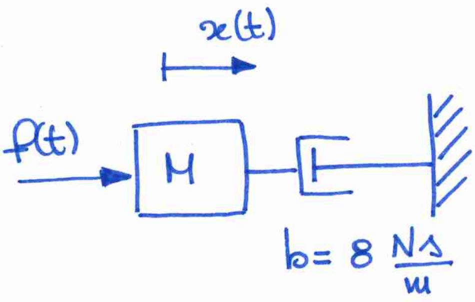
EX. 1

a) Find an equation to relate settling time of $\dot{x}(t)$ to the mass, M.

b) Relate rise time of the velocity $\dot{x}(t)$ to the mass, M.



SOLUTION:

Equation of motion: $M\ddot{x} + b\dot{x} = f(t) \xrightarrow[\text{transform}]{\text{Laplace}} (Ms^2 + 8s)X(s) = F(s)$

Transfer function: $G(s) = \dfrac{X(s)}{F(s)} = \dfrac{1}{Ms^2 + 8s}$

To have the transfer function in terms of velocity, $\dot{x}(t)$:

$$\frac{sX(s)}{F(s)} = \frac{1}{Ms+8} = \frac{1/M}{s + 8/M}$$

↳ 1º ORDER SYSTEM

For a first order system:

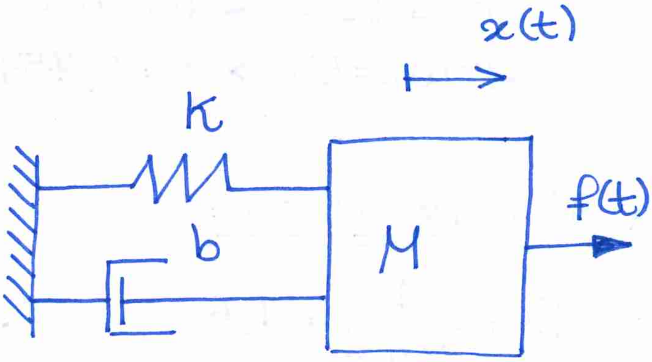
a) $\boxed{T_s = \dfrac{4}{\omega}} = \dfrac{4}{8/M} = \dfrac{1}{2}M$ SETTLING TIME

b) $\boxed{T_r = \dfrac{2.2}{\omega}} = \dfrac{2.2}{8/M} = 0{,}275\,M$ RISE TIME

EX. 2

Find the:
- damping ratio, ζ
- natural frequency, ω_n
- percent overshoot, %OS
- settling time, T_s
- peak time, T_p

for the 2º order system.



$M = 3$ kg
$K = 33$ N/m
$b = 15$ Ns/m

SOLUTION:

Equation of motion: $M\ddot{x} + b\dot{x} + Kx = f(t)$

Laplace domain: $s^2 MX(s) + sbX(s) + KX(s) = F(s)$

$\hookrightarrow$ transfer function: $G(s) = \dfrac{X(s)}{F(s)} = \dfrac{1}{Ms^2 + bs + K} =$

$$= \frac{1}{3s^2 + 15s + 33}$$

We can rewrite the TF as:

$G(s) = \dfrac{1/3}{s^2 + 5s + 11}$ to relate to the generalized form: $G(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta s\omega_n + \omega_n^2}$

From the TF we know that:

1) Poles: $s_{1,2} = -\dfrac{5}{2} \pm \dfrac{\sqrt{25-44}}{2} = -\dfrac{5}{2} \pm 2{,}18\,i$ complex conjugated poles

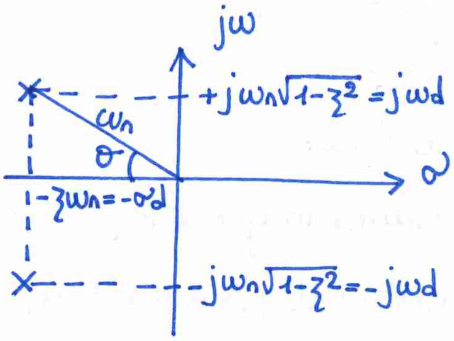
$\longrightarrow$ stable system ($\mathrm{Re} < 0$)

$\longrightarrow$ underdamped $\longrightarrow$ oscillatory (complex poles)

2) TF denominator:

$\omega_m^2 = 11 \longrightarrow \omega_m = \sqrt{11}$ NATURAL FREQUENCY

$2\zeta\omega_m = 5 \longrightarrow \zeta = 0,754$ DAMPING RATIO



For a 2º order system:

$$\boxed{T_s = \frac{4}{\zeta\omega_m}} = 1,6\,s$$ SETTLING TIME

PEAK TIME $$\boxed{T_p = \frac{\pi}{\omega_m\sqrt{1-\zeta^2}}} = 1,44\,s = \frac{\pi}{\omega_d}$$

$\omega_d \equiv$ damped natural frequency

$\omega_d = \omega_m\sqrt{1-\zeta^2}$

PERCENT OVERSHOOT $$\boxed{\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100} = 2,7\%$$

$\longrightarrow \frac{C_{max} - C_{final}}{C_{final}} \times 100$

$\boxed{\text{EX. 3}}$ Find the transfer function of each system:

a) First-order system: $G(s)=\dfrac{K}{s+a}$

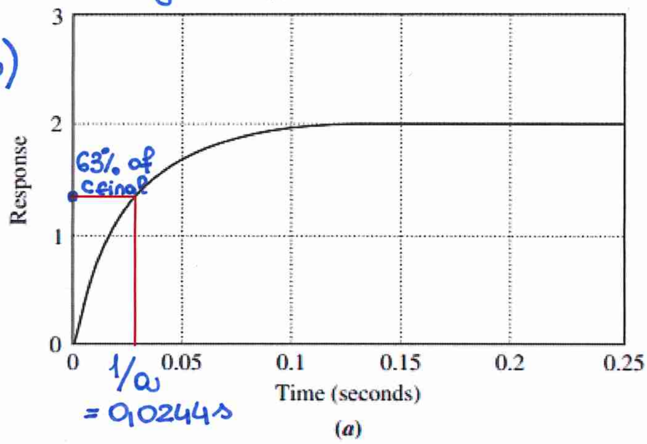
time constant $=\dfrac{1}{a}=0{,}0244s \rightarrow a=\dfrac{1}{t.c.}=40{,}984$

we know that the steady-state response settles at 2: $\rightarrow$ at steady-state $s=0$:

$$G=\frac{K}{a}=2 \rightarrow K=81{,}967$$

TF: $G(s)=\dfrac{81{,}967}{s+40{,}984}$

a)



(a)

b) Second-order system: $G(s)=\dfrac{K}{s^2+2\zeta\omega_n s+\omega_n^2}$

$$\zeta=\frac{-\ln(\%OS/100)}{\sqrt{\pi^2+\ln^2(\%OS/100)}};$$

$\%OS=\dfrac{C_{max}-C_f}{C_f}\times 100=25{,}3\% \longrightarrow \zeta=0{,}4$

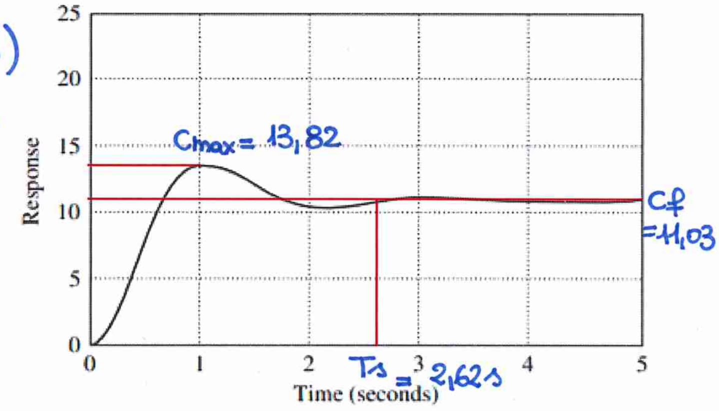
$T_s=\dfrac{4}{\zeta\omega_n}=2{,}62s \longrightarrow \omega_n=3{,}82$

At steady state $(s=0)$: $G(s)=\dfrac{K}{\omega_n^2}=C_f$

$\rightarrow K=C_f\cdot\omega_n^2=160{,}95$

$$G(s)=\frac{160{,}95}{s^2+0{,}458s+14{,}59}$$

b)



(b)

c) Second-order system: $G(s)=\dfrac{K}{s^2+2\zeta\omega_n s+\omega_n^2}$

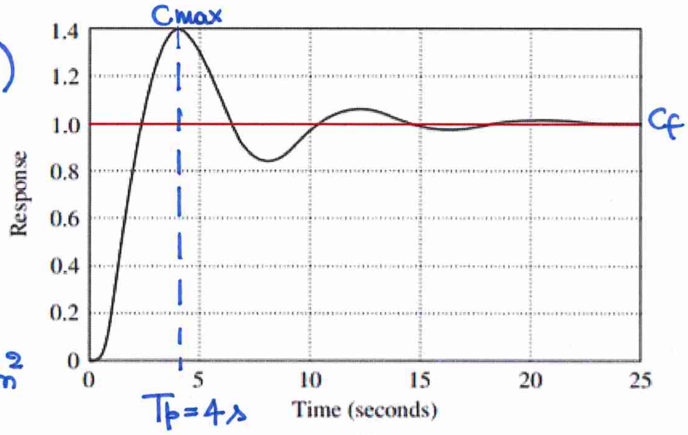
$\%OS=\dfrac{C_{max}-C_{fin}}{C_{fin}}\times 100=40\% \longrightarrow \zeta=0{,}28$

$T_p=4s=\dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} \rightarrow \omega_n=0{,}818$

At steady-state: $G=\dfrac{K}{\omega_n^2}=1=C_f \rightarrow K=\omega_n^2=0{,}669$

$$G(s)=\frac{0{,}669}{s^2+0{,}458s+0{,}669}$$

c)



(c)

$\boxed{EX. 4}$

Given the following response functions: determine if pole-zero cancellation is applicable.
If so, find %OS, T_s, T_p
↳ possibility to approximate as a 2° order system

a) $C(s) = \dfrac{s+3}{s(s+2)(s^2+3s+10)}$

b) $C(s) = \dfrac{s+2{,}1}{s(s+2)(s^2+s+5)}$

c) $C(s) = \dfrac{s+2{,}01}{s(s+2)(s^2+5s+20)}$

SOLUTION:

a) 1 zero: $s=-3$

poles: $s=0$

$s=-2$

$s^2+3s+10=0 \longrightarrow$ dominant poles $\quad \dfrac{-3\pm\sqrt{9-40}}{2} = -\dfrac{3}{2}\pm 2{,}78\,i$

Running the partial fraction expansion: (PFE)

$$\frac{A}{s}+\frac{B}{s+2}+\frac{Cs+D}{s^2+3s+10} = \frac{s+3}{s(s+2)(s^2+3s+10)}$$

$$A(s+2)(s^2+3s+10)+Bs(s^2+3s+10)+(Cs+D)s(s+2) = s+3$$

substituting the roots:

$s=0 \longrightarrow 20A=3 \longrightarrow A = 3/20$

$s=-2 \longrightarrow -2B(4-6+10)=1 \longrightarrow B=-1/16$

$s=-\dfrac{3}{2}\pm 2{,}78\,i \longrightarrow C = 7/80\,;\ D = 31/80$

$$C(s) = \frac{3}{20}\cdot\frac{1}{s}+\frac{-1/16}{s+2}+\frac{1}{80}\,\frac{7s+31}{\left(s+\frac{3}{2}\right)^2+\frac{31}{4}}$$

The pole $s=-2$ is the closest to the zero $s=-3$. The residue of the pole $s=-2$ (B) is of the same order of magnitude as the dominant poles (C and D residues).

$\Rightarrow$ pole-zero cancellation can not be assumed

b) PFE: $C(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{Cs+D}{s^2+s+5}$;

$A = 0{,}21$
$B = -0{,}0071429$
$C = -0{,}20286$
$D = 0{,}21714$

The residue of the pole at $s=-2$ is negligible compared to the dominant poles.

$\Rightarrow$ pole-zero cancellation can be assumed and the system can be considered (approximated) as 2º order:

$2\zeta\omega_n = 1$

$\omega_n^2 = 5 \longrightarrow \omega_n = \sqrt{5}$; $\zeta = \frac{1}{2\omega_n} = 0{,}224$

$\%OS = 48{,}64\% = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100$

$T_s = \frac{4}{\zeta\omega_n} = 8\,s$

$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 1{,}44\,s$

c) PFE: $C(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{Cs+D}{s^2+5s+20}$;

$A = 0{,}05025$
$B = -0{,}0071429$
$C = 0{,}20286$
$D = 0{,}21714$

Residue at $s=-2$ is an order of magnitude smaller than the dominant poles.

$\Rightarrow$ pole-zero cancellation can be assumed and the system can be approximated as 2º order.

$2\zeta\omega_n = 5$

$\omega_n = \sqrt{20}$; $\zeta = 0{,}559$; $\%OS = 12{,}03\%$

$T_s = 1{,}6\,s$

$T_p = 0{,}847\,s$

EX. 5

Given the state-space representation:

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 6 & 5 \\ 1 & 4 & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(t) \; ; \quad y = [1 \;\; 2 \;\; 0] \vec{x}$$

Find: a) the characteristic equation

b) the poles of the system

SOLUTION:

we know that the eigenvalues of the state-space system are the poles of the transfer functions: $|\lambda I - A| = 0 \longrightarrow |sI - A| = 0$

a) by definition the characteristic equation is:

$$|sI - A| = \begin{vmatrix} s & -2 & -3 \\ 0 & (s-6) & -5 \\ -1 & -4 & (s-2) \end{vmatrix} = s[(s-6)(s-2)-20] - 1(10+3(s-6))$$

$$= s(s^2 - 8s - 8) - 10 - 3s + 18$$

$$= \underline{s^3 - 8s^2 - 11s + 8}$$

Characteristic equation

b) poles $\Rightarrow s^3 - 8s^2 - 11s + 8 = 0$

Factoring the polynomial:

$$\begin{cases} s_1 = -1{,}6448 \\ s_2 = 0{,}5338 \\ s_3 = 9{,}111 \end{cases}$$

EX. 6

Given the state-space representation:

$$\dot{\vec{x}} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -6 & 1 \\ 0 & 0 & -5 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \underbrace{u(t)}_{\text{STEP}} ; \quad y = [0 \ 1 \ 1]\vec{x}; \quad \vec{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solve for $y(t)$ using the Laplace transform method.

SOLUTION:

$$\vec{X}(s) = (sI-A)^{-1}(X(0)+BU(s)) =$$

$$= \left\{ \underbrace{\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} -3 & 1 & 0 \\ 0 & -6 & 1 \\ 0 & 0 & -5 \end{bmatrix}}_{} \right\}^{-1} \cdot \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \frac{1}{s} \right\} = \begin{bmatrix} \frac{1}{s(s+3)(s+5)} \\ \frac{1}{s(s+5)} \\ \frac{1}{s(s+5)} \end{bmatrix}$$

$$\frac{\operatorname{adj}(sI-A)}{\det(sI-A)}$$

$$\dot{x} = Ax + Bu \rightarrow sX(s) - X(0) = AX(s) + BU(s)$$
$$y = Cx + Du$$
$$(sI-A)X(s) = X(0) + BU(s)$$
$$\rightarrow X(s) = (sI-A)^{-1}x(0) + (sI-A)^{-1}BU(s) = \frac{\operatorname{adj}(sI-A)}{\det(sI-A)}[x(0)+BU(s)]$$
$$\rightarrow Y(s) = CX(s) + DU(s)$$

PFE $\rightarrow$

$$\vec{X}(s) = \begin{bmatrix} \frac{1/15}{s} - \frac{1/6}{s+3} + \frac{1/10}{s+5} \\ \frac{1}{s(s+5)} \\ \frac{1}{s(s+5)} \end{bmatrix}$$

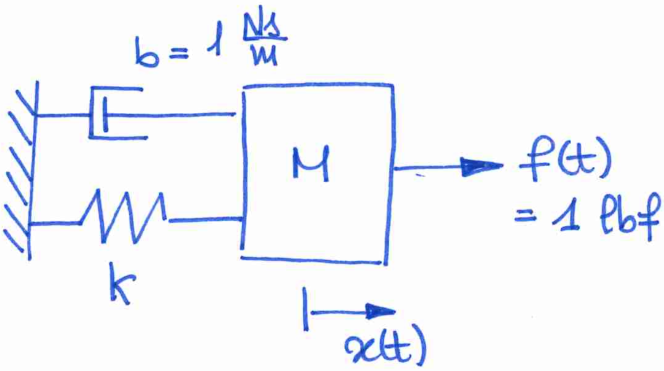
taking the inverse Laplace transform:

$$\mathcal{L}^{-1} \quad \vec{x}(t) = \begin{bmatrix} \frac{1}{15} - \frac{1}{6}e^{-3t} + \frac{1}{10}e^{-5t} \\ \frac{1}{5} - \frac{1}{5}e^{-5t} \\ \frac{1}{5} - \frac{1}{5}e^{-5t} \end{bmatrix}$$

$$\rightarrow y(t) = [0 \ 1 \ 1]\vec{x}(t) = \frac{2}{5}\left(1 - e^{-5t}\right)$$

$\boxed{\text{EX. 7}}$

Given the mechanical system:
Find K and M such that the response demonstrates a $T_s = 2s$ and $T_p = 1s$.
Find the %OS.



SOLUTION:

Equation of motion: $M\ddot{x} + b\dot{x} + kx = f(t)$

$$\xrightarrow{\mathcal{L}} (Ms^2 + bs + k)X(s) = F(s)$$

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2+bs+k} = \frac{1/M}{s^2 + \frac{b}{M}s + k/M}$$

2º order system: mass-spring-damper

$$2\zeta\omega_n = b/M \quad ; \quad \omega_n = \sqrt{k/M}$$

- $T_s = 2s = \dfrac{4}{\zeta\omega_n} = \dfrac{4}{b/2M} = 8M \rightarrow M = 1/4$

- since $b = 1\,\frac{Ns}{m}$: $\quad G(s) = \dfrac{4}{s^2+4s+4k} \xrightarrow{\text{poles}} s_{1,2} = -2 \pm 2\sqrt{k-1}\,j$
  $-\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}$

$$T_p = 1s = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d} = \frac{\pi}{2\sqrt{k-1}}$$

$$\rightarrow k = 3{,}467\ N/m$$

- $\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 \quad ; \quad \zeta = \dfrac{b}{2M\omega_n} = 0{,}537$

$$\rightarrow \%OS = 13{,}53\%$$