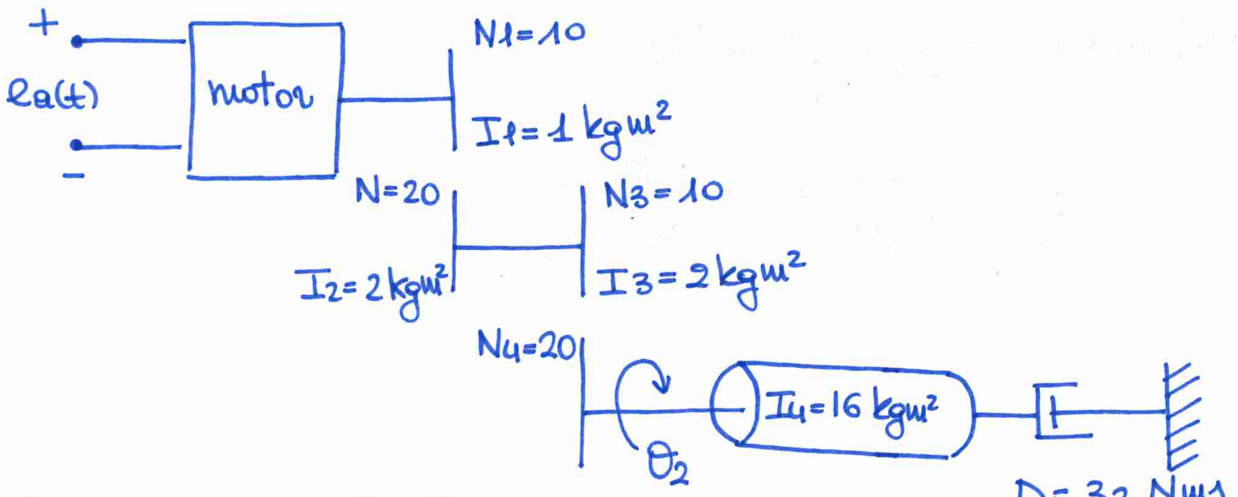
TRANSFER FUNCTION OF SYSTEM: ELECTRIC DC MOTOR-LOAD (SEE THEORY)

$$\frac{\Theta_m(s)}{E_a(s)} = \frac{1/2}{s(s+5/3)} \xrightarrow{\Theta_L=\frac{1}{3}\Theta_m} \boxed{G(s)=\frac{1/6}{s(s+5/3)}}$$

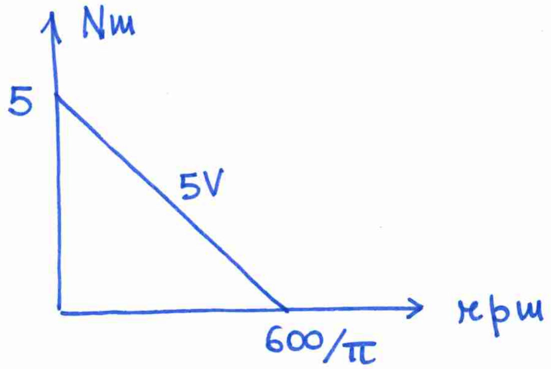
$\boxed{\text{EX. 8}}$ × pag. 121 n° 43 (ELECTROMECHANICAL SYSTEM)

Solve for the transfer function: $G(s) = \dfrac{\theta_2(s)}{E_a(s)}$

system: motor with gear trains (this time with inertia)



Speed-torque curve for the motor:



SOLUTION:

1) Find motor characteristics:

$$\frac{k_t}{R_a} = \frac{T_{stall}}{E_a} = \frac{5}{5} = 1 \; ; \qquad k_b = \frac{E_a}{\omega_{NL}} = \frac{5}{\underbrace{\frac{600}{\pi}\left[2\pi/60\right]}_{\to\ rad/s \text{ from rpm}}} = \frac{1}{4}$$

2) Find the equivalent impedances:

$$I_m = I_4 \left(\frac{N_3}{N_4}\right)^2 \left(\frac{N_1}{N_2}\right)^2 + \left(I_2 + I_3\right)\left(\frac{N_1}{N_2}\right)^2 + I_1 = 3\ kgm^2$$

$$D_m = D \left(\frac{N_3}{N_4}\right)^2 \left(\frac{N_1}{N_2}\right)^2 = 2\ \frac{Nms}{rad}$$

3) Transfer function:

$$\frac{\Theta_m(s)}{E_a(s)} = \frac{k_t/R_a I_m}{s\left[s+\frac{1}{I_m}\left(D_m+\frac{k_t k_b}{R_a}\right)\right]} \rightarrow \frac{1/3}{s(s+3/4)}$$

Considering: $\Theta_2(s) = \frac{1}{4}\Theta_m(s)$

$$\Rightarrow \boxed{\frac{\Theta_2(s)}{E_a(s)} = \frac{1/12}{s(s+3/4)}}$$