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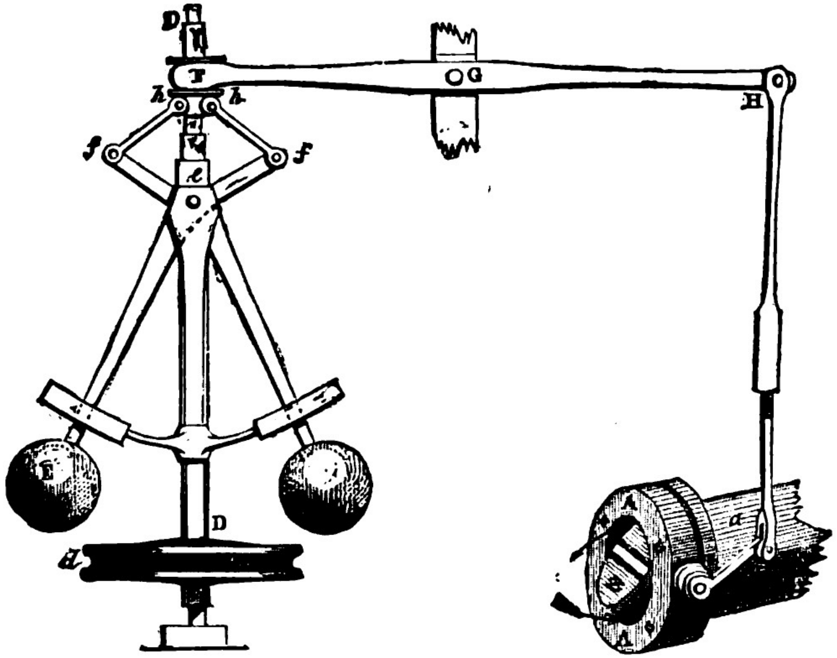
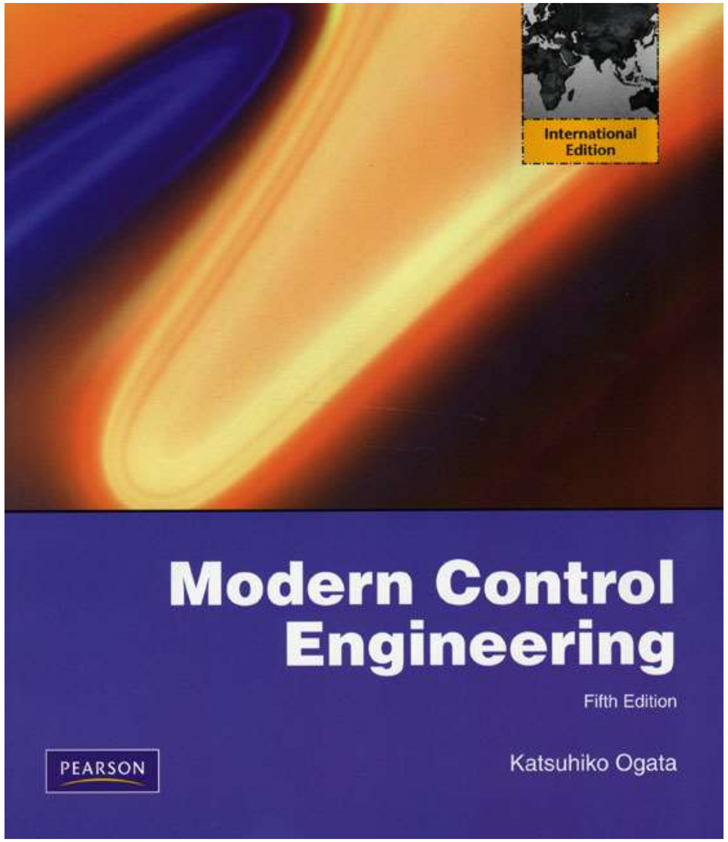


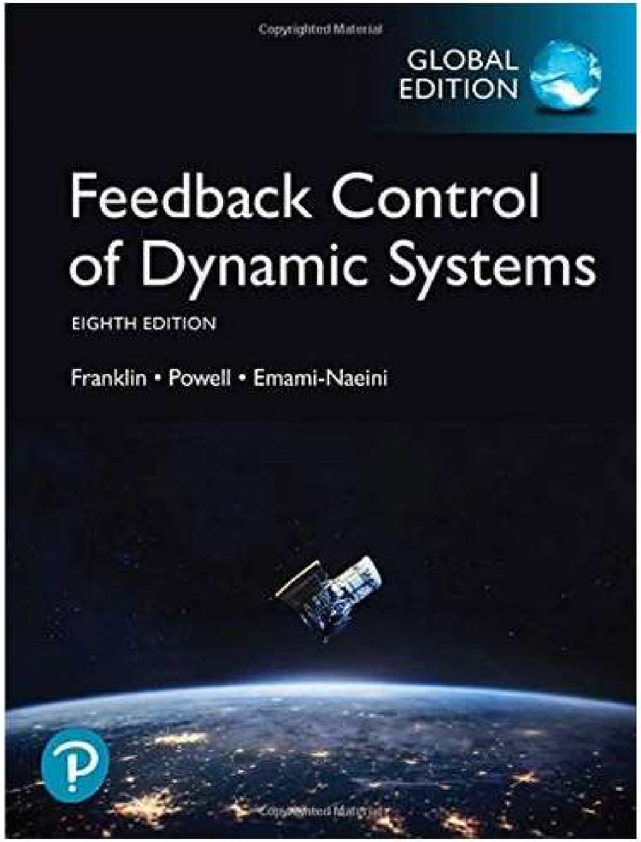
Control System Design

Prof Dr Lorenzo Masia

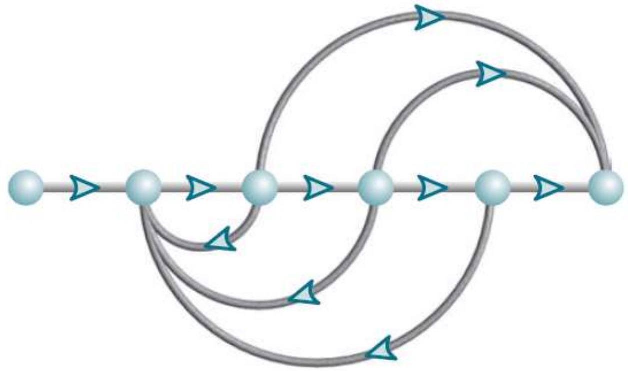
Lorenzo.masia@ziti.uni-Heidelberg.de

Advises for a good reading



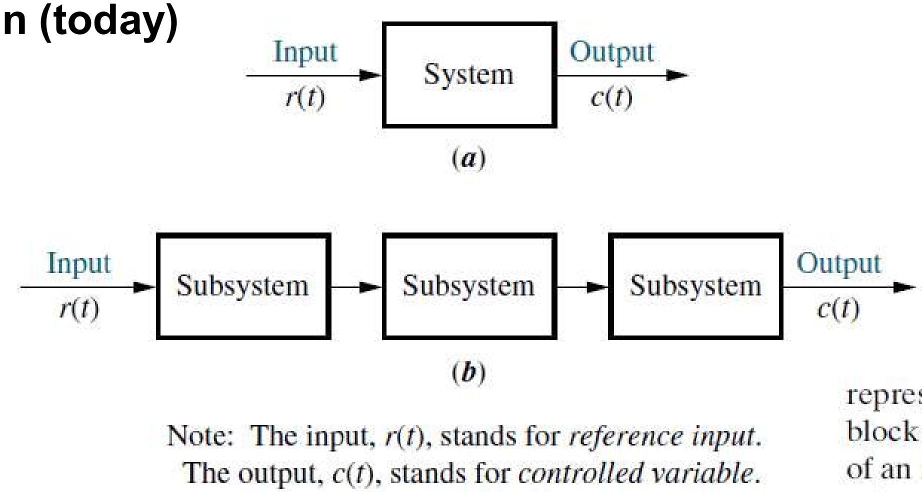


Modelling in the Frequency Domain



The goal is to develop mathematical models from schematics of physical systems.

We will discuss two methods:
(1) transfer functions in the frequency domain (today)
(2) state equations in the time domain.



a. Block diagram representation of a system; **b.** block diagram representation of an interconnection of subsystems

Laplace Transform Review

A system represented by a **differential equation is difficult to model as a block diagram**.
Thus, we now lay the groundwork for the **Laplace transform**, with which we can **represent the input, output, and system as separate entities**.
Further, their interrelationship will be **simply algebraic**.

The Laplace transform is defined as

$$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$$

$$s = \sigma + j\omega,$$

The inverse Laplace transform, which allows us to find f(t) given F(s), is

$$\mathscr{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds = f(t)u(t)$$

$$\begin{aligned} u(t) &= 1 \qquad t > 0 \\ &= 0 \qquad t < 0 \end{aligned}$$

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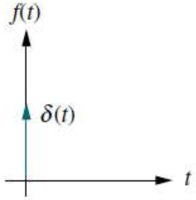
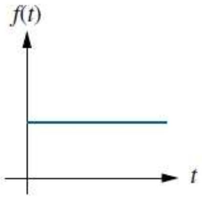
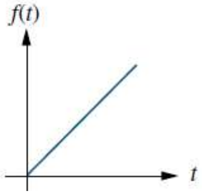
Laplace Transform Review

There are functions which will be recursive in our class and will be used as INPUTS to our control systems. Below the most important ones and how they look in the two domains of Time (*t*) and Laplace (*s*)

Test waveforms used in control systems

Input	Function	Description	Sketch	Use
Impulse	$\delta(t)$	$\delta(t) = \infty$ for $0- < t < 0+$ $= 0$ elsewhere $\int_{0-}^{0+} \delta(t)dt = 1$		Transient response Modeling
Step	$u(t)$	$u(t) = 1$ for $t > 0$ $= 0$ for $t < 0$		Transient response Steady-state error
Ramp	$tu(t)$	$tu(t) = t$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$		$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n,\ n = 1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p,\ p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\ n = 1,2,3,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at) - at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at) + at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at) - at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at) + at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\frac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$

Laplace Transform Review

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Example 2.1

Laplace Transform of a Time Function

PROBLEM: Find the Laplace transform of $f(t) = Ae^{-at}u(t)$.

SOLUTION: Since the time function does not contain an impulse function, we can replace the lower limit

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty Ae^{-at}e^{-st}\,dt = A\int_0^\infty e^{-(s+a)t}\,dt$$

$$= -\frac{A}{s+a}e^{-(s+a)t}\Big|_{t=0}^{\infty} = \frac{A}{s+a}$$

TABLE 2.1 Laplace transform table

5.	$e^{-at}u(t)$	$\frac{1}{s+a}$

Laplace Transform Review

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TABLE 2.2 Laplace transform theorems

Item no.	Theorem		Name
1.	$\mathscr{L}[f(t)] = F(s)$	$= \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)]$	$= kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)]$	$= F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)]$	$= F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)]$	$= e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)]$	$= \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right]$	$= sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right]$	$= s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right]$	$= s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right]$	$= \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty)$	$= \lim_{s\to 0} sF(s)$	Final value theorem
12.	$f(0+)$	$= \lim_{s\to\infty} sF(s)$	Initial value theorem

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Properties of Laplace Transforms

Linearity of Laplace Transforms

- Let functions $f(t)$ and $g(t)$ both be Laplace transformable.
- And $h(t) = f(t) + g(t)$

- Then $\mathcal{L}(h(t)) = \mathcal{L}(f(t) + g(t)) = \mathcal{L}(f(t)) + \mathcal{L}(g(t))$
- Or, $H(s) = F(s)+G(s)$

Laplace Transform of Derivative of $f(t)$

$$\mathcal{L}\left[\frac{d}{dt} f(t)\right] = sF(s) - f(0)$$

- Example: Laplace Transform of $\cos(\omega t)$

$$\mathcal{L}[A\cos(\omega t)] = \frac{A}{\omega}\mathcal{L}\left[\frac{d}{dt}\sin(\omega t)\right]$$

$$= \frac{As}{\omega}\mathcal{L}[\sin(\omega t)] - \frac{A\sin(0)}{\omega}$$

$$= \frac{As}{s^2 + \omega^2}$$

TABLE 2.1 Laplace transform table

Laplace transform table

Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^n + 1}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Laplace Transform of higher derivatives

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- Laplace Transform of second derivative of $f(t)$

$$\mathcal{L}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0) - \dot{f}(0)$$

- Laplace transform of n^{th} derivative of $f(t)$

$$\mathcal{L}\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}\dot{f}(0) - \ldots \overset{n-1}{f}(0)$$

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Laplace Transform of Integral of $f(t)$

$$\mathcal{L}\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \int f(t)\,dt\Big|_{t=0}$$

Original given function

- Example, Laplace Transform of Ramp function

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ t & \text{for } t \geq 0 \end{cases}$$

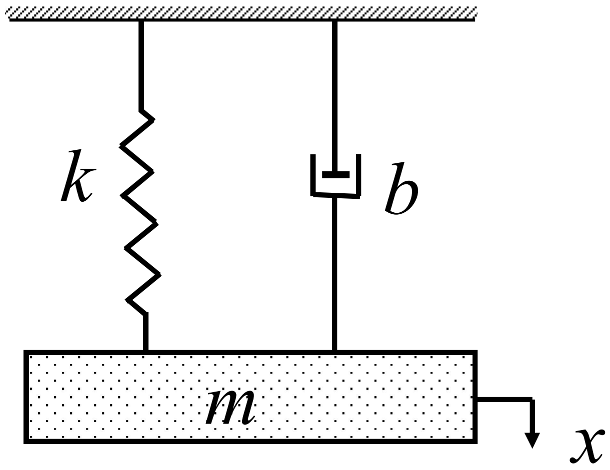
$$\mathcal{L}[t] = \mathcal{L}\left[\int 1\,dt\right] = \frac{\mathcal{L}(1(t))}{s} + \int 1(t)\,dt\Big|_{t=0}$$

$$= \frac{1}{s^3}$$

Laplace transform table

Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^n + 1}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Example



- Equation of motion is

$$m\ddot{x} + b\dot{x} + kx = 0$$

- Laplace Transform is

$$m\left(s^2X(s) - sx(0) - \dot{x}(0)\right)$$

$$+b\left(sX(s) - x(0)\right) + kX(s) = 0$$

Functions in the time (t) domain

$$x(0) = x_i$$

$$\dot{x}(0) = v_i$$

$$\Rightarrow X(s) = \frac{m(sx_i + v_i) + bx_i}{ms^2 + bs + k}$$

Final-value theorem (FVT)

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

- Relates steady-state behavior of $f(t)$ to behavior of $sF(s)$ in neighborhood of $s = 0$
- Applies only if $\lim_{t \to \infty} f(t)$ exists
- *i.e.*, all **poles** of $F(s)$ lie in the left half s-plane (and at most one pole on the imaginary axis)

Example

$$\mathcal{L}[f(t)] = F(s) = \frac{2}{s(s+2)}$$

By FVT: $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{2}{s+2} = 1$

Now consider, $\mathcal{L}[f(t)] = F(s) = \frac{2}{s(s-2)}$

By FVT: $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{2}{s-2} = -1$

However, as $\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] = e^{2t}$ $\quad e^{2t} \to \infty \quad t \to \infty$

Thus, FVT cannot be applied here…

Initial Value Theorem (IVT)

$$f(0+) = \lim_{s \to \infty} sF(s)$$

- $f(t)$ and $df(t)/dt$ are Laplace transformable, and $\lim_{s\to\infty} sF(s)$ exists
- No restriction on poles location of $sF(s)$, can be used for sinusoidal functions
- Value of $f(0+)$ can be obtained from the LT of $f(t)$

Remarks on FVT and IVT

- Enable us to predict the system behavior in time domain without transforming function from s back to time domain

- Both theorems provide a convenient check on the solution

Convolution

- The convolution of functions f and g, denoted $h = f * g$, is the function
  - same as $h(t) = \int_0^t g(t-\tau) f(\tau) d\tau$
  - $f * g = g * f$

Laplace Transform of Convolution Integral

If, $$h(t) = g * f = \int_0^t g(t-\tau) f(\tau) d\tau$$

$$H(s) = F(s)G(s),$$

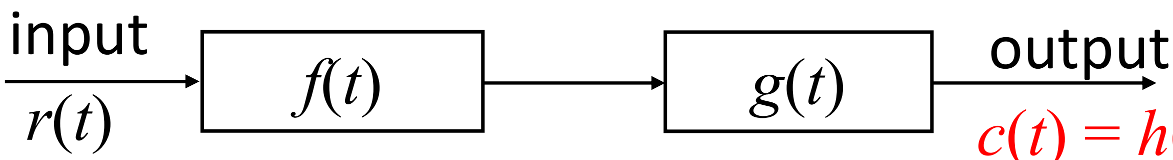
or equivalently,

$$\mathcal{L}^{-1}(\, F(s)G(s)\,) = f * g$$

Time-domain convolution ⇒ Frequency-domain multiplication

Implications of Laplace of Convolution Integral

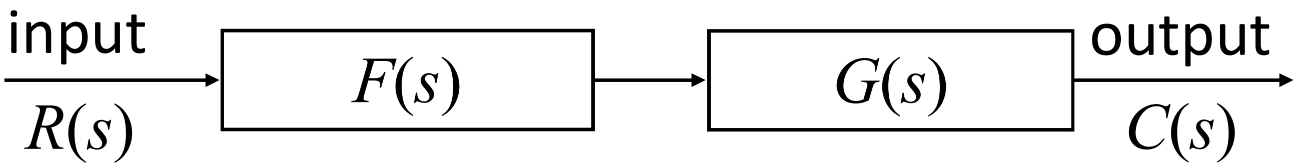
- In time domain, finding $c(t)$ is difficult:



$$c(t) = h(t) * r(t)$$

$$h(t) = f * g$$

- In s-domain:



$$C(s) = G(s)F(s)R(s)$$

$$c(t) = \mathcal{L}^{-1}\left[C(s)\right]$$

Concluding Comments on Laplace Transform

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Patterns in Laplace Transform

- While the details differ, one can see some interesting symmetric patterns between
  - ❑ the time domain (*i.e.*, signals, functions), and
  - ❑ the frequency domain (*i.e.*, their Laplace transform)

- *Differentiation* corresponds to *multiplication* by the variable s

$$\mathcal{L}\left[\frac{d^n}{dt^n} f(t)\right] = s^n F(s)$$

Initial conditions are zero

- *Integration* corresponds to dividing by the variable s

$$\mathcal{L}\left[\int \dots \int f(t)dt\right] = \frac{F(s)}{s^n}$$

Initial values are zero

- *Multiplication* by an exponential corresponds to *shift* (or *delay*) in the other

$$\mathcal{L}\left[e^{-at} f(t)\right] = F(s+a)$$

- *Convolution* in one domain is *multiplication* in another domain

$$\mathcal{L}\left[\int_0^t g(t-\tau) f(\tau) d\tau\right] = F(s)G(s)$$

Laplace Transform Review

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Example 2.1

Laplace Transform of a Time Function

PROBLEM: Find the Laplace transform of $f(t) = Ae^{-at}u(t)$.

SOLUTION: Since the time function does not contain an impulse function, we can replace the lower limit of Eq. (2.1) with 0. Hence,

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty Ae^{-at}e^{-st}\,dt = A\int_0^\infty e^{-(s+a)t}\,dt$$

$$= -\frac{A}{s+a}e^{-(s+a)t}\Big|_{t=0}^{\infty} = \frac{A}{s+a} \qquad (2.3)$$

Eq. (2.1)

$$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$$

$$\mathscr{L}[e^{-at}f(t)] \qquad = F(s+a)$$

$$tu(t) \qquad \frac{1}{s^2}$$

Laplace Transform Review

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Example 2.2

Inverse Laplace Transform

PROBLEM: Find the inverse Laplace transform of $F_1(s) = 1/(s+3)^2$.

SOLUTION: For this example we make use of the frequency shift theorem, Item 4 of Table 2.2, and the Laplace transform of $f(t) = tu(t)$, Item 3 of Table 2.1. If the inverse transform of $F(s) = 1/s^2$ is $tu(t)$, the inverse transform of $F(s+a) = 1/(s+a)^2$ is $e^{-at}tu(t)$. Hence, $f_1(t) = e^{-3t}tu(t)$.

4.	$\mathscr{L}[e^{-at}f(t)]$	$= F(s+a)$	Frequency shift theorem
3.	$tu(t)$	$\frac{1}{s^2}$	

$$\boxed{\mathscr{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds = f(t)u(t)}$$

$$\begin{aligned} u(t) &= 1 \qquad t>0 \\ &= 0 \qquad t<0 \end{aligned}$$

Laplace Transform Review

Partial-Fraction Expansion

To find the inverse Laplace transform of a complicated function, **we can convert the function to a sum of simpler terms for which we know the Laplace transform of each term**.

The result is called a partial-fraction expansion. If $F_1(s)$= N(s)/D(s), where the order of N(s) is less than the order of D(s), then a partial-fraction expansion can be made

If the order of N(s) is greater than or equal to the order of D(s), then N(s) must be divided by D(s) successively until the result has a remainder whose numerator is of order less than its denominator.

$$F_1(s) = \frac{s^3 + 2s^2 + 6s + 7}{s^2 + s + 5}$$

$$F_1(s) = s + 1 + \frac{2}{s^2 + s + 5}$$

$$f_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + \mathscr{L}^{-1}\left[\frac{2}{s^2 + s + 5}\right]$$

Laplace Transform Review

Case 1. Roots of the Denominator of F(s) Are Real and Distinct

An example of an F(s) with real and distinct roots in the denominator is

$$F(s) = \frac{2}{(s+1)(s+2)} \qquad F(s) = \frac{2}{(s+1)(s+2)} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)}$$

Letting s approach -1 eliminates the last term and yields $K_1 = 2$.

$$\frac{2}{(s+2)} = K_1 + \frac{(s+1)K_2}{(s+2)}$$

Similarly, K_2 can be found by multiplying Eq. by $(s+2)$ and then letting s approach -2; hence, $K_2 = -2$.

$$f(t) = (2e^{-t} - 2e^{-2t})u(t)$$

Laplace Transform Review

In general, then, given an F(s) whose denominator has real and distinct roots, a partial-fraction expansion, can be made if the order of N(s) is less than the order of D(s).

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_m)\cdots(s+p_n)} = \frac{K_1}{(s+p_1)} + \frac{K_2}{(s+p_2)} + \cdots + \frac{K_m}{(s+p_m)} + \cdots + \frac{K_n}{(s+p_n)}$$

To evaluate each residue, K_i, we multiply Eq. above by the denominator of the corresponding partial fraction.

Thus, if we want to find K_m, we multiply the Eq. by $(s + p_m)$ and get:

$$\begin{aligned}(s+p_m)F(s) &= \frac{(s+p_m)N(s)}{(s+p_1)(s+p_2)\cdots(s+p_m)\cdots(s+p_n)} \\ &= (s+p_m)\frac{K_1}{(s+p_1)} + (s+p_m)\frac{K_2}{(s+p_2)} + \cdots + K_m + (s+p_m)\frac{K_n}{(s+p_n)}\end{aligned}$$

Laplace Transform Review

If we let s approach p_m, all terms on the right-hand side of the Eq. go to zero except the term K_m, leaving

$$\left.\frac{\cancel{(s+p_m)}N(s)}{(s+p_1)(s+p_2)\cdots\cancel{(s+p_m)}\cdots(s+p_n)}\right|_{s\to -p_m} = K_m$$

Example 2.3

Laplace Transform Solution of a Differential Equation

PROBLEM: Given the following differential equation, solve for $y(t)$ if all initial conditions are zero. Use the Laplace transform.

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32u(t)$$

Written demonstration.

Laplace Transform Review

Case 2. Roots of the Denominator of F(s) Are Real and Repeated

An example of an F(s) with real and repeated roots in the denominator is

$$F(s) = \frac{2}{(s+1)(s+2)^2}$$

$$F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)}$$

then $K_1 = 2$, which can be found as previously described. K_2 can be isolated by multiplying the Eq. by $(s+2)^2$ yielding:

$$\frac{2}{s+1} = (s+2)^2 \frac{K_1}{(s+1)} + K_2 + (s+2)K_3$$

Letting s approach 2; $K_2 = 2$.

To find K_3 we see that if we differentiate the Eq. with respect to s, $K_3 = 2$.

Each component part of Eq. is an F(s) in Table 2.1; hence, f(t) is the sum of the inverse Laplace transform of each term, or

$$f(t) = 2e^{-t} - 2te^{-2t} - 2e^{-2t}$$

Laplace Transform Review

Case 3. Roots of the Denominator of F(s) Are Complex or Imaginary An example of F(s) with complex roots in the denominator is

$$F(s) = \frac{3}{s(s^2+2s+5)}$$

This function can be expanded in the following form:

$$\frac{3}{s(s^2+2s+5)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2+2s+5}$$

K_1 is found in the usual way to be $\frac{3}{5}$.

K_2 and K_3 can be found by first multiplying Eq. by the lowest common denominator, $s(s^2+2s+5)$, and clearing the fractions. After simplification with $K_1 = \frac{3}{5}$, we obtain

$$3 = \left(K_2 + \frac{3}{5}\right)s^2 + \left(K_3 + \frac{6}{5}\right)s + 3$$

Balancing coefficients, $\left(K_2 + \frac{3}{5}\right) = 0$ and $\left(K_3 + \frac{6}{5}\right) = 0$. Hence $K_2 = -\frac{3}{5}$ and $K_3 = -\frac{6}{5}$. Thus,

$$F(s) = \frac{3}{s(s^2+2s+5)} = \frac{3/5}{s} - \frac{3}{5}\,\frac{s+2}{s^2+2s+5}$$

Laplace Transform Review

$$F(s) = \frac{3}{s(s^2+2s+5)} = \frac{3/5}{s} - \frac{3}{5}\,\frac{s+2}{s^2+2s+5}$$

$$\mathscr{L}[Be^{-at}\sin\omega t] = \frac{B\omega}{(s+a)^2+\omega^2}$$

$$\mathscr{L}[Ae^{-at}\cos\omega t] = \frac{A(s+a)}{(s+a)^2+\omega^2}$$

Adding the Eqs. we get

$$F(s) = \frac{3/5}{s} - \frac{3}{5}\frac{(s+1)+(1/2)(2)}{(s+1)^2+2^2}$$

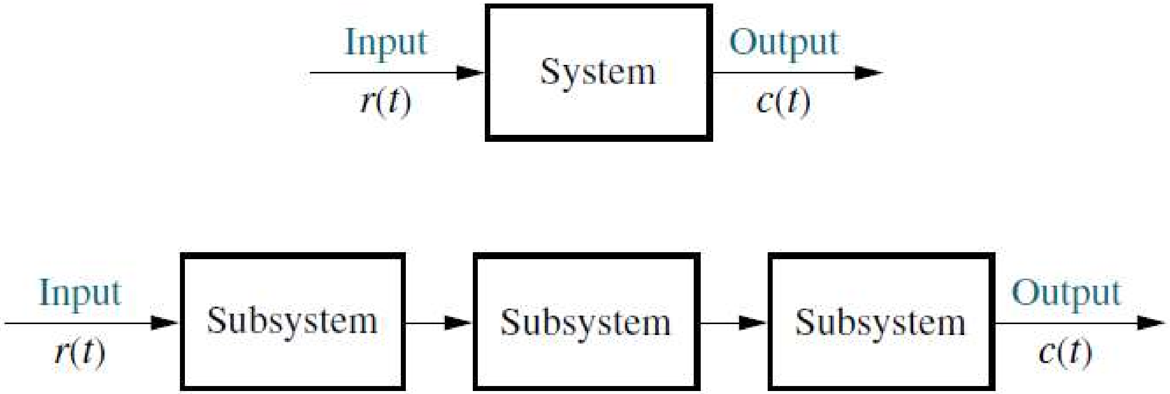
Comparing the Eq. above to Table 2.1

$$f(t) = \frac{3}{5} - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)$$

The Transfer Function

In the previous section we defined the Laplace transform and its inverse.
We presented the idea of the partial-fraction expansion and applied the concepts to the solution of differential equations.

We are now ready to formulate the system representation shown in Figure by establishing a viable definition for a function that algebraically relates a system's output to its input.



Note: The input, $r(t)$, stands for *reference input*.
The output, $c(t)$, stands for *controlled variable*.

The Transfer Function

Let us begin by writing a general nth-order, linear, time-invariant differential equation,

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

where **c(t) is the output, r(t) is the input**

Taking the Laplace transform of both sides,

$$\begin{aligned} & a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \cdots + a_0 C(s) + \text{initial condition terms involving } c(t) \\ & = b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \cdots + b_0 R(s) + \text{initial condition terms involving } r(t) \end{aligned}$$

If we assume that all initial conditions are zero

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0) C(s) = (b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0) R(s)$$

$$\boxed{\frac{C(s)}{R(s)} = G(s) = \frac{(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0)}}$$

The Transfer Function

We call this ratio, G(s), the transfer function and evaluate it with zero initial conditions.

$$R(s) \longrightarrow \boxed{\frac{(b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0)}} \longrightarrow C(s)$$

Also, we can find the output, C(s) by using $\boxed{C(s) = R(s)G(s)}$

Example 2.4

Transfer Function for a Differential Equation

PROBLEM: Find the transfer function represented by

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

SOLUTION: Taking the Laplace transform of both sides, assuming zero initial conditions, we have

$$sC(s) + 2C(s) = R(s)$$

The transfer function, $G(s)$, is

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2}$$

The Transfer Function

System Response from the Transfer Function

PROBLEM: Use the result of Example 2.4 to find the response, $c(t)$ to an input, $r(t) = u(t)$, a unit step, assuming zero initial conditions.

SOLUTION: To solve the problem, we use Eq. $\boxed{C(s) = R(s)G(s)}$ where $G(s) = 1/(s+2)$

Since $r(t) = u(t)$, $R(s) = 1/s$,

$$C(s) = R(s)G(s) = \frac{1}{s(s+2)}$$

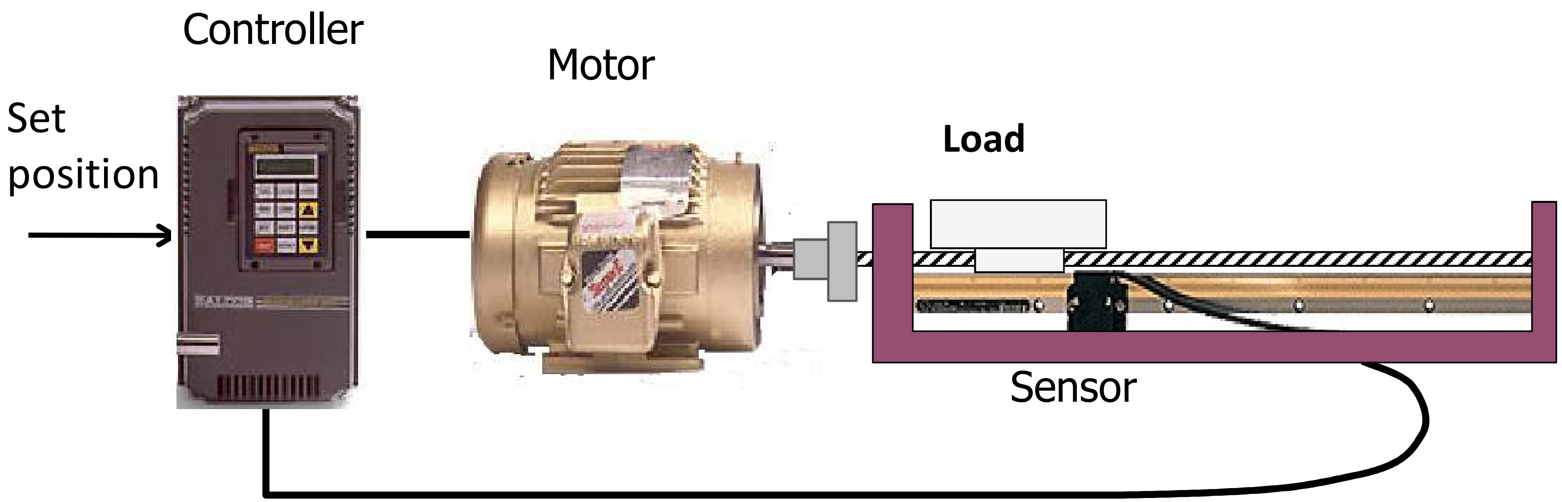
Expanding by partial fractions, we get

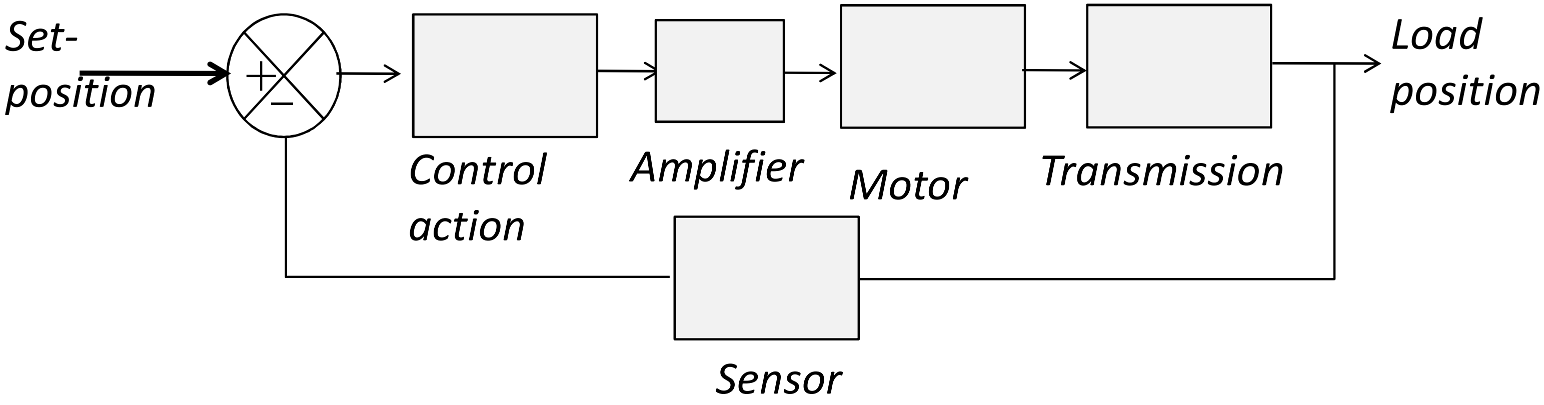
$$C(s) = \frac{1/2}{s} - \frac{1/2}{s+2}$$

Finally, taking the inverse Laplace transform of each term yields

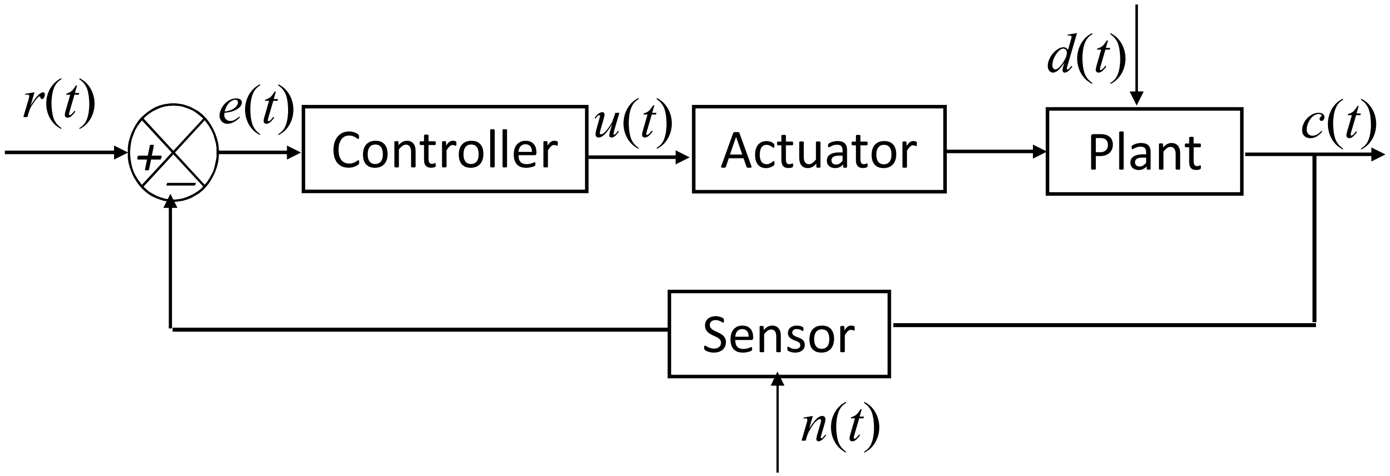
$$c(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$$

Feedback Control- Block Diagram

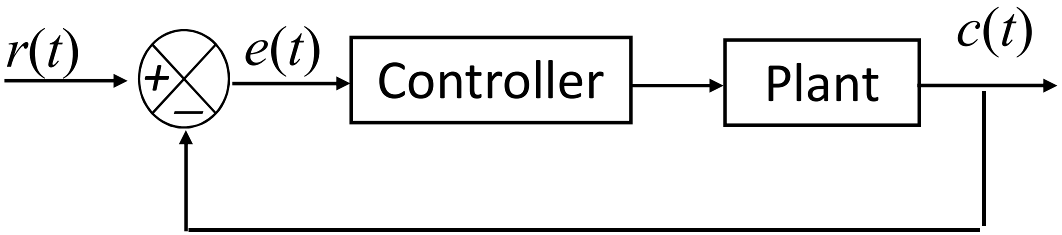




Block Diagram for Feedback Control



A simplified version – sufficient for most cases of study



The overall system is a mx between electrical and mechanical components.

Electrical Network Transfer Functions

- we formally apply the **transfer function to the mathematical modelling** of electric circuits including passive networks and operational amplifier circuits.

- We now combine electrical components into circuits, decide on the input and output, and find the transfer function

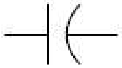
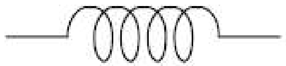
- Our guiding principles are **Kirchhoff's laws**.
- We sum voltages around loops or sum currents at nodes, depending on which technique involves the least effort in algebraic manipulation, and then equate the result to zero.

- From these **relationships we can write the differential equations for the circuit**.

- Then **we can take the Laplace transforms of the differential equations and finally solve for the transfer function**.

Electrical Network Transfer Functions

TABLE 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
 Capacitor	$v(t) = \frac{1}{C}\int_0^1 i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^1 v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ – V (volts), $i(t)$ – A (amps), $q(t)$ – Q (coulombs), C – F (farads), R – Ω (ohms), G – Ω (mhos), L – H (henries).

Electrical Network Transfer Functions

Example 2.6 Transfer Function—Single Loop via the Differential Equation

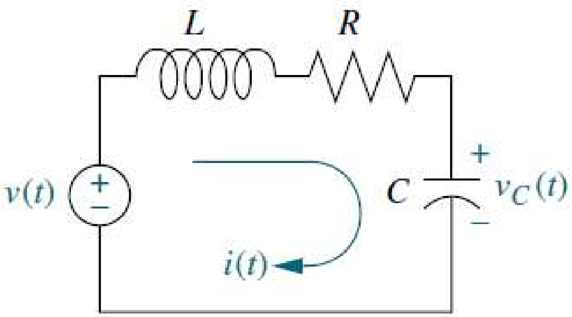


FIGURE 2.3 *RLC* network

PROBLEM: Find the transfer function relating the capacitor voltage, $V_C(s)$, to the input voltage, $V(s)$ in Figure 2.3.

Summing the voltages around the loop, assuming zero initial conditions, yields the integro-differential equation for this network as:

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t)$$

Changing variables from current to charge using $i(t) = dq(t)/dt$ yields

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = v(t)$$

From the voltage-charge relationship for a capacitor in Table 2.3, $q(t) = Cv_C(t)$

$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v(t)$$

Electrical Network Transfer Functions

$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v(t)$$

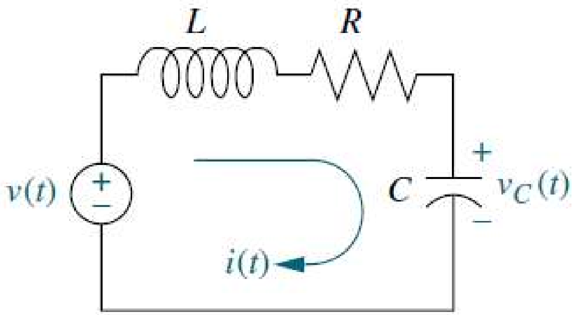


FIGURE 2.3 *RLC* network

Taking the Laplace transform assuming zero initial conditions, rearranging terms, and simplifying yields

$$(LCs^2 + RCs + 1)V_C(s) = V(s)$$

Solving for the transfer function, $V_C(s)/V(s)$, we obtain

$$\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

as shown in Figure

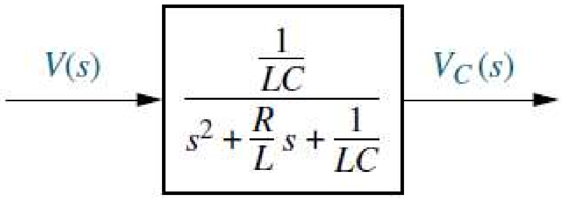


FIGURE Block diagram of series *RLC* electrical network

Electrical Network Transfer Functions

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For the capacitor, $\boxed{V(s) = \frac{1}{Cs} I(s)}$

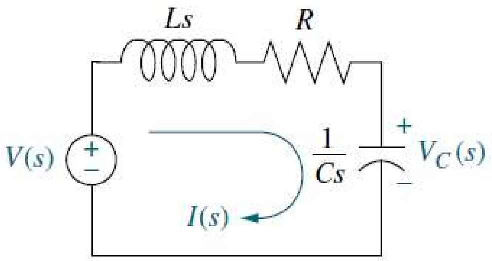
For the resistor, $\boxed{V(s) = RI(s)}$

For the inductor, $\boxed{V(s) = LsI(s)}$

Now define the following transfer function:

$$\boxed{\frac{V(s)}{I(s)} = Z(s)}$$

We call this particular transfer function ***impedance.***



$$\left(Ls + R + \frac{1}{Cs}\right) I(s) = V(s)$$

$$[\text{Sum of impedances}]\, I(s) = [\text{Sum of applied voltages}]$$

Complex Circuits via Mesh Analysis

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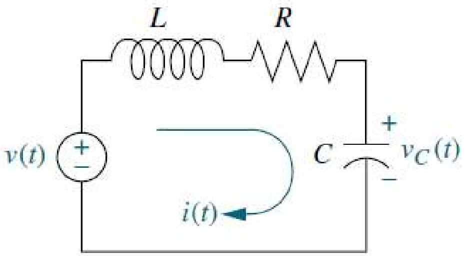
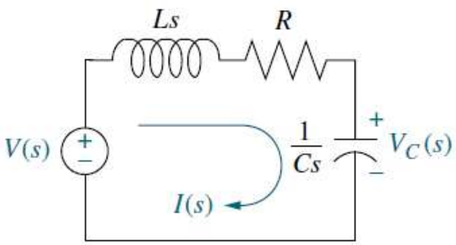
To solve complex electrical networks—those with multiple loops and nodes—using mesh analysis, we can perform the following steps:

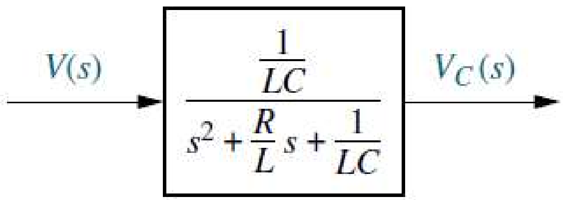
1. Replace passive element values with their impedances.
2. Replace all sources and time variables with their Laplace transform.
3. Assume a transform current and a current direction in each mesh.
4. Write Kirchhoff's voltage law around each mesh.
5. Solve the simultaneous equations for the output.
6. Form the transfer function.





$$[\text{Sum of impedances}]I(s) = [\text{Sum of applied voltages}]$$

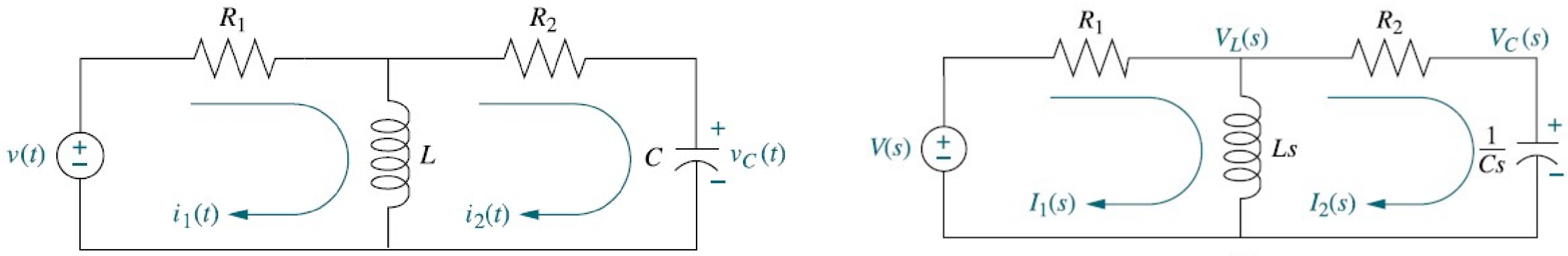
$$\left(Ls + R + \frac{1}{Cs}\right)I(s) = V(s)$$



Example

Transfer Function—Multiple Loops

PROBLEM: Given the network of Figure 2.6(a), find the transfer function, $I_2(s)/V(s)$.



(*a*) (*b*)

SOLUTION:

two simultaneous equations to solve for the transfer function. These equations can be found by summing voltages around each mesh through which the assumed currents, $I_1(s)$ and $I_2(s)$, flow. Around Mesh 1, where $I_1(s)$ flows,

$$R_1 I_1(s) + LsI_1(s) - LsI_2(s) = V(s)$$

Around Mesh 2, where $I_2(s)$ flows,

$$LsI_2(s) + R_2 I_2(s) + \frac{1}{Cs} I_2(s) - LsI_1(s) = 0$$

$$V(s) \longrightarrow \boxed{\dfrac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1R_2C + L)s + R_1}} \longrightarrow I_2(s)$$

(*c*)

$$R_1I_1(s) + LsI_1(s) - LsI_2(s) = V(s)$$
$$LsI_2(s) + R_2I_2(s) + \frac{1}{Cs}I_2(s) - LsI_1(s) = 0$$

$$(R_1 + Ls)I_1(s) \qquad - LsI_2(s) = V(s)$$
$$- LsI_1(s) + \left(Ls + R_2 + \frac{1}{Cs}\right)I_2(s) = 0$$

$$\Delta = \begin{vmatrix} (R_1 + Ls) & -Ls \\ -Ls & \left(Ls + R_2 + \dfrac{1}{Cs}\right) \end{vmatrix}$$

$$I_2(s) = \frac{\begin{vmatrix} (R_1 + Ls) & V(s) \\ -Ls & 0 \end{vmatrix}}{\Delta} = \frac{LsV(s)}{\Delta}$$

$$\boxed{G(s) = \frac{I_2(s)}{V(s)} = \frac{Ls}{\Delta} = \frac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1R_2C + L)s + R_1}}$$

General rule

$$R_1 I_1(s) + LsI_1(s) - LsI_2(s) = V(s)$$

$$LsI_2(s) + R_2 I_2(s) + \frac{1}{Cs} I_2(s) - LsI_1(s) = 0$$

$$\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{around Mesh 1} \end{bmatrix} I_1(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to the} \\ \text{two meshes} \end{bmatrix} I_2(s) = \begin{bmatrix} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 1} \end{bmatrix}$$

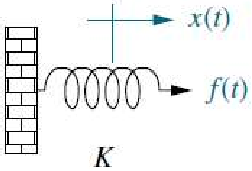
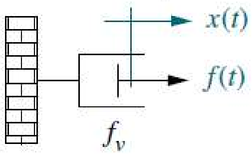
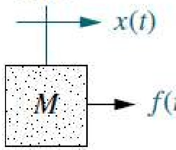
$$-\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to the} \\ \text{two meshes} \end{bmatrix} I_1(s) + \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{around Mesh 2} \end{bmatrix} I_2(s) = \begin{bmatrix} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 2} \end{bmatrix}$$

Translational Mechanical System Transfer Functions

We have shown that electrical networks can be modelled by a transfer function, G(s), that algebraically relates the Laplace transform of the output to the Laplace transform of the input.

Now we will do the same for mechanical systems.

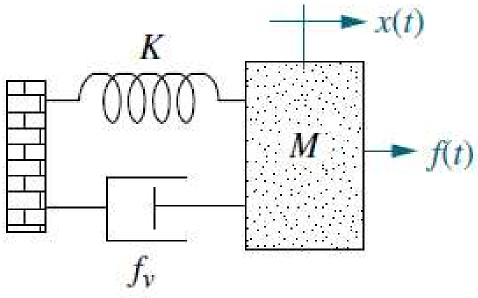
TABLE 2.4 Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

Component	Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K\int_0^t v(\tau)d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m(newton-seconds/meter), M = kg (kilograms = newton-seconds^2/meter).

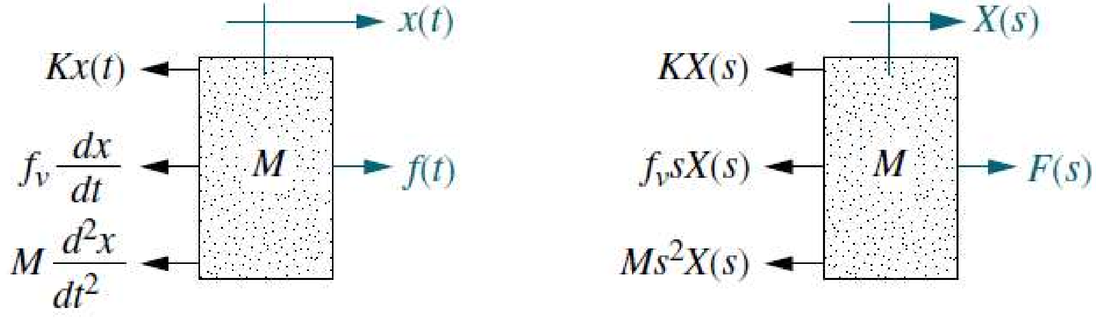
Example

PROBLEM: Find the transfer function, $X(s)/F(s)$, for the system of Figure 2.15(a).



(a)

SOLUTION:



(a) (b)

$$M\frac{d^2x(t)}{dt^2} + f_v\frac{dx(t)}{dt} + Kx(t) = f(t)$$

Taking the Laplace transform, assuming zero initial conditions,

$$Ms^2X(s) + f_v sX(s) + KX(s) = F(s)$$

$$(Ms^2 + f_v s + K)X(s) = F(s)$$

$$\boxed{G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}}$$

Translational Mechanical System

Now can we parallel our work with electrical networks by circumventing the writing of differential equations and by defining impedances for mechanical components?

for the spring, $\boxed{F(s) = KX(s)}$

for the viscous damper, $\boxed{F(s) = f_v sX(s)}$

and for the mass, $\boxed{F(s) = Ms^2X(s)}$

$$(Ms^2 + f_v s + K)X(s) = F(s)$$

If we define impedance for mechanical components as

$$\boxed{Z_M(s) = \frac{F(s)}{X(s)}}$$

$$\boxed{F(s) = Z_M(s)X(s)}$$

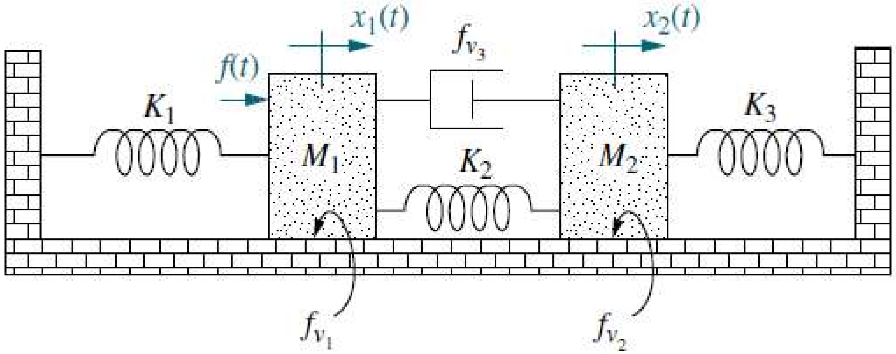
$$[\text{Sum of impedances}]X(s) = [\text{Sum of applied forces}]$$

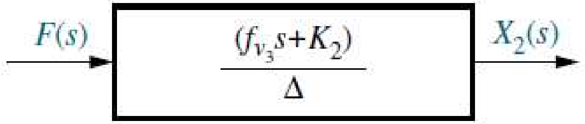
Translational Mechanical System

In mechanical systems, the number of equations of motion required is equal to the number of linearly independent motions.

Another name for the number of linearly independent motions is the number of ***degrees of freedom***.

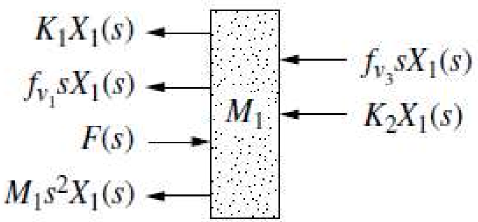
Example **PROBLEM:** Find the transfer function, $X_2(s)/F(s)$, for the system of Figure





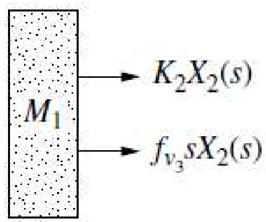
Translational Mechanical System

SOLUTION: The system has two degrees of freedom, since each mass can be moved in the horizontal direction while the other is held still. Thus, two simultaneous equations of motion will be required to describe the system. The two equations come from free-body diagrams of each mass. Superposition is used to draw the free-body diagrams. For example, the forces on M_1 are due to (1) its own motion and (2) the motion of M_2 transmitted to M_1 through the system. We will consider these two sources separately.

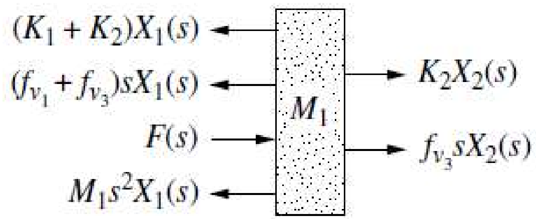


(1) If we hold M_2 still and move M_1 to the right

(2) If we hold M_1 still and move M_2 to the right



(1)+(2)=(3)

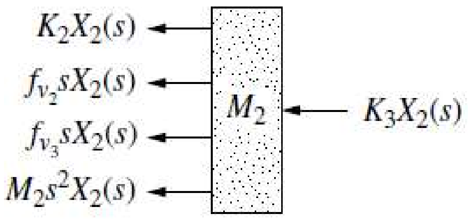


(3) This result is the overall forces

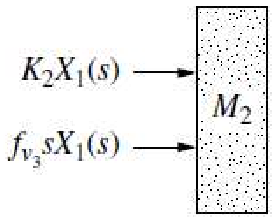
Translational Mechanical System

ForM$_2$, we proceed in a similar fashion:
(1) First we move M$_2$ to the right while holding M$_1$ still;
(2) then we move M$_1$ to the right and hold M$_2$ still.
For each case we evaluate the forces on M$_2$.

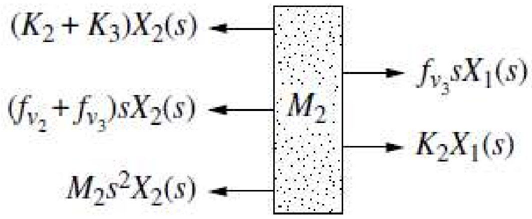
(1)



(2)



(3)



The Laplace transform of the equations of motion can now be written

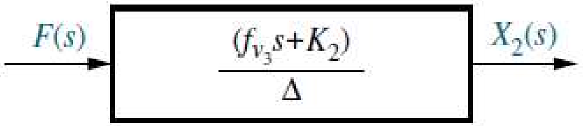
$$\left[M_1s^2(f_{v_1} + f_{v_3})s + (K_1 + K_2)\right]X_1(s) - (f_{v_3}s + K_2)X_2(s) = F(s)$$

$$-(f_{v_3}s + K_2)X_1(s) + \left[M_2s^2 + (f_{v_2} + f_{v_3})s + (K_2 + K_3)\right]X_2(s) = 0$$

Translational Mechanical System

From this, the transfer function, $X_2(s)/F(s)$, is

$$\frac{X_2(s)}{F(s)} = G(s) = \frac{(f_{v_3}s + K_2)}{\Delta}$$

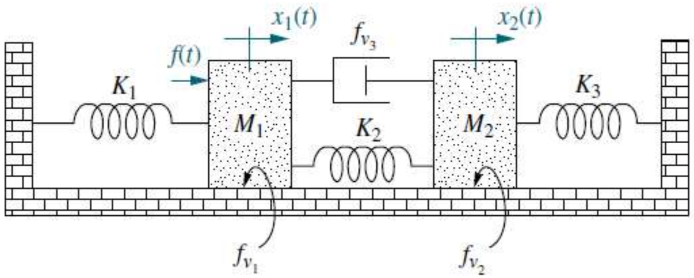


as shown in Figure where

$$\Delta = \begin{vmatrix} [M_1 s^2 + (f_{v_1} + f_{v_3})s + (K_1 + K_2)] & -(f_{v_3}s + K_2) \\ -(f_{v_3}s + K_2) & [M_2 s^2 + (f_{v_2} + f_{v_3})s + (K_2 + K_3)] \end{vmatrix}$$

$$[M_1 s^2 + (f_{v_1} + f_{v_3})s + (K_1 + K_2)]X_1(s) - (f_{v_3}s + K_2)X_2(s) = F(s)$$

$$-(f_{v_3}s + K_2)X_1(s) + [M_2 s^2 + (f_{v_2} + f_{v_3})s + (K_2 + K_3)]X_2(s) = 0$$

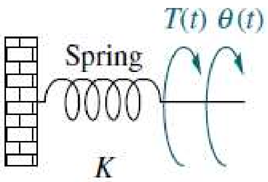
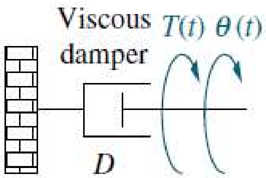
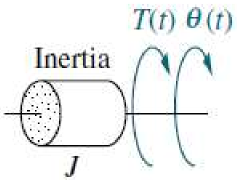


$$\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } x_1 \end{bmatrix} X_1(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ x_1 \text{ and } x_2 \end{bmatrix} X_2(s) = \begin{bmatrix} \text{Sum of} \\ \text{applied forces} \\ \text{at } x_1 \end{bmatrix}$$

$$-\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ x_1 \text{ and } x_2 \end{bmatrix} X_1(s) + \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } x_2 \end{bmatrix} X_2(s) = \begin{bmatrix} \text{Sum of} \\ \text{applied forces} \\ \text{at } x_2 \end{bmatrix}$$

Rotational Mechanical System Transfer Functions

TABLE Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

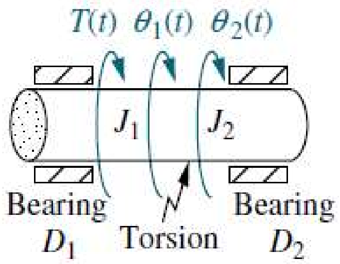
Component	Torque-angular velocity	Torque-angular displacement	Impedence $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\dfrac{d\theta(t)}{dt}$	Ds
	$T(t) = J\dfrac{d\omega(t)}{dt}$	$T(t) = J\dfrac{d^2\theta(t)}{dt^2}$	Js^2

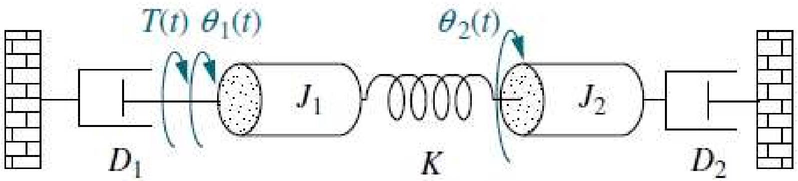
Note: The following set of symbols and units is used throughout this book: $T(t)$ – N-m (newton-meters), $\theta(t)$ – rad(radians), $\omega(t)$ – rad/s(radians/second), K – N-m/rad(newton- meters/radian), D – N-m-s/rad (newton- meters-seconds/radian). J – kg-m^2(kilograms-meters^2 – newton-meters-seconds^2/radian).

Rotational Mechanical System

Example Transfer Function—Two Equations of Motion

PROBLEM: Find the transfer function, $\theta_2(s)/T(s)$,



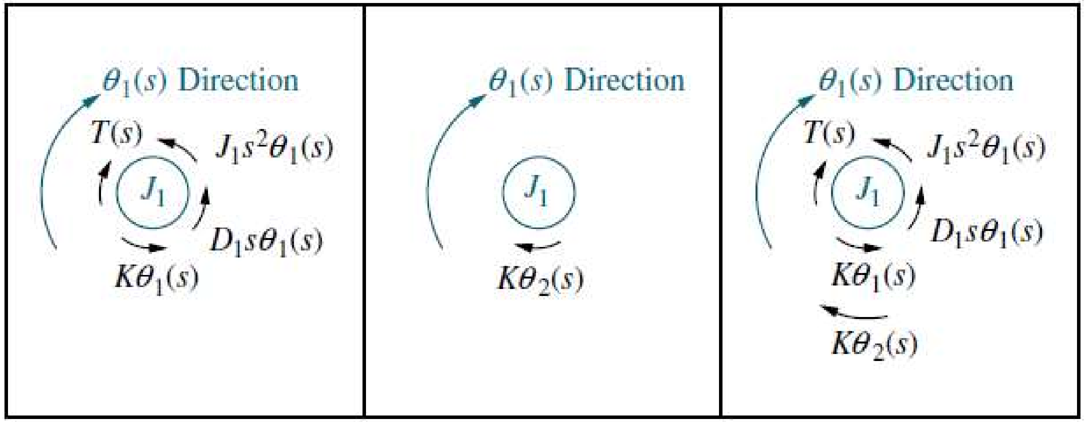


free-body diagram of J_1, using superposition

(a) torques on J_1 if J_2 is held still and J_1 rotated

(b) torques on J_1 if J_1 is held still and J_2 rotated

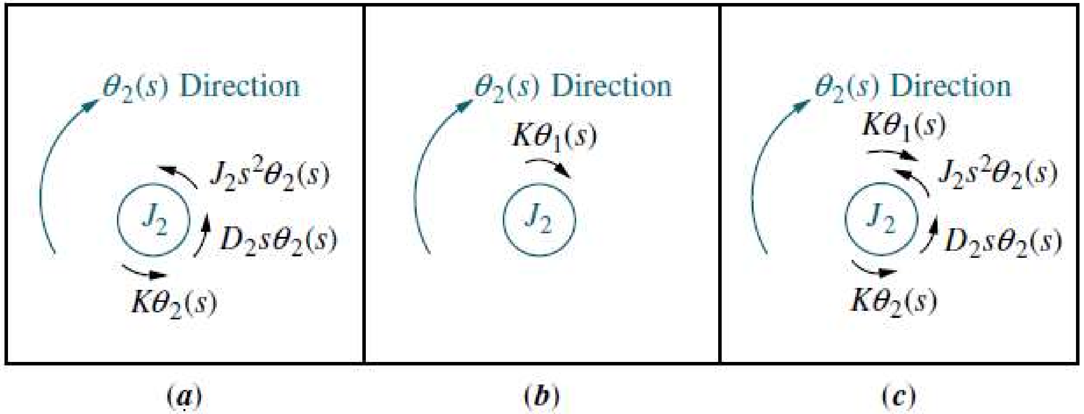
(c) sum of Figures/Forces



(a) (b) (c)

Rotational Mechanical System

FIGURE **a.** Torques on J_2 due only to the motion of J_2; **b.** torques on J_2 due only to the motion of J_1; **c.** final free-body diagram for J_2

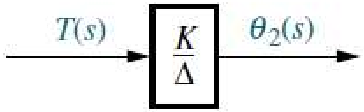


Summing torques respectively from Figures

$$(J_1 s^2 + D_1 s + K)\theta_1(s) \qquad - K\theta_2(s) = T(s)$$

$$-K\theta_1(s) + (J_2 s^2 + D_2 s + K)\theta_2(s) = 0$$

$$\frac{\theta_2(s)}{T(s)} = \frac{K}{\Delta}$$



$$\Delta = \begin{vmatrix} (J_1 s^2 + D_1 s + K) & -K \\ -K & (J_2 s^2 + D_2 s + K) \end{vmatrix}$$

Rotational Mechanical System

$$(J_1 s^2 + D_1 s + K)\theta_1(s) \qquad - K\theta_2(s) = T(s)$$

$$-K\theta_1(s) + (J_2 s^2 + D_2 s + K)\theta_2(s) = 0$$

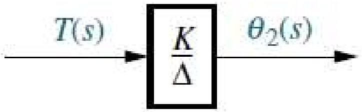
$$\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } \theta_1 \end{bmatrix} \theta_1(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \theta_1 \text{ and } \theta_2 \end{bmatrix} \theta_2(s) = \begin{bmatrix} \text{Sum of} \\ \text{applied torques} \\ \text{at } \theta_1 \end{bmatrix}$$

$$-\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \theta_1 \text{ and } \theta_2 \end{bmatrix} \theta_1(s) + \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } \theta_2 \end{bmatrix} \theta_2(s) = \begin{bmatrix} \text{Sum of} \\ \text{applied torques} \\ \text{at } \theta_2 \end{bmatrix}$$

Use the following MATLAB and Symbolic Math Toolbox Statements……..

```
syms s J1 D1 K T J2 D2...
  theta1 theta2
A =[(J1*s^2+D1*s+K) -K
    -K (J2*s^2+D2*s+K)];
B =[theta1
    theta2];
C =[T
    0];
B=inv(A)*C;
theta2=B(2);
'theta2'
pretty(theta2)
```

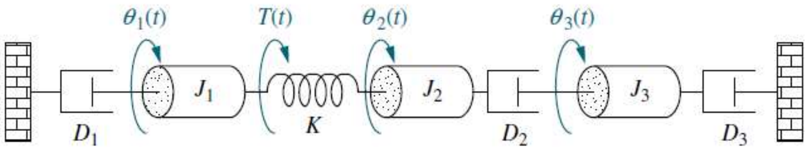
$$\frac{\theta_2(s)}{T(s)} = \frac{K}{\Delta}$$



Rotational Mechanical System

Example

PROBLEM: Write, but do not solve, the Laplace transform of the equations of motion for the system shown in Figure



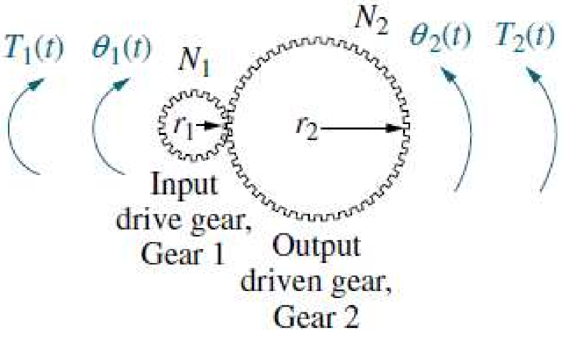
$$\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } \theta_1\end{bmatrix}\theta_1(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_1 \text{ and } \theta_2\end{bmatrix}\theta_2(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_1 \text{ and } \theta_3\end{bmatrix}\theta_3(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied torques}\\ \text{at } \theta_1\end{bmatrix}$$

$$-\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_1 \text{ and } \theta_2\end{bmatrix}\theta_1(s) + \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } \theta_2\end{bmatrix}\theta_2(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_2 \text{ and } \theta_3\end{bmatrix}\theta_3(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied torques}\\ \text{at } \theta_2\end{bmatrix}$$

$$-\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_1 \text{ and } \theta_3\end{bmatrix}\theta_1(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_2 \text{ and } \theta_3\end{bmatrix}\theta_2(s) + \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } \theta_3\end{bmatrix}\theta_3(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied torques}\\ \text{at } \theta_3\end{bmatrix}$$

$$\begin{aligned}
(J_1s^2 + D_1s + K)\theta_1(s) \quad & -K\theta_2(s) \quad & -0\theta_3(s) &= T(s)\\
-K\theta_1(s) \quad & +(J_2s^2 + D_2s + K)\theta_2(s) \quad & -D_2s\theta_3(s) &= 0\\
-0\theta_1(s) \quad & -D_2s\theta_2(s) \quad & +(J_3s^2 + D_3s + D_2s)\theta_3(s) &= 0
\end{aligned}$$

Transfer Functions for Systems with Gears



Gears provide mechanical advantage to rotational systems. Anyone who has ridden a 10-speed bicycle knows the effect of gearing.
Going uphill, you shift to provide more torque and less speed. On the straightaway, you shift to obtain more speed and less torque.
Thus, gears allow you to match the drive system and the load—a trade-off between speed and torque

$$\boxed{\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}}$$

$$\boxed{\frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}}$$

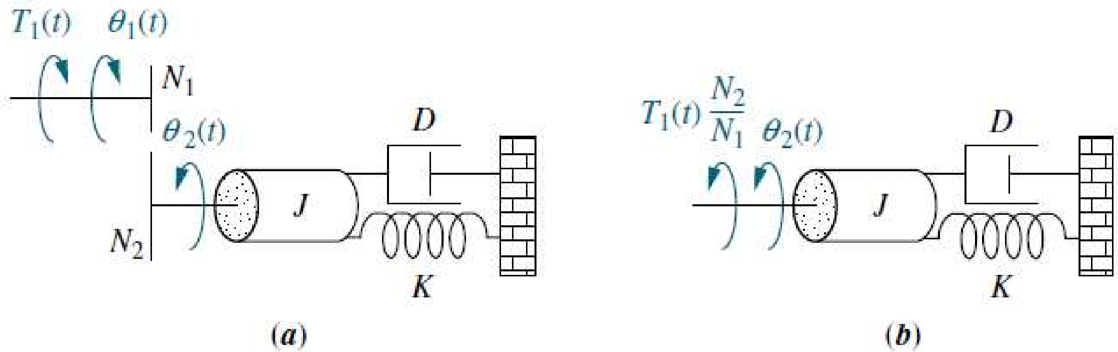
$$\boxed{T_1\theta_1 = T_2\theta_2}$$

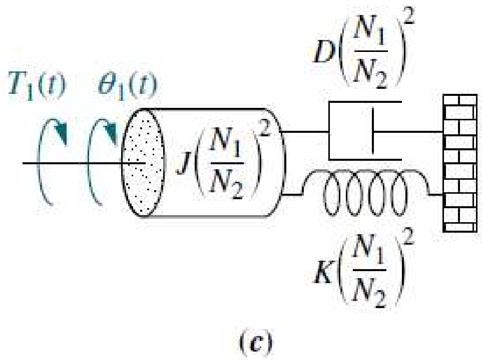
$$\theta_1 \longrightarrow \boxed{\frac{N_1}{N_2}} \longrightarrow \theta_2 \qquad T_1 \longrightarrow \boxed{\frac{N_2}{N_1}} \longrightarrow T_2$$

Transfer Functions for Systems with Gears

Let us see what happens to mechanical impedances that are driven by gears

FIGURE **a.** Rotational system driven by gears; **b.** equivalent system at the output after reflection of input torque; **c.** equivalent system at the input after reflection of impedances





How can the mechanical impedances be reflected from the output to the input, thereby eliminating the gears?

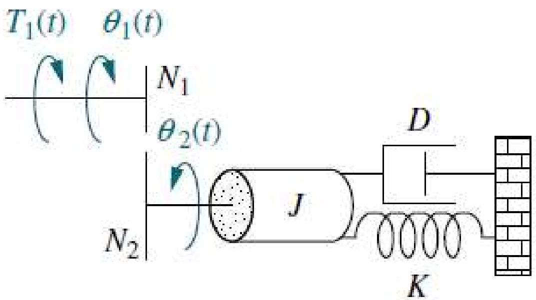
Transfer Functions for Systems with Gears

We want to represent Figure (a) as an equivalent system at θ_1 without the gears.

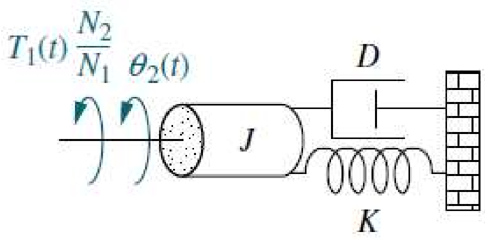
$$\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}$$

$$\frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}$$

$$T_1\theta_1 = T_2\theta_2$$



(a)



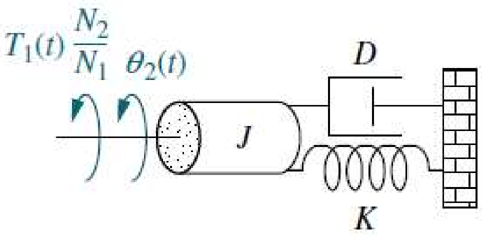
(b)

From Figure (a), T_1 can be reflected to the output by multiplying by N_2=N_1. The result is shown in Figure (b), from which we write the equation of motion as:

$$(Js^2 + Ds + K)\theta_2(s) = T_1(s)\frac{N_2}{N_1}$$

Transfer Functions for Systems with Gears

Now convert θ_2 into an equivalent θ_1, Using Figure (b) to obtain θ_2 in terms of θ_1, we get



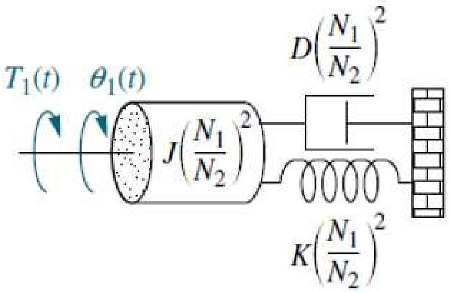
(b)

$$(Js^2 + Ds + K)\frac{N_1}{N_2}\theta_1(s) = T_1(s)\frac{N_2}{N_1}$$

With few adjustments:

$$\left[J\left(\frac{N_1}{N_2}\right)^2 s^2 + D\left(\frac{N_1}{N_2}\right)^2 s + K\left(\frac{N_1}{N_2}\right)^2\right]\theta_1(s) = T_1(s)$$

And we have the equivalent system:



(c)

Thus, the load can be thought of as having been reflected from the output to the input.

Transfer Functions for Systems with Gears

Generalizing the results, we can make the following statement:
Rotational mechanical impedances can be reflected through gear trains by multiplying the mechanical impedance by the ratio.

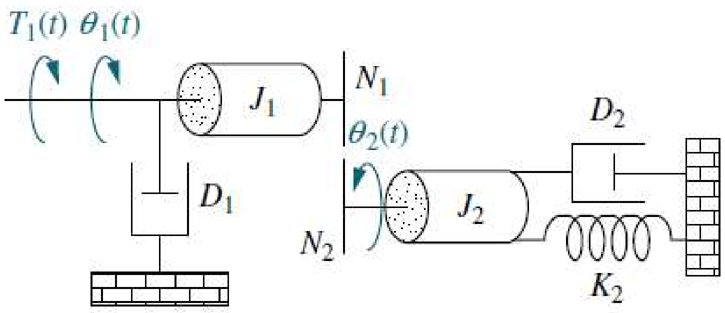
$$\left(\frac{\text{Number of teeth of gear on } \textit{destination} \text{ shaft}}{\text{Number of teeth of gear on } \textit{source} \text{ shaft}}\right)^2$$

where the impedance to be reflected is attached to the source shaft and is being reflected to the destination shaft.
The next example demonstrates the application of the concept of reflected impedances as we find **the transfer function of a rotational mechanical system with gears.**

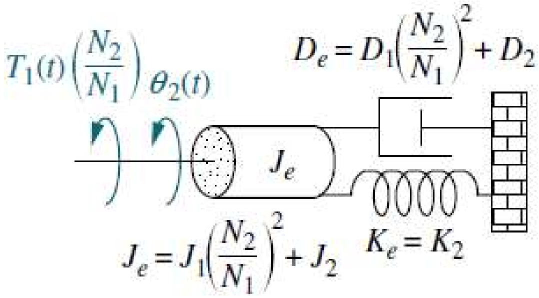
Transfer Functions for Systems with Gears

Example Transfer Function—System with Lossless Gears

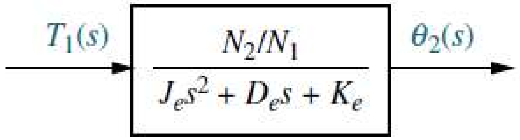
PROBLEM: Find the transfer function, $\theta_2(s)/T_1(s)$, for the system of Figure (*a*).



(*a*)



(*b*)

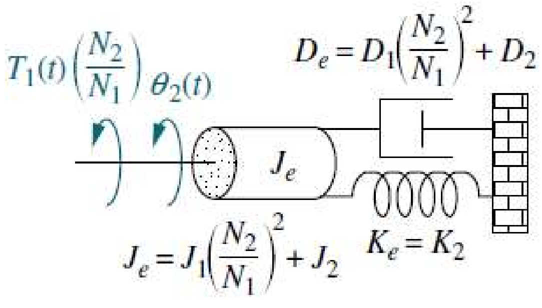


(*c*)

Transfer Functions for Systems with Gears

It may be tempting at this point to search for two simultaneous equations corresponding to each inertia. ***The inertias, however, do not undergo linearly independent motion, since they are tied together by the gears.***
Thus, there is only one degree of freedom and hence **one equation of motion**!!!!

Let us first reflect the impedances (J_1 and D_1) and torque (T_1) on the input shaft to the output as shown in Figure (b), where the impedances are reflected by $(N_2/N_1)^2$ and the torque is reflected by (N_2/N_1).
The equation of motion can now be written as:



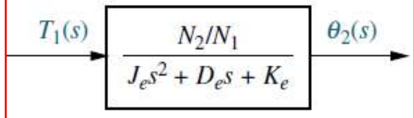
(b)

$$(J_e s^2 + D_e s + K_e)\theta_2(s) = T_1(s)\frac{N_2}{N_1}$$

$$J_e = J_1\left(\frac{N_2}{N_1}\right)^2 + J_2; \quad D_e = D_1\left(\frac{N_2}{N_1}\right)^2 + D_2; \quad K_e = K_2$$

Solving for $\theta_2(s)/T_1(s)$, the transfer function is found to be

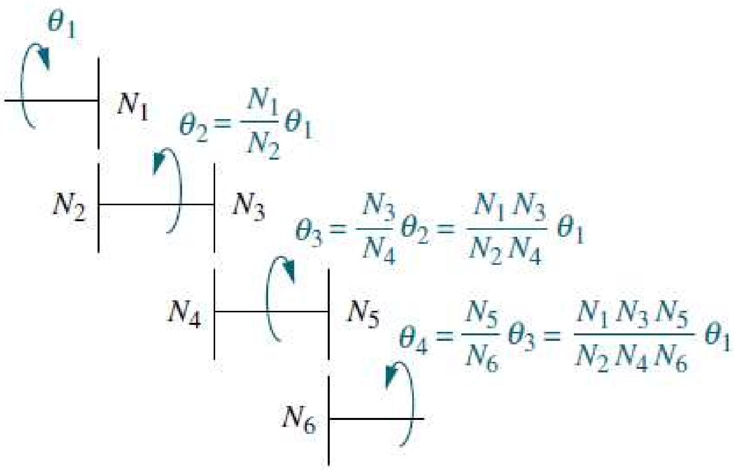
$$\boxed{G(s) = \frac{\theta_2(s)}{T_1(s)} = \frac{N_2/N_1}{J_e s^2 + D_e s + K_e}}$$



(c)

Transfer Functions for Systems with Gears

In order to eliminate gears with large radii, a gear train is used to implement large gear ratios by cascading smaller gear ratios.



$$\theta_4 = \frac{N_1 N_3 N_5}{N_2 N_4 N_6}\theta_1$$

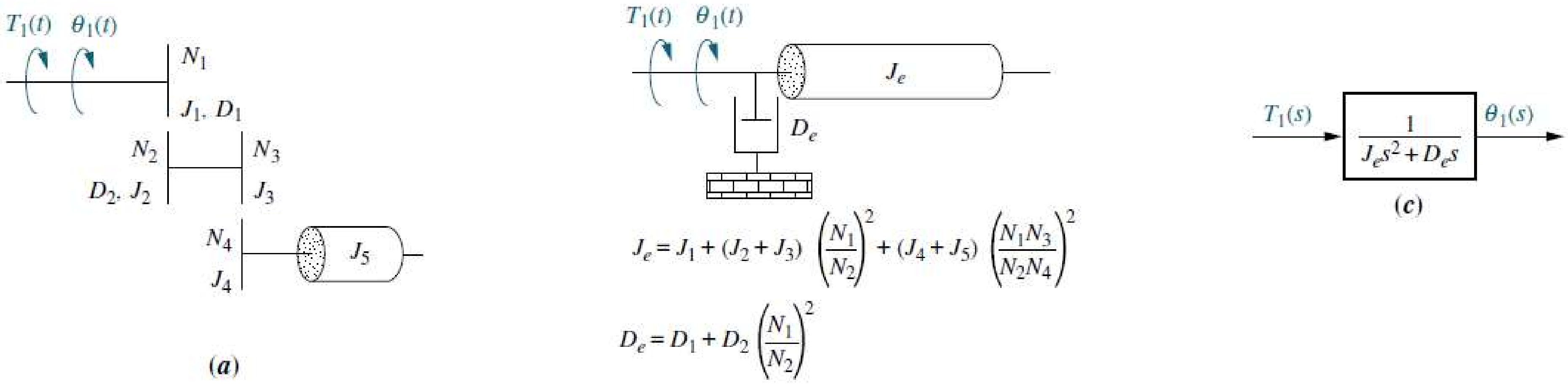
For gear trains, we conclude that the equivalent gear ratio is the product of the individual gear ratios.

We now apply this result to solve for the transfer function of a system that does not have lossless gears (next example).

Transfer Functions for Systems with Gears

Example Transfer Function—Gears with Loss

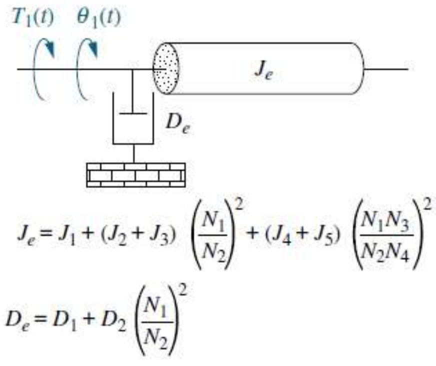
PROBLEM: Find the transfer function, $\theta_1(s)/T_1(s)$, for the system of Figure (*a*).



(*a*) (*b*) (*c*)

Transfer Functions for Systems with Gears

SOLUTION: This system, which uses a gear train, does not have lossless gears.
All of the gears have inertia J, and for some shafts there is viscous friction D.
To solve the problem, we want to reflect all of the impedances to the input shaft, θ_1.



(*b*)

The gear ratio is not the same for all impedances.

- D_2 is reflected only through one gear ratio as $D_2(N_1/N_2)^2$,
- whereas J_4 plus J_5 is reflected through two gear ratios as $(J_4+J_5)[(N_3/N_4)(N_1/N_2)^2$

The result of reflecting all impedances to θ_1 is shown in Figure (b), from which the equation of motion is:

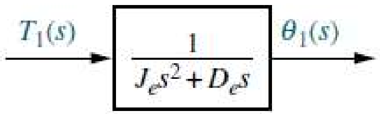
$$(J_e s^2 + D_e s)\theta_1(s) = T_1(s)$$

where

$$J_e = J_1 + (J_2 + J_3)\left(\frac{N_1}{N_2}\right)^2 + (J_4 + J_5)\left(\frac{N_1 N_3}{N_2 N_4}\right)^2 \qquad D_e = D_1 + D_2\left(\frac{N_1}{N_2}\right)^2$$

And the requested transfer function is:

$$G(s) = \frac{\theta_1(s)}{T_1(s)} = \frac{1}{J_e s^2 + D_e s}$$



(*c*)

Electromechanical System Transfer Functions

Now, we move to systems that are hybrids of electrical and mechanical variables, the electromechanical systems.

Applications for systems with electromechanical components are robot controls, sun and star trackers, and computer tape and disk-drive position controls.

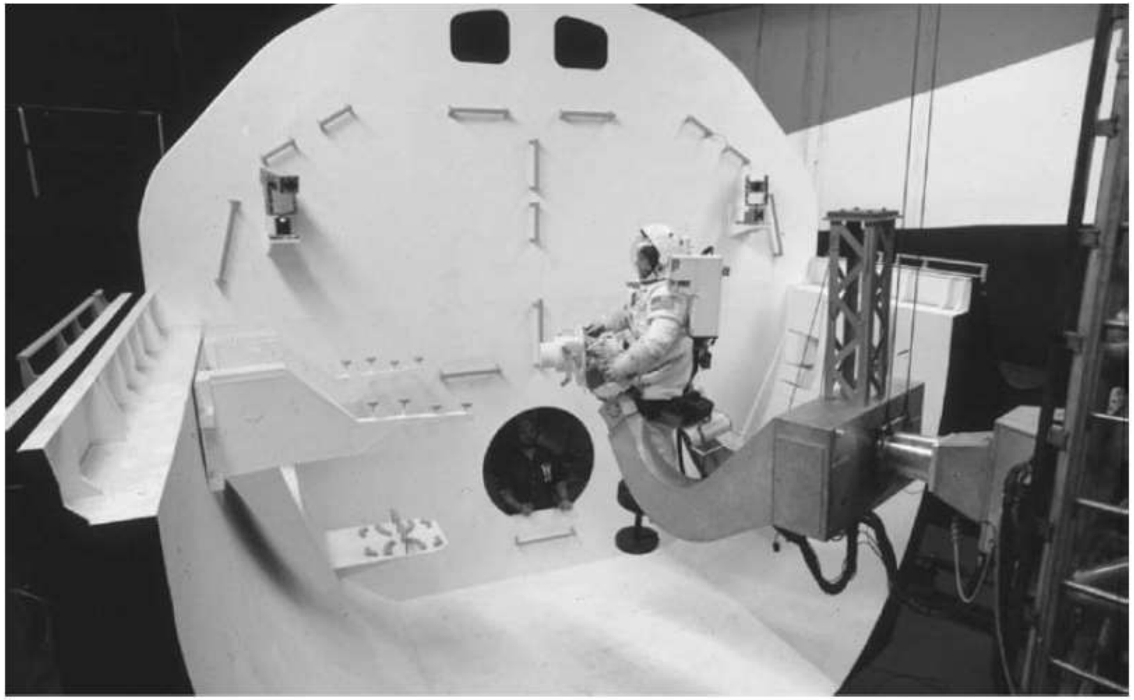
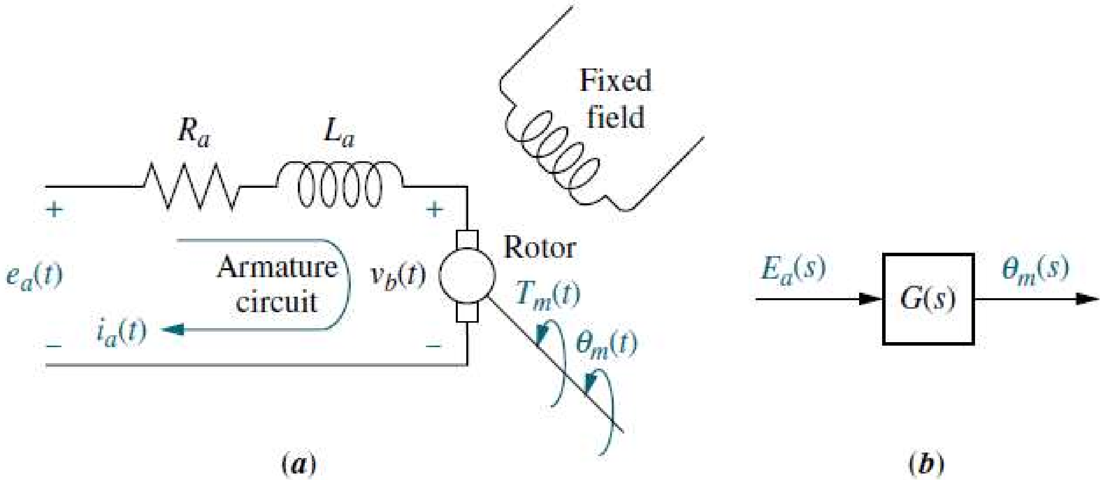


FIGURE NASA flight simulator robot arm with electromechanical control system components.

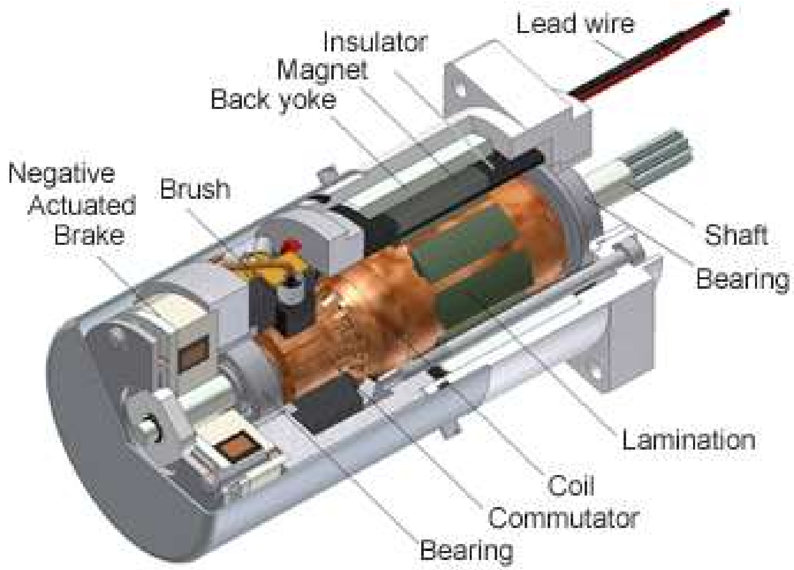
Electromechanical System Transfer Functions

We will derive the transfer function for one particular kind of electromechanical system, the armature-controlled dc servomotor (brushed).

The motor's schematic is shown in Figure (a), and the transfer function we will derive appears in Figure(b).



DC motor: **a.** schematic; **b.** block diagram

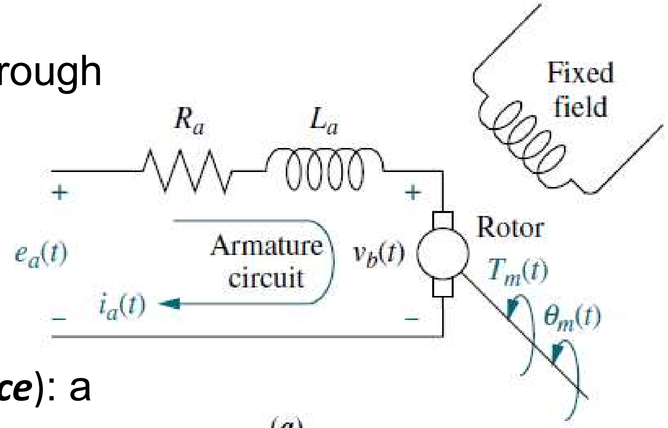


Electromechanical System Transfer Functions

A magnetic field is developed by stationary permanent magnets or a stationary electromagnet **called the fixed field.**

A rotating circuit called the armature, through which current $i_a(t)$ flows, passes through this magnetic field at right angles and feels a force, $\boldsymbol{F = B\, l\, i_a(t)}$, where $\boldsymbol{B}$ is the magnetic field strength and $\boldsymbol{l}$ is the length of the conductor.

The **resulting torque turns the rotor, the rotating member of the motor.**

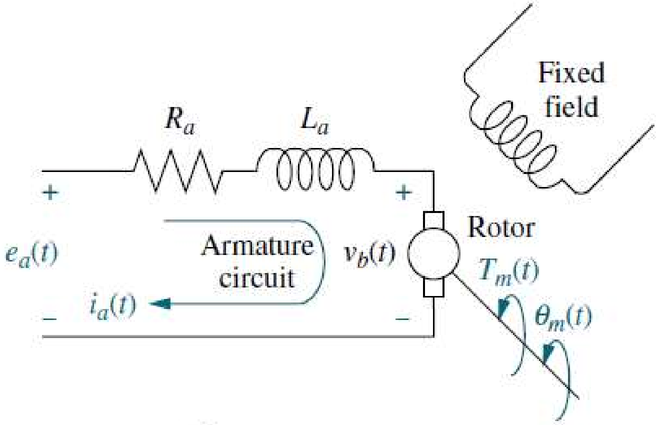


(a)

There is another phenomenon that occurs in the motor (***back electromotive force***): a conductor moving at right angles to a magnetic field generates a voltage at the terminals of the conductor equal to $\boldsymbol{v_b}$ proportional to the velocity of the conductor normal to the magnetic field.

$$v_b(t) = K_b \frac{d\theta_m(t)}{dt} \quad \rightarrow \text{Laplace} \rightarrow \quad V_b(s) = K_b s \theta_m(s)$$

Electromechanical System Transfer Functions



$$R_a I_a(s) + L_a s I_a(s) + V_b(s) = E_a(s)$$

The torque developed by the motor is proportional to the armature current

$$T_m(s) = K_t I_a(s)$$

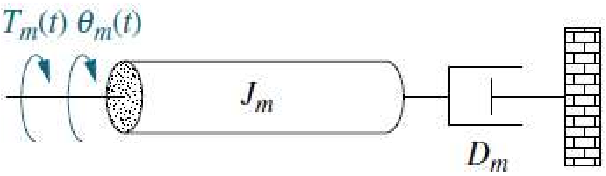
$$I_a(s) = \frac{1}{K_t} T_m(s)$$

$$\frac{(R_a + L_a s) T_m(s)}{K_t} + K_b s \theta_m(s) = E_a(s)$$

$$V_b(s) = K_b s \theta_m(s)$$

Now we must find $T_m(s)$ in terms of $\theta_m(s)$ if we are to separate the input and output variables and ***obtain the transfer function, $\theta_m(s)/E_a(s)$.***

Electromechanical System Transfer Functions



Typical equivalent mechanical loading on a motor

$$T_m(s) = (J_m s^2 + D_m s)\theta_m(s)$$

$$\frac{(R_a + L_a s)T_m(s)}{K_t} + K_b s\theta_m(s) = E_a(s)$$

$$\frac{(R_a + L_a s)(J_m s^2 + D_m s)\theta_m(s)}{K_t} + K_b s\theta_m(s) = E_a(s)$$

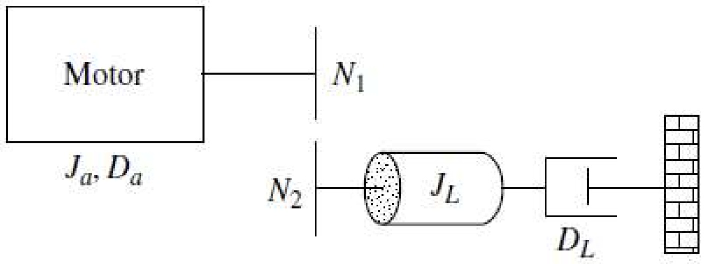
If we assume that the armature inductance, L_a, is small compared to the armature resistance, R_a, which is usual for a dc motor:

$$\left[\frac{R_a}{K_t}(J_m s + D_m) + K_b\right] s\theta_m(s) = E_a(s)$$

$$\boxed{\frac{\theta_m(s)}{E_a(s)} = \frac{K_t/(R_a J_m)}{s\left[s + \frac{1}{J_m}\left(D_m + \frac{K_t K_b}{R_a}\right)\right]}}$$

Electromechanical System Transfer Functions

Example **Transfer Function**



DC motor driving a rotational mechanical load

Assuming that all inertia and damping values shown are known, J_L and D_L can be reflected back to the armature as some equivalent inertia and damping to be added to J_a and D_a, respectively.

Thus, the equivalent inertia, J_m, and equivalent damping, D_m, at the armature are:

$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2; \; D_m = D_a + D_L\left(\frac{N_1}{N_2}\right)^2$$

Combining these 3 and neglecting L_a

$$I_a(s) = \frac{1}{K_t} T_m(s)$$

$$V_b(s) = K_b s\theta_m(s)$$

$$\frac{(R_a + L_a s)T_m(s)}{K_t} + K_b s\theta_m(s) = E_a(s)$$

$$\frac{R_a}{K_t} T_m(s) + K_b s\theta_m(s) = E_a(s)$$

→ L^{-1} →

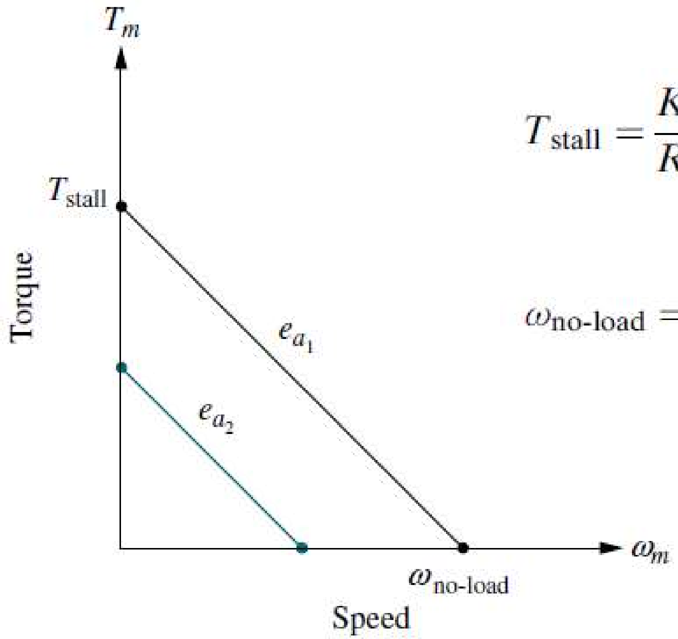
$$\frac{R_a}{K_t} T_m(t) + K_b \omega_m(t) = e_a(t)$$

Electromechanical System Transfer Functions

$$\frac{R_a}{K_t}T_m(t) + K_b\omega_m(t) = e_a(t)$$

$$\frac{R_a}{K_t}T_m + K_b\omega_m = e_a$$

$$\boxed{T_m = -\frac{K_bK_t}{R_a}\omega_m + \frac{K_t}{R_a}e_a}$$



$$T_{\text{stall}} = \frac{K_t}{R_a}e_a$$

$$\omega_{\text{no-load}} = \frac{e_a}{K_b}$$

The electrical constants of the motor's transfer function can now be found

$$\boxed{\frac{K_t}{R_a} = \frac{T_{\text{stall}}}{e_a}} \qquad \boxed{K_b = \frac{e_a}{\omega_{\text{no-load}}}}$$

$$\boxed{\frac{\theta_m(s)}{E_a(s)} = \frac{K_t/(R_aJ_m)}{s\left[s + \frac{1}{J_m}\left(D_m + \frac{K_tK_b}{R_a}\right)\right]}}$$

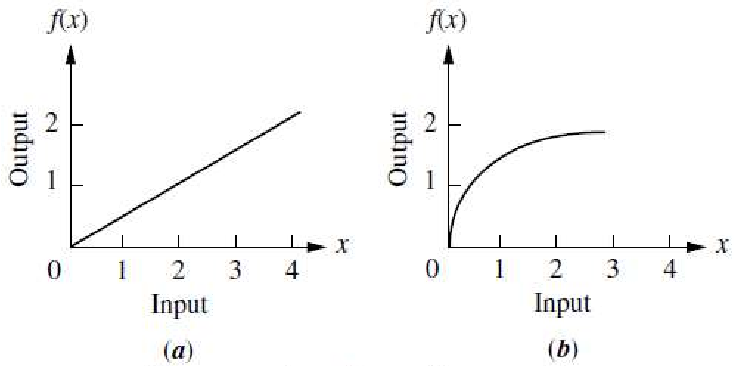
Nonlinearities

The models thus far are developed from systems that can be described approximately **by linear, time-invariant differential equations**

A linear system possesses two properties: **superposition** and **homogeneity**

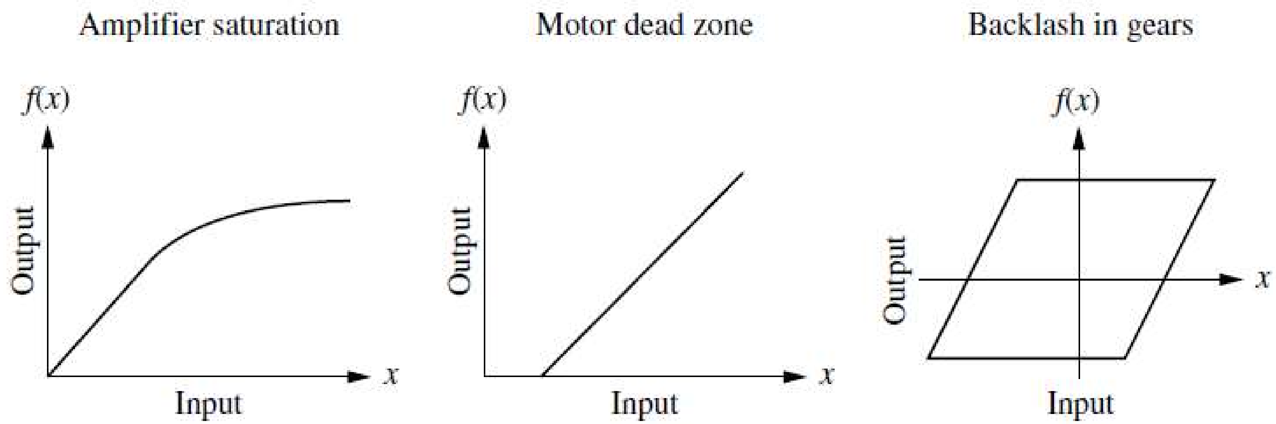
- $r_1(t)$ yields an output of $c_1(t)$ and an input of $r_2(t)$ yields an output of $c_2(t)$, then an input of $r_1(t) + r_2(t)$ yields an output of $c_1(t) + c_2(t)$.

- for an input of $r_1(t)$ that yields an output of $c_1(t)$, an input of $Ar_1(t)$ yields an output of $Ac_1(t)$



a. Linear system; **b.** nonlinear system

Physical nonlinearities



An electronic amplifier is linear over a specific range but exhibits the nonlinearity called saturation at high input voltages.

A motor that does not respond at very low input voltages due to frictional forces exhibits a nonlinearity called dead zone.

Gears that do not fit tightly exhibit a nonlinearity called backlash: The input moves over a small range but the output gear does not respond cause there is a gap between the teeth.

Linearization

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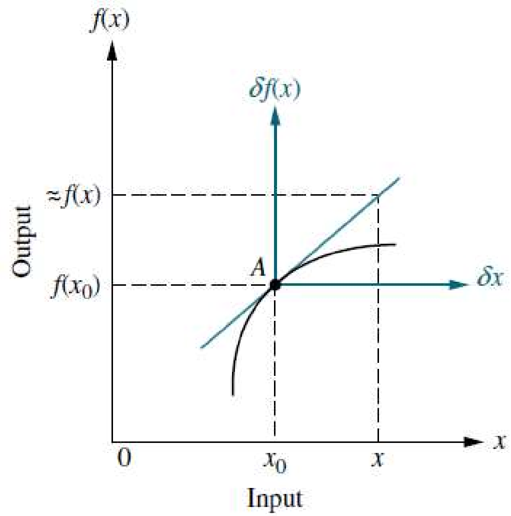
The first step is to recognize the nonlinear component and write the nonlinear differential equation.

If we assume a nonlinear system operating at point A, [x_0; f (x_0)]



small changes in the input can be related to changes in the output about the point by way of the slope of the curve at the point A.

Thus, if the slope of the curve at point A is m_a, then small excursions of the input about point A, δ_x, yield small changes in the output, $\delta f(x)$, related by the slope at point A.

$$[f(x) - f(x_0)] \approx m_a(x - x_0)$$

$$\delta f(x) \approx m_a \delta x$$

$$f(x) \approx f(x_0) + m_a(x - x_0) \approx f(x_0) + m_a \delta x$$

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Few examples:

Paper and pen!