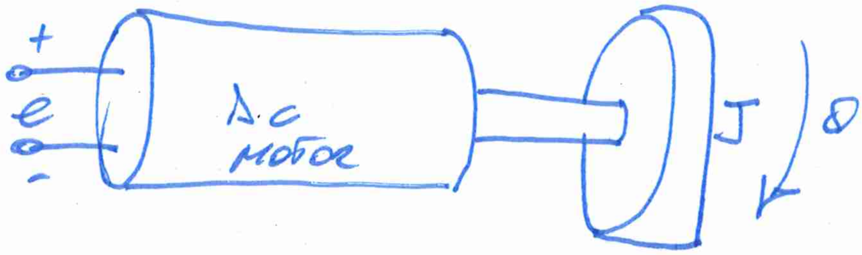
ELECTRIC MOTOR WITH INERTIAL LOAD

EXAMPLE 1



UNDER IDEAL ASSUMPTIONS

$$\begin{cases} \tau = k_1 i \\ v = k_2 \omega \end{cases}$$

k_1 = TORQUE CONSTANT

k_2 = BACK EMF CONSTANT

LET'S WRITE THE STATE SPACE EQUATIONS OF THE SYSTEM.

#DEMONSTRATION

1) THE ELECTRICAL POWER OF THE MOTOR IS:

$$P_e = V i = k_2 \omega \, \tau / k_1 \quad \#1$$

2) THE MECHANICAL OUTPUT POWER OF THE MOTOR IS:

$$P_m = \omega \, \tau \quad \#2$$

COMBINING #1 WITH #2 $\quad P_e = \dfrac{k_2}{k_1} P_m$

IF THE ENERGY CONVERSION IS 100% EFFICIENT $k_1 = k_2 = k$

BUT IN REAL OPERATION $k_2 / k_1 > 1$.

PAG → CONTINUES

#2)

#3 TO SPECIFY THE BEHAVIOR OF THE SYSTEM WE NEED THE RELATIONSHIP BETWEEN THE INPUT VOLTAGE "e" AND THE INDUCED "EMF", AND BETWEEN τ AND ω.

$$e - v = Ri \quad (3) \qquad \tau = J\frac{d\omega}{dt} \quad (4)$$

WE CAN COMBINE (1), (2), (3) AND (4)

$$J\frac{d\omega}{dt} = k_1 i = \frac{k_1}{R}(e - v)$$

$$J\frac{d\omega}{dt} = \frac{k_1}{R}e - \frac{k_1 k_2}{R}\omega$$

$$\frac{d\omega}{dt} = -\frac{k_1 k_2}{JR}\omega + \frac{k_1}{JR}e$$

FIRST ORDER EQUATION WITH ANGULAR VELOCITY "ω" AS STATE VARIABLE, AND "e" AS INPUT.

WE CAN POSE $\frac{d\theta}{dt} = \omega$

AND WE CAN OBTAIN THE STATE SPACE REPRESENTATION.

3)

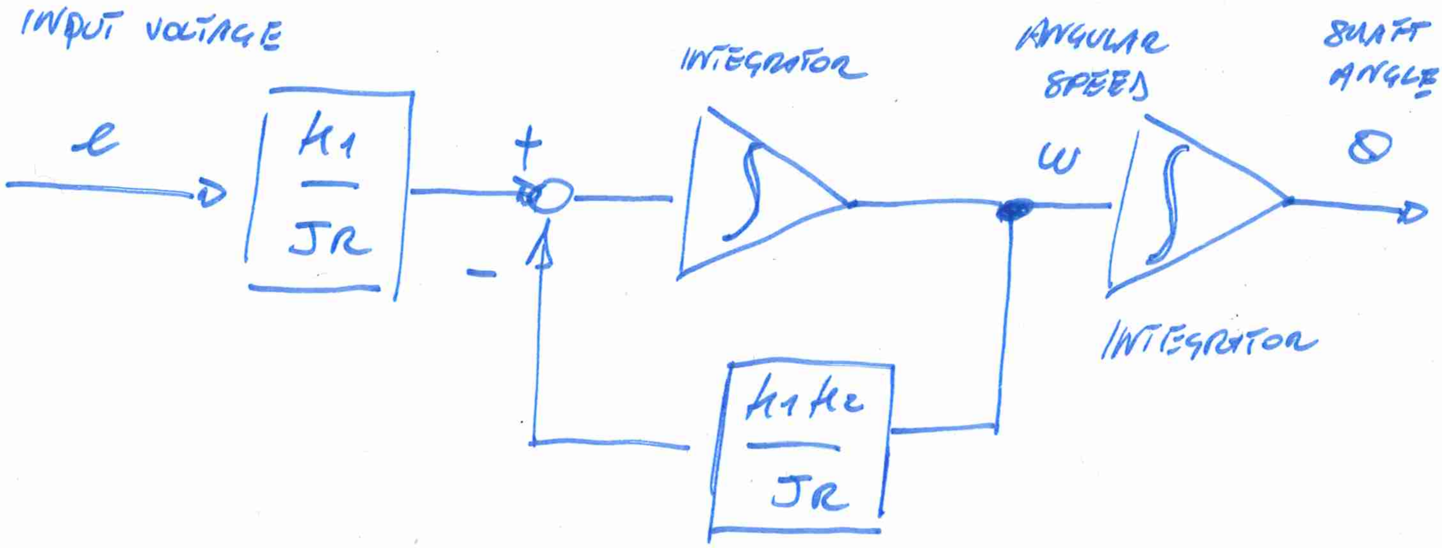
POSING $x_1 = \theta$ $x = \begin{bmatrix} \theta \\ \omega \end{bmatrix}$

$x_2 = \omega$

$$\frac{d}{dt}\begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -k_1k_2/JR \end{bmatrix}\begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ k_1/JR \end{bmatrix} e$$

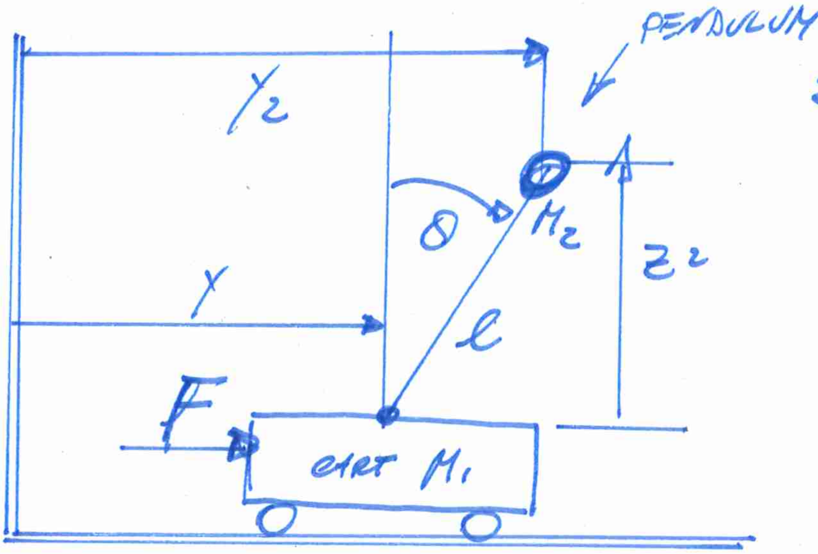
$$\boxed{\dot{x} = A x + B u}$$

WE CAN ALSO WRITE DOWN A BLOCK DIAGRAM



XAMPLE 2

INVERTED PENDULUM ON A MOVING CART



$\#$ WE WILL USE THE LAGRANGIAN FORMULATION

$$L = T - V$$

KINETIC POTENTIAL

ENERGIES

HE KINETIC ENERGY OF THE WHOLE SYSTEM IS DEFINED BY THE SUM OF ALL THE KINETIC ENERGIES

$$T_{CART} = T_1 = \frac{1}{2} M_1 \dot{Y}^2 \qquad T_{PEND} = T_2 = \frac{1}{2} M_2 (\dot{Y}_2^2 + \dot{Z}_2^2)$$

WE CAN EXPRESS

$$\begin{cases} Y_2 = Y + \ell \sin\theta \\ Z_2 = \ell \cos\theta \end{cases} \qquad \begin{cases} \dot{Y}_2 = \dot{Y} + \ell\dot{\theta}\cos\theta \\ \dot{Z}_2 = -\ell\dot{\theta}\sin\theta \end{cases}$$

$$T_{TOT} = \sum_i T_i = T_1 + T_2 =$$

$$= \frac{1}{2} M_1 \dot{Y}^2 + \frac{1}{2} M_2 \left[(\dot{Y} + \ell\dot{\theta}\cos\theta)^2 + \right.$$

$$\left. + \ell^2 \dot{\theta}^2 \sin^2\theta \right]$$

$$T_{TOT} = T_1 + T_2 = \frac{1}{2} M_1 \dot{y}^2 + \frac{1}{2} M_2 \left[\dot{y}^2 + 2\dot{y}\dot{\theta}\, \ell \cos\theta + \ell^2 \dot{\theta}^2 \right]$$

POTENTIAL ENERGY PENDULUM

$$V = M_2\, g\, z_2 = M_2\, g\, \ell \cos\theta$$

HENCE THE LAGRANGIAN IS:

$$\mathcal{L} = T - V = \frac{1}{2}(M_1 + M_2)\dot{y}^2 + M_2\, \ell \cos\theta\, \dot{y}\dot{\theta} + \frac{1}{2} M_2 \ell^2 \dot{\theta}^2 - M_2\, g\, \ell \cos\theta$$

WE SELECT (y, θ) AND WE CAN WRITE THE LAGRANGIAN IN TWO COORDINATES

DISSIPATIVE FORCES

$$\left.\begin{aligned} y)\quad & \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial \mathcal{L}}{\partial y} = F \\ \theta)\quad & \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \end{aligned}\right\} \#1$$

LET'S SOLVE OR WRITE ALL THE SINGLE TERMS.

$$\frac{dL}{d\dot{y}} = (M_1+M_2)\dot{y} + M_2 \ell \dot{\theta} \cos\theta$$

$$\frac{dL}{dy} = 0$$

$$\frac{dL}{d\dot{\theta}} = M_2 \ell \dot{y} \cos\theta + M_2 \ell^2 \dot{\theta}$$

$$\frac{dL}{d\theta} = M_2 g \ell \sin\theta - M_2 \ell \dot{y} \dot{\theta} \sin\theta$$

THE LAGRANGIAN #1 BECOMES:

$$\begin{cases} (M_1+M_2)\ddot{y} + M_2 \ddot{\theta} \ell \cos\theta - M_2 \ell \dot{\theta}^2 \sin\theta = F \\ M_2 \ell \ddot{y} \cos\theta + M_2 \ell^2 \ddot{\theta} - M_2 g \ell \sin\theta = 0 \end{cases}$$

\# LET'S APPROXIMATE FOR SMALL OSCILLATIONS.

$$\begin{cases} \cos\theta \simeq 1 & \ddot{y} \to \simeq 0 \\ \sin\theta \simeq \theta & \ddot{\theta} \quad \simeq 0 \end{cases}$$

7)

#2
$$\begin{cases} (M_1+M_2)\ddot{y} + M_2 l\ddot{\theta} = F \\ M_2\ddot{y} + M_2 l\ddot{\theta} - M_2 g\theta = 0 \end{cases}$$

DEFINING THE STATE VARIABLE $X \begin{bmatrix} y \\ \theta \\ \dot{y} \\ \dot{\theta} \end{bmatrix}$

$$\frac{dy}{dt} = \dot{y}; \quad \frac{d\theta}{dt} = \dot{\theta}$$

THEN WE CAN OBTAIN TWO MORE EQUATIONS SOLVING FOR $\ddot{y}$ AND $\ddot{\theta}$ THE #2

$$\begin{cases} \frac{d}{dt}(\dot{y}) = \ddot{y} = \frac{F}{M} - \frac{M_2 g}{M_1}\theta \\ \frac{d}{dt}(\dot{\theta}) = \ddot{\theta} = -\frac{F}{M_1 l} + \left(\frac{M_1+M_2}{M_1 l}\right) g\theta \end{cases} \qquad M = M_1 + M_2$$

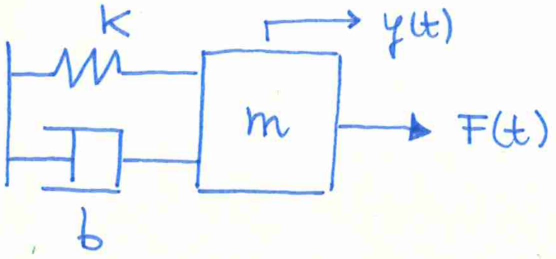
WHERE

$u = F$

HENCE WE CAN WRITE IN THE FORM $\dot{X} = AX + Bu$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -M_2 g/M_1 & 0 & 0 \\ 0 & (M_1+M_2)g/M_1 l & 0 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ 1/M_1 \\ -1/M_1 l \end{bmatrix};$$

###



$y(t)$ is the output
$F(t)$ is the input

$$m\ddot{y} + b\dot{y} + ky = F(t)$$ equation of motion

$$\ddot{y} = \frac{F}{m} - \frac{b}{m}\dot{y} - \frac{k}{m}y$$

State-space variable choice:

$$\begin{cases} x_1 = y \\ x_2 = \dot{y} \end{cases} \xrightarrow[\dot{x}]{} \begin{cases} \dot{x}_1 = x_2 = \dot{y} \\ \dot{x}_2 = \ddot{y} = \dfrac{F}{m} - \dfrac{b}{m}\dot{y} - \dfrac{k}{m}y \end{cases}$$

Vector-matrix form:

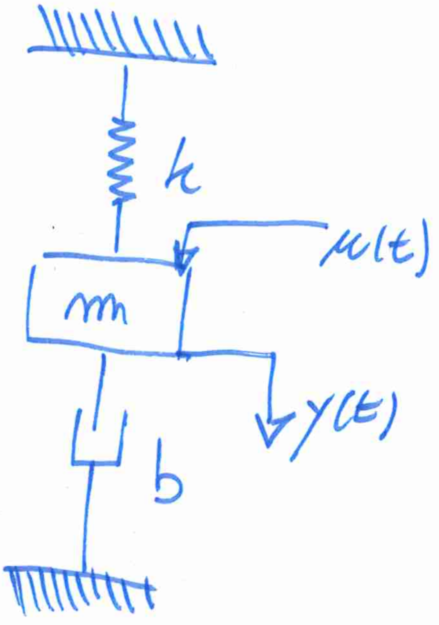
$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B} \underbrace{F}_{u}$$

$$\dot{x} = A \quad x + B \quad u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$y = C \quad x$$

EXAMPLE 9



CONSIDER THE SYSTEM IN FIGURE
WE ASSUME AT $t > 0$ A FORCING
INPUT $u(t)$ IS APPLYING

$$m\ddot{y}(t) + b\dot{y}(t) + k\,y(t) = u(t)$$

WE CAN DEFINE THE STATE
COORDINATES:

$$\begin{cases} x_1(t) = y(t) \\ x_2(t) = \dot{y}(t) \end{cases}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{1}{m}\left(-ky - b\dot{y}\right) + \dfrac{1}{m}u \end{cases}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{k}{m}x_1 - \dfrac{b}{m}x_2 + \dfrac{1}{m}u \end{cases}$$

STATE SPACE EQNS.

OUTPUT

$$y = x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix}}_{\bar{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\bar{x}} + \underbrace{\begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix}}_{\bar{B}} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\bar{C}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\bar{x}}$$

$$\begin{cases} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ y = \bar{C}\bar{x} + Du \end{cases}$$

#