Edmond Saliklis

Structures: A Geometric Approach

Graphical Statics and Analysis

Springer

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This book is dedicated to my lovely wife Ruta,
to whom I owe everything.

Preface

Viewing the world of statics and structural analysis through the prism of graphical techniques is truly transformative. Students who have been introduced to these techniques are empowered by the elegance and ease of performing structural calculations graphically. Practitioners who have embraced these methods, from the Nineteenth century to the present, have created efficient structural forms that express the flow of forces. Present-day researchers at the PhD level have made enormous strides on three-dimensional graphic statics, and this has led to stunning compression-only shells constructed with masonry or with reinforced concrete.

Even with the newfound energy and interest surrounding this centuries-old technique, very few structures courses around the world focus on graphical techniques. Some may ask “if graphic statics is so good, why isn’t it taught anymore?” The answer can be found in the simultaneous changes that have occurred in structural pedagogy over the past 50 years. More and more efforts have been placed on computational and matrix methods; less and less attention has been given to visual thinking and hand drawing. This combination has led to the near disappearance of graphical techniques from structures courses.

However in the Twenty-first century this is certainly going to change. The availability of so many drawing packages that have programmable features in 2D and 3D makes graphic statics an extremely attractive tool today. Students can now immerse themselves in visual thinking in a way that previous generations could not have experienced because the tools are so approachable and intuitive. They can even graphically solve these problems on their mobile phones. This will lead to revitalization of how structural courses are taught around the world.

The techniques presented in this book can be used in introductory courses to architecture and engineering students, and they can be studied at the graduate level as well. The methods are exact and rigorous, yet they are full of visual insights and surprising aesthetic power. The drawings can be personalized, and the logic used to solve problems is fun and energizing. Graphic statics can bring together left brain and right brain thinking and can reunite the long lost siblings of architecture and engineering.

San Luis Obispo, CA, USA — Edmond Saliklis

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Wise and helpful guides have marked the path of this book's creation. Masoud Akbarzadeh has been extremely influential; his patience and counsel are deeply appreciated. Edward Allen has been endlessly supportive. William Baker provided much needed guidance and encouragement. Allan McRobie has been an inspiration to me. Phillipe Block and Corentin Fivet have been gracious and supportive throughout. The wonderful professional society IASS has been instrumental in my progress, allowing me to share ideas with brilliant people all around the world. Graham Archer has patiently provided advice and feedback. I am thankful for the trust that Paul Drougas had in me to create this book. My students at California Polytechnic State University have helped me in researching various aspects of graphic statics; they include Evan Gerbo, Jonas Houstan, Jared Parker, Kion Nemati, and Sydney Gallion. Finally, my hero David Billington has influenced me beyond measure.

Contents

1 Introduction

This textbook was written for students and practitioners and professors who are interested in learning how to use graphic statics, after having already studied statics and elementary mechanics of beams. The history of graphic statics is a long and rich story, with a baffling mystery thrown in during the mid-twentieth century. Many papers have recently been written during the current "renaissance" of graphic statics research that is occurring today. These papers neatly summarize the traditional arc of the story of graphic statics, starting with Varignon, moving through Cremona, Cullman and others. Graphic statics was part of every structural engineer's curriculum up till the 1930s or so, and those engineers became comfortable with very accurate drawing solutions, and they were able to grasp geometric principles readily. Yet, as detailed algebraic solutions became more and more widespread in the mid-twentieth century, and certainly as the computer made inroads into the structural engineer's toolkit, graphic statics rapidly disappeared from almost all structural engineering curricula around the world. Today, most structural engineers have no first-hand knowledge of graphic statics, even if they have heard of the technique because of the recent spate of scholarly articles on the topic. This mystery of the sudden demise of graphic statics education, can be partially explained by the rise of intricate algebraic methods, and the reliance on computers. Yet, that does not fully explain how something so beautiful and profound disappeared so quickly.

The science of graphic statics is an exact method for solving two dimensional and three dimensional equilibrium problems. The power of graphic statics manifests itself when such analysis methods are then used as design methods, by means of manipulating the geometry of the problem. Such iterations of geometry are extremely rapid if the drawings are done on the computer in an environment that allows for manipulation of parameters. In Chap. 2, General Guidelines, such a drawing environment is introduced, with suggestions of how to order and formalize drawings.

Concurrent Co-planar Forces

The study of graphic statics in this textbook begins with co-planar forces, i.e. forces that can be modeled in two dimensions. A force is a vector, distinguished by three characteristics; magnitude, direction of action, point of application. The starting point of a vector is called its origin, and the end point with the arrow is known as the terminus. In Fig. 1.1, two co-planar forces are acting compressively on a point, i.e. each vector's terminus touches the same point. The force that makes a known angle α with the horizontal has a known magnitude of F. A second force is twice the magnitude of

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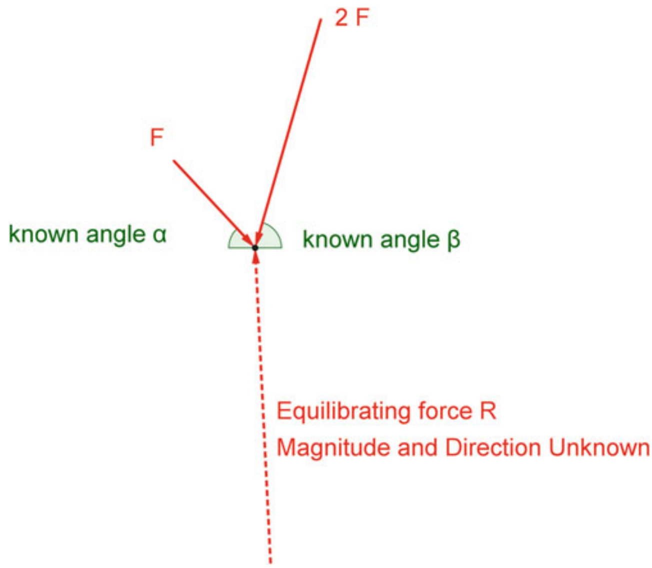
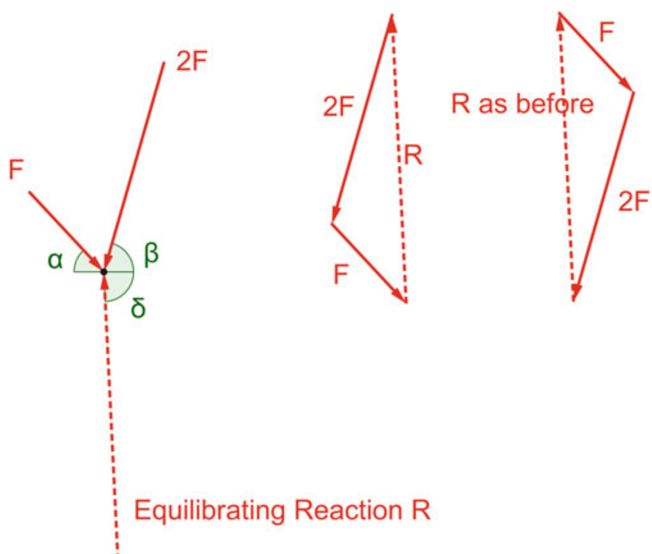
the first force, and it makes a known angle β to the horizontal. What force R would equilibrate the point being loaded by F and $2F$? This unknown force is shown as a dashed vector in Fig. 1.1, but of course its magnitude and direction is only hinted at in Fig. 1.1, as they are both still unknown.

Fig. 1.1 Form diagram with two unknowns



Algebraically or graphically, this problem has two unknowns, the magnitude and direction of the equilibrating force. It is clear that the terminus of the equilibrating force must be applied to the point being loaded by F and $2F$. The equilibrating force R must have a magnitude equal to the resultant of the two applied forces, and it must align with this resultant, with an opposite direction. It is good practice to consider the original problem as the Form Diagram, i.e. a depiction of the geometry of the original problem. Here the form is rather simple, only a point being loaded, but the directions of the two loads is essential information that is captured on the Form Diagram. Force Diagrams corresponding to an equilibrium solution of the Form Diagram are shown in Fig. 1.2.

Fig. 1.2 Force diagrams show equivalent solutions

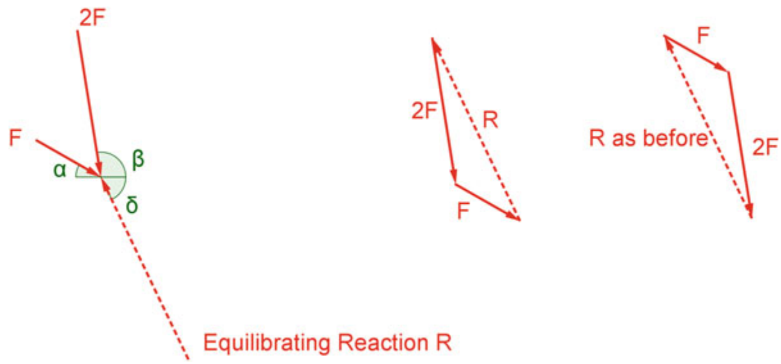


Two solutions are provided because the order of operations does not matter. The terminus of F can be applied to the point, and then the terminus of $2F$ can be applied to the origin of F. Or the opposite order can occur, the terminus of $2F$ can be applied to the point and the terminus of F can be applied to the origin of $2F$. Either way, the resultant R is found.

Drawing the force diagram to some arbitrary scale is a critical idea. The scale does not matter, but the force vectors must be drawn to a known, comfortable scale. The magnitude of the resultant is the length of the dashed line on the right side of Fig. 1.2, the Force Diagram, multiplied by this scale factor. In summary, the key ideas of Fig. 1.2 are the Form Diagram on one side of the page, and a Force Diagram on the same page. Both diagrams are drawn to independent scales, but this initial example had a simple Form Diagram so its scale was not really an issue.

Figure 1.3 demonstrates some of the power of creating these geometric solutions in a programmable drawing environment. Changing the magnitude of the force F, and changing the angles α and β presents no difficulty in an immediate re-solving for the equilibrating force R.

Fig. 1.3 Immediate update of force diagram when input is varied



Another elementary equilibrium of a point example shows a few more graphic statics ideas. In this problem, shown in Fig. 1.4, one force is compressing a point on the ground. The question is to determine what are the x and y components of the equilibrating reaction.

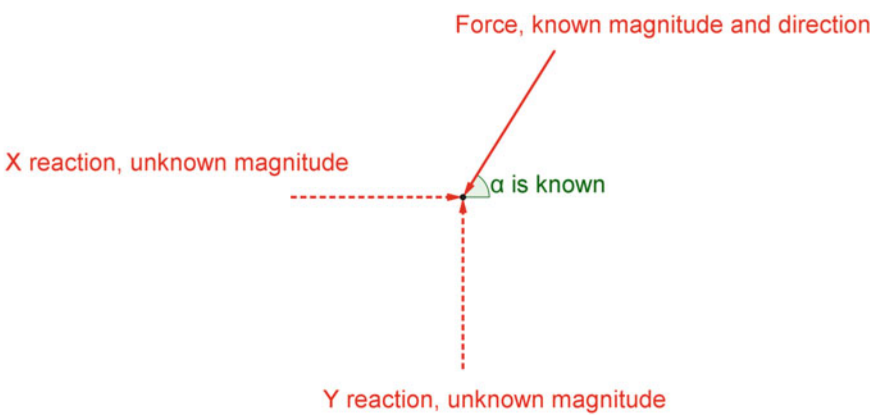


Fig. 1.4 Unknown magnitudes of X reaction and of Y reaction

Once again, consider the original problem drawing to be the Form Diagram, and create a Force Diagram to some suitable scale to solve the problem. In the right side of Fig. 1.5, the original force vector is drawn to a convenient scale, and a vertical line is passed through its terminus, and a horizontal line is passed through its origin.

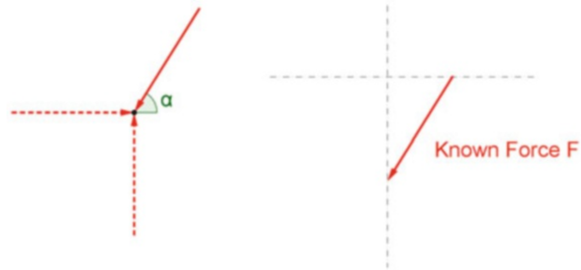


Fig. 1.5 First step to solve for reaction X and reaction Y

The vertical *Y* reaction must align with this vertical line, and the horizontal *X* reaction must align with the horizontal line as shown in Figs. 1.5 and 1.6. If the original loading force had 1000 units, and if its angle to the horizontal was 58°, then the *X* reaction is 530 units of force, and the *Y* reaction is 848 units of force as shown in Fig. 1.6.

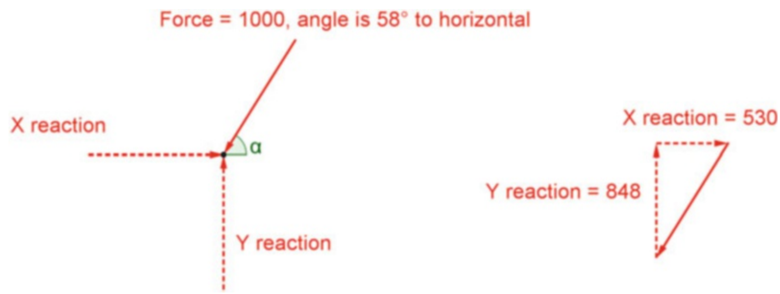


Fig. 1.6 Final solution for reaction X and for reaction Y

The previous problem can be changed somewhat, to introduce two applied co-planar loads that must be equilibrated by distinct X direction and Y direction reactions. Figure 1.7 shows this problem qualitatively.

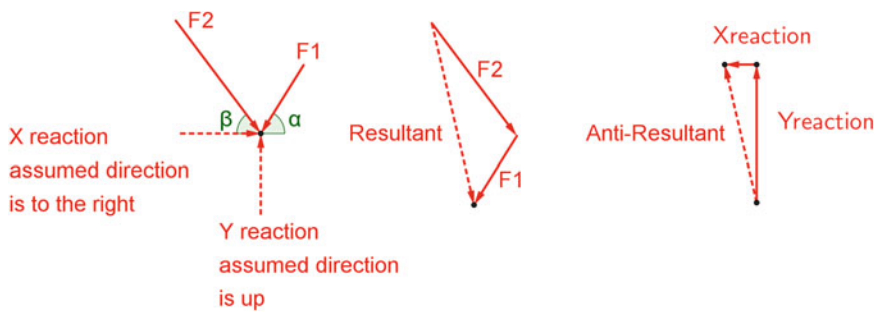
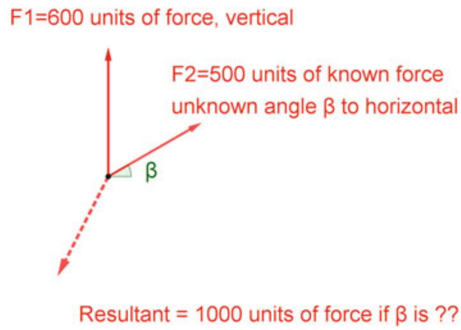


Fig. 1.7 Two known forces and two unknown reaction magnitudes

Here, one load has a given magnitude and it makes an angle α to the horizontal. A second load has a different magnitude and it makes an angle β to the horizontal. As in Figs. 1.2 and 1.3 a force diagram is drawn to the side of the original form diagram. The first step is to calculate the resultant of the two applied loads. The second step is to break up the anti-resultant, shown as a dashed vector, into *X* and *Y* components. The reactions in the *X* and *Y* directions will be the two components of the anti-resultant.

Another example seeks a different unknown. In this example, a vertical load of 600 units of force is pulling on a point. A second load of 500 units of force is pulling at the same point, but this second force makes an unknown angle β to the horizontal. What angle β would induce a reaction of 1000 units of force? The problem is shown in Fig. 1.8

Fig. 1.8 Magnitudes of forces and reaction are known but geometry is variable



One way of solving this graphically within a parametric drawing environment is start at some angle β, create the force diagram, then slowly change the angle β till the resultant exactly matches the required value of 1000 units of force. To actually implement the steps in a drawing environment, refer to Chap. 2 General Guidelines. For now, look at Fig. 1.9 and see if the logic is clear and unambiguous.

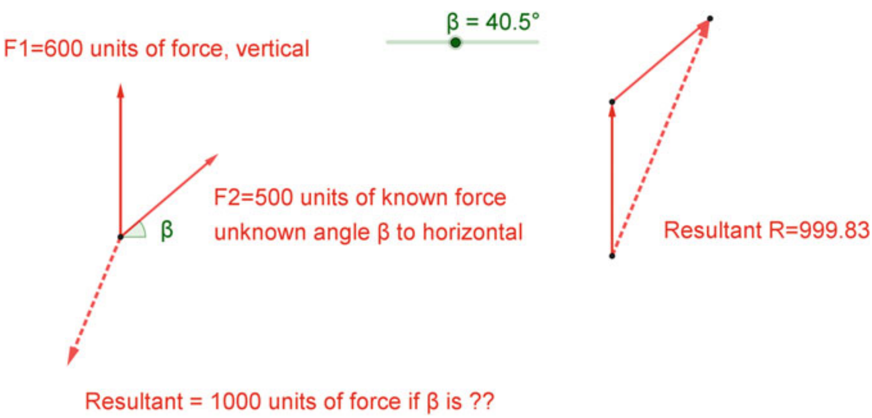


Fig. 1.9 Iterative solution found by varying angle β

If more than two co-planar forces are acting concurrently at a point, a force polygon should be created and the resultant is found from the final force's terminus, to the first force's origin. The order of operations does not matter.

In Fig. 1.10, a column shown in brown, is fixed at its base, and at its top it supports four tensile cables meeting at point A. The cables hold up a roof shown in blue. The force in each cable is known, and of course, the angles are known also because the original geometry is given. Cartesian coordinates of the five critical points are shown. Find the magnitude and direction of the reaction at the top of the column.

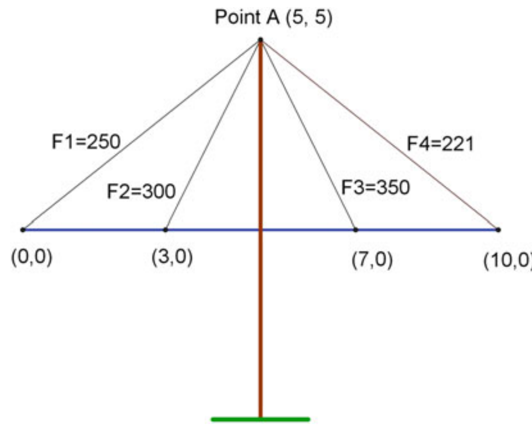


Fig. 1.10 One column supporting four cable loads

A beautiful insight occurs at the onset of this problem. The Cartesian coordinates are not needed! The form diagram in this problem is more complicated than previous examples, but the power of graphic statics is such that transferring parallel lines from the form diagram to the force diagram ensures that the inclinations of the four cable forces will be correct.

A force polygon is created as the force diagram. The order of application is inconsequential, but note that all force are pulling on Point A (Fig. 1.11).

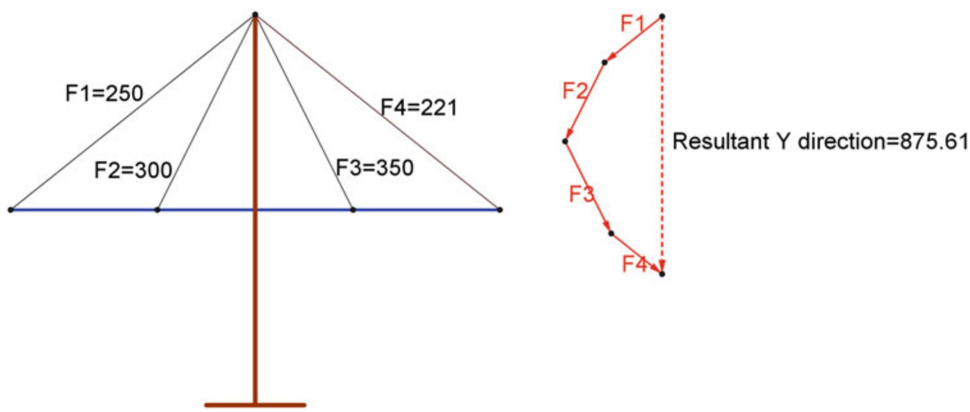


Fig. 1.11 Solution to four cable problem

The previous problem could not have had a pinned-base column. Why not? And what happens if the resultant is not vertical?

Non-concurrent Co-planar Forces

When two or more co-planar forces are applied such that they do not coincide on a point, they are called non-concurrent forces. Such forces induce a moment about some axis perpendicular to the plane of the forces. In this section, some classical ideas about such non-concurrent forces are established, then the powerful idea of the "funicular" is introduced. These ideas are presented qualitatively here. After reading Chap. 2, the user can re-create these exercises quantitatively.

For example, two co-planar parallel forces, both pointing downwards (i.e. having the same sense) are some known distance apart. To solve for the resultant of these two forces necessarily requires the magnitude and location of the equivalent force (Fig. 1.12).

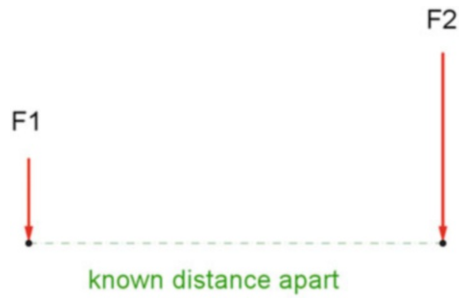


Fig. 1.12 Two known non-concurrent forces

The first technique used to solve this problem is a classical one described by Wolfe in his 1921 textbook, called the "inverse axis method". This method is handy if seeking the resultant of two loads. Yet it also provides insights into a more generally useful technique. At the original location of *F1*, draw a scaled vector *F2*. Then at the original location of *F2*, draw a scaled vector *F1* using the same scale factor used for *F2*. Straight lines are drawn from the terminus of one vector to the origin of the other vector. Two such lines are drawn and the intersection is the location of the centroid of the loads. The resultant force, here *F1+F2* is passed through this centroid. This is shown in Figs. 1.13 and 1.14.

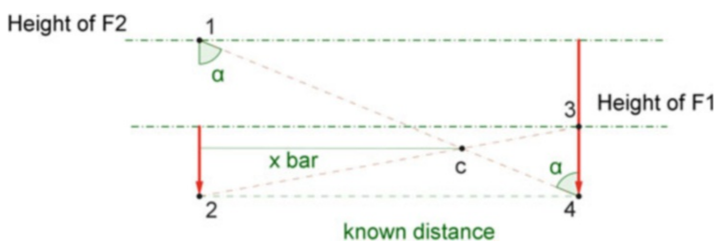


Fig. 1.13 Inverse axis method to find resultant of co-planar forces

In Fig. 1.13, the height of the vector representing F2 is established from Point 1 to Point 2. The height of the scaled vector representing F1 is established from Point 3 to Point 4. Of course, both vectors are drawn to the same scale. Now by similar triangles, $\frac{c\ to\ 4}{3\ to\ 4} = \frac{c\ to\ 1}{1\ to\ 2}$ therefore $(c\ to\ 4) \cdot (1\ to\ 2) = (3\ to\ 4) \cdot (c\ to\ 1)$

Yet it is known that moving from 1 to 2 captures F2, and moving from 3 to 4 captures F1, thus:

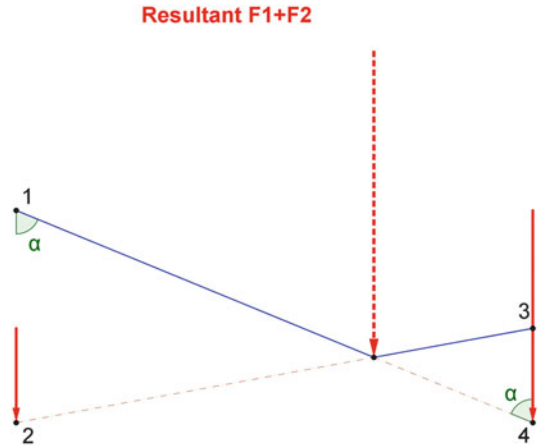
$$(c\ to\ 4) \cdot F2 = F1 \cdot (c\ to\ 1)$$

Looking at Fig. 1.13, it is clear that $(c\ to\ 4) \cdot F2 \cdot \sin(\alpha) = Moment\ of\ F2\ about\ c$

And that $(c\ to\ 1) \cdot F1 \cdot \sin(\alpha) = Moment\ of\ F1\ about\ c$. Equating these two expressions and seeing that *sin(α)* cancels reveals that $Moment\ of\ F2\ about\ c - Moment\ of\ F1\ about\ c = 0$. This is as it should be, passing the resultant of loads through the centroid of the loads results in no moment about that point.

In Fig. 1.13, the kinked line *1-c-3* is interesting to investigate, and it leads to the more general discussion of funiculars.

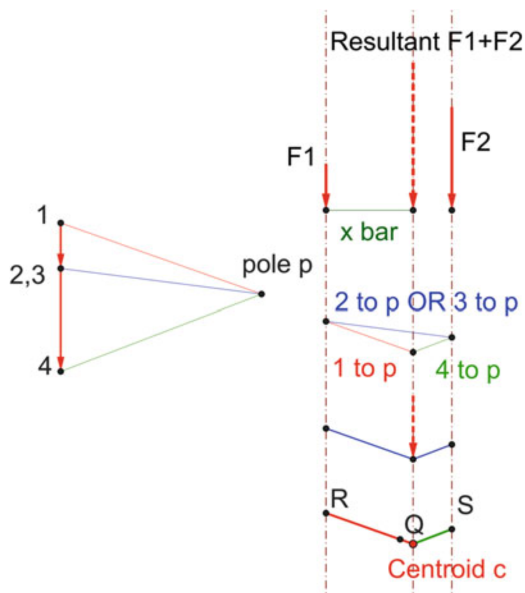
Fig. 1.14 The first look at a funicular



Taking line *1-c-3* from Fig. 1.13, and making it blue in Fig. 1.14 and superimposing the resultant of *F1* and *F2* shows the magic of the funicular! The blue line is the shape that a cable would take under a single concentrated load. In almost all of the subsequent chapters, this concept will be used again and again. Here, the concepts are presented somewhat qualitatively so that the user need not get bogged down in the drawing details. Such details will be presented in Chap. 2. For now, look at Fig. 1.14, and imagine any number of possible solutions, each solution having a different "length of cable" but each solution has the kink precisely at the centroid of the two loads.

Figure 1.15 may seem very complicated at first glance. It deserves further study later after some quantitative experience with the creation of drawings.

Fig. 1.15 A pole establishes rays of the funicular



Beginning with the force diagram on the left side, a vertical line is used to draw the two vertical force vectors *F1* and *F2* to some scale. The distance from *1* to *2* on this vertical line captures the magnitude of *F1* to some scale. *F2* begins where *F1* ends, thus point *2* coincides with point *3* here.

Then point *3* to point *4* captures the magnitude of *F2*. Next, an arbitrary pole *p* is drawn somewhere to the right of this vertical load line. Rays are drawn from *1* to *p* (in red), from *2* to *p* (or *3* to *p*) (in blue) and from *4* to *p* in green. A line parallel to *1p* is drawn, starting at some point *Q* between the two loads. Where this line intersects the path of *F1* mark point *R*. Next, the slope of the blue line (*2p* or *3p*) is copied, starting at *R* until it intersects the load path of *F2*, this point is marked as *S*. Finally, the slope of *4p* is copied, starting at point *S* until it intersects the first line *QR*. This point of intersection is marked as the centroid *c*, and one can see that it aligns perfectly with the location of the resultant load. This funicular is redrawn more simply as a kinked blue line directly above the *QRS* construction. The same *QRS* construction is redrawn as simple red, blue and green segments in the drawing above the blue funicular. Subsequent constructions of the funicular will be much simpler than this, this example is meant to show the underpinnings of the relationship between the funicular and the centroid of co-planar parallel loads.

If two parallel loads are non-concurrent, and they have opposite senses, the inverse axis method can still be used to locate the centroid of the two loads, but the following changes must be implemented (Fig. 1.16).

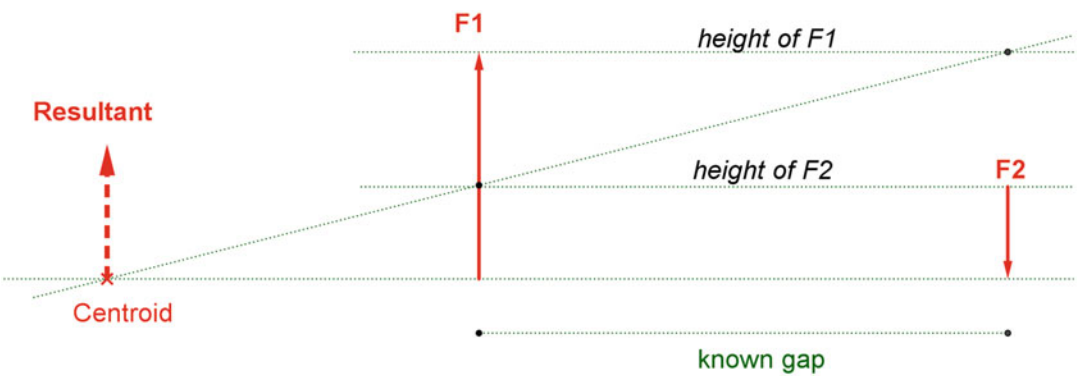
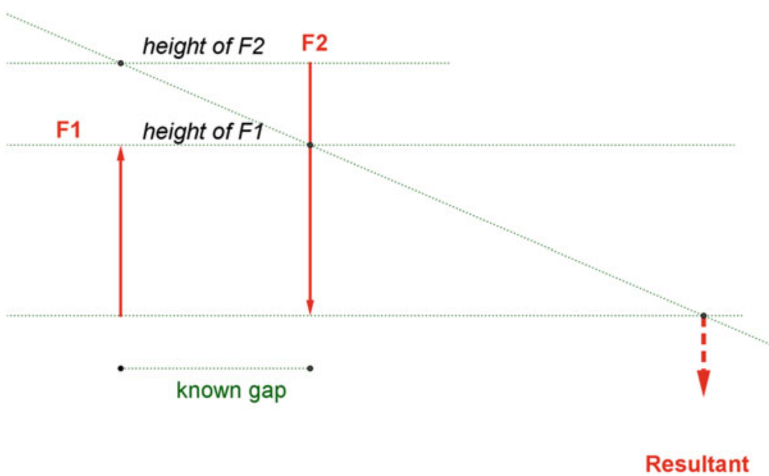


Fig. 1.16 Resultant of two non-concurrent forces with opposite senses

The resultant will lay on the outside of the larger force. Figure 1.17 shows different magnitudes and senses of the two forces. Here, the centroid of the loads lies to the right.

Fig. 1.17 Resultant lays outside of the larger force



When a force is in 3D, the techniques remain the same to decompose the known force into components along some axis. For example, a known force is shown in Fig. 1.18. The components along the X, Y and Z axes of this force are sought.

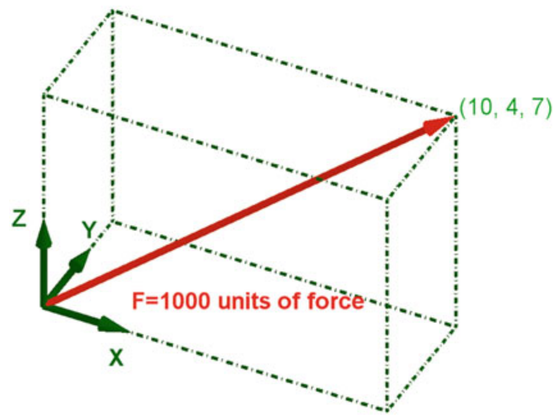


Fig. 1.18 Decomposing a 3D force into orthogonal components

The force scale used to interpret the original force vector is important to grasp. It was drawn to some size on the page, and coordinates (0, 0, 0) and (10, 4, 7) were used to represent it. But it is a force, not a distance, thus

$$ForceScale = \frac{Magnitude\ of\ Force}{Length(represented\ vector)}$$

In this example, the magnitude is 1000 and the length of the vector used to represent the force F is 12.85. Thus

$$ForceScale = \frac{1000}{12.85} = 77.85$$

Projecting components of a force is perhaps more easily understood if a triangle is drawn in 3D space. The original force is always the hypotenuse of such a triangle, and the components are smaller than the original force. Figure 1.19 shows Fx which is quantified as

$$F_x = Length\ of\ Leg \cdot ForceScale = 778.5$$

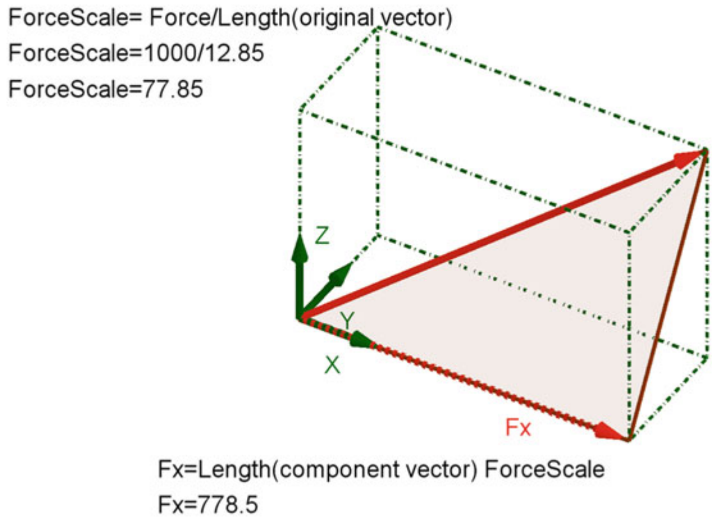


Fig. 1.19 Force scale relates length to magnitude of force

To highlight the interplay between the coordinates (10, 4, 7) and the magnitude of the force, consider the same orientation of force, but now suppose this force is 675 units of force (Fig. 1.20). What are its X, Y, Z components?

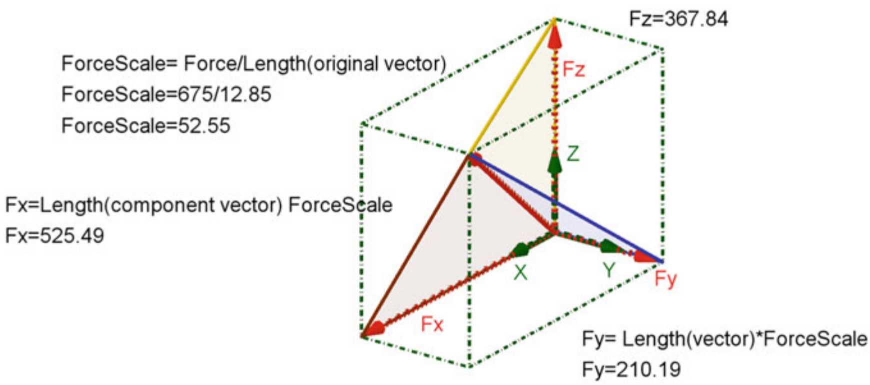


Fig. 1.20 One force broken in to three components

Finding the component of a force along some arbitrary axis *q-q* uses the exact same technique. Figure 1.21 shows this problem. But caution must be used! Improper constructions that do not use the original force as the hypotenuse of a triangle cannot be created. Figure 1.21 shows such an incorrect construction.

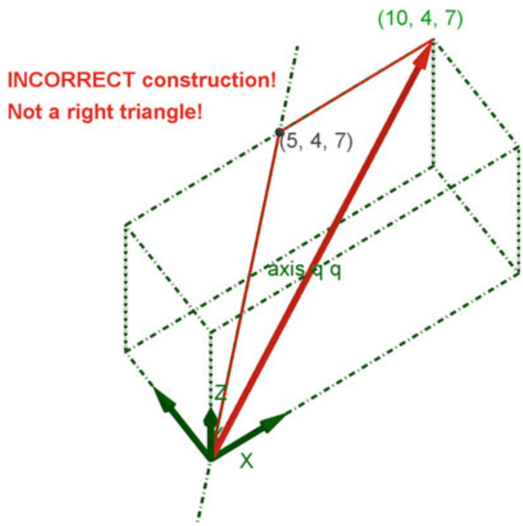


Fig. 1.21 Components in 3D are not constructed arbitrarily

One method of correctly constructing the right triangle needed for the solution is shown in Fig. 1.22. A plane perpendicular to axis qq is passed through point that is the terminus of the original force F. The intersection of that plane with axis qq establishes the terminus of the vector Fqq, which is a component of the original force F.

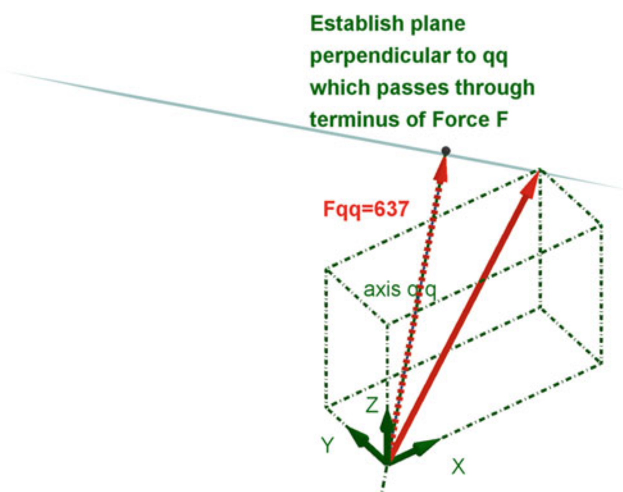


Fig. 1.22 Establishment of plane perpendicular to sought-after component

Forces can be added vectorially in 3D space. Figure 1.23 shows two forces in 3D space that are connected "head to tail". The origin of one vector is the terminus of another vector.

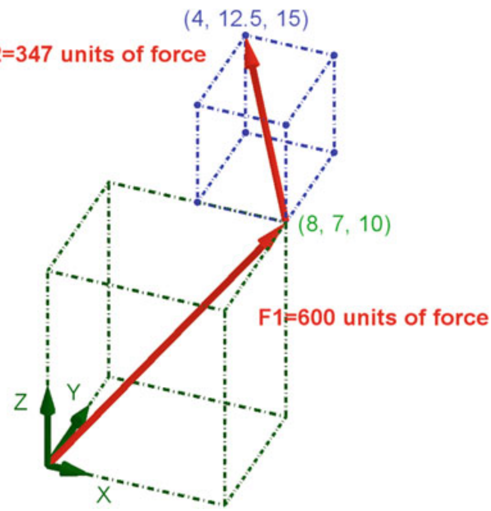


Fig. 1.23 Graphical addition of vectors in 3D

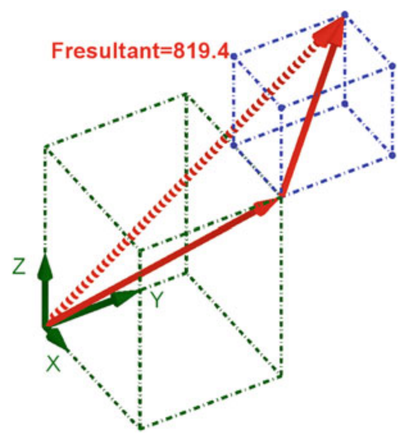


Fig. 1.24 Graphically finding the resultant of vectors in 3D

The problem presented in Figs. 1.23 and 1.24 is extended in Fig. 1.25. Here, the problem is: given a third force $F3 = 650$ units of force, that begins at the terminus of $F2$, what is the orientation of $F3$ such that the resultant is vertical? To create a vertical resultant, the terminus of $F3$ must clearly lie on $X = 0$ and $Y = 0$, but what Z value is needed?

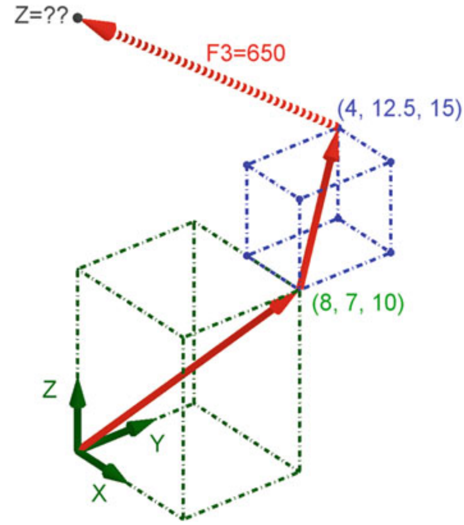


Fig. 1.25 3D problem with one remaining unknown

One way of solving this problem is to create a sphere at the terminus of F2, a sphere of radius 650/*ForceScale* (Fig. 1.26). Where the sphere intersects a vertical line through X = 0, Y = 0 provides the solution. Here, two solutions exist!

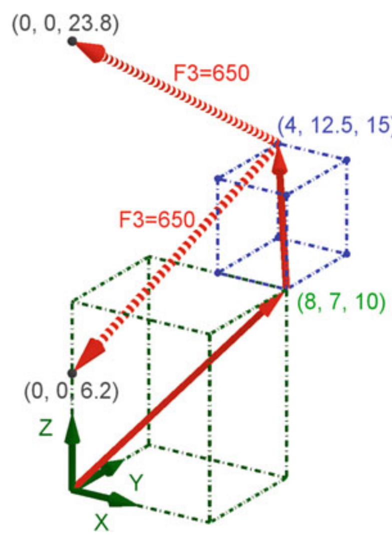
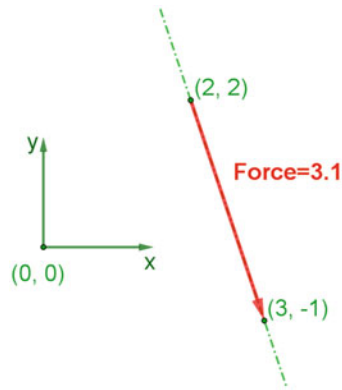


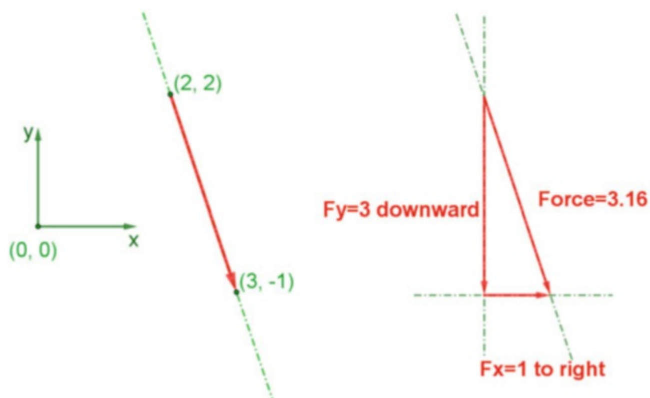
Fig. 1.26 Two solutions exist which ensure verticality of resultant

Chapter 1 Exercises

Exercise 1.1 The magnitude and direction of Force F are shown. Determine the x and y components of F.

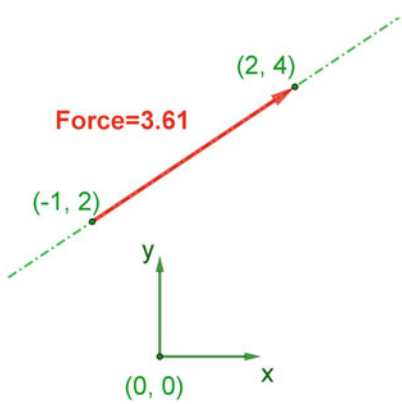


Exercise 1.1 solution

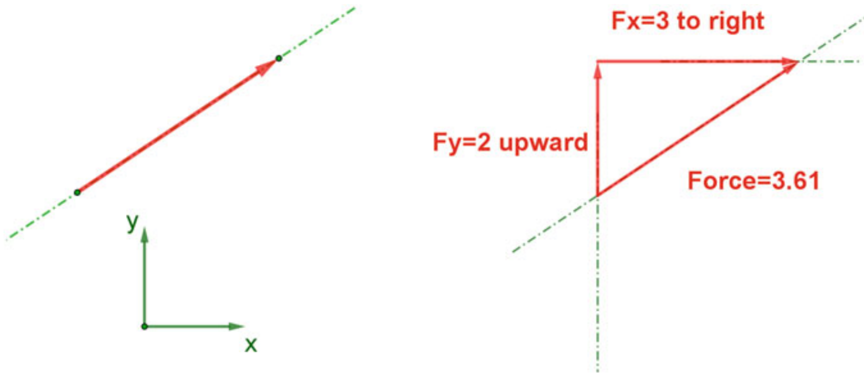


The coordinates are never needed, but could be used for an algebraic check.

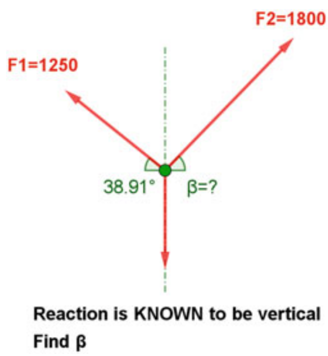
Exercise 1.2 The magnitude and direction of Force F are shown. Determine the x and y components of F.



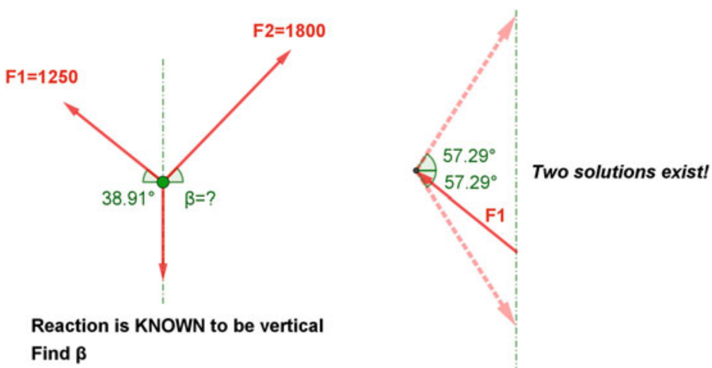
Exercise 1.2 solution



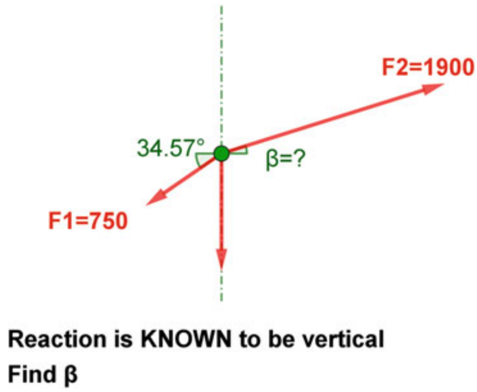
Exercise 1.3 The magnitudes of F1 and F2 are known. The direction of F1 is known. Given that the reaction is vertical, find β.



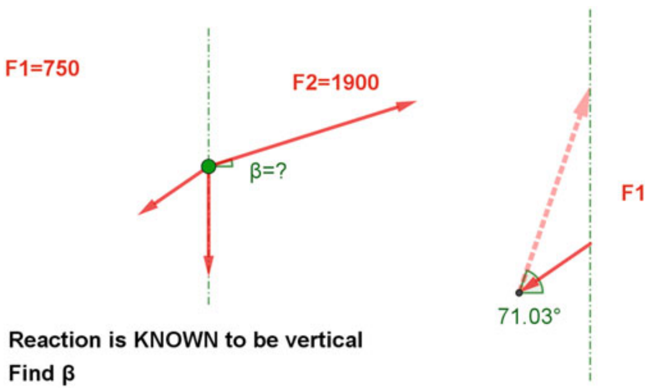
Exercise 1.3 solution



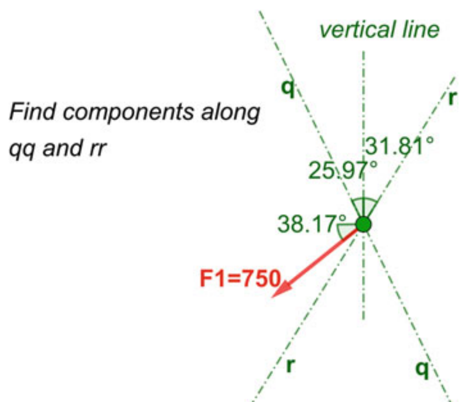
Exercise 1.4 The magnitudes of F1 and F2 are known. The direction of F1 is known. Given that the reaction is vertical, find β.



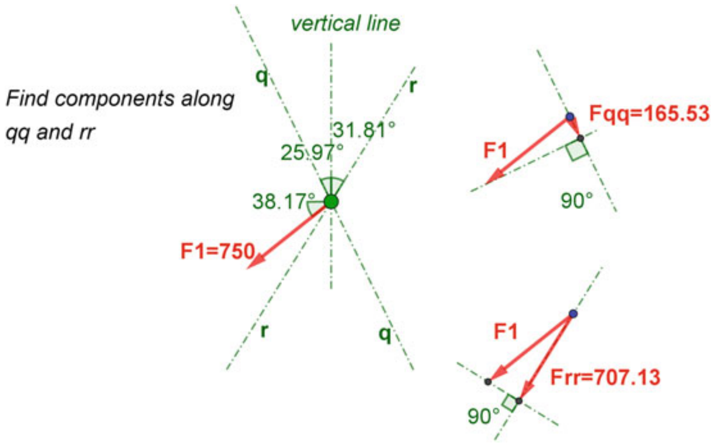
Exercise 1.4 solution



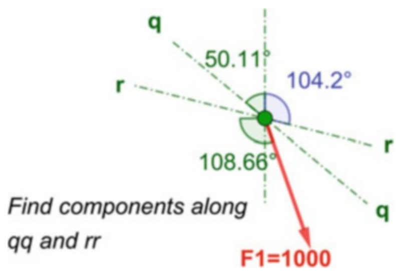
Exercise 1.5 Given the magnitude and direction of the 2D Force F1, find its components along co-planar axes qq and rr. These two axes are arbitrary, they are not perpendicular to each other. This is a 2D problem.



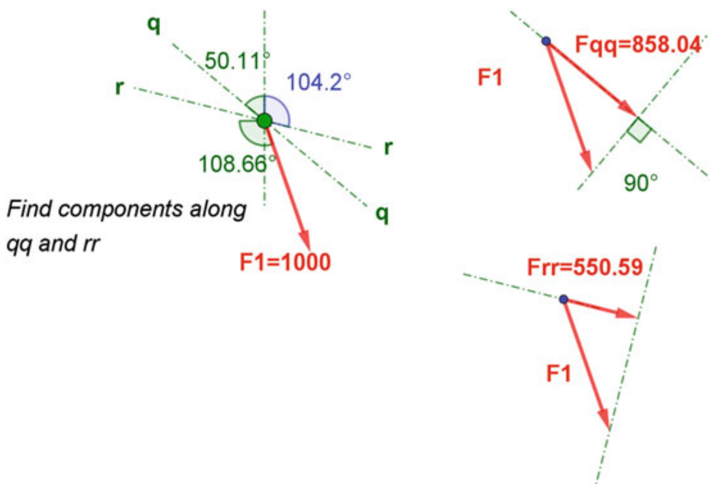
Exercise 1.5 solution



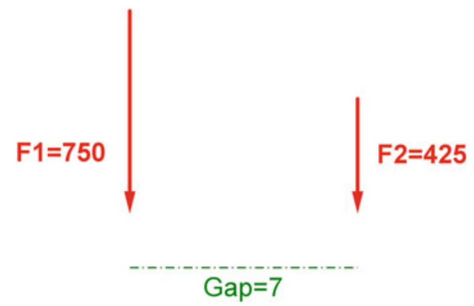
Exercise 1.6 Given the magnitude and direction of the 2D Force F1, find its components along co-planar axes qq and rr. These two axes are arbitrary, they are not perpendicular to each other. This is a 2D problem.



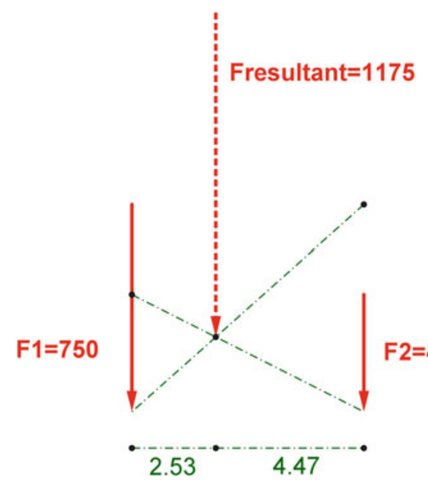
Exercise 1.6 solution



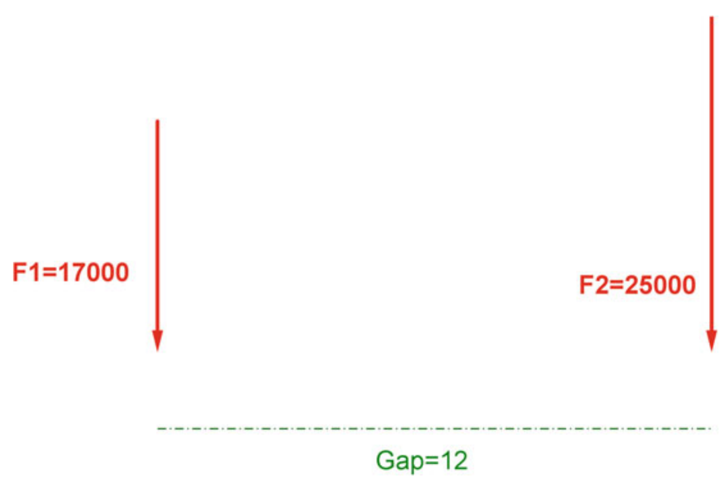
Exercise 1.7 Given the following two loads and the distance between them, calculate the magnitude and the location of the resultant. Clearly the resultant will be vertical since both loads are vertical.



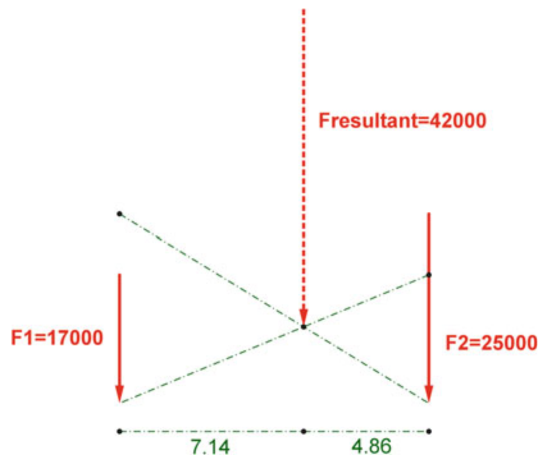
Exercise 1.7 solution Using the inverse-axis method allows for the location of the resultant. The magnitude of the resultant will be F1+F2



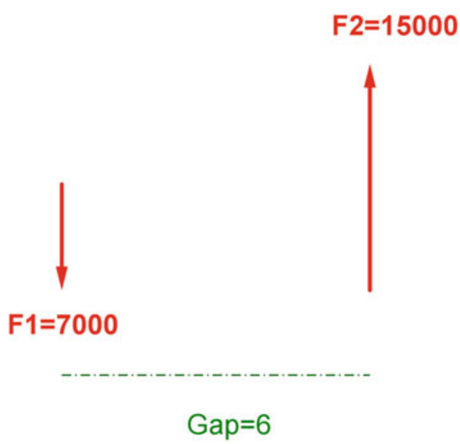
Exercise 1.8 Given the magnitude of two vertical loads, and the gap between them, locate the magnitude and location of the resultant.



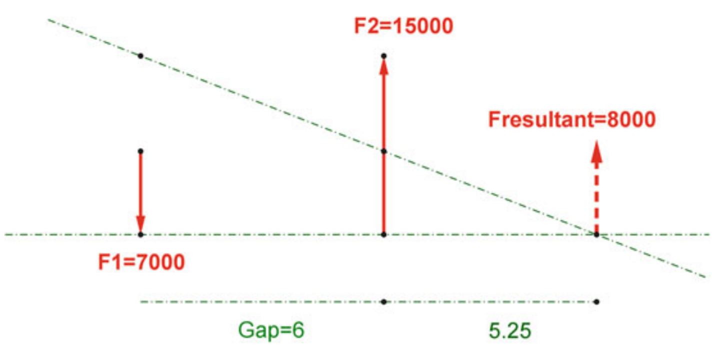
Exercise 1.8 solution



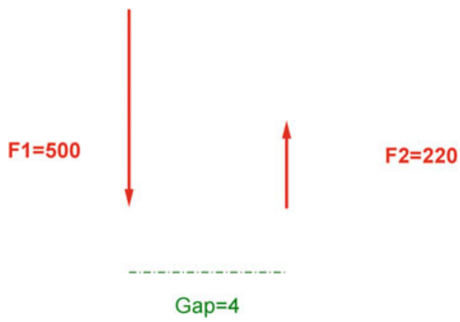
Exercise 1.9 Given the magnitude of two vertical loads of opposite sense, and the gap between them, locate the magnitude and location of the resultant.



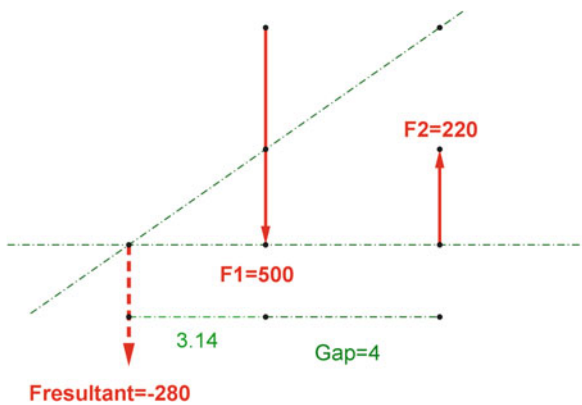
Exercise 1.9 solution If the loads are parallel but of opposite sense (i.e. one is upwards and one is downwards), then the inverse axis is modified.



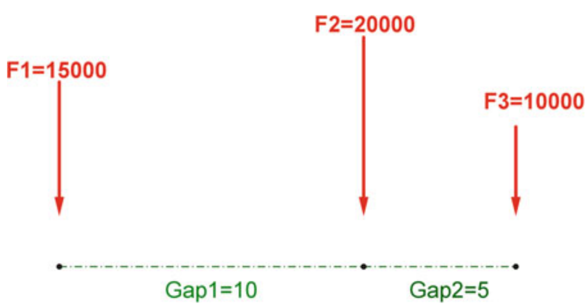
Exercise 1.10 Given the magnitude of two vertical loads of opposite sense, and the gap between them, locate the magnitude and location of the resultant.



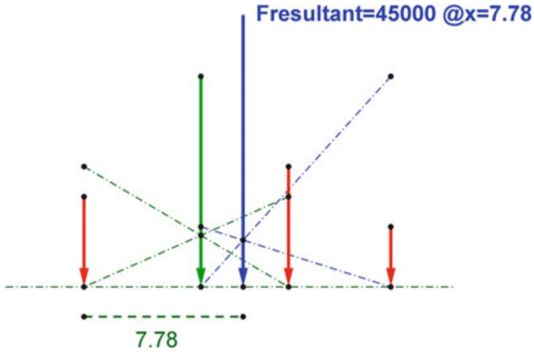
Exercise 1.10 solution



Exercise 1.11 Three vertical downward loads are shown. Find the magnitude and the location of the overall resultant by using the inverse axis method twice.



Exercise 1.11 solution



2 General Guidelines for Creating Efficient and Elegant Programs

There are many drawing environments available today which rapidly create worksheets with graphics that can be easily manipulated. One program that stands out is GeoGebra, it is extremely powerful and very simple to use. At times, it seems like it was created for graphic statics! When creating graphic statics solutions in GeoGebra, a consistently clean, minimal workspace greatly enhances the final product. Of course, each designer could put his or her own mark or style to such solutions, yet certain practices will improve the clarity and user-friendliness of any program. The following two paragraphs are suggested practices, they need not be strictly followed and in fact, you may be able to improve upon them as you get familiar with the program. If the following two paragraphs seem confusing at first, they should be reviewed later after a few exercises have been completed. Revisiting this section later will improve the understanding of these suggestions, and will allow for the personalization of any programs. It is also highly recommended to try to re-create these exercises in other parametric programs such as Grasshopper. The logic is what is important, not any one particular platform. Yet, to begin elementary programming, a few basics of GeoGebra will be described in detail.

The geometry of the form is drawn at the following scale, one unit of GeoGebra length equals one unit of the form's length (feet or meters). So whether the problem is 10 units of length long, or 700 units of length long, draw points at 10, or 700 as need be. A *ForceScale* slider can be used to adjust the size of the force polygons. Other choices exist, but it seems logical to scale only one part (either the form or the forces), and here it is suggested to adjust the size of the force polygon to comfortably match the size of the form diagram. In general, all new objects receive *"No Label"* to avoid clutter, axes are turned off, sliders are kept to a minimum and variables are simply entered in the algebra window. Points are typically small black circles, segments such as parts of the force polygon or the funicular form, are drawn as solid lines. Vectors representing forces are red. Lines that are parallel, or perpendicular to key points are typically dashed and more transparent. Lines that demarcate higher order entities such as the locations of centroids of the moment diagram, are colored and dashed and even more transparent. The modulus of elasticity and cross sectional properties of any element is entered in units corresponding to units of length in the original form diagram, (meters or feet). Variables use clear descriptive names such as *Modulus*, *Force1*, etc, and variables are case sensitive.

A useful technique is the creation of circles of a known diameter to establish the size of any vector, for example gravity force vectors, which typically begin at the top of such a circle. Then, the user hides (not deletes!) all of the construction lines and circles, leaving only the information used to create the original problem. Deleting a key item, such as a line used to lay out forces, deletes all the offspring tied to that item, so never delete to avoid clutter, always hide.

E. Saliklis, *Structures: A Geometric Approach*, https://doi.org/10.1007/978-3-319-98746-0_2

A typical exercise is to find the resultant of three known loads, Forces 1 and 2 are of known magnitude and orientation, Force 1 is 1000 units of force, horizontal to the right, Force 2 is 3000 units of force, vertical downward. Force 3 will have a variable magnitude and direction. Notice that the user must ensure consistency of units, they are not built into GeoGebra. To enter a variable and its associated value, simply type in the lower toolbar, the equal sign is the way to assign a value to a variable. Such typical assignments are shown in Fig. 2.1.

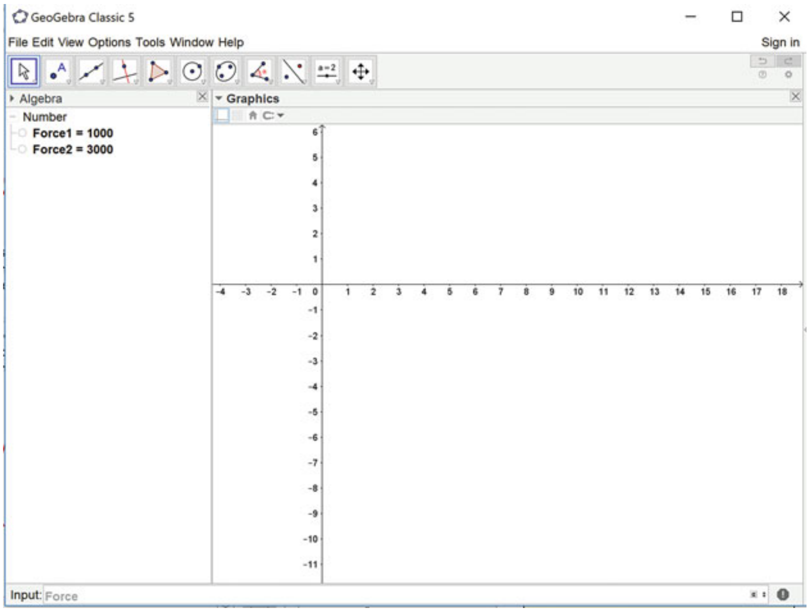


Fig. 2.1 Initial steps to create parameters of a problem

In this example, *Force3* is of a variable magnitude and orientation. It is convenient to place each of these two values on their own slider. Numbers have their own drop down menus, as do angles, you can choose degrees in such a menu (Fig. 2.2).

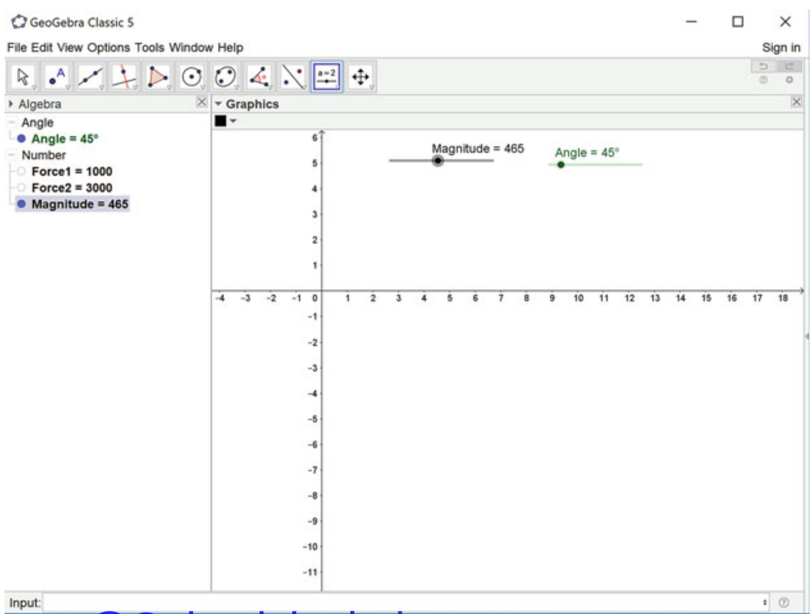


Fig. 2.2 Minimal use of sliders will enhance program flexibility

Arbitrarily start the first vector from the origin. To get accustomed to using circles of a known radius to describe a vector, create a circle of radius (*Force1*/*ForceScale*), where *ForceScale* is a slider that ensures a comfortable size of the force vectors. Use the Point icon to select a point at 3o'clock on this circle, this will establish the end of the *Force1* vector (Fig. 2.3).

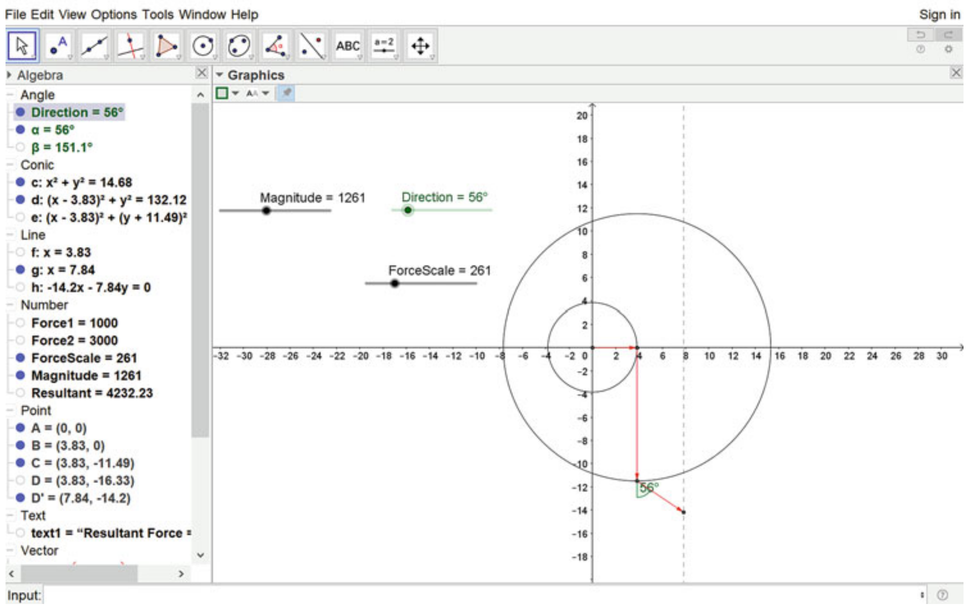


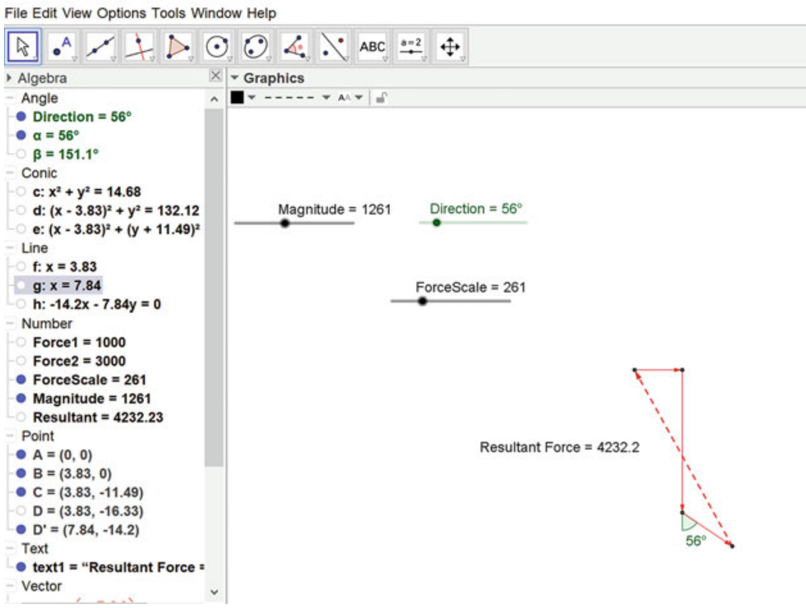
Fig. 2.3 Use of circles to create vectors

The process is repeated for the second force vector. The origin of this second force vector, and thus the center of the second circle, begins at the terminus of the first force vector. Create a second circle, radius *Force2*/*ForceScale*, and locate 6o'clock on that circle. The third force vector is a bit more involved, because it is to be of variable magnitude and variable direction, both controlled by the sliders. The angle is somewhat arbitrarily created by picking the 6o'clock intercept as the leg of the angle, the origin of the third circle as the vertex, and the variable "*Direction*" as the angle. *Hint*: avoid calling the variable "*Angle*" because that is the name of a built-in function in GeoGebra. Try to avoid those conflicts, (for example "*Length*").

Adjusting the sliders, allows for immediate feedback on the effect of the adjustments.

The next step, that of actually finding the resultant should be easy to guess. Try to articulate what steps are needed to find the resultant, then look at the next step when ready (Fig. 2.4).

Fig. 2.4 Hide extra entities and find resultant



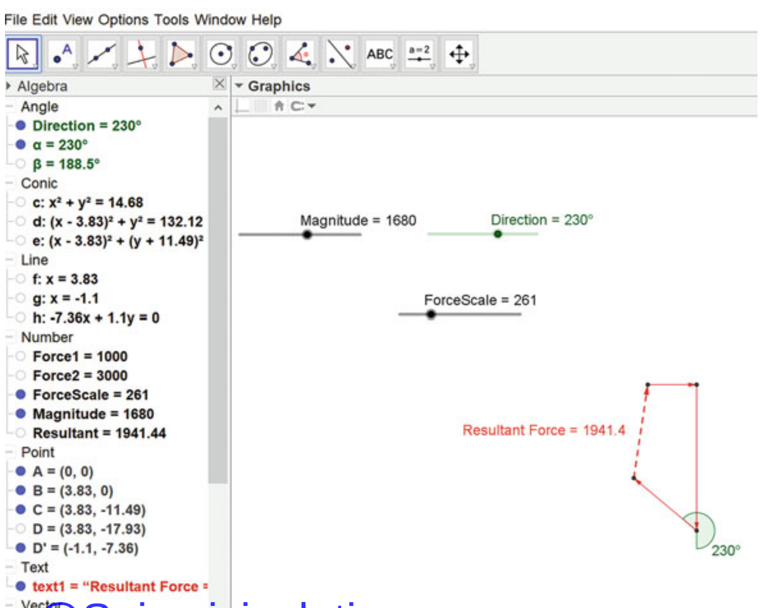
The answer is to add one remaining vector that joins the terminus of *Force3*, back to the origin of *Force1*. Here, a heavier dashed line is used for this resultant vector. The magnitude of the vector is found from the command:

$$Resultant = Length[a] \cdot ForceScale$$

where "*a*" symbolizes the resultant vector's pre-chosen name. The name of all such entities can be displayed by hovering the mouse over the entity. If an entity had a name with a subscript such as a_2, then in the command window it is referred to as a_2.

Notice how the clarity dramatically improves when the workspace is made uncluttered. Finally, since the variable *"Resultant"* has been found, it can be displayed on the final workspace by using the so-called dynamic text option.

Fig. 2.5 Immediate solution found by changing parameters



Text is simply entered into the text box, but to display the variable, insert the *Object* of interest. Note in Fig. 2.5 that the *Resultant* in the Algebra window has two decimal places, but it can be rounded for purposes of display to one decimal place. Also note that the *Direction* of the third force has been altered in Fig. 2.5, only to show how easy it is to manipulate sliders.

Hopefully, the added benefits of consistent and logically designed line weight, style and color are becoming self-evident.

To display the direction of the Resultant, one must identify two lines or three points to use the function *"Angle"*. Putting extra points on the workspace seems unsightly, so establish two lines, one parallel to the vertical and a second parallel to the resultant. After choosing these two lines, the opacity and color of the display of the Resultant's angle can be adjusted and the lines used to calculate this *Resultant* angle can be hidden. Certainly there are alternate ways of finding, or defining this angle.

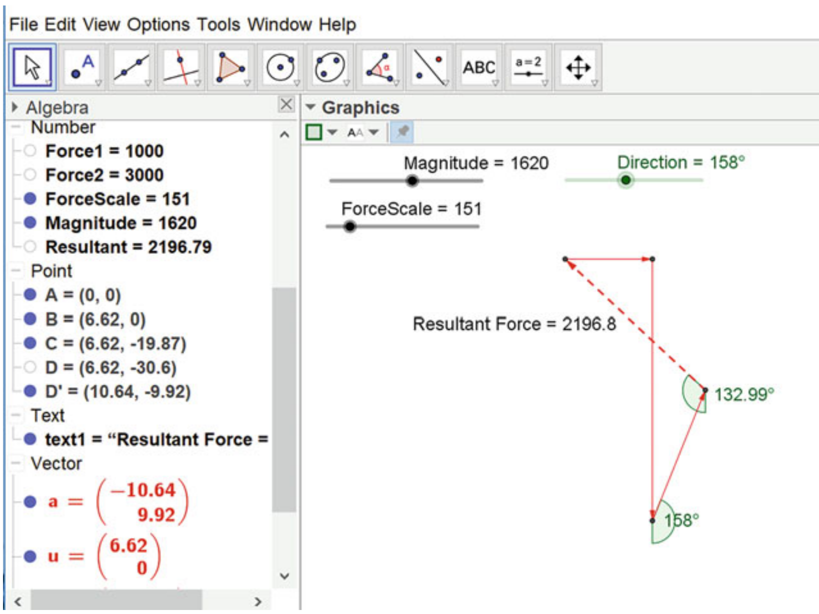


Fig. 2.6 Highlighting angle of resultant

Here is one way of displaying the angle of the Resultant. The vertical line remains visible, but the line parallel to the Resultant is now hidden. Notice subtle variations in opacity help the viewer. In Fig. 2.6, the Magnitude and Direction of the third force have been varied as has the *ForceScale*.

The next example will give details of how Fig. 1.11 was created. First, to establish the key points, type in a variable name for the point, such as *A* and provide the Cartesian coordinates as *A = (0,0)* Repeat for the other points *B = (3,0) C = (7,0)*, etc. Text can be added either as a text box, or as a Caption for the points, found under *"Object Properties"*, *"Basic"* (Fig. 2.7).

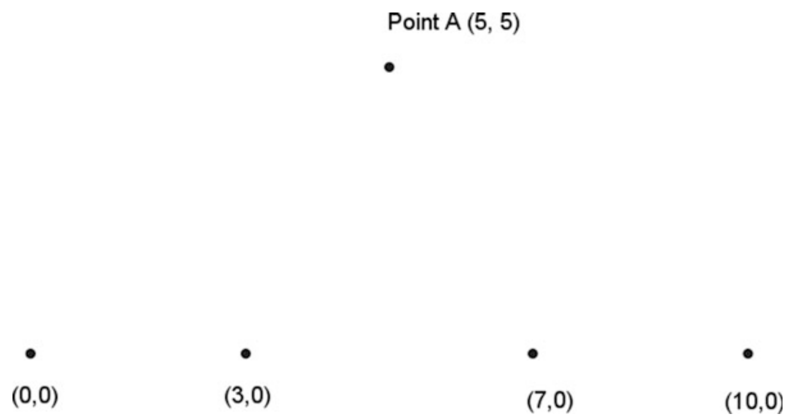


Fig. 2.7 Initial creation of key points for Fig. 1.11

Segments are then added, with color and line weight to distinguish the hierarchy of elements. Points established to create the fixed base line eventually will be hidden, not deleted, since deleting them would delete their offspring as well (the segment depends on the end points) (Fig. 2.8).

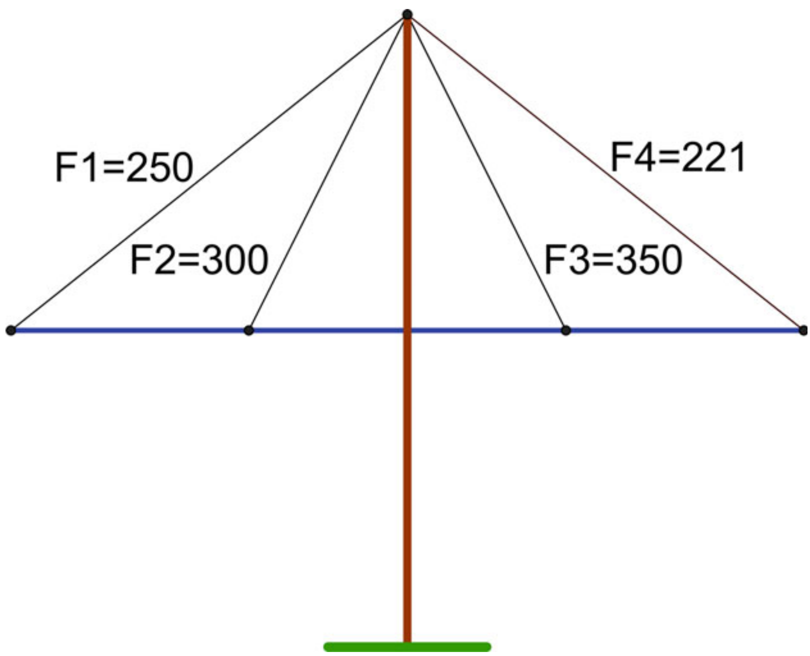


Fig. 2.8 Hints on drawing form diagram

Having the form diagram more or less complete allows one to begin construction of the force diagram. While the form diagram could be drawn at a 1:1 scale, the force diagram absolutely must be scaled to some comfortable size. A *ForceScale* slider neatly does this. To establish the force diagram, begin with a vertical line (a line can be drawn parallel to the *y* axis for example), then draw a circle of radius *F1*/*ForceScale*, then draw a line parallel to the *F1* cable, and create a point at the intersection of that line and the circle. This is the intersection of interest, roughly 7o'clock since *F1* is tensile, it pulls away from point *A* down and to the left. Draw a vector from the center of the circle to that point, and this is a scaled representation of the first cable force. Repeat for the other cables (Fig. 2.9).

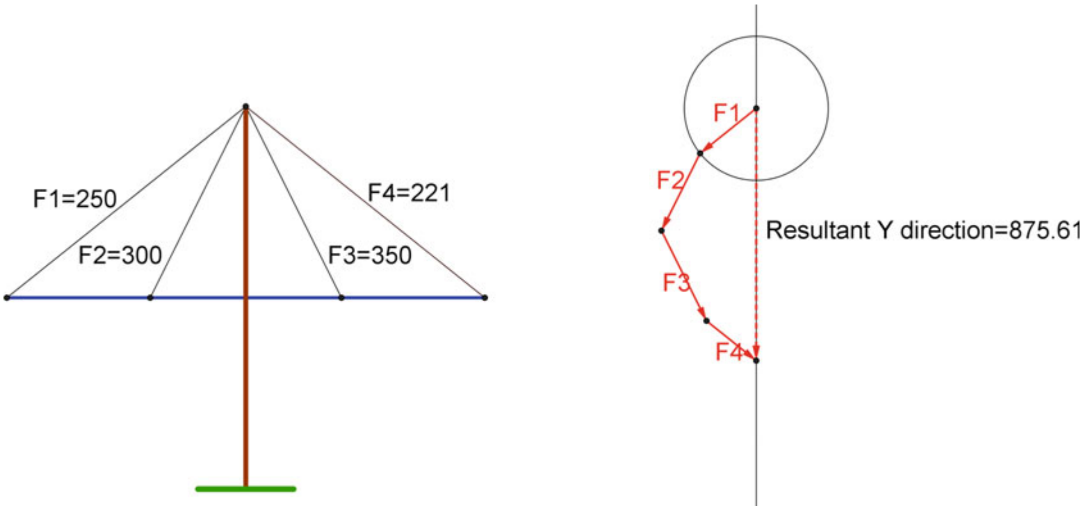


Fig. 2.9 Hints on drawing force diagram

As always, the resultant of all the forces is a vector that begins at the origin of the first vector and ends at the terminus of the final vector. The order of placement of the vectors is inconsequential.

It may be very helpful to limit the number of sliders in your workspace, and it always improves clarity if extraneous construction elements like lines and circles are hidden, not deleted. Judicious use of text, line weight and color all will improve clarity of the final work.

There is another pleasant surprise here. Namely, that one can personalize these drawings and make them one's own in a manner that is nearly impossible to do with traditional computer aided drafting programs. Approaching this as a program, not simply as a drawing, allows for logical computational ideas to be repeated in subsequent programs, steps that are personal to the creator.

3 Non-concurrent Forces and the Funicular

The funicular is a central concept in graphic statics constructions. It is both abstract and concrete, it is visual and analytical. It has been studied for centuries, yet today, very few engineering or architecture students are familiar with its construction or its implications. In its most prosaic form, a funicular is the shape that a hanging chain or cable takes when it is subjected to loads. An infinite amount of funiculars exist for a given loading condition, because an infinite number of lengths of chain could be used to carry such loads.

From a more mathematical viewpoint, a segment of a funicular represents the link between form and forces, in the form diagram of the funicular, the straight-line orientation of each segment must align perfectly with the orientation of the corresponding ray in the force diagram. A tantalizing feature of the funicular is that it can rapidly locate the location of the resultant of forces, in other words, it can find the placement of the centroid of forces. This was introduced qualitatively in Chap. 1. The funicular also links directly to the internal bending moment, which will be explored later. And it can be a powerful design tool when the geometry is manipulated in a controlled manner.

The first two examples will deal with non-concurrent, co-planar parallel forces. This restriction is actually very common, when dealing with gravity loads. In the first example, there are downward gravity loads and upwards, suction loads.

Figure 3.1 shows the problem of constructing a funicular for seven parallel, vertical, co-planar forces that are NOT acting on a beam, but their locations are all known, as are their magnitudes. Along a horizontal line, at $x = 0, 2, 5, 6, 8, 11$ and 13 there are forces as shown $F1 = 100, F2 = 40, F3 = 75, F4 = 90$ upwards, $F5 = 300$, $F6 = 125$ upwards, and $F7 = 80$.

E. Saliklis, *Structures: A Geometric Approach*, https://doi.org/10.1007/978-3-319-98746-0_3

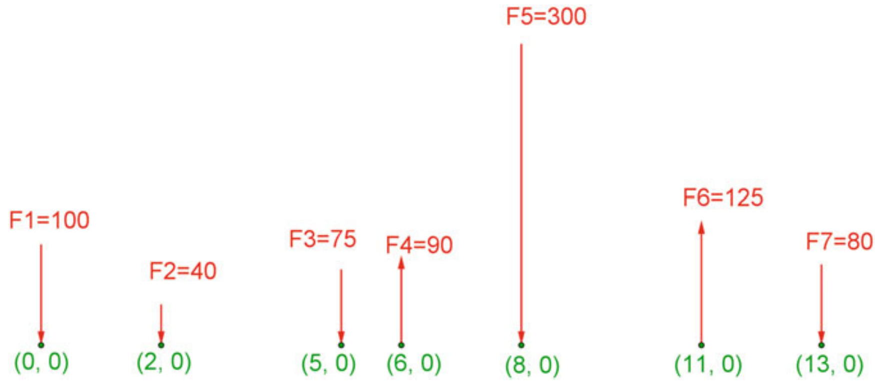
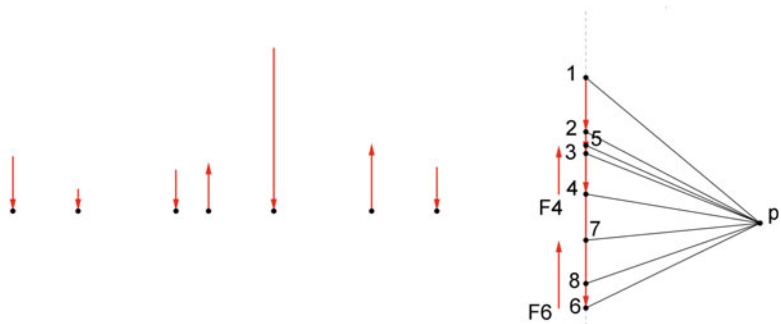


Fig. 3.1 Seven parallel vertical co-planar forces acting in 2D space

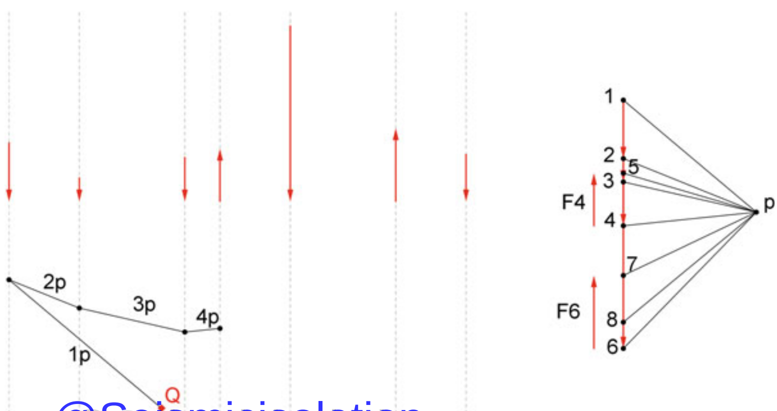
While five of the loads are downward, and two are upwards, all seven are vertical. Thus, the force diagram of Fig. 3.2 is begun by the creation of a vertical load line and each force vector is scaled by the same, comfortable factor. Note that *F4* and *F6* move upwards on this load line. The drop from Point *1* to Point *2* captures *F1* to some scale, the drop from Point *2* to Point *3* captures *F2* and so on, and the jump up from *4* to *5* captures *F4*.

Fig. 3.2 Force diagram drawn to a comfortable scale with an aribitrary pole *p*



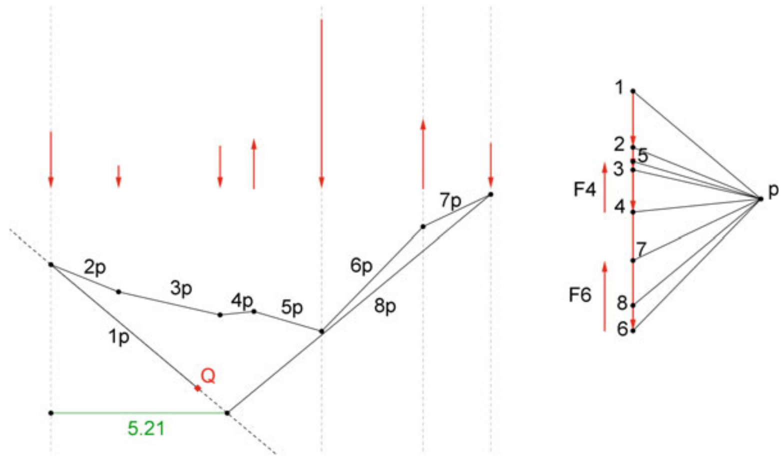
The force diagram is shown on the right hand side of Fig. 3.2. Note that in Fig. 3.2, the vectors *F4* and *F6* are drawn off of the vertical load line to clarify their role in the construction. An arbitrary pole *p*, establishes rays to the pole *p* and one can begin the creation of the funicular. But take note! The funicular is begun at some point within the loads of the form diagram. This was first demonstrated in Fig. 1.15 so review that construction if it seems unfamiliar. The beginning of the funicular is shown in Fig. 3.3.

Fig. 3.3 Funicular begins at arbitrary interior point *Q*



Notice that the funicular began at some arbitrary, interior point *Q*, just as it did in Fig. 1.15. The construction of the funicular absolutely must follow the serial order of the numbers of the force diagram. Later, the notation describing points on the force diagram will be refined. Complete the funicular as shown in Fig. 3.4.

Fig. 3.4 Completion of funicular



Note that Point *Q* is interior and arbitrarily placed. The serial order of the construction is apparent when segments of the funicular are labeled as in Fig. 3.4. The final ray, *8p*, begins from the terminus of *7p* and it continues till it intersects the slope of the original ray *1p*, here shown as a dashed blue line. The distance of the final point from the left end is *5.21* units of length. This precisely marks the location of the centroid of all the loads. The magnitude of the resultant load is captured by a vector spanning from Point *1* on the force diagram, to Point *8*. This vector must be multiplied by the *ForceScale* used, to establish the resultant's magnitude.

A similar problem, shown in Fig. 3.5, is to construct a funicular for an arbitrary set of four non-concurrent, vertical co-planar forces which are acting on a beam *10* units of length long. Assume that the magnitude and the placement of these four forces is known. Here, points on the *X* axis at *0, 2, 3, 5, 7* and *10* are created. As before, this is drawn so that one unit in the parametric drawing environment corresponds to one unit of length in the real world, perhaps feet or meters. Let points at $X = 0$ and *10* represent the start and end of a beam subjected to these four loads. Make lines parallel to the *Y* axis through these six points. This initial horizontal form diagram is shown in Fig. 3.5.

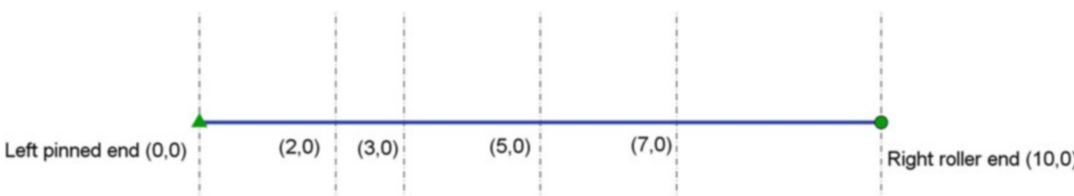


Fig. 3.5 Funicular on a loaded beam, form diagram

Then, as before, create a comfortable *Force Scale* slider, draw circles with each radius corresponding to each of the four forces divided by the *Force Scale*, and locate *12o clock* on each circle. Then draw vectors representing the four loads. The construction of this is shown in Fig. 3.6.

Afterwards, hide the unnecessary elements, do not delete them. Keep visible the vertical lines through the vectors however, their importance will become clearer in the subsequent step.

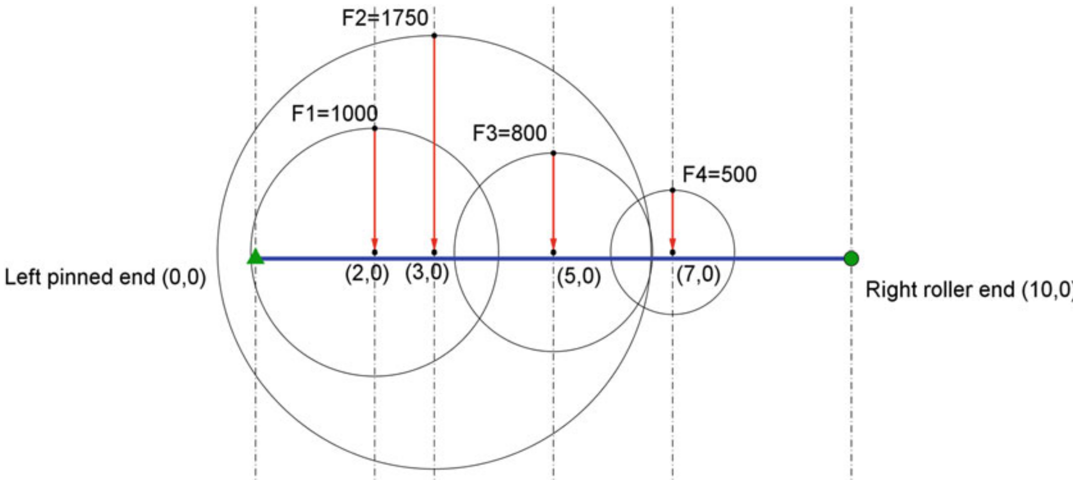
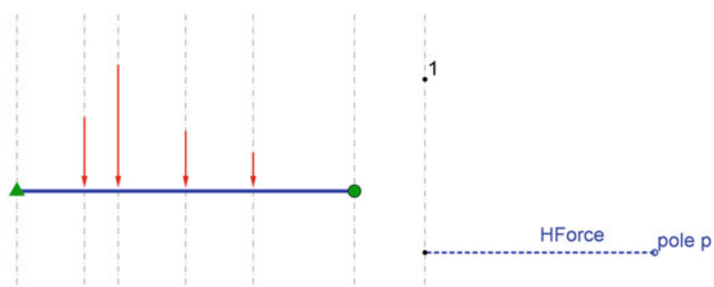


Fig. 3.6 Using circles to draw the applied loads on form diagram

Once the four vectors have been drawn to some comfortable scale, the next step is to create a funicular, which is based on an arbitrary pole *p*. It will be useful to capture the horizontal distance that the pole has from a vertical load line, this distance has a physical meaning, and thus it is labeled as *HForce*, which stands for *Horizontal Force*, to be explained shortly. One way of monitoring the magnitude of *HForce* is to draw a horizontal segment from the pole to the load line, and then to capture the length of that segment using the *Length* command. An open circle demarcates the pole. The location of the pole is arbitrary. The pole is a point from which rays emanate, rays which will shape the funicular. A larger *HForce* will cause less "sag" in the funicular, a lower *HForce* will induce greater sag. Establishing the pole is shown in Fig. 3.7.

Fig. 3.7 Arbitrary pole *p* at distance *HForce* from load line



Now create a force diagram. Along the vertical load line, which established the force diagram, create four circles each of radius *Force i*/*ForceScale*, *i* going from *1* to *4*. Use segments to create the five rays representing the force polygon as shown in Fig. 3.8.

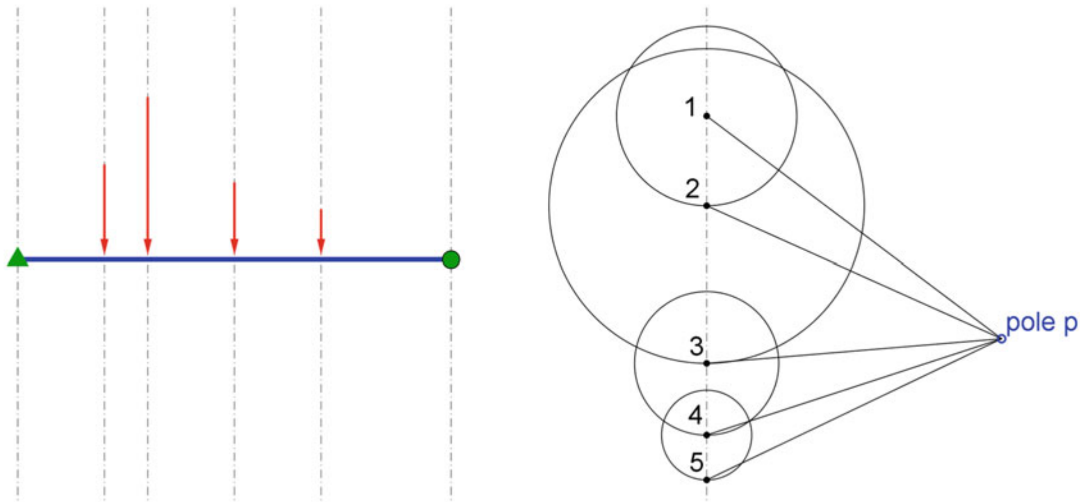
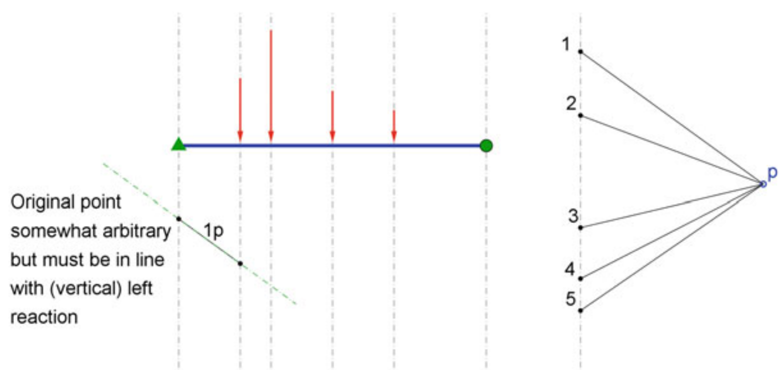


Fig. 3.8 Using circles to create scaled force diagram

The rays of the force polygon correspond precisely to the slopes of each segment of the funicular. Recall that the inclination (or sag) is arbitrarily controlled by the variable *HForce*. The funicular will locate the resultant load's location. The magnitude of the resultant will be the summation of all four forces.

Begin at some point directly below the left end of the form diagram. The exact placement of this first point is somewhat arbitrary, but it must be on the first dashed line which aligns with the left reaction. Locate where the sloping funicular line intercepts the next vertical line, which corresponds to the trajectory of the first load, *F1*. This is shown in Fig. 3.9.

Fig. 3.9 Funicular segments have controlled starting and stopping points



Repeat for the remaining four segments. Feel free to adjust the location of the pole, and the magnitudes of *HForce* and *ForceScale*. Along the vertical left end and right end lines, it does not matter where the funicular starts or ends, as long as the kinks of the funicular align with the vertical lines demarcating lines of action of the forces, here dashed vertical lines. Two scenarios are shown with varying pole location which of course immediately changes *HForce*, these are Figs. 3.10 and 3.11. Note that the start and the end of the funicular need not be on a horizontal line.

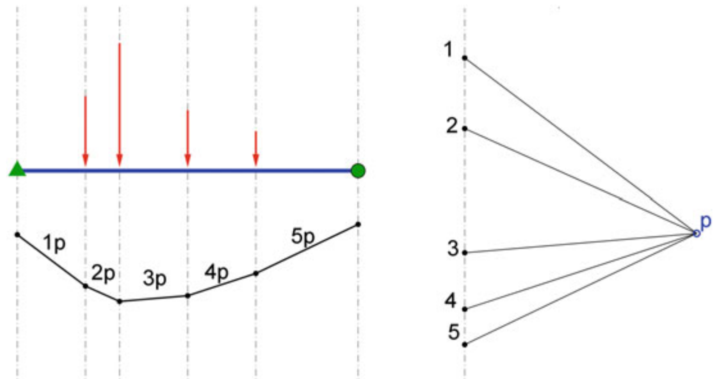


Fig. 3.10 Completed funicular for some placement of pole *p*

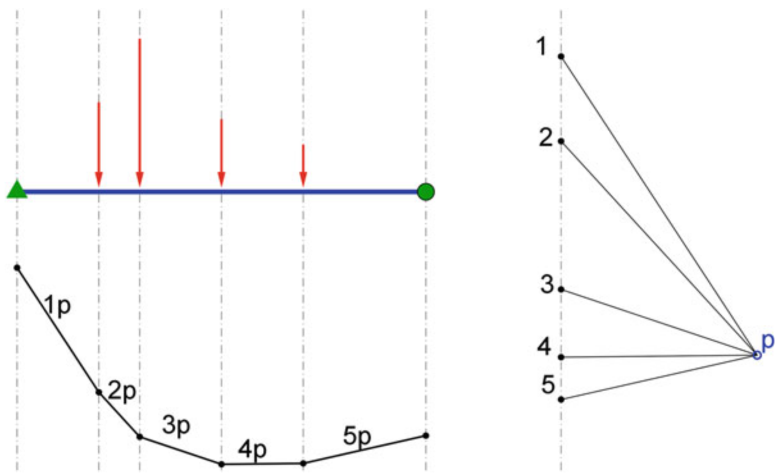


Fig. 3.11 Completed funicular for alternate placement of pole *p*

There are an infinite number of funiculars for a given load, and it is also known that the inclination of the starting and ending points of the funicular differ according to the arbitrary placement of the pole. Using this knowledge, complete the very important closing line, a line that connects the start and end of the funicular. This closing line will be especially important when internal bending moments are calculated.

For now, draw this closing line, and draw lines parallel to the first and last segments of the funicular, and find where these two lines intersect (Fig. 3.12).

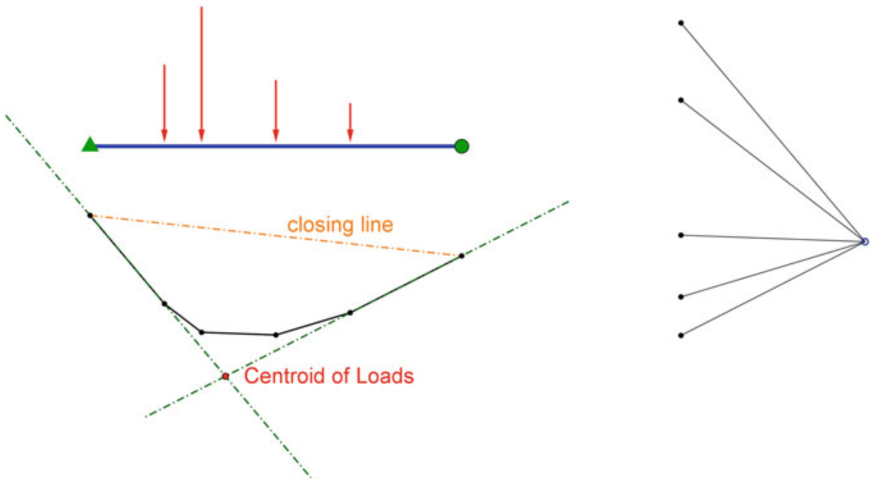


Fig. 3.12 Closing line and initial and final slopes shown on funicular

Regardless of the shape of the funicular, the red dot, which denotes the intersection of the starting and ending slopes, remains at the same horizontal distance from the supports. This red dot is the precise location of the centroid of the loads. This example can be taken a few steps further. The magnitude and the location of the resultant of all four forces are now known. This information can be displayed in a bit more detail as follows.

Draw a vector capturing the magnitude of all four of the loads on the force diagram. The magnitude of the resultant is the length of that force vector multiplied by the force scale. The location of this resultant is at the red dot. To quantify how far this is from one side, for example from the left end, create a horizontal segment that starts at the red dot and ends at the leftmost dashed vertical line (Fig. 3.13).

Fig. 3.13 Locating the force resultant

There is yet another remarkable feature of the funicular. The closing line of the funicular allows for the calculation of the reactions of a beam. If this example was really a 10 unit long, simply supported beam, subjected to four point loads, the reactions at each end are readily found.

Draw a line parallel to the closing line. Pass that line through the pole. Identify where this line intersects the vertical line on the force diagram. In Fig. 3.14, that point is marked by an *X*.

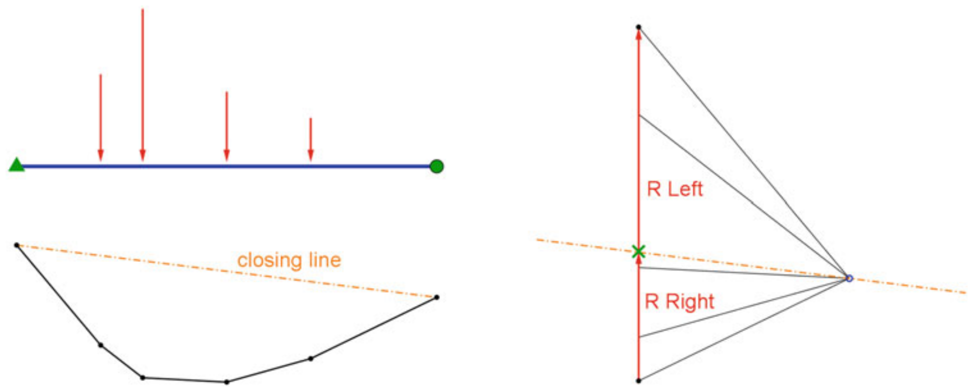


Fig. 3.14 Using the closing line to establish the reactions

This clean, minimal drawing captures the reactions. Adjusting the vertical location of the pole, the magnitude of *HForce*, etc. will demonstrate that the reactions are independent of these factors. A common mistake is to simply forget to include *ForceScale* when solving for the lengths of these two reaction vectors. Don't do this!

While this is not a general program, it is not difficult to imagine efficient ways of controlling the magnitude and locations of each of the four loads, making it a bit more general and informative.

Whereas the first two examples of this chapter dealt with non-concurrent parallel loads, the next example will address co-planar (i.e. 2 dimensional) but non-parallel, non-concurrent forces. Of course, these forces are all in the same plane, but are neither parallel, nor do they all meet at the same point. Wind and other lateral loads fall into this category.

Four loads are acting as shown in Fig. 3.15. Find the magnitude and location of the resultant of these loads. Include variability in the magnitude and angle of each load in your program.

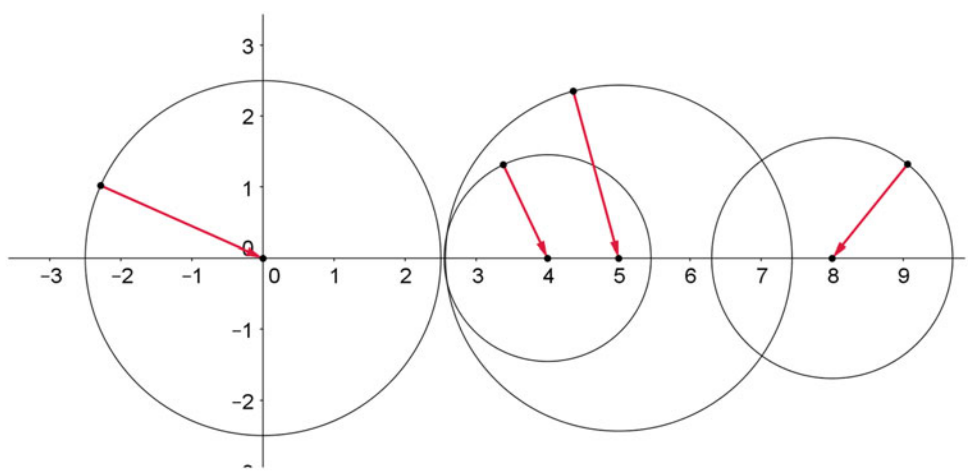


Fig. 3.15 Four non-parallel, non-concurrent loads in 2D space

Here, four loads are created, with the terminus of each touching $X = 0, 4, 5$ and 8 respectively. To allow for variability of the magnitude and the angle of each load, a circle is drawn around each of the four X points and a point is placed somewhere on the circle.

The form diagram will be enhanced a bit, by introducing Bow's Notation for capturing the loads. This notation is a classical one, introduced by Robert H. Bow in the 1870s. It is sometime called an interval notation because capital letters are placed in the intervals, or spaces, between external forces. Therefore, we no longer will call the first force $F1$, rather, we will call it AB if we move from space A to space B, or perhaps we will call it BA if we move from space B to space A. More on the order of letters will be presented later. For now, move from left to right, thus the third force will now be called CD (Fig. 3.16).

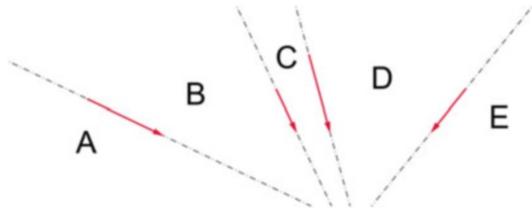


Fig. 3.16 Introducing Bow's notation in the intervals between loads

As was done in Fig. 3.3, begin the funicular at some interior point Q. Draw an arbitrary pole, p near the scaled force vectors, this is known as the force diagram. Bow's Notation traditionally uses capital letters on the form diagram and corresponding lower case letters on the force diagram. Rays emanating from each lower case letter to the pole are the slopes of the funicular to be drawn on the form diagram. Draw segments of the funicular, corresponding to the slopes of the rays of the force polygon, till those rays intersect the line of action of the following load. This is shown in Fig. 3.17.

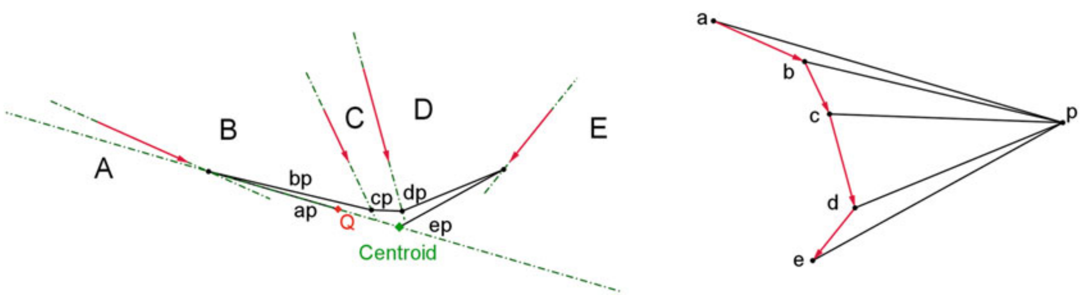


Fig. 3.17 The completed force diagram and corresponding funicular

It is safe to say that Fig. 3.17 has an aesthetic power, as well as a profound engineering insight linking the funicular to the centroid of the loads. And the solution is exact, it can be checked against an algebraic solution within the same programming environment. Figure 3.18 shows the overall force resultant and its components in the X and Y directions.

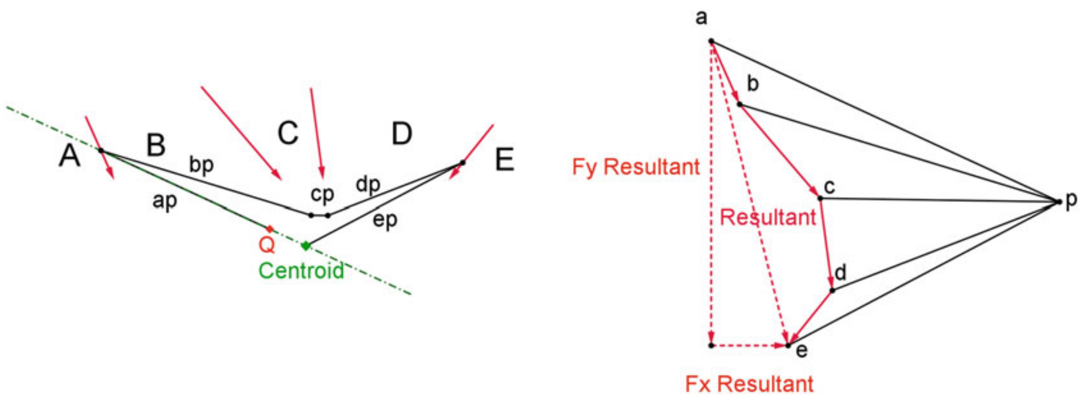


Fig. 3.18 Breaking the force resultant into two components

Furthermore, having programmed this solution in a parametric environment allows for instant solutions of alternate loading situations, since the magnitude and direction of each load was variable.

In the previous examples, the funicular had a physical immediacy to it, it looked like the shape a chain would take when subjected to the given loads. But there is no chain, even if it is tempting to imagine a simply supported beam deforming into roughly the shape of the funicular. To emphasize this unique shape of the funicular, consider a cantilever beam subjected to a couple near its free end. For quantitative purposes, imagine a cantilever beam *5* units of length long, with a couple centered at *4.75* units of length. Each discrete force F_{couple} is *5000* units of force and the gap between the two forces is *0.5* units of length. This is shown in Fig. 3.19.

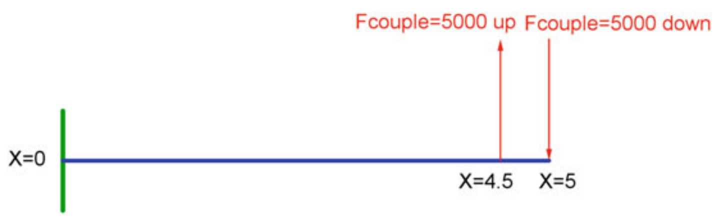


Fig. 3.19 Cantilever subjected to two forces

It is known from theory that the magnitude of the moment induced by the couple is one of the forces multiplied by the gap between the two forces. Here the magnitude of the moment is

$$M_{induced} = 5000 \cdot 0.5 = 2500\, force\ length$$

The moment at the fixed support ($X = 0$) can also be calculated theoretically including the sign, it is

$$M_{induced} = 5000 \cdot 4.5 - 5000 \cdot 5 = 5000 \cdot (-0.5) = -2500\, force\ length$$

The negative side means tension on the top of the cantilever. Of course, the moment does not need a moment arm, it is already a moment. Thus the moment anywhere between $X = 0$ and $X = 4.5$ in this problem must be a constant

$$M_{@any\ X} = -2500\, force\ length$$

This can be shown graphically as seen in Fig. 3.20. Chapter 4 will explore reactions of beams and the bending moment diagram more thoroughly. In this example, simply look at the shape of the funicular and see how it reflects the shape of the bending moment diagram of the beam.

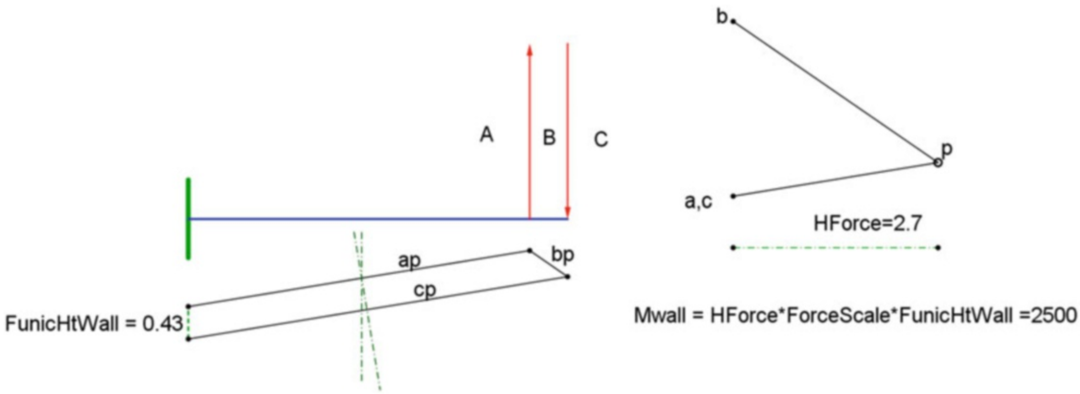


Fig. 3.20 Force diagram and corresponding funicular for cantilever problem

Here, in the force diagram on the right side, a vertical load line captures the vertical forces by drawing them to some scale. In the force diagram, the jump from a to b represents the force captured by moving from A to B in the form diagram. Then force B to C must be captured, and it has been so captured in the force diagram, where c coincides with a. An arbitrary pole p is drawn and rays connecting a, b and c to the pole p are drawn. Slopes of these rays are then transferred to the form diagram, starting at the fixed support, and continuing till it intersects the path of the next applied load.

In this example, an arbitrary pole distance from the vertical load line ab to p, called $HForce$ is 2.7, $ForceScale$ is 2161 and the height of the funicular at the wall is 0.43, thus

$$M_{wall} = HForce \cdot ForceScale \cdot FunicHtWall = 2500$$

as it must be, the sign will be explored in the Chap. 4 which covers bending moment diagrams in beams. A few other insights pop out as well. The moment is constant anywhere in space A, and this is captured graphically by the constant vertical height between ap and cp. Don't make the mistake of measuring perpendicular to ap or cp, the dashed lines show that such a measurement would be incorrect. An even deeper insight is captured in the idea that a concentrated moment is described by a gap between segments of the funicular. This can be linked to Airy Stress Functions, a current area of intense research in the field of Graphic Statics.

One more complexity will be added to the previous example. Suppose that in addition to the couple, there was a concentrated downward load F_{pt} applied to the free end of the cantilever. This is shown in Fig. 3.21, as well as the funicular generated by the corresponding force diagram.

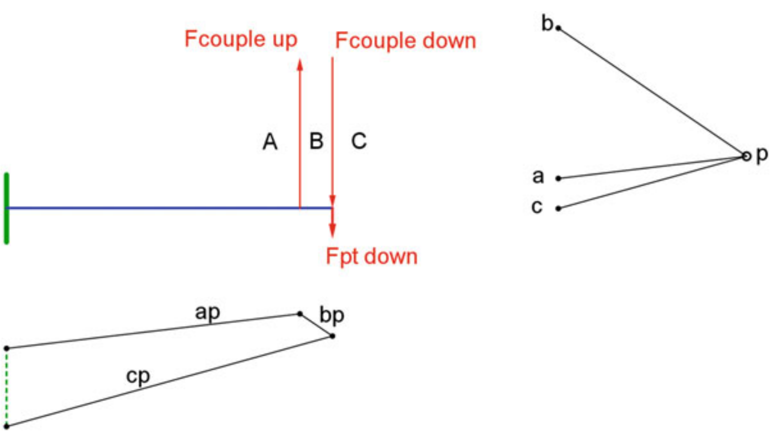


Fig. 3.21 A force couple plus a point load acting on a cantilever

Here, it is extremely convenient to simply think of the two loads at the free end as one single load, magnitude $F_{couple} + F_{pt}$. By doing this, the space is preserved as it was before. The force diagram captures the upward force *AB* by an upward movement *a* to *b*. Then in the form diagram, moving from *B* to *C* captures $F_{couple} + F_{pt}$, thus a downward jump from *b* to *c* in the force diagram means that *c* no longer coincides with *a* as it did in the previous example. Mapping rays to the funicular as before shows that *ap* and *cp* are no longer parallel. They would be parallel if F_{pt} was zero, but now, a triangular effect of F_{pt} multiplied by the distance from the free end of the cantilever is added. Consequently, the bending moment gets worse as it approaches the fixed support. The parametric drawing environment allows for quick changes of variables such as the magnitude of the loads. This is demonstrated in Fig. 3.22.

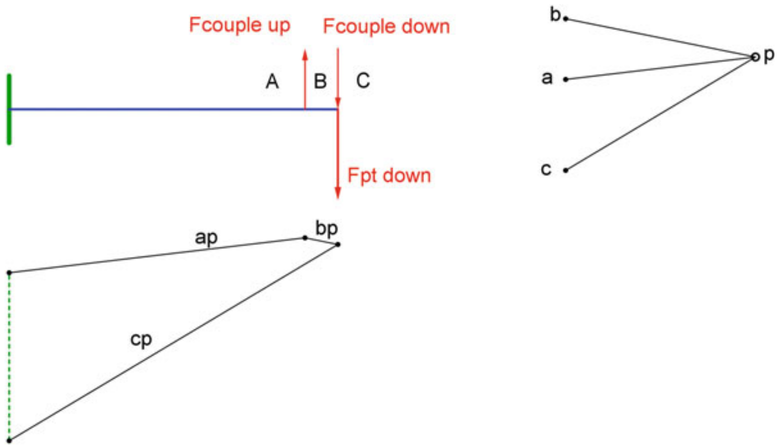


Fig. 3.22 Previous problem with new force parameters

The following example shown in Fig. 3.23 begins to explore in detail the concept of inclined loads on structures. Horizontal beams rarely experience such lateral plus vertical loads, yet an elementary example of one introduces some of the subtleties of the funicular associated with this loading.

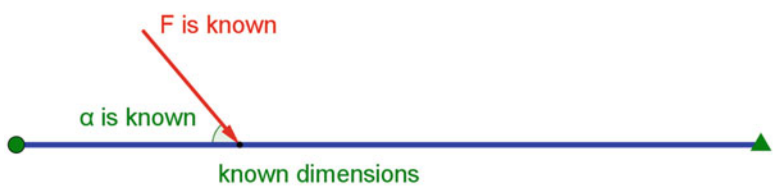


Fig. 3.23 Inclined load applied to a beam

In Fig. 3.23, notice that the pinned support is on the right side of the beam. This is important for the initial step of the construction of the funicular. The slope, not the magnitude, of the left side reaction is known, it must be vertical. But the slope as well as the magnitude of the right side reaction is completely unknown. All that is known about this reaction is that one point on its line of action must pass through the right hand support. This fact necessitates that the funicular begin at the right hand pin. Any arbitrary pole p' is used on the force diagram as shown in Fig. 3.24. Ray bp' is begun at the right support of the form diagram. Where this ray intersects the path of the load, ray ap' is begun. Ray ap' terminates along the line of action of the left support, since this reaction is known to be vertical it must terminate directly above or below *A* depending on where p' is placed. This is a key idea, namely that the slope of one reaction is fully known, here the left reaction's slope is vertical. The black dashed lines form the trial funicular. It is only a trial funicular since it was based upon the trial pole. In Chap. 4, the correct pole p will be established, but for now it is not needed. The slope of the trial funicular's closing line is passed through the trial pole p', here shown as a purple line. The point where this slope intersects the load line (note, here that load line's slope is vertical, for an inclined roller it will be normal to the roller surface) establishes point c and consequently, the reactions. Force bc can be shown as a resultant vector, or it can be shown with vertical and horizontal components. A note of caution: this funicular cannot be used for the final bending moments, that is why the pole is referred to as a trial pole, p'. Moving the pole till the left end of the funicular matches the boundary condition location (i.e. funicular ends at the roller and a horizontal closing line), establishes the final pole p and that final funicular can be used to calculate the bending moments.

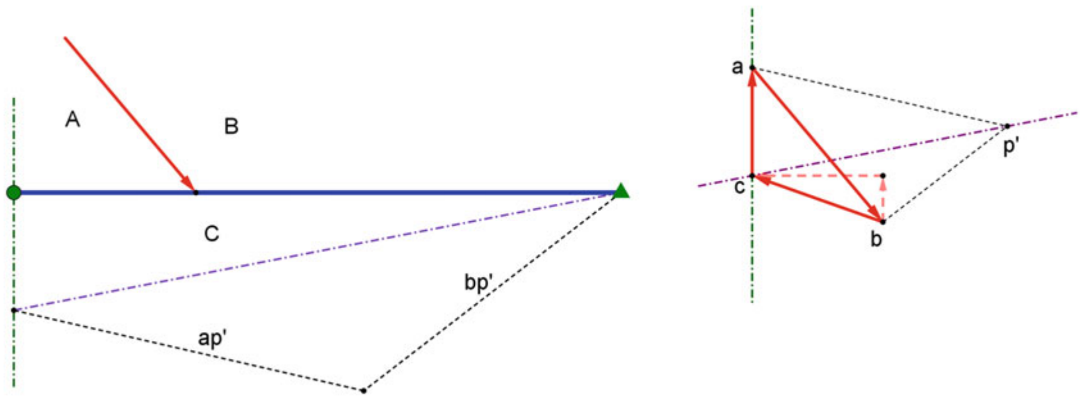
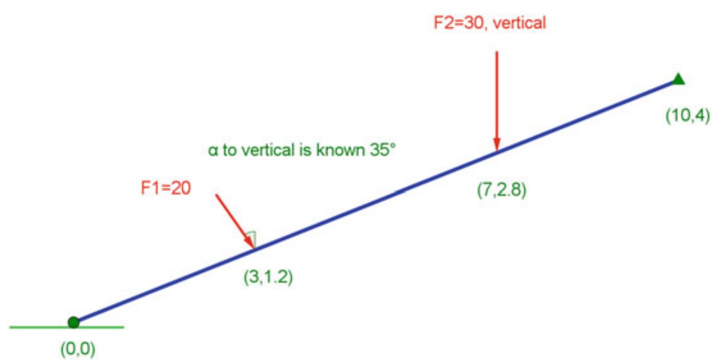


Fig. 3.24 Ray bp' is drawn first in funicular

In Fig. 3.25, a roller support on a horizontal surface is the left reaction, and a pinned support is the right reaction of an inclined beam. Angles, loads and dimensions are all shown.

Fig. 3.25 An inclined beam with arbitrary loads



A force diagram, as shown in Fig. 3.26, is drawn to some convenient scale and an arbitrary pole p' is established as usual. Rays to p' create the slopes of a trial funicular. As before, the trial funicular must start at the pin, not at the roller, thus lay down ray cp' first. A trial closing line is drawn from the ends of the funicular, and that trial closing line, shown in purple, is passed through p'. Notice the next idea; the path from D to A in the form diagram must be vertical, thus the trial funicular slope in the force diagram passing through p' must be intersect a vertical line in line with a. This point d could end up above or below a, depending on the problem, but the fact that the line of action from d to a is known is a key idea. Furthermore, in the force diagram of Fig. 3.26, notice that a single ray ap' spans from a vertical line passing through a, whereas two rays, bp' and cp' span from a vertical line passing through b. This will be important in Chap. 4 where the bending moments are established. An extremely rapid way of finding the moment is to first capture the verticality of applied loads and then the horizontal moment arm of each. Of course, an alternate manner is of finding the bending moment is to find the perpendicular distance of the cut from the path of load. However, in this example, the point is to show that any trial funicular will establish support reactions. This is shown in Fig. 3.26.

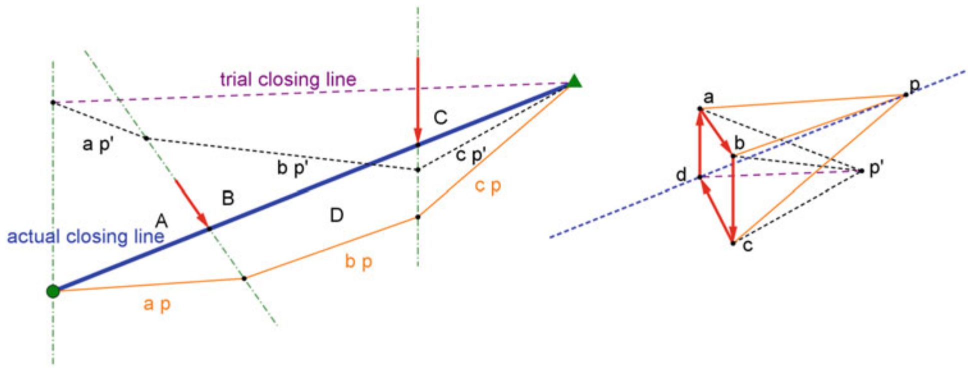
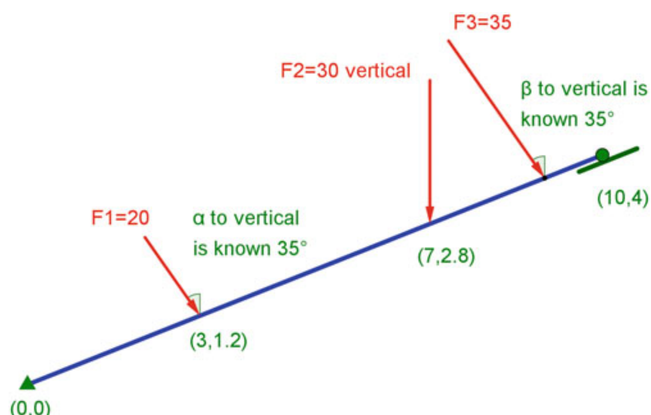


Fig. 3.26 Trial funicular and final funicular for inclined beam

Figure 3.26 also points to the generation of the final bending moment of this beam. The trial closing line established a trial funicular which originated at the hinged support because two unknowns are at this point yet they both must pass through this support. The final closing line is parallel to the inclined beam since the funicular must end at each support for inclined load problems. The final funicular can be established by simply moving the trial pole p' until the funicular touches the roller support, or a line can be drawn parallel to the inclined beam, and a final pole p can be established anywhere on that line. The final funicular is shown in orange in Fig. 3.26, and it is the final bending moment diagram which will be described in Chap. 4.

In Fig. 3.27, the geometry of the beam is the same as in Fig. 3.25, but the support conditions are different. Here, the pinned support is at the left end, and a roller support is on the right end, but this roller rests on a surface that is parallel to the beam. Also, in Fig. 3.27 a third load is applied at a known inclination to vertical.

Fig. 3.27 Inclined beam with inclined support



The final funicular, and thus the final bending moment are not needed to establish the correct reactions. This is shown in Fig. 3.28. If the funicular were to end at the roller support, by means of adjusting the trial pole p', or by realizing that the closing line is parallel to the inclined beam, then this would be the final funicular as well as the bending moment diagram.

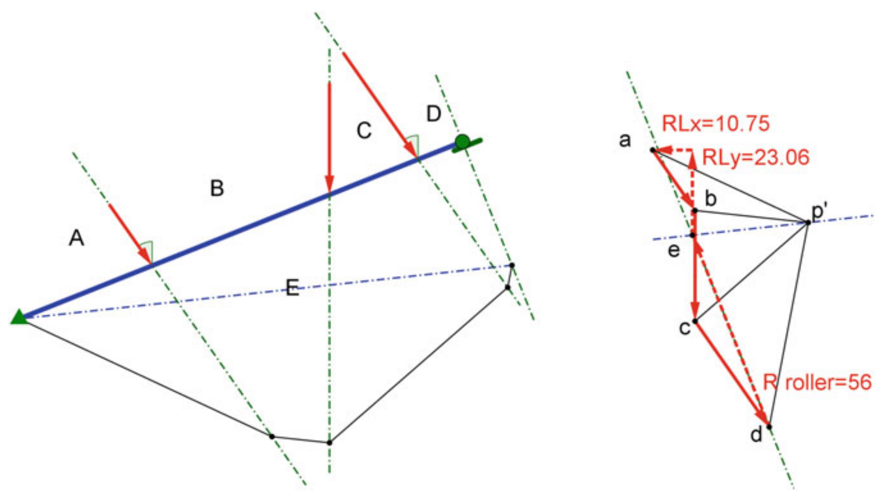


Fig. 3.28 Reactions can be found from trial, not final, funicular

A full discussion of bending moment diagrams will occur in Chap. 4. But a hint of what lies ahead is shown in Fig. 3.29. When lateral loads are present, the funicular must be adjusted such that each end of the funicular lands on the boundary (support) conditions of the structure. The funicular can be moved via adjustment of the pole p', till it lands on a line parallel to the beam which de facto spans from support to support. The moment in the beam anywhere in Region B (between the first and second loads), is established as follows:

$$M_{RegionB} = RegionBheight \cdot HForce \cdot ForceScale$$

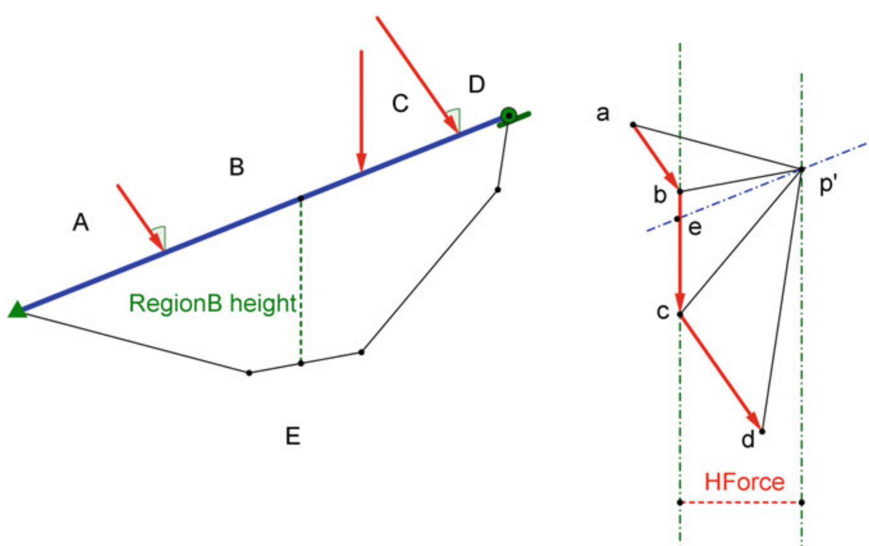


Fig. 3.29 Only the final funicular can be used to calculate bending moments

One surprise of the study of resultants and reactions of beams is the utter simplicity of the calculations when the problem is solved graphically. Multiple forces can be reduced to a single, statically equivalent load, either using the inverse axis method, or the funicular. For example, if three forces act on a simply supported beam, the inverse axis method could be used twice in succession to calculate the resultant. An unusually simple method of finding the reactions of a beam similarly uses the inverse axis method, but it uses it in an "inverted" manner, thus it could be called the Inverse Inverse Axis Method.

Consider the following situation, a beam with one known force and two reactions with known location but unknown magnitudes (Fig. 3.30). Note that the single force might have arrived from the resultant of several loads as described previously.

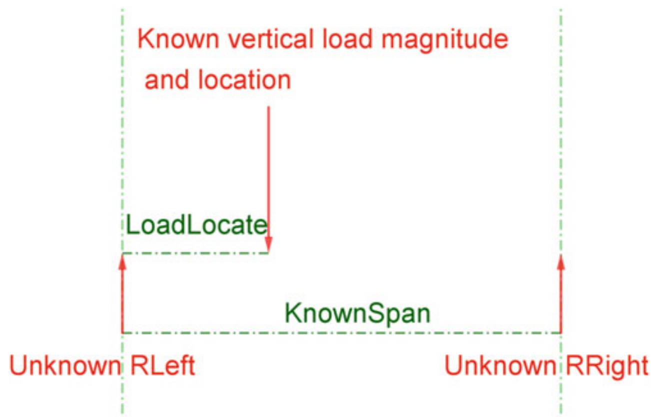


Fig. 3.30 Inverse Inverse Axis method to find beam reactions, problem setup

The applied load (or the resultant of several loads), is reproduced to scale at the locations of the supports, but only the origin and the terminus of the load vector are necessary. A line is drawn from the top left of the diagram to the lower right of the diagram. An X marks where this line intersects the applied load (Fig. 3.31).

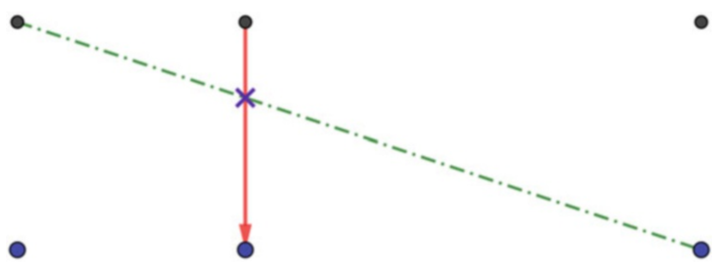


Fig. 3.31 Magnitude of known load is re-created at each support, diagonal line drawn

A horizontal line is passed through the intersection X and where this line intersects the left and right paths of the supports marks the reactions. For example, if the span between the reactions was 9 units of length, and an applied load of 30 units of force was applied at 1/3 the span, the reactions are found as shown in Fig. 3.32.

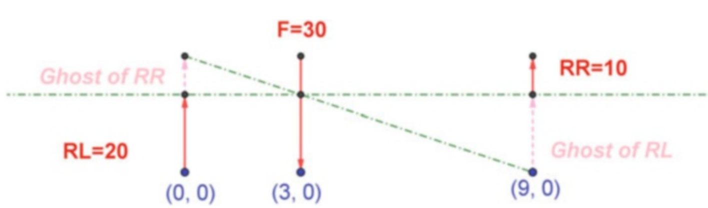


Fig. 3.32 Inverse Inverse Axis Method used to find beam reactions

The Inverse Inverse Axis Method works for loads to the right of the centerline as well. This is shown in Fig. 3.33.

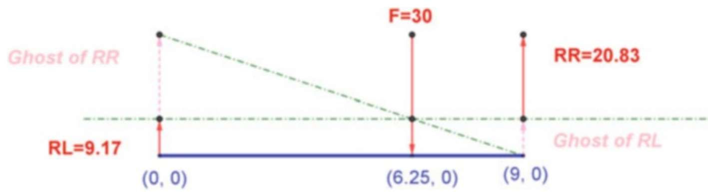


Fig. 3.33 Inverse Inverse Axis Method for load right of center

The final example in this chapter shows two vertical loads applied to a beam at known locations. Supports at the ends of the beam are also at known locations. It is desired to find these reactions using the Inverse Inverse Axis Method. The problem is shown in Fig. 3.34.

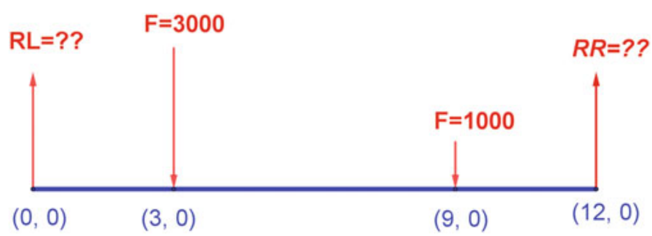


Fig. 3.34 Inverse Inverse Axis Method, beam with two point loads

The first step is to use the Inverse Axis Method to find the resultant of the two applied loads. Its magnitude and location is shown in Fig. 3.35. Of course, the magnitude is simply the sum of the two applied vertical loads.

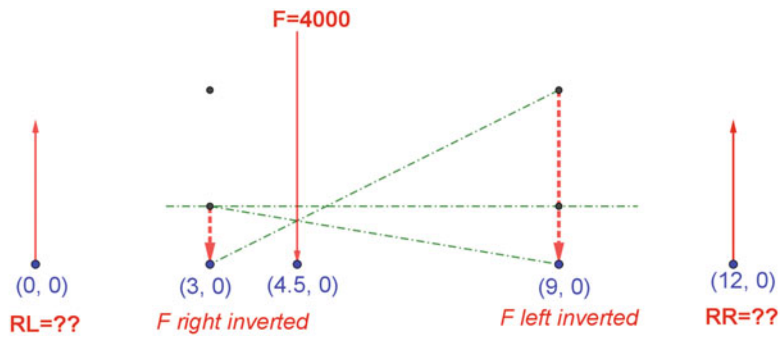


Fig. 3.35 Step 1, find the resultant of the applied loads

Then the Inverse Inverse Axis Method is shown in Fig. 3.36. This requires recreating the scale of the resultant load at each of the two ends. A line drawn from the top left to the bottom right intersects the load as the point X shown.

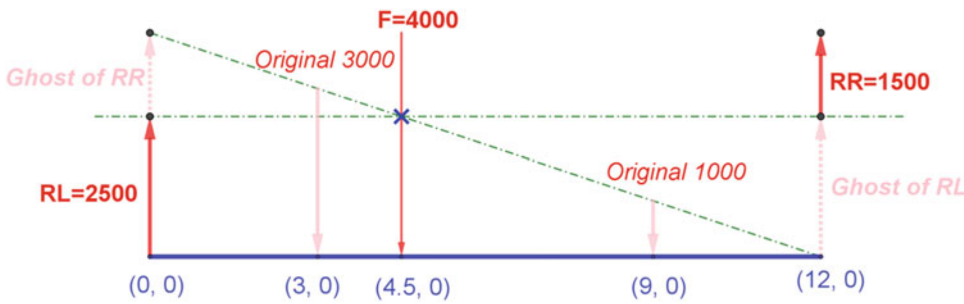


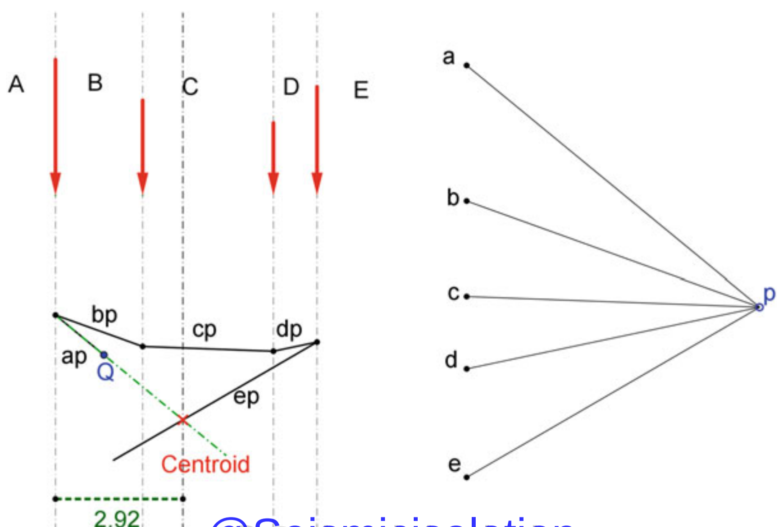
Fig. 3.36 Magnitude of each vector at each end is the resultant of the applied loads

Note how this line used to mark the intersection X traces the magnitudes of the original loads! This hints at the deep relationship between centroids of loads and equilibrating reactions of loads. The analysis is completed when a horizontal line is passed through the intersection X. This denotes the magnitude of the left and right reactions as shown.

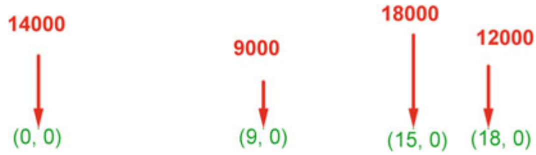
Chapter 3 Exercises

Exercise 3.1 Given known locations of several parallel, vertical loads, establish the magnitude and the centroid of the resultant.

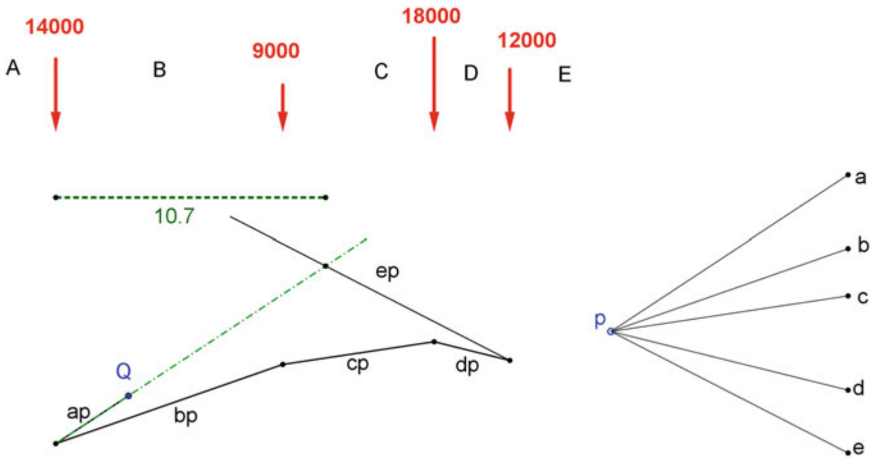
Exercise 3.1 solution



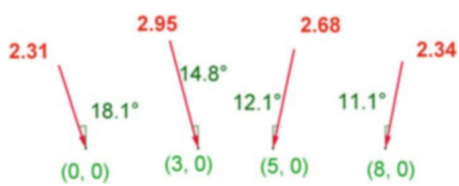
Exercise 3.2 Given known locations of several parallel, vertical loads, establish the magnitude and the centroid of the resultant.



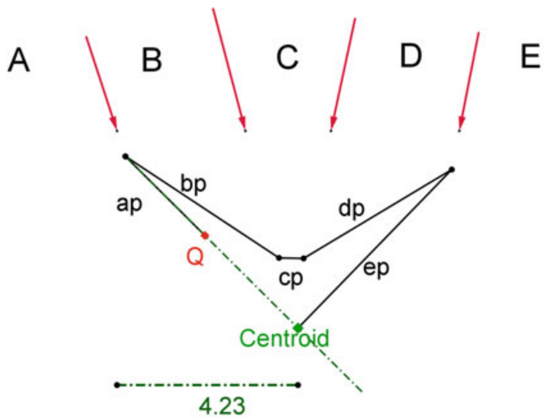
Exercise 3.2 solution The solution shown has the pole on the left of the load line. It can be on either side. Q is arbitrary but it must be within the cluster of loads.

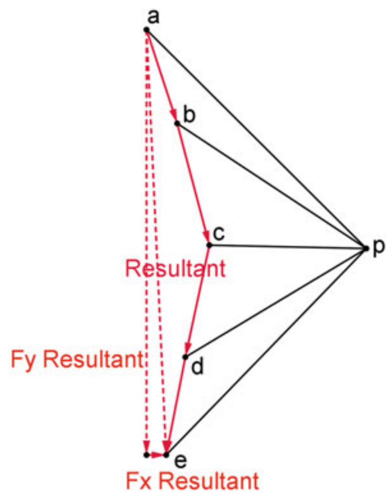


Exercise 3.3 Given known locations of several arbitrarily oriented loads, establish the magnitude and the centroid of the resultant. Then break up the resultant into X and Y components.

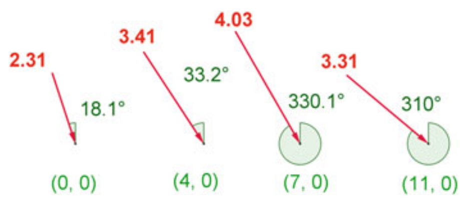


Exercise 3.3 solution

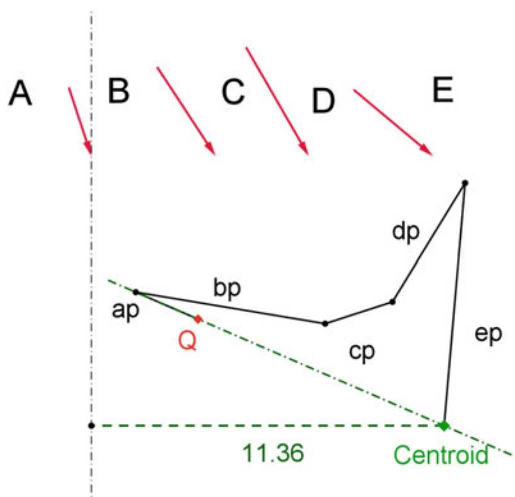


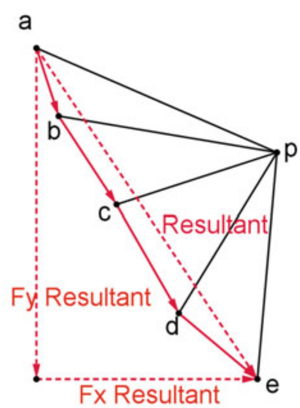


Exercise 3.4 Given known locations of several arbitrarily oriented loads, establish the magnitude and the centroid of the resultant. Then break up the resultant into X and Y components.



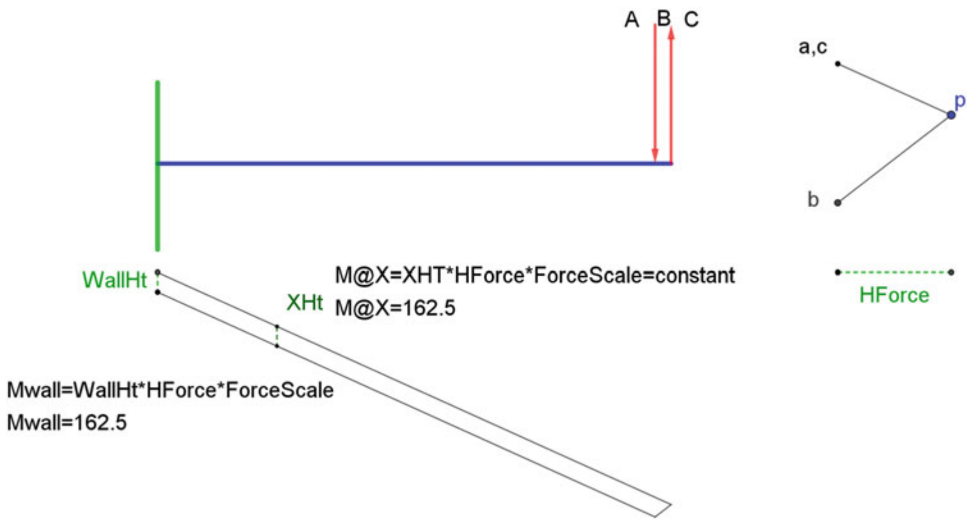
Exercise 3.4 solution





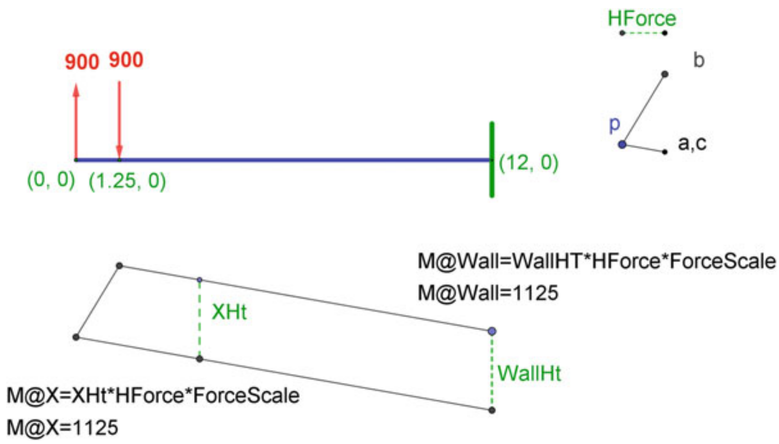
Exercise 3.5 A cantilever beam is subjected to a known couple. Draw the funicular. Calculate the internal bending moment at any X.

Exercise 3.5 solution

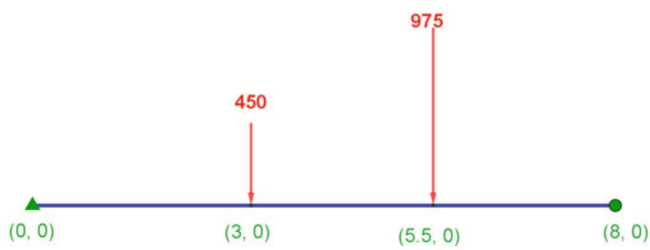


Exercise 3.6 A cantilever beam is subjected to a known couple. Draw the funicular. Calculate the internal bending moment at any X.

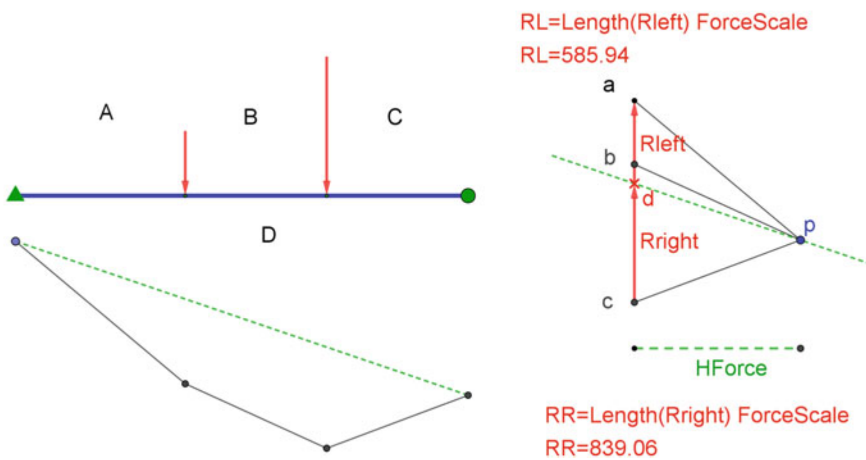
Exercise 3.6 solution



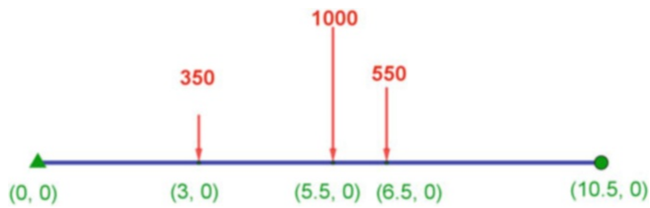
Exercise 3.7 A simply supported beams has loads as shown. Draw the funicular, the closing line and determine the reactions.



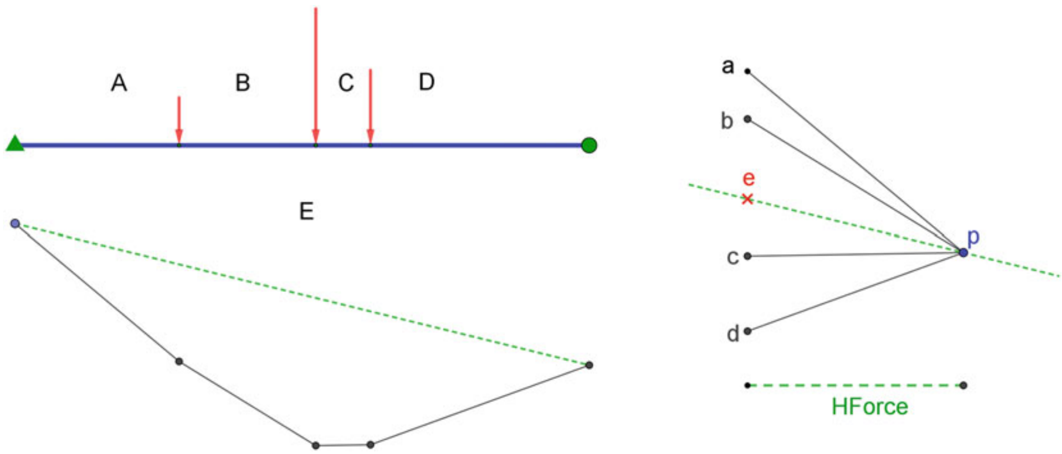
Exercise 3.7 solution



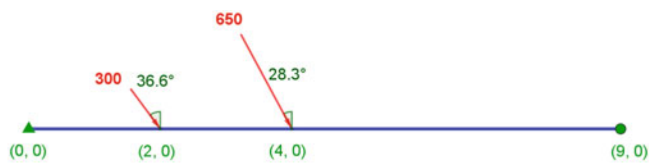
Exercise 3.8 A simply supported beams has loads as shown. Draw the funicular, the closing line and determine the reactions.



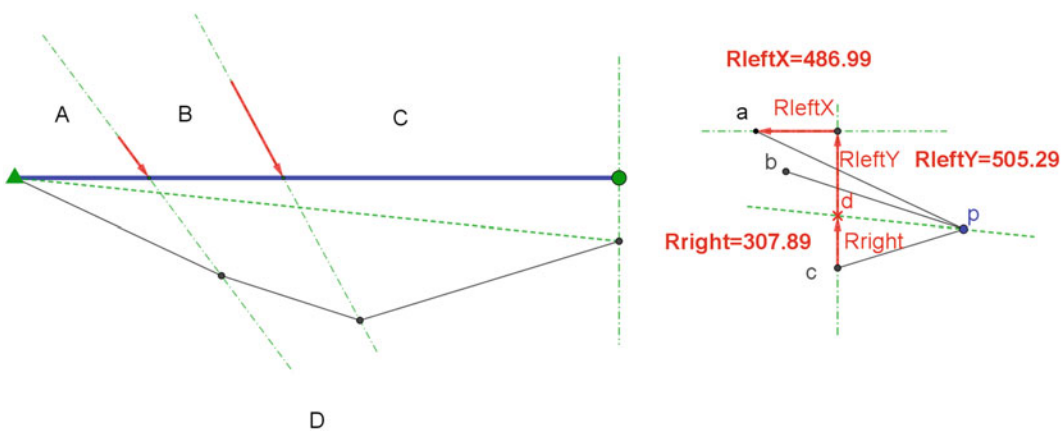
Exercise 3.8 solution



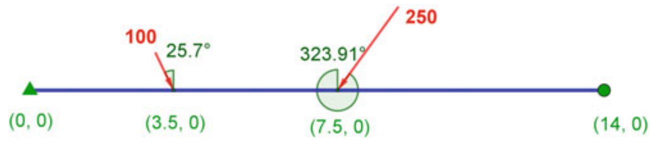
Exercise 3.9 A simply supported beams has loads as shown. Draw the funicular, the closing line and determine the reactions.



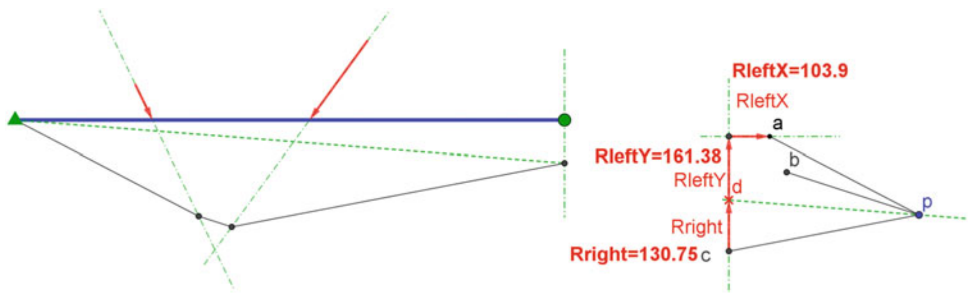
Exercise 3.9 solution



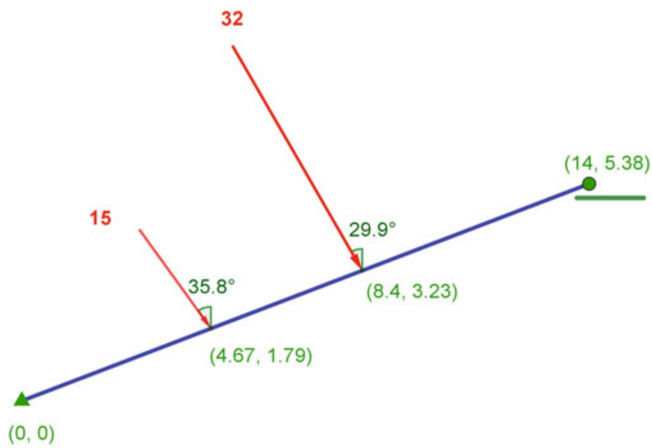
Exercise 3.10 A simply supported beams has loads as shown. Draw the funicular, the closing line and determine the reactions.



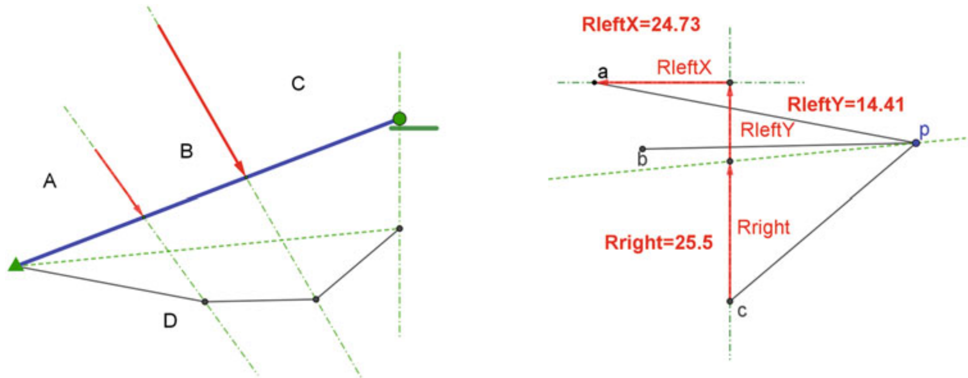
Exercise 3.10 solution



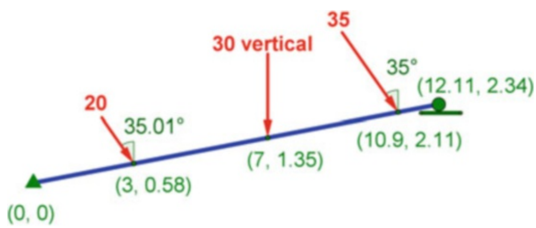
Exercise 3.11 A simply supported beams has loads as shown. Draw the funicular, the closing line and determine the reactions.



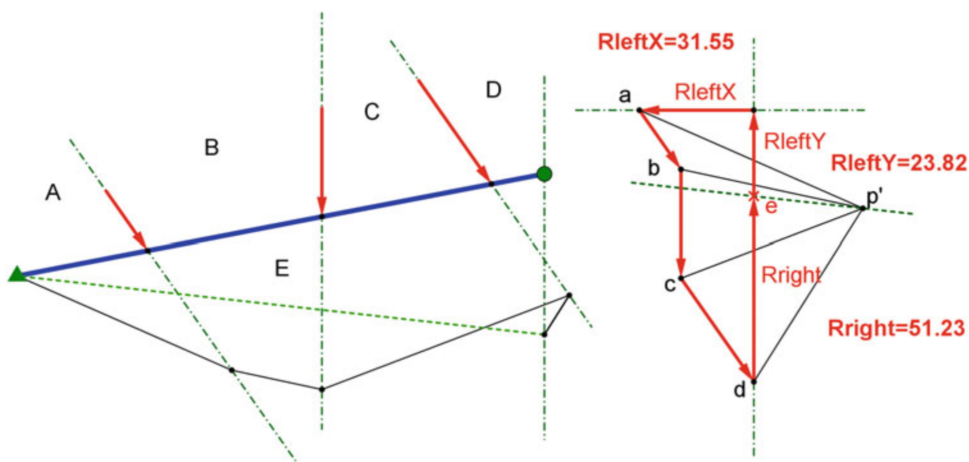
Exercise 3.11 solution



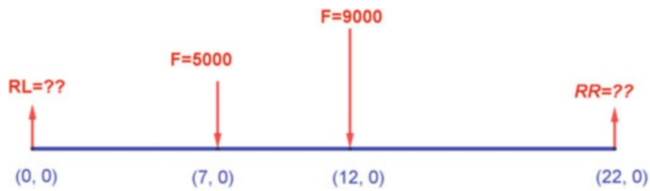
Exercise 3.12 A simply supported beams has loads as shown. Draw the funicular, the closing line and determine the reactions.



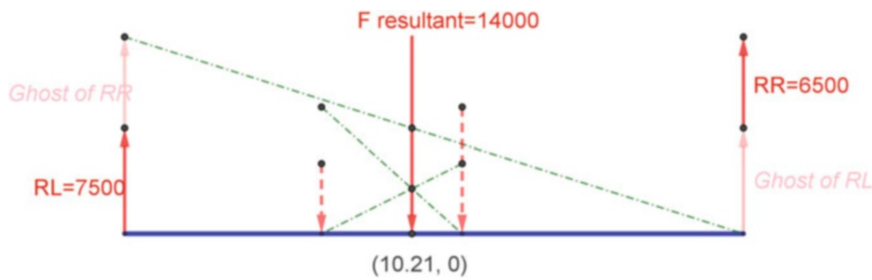
Exercise 3.12 solution



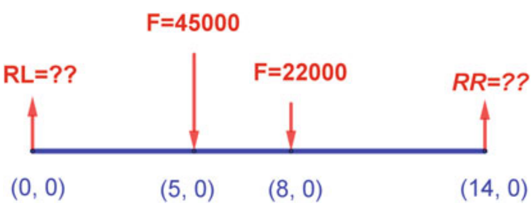
Exercise 3.13 A simply supported beams has loads as shown. Use the Inverse Inverse Axis Method to determine the reactions.



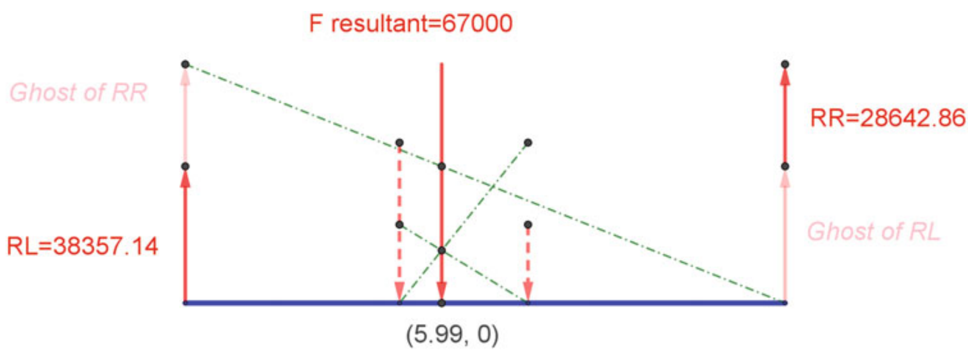
Exercise 3.13 solution



Exercise 3.14 A simply supported beams has loads as shown. Use the Inverse Inverse Axis Method to determine the reactions.



Exercise 3.14 solution



4 The Funicular and Moments

In Chap. 3 it was shown that the funicular is a powerful tool that has multiple levels of meaning. It has a physical interpretation, namely the shape that a chain would take on due to multiple loads. That shape can then be used to identify the precise location of the resultant of these loads, by intersecting the starting and ending slopes of the funicular. But in this chapter, yet another meaning of the funicular is revealed and explained. The funicular automatically generates a bending moment diagram. This was introduced in Chap. 3. It will be helpful to summarize what a bending moment actually is and how it links to a funicular. In Fig. 4.1, consider a force F and find its moment about point O. This diagram in Fig. 4.1 is the form diagram, although there is very little information other than verticality of F and distance to O.

Fig. 4.1 Form diagram of cantilever subjected to point load

The force diagram requires a force vector F drawn at some convenient scale, parallel to the original force F, and any convenient pole p. Use points a and b as the start and end points of the loading vector where the drop a to b captures the magnitude of the force in the scaled force diagram. Rays are drawn from a to p and from b to p. Both the form and the force diagrams are shown in Fig. 4.2. Also shown in Fig. 4.2 is an alternate understanding of the points a and b. In the form diagram, we use capital letters between the loads. The letters identify the overall global forces. Corresponding lower case letters are used in the force diagram to identify each force, name the load F and the external vertical reaction at O. Finally in Fig. 4.2, the arbitrary distance from the load line ab to the pole p is labeled as $HForce$. The meaning of this name will be made clear shortly.

E. Saliklis, *Structures: A Geometric Approach*, https://doi.org/10.1007/978-3-319-98746-0_4

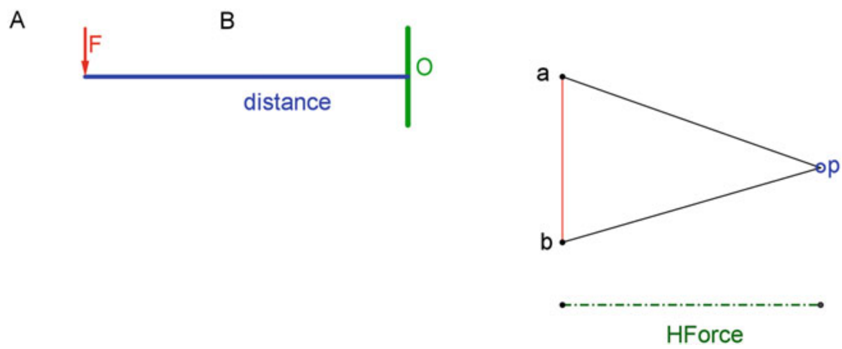
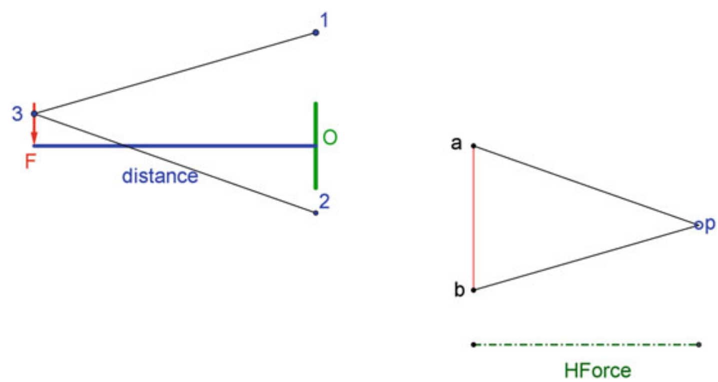


Fig. 4.2 Force diagram begins with arbitrary pole *p*

Of course, the moment M of F about point O is $M = F \cdot distance$ where $distance$ is the perpendicular distance from the line of action of F to a line parallel to F through O.

Transferring the line of action of rays in the force diagram over to the form diagram, creates funicular segments, thus forming a funicular. To do this, choose any point *3* on the line of action of the force F and draw line *1-3* which is parallel to *bp* and line *2-3* which is parallel to *ap*. This creates two similar triangles, *123* and *abp*. Resizing the diagram and changing the Force Scale is one way to demonstrate that they are indeed similar.

Fig. 4.3 Rays of force diagram transferred back to form diagram



Since they are similar, we can state that $\frac{F}{HForce} = \frac{1\ to\ 2}{distance}$

Which means that $F \cdot distance = 1\ to\ 2 \cdot HForce = M$

This important insight tells us that the bending moment M at the wall point O is equal to *length 1-2* multiplied by *force HForce*, scaled by the force scale that the force diagram was drawn at. This may seem laborious in this example, but subsequent examples will demonstrate that the method is extremely powerful and easy to use.

The following example has four parallel, vertical forces. Suppose that they are acting downward on a beam simply supported at its ends. The bending moment in the beam, at a section defined by the line with a hollow red circle between *F3* and *F4*, is to be calculated. Figure 4.4 shows many steps simultaneously, it is worth studying in detail.

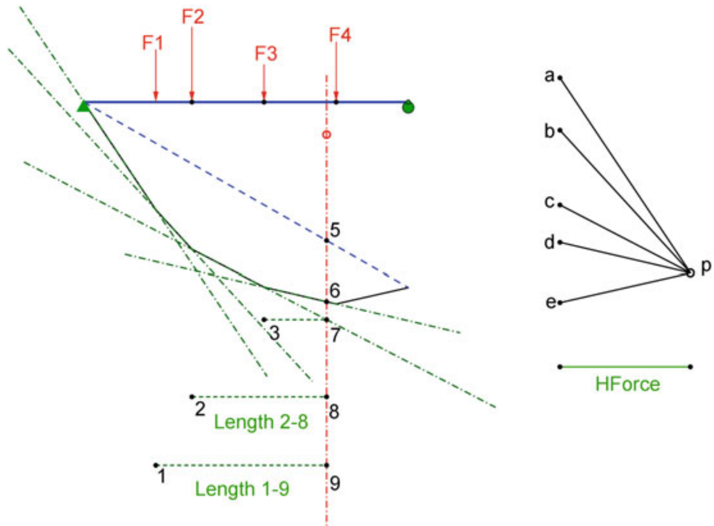


Fig. 4.4 Rays of force diagram become a funicular on form diagram

Forces $F1$, $F2$, $F3$, and $F4$ are drawn to some convenient scale in the force diagram, $F1$ shown by the drop ab in the force diagram, $F2$ shown by the drop bc, $F3$ by cd, and $F4$ by de. The location of the pole is arbitrary, but it is necessary to capture the horizontal span of the vertical load line to the pole and label this force $HForce$. Ray ap is transferred back to the form diagram, it is the first segment of the funicular, it starts at the left end of the beam and continues until it passes the line of action of $F1$. Ray bp is the second segment of the funicular, it starts where the first segment ended, and it continues till it intersects the line of action of $F2$. This continues for all five rays. Notice that the blue dashed line connecting the start and end of the funicular is not horizontal, this is due to the arbitrary location of the pole. This line need not be horizontal, even for a horizontal beam, if the loads are all vertical. This line is known as the closing line, and the distance that the funicular lies from the closing line defines the magnitude of the moment. This "secret" was known centuries ago by masons who built grand masonry arches, albeit in a different knowledge set. More on that topic later. For now, recall what was established in Fig. 4.3, namely that the moment of a force about a point can be readily calculated graphically. In Fig. 4.4, the moment induced by $F1$ about the section defined by the red open circle o must be $M_{F1} = F1 \cdot 1 \; to \; 9$ it also must be $M_{F1} = 9 \; to \; 8 \cdot HForce \cdot ForceScale$ Similarly, the moment of $F2$ about o must be $M_{F2} = F2 \cdot 2 \; to \; 8$, yet it also must be ... what? Hopefully it is clear that $M_{F2} = Segment \; 87 \cdot HForce \cdot ForceScale$. Finally, the moment of $F3$ about o must be $M_{F3} = Segment \; 76 \cdot HForce \cdot ForceScale$

Combining these previous equations, we can logically conclude that the moment due to $F1$ and $F2$ and $F3$ about o is

$$M_{F1F2F3} = HForce \cdot (9 \; to \; 8 + 8 \; to \; 7 + 7 \; to \; 6) \cdot ForceScale$$

Or

$M_{F1F2F3} = HForce \cdot Segment96 \cdot ForceScale$. It is positive because the rotation is about an axis going out of the page, i.e. in the positive z direction.

This is NOT the internal bending moment at o of a simply supported beam, it is the moment at o induced by $F1$, $F2$ and $F3$. Something is missing for it to be the final moment at o. What is missing? The answer is that the effect of the left reaction is missing! In Fig. 3.14, the graphical calculation of reactions of a simply supported beam was introduced. However, the actual value of the left vertical reaction is not needed here, because moment of a force is fully captured by the scaled horizontal segment spanning from the line of action of the forces to the pole, called $HForce$, multiplied by the tangential deviation distance (segment 9 to 6 for example in Fig. 4.4). Thus, the moment induced by R_{Left} at a section defined by the red o in Fig. 4.4 is $M_{from\ RLeft} = -1 \cdot HForce \cdot ForceScale \cdot segment\, 9\, to\, 5$. It is negative because the moment is about an axis going in to the page, i.e. the $-z$ axis.

Summing up the net effect of the three loads' moment about o with the left reaction's moment about o gives the following answer and important insight.

$M_{net\ at\ o} = HForce \cdot ForceScale \cdot ((9\ to\ 6) - (9\ to\ 5)) = -HForce \cdot ForceScale \cdot (6\ to\ 5)$, negative because the net moment is into the page, or compression on top.

This is a profound idea, so take some time absorbing it. The distance, from one of the infinite number of funiculars to the corresponding closing line, precisely captures the bending moment in the element, scaled by $HForce \cdot ForceScale$. This idea is a centuries old one, yet it can be used in the twenty first century to provide design ideas. Such design ideas today are the domain of the rigorously trained engineer. Even for most engineers today, this concept is not part of traditional engineering curricula. Thus, the calculation of the internal bending moment at any point in the beam is readily calculated as the distance of the funicular from the closing line, multiplied by the scaled force $HForce$ which establishes the sag of the arbitrary funicular. The sign of the bending moment will be quantified more generally in subsequent examples.

Figure 4.5 shows a new pole location, and a slightly different location for the section being investigated (marked by the red O). It is a worthwhile exercise to actually program the theoretical internal bending moment in the beam at this section, defined by the red o. This can be readily programmed once the distance to the cut is captured.

The key idea that must be remembered, here in this chapter and again in Chap. 8, is that

$$M_{@cut} = HForce \cdot (y\ to\ closing) \cdot ForceScale$$

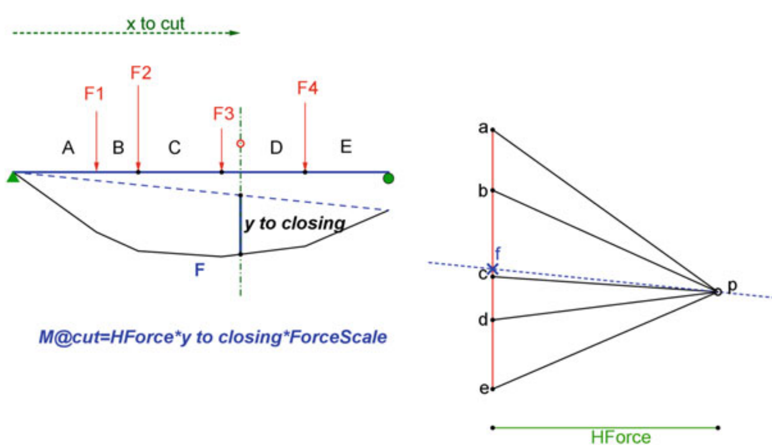
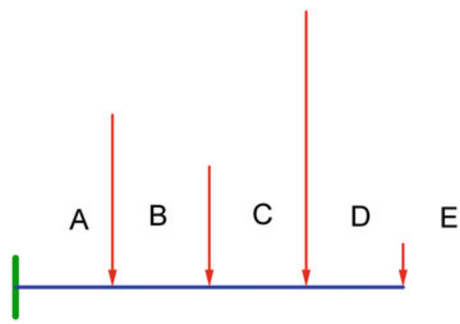


Fig. 4.5 Graphical definition of bending moment anywhere

In Fig. 4.5, a line parallel to the closing line is drawn through the arbitrary pole p. Where this line intersects the load line in the force diagram establishes the key to finding the magnitude of the reactions supporting the beam. The segment from point e to this intersection captures the right reaction, and the segment from this intersection to a captures the left reaction. Of course, both of these forces (e *intersection* and *intersection* a) must be scaled by the force scale factor. Having a quantified value for the reactions allows for the theoretical computation of the bending moment. Because the loads are discrete point loads, the agreement between the theoretical and the graphical bending moment values is exact. If the load is distributed, some approximations must be made, because the funicular will still be assumed to take on a piecewise linear form. It is worth repeating that the closing line need not be horizontal, and that for VERTICAL loads, the distance necessary for moment calculation (here segment *y to closing*) is a vertical distance, not a distance perpendicular to the closing line.

The following example will add a bit of knowledge to the user's toolkit and will enhance the understanding of the graphical construction of the bending moment diagram. Figure 4.6 shows a cantilever beam subjected to multiple vertical point loads.

Fig. 4.6 Cantilever beam with multiple vertical point loads



In Fig. 4.7, rather than describing the forces as *F1*, *F2*, etc, it is preferred to describe the forces with a two-letter index, capital letters in the spaces between the loads. This is known as Bow's Notation. Thus, the second load from the left will be referred to as *BC*

The force diagram can used to find the shear diagram, this is shown in Fig. 4.7. The heights of the graph areas represent the magnitude of the shear *V*, the shear in spaces *A* and *B* is shown qualitatively.

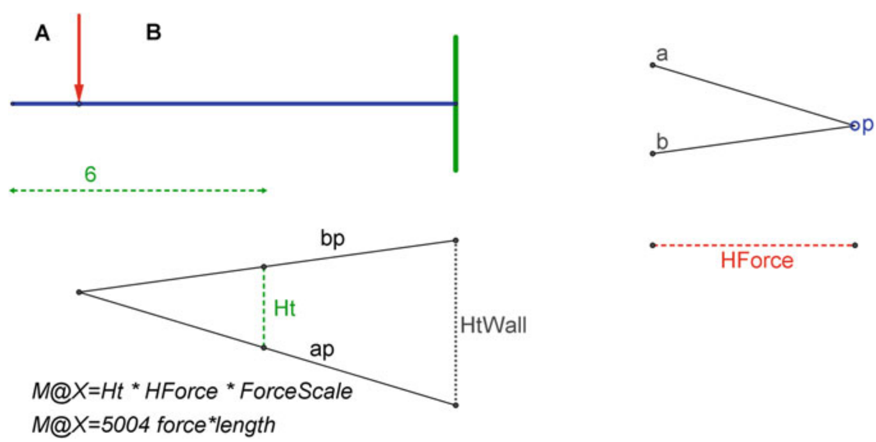


Fig. 4.7 Graphical construction of shear diagram

The shear is constant between the point loads, and the shear is undefined directly in the line of action of any point load. To calculate the bending moment diagram, the funicular must be drawn as was described previously, but because this is a cantilever, a line parallel to the fifth ray *ep* is drawn from the end of the fourth funicular segment, back till it intersects a point in line with the start of the first funicular segment. The vertical distance between elements of the funicular is one part of the internal bending moment, which is completed when multiplied by *HForce*, and scaled by the force scale factor. For example, to capture the internal bending moment at a section at the left end fixed support, calculate the distance between points *1* and *2* in Fig. 4.8. To capture the moment in a section directly under the first load AB, calculate the distance between points *3* and *4*. It is worthwhile to compare these graphically calculated moments to the theoretical moments, to prove that they indeed match exactly. To quantify this example, set the cantilever beam equal to *8* units of length, with loads at lengths *2, 4, 6* and *8*. Load *AB* is *1000* units of force, load *BC* is *700*, load *CD* is *1600* and load *DE* is *250* units of force. With these quantities, the moments at the left support (*Mwall*) and directly under the *1000* unit load (*M@AB*) are shown quantitatively in Fig. 4.8. Both these moments will induce tension at the top of the cantilever.

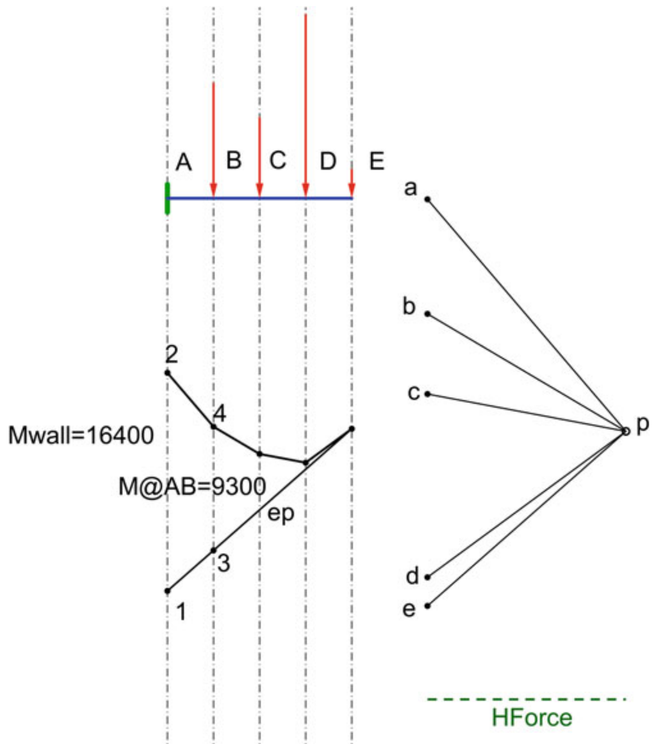


Fig. 4.8 Bending moment found quantitatively at two points

The next example demonstrates a simply supported beam with an overhang. The beam is subjected to a uniformly distributed load, which must be modeled as a series of discrete point loads. Such modeling will induce some error when comparing the graphical answer to theoretical ones. Error is reduced as the number of discrete point loads is increased. A good rule of thumb is to divide the load into at least ten segments.

This example will begin with quantitative information to highlight a few important features about the discretization of loads. Suppose the beam is 10 units of length long, with the first 3 units of length being the overhang. Also suppose that the beam is subjected to a uniformly distributed load of 975 units of force per length. The load is somewhat arbitrarily discretized into ten parts, each part carefully placed at the centroid of each discretized segment. The magnitude of each point load is

$$975\,\frac{force}{length}\cdot\frac{10\;length}{10}=975\,force$$

The centroidal placement of each point load is shown in Fig. 4.9.

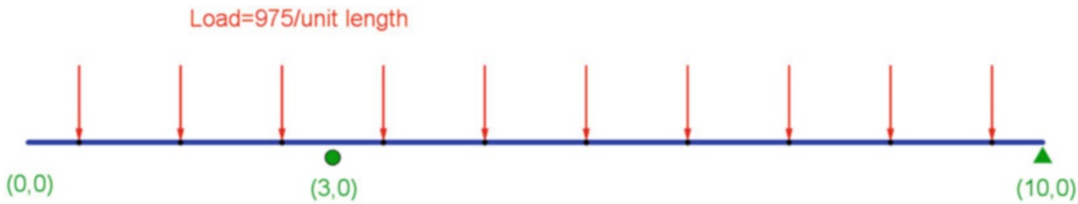


Fig. 4.9 Discretizing a uniform load into ten point loads, each point load is at the centroid of the segment

Using Bow's Notation, capital letters are placed in the spaces between loads. This includes placing a letter that captures the as yet unknown reactions. A force diagram is drawn to some convenient scale, an arbitrary pole is created and rays are drawn. The force diagram uses corresponding lower case letters. Noteworthy in Fig. 4.10 is the fourth ray, *dp*. It does not stop when it crosses the blue vertical line which is in line with the left reaction. It stops when it enters space *E*. The first ray *ap* must not be improperly placed, it captures space *A*, which begins to the left of the left-most support and space *A* continues until the first point load is placed on the beam. Thus, the funicular ray *ap* begins along the line of action of the left-most support, here a vertical line, and it terminates at the line of action of load *AB*. A blue dashed closing line is drawn between the start and end of the funicular. That line is transferred over to the force diagram through the pole, and the point where this line intersects the vertical load line is noted as Point *l*, shown as a red *x*. Having Point *l* which corresponds to space *L* unlocks the beam reactions.

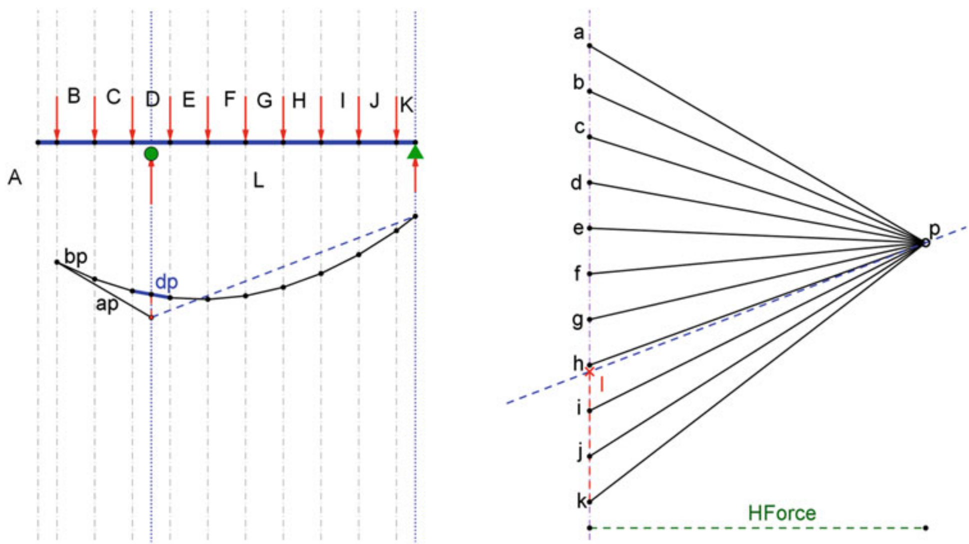


Fig. 4.10 Funicular of beam with an overhang

The moment in the beam directly in line with left support is captured by the vertical distance between the funicular and the closing line at that section. In Fig. 4.11 this is shown as the distance between points *1* and *2*. This distance must be multiplied by *HForce* (the horizontal distance of the pole *p* to the vertical load line) multiplied by the force scale. In this example, that bending moment matches theory exactly. To capture the bending moment elsewhere, say at $x = 8$ units of length (or 2 units of length from the right pinned support), the bending moment is the distance between points *3* and *4*, multiplied by *HForce*, multiplied by the force scale. Here, the bending moment does not exactly match the theory because the bending moment diagram is theoretically parabolic, whereas this technique models it as a piecewise linear curve. Another beautiful insight arises from the study of the moment at *12* versus the moment at *34*. At *12*, the funicular is above the closing line, at *34* the funicular is below the closing line. This demonstrates that the moments are of opposite sense. At *12*, there is compression on the bottom of the beam, at *34* there is compression at the top.

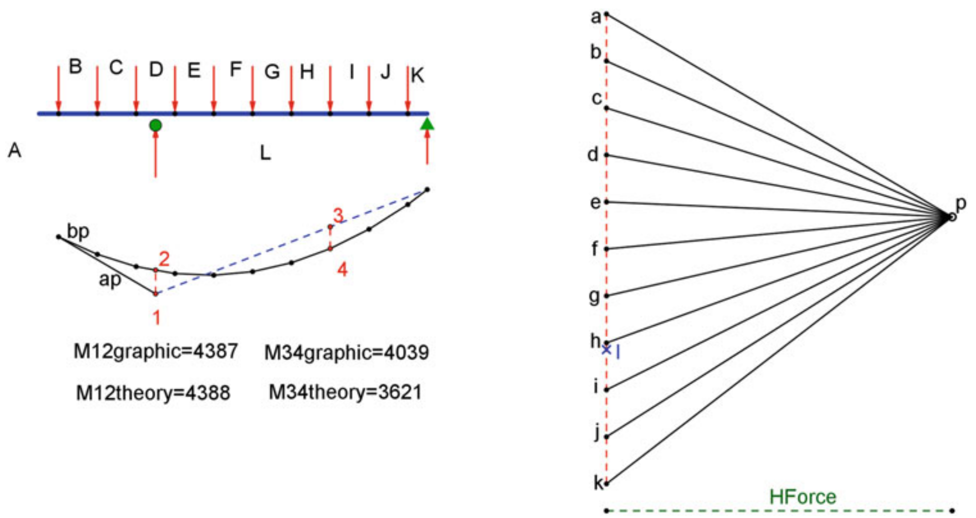
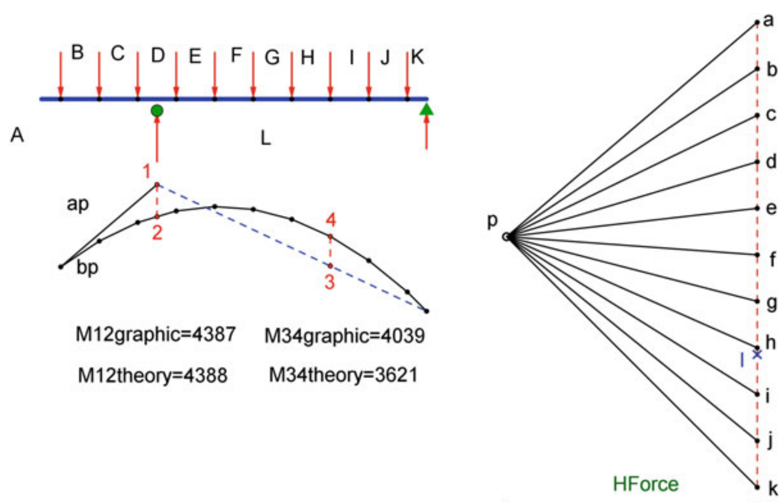


Fig. 4.11 Quantifying the bending moment graphically

The choice of how to present the bending moment diagram, either on the compression side or the tension side, is arbitrary. It just "depends on how your parents raised you"... . Figure 4.12 shows how moving the pole over to the other side of the load line flips the presentation of the bending moment diagram.

Fig. 4.12 Alternate presentation of previous bending moment diagram



The following example analyzes a statically determinate beam which has an internal hinge. For quantitative purposes, the beam has a pin support at the left end, a vertical support at lengths $x = 6$ and *14* units of length. The internal hinge is at $x = 9$ units of length. The beam is subjected to a uniformly distributed load of *750 force/length*. This is shown in Fig. 4.13.

Fig. 4.13 Uniform load on beam with an internal hinge

To discretize the uniform load, suppose that the total load is divided into ten pieces, but the discretization captures two distinct portions of the beam. The first discretization takes the 9 unit length segment and divides it into five pieces. The second discretization takes the 5 unit length and divides it into five pieces. This is statically equivalent to the original loading (Fig. 4.14). In fact, two statically equivalent loads could have modeled the original loading, a $9\ \mathit{length} \cdot 750\ {}^{\mathit{force}}/_{\mathit{length}} = 6750\ \mathit{force}$ at x = 4.5 and a $5\ \mathit{length} \cdot 750\ {}^{\mathit{force}}/_{\mathit{length}} = 3750\ \mathit{force}$ at x = 11.5

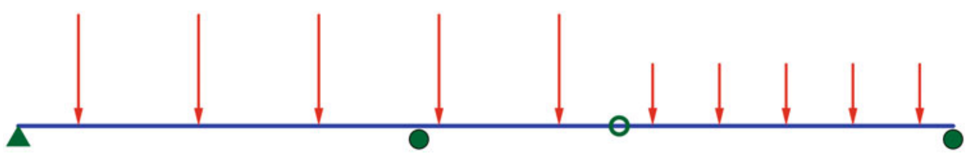


Fig. 4.14 Discretizing uniform load into ten point loads, five on one side of hinge, five on the other

Using Bow's notation, meaning capital letters in the spaces between loads, and lower case letters on the scaled force diagram allows for the creation of rays in the force diagram which meet at the arbitrary pole *p*, shown in Fig. 4.15 left of the load line.

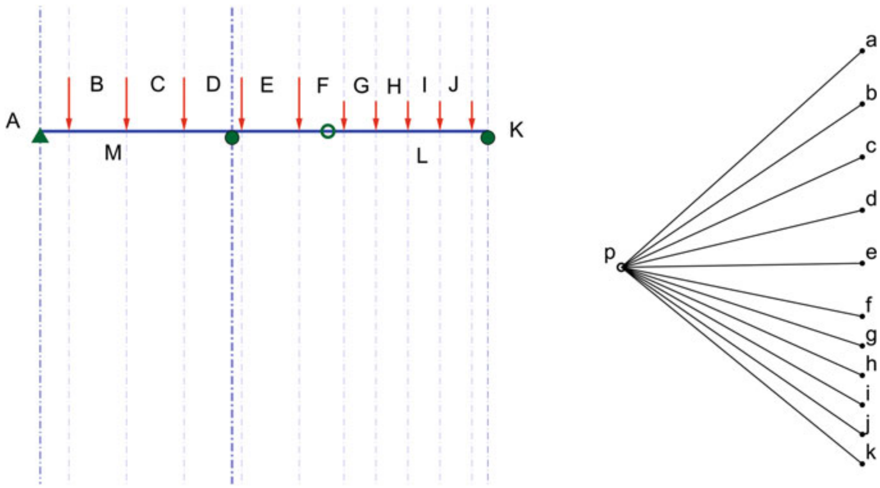


Fig. 4.15 Force diagram rays and key vertical lines on form diagram

Noteworthy is space *D*. It spans between the loads, it does not terminate in line with the roller support, that line is shown as a bolder blue dashed line.

Draw the eleven rays (*ap* through *kp*) as segments of the funicular which correspond to spaces *A* through *K*. At the termination of the final funicular segment corresponding to space *K*, draw a line that intersects the funicular at the abscissa corresponding to the internal hinge. This is shown as a dashed red line, and it is known as a "partial closing line" or more simply a part of the closing line. Locate where this partial closing line intersects the next support path. Connect that intersection to the leftmost support path, again with a red dashed line, this is the second partial closing line. The segmental line, connecting the end of the funicular, passing through the hinge abscissa, and connecting the support paths is the kinked closing line. Any deviation of the funicular from this closing line denotes internal bending moment. This explains why the closing line must intersect the funicular at the abscissa of the hinge, at that spot there can be no internal bending, the funicular and the closing line match there. Thus, the moment in the beam directly over the central support is found by the segment length *1* to *2*, multiplied by the *Force Scale* and multiplied by the *HForce* value (Fig. 4.16).

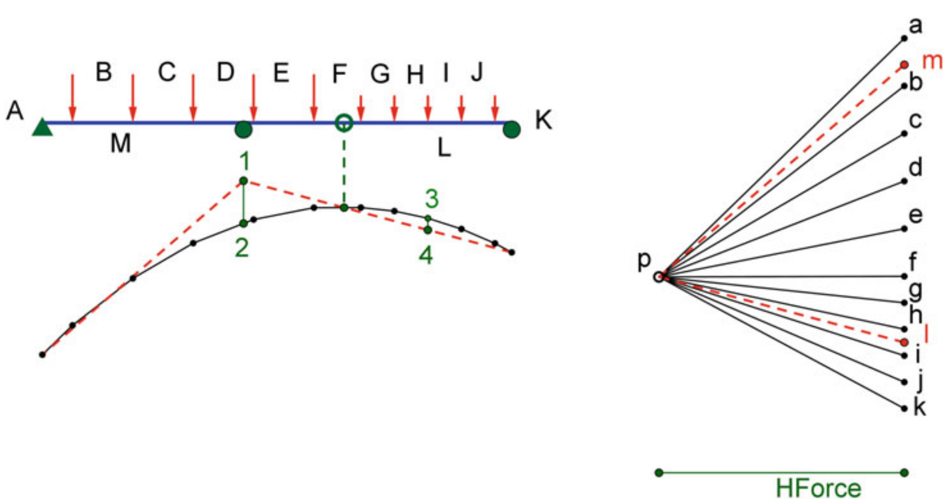


Fig. 4.16 Closing line must pass through hinge, thus it starts at *K*

The two partial closing lines are transferred back to the force diagram to locate points *l* and *m*. Now, a segment from *k* to *l* represents the right reaction, a segment from *l* to *m* represents the central reaction, and a segment from *m* to *a* represents the left reaction.

Figure 4.17 shows the effect of moving the pole to the right of the load line. Nothing changes, except for the style of the bending moment diagram. It is worth noting that in this example, the FIRST partial closing line to be drawn utilizes the free body that has only one support. That partial closing line starts at the support and passes through the internal hinge. This is totally analogous to the technique of breaking apart the structure to use algebraic statics, and then analyzing the "simpler" free body, i.e. the free body with one, not two unknown supports.

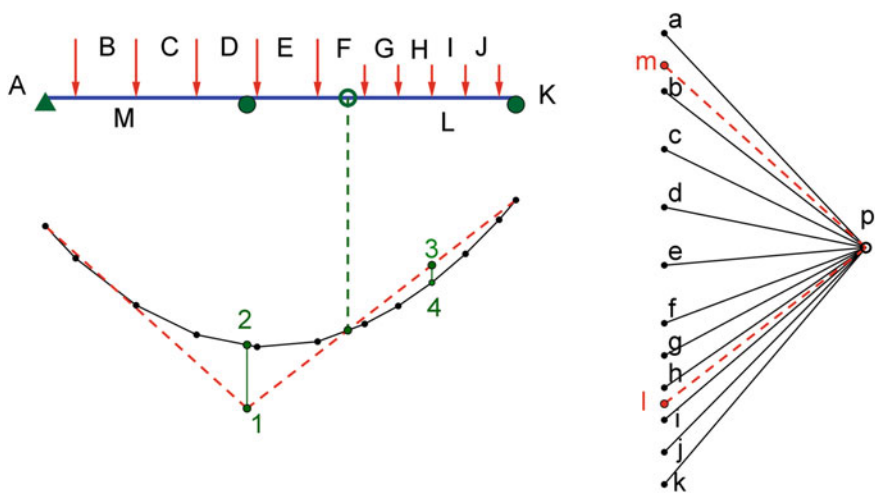


Fig. 4.17 Changing the presentation of the previous bending moment diagram

In the following example, a beam has a fixed support at its right end, a roller support at its left end, and an internal hinge as shown. The beam is subjected to two point loads. The force diagram is drawn to some convenient scale and an arbitrary pole *p* is drawn which allows for three rays to be drawn (Fig. 4.18).

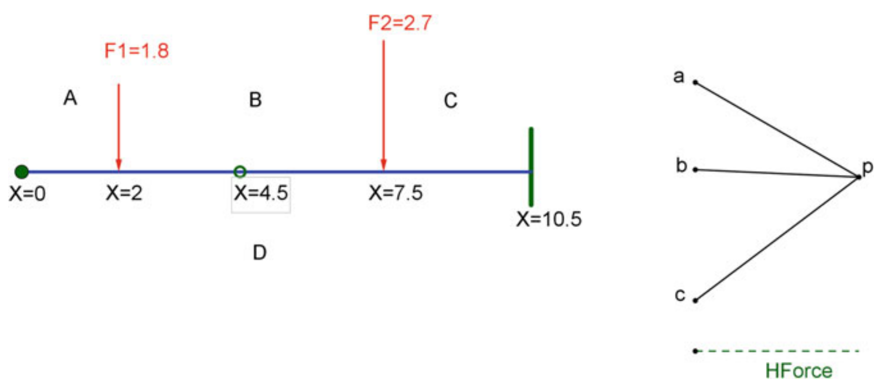


Fig. 4.18 Propped cantilever with internal hinge and two point loads

The three segments of the funicular are laid down in Fig. 4.19, corresponding to the slopes of the rays of the force diagram. After creating the three segments of the funicular, note that just as in algebraic statics, the next step must be to start with the “simpler free body”, here that is from the left support to the internal hinge, not from the fixed wall to the hinge. Thus, draw a line from the left end of the funicular and pass it through the internal hinge. Note where that line intersects the path of the right vertical reaction, here that is point *2*. The green dashed line is the closing line, and as always, the distance from the closing line to the funicular is the magnitude of the internal bending moment of the beam. Here, the distance between points *1* and 2, multiplied by the *Force Scale*, multiplied by *HForce* gives an internal moment at the fixed support of *12.9 force length*. The internal bending moment directly under the force *F1* is the distance between points *3* and 4, multiplied by the *Force Scale*, multiplied by *HForce* gives an internal moment at the fixed support of *−2 force length* i.e. it is of opposite sign from the moment in the wall because the funicular is on the opposite side of the closing line (Fig. 4.19).

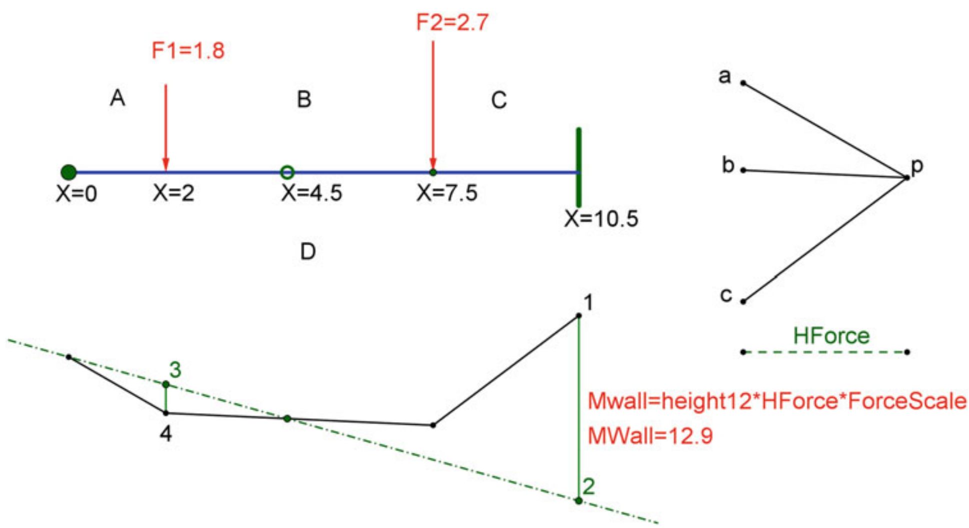


Fig. 4.19 Quantifying the bending moment diagram

As always, moving the pole has no effect on the answers, it simply presents a different representation of the bending moment, drawing on the compression side versus the tension side of a beam (Fig. 4.20). A fairly astonishing feature of this technique is that the reactions are not needed to establish the bending moment anywhere in the beam! Notice that lower case letter *d* was never needed in this example because the reactions were never calculated!

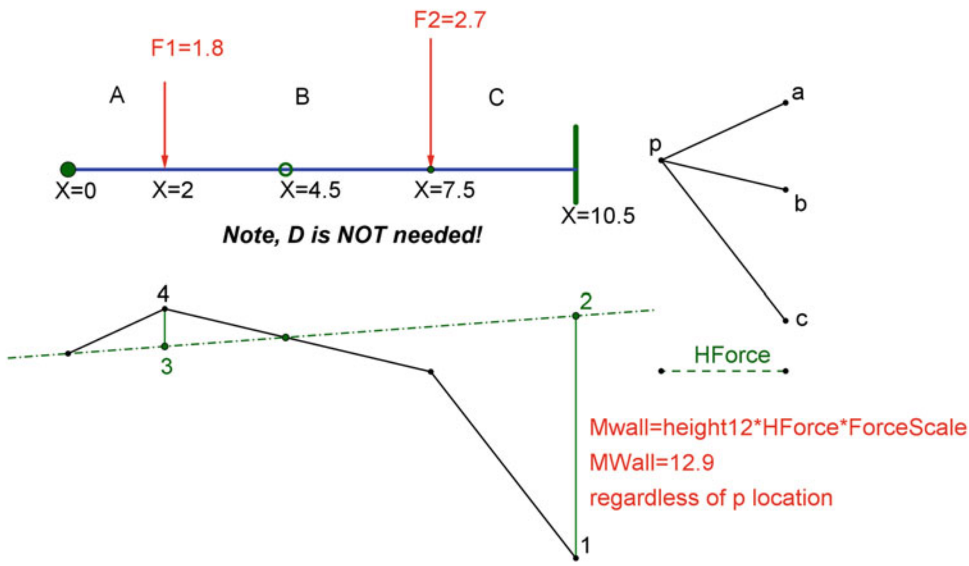
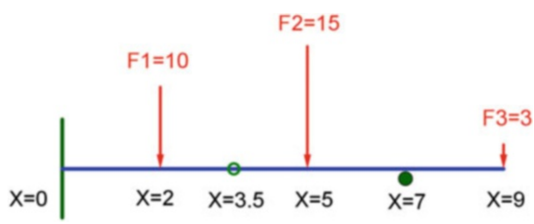


Fig. 4.20 Alternate presentation of previous bending moment diagram

The following example combines elements of the previous examples. A beam has a fixed support at the left end, and a small cantilever at the right end. An internal hinge is placed as shown in Fig. 4.21.

Fig. 4.21 Propped cantilever with internal hinge and an overhang



Using Bow's Notation, capital letters are placed between the loads. Notice that space *D* in Fig. 4.22 captures everything to the right of the tip load, and the underside of the beam, until a vertical line which passes through the roller support. To the left of that support, but before the fixed support is space *E*. As shown in the previous example, no reactions are needed, simply draw the funicular based on the force polygon rays, then starting with the "simpler free body", (here to the right of the internal hinge), pass a closing line from the terminus of the funicular through the hinge, noting where it changes from space *E* to space *A*.

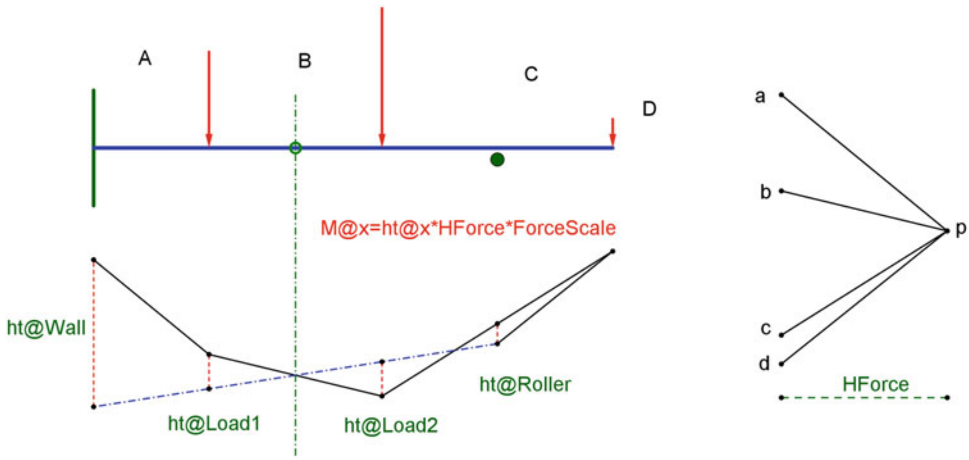


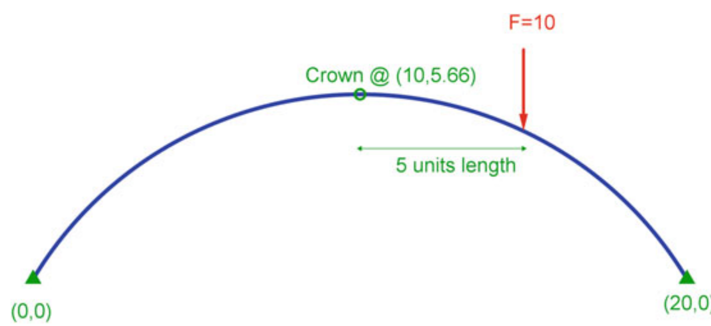
Fig. 4.22 Qualitative calculation of bending moment at multiple cross sections

The link between the bending moment diagram and the deflected shape of the structure will be explored deeply in Chap. 5.

It is not difficult to see how the statically determinate beam with an internal hinge is analogous to the classic "three hinged arch" problem. Such structures have been used to span enormous distances, even in the nineteenth century. They typically are pinned at each support, and there is an internal hinge at or near the "crown" of the structure, thus the name "three hinged arch". They differ from a two-element truss because three hinged arches may carry transverse loads on each member, thus they experience bending as well as axial forces.

The following example, shown in Fig. 4.23 is just such a three hinged arch. It is a circular arc segment, with a 11.66 unit length radius, centered at (10, −6). It is subjected to a force of 10 units, at a point 5 units of length from the crown.

Fig. 4.23 Three hinged arch with one span loaded



Analogous to the previous examples of beams with an internal hinge, here the technique is the same, i.e. analyze the "simpler" free body first. In this problem, the simpler portion is from the left support to the crown, since there are no external loads on this portion of the structure. That means that the resultant of the left reaction must pass through the crown itself. If the resultant of the left support did not pass through the crown, it would induce a net moment at the crown, which is impossible since the crown does not transfer any moment to the right hand free body. This incorrect assumption is shown in Fig. 4.24 to emphasize this point.

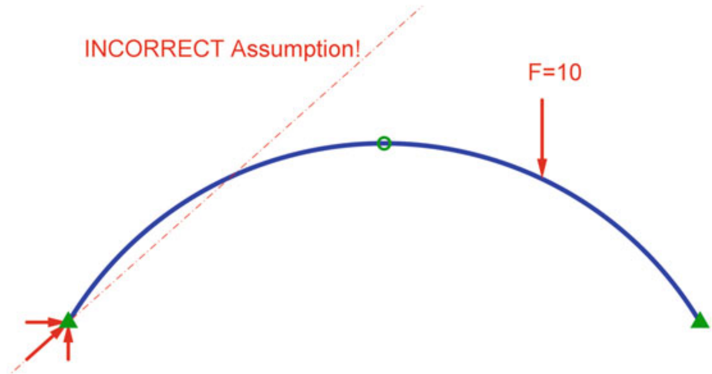


Fig. 4.24 Incorrectly placed funicular segment

It is clear from Fig. 4.24 that only if the resultant of the two reactions on the "simpler side" reaction passes through the crown hinge, no moment will be induced about the crown hinge. Use this knowledge to establish the correct slope of this net reaction in the force diagram. Then, graphically find the net right-hand reaction based on the slope of the applied load and where the left side reaction intersects this path. Essentially, the problem has been reduced to equilibrium of a point. This is shown in Fig. 4.25. The slopes of the closing lines establish the slopes of the reactions on the force diagram.

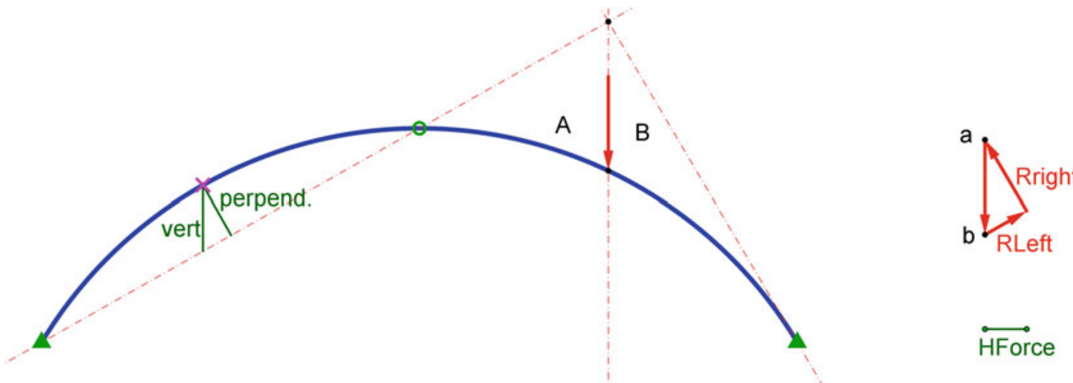


Fig. 4.25 Three hinged arch reduced to equilibrium of a point

In the force diagram of Fig. 4.25, the applied force *AB* is captured by the vector *ab*. On the form diagram, first the slope on the "easier" free body is established (*Rleft*) by passing a line from the support through the crown hinge, and then the point where this line intersects the load path is found, allowing for the *Rright* slope to be readily found. These slopes then are transferred to the force diagram and the magnitudes of the reactions are immediately known. Capture the width of the force diagram and as always, call it *HForce*. This width will be used to calculate the bending moment at any cross section.

The red lines passing through the supports of Fig. 4.25 actually establish the funicular. Recall from Fig. 3.12 that the starting and ending slopes of the funicular represent the slopes of edge forces, pointing to the centroid of loads, essentially the same "equilibrium of a point" idea. In the three hinged arch problem, the funicular is established immediately on the form diagram since the funicular must pass through the internal hinge. This fact allows for the immediate calculation of moment, as the distance of the form from the "centerline" or *neutral axis of the structure creating the from* this funicular defines the magnitude of bending in the form. Notice the subtlety of the previous sentence, in this problem the closing line is the form of the arch itself! The closing line always spans from the origin to the terminus of the funicular, and here it does so, but it curves along the shape of the arch in the span between the ends. In Fig. 4.26, the bending moment at any point left of the crown can be calculated three different ways. One way is to the multiply the vertical left reaction by the horizontal distance to the cut (marked by a pink X) and from this subtract the horizontal left reaction multiplied by the vertical distance to the cut. The second way is to multiply the net left reaction by the perpendicular distance of the closing line (the arch form) to the funicular. The third way is perhaps the most elegant and it echoes the previous techniques. This method uses the vertical distance from the funicular to the closing line, multiplied by $HForce$, multiplied by the $Force\ Scale$, i.e.

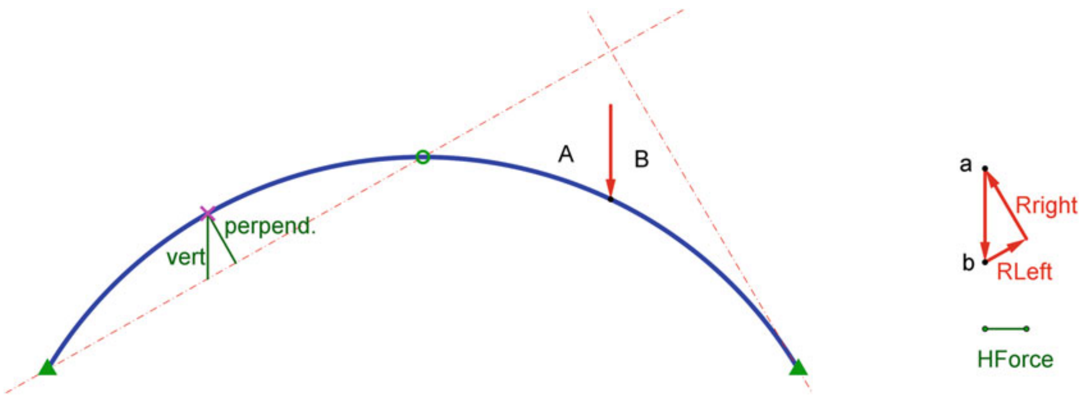


Fig. 4.26 Multiple ways of establishing the bending moment

$$M_{cut\ left} = vert \cdot HForce \cdot Force\ Scale$$

The moment in the arch at a point to the right of the crown but to the left of the applied load is found by the just described method, namely the vertical distance *vert2* from the cut to the funicular, multiplied by $HForce$, multiplied by the $Force\ Scale$. Similarly, the bending moment in the arch to the right of the load until the right support uses *vert3* rather than *vert2*. Notice the signs of the bending moment change at different points in the arch, with no moment as required at the three hinges (Fig. 4.27).

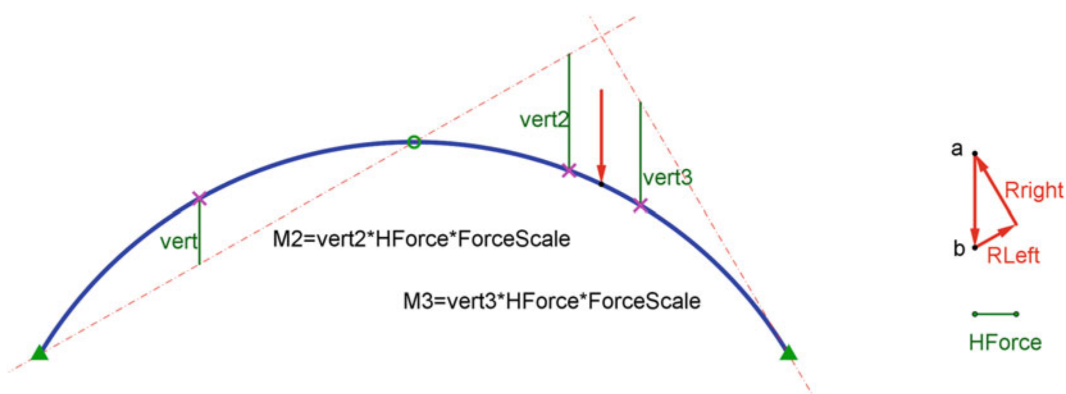


Fig. 4.27 Using *HForce* and the vertical distance to establish bending moment

The next example uses the previous example's geometry, but adds a second load (Fig. 4.28).

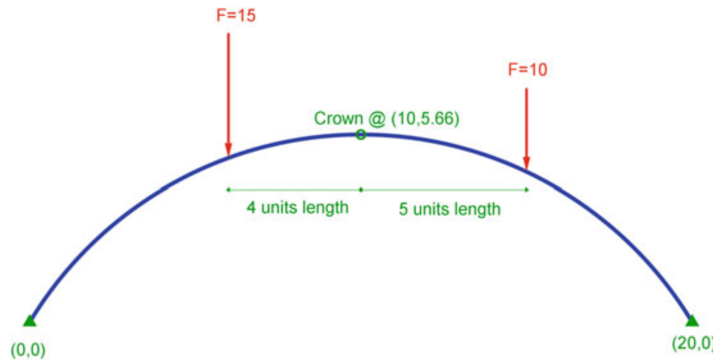


Fig. 4.28 Three hinged arch with each span loaded

Attack this problem in two steps, each step being identical to the previous example, namely assume that the right side load is NOT present, pass a straight line from the right support, through the hinge at the crown till it intersects the path (vertical straight line) of the left-hand load, then a second straight line from that point to the left support. All that is shown in blue. Then repeat with the right hand side, shown in red. These steps are shown in Fig. 4.29.

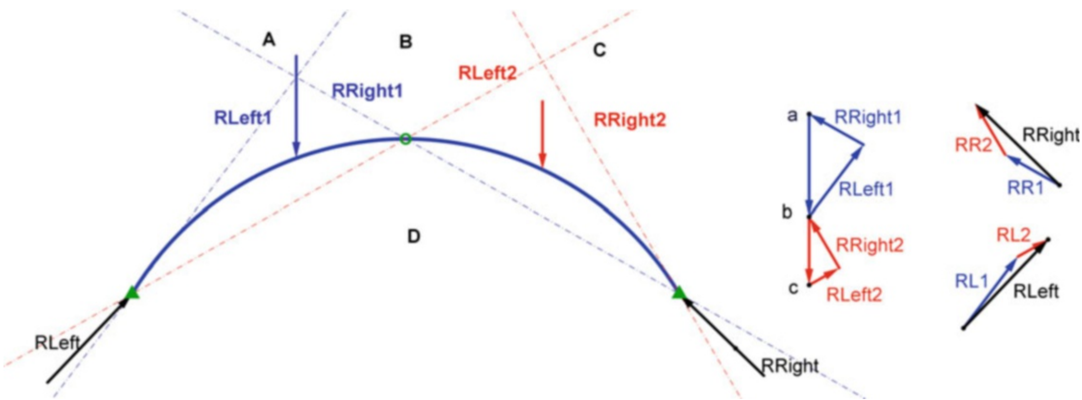


Fig. 4.29 Two step process assuming one span is unloaded each time

This establishes the slopes of each reaction, as if there were only one load. Thus, case 1 (in blue) has a force polygon with a vertical drop from *a* to *b* of 15 units of force. And case 2 (in red) has a second force polygon with a vertical drop from *b* to *c* of 10 units of force. The final reactions are the superposition of the two individual cases, thus the right reaction of case 1 (*RR1*) is vectorially added to the right reaction of case 2 (*RR2*) to establish the final right reaction. Similar calculations are repeated for the left reaction. This is all shown in Fig. 4.29.

In Fig. 4.30, the final resultant reactions are placed on the form diagram. Projecting these reactions till they intersect the path of the loads, and then drawing a straight line through the crown hinge creates the final funicular. Note how much less bending is present in this situation compared to the previous example denoted by the vertical deviation from the funicular. Speculate why that is the case. Speculate how to remove even more bending!

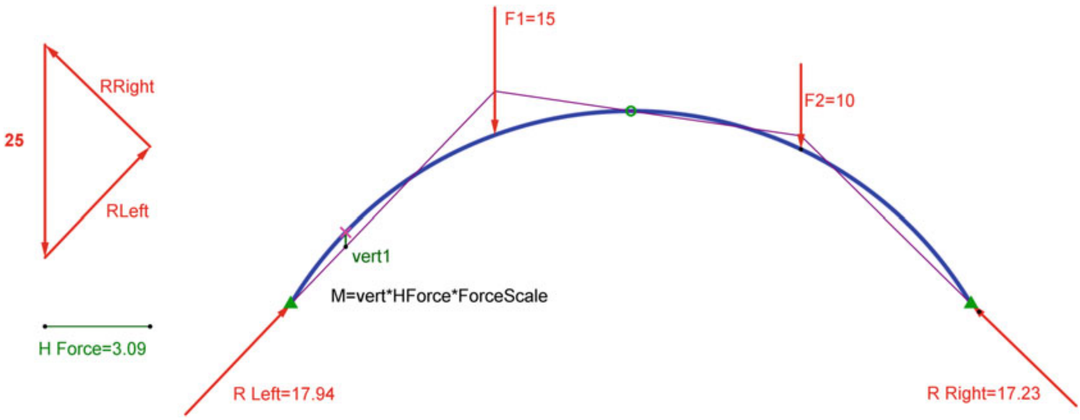
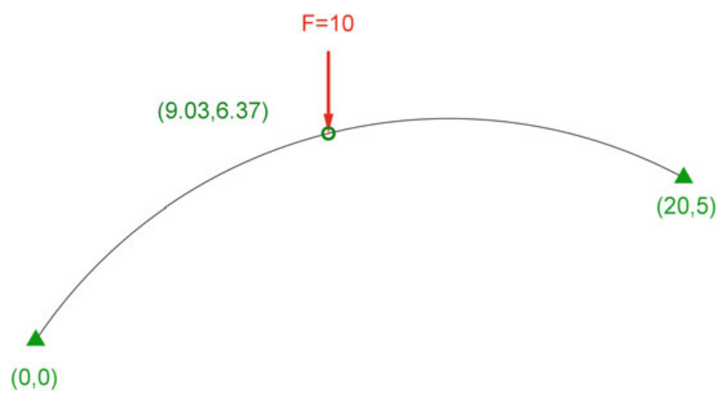


Fig. 4.30 Final funicular is quite close to the three hinged arch centerline

The following three hinged arch example adds two new features. One is that the supports do not land on the same elevation. The second is that an applied load is applied at the crown hinge. In Fig. 4.31, such a three hinged arch is shown, it is a circular segment of radius 15.32 units of length, spanning from left (0, 0) to right (20, 5) centered on point (12.75, −8.5). This places the crown, which bisects the arch, at (9.03, 6.37). A force of 10 units is applied downward at the crown.

Fig. 4.31 Three hinged arch with varying support locations and with point load on crown hinge



If this problem were to be solved algebraically, the vertical reactions could not be found immediately from a free body of the entire structure, thus the arch would have to be broken into two free bodies, separated at the crown with "equal but opposite" axial and shear forces at the crown. But graphically, the slope of the support line poses no additional work. Yet the concept of breaking the arch into two free bodies is helpful when considering the pesky load at the crown. The preferred method is to assume that the load is either a tiny distance to the left, or to the right of the crown. Then proceed as before (Fig. 4.32).

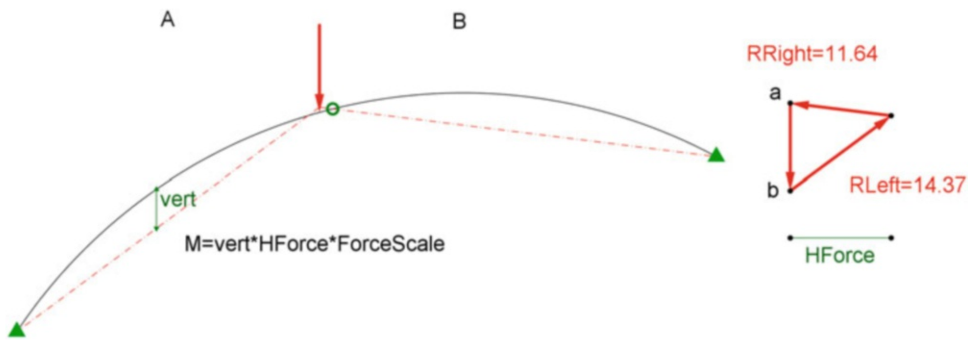


Fig. 4.32 Solution to the pesky load at the crown hinge

The following is a three hinged arch that appears to be a truss with inclined loads applied normal to the top chord. Any such truss will have an overall global deformation as that of a solid three hinged arch, yet each individual truss member will remain straight throughout the deformation if each element is indeed pinned-pinned. It will be shown that the structure's variation from the funicular directly corresponds to the deformed shape found from a finite element analysis (Fig. 4.33).

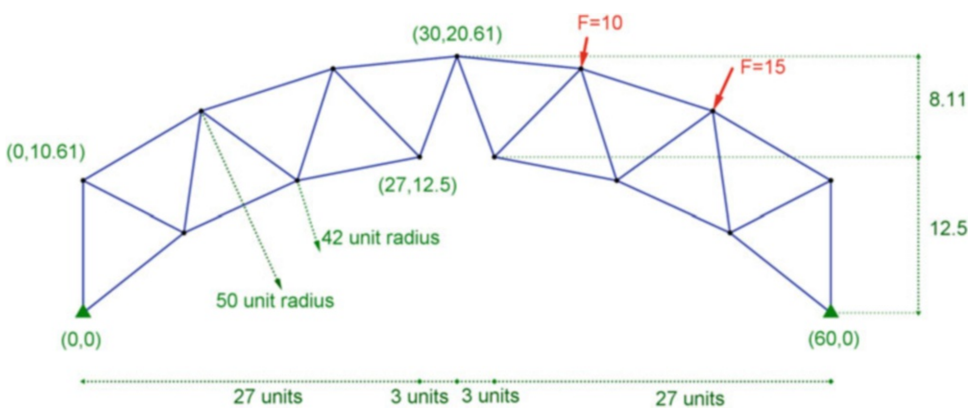
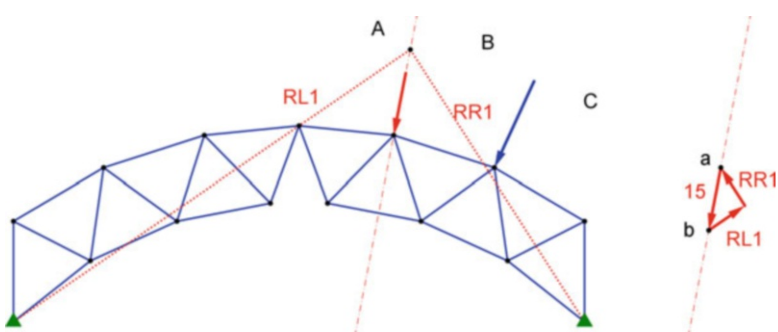


Fig. 4.33 Truss in a three hinged arch form

Since the left side of the three hinged arch is unloaded, this side constitutes the "simpler free body" to start with. As such, draw a line from the left hinged support through the hinge at the crown. That line is extended till it intersects the path of the first load on the right side of the arch. Note that that first load is inclined, not vertical. This is shown in Fig. 4.34.

Fig. 4.34 First step of solution ignores the second load



From that intersection, draw a second straight line to the right support. This is also shown in Fig. 4.34. If there were only one load, this would constitute a solution for the reactions. Yet there is a second load, thus the process is repeated for the second load as shown in Fig. 4.35.

Next, as was done in Fig. 4.29, a superposition of the first case (red) and the second case (blue) leads to the final reactions. The first load case scales Reaction Right 1 (RR1) and Reaction Left 1 (RL1) by 10 units of force since that is the magnitude of the first load, and its inclination is known. Similarly, the second load case finds RR2 and RL2 by scaling to 15 units of force at the known inclination of load 2. These steps are shown in Fig. 4.35.

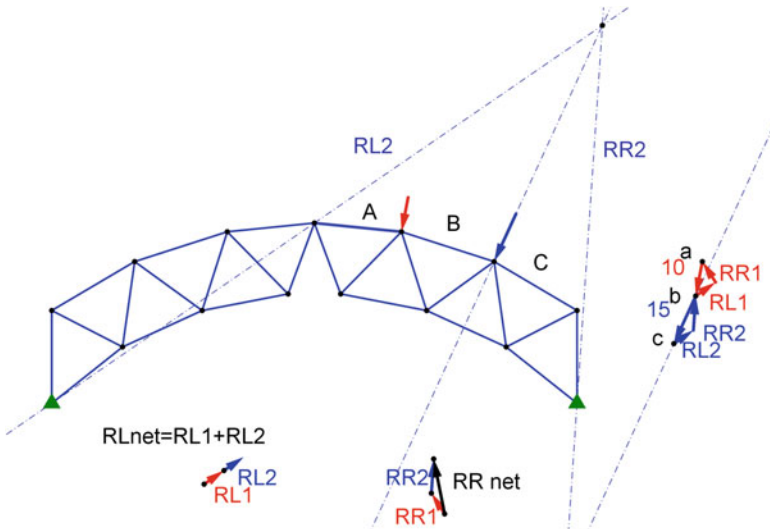


Fig. 4.35 Second step of solution ignores first load, then superpose the results

The bottom of Fig. 4.35 shows the magnitude and orientation of the net reactions on the left and on the right. The funicular begins and ends at these slopes of RLnet and RRnet, and the funicular must pass through the supports, and must intersect the path of each load. This is shown in Fig. 4.36. The blue dashed line in Fig. 4.36 approximates the "neutral axis" of this structure, in other words it represents the closing line. As always, the distance from the funicular to the closing line captures the magnitude of the internal bending.

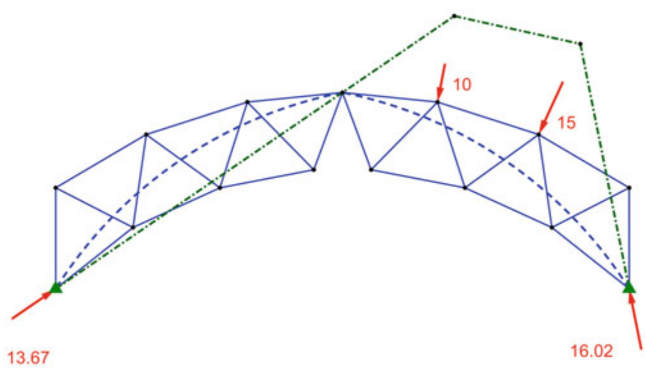


Fig. 4.36 Final funicular and its deviation from the approximate neutral axis of the form

Figure 4.37 depicts the axial load in each truss element of this three hinged arch, information that was obtained from a finite element program. Since this three hinged arch is truly a truss, each element in the truss is only axially loaded, no bending exists within each element. In Fig. 4.36, the distance between the funicular and the closing line is much larger on the right hand side compared to the left hand side, thus it is logical to see much larger axial forces in the truss elements on the right hand side.

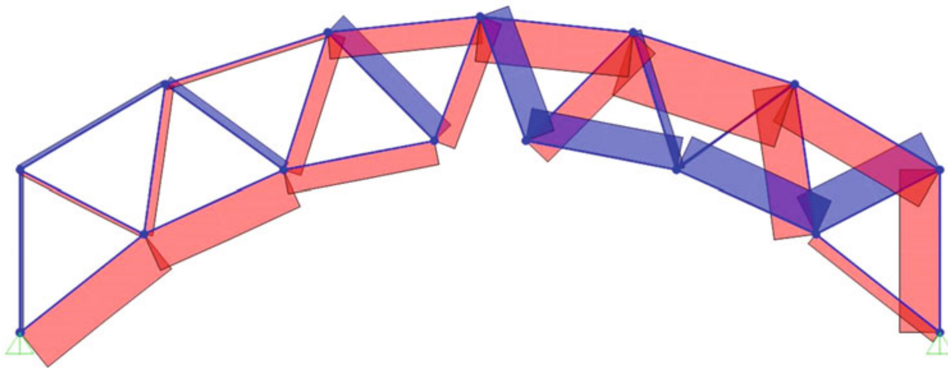


Fig. 4.37 Finite element qualitative representation of truss bar forces

Figure 4.36 also captures positive (concave up) bending and negative (concave down) bending. This is captured by the closing line being one side, then another side of the closing line. A finite element depiction of the deformation of this three hinged arch verifies these insights obtained graphically. This is shown in Fig. 4.38.

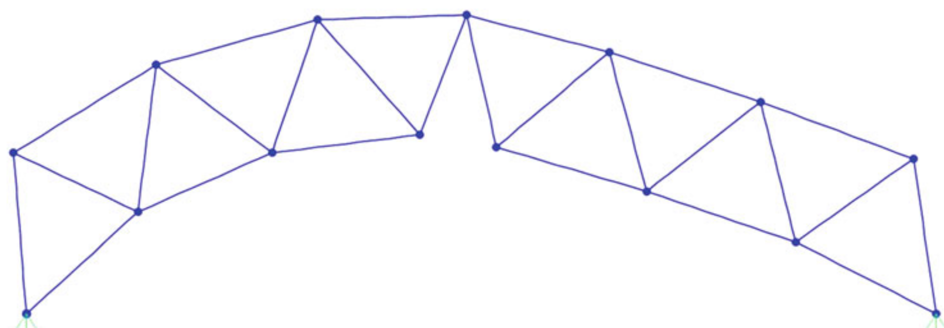
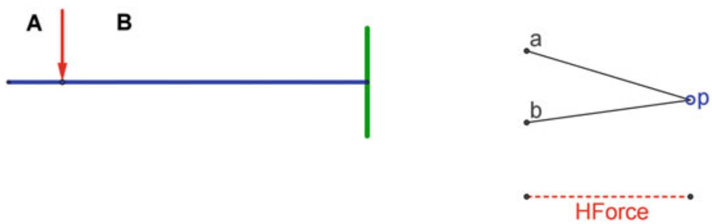


Fig. 4.38 Finite element representation of deformed truss showing positive and negative overall bending

Chapter 4 Exercises

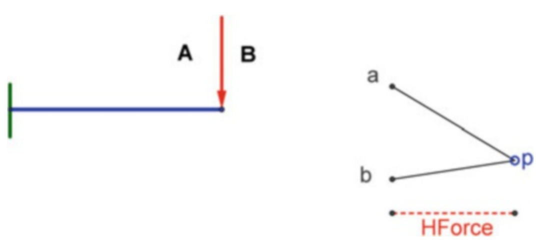
Exercise 4.1 Draw the force polygon, and capture *HForce*

Exercise 4.1 solution Notice that *a* to *b* captures the applied force. The location of the pole *p* is arbitrary.

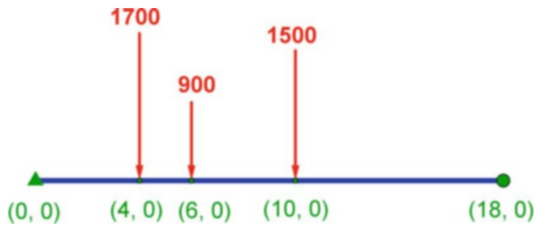


Exercise 4.2 Draw the force polygon, and capture *HForce*

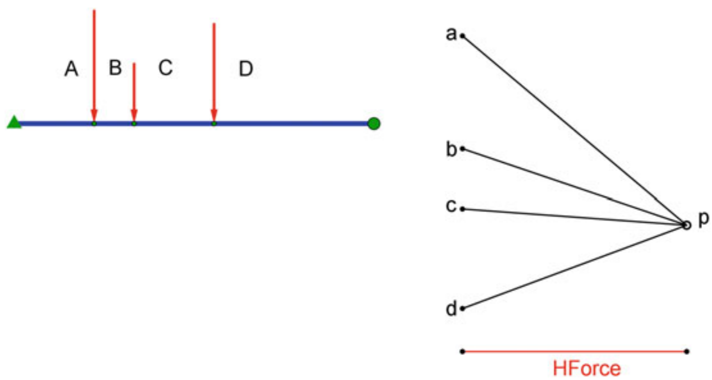
Exercise 4.2 solution Notice that *a* to *b* captures the applied force. The location of the pole *p* is arbitrary.



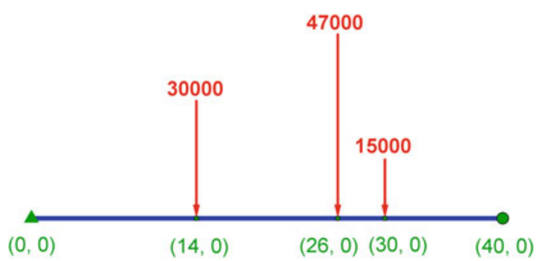
Exercise 4.3 Draw the force polygon, and capture *HForce*



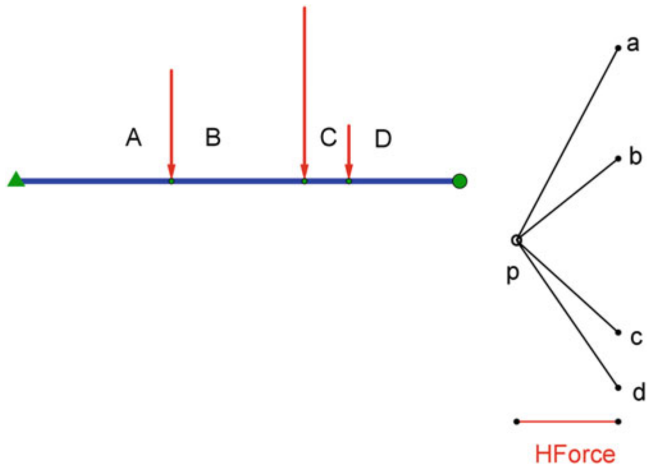
Exercise 4.3 solution Notice that the labels for magnitude of forces, and for locations of points are no longer useful. The pole position is arbitrary.



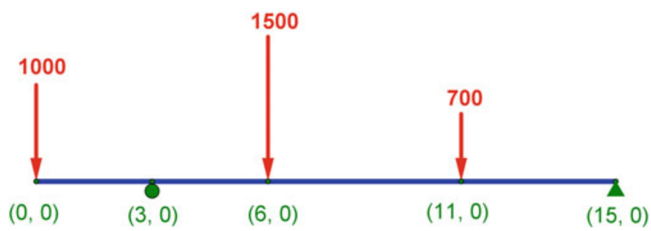
Exercise 4.4 Draw the force polygon, and capture *HForce*



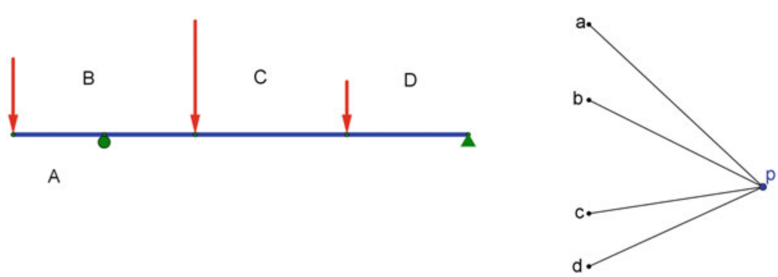
Exercise 4.4 solution Notice that the labels for magnitude of forces, and for locations of points are no longer useful. The pole position is arbitrary.



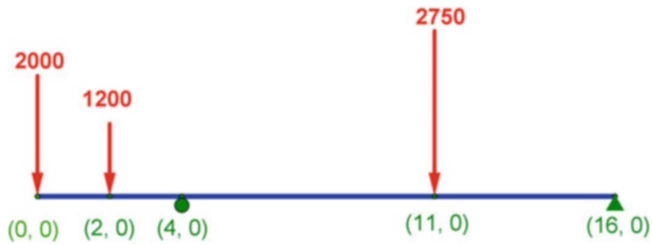
Exercise 4.5 Draw the force polygon, and capture *HForce*



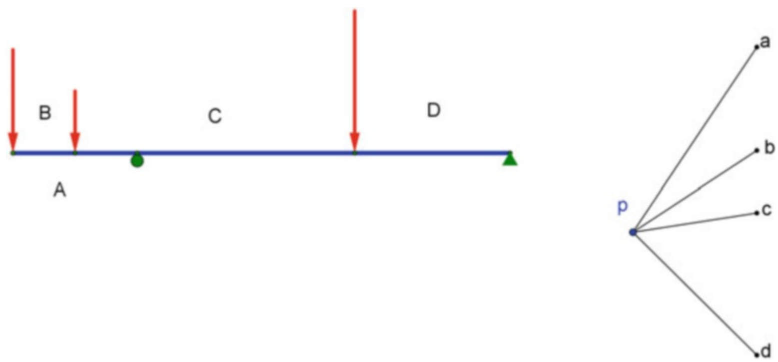
Exercise 4.5 solution Notice that the labels for magnitude of forces, and for locations of points are no longer useful. The pole position is arbitrary. The first ray *ap* captures Space *A*, to the left of the roller, but before the first load.



Exercise 4.6 Draw the force polygon, and capture *HForce*

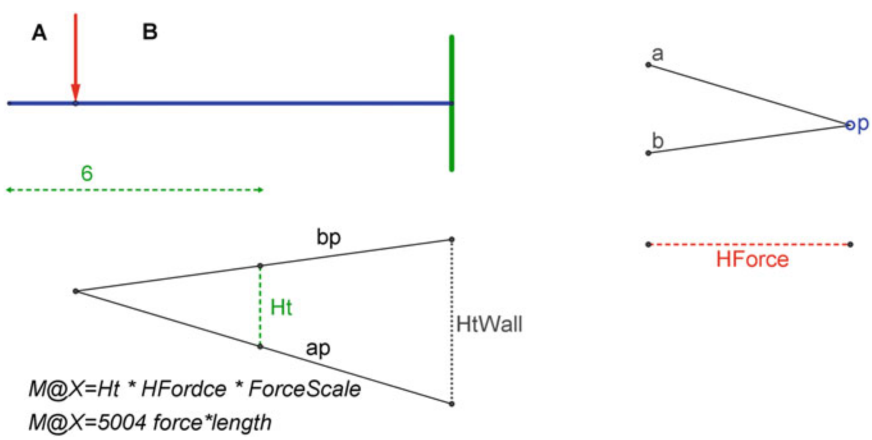


Exercise 4.6 solution Notice that the labels for magnitude of forces, and for locations of points are no longer useful. The pole position is arbitrary. The first ray *ap* captures Space *A*, to the left of the roller, but before the first load.



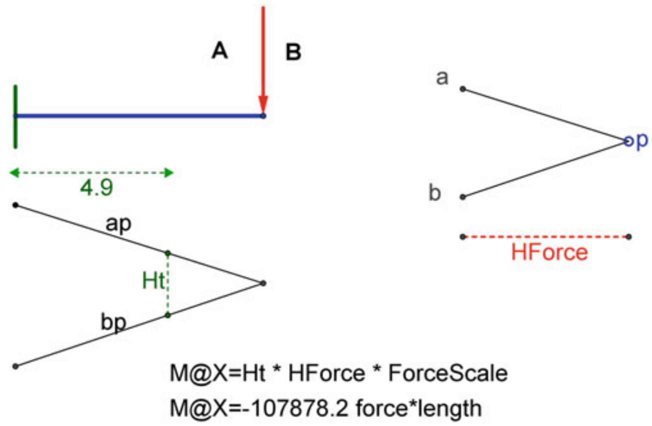
Exercise 4.7 For the problem shown in 4.1, draw the bending moment and display the internal moment at any value of x.

Exercise 4.7 solution



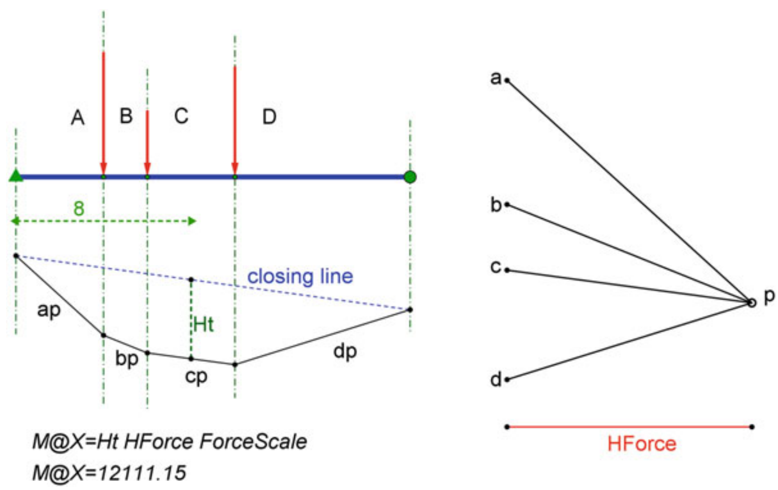
Exercise 4.8 For the problem shown in 4.2, draw the bending moment and display the internal moment at any value of x.

Exercise 4.8 solution



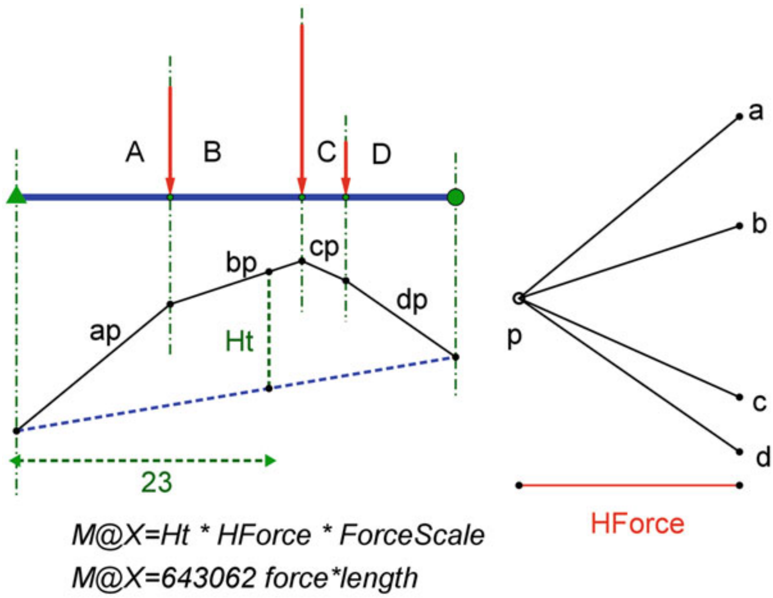
Exercise 4.9 For the problem shown in 4.3, draw the bending moment and display the internal moment at any value of x.

Exercise 4.9 solution



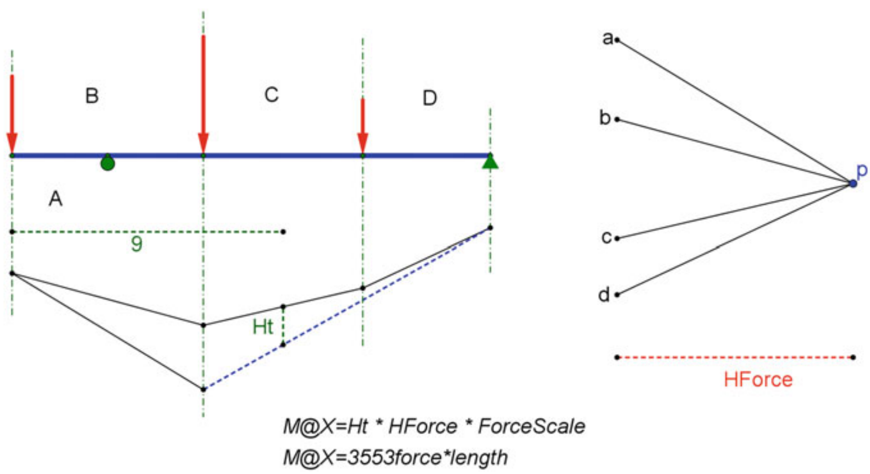
Exercise 4.10 For the problem shown in 4.4, draw the bending moment and display the internal moment at any value of x.

Exercise 4.10 solution



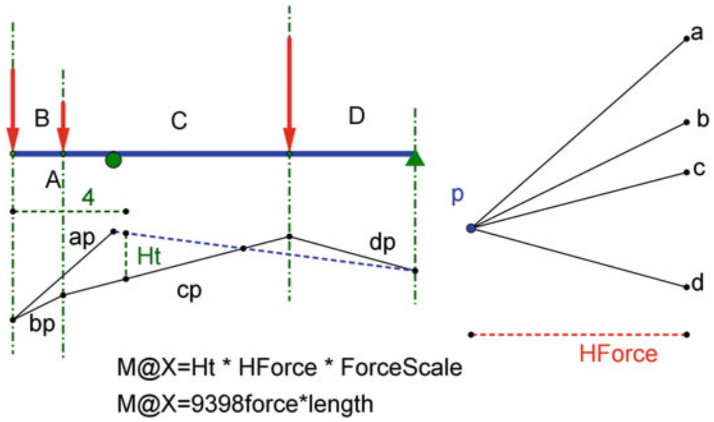
Exercise 4.11 For the problem shown in 4.5, draw the bending moment and display the internal moment at any value of x.

Exercise 4.11 solution

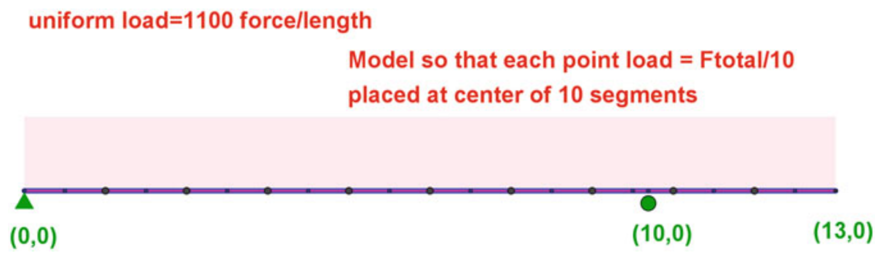


Exercise 4.12 For the problem shown in 4.6, draw the bending moment and display the internal moment at any value of x.

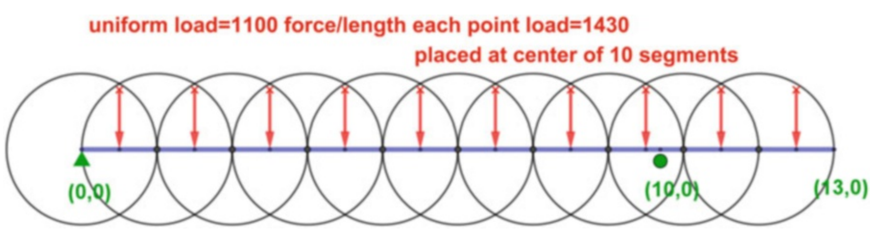
Exercise 4.12 solution



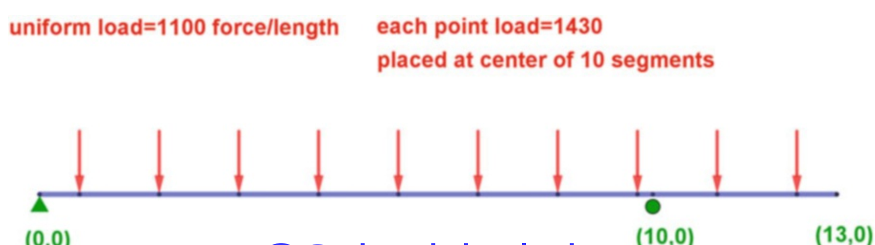
Exercise 4.13 For the following beam with an overhang, change the uniform load to 10 discrete point loads. Place each point load at the centroid of 10 equally sized segments.



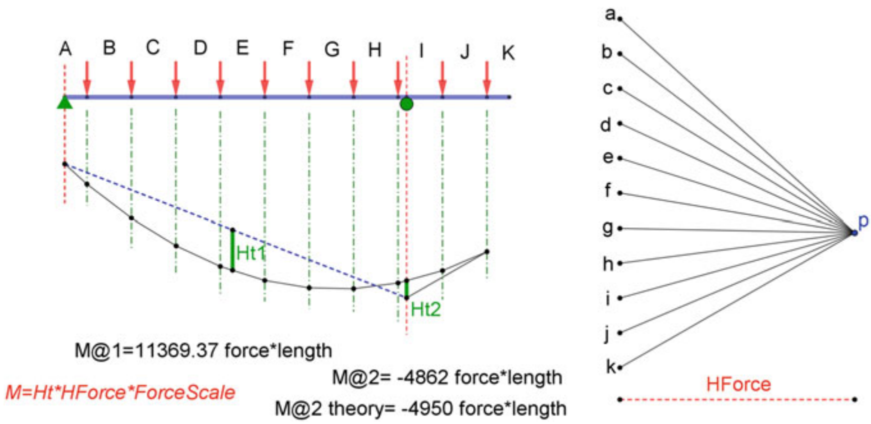
Exercise 4.13 solution



Exercise 4.14 For the discrete loads shown in the solution of Problem 4.13, calculate the moment in the beam over the prop support and calculate the moment near the peak positive bending cross section.

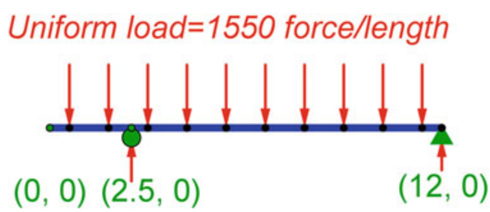


Exercise 4.14 solution

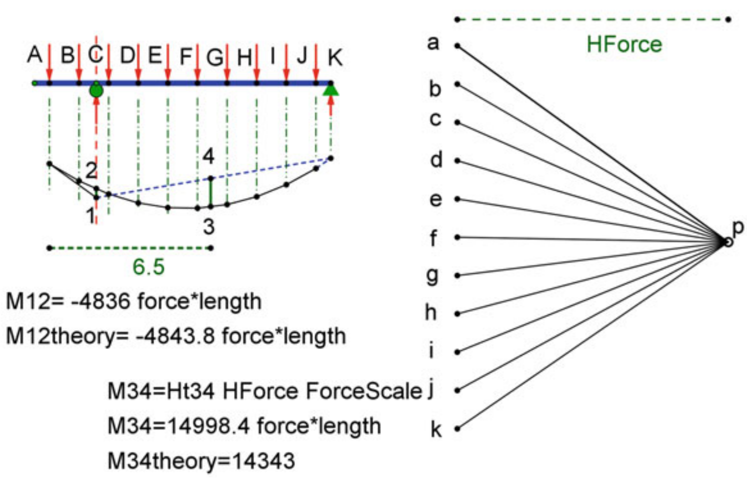


The difference between theoretical and graphical solutions can be reduced if more point loads are used to simulate the uniformly distributed load. Also, the centroidal placement of loads prevented a point load from being applied to the free end.

Exercise 4.15 For the following beam with an overhang, draw the bending moment and display the internal moment near the peak positive and above the left support to capture the peak negative moment.

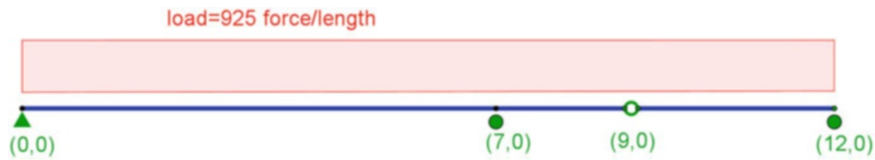


Exercise 4.15 solution

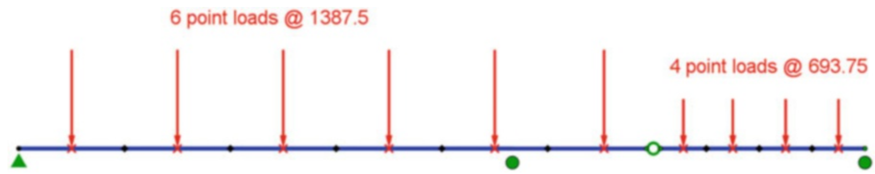


The difference between theoretical and graphical solutions can be reduced if more point loads are used to simulate the uniformly distributed load.

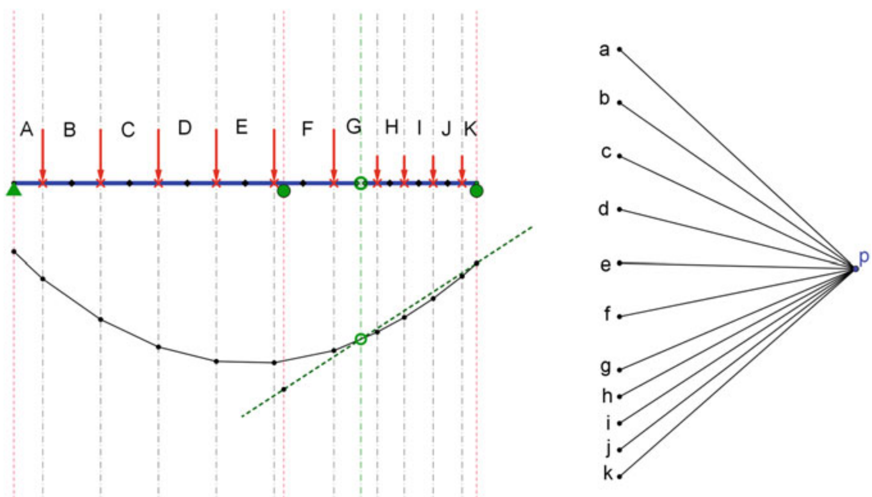
Exercise 4.16 A beam is pinned at the left end, and it has two roller supports and an internal hinge as shown. Break up the uniform load into 6 point loads applied centroidally to the left of the hinge, and 4 point loads applied centroidally to the right of the hinge.



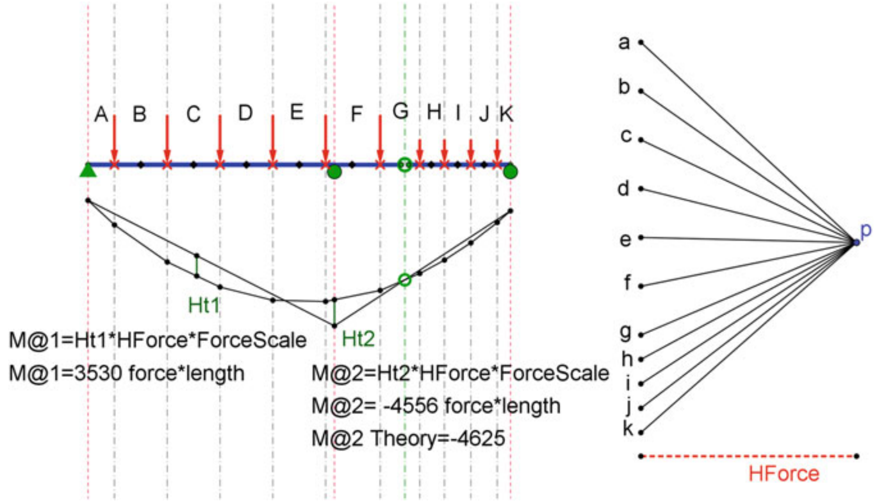
Exercise 4.16 solution



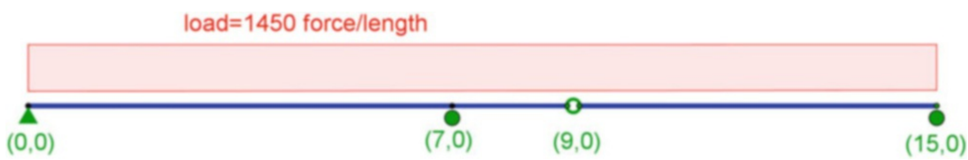
Exercise 4.17 Given the discretization of loads from problem 4.16, calculate the bending moment over the central roller support and at a place near the peak positive moment.
Exercise 4.17 solution Part 1



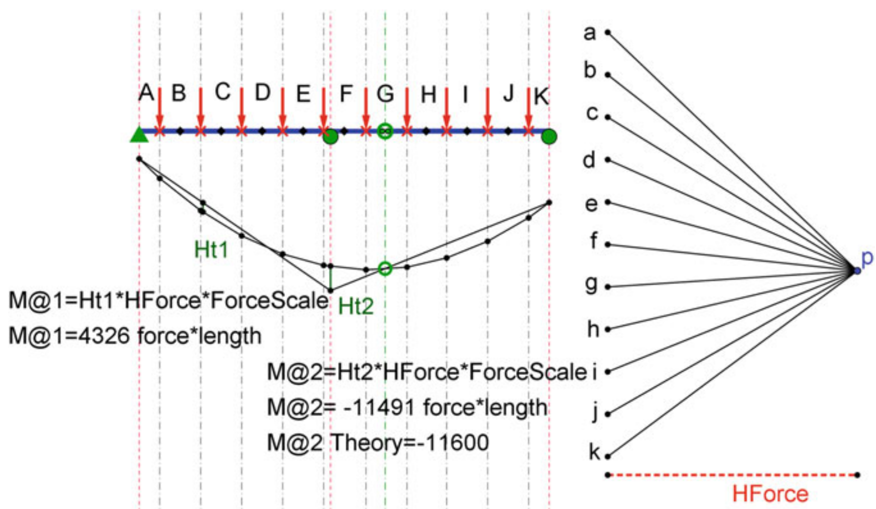
Exercise 4.17 solution Part 2



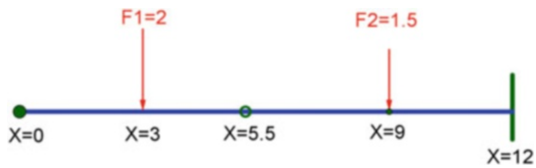
Exercise 4.18 A beam is pinned at the left end, and it has two roller supports and an internal hinge as shown. Break up the uniform load into 10 point loads applied centroidally. Then calculate the moment in the beam over the interior roller and near the region of maximum positive moment.



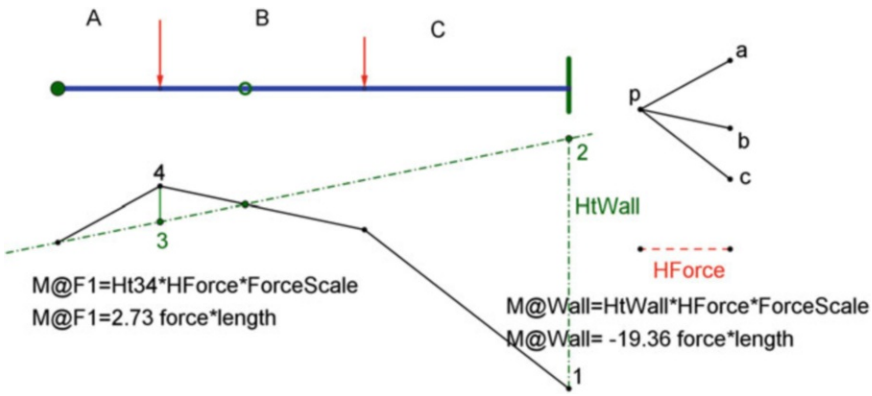
Exercise 4.18 solution



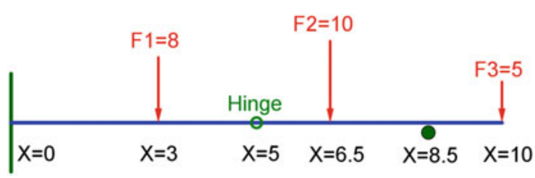
Exercise 4.19 A beam has a roller support at the left end, an internal hinge, and a fixed support at the right end. Calculate the bending moment directly beneath the first load F1 and at the fixed right end.



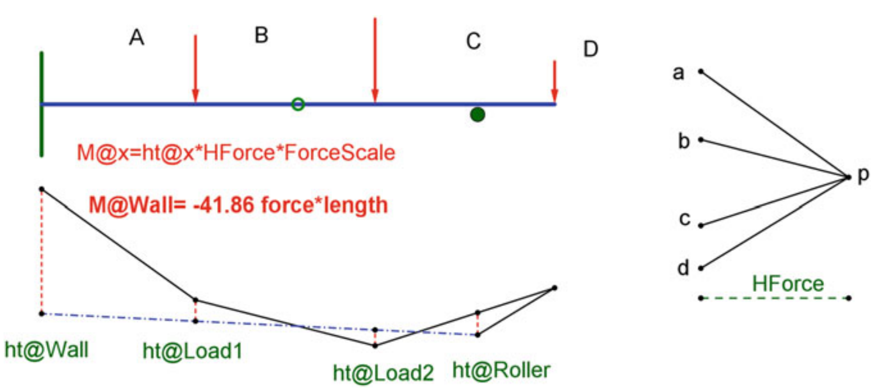
Exercise 4.19 solution



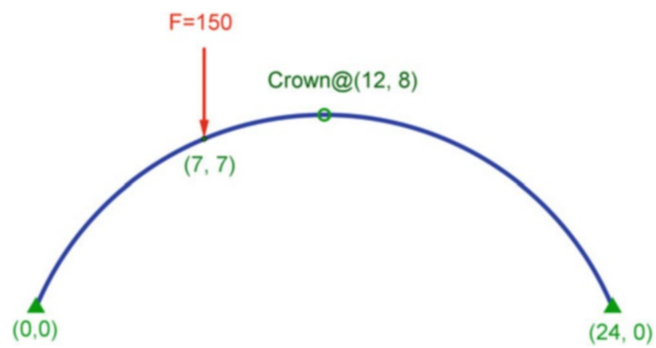
Exercise 4.20 A beam has a fixed support at its left end, an internal hinge, and a roller support. Calculate the bending moment under the left two loads and above the roller support.



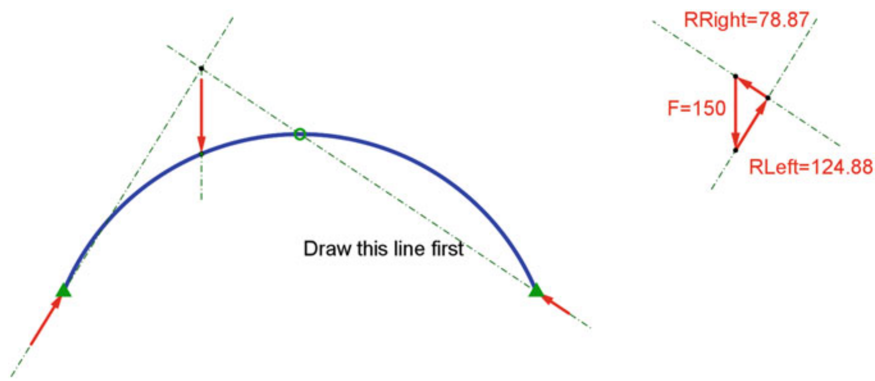
Exercise 4.20 solution



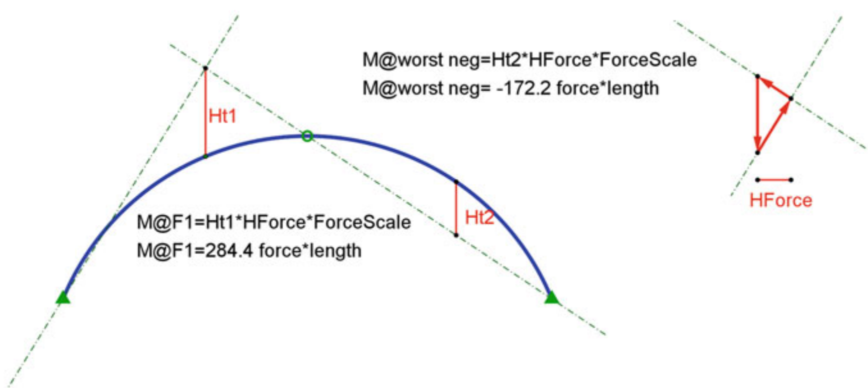
Exercise 4.21 A circular, three-hinged arch is subjected to a single downward load as shown. Calculate the reactions and establish the bending moment near the area of most extreme negative bending.



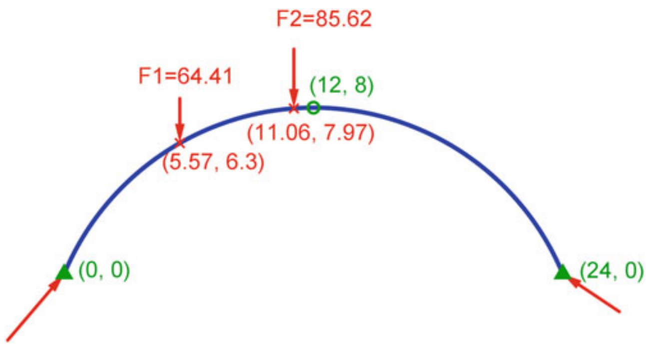
Exercise 4.21 solution Part 1



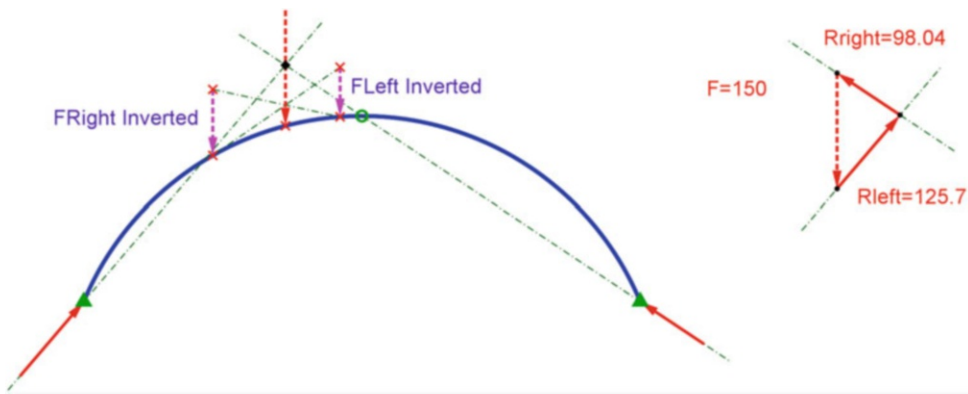
Exercise 4.21 solution Part 2



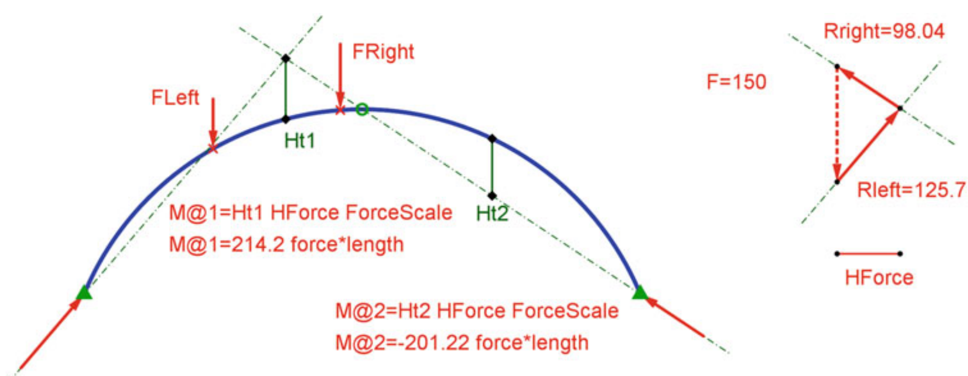
Exercise 4.22 A circular, three-hinged arch is subjected to two downward loads as shown. Calculate the reactions and establish the bending moment near the area of most extreme bending.



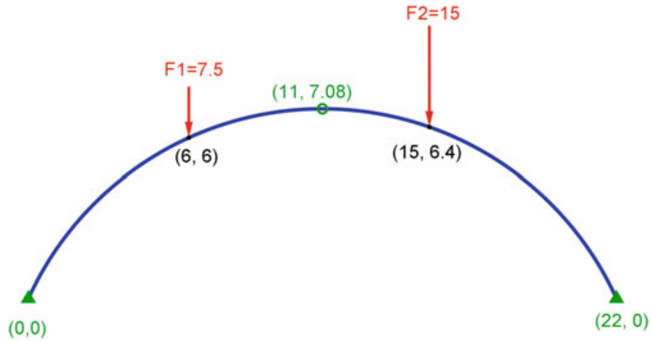
Exercise 4.22 solution Part 1



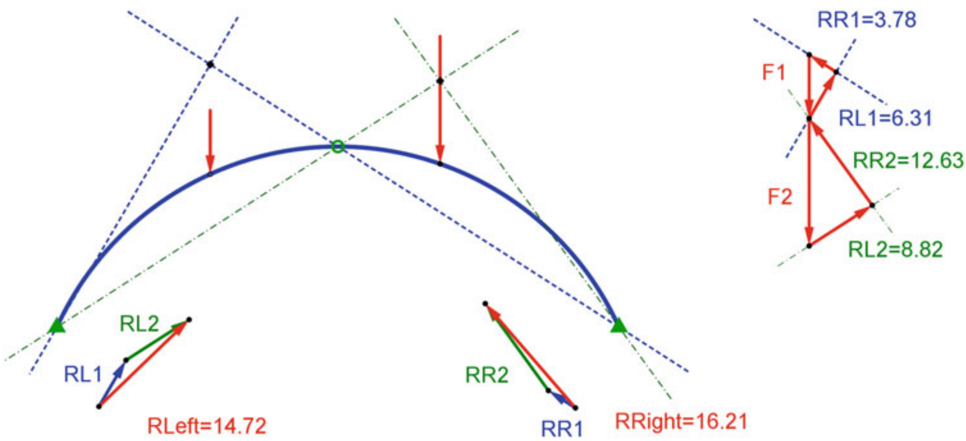
Exercise 4.22 solution Part 2



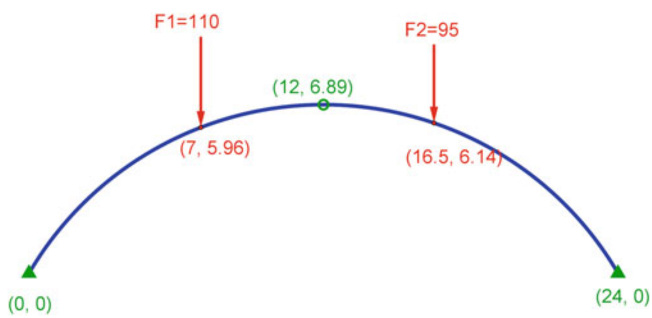
Exercise 4.23 A three hinged circular arch is subjected to two point loads as shown. Calculate the reactions.



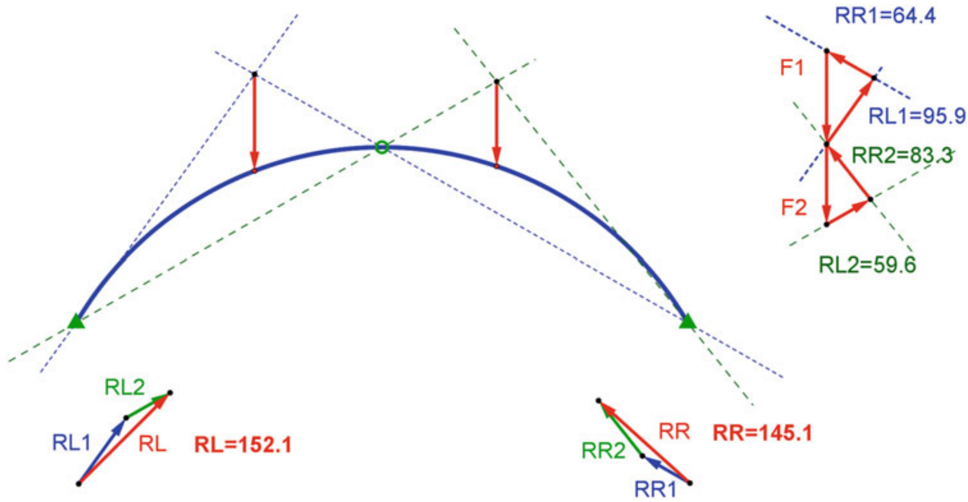
Exercise 4.23 solution



Exercise 4.24 A three hinged circular arch is subjected to two point loads as shown. Calculate the reactions.

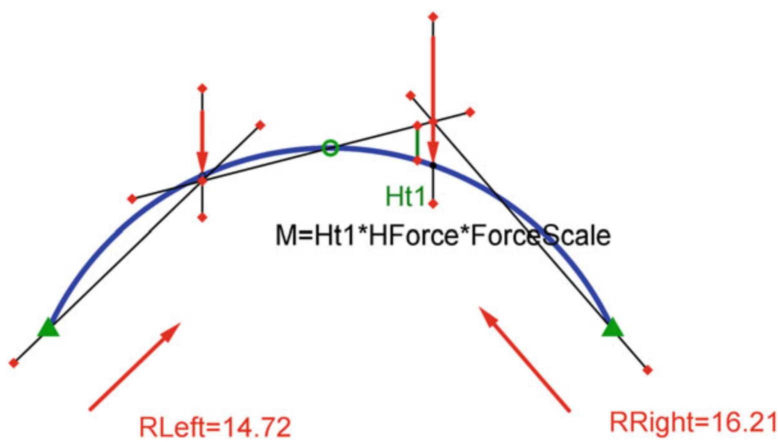


Exercise 4.24 solution



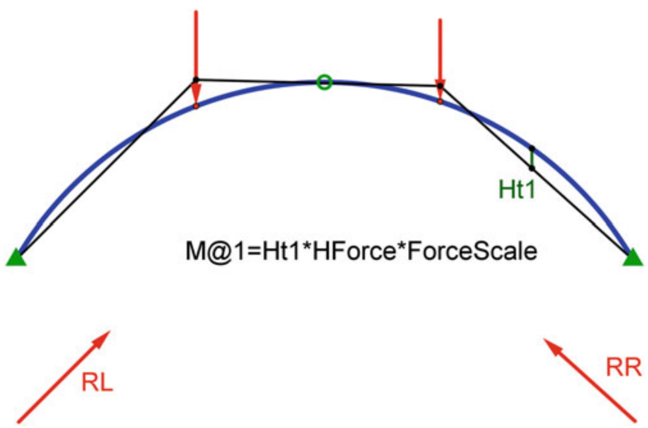
Exercise 4.25 For the three hinged circular arch shown in Problem 4.22 draw the bending moment diagram and quantify the moment in several regions.

Exercise 4.25 solution

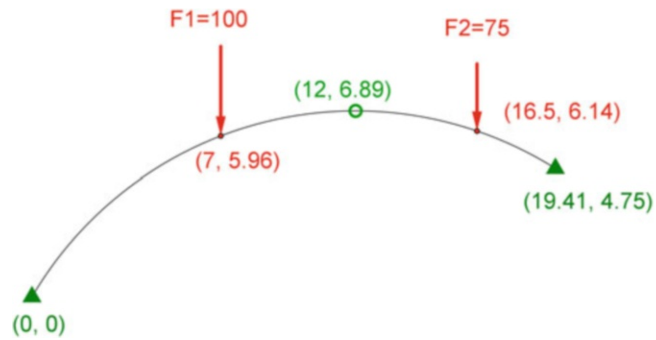


Exercise 4.26 For the three hinged circular arch shown in Problem 4.23 draw the bending moment diagram and quantify the moment in several regions.

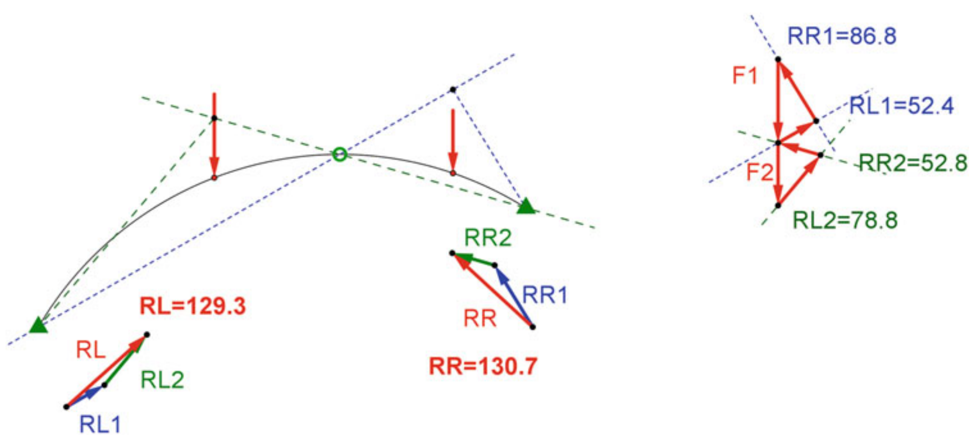
Exercise 4.26 solution



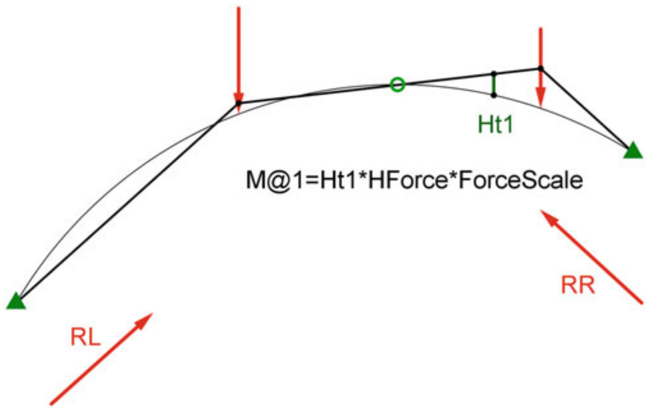
Exercise 4.27 A three hinged circular arch is subjected to two point loads as shown. Calculate the reactions.



Exercise 4.27 solution



Exercise 4.28 For the problem shown in 4.26, show the bending moment diagram.
Exercise 4.28 solution



Truss Analysis and Design

5

The study of trusses begins by looking at these structures simply as deep beams, with the top and bottom chords playing the role of flanges, and the diagonals behaving as a beam web. As such, the first step requires finding only the external reactions, which is a review at this point, but the examples appear different only because the trusses seem more complicated than beams. Lateral loads will be studied in this chapter, and a special case of externally indeterminate supports will also be presented.

As was shown in Chap. 4, calculating the reactions for vertical loads is extremely quick. It does not matter where the pole is placed, but as has been shown, the funicular takes the shape of the bending moment diagram, and the choice of which side the pole is placed on affects whether the bending moment is depicted on the tension side, or on the compression side. Figure 5.1 shows a truss that carries vertical loads of 100, at $x = 12$ and $x = 24$, and vertical loads of 150 at $x = 36$ and $x = 48$. There are five panels, each panel is 12 units of length wide and 6 units of length tall.

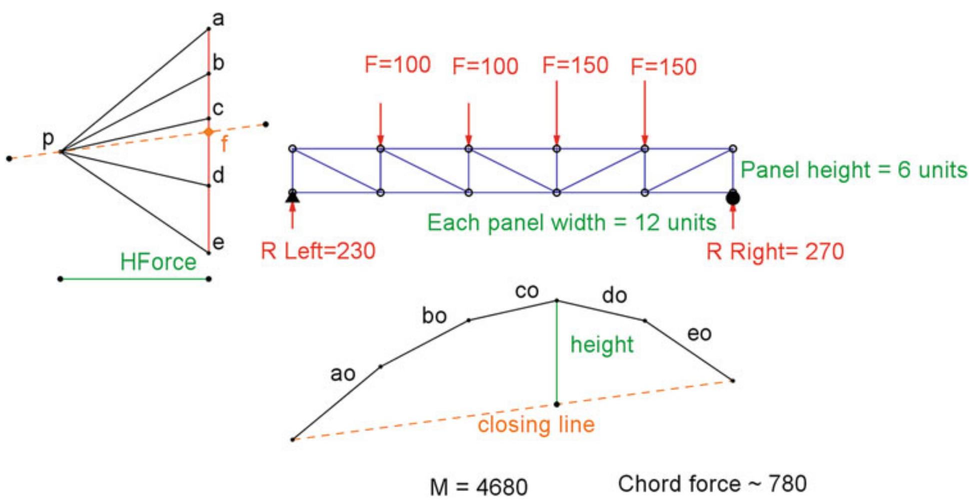


Fig. 5.1 A truss with vertical loads, reactions are sought

E. Saliklis, *Structures: A Geometric Approach*, https://doi.org/10.1007/978-3-319-98746-0_5

As before, Bow's notation is used, wherein capital letters are placed between loads on the form diagram, and lower case letters correspond to these on the force diagram. Figure 5.2 shows the force diagram drawn to some convenient scale, with an arbitrary pole and with rays drawn from each letter to the pole. These rays are transferred back to the form diagram, but they are truncated so that each funicular segment ends at the lines (here vertical) denoting transitions of zones in Bow's notation.

A closing line is drawn from the end of the funicular, back to the start, here shown in orange. That orange line is then drawn through the pole *p* on the force diagram to locate where it intersects the vertical load line. Here, that point of intersection is marked with a different point symbol, and it is point *f*. Ray *ef* multiplied by the force scale gives the right reaction, ray *fa* multiplied by the force scale gives the left reaction.

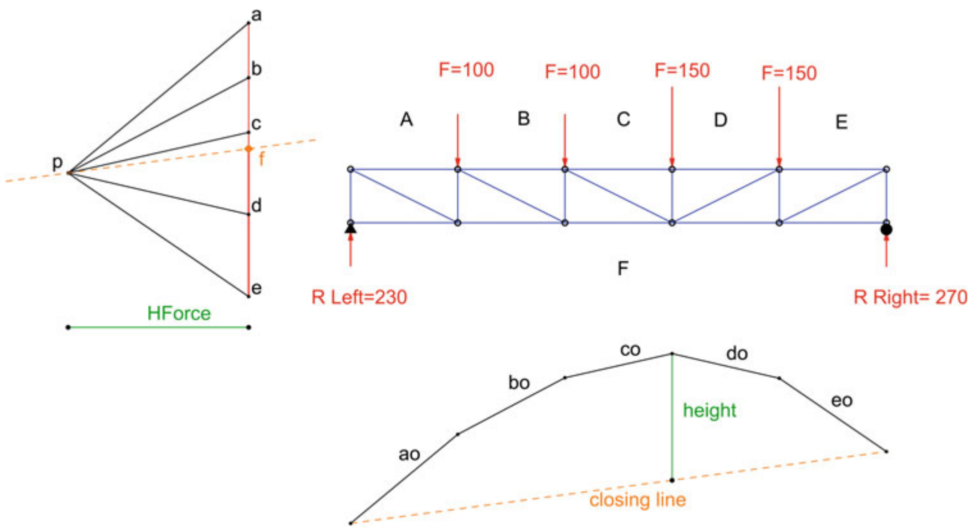


Fig. 5.2 External support reactions found graphically

One structural insight becomes immediately available: that is an estimate of the worst chord force, based on the bending moment diagram. But before proceeding, take a moment (ha ha!) to fully understand the mechanics here. The truss is acting in a global sense as a beam. Since it is bending concave upwards, chords on the top are in compression and those on the bottom are in tension. Yet it is still a truss, meaning that each member is assumed to be pinned-pinned, thus, each individual member experiences no bending. There is bending on a global scale, but no bending on the elemental level. The bending moment at any cross section is quickly found, but what is an estimate of the peak chord force? In Fig. 5.2 an analysis for the approximate value of the peak chord force is shown. A vertical segment is drawn in red, near the point where the bending moment diagram appears to be worst. This segment extends from the funicular to the closing line. Multiplying the length of that segment by the *HForce* value and by the force scale value establishes a moment of *4680 force length.*

This quick calculation assumes that the moment is composed solely of the flange forces separated by the web distance *d*. Figure 5.3 is a qualitative plot of the axial forces (notice how web forces decrease as shear approaches zero). Then

$$M = F \cdot d \ thus \ F = {}^{M}\!/_{d} = {}^{4680}\!/_{6} = 780 \ force.$$

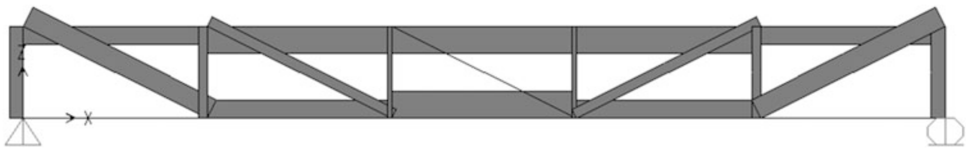
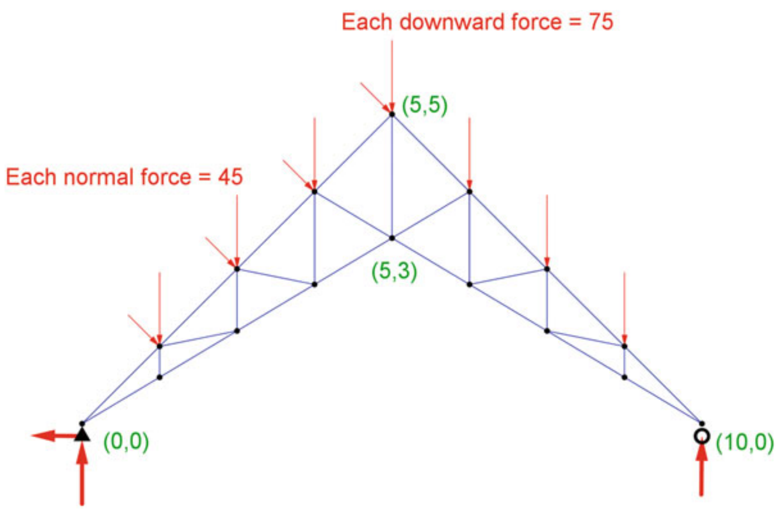


Fig. 5.3 Qualitative plot of each bars axial force, found via finite element method

Of course, this estimate is not perfectly accurate, the web member may have some small axial force even in this central panel, but it is a quick and useful method of getting a rough idea of chord forces.

The next example calculates the external reactions of a truss subjected to vertical and lateral loads. This example was beautifully drawn in Wolfe's textbook from 1929 as Figures 46 and 47, but there all three cases are combined into a single drawing. Here, each support condition case will be treated separately for greater clarity. In this example, a compressive normal force of 45 units of force is applied to nodes on the top left chord, and simultaneously, a downward force of 75 units of force is applied to each of the top chord nodes. No suction is applied to the right top chord in this example. This is shown in Fig. 5.4. The truss has a pinned reaction at the left support (0, 0) and a roller reaction at the right (10, 0).

Fig. 5.4 Truss with normal loads on one side and downward loads along top



Certainly, Bow's notation could be used to place letters between each of these loads, but it would be a bit cumbersome to do so on the left top chord, placing a letter between each normal and the vertical load hitting the same node. Instead, vectorially add these two, and use only the resultant.

This was done in Fig. 5.5. The vertical and lateral loads on the left top chord were combined and shown as a dashed vector. Lines of action of each of the loads were then drawn through the point of application. In this example the horizontal component of the left reaction (Force *JA*) was initially assumed to be leftwards, but such assumptions won't affect the final result. The roller reaction at the right must be in a vertical direction. Neither the magnitude or the direction of the reaction at the left is known, all that is known about this reaction is that it must pass through the origin (0, 0) of the geometry. This fact alone demands that the start of the funicular begin the left support as shown in Fig. 5.5.

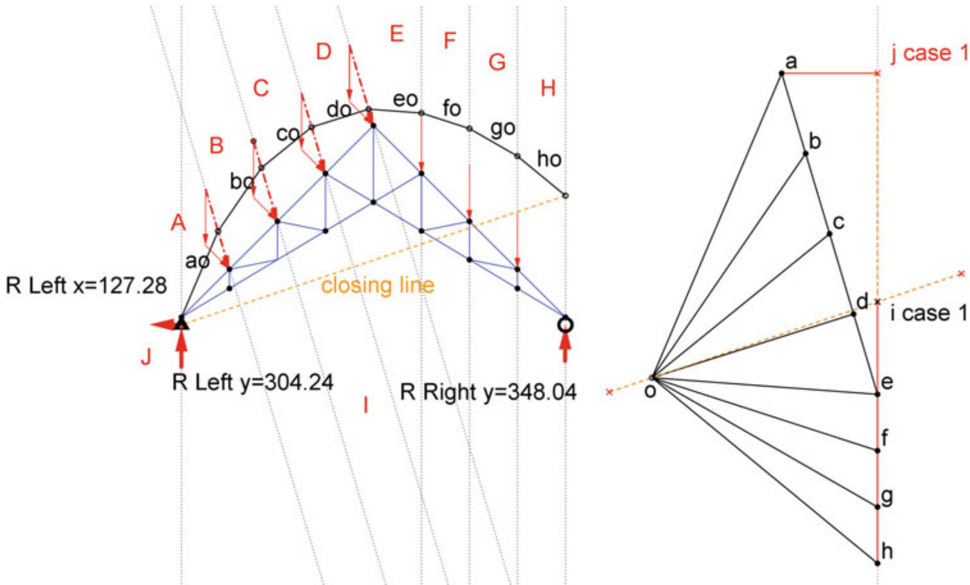


Fig. 5.5 External reactions of previous truss found graphically

Each funicular segment is truncated at the line of actions of forces which separate the spaces. Care must be taken when transferring the closing line from the form diagram to the force diagram. In this *Case 1*, the support is pinned at the left, and the support at the right is a roller. Thus, when seeking force *hi* on the force diagram, the jump is purely vertical. Where it intersects the orange closing line identifies *point i case 1*. For this point, the use of an alternate symbol, here an *x*, helps avoid confusion with terminal points of rays. Space *I* to *J* in the form diagram is also vertical, thus the force diagram takes another vertical jump, until the next step which is a purely horizontal step back to *a*. The reactions are shown in Fig. 5.5 for this first case of a pin at the left.

The key step in this first case is that since the reaction at the left has an unknown net direction and unknown net magnitude, the funicular started at the left reaction, as it must pass through that point.

Use that insight, and answer what would happen in this problem if the left support was a roller, and the right support was a pin?

In this *Case 2* situation, since the right support has two unknowns, but only one known location, the funicular must start at that point where the location is known. Note, that this will be nearly the same funicular as in *Case 1*, but care must be taken to ensure that the first ray laid down as a funicular segment is *ho*. Then, *go*, *fo* etc. This is shown in purple in Fig. 5.6. Although the funiculars for *Case 1* and *Case 2* are extremely similar, there is a small difference, note the length of funicular segment *eo* for instance.

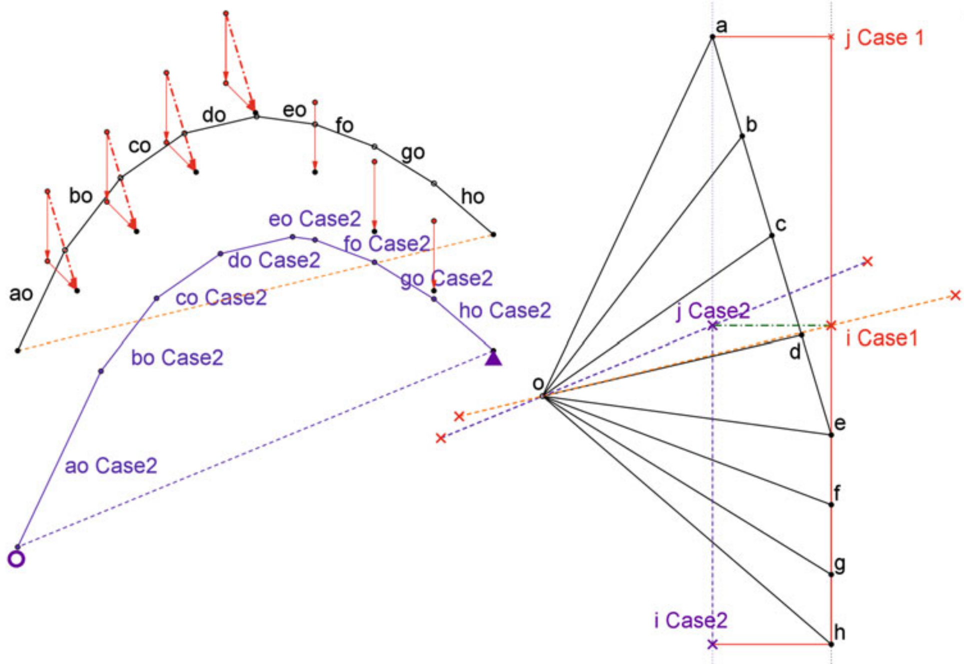


Fig. 5.6 External reactions found for previous truss but support conditions flipped

A few more insights become apparent after studying Fig. 5.6. Certainly, the horizontal reaction can be immediately found, by considering Space *HI* in the form diagram and force *h to iCase2* in the force diagram. But consider further, that *h to jCase2* in the force diagram captures the net right reaction in *Case 2*. Similarly, *iCase1 to a* in the force diagram captures the net left reaction in Case 1. Both of these scenarios are shown in Fig. 5.6, with *Case 1* (pinned at left support) shown in black and Case 2 (pinned at right support) shown in purple.

So what might *Case 3* be? Imagine *Case 3* requiring that both reactions have the same magnitude in the horizontal direction. One practical reason for desiring such similarity is that the design of the connections could be then repeated for both supports. Clearly this means a statically indeterminate problem! But it is special, in that the direction of the two identical reactions are prescribed a priori. How to do this? Figure 5.7 shows the answer.

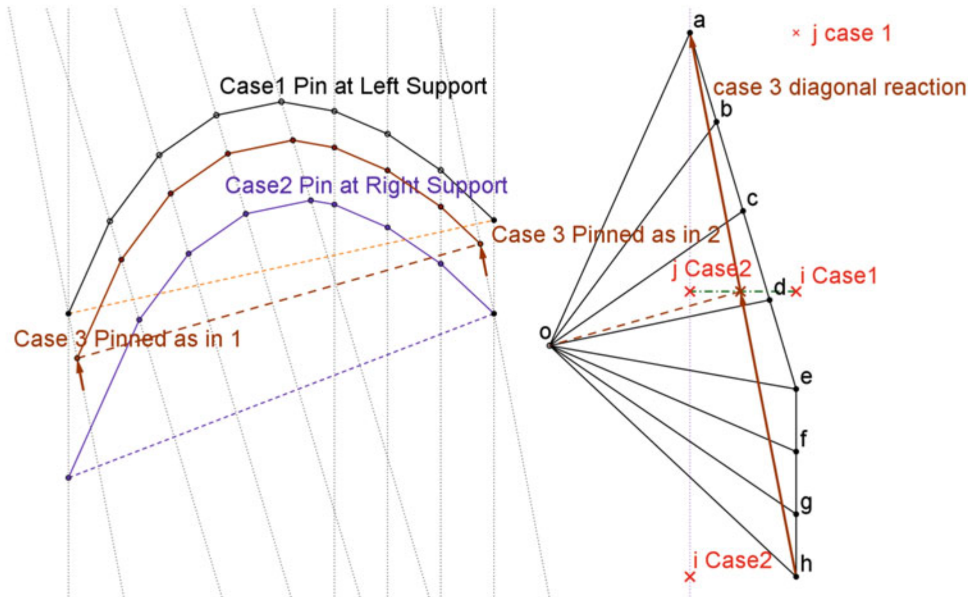


Fig. 5.7 Case 3 is a indeterminate but horizontal reaction must be the same on each support

In Fig. 5.7, ensuring that each reaction picks up an equal share of the horizontal load means locating the mid-point between *iCase1* and *jCase2* in the force diagram. This is shown in brown, and the slope of the brown line from *h* to *a* provides the slope of the net reactions in the form diagram. This slope is critical as this is the start of the funicular, on a line *parallel to* the net left reaction (alternatively, the funicular could have started with the right reaction). The funicular ends on a line *parallel to* the net right reaction. The closing line of *Case 3* is then run through the pole and it bisects the horizontal line between *jCase2* and *iCase1*. The line passing from *jCase2* to *iCase1* is horizontal because the reactions are both on a single horizontal plane. The closing line and funicular for *Case 3* are shown as brown.

Internal Member Forces in Trusses

Till now, only the determination of forces outside of the structure, i.e. the reactions have been studied. In this example, analysis for the internal bar forces is presented. The example shown in Fig. 5.8 is a truss, with a pinned support at the left end (0, 0), a roller support at the right end (12, 0). The crown is at (6, 3). Diagonals intersect the chords at midspan. Three loads are applied as shown (500, 500, 350 units of force)

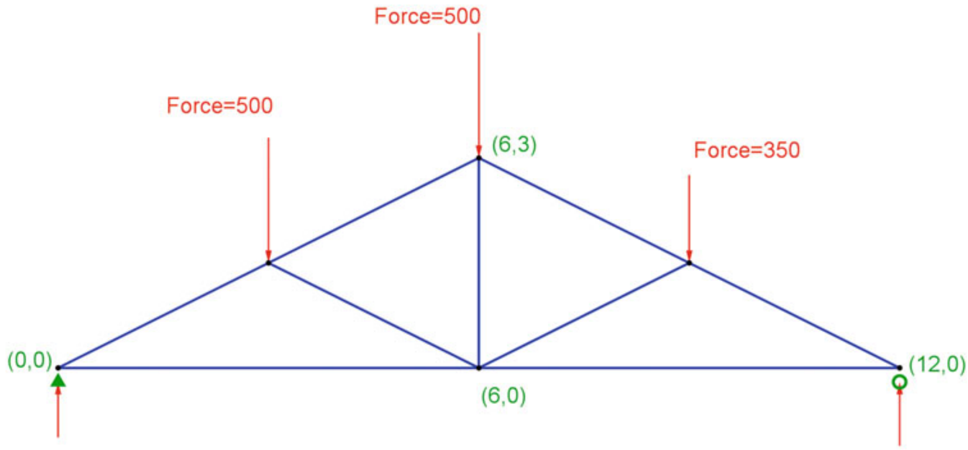


Fig. 5.8 Statically determinate truss internal bar forces are sought

The initial steps are as before. Use Bow's notation to place letters in the spaces between loads. Draw a force diagram at some convenient scale, transfer rays of the force diagram back to the form diagram, draw a closing line and pass that closing line through the origin of the force diagram. This is all shown in Fig. 5.9. Point *e* on the force diagram is now known. This means that the reactions are fully known.

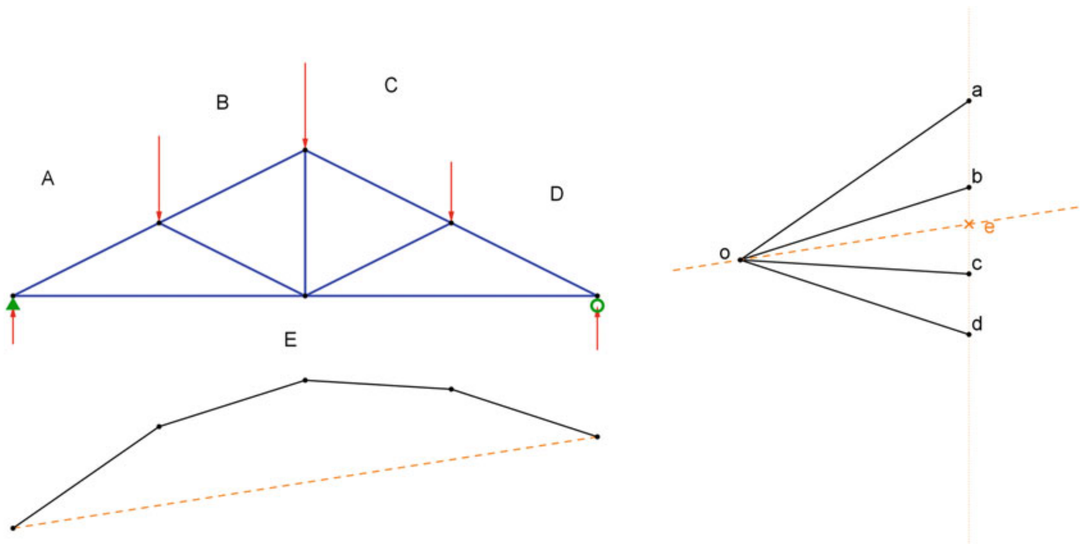


Fig. 5.9 Closing line establishes external reactions of truss

Next, the internal bar forces can be found, both their magnitude and whether they are in tension or compression. Notice that the direction of each of these bar forces is already known. Because they are pinned-pinned, the axial force in each bar must align with the orientation of each bar. This key piece of information is needed to take the next steps. Consider for example the bar directly below space *A* in Fig. 5.9. The angle of the force is known, it aligns with the bar, but more information is needed, as would be in algebraic statics. Thus, the analyst might seek a joint that allows for the solution of one equation with one unknown. First of all, a method is needed to name the force in any bar, with a two-character designation, just as the external loads were called *AB*, *BC* etc, now there is a need for a second index to name each bar force. Traditionally, Bow's notation inserts numbers, not letters, in the interior panels. It doesn't matter what order you put the numbers in, most people start with *1* on the left-most interior panel as shown in Fig. 5.10. Thus, the lower left top chord bar will be called *A1* or *1A*. The order of naming will be described shortly. As in algebraic statics, start somewhere "easy", here either find point *1* or point *4*.

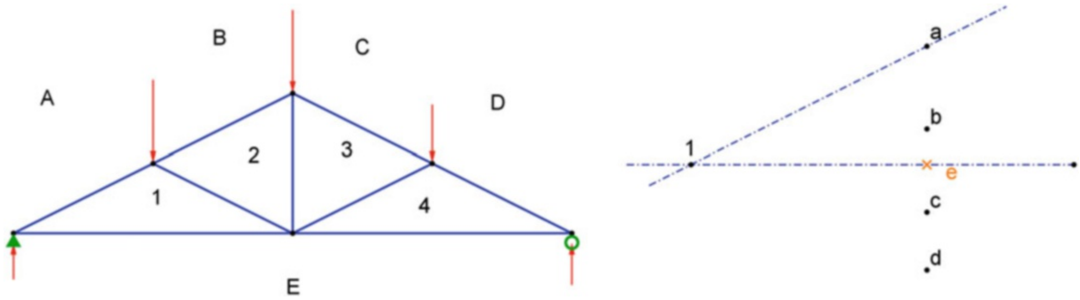


Fig. 5.10 Finding the first interior number

The reason why it is necessary to seek Point *1* or Point *4* is that it is possible to move from *A* to *1* and cross only one bar, yet point *a* is previously known on the force diagram, or moving from *E* to *4* (remember *E* is the space between the supports under the truss), with point *e* already known on the force diagram. It is not possible to start by finding point *2* for example, because while *B* to *2* has one known (point *b*), *2* to *3* and *2* to *1* don't help as *1* and *3* are not yet identified. It is impossible to go from *2* to *E* without crossing more than one bar. The force in bar *A1* is unknown, but its slope is known, thus copy that slope and pass it through point *a* on the force diagram. The force in bar *1E* is unknown, but its slope is known (horizontal) thus pass a horizontal line through point *e*. Where these two intersect, locates point *1* as shown in Fig. 5.10.

Similarly, seek another point. It is now possible to find point *2* because 2 is common to *B* through the top chord on the left, and *2* is common to *1* through the diagonal web. Transfer the *B2* slope through *b*, and transfer the *21* slope through *1* to find point *2*. This is shown in Fig. 5.11.

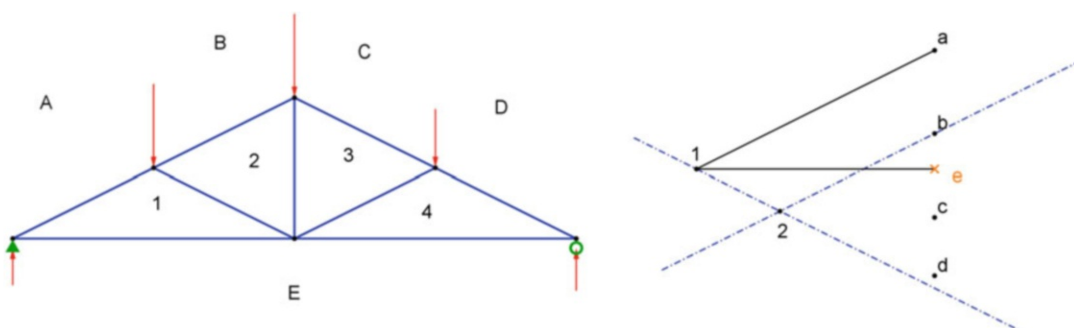


Fig. 5.11 Finding the second interior number

The find *3* observe that *3* is common to *C* through the top right chord, and *3* is common to *2* through the vertical web member. Note, *3* is common to *4* also, but *4* hasn't been found yet. This is completely in the style of method of joints, use one piece of information to find the subsequent piece. At the end, there will also be available a check of the work, i.e. one equation no unknowns, as in method of joints. Figure 5.12 shows the location of point *4* in the force diagram.

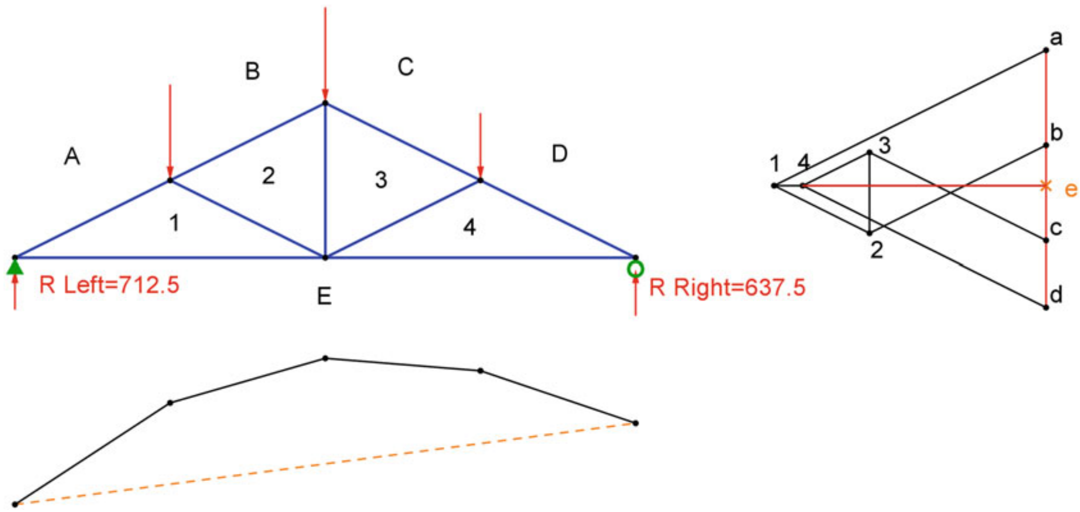


Fig. 5.12 Finding point 3 and then point 4

Notice that *4* is common to *3* so a line is passed through known point *3*, having the slope of bar *34*. *4* is also common to *E* so a line is passed with the known (horizontal) slope of *4E* through point *e*. These two conditions are sufficient to locate point *4* in the force diagram. But notice that *4* is also common to *D*, and as a check, run a line parallel to *D4* through point *d* and celebrate the fact that it falls on point *4* as well. This is the check of your work.

At first glance, the force diagram of a truss seem baffling. Hopefully this diagram is visually intriguing as well, these force diagrams have a deep and powerful meaning, as well as aesthetic power. In a very short time such diagrams will be completely understandable.

Focusing on the force diagram shown in Fig. 5.12 will provide insights. The length of each ray in the force diagram corresponds exactly to the magnitude of the load in the matching bar, of course multiplied by the force scale. So the measurement of the length of ray *a1*, multiplied by the force scale, gives the exact answer for the force in the top left chord.

The length of ray *12*, multiplied by the force scale provides the magnitude of the left diagonal web member. These values are independent of the pole location and independent of the force scale, as they must be. The determination of whether each force is tensile or compressive will be explained shortly, but now is a good time for a "reality check". A useful exercise is to recreate the magnitudes of forces shown in Fig. 5.12. It is good practice to label the forces using "dynamic text" on the form diagram, rather than on the force diagram, because the force diagram is graphically already a description of the forces. Thus, textual descriptions of the forces are not useful in the force diagram. Note, segment *1e* in Fig. 5.12 captures the left, lower chord force. Segment *4e* captures the right lower chord force. Since these are coincident for some space in Fig. 5.12, segment *4e* was drawn in red to help distinguish it from segment *1e*.

Figure 5.12 shows the magnitudes, but not the senses (tension or compression) of the bar forces. A visual clue of the overall efficiency of the members is implicit in the force diagram in Fig. 5.12, longer segments mean more force, shorter segments such as *23* and *34* denote less force in those members.

The final step is to determine the sense of each force. An immediate observation is that *de* is upward on the force diagram, so moving from *D* to *E* on the form diagram captures an upward force, i.e. the right reaction is upwards. Similarly, *e* to *a* on the force diagram is upward, thus *E* to *A* captures an upward reaction on the form diagram, meaning the left reaction is upward. This idea is useful when capturing the sense of forces in internal members.

Consider the left top chord member *A1*, or *1A*. This subtle distinction only takes a few moments to clearly understand, so spend some time with it. Naming a member actually defines which joint the member connects to. Bow's notation insists on a clockwise rotation, thus calling the member *A1* immediately requires you circle around the joint at the left support, as that is clockwise. If the member is called *A1* then you move from a to 1 in the force diagram, this is down and to the left, which means you are pushing on the joint or node, this is compression. It is shown in Fig. 5.13. However, you may choose to call this bar force *1A*. If you call it *1A*, it is impossible to move about the left reaction joint in a clockwise manner as required by Bow's notation. *1A* in a clockwise path means that the joint of investigation is the joint directly underneath force *AB*, the 500 unit force. Moving from *1* to *a* in the force diagram is a movement up and to the right, thus the force in this free body diagram is up and to the right, which of course, is compression. No contradictions can exist. This is shown in Fig. 5.13.

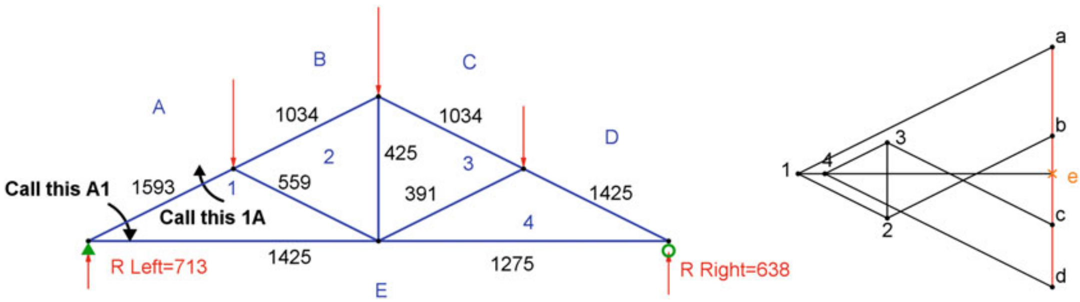


Fig. 5.13 Clockwise rotations around a node determine tension or compression of member framing into node

Using this clockwise Bow's notation allows us to doubly check the sense of each force. For example, consider the leftmost diagonal web member. If we call it *12*, we must go clockwise from *1* to *2* around a joint that touches this member. The only joint that qualifies is the center joint on the bottom chord. Moving from *1* to *2* on the force diagram means moving down and to the right. Thus, the force is compressive as it is pushing on this joint. If we call it *21*, moving from *2* to *1* clockwise around a joint that touches this member necessarily means we are circling around the joint below force *AB*. On the force diagram, this means moving from *2* to *1* is up and to the left, thus the force is up and to the left on that joint, again showing compression.

It may be helpful to use the following summary of Bow's notation for the sense of the force if you are still confused:

- Pick a member
- Pick a joint that the member touches
- Go clockwise around that joint
- Define the capital letter and number associated with that movement
- See the direction of the segment with corresponding lower case letter and number on the force diagram
- Apply that force direction to the joint being investigated, not to the member!

The last bullet is very important and a bit subtle. You are taking a free body diagram of the joint, not of the member! Pushing on a joint or a node is always compression. Pulling on any node always denotes tension. Tension is not up or down, not left or right, it is simply pulling.

While there are visual insights into the graphic description of forces in the force diagram, i.e. larger segments means larger forces, the usefulness would be limited if that were the end of the story. Let's take this example and literally push and pull it a bit further. The graphic statics analysis immediately can become a design tool. Imagine the same truss we have just looked at, but this time, suppose a client or a designer wanted a slimmer elevation. How will that impact the forces in the truss? In Fig. 5.14, the top node on the top chord was moved from its previous location of (6, 3) to (6, 1.5). This has a dramatic effect on the member force, they are markedly increased. Perhaps this is acceptable, perhaps not, but it is a very quick process to adjust the profile of the truss and to see the impact of this design decision.

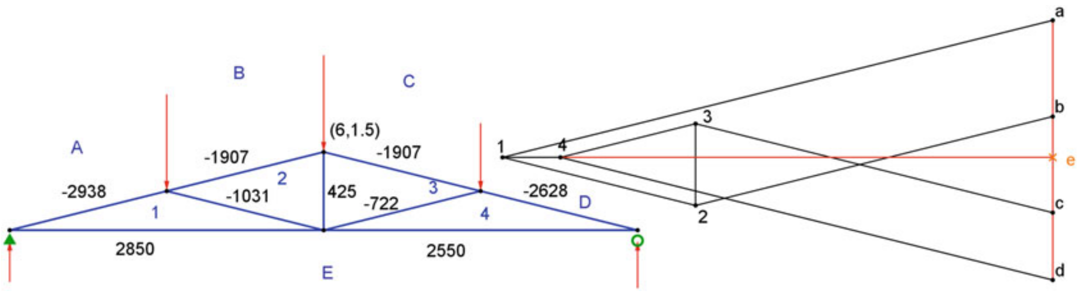


Fig. 5.14 Immediate representations of increased internal bar forces for decreased truss depth

In Fig. 5.15 another design possibility is presented. Suppose a design constraint was the upper limit of the force capacity of any member, either because of the material, the cross-section, or because of some standardized connection. Assume the upper limit is 2000 units of force for any member in the truss. What should the profile be to control the forces to remain under 2000? Without graphic statics, this would be an iterative process, try something, analyze, try again. But now, design changes can be rapidly made using parametric modeling.

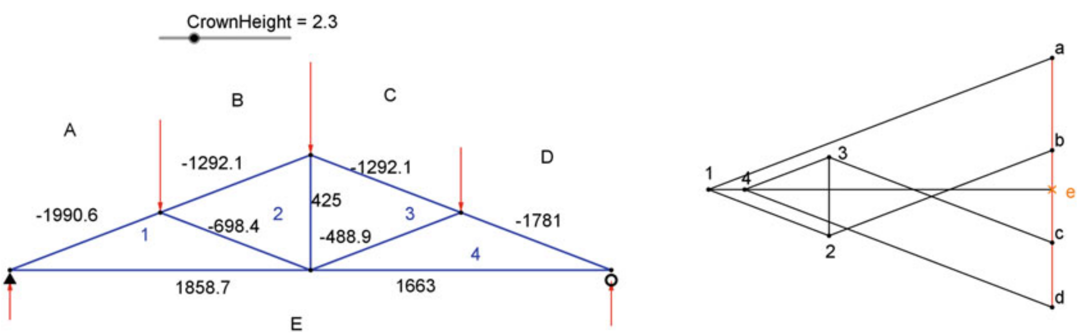


Fig. 5.15 Designing truss height based on some peak internal bar force

Here, a slider was inserted which contains the height of the top chord center node. Adjusting the position of the slider instantly updates the form diagram as well as the force diagram. Here the slider was adjusted till the peak force was acceptable, with the worst force being −1990.6 units of force for a crown height of 2.3 units of length.

Another example truss is shown in Fig. 5.16. This figure deliberately shows the construction of the loading vectors on the form diagram, to help the user understand one way of setting up the analysis/design program in a parametric modeling environment. Of course, other methods also exist, but the use of parallel lines, perpendicular lines and circles of a known radius is an efficient method of setting up the workspace. In this problem, the span of the truss is 24 units of length, the height of the truss is 8 units of length. Normal forces of 500 units of force are applied to the left top chord. Gravity forces, of 1000 units are applied vertically to nodes on the top chord.

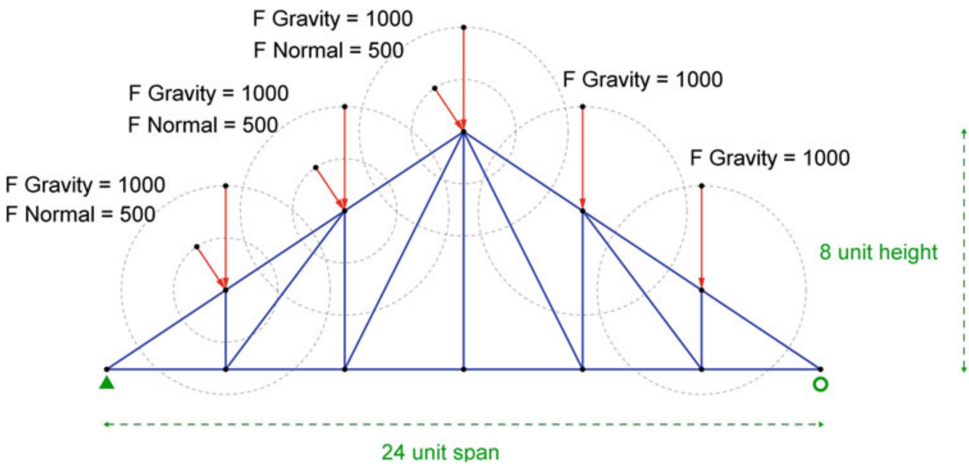


Fig. 5.16 Truss with normal loads on one side and downward loads along top chord

In the previous example with lateral loads, shown in Fig. 5.5, the normal and gravity loads were combined into a single resultant. In this current example, that will not be done, rather, as shown in Fig. 5.17, Bow's notation will account for the normal and gravity loads separately on the left top chord. Bow's notation is shown with the traditional capital letters on the form diagram. The corresponding lower case letters will be used on the force diagram, this establishes overall global equilibrium and is shown in Fig. 5.17 which also shows the force polygon. When transferred over to the form diagram, this will create a funicular, a closing line, and ultimately the location of points *j* and *k* back on the force diagram.

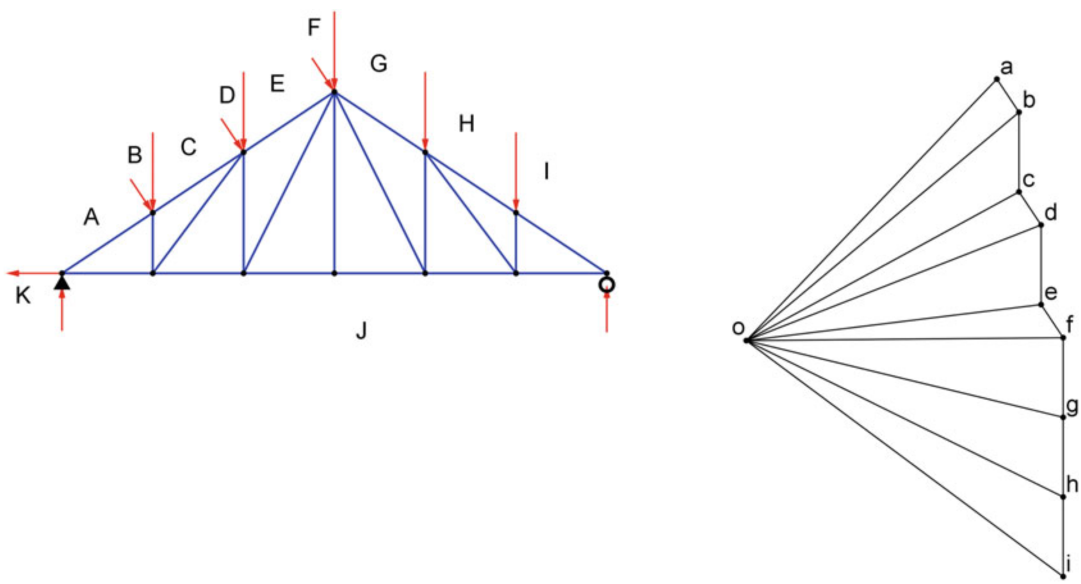


Fig. 5.17 Force diagram for previous truss

The funicular is shown in Fig. 5.18 for some arbitrary pole placement. The right reaction is the length of segment *ij*, multiplied by the force scale. The left vertical reaction is the length of segment *jk*, multiplied by the force scale. The left horizontal reaction is the length of segment *ka*.

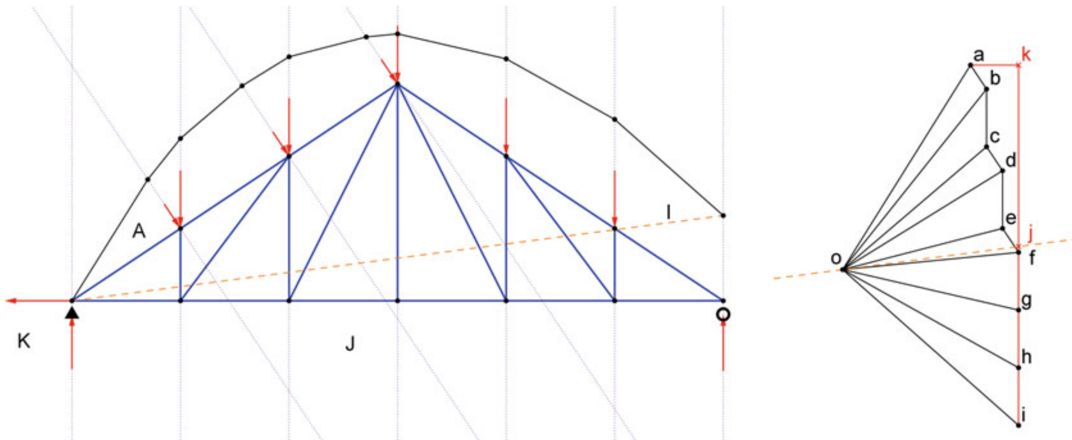


Fig. 5.18 Closing line of funicular establishes support reactions

The first few steps needed to find the internal bar forces of this truss will be shown. As described before, numbers are inserted in the interior panels allowing for a two-index designation of each bar force. The order of the numbering is arbitrary, it is convenient to start with numbers *1, 2*, etc. from the leftmost end. Notice that the original rays used to construct the funicular are no longer needed, they were used to locate points *j* and *k*. They will remain hidden for the remainder of this example (Fig. 5.19).

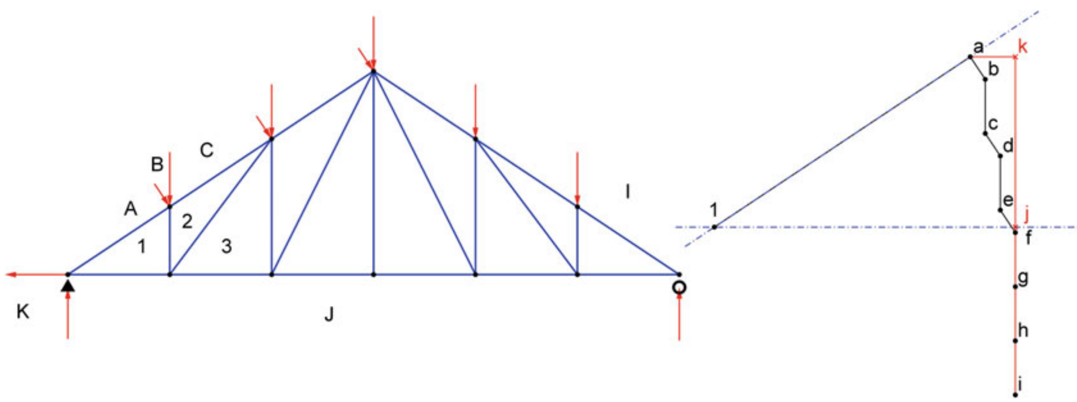


Fig. 5.19 Initial construction of force polygon to locate point 1

One possible option is to find point *1*, since point *1* is common to *A* (and *a* is known) and *1* is common to *J* (and *j* is known). This essentially is one equation with one unknown. Alternatively, one could have started with the number in the far right interior panel. Draw a line parallel to *A1* through *a*, and a line parallel to *1J* through *j*. The intersection of these lines locates point *1*. The magnitude of the load is the length of segment *a1* multiplied by the force scale. Be careful! Ensure that the horizontal line passes through *j*, and not through the nearby point *f*.

Using the Bow's Notation rules of "clockwise around a joint", the sense (tension or compression) of the force is immediately apparent. Recall the handy technique "pick a member, then pick a joint". Choosing the leftmost top chord member, and then picking the joint that is pinned at the left end, the member must be called *A1*, since swinging clockwise around that joint starts at *A* and moves to *1*. Thus, moving from *a* to *1* on the force diagram is a downward, leftward movement, meaning that the force is pushing on the left joint, thus it is compression. Alternatively, picking the same member, but picking the other joint would designate the member as *1A*, since a clockwise rotation about that joint begins at *1* and moves to *A*. Then, moving from *1* to *a* on the force diagram is up and to the right. Looking at the force exerted on that joint, a force up and to the right is pushing on that joint. Again, this means compression, no contradictions can exist.

The next step is to find point *2*. *2* is common to *1* through the vertical web member, and *2* is common to *C* through the diagonal top chord member. *2* is also common to *3* through the diagonal web member but this is not useful since point *3* has not been located yet. Thus draw a vertical line through *1* and a diagonal line through *c* to locate *2*. The magnitude of this force is the length of segment *12* multiplied by the force scale (Fig. 5.20).

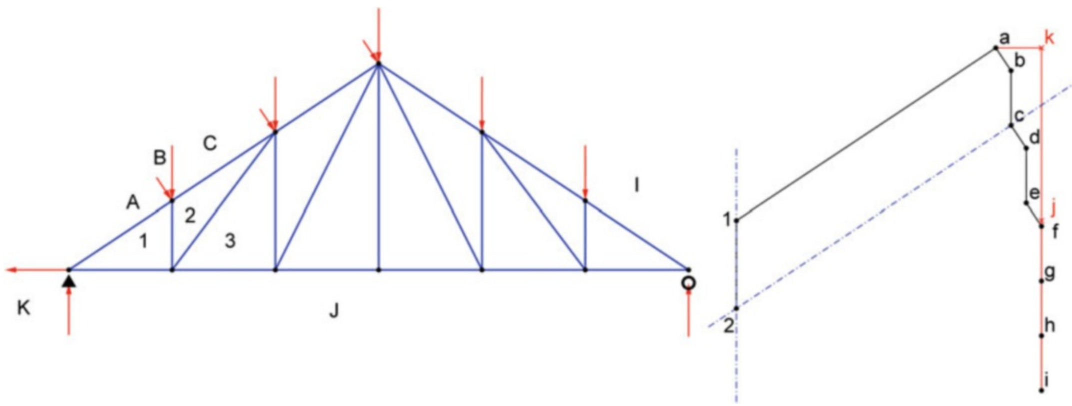


Fig. 5.20 Locating the second interior panel point

The sense of this force is found by picking a joint. If the top joint is chosen, the member is called *21*. If the bottom joint is chosen, the member is called *12*. Moving from *2* to *1* in the force diagram is upwards vertically, and an upwards vertical force on that top joint of *21* in the form diagram means pushing on the joint, i.e. compression.

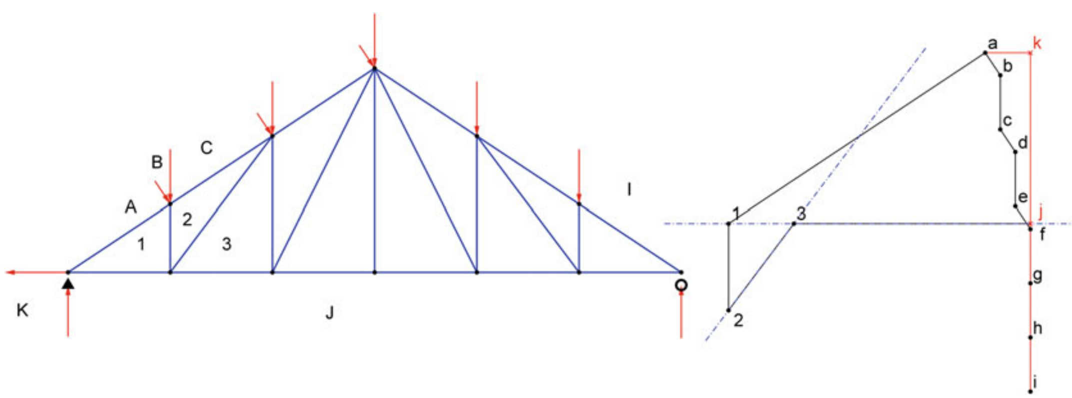


Fig. 5.21 Locating the third interior panel point

To locate point *3* as shown in Fig. 5.21, a line parallel to *23* is passed through *2* and a line parallel to *3J* is passed through *j*. Choosing the lower joint of this member necessitates calling it *23*, and moving from *2* to *3* on the force diagram is up and to the right, thus it is pulling on the joint, meaning tension.

The following example is of a truss with an overhang. The first panel is 10 units of length wide, the other two panels are 15 units of length wide. Four point loads of 1000 units of force are applied as shown in Fig. 5.22. The angle forming the top chord is 60° to the horizontal. Bow's notation is shown as well as the roller and pin supports.

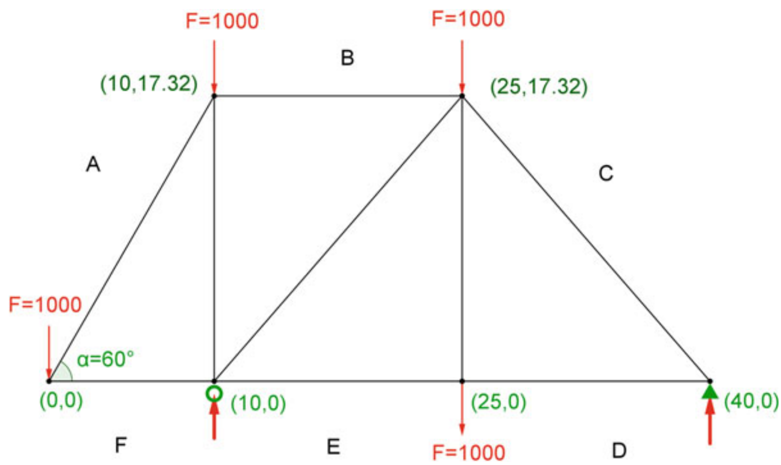


Fig. 5.22 Example of truss with an overhang

There are several stumbling blocks in this problem. The first is the overhang in area *F*. The second is that some loads are pushing down on the top of the truss, and one load is pulling from the bottom. A concise manner of handling the two co-linear loads is to simply combine them into one load. Thus the region between *B* and *C* has a 2000 force drop, and the transition from space *D* to *E* captures the upwards left reaction. That way, letter *F* is no longer needed. This is technically correct from the external reactions' point of view, it is analogous to replacing several loads with a single, statically equivalent load. Yet it is not correct for an analysis of the interior bar forces. Again, an analogy can be made with free body cuts that no longer capture forces that were replaced by an equivalent static load. This idea will be reinforced in Fig. 5.25.

The static simplification technique is shown in Fig. 5.23 with a darker line denoting the statically equivalent drop from *B* to *C*. Notice that *b* to *c* is twice the other drops in the force diagram.

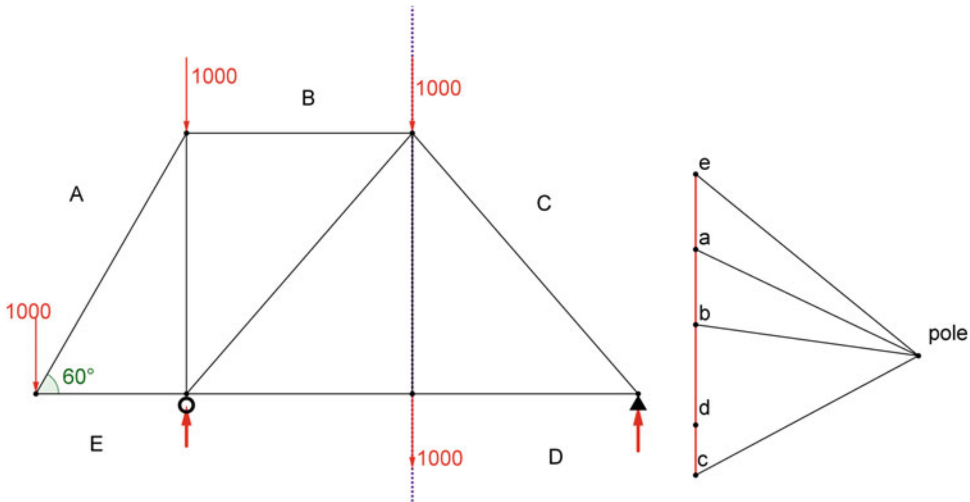


Fig. 5.23 Combining two colinear loads into one

The overhang presents a small challenge. To overcome this challenge, begin the funicular at the left support in region *E*, using the ray *e-pole*, till it intersects into region *A* which occurs at the extreme left end of the truss. Then lay down ray *a-pole*, *b-pole*, *c-pole* till the region of *D*. A closing line is drawn between the start and the end of the funicular as always, here shown in Fig. 5.24 with an orange dashed line. Transferring that line through the pole of the force diagram locates point *d* on the vertical load line. Note that in Fig. 5.24 region *D* in the form diagram extends below the truss from the right pinned support until the left roller support because both downward loads were incorporated into a single jump from *B* to *C* as if they were both acting on the top of the truss.

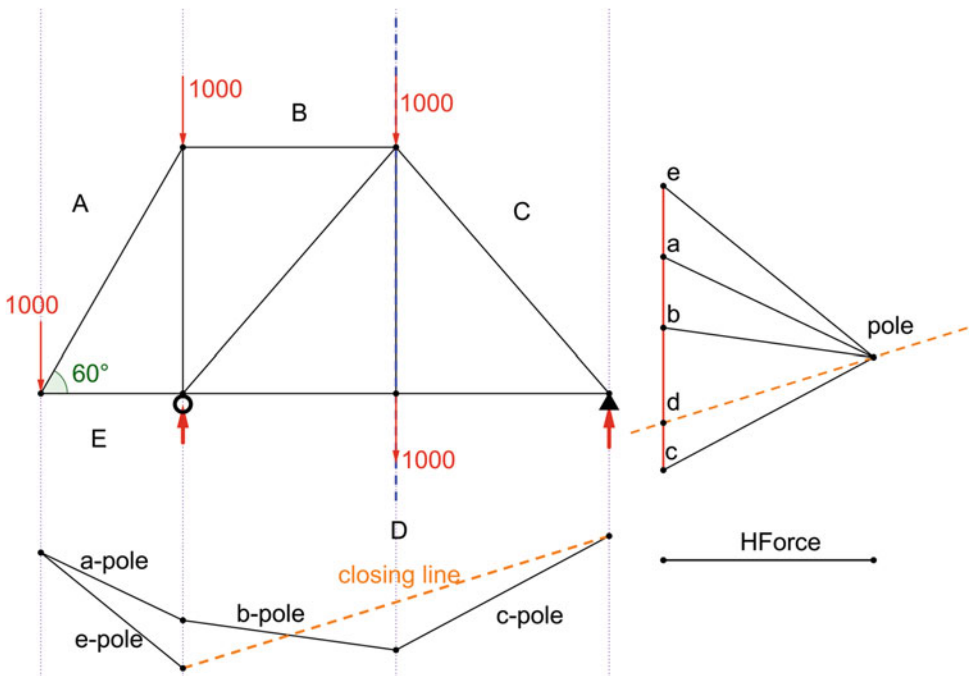


Fig. 5.24 Establishing the support reactions of the truss

Having point *d* on the force diagram completes the external equilibrium calculations, because the length of a segment from *c* to *d*, multiplied by the force scale is the right reaction, and the length of a segment from *d* to *e*, multiplied by the force scale is the left reaction. Changing the location of the pole, or changing the magnitude of *HForce* has no effect on this calculation. Recall that this calculation was done with the simplification of combining two external loads into one.

To demonstrate that this statically equivalent simplification is not useful for determining internal bar forces (it of course was very useful in establishing the external reactions), consider the internal bar force calculations for the truss with region *F* removed (Fig. 5.25).

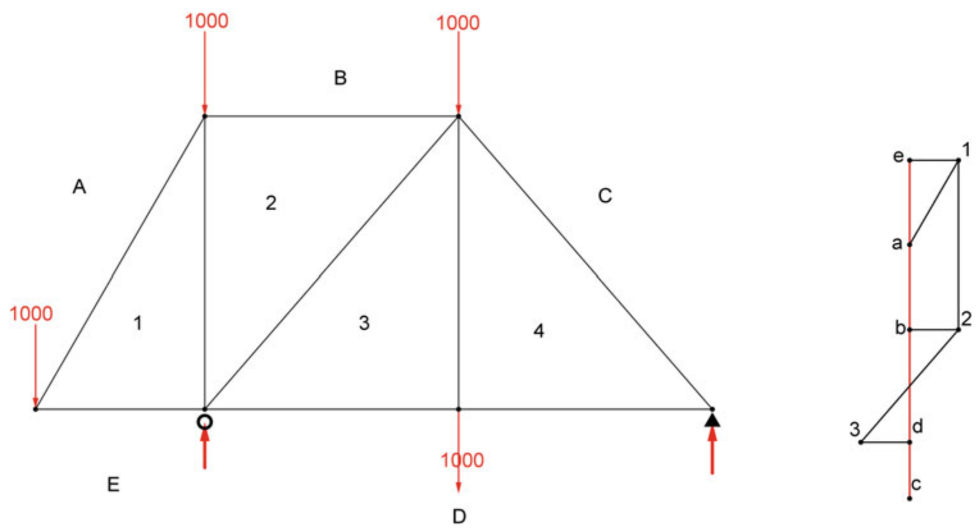


Fig. 5.25 Combining two external loads into one leads to incorrect analysis of individual bar forces

Here is the difficulty mentioned before, the bar forces *A1*, *12*, *23* are all correct, but now bar *34* appears to be a zero force member, and it clearly is not. The reason for this is that the 1000 unit force left of space *D* was removed and it was put it between *B* and *C*. If this is not perfectly clear, imagine an analogous situation. Suppose all four original loads were replaced by a single, statically equivalent load (Fig. 5.26).

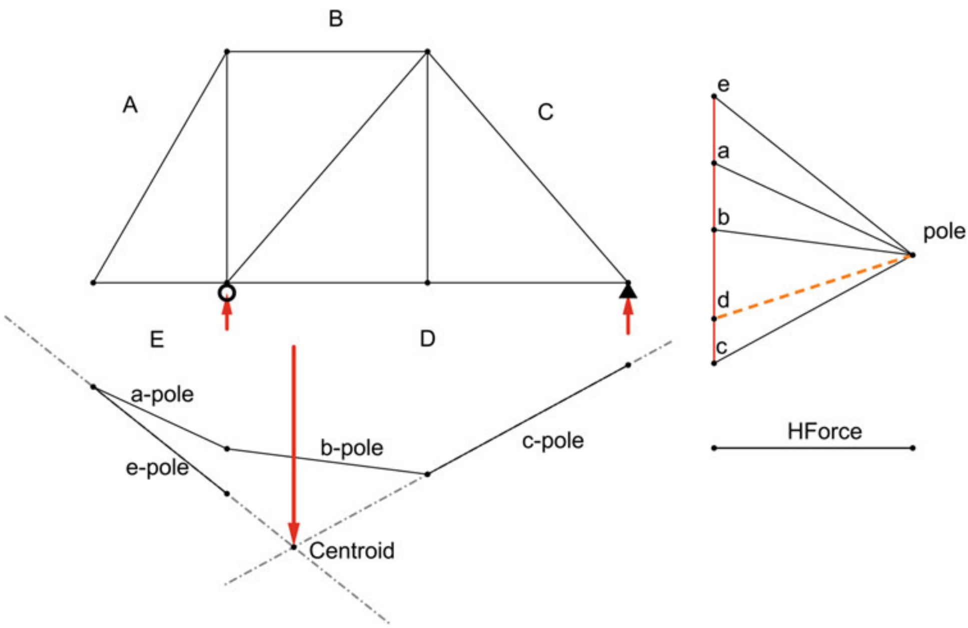
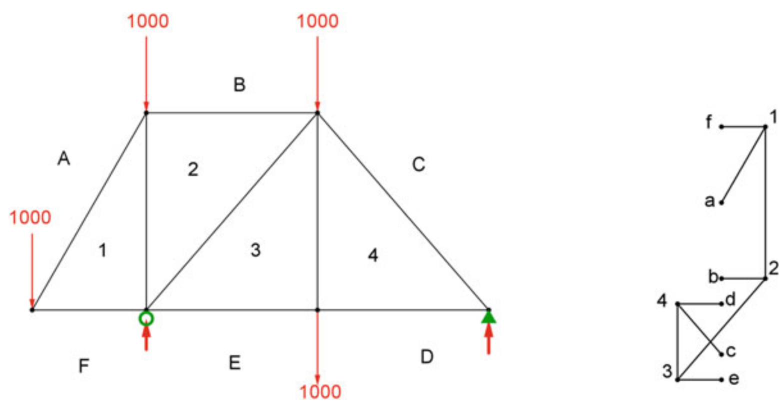


Fig. 5.26 Combining all external loads is correct for external reactions, but inappropriate for internal bar force calculations

Recall that a very quick way of locating the centroid of the loads is to intersect the start and the end slopes of the funicular. This is the centroid. It, of course, can be used to calculate the external support reactions, but it is not at all useful for calculating internal member forces. The exact same thing happened in Fig. 5.25, but on a smaller scale. Two loads were combined, preventing accurate internal force calculations in the region where they were combined.

To resolve this dilemna, use any number of statically equivalent consolidations needed to calculate the external reactions, but then return to the original configuration of applied loads to calculate internal bar forces. This is shown in Fig. 5.27.

Fig. 5.27 Use all original loads without combining when calculating bar forces



This next example appears to be a slightly complicated problem because of the complicated angles of the elements, as well as the varied loads, some of which are inclined. The technique used to solve this problem recalls the first methods shown in Chap. 1, namely, quickly finding resultants and equilibrium of a point subjected to three loads. Furthermore, the multiple loads can be readily handled by breaking them up into two parts; part 1 accounts for the three inclined loads, and part 2 accounts for the two vertical loads. Using such quick, geometric insights whenever possible sharpens one's ability to see if the answers make sense or not (Fig. 5.28).

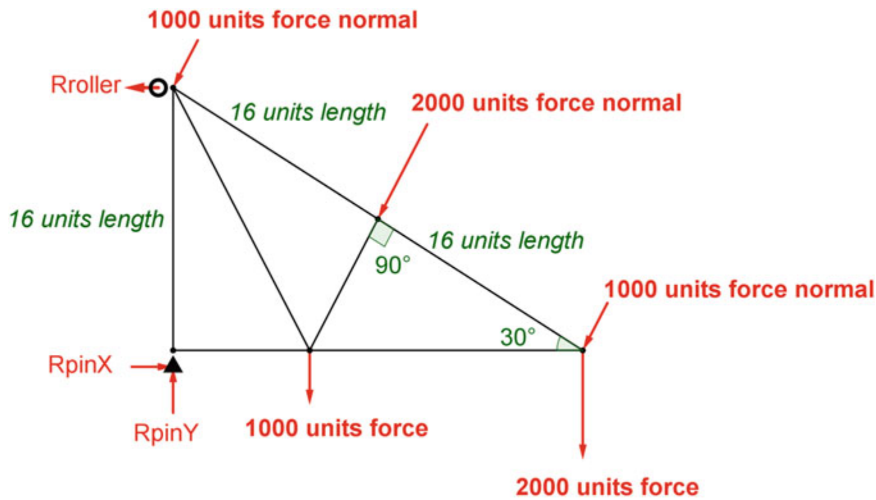
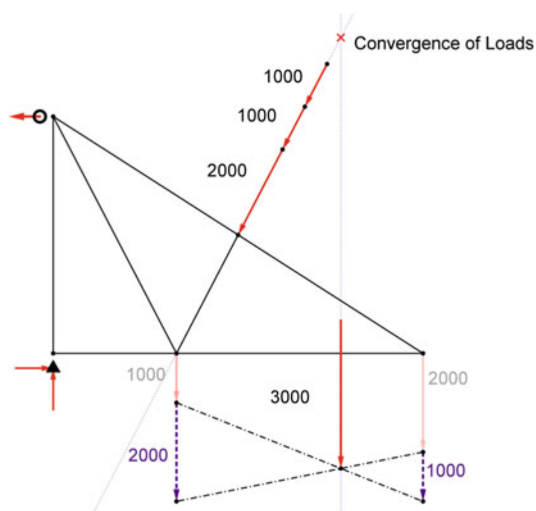


Fig. 5.28 Truss supported on its side, subjected to multiple inclined and vertical loads

Here, the diagonal loads on the top chord can be immediately combined into one resultant, due to symmetry. The two vertical loads hanging off the lower chord can be combined into one resultant using the "inverse axis method" described in Fig. 1.13. The "inversed" lower vertical loads are shown in purple. Geometrically find the resultant of these two resultants to get the overall resultant of all of the loads. This is shown in Fig. 5.29

Fig. 5.29 Three top chord loads are combined into one and two bottom chord loads are combined into one



Now the problem is elegantly viewed as *equilibrium of a point*. All of the external loads are reduced to a single resultant. The roller reaction must be horizontal, thus a horizontal line is passed through this support. The magnitude and direction of the pinned reaction are both unknown, yet the net pinned reaction must pass through the pin itself! Thus, equilibrium of a point is drawn graphically and since the external load resultant is fully known, drawing it to some convenient scale immediately establishes the magnitude of the two external reactions (Fig. 5.30).

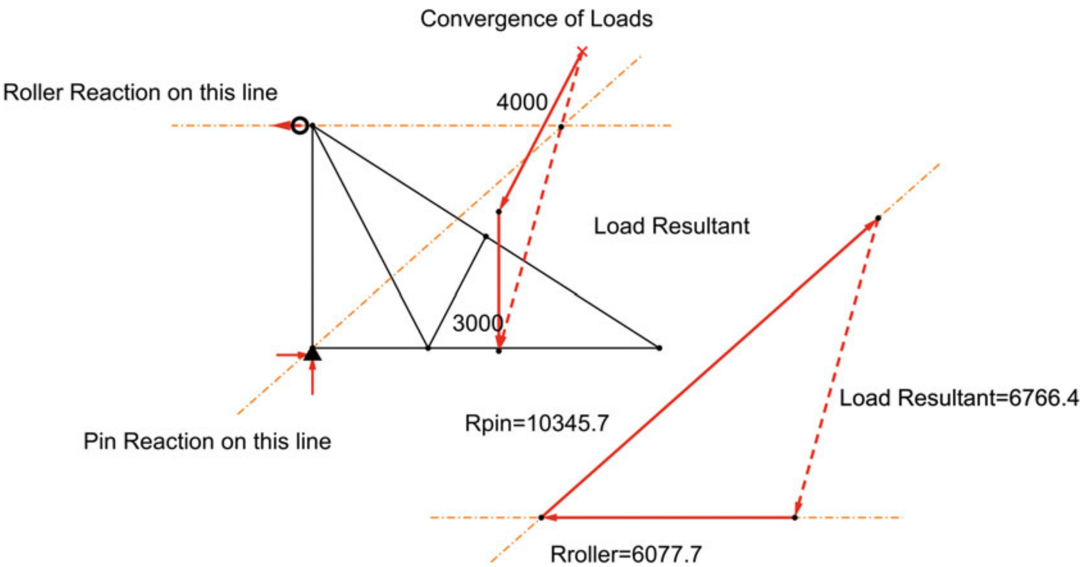


Fig. 5.30 External loads resultant converges on *x*, and *x* must be on path that intersects roller to find dot, and then to pin

Having the external reactions allows for the creation of the complete force diagram, including the location of the interior points *1*, *2* and *3*. This is shown in Fig. 5.31. Solving for these interior panel points allows for the calculation of bar forces, for example bar force *G1* (or *1G*) is the length of the segment between *g* and *1* multiplied by the force scale. That value is 6464.1 units of force.

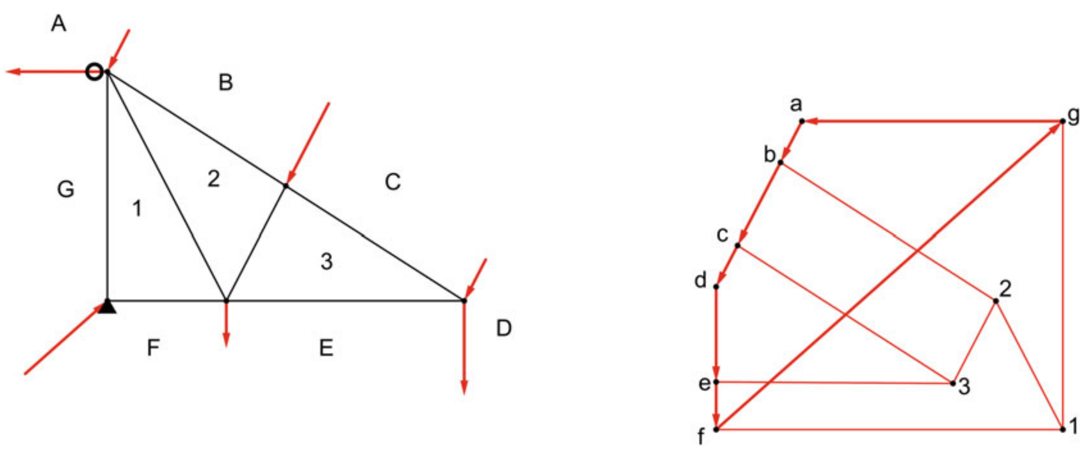


Fig. 5.31 All bar forces can be calculated once external reactions are found

Method of Sections

The method of sections analyzes only a free body diagram of a portion of a truss. It is used for the situation where only a few selected truss member forces are needed. As such, the cut needed to create the free body diagram should pass through the members of interest. Then, either the left free body, or the right free body can be analyzed. In the following example, very small steps are taken to provide insights into the statics of such a problem. A moment equilibrium equation could have been used to solve for one of the unknowns, but here, a careful construction of equilibrium of a point is slowly developed.

In the following example, suppose one was interested in only the members between the first two 2000 unit external loads. The truss shown in Fig. 5.32 has a constant 30° slope for its top chord, and the truss is 60 units of length long with loads applied as shown, with vertical reactions shown after being solved for by inspection. The horizontal reaction at one of the pinned supports must be zero here, as there are no lateral loads.

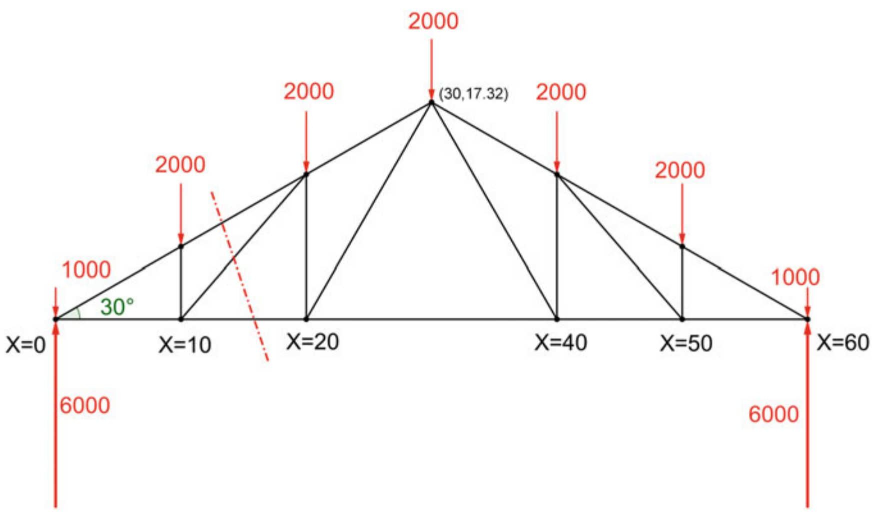
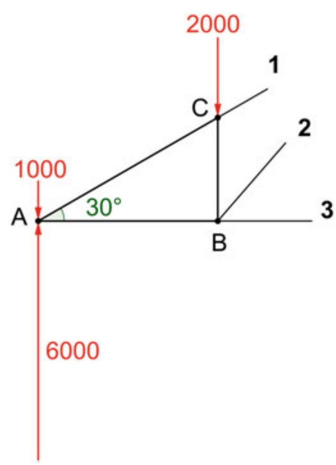


Fig. 5.32 Initial problem statement demonstrating method of sections

In Fig. 5.33, the free body on the left side is shown, with the two given applied loads, the one known vertical reaction and the three unknown bar forces exposed by the cut.

Fig. 5.33 Free body to the left of the cut



From the study of Fig. 5.33, it is clear that the cut members *1*, *2* and *3* must add another net 3000 units of force vertically to equilibrate this fee body. Furthermore, the net horizontal of the three exposed struts *1*, *2* and *3* must have zero net horizontal force, as the pin support has no horizontal reaction in this problem. Is there a way of elegantly reducing this problem to equilibrium of a point, rather than equilibrium of a section that experiences rotation? The answer is "Yes!".

Step 1 is to use the "inverse axis method", previously described. Lay off 3000 units of force in reverse order, (i.e. under line *BC*) and determine where the 2000 unit force is applied such that there is no rotation about point *A*. This is shown in Fig. 5.34. This step is necessary to locate point *Q*, which is where the equilibrating 3000 unit force must be applied to remove rotation of this free body. The height of segment *QR* reflects 2000 units of force since this is the inverse axis method.

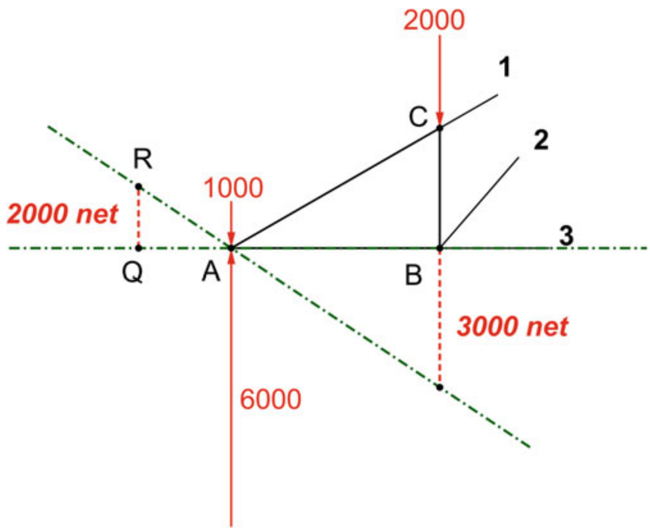


Fig. 5.34 Determining where to place extra load so that no rotation occurs

In Fig. 5.35, the 3000 unit net force is placed on the line established by *QR* vertically through points *Q* and *R*, thus begins the equilibrium of a point study. The force of bar *1* must pass through the line of action of the 3000 unit force, that intersection is shown as point *T*.

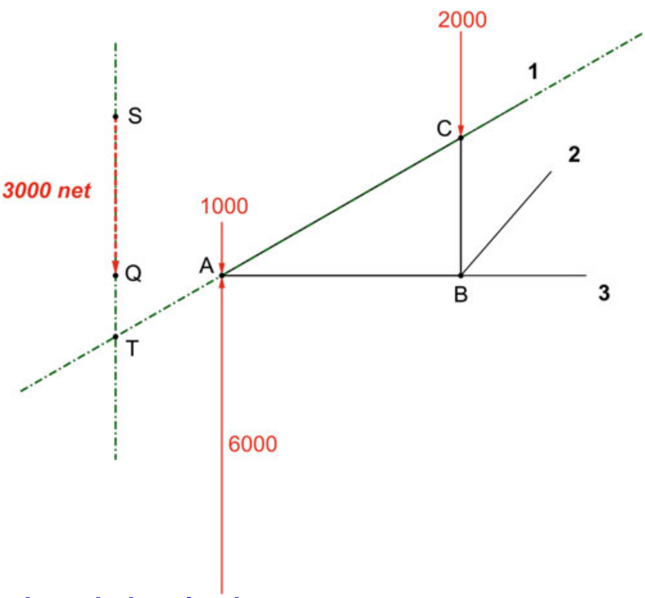


Fig. 5.35 Actually placing the equilibrating load in the vertical path previously found

The resultant of bar forces *2* and *3* must pass through Point *B*, yet equilibrium of a point also necessitates that this resultant pass through point *T* as shown in Fig. 5.36. It is this insight that allows for the final calculation of equilibrium of a point.

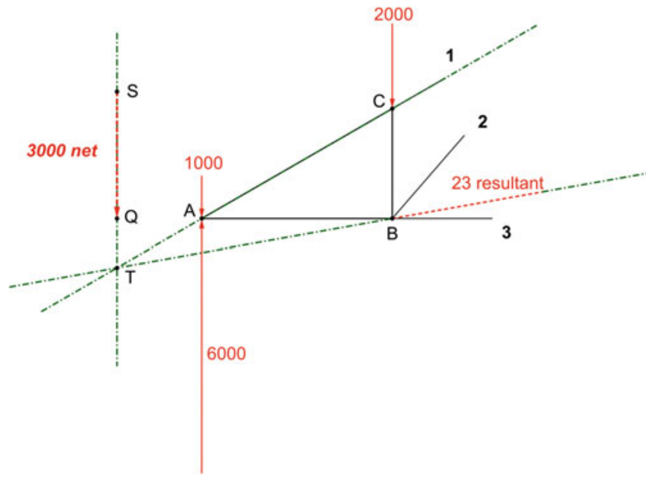


Fig. 5.36 Slope of resultant of bars 2 and 3 is found

The resultant of bars *2* and *3* and the bar force of *1* are laid down in proportion to the 3000 unit force imbalance. Then, the *23* resultant can be immediately broken down into components *2* and *3* (Fig. 5.37).

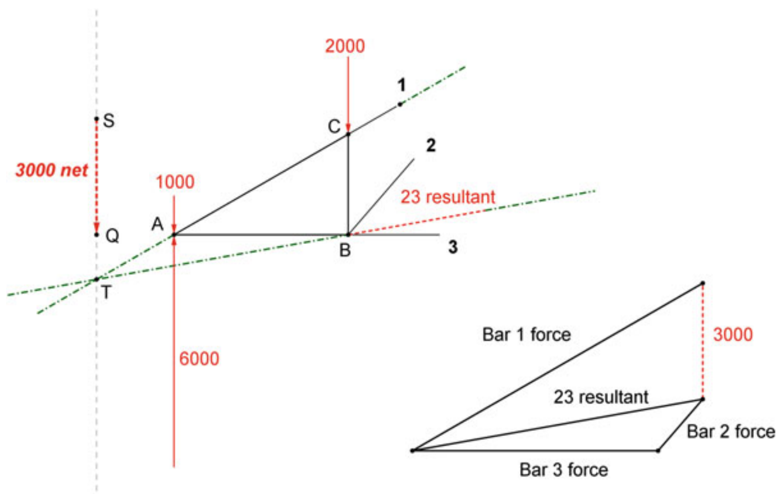


Fig. 5.37 Resultant of forces from bars 2 and 3 is decomposed back into bar force 2 and bar force 3

Figure 5.38 shows the original data for another example of a truss problem. Here, information is required only at the shown section cut. This is typically done with method of sections or method of joints, but here the problem will be solved by analyzing a free body as would be done in method of sections, but the problem will be finished by method of joints. This example deliberately chooses a truss geometry that at first glance does not seem overly complex, but upon closer inspection reveals many angles that would have to be calculated if this problem were to be done algebraically. Note for example, that the inclined loads are perpendicular to the top chord, but the web members are not. The load on the overhanging tip is vertical.

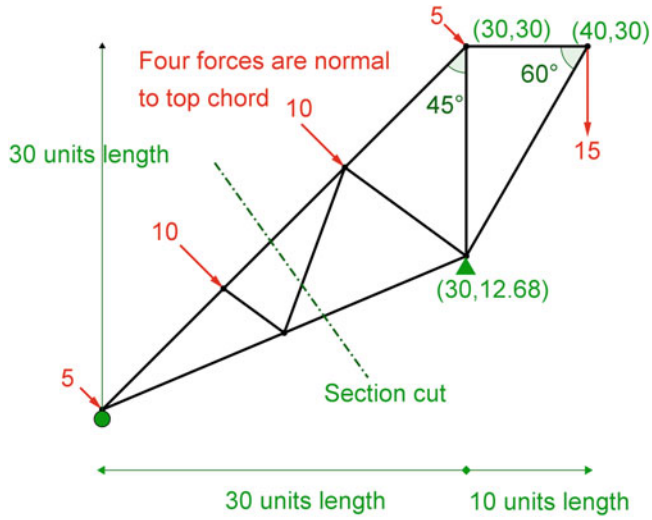


Fig. 5.38 Complex geometry of truss, method of sections and method of joints problem

As in algebraic statics, when investigating the forces exposed by a section cut, one could analyze either the left side or the right side of the cut. Both sides must provide exactly the same answer for the exposed forces. In this example, the left side free body will be analyzed as it is bit simpler than the right side. Step one is to find the reactions. To simplify the process, the four normal loads are replaced by an equivalent static load of 30 units of force centrally located on the top chord. This was done by inspection. Bow's Notation was used to place capital letters between loads on the form diagram, using a single resultant reaction at the hinged right support, thus only four letters are needed. Five letters would have been needed if the X and Y components of the right support were found separately. Any arbitrary pole *p* is established on the force diagram which is drawn to some convenient scale. Three rays *ap*, *bp* and *cp* are drawn. Nothing is known about the orientation of the net reaction on the right, only that the funicular must pass through this point. Thus, ray *cp* is laid down first on the funicular shown back on the form diagram. Ray *cp* is extended till the funicular passes into zone *B*, marked by a vertical line through the 15 unit applied load. Then ray *bp* is laid down, till zone A is entered, the transition is marked by the path of the 30 unit load. As before, a closing line connects the start to the end of the funicular. That closing line is transferred to the force diagram (again orange dashes) to locate point *d*, and consequently the reactions are unlocked (Fig. 5.39).

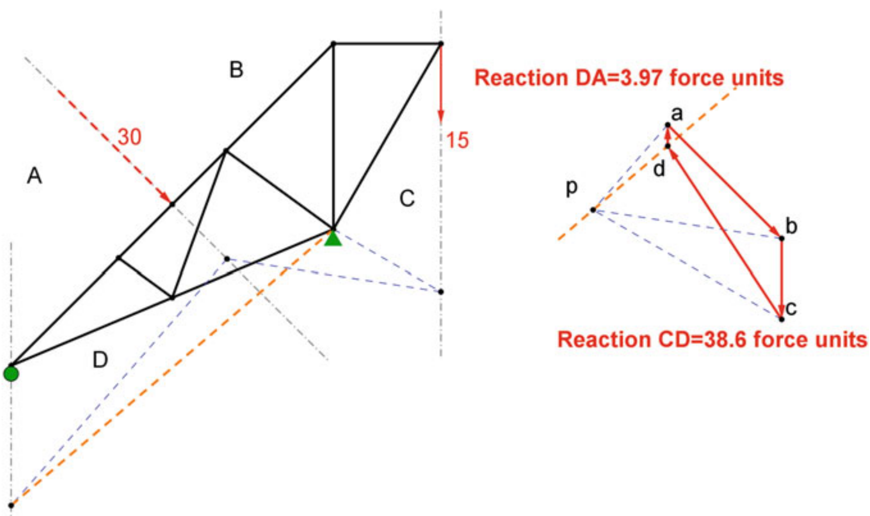


Fig. 5.39 Funicular is established by starting with ray *cp*, then closing line is drawn to find *d*

To demonstrate that the location of the pole *p* has no effect on the identification of point *d*, Fig. 5.40 shows an alternate solution for the external reactions of the structure. Notice how the funicular looks different than it did in Fig. 5.39, but the transitions or kinks are still defined by the same paths of loads or known roller reaction trajectories.

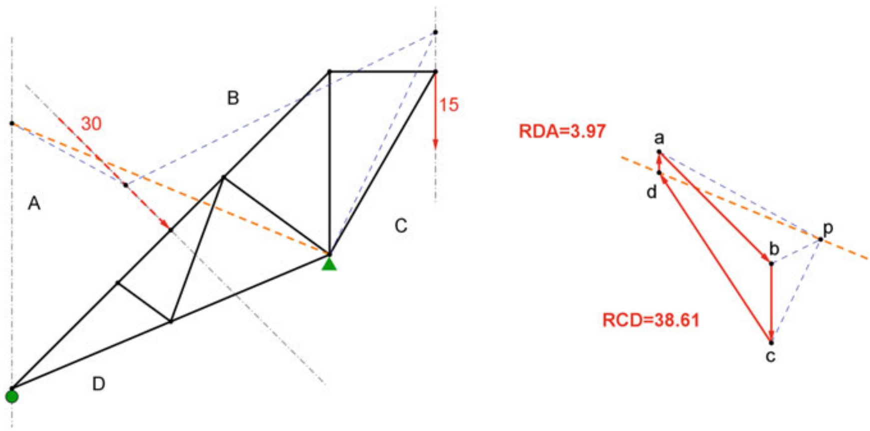


Fig. 5.40 Alternate but equivalent funicular compared to previous figure

Having the reactions allows for solution of the individual bar forces in the traditional manner, essentially a method of joints step by step solution. But notice that such a solution requires the location of Points *b, c, d, e* and *f*. In other words, the equivalent resultant 30 unit load cannot be used for analysis of individual bar forces, it can only be used to find the external supporting reactions. This is shown in Fig. 5.41 and was previously presented in Fig. 5.26. This more detailed setup will be used for a method of joints solution later. What was Point *d* in Fig. 5.40 has become Point *g* in Fig. 5.41.

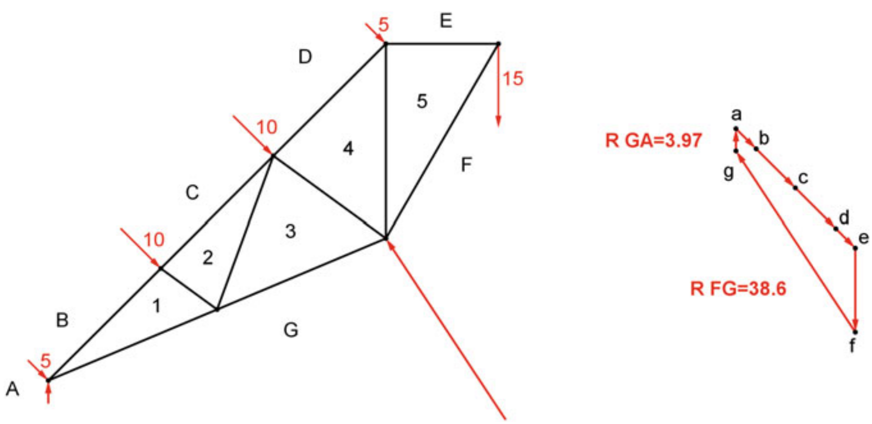


Fig. 5.41 Redrawing the form diagram, using what was point *d* but now is point *g*

To solve this problem via method of sections isolating the members left of the cut in Fig. 5.38, first find the resultant of loads *AB* and *BC* (the 5 unit and 10 unit forces). These act with the vertical left reaction $R\ GA = 3.97$. The 15 unit load and 3.97 unit reaction converge on Point *O1*. The resultant of the left reaction $R\ GA = 3.97$ and the 15 unit applied force is shown in Fig. 5.42 and is labeled as

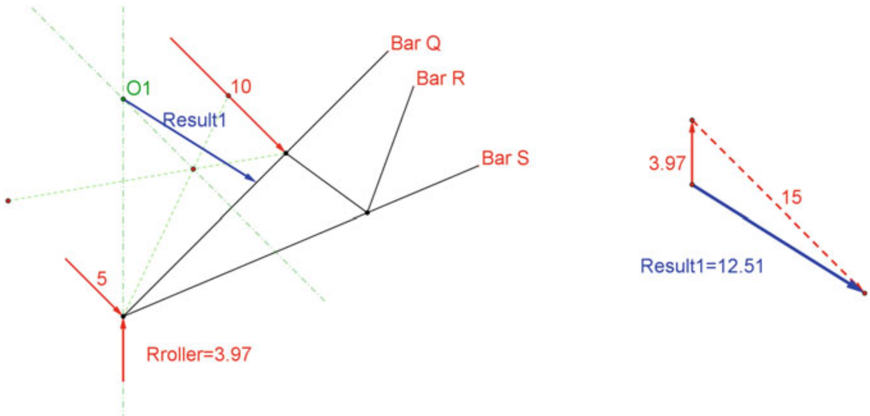


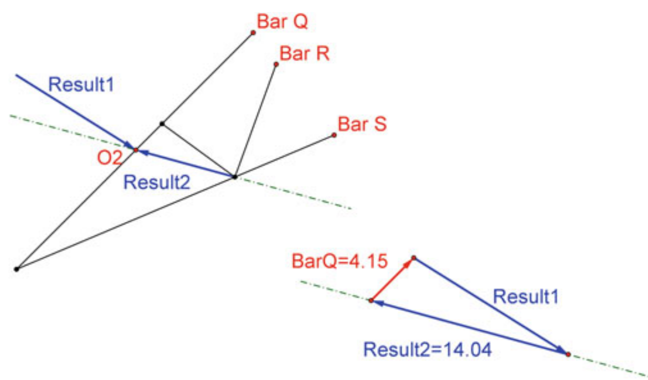
Fig. 5.42 Method of sections, free body left of the cut

Result1. The magnitude of this resultant is 12.51. *Result1* must be equilibrated by *Q*, *R* and *S*, or more compactly, by *Q* and the resultant of *R* and *S*.

Result1, *Bar Q* and the resultant of *R* and *S* must meet at *O2*, since *O2* is in line with *Result1* and with *Q*.

We know that *Result2* is the resultant of *R* and *S* and that it passes through *02* as well as the juncture point of *R* and *S*. Thus, establish the magnitude of *Result2* since the slopes of *Result1* and *BarQ* are known (Fig. 5.43).

Fig. 5.43 Method of sections reduces to equilibrium of a point



Finally, decompose *Result2* into *BarR* and *BarS* as shown in Fig. 5.44.

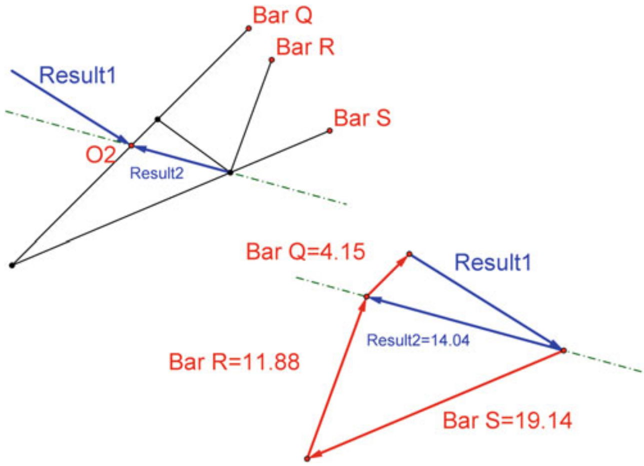


Fig. 5.44 Decomposing intermediate result into final bar forces

Typically, method of sections is faster to perform than method of joints because when done algebraically, a single moment equilibrium equation is often employed. In the previous example, the method of sections was performed solely using force equilibrium, essentially three equations, three unknowns. Repeating the problem to obtain forces in Bars *Q*, *R* and *S* via a graphical method of joints is quite efficient. The solution is shown in Fig. 5.45 and of course it matches the previous solution.

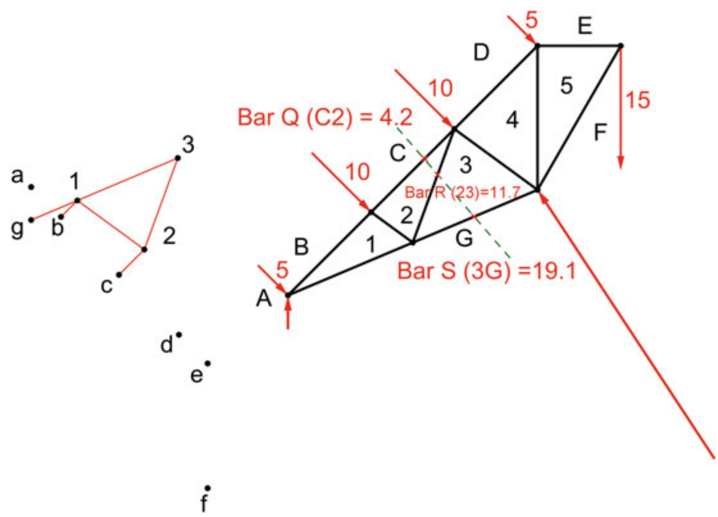
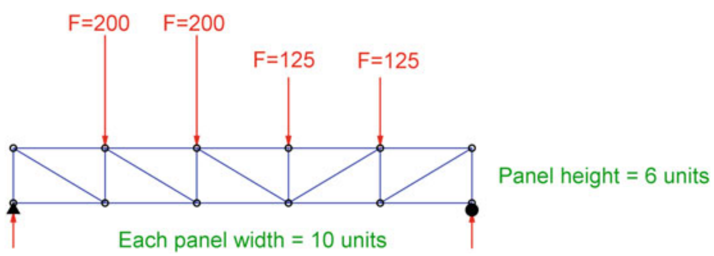


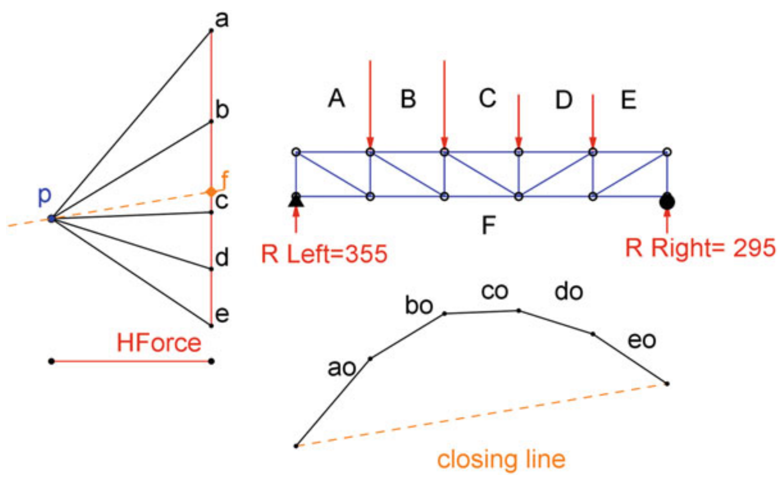
Fig. 5.45 Alternate, but complete solution, using method of joints

Chapter 5 Exercises

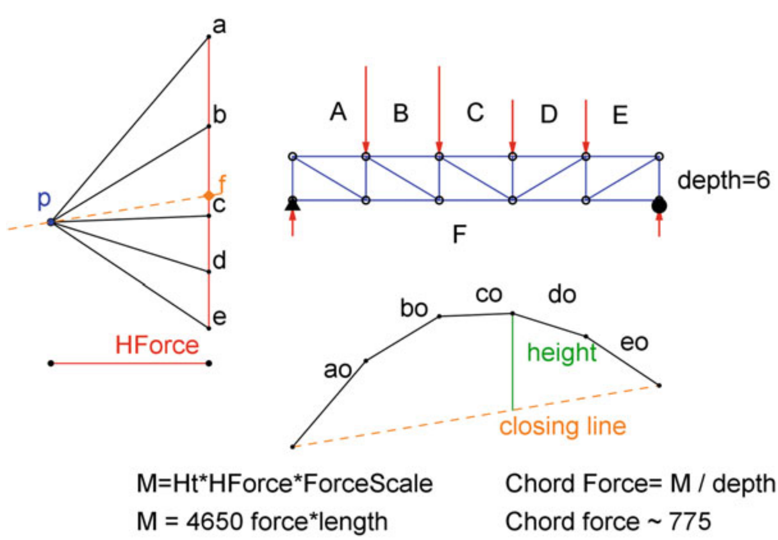
Exercise 5.1 Calculate the reactions at the ends of this simply supported truss.



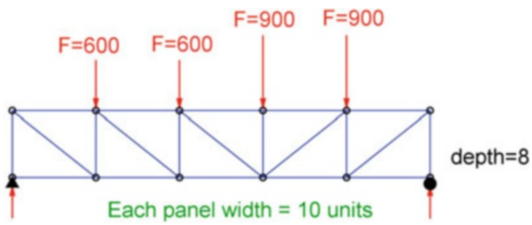
Exercise 5.1 solution



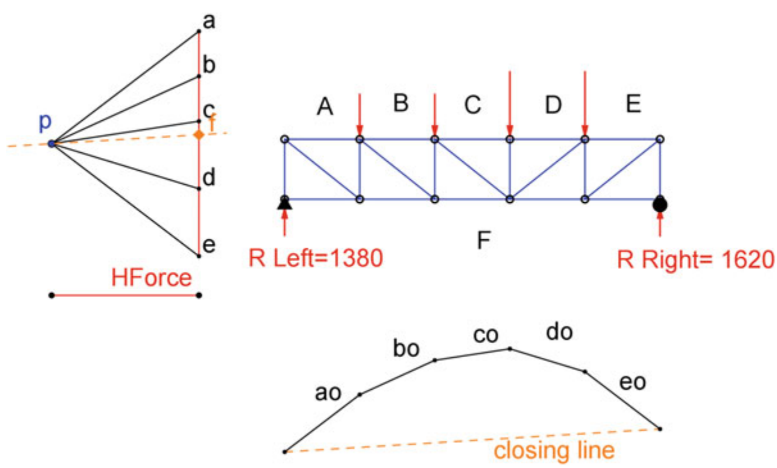
Exercise 5.2 For the truss in Problem 5.1, estimate the peak chord force.
Exercise 5.2 solution



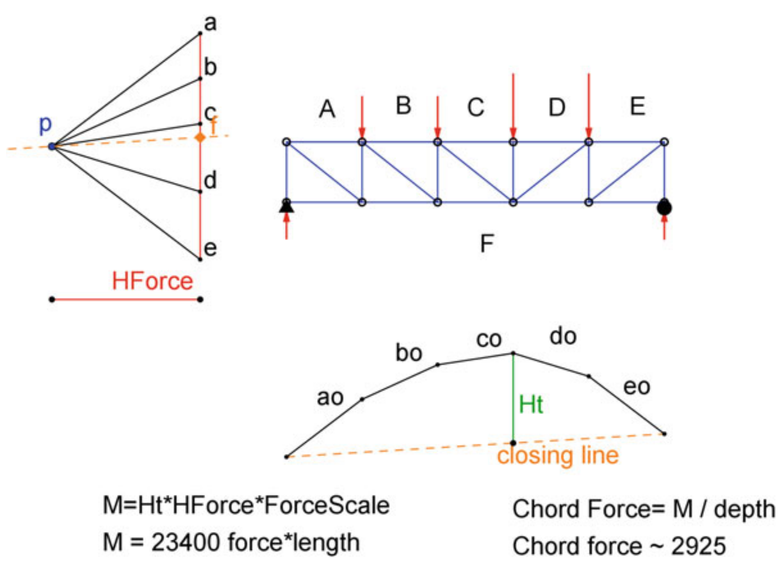
Exercise 5.3 Calculate the reactions at the ends of this simply supported truss.



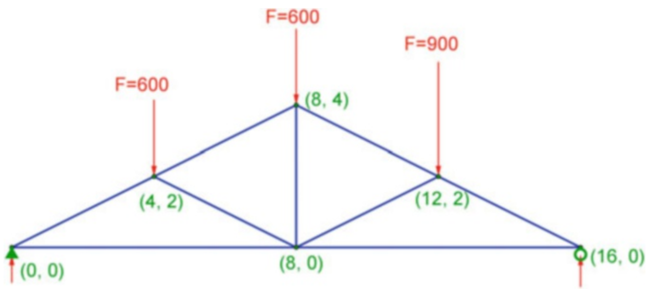
Exercise 5.3 solution



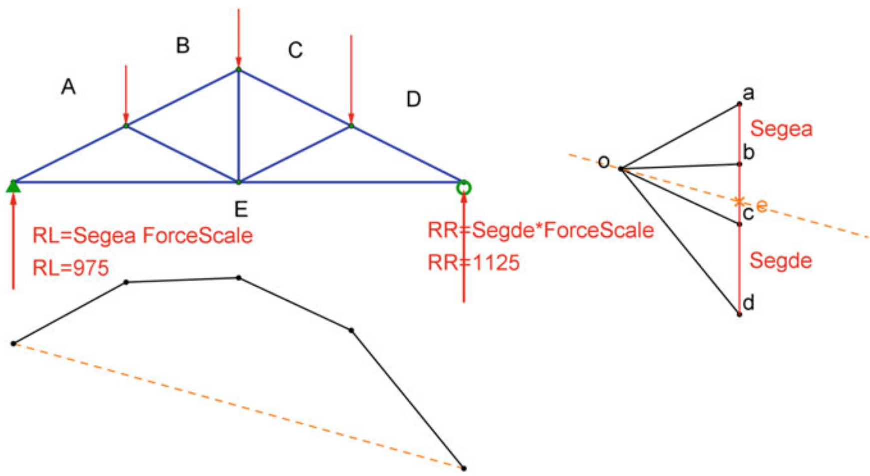
Exercise 5.4 For the truss in Problem 5.3, estimate the peak chord force.
Exercise 5.4 solution



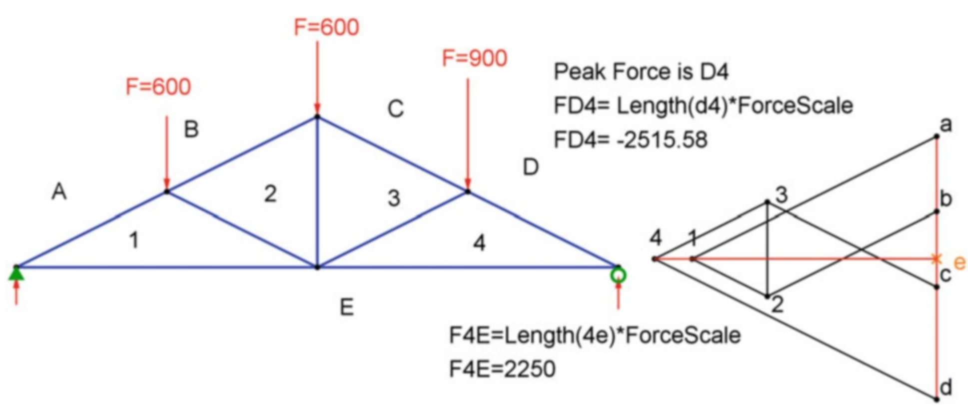
Exercise 5.5 For the simply supported truss, calculate the reaction forces.



Exercise 5.5 solution

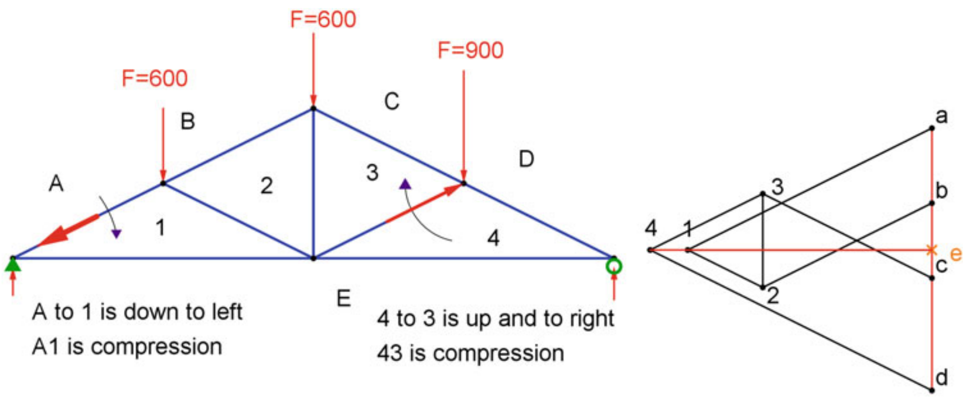


Exercise 5.6 For the truss shown in Problem 5.5, calculate the extreme bar forces.
Exercise 5.6 solution

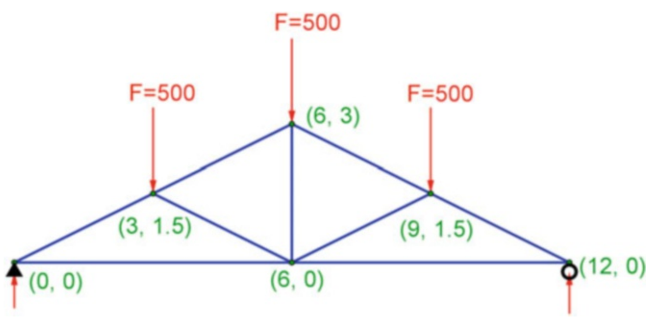


Exercise 5.7 For the truss shown in Problem 5.6, determine the sign (positive = tension, negative = compression) for several bars.

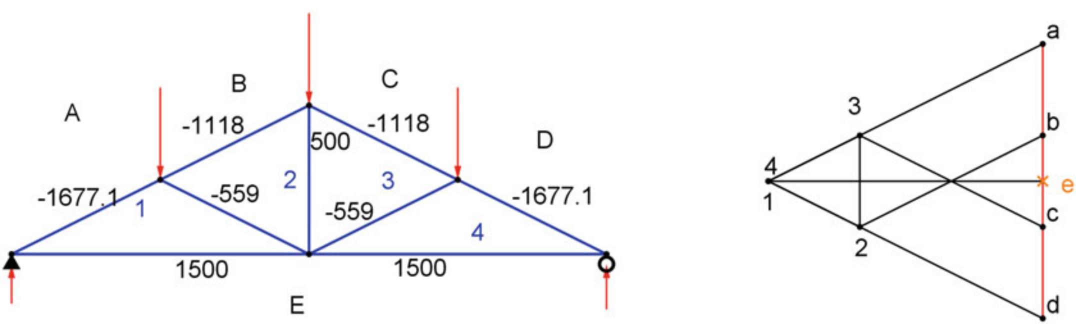
Exercise 5.7 solution



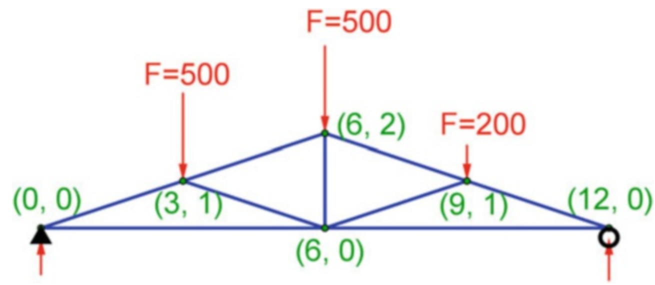
Exercise 5.8 For the following simply supported truss, calculate each bar force, including the sign (tension or compression).



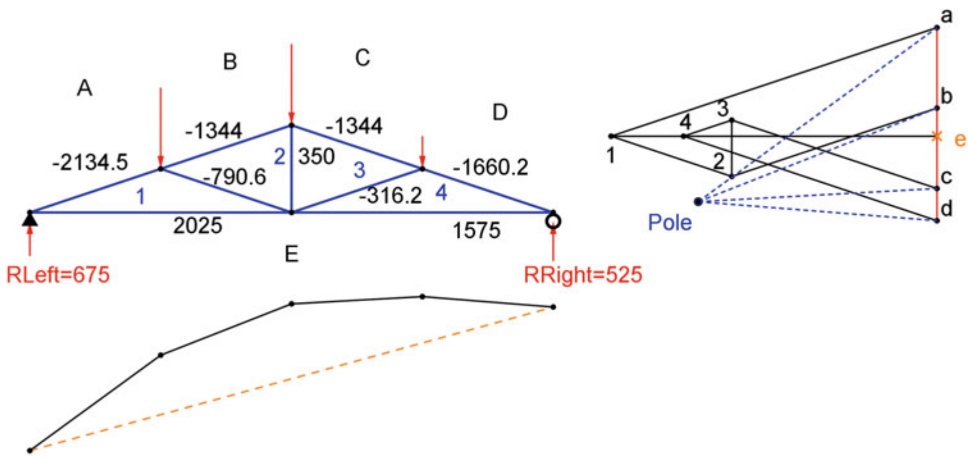
Exercise 5.8 solution



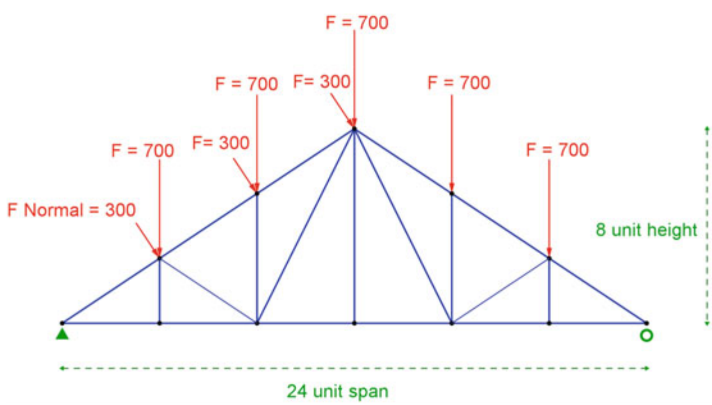
Exercise 5.9 For the following simply supported truss, calculate the reactions then each bar force, including the sign (tension or compression).



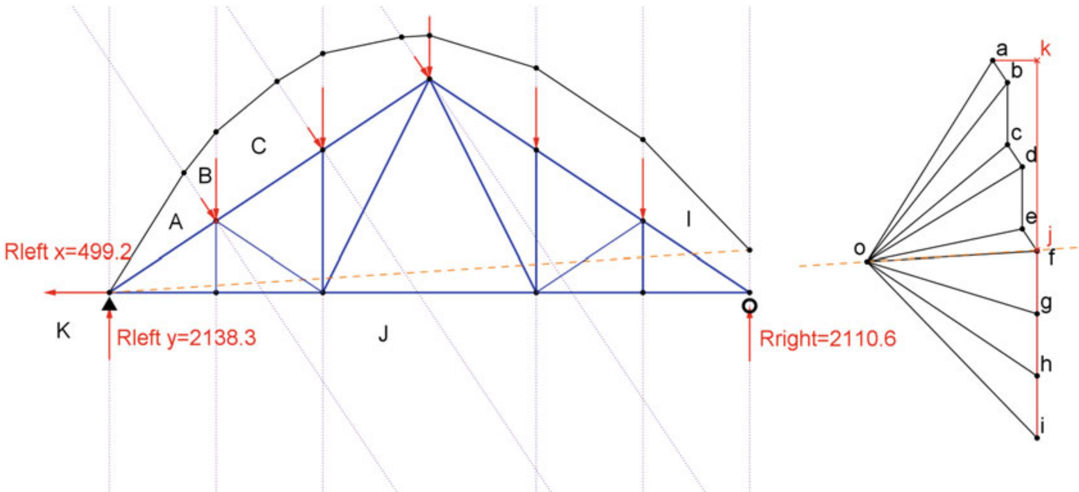
Exercise 5.9 solution



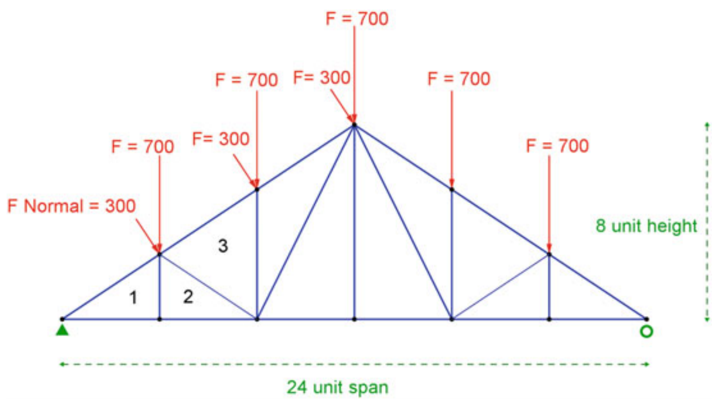
Exercise 5.10 Wind loads are normal to the windward surface. Gravity loads are applied to each top chord node. Calculate the reactions of this simply supported truss.



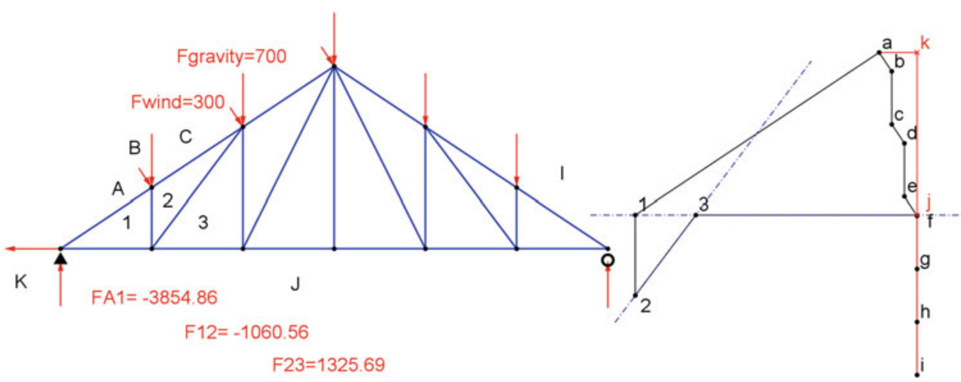
Exercise 5.10 solution



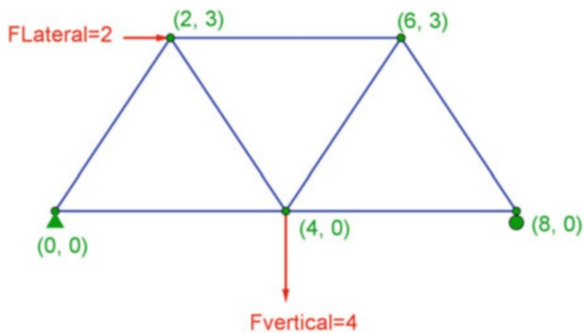
Exercise 5.11 For the truss of Problem 5.10, identify the location of Points 1, 2 and 3 in the force diagram.



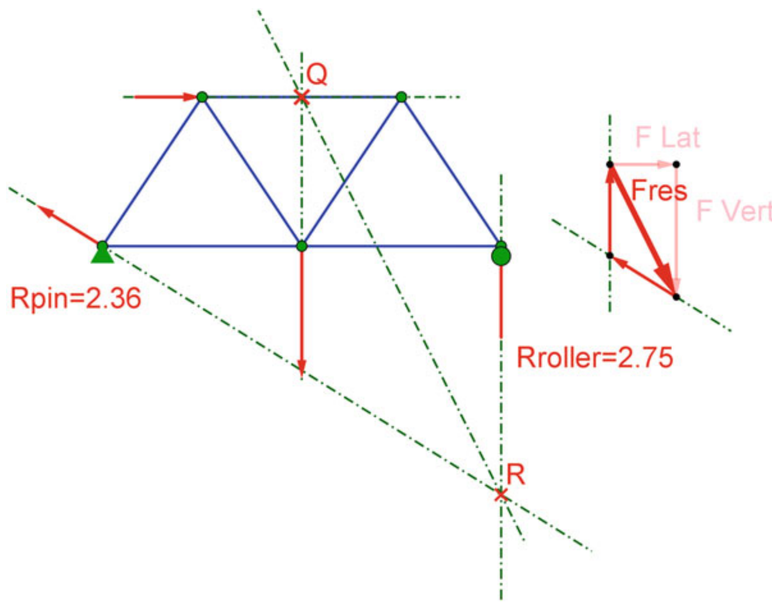
Exercise 5.11 solution



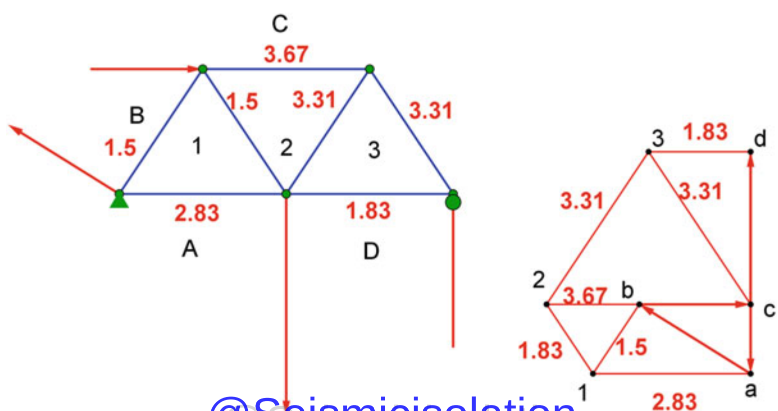
Exercise 5.12 Calculate the equilbrating reactions of this simply supported truss.



Exercise 5.12 solution

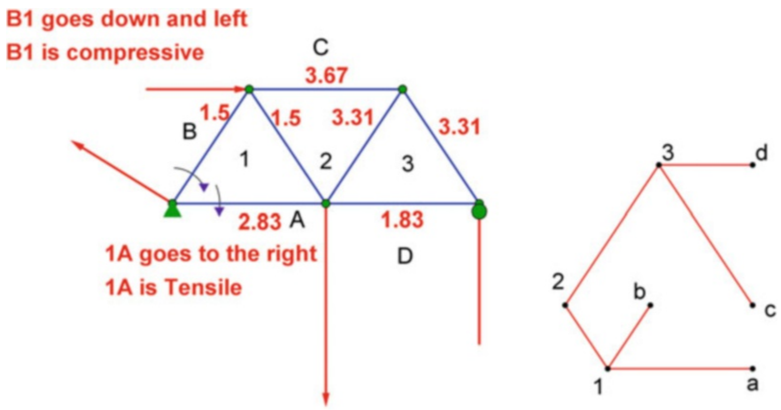


Exercise 5.13 For the truss of Problem 5.12, calculate the magnitude of each bar force, but ignore the sign (positive or negative).
Exercise 5.13 solution

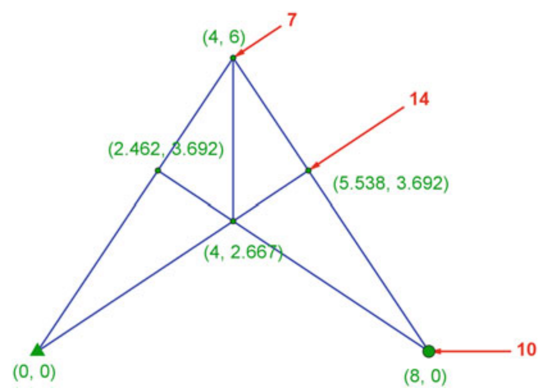


Exercise 5.14 For the truss shown in Problem 5.13, determine the sign (positive or negative) of several bar forces.

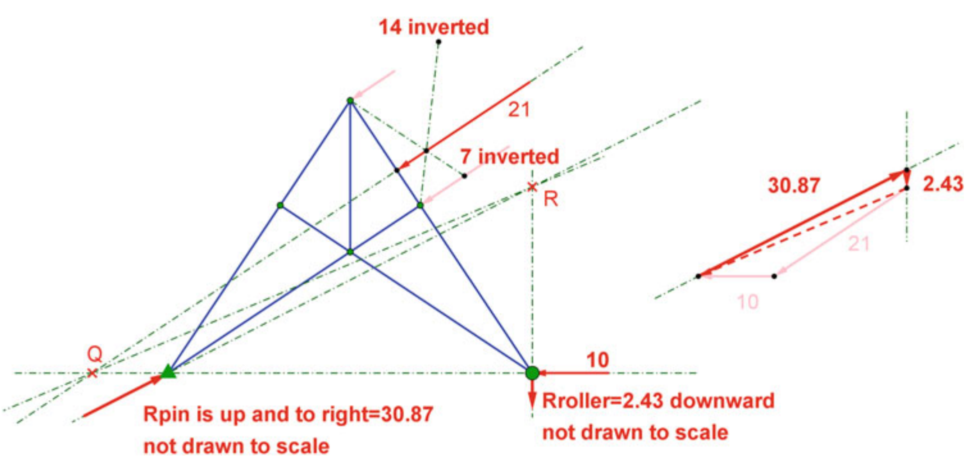
Exercise 5.14 solution



Exercise 5.15 For the truss shown, calculate the reactions. The 7 unit load and the 14 unit load are perpendicular to the side of the truss.

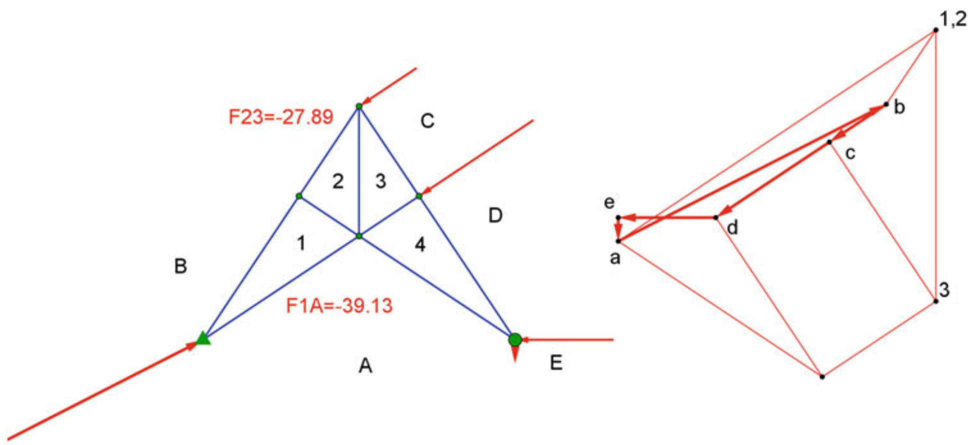


Exercise 5.15 solution

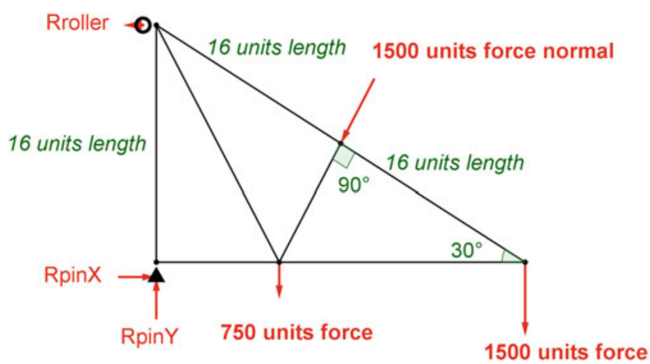


Exercise 5.16 For the truss of Problem 5.15, calculate the largest bar force as well as the force in the vertical member.

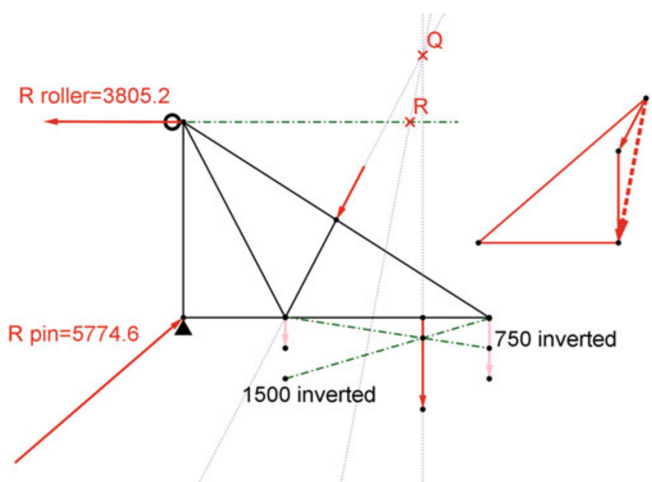
Exercise 5.16 solution



Exercise 5.17 For the truss shown, calculate the reactions.

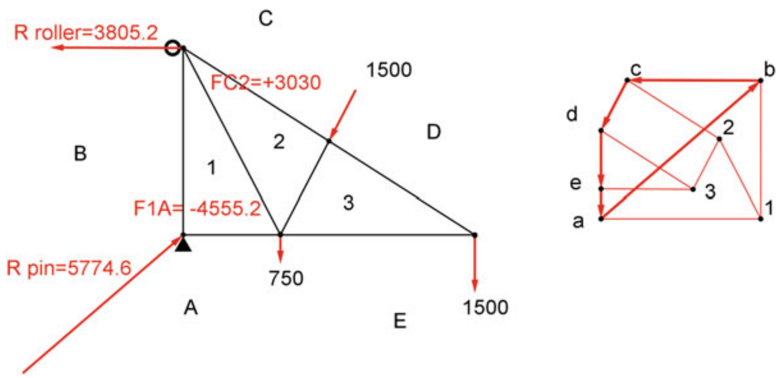


Exercise 5.17 solution

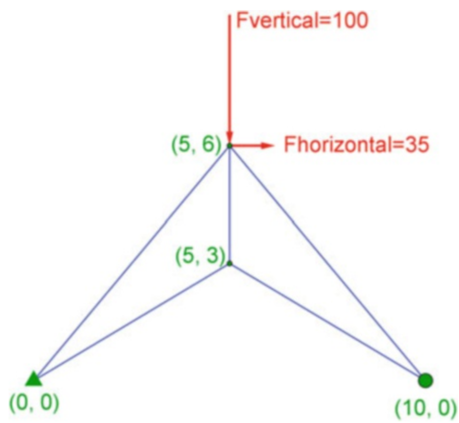


Exercise 5.18 For the truss shown in 5.17, calculate the most extreme bar force. Note that the original loads must be used, not the resultant.

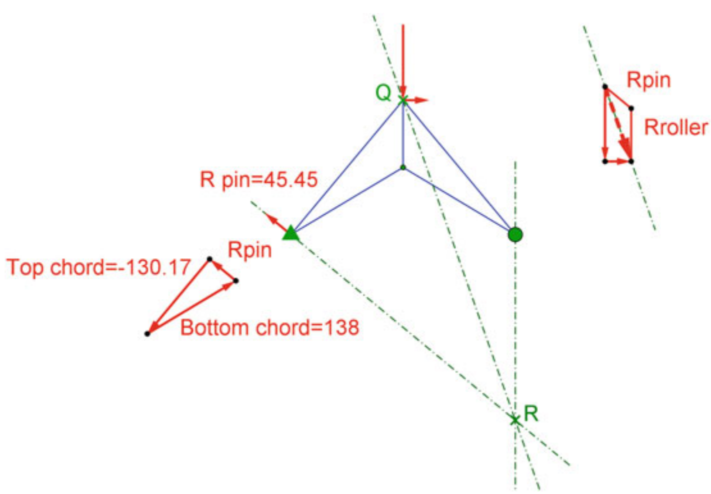
Exercise 5.18 solution



Exercise 5.19 For the truss shown, use the method of sections to find the two left bar forces.

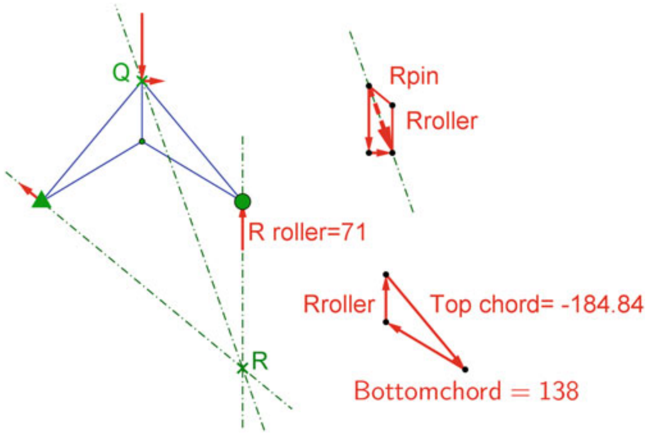


Exercise 5.19 solution

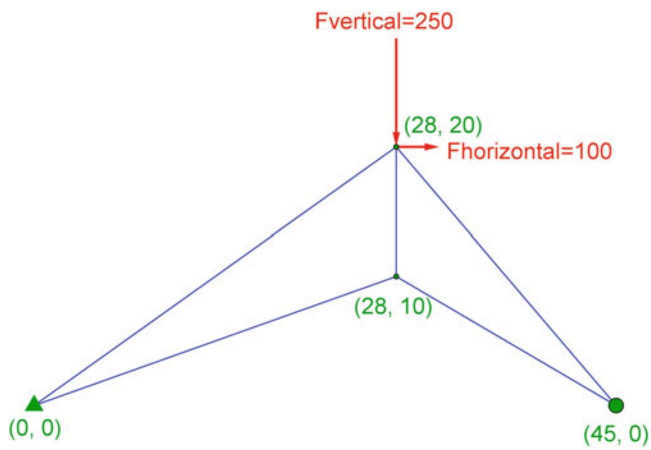


Exercise 5.20 For the truss shown in Problem 5.19, calculate the two right side bar forces using method of sections.

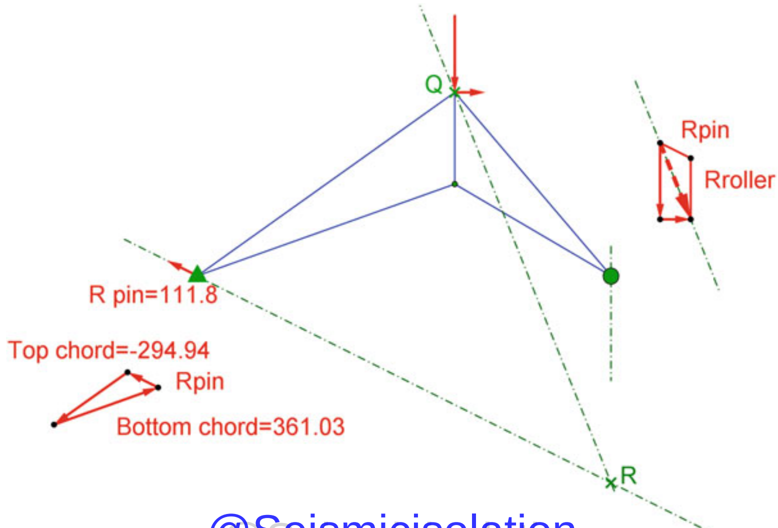
Exercise 5.20 solution



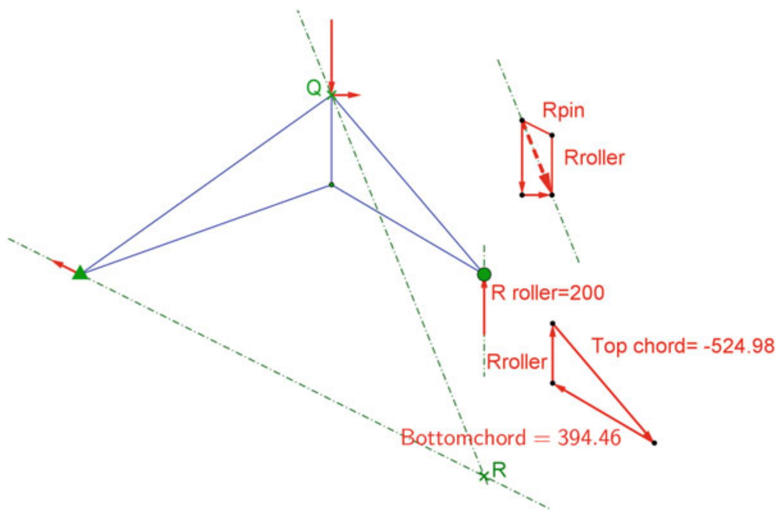
Exercise 5.21 For the truss shown, use the method of sections to find the two left bar forces.



Exercise 5.21 solution



Exercise 5.22 For the truss shown in 5.21, use the method of sections to find the two right bar forces.
Exercise 5.22 solution



6 Frames

Frames are assemblages of members that are similar to a truss, but differ insofar as individual frame members can carry axial load as well as bending load. Members in a truss carry only axial loads because the academic classification of "truss" typically requires pinned-pinned elements that are loaded only at the joints. It is important to recognize that some frame members are forced to be axial only if such a member is pinned at each end AND it carries no transverse loads. In the following example note that the horizontal member and vertical members are continuous and subject to bending, whereas the diagonal is pin-ended without transverse loads, thus there is no bending in it (Fig. 6.1).

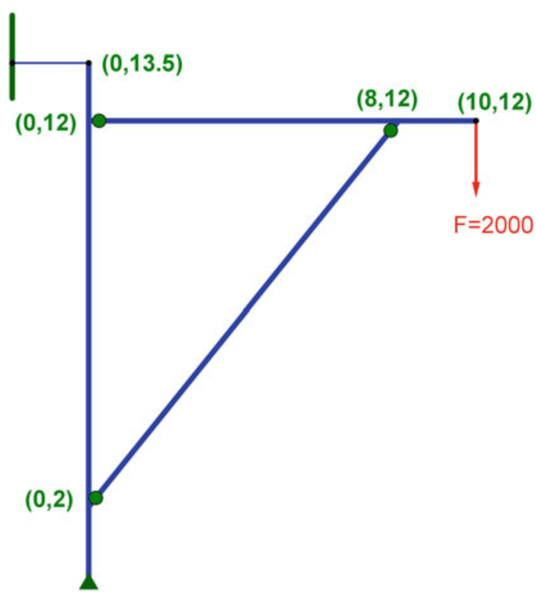


Fig. 6.1 Frame with one member acting solely axially

E. Saliklis, *Structures: A Geometric Approach*, https://doi.org/10.1007/978-3-319-98746-0_6

Figure 6.2 shows a line through *F* along its line of action (vertical) and a line through *D* along its line of action (horizontal) locate point *K*. Nothing is yet known about the magnitude or the direction of the pinned reaction at *A*, but the vector must pass through *A* itself. Thus, using the slope of *AK* and the slope of *DK* and the magnitude and direction of the external applied load allows for a rapid solution of the reactions. Essentially this looks at the frame problem as equilibrium of a point (Point *K*).

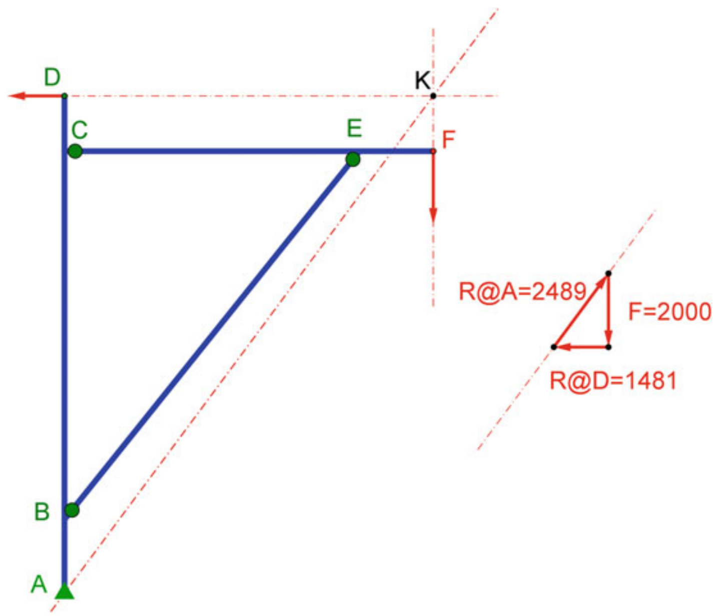


Fig. 6.2 Force diagram establishes external support reactions

CEF is one continuous bent member, but there is a pin at *C*. Pins carry two orthogonal forces, these can be imagined as shear (transverse) and axial (normal) forces which are perpendicular to each other. The perpendicular pin forces can be vectorially combined to create one net pin force. The strategy in any such frame that has bending members is to create free bodies of each individual member, expose the two pin forces on one member and place two opposing pin forces, equal in magnitude but opposite in direction on the adjoining member. Either algebraically or graphically, it is best to begin on a member that has no more than three unknown forces acting on it as a free body. Member *CEF* is a logical starting point in this example. The key idea when creating a free body of member *CEF* is to recognize that the diagonal member *BE* is pin-ended and it is not subjected to any transverse applied loads. Thus, the force at pin *B* and the force at pin *E* must have a resultant that aligns with the axis of member *BE*. This means that the direction of the force acting on *CEF* from the pin at *E* is known. The magnitude of this load can then be quickly found from the construction shown in Fig. 6.3. Equilibrium of member *CEF* has been reduced to equilibrium of a point, namely point *L*.

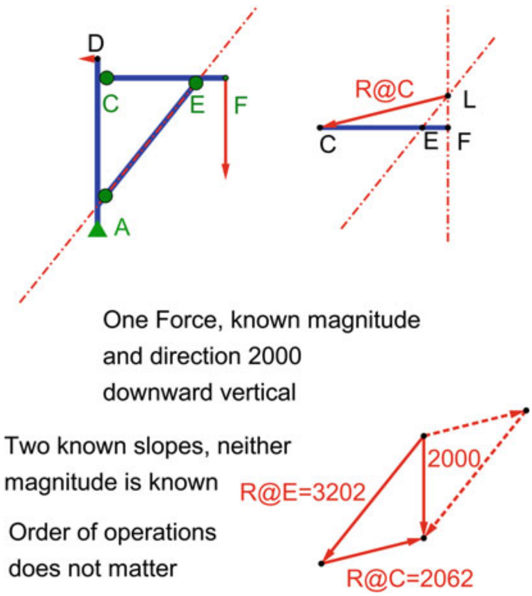


Fig. 6.3 Equilibrium of member *CEF*

The solved for net forces at C and E can be applied to the vertical member AD as a check of completed work, since the reactions at D and at A have been already solved for (Fig. 6.4).

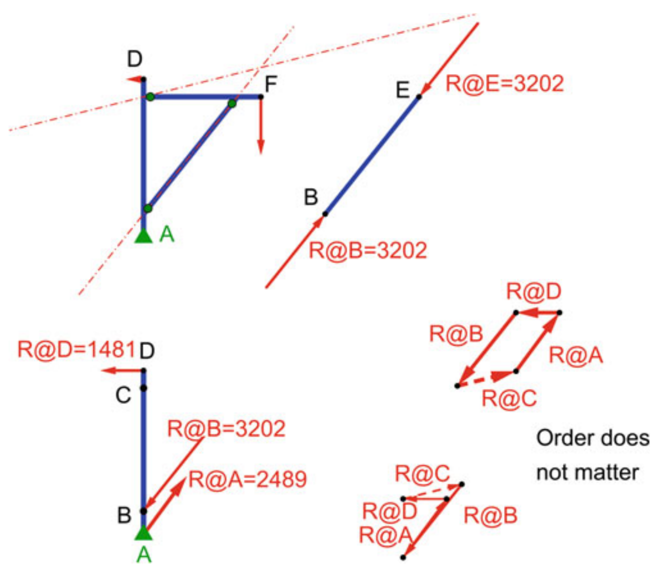


Fig. 6.4 Checks of equilibrium of individual member

Figure 6.5 shows a new example that has the same overall geometry and external loads as does Fig. 6.1, but this time a motorized pulley is attached to the boom (diagonal member), to hoist the load.

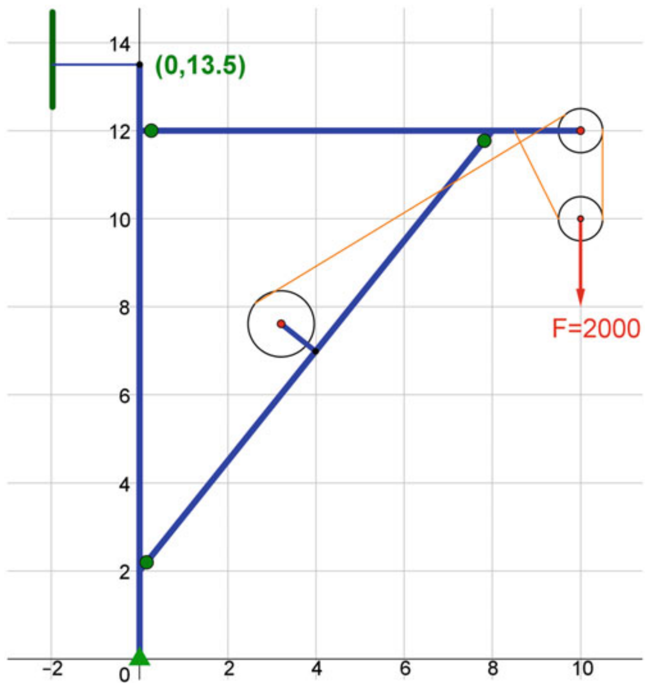


Fig. 6.5 Previous frame with additional pulley

Figure 6.6 proves that the three external reactions are the same as before.

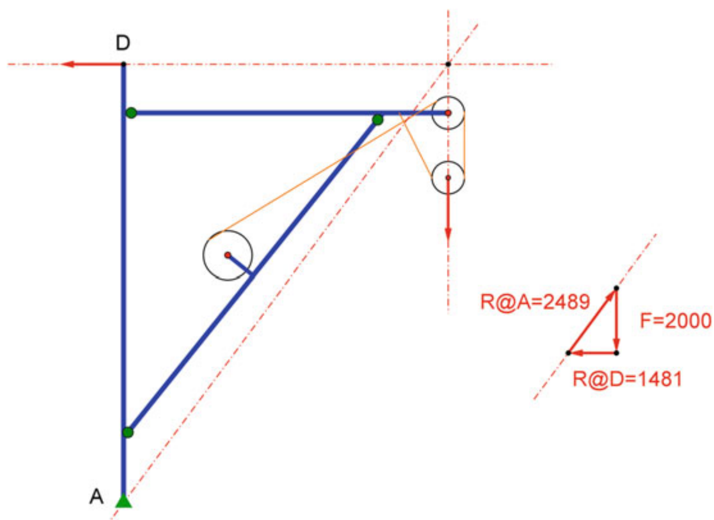


Fig. 6.6 External equilibrium of previous frame

Yet the diagonal boom is now subjected to bending because of the presence of the tensile cable force, which passes through the pulley, through the pin of the pulley and through the strut connecting the pulley to the boom. The horizontal member *CEF* is subjected to more complicated bending than it previously experienced in Fig. 6.2 due to the added effect of the cable. Thus, all three frame members are now bending members, and they could each be analyzed as beam elements using the techniques described in Chap. 5.

Figure 6.7 shows a different frame. It is cantilevering horizontally off of some structure and it supports one downward load. The connections at A and B are pinned to the supporting structure.

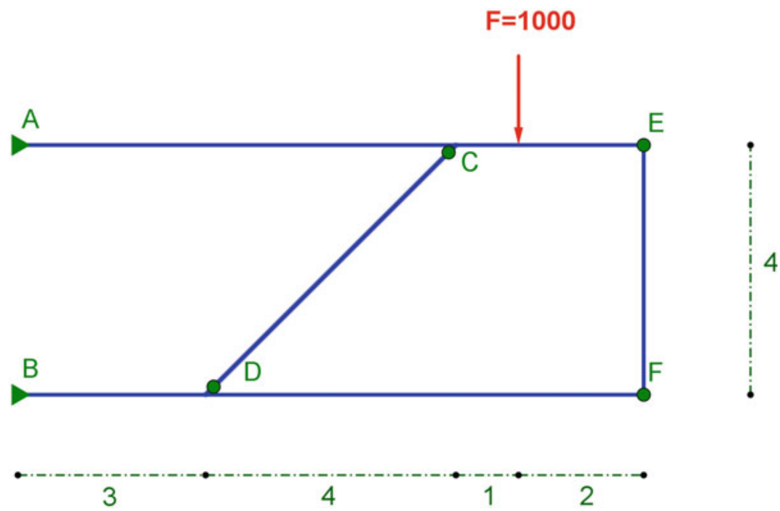


Fig. 6.7 Frame with two axial only elements

A key first step is to note that member CD and member EF are both pinned-pinned and neither is subject to any transverse loads, thus each is an axially loaded member. The axial force in each must be aligned with the member axis and there is no bending present. Of course, this assumes that self-weight of the structure can be neglected. This allows for an analysis of either member ACE or member BDF which are both continuous and subject to bending.

Isolating the free body diagram of BDF allows for known trajectories of the pinned-pinned forces from CD and EF. The trajectory of the net force at pin B is unknown, but it must pass through point B. This construction is shown in Fig. 6.8.

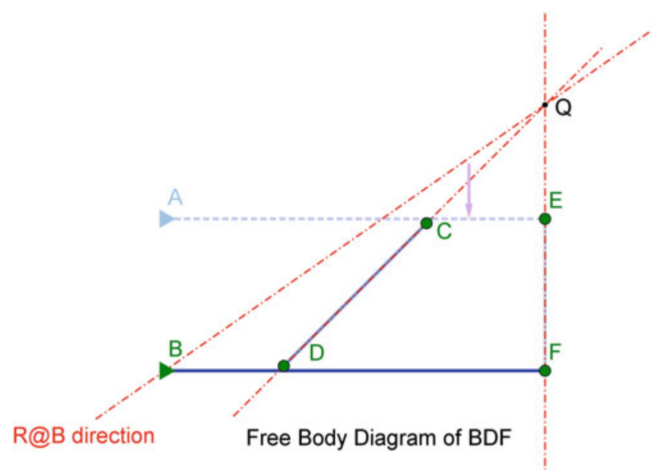


Fig. 6.8 Free body of member BDF calculates direction, not magnitude, of net reaction at B

The direction of the net reaction at support *B* is now known, but its magnitude is unknown. To find its magnitude, consider that the overall structure subjected to the downward external force, the net force at *B* whose trajectory is now known, and the net force at *A*. These three forces all meet at Point *S* in Fig. 6.9. A critical piece of information is the given magnitude of the external load, it will create a scale for the magnitude of the net reaction forces. The construction which identifies the magnitude of the reactions is shown in Fig. 6.9.

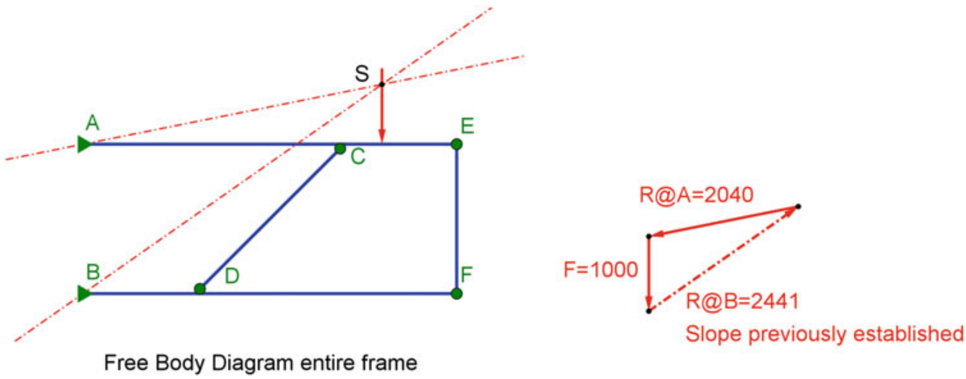
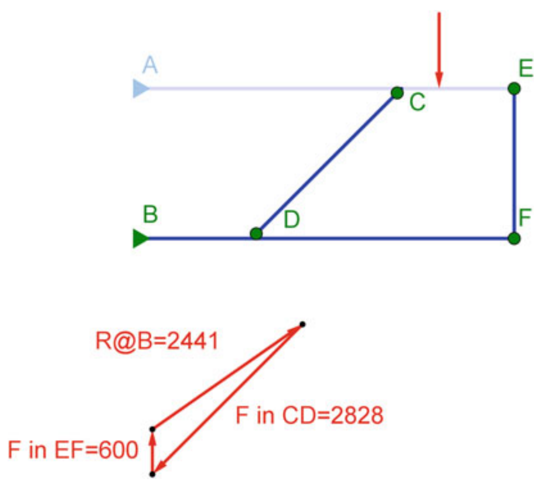


Fig. 6.9 Overall problem reduces to equilibrium of point *S*

The problem has now essentially been unlocked. The magnitude of the forces in the axial members *CD* and *EF* can be solved for by using the information that was just found, applied to a free body diagram of member *BDF* or to member *ACE*. The final force calculations are shown on a free body diagram of member *BDF* in Fig. 6.10.

Fig. 6.10 Back calculating individual member forces



The following frame shown in Fig. 6.11 has a pulley of known dimensions, which is hoisting a known load. The horizontal member and the main diagonal member are continuous. The interior diagonal member is pinned-pinned without transverse loads, thus it is an axial only member.

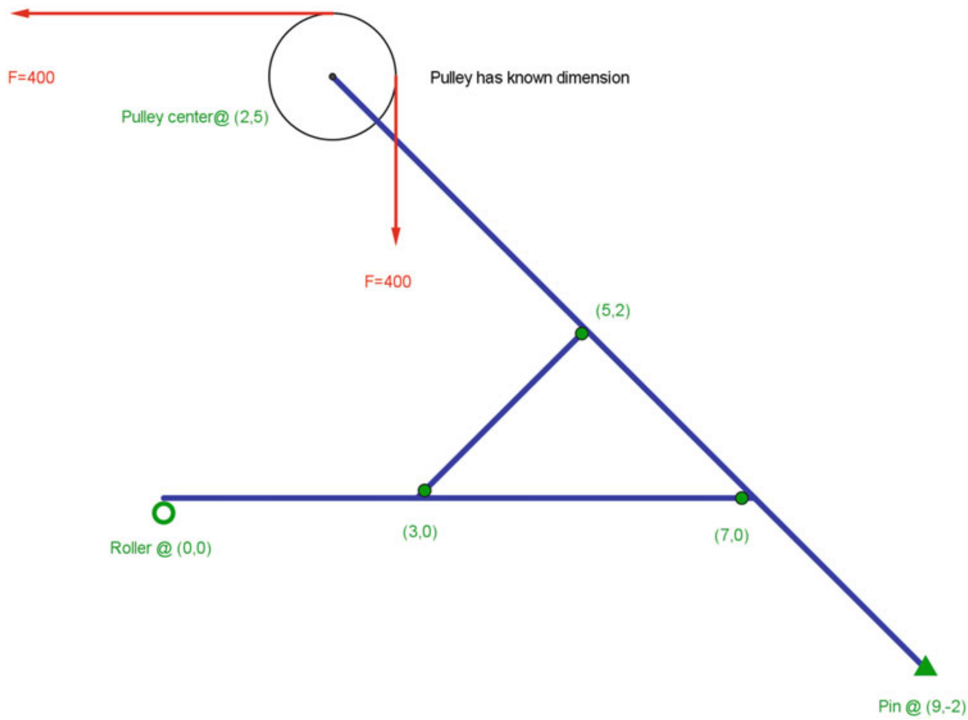


Fig. 6.11 Frame with one axial only member

The direction of the reaction at the left roller support must be vertical. The direction of the reaction at the right pinned support is unknown, but this net reaction must pass through the pinned support. Yet the net direction and magnitude of the load from the pulley is known as the vectorial sum of the horizontal cable force and the vertical cable force. Overall equilibrium of the entire structure can be calculated as three forces passing through point Q (Fig. 6.12).

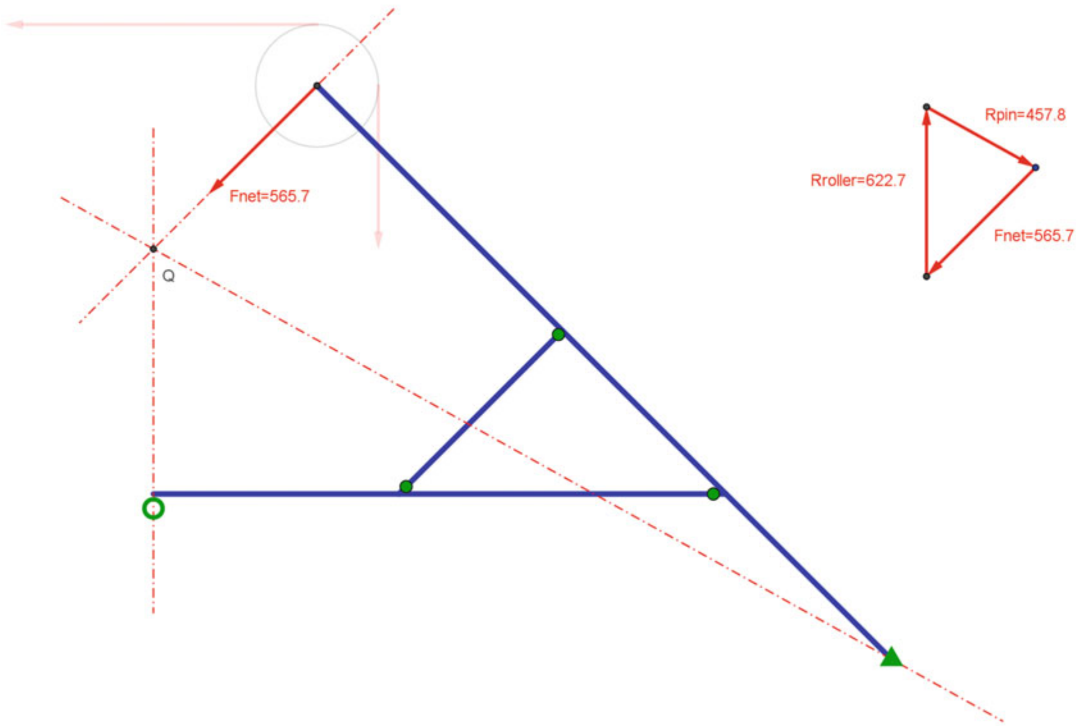


Fig. 6.12 Overall frame equilibrium reduces to equilibrium of point Q

Now that the external support reactions are fully known, an analysis of the individual frame elements can begin. It is worth noting that the force diagrams for the individual members look nothing like the individual members themselves, it is the forces acting on the members that are critical. In Fig. 6.13, a free body diagram of the horizontal member is shown. An analysis of this member defines the location of Point R, which is the intersection of the path of two known forces acting on this horizontal member. The establishment of Point R allows for the determination of the slope of the net internal pin force acting at the pinned right end of the horizontal member. Recall that the internal pinned connection has two orthogonal forces, which accounts for the slope of its net force.

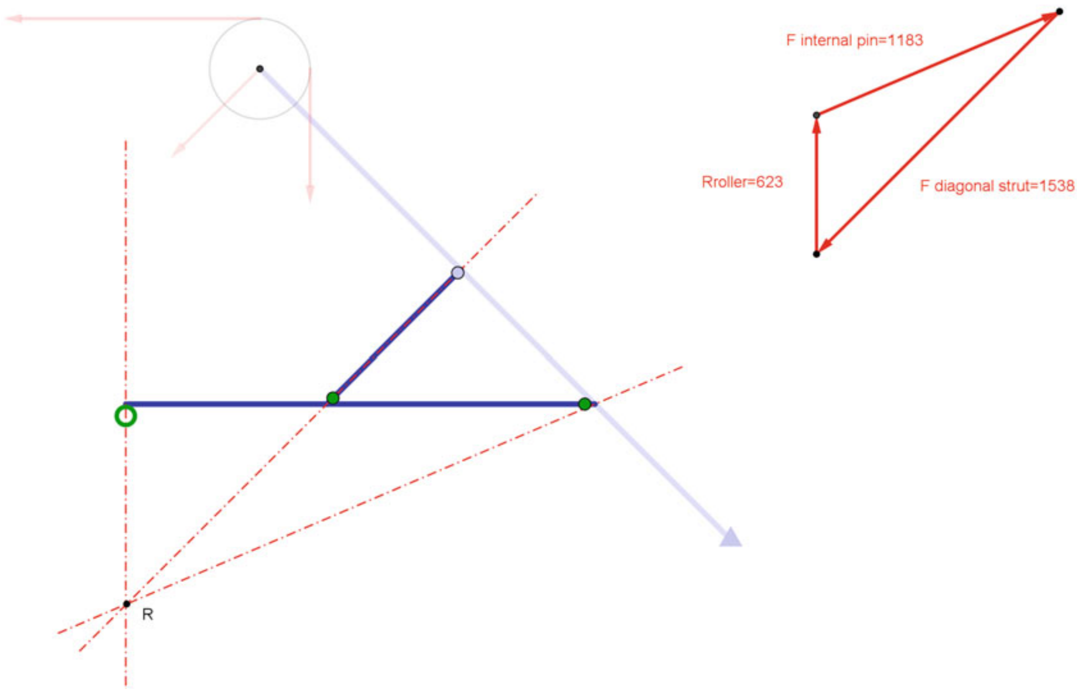


Fig. 6.13 Free body of horizontal member

A check of the work can be done at this point by an analysis of the main diagonal member which spans from the pulley to the right pinned support. All forces acting on this member are now fully known, and it is comforting to see that the force diagram closes in on itself, ensuring equilibrium of this sub-structure. This is shown in Fig. 6.14. Notice how the vectors in the force diagram act on the diagonal piece. The internal diagonal strut and the internal pin force are acting in opposite senses than they were when a free body was taken of the horizontal piece.

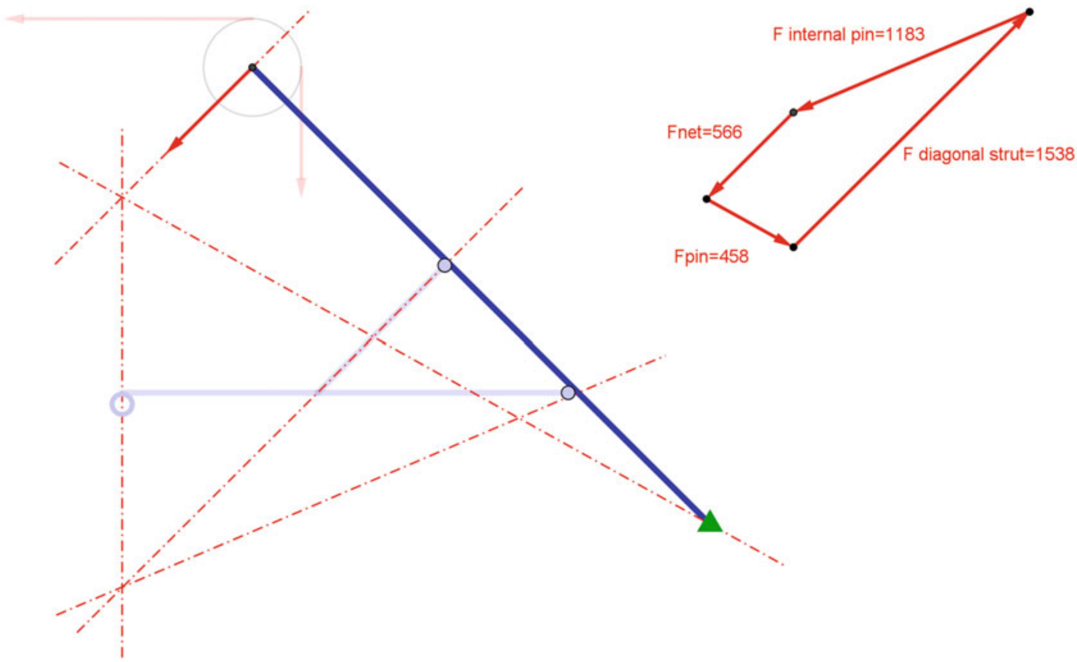
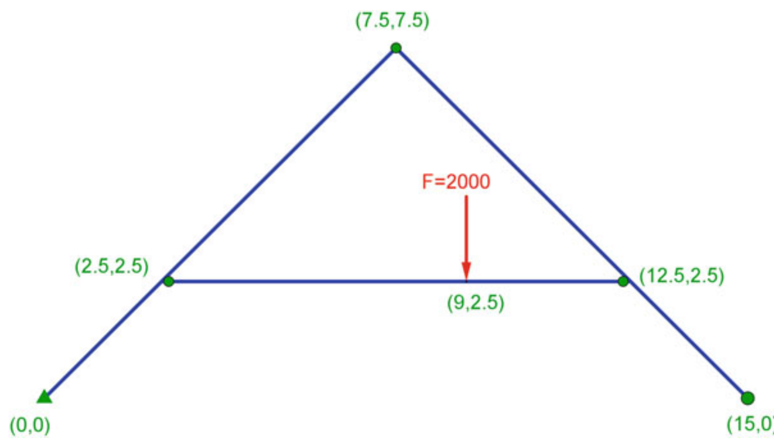


Fig. 6.14 Checking results by analyzing substructure

Figure 6.15 shows a frame made up of three individual members. The diagonal members are sloped at 45°. The horizontal member is pinned-pinned, yet because it carries a transverse load of 2000 units of force, this member experiences bending.

Fig. 6.15 Frame where each member bends



The overall support reactions can be readily found via a force diagram and a trial pole. This is shown in Fig. 6.16. By inspection the horizontal reaction at the left is zero.

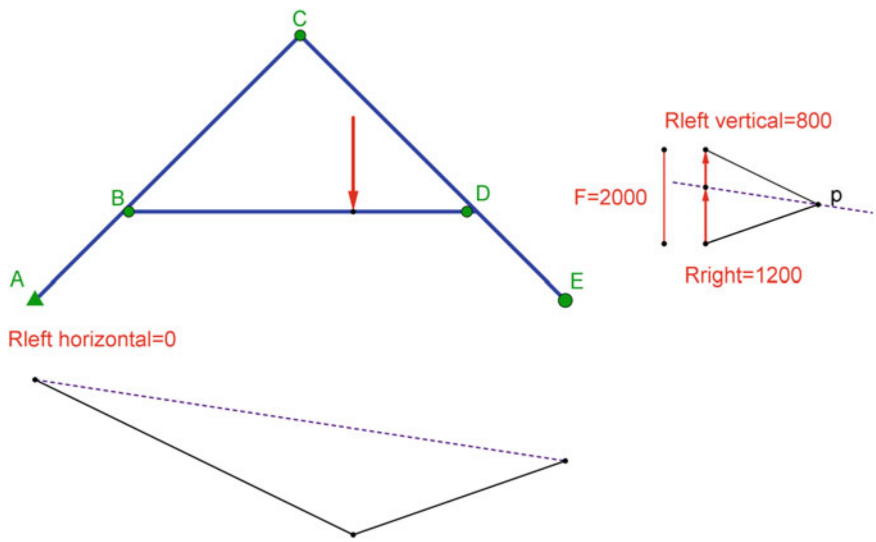
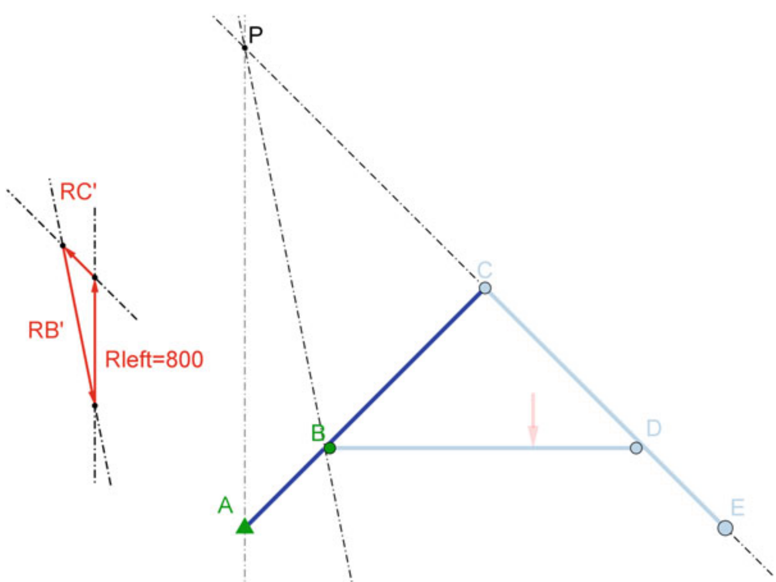


Fig. 6.16 Vertical, not horizontal reactions found by any trial funicular

Isolating any of the three members as free bodies still shows four unknown forces. For example, a free body diagram of the left diagonal piece *ABC* has one known vertical reaction, and two orthogonal (mutually perpendicular) unknown forces at each of the two pins.

Making the incorrect assumption that the right diagonal member acts through the crown pin C along an axis parallel to the right member *CDE* provides a temporary answer for the net pin force at *B*. The starting calculation is to find where the assumed force at *C* coincides with the known reaction at *A*. This is Point *P* in Fig. 6.17. The pin force at *B* is unknown, but the force must pass through Point *B* whatever its magnitude and direction might be. Thus, a line from *P* through *B* establishes the slope of the temporary force at *B*. These incorrect temporary answers will be denoted as RC' and RB' as shown in Fig. 6.17.

Fig. 6.17 Partial solution, using incorrect assumption of path of reaction at *E*



A similar temporary solution is posed for the second diagonal. Essentially this sets up the solution of simultaneous equations which would be necessary if this problem were to be solved algebraically (Fig. 6.18)

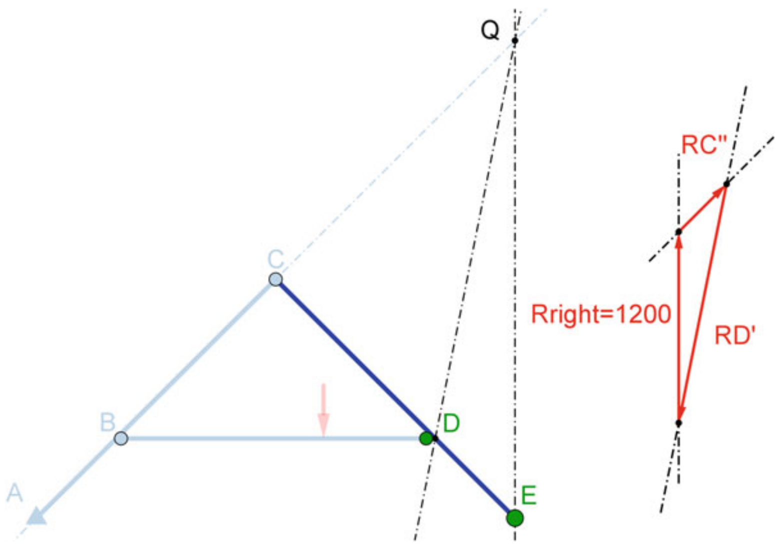


Fig. 6.18 Second partial solution, using incorrect assumption of path of reaction at *A*

The final pin force at B is the superposition of:

- the fictitious pin force at B taken from the left free body (*RB′*)
- plus the left reaction (*Rleft*)
- plus the effect of the pin force at *C* taken from the left free body (*RC′*)
- plus the effect of the pin force at *C* taken from the right free body but applied to the left equal in magnitude yet opposite in direction (<*RC″*>).

A similar operation finds the pin force at *D*.

$$R_B = R_B' + R_{left} + R_C' + \langle R_C'' \rangle$$

$$R_D = R_D' + R_{right} + R_C'' + \left\langle R_C' \right\rangle$$

Notice in Fig. 6.19 that the horizontal member is in equilibrium because the three forces acting on it, *RB*, *RD* and the applied load all converge on a single point.

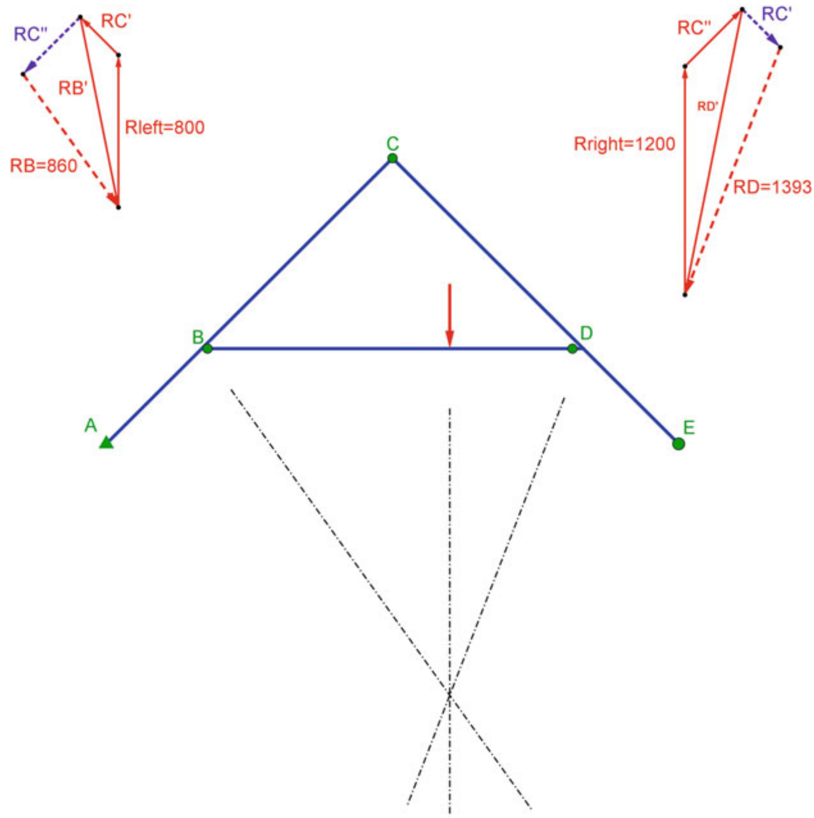


Fig. 6.19 Superposing previous partial solutions to arrive at external reactions

The final pin force at C can now be found. Isolating either diagonal member as a free body allows for the calculation of this pin force at C. Notice in Fig. 6.20 that the pin force at C acts down and to the left on member ABC, and it acts up and to the right on member CDE.

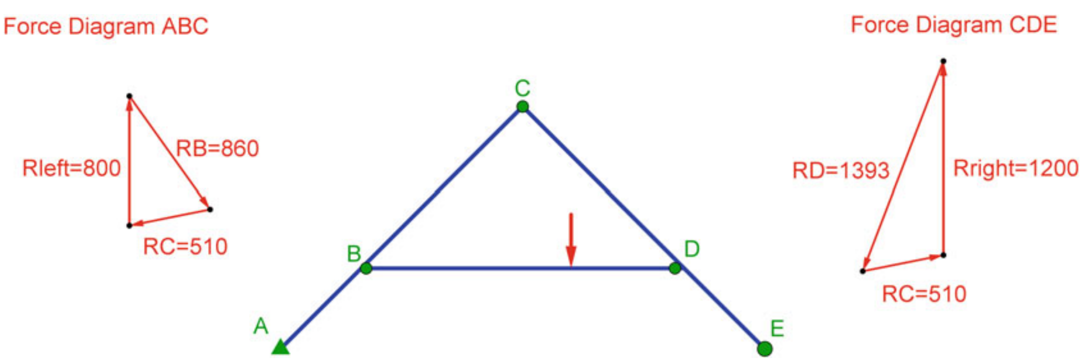


Fig. 6.20 Separate free bodies of diagonal members

The following example uses a similar technique to graphically superpose cases to solve for unknown forces. This superposition replaces algebraic simultaneous equations. This example is a three hinged arch which supports two downward loads as shown in Fig. 6.21.

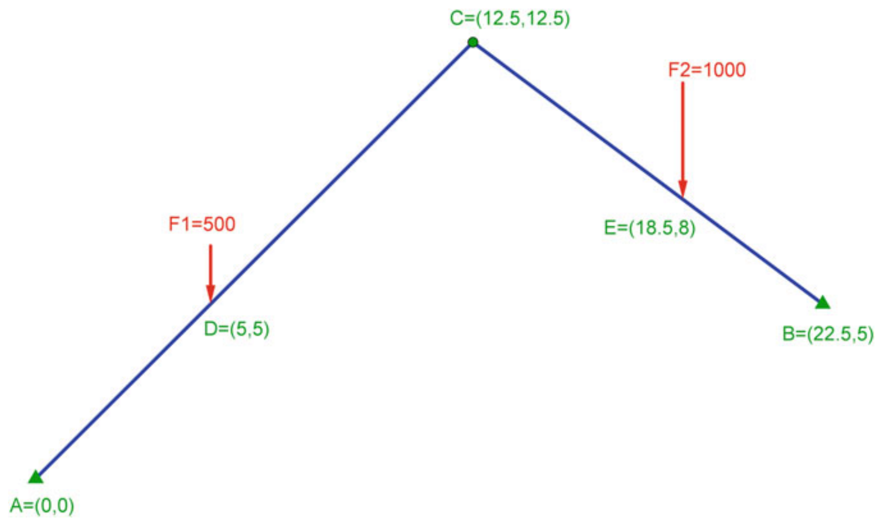


Fig. 6.21 Three hinged arch with two vertical loads, using superposition

Figure 6.22 shows the beginning of the solution. Neither the direction nor the magnitude of either the right or the left reaction are known. A free body diagram of member *ADC* assumes that the reaction at *C* coming from *CEB* is aligned with the axis of member *CEB*. Equilibrium of *ADC* requires that the assumed pin force and the applied force *F1* are coincident, here that point of coincidence is Point *P*. Nothing is known about the force at *A* other than it must pass through Point *A*, thus a slope is drawn from Point *P* to Point *A* and this is the temporary force *RA′*. Similar operations on the free body *CEB* result in a temporary force *RB′*. These steps are also shown in Fig. 6.22.

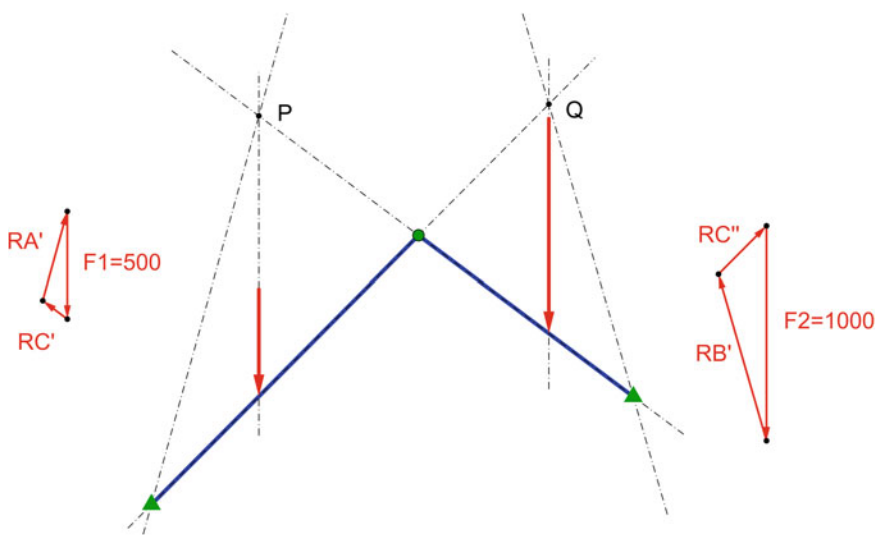


Fig. 6.22 Two incorrect partial solutions, each assuming known direction of support reaction resultants

Figure 6.23 shows the final solution for the reactions at A and at B, as well as two calculations for the pin force at C, derived from the two separate free bodies. Note that the values of RC match and the directions oppose each other as they should on the two free bodies. Another noteworthy item in Fig. 6.23 is the placement of the temporary reaction forces RA' and RB'. Note how they nestle into the larger force polygon, as they are truly an intermediate step.

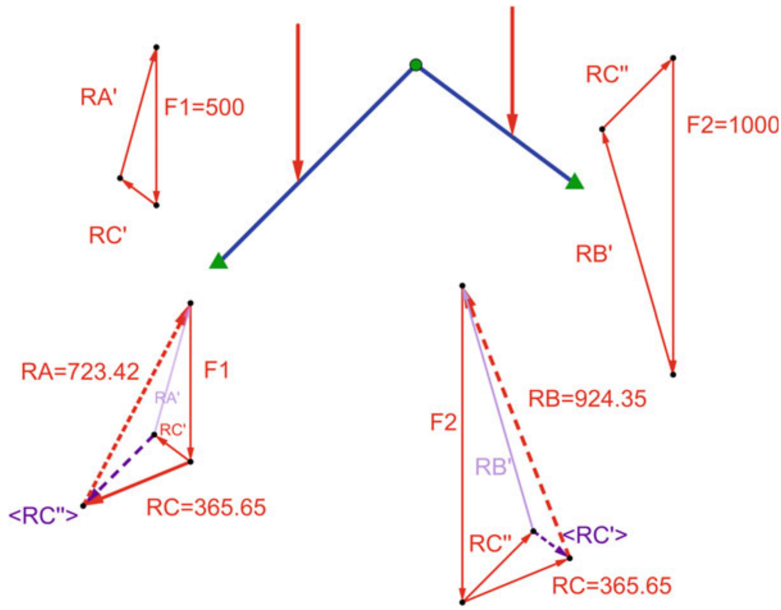


Fig. 6.23 Final reactions found from superposition of previous partial solutions

An elementary frame is shown in Fig. 6.24. Such a structure experiences bending of individual elements, unlike a truss.

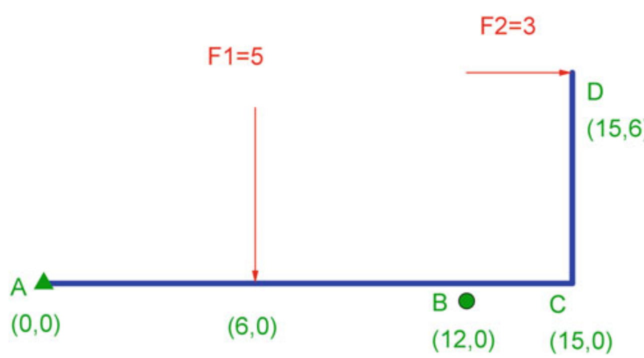


Fig. 6.24 Frame with a vertical and with a horizontal load

The overall reactions are quickly found by using a single equivalent applied load (Fig. 6.25).

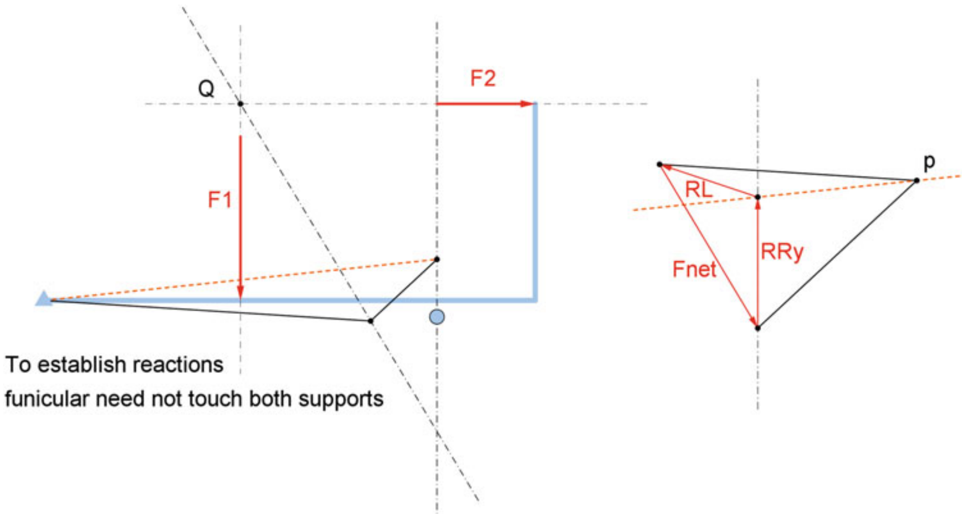


Fig. 6.25 External reactions found by any trial funicular based on resultant of applied loads

To find the bending moment at any point, the equivalent applied load can no longer be used. The actual loads must be applied to investigate final internal bending moments. A trial funicular is shown in Fig. 6.26.

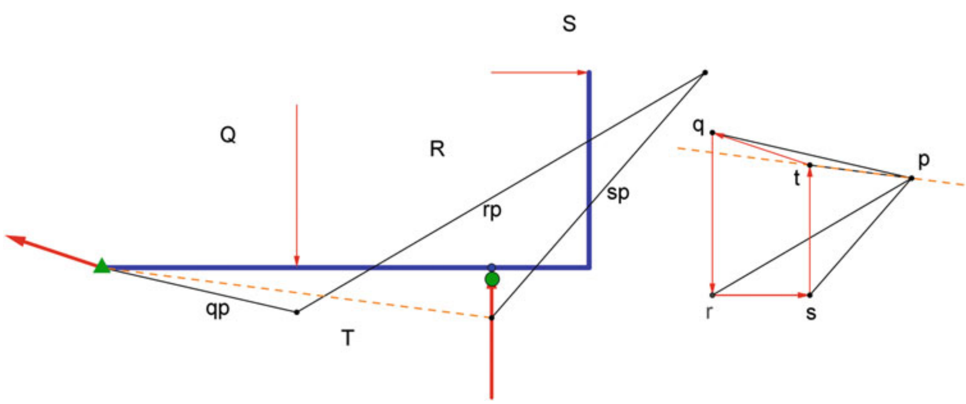


Fig. 6.26 Force diagram incorporates actual reactions, but trial funicular still

This trial funicular cannot be used to establish the final bending moments because the funicular does not pass through the support boundary conditions. This example is different than the previous examples in Chap. 4 because of the lateral load. When only vertical loads are applied, the funicular need not satisfy both boundary conditions. But when lateral loads are present, the trial funicular must be adjusted until it passes through both supports.

First, to calculate the bending moments between the two supports, the funicular must pass through the supports themselves. Figure 6.27 is correct for any moment between the two supports. For example, to calculate the bending moment under the vertical load, use the following formula but recall that the variable *HForce* is not an external load, but rather, is the width on the force diagram from the pole p to the load line.

$$M_{@F1} = d1 \cdot HForce \cdot ForceScale$$

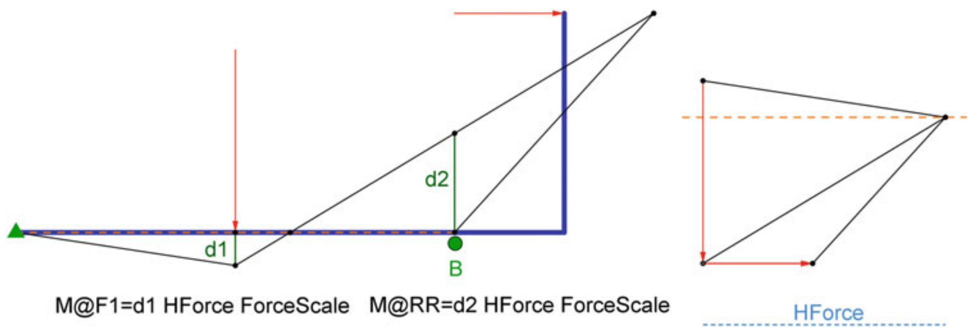


Fig. 6.27 Move pole *p* till funicular is final, i.e. it passes through both supports

However, the moment is constant in the horizontal overhang portion to the right of *B*, and it is clear that the solution shown in Fig. 6.27 will not be applicable for this portion. To investigate any cut to the right of *B*, move the funicular such that the GAP in the funicular is in line with a point of interest on the frame, but use *VForce* and horizontal widths of the funicular to calculate moment because this portion is governed by the horizontal load only. Notice that the variable *VForce* is not an external load, but rather, is the vertical load vector on the force diagram from the pole to the load line. This is shown in Fig. 6.28.

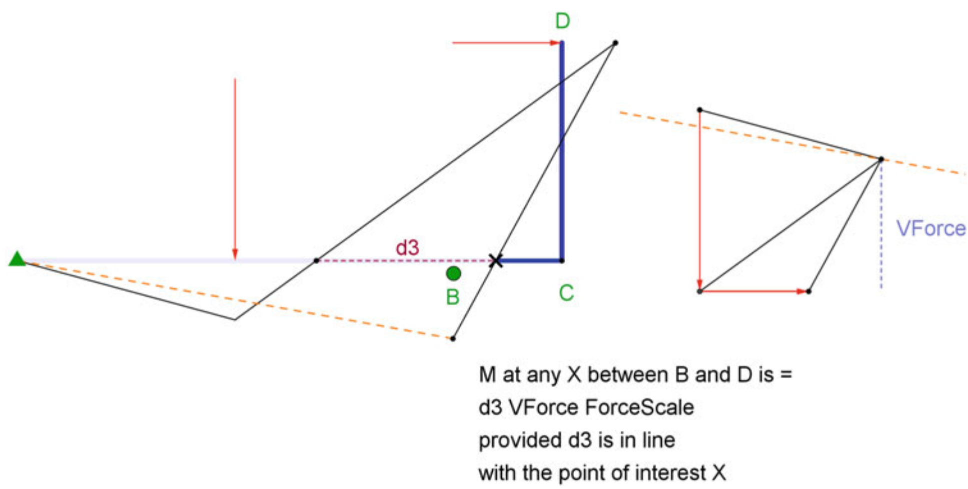


Fig. 6.28 Use variable *VForce* in conjunction with distance *d3*, noting that *d3* is in line with cut

Placing the cut *X* such that it coincides with point *C* allows for the calculation of the moment at *C*. Again, this calculation uses horizontal distances and force *VForce*.

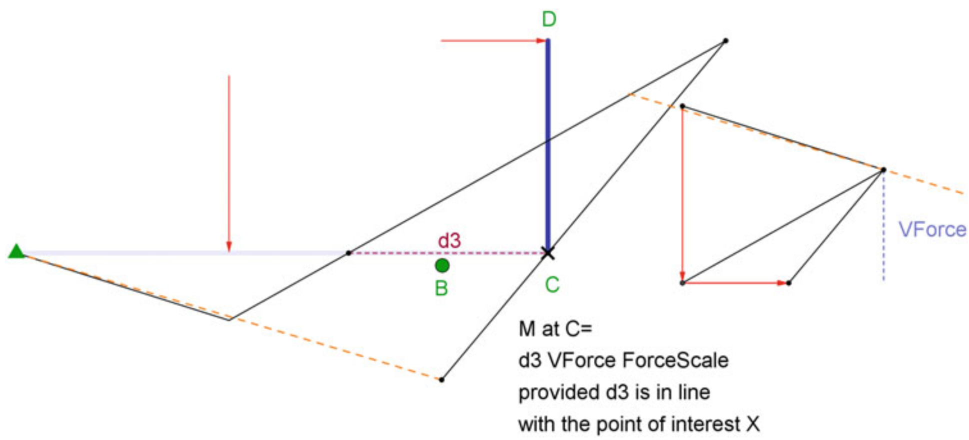


Fig. 6.29 Always move funicular such that *d3* is in line with cut

The same funicular of Fig. 6.29 or of Fig. 6.30 can be used for any cut through the vertical portion *CD*. Figure 6.30 shows how to calculate internal moment anywhere along *CD*. Point *X* along the vertical member remains in line with the distance *d3*.

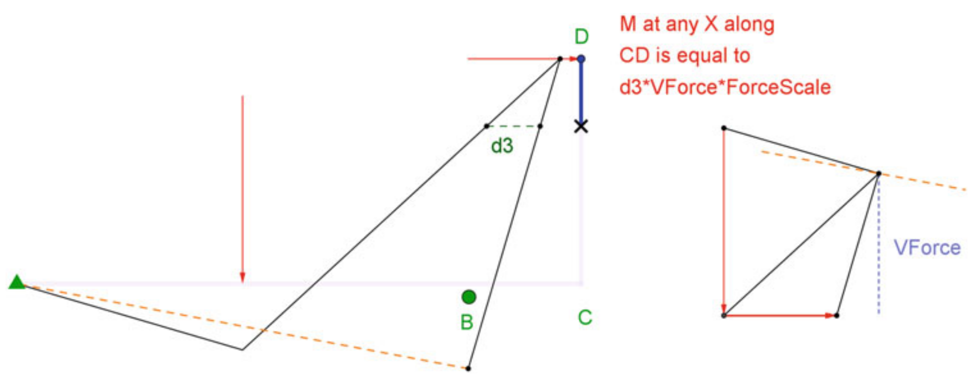
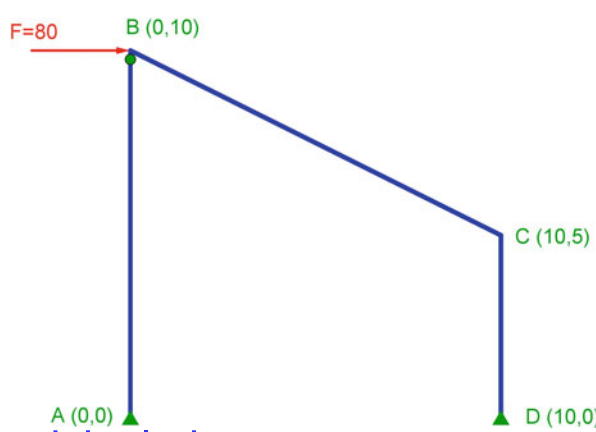


Fig. 6.30 Using *VForce* and *d3* which is in line with cut

The following frame shown in Fig. 6.31 has a pinned-pinned vertical member at its left side which experiences no transverse loads, thus this member has only an axial force, no bending. The other two members experience bending.

Fig. 6.31 Frame with one axial only member



Since it is known that member *AB* experiences no bending, the applied force and the line of action of the force in *AB* coincide at a known point, here Point *B*. Nothing is known of the magnitude or the direction of the force at *D*, other than the fact that the force must pass through *D* itself. Thus, a line from the coincident point *B* and *D* establishes the slope of the reaction at *D* (Fig. 6.32). A force diagram readily solves for the magnitude of the reaction forces as shown in Fig. 6.32.

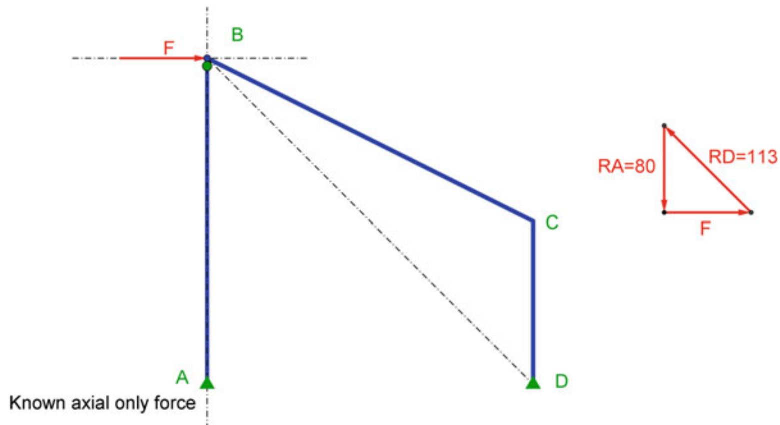


Fig. 6.32 Reactions are found via equilibrium of point *B*

The moment along the bent continuous member *BCD* can be found a number of different ways as described in Fig. 6.33. The moment is found as:

- $M = R_D \cdot perpdistance$
- $M = HForce \cdot vertdistance \cdot ForceScale$
- $M = VForce \cdot horizdistance \cdot ForceScale$ where horizontal distance is either to the inclined member or to the vertical member as shown in Fig. 6.33.

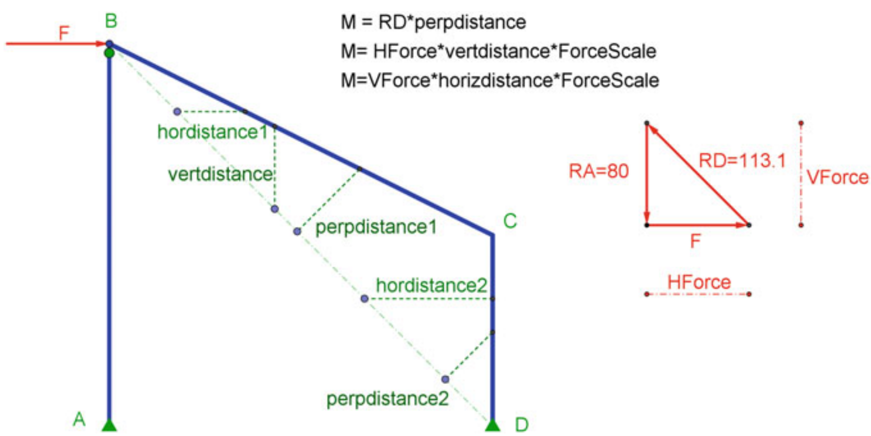


Fig. 6.33 Multiple ways of finding the internal moment anywhere in member *BCD*

Another frame, which is the essentially the same geometry as the previous example but now with an additional load at the midpoint of the sloping member and the vertical member on the left is continuous. The right support is a roller keeping the frame statically determinate as there are no internal hinges (Fig. 6.34).

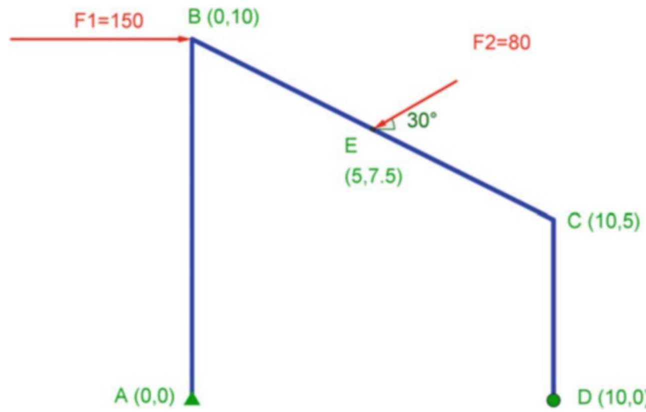


Fig. 6.34 Statically determinate frame with no internal hinges

Figure 6.35 shows a trial funicular based on an arbitrary trial pole p'. Point t is located by the intersection of the trial closing line (green dashed) and a vertical line denoting the path of the right reaction that is identified as the transition between space S and space T. Once point t is located, the external reactions are known. The true pole p must lie on a line parallel to the true closing line (purple dashed) which passes through t.

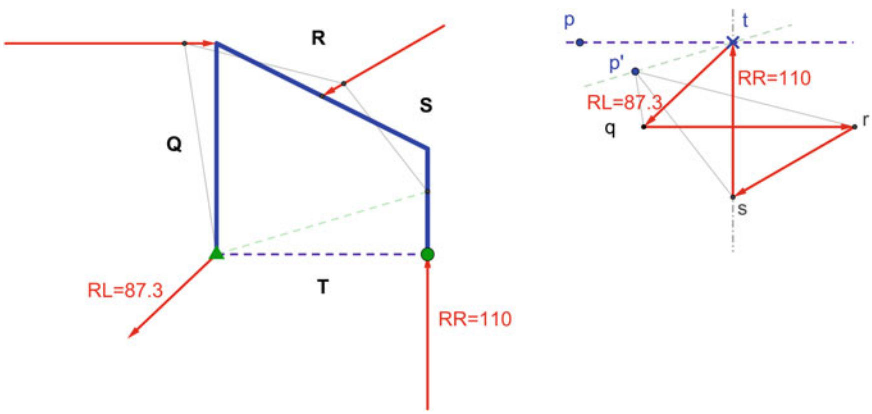


Fig. 6.35 The true pole p must lie somewhere on the dashed purple line

Figure 6.36 shows a funicular for some point p on the true closing line. But recall from Chap. 3 and a figure such as Fig. 3.4 that the START and the END of the funicular captures the slopes of the support reactions. Figure 6.36 shows incorrect slopes, namely that ST is not vertical, whereas it really must be because the reaction at the right roller support is vertical.

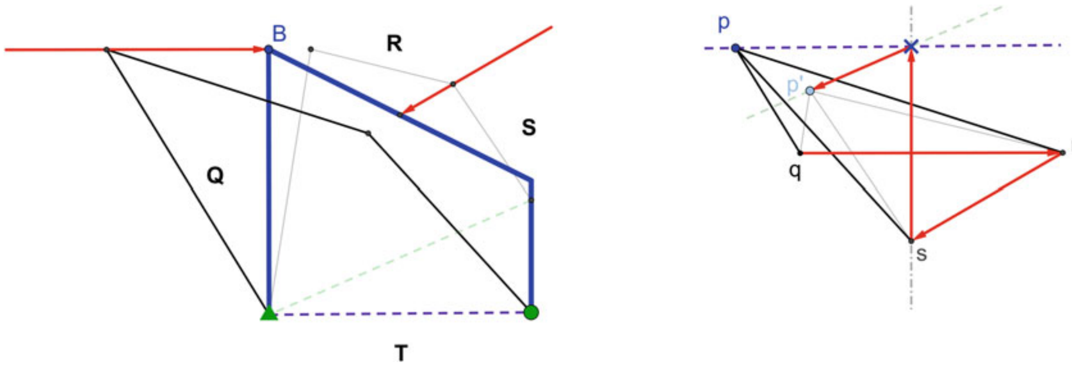


Fig. 6.36 Incorrect funicular since slope at right support is not vertical

Moving the pole till the ending slope coincides with the slope of the known reaction at the right, namely vertical. This correct funicular allows for the determination of bending moments anywhere in the frame. For example, in the vertical member that is in the *Q space* of the form diagram, the moment is

$$M_{Q\ space} = QspaceDistance \cdot QspaceForce \cdot ForceScale$$

The red *X* in Fig. 6.37 shows the cut and the free body diagram is highlighted by the dark blue frame line. An important note is that the pole p must align with Point t in order for the funicular to be vertical at the right roller support.

QspaceForce is defined as the segment on the Force Diagram from the closing line to the force *qr* multiplied by *ForceScale*.

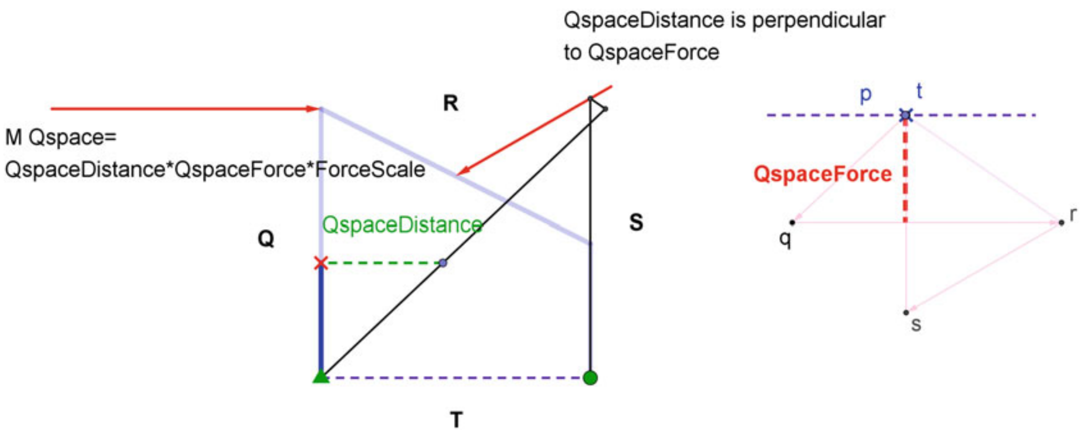


Fig. 6.37 Finding moment in left leg of frame

The moment at the top of the vertical column in *Q* space must match the moment at the left end of the diagonal in *R space*. In *R space*, the moment is found from the exact same force polygon as before, but different quantities are used to find this moment. An analogy can be made with algebraic free body calculations; a cut in *R* space would have two external forces (net reaction plus one applied load), whereas a cut in *Q* space would have only one (the net reaction). *RspaceForce* is defined as the segment from the closing line to force *rs*, multiplied by *ForceScale* and it is measured along a path perpendicular to *RspaceDistance* (Fig. 6.38).

$$M_{R\ space} = RspaceDistance \cdot RspaceForce \cdot ForceScale$$

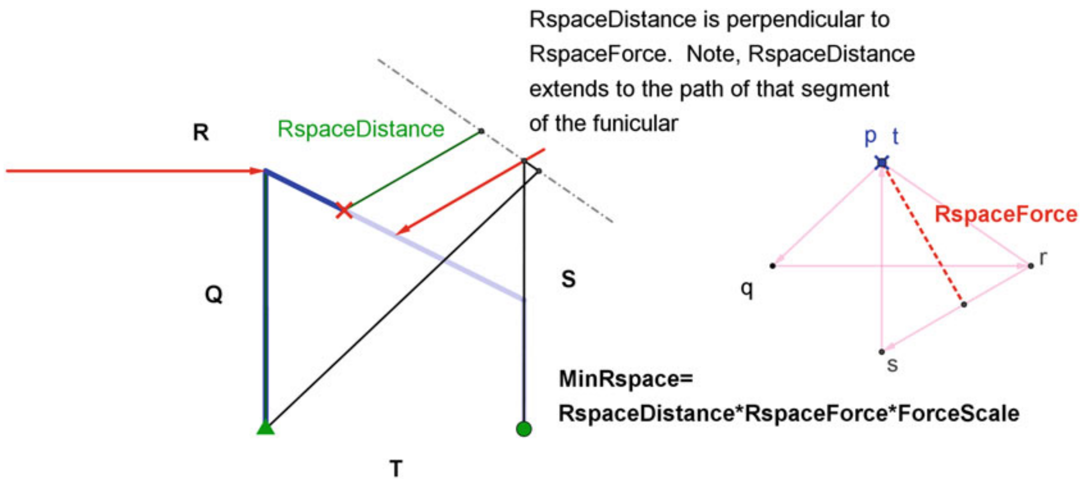


Fig. 6.38 Finding moment in diagonal frame member, left of load on that member

Finally, a third free body has yet another calculation, just as it would in algebraic statics, to account for the new cut in *S* space. This is shown in Figs. 6.39 and 6.40.

$$M_{S\ space} = SspaceDistance \cdot SspaceForce \cdot ForceScale$$

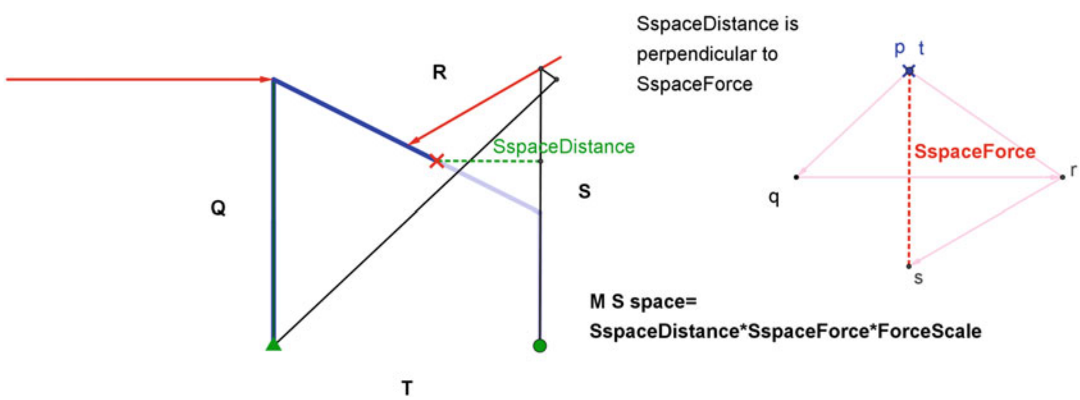


Fig. 6.39 Finding moment in the frame anywhere to the right of load on top

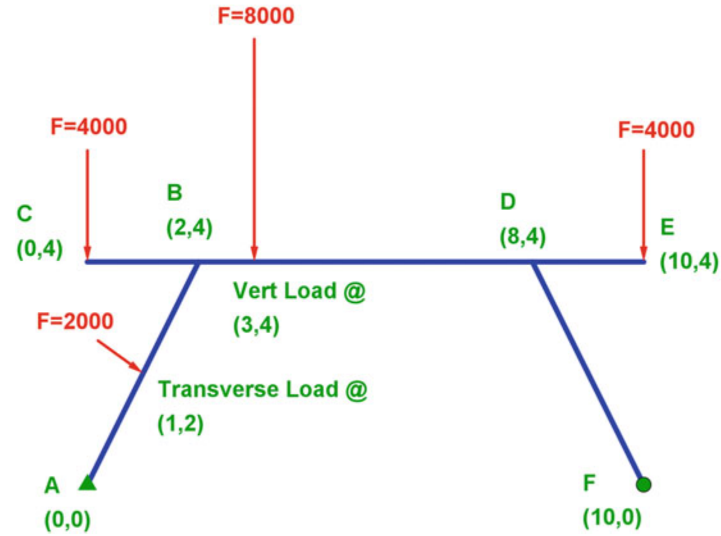


Fig. 6.40 Statically determinate frame with cantilever portions as well as a lateral and vertical loads

Figure 6.40 shows a slightly more complicated frame. The top horizontal member has vertical loads on it, while the left leg has a lateral load that is perpendicular to that leg. Figure 6.41 demonstrates how to locate Point v on the force diagram, and to unlock the reactions, which are segment lengths uv and vq, each multiplied by the $ForceScale$.

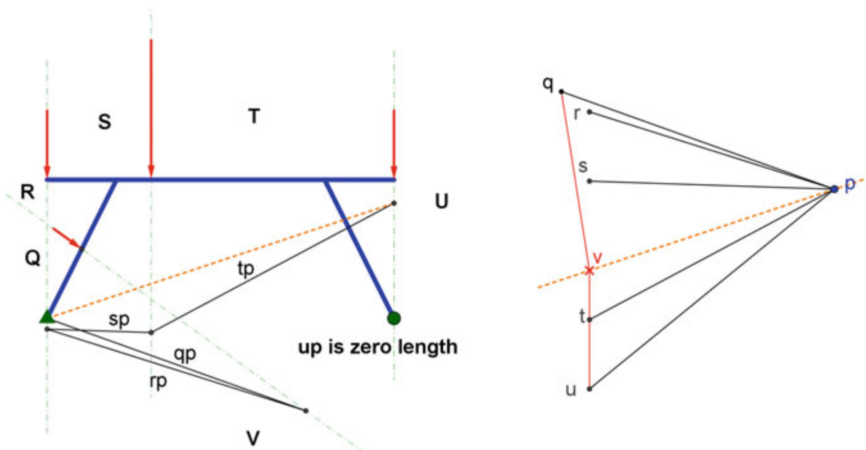


Fig. 6.41 Force diagram establishes slopes of funicular which unlock reactions

The approach shown in the following figures combines algebraic statics thinking with the visual capabilities of graphical analysis. For example, since the magnitude and direction of the left pinned reaction are known, the moment anywhere along the left diagonal leg within *Space Q* is found as:

$$M_{along\ Q} = R_{pin} \cdot QSpaceDistance$$

where $Rpin$ is the $Force\ VQ$ and $QSpaceDistance$ is the perpendicular distance from the force trajectory to the cut shown as a red X. This is shown in Fig. 6.42.

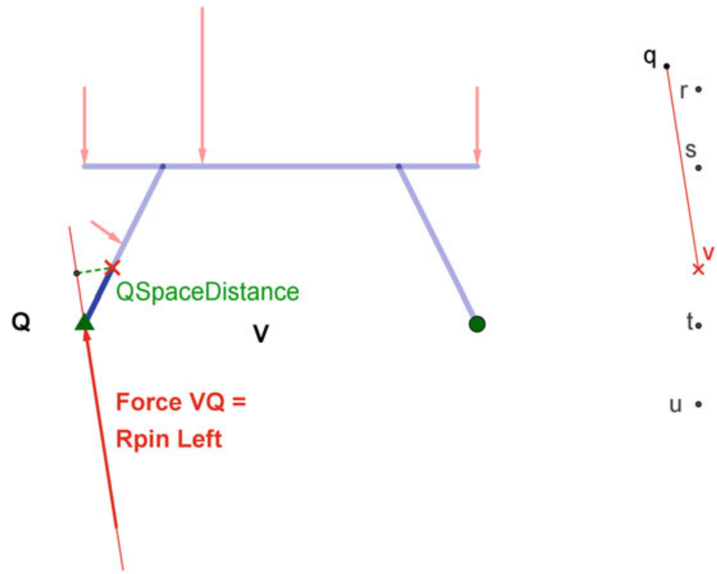


Fig. 6.42 Calculating the moment in *Q Space*

To find the moment in the diagonal leg within *Space R*, first vectorially combine the left pin reaction force with the applied diagonal force of 2000. The resultant of these two forces ends up being vertical because there are no more inclined loads on the structure. The perpendicular (horizontal) distance from this trajectory to the cut *X* on the left leg is readily captured (Fig. 6.43). Then:

$$M_{along\ R} = RspaceNetForce \cdot RSpaceDistance$$

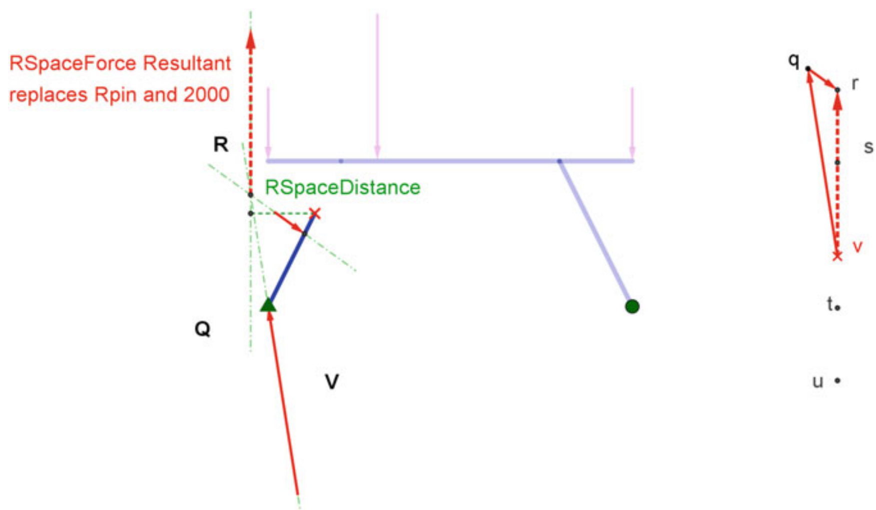


Fig. 6.43 Calculating the moment in *R Space*

The bending moment in the overhanging left portion in *Space S* is immediately found by

$$M_{along\ S} = 4000 \cdot SspaceDistance$$

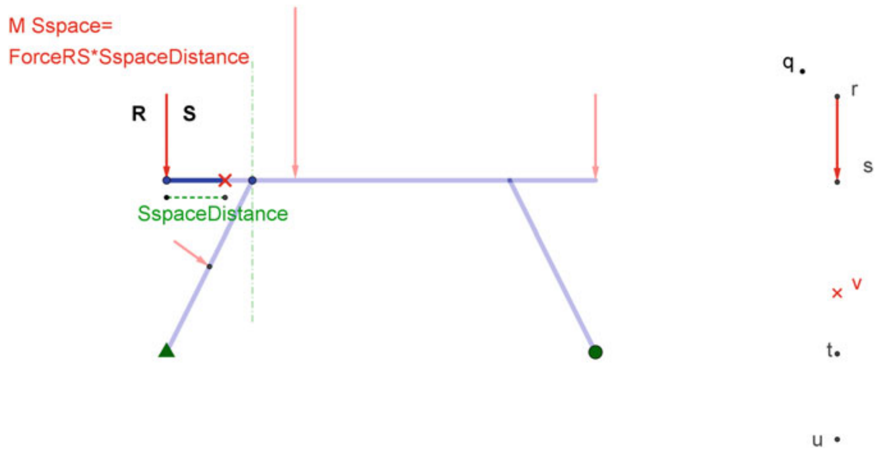


Fig. 6.44 Calculating the moment in *S Space first part*

A different free body is required for moment still within *Space S* but to the right of the inclined supporting leg, since that force affects the moment in this region (Fig. 6.44). Thus, it is helpful to label this space as *S2*.

Certainly, two forces could be used to find the moment in space *S2* algebraically, but it is rather elegant to combine the two forces, namely the previous resultant in *Space R* and the 4000, shown in Fig. 6.45. Using the Inverse Axis Method described in Fig. 1.13, note that the centroid will be on the outside of the larger load. The resultant is shown in Fig. 6.46.

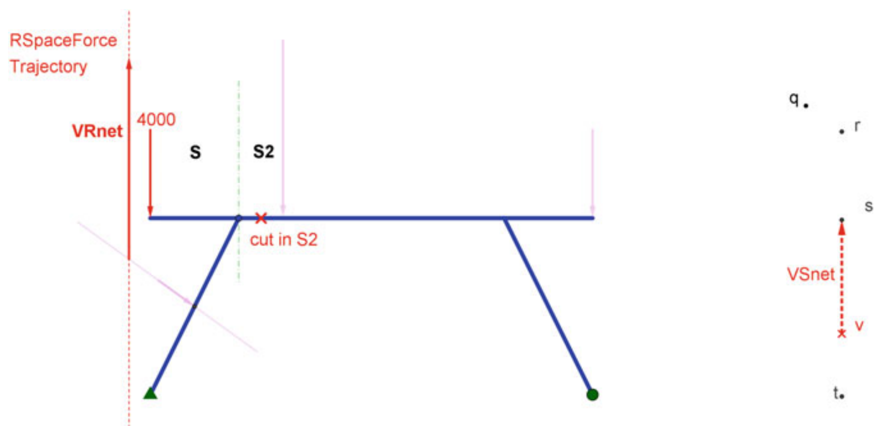


Fig. 6.45 Calculating the moment in *S Space second part*

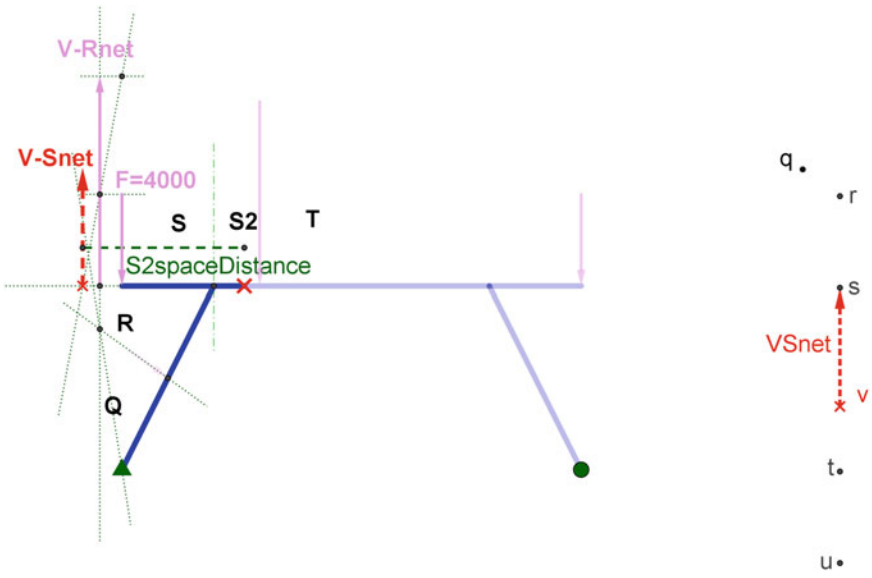


Fig. 6.46 Calculating the net force acting on a cut

Figure 6.47 shows the calculation for moment in *Space T*. All of the loads to the left of the cut *X* have been combined into a single resultant *V T net*. The centroid of this resultant *V T net* lies to the right of the 8000 unit force, and it coincides with the extreme right end of the structure for this problem. Note that the free body is still everything to the left of the cut.

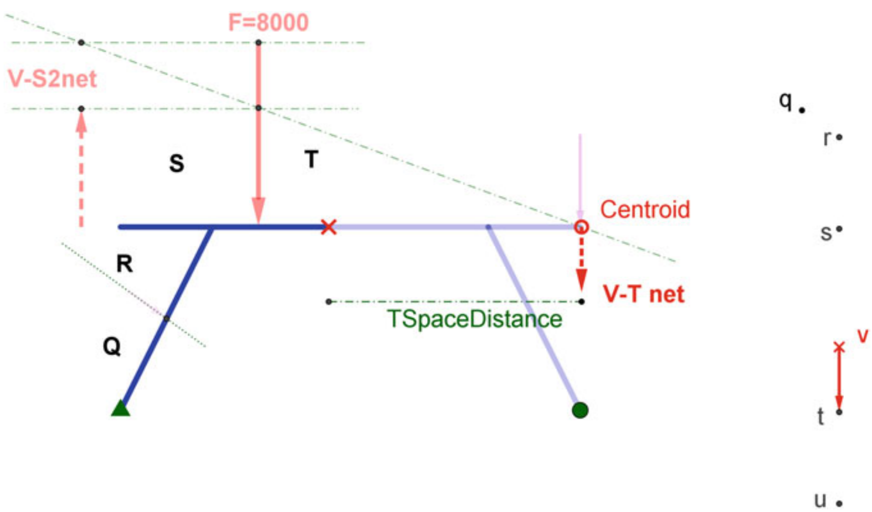


Fig. 6.47 Calculating the moment in *Space T*

Finally, two elementary free body diagrams are combined in Fig. 6.48. The moment in *T2Space* is immediately found due to the 4000unit force, and the moment in *USpace* is found based solely on the right roller support and the horizontal distance to that force's trajectory.

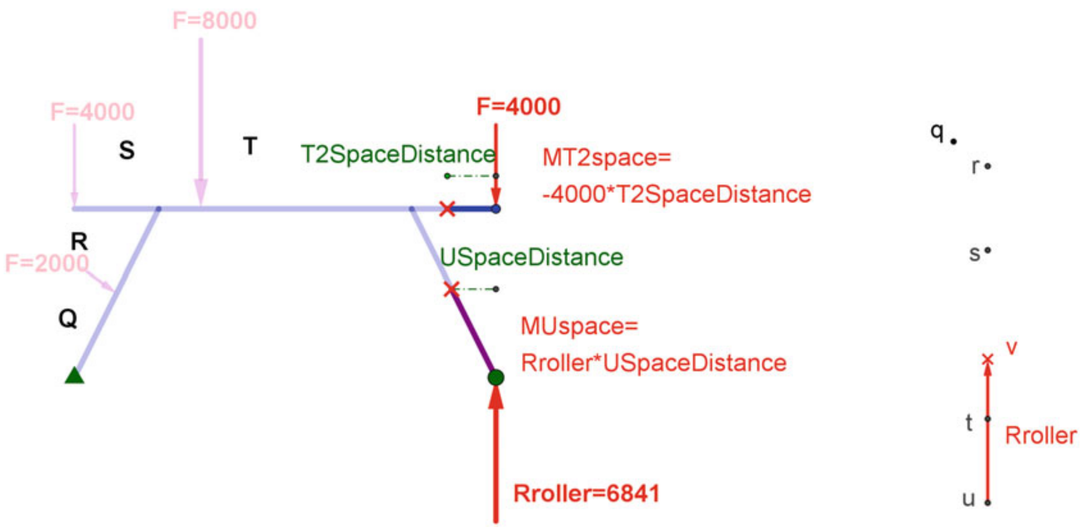


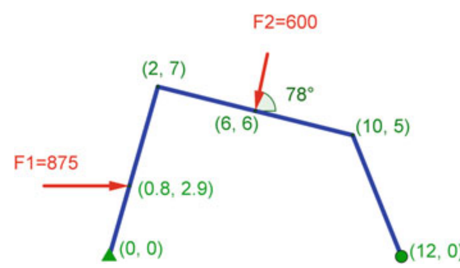
Fig. 6.48 Calculating the moment along right leg

One insight can be gained from Figs. 6.38, 6.39, and 6.46. Namely, that for frames with inclined members, the simplest method of calculating bending moment is to:

- Graphically determine the reactions without any funicular
- Determine the resultant of the loads acting on some section of the frame
- Determine the distance from the line of action of that load to the frame
- Calculate the product of that resultant and that distance, this is the moment

One final example will demonstrate the simplicity of this approach. No funicular is needed to calculate the reactions. The problem is shown in Fig. 6.49.

Fig. 6.49 Frame with sloping members and lateral loads



First, the loads are combined into one resultant. The intersection of the paths of the original loads will determine the location of the resultant, Point Q. A force diagram will determine the direction of the resultant. Where the path of this resultant intersects the path of the roller support identifies point R. A line from R to the pinned support determines the direction of the pin support force. The force diagram will calculate the magnitudes of the roller support and the pin support forces. This is shown in Fig. 6.50.

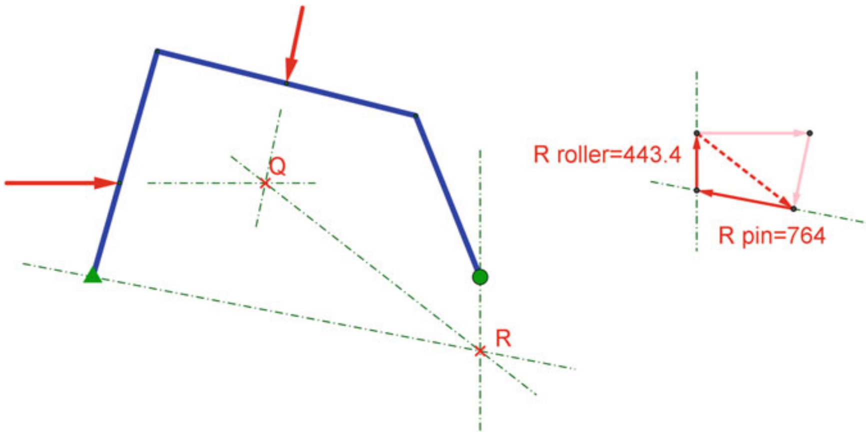


Fig. 6.50 No funicular needed to calculate reactions

Having the reactions allows for very rapid calculation of moments. But care must be taken to ensure that the magnitude of the net force acting on a free body is properly identified. Also, the location of the net force must also be identified. This will be at the intersection of the paths of the loads needed. For example, in Fig. 6.50 there are three distinct free body zones. These will be labeled as Zones *A*, *B* and *C*. In Zone *A*, only the pin force is needed, or more descriptively, Force *D A* is needed. In Zone *B*, the resultant of the pin force and the lateral force are needed, or Force *D B*. In Zone *C*, only the roller force is needed, or alternatively, the resultant of the pin force, the lateral force and the force on the top chord or Force *D C* (Fig. 6.51).

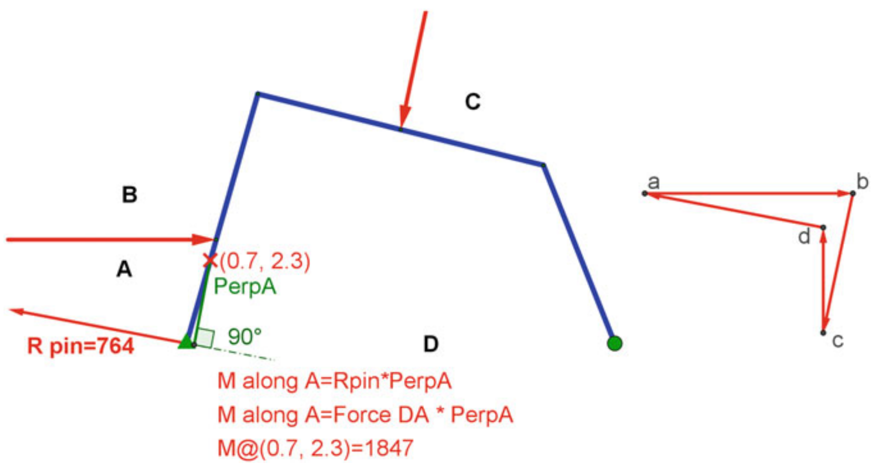


Fig. 6.51 Calculating the moment in *A Space*

For Zone *B*, the resultant of *Rpin* and the *Lateral Force* must be calculated but this is very quick to do since the force diagram is available. A useful tool is to label this resultant force as *DB* since it captures Zone *D* to Zone *A*, and then Zone *A* to Zone *B*. This greatly simplifies the calculation of the resultant which must go from Point *d* on the force diagram to Point *b* (Fig. 6.52).

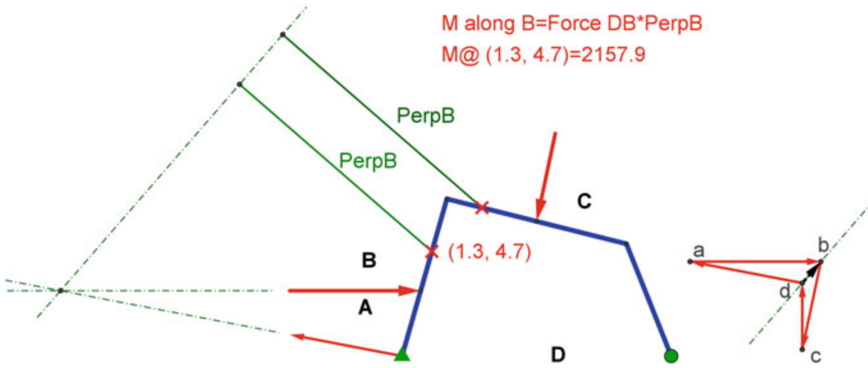
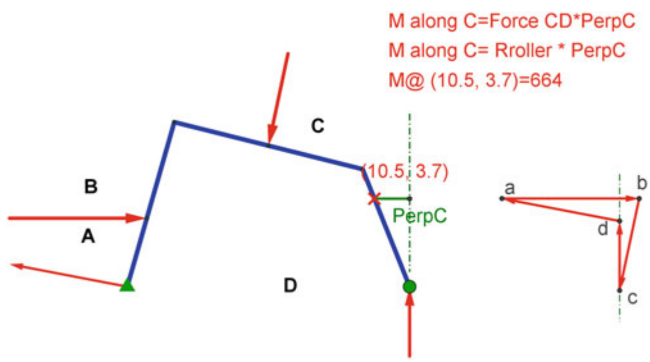


Fig. 6.52 Calculating the moment in *B Space*

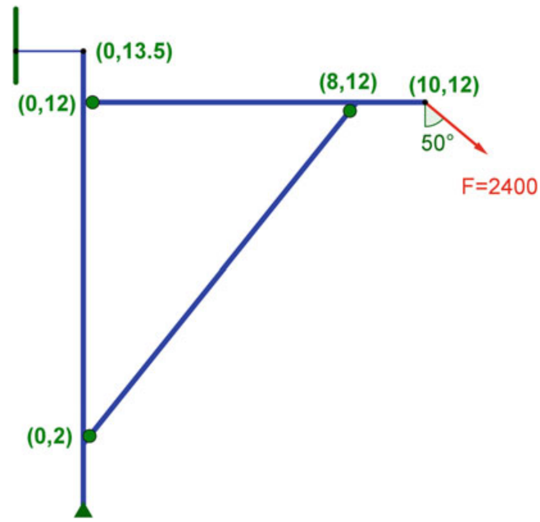
The final step is to capture the force in Zone C. This is Force CD, in other words, the roller reaction. In this problem, that force (or reaction) is vertical. This is shown in Fig. 6.53.

Fig. 6.53 Calculating the moment in *C Space*

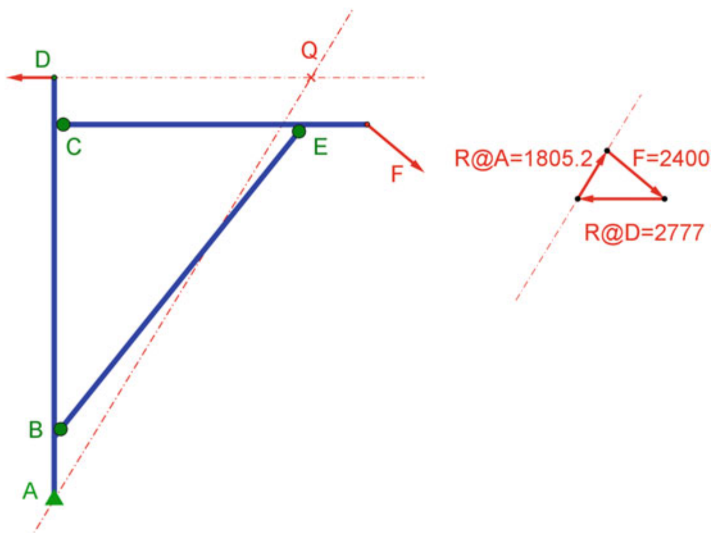


Chapter 6 Exercises

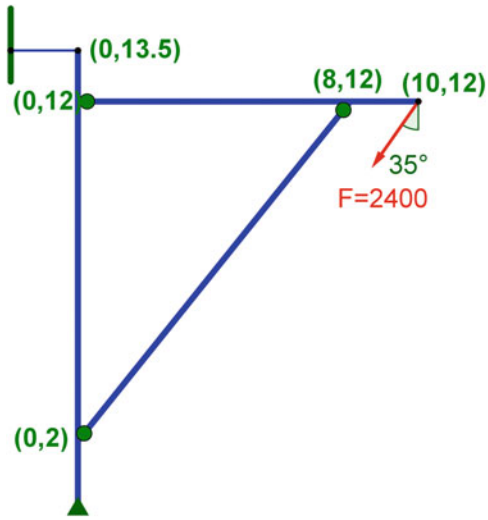
Exercise 6.1 Calculate the reactions for the following frame. Exercise 6.2 Calculate the reactions for the following frame



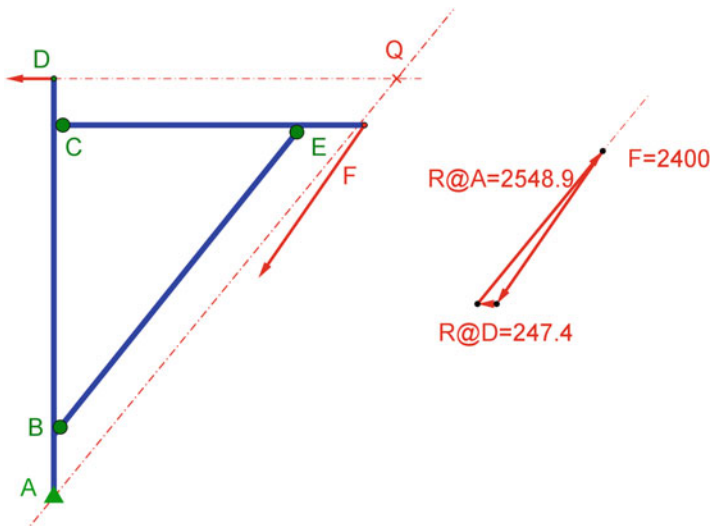
Exercise 6.1 solution



Exercise 6.2 Calculate the reactions for the following frame

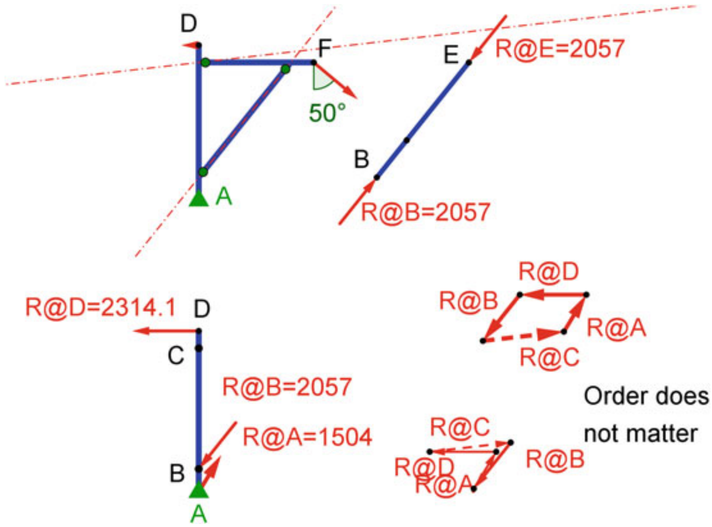


Exercise 6.2 solution



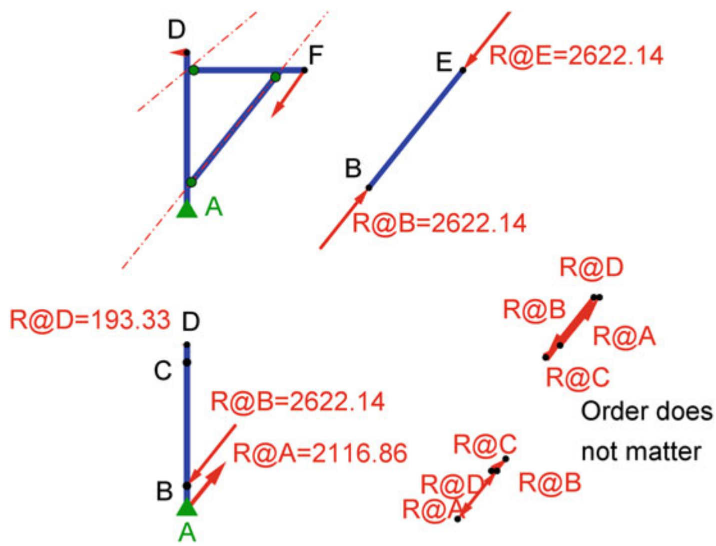
Exercise 6.3 For the frame shown in Problem 6.1, complete the free body analysis of individual members.

Exercise 6.3 solution

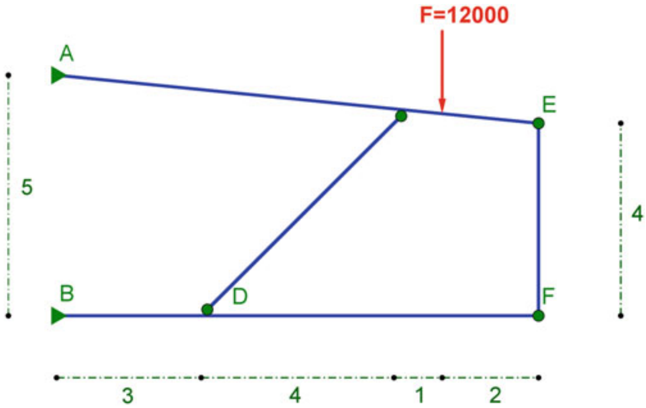


Exercise 6.4 For the frame shown in Problem 6.2, complete the free body analysis of individual members.

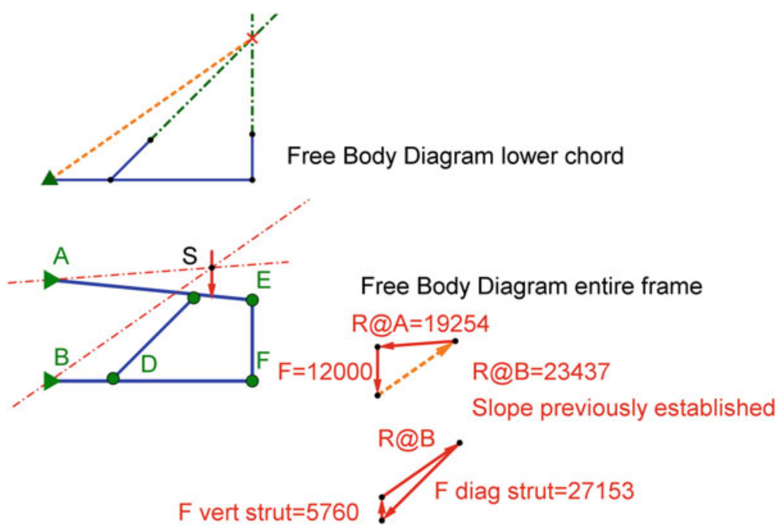
Exercise 6.4 solution



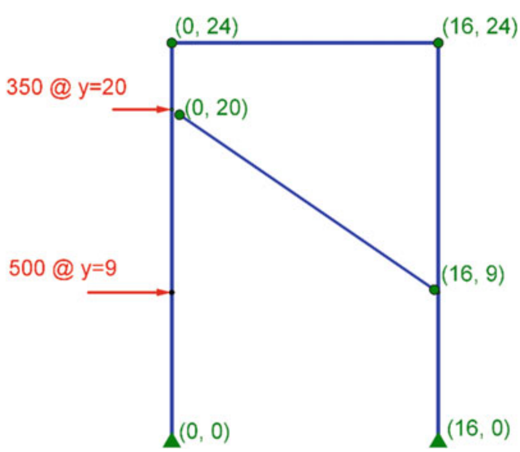
Exercise 6.5 For the following frame, complete the free body analysis of individual members.



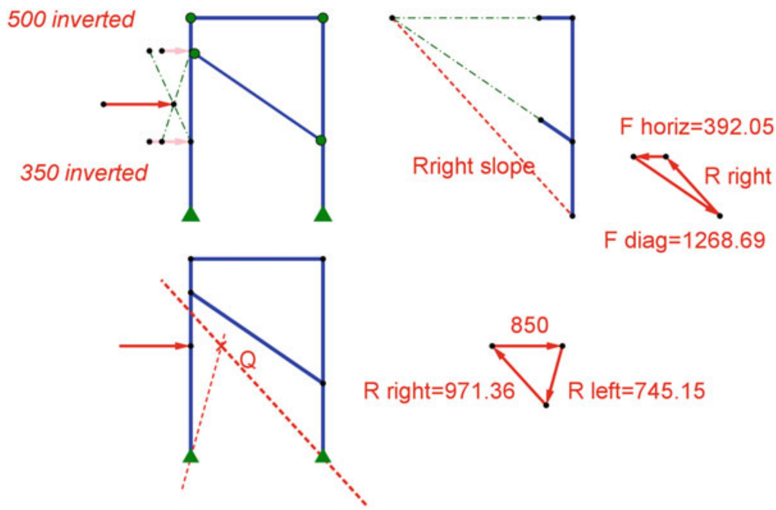
Exercise 6.5 solution



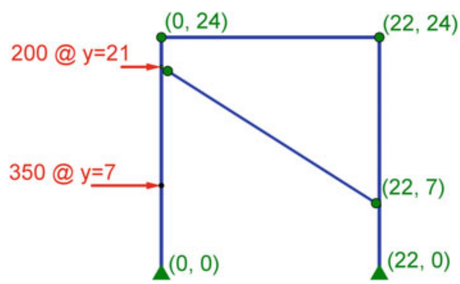
Exercise 6.6 For the following frame, complete the free body analysis of individual members.



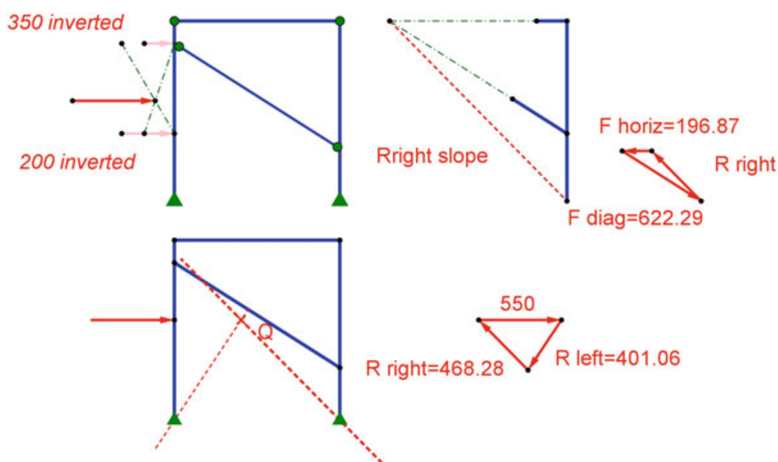
Exercise 6.6 solution



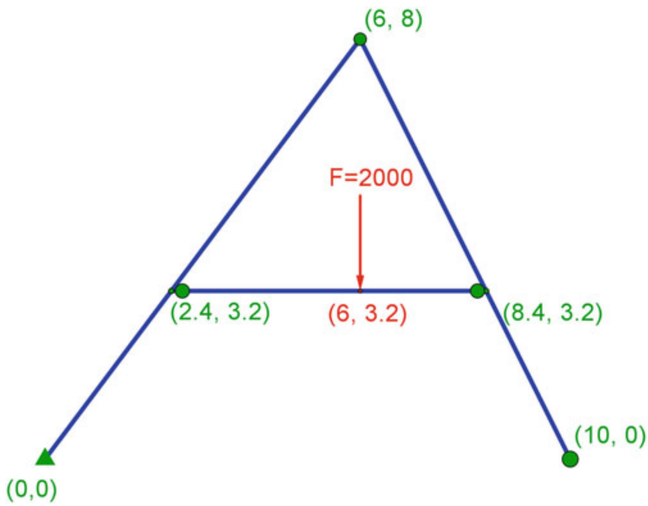
Exercise 6.7 For the following frame, complete the free body analysis of individual members.



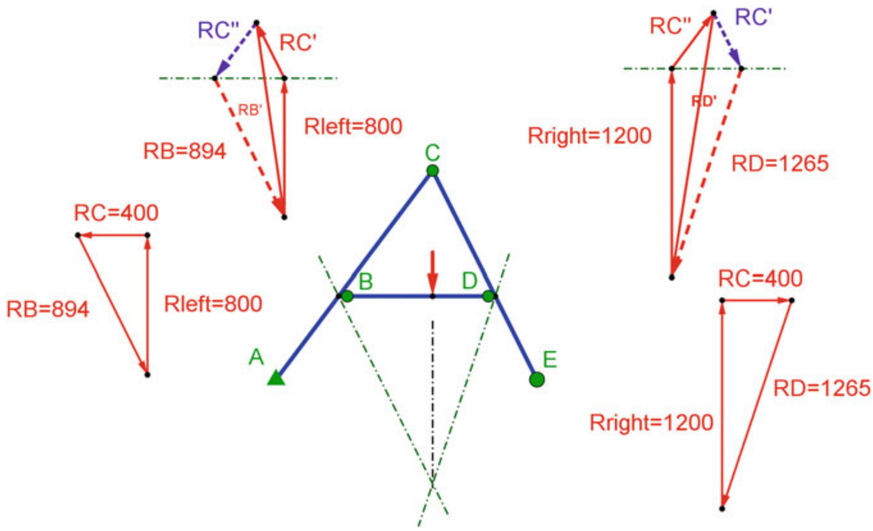
Exercise 6.7 solution



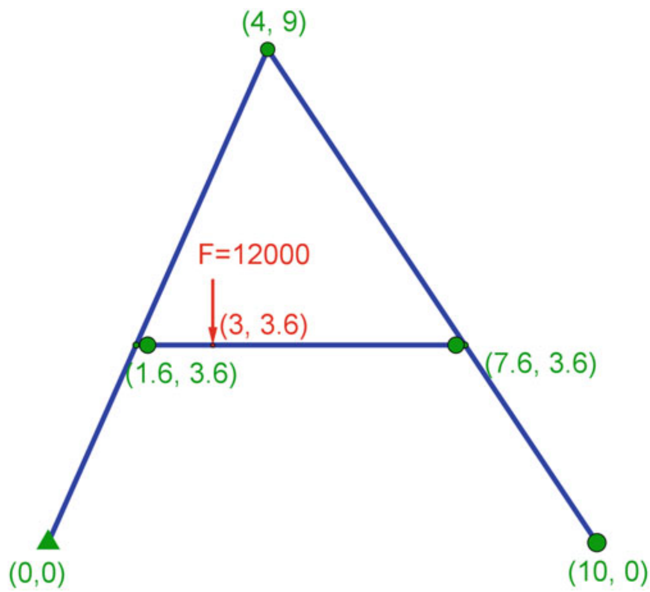
Exercise 6.8 For the following frame, complete the free body analysis of individual members.



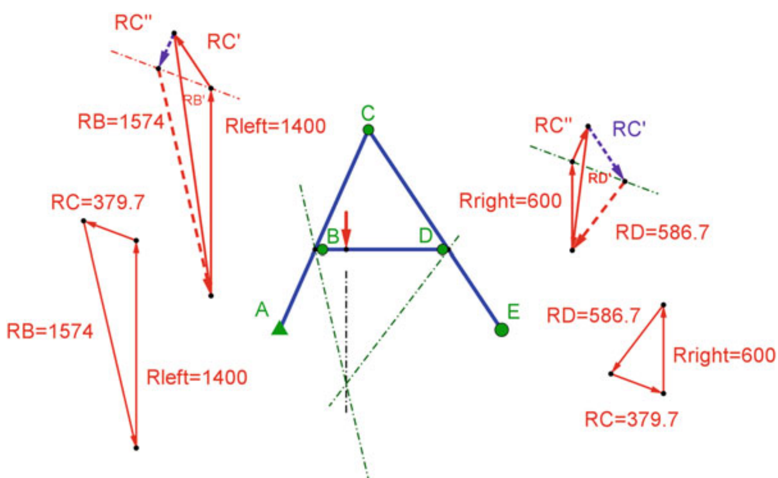
Exercise 6.8 solution



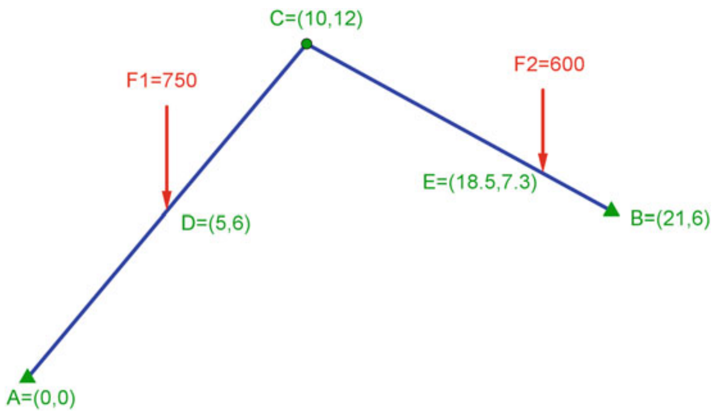
Exercise 6.9 For the following frame, complete the free body analysis of individual members.



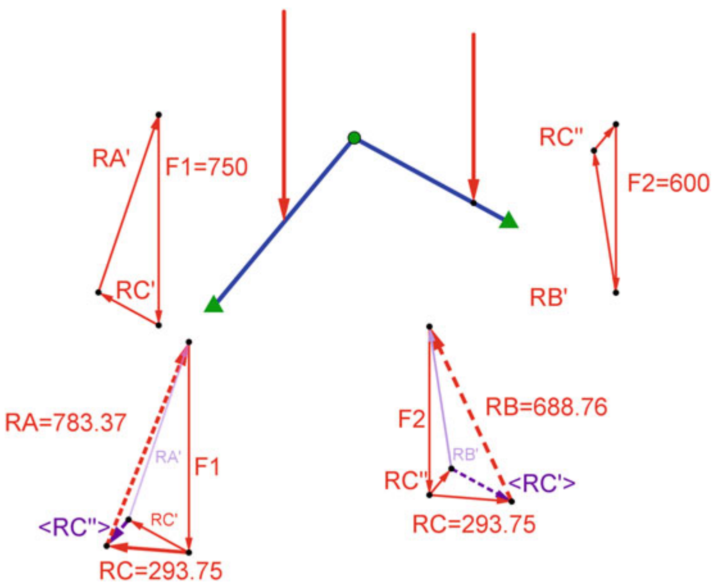
Exercise 6.9 solution



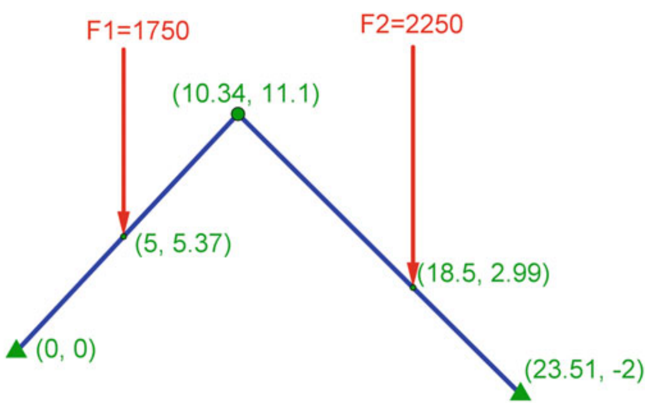
Exercise 6.10 For the following frame, complete the free body analysis of individual members.



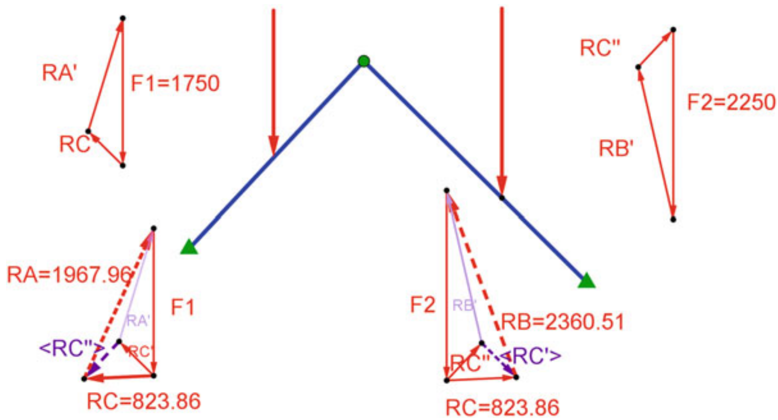
Exercise 6.10 solution



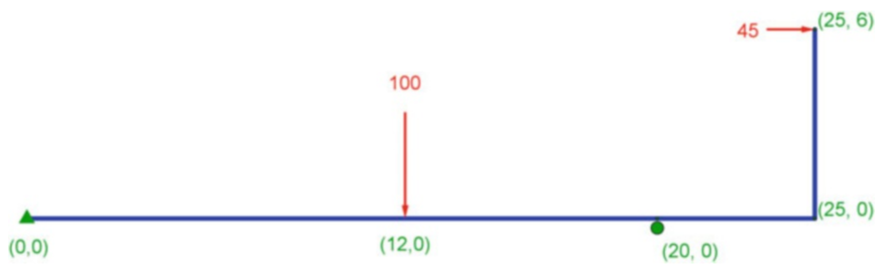
Exercise 6.11 For the following frame, complete the free body analysis of individual members.



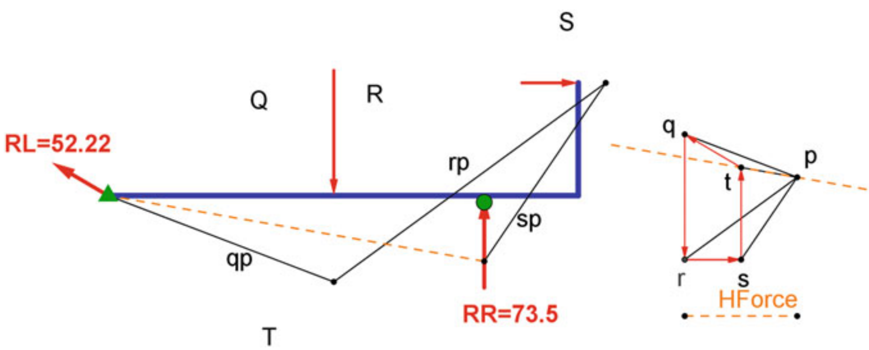
Exercise 6.11 solution



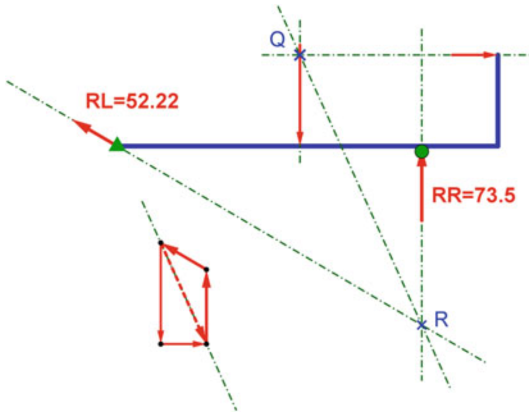
Exercise 6.12 Calculate the reactions using a trial funicular in the following beam.



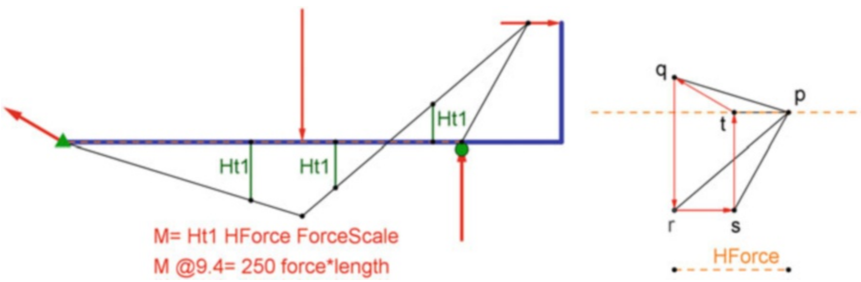
Exercise 6.12 solution



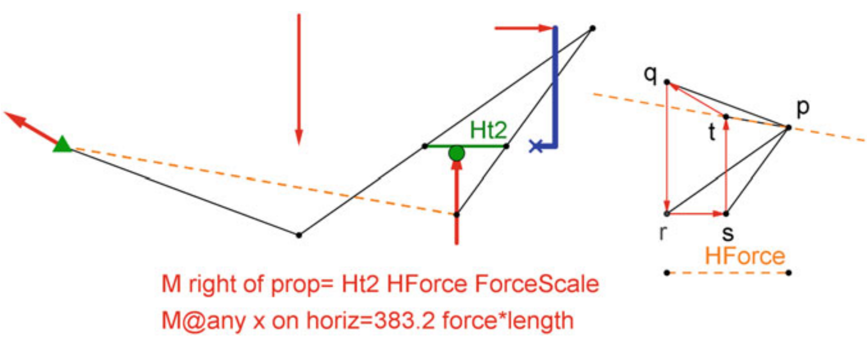
Exercise 6.13 For the beam of Problem 6.12, calculate the reactions using only equilibrium of a point.
Exercise 6.13 solution



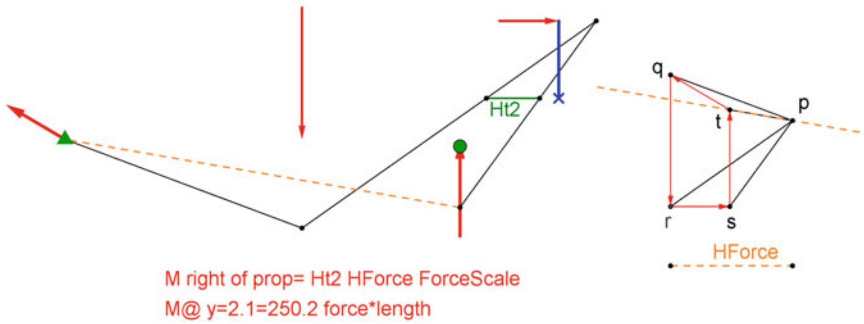
Exercise 6.14 For the beam of Problem 6.12, calculate the bending moment at any cross section between the two supports.
Exercise 6.14 solution



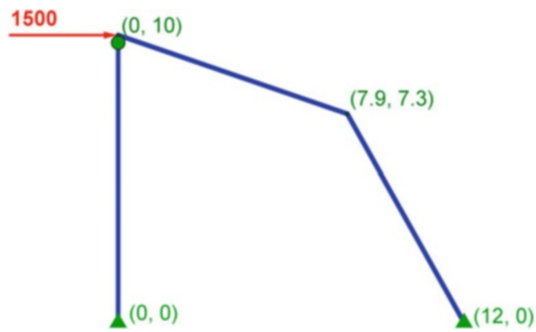
Exercise 6.15 For the beam of Problem 6.12, calculate the bending moment at any cross section to the right of the roller support.
Exercise 6.15 solution part 1



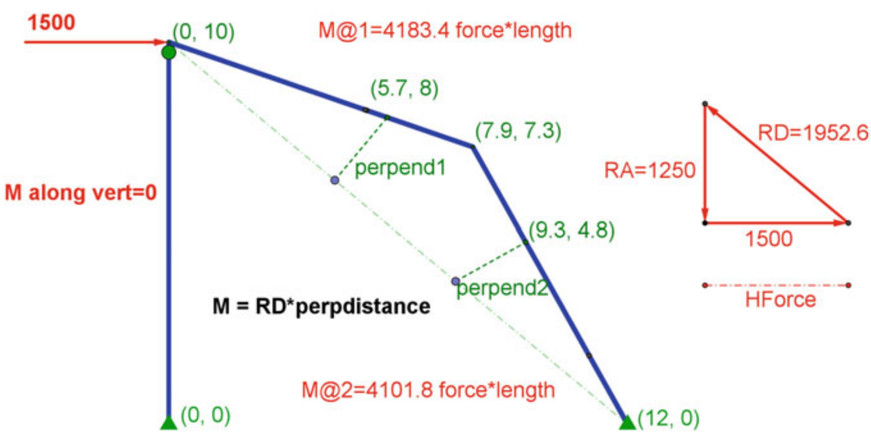
Exercise 6.15 solution part 2



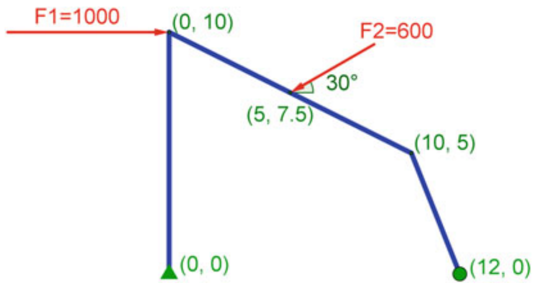
Exercise 6.16 For the following frame calculate the bending moment at any cross section



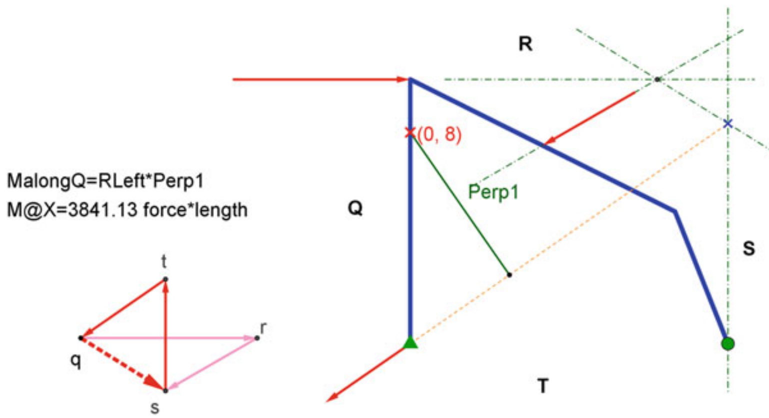
Exercise 6.16 solution



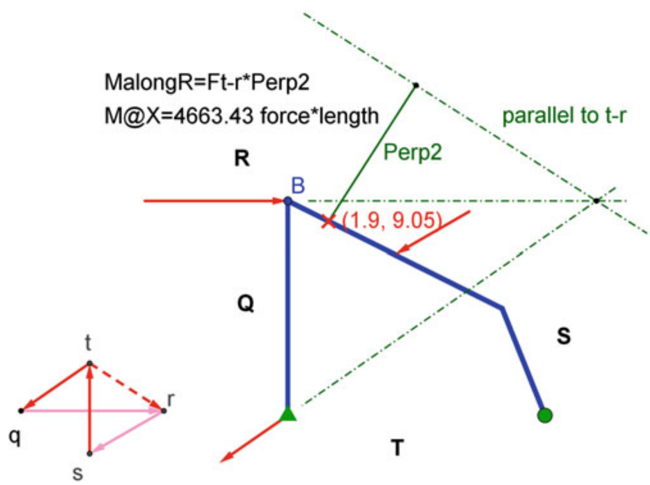
Exercise 6.17 Calculate the bending moment at any cross section without the use of any funiculars.



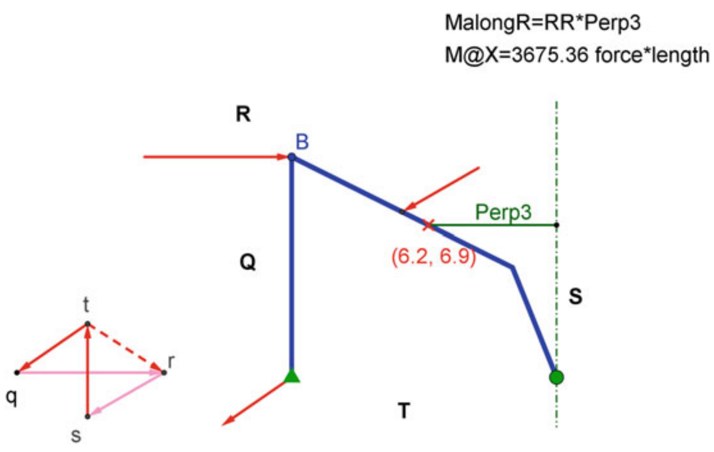
Exercise 6.17 solution part 1



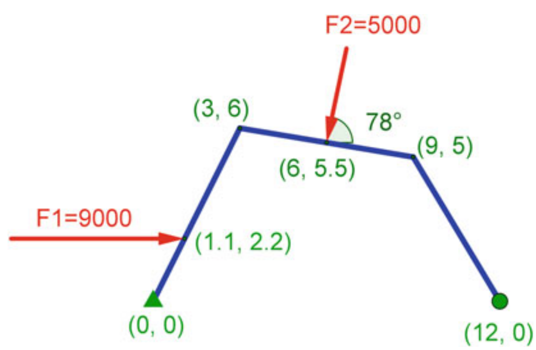
Exercise 6.17 solution part 2



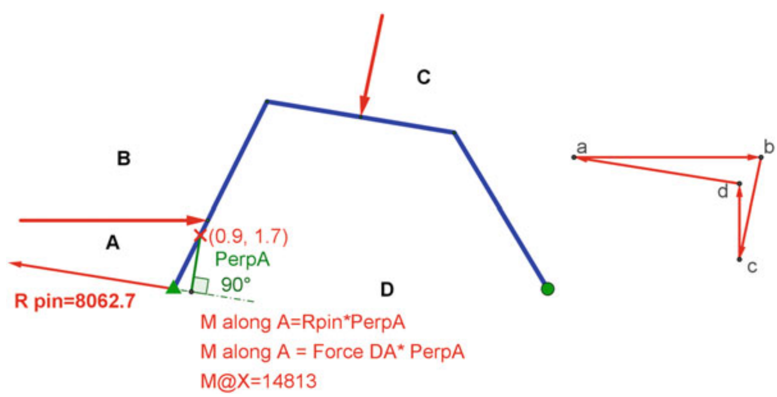
Exercise 6.17 solution part 3



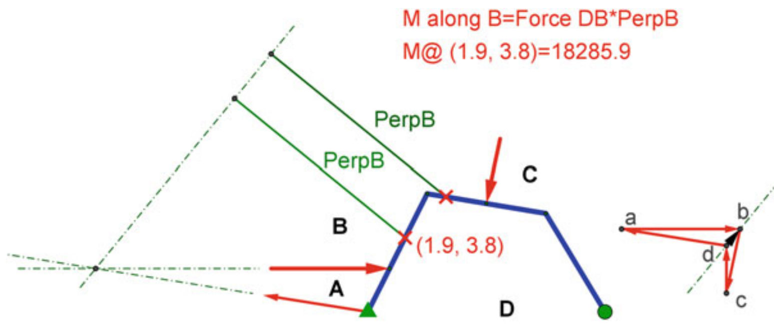
Exercise 6.18 Calculate the bending moment at any cross section without the use of any funiculars.



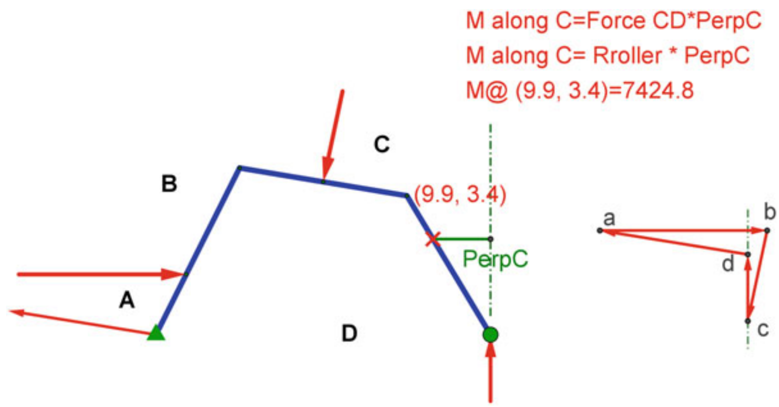
Exercise 6.18 solution part 1



Exercise 6.18 solution part 2



Exercise 6.18 solution part 3



7 Stability in 3D Space

STABILITY OF A POINT Recall from elementary statics or physics that a point in 3D space has three degrees of freedom, it can only translate in the X, Y, Z directions (or any other mutually orthogonal set of axes). This is visualized by imagining a ball flying through the air, the ball is free to travel in any direction, it is unstable. The point is infinitely small, all loads hit it directly at its centroid. This explains why there is no rotational degrees of freedom for a point in space, thus the ball does not spin. Loads have no eccentricity to the point's centroid. Since the point is free to translate in three directions, the point requires three supports to stabilize it. Of course the supports must be able to carry tension or compression. Traditionally, a pinned support (boundary condition) is assumed to carry three mutually perpendicular (orthogonal) forces as reactions. In this first example it is noted that each strut is pinned-pinned without any transverse loads, thus the net reaction must align with the axis of each supporting strut as shown in Fig. 7.1.

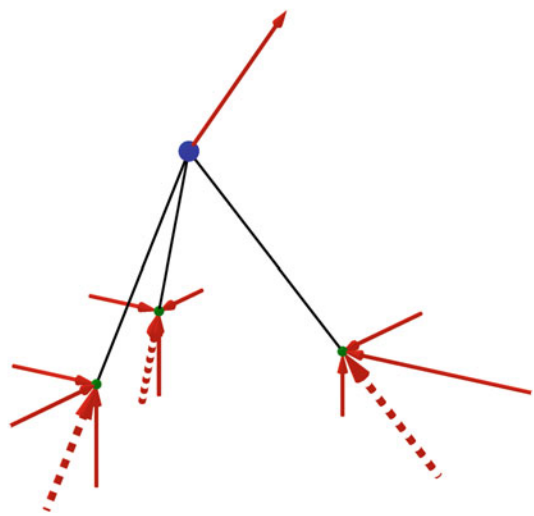


Fig. 7.1 A point in space subjected to some load needs three axial forces for stabilization

The placement of the three supporting pinned-pinned struts is not arbitrary. In fact, the point can still be unstable if the third strut is placed in a position that is co-planar with the other two supports. This unstable configuration is shown in Fig. 7.2.

E. Saliklis, *Structures: A Geometric Approach*, https://doi.org/10.1007/978-3-319-98746-0_7

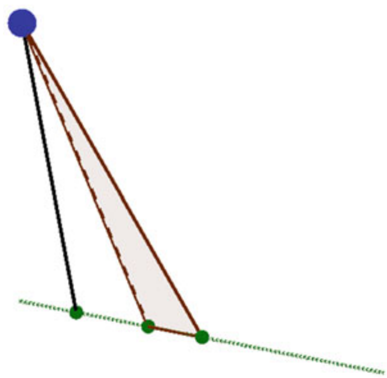


Fig. 7.2 Unstable arrangement of three supporting axial forces

DOFs of a Line in 3D Space

A single line, or axial element like a pinned-pinned strut in 3D space has five, not six degrees of freedom. Such a line clearly has the three translations that a point in 3D does, but what else happens? Imagine throwing such a stick through the air, it tumbles, i.e. it rotates. Such a line in 3D space is free to rotate about any two orthogonal axes perpendicular to the stick. For example it can twirl like a propeller about 12’o clock or about 9’o clock. But it does NOT spin about its own longitudinal axis. The reason it does not spin is completely analogous to why the point in space does not spin. All applied loads touch the center-line of the stick, loads cannot be eccentrically applied to the line. Thus, a line in space has 5 DOF, there is no “drilling” rotational degree of freedom (Fig. 7.3).

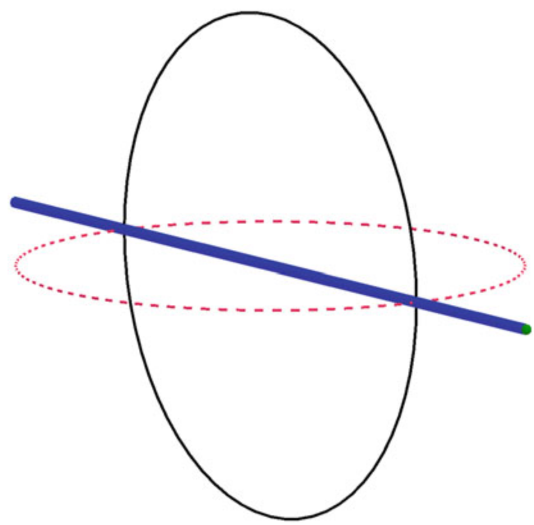


Fig. 7.3 A line in space has five degrees of freedom, not six

The final image to consider is that of a plate or a plane moving through space. Such a plane in 3D space clearly has all the previous DOFs of the line, but now it also must have the “drilling” DOF. It is helpful to think of planes first before thinking of 3D solids, because building designers generate so many planes in structures. Planes exists as shear walls, as roof diaphragms, and as cladding on a building. All the planes have some thickness, so we have described the final body we need to stabilize namely, a solid (or a plane of some finite thickness). It is important to remember that this plane can be considered as rigid for statics calcs! Thus it would not deform due to its own weight or any other forces, it does not bend, or distort. It maintains its original shape after loading! A plane (or solid) has 6 DOFS (Fig. 7.4).

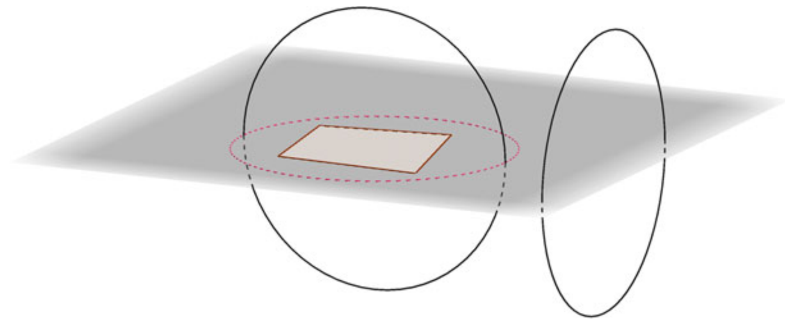


Fig. 7.4 A solid in 3D space has six degrees of freedom

Stabilizing a Line in 3D Space

It has been established that a line has 5 DOFs, thus it is intuitive to realize that if a line (or a two dimensional element) in space subjected to some loads, 5 supporting struts would be needed to stabilize this line against any (reasonable) loads. The word “reasonable” is subtly important because there is no spinning moments that rotate the element about its own longitudinal axis. Yet there is another way of viewing this problem. Suppose that the line in space was connected to a stable point at one of its ends. Figure 7.5 shows one end of the line anchored in space by a 3D pin. That one end cannot move in X, or Y or Z. How many DOFs remain free? An analogous design question is how many struts are now needed to stabilize the line?

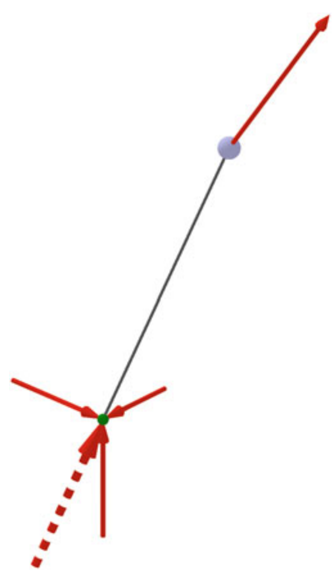


Fig. 7.5 A line in space can be imagined as one of the axial restraints of a point in space

Of course the answer is "two" struts are needed to stabilize the point, but an insightful alternate manner of solving this problem is to recognize that the line had 5 DOF originally, and three were removed due to the pin at the base. 5 – 3 = 2 DOF remain.

The Four Node Model

Consider the following very simple "building" shown in Fig. 7.6. Perhaps it is a bit much to call this a building but it begins to look like a conventional structure. Four nodes are in the plane of the roof. How many DOFs are currently present? What arrangement can be created to stabilize these points? The answer to the first question is unambiguous, the answer to the second question reveals the infinite variety of structural design! There are always new and interesting ways of solving structural problems.

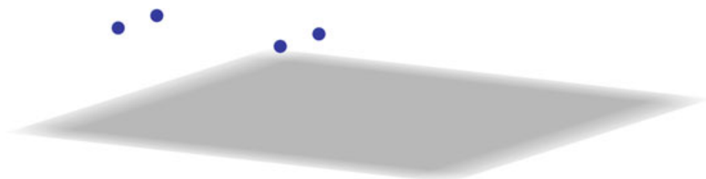


Fig. 7.6 Four unsupported nodes in space, stabilization choices are endless, but DOFs are unambiguous

Each of the four nodes has 3 DOF, thus there are twelve DOFs which must all be stabilized, i.e. locked or prevented from movement by twelve rigid members. The vertical (Z) movements of each of the nodes have an immediately obvious solution. Inserting a rigid pinned-pinned column under each node clearly prevents the four points from moving vertically, thus they are stabilized against translation in the vertical, Z, direction. 12 – 4 = 8 DOFs remain unrestrained. Four such axial elements are shown under the nodes in Fig. 7.7.

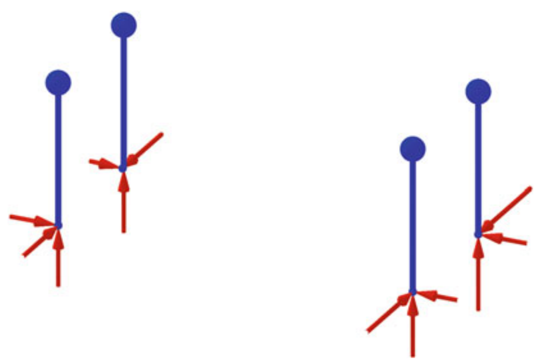


Fig. 7.7 Removing Z DOF of each node, or alternatively, four lines in space with 12 DOF removed

Figure 7.7 shows the insertion of four vertical struts, but it also allows for an alternate way of looking at this problem. The image was originally thought of as 4 nodes in air suspended by four vertical struts. Now consider the very same image as four struts, each of which have 5 DOFs resulting in 4*5 = 20. But notice that each of the four struts is stabilized at one end, removing 3DOFs at that bottom pinned end. So 20 − (4*3) = 8 resulting in the 8DOFs previously found.

Figure 7.8 shows a reasonable next step. The Y motion of points Q and R have arbitrarily been chosen to be stabilized. Since two more DOFs have been stabilized, 8 – 2 = 6 so there are 6 DOF remaining.

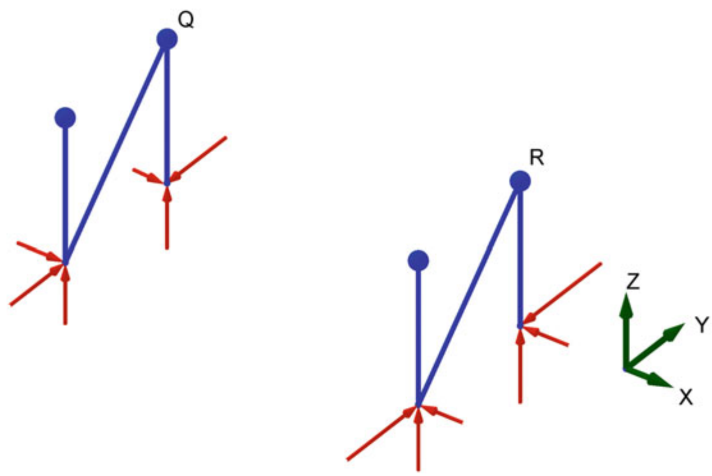


Fig. 7.8 Stabilizing points Q and R against Y translation

Figure 7.9 shows a subtle but important idea, namely that a designer can take advantage of the fact that the points Q and R are now stable in the Y and Z directions, they are not allowed to move in those two directions. Subsequent nodes can be stabilized in the Y direction by tying them to these two nodes with a rigid strut. This powerful but simple idea will be used in all of the stabilization exercises. A subsequent point's particular DOF can be stabilized by connecting it to a point that has already been prevented from moving in that DOF. In Fig. 7.9, all four points have their Z motion prevented, but Points S and T also have their Y motion restricted solely because points Q and R have been stabilized in Y.

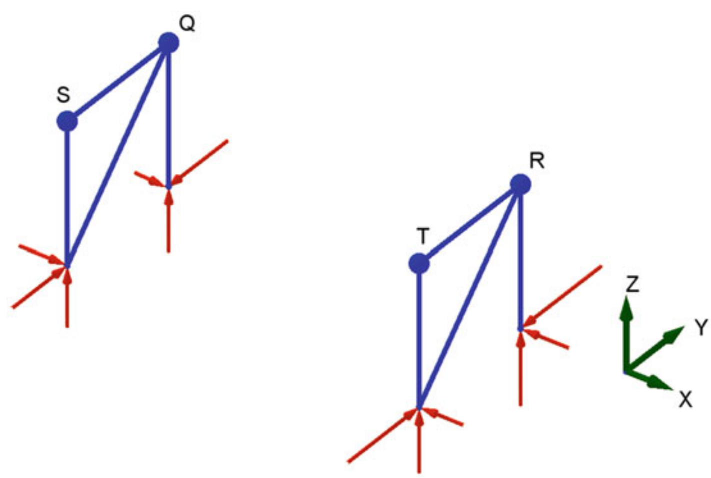


Fig. 7.9 Four DOF remain to be stabilized

A reasonable next step is to arbitrarily stabilize the last DOF of the node Q, namely translation in the X direction. That can be accomplished with a diagonal in the XZ plane. Now there are 3 DOF remaining to be stabilized, namely the X movements of nodes R, S and T (Figs. 7.10 and 7.11).

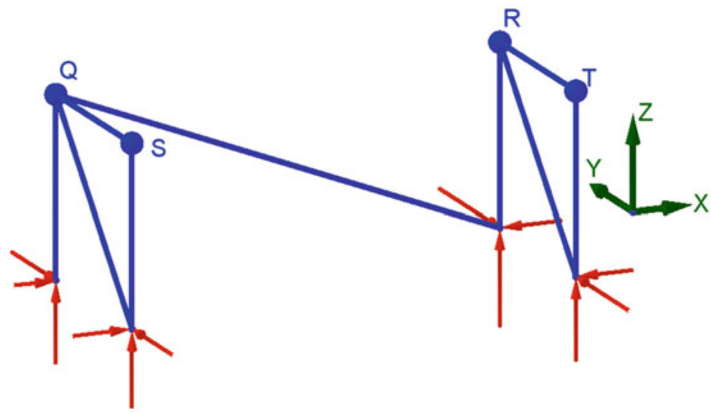


Fig. 7.10 Three DOF remain to be stabilized

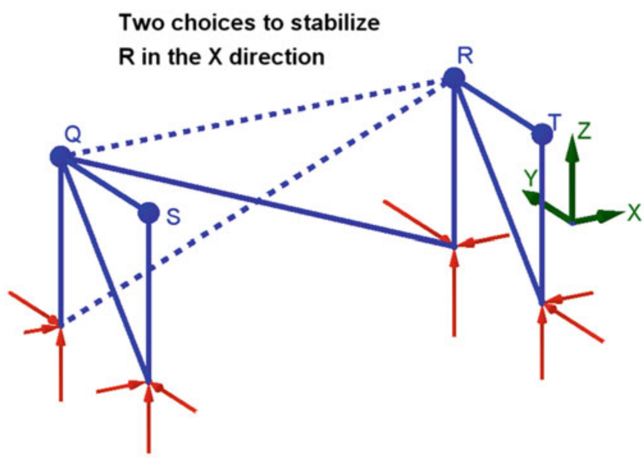


Fig. 7.11 Two choices to stablize point *R* in the X direction

Either choice is acceptable, both architecturally and structurally as that bay already had one diagonal passing through it already. Choose the option of a horizontal strut from *Q* to *R*. Now two DOF remain, namely the X movements of Points *S* and *T*.

Point *S* can be stabilized in the X direction by tying it to Point *R* which is already stable in X (and Y and Z) (Fig. 7.12).

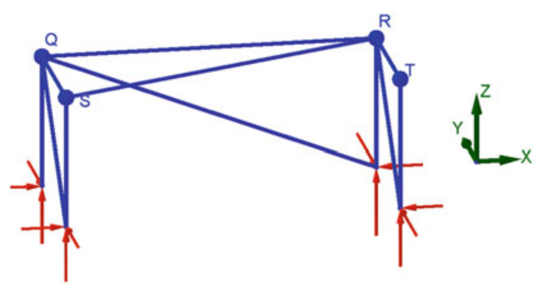


Fig. 7.12 Only one DOF remains to be stabilized

The final step is to stabilize Point T in the X direction. Using a diagonal brace in the large bay is not acceptable architecturally, thus, add a second diagonal from Point Q to Point T which leaves the front bay open and accessible. This is shown in Fig. 7.13.

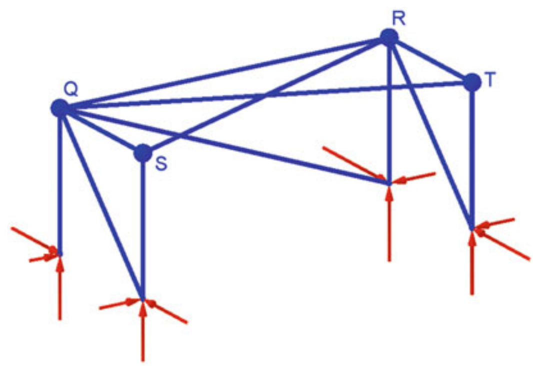


Fig. 7.13 It is preferred to stabilize point T via the roof, rather than with an obtrusive brace to the ground

Next, consider the roof of the building in Fig. 7.13 as a rigid plane. Recall that a plane of some thickness has six DOFs In Fig. 7.14, all of the line elements in the roof of Fig. 7.13 have been replace by a single rigid plane of some thickness.

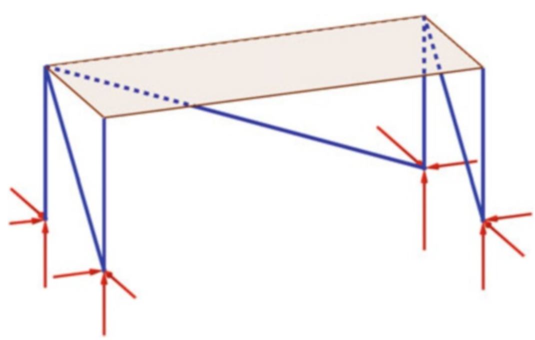


Fig. 7.14 Roof braces are replaced with a rigid plane

The presence of seven members shown in blue in Fig. 7.14, supporting the rigid roof indicates that there is a redundant member, as only six are needed for stability. Before changing the structure into a statically determinate form, note that the reason for stabilizing these problems with the minimum number of struts is to fully understand the unrestrained motion of each specific problems. In real design, redundant elements must always be added, but the designer needs to understand what is minimum and what is redundant.

To demonstrate the importance of the placement of the minimum number of struts, consider the following poor decision to remove one strut from Fig. 7.14. The resulting structure shown in Fig. 7.15 is unstable for moment about a vertical line through Point Q. This is so because vertical pinned-pinned columns with no transverse loads are axial only elements, they have no lateral load resistance.

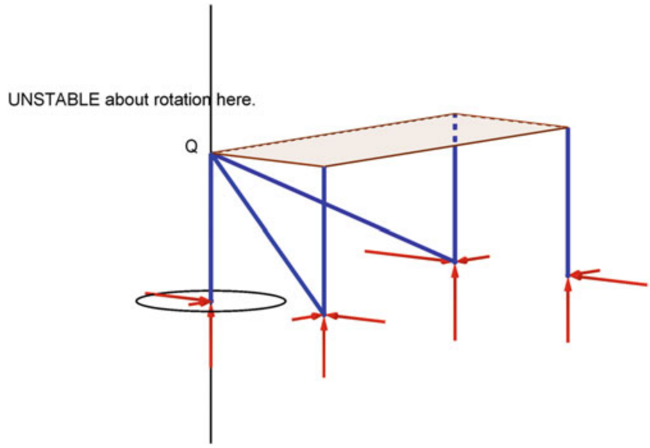


Fig. 7.15 Incorrect design choice, structure is unstable

The structural truth shown in Fig. 7.15 is that the plane of some thickness has six DOFs but that these six supports cannot be arbitrarily placed. A stable configuration would place a third diagonal such that it is not concurrent with the other two diagonals. Figure 7.16 shows one such solution that still leaves the front bay open and utilizes only 6 supporting struts.

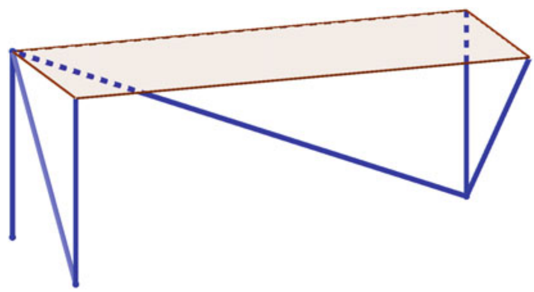


Fig. 7.16 Rigid roof is stabilized by six struts

It may seem odd to configure this roof as shown in Fig. 7.17 because of the cantilevered corner, but such an arrangement is stable and acceptable because the roof is assumed to be rigid, it does not deform under its own weight or under additional loads. Thus, the configuration in Fig. 7.17 could be used instead of the one shown in Fig. 7.16.

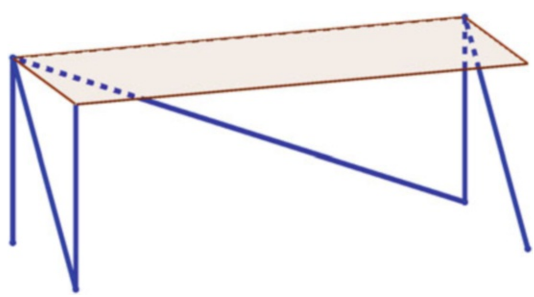


Fig. 7.17 Rigid roof is stabilized by another six strut arrangement

A very important point about the statics of such rigid roof must be highlighted. It is perfectly appropriate to assume that:

- Vertical Columns carry gravity loads but not lateral loads
- Diagonal Braces carry lateral loads but not gravity loads

These two assumptions will be highlighted again during the statics calculations examples of this chapter. These two assumptions allow the structure to remain statically determinate, consequently hand calculations can and will be performed on such structures.

Next, suppose there was a need to add a smaller second roof to the building. The smaller roof is connected to the already stable Point Q. The appendage itself has six degrees of freedom, but three have been removed due to the anchorage at stable Point Q. This is shown in Fig. 7.18.

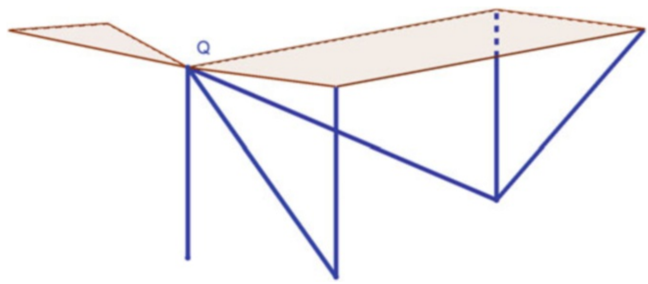


Fig. 7.18 Appendage has three DOF remaining to be stabilized

Three degrees of freedom must be stabilized for the new appendage, thus three additional struts are necessary to stabilize the new additional roof. It is reasonable to place vertical supports at the two free corners, but perhaps it is not immediately obvious what the remaining degree of freedom actually is. Figure 7.19 shows the addition of the two supports, as well as the remaining instability of the appendage.

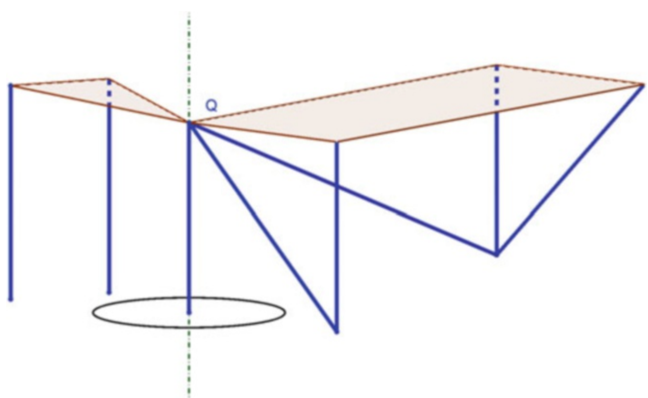


Fig. 7.19 Appendage has one remaining DOF to be stabilized

The fact that the two added supports have no lateral force capabilities means that rotation about a vertical axis passing through Point Q is still possible. Thus, one additional diagonal is needed, one that is not concurrent with that vertical axis passing through Q. Figure 7.20 shows one possible solution that is stable.

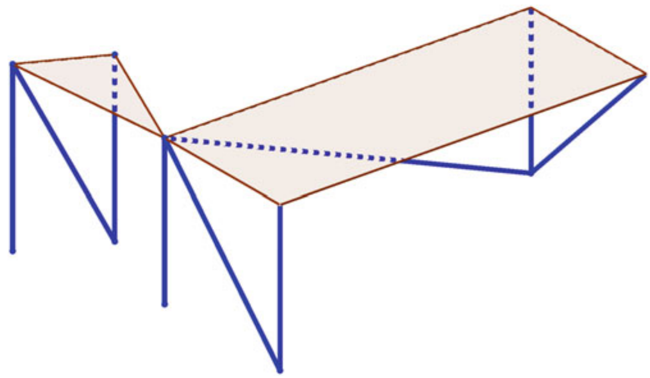


Fig. 7.20 A stable configuration

Note that as before, an incorrect arrangement of struts with the correct number of struts is possible, even on a fairly simple problem such as the stabilization of the appendage roof. Figure 7.21 shows an unstable arrangement, as the moment about Node Q is still not being resisted.

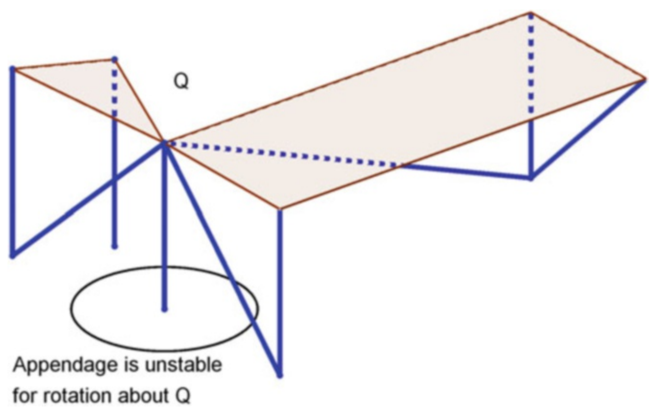


Fig. 7.21 An unstable configuration

The Nine Node Model

The same principals are applied to more complicated structures. Suppose nine nodes represented a roof plane in 3D space. Clearly there are 27 degrees of freedom in such a structure and consequently, 27 pinned-pinned struts are the minimum number needed to stabilize the structure. In Fig. 7.22, the nine nodes are shown, with a vertical pinned-pinned element under each node, as well as six horizontal and three diagonal elements. Thus, 18 members have been used, and 9 remain to be used (Fig. 7.22).

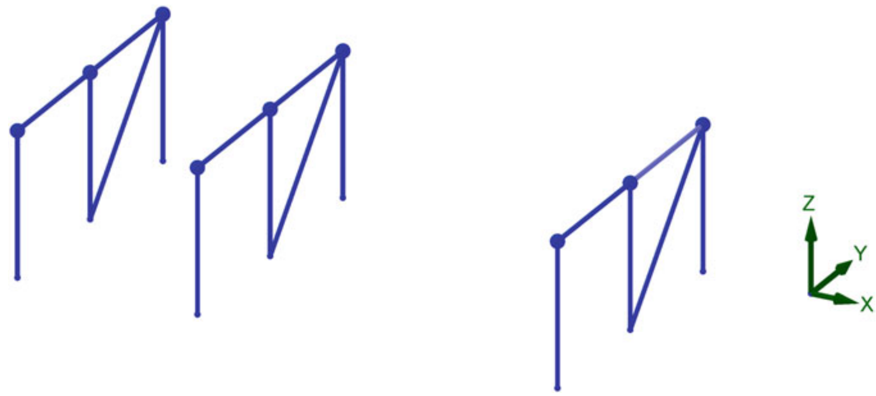
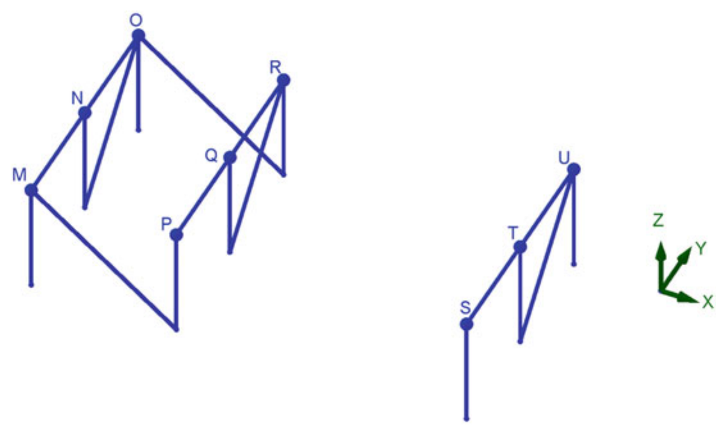


Fig. 7.22 Nine DOF remain to be stabilized

Stability in the X direction is needed, but dropping down many more diagonals is architecturally not viable. Two diagonals are added in the shorter bay to provide rigidity against X translation of Nodes *M* and *O*, this is shown in Fig. 7.23. Now, 20 struts have been used and 7 still remain.

Fig. 7.23 Seven DOF remain to be stabilized



To stabilize Nodes *R* and *U* in the X direction without adding an additional diagonals to the ground, horizontal elements *OR* and *RU* can be added to the roof line, since Node *O* is stable in the X (and Y and Z) direction. If Node *U* is stable in the X direction, then Node *Q* can be stabilized in the X direction by another horizontal member *QU* in the roof line. This is shown in Fig. 7.24. Now 23 struts have been used and 4 remain to be used.

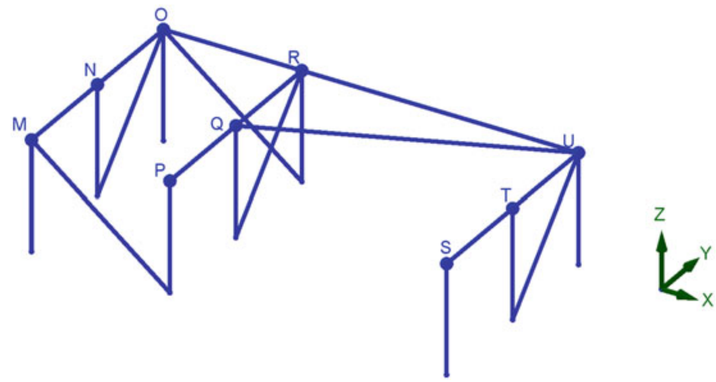


Fig. 7.24 Four DOF remain to be stabilized

The X motion of Nodes *N* and *T* can be stabilized by horizontal struts *NQ* and *QT* since Node *Q* is now stable in the X direction (as well as the Y and Z). Similarly, the X motion of Nodes *P* and *S* are restrained by tying them to Node *M* which is stable in X (as well as Y and Z). This is shown in Figs. 7.25 and 27 members have been used, the minimum needed to stabilize the nine original nodes.

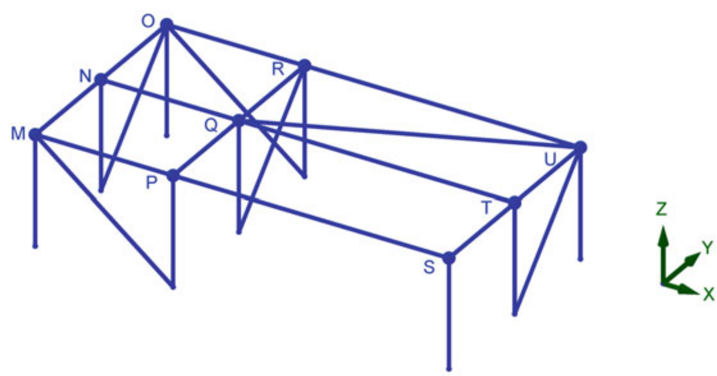


Fig. 7.25 A stable configuration using 27 struts

The diagonal from Node *M* down to the ground could be removed for architectural purposes. The X stability of Node *M* could be ensured by tying it to Node *Q* with a horizontal member in the roof line, as shown in Fig. 7.26.

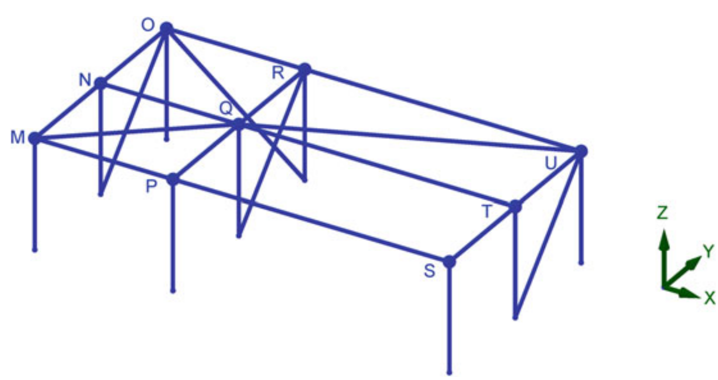


Fig. 7.26 Alternate stable configuration, one less strut goes to ground

Next, suppose that the column under Node *P* was to be removed for architectural reasons. Node *P* would be unstable in the Z direction after such a decision, this is shown in Fig. 7.27. The math also denotes an instability in Fig. 7.27, as there are 9 nodes with 3DOF each, but only 26 supporting members.

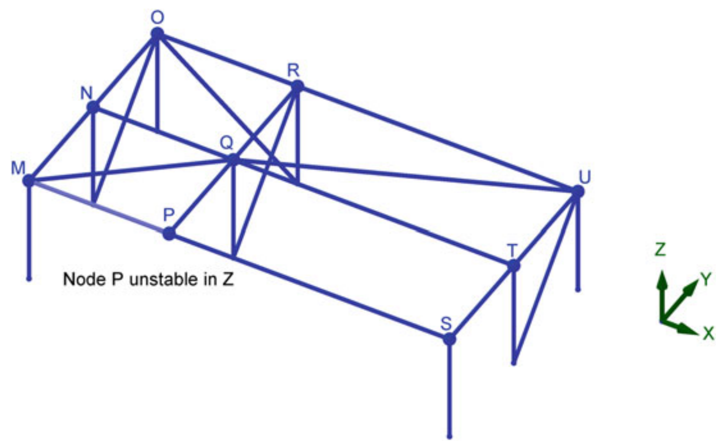


Fig. 7.27 Removing vertical column under node *P* makes it unstable in Z

One way of stabilizing Node *P* in the Z direction is to create a truss in the plane of *MPS*. This adds three members, creating a total of 29. Does this seem correct? Or is there something amiss? The situation is shown in Fig. 7.28.

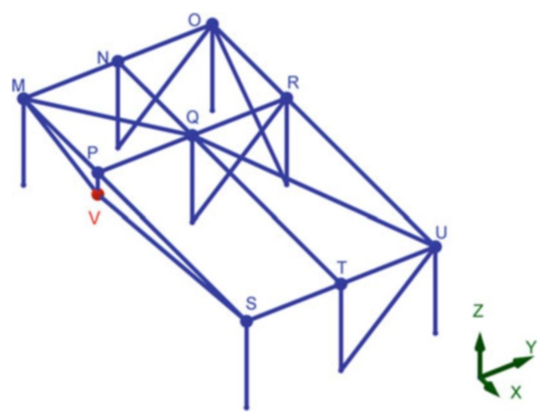


Fig. 7.28 Adding a floating column under node *P* stabilizes it in Z, but not in Y

The seeming paradox of 29 struts is resolved when it is noted that the Y movement of Node *V* is still unrestrained. The "removal" of the column under Node *P* was actually a dramatic shortening of said column, this created a new tenth node, Node *V*. Thus, 30 DOF exist, which explains the need for the 30th strut which can stabilize the Y movement of Node *V*. A solution is shown in Fig. 7.29.

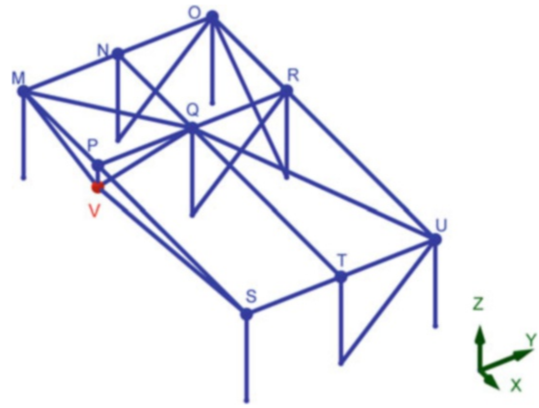


Fig. 7.29 A stable configuration but now there are 10 nodes

The following example analyzes a rigid roof supported by six pinned-pinned struts. The building is 5 units tall in the Z direction, 10 units wide in the X direction and 4 units deep in the Y direction. Lateral loads may hit the walls of the building but it is always assumed that such lateral loads are passed through the horizontal rigid roof diaphragm. Thus, three lateral loads are applied to the roof as shown in Fig. 7.30.

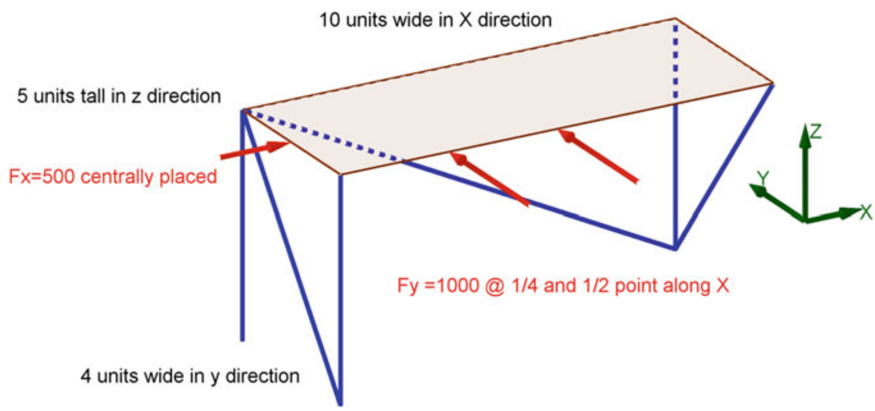


Fig. 7.30 Rigid roof subjected to lateral loads

Although this appears to be a three-dimensional problem, it can really be thought of as a two-dimensional problem for equilibrium of the roof in plan view, and then again as two-dimensional in the planes of the diagonal struts. Perhaps such a problem is best described as "*2.5 D*" as coined by Professor Nordenson of Princeton University.

Recall the assumption which allows the structure to be analyzed as a statically determinate form:

- Vertical Columns carry gravity loads but not lateral loads
- Diagonal Braces carry lateral loads but not gravity loads

Figure 7.31 shows the force polygon placed off to the side, but somewhere in the plane of the roof. The three applied loads are equivalent to a single resultant force shown as a heavier dashed vector. This net resultant of the external forces must pass through Point *Q* which is found from the intersection of the horizontal and vertical external forces. One step was skipped in Fig. 7.31, namely the location of Point Q with respect to Force BC and Force CD. That location is found from the resultant of those two forces, and it is fully described in Fig. 7.32.

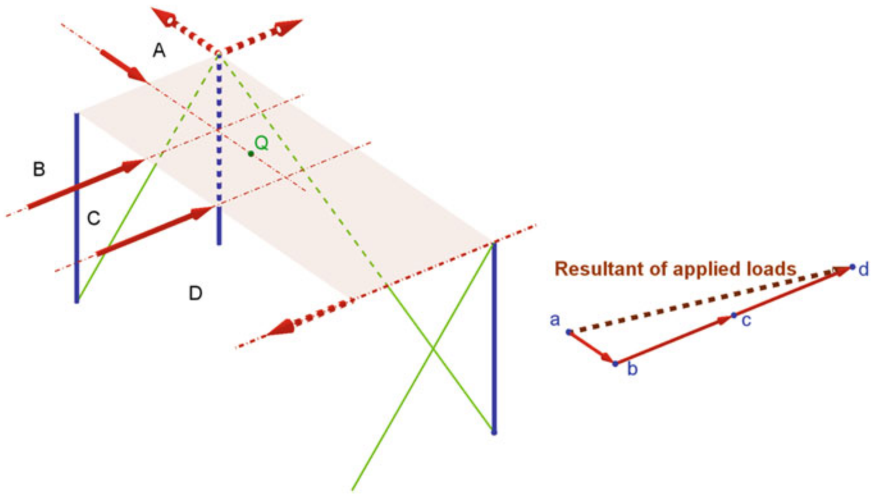
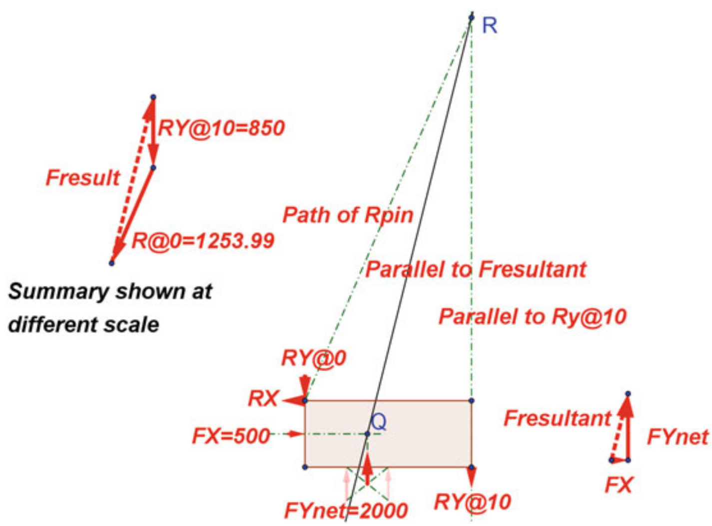


Fig. 7.31 First find the resultant of the applied loads, then pass that through Q

Figure 7.32 shows the same situation but solely in plan view, thus the rigid roof appears to be a 2D free body. Note here that the Inverse Axis Method was used to quickly establish the centroid of the two 1000unit loads, clearly symmetry could have also been used. Figure 7.32 proves that such problems can be viewed as equilibrium of a point. No funicular is needed to establish the reactions, since the resultant of all the loads, *Fresultant*, must intersect the path of the "roller" *RY@10*. They intersect at Point R. Nothing is yet known about the magnitude or the direction of the "pin" at *RY@0* and *RX*, but the net resultant vector must pass through this point. Thus, connecting Point R to the point of the "pin" at *RY@0* and *RX* determines the slope of the net "pin" reaction, *R@0*.

Fig. 7.32 Plan view of roof which shows solution of equilibrating reactions



The construction of Fig. 7.32 is the quickest means of obtaining the net reactions. However, a trial funicular can also be used to find the reactions. The answer obtained is identical to the one found in Fig. 7.32, but for the sake of familiarity with previous exercises, Fig. 7.33 shows how the reactions can be found from a trial funicular solely from a 2D perspective. Figure 7.33 may appear a bit complicated, but it is essentially the same as Fig. 6.25. The only difference is that in Fig. 7.33, three loads are applied. Thus, the centroid of the two Y direction loads is calculated, and the intersection of this centroidal load in Y and the lateral load in X is located. This is Point *Q*. A line parallel to the overall net force, shown as a dashed heavy line, must pass through *Q* as it did in Fig. 6.25. A trial funicular begins at the pinned support and ray *ap* spans from that "pinned" support to net reaction line passing through Point *Q*. Ray *dp* of the trial funicular terminates at the path of the Y force for the "roller" support. It is the perpendicular distance that is key, much like the algebraic moment arm. The closing line of the trial funicular, shown as a dashed yellow line is transferred to the force diagram. This allows for the location of Point *e*, which unlocks all of the net reactions of the rigid roof.

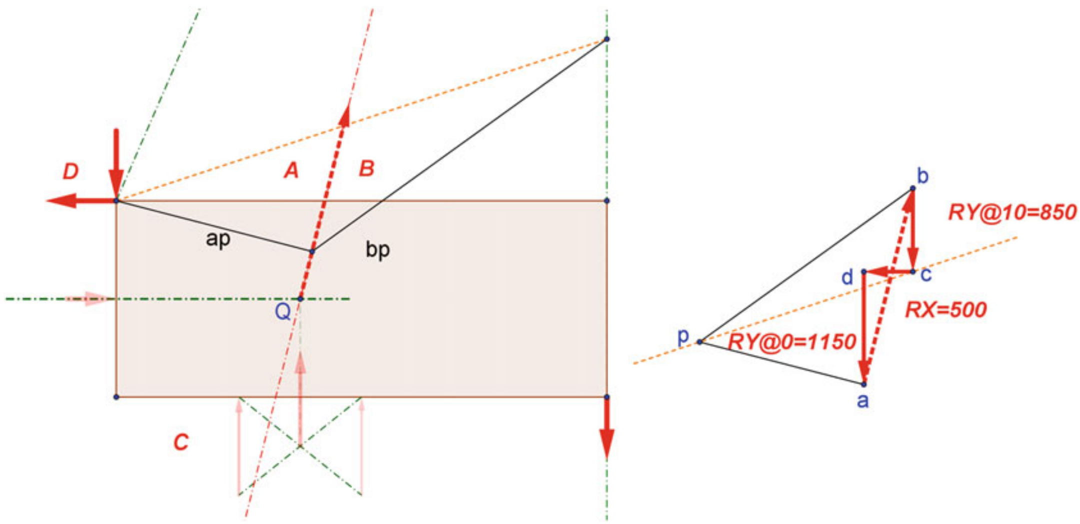


Fig. 7.33 Previous answers replicated via funicular

Figure 7.34 shows the exact same technique, but the construction is done on a horizontal plane within the *3D* diagram. Intermediate letters are placed on the form diagram to show another way of interpreting the force polygon.

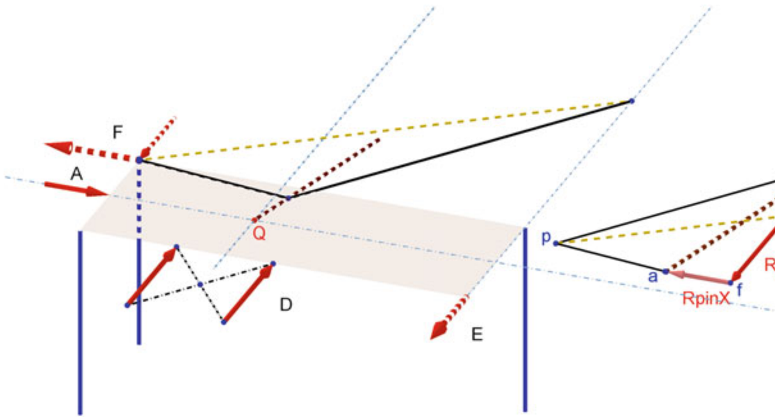


Fig. 7.34 Same funicular technique as before, but shown on horizontal plane within 3D image

Locating Point *e* on the force diagram was the key to solving for all of the net "reactions" of the rigid roof in *2D*. Each of these reactions, shown as dashed vectors in Fig. 7.35 are easily decomposed into the three strut forces as shown. For example, *Rypin* is simply the Y component of the *Strut3 Force*, *Rxpin* is the X component of *Rxpin*. These can be drawn in *2D* planes adjacent to the original structure for clarity of thought and ease of calculations (Figs. 7.36 and 7.37).

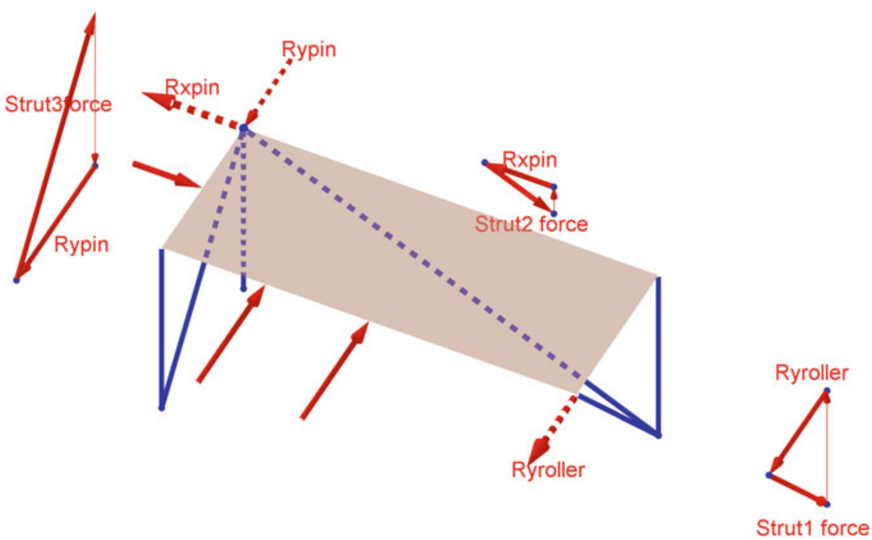


Fig. 7.35 Reactions are immediately decomposed into strut forces in the plane of the struts

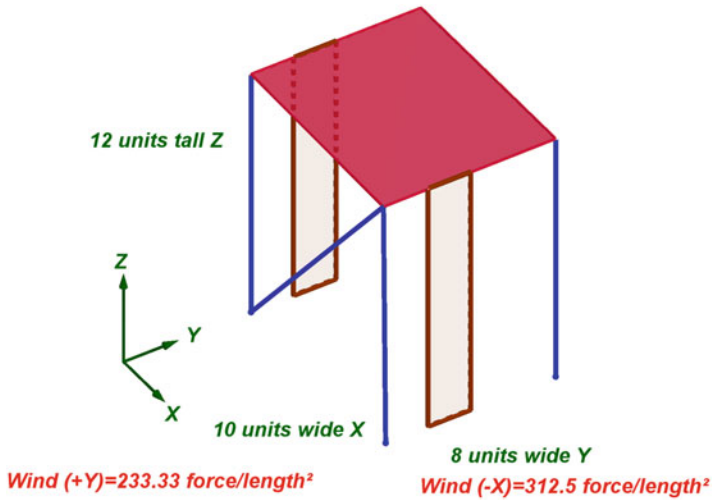


Fig. 7.36 Wind load hits two sides of a building

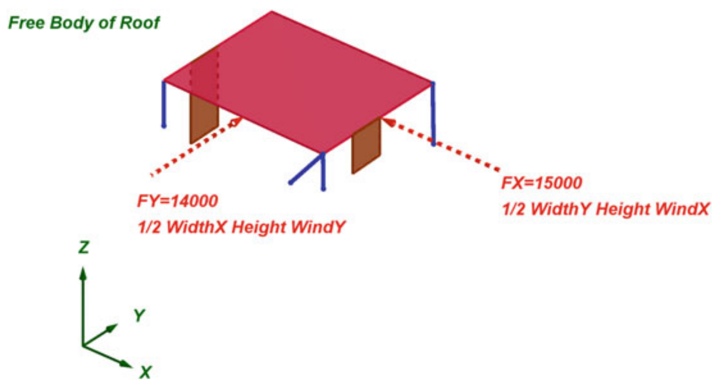


Fig. 7.37 Free body of rigid roof, half of the wind goes to foundation

The pinned-pinned vertical columns have no lateral load carrying capabilities. The shear walls are assumed to have no stiffness out-of-plane. Thus, the two shear walls each provide a lateral restraining force solely in the Y direction. The diagonal brace provides a restraining force in the X direction (Fig. 7.38).

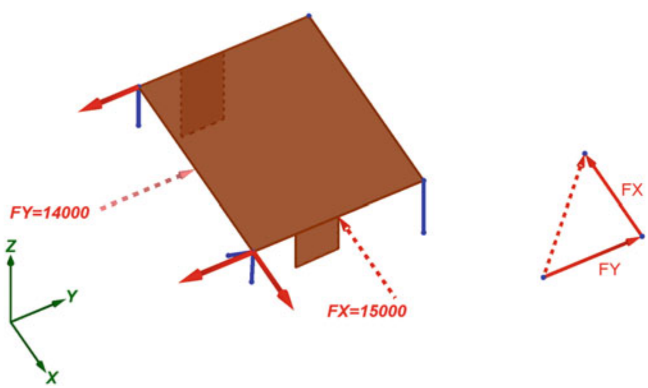


Fig. 7.38 External wind loads are combined into net resultant load

As was done in Fig. 6.25 and again in Fig. 7.32, Point Q in the plane of the roof is established which facilitates analyzing the roof as a problem of equilibrium of a single point, Point Q. Point Q must be in line with the overall applied forces FX and FY, and Point Q must also lie on the path of $Fnet$. Figure 7.39 shows a free body diagram of the rigid roof, with the supporting elements grayed out to demonstrate that there are three stabilizing reactions, two are aligned with the shear walls in the Y direction and one is aligned with the diagonal brace in the X direction.

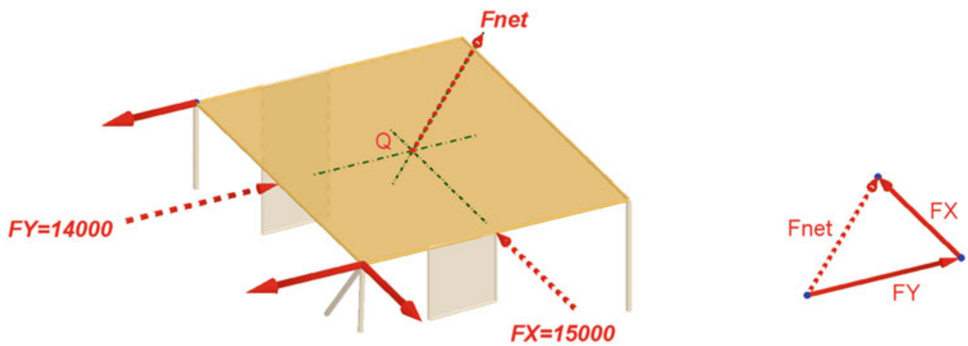


Fig. 7.39 Free body of rigid roof and net wind load

Figure 7.40 solves the rigid roof as a study of equilibrium of Point Q. The path of the net external forces $Fnet$ intersects the path of the single "roller" support, this intersection occurs at Point P. Then the path from Point P to the "pinned" support is found and is transferred to the force diagram. This locates the critical Point d. From b to c in the force diagram the path must be pure X, the path from d to c must be pure Y, thus Point c is located and all of the external reactions are now known.

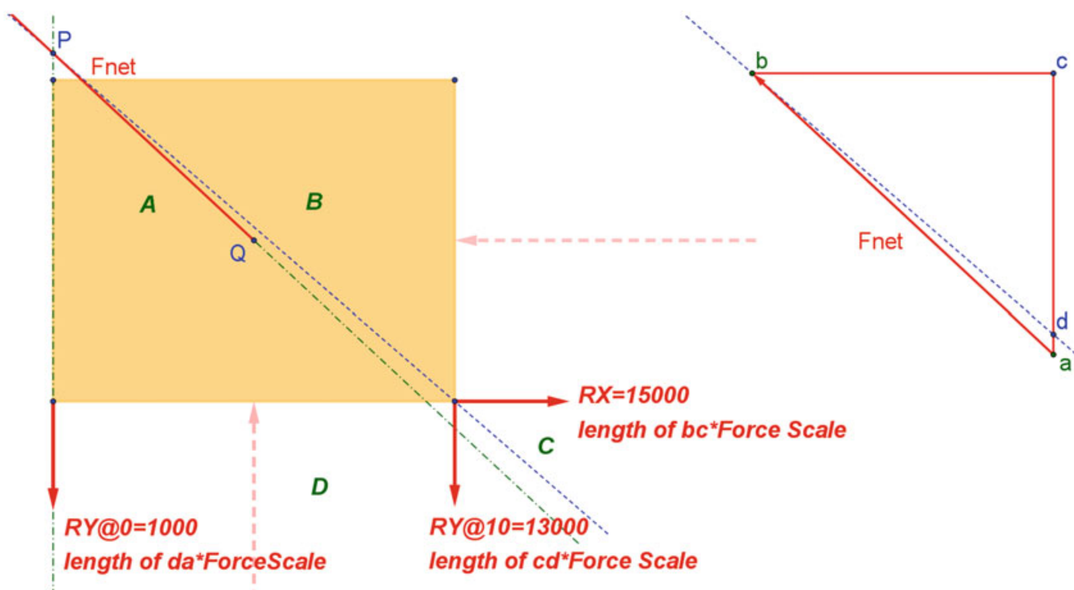


Fig. 7.40 Reactions found via equilibrium of point Q

Figure 7.41 demonstrates that the previous "equilibrium of a point" method of Fig. 7.40 for finding reactions is equivalent to doing so via a trial funicular. As in Fig. 6.27, the funicular must begin at the "pinned support" which is the intersection of the two lateral load-resisting paths, namely the path of the diagonal brace and the path of the plane of one shear wall. The reason for this is that neither the magnitude nor the direction of the net reaction resultant at the "pinned" support is known, all that is known is that this reaction must pass through this point. Of course, no such pin exists, the location is determined from the intersection of the path of one shear wall and the path of the diagonal brace. This intersection occurs at one corner of the roof and it is equivalent to a pinned support. The funicular extends from this "pinned support" until it intersects the path of the resultant load Fnet. This is ray *bp*. From there, ray *ap* is laid down till it intersects the path of the other shear wall. A trial closing line from the form diagram, shown as an orange dashed line, is transferred to the force diagram through the pole *p*. The intersection of the trial closing line and a line parallel to the shear wall force *AB* (here a vertical line through a on the force diagram) locates point *d*. This is precisely the same location of Point *d* that was established by the equilibrium of a point method shown in Fig. 7.40.

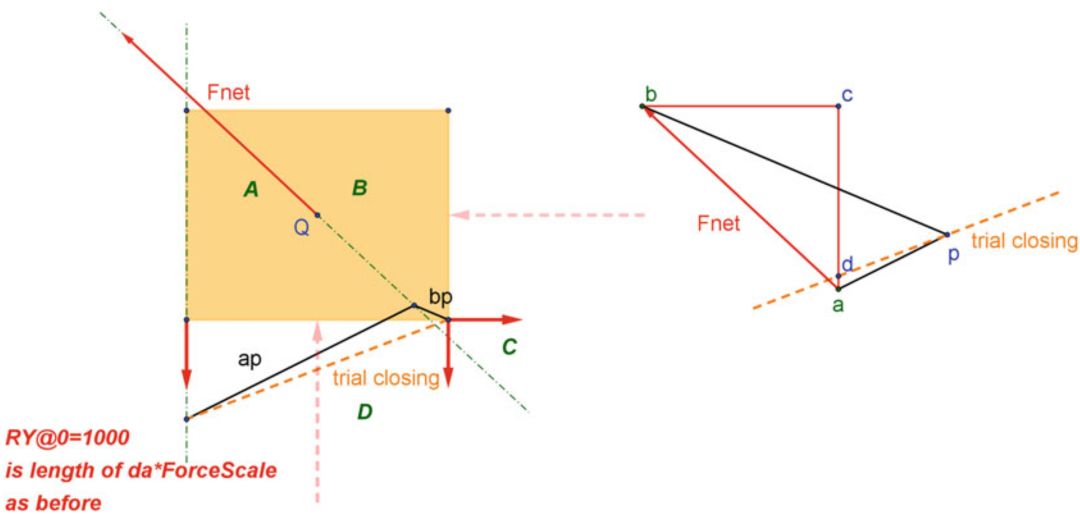


Fig. 7.41 Equilibrium solution repeated via funicular

The creation of 2D plane parallel to the diagonal brace immediately solves for the brace force as shown in Fig. 7.42.

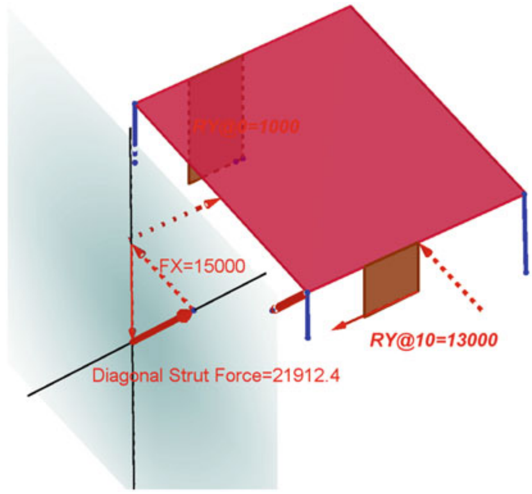


Fig. 7.42 Reactions decomposed into strut forces in the plane of the struts

It was noted several times that:

- Vertical Columns carry gravity loads but not lateral loads
- Diagonal Braces carry lateral loads but not gravity loads

This final example shows both calculations performed graphically.

In Fig. 7.43, a rigid roof is supported by three columns which carry only gravity loads, and three diagonal braces which carry only lateral loads. The roof is uniformly loaded and the lateral load in either direction is 25% of the gravity load. Calculate the force in each of the six supporting struts.

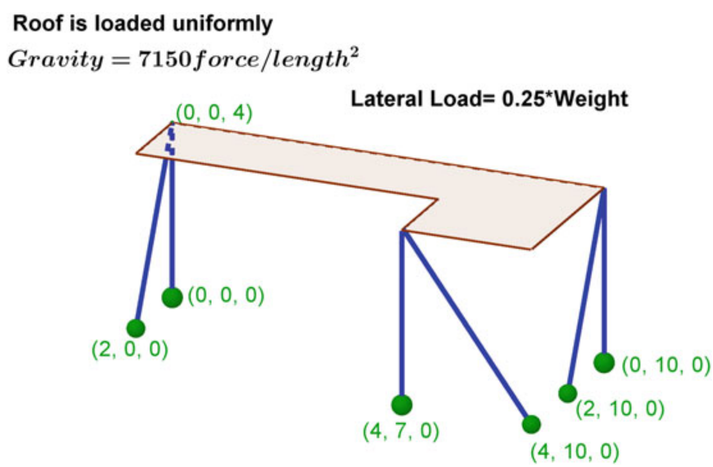


Fig. 7.43 Rigid roof subjected to gravity load and to two earthquake loads

No funiculars will be used to calculate the lateral reaction. Rather, the lateral loads will be combined into one resultant, than the restraining forces will be found graphically, assuming one lateral force resisting system acts as a "roller" and the other two act as a "pin".

Arbitrarily choose to study the lateral first. This requires the calculation of the weight of the roof, then at the roof's centroid, two lateral loads are applied. One load represents an East/West earthquake and the other represents a North/South earthquake (Fig. 7.44).

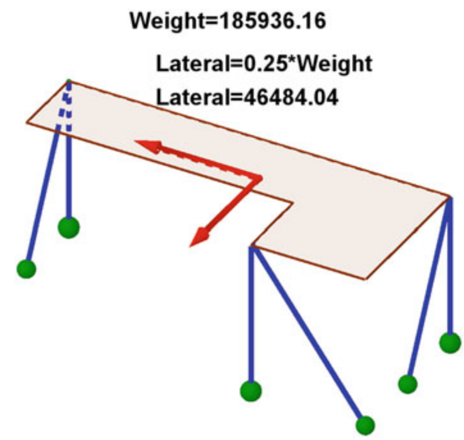


Fig. 7.44 Earthquake loads are applied at the center of mass of the rigid roof

On the thinner end of the building, assume that the single brace acts as a "roller" in the North/South direction. At the opposite corner, assume both diagonals converge to act as a "pin". From the centroid, pass the resultant force till it intersects with the path of the "roller". Call this Point *R*. From Point *R*, draw a trajectory to the "pin". This establishes the "pin's" direction of force, and that force can be quickly broken up into a N/S and E/W component. This is shown in Fig. 7.45.

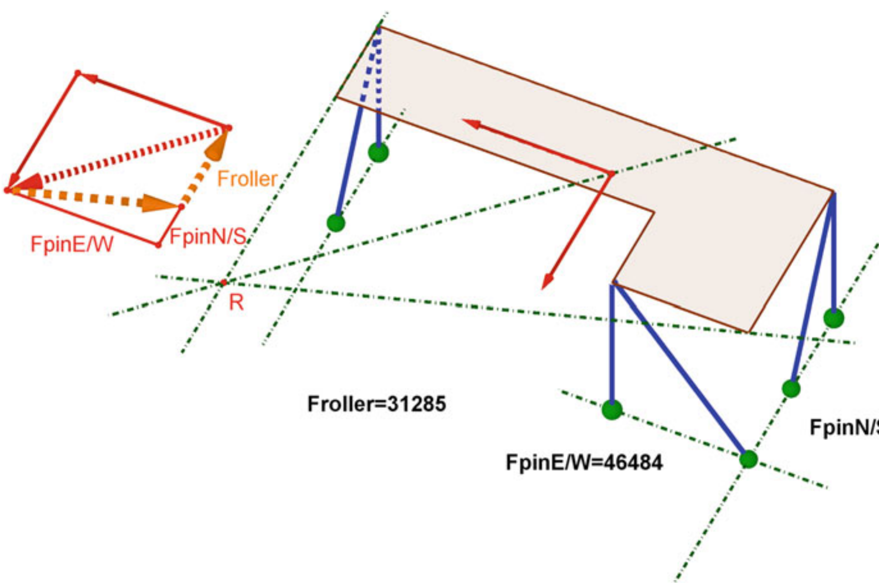


Fig. 7.45 Reactions are found via equilibrium of point *R*

Figure 7.46 shows an alternate view of the same statics analysis as was shown in Fig. 7.45. Note how the path of one arbitrarily chosen brace is assumed to act as the "roller" and how the paths of the other two braces intersect at a point which acts as the "pin". It is the second point, the "pin" that establishes the direction from *R* which is in the plane of the roof, to a point directly above the "pin", call that Point *S*.

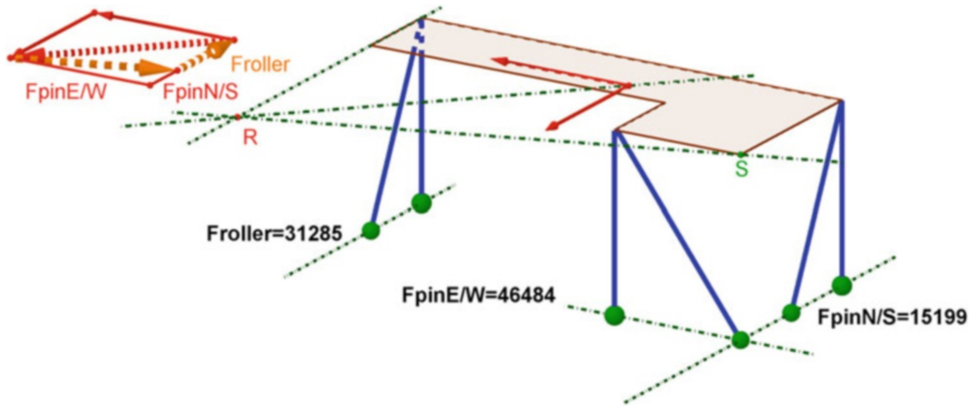


Fig. 7.46 Previous solution shown in alternate view

Each reaction here, Froller, FpinE/W and FpinN/S are simply the horizontal components of the diagonal braces. The calculation of the brace forces themselves is extremely rapid, the brace is the hypotenuse of the triangle and the reaction is the horizontal leg. This is shown in Fig. 7.47.

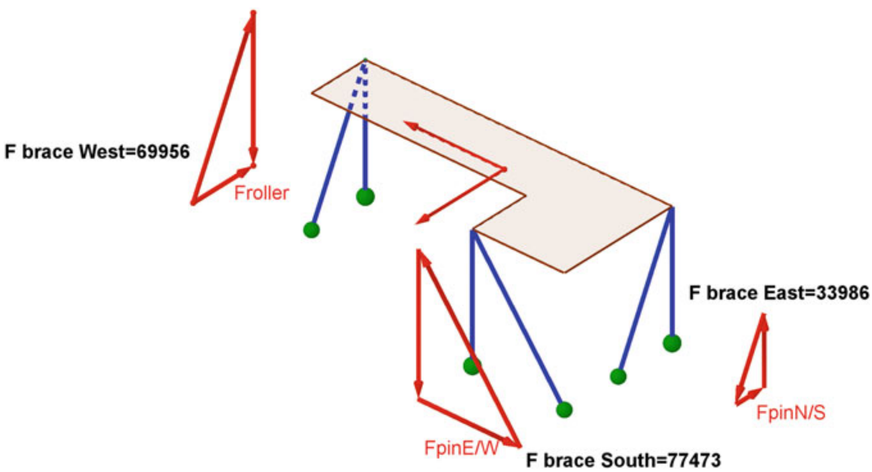


Fig. 7.47 Reactions decomposed into strut forces in the planes of the struts

The gravity load calculation is extremely efficiently performed if a single moment equation is taken about an axis going through two of the columns. Recall that this uses the assumption of vertical columns carrying gravity load, not the inclined struts. Furthermore, the centroid of the roof is immediately found from the graphical parametric drawing program used. Figure 7.48 shows the first calculation which is done algebraically.

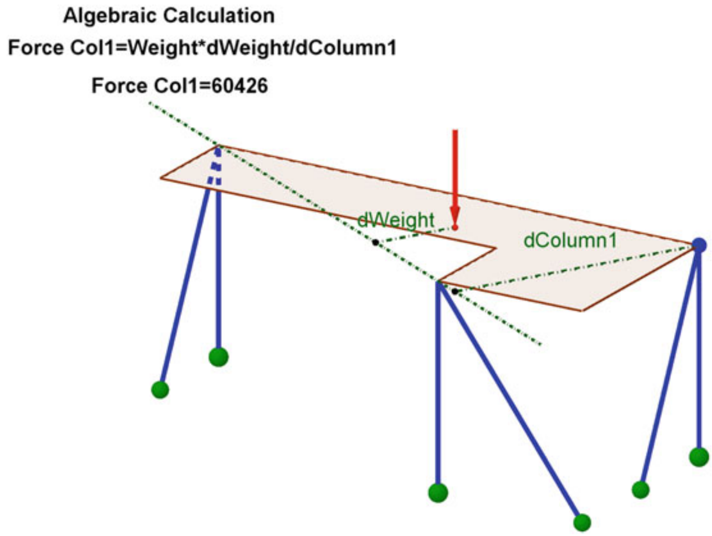


Fig. 7.48 Force in one vertical column found algebraically

The same result can be found graphically, using the Inverse-Inverse Axis Method. Figure 7.49 shows this technique.

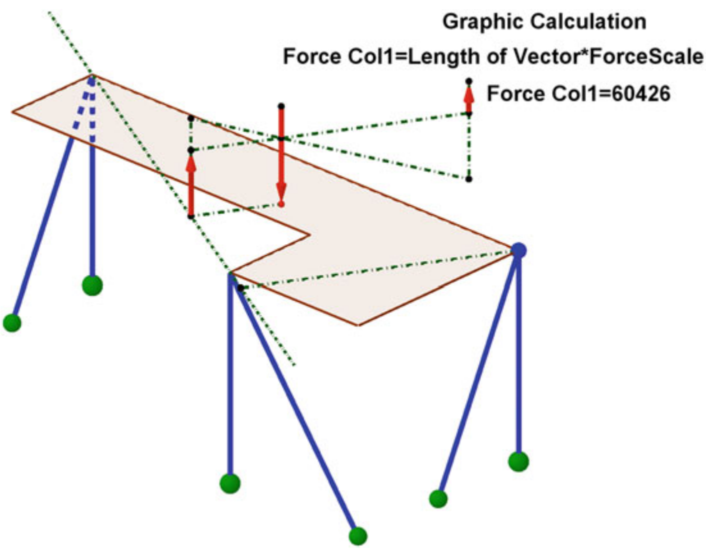
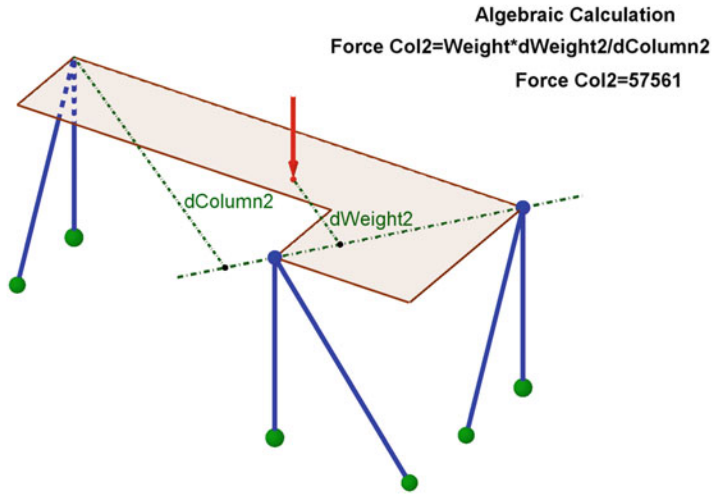


Fig. 7.49 Graphical recreation of previous vertical column load calculation from gravity

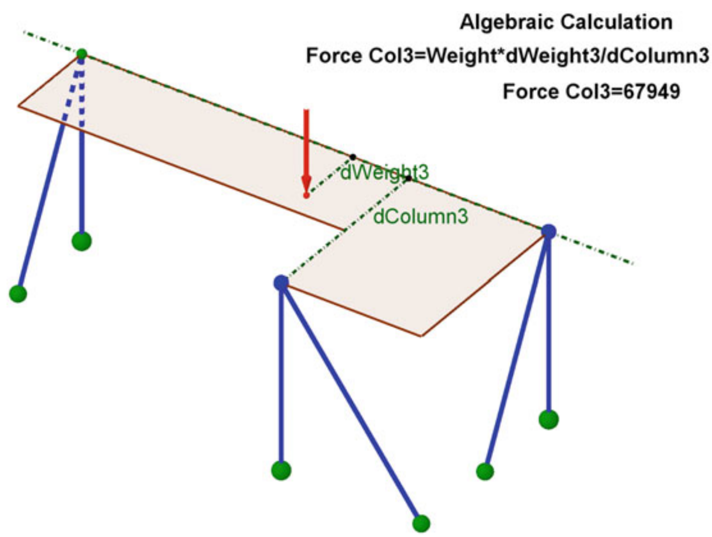
Analogous calculations are made for the other two vertical columns. For example, Figure 7.50 shows the calculation for the second vertical column. Recall that the diagonal braces are assumed to not carry gravity load.

Fig. 7.50 Algebraic calculation for next vertical column force due to gravity



The third vertical column calculations are shown in Fig. 7.51. Of course, the three vertical column forces add up to the total weight of the roof.

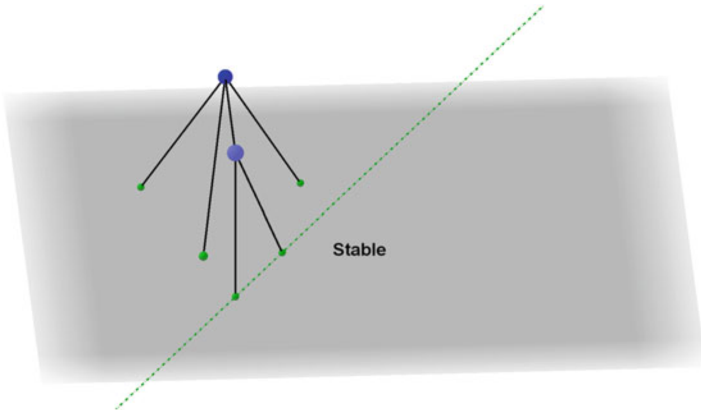
Fig. 7.51 Algebraic calculation for third vertical column force due to gravity



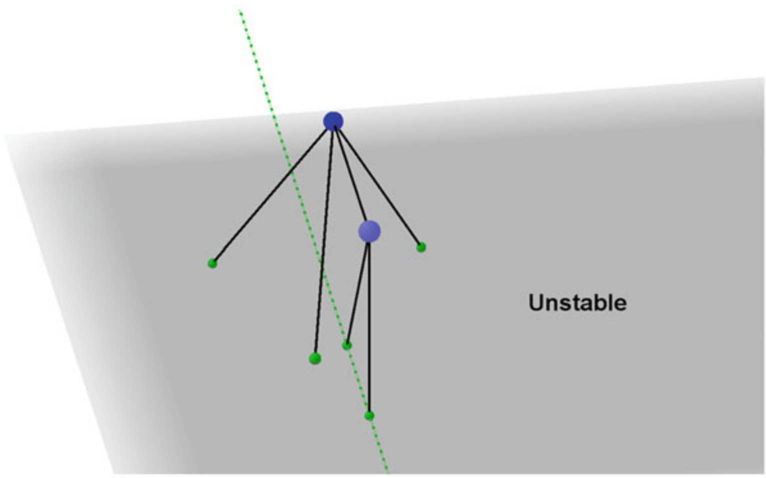
Chapter 7 Exercises

Exercise 7.1 Two nodes are at some known locations in 3D space. Design a stable arrangement, with the minimum number of struts needed for this problem, and which connects the two nodes as in a roof. Then design a similar, but unstable arrangement for the minimum number of struts needed for this problem.

Exercise 7.1 solution part 1

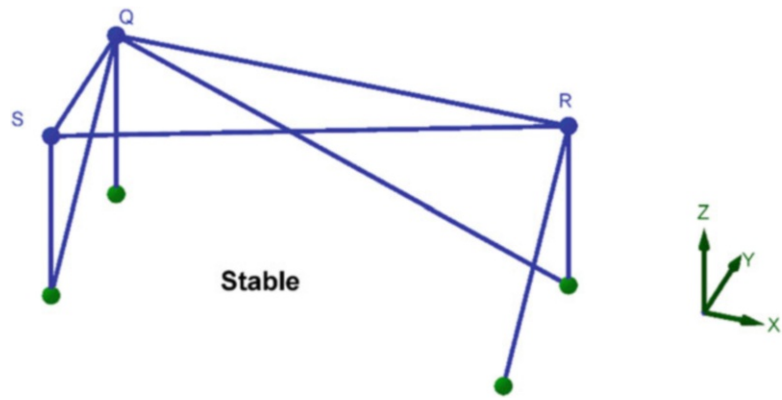


Exercise 7.1 solution part 2

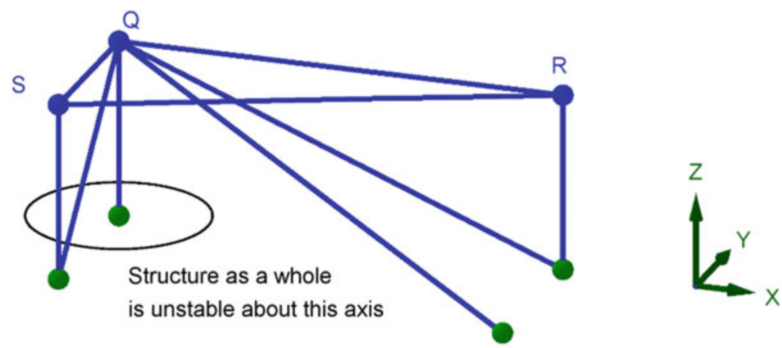


Exercise 7.2 Three nodes are at some known locations in 3D space. Design a stable arrangement, with the minimum number of struts needed for this problem, and which connects the three nodes as in a roof. Then design a similar, but unstable arrangement for the minimum number of struts needed for this problem.

Exercise 7.2 solution part 1

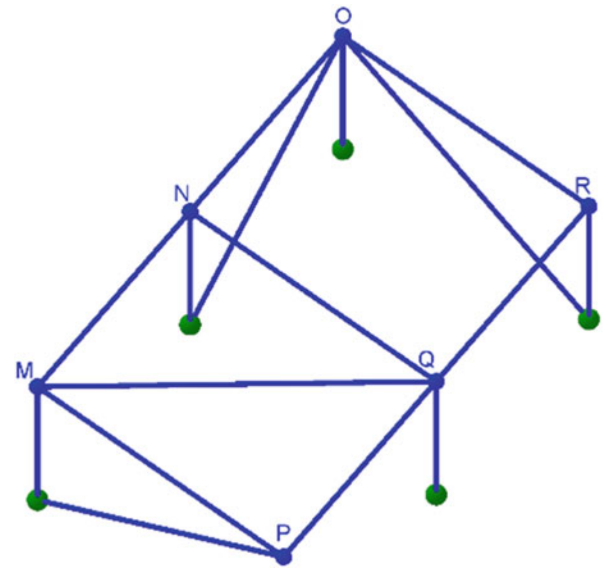


Exercise 7.2 solution part 2

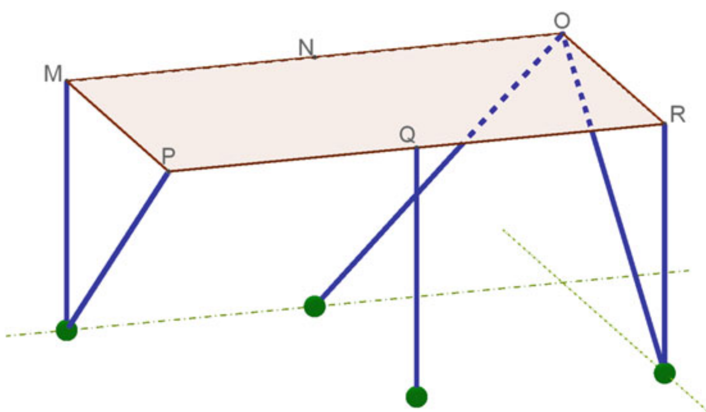


Exercise 7.3 Six nodes are at some known locations in 3D space. Design a stable arrangement, with the minimum number of struts needed for this problem, and which connects the six nodes as in a roof. Then replace the roof elements with a single rigid slab and support it with three vertical load carrying columns and three lateral-only load carrying diagonal members.

Exercise 7.3 solution part 1

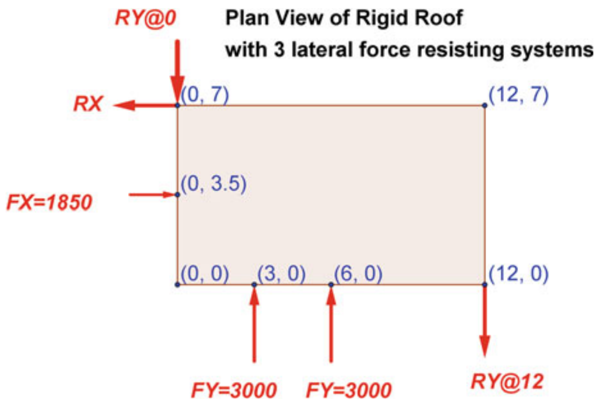


Exercise 7.3 solution part 2

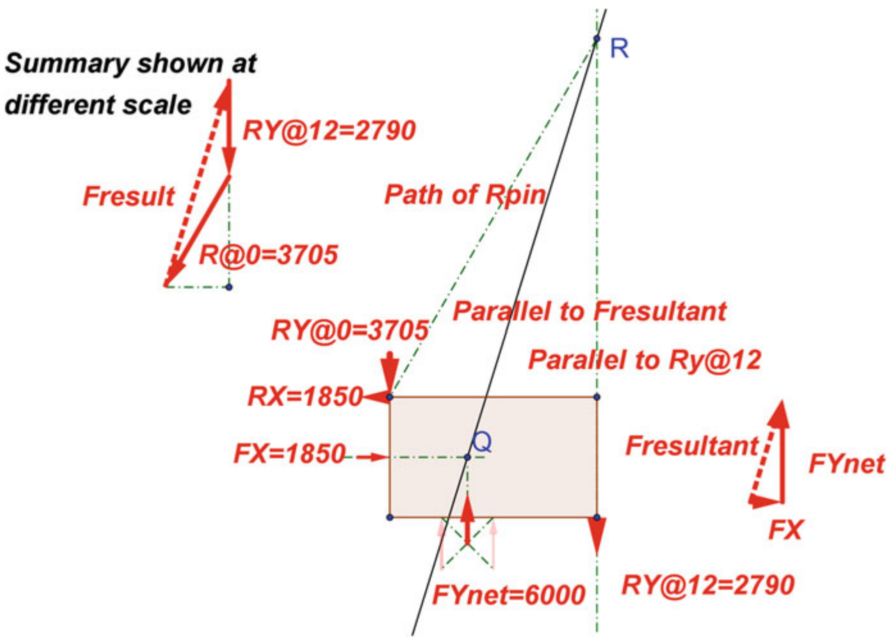


Exercise 7.4 A rigid roof with three lateral force resisting systems is shown in plan view. Calculate the horizontal equilibrating reactions in each of the three supports.

Exercise 7.4

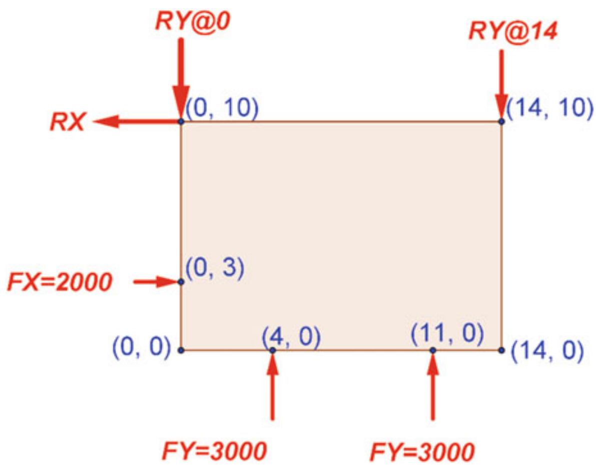


Exercise 7.4 solution

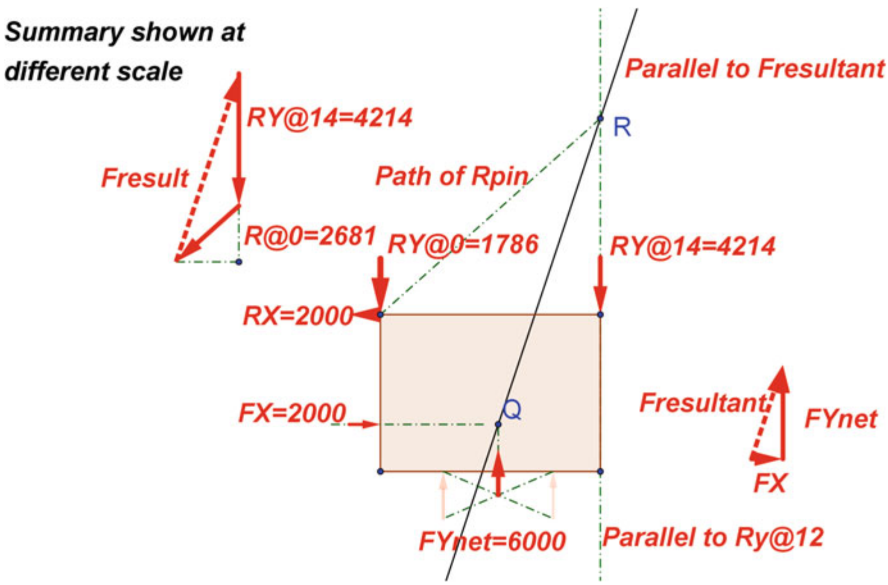


Exercise 7.5 A rigid roof with three lateral force resisting systems is shown in plan view. Calculate the horizontal equilibrating reactions in each of the three supports.

Exercise 7.5

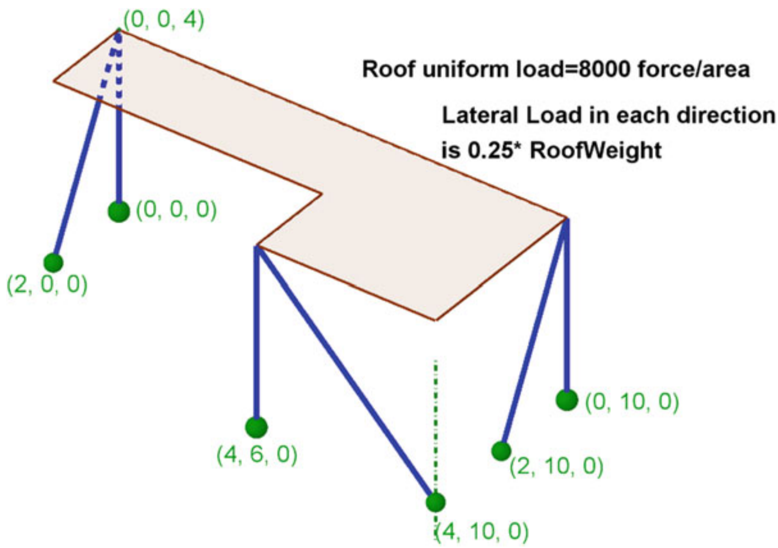


Exercise 7.5 solution

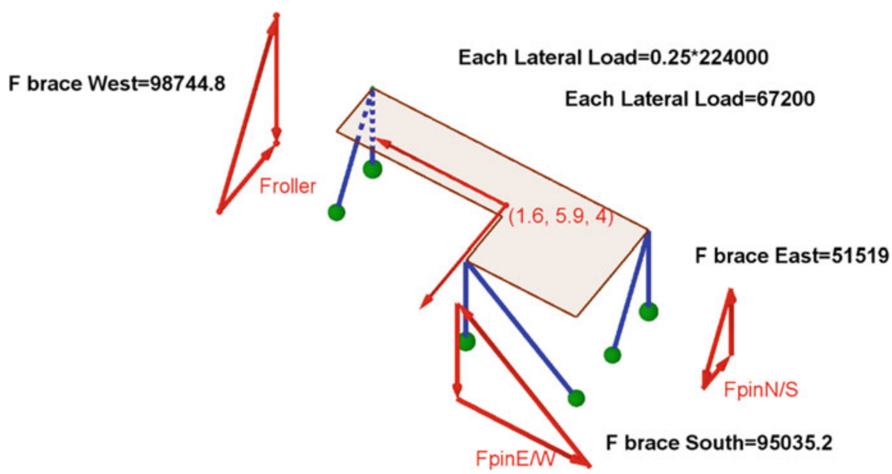


Exercise 7.6 A rigid roof is subjected to a uniform dead load of 8000 force/area. If 25% of the dead load is applied laterally in each orthogonal direction, calculate the equilibrating forces in each of the three braces.

Exercise 7.6

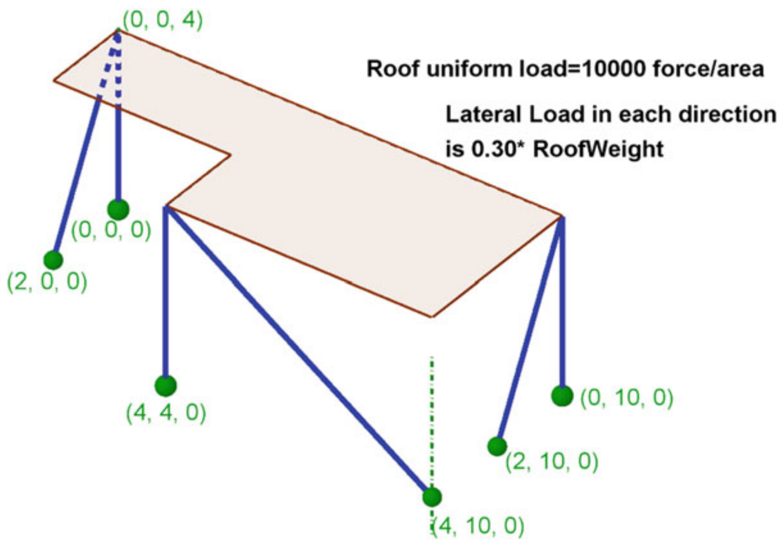


Exercise 7.6 solution

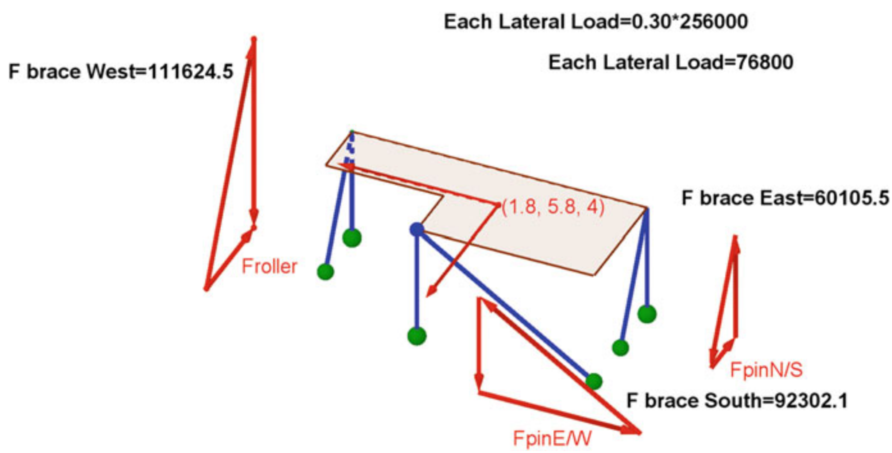


Exercise 7.7 A rigid roof is subjected to a uniform dead load of 10,000 force/area. If 30% of the dead load is applied laterally in each orthogonal direction, calculate the equilibrating forces in each of the three braces.

Exercise 7.7



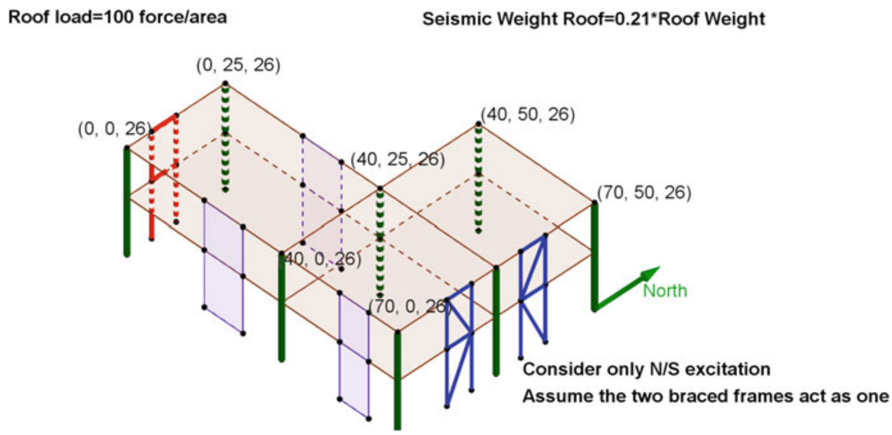
Exercise 7.7 solution



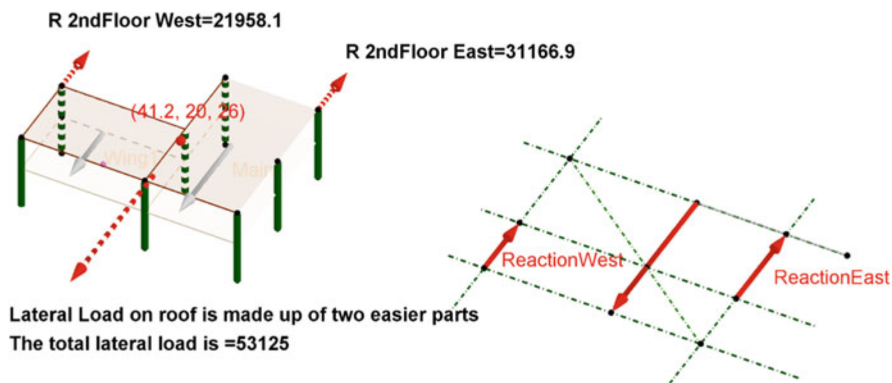
Exercise 7.8 Consider only North/South excitation. Assume that the rigid roof has a dead load of 100 force/area. Assume that the earthquake load in the plane of roof diaphragm is 0.21*RoofWeight. Assume that the two braced frames on the East end act as one lateral force resisting system. Ignore any out of plane forces of the shear walls.

Equilibrate the roof using the Inverse-Inverse Axis method and show the forces acting at the top of the two N/S lateral force resisting systems.

Exercise 7.8



Exercise 7.8 solution



8 Deflections of Beams and Indeterminate Beam Analysis

The theory of the deflection of beams is well known, but little published information exists on the deflections of beams using graphical analysis. Calculating the deflections of the elastic curve of a beam is a necessary step in the analysis of indeterminate beams because the movement or the rotation at a redundant support must be calculated and eventually be set to zero through manipulation of the pole of the force diagram.

The *deflection* of a horizontal beam at any section is the vertical movement of the elastic curve tracing the beam's *neutral axis*. If the deformed elastic curve is plotted in x-y space, the deflection is the ordinate of the curve (y value) and the cut at the beam's cross section being investigated is the abscissa (x value). The curvature of segments of this elastic curve is based on the assumption that movement is small, thus circular segments can adequately model the arc lengths of the deformed elastic curve. The relationship between internal bending moment and deflection of the elastic curve is well known. It is a second order differential equation. This equation relates the bending moment as a function of x, $M(x)$, to the deflection y. The equation captures the modulus of elasticity E which is usually not a function of x, and thus is a constant, as well as the moment of inertia $I(x)$ which may or may not be a function of x. That relationship is:

$$\frac{d^2y}{dx^2} = \frac{M(x)}{E \cdot I(x)}$$

It is convenient to complete the following steps for a prismatic, i.e. uniform cross section beam, thus the moment of inertia I is held constant.

$$\frac{d^2y}{dx^2} = \frac{M(x)}{E \cdot I}$$

Furthermore, it is well known that taking the derivative of both sides of the above expression provides the shear as a function of x, $V(x)$.

$$\frac{d^3y}{dx^3} = \frac{\frac{dM(x)}{dx}}{E \cdot I} = \frac{V(x)}{E \cdot I}$$

One more derivative with respect to x gives an expression for the load as a function of x. There is a sign switch because loads are typically downward in architectural engineering applications.

E. Saliklis, *Structures: A Geometric Approach*, https://doi.org/10.1007/978-3-319-98746-0_8

$$\frac{d^4y}{dx^4} = \frac{\frac{dV(x)}{dx}}{E \cdot I} = \frac{-w(x)}{E \cdot I}$$

Note that integrating the load *w(x)* twice provides the moment $M(x)$. This is precisely what has been done in all graphical analysis work, namely, creating a force polygon, transferring the rays of the force polygon to the form diagram, and adjusting the closing line is completely the same as integrating twice and applying appropriate boundary conditions. That insight leads to the following key idea, namely that "loading" the beam with the M/EI diagram, creating a "force diagram" based on this "load" and creating a funicular based on those rays leads to another double integration. The insight is as follows:

1. Create the moment diagram graphically
2. Discretize the moment diagram and apply these as discrete "loads" to the beam and analyze graphically
3. The result of step 2 will be the deformation of the elastic curve (Fig. 8.1).

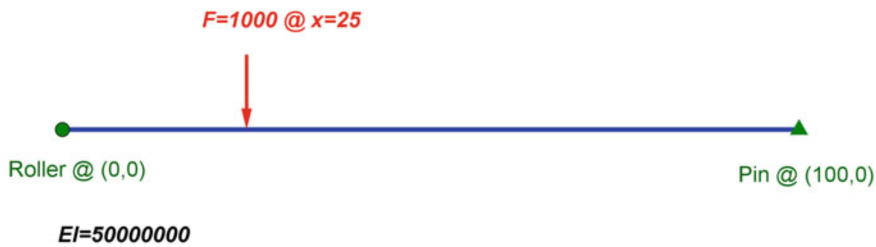


Fig. 8.1 Qualitative example of simply supported beam deflection initial steps

Here the moment diagram will be generated through the force diagram (Figs. 8.2 and 8.3).

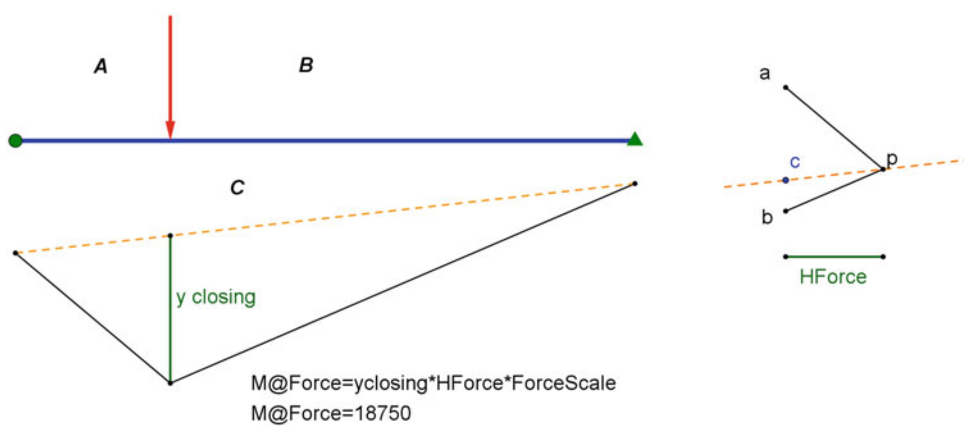


Fig. 8.2 Generate moment diagram from funicular

Now discretize the moment diagram, noting the centroid of each discretization.

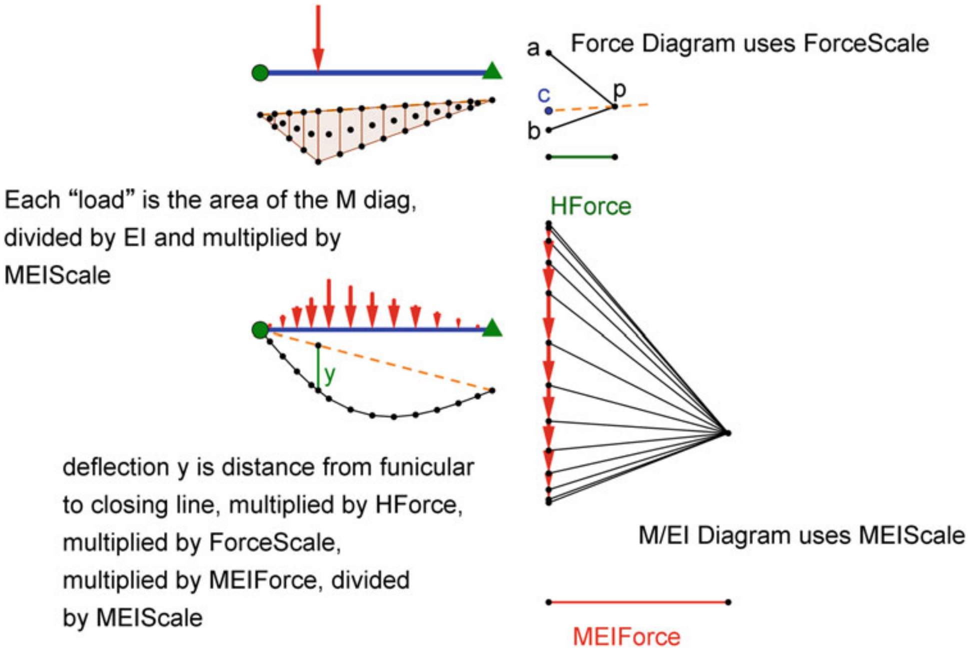
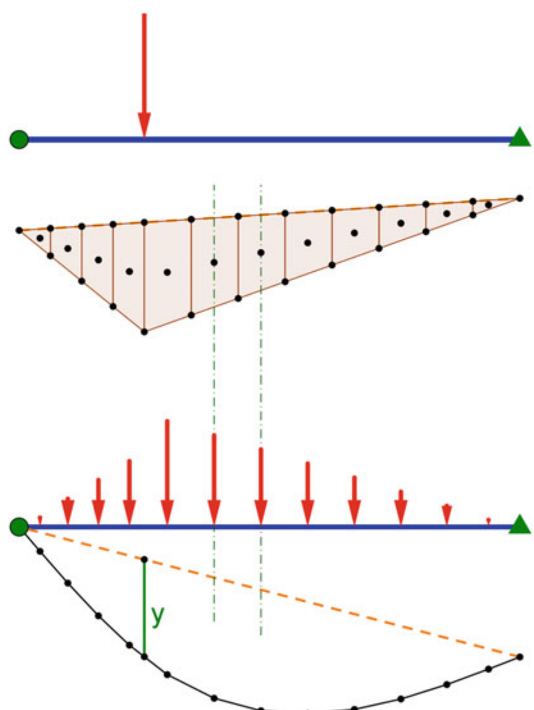


Fig. 8.3 Moment diagram discretized

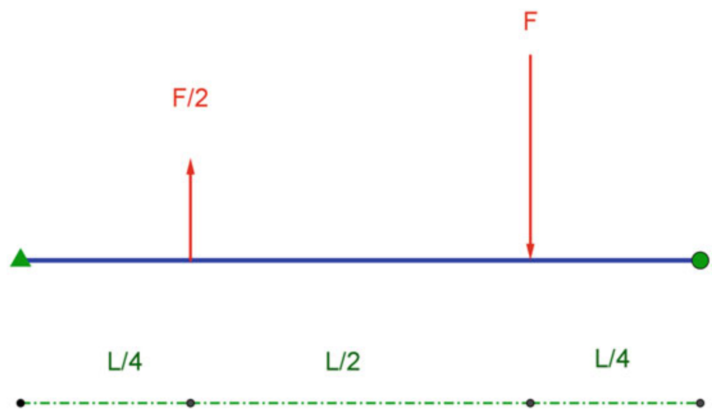
Figure 8.4 shows a closeup detail of Fig. 8.3, to highlight the placement of the *M*/*EI* "loads" on the beam. Each "load" must be placed at the centroid of each polygon of the *M*/*EI* diagram. Fortunately, graphical programming environments such as GeoGebra and Grasshopper can automatically calculate the areas and the centroids of polygons.

Fig. 8.4 Detail of previous figure showing centroids of each piece



The following example addresses the issue of positive and negative curvature of a continuous, prismatic (i.e. moment of inertia I does not change) beam. The beam shown in Fig. 8.5 is simply supported, length L, and is subject to an upward load of $F/2$ at the quarter point and a downward load of F at the three-quarter point.

Fig. 8.5 Detailed example of beam deflection



Using Bow's notation on the form diagram allows for the creation of a force diagram of width $HForce$ (Fig. 8.6). The bending moment in any cross section of the beam is readily found by:

$$M = height_{to\ closing} \cdot HForce \cdot ForceScale$$

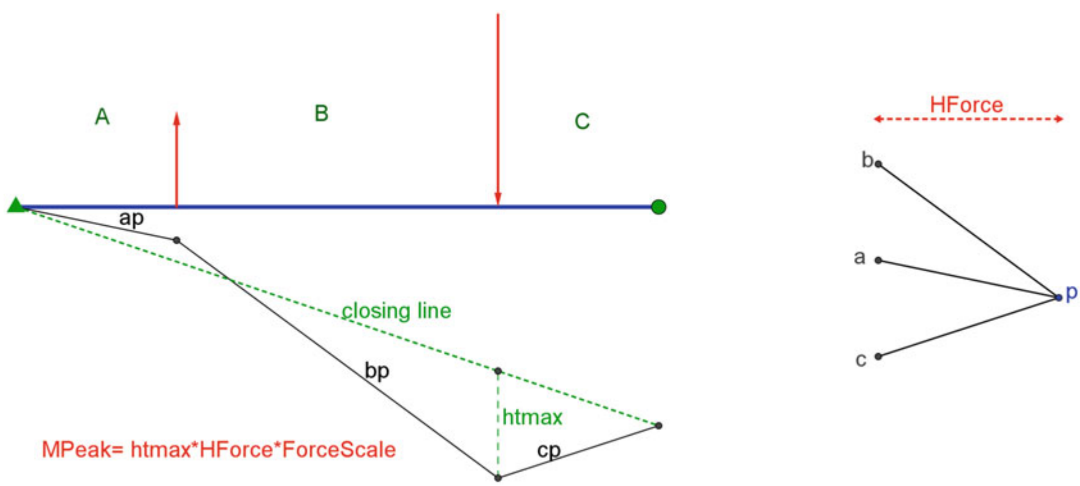
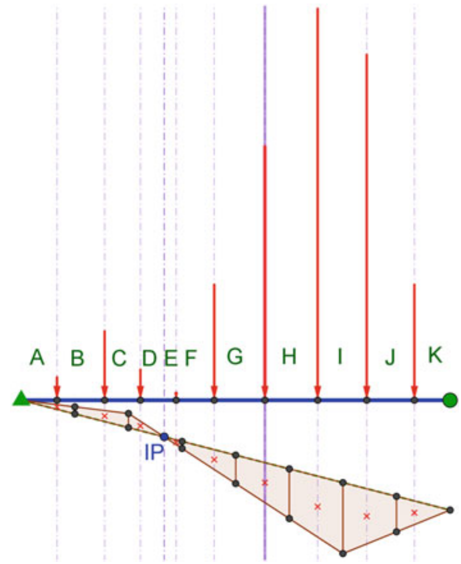


Fig. 8.6 Evaluating the bending moment at any cross section

The bending moment is now discretized into a number of shapes. More point loads will give a better approximation of a parabolic elastic curve. Eight to ten elements should be sufficient. Each element is the area of the bending moment sub-polygon shape, divided by EI for that portion of the beam. If the moment of inertia I varies, its influence appears in the denominator of each "load" generated by the moment diagram. Since EI is a large number, each "load" must be scaled by an arbitrary scale factor known as $MEIScale$. Bow's notation can then be applied to the form diagram which shows the "loads" applied at the centroid of each polygon. Fortunately, graphical programming languages immediately can calculate the area and the centroid of polygons (Fig. 8.7).

Fig. 8.7 Area and centroid of each segment graphically calculated



An area of positive moment adjacent to an area of negative moment will induce concave up curvature and concave down curvature, meeting at the inflection point IP. At the inflection point, the slope of a continuous beam is the same on either side of the inflection point. This is a very important idea as it lays out the premise for the construction of the subsequent "load" diagram.

Fig. 8.8 Slope is exactly the same on either side of an inflection point

At the inflection point (*IP*) the slope is the same to the immediate left and to the immediate right of the *IP* as shown in Fig. 8.8. A straight line is passed through one pole ($p1$) through the inflection (*IP*) and on the other side of *IP*, the second pole ($p2$) is located. The second pole $p2$ must be the same horizontal distance $MEIForce$ from the load line that the first pole $p1$ is, otherwise the scaling of the problem will be incorrect. But the magnitude of $MEIForce$ itself is inconsequential. Figure 8.9 shows the beginning of the construction of the "loads" that are based on M/EI segments.

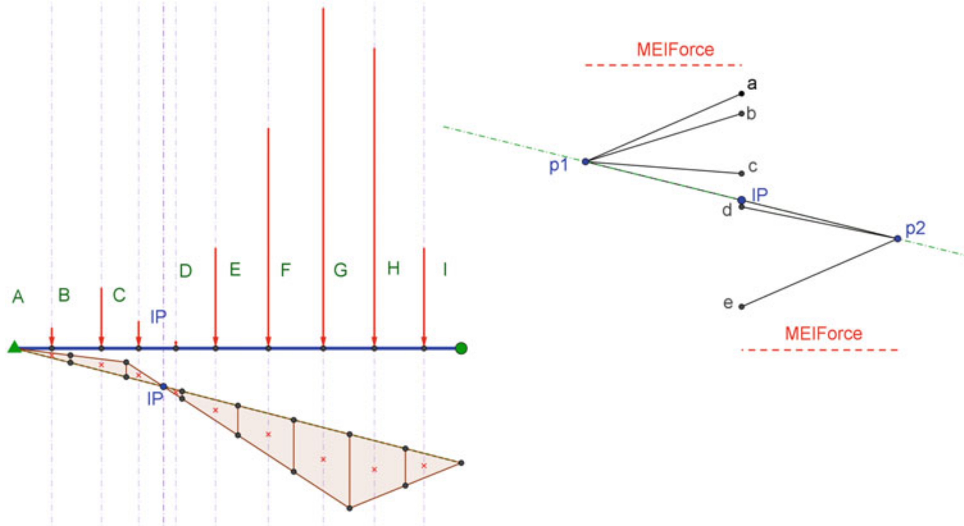
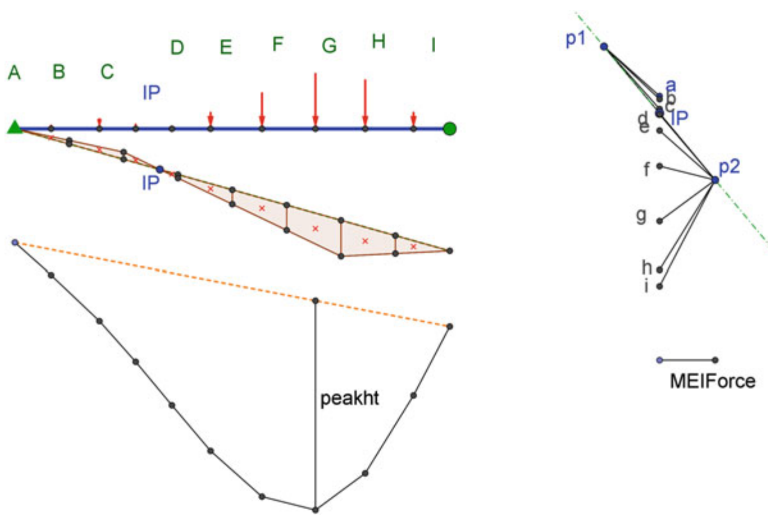


Fig. 8.9 Start of construction of "loads" based on M/EI segments

The elastic curve of the deformed neutral axis of the beam is constructed from the *M*/*EI* "loads". One of the poles can be adjusted, the other pole will automatically be forced to adjust, and the elastic curve can be placed precisely on the supporting boundary conditions, i.e. the pin and roller support. Or the elastic curve can be shifted elsewhere and the distance between the closing line and the elastic curve is the true, final displacement. This alternate representation of the elastic curve is shown in Fig. 8.10.

Fig. 8.10 Elastic curve is placed somewhere below the beam



The efficiency of the graphical method is highlighted not by examples with a single point load, but with examples containing multiple loads. The algebraic method becomes tedious quickly, but the complexity of loads does not add any additional difficulty to the graphical method.

Consider the beam shown previously in Fig. 4.19. It has a roller support at the left end, a fixed support at the right end, an internal hinge and two point loads as shown in Fig. 8.11. The modulus of elasticity is constant, but the moment of inertia varies as shown.

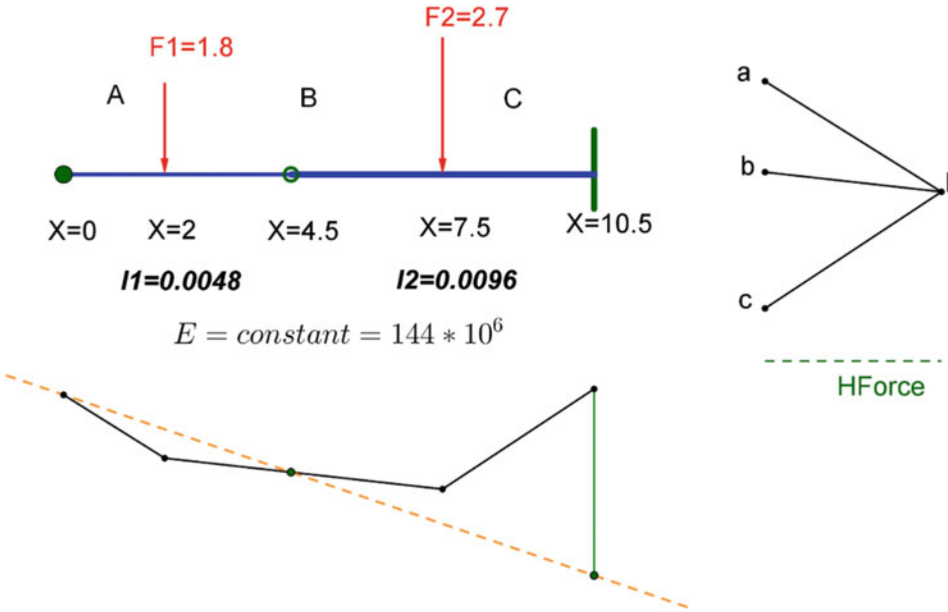
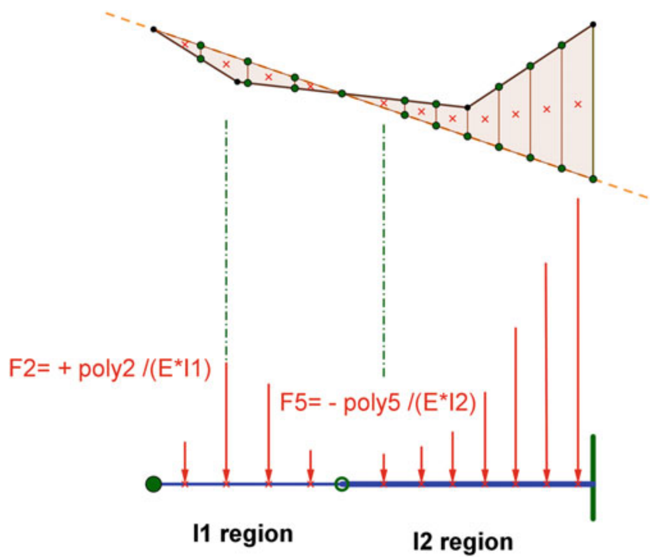


Fig. 8.11 Propped cantilever beam with an internal hinge

The vertical distance between the funicular and the orange dashed closing line, multiplied by *HForce*, multiplied by the *ForceScale* gives the magnitude of bending moment at any cross section.

This moment diagram will be discretized and then divided by the *EI* appropriate to each section. Note that polygon 1 is a triangle and polygon 5 is a pentagon. The drawing program can readily calculate the area and the centroid of each of these (Fig. 8.12).

Fig. 8.12 Graphical discretization of bending moment diagram



Note that curvature drives the deflection of beams. Thus, negative moment means concave downward curvature, consequently the "loads" from positive versus negative M/EI portions use different poles to reflect this change of concavity of the deformed beam as described in Fig. 8.9. But, whereas in Fig. 8.9 the slope of the continuous beam was *constant* through the inflection point IP, in a beam with an internal hinge such as the one shown in Fig. 8.11, the slopes are *completely different* on either side of the internal hinge (Fig. 8.13).

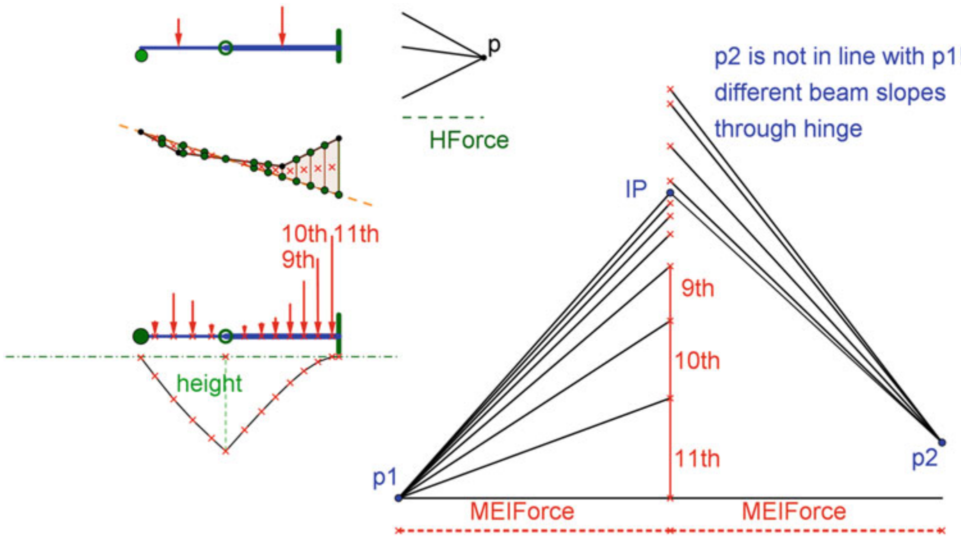


Fig. 8.13 Unlike continuous beam, slopes are completely different on either side of internal hinge

To solve for any displacement to the right of the hinge, only the elastic curve from the fixed wall to the internal hinge needs to be constructed. Furthermore, the closing line used for such a curve need not be horizontal, but the distance from the elastic curve to any such closing line must be captured. This is shown in Fig. 8.14.

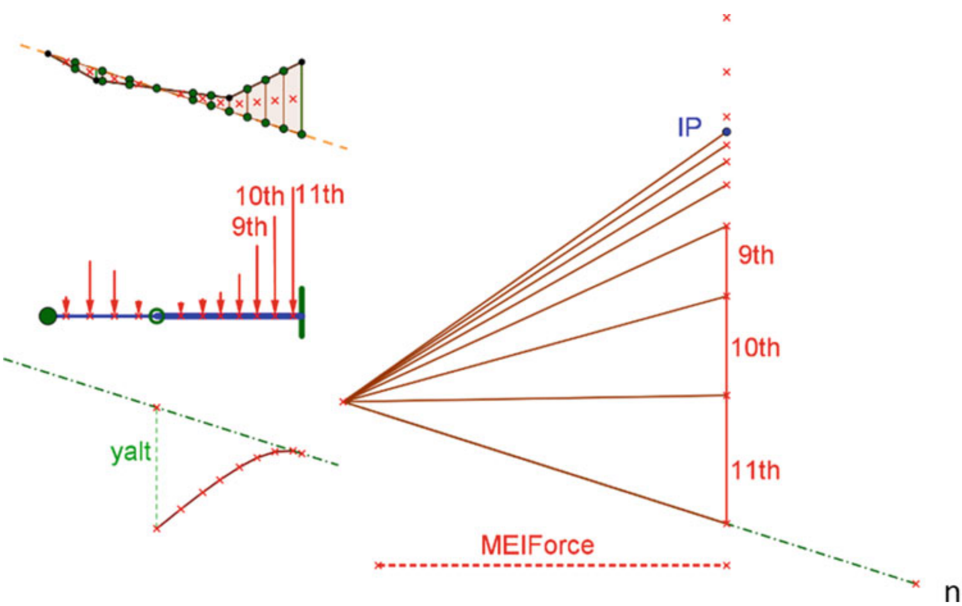


Fig. 8.14 Deflected shape can be constructed from portion extending from fixed support

Indeterminate Beams

Having the ability to generate the elastic curve based solely on the *M*/*EI* diagram is a powerful tool in the analysis of indeterminate beams. Consider the beam shown in Fig. 8.15, with a roller support at its left end and a fixed support at its right end. This is commonly referred to as a propped cantilever.

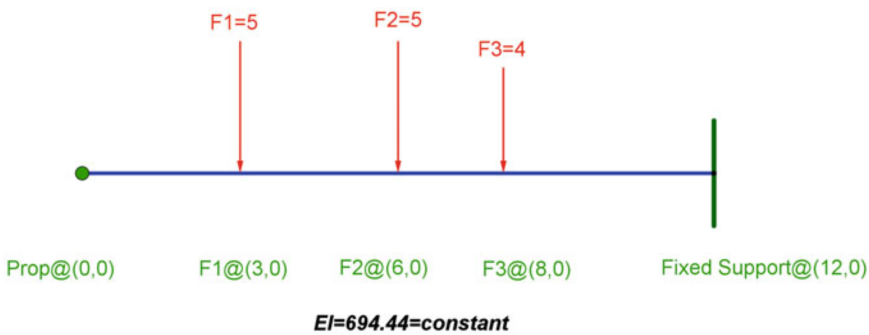


Fig. 8.15 Propped cantilever beam is indeterminate

The loads are drawn on the force diagram as usual, a funicular is generated and if a closing line is drawn from end to end of the funicular, the moment diagram would be that of a simply supported beam as shown in Fig. 8.16. Such an incorrect closing line is shown in Fig. 8.16 as *"if simply supported"*. If however, a "reasonable" Point *Fixed* is identified, a temporary closing line is passed through Point *Fixed* to the other end of the funicular. Accumulated experience with deflected shapes of beams will help to roughly approximate the location of the inflection point, *IP*. Correctly locating the inflection point exactly solves the problem of finding Point *Fixed*. Approximately locating the inflection point provides an approximate solution to the indeterminate problem.

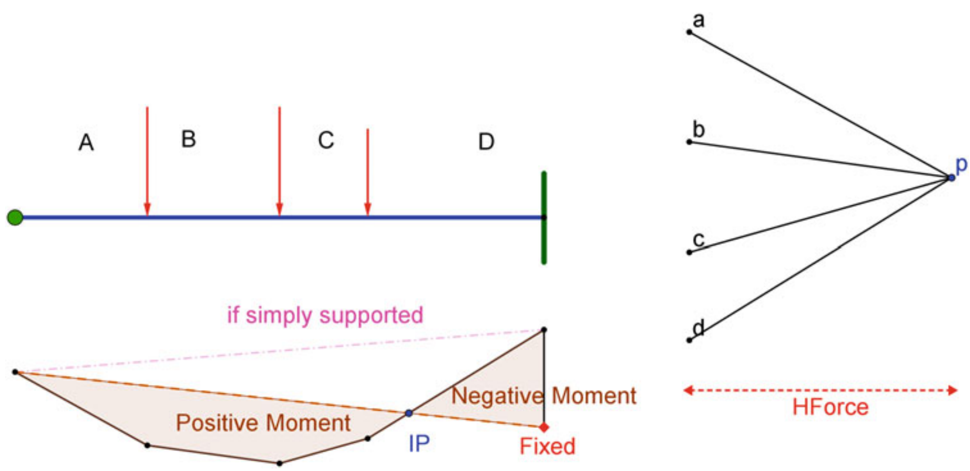


Fig. 8.16 Unreasonable closing line "simply" and reasonable closing line "fixed"

Once a reasonable "*Fixed*" point is established, the bending moment is discretized into at least 9 or 10 sections. Each portion of the *M* diagram is divided by the matching *EI* associated with that segment of the beam. In this example *EI* is constant, thus each *M* portion is divided by a constant *EI* value.

These “loads” are passed on to the beam through the centroidal line of action of each segment. As stated in Figs. 8.8 and 8.9, when curvature changes, a new pole is needed in the “load” diagram. Each pole is a constant distance away from the load line, a proper name for that distance is $MEIForce$. A helpful technique to use in this problem is to ensure that the slope in Space Y is horizontal. Thus, begin the funicular construction with ray $y\text{-}p1$, and establish that initial ray as horizontal. A straight line passes from $p1$ through IP and on that line $p2$ is located. Recall that $p2$ must be the same fixed horizontal distance $MEIForce$ away from the load line. This ensures the same scaling effect occurs to all rays, and that the ray through the IP has the same slope on either side of the IP.

Note that the magnitude of the force X to IP is fully known, thus the location of IP on the force diagram is known, it is the step from X to IP, just as the previous step was Y to X.

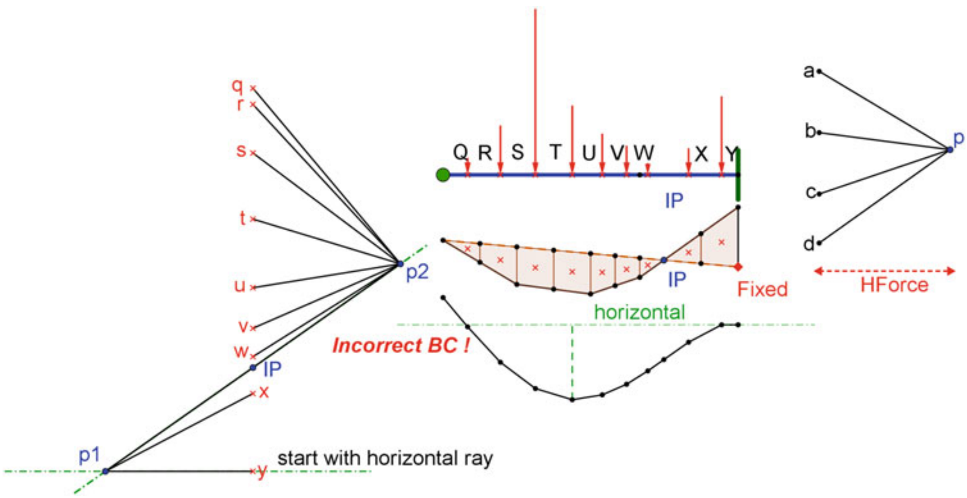


Fig. 8.17 Nearly correct deflected shape

It is clear from Fig. 8.17 that the left roller support does not land on a horizontal line passing from the right fixed support. This means that an adjustment in the position of Point $Fixed$ is required.

Figure 8.18 shows a correct solution, since the elastic curve ends at the same elevation as which it started.

The moment anywhere along the beam can be calculated as

$$M_{cut} = closing\ distance \cdot HForce \cdot ForceScale$$

Where $closing\ distance$ is the vertical distance from the funicular to the closing line. Notice once again that the reactions are never needed to perform this statics calculation. The deflection of the elastic curve is calculated at any cut as:

$$y_{elastic\ curve} = height \cdot HForce \cdot ForceScale \cdot MEIForce / MEIScale$$

Where $height$ means the vertical height of a segment from the elastic curve to the reference closing line.

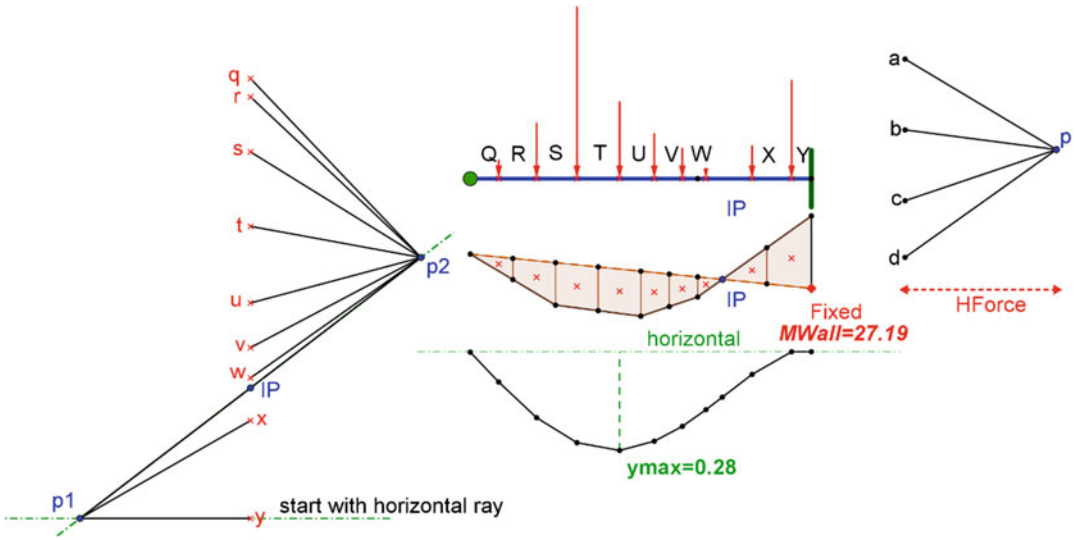


Fig. 8.18 Correct deflected shape allows for final quantitative answers

The vertical reactions in the left and right supports are quickly found. A line parallel to the closing line of the moment diagram is passed through the pole of the force diagram. Where this line intercepts the load line establishes point *e*, and thus the wall vertical reaction is segment *de*, and the roller vertical reaction is segment *ea* (Fig. 8.19).

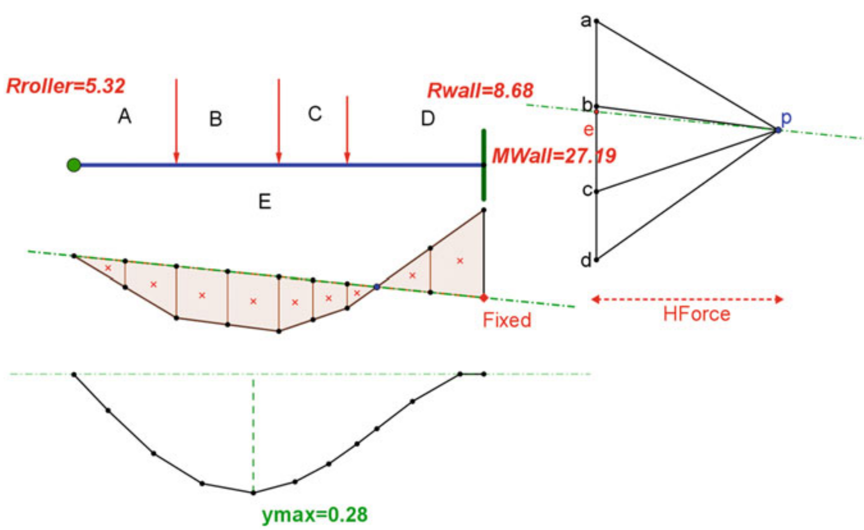


Fig. 8.19 Summary of final quantitative answers

The following examples shows a beam fixed at each end, subjected to some loads, yet this beam has two distinct cross sections (Fig. 8.20).

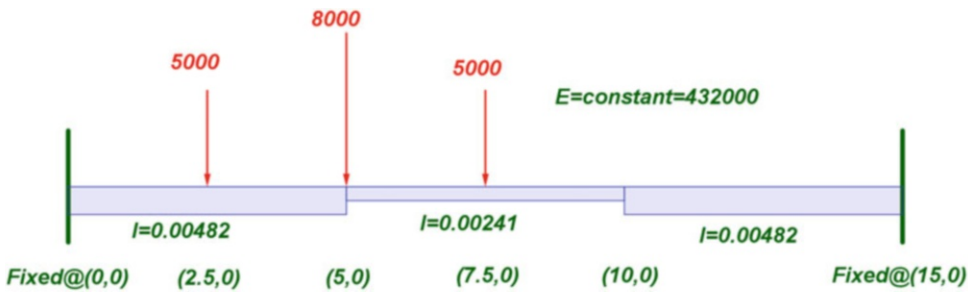


Fig. 8.20 Highly indeterminate, non-prismatic beam

Use Bow's notation with capital letters on the form diagram, and corresponding lower case letters on the force diagram. The reactions (i.e. point *e*) are not needed to proceed with the analysis. A funicular is drawn, and its closing line, extending from end-to-end at some slope, would be the shape of the bending moment diagram of a simply supported beam. Sliding the Point *FixedLeft* down to some reasonable amount introduces negative moment at the left side, and sliding the Point *FixedRight* down to another reasonable amount introduces negative moment on the right. The temporary closing line shown in Fig. 8.21 is a straight line between these two variable points.

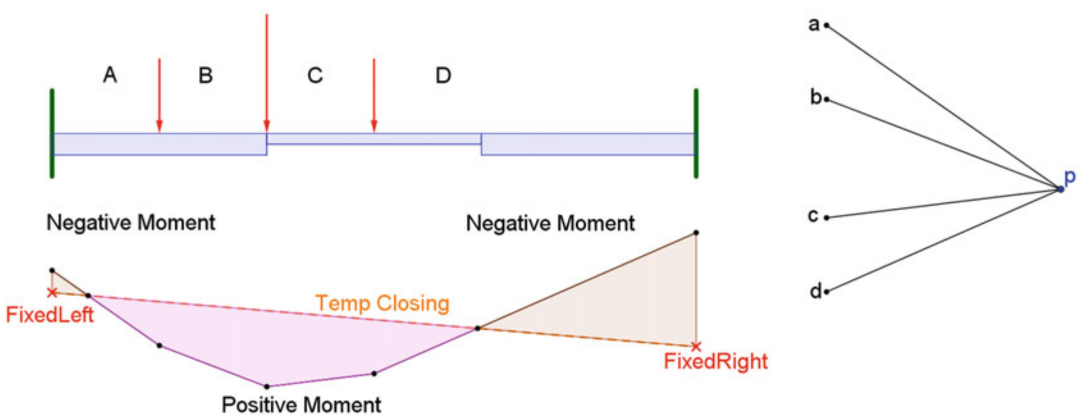


Fig. 8.21 Points *FixedLeft* and *FixedRight* are variables along given vertical lines

As before, the bending moment is discretized, and each discrete portion is divided by its appropriate *EI* value. Some decisions must be made with care in situations when it seems convenient to create a polygon based on M, but that polygon needs further subdivision because of two distinct moments of inertia. For example, in Fig. 8.22 immediately to the right of inflection point 2 (*IP2*), the areas needed to be broken up to capture the smaller moment of inertia (*Ismaller*).

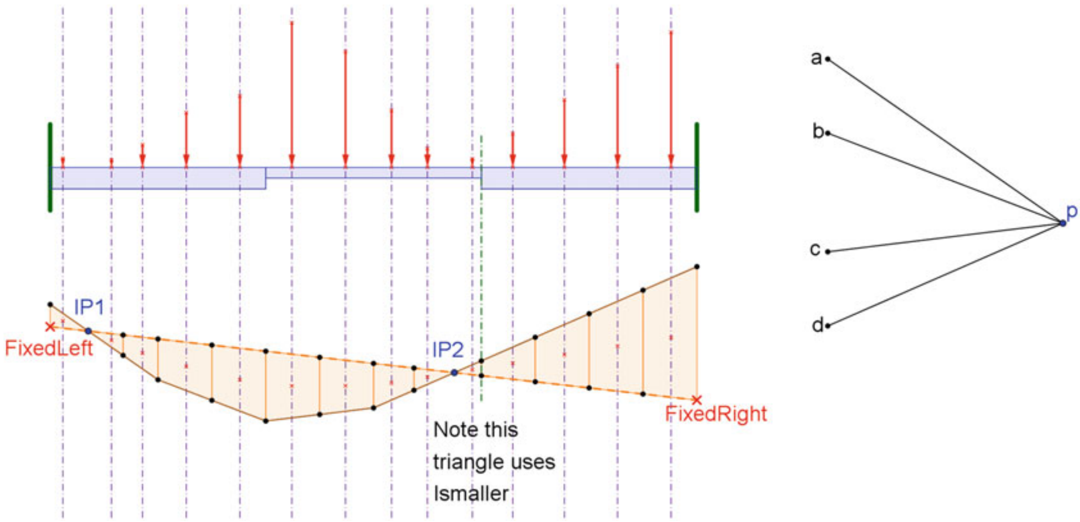
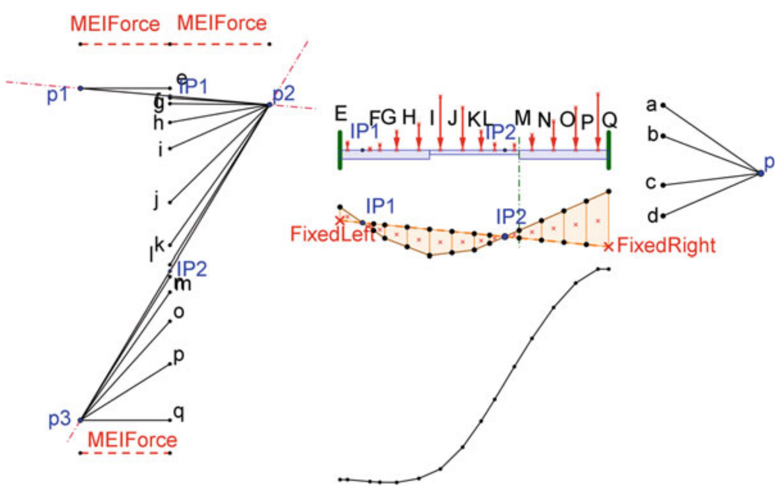


Fig. 8.22 Graphically locate centroid of each bending moment diagram segment

The area of each M subdivision, divided by the appropriate EI factor creates discrete "loads" that are applied to the beam at the centroid of each M subdivision. A new "load" line is created at some scale (*MEIScale*), but since there are two inflection points in this problem, two switches of poles will be required on the "load" line establishing three poles. Pole 1 (*p1*) is at some horizontal distance *MEIForce* from the vertical load line. A straight line through *p1* and the first inflection point (*IP1*) ensures continuity of slope through that inflection point. The second pole (*p2*) must be the same horizontal distance *MEIForce* from the vertical load line. Another straight line from *p2* through *IP2* ensures slope continuity through the second inflection point. Again, *p3* is the same *MEIForce* distance away from the load line to ensure consistent scaling. The entire *M*/*EI* load line is scaled by a factor *MEIScale*. Figure 8.23 shows the generated elastic curve from the first iteration of the trial *FixedLeft* and *FixedRight* points. Clearly, this elastic curve does not satisfy the required boundary conditions.

Fig. 8.23 Elastic curve does not yet satisfy all boundary conditions



If the *M*/*EI* model is too coarse, a straight line passing from *p2* through *IP2* may not precisely end a horizontal distance *MEIForce* away from the terminal point on the load line. If that happens, it is best to stay true to *p2* and *p3* being a fixed *MEIForce* distance away from the load lines, and sacrifice a bit on the precise location of *IP2* on the vertical load line. This is so because a coarse *M*/*EI* discretization will result in small errors in location of *IP2*. The impact of this placement of the inflection point will be very small and it is better to maintain the *MEIForce* scale on both sides of the load line.

Figure 8.24 shows just such an adjustment. The line from *p2* to *p3* is extremely close to the second inflection point *IP2*. Figure 8.24 shows how quickly the answer converges to a correct solution. Convergence is assured when the two ends of the elastic curve both land on the horizontal line that pre-defined the zero slope at both ends.

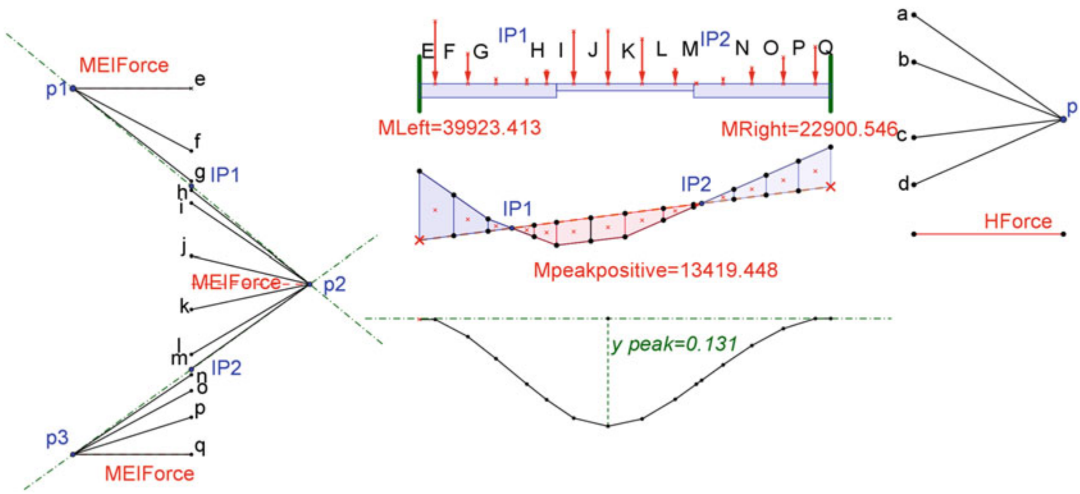
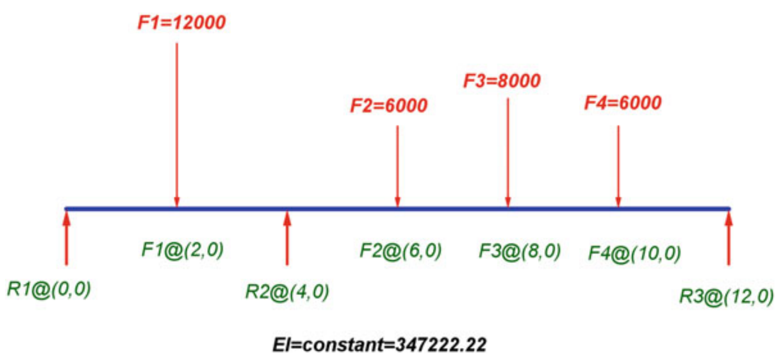


Fig. 8.24 Line *p2*-*p3* barely misses inflection point *IP2*

The following example is a different type of indeterminate beam. This example is of a simply supported beam with an intermediate support, creating two spans of structure. This type of problem will be solved for in two different, but analogous ways. The first way will be to assume a funicular based on the far left and far right supports, as a simply supported beam would appear to be. Then a judgment is made about the location of Point *X*, which defines the magnitude of negative bending in the beam directly above the intermediate support. Two straight closing lines are formed from Point *X* to the ends of the funicular, which establishes a reasonable bending moment diagram. This is shown in Figs. 8.25 and 8.26. Notice, that no reactions are needed to pursue the problem.

Fig. 8.25 Prismatic beam, continuous and indeterminate



The trial moment diagram is immediately generated, and the magnitude of the bending moment at any cross section is found from:

$$M_{any\ cut} = distance\ to\ closing \cdot HForce \cdot ForceScale$$

Where $ForceScale$ is used to make a comfortable force diagram.

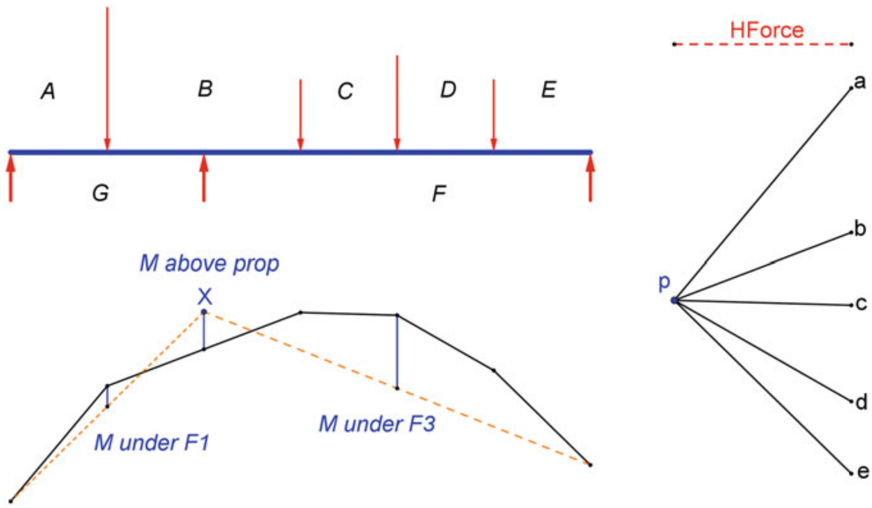
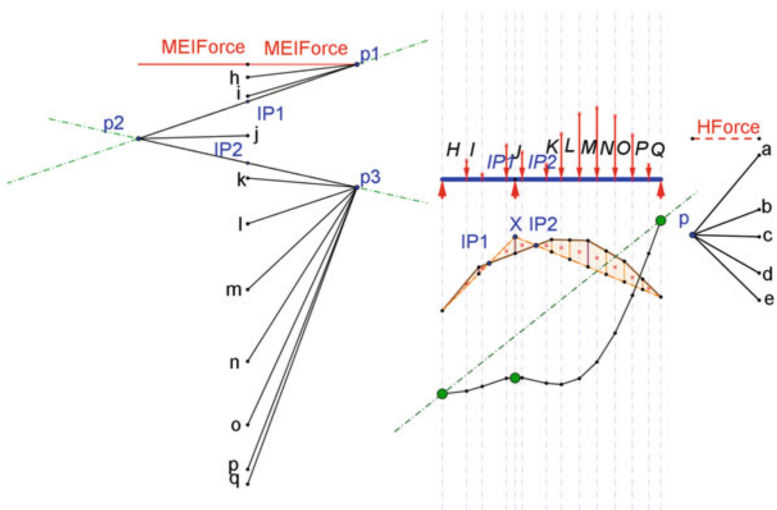


Fig. 8.26 Trial moment diagram generated by location of point X above interior support

As in previous examples, the next step is to discretize the bending moment diagram. Since EI is constant, all areas of the discretized moment diagram will be divided by the same constant EI.

Figure 8.27 shows an elastic curve that does not satisfy the boundary conditions of having all three supports touching the same closing line.

Fig. 8.27 Incorrect elastic curve does not satisfy boundary conditions on line



Small adjustments to the location of Point X make marked changes in the elastic curve. A solution is rapidly found wherein a straight line passes through all three supports. Note that this closing line need not be horizontal as is shown in Fig. 8.28. Once the solution is found, all of the moments can be immediately calculated. The vertical reactions are found by using the closing lines of the moment diagram which pass through the correct Point X. Lines parallel to these two closing lines are passed through the pole p on the force diagram to establish points f and g, and consequently the vertical reactions EF, FG and GA.

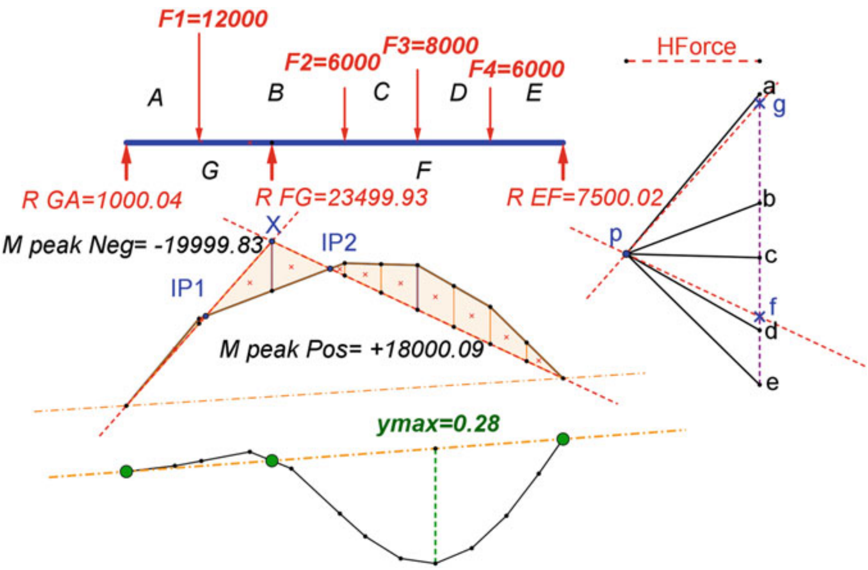


Fig. 8.28 Closing line need not be horizontal but boundary conditions must be satisfied

Figure 8.29 shows an alternate solution of the problem shown in Fig. 8.26. Here, two distinct funiculars are drawn, as if the two spans were independent, simply supported beams. Then point Y is chosen to capture the negative bending moment in the beam above the interior support. Quick manipulations bring about practically the same answers as were obtained in Fig. 8.28. Figure 8.29 shows the completed statics only, not the elastic curve which would be identical to the previous solution.

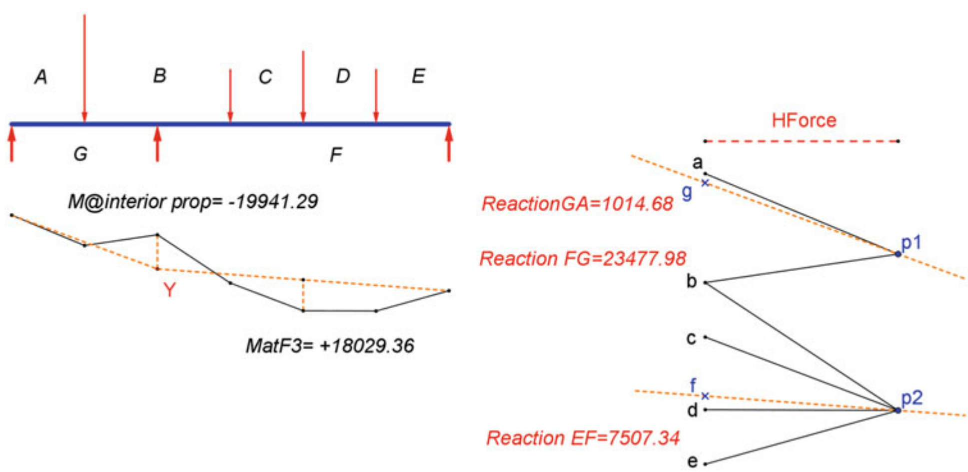


Fig. 8.29 Beam from Fig. 8.25 solved via two distinct funiculars, not one as before

This technique can be applied to an indeterminate beam having three spans. Such a beam is shown in Fig. 8.30. It is indeterminate to the second degree, subjected to a linearly increasing load, and has a constant *EI* value throughout (Figs. 8.31 and 8.32).

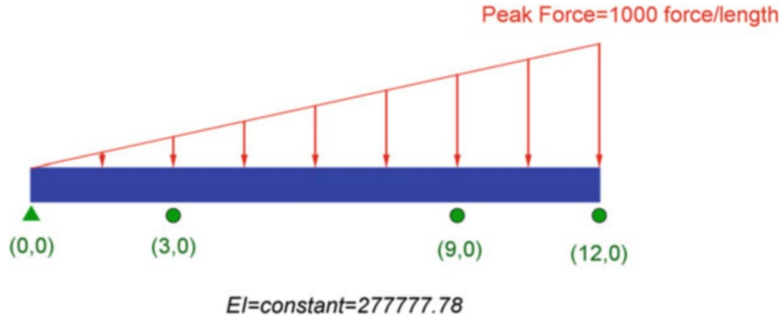


Fig. 8.30 Indeterminate, continuous beam with three spans

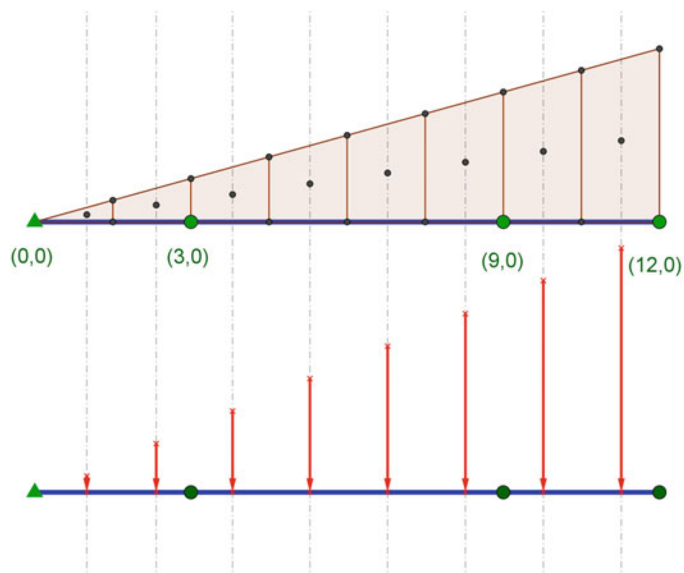


Fig. 8.31 Discretization of bending moment diagram

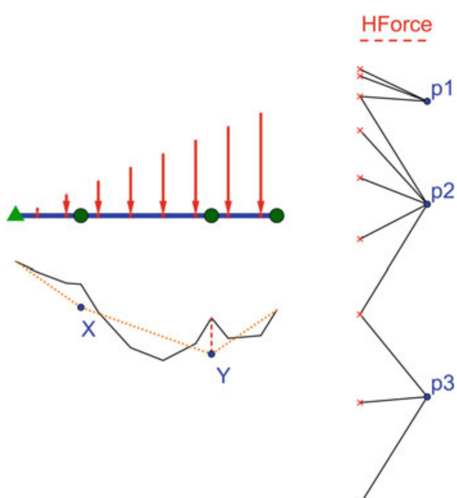


Fig. 8.32 Trial closing lines based on one funicular

A serious flaw can occur if the wrong number of inflection points is assumed. In Fig. 8.33, arbitrarily choosing some side of the load line for the pole p4 results in a deflected elastic curve that may be upside down, or may be right side up. Either approach will still provide answers that are nearly theoretically correct.

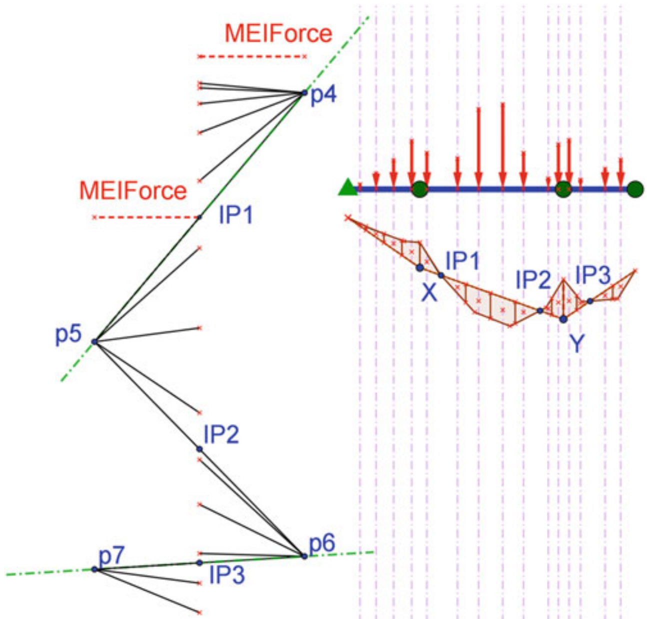


Fig. 8.33 An upside down deflected curve is inconsequential, numbers will still be valid

Figure 8.34 shows the generated elastic curve, which happens to be upside down.

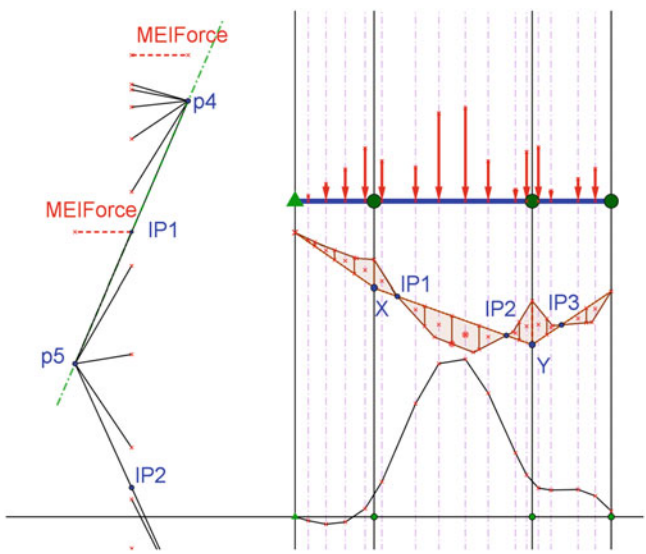
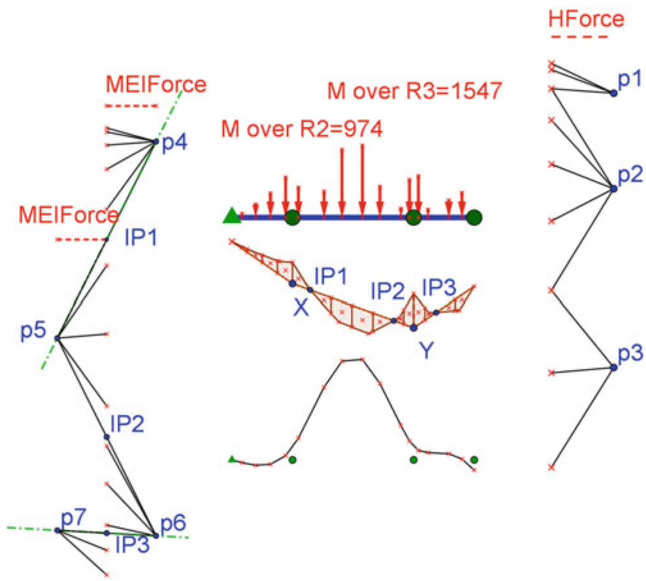


Fig. 8.34 Upside down or right side up is based on placement of poles

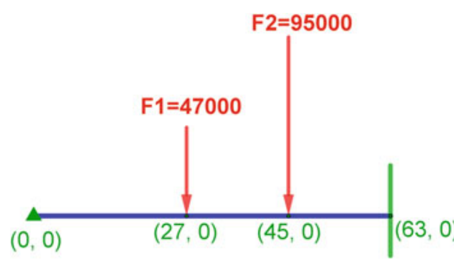
Figure 8.35 shows an approximately correct displaced shape, insofar as the elastic curve nearly touches all of the supports. Of course, the deflected shape is upside down in this figure. The coarseness of the *M*/*EI* model prevents a precise solution, but even at this level of discretization, the moments in the beam over the two interior supports are extremely close to the exact theoretical answers.

Fig. 8.35 Approximately correct deflected shape



Such approximately correct interpretations of the elastic curve lead to very good results of the final bending moments as well as the reactions in the indeterminate beam. Yet, the detailed fussiness of constructing such curves may be off-putting to students of architecture or construction management. There is however, another way of obtaining approximately correct bending moments that skips this last step and solely uses intuition of where the inflection points occur. If the location of the inflection points are guessed perfectly correctly, then the answers will be exact. The key idea is to learn how such beams deform, this occurs over time and repetition (Figs. 8.36, 8.37, and 8.38).

Fig. 8.36 A faster, but approximate method of obtaining the indeterminate bending moment diagram example



The first step is to intuit, then sketch the deflected shape ensuring that boundary conditions are all satisfied. Then estimate the location of the inflection points.

Fig. 8.37 Estimate of inflection point is key

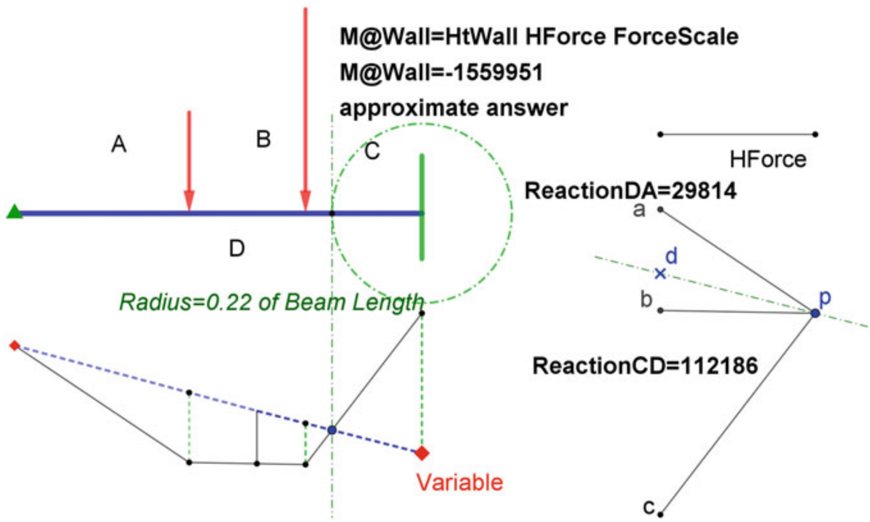


Fig. 8.38 Adjust closing line so there is no moment at estimated inflection point

Notice the three straightline segments of the bending moment diagram. These correspond perfectly to rays, *a-p, b-p* and *c-p*, albeit at a differing scale (Figs. 8.39 and 8.40).

Fig. 8.39 Finite element output of bending moment diagram

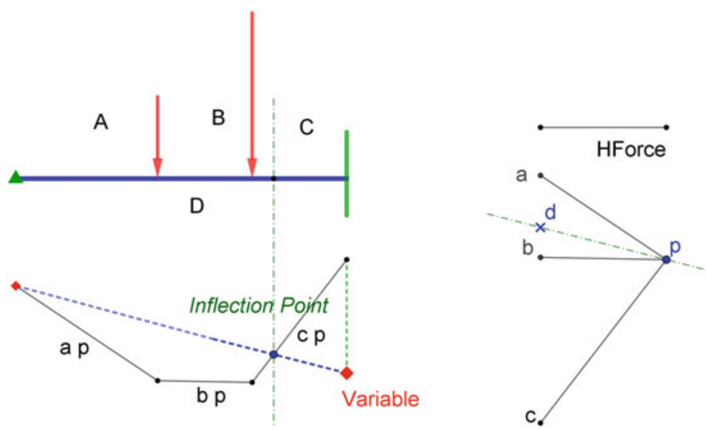


Fig. 8.40 Here, one *Variable* fixed end point affects where closing line crosses funicular

The insight here is that:

- The funicular establishes the bending moment diagram for the simply supported case
- Moving the closing line manually through the use of one variable point, such that the closing line intersects the funicular at a reasonable point (the Inflection Point), gives the indeterminate bending moment diagram

This technique begins with sketching the assumed deformed shape of the beam. Without an inflection point, there is no sure way of knowing where to place the variable end of the closing line. Notice that the closing line need not be horizontal. Also notice that moving the pole *p* simply changes the scale of the bending moment diagram, not the values themselves. Sign changes are clearly dependent on which side of the closing line the funicular lies.

The bending moment at any point is:

$$M = Height \cdot HForce \cdot ForceScale$$

A startling feature of this technique is the fact that the reactions were *never needed* to solve this problem!

Is this technique exact? No. It clearly depends on identifying the correct inflection point which is not possible to do perfectly simply by inspection. A very inaccurate estimate of the inflection point's location will get a largely incorrect answer, but the problem will be "somewhat correct" insofar as general signs and rough magnitudes of all of the moments.

For a single beam that is pinned at one end and fixed at the other end, a good rule of thumb is to estimate that the inflection point will be *25% of the span distance* away from the fixed end.

Figure 8.41 shows another indeterminate beam. This time, both ends are fixed. Notice that this is the same beam as was shown in Fig. 8.36, it is 63 units of length long, with a 47000 units of force at 27 units from the left end, and a 95000 unit force, at 45 units from the left end. The deformed shape of the fixed-fixed beam is similar to the shape of the pinned-fixed beam, but of course now, there is "sadness" i.e. concave downward near the left support. This is shown in Fig. 8.42.

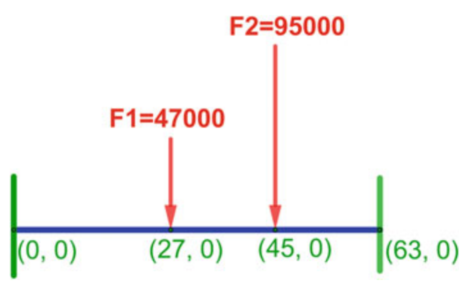


Fig. 8.41 Highly indeterminate beam, two inflection points will be needed

Fig. 8.42 Sketch of deformed shape allows for estimated location of inflection points

Place an inflection point approximately 25% from the left end. But notice that the loads are not equal and symmetric. The 94437 unit load is approximately twice as large as the 47000 unit load, and the right load is closer to the support than the left load. The asymmetry of loads means there will be asymmetry of the deformed shape, i.e. the distance of the left inflection point from the left fixed support will differ from the distance of the right inflection point to the right fixed support. It is difficult to discern what this difference will be. In general, loads further from the fixed wall will result in a "gentler, loopier" shape, loads closer to the support will result in a "tighter" more constrained shape. Think of a diving board, one really sees the gentle loopy curve of the board only when the diver is far from the support. So for lack of anything better, set the right inflection point somewhat closer to the right wall, perhaps 20% of the span. Note that all of this discussion is approximate, and it actually depends on the material properties and the cross sectional properties which we have not even incorporated yet. Nevertheless, this begins to establish a "feeling" for the moments in this indeterminate structure. Figure 8.43 shows this difficult calculation in a remarkably simple manner. No reactions at all are needed, point *d* was not used, but nevertheless it was easily located.

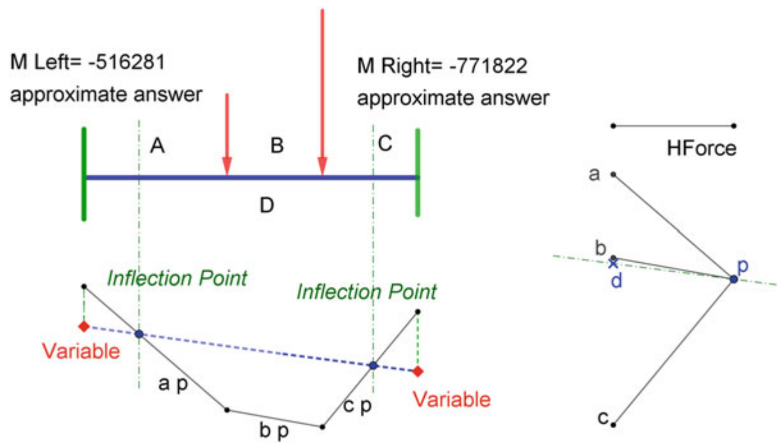


Fig. 8.43 Two variable points establishing crossing, reactions never needed

The next example is a different type of indeterminate beam. It is known as a continuous beam over two spans. The geometry and loads are the same as before, but now the ends are pin supported and an intermediate prop is placed between the two point loads, i.e. at 36 units from the left end. This is shown in Fig. 8.44.

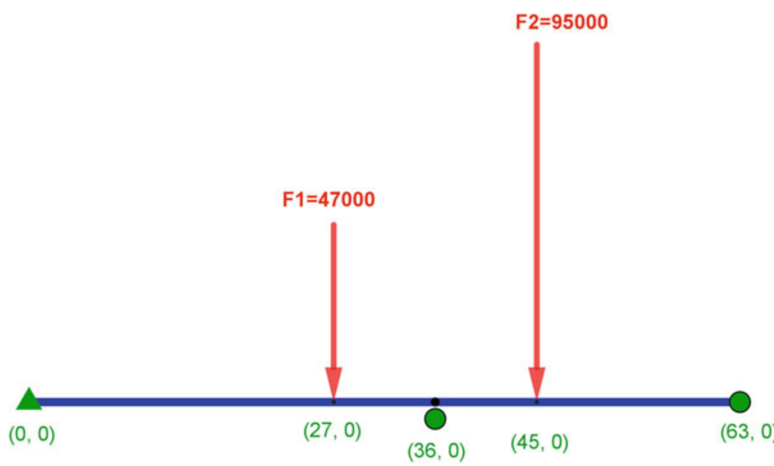


Fig. 8.44 Continuous beam, inflection points will be estimated

Figure 8.45 shows the deformed shape of the beam. Where curvature changes from concave up to concave down is the location of an inflection point.

Fig. 8.45 Sketch of deformed continuous beam

An approximate analysis of the beam is quickly performed graphically. An inflection point is placed somewhere not too far to the left of the interior prop, and a second inflection point is placed somewhere not too far to the right of the interior prop.

Hopefully it is immediately obvious that the variable positions of the end lines must coincide with the ends of the funicular, so that the gap between the two is zero meaning that there is zero moment at each end. But how might one have the concave up and concave down curvature exhibited in this beam and still satisfy the zero end moment at each end? The answer is a bent closing line! Introduce a third variable point that is in line with the support. The closing lines start at the ends of the funicular and meet at the interior variable point as shown in Fig. 8.46. Recall that there are three variable points here, but the two at the ends coincide with the start and with the end of the closing line to ensure zero moment at each end.

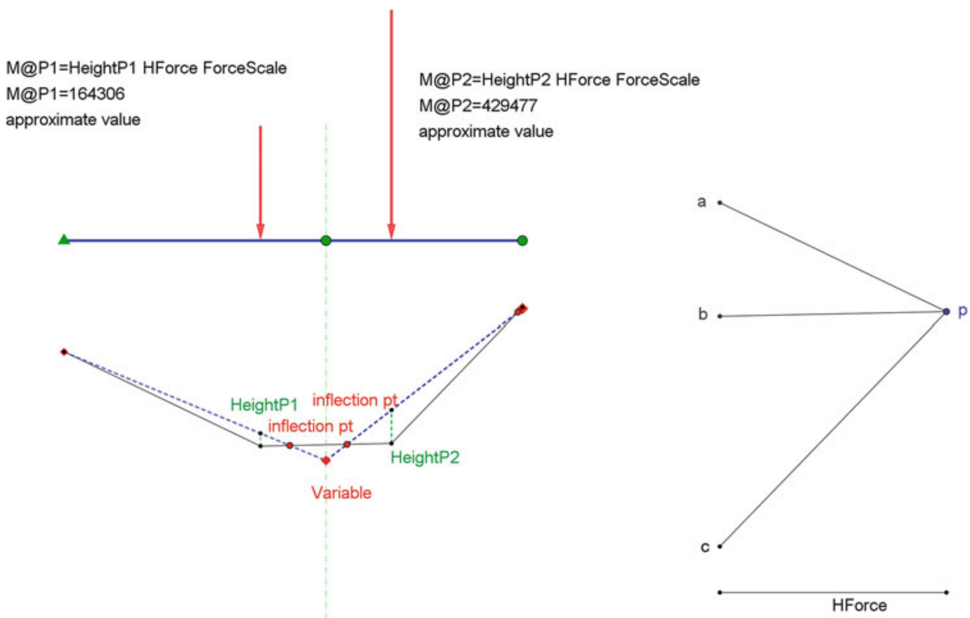


Fig. 8.46 Truly only one variable needed to establish two crossing points, end conditions are known

The last example is extremely quick to perform. Figure 8.47 shows the same beam as above, but this time both ends are fixed and the intermediate prop is also in place. This is a highly indeterminate problem! But the algorithm used above can be immediately applied to this problem as well. Its efficacy solely depends on correctly placing the inflection points. If they are accurately identified, the answer will be exactly correct. If they are inaccurately placed, the general shape of the bending moment will still be correct but the magnitudes of the answers will be incorrect (Fig. 8.48).

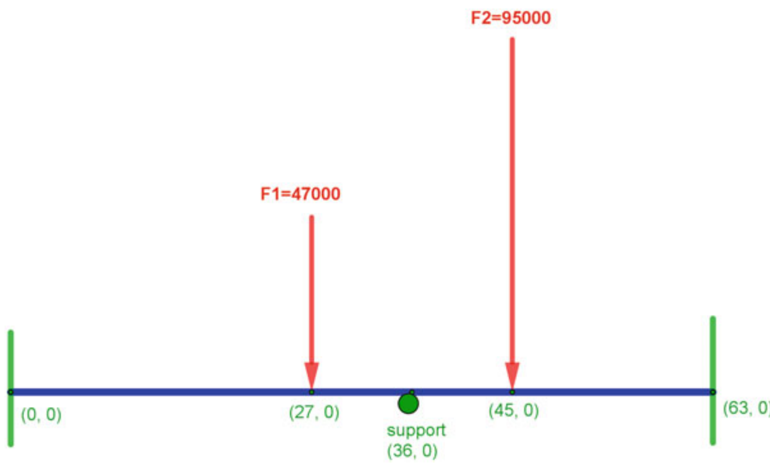


Fig. 8.47 Highly indeterminate beam but simple to solve approximately

Fig. 8.48 Intuit displaced shape and locate inflection points

The only change needed from the solution arrived at in Fig. 8.46 is to adjust the position of the two points at the left and right wall, such that a negative bending moment is present at each support. Small adjustments can also be made to the variable point at the intermediate support, but any tweaking would be based on a solid sense of where the inflection points really are. With little to go on, one can still note that the moment at the right wall will be a bit more severe than the moment in the left wall due to the larger load on the right span. Furthermore, the beam is highly distorted near the intermediate support, thus these inflection points are somewhat near the left and right sides of the intermediate support. This is shown in Fig. 8.49.

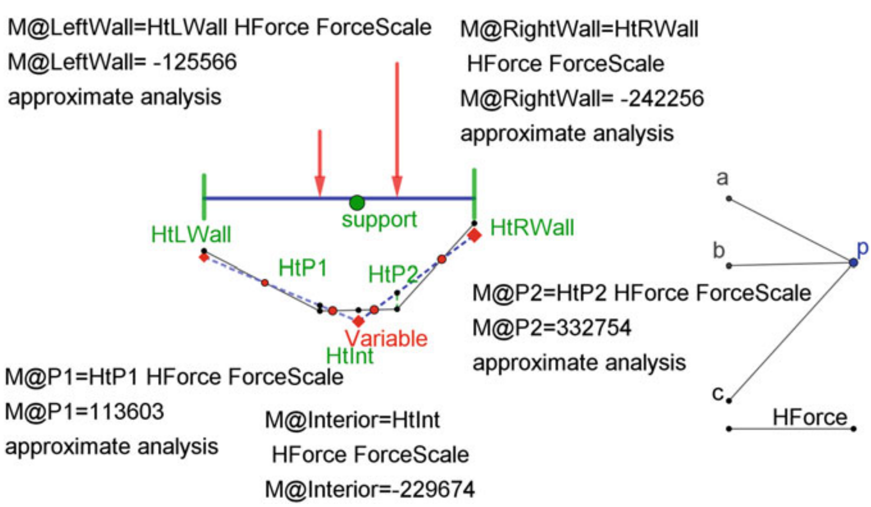
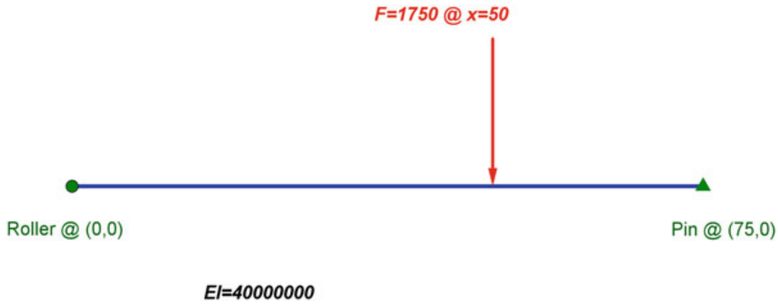


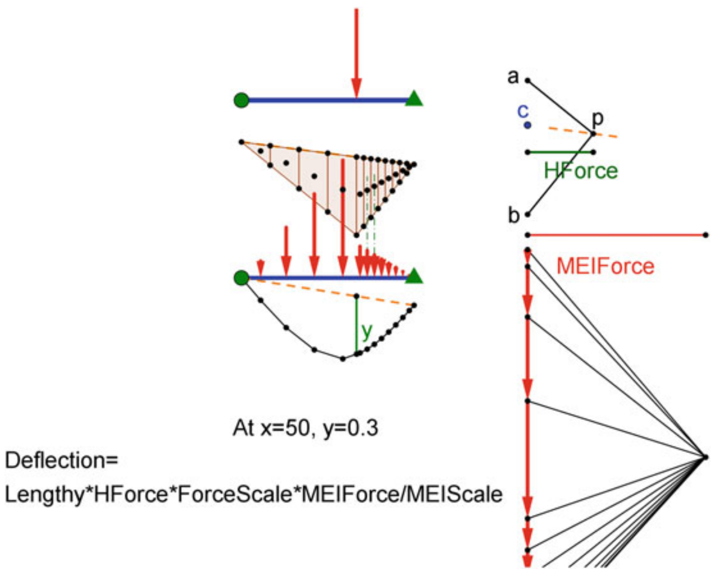
Fig. 8.49 Three variable points shown as red diamonds, these cause closing line to cross funicular four times

Chapter 8 Exercises

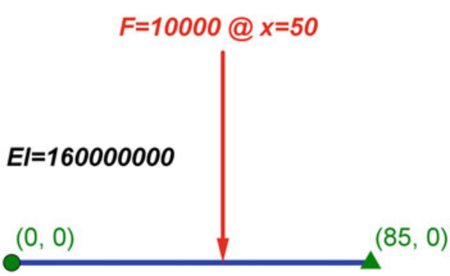
Exercise 8.1 A simply supported beam is subjected to a single point load. Calculate the vertical movement of the beam at a cross section directly beneath the applied load.



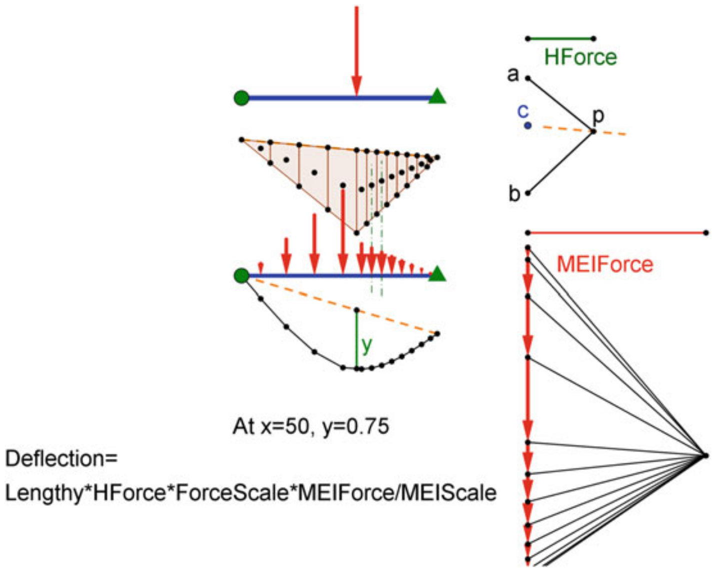
Exercise 8.1 solution



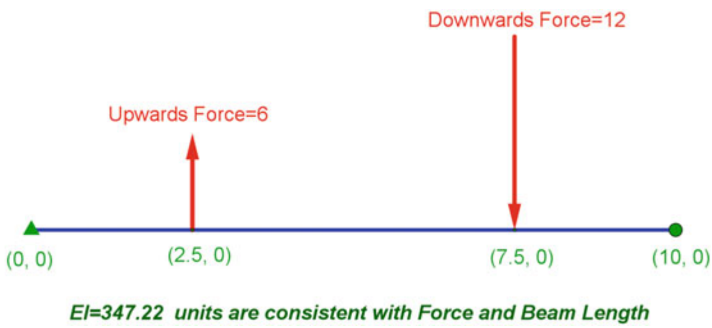
Exercise 8.2 A simply supported beam is subjected to a single point load. Calculate the vertical movement of the beam at a cross section directly beneath the applied load.



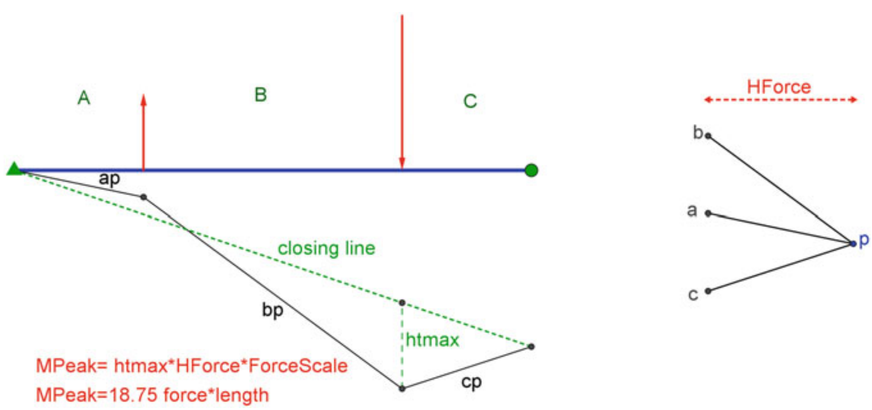
Exercise 8.2 solution



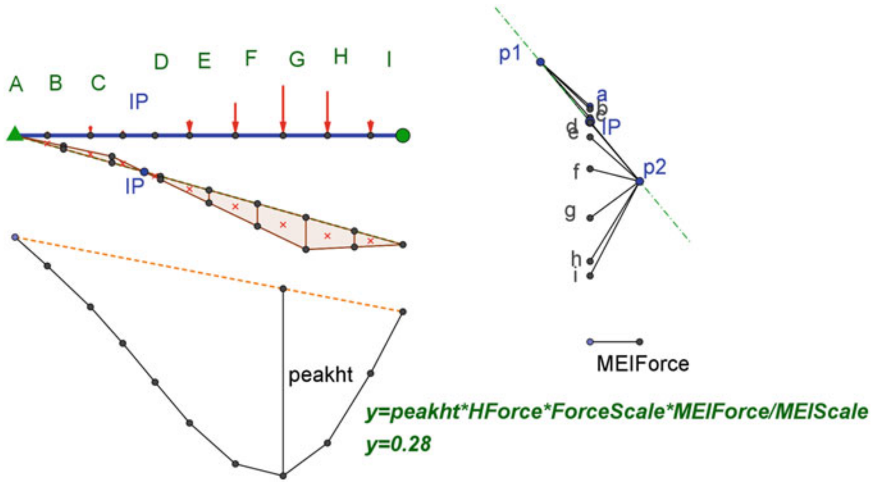
Exercise 8.3 A simply supported beam is subjected to two point loads. Calculate the bending moment in the beam at a cross section directly beneath the downwards applied load. Then calculate the deflection at this cross section.



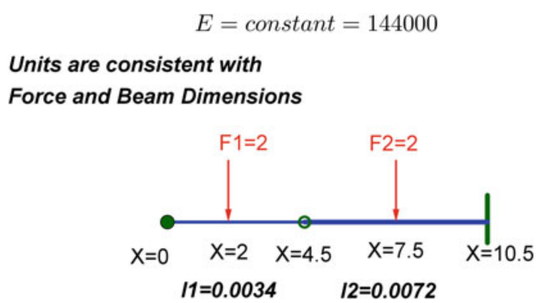
Exercise 8.3 solution part 1



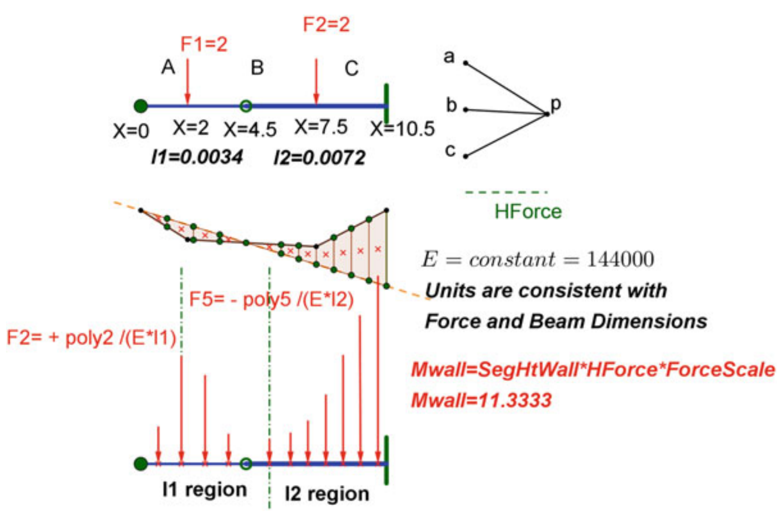
Exercise 8.3 solution part 2



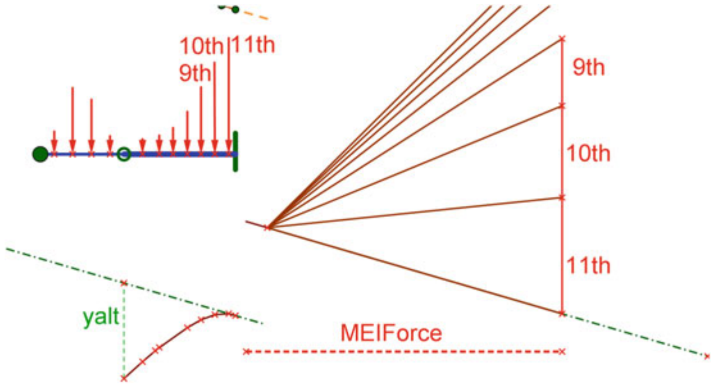
Exercise 8.4 A beam with variable moment of inertia is propped at its left end, fixed at its right end and it has an internal hinge. The beam is subjected to two point loads. Calculate the bending moment in the beam at the right fixed support and discretize the bending moment diagram into at least ten pieces. Then calculate the deflection at the hinge.



Exercise 8.4 solution part 1



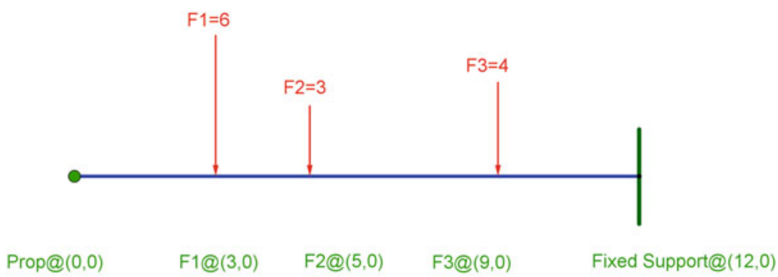
Exercise 8.4 solution part 2



yhinge=yalt*HForce*ForceScale*MEIForce/MEIScale

yhinge=0.103

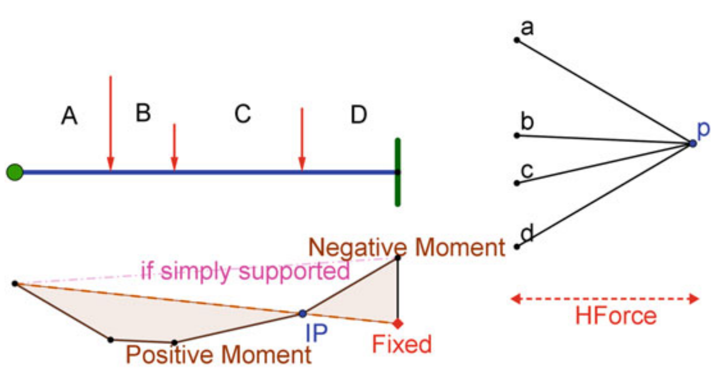
Exercise 8.5 A beam with a constant moment of inertia is propped at its left end and is fixed at its right end. The beam is subjected to three point loads. Calculate the bending moment in the beam at the right fixed support if the inflection point is assumed to be 1/3 the total span distance from the fixed support. Then discretize the bending moment diagram into at least ten pieces. Then calculate the deflection at the hinge after satisfying the boundary conditions.



EI=520.83=constant

Units are consistent with forces and with beam dimensions

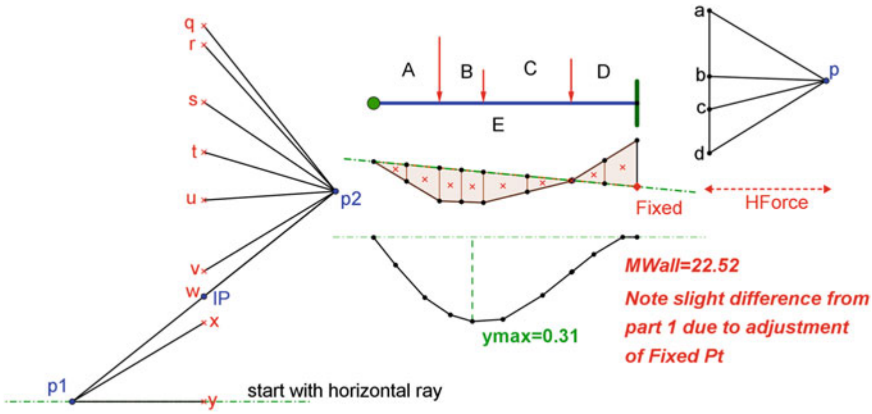
Exercise 8.5 solution part 1



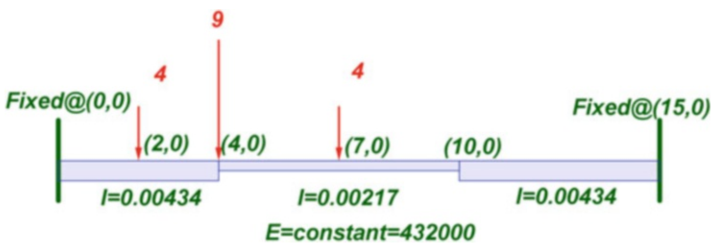
MWall=Ht@Wall*HForce*ForceScale

MWall=22.82

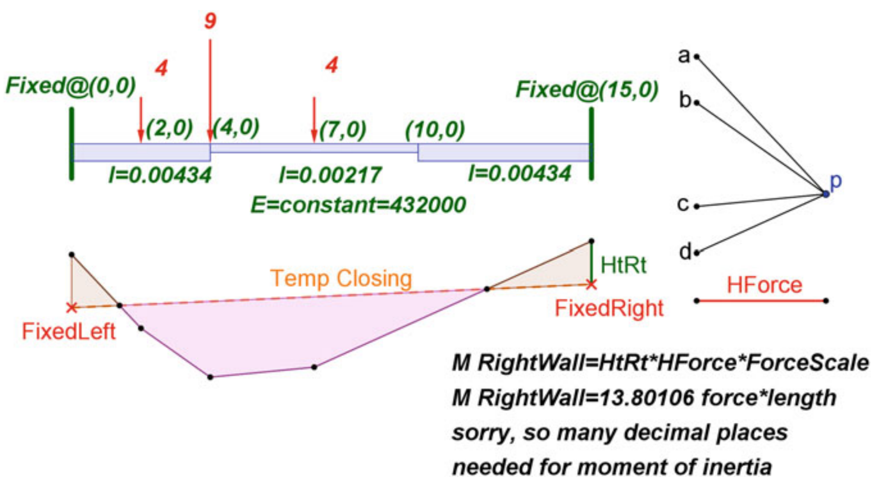
Exercise 8.5 solution part 2



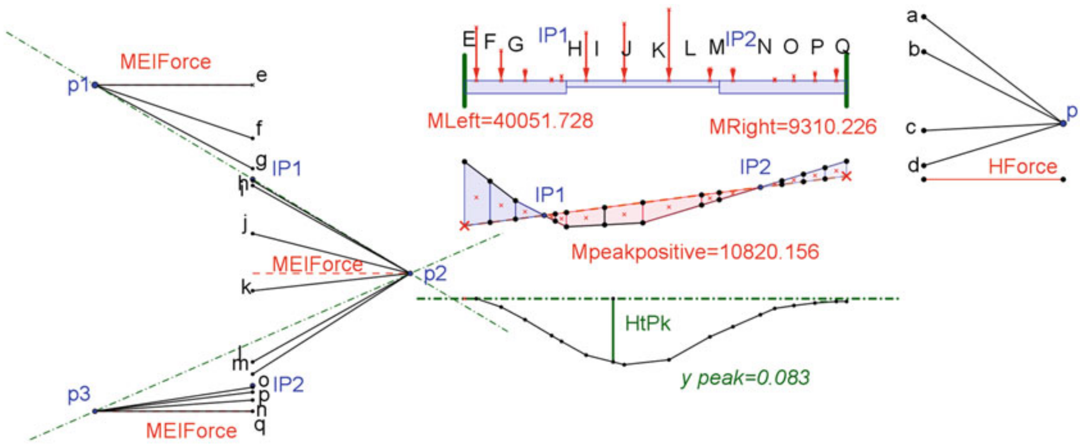
Exercise 8.6 A beam with a varied moment of inertia is fixed at its left end and is fixed at its right end. The beam is subjected to three point loads. Calculate the bending moment in the beam at the right fixed support if the inflection points are assumed to be somewhere near each fixed support. Then discretize the bending moment diagram into at least ten pieces. Then calculate the worst deflection after satisfying the boundary conditions.



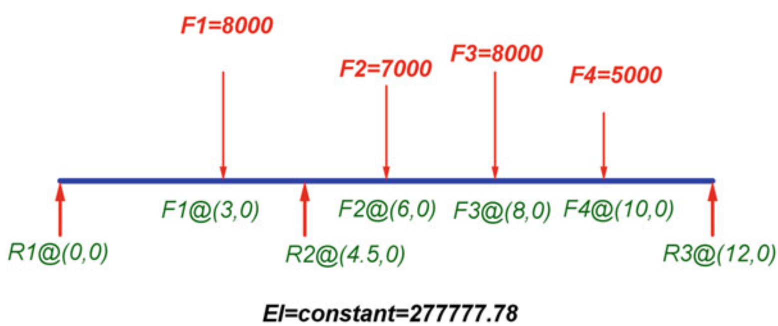
Exercise 8.6 solution part 1



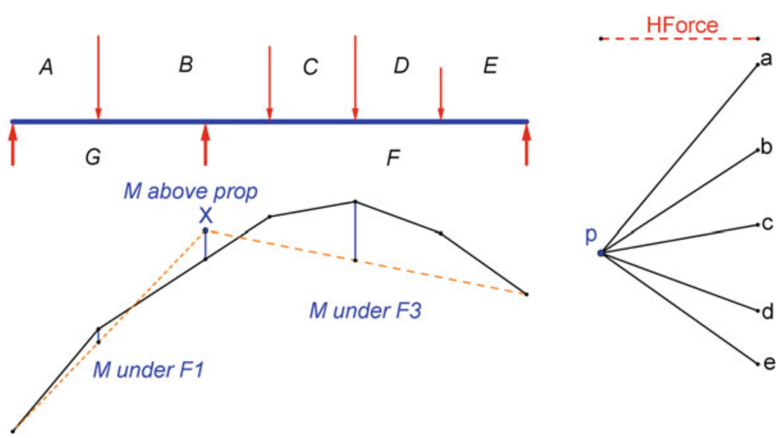
Exercise 8.6 solution part 2



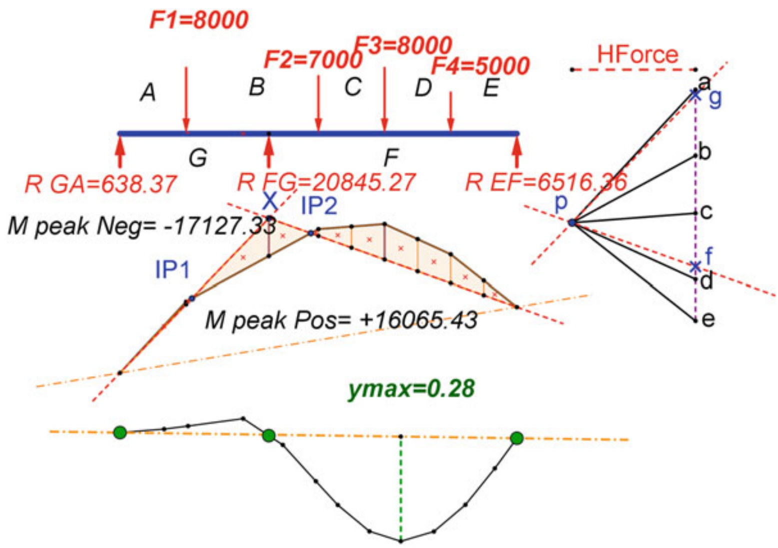
Exercise 8.7 The following indeterminate beam has multiple point loads and three vertical supports. There are no lateral loads. Graphically create a reasonable bending moment diagram assuming one funicular initially passes from the left end to the right end. Then discretize the bending moment to apply the pieces as loads to the beam, and solve for peak bending moment and peak deflection.



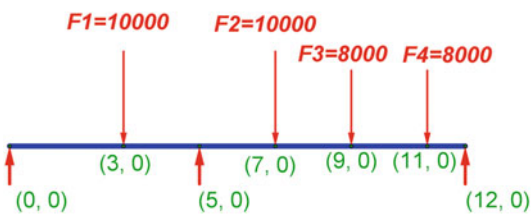
Exercise 8.7 solution part 1



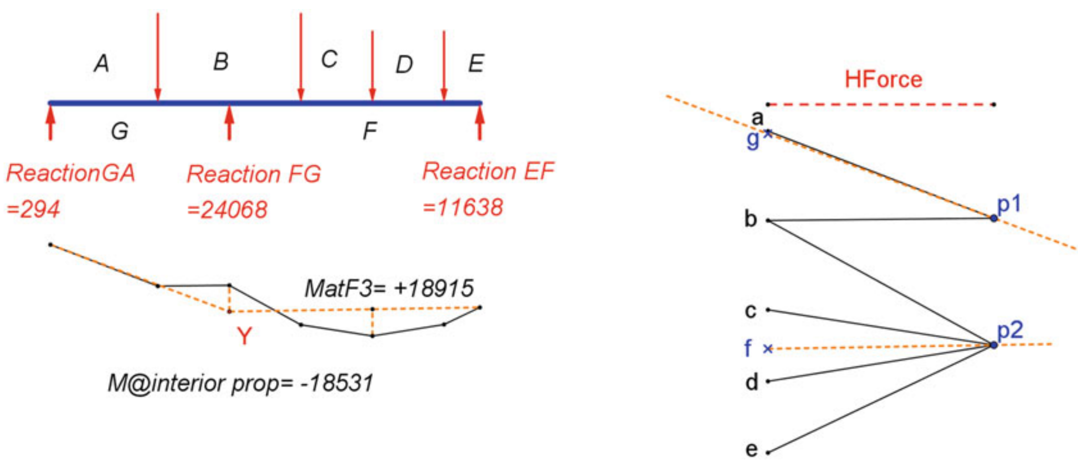
Exercise 8.7 solution part 2



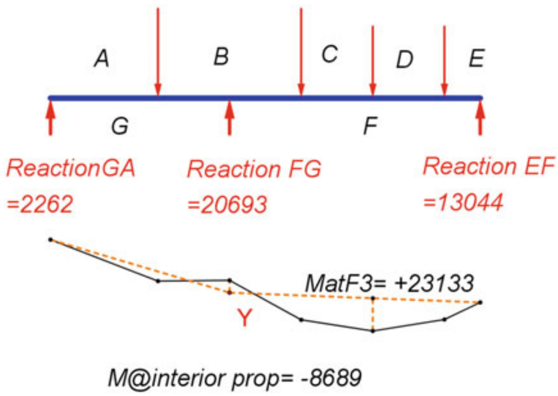
Exercise 8.8 The following indeterminate beam has multiple point loads and three vertical supports. There are no lateral loads. Graphically create a reasonable bending moment diagram assuming two funiculars initially, one in each of the interior spans. Provide a reasonable estimate of the peak bending moment by moving the variable closing line kink point, Point Y. Calculate the magnitudes of the three reactions. Then investigate the effect of moving Point Y.

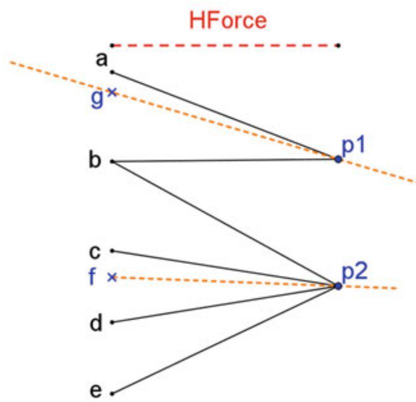


Exercise 8.8 solution part 1

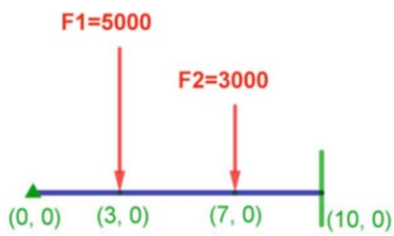


Exercise 8.7 solution part 2

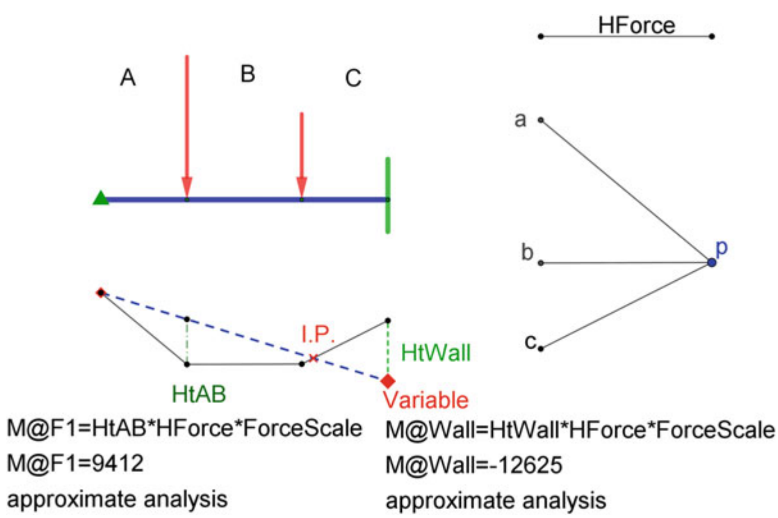




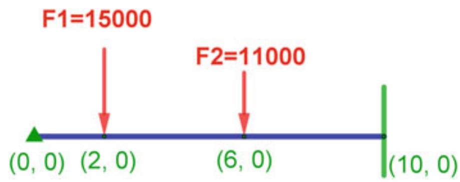
Exercise 8.9 A beam is propped at its left end, fixed at its right end and subject to two point loads. The moment of inertia is constant, but is not specified. Intuit where any inflection point or points will be and capture the approximate moment reaction at the fixed support.



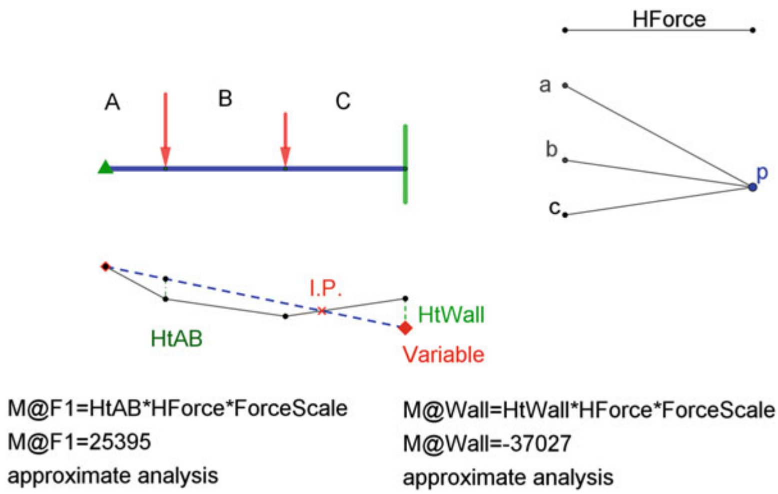
Exercise 8.9 solution



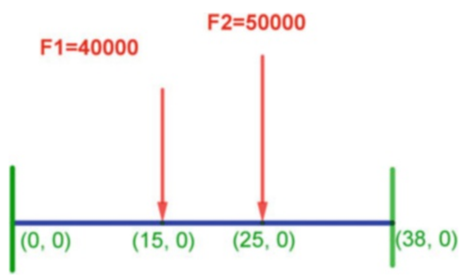
Exercise 8.10 A beam is propped at its left end, fixed at its right end and subject to two point loads. The moment of inertia is constant, but is not specified. Intuit where any inflection point or points will be and capture the approximate moment reaction at the fixed support.



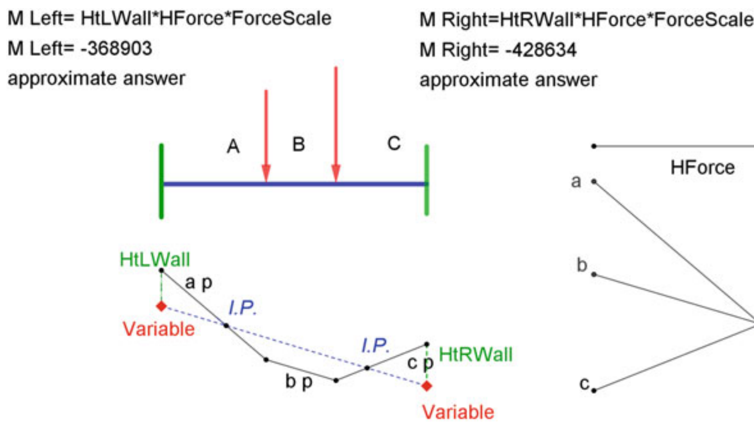
Exercise 8.10 solution



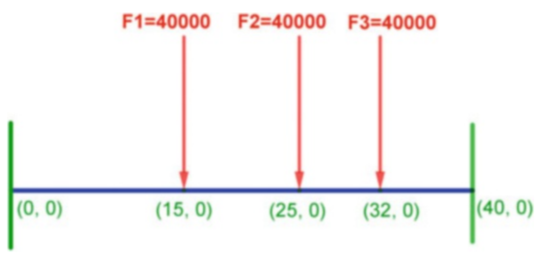
Exercise 8.11 A beam is fixed at its left end, fixed at its right end and subject to two point loads. The moment of inertia is constant, but is not specified. Intuit where any inflection point or points will be and capture the approximate moment reaction at the fixed supports.



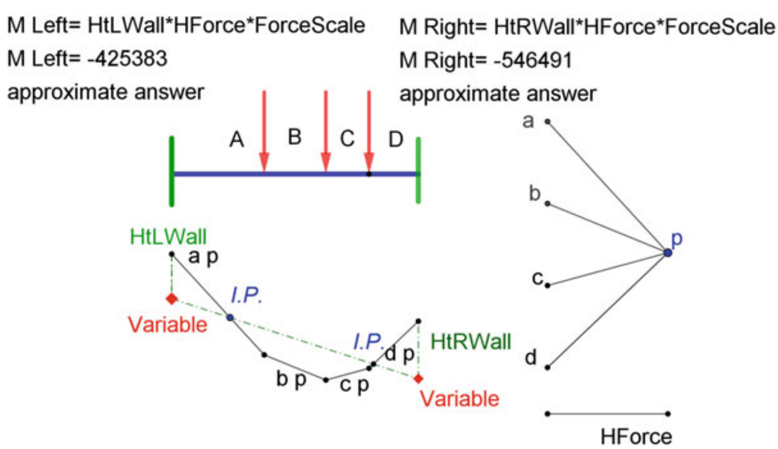
Exercise 8.11 solution



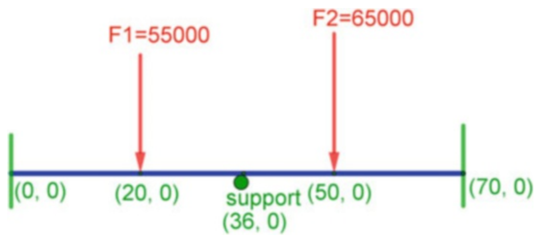
Exercise 8.12 A beam is fixed at its left end, fixed at its right end and subject to three point loads. The moment of inertia is constant, but is not specified. Intuit where any inflection point or points will be and capture the approximate moment reaction at the fixed supports.



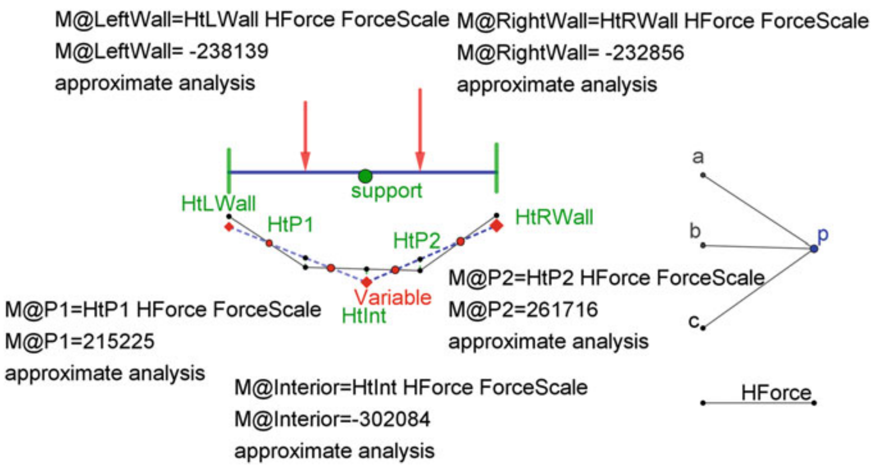
Exercise 8.12 solution



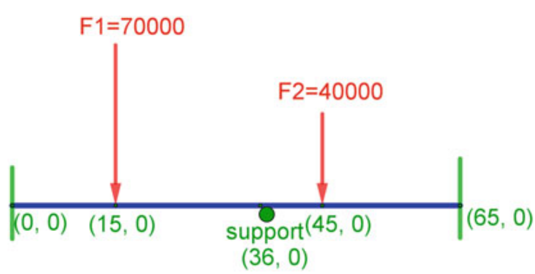
Exercise 8.13 A beam is fixed at its left end, fixed at its right end propped up at an intermediate point. It is subject to two point loads. The moment of inertia is constant, but is not specified. Intuit where any inflection point or points will be and capture the approximate moment reaction at the fixed supports. Also calculate the bending moment at the points of load application and directly above the interior support.



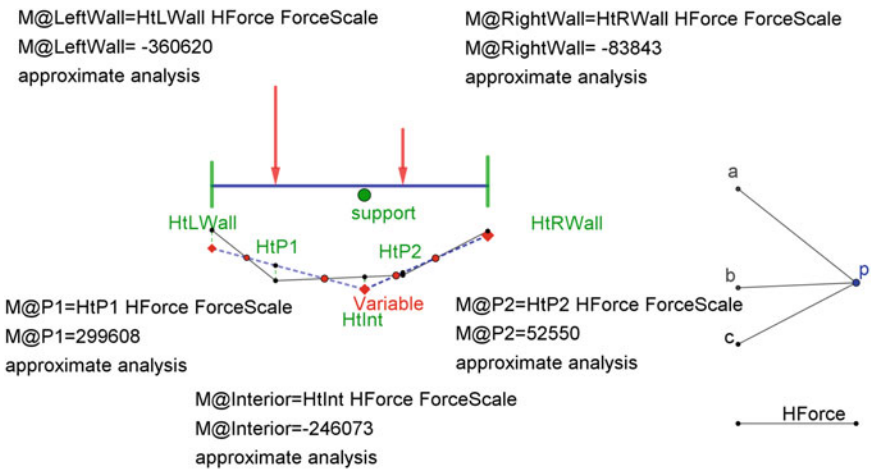
Exercise 8.13 solution



Exercise 8.14 A beam is fixed at its left end, fixed at its right end propped up at an intermediate point. It is subject to two point loads. The moment of inertia is constant, but is not specified. Intuit where any inflection point or points will be and capture the approximate moment reaction at the fixed supports. Also calculate the bending moment at the points of load application and directly above the interior support.



Exercise 8.14 solution



9 Indeterminate Truss Analysis

As was done in this chapter for beams, a kinematic analysis is necessary to calculate reactions of a statically indeterminate truss. This chapter will present the Williot-Mohr method of calculating nodal displacements. Initially, kinematic analyses of statically determinate trusses will be presented. The Williot-Mohr method establishes the final displacement of all the nodes, not just a single node's X or Y movement as is done with Virtual Work, and the Unit Load Method.

Displacement analysis of trusses by means of purely geometric, kinematic tools was introduced in the United States by Molitor in 1894. This method was first presented by Williot and modified by Mohr, hence it came to be known as a Williot Diagram with a Mohr Correction. The method was widely described in textbooks devoted to structural analysis, but in a story that parallels that of Graphic Statics, the method was abandoned and nearly forgotten. In the 1894 paper by Molitor entitled "The Graphical Solution of the Distortion of a Framed Structure", the author asserts that the graphical method for finding displacements of a truss are superior to the algebraic methods because they are "accurate and expeditious". He adds that the graphical methods proposed by Williot and refined by Culmann provide displacements of all the nodes in the planar 2D truss, whereas the algebraic method, now known as Virtual Work, provides only one displacement at a time. Molitor cites the thoroughness of the 1894 textbook "Modern Framed Structures", and it is true that this book does a remarkable job of summarizing, not introducing, extremely sophisticated state-of-the-art algebraic methods since it was then "less than 50 years since the first successful attempt was made to correctly analyze the stresses in a framed structure". That credit goes to Squire Whipple of Albany, New York who wrote an astonishing booklet in 1847, a work that correctly analyzed bridges for both static and moving loads, and for computing the total "strain lengths", i.e. deformations, of various styles of trusses.

For indeterminate trusses, the kinematic analysis will be performed twice, once on a truss made statically determinate due to the judicious removal of a redundant. The second part of such analyses is typically much faster than the first part due to the presence of many zero force members, or symmetry, or the ability to solve for statics by inspection. Enforcement of all boundary conditions leads to a final solution.

A three bar truss shown in Fig. 9.1 is subject to a vertical load at its crown. The area (A) of each bar and the modulus of elasticity (E) are constant, and the product EA is 2,000,000 units of force. The truss is 10 units wide, 4 units tall, with a 50,000 force load applied at the crown which is at (4, 4) in Cartesian Coordinates. The reactions are found graphically, and each individual bar's axial deformation is calculated. The truss is statically indeterminate because of the presence of an unknown restraining force at the right end. This force is such that the roller does not move horizontally.

E. Saliklis, *Structures: A Geometric Approach*, https://doi.org/10.1007/978-3-319-98746-0_9

This is shown in Fig. 9.1. Of course, if the truss was pin supported at each base, the bottom chord will be a zero force member and the truss reduces to a three hinged arch. Thus, this redundant force really plays the role of a horizontal pin force.

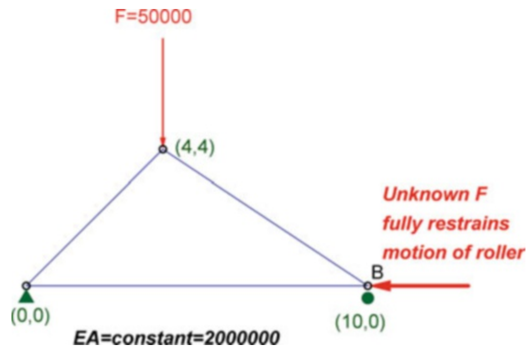


Fig. 9.1 Externally indeterminate truss

As would typically be done in algebraic analyses, one reaction is termed "Redundant" and it is temporarily removed to make the truss statically determinate. Then, the displacement in the direction of the redundant reaction is calculated. Here, such calculations will be performed graphically using the Williot-Mohr technique.

Figure 9.2 shows the removal the right side X direction reaction. Now the truss is statically determinate and the reactions *bc* and *ca* can be found graphically.

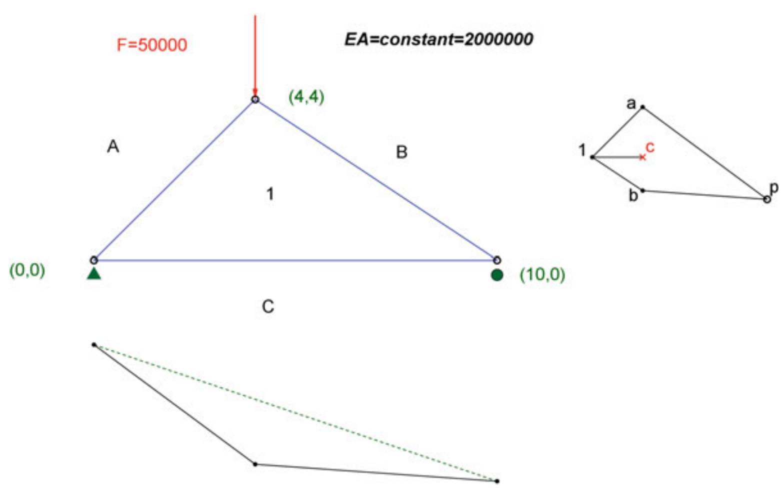


Fig. 9.2 Remove horizontal reaction on right as redundant

Step 1 is to assume that some bar, for example *A1* does not rotate. It simply shortens but its orientation does not change from the original state. The new location of the crown which temporarily sits in line with *A1* is established based on this premise. One way of drawing this is to create a circle of radius

$$r1 = {}^{(L_{A1} - \delta_{A1})} \Big/ {}_{Deformation\ Scale}$$

Where the axial deformation δ is defined as $\delta_i = \frac{F_i \cdot L_i}{A_i \cdot E_i}$

Another circle centered on this temporary crown,

$$r2 = {}^{(L_{B1} - \delta_{B1})} \Big/ {}_{Deformation\ Scale}$$

and a circle centered from the lower left pinned support, of radius

$$r3 = {}^{(L_{C1} + \delta_{C1})} \Big/ {}_{Deformation\ Scale}$$

will help locate the temporary roller. Where these two circles intersect is the temporary location of the right hand roller support. This is shown in Fig. 9.3.

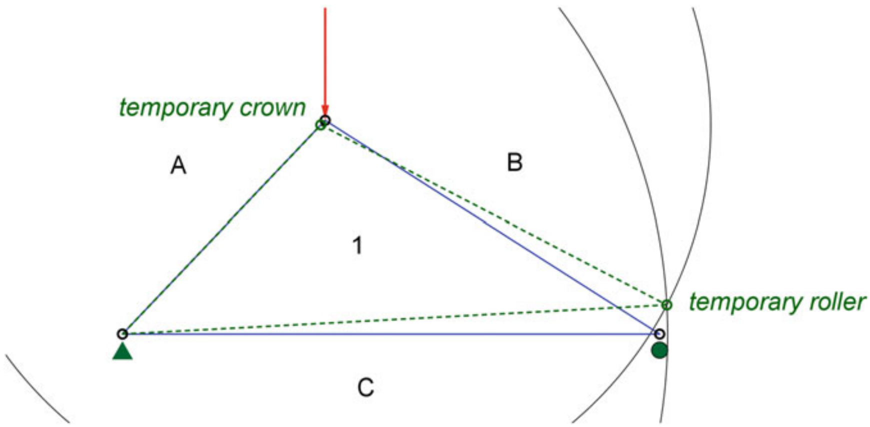


Fig. 9.3 Temporary but incorrect position of one support

The lengths are now correct, but a correction must be made to account for the requirement that the roller cannot move vertically.

Rotating the green dashed truss means moving along a line perpendicular to the original A1 position.

Using similar triangles

$$\frac{temp\ crown - final\ crown}{Length\ A1_{new}} = \frac{temp\ roller - final\ roller}{Length\ C1_{new}}$$

allows for the calculation of the distance

$$temp\ crown - final\ crown$$

to be found since all the other values are known. The angle swung through, when moving the temporary roller to the final roller, (the path is tangent to the bottom chord) is the same angle as swinging the temporary crown to the final crown (a path tangent to the top left chord).

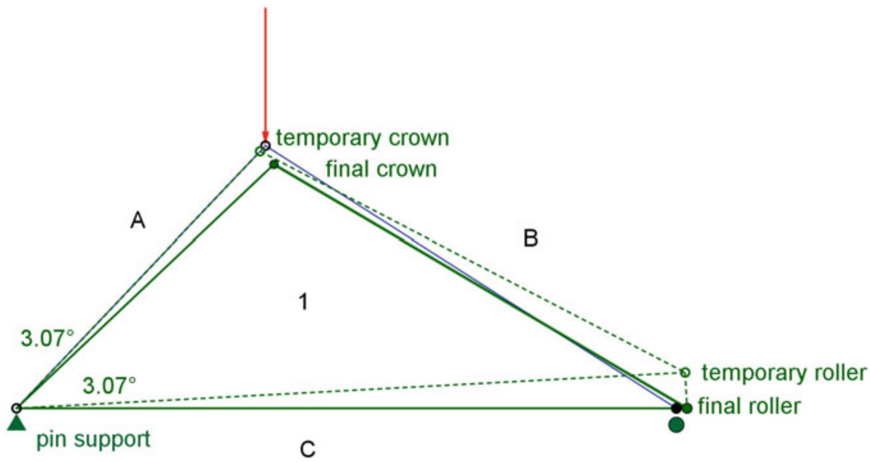


Fig. 9.4 Establishing the final roller position

The Williot Diagram repeats these same steps, but it does so in a more elegant and compact manner. In the Williot Diagram, only the deformations are plotted, not the deformations added to the original lengths. As before, one end is a stable reference point and a bar associated with this reference point is assumed to initially not change orientation. Thus, in this example, the left pinned support is taken as the reference point and only the axial deformation of bar *A1*, $\delta A1$ is plotted. To calculate the axial deformation of the element, it is assumed not to rotate. In Fig. 9.4, the reference element *A1* is in compression, thus this axial deformation moves from the crown, toward the pin support, i.e. down and to the left as shown in Fig. 9.5, enlarged to some known scale.

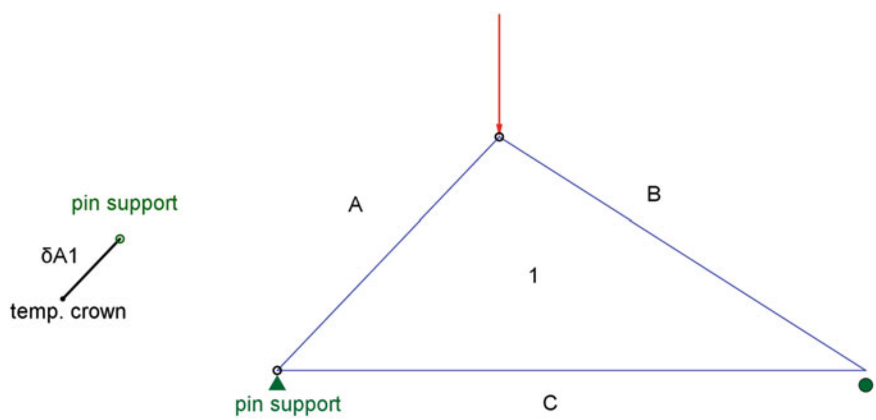


Fig. 9.5 Crown moves down and left with respect to left support

Now the subsequent element connected the initial reference point, (the pinned support) can be analyzed. In Fig. 9.6, that element is *C1*. *C1* elongates an amount $\delta C1$, but the reference point location is fixed, thus a lengthening of element *C1* is captured by the far end roller support moving away from the pinned support reference point. This means the next step on the Williot diagram is horizontally to the right from the reference point. This new position of the roller support is actually the final position because the roller support cannot move vertically.

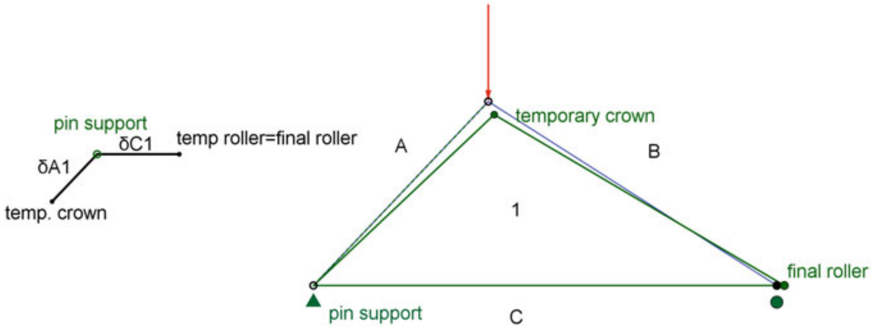


Fig. 9.6 Roller moves horizontally and to the right with respect to pin support

Next, the temporary crown position is found as it relates to element *B1*, this is a move down and to the right from the roller support because that element shortens, i.e. the crown moves towards the roller support. The two temporary crown locations must be reconciled through a tangential swing and the final position of the crown is established. The final movement of the crown is denoted by the dashed green lines in Fig. 9.7.

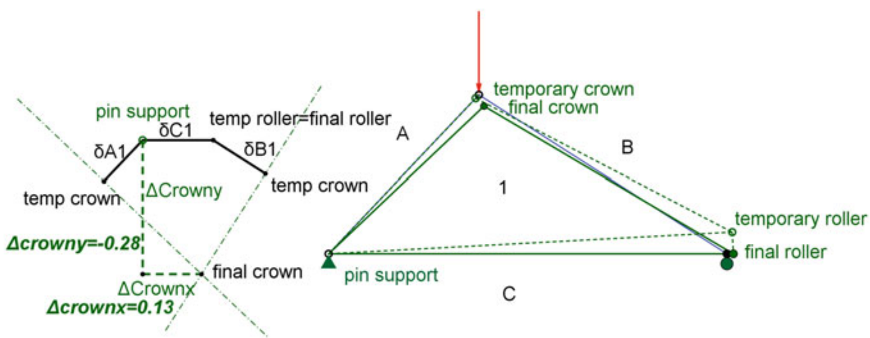


Fig. 9.7 Final roller position is found from intersecting arc lengths as perpendicular lines

Calculating the redundant force takes almost no effort. The movement of the roller is equal to δC1. Thus, moving that node back the same amount sets the final displacement to zero. The force causing this fictitious movement back is the applied redundant (Fig. 9.8). Only the redundant is applied, not the original loads. Then since:

$$\delta = \frac{Force \cdot Length}{EA}$$

The redundant force is immediately found from:

$$RedundantForce = \frac{\delta C1 \cdot EA}{LengthC1}$$

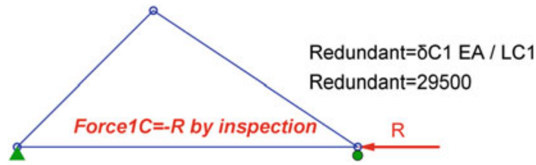


Fig. 9.8 Redundant force immediately found

In the following example, the forces are not given, rather, a series of known fabrication errors induced known axial deformations of each element. The truss is externally statically indeterminate due to the presence of four unknown reactions, two at A, one at B and one at C. This is shown in Fig. 9.9.

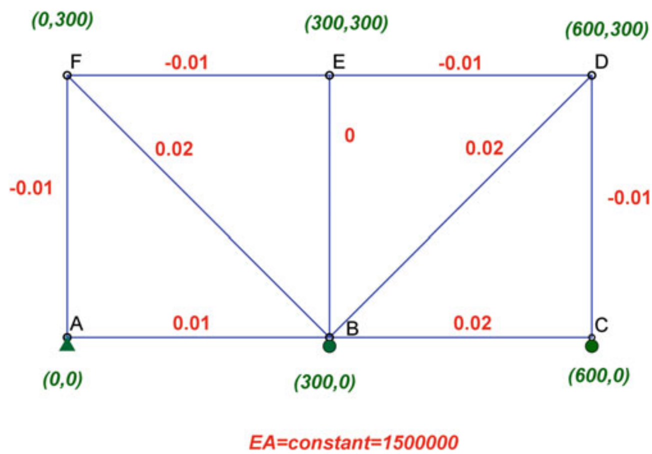


Fig. 9.9 Externally indeterminate truss

In this example one can assume that the vertical reaction at B is redundant and consequently it is temporarily removed to make the truss statically determinate. To begin the kinematic analysis, one can assume that the lower left bar will not rotate, i.e. it is assumed to remain horizontal. That is why it is shown in red in the Williot Diagram of Fig. 9.10. Then, the pinned support at A is a good starting point for the construction of the Williot Diagram. It is shown as point a, and symbolized by a red pinned support triangle in Fig. 9.10. Member AB has an elongation deformation, thus point b is to the right of point a. Member AF has a negative deformation, thus point f moves down with respect to point a. The temporary location of node B, called b', as the Mohr correction (rotation) has not yet been applied, is shown. Since member BF has an elongation deformation, point f moves up and to the left with respect to point b'. This is shown in Fig. 9.10. The resolved location of f' is found by (arcs) tangents to δBF and to δAF.

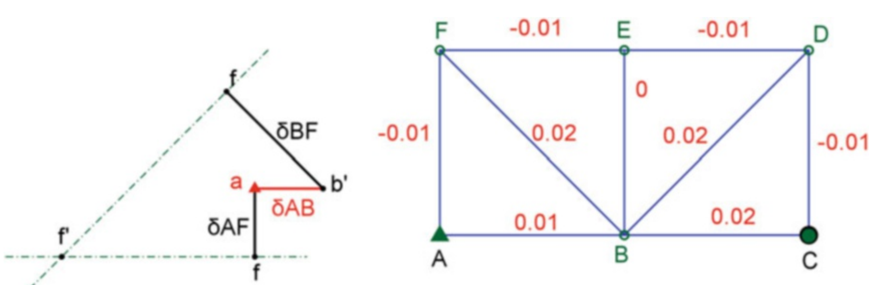


Fig. 9.10 Initial steps of Williot Diagram

Figure 9.11 shows subsequent additions to the Williot Diagram, all these are the deformations, shown at some convenient scale. Working from b', note that δBD, and δBC each move rightward from b' as these two elements have positive (elongation) deformations. From f', draw δEF which moves leftward (i.e. E moves left relative to F due to compression). Now there are two pieces of information linking E, but one of them is subtle. Draw a tangent from the end of δEF and then note that element EB is vertical, but δEB is zero, thus a tangent to this zero length vertical element must be horizontal. Those two tangents intersect at e'.

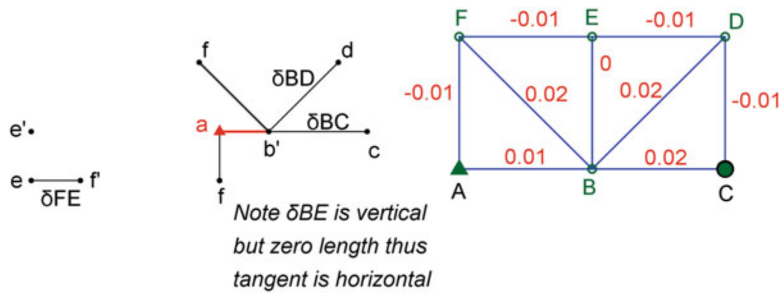


Fig. 9.11 Subsequent steps in Williot Diagram

From e', construct δED noting that D moves leftwards. And the locations of c' and d' still need to be established. This is done in Fig. 9.12. The location of d' is at the intersection of arcs (tangents) to δBD and to δED. Note that it is helpful and convenient to label these deformations such that the second index (here letter D) is the unknown. Thus label these as δBD and to δED rather than the less helpful δDB and to δDE. Points c' and d' are shown in Fig. 9.12.

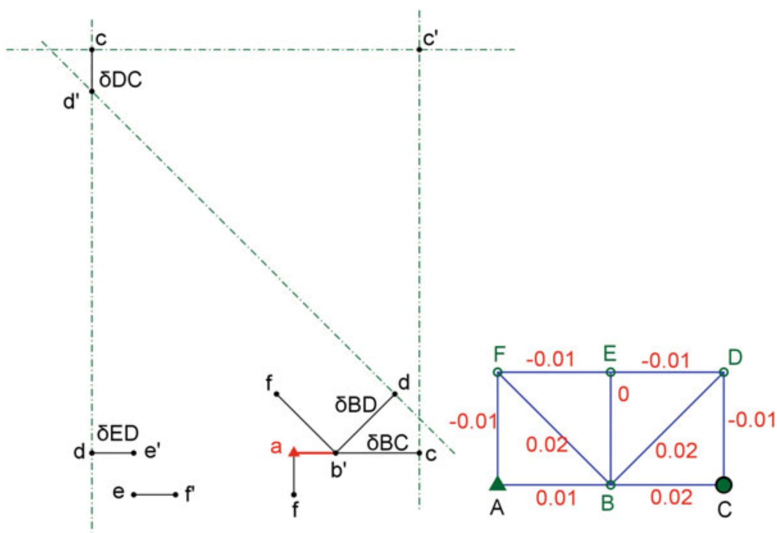


Fig. 9.12 All points identified in Williot Diagram

All the points have a temporary location now. Yet the construction of Fig. 9.12 was based on the assumption that the lower left bar AB remains horizontal. This assumption resulted in c' having a vertical and horizontal displacement with respect to A, since c' is up and to the right of a in Fig. 9.12. This cannot be true, as C is constrained to move only horizontally.

Thus, a correction is needed which is a pure rotation about a. This correction is shown in Fig. 9.13. Because of the small angle assumption, the arc of this motion is line perpendicular to AC, in other words a vertical line, since line AC is horizontal. The downward vertical vector through a, providing the movement of C for rotation about A is added to the diagonal (up and to the right) vector of the originally assumed movement of C with respect to A. The angle between these two vectors is α. The resultant of these two vectors is the final, true movement of C, shown as δC and the final location of the point uses the double prime notation, c″.

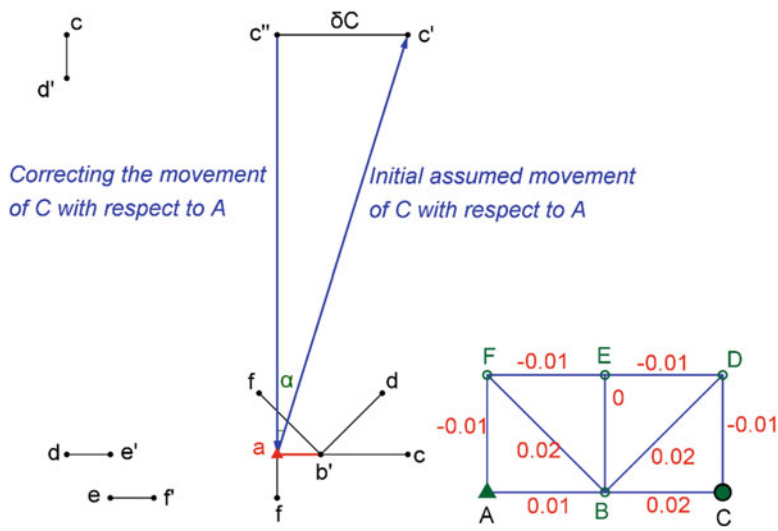


Fig. 9.13 Correcting the movement of point C with respect to point A

Thus far, only c'' has been established. It was found by adding the correction of rotation about the reference point a. During such rotation about a, not only joint C, but all the other joints are assumed to move in a tangential path normal to the vectors connecting each node to the reference pivot point A. The angle of rotation is α is small, thus perpendicular straight line movements replace arc sectors.

Now something remarkable happens! The distance from is c'' to a on the Williot Diagram mimics the length of C to A on the original truss. Draw the remaining truss elements to this scale, but rotate it 90° about the reference point a. Superimposing this new correction onto the original Williot diagram establishes the final point b'', c'' etc. This superimposed diagram is called the Mohr correction diagram. The true deflection of any joint in real life is found, with both magnitude and direction, from each vector drawn from the double prime point on the Mohr diagram to the corresponding single prime point on the Williot diagram, divided by the *ScaleFactor* used to create the original Williot diagram. For example, Fig. 9.14 shows how to establish the location of b''.

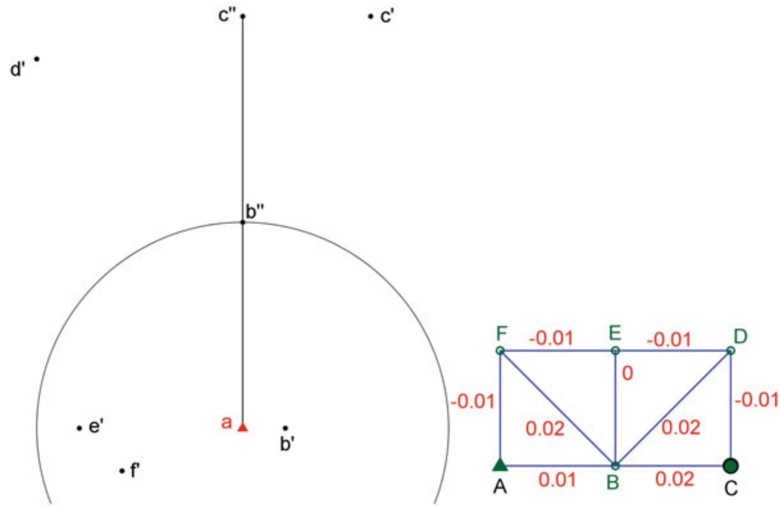


Fig. 9.14 Double apostrophe denotes the Mohr Correction to points

The previous step shown in Fig. 9.14 was meant to explain the process of obtaining the final points. In reality, it is extremely quick to make this final step. Simply use this scale to draw the original truss which is the Mohr diagram, superimposed upon the existing Williot diagram, but the Mohr truss is rotated 90° about the reference point *a*.

All of the final points are established in Fig. 9.15.

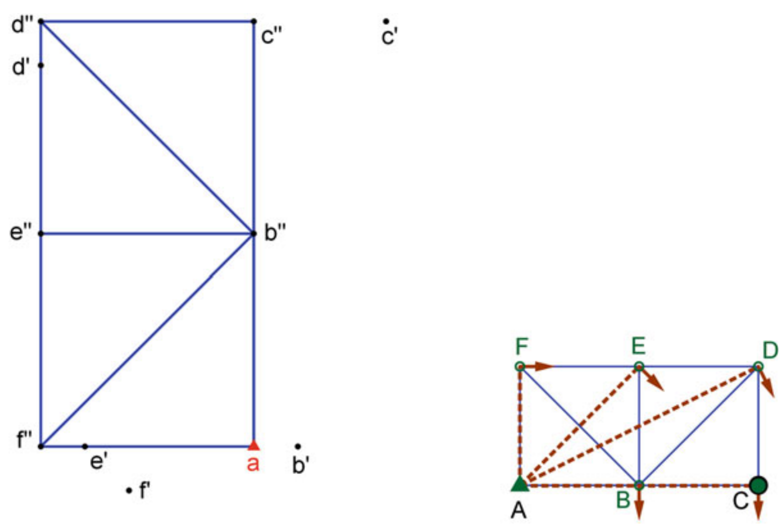


Fig. 9.15 Remarkably, the Mohr Correction mimics the shape of the original truss

The actual movement of each node in real life is captured by the vectors drawn from the Mohr diagram double prime point, to the Williot diagram single prime point, divided by the scale used to draw the Williot diagram. These are shown as green dashed vectors in the following Williot-Mohr diagram displayed in Fig. 9.16.

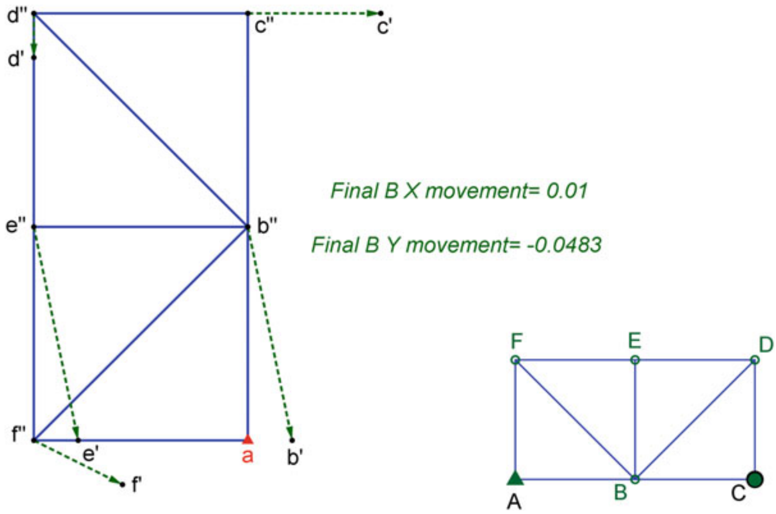


Fig. 9.16 Double apostrophe to single apostrophe captures real life movement

Note that since By was removed as a support from the original truss shown in Fig. 9.9, only the vertical movement at B is needed to solve for the unknown redundant RBy. No intuition can be used to estimate the magnitude of this force. Yet, because the analysis is linear, any guess for the magnitude of RBy can be used. The direction must absolutely be vertical, but it is not strictly necessary to use an RBy that is upwards. In Fig. 9.17, a force of 1000 units of force has been applied to B vertically upward. The goal of this analysis is to establish the relative vertical movement between point G (or I) to point H. As shown in Fig. 9.17, for 1000 units of force at B, this relative movement is 0.48 units of length which is approximately 100 times the required movement of 0.0483 units of length shown in Fig. 9.16. Thus, calculate the exact ratio of the movement of H due to the redundant of 1000, divided by the movement of B due to the fabrication errors. That ratio for is 10. Thus, the actual redundant force at By is $1000/10 = 100$ units of force. This is the exact answer.

In these second Williot analyses in which some magnitude of redundant load is applied, it will be helpful to use a second force scale, independent of the force scale used to represent the original loads, and it will also be useful to use a second Williot scale, as the redundant loading problem is completely independent of the original loading problem (Fig. 9.17).

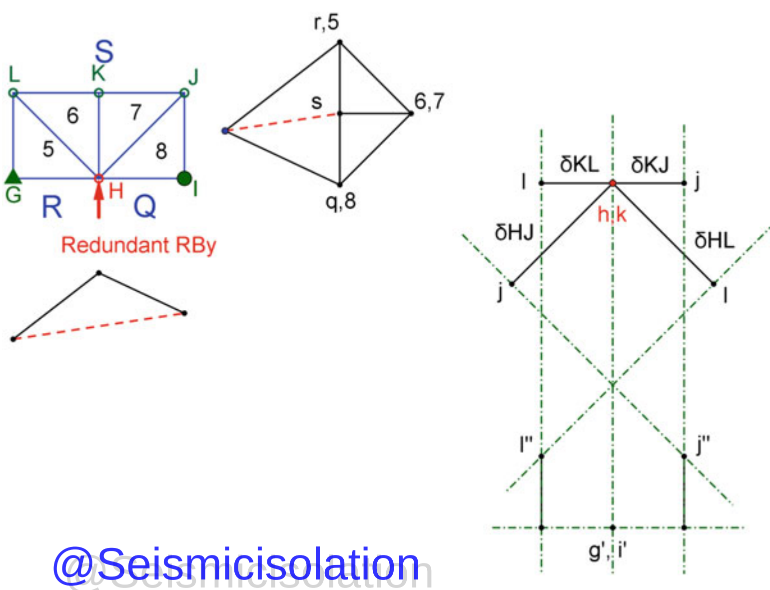


Fig. 9.17 An independent scale for the Williot diagram is helpful in capturing final bar forces, independent of original load scale

To further exemplify the technique of the Williot-Mohr method, revisit the previous three member truss shown in Fig. 9.1 and include the Mohr correction. For simplicity of this example, assume the truss is simply supported and thus statically determinate. Assume that the top left member, here shown as member QR remains at its original orientation. It is shown in red to emphasize this initial assumption. The first move is to capture the deformation of member QR. Here, that deformation is compressive, thus R moves down and to the left with respect to Q. This establishes r', the prime notation being from the initial assumption of the orientation of QR. From r', capture the deformation of RS, S moves up and to the left with respect to R. And from q, capture the deformation of member QS, here S moves horizontally to the right with respect to Q. Resolve δRS and δQS to establish s'. Then redraw the original truss, rotated 90°about q, such that s'' to q is a vertical line, and s'' to s' is a horizontal line (no vertical displacement allowed here). This establishes s'' as well as r'' through the re-drawing of the truss. The final deformation of the crown R is shown as the vector from r'' to r' (δR), and the final deformation of support S is shown as the vector from s'' to s' (δS). This is all shown in Fig. 9.18.

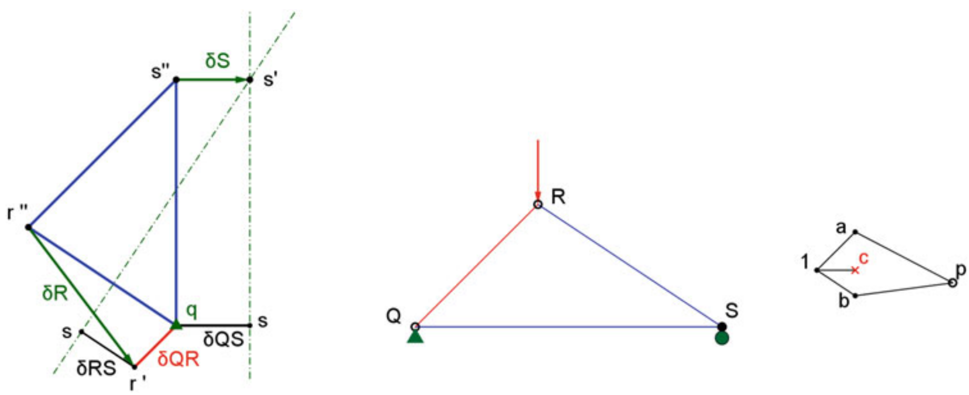


Fig. 9.18 Truss of Fig. 9.1 revisited using Williot technique

It is not necessary to always rotate about a pinned support. For example, in the following truss, the process can be started by assuming that member cC remains vertical. The truss is pinned at the left end a, roller supported at an interior point d, and roller supported at the right end e. It has a span of 32 units of length, and a height of 8 units of length. It is subjected to a single point load downward from b. Let the force at b equal 200 units of force. The truss is externally statically indeterminate, with vertical reactions at a, d and e. The horizontal reaction at a is zero. This is all shown in Fig. 9.19.

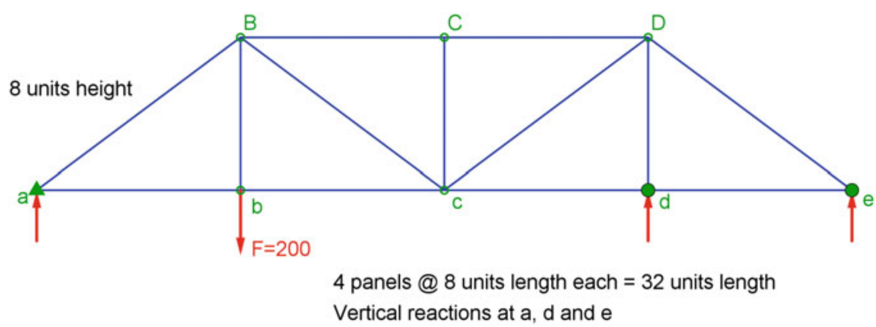


Fig. 9.19 New example where bar cC is assumed to remain vertical

In this example, the vertical reaction at d was chosen as redundant. Temporarily removing this reaction creates a stable, determinate truss. The vertical movement at d must be calculated, then only a redundant force will be applied to d vertically, and the two movements will cancel each other, which solves for the final force at d. First, the reactions for the temporarily determinate truss and all the bar forces are graphically obtained. This is done with a force diagram. Reactions are rapidly found, shown in Fig. 9.20.

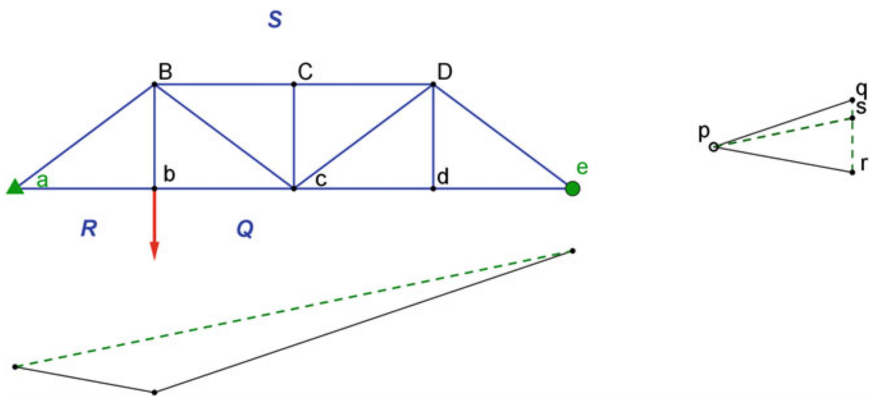


Fig. 9.20 Vertical reaction at d was chosen as redundant

Bar forces are also quickly found as shown in Fig. 9.21.

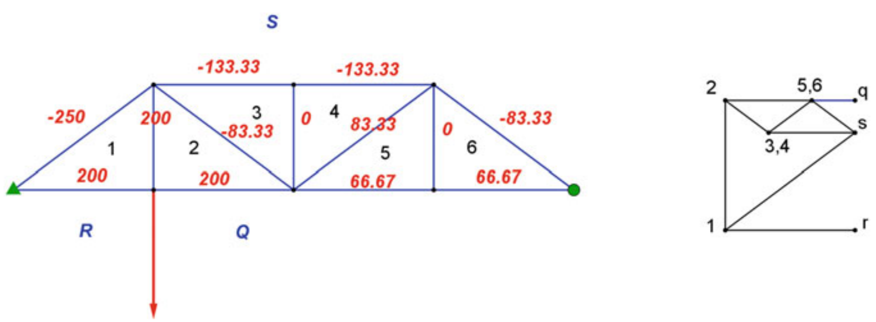


Fig. 9.21 Bar forces quickly found graphically

The Williot Diagram is begun based on the assumptions that cC remains vertical. Here are a few first steps. Locate D' first, resolving the ends of δcD and δCD. Then locate d′ noting that element Dd is vertical, but of zero length (here symbolized by the orange segment), and use that segment and segment δcd to resolve d'. This is shown in Fig. 9.22.

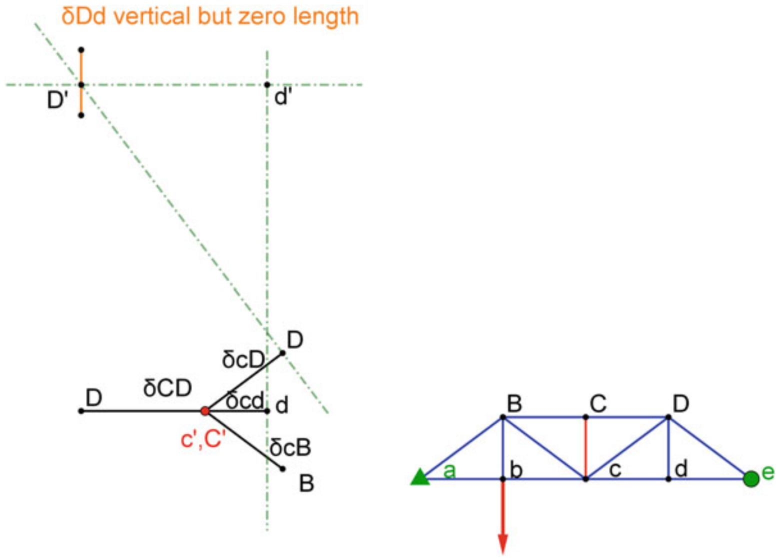


Fig. 9.22 Initial steps to create Williot diagram

Figure 9.23 shows how to locate e'

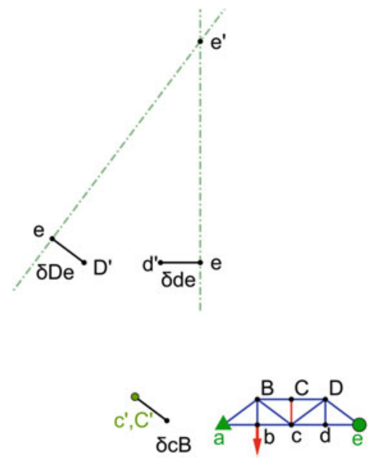
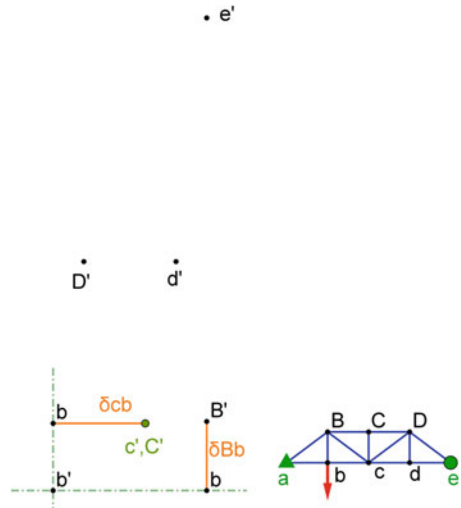


Fig. 9.23 Location of point e'

Figure 9.23 also shows the recommended good practice of hiding (not deleting!) the elements that have been used and are no longer needed. B' is found next, from δbB and δcB. Note, it is more useful to name these deformations with the second index as the sought after term, so label as δbB and δcB rather than δBb and δBc. Leaving the unknown letter as the second index helps prevent programming errors.

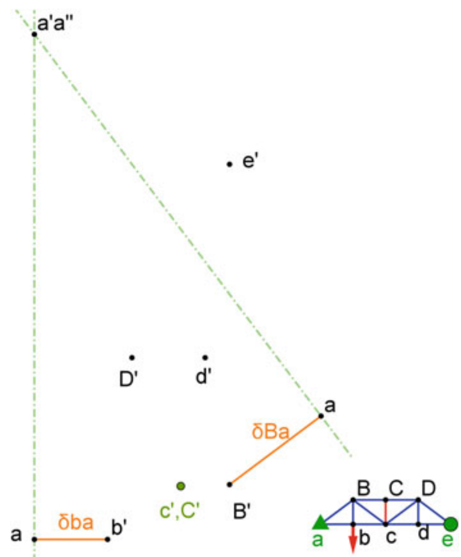
Find b' shown in Fig. 9.24 and then a'.

Fig. 9.24 Subsequent steps use second index as sought after item, this helps reduce errors



Note that a' will also be a'' as the distance in the final Williot-Mohr diagram between a'' and a' must be zero. This is shown in Fig. 9.25.

Fig. 9.25 Completion of Williot diagram



Having completed the Williot Diagram (single prime notation) allows for the creation the Mohr Correction Diagram (double prime notation). This is established by realizing that the distance between e'' (as yet not found) and e' must be pure horizontal, it can have no vertical component. Draw a horizontal line through e, and a vertical line through a'' till it intersects this horizontal line. This vertical line will be the Mohr Diagram capturing the lower chord of the real truss, in other words, a 90° rotation. In this example, the Mohr Diagram rotates the original truss clockwise. Of course, in the Mohr Diagram, this lower chord must be vertical. Take note of the scale of this segment and use that scale to recreate the rest of the truss. Final displacements of any point, say B, are calculated by tracking the double prime location (B'' for example) to the single prime location (B'). In the final

Williot-Mohr Diagram, these displacements match the truss displacements exactly. In this example, the downward vertical movement of Point *B* is nearly three times the rightward horizontal movement of *B* (Fig. 9.26).

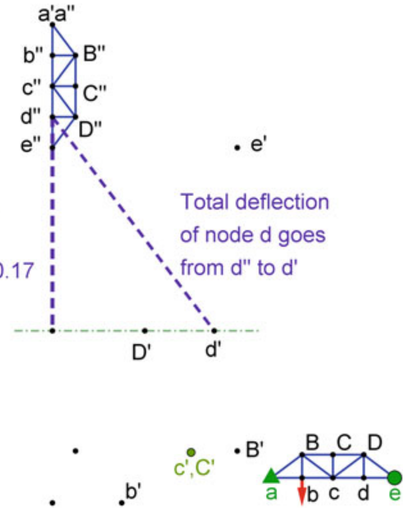


Fig. 9.26 Immediate Mohr Correction is found by replicating truss shape

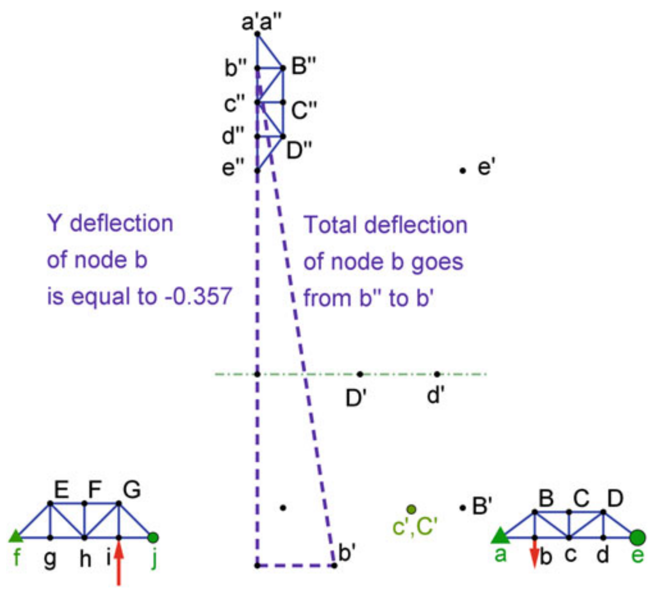


Fig. 9.27 Real life deflections captured by going from double apostrophe to single apostrophe

To avoid solving this problem once again, this time for an upward load at *i* as shown in Fig. 9.27, note that the original loads can be used in a clever manner. By viewing the original truss from behind, node *b* would one interior panel space from the right side. Thus, use the original load at *b*, but calculate the displacement at *b*, this would be equivalent, but opposite in sign, to an upward load at node *i*.

The original downward deflection at node *d* due to the load of 200 at *b*, is negated by an upward load at *i* (here use load at *b* and switch sign), and use 200 lb (Fig. 9.28). The deflection at *d* due to load at b is -0.17 units of length. If 200 units of force are applied at i (at b from behind), then the deflection at i (at b from behind) is +0.357.

Thus

$$\Delta = \Delta@d_{due\ to\ external\ load} + Redundant@d \cdot \Delta@d_{due\ to\ redundant\ load} = 0$$

$$Redundant = \frac{0.17}{0.357} 200 = 95.6\ units\ of\ force$$

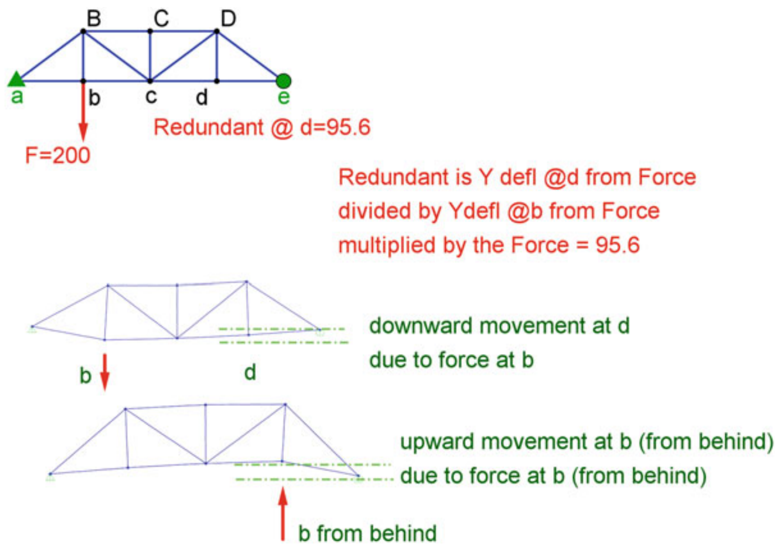


Fig. 9.28 Visual thinking allows for quick simplifications

The Williot-Mohr method can be used on three hinged arch/trusses, but prior to solving such a problem, it is useful to review the graphic statics of the simple three hinged arch. The following arch is pinned at each end and the span is 8 units of length. An internal hinge is at the center 2 units of length above the supports. A downward force of 8000 units of force is applied to the left side of the arch, 2 units of length from the central hinge. This is shown in Fig. 9.29.

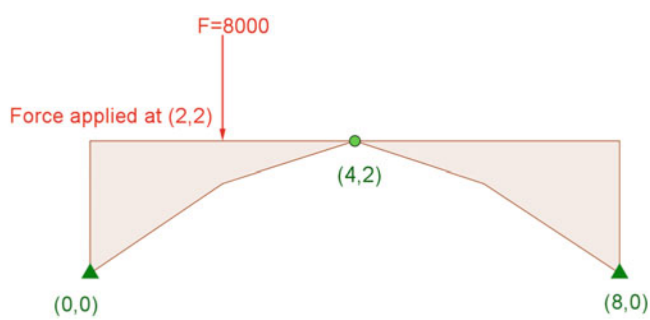


Fig. 9.29 Review of statics of three hinged arch which will be made into a truss

Although there are four external reactions, the system is statically determinate due to the extra equation stating that the internal moment at the hinge must be zero. In all such three hinge arch problems, it is necessary to "break apart" the arch into two pieces separated by the hinge. If there is a load placed directly on the hinge, arbitrarily place it just a bit to the right or to the left of the hinge.

In this example, since the right hand side has no external loads, the net reaction at the right pin support is the only force acting on the right hand side free body, and in order for there to be no bending moment at the internal hinge, the line of action of the right pin support must pass through the hinge. Thus, the direction of the right hand reaction is known. This is shown in Fig. 9.30.

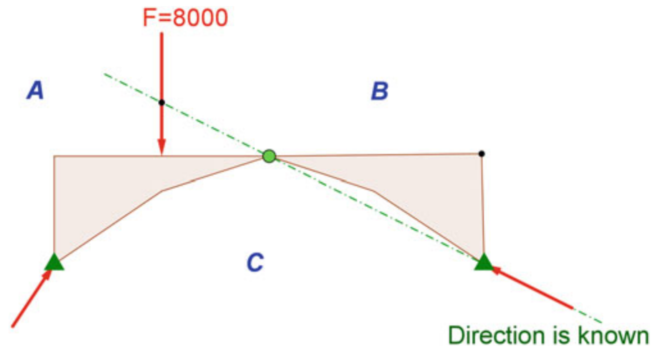


Fig. 9.30 Always start with easier side

The reactions are solved for quickly if the problem is viewed as equilibrium of a point. The left reaction must pass through this point, and this point must be on the line of action of the external force F. A scaled force polygon immediately establishes the magnitude of the reactions, as their orientations are already known as is shown in Fig. 9.31. The directions of the reactions by default establish the location of point c on the force diagram.

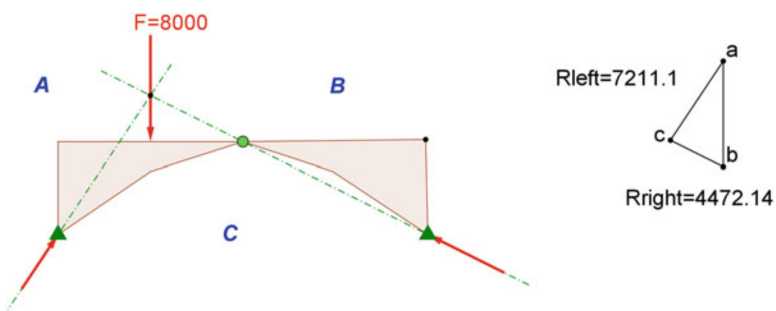


Fig. 9.31 Reactions are found via equilibrium of a point, not via funicular

The next example is more involved as it breaks up the arch into two distinct free bodies, each of which is loaded. The geometry of the arch is the same as before, but loads $F = 2000$, 1500, 2500 and 1000 units of force are applied at $x = 1$, 2, 5, and 7 units of length respectively. The internal hinge is still at $x = 4$ units of length. See Fig. 9.32

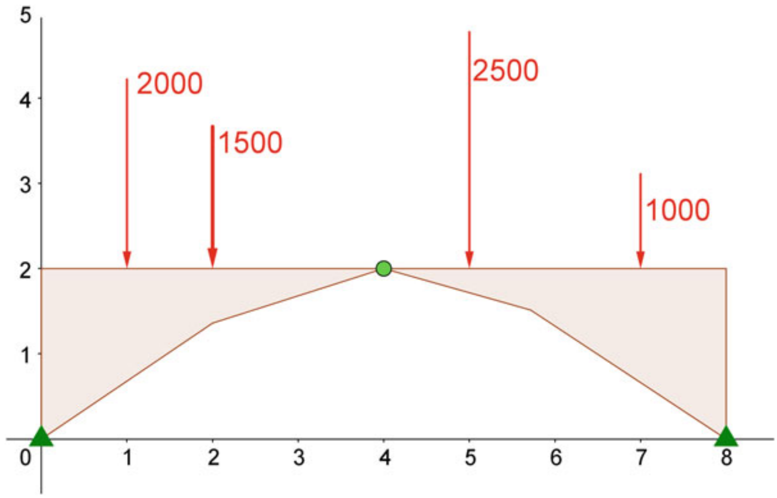


Fig. 9.32 Three hinged arch will morph into a three hinged truss

Figure 9.33 shows the force diagram capturing scaled drops representing the four external forces. Since neither side is unloaded as was one side in the previous example, it is necessary to establish a pole, *p1*, and use it to draw two separate funiculars. One funicular starting at *A* and terminating at the line demarcating the internal hinge. Then the other funicular starting at *B* and terminating at this same vertical line. A closing line for each funicular is drawn, and each of those closing lines are passed through the pole *p1*. The first closing line locates point *q1*. The second closing line locates point *q2*.

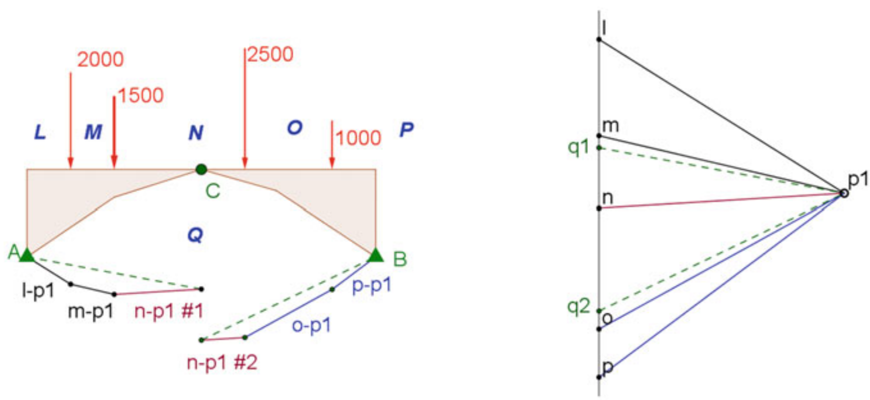


Fig. 9.33 Two separate funiculars are drawn, noting the gap between them

Having the points *q1* and *q2* allows for the completion of the analysis, which is called "passing a funicular through three points". Figure 9.34 shows how the final pole is established. For convenience it is located left of the load line. A line parallel to *AC* is passed through *q1*, and a line parallel to *CB* is passed through *q2*. Where these lines intersect locates the final pole.

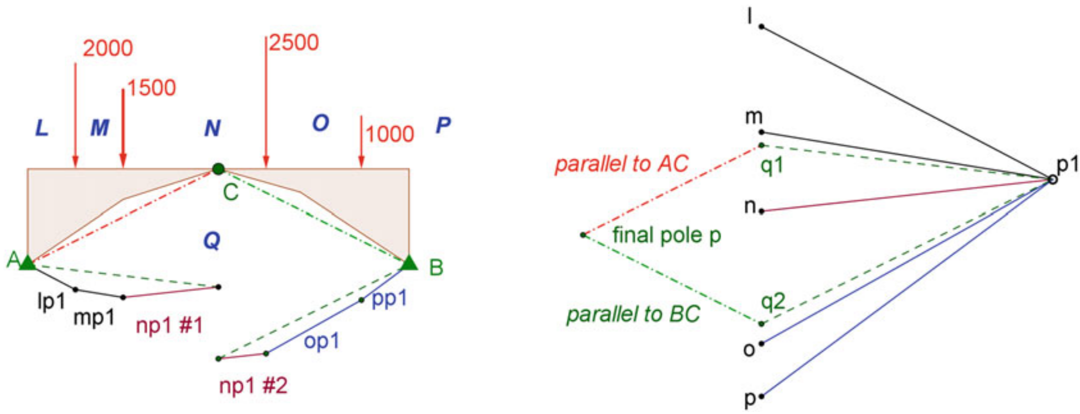


Fig. 9.34 Establishing the final pole

The final funicular must pass through the internal hinge to have zero moment there. A check of the work is to see if the funicular passes through all three prescribed points, if so then the answer is correct. Notice that as was shown in Fig. 9.31, here the final pole is the location of Point *q* on the force diagram.

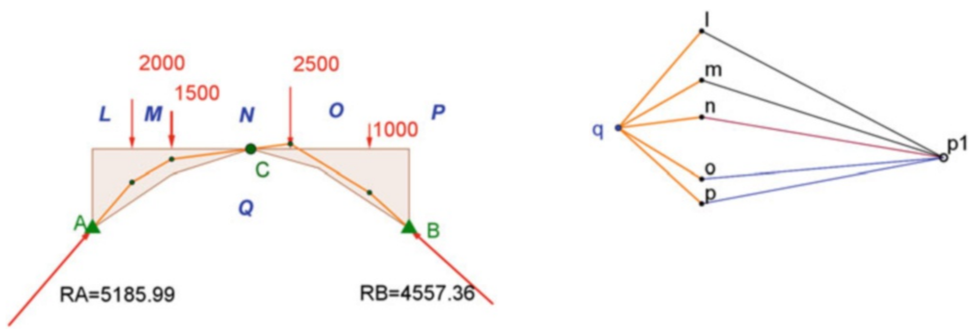


Fig. 9.35 Final funicular is not fully contained within the arch form

There is a beautiful secret partially revealed in the previous diagram. The funicular is nearly contained within the arch itself. This ancient secret was known by master builders of Gothic cathedrals. This arch is not perfectly designed because the funicular slips out a little bit, at the right of the internal hinge. This means that some bending will occur in this form.

Another small surprise awaits within Fig. 9.35. A truss constructed from the path of the funicular will exhibit zero force members for every member off of the funicular. For example, Fig. 9.36 shows one such truss. The first few steps of locating interior points are shown, and more such steps will also result in every member off of the funicular being a zero force member. Of course, if the loads change, then these other members will begin to pick up loads.

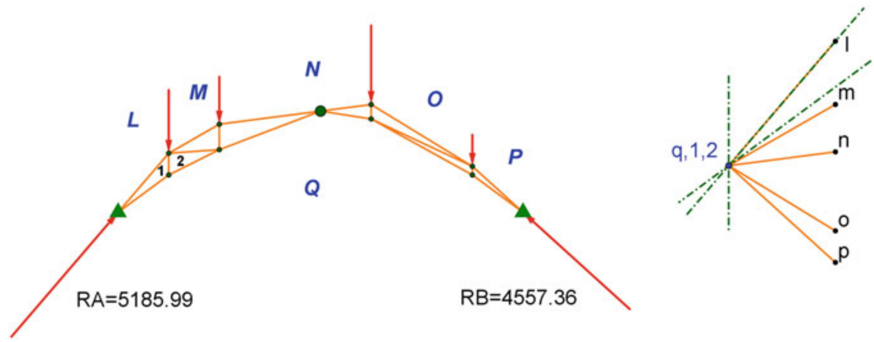
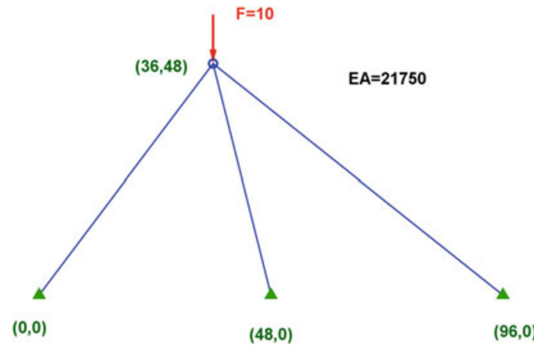


Fig. 9.36 Arch can morph into a truss

Such a truss could be made statically indeterminate, and then the Williot-Mohr Method would be employed to solve for the redundant.

This chapter on indeterminate truss analysis concludes with a truss that is internally statically indeterminate. The truss is shown in Fig. 9.37.

Fig. 9.37 Internally indeterminate truss



The technique shown here parallels the traditional "unit load" method of virtual work, namely an interior member will be "cut" (i.e. removed from force carrying system), then the movement along the cut bar will be captured. Then a force will be applied to the cut member until the gap across the cut bar is zero. This force is then the redundant member force. Algebraically this is:

$$\Delta + F_{redundant} \cdot \delta = 0$$

$$\sum_{i=1}^{n} \frac{F_i \cdot f_i \cdot L_i}{AE} + F_{redundant} \cdot \sum_{i=1}^{n} \frac{f_i \cdot f_i \cdot L_i}{AE} = 0$$

Arbitrarily remove the interior member to create a stable, determinate truss. The movement of the gap across the cut is captured by the relative movement of the crown (Point R) compared to the support of the cut member. This movement is readily found using the Williot technique. Figure 9.38 establishes the forces in the determinate truss due to the applied 10 unit load.

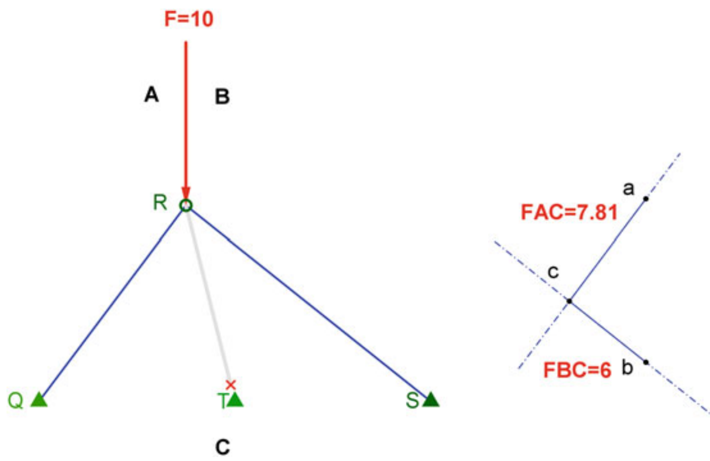


Fig. 9.38 Temporarily statically determinate truss due to removal of redundant strut

Figure 9.39 shows two analogous ways of capturing the movement of the crown Point *R* which is the movement sought after namely *ΔRT*. Swinging two arcs whose radii reflect the shortened lengths of the members allows for the location of the intersection of the arcs, thus Point *Rnew*. Alternatively, the Williot Diagram only shows deformations magnified to some comfortable *WilliotScale*. Perpendicular lines replace arc lengths and *Rnew* is found. Note that the deformation along the path of the redundant interior strut is needed! This is a component of the deformation of the crown. A helpful hint is to label each Williot extension with the second letter of the two-letter index being the sought after variable. Thus, it is more helpful to label segments as *δQR* and *δSR* rather than *δRQ* and *δRS*.

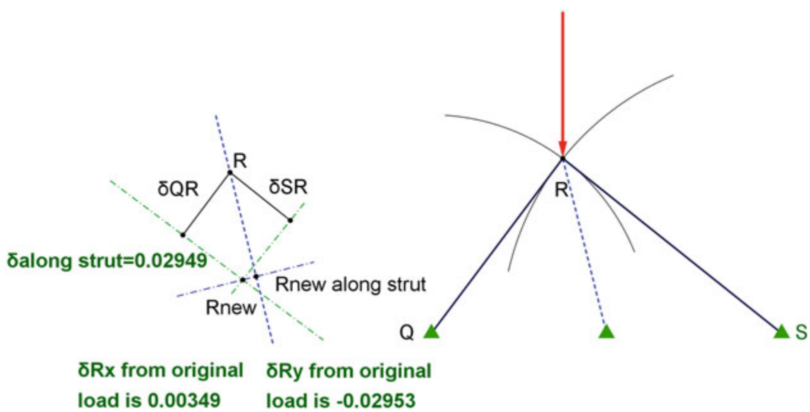


Fig. 9.39 Two analogous ways of establish position of crown *R*

If the interior bar was cut with an infinitely thin blade, the gap between the two cut portions would be Δ due to the applied 10 unit load shown in Fig. 9.40. The gap would be δ due to some redundant strut force added to the cut end of the interior bar itself. Compatibility requires that

$$\Delta + \delta = 0$$

Where:

- Δ is deformation of the crown along the axis of the redundant strut, due to the applied load
- δ is deformation of the crown along the axis of the redundant strut, due to the redundant force *Fstrut*, plus the deformation of the strut itself

For example, if $Fstrut = 3.4$ units of force was arbitrarily chosen as the redundant force, the statics of the truss are immediately solved for via equilibrium of a point as described in Chap. 1, Fig. 1.3 for example.

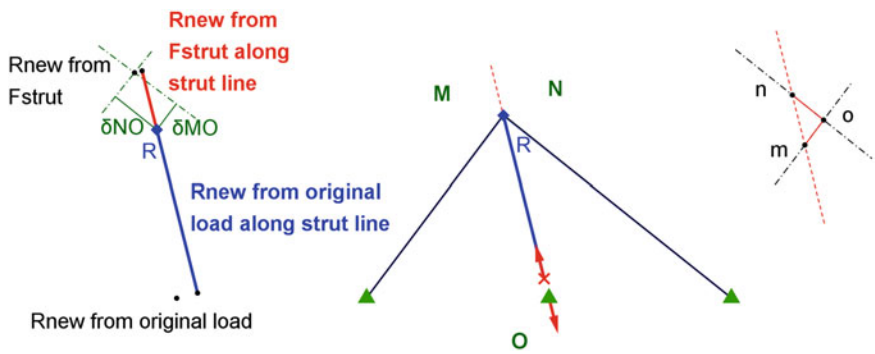
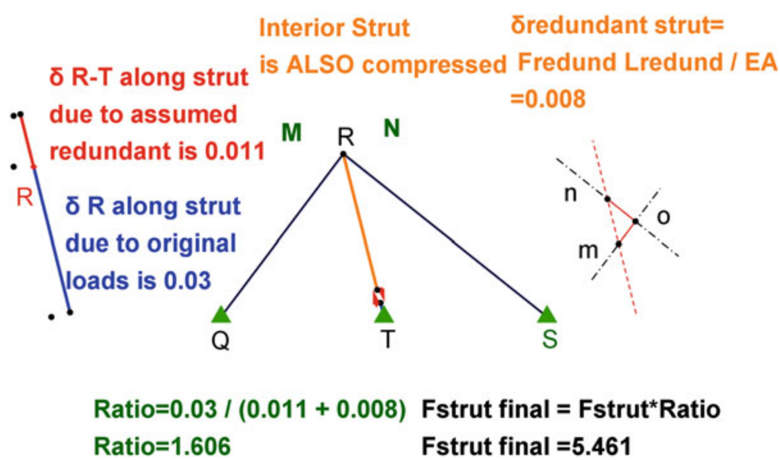


Fig. 9.40 Chosen value for redundant force irrelevant, value can be 1.0 or something similar

The deformation of the crown along the axis of the interior strut is found as before. For the guessed value of $Fstrut = 3.4$ the crown moves up and to the left, but the deformation of the redundant strut will also have to be taken into account. Think of this as springs in series, the deformation of the interior strut must be included. This is also true when approaching this problem as a virtual work exercise. Of course, both the deflections due to the applied 10 unit load and the deflections due to the assumed strut force must be drawn to the same Williot scale (Fig. 9.41).

Fig. 9.41 Final strut forces quickly found graphically



The final redundant strut force is found from:

$$F_{redundant\ final} = F_{strut\ assumed} \cdot \frac{\delta_{RT\ due\ to\ loads}}{\delta_{RT\ due\ to\ Fstrut} + \delta_{strut}}$$

The final example is an indeterminate truss because of the redundant internal strut. The truss is subjected to two external loads as shown in Fig. 9.42.

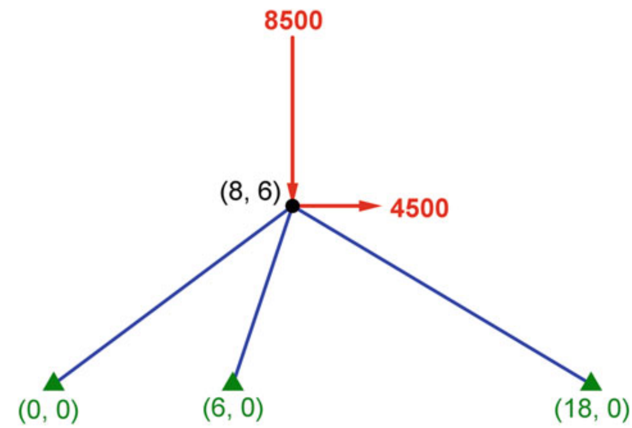


Fig. 9.42 Internally indeterminate structure with gravity plus lateral loads

As was done in Fig. 9.38, the internal strut is temporarily removed and it is labeled as the redundant. The movement of the crown node *R* is readily found via the Williot Method. But the critical piece of information needed is the movement of node *R* *along the path of the redundant* (Fig. 9.43).

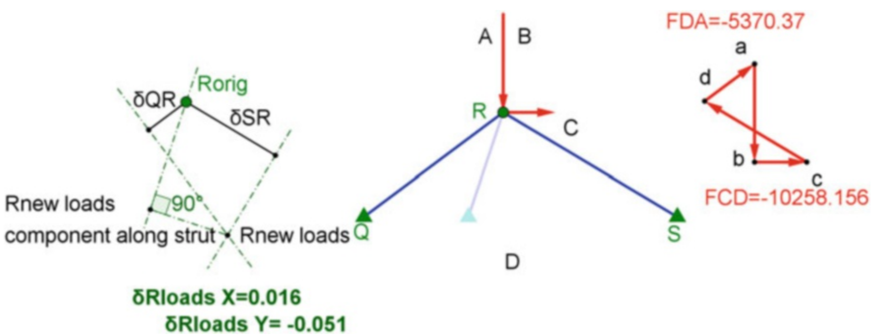


Fig. 9.43 Temporarily removing strut to make truss determinate

Then, as was done in Fig. 9.40, some redundant force is applied along the axis of the redundant. The magnitude of this assumed load is inconsequential, it could be a unit load 1, or it could be some other assumed load as in Fig. 9.44 where the strut force was assumed to be 1000 units of force. The Williot diagram immediately provides the final position of node R due to this assumed load. Again, it is critical to capture the movement of node *R* along the path of the redundant strut.

It is extremely important to note that three members are being strained by this redundant force, not two. There is strain energy in the redundant strut itself. This movement is captured as δRT in Fig. 9.44.

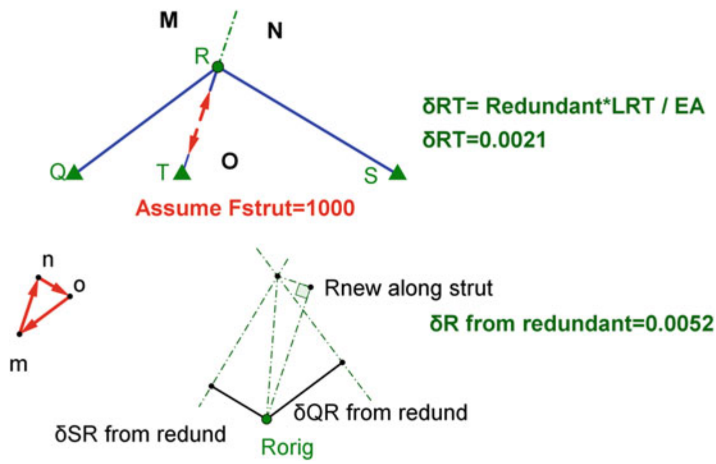


Fig. 9.44 Assume some arbitrary value for the redundant force

The final redundant strut force is found from:

$$F_{redundant\ final} = F_{strut\ assumed} \cdot \frac{\delta_{RT\ due\ to\ loads}}{\delta_{RT\ due\ to\ Fstrut} + \delta_{strut}}$$

This is shown in Fig. 9.45.

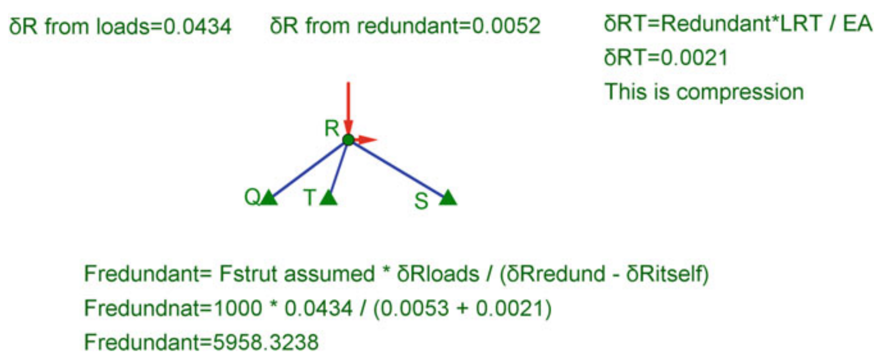
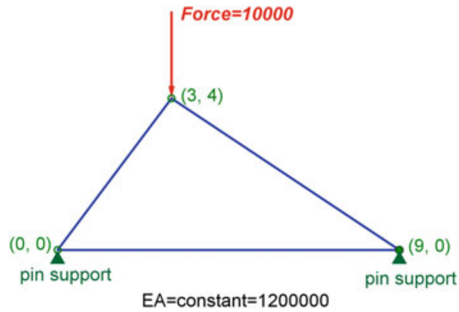


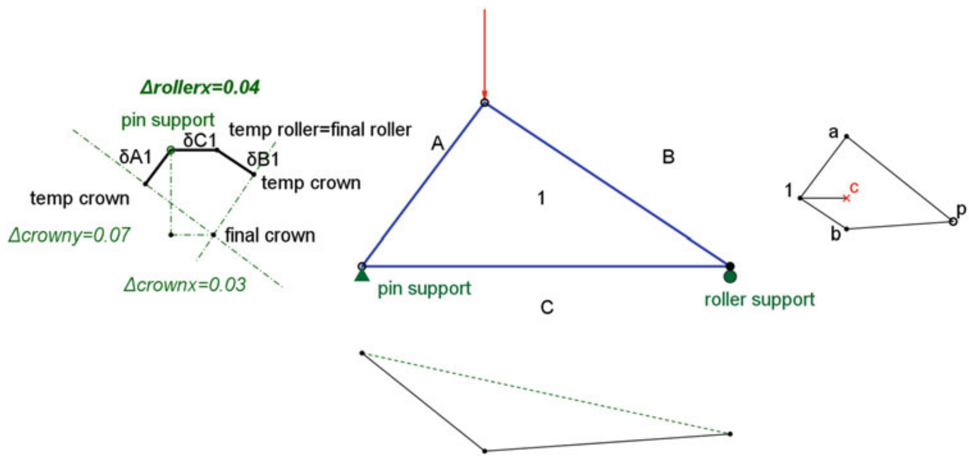
Fig. 9.45 Final redundant force found by enforcing compatibility

Chapter 9 Exercises

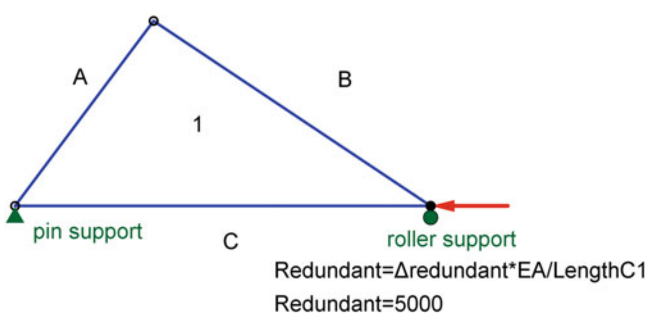
Exercise 9.1 A statically indeterminate truss is pin supported at each end. First, remove the horizontal reaction at the right end and calculate the final nodal movements. Then calculate the redundant horizontal reaction.



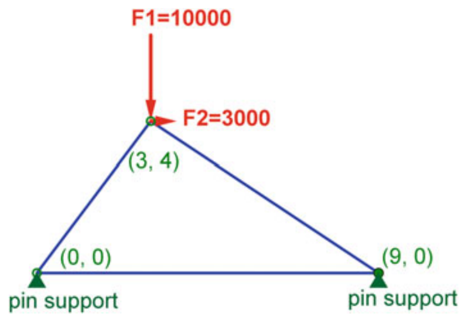
Exercise 9.1 solution part 1



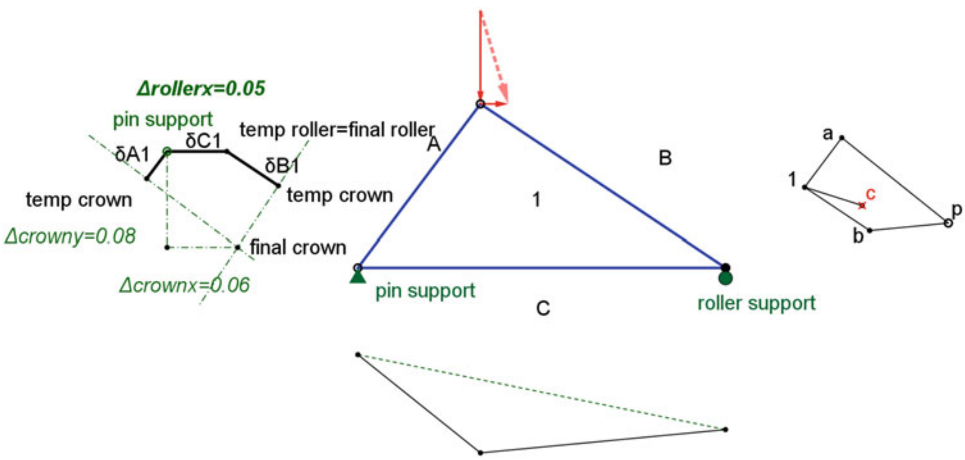
Exercise 9.1 solution part 2



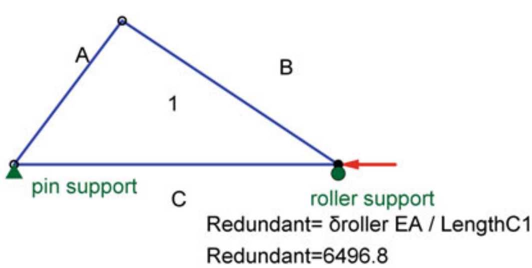
Exercise 9.2 A statically indeterminate truss is pin supported at each end. First, remove the horizontal reaction at the right end and calculate the final nodal movements. Then calculate the redundant horizontal reaction.



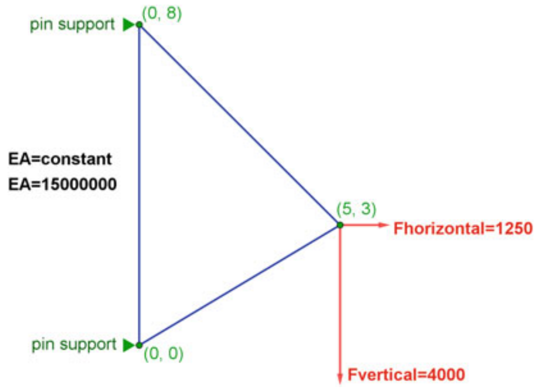
Exercise 9.2 solution part 1



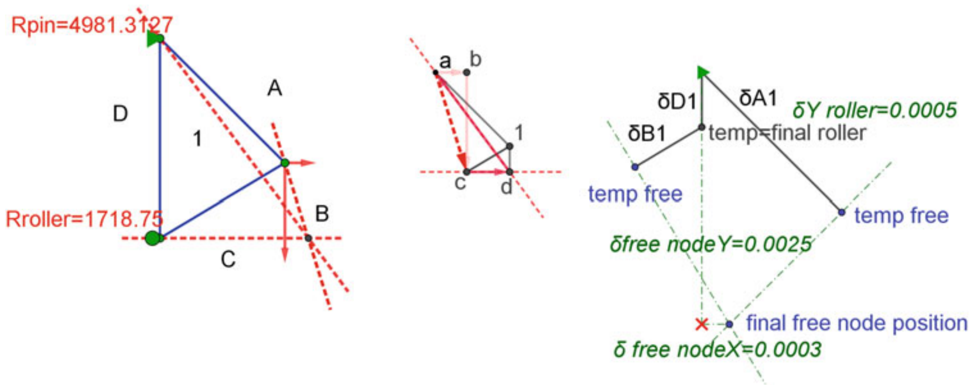
Exercise 9.2 solution part 2



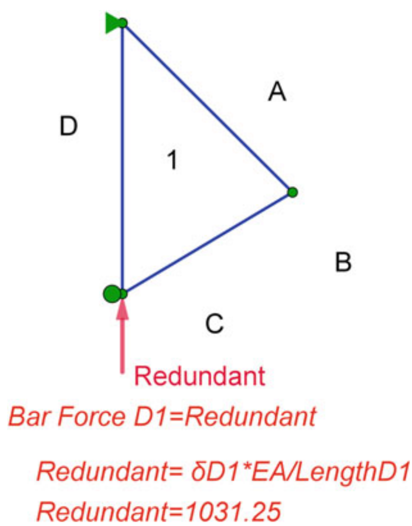
Exercise 9.3 A statically indeterminate truss is pin supported at each end. First, remove the vertical reaction at the bottom end and calculate the final nodal movements. Then calculate the redundant vertical reaction. Perform global statics graphically, but without a funicular.



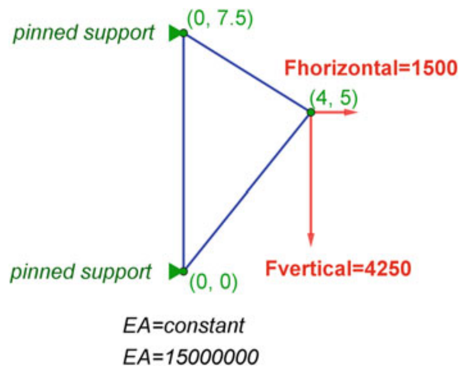
Exercise 9.3 solution part 1



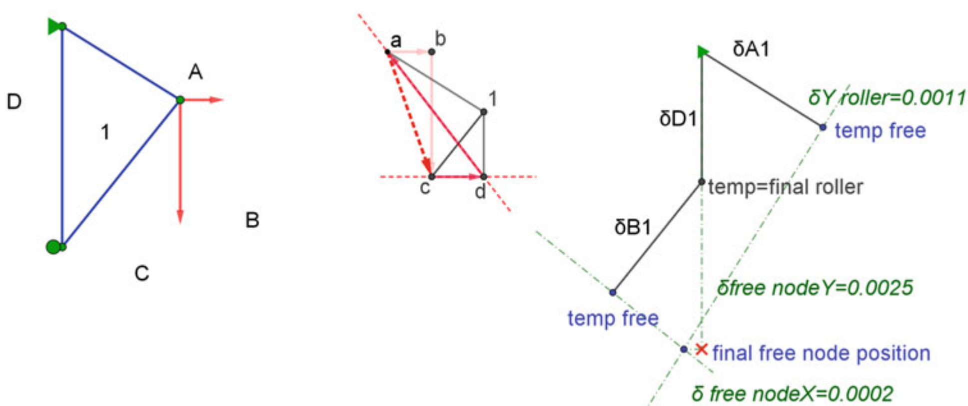
Exercise 9.3 solution part 2



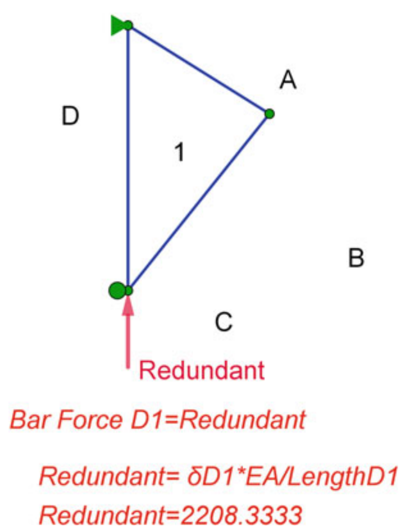
Exercise 9.4 A statically indeterminate truss is pin supported at each end. First, remove the vertical reaction at the bottom end and calculate the final nodal movements. Then calculate the redundant vertical reaction. Perform global statics graphically, but without a funicular.



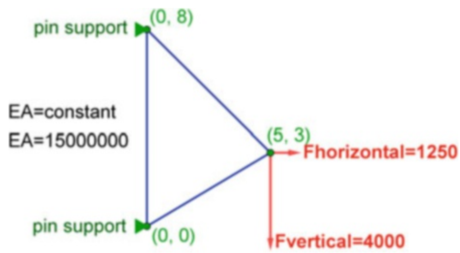
Exercise 9.4 solution part 1



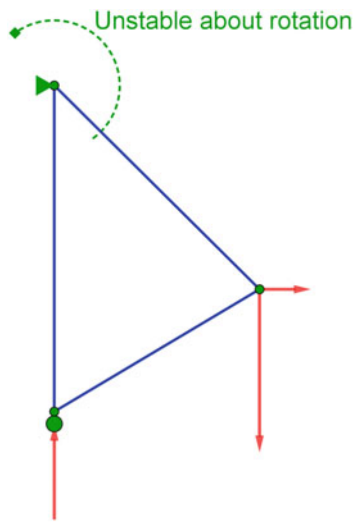
Exercise 9.4 solution part 2



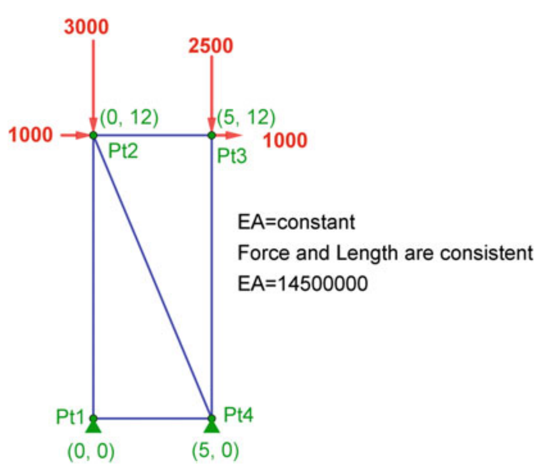
Exercise 9.5 A statically indeterminate truss is pin supported at each end. Explain whether or not it is possible to solve for the horizontal reaction at the bottom of the truss as the redundant.



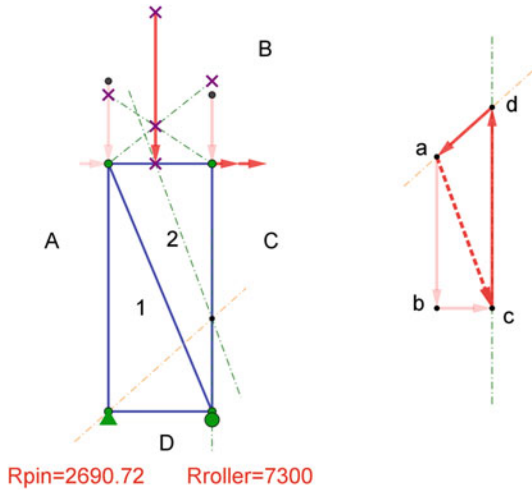
Exercise 9.5 solution No, it is not possible to choose the horizontal reaction at the bottom (0, 0) as redundant because the sub-structure is unstable.



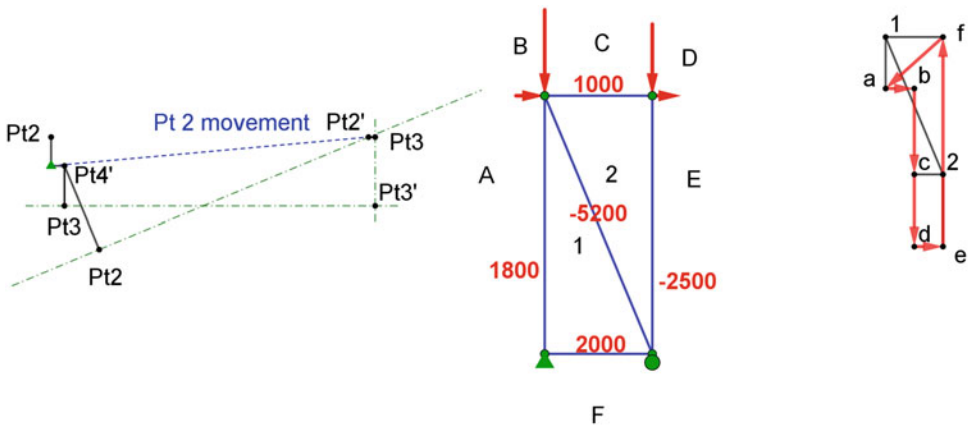
Exercise 9.6 The following statically indeterminate truss is subjected to external loads. Assume that the horizontal reaction at the right end (Pt 4) is redundant. First, use the Inverse axis method to reduce the gravity loads to one load, then solve for the reactions without a funicular after removing the redundant reaction. Then, calculate the bar forces for this temporarily determinate truss. Then, solve for the redundant force. Then calculate the final bar forces and the final movement of Pt 2.



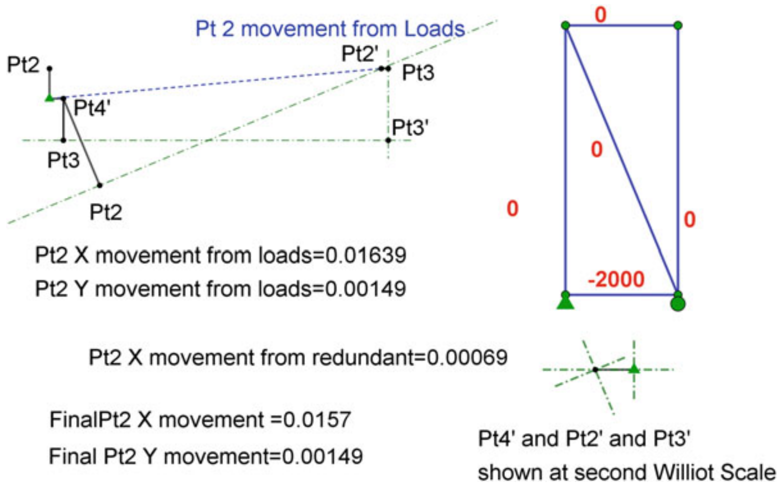
Exercise 9.6 solution part 1



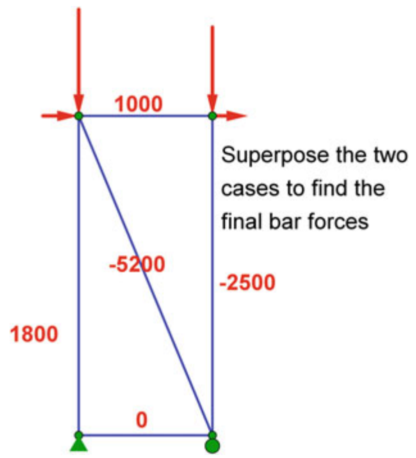
Exercise 9.6 solution part 2



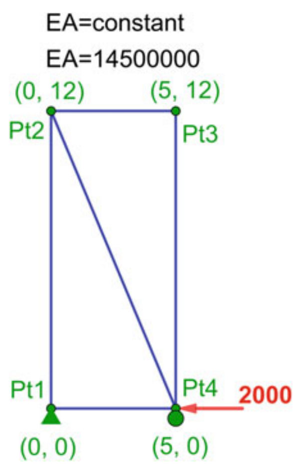
Exercise 9.6 solution part 3



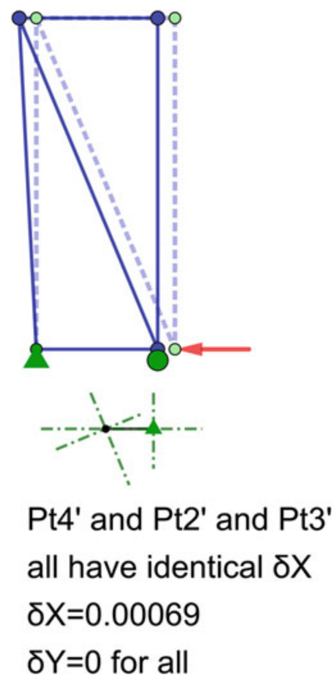
Exercise 9.6 solution part 4



Exercise 9.7 Perform a detailed analysis of Part 3 of Problem 9.6. Namely, draw the deformed statically determinate truss subjected to a single horizontal load at the right base roller support. Interpret the results.

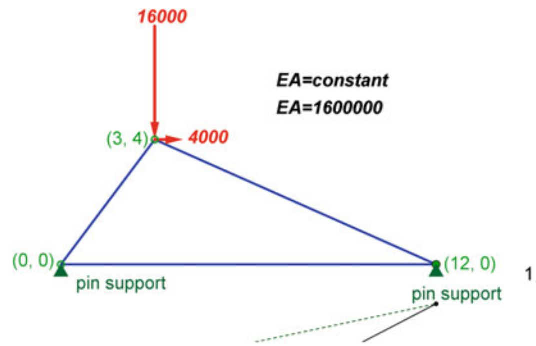


Exercise 9.7 solution

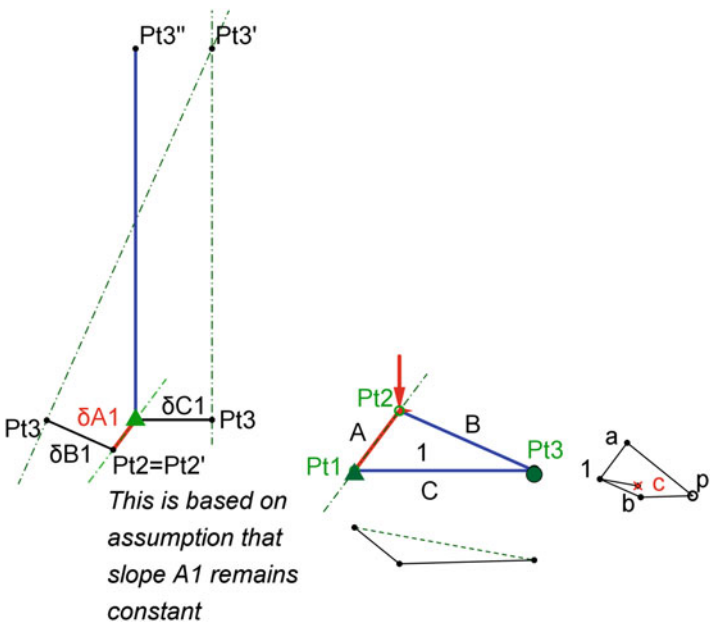


Interpretation: The only non-zero bar force is the lower horizontal bar. Yet surprisingly, three nodes move horizontally. The right bar is remains vertical, but the left bar is slightly rotated counter-clockwise.

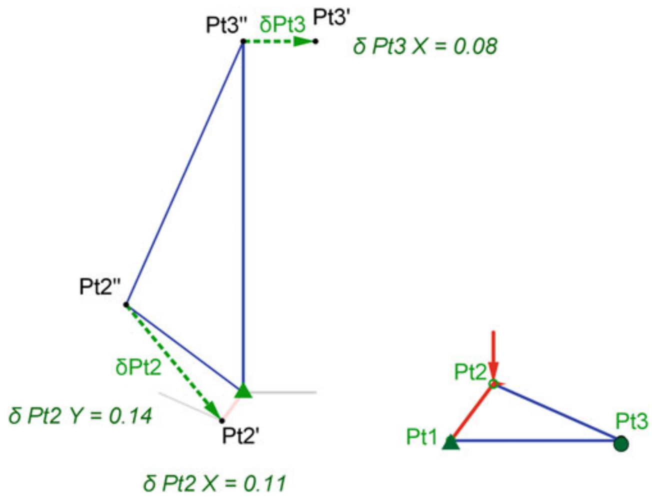
Exercise 9.8 A truss is statically indeterminate due to a redundant reaction. First, assume that the horizontal reaction at the right end is redundant. Then create a Williot Diagram based on the assumption that the left top chord does not change orientation. Then perform the Mohr Correction. Finally, solve for the redundant reaction force.



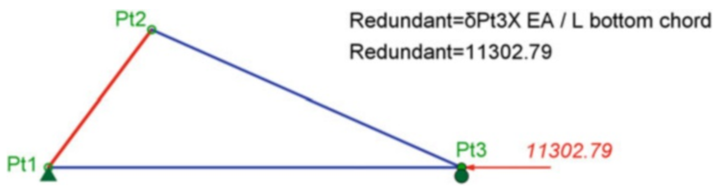
Exercise 9.8 solution part 1



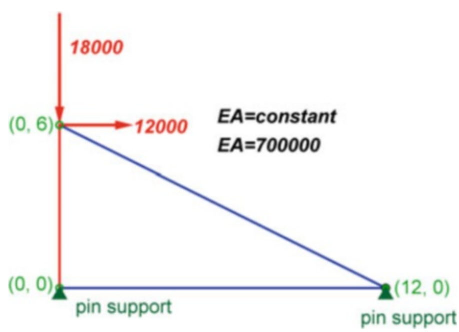
Exercise 9.8 solution part 2



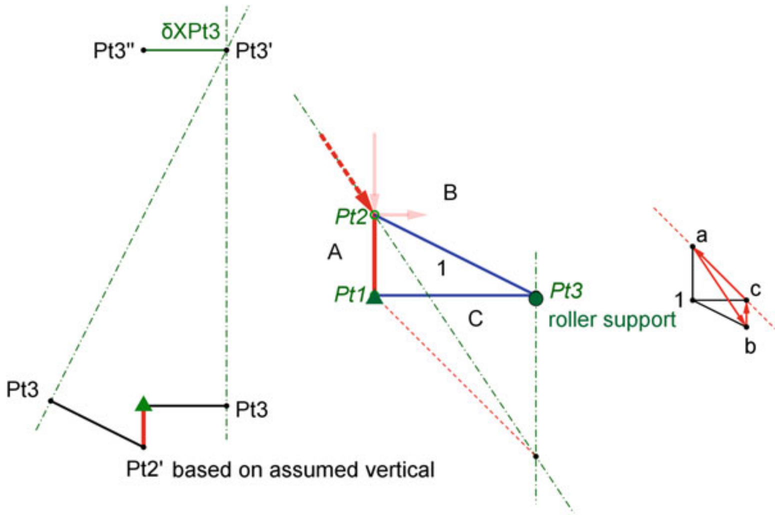
Exercise 9.8 solution part 3



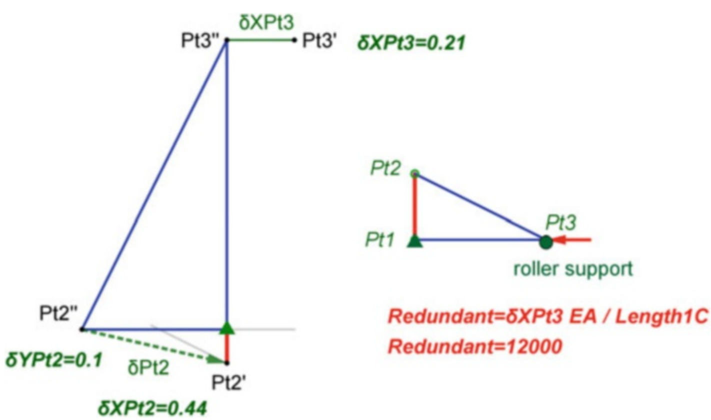
Exercise 9.9 A truss is statically indeterminate due to a redundant reaction. First, assume that the horizontal reaction at the right end is redundant. Then create a Williot Diagram based on the assumption that the vertical bar does not change orientation. Then perform the Mohr Correction. Calculate the movement of the free node and solve for the redundant reaction force.



Exercise 9.9 solution part 1

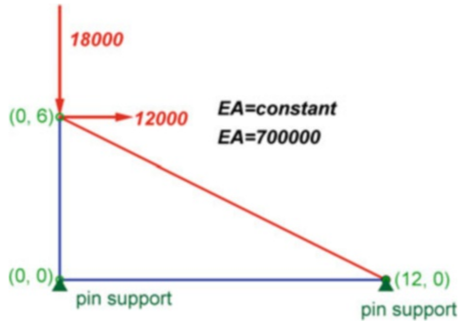


Exercise 9.9 solution part 2

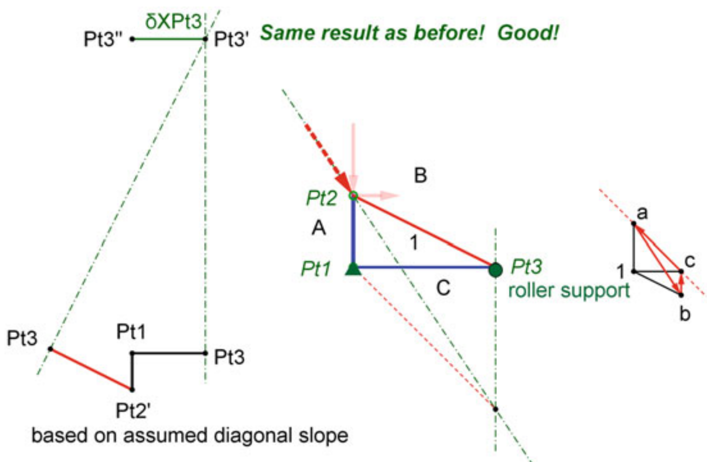


The solved for redundant makes perfect sense as the bottom chord must be a zero force member.

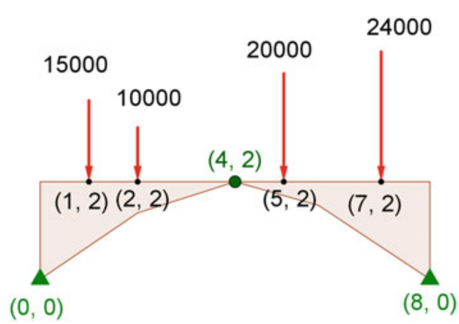
Exercise 9.10 Re-analyze the truss of Problem 9.9. Again, assume that the horizontal reaction at the right end is redundant. Then create a Williot diagram based on the assumption that the diagonal bar does not change orientation. Find the horizontal movement of Pt 3 and compare this result to that of Problem 9.9.



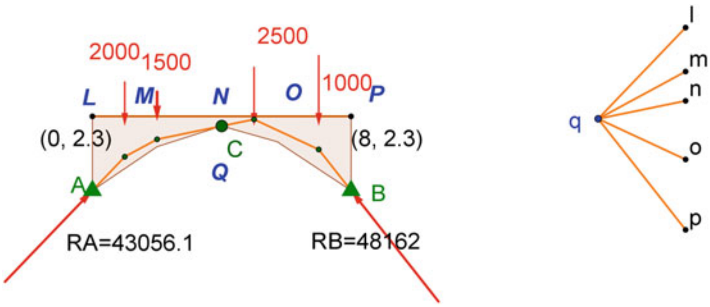
Exercise 9.10 solution



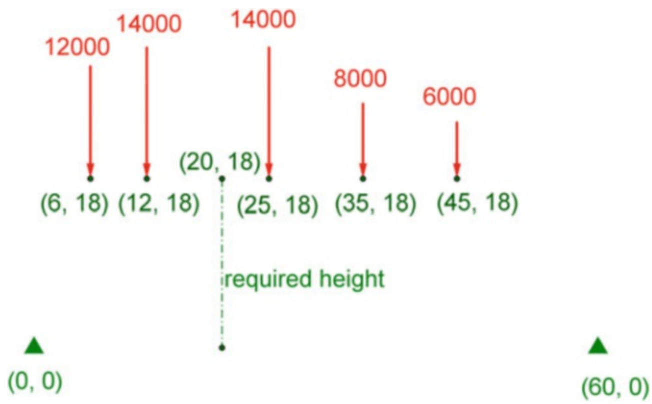
Exercise 9.11 Four concentrated loads will be applied to a structure of known span. One suggestion is to design a three hinged arch to carry the loads. Design such a solid arch so that it is compression only. A suggested shape is shown in Fig. 9.11.



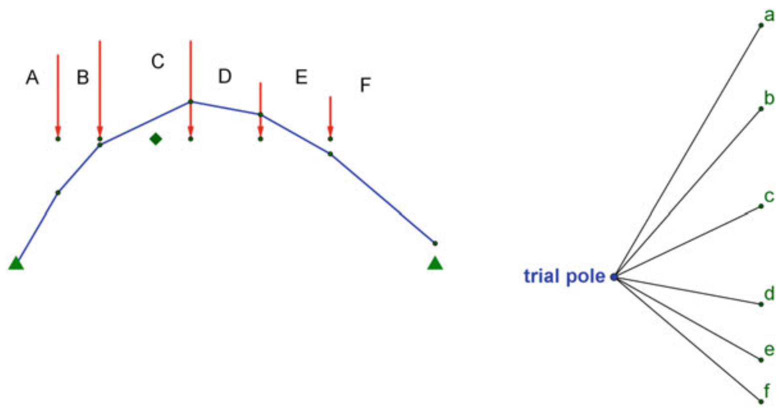
Exercise 9.11 solution part 1 An arch that is 2.3 units tall would be sufficient. The depth of the arch at the hinge has a very short height, here 0.3 units of length.



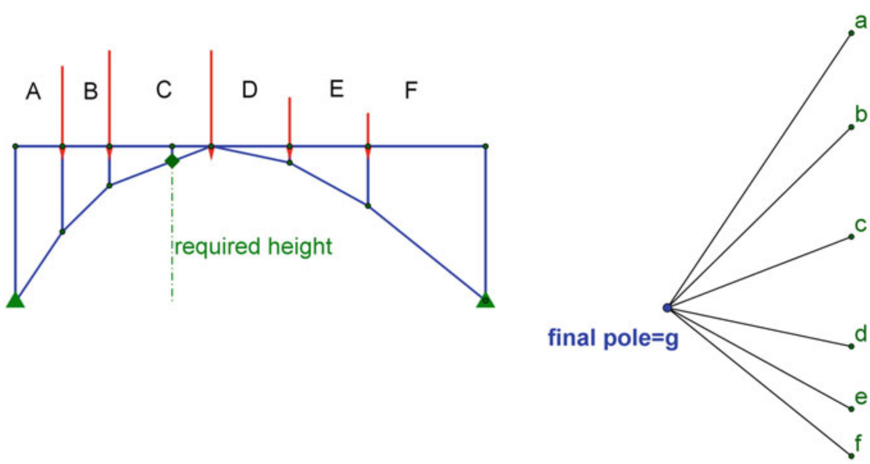
Exercise 9.12 The magnitude and position of five loads are known. Also, the starting point (0, 0) and ending point (60, 0) of the structure are known. Design a Vierendeel-like truss that efficiently carries those loads such that the truss has a required clearance through a prescribed point (20, 18). Comment on whether or not the required height is the maximum clearance of the structure, or not. Also comment on whether or not the structure is determinate and stable.



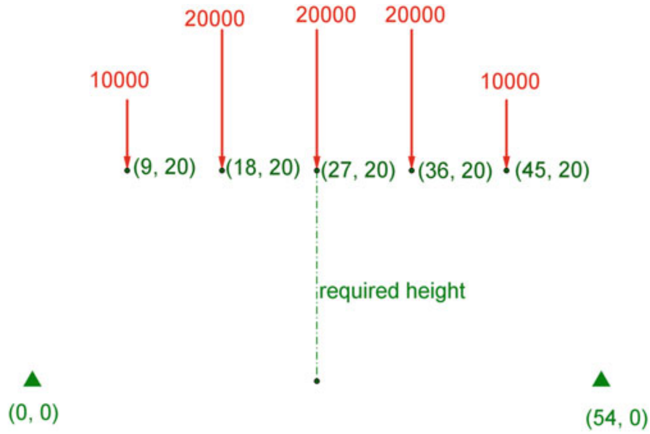
Exercise 9.12 solution part 1 Placing the trial pole on the left of the load line creates an arch-like structure. An arbitrary trial pole will not satisfy the constraints of passing through the three prescribed points.



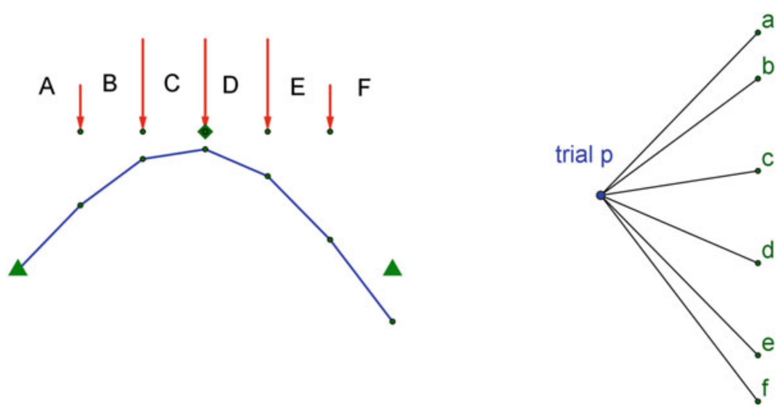
Exercise 9.12 solution part 2 Quick manipulations of the trial pole immediately find a solution that passes through all three prescribed points. The required height is NOT the maximum clearance of the structure. A horizontal top chord, slightly higher than the original 18 units of length will accept the five downward loads. If the structure is made of pin-ended elements, it appears to be extremely unstable, but theoretically, for the applied loads ONLY, the structure is stable. Of course, if the joints can carry moments, the structure is a Vierendeel Girder and it is stable and highly indeterminate. All elements other than the funicular arch are zero force members. Notice the reactions are never calculated!



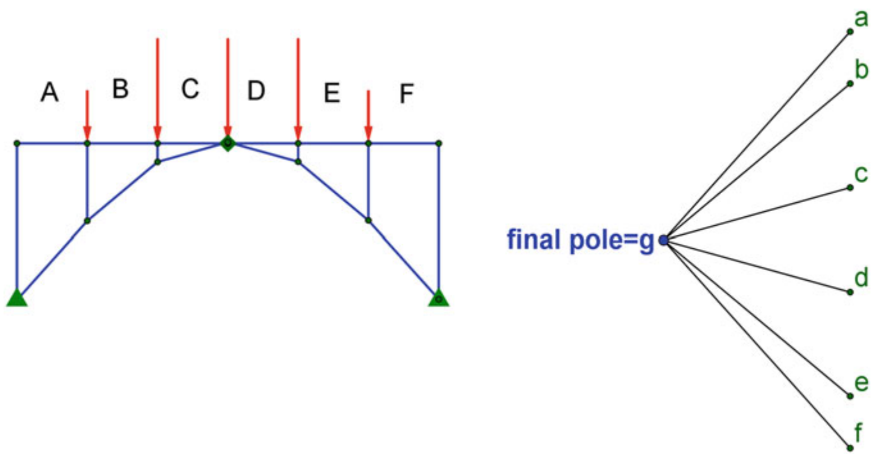
Exercise 9.13 The magnitude and position of five loads are known. Also, the starting point (0, 0) and ending point (54, 0) of the structure are known. Design a Vierendeel-like truss that efficiently carries those loads such that the truss has a required clearance through a prescribed point (27, 20). Comment on whether or not the required height is the maximum clearance of the structure, or not. Also comment on whether or not the structure is determinate and stable.



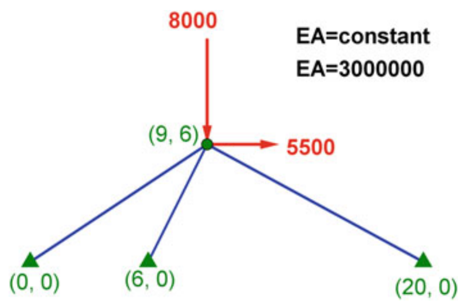
Exercise 9.13 solution part 1 Placing the trial pole on the left of the load line creates an arch-like structure. An arbitrary trial pole will not satisfy the constraints of passing through the three prescribed points.



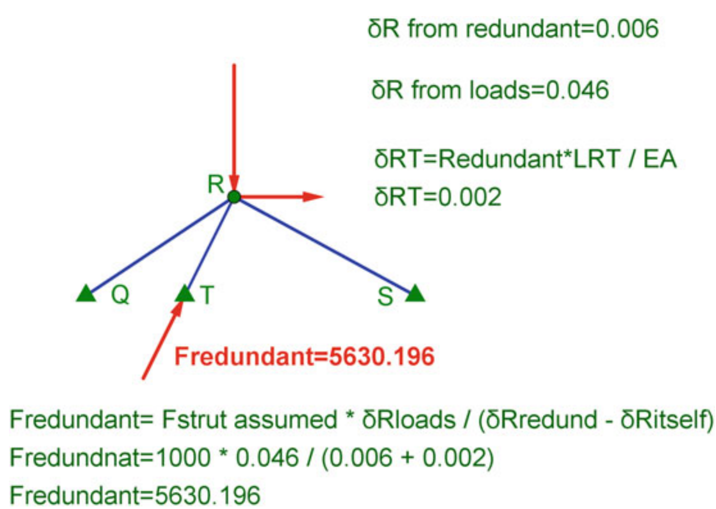
Exercise 9.13 solution part 2 Quick manipulations of the trial pole immediately find a solution that passes through all three prescribed points. The required height is the maximum clearance of this structure. A horizontal top chord, at the elevation of the original 20 units of length will accept the five downward loads. If the structure is made of pin-ended elements, it appears to be extremely unstable, but theoretically, for the applied loads only, the structure is stable. Of course, if the joints can carry moments, the structure is a Vierendeel Girder and it is stable and highly indeterminate. All elements other than the funicular arch are zero force members. Notice the reactions are never calculated!



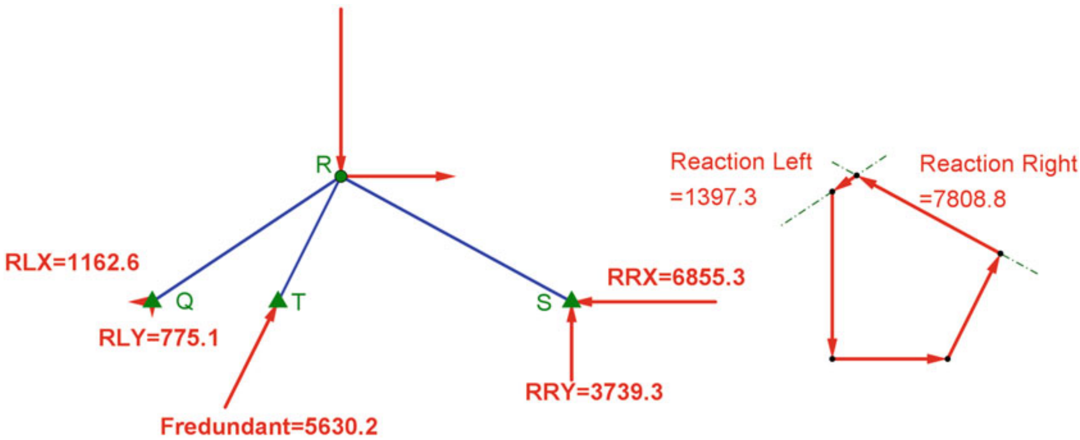
Exercise 9.14 A truss is indeterminate because of the redundant internal strut. The truss is subjected to two external loads as shown. Calculate the final force in the redundant internal strut. Then solve for the forces in the two external members.



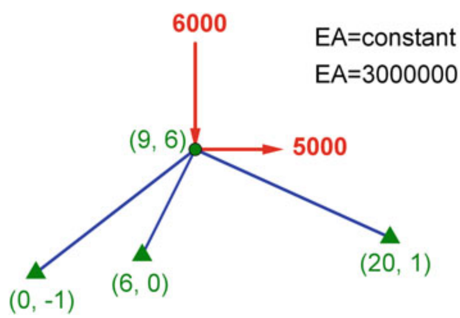
Exercise 9.14 solution part 1



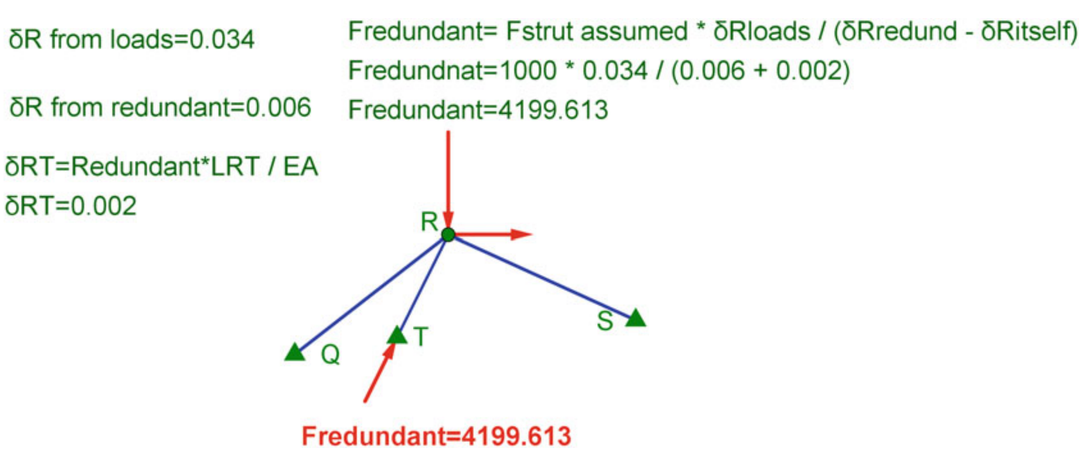
Exercise 9.14 solution part 2



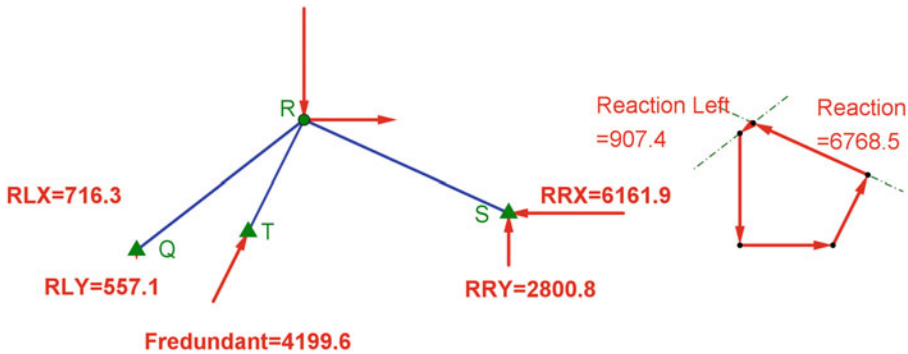
Exercise 9.15 A truss is indeterminate because of the redundant internal strut. The truss is subjected to two external loads as shown. Calculate the final force in the redundant internal strut. Then solve for the forces in the two external members.



Exercise 9.15 solution part 1



Exercise 9.15 solution part 2



10 Forces in Space

Three or more forces which act through a common point, but are not all in the same plane are called non-coplanar concurrent forces. Such a loading situation necessitates a three dimensional (3D) analysis. Algebraically, the solution is well known with the resultant magnitude described as:

$$F_{resultant} = \sqrt{\left(F_x^2 + F_y^2 + F_z^2\right)}$$

where Fx, Fy and Fz are mutually orthogonal components, sometimes called the *i, j, k* vectors. These three components need not align with any pre-determined *x, y, z* axes, but they must be mutually perpendicular, i.e. they must have 90° angles between them.

A two-step process of solving for this resultant force graphically is to choose any two forces, establish their co-planar resultant, and then to add this intermediate resultant to the next force, and so on for any subsequent forces. Of course, the original forces can take on any combination of orientations, but in this first example, there are three original forces, each of which lies on the *X, Y* and *Z* axes respectively, this conveniently matches the *i, j, k* components in the previous equation. These may represent net *X, Y, Z* resultants of a large number of forces acting through a point as shown in Fig. 10.1.

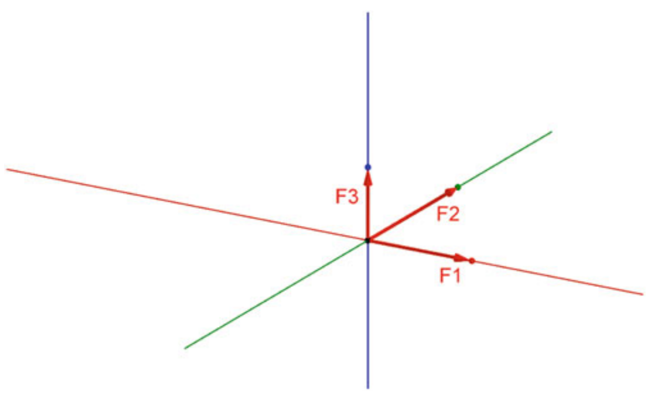
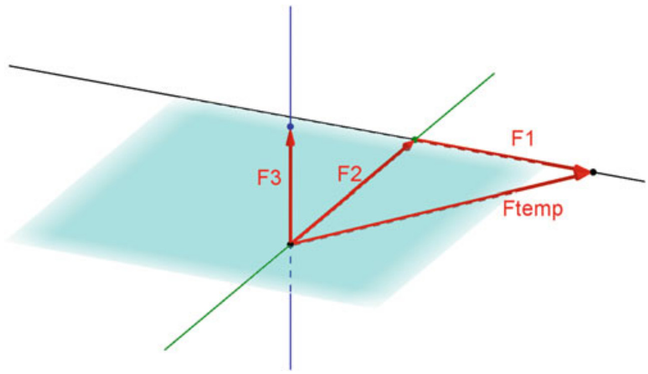


Fig. 10.1 Three mutually perpendicular forces in space

In this example, the resultant of *F1* and *F2* will be established as *Ftemp* since it is a temporary resultant that will be used in the subsequent step (Fig. 10.2).

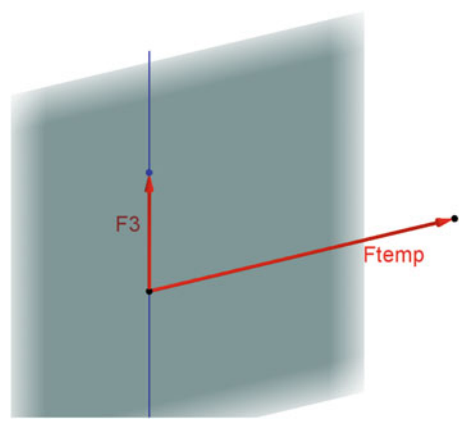
E. Saliklis, *Structures: A Geometric Approach*, https://doi.org/10.1007/978-3-319-98746-0_10

Fig. 10.2 First find one resultant of any two forces



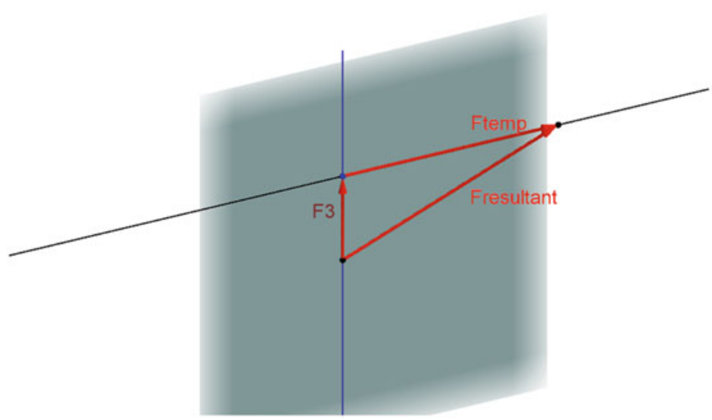
This operation is repeated, but this time in the plane of *Ftemp* and *F3*. The resultant of these two forces will be the net resultant of the original three forces. The alternate placement of *F1* is there to demonstrate the vector addition process of terminus-to-origin alignment (Fig. 10.3).

Fig. 10.3 Combine third force with resultant of first two



The magnitude of this final resultant matches the algebraic result exactly (Fig. 10.4).

Fig. 10.4 Final resultant force is exactly found



In the following example shown in Fig. 10.5, three forces are acting at a point as shown, and the three forces are not mutually orthogonal. The resultant of all three forces is sought.

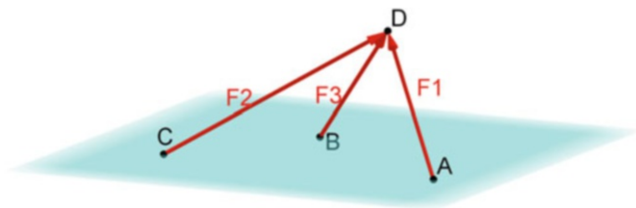


Fig. 10.5 Three forces in space which are not mutually perpendicular

The traditional algebraic approach to solving this problem is to first calculate the net force in each of three mutually orthogonal directions, for example the *X, Y* and *Z* directions. This could be done by calculating *dx, dy, dz* for each of the vectors. For example, the *x* component of *F1* is:

$$d_{x1} = x(D) - x(A)$$

Similar operations apply to *F2* and *F3* then F_x is the summation of d_{x1}, d_{x2} and d_{x3}.

Graphically however, the two-step method avoids all such clumsy calculations. All that is needed is to find one temporary resultant, co-planar with any two of the forces, and then to perform a subsequent step combining this temporary resultant with the third known force.

In Fig. 10.6, arbitrarily choosing to find the resultant of *F1* and *F3*, creates the temporary resultant *Ftemp*. Notice how vector addition requires placement of the terminus of one vector adjacent to the origin of the next vector.

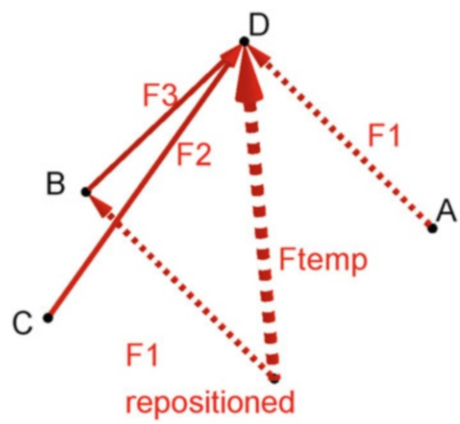


Fig. 10.6 Choose any two forces and find their resultant

Resolving *Ftemp* and *F2*, establishes the overall resultant of the forces (Fig. 10.7).

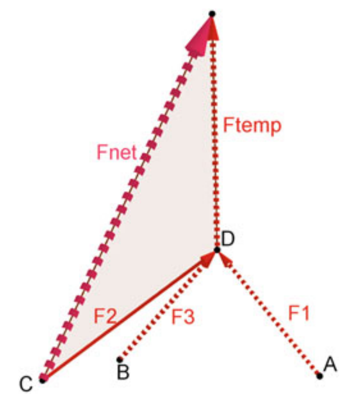


Fig. 10.7 Add the third force to the resultant of the first two

A faster way is to simply draw all three vectors in 3D space terminus to origin, and then create a resultant vector spanning from the origin of the first vector to the terminus of the third vector as shown in Fig. 10.8.

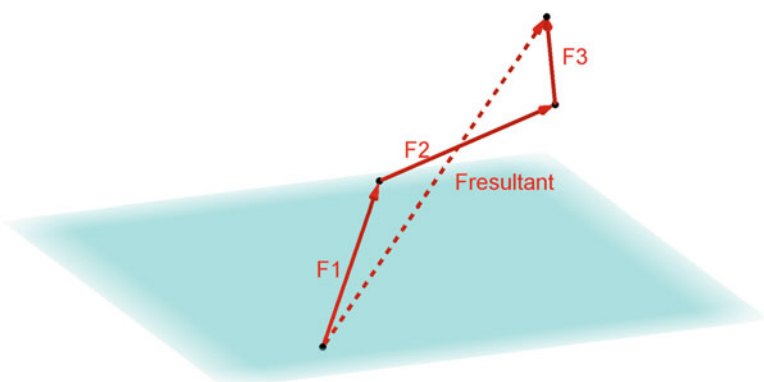


Fig. 10.8 Extremely fast 3D solution graphically, no temporary resultants are needed

Figure 10.9 shows the same solution from a different angle to highlight the three dimensional nature of the solution.

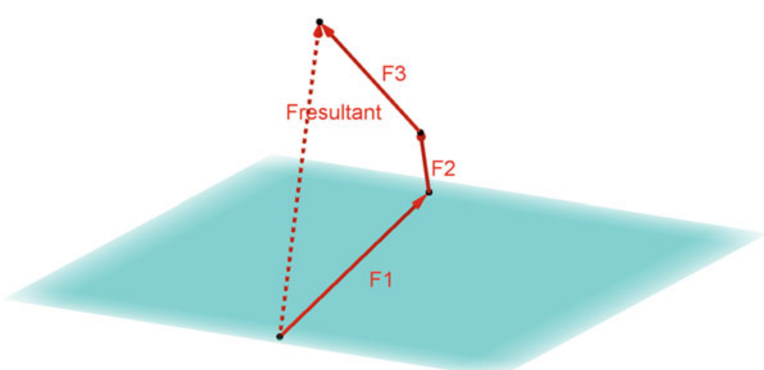


Fig. 10.9 Same solution as 10.8 shown from different angle

As always, equilibrium of a point requires that the reaction be equal in magnitude and opposite in direction to the net resultant of all the forces.

The following example explores equilibrium of a point in 3D space.

In Fig. 10.10 a tripod is made up of three equal-length legs, each leg is 8 units of length long, and all legs rest on horizontal ground. Each of the legs make an angle $\beta = 60°$ to the horizontal as shown. At the crown of the tripod, a downward load of 100 units of force is applied. Find the force in each supporting leg.

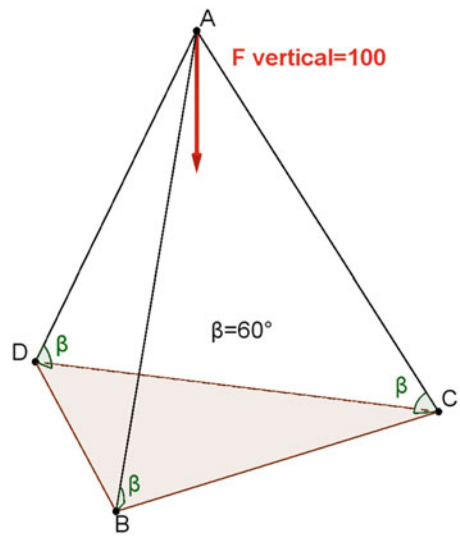


Fig. 10.10 One known force and three unknown but symmetric reactions

Exploiting the three dimensional capabilities of GeoGebra shown in Fig. 10.9 and taking advantage of the fact that due to symmetry, each leg force will be identical, a three dimensional construction is shown of three vectors, each of some magnitude *Funknown*, placed on a slider scale. The target of the final terminus must be on a vertical line through the crown, and must equal 100 units of force (Fig. 10.11).

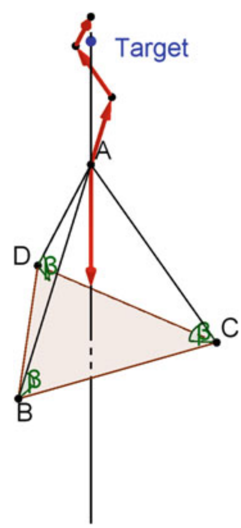


Fig. 10.11 Symmetry allows for rapid graphical iterative solution

In Fig. 10.11, the target is missed by assuming too large of a leg force. Having the unknown leg force on a slider allows for a quick solution to the problem, visually affirmed by having the final terminus align with the target blue dot. Figure 10.11 shows that the unknown leg force in each leg is 38.48 units of force.

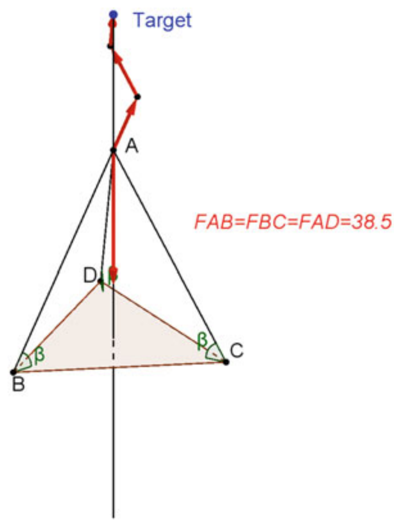


Fig. 10.12 Iterative solution quickly found by using a slider in the graphical program

The same problem can be solved in a two-step process, but this process becomes extremely elegant when the power of the three dimensional graphics is harnessed. As shown in Fig. 10.13, first create a plane with the applied load and one of the unknown legs, for example with leg AC. Create a second plane from the other two unknown legs. The temporary force *FAE* lies at the intersection of these two planes. Solve for the fictitious force and the first unknown leg. Then immediately decompose the fictitious force into the second and third unknown leg forces. In summary: the temporary force must be on a line that intersects *two planes*, plane 1 contains the known force and one unknown, plane 2 contains the remaining two unknown forces. All of these steps are shown in Fig. 10.13.

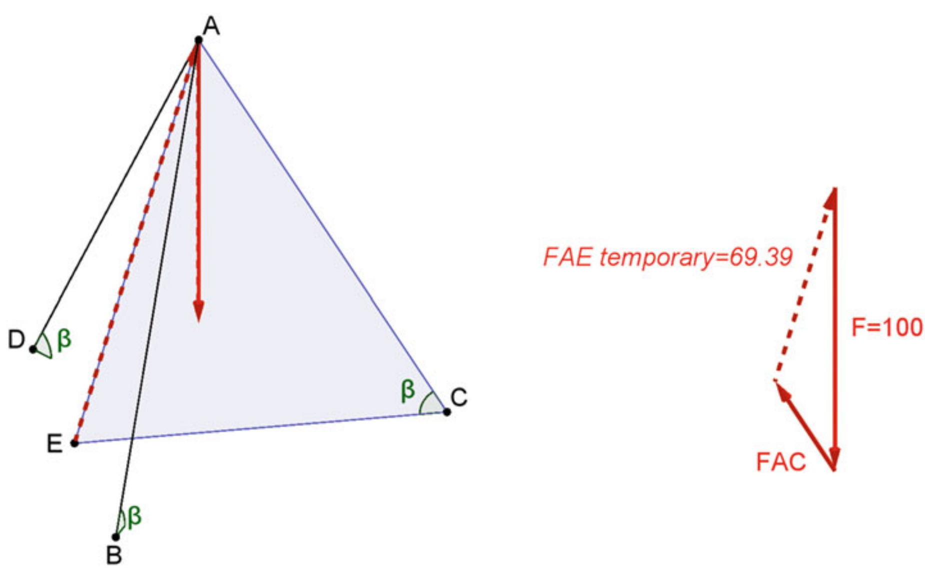


Fig. 10.13 Two step process drawn immediately in 3D

A powerful feature of three dimensional graphic tools is that a line parallel to *AE* and to *AC* can be immediately drawn in 3D space. This avoids the traditional technique of projecting 2D drawings onto multiple planes. The impact of this feature needs to be highlighted. In Fig. 10.14 for example, the line *DA* and the vector *FAD* both appear as two dimensional entities sloping up and to the right. In a 3D drawing environment, they do indeed appear that way to the human eye looking at a computer screen. But in reality they are both diving out of the screen. In the twentieth century, such a drawing was impossible to do because engineers were limited to drawing on paper. Now, transferring a parallel line such as *AD* from the form diagram to the force diagram captures a true 3D line.

This technique is then used to decompose *FAEtemporary* into two forces *FAB* and *FAD*. There is no need to project onto separate planes, these lines can be drawn in 3D space as shown in Fig. 10.14.

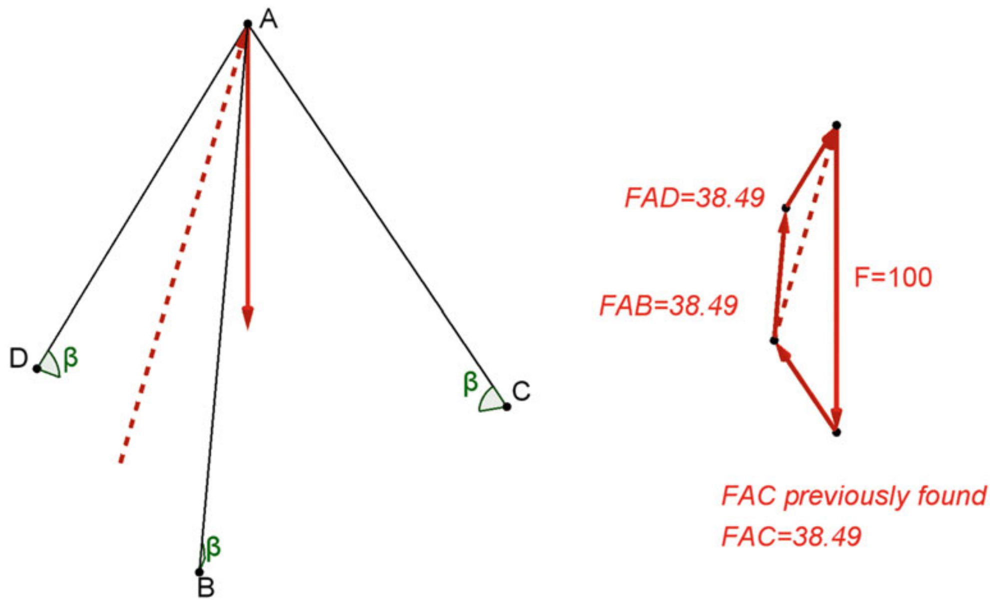


Fig. 10.14 Two step process drawn in one single programming environment

The same solution as was found in Fig. 10.12 is once again obtained, the magnitude of vectors *FAC*, *FAB* and *FAD* are all 38.49 units of force. Multiple rotated views show that these vectors are indeed the same length. This is shown in Fig. 10.15.

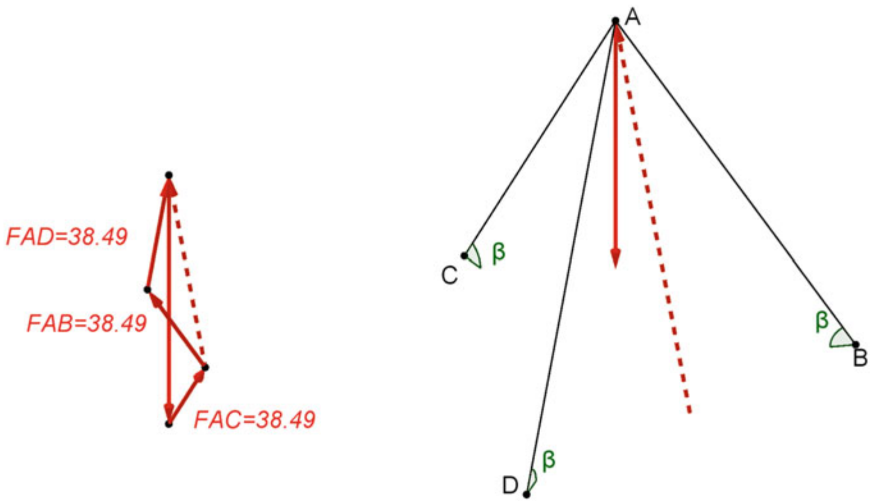
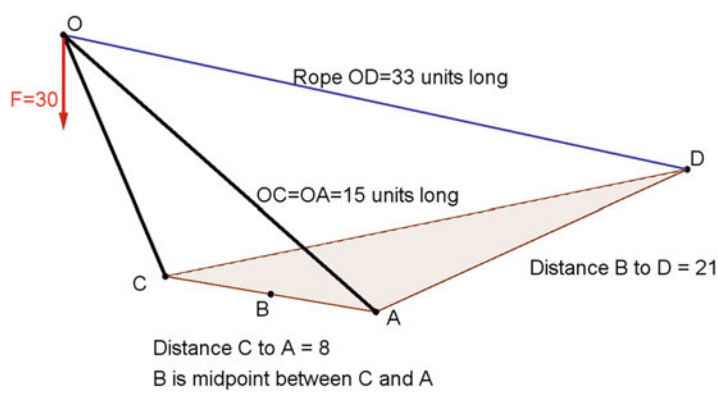


Fig. 10.15 Alternate view of Fig. 10.14

In this next example, the problem is stated from the perspective of a builder faced with a structure of known dimensions on the ground, rather than having the Cartesian coordinates of all points given a priori.

Here, a pair of pin-ended legs are used to raise a load. Each leg is 15 units of length long. The feet of the legs are spread 8 units apart on horizontal ground. Point B is centrally located between A and C. A guy rope 33 units of length long is attached to the ground 21 units from point B, the middle point of the line joining the two feet. Find the force in each leg and the tensile force in the guy rope, when a load of 30 units of force is suspended as shown in Fig. 10.16.

Fig. 10.16 3D problem without Cartesian coordinates given at onset



The question immediately arises: "Where is point O located"? One way of precisely locating point O is to draw a circle in the plane of OBD, of radius 33 (the length of the rope), and a second circle in the plane of OCA of radius $OA = 15$ ($OA^2 = OB^2 + BA^2$). This is shown in Fig. 10.17.

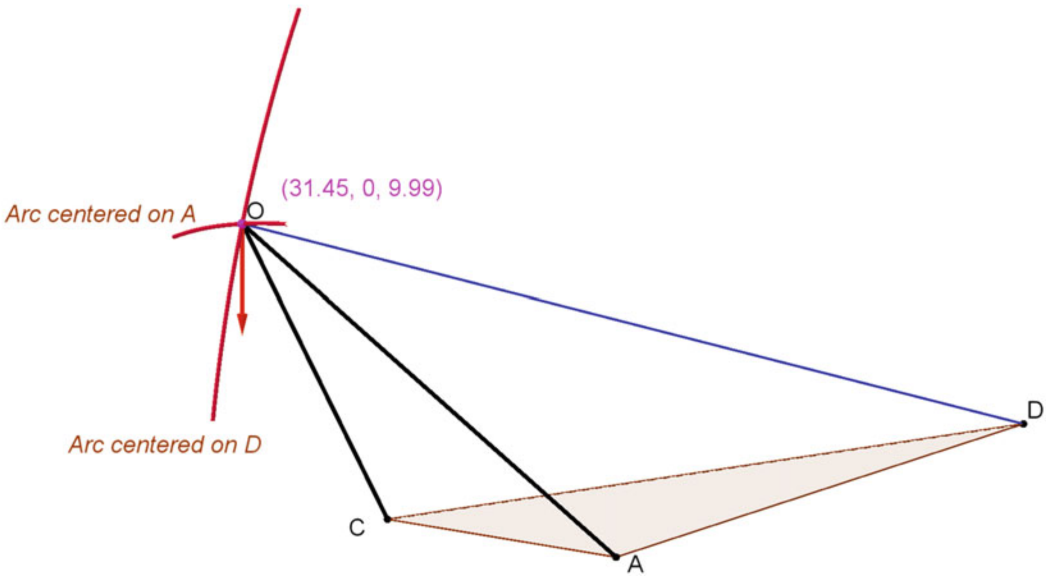


Fig. 10.17 Intersection of two circles locates point *O*

Using the two-step process described in Figs. 10.14 and 10.15, the force in the rope and the force in the fictitious strut *OB* are immediately found. The temporary force must be on a line that intersects *two planes*, plane 1 contains the known force and one unknown, plane 2 contains the remaining two unknown forces as shown in Fig. 10.18.

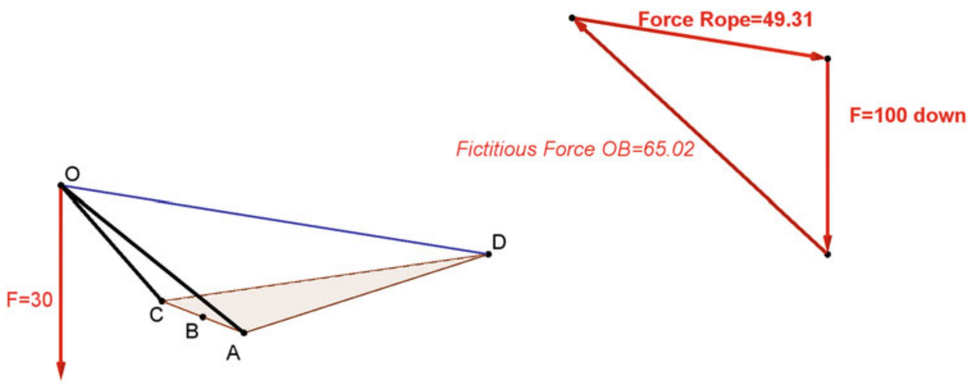


Fig. 10.18 Two step process applied in 3D space

Then simply drawing lines parallel to *OC* and to *OA* decomposes the fictitious force *OB* into the final leg forces. Figure 10.19 shows one perspective of the solution.

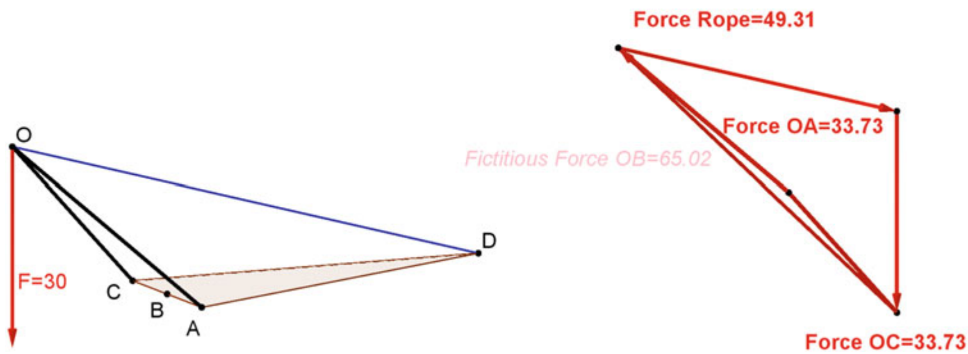
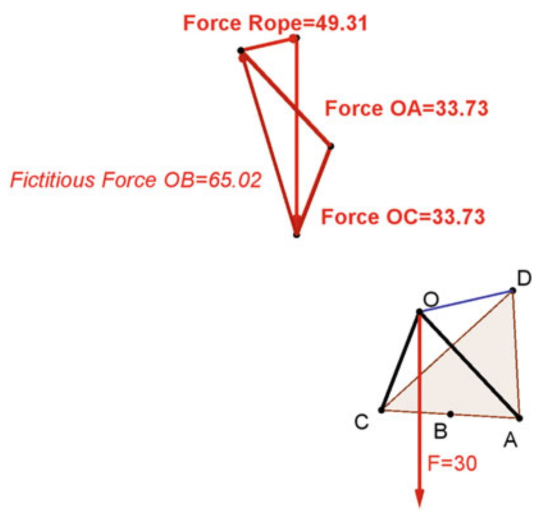


Fig. 10.19 One view of solution to problem of Fig. 10.16

Rotating the drawing demonstrates that the components do add properly and vectorially. This is shown in Fig. 10.20.

Fig. 10.20 Alternate view of solution to problem of Fig. 10.16



In the following example, the problem again shows two pin-ended legs supported by a rope, but this time, the geometry is known through given Cartesian coordinates of all the key points. Also, this example is asymmetric. Point T is in line with Points P and Q on a horizontal plane. The distance from P to T is 3 units of length, the distance from T to Q is 5 units of length, the distance from T to R is 6 units of length. Point O' supports a downward load of 6 units of force. Point O' is directly above Point T', which is 2.67 units from T as shown in Fig. 10.21.

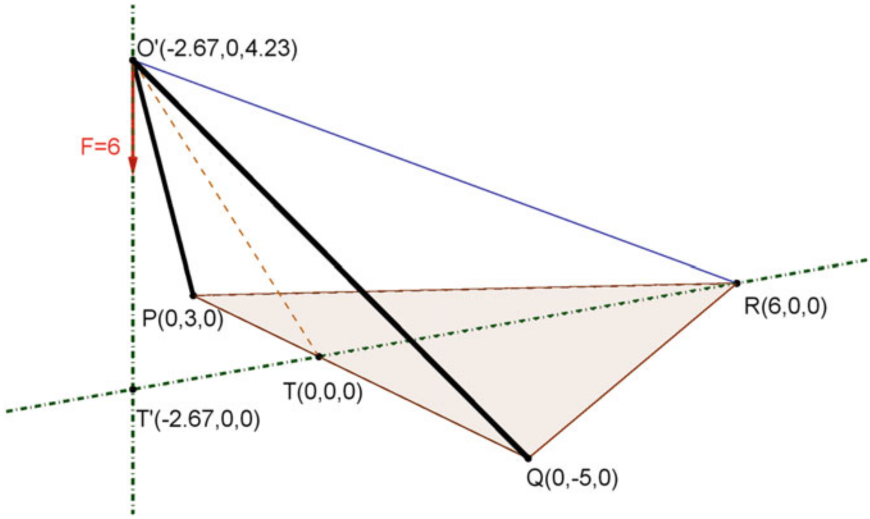


Fig. 10.21 3D problem with Cartesian coordinates given

Again, using the two-step process in 3D, first find the fictitious force $O'T$, and the rope force $O'R$. This is really solving two equations in two unknowns, a well-known algebraic technique. The temporary force must be on a line that intersects *two planes*, plane 1 contains the known force and one unknown, plane 2 contains the remaining two unknown forces (Fig. 10.22).

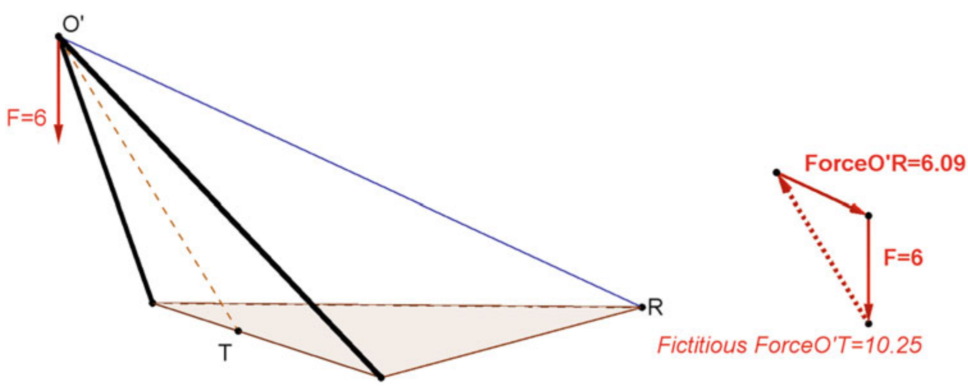


Fig. 10.22 Temporary fictitious force found

Then, as was done in Figs. 10.19 and 10.20, the fictitious force is resolved into two real leg forces, but done so immediately in 3D in Fig. 10.23.

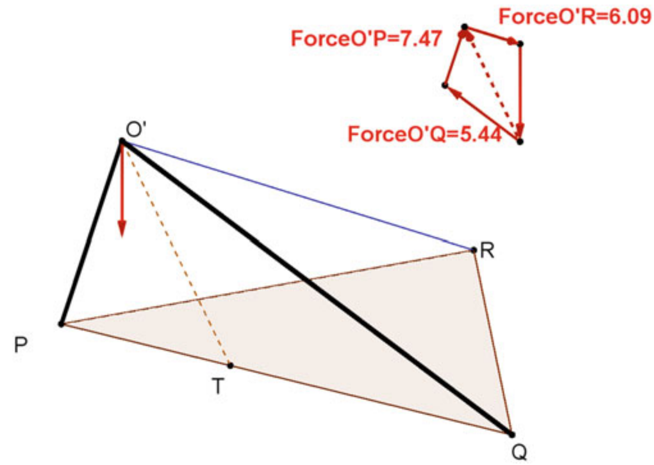


Fig. 10.23 Decomposing fictitious force into structure strut and tie forces

Having the forces calculated, a kinematic analysis can proceed. This example uses only the Williot Diagram discussed in Chap. 9, no Mohr correction is needed because only O' is free to translate. The exact same technique used in 2D can be replicated in 3D, yet in 3D, perpendicular planes are drawn instead of perpendicular lines. Of course, in the early twentieth century such drawings were impossible.

To perform a kinematic analysis, material properties and cross sectional areas are needed (Fig. 10.24). Suppose E of each of the pin-ended legs is $E_{leg} = 1000\frac{force}{length^2}$ and $E_{rope} = 500\frac{force}{length^2}$

Let $A_{PO'} = 3\ length^2$ and $A_{QO'} = 0.5\ length^2$ and $A_{rope} = A_{RO'} = 0.5\ length^2$

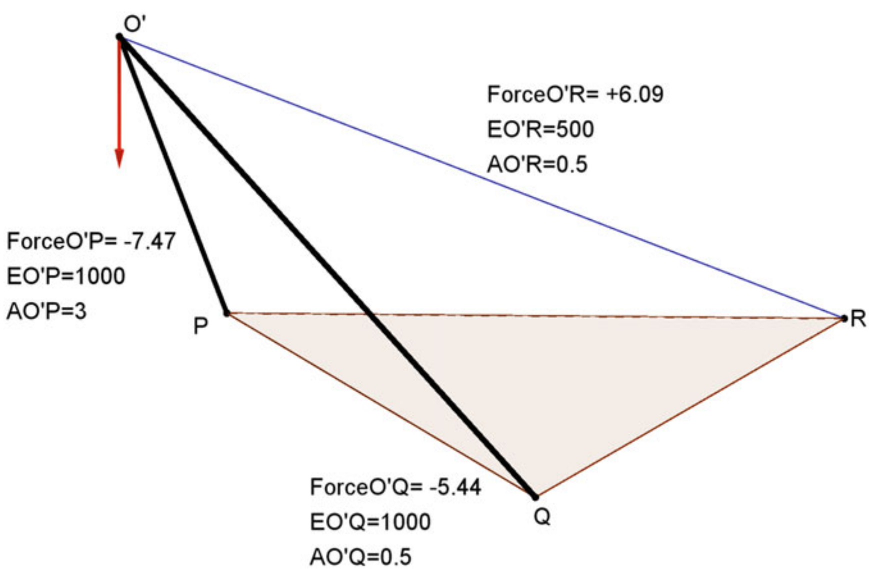


Fig. 10.24 Material properties and cross sectional areas are needed for kinematic analysis

Having the force in each member, the cross sectional areas, modulus and lengths allows for the calculation of axial unrestrained displacements δ. The Williot Diagram shows only the δ values for each member.

Fig. 10.25 Williot Diagram shows axial elongations of the three elements

Note that in Fig. 10.25, it is good practice to label an axial deformation with the second index referring to the sought after quantity. Thus, use $\delta QO'$ rather than $\delta O'Q$.

Create a plane perpendicular to two of the axial deformations, here $\delta RO'$ and $\delta QO'$. These planes are completely analogous to the perpendicular lines (small tangential arcs) in the 2D Williot Diagram. Capture the line at the intersection of these two planes with a red intersecting line. This is shown in Fig. 10.26.

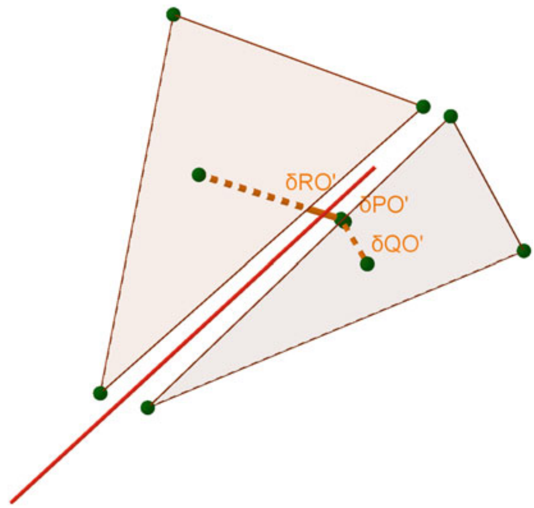


Fig. 10.26 Intersection of two planes is necessary in 3D

Repeat this technique with two other planes, here a blue line intersects the green and blue planes (Fig. 10.27).

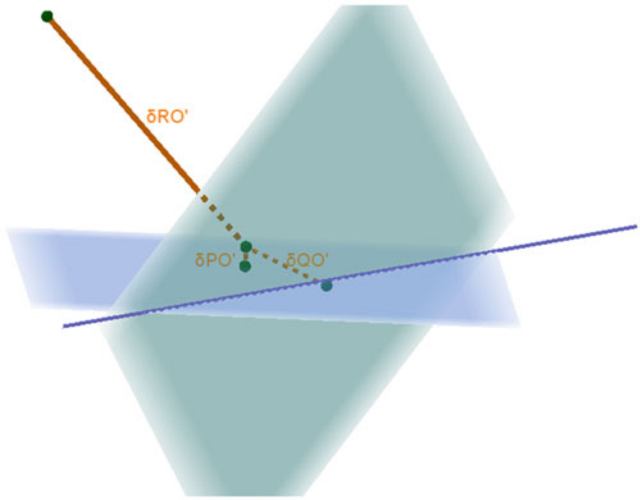


Fig. 10.27 Temporarily view normal planes to establish intersecting line

Now find the intersection of the two lines (red and blue), this locates the loaded tip's displaced position.

Use the (x(Intersection) − x(Origin))/WillScale to find the x displacement of the tip for example (Fig. 10.28).

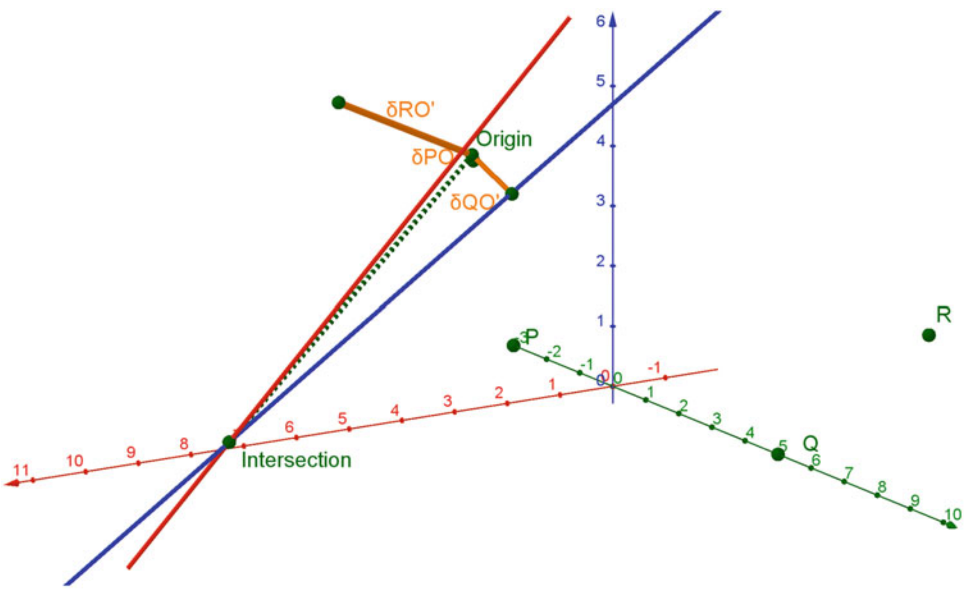


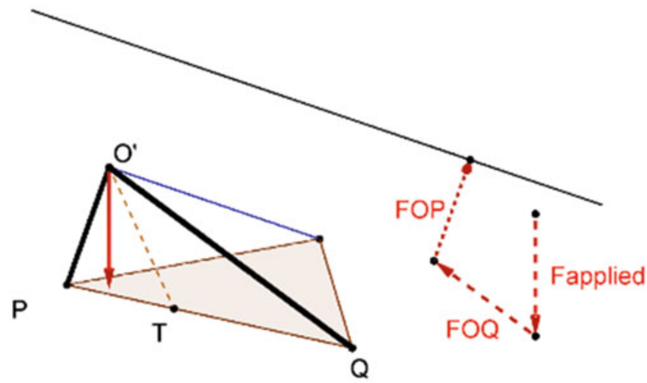
Fig. 10.28 Intersection of the red line and the blue line easily found graphically

The tip displacement matches theory exactly. An alternate approach to solving this problem would be to find, at the onset of the problem, the intersection of the three perpendicular planes. That way, the two step approach is not needed.

The supporting member forces of the previous problem can be solved for using a trial and error method, which rapidly converges. To solve this problem iteratively, a force diagram is established in

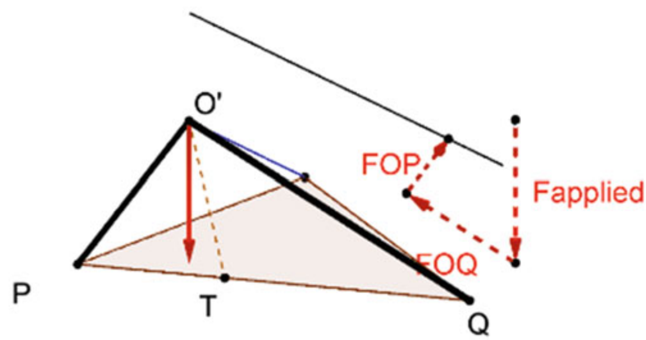
3D, with parametric sliders controlling the magnitude of the forces in each of the struts ($O'Q$ and $O'P$), but no slider is needed for the rope force (OR) because this answer will appear at the end of the iterations. The slope and the magnitude of the applied load are known and the slopes of all three reactions are known, yet their magnitudes are unknown. Sliders control the magnitude of vectors $O'Q$ and $O'P$. To find the target of a solution, a line parallel to the rope $O'R$ is shown. Moving the sliders of the shear leg force magnitudes rapidly shows how the line parallel to the rope will close the force diagram. In this first attempt, the line is well above the origin of the applied load vector (Fig. 10.29).

Fig. 10.29 Rapid iterative solution in 3D space using one force polygon



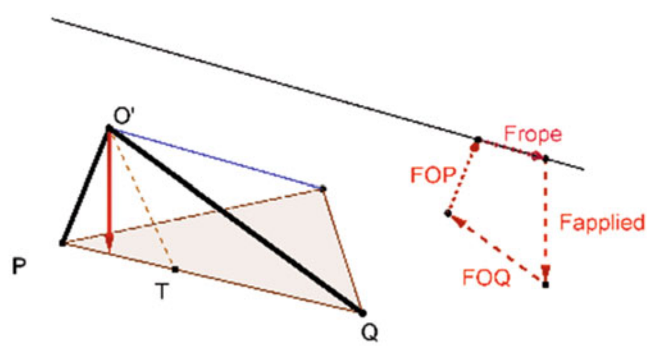
A few quick manipulations of the sliders show the line parallel to the rope is now below the target (Fig. 10.30).

Fig. 10.30 Sliders for force magnitudes allow for rapid convergence



And then it is not difficult to converge to the exact answer (Fig. 10.31).

Fig. 10.31 Solution is quickly found and visually satisfying



One way of finding the magnitude of the rope force is to measure the error between the terminus of this third force and the origin of the applied force. When the error approaches zero, the system is in equilibrium and the rope force will be known.

Figure 10.32 shows a 3D truss with three free nodes, six pinned nodes at $z = 0$, and nine struts, since each free node has three degrees of freedom which must be stabilized. Two of the nodes are subjected to a force resultant of 17.32 units of force as shown ($Fx = Fy = Fz = 10$). The Cartesian coordinates of each node are also shown.

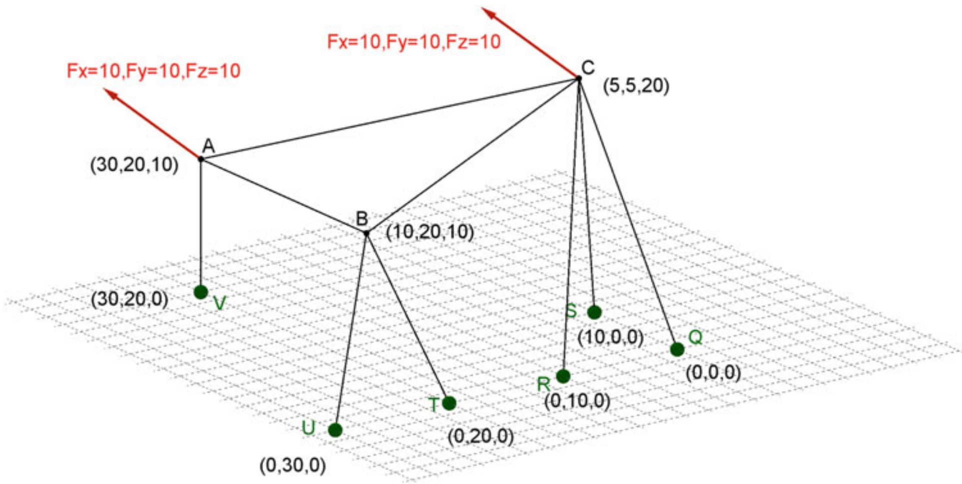
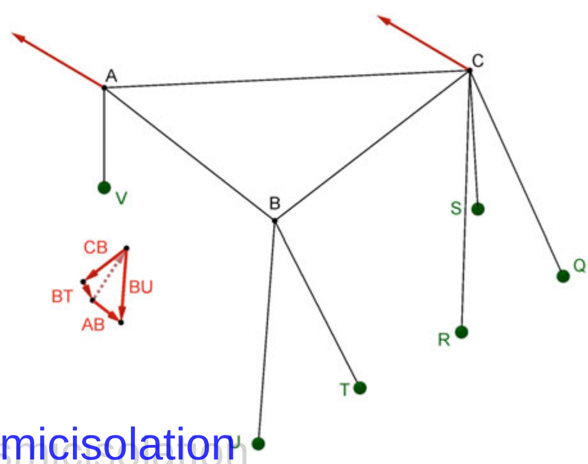


Fig. 10.32 A more complicated 3D truss subjected to various loads

The two-step process is applied to the static equilibrium calculation of joint A. Joint A is chosen because it has only three unknowns. Arbitrarily choose a plane made by the resultant applied force and member AV, and a second plane made by members AB and AC. The intersection of those two planes establishes the location of the fictitious force. Solve for that force, as well as the force in AV. Then immediately resolve the fictitious force into AB and AC.

Next, move to Joint B since that now has only three unknowns. Arbitrarily create a plane from member AB (now known) and member BU. Create a second plane from members BC and BT. The intersection of those two planes establishes the slope of the fictitious force. Note, all that is needed is the slope of this line. Solve for BU and for the fictitious force, then immediately decompose the fictitious force into BC and BT. This is shown in Fig. 10.33.

Fig. 10.33 Selectively moving joint to joint, establishing fictitious temporary forces as needed



Then, the applied force at *C* must be vectorially combined with the now known forces *CB* and *CA*. Figure 10.34 shows how the new load applied to joint *C* is different than the original load.

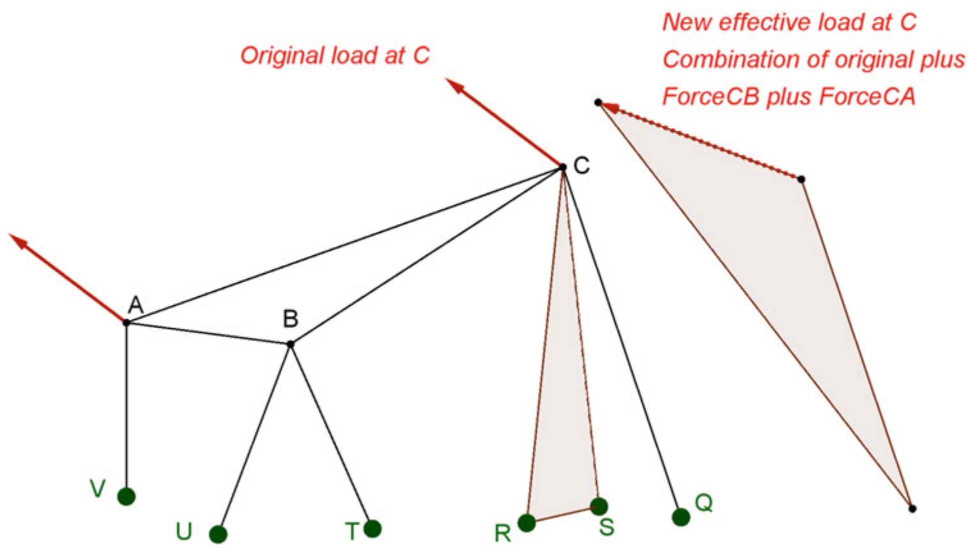
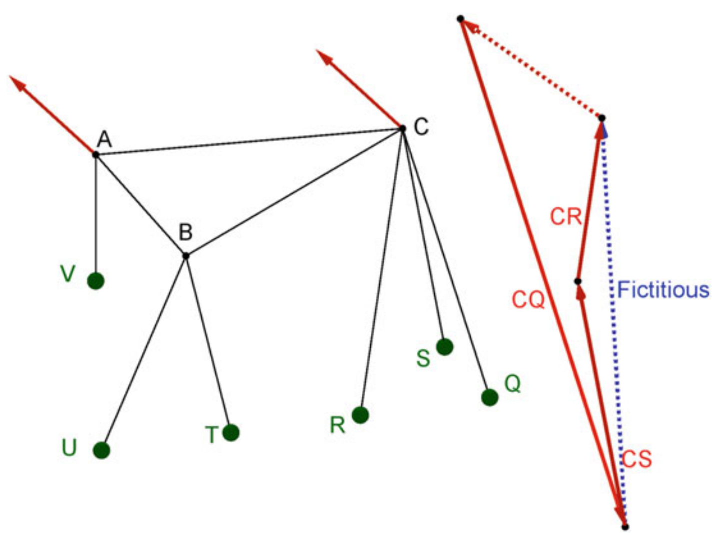


Fig. 10.34 No more than three unknowns allowed at a joint, note that the load is known

Then, arbitrarily make a plane with this net force and member *CQ*. Make a second plane with *CR* and *CS*. The intersection of these two planes defines the fictitious strut slope. The two planes are shown in Fig. 10.34. Solving for the force in *CQ* and solving for the fictitious force allows for the resolution of the fictitious force into *CR* and *CS*. This is shown in Fig. 10.35. This completes the static analysis of the problem.

Fig. 10.35 Static analysis is completed at joint C



The kinematic analysis proceeds as follows. Begin the analysis on node *C*, since there are three struts down to the ground stabilizing this point. Recall that the statics began at *A* (one strut to ground), proceeded to *B* (two struts to ground) and ended at *C* (three struts to ground), whereas the kinematics goes in reverse order. After taking into account the original load and the strut forces *CA* and *CB*, the three stabilizing struts *CQ*, *CR* and *CS* are sufficient to complete the kinematic analysis of *C*. Draw the Williot diagram and resolve the final location of *C*.

Then move to node *B*, but conditionally stabilize it using struts *BU*, *BT* and *BC* where Point *C* is considered pinned in space, not on the ground. Furthermore, a correction analogous to the Mohr Correction in 2D can be immediately applied. The correction is made by adding the "settlement", or already known movement of Point *C*. To do this, create one origin of *B* based on *BU* and *BT*. Then shift from this origin to a second origin, based on the known displacements of *C*. Then draw the Williot extension δCB. Create perpendicular planes to δCB, δTB and δUB. Draw a vector from Origin 1 (the assumed point), not from Origin 2 (which accounts for the correction needed due to the movement of the previous node). This final vector gives the displacement of the second node. This is shown in Fig. 10.36.

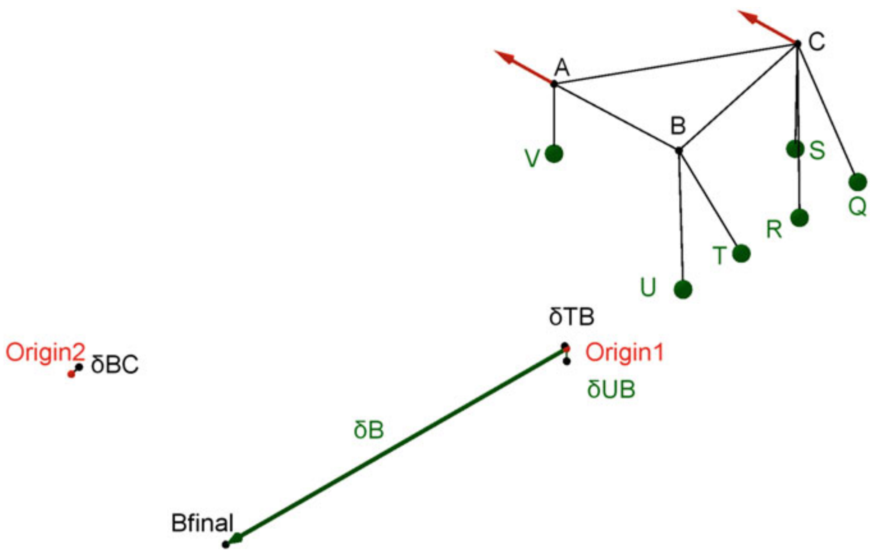


Fig. 10.36 Kinematics begin at most highly constrained joint whereas statics began at least constrained joint

For the third node, Node *A*, create three origins, one assumed on the strut going down to the ground, one based on the movement of *B*, and one based on the movement of *C*. The vector from the first origin to the final resolved point (from three perpendicular planes), is the final displacement of that node.

This chapter on 3D graphic statics ends with a tutorial on the preliminary design of thin shell structures. A series of funiculars in various 2D planes can be used to create a 3D compression only shell structure.

Some simplifications can speed up the modeling process. A strip of unit width, known shell thickness and known concrete density is modeled to create a known force/length load. A circular arc is used to estimate the length of the concrete shell passing through three pre-defined points. This is not perfectly accurate but is acceptable for initial design.

A plane is passed through the three pre-defined points, points that establish the overall shape of the shell based on architectural, programmatic concerns. A plane is created through three target points and on that plane a force diagram captures the entire load of the strip of concrete. The total vertical drop on the force diagram which captures the weight of the strip is discretized into eight sub-drops because the bisect tool can be used very quickly to break up the load diagram. Similarly, the span from the start to end of the strip in the form diagram is discretized into eight segments, and the center of each segment is shown in a different color, to create the nine segments of the funicular. It is convenient to establish the mid-point of each of the eight segments, creating a half-span funicular segment at the very start and very end. This allows for nine funicular segments to be established, matching the nine rays of the force diagram. This is shown in Fig. 10.37.

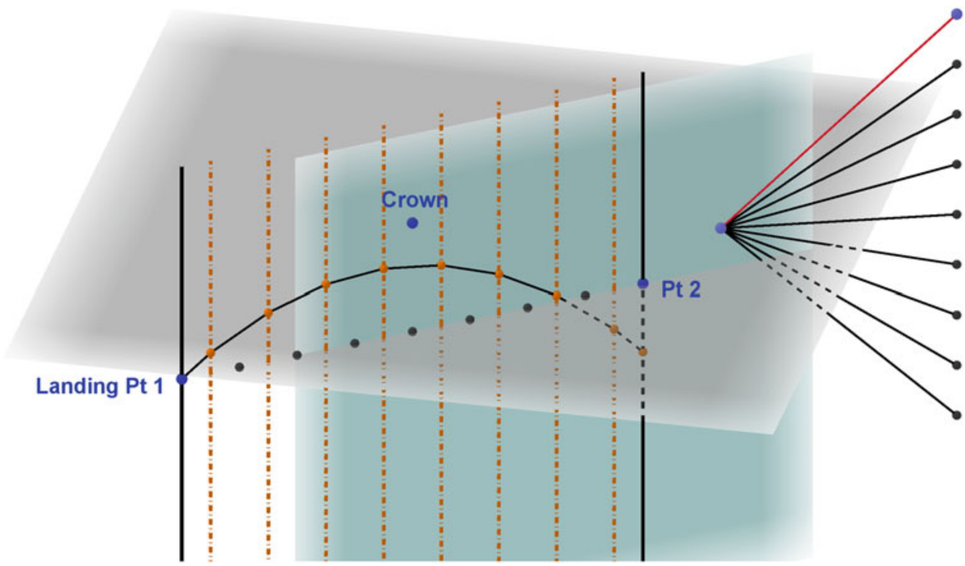


Fig. 10.37 Passing a funicular through three predetermined points on a 2D plane in 3D space

Two other pre-defined landing points of the shell are shown on the orange horizontal line of Fig. 10.38. The next funicular should also pass through the crown of the first funicular. But in order to ensure that the second plane capturing the second funicular is vertical, a vertical line is dropped through the crown of the first funicular. The verticality of the plane is not strictly necessary, but it is very convenient when tying the funicular to the gravity loads of the second concrete strip. Then a plane is established with the crown, one landing point, and the point directly below the crown.

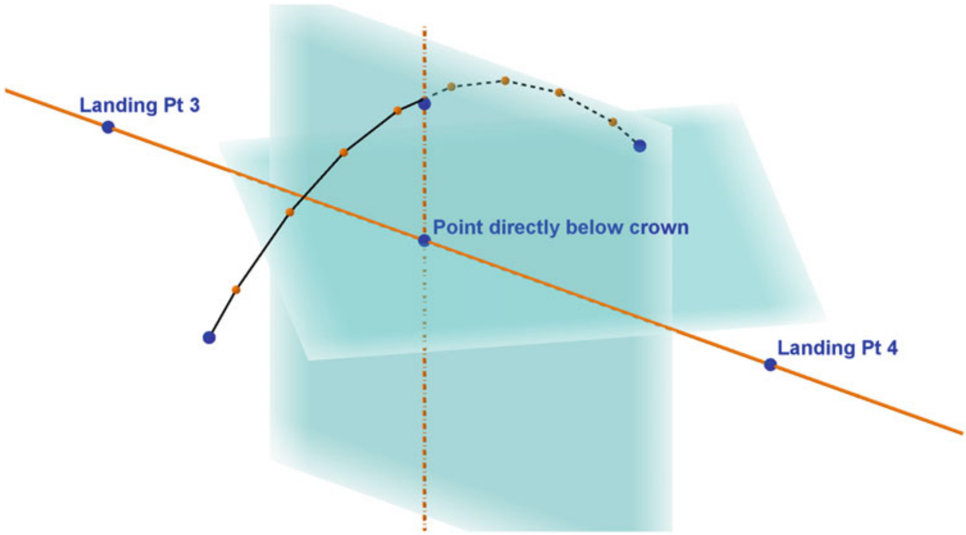
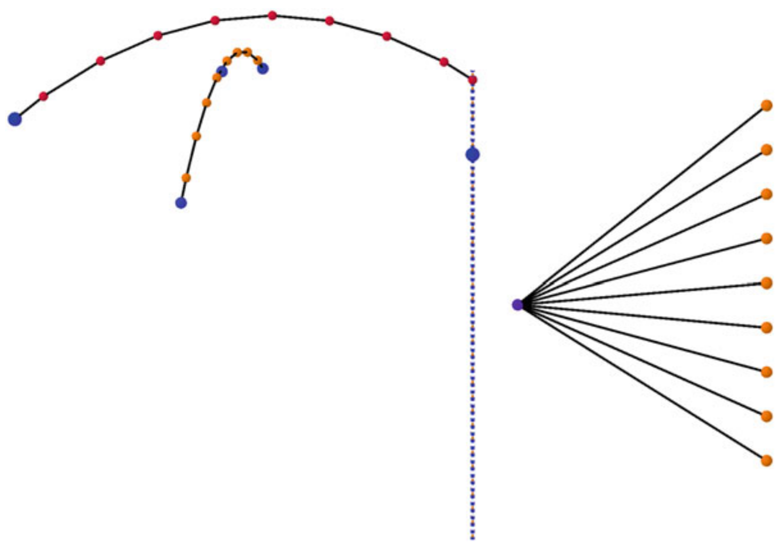


Fig. 10.38 First ensure verticality under the crown, then establish a plane capturing that line

The second funicular begins at Landing Point 3 and is in line with the newly established plane. Initial attempts with a trial pole will not satisfy the three boundary conditions, namely Landing Point 3, the Crown, and Landing Point 4 (Fig. 10.39).

Fig. 10.39 Adjust pole till second funicular touches crown as well as opposite landing point



Manipulating this second pole rapidly adjusts the second funicular until it passes through the three required points. Two funiculars which satisfy the pre-defined programmatic needs are shown in Fig. 10.40.

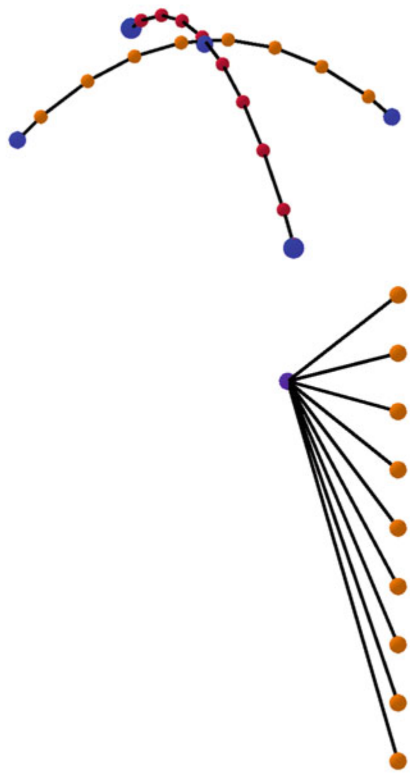


Fig. 10.40 A satisfactory solution of two funiculars that meet at crown and on prescribed landing points

Programmatic needs may establish more formal ideas for the shell design. For example, one plane of the shell could roughly bisect the already established funiculars, and a new shape could be created that still matches the previously established crown, which is no longer the highest point in the shell (Fig. 10.41).

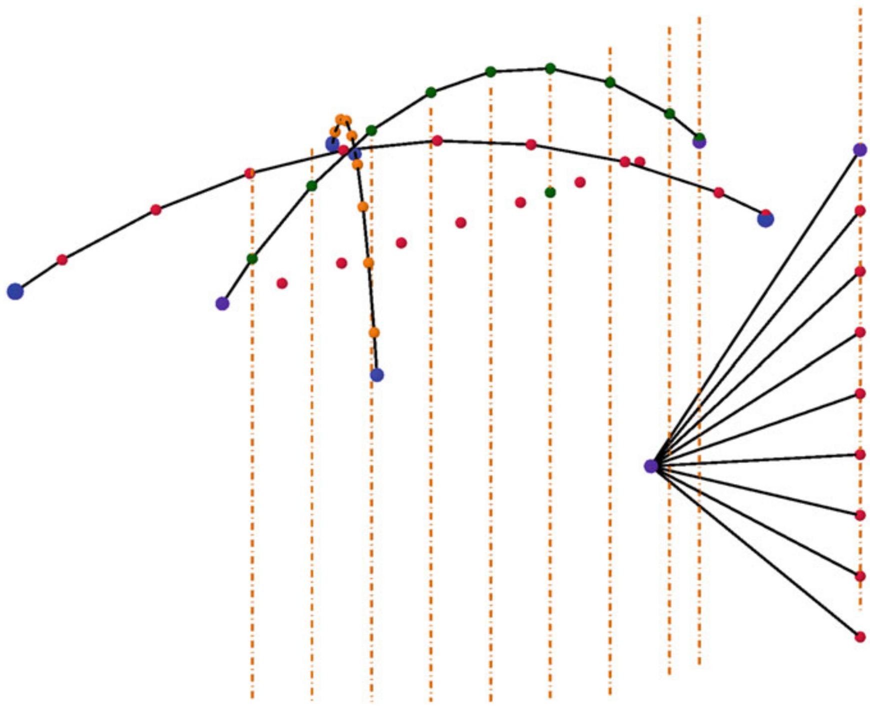


Fig. 10.41 Playful form finding with third funicular which establishes a new peak height

Splines can be passed through horizontal planes to create a curvilinear shell outline (Figs. 10.42 and 10.43).

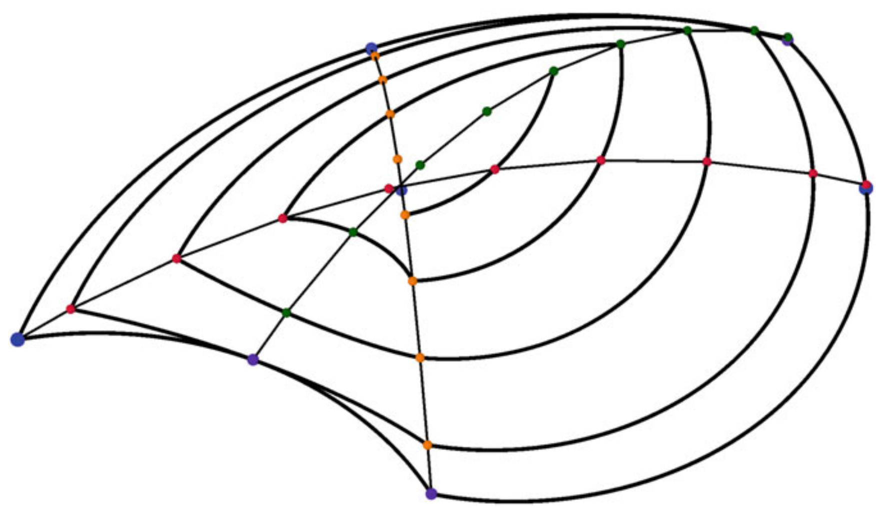


Fig. 10.42 Horizontal splines quickly establish an overall form of the shell

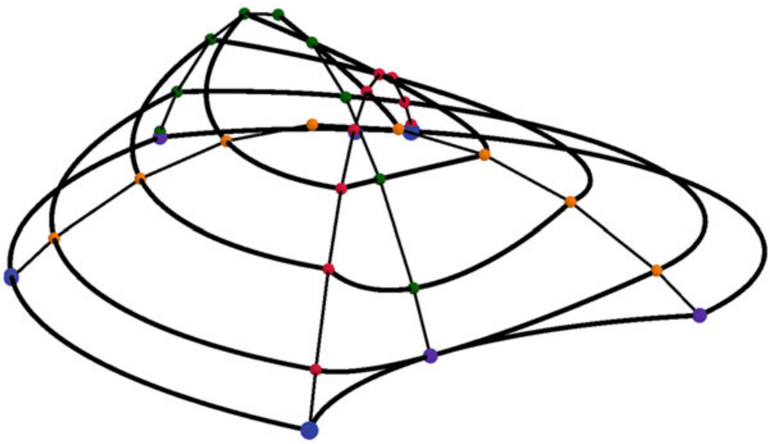
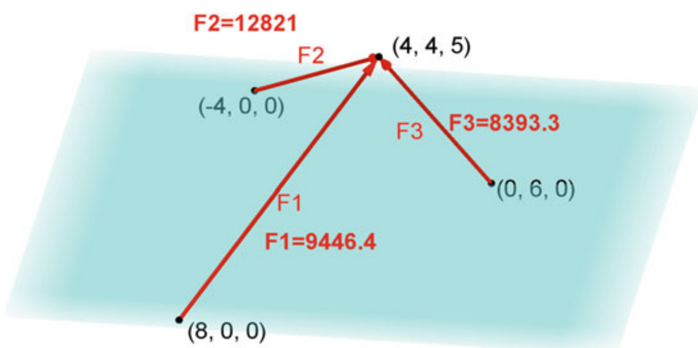


Fig. 10.43 Splines are simply convenient, not funicular

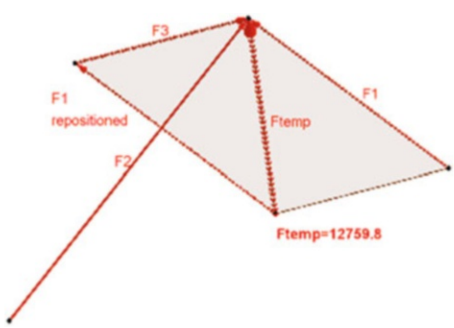
Such a shell will be nearly all compression-only. As such, it can be made extremely thin, but such thinness may cause buckling concerns.

Chapter 10 Exercises

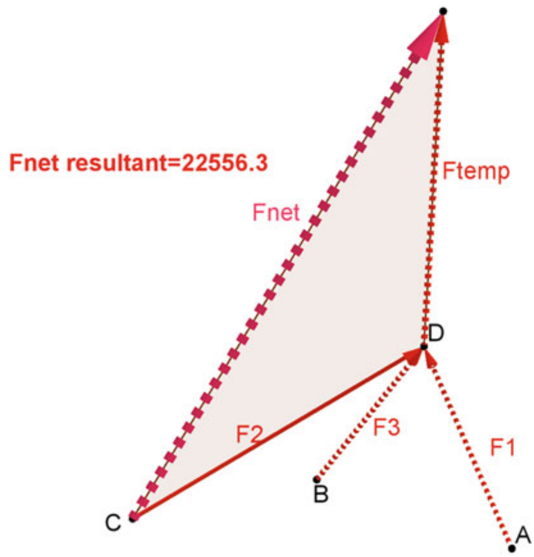
Exercise 10.1 Three forces all act on a single point, but the forces are not co-planar. Find the resultant of these three forces two different ways. First, by the two-step method, and then again directly in 3D with a single superposition.



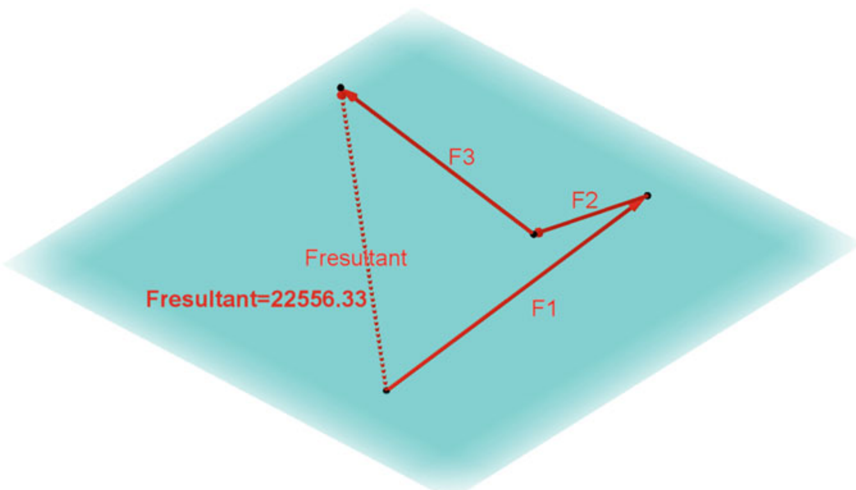
Exercise 10.1 solution part 1



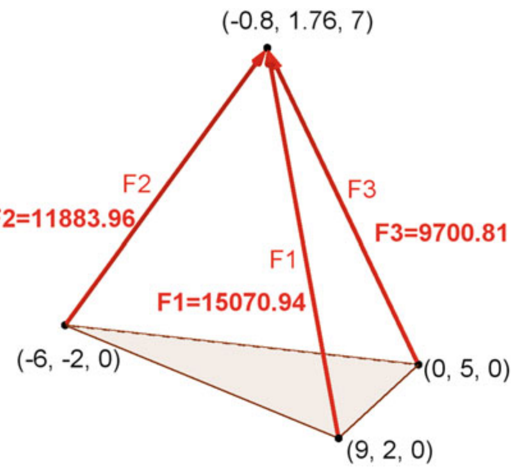
Exercise 10.1 solution part 2



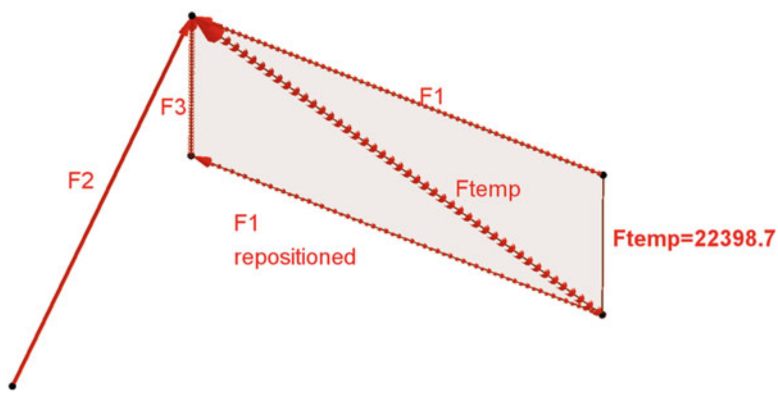
Exercise 10.1 solution part 3



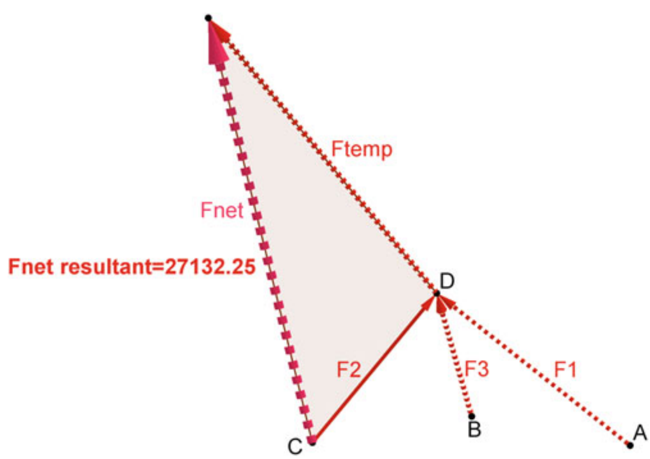
Exercise 10.2 Three forces all act on a single point, but the forces are not co-planar. Find the resultant of these three forces two different ways. First, by the two-step method, and then again directly in 3D with a single superposition.



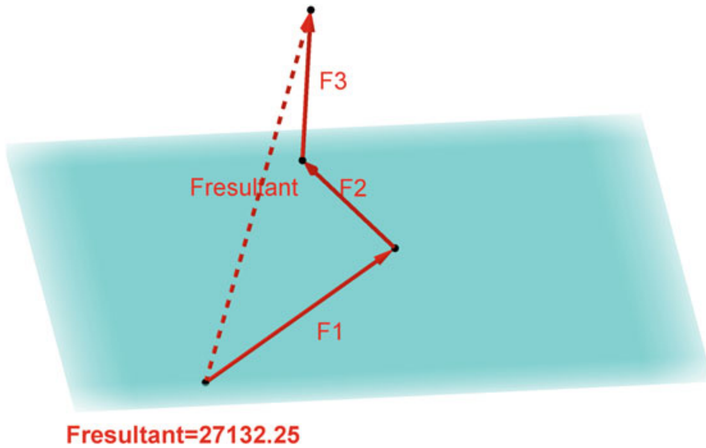
Exercise 10.2 solution part 1



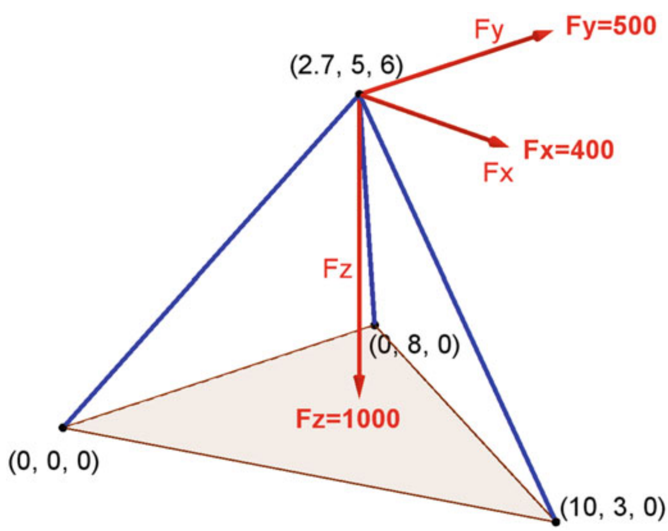
Exercise 10.2 solution part 2



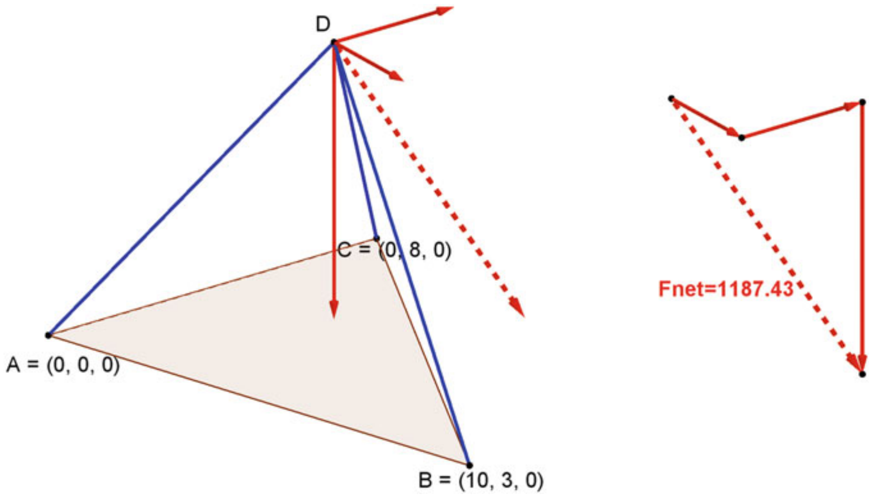
Exercise 10.2 solution part 3
Exercise 10.2 solution iteratively



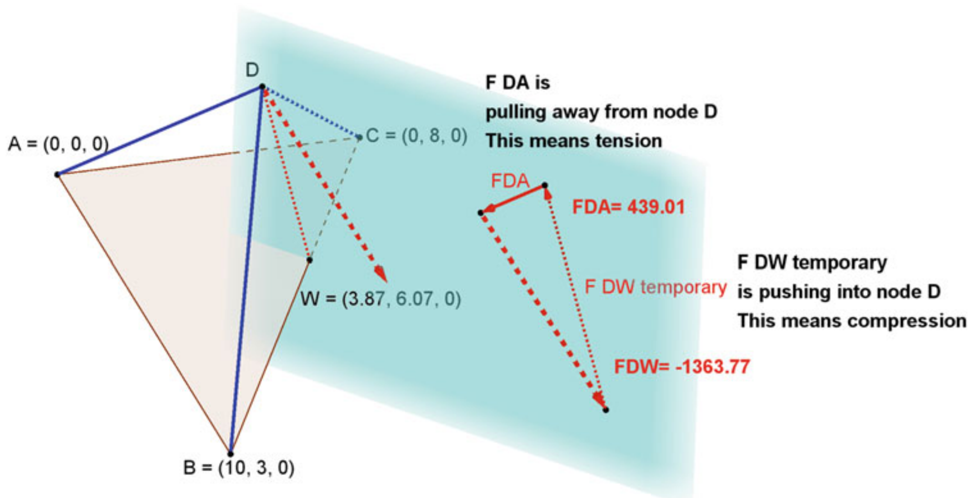
Exercise 10.3 A three dimensional, pin-ended truss is subjected to three loads. First, combine the three applied loads into one statically equivalent net resultant load. Then choose one temporary force that is co-planar with two of the unknown members. Explain clearly why the temporary force and the third force are either tension or compression.



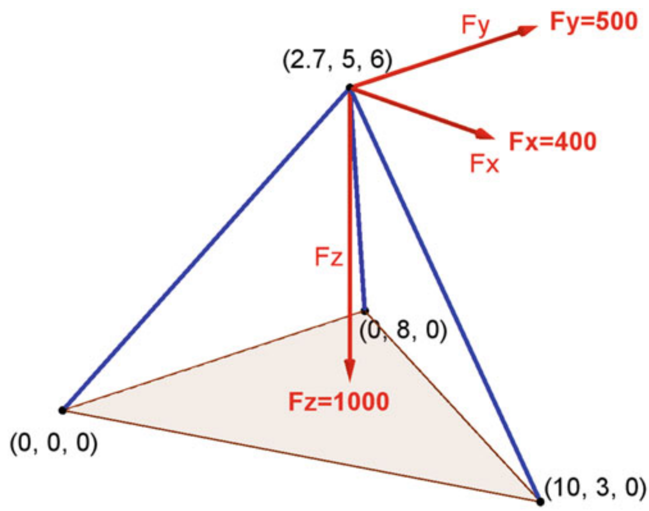
Exercise 10.3 solution part 1



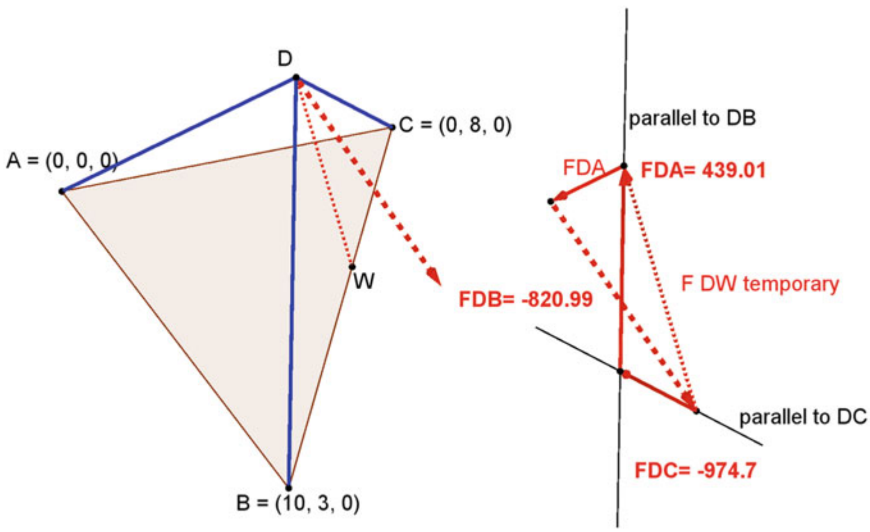
Exercise 10.3 solution part 2



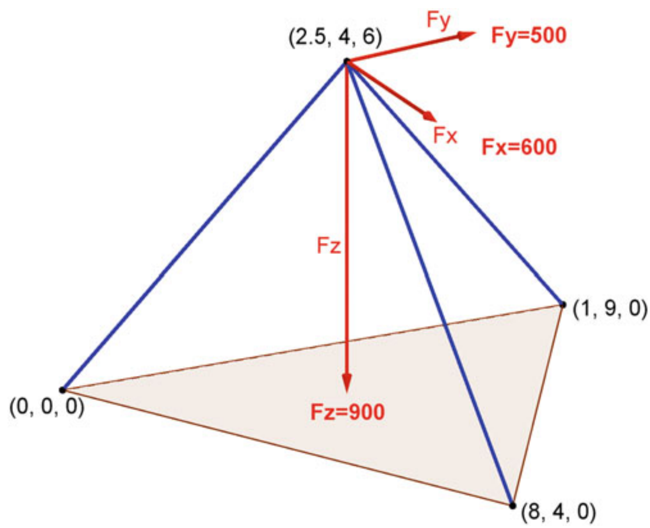
Exercise 10.4 The three dimensional, pin-ended truss of Problem 10.3 is subjected to three loads. Calculate the equilibrating forces in the truss members.



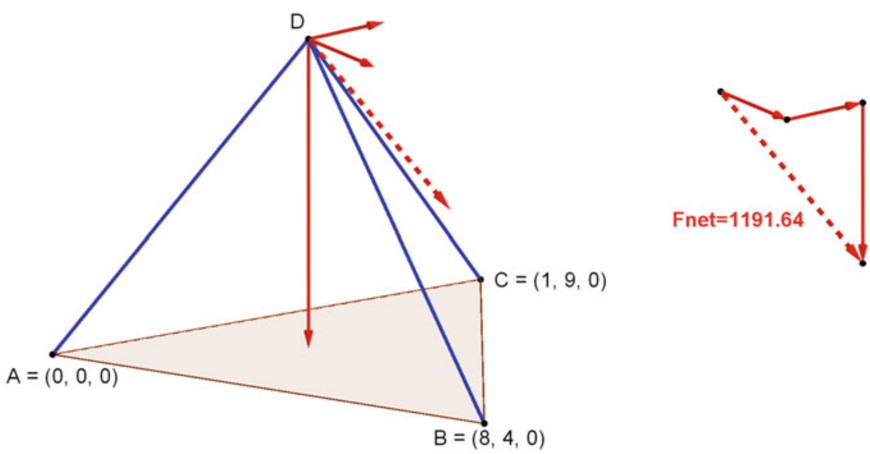
Exercise 10.4 solution



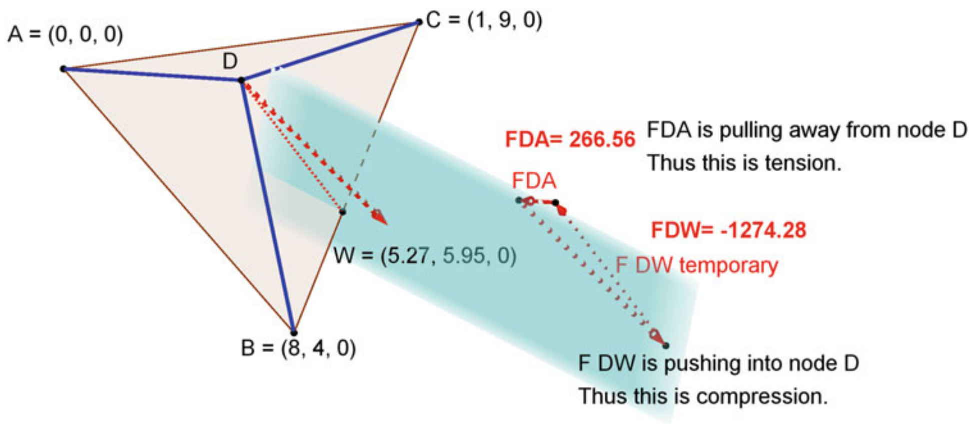
Exercise 10.5 A three dimensional, pin-ended truss is subjected to three loads. First, combine the three applied loads into one statically equivalent net resultant load. Then choose one temporary force that is co-planar with two of the unknown members. Explain clearly why the temporary force and the third force are either tension or compression.



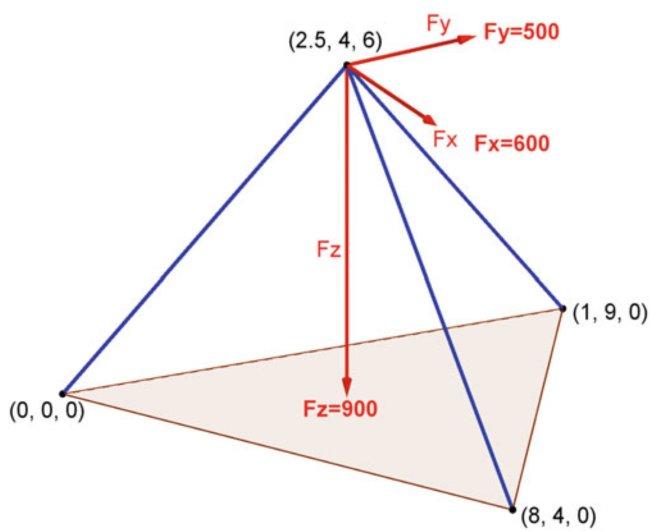
Exercise 10.5 solution part 1



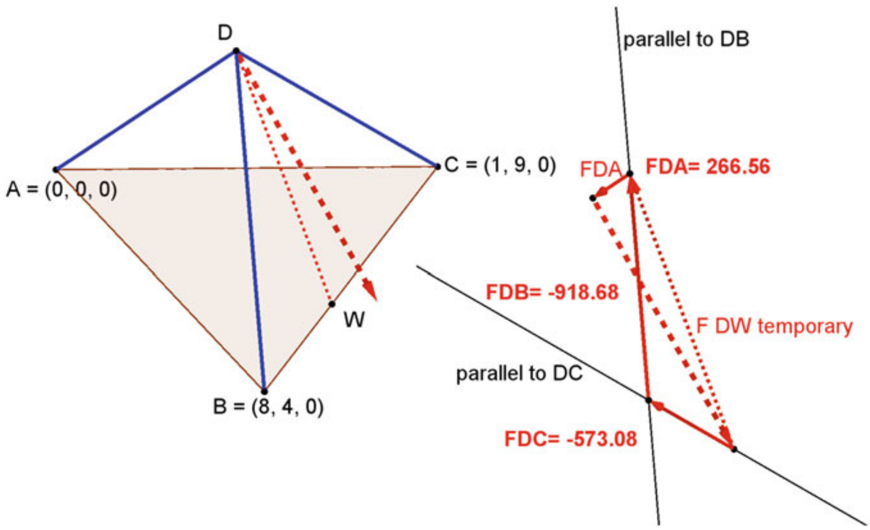
Exercise 10.5 solution part 2



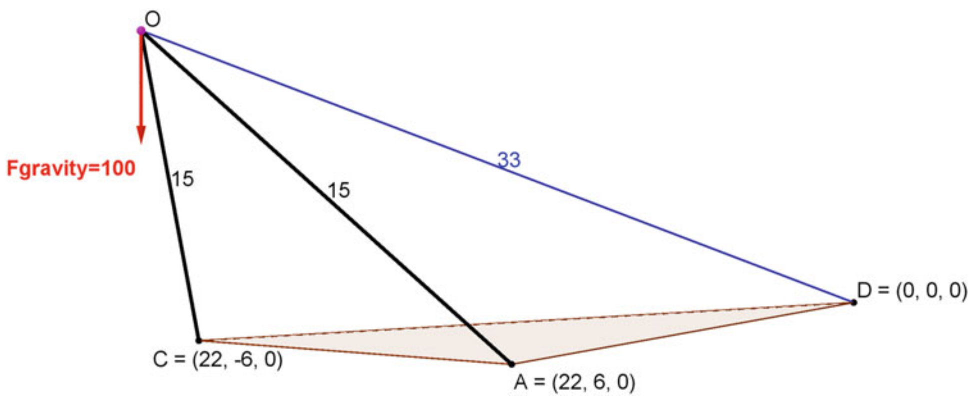
Exercise 10.6 The three dimensional, pin-ended truss of Problem 10.5 is subjected to three loads. Calculate the equilibrating forces in the truss members.



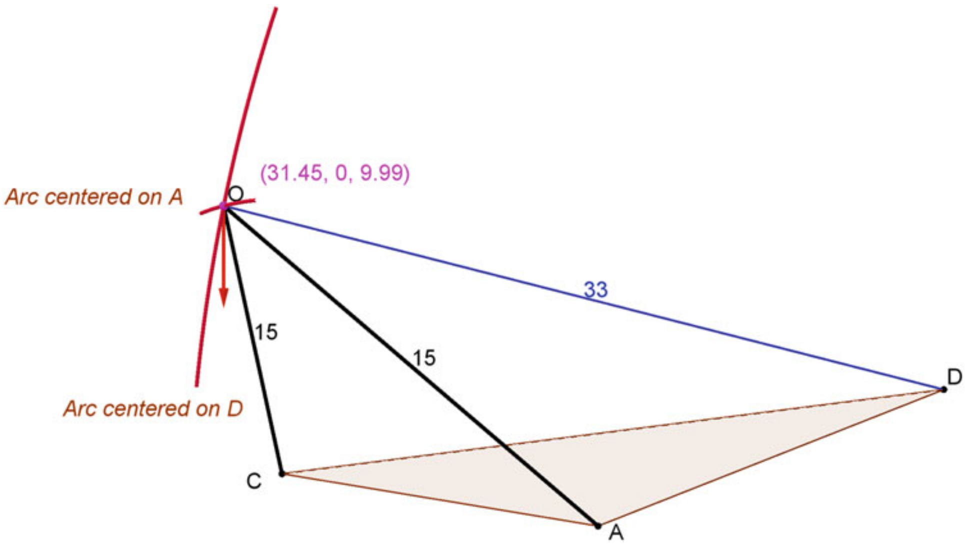
Exercise 10.6 solution



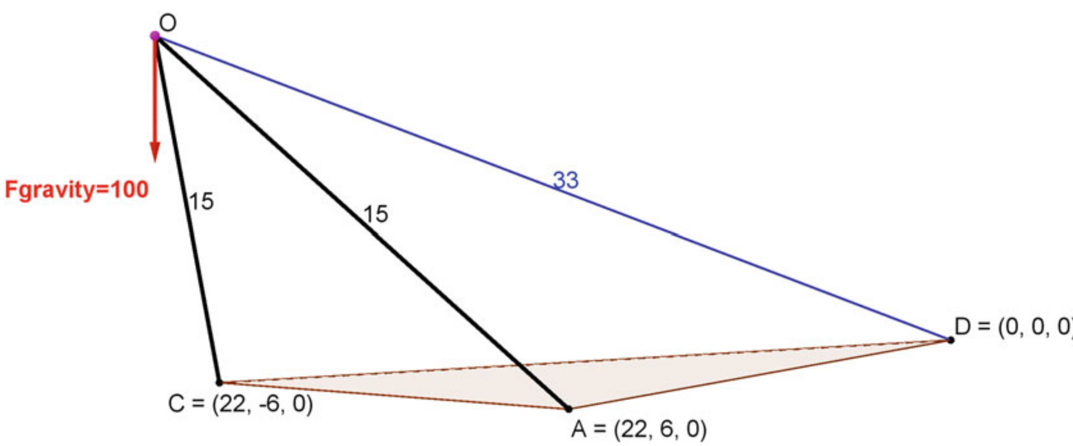
Exercise 10.7 Two pin-ended legs and one rope support a load. Given the length of each leg and the position where each element is anchored to the ground, establish where in 3D space point O is.



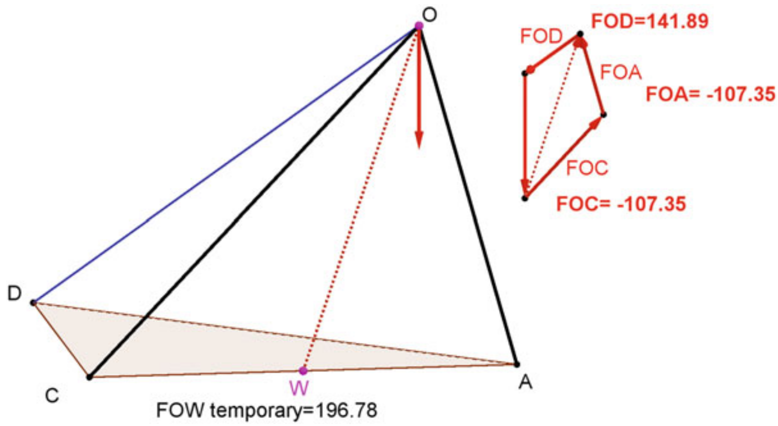
Exercise 10.7 solution



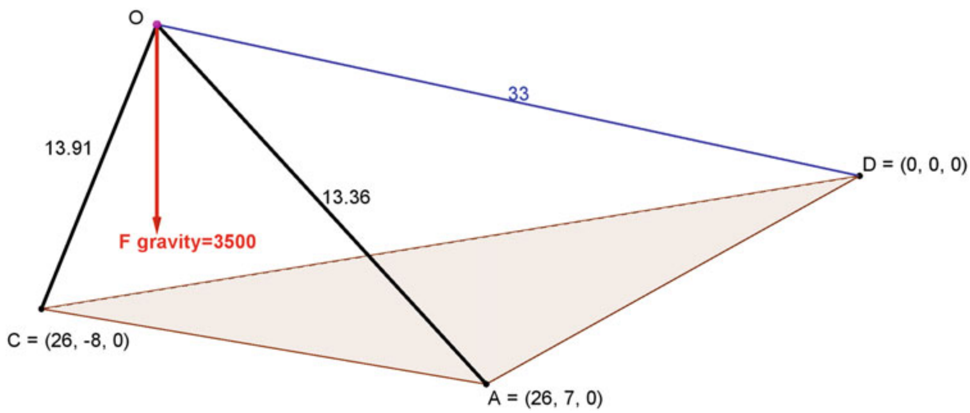
Exercise 10.8 The problem shown in 10.7 is to be completed. Two pin-ended legs and one rope support a load. Given the length of each leg and the position where each element is anchored to the ground, calculate the force in each leg and in the rope.



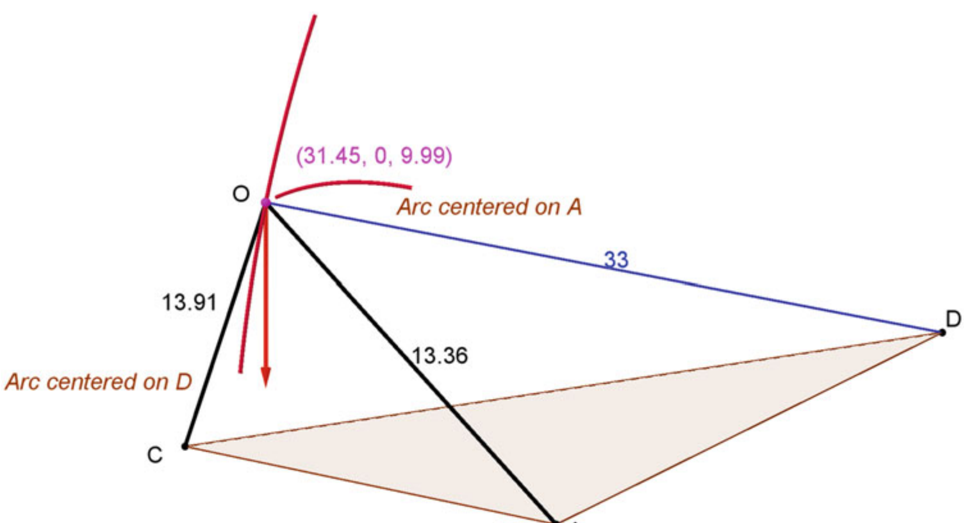
Exercise 10.8 solution



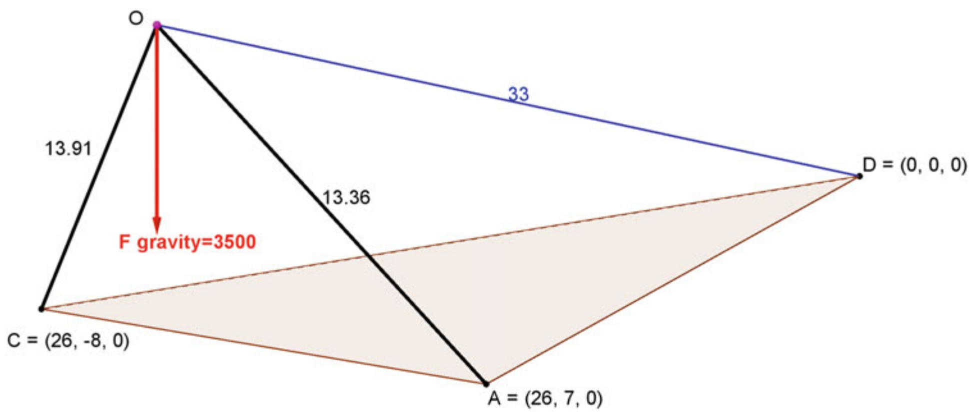
Exercise 10.9 Two pin-ended legs and one rope support a load. Given the length of each leg and the position where each element is anchored to the ground, establish where in 3D space point O is.



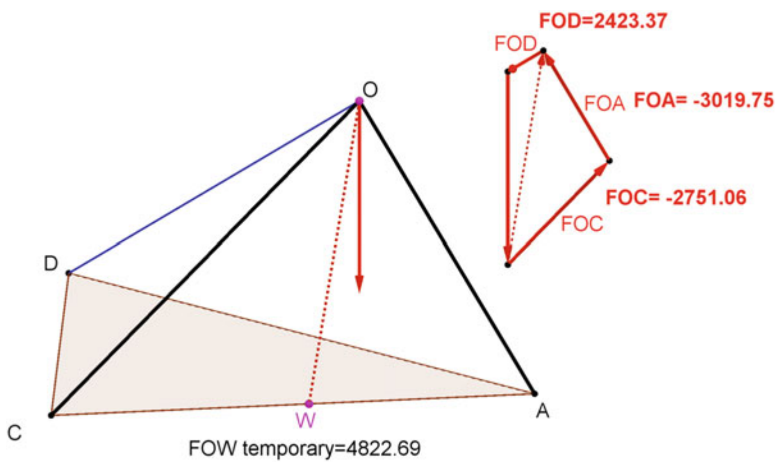
Exercise 10.9 solution



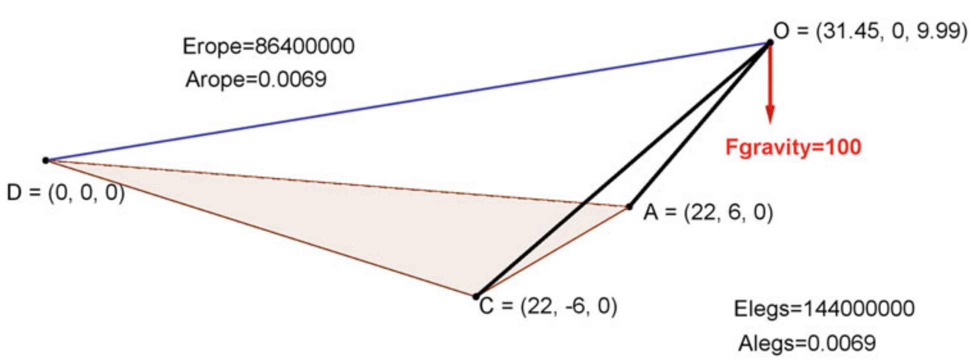
Exercise 10.10 The problem shown in 10.9 is to be completed. Two pin-ended legs and one rope support a load. Given the length of each leg and the position where each element is anchored to the ground, calculate the force in each leg and in the rope.



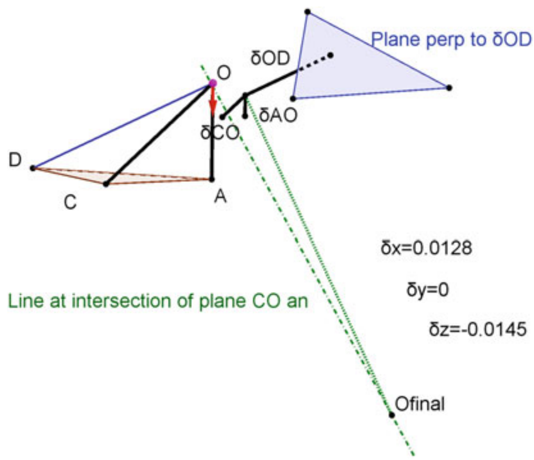
Exercise 10.10 solution



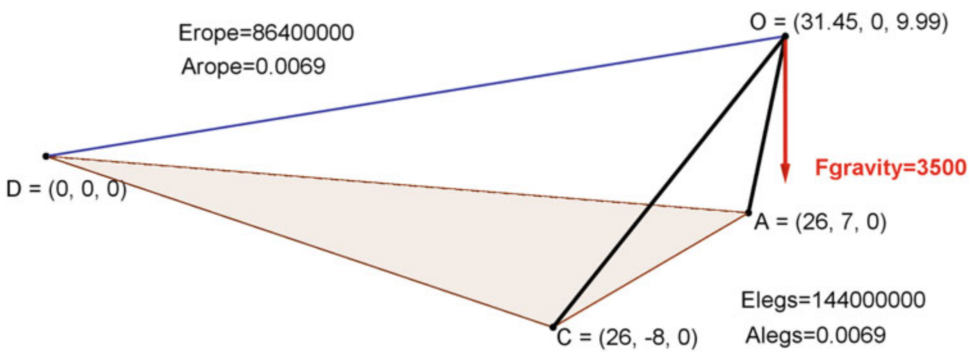
Exercise 10.11 The following structure has loads, material properties and geometry as given. Find the final position of point O using graphical calculations.



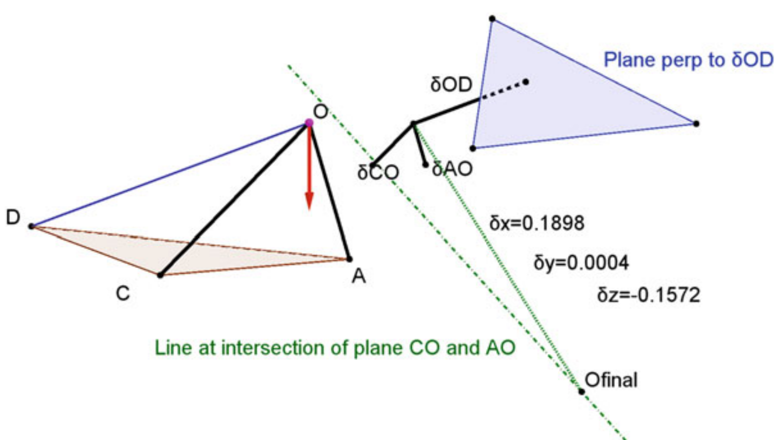
Exercise 10.11 solution



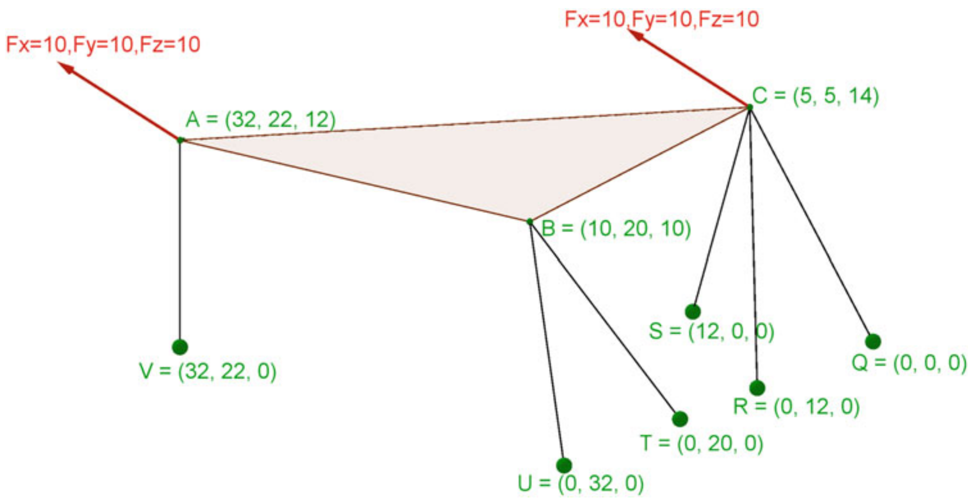
Exercise 10.12 The following structure has loads, material properties and geometry as given. Find the final position of point O using graphical calculations.



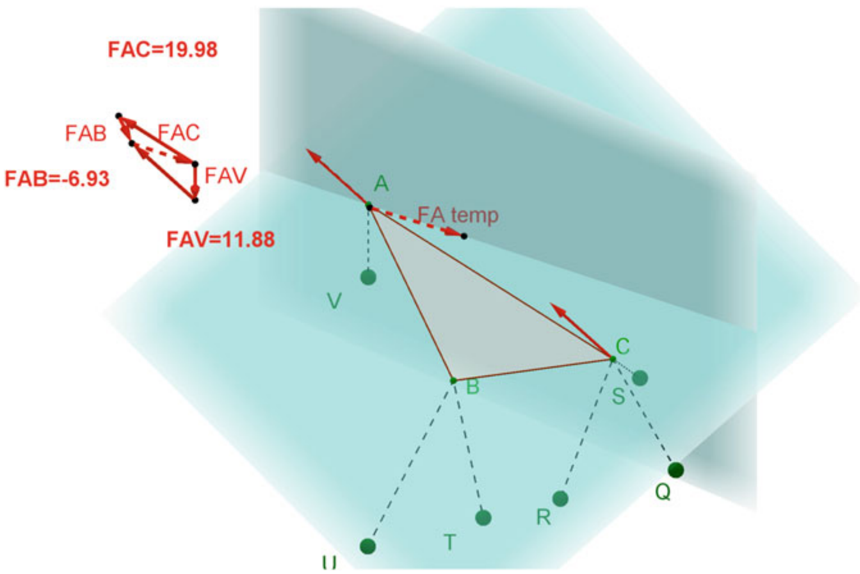
Exercise 10.12 solution



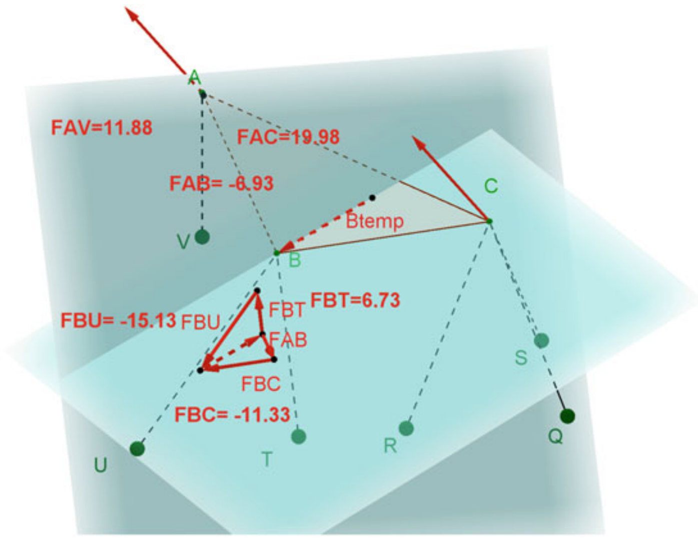
Exercise 10.13 A truss made up of nine pin-ended members is subjected to multiple loads. Sequentially solve for each member force.



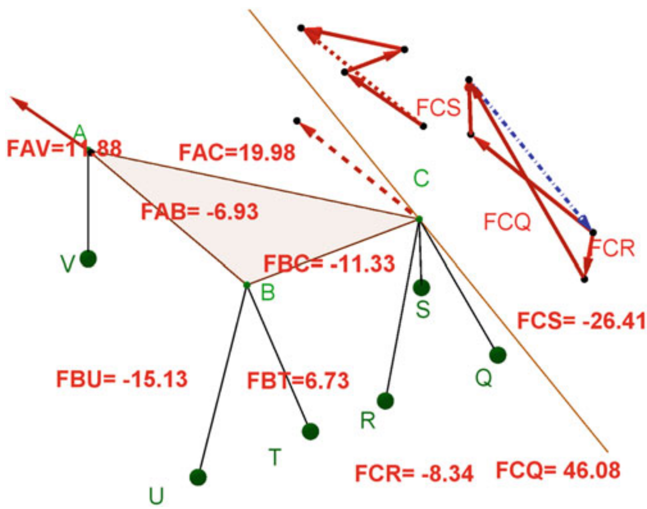
Exercise 10.13 solution part 1: The temporary force must be on a line that intersects the two planes, plane 1 contains the known force and one unknown, plane 2 contains the remaining two unknown forces.



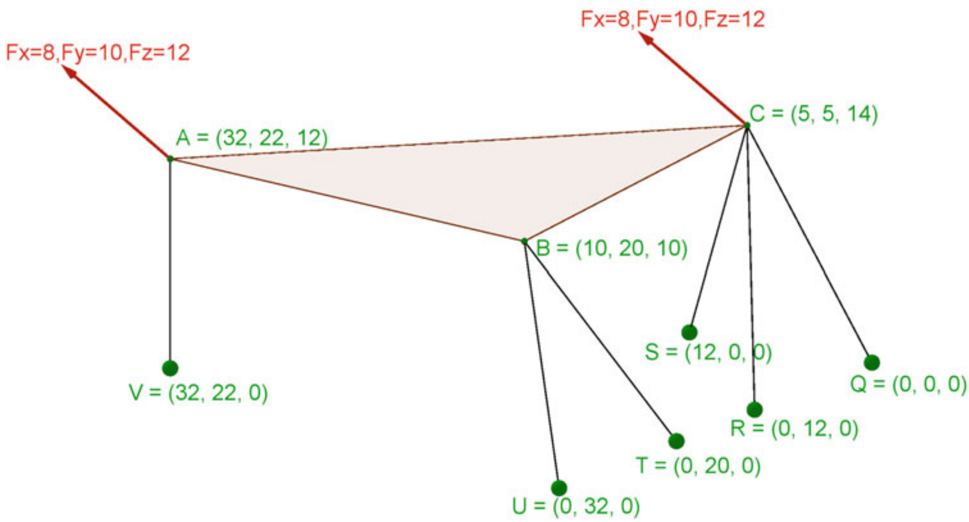
Exercise 10.13 solution part 2: The temporary force must be on a line that intersects the two planes, plane 1 contains the known force and one unknown, plane 2 contains the remaining two unknown forces.



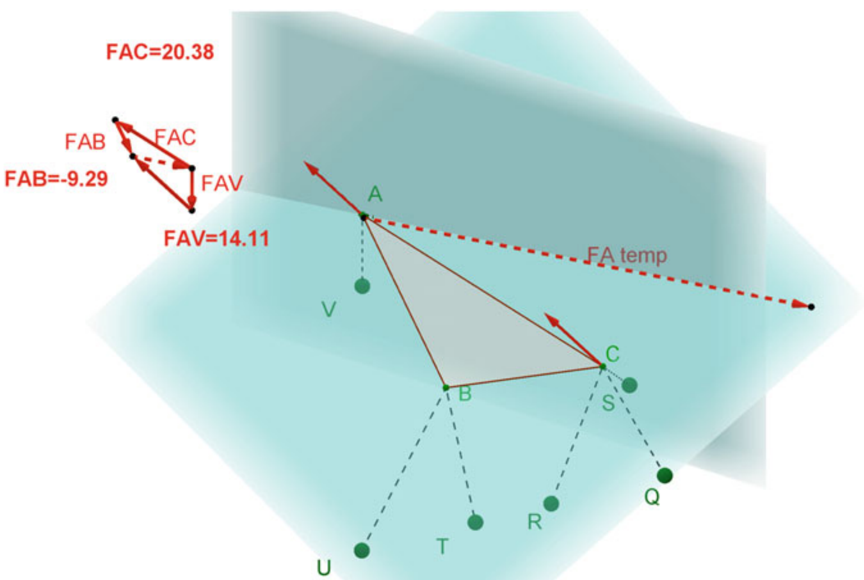
Exercise 10.13 solution part 3: First combine FBC with a FAC and with the external load at C. Notice that this new load, not the original load, is shown in the next step of the solution. Then create a temporary force that is co-planar with the other two unknown forces.



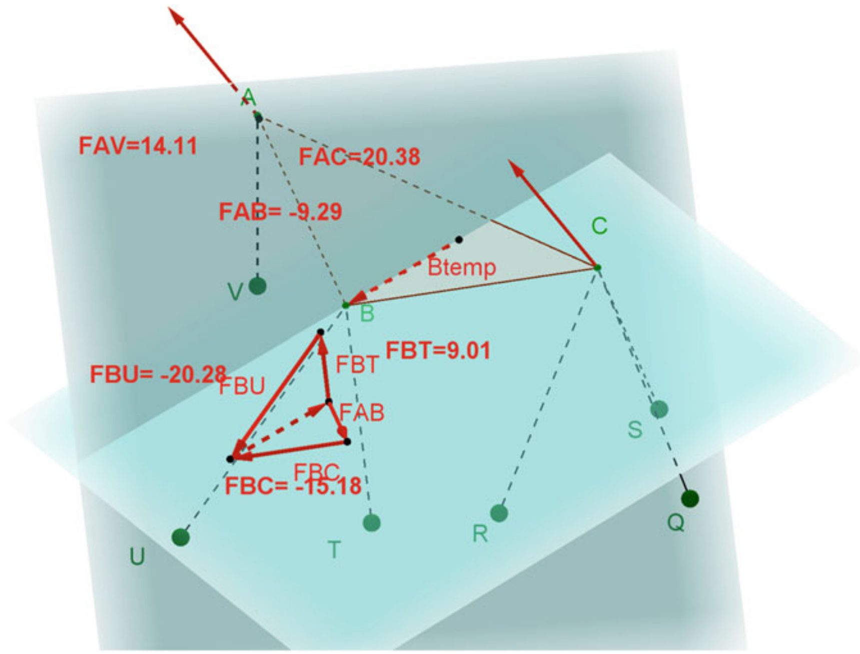
Exercise 10.14 A truss made up of nine pin-ended members is subjected to multiple loads. Sequentially solve for each member force.



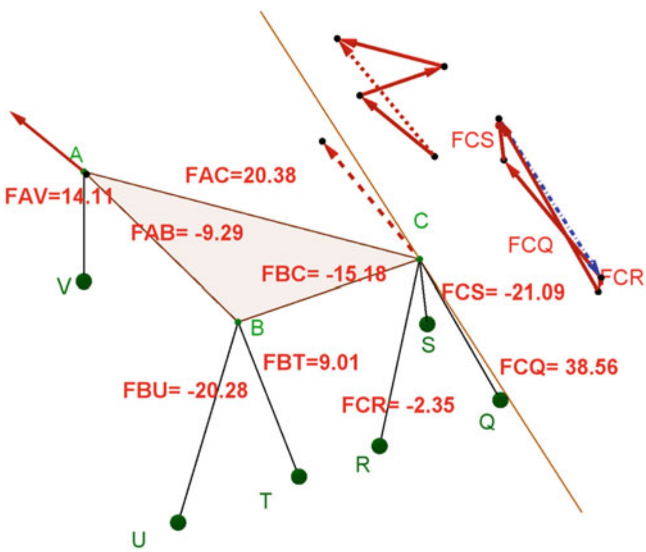
Exercise 10.14 solution part 1: The temporary force must be on a line that intersects the two planes, plane 1 contains the known force and one unknown, plane 2 contains the remaining two unknown forces.



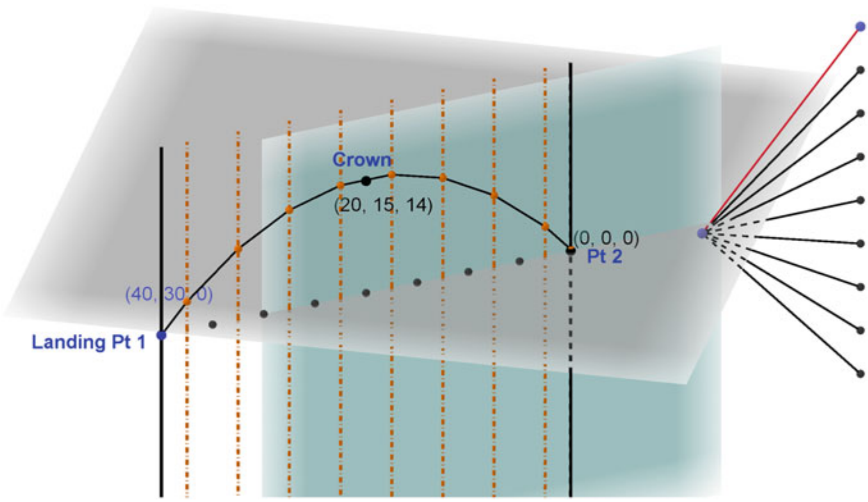
Exercise 10.14 solution part 2: The temporary force must be on a line that intersects the two planes, plane 1 contains the known force and one unknown, plane 2 contains the remaining two unknown forces.



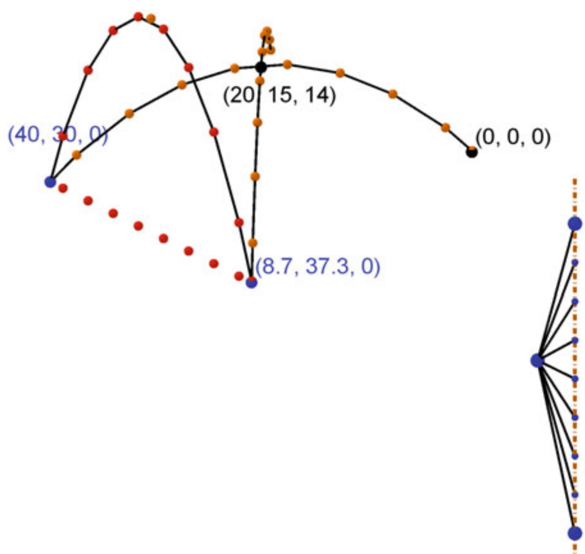
Exercise 10.14 solution part 3: First combine FBC with a FAC and with the external load at C. Notice that this new load, not the original load, is shown in the next step of the solution. Then create a temporary force that is co-planar with the other two unknown forces.



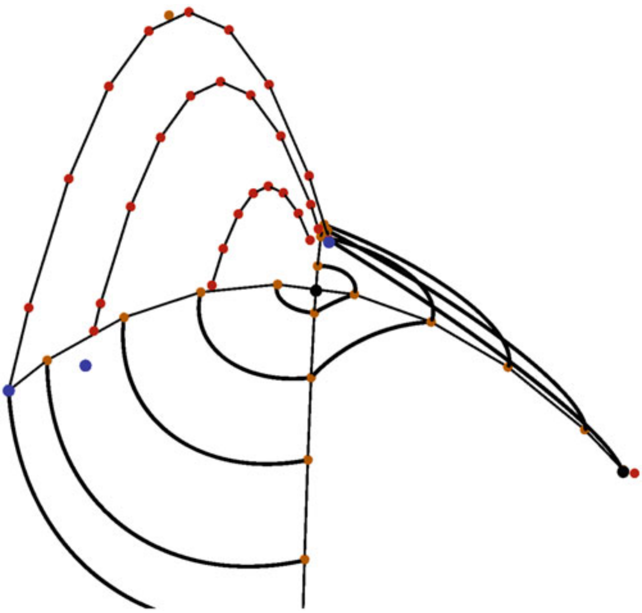
Exercise 10.15 Design a compression-only shell that is made up of symmetrical portions, roughly X Shaped in plan, 40 units across, 15 units height. At one end of the shell, a soaring space at least 30 units of height captures the legs of one end of the X.



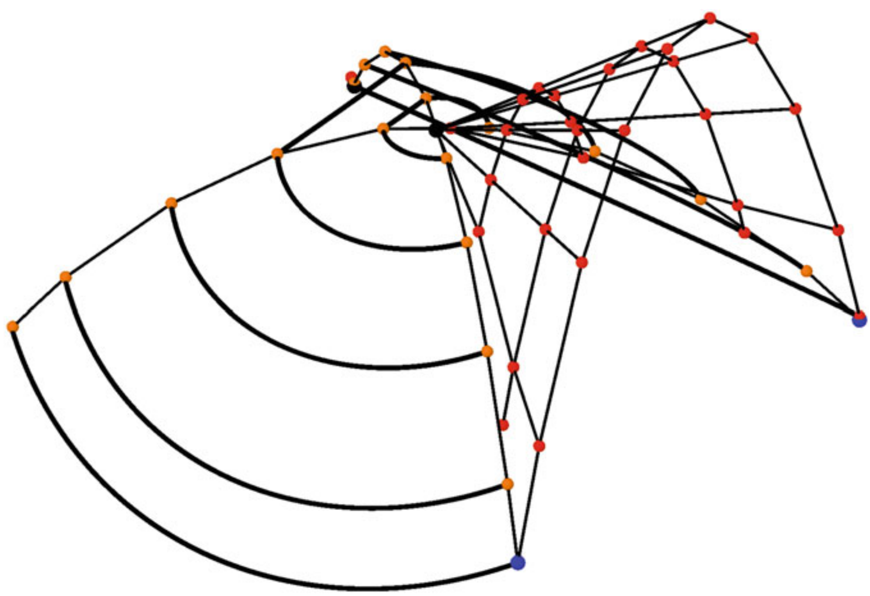
Exercise 10.15 solution part 1: Use symmetry to establish the second axis of the X. Create a new funicular on one tip of the X, have it pass through a height of approximately 30 units at its highest point.



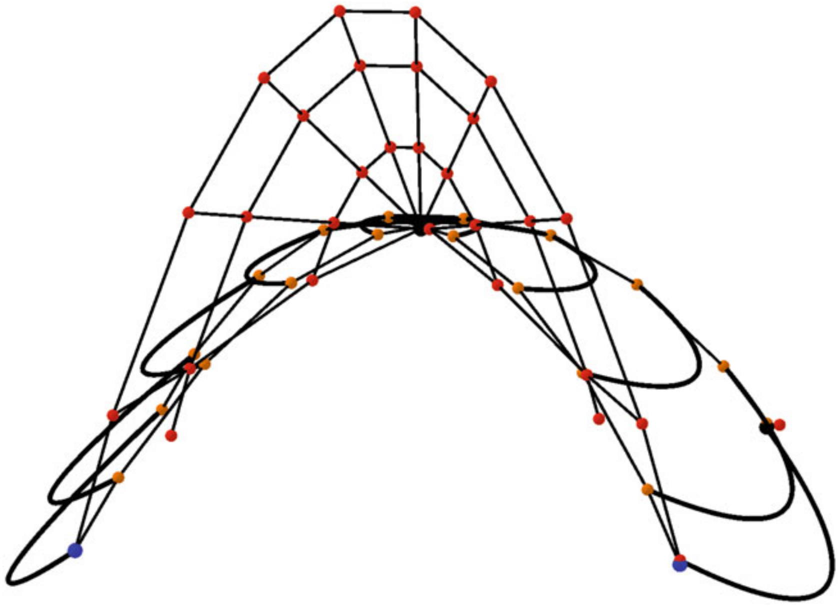
Exercise 10.15 solution part 2: Repeat with subsequently smaller funiculars.



Exercise 10.15 solution part 3: Flesh out remaining portions.



Exercise 10.15 solution part 4: Here is an alternate view.



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