Elementary Mechanics of Solids

BY

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PERGAMON PRESS

OXFORD · LONDON · EDINBURGH · NEW YORK

PARIS · FRANKFURT

Pergamon Press Ltd., Headington Hill Hall, Oxford
4 & 5 Fitzroy Square, London W.1
Pergamon Press (Scotland) Ltd., 2 & 3 Teviot Place, Edinburgh 1
Pergamon Press Inc., 44-01 21st Street, Long Island City, New York 11101
Pergamon Press GmbH, Kaiserstrasse 75, Frankfurt-am-Main
Pergamon Press S.A.R.L., 24 rue des Écoles, Paris 5e

First edition 1965
Library of Congress Catalog Card No. 65-27367

Printed in Great Britain by Bell and Bain Ltd., Glasgow

(2220/65)

PREFACE

THIS book has been written for the student commencing a first course in Mechanics of Solids (or Strength of Materials) for a university degree in engineering, diploma of technology, or higher national certificate. The content should be adequate for the first two years of a lecture course in the subject at most establishments.

Mechanics of Solids might be defined as the analysis of load, displacement, stress and strain which occur in solid bodies. The material may be in a state of elasticity or plasticity, but this book is only concerned with the former. The subject is based on three fundamental principles, namely, equilibrium of forces, geometry and compatibility of deformations, and a stress–strain relationship. The aim of the book is to engrain these principles firmly into the mind of the reader, along with the concept of simplifying assumptions about behaviour, in order to obtain the simpler engineering solutions in Solid Mechanics, as compared with the more rigorous treatment in the Theory of Elasticity. Emphasis is also laid on the recognition of problems which are statically determinate and those which are statically indeterminate. These concepts are introduced in relation to simple uniaxial stress and strain systems in Chapters 1 and 2. This leads into the detailed treatment in Chapter 3 of biaxial stress and strain systems and the generalized stress–strain relationships. The basic principles discussed earlier are applied in Chapters 4 and 5 to develop a theory for bending, and in Chapter 6 a theory for torsion. The final chapter discusses some cases in which the combined effects of bending and torsion occur. There is a number of worked examples throughout the text to assist the reader in applying the relationships developed, and there are unworked problems at the end of each chapter with answers at the end of the book.

The author wishes to express his thanks to the Syndics of the Cambridge University Press, to the Senate of the University of London and to the Institution of Mechanical Engineers for permission to reproduce examination questions. None of these bodies is committed to approve the solutions or answers given.

Professor Neal of Imperial College and Professor Charlton of Queen's University devoted some time to reading the manuscript and their valuable comments and suggestions were much appreciated. Grateful acknowledgement is also due to Miss R. Burgess (Imperial College) and Miss J. Cooke (Queen's University) who produced the typed manuscript, and Mr. H. Wilson and Miss E. Gent (Queen's University) for drawing the diagrams.

P. P. BENHAM

CHAPTER 1

CONCEPTS OF FORCE AND STRESS

1.1. Introduction

Everything around us, and also ourselves are subjected to *forces*. A body under the action of forces will deform in some way depending on the position and magnitude of the forces, the shape of the material, and the condition of the material. Many materials and in particular metals have the ability, within certain limits, to be deformed under applied force and recover completely in shape and size when the force is removed. This concept is broadly what is meant by *elasticity*. The fact that force and deformation are linearly related as a *property* of many materials enables us to build up a mathematical structure which is *independent* of the type of material so long as it obeys the laws of linear elasticity.

The end products of the theoretical arguments to be developed in this book are quantities termed *stress*, *strain* and *displacement*. These quantities to a large extent dictate the strength, safety and efficient working of a component or machine. The mechanics of solids is developed in relation to common modes of deformation, e.g. stretching, bending and twisting, which are exhibited in elements such as cables, beams, springs and shafts.

The traditional, but somewhat misleading, title for the above subject has been *strength of materials*.

1.2. Statically Determinate Problems

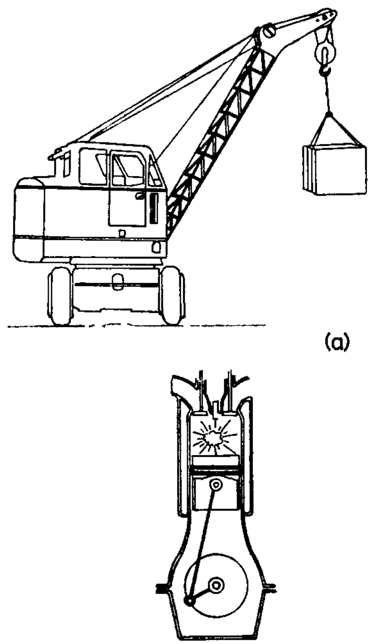
A problem in which all the internal forces can be determined solely through consideration of equilibrium is termed *statically determinate*. If internal forces can only be analysed by studying

A*

the geometry of deformation as well as equilibrium then it is said to be *statically indeterminate*. This chapter deals with problems of the former type.

1.3. Force and Moment

Force can be defined as the directed interaction between elements and bodies. A simple example is illustrated in Fig. 1.1(*a*)



(a)

(b)

FIG. 1.1

in which a *load* is hanging stationary on a crane hook. This load sets up a system of forces, via the crane hook and the cable, in the members of the crane, by reactions at the various joints and bearings. The second example in Fig. 1.1(*b*) shows the cylinder of an internal combustion engine at the point when the fuel mixture has ignited and the heated gas is expanding. The walls of the cylinder and end of the piston are containing the gas and resisting expansion. Forces are therefore exerted by the gas on the cylinder and piston and equal and opposite forces by the cylinder and piston on the gas.

Forces can arise owing to various physical conditions, e.g. weight, inertia, friction, centrifugal. A body subjected to external

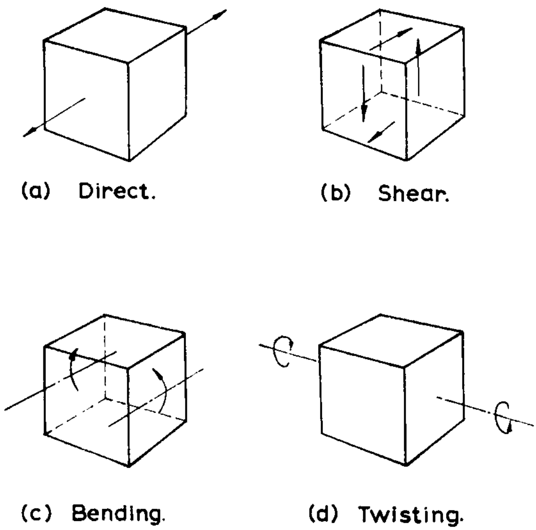


FIG. 1.2

force which is at rest is said to be in a *static* condition, but if forces give rise to velocity and acceleration then the body is in a *dynamic* state.

Forces can also produce moment or torque about some point or axis at a distance from the line of action of the force. The magnitude of the moment is the product of the perpendicular distance from the point to the line of force and the magnitude of the force.

When considering the effect of external forces on a body, which may be irregular in shape, owing to the variation of internal reaction through the material it is necessary to study the behaviour of a small block or element within the body and subsequently integrate to a larger volume.

Various kinds of deformation produced by forces acting on an element of material are illustrated in Fig. 1.2(*a*)–(*d*). A *direct* or *normal* force acts perpendicular to the surface of the element and produces extensions or contractions, Fig. 1.2(*a*); a *shear* force acts parallel to a surface and produces a sliding action between various planes in the element, Fig. 1.2(*b*); a moment is usually associated with *bending* and a torque with *twisting*, Fig. 1.2(*c*) and (*d*).

1.4. Equilibrium

One of the basic concepts in solid mechanics is the requirement of statical equilibrium of all external forces and moments. When considering a single particle it can be said that it is in a state of equilibrium if the vector sum of the applied forces $F_1, F_2, \ldots F_n$ is zero and also the vector sum of all moments or couples $M_1, M_2, \ldots M_n$ acting on the particle is also zero, or

$$F_1 + F_2 + \ldots + F_n = \sum_r F_r = 0$$

and

$$M_1 + M_2 + \ldots + M_n = \sum_r M_r = 0.$$

In the previous section a body was shown in Fig. 1.2(*a*) subjected to externally applied direct forces. There is nothing surprising about that illustration yet it is pertinent to ask why the material stays in one piece and does not separate into two or more parts under the action of the forces. It can only be because the structure of the material can resist being pulled apart by the attractive forces between atoms. It is now evident that there are *internal forces* in a material which resist external forces. The external forces must in themselves be in equilibrium for the body to remain at rest, and the internal forces must be in equilibrium for continuity of the material, that is any part or element of the body must in itself be in equilibrium. One further arrives at the conclusion that internal and external force equilibrium can be equated and therefore

$$\sum F_{\text{ext}} = \sum F_{\text{int}} = 0$$

and

$$\sum M_{\text{ext}} = \sum M_{\text{int}} = 0.$$

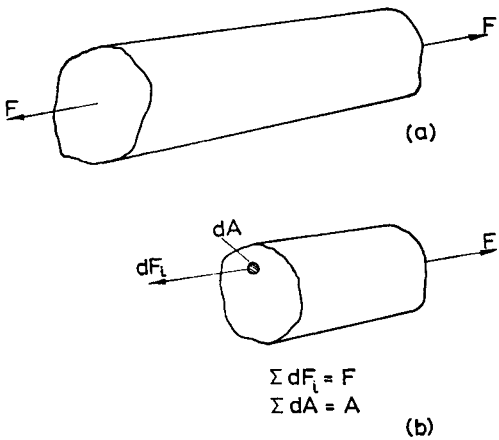


FIG. 1.3

1.5. Stress

In order to be able to describe how two pieces of material of different size will react to the application of a particular value of external force on each it is necessary to consider the internal force in a more quantitative manner. In Fig. 1.2(*a*), the former is represented by a single arrow, i.e. a resultant force; however, just as this is unrepresentative of how a force could be applied, it is also unlikely that the internal force will only occur and react along the line of the arrow, since the material is continuous and will transmit the force throughout its bulk. Figure 1.3(*b*) shows *part* of the body of Fig. 1.3(a) in equilibrium under a force-reaction system, and this is termed a *free-body* diagram. In Fig. 1.3, a small element of the cross section within the material of area dA supports a small component of internal force dF_i.

The internal force per unit area is

$$\mathrm{Lt}_{\mathrm{d}A\to 0}\frac{\mathrm{d}F_i}{\mathrm{d}A}$$

and this quantity is termed *stress.*

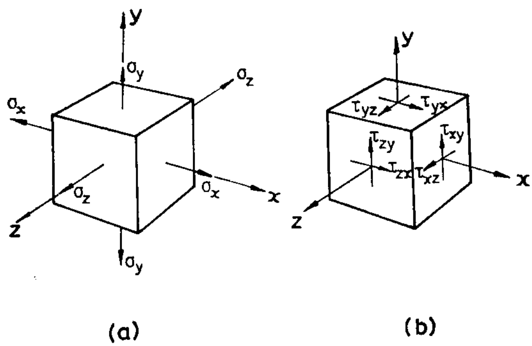


FIG. 1.4

The purpose of the quantity, " stress ", is to enable a comparison to be made of the reaction to external loading of objects of different size. This concept will become more apparent when deformations are studied in the next chapter.

Direct or normal stresses are those which have a direction perpendicular to the plane on which they act. If the direction is outwards from the plane the stress is termed tensile and is positive in sign. A direct compressive stress acts towards a plane and is negative in sign. Direct or normal stress is designated by the symbol, σ, and its direction will be denoted by a suffix x, y or z for Cartesian coordinate axis as shown in Fig. 1.4(a).

A force which acts parallel to a plane is called a shear force. Internal shear force expressed per unit area also acts parallel to planes within the material and is termed *shear stress.* This is denoted by the symbol, τ, and in general a double suffix notation xy, xz, yz, etc. is used as shown in Fig. 1.4(b). The first suffix denotes the direction of the normal to the plane and the second suffix the direction of the shear stress component. These directions are taken as positive or negative in the same sense as the coordinate axes directions. It is also now appropriate to mention the requirement of *complementary shear stresses.* It may be seen that in the element of unit thickness in Fig. 1.5 for rotational equilibrium of forces about a z-axis through the centre of the element

$$2\tau_{xy} \times \delta y \times 1 \times \delta x/2 = 2\tau_{yx} \times \delta x \times 1 \times \delta y/2$$
$$\text{or } \tau_{xy} = \tau_{yx}$$

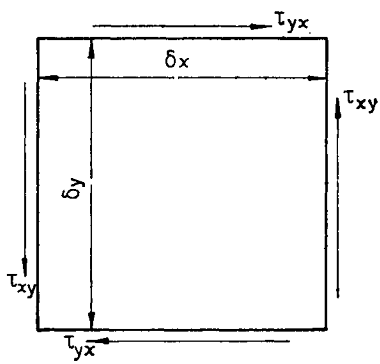


FIG. 1.5

thus a shear stress on one plane is *always* accompanied by a complementary shear stress of the same sign and magnitude on a perpendicular plane.

A special state of direct stress that should be mentioned although it does not often occur is that known as hydrostatic stress. This may be represented by the stress set up in a body immersed at great depth in a fluid, and the hydrostatic compressive stress, σ, in the body would be the same in all directions and equal to the external applied pressure.

1.6. Statically Determinate Stress Systems

Now that force and stress have been defined and the requirements laid down for statical equilibrium in a body, it is possible to analyse the stress conditions in a few components having a simple shape and external loading. Problems in which stresses can be found purely by consideration of the conditions of equilibrium and without studying any deformations that occur, are few in number and are termed *statically determinate*. The following problems will illustrate the latter condition and also various types of external loading.

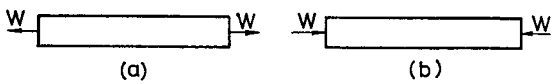


FIG. 1.6

1.6.1. Tie and Strut

A bar or member which prevents two components to which it is joined at each end from moving apart is subjected to uniaxial tension and may be termed a *tie* bar, Fig. 1.6(*a*). This is the simplest case of a statically determinate problem and equilibrium of external and internal forces is represented by $W = \sigma a$,

where W = external applied tension

σ = direct stress acting over the cross-sectional area a of the bar, assumed uniform.

Hence $\sigma = W/a$.

A member which prevents two bodies from moving or deforming towards each other is subjected to uniaxial compression and is termed a *column* or *strut*, Fig. 1.6(*b*). For equilibrium of internal and external forces we have once more $-W = \sigma a$ where the symbols are as above, and thus the compressive stress is $\sigma = -W/a$.

One point which is of some importance relates to the stress conditions in the immediate vicinity to the point of application of a force, where the relationships for equilibrium and deformation which are derived for the main bulk of a body will not be true. Calculation of the local disturbance of stress distribution near an applied force is beyond the scope of this book. However, it has been demonstrated by St. Venant that local effects die away quite rapidly and that the main stress distribution at a relatively short distance away from the external force is unaffected.

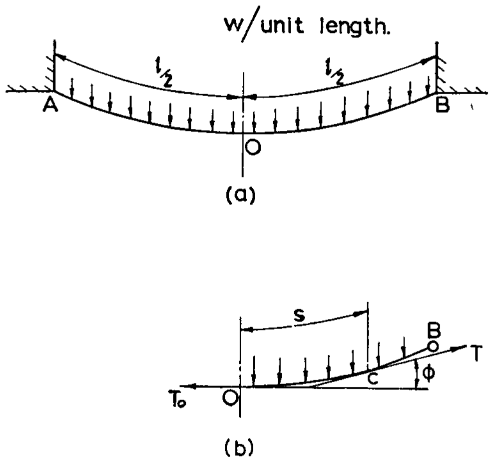


FIG. 1.7

1.6.2. Cables and Wires

In the previous example the equilibrium relationship between internal force and external load was readily apparent. However, in the case of freely suspended cables and wires, more than one equilibrium equation is required to solve for the unknown force, and in addition this is not constant along the length of the wire as in the problem above.

Referring to Fig. 1.7(*a*), a wire is freely suspended between two fixed points *AB* and is subjected to uniform loading, *w* per unit length of the wire, which may be simply the weight of the wire itself or some additional external load such as snow. The action of the loading on the wire is such as to cause tension to be set up as the internal reaction. The wire cannot sustain any bending, so that the axis of the wire must at all points coincide with the direction of the resulting tension. Referring to the free-body diagram, Fig. 1.7(*b*), the horizontal tension at the lowest point O is T_o and the tension at a point C a distance s *along* the wire from O is T.

Horizontal equilibrium:

$$T\cos\phi = T_o. \tag{1.1}$$

Vertical equilibrium:

$$T\sin\phi = ws. \tag{1.2}$$

Eliminating the slope of the wire, ϕ, at C gives

$$\frac{T_o^2}{T^2} + \frac{w^2s^2}{T^2} = 1$$

or

$$T^2 = T_o^2 + w^2s^2 \tag{1.3}$$

which is the complete equilibrium equation for the wire.

Eliminating T from equations (1.1) and (1.2), an equation for the shape of the wire is obtained,

$$s = (T_o/w)\tan\phi \tag{1.4}$$

which describes the *common catenary.*

It is apparent from equation (1.3) that the maximum tension occurs at the ends A and B and if l is the length of the wire then $T = T_{max}$ at $s = l/2$ and $T_{max} = [T_o^2 + (w^2l^2/4)]^{1/2}$. Also if a is the cross-sectional area of the wire then

$$\sigma_{max} = \frac{T_{max}}{a} = \frac{1}{a}[T_o^2 + (w^2l^2/4)]^{1/2}. \tag{1.5}$$

If T_o, which is also equal to the horizontal component of the tension at the ends of the wire, is not known, but the slope ϕ_A

or ϕ_B is defined then T_{max} can be found from equation (1.2). Thus

$$T_{max} = \frac{wl}{2 \sin \phi_A}.$$

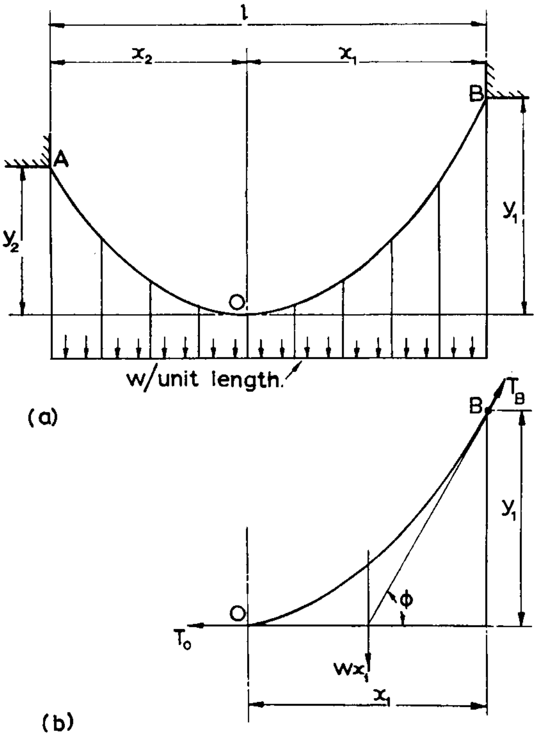


FIG. 1.8

Another common form of loading on a cable, for example in suspension bridges, is shown in Fig. 1.8(*a*). In contrast to the catenary, here the loading *w* per unit length is distributed uniformly on a *horizontal* base, the weight of the cable being neglected. In this particular example, the ends *A* and *B* are set at different heights above the lowest point. It is useful for the analysis to cut the cable at O and insert a reaction at that point and consider the equilibrium of the right-hand part of the cable. The free-body diagram in Fig. 1.8(b) shows that equilibrium is satisfied by the triangle of forces T_B, T_o and wx_1. The position of the lowest point O and hence the distance x_1 is not known. The distance x_1 can be determined by equilibrium of moments of the forces for either part of the cable. Thus taking moments about B we have

$$T_o y_1 - wx_1 \times \frac{x_1}{2} = 0, \tag{1.6}$$

or for the left-hand part moments about A gives

$$-T_o y_2 + w(l - x_1)\frac{(l - x_1)}{2} = 0. \tag{1.7}$$

Inspection of equation (1.6) shows that for any point on the cable having coordinates (x, y) relative to O,

$$y = \frac{wx^2}{2T_o} \tag{1.8}$$

which is an equation for a parabola.

Returning to equations (1.6) and (1.7) and eliminating T_o gives

$$\frac{y_1}{y_2} = \frac{x_1^2}{(l - x_1)^2}$$

from which

$$x_1 = \frac{l(y_1/y_2)^{1/2}}{1 + (y_1/y_2)^{1/2}}. \tag{1.9}$$

If the ends A and B are at the same level then $y_2 = y_1$ and $x_1 = l/2$.

The maximum tension in the cable will naturally occur at end B since this side of the cable supports the greater proportion of the load. The force T_B can be determined from equilibrium of the triangle of forces T_o, wx and T_B.

Vertical equilibrium:

$$T_B \sin\phi - wx_1 = 0. \tag{1.10}$$

But

$$\sin\phi = y_1/\{y_1^2 + (x_1/2)^2\}^{1/2}$$

therefore

$$T_B = wx_1\{y_1^2 + (x_1/2)^2\}^{1/2}/y_1. \tag{1.11}$$

Similarly

$$T_A = w(l - x_1)\left\{y_2^2 + \left(\frac{l - x_1}{2}\right)^2\right\}^{1/2}\bigg/ y_2, \tag{1.12}$$

where

$$x_1 = l(y_1/y_2)^{1/2}/\{1 + (y_1/y_2)^{1/2}\}.$$

The minimum tension in the cable is T_o which is obtained from equation (1.6) and

$$T_o = \frac{wx_1^2}{2y_1}. \tag{1.13}$$

Any required stress values are determined by dividing the appropriate tension by the cross-sectional area a of the cable, since it is assumed that the stress is uniformly distributed across the section and thus $\sigma = T/a$.

1.6.3. Thin Shells Subjected to Pressure

Vessels or containers in the form of cylinders and spheres are widely used in engineering. They may be classified in two ways—thick walled or thin walled. The dividing line will be taken arbitrarily as a wall thickness of one-tenth of the overall cross-sectional dimension. The reasons for differentiating between thick- and thin-walled vessels are based on the stress conditions set up in the material when subjected to internal pressure (or external which is very rare). For the case of a close-ended cylinder subjected to internal pressure there will be a tensile stress parallel to the axis of the cylinder, a tensile stress circumferentially in the material and compressive stress through the thickness of the wall. In a thick-walled vessel the latter two stresses vary through the thickness of the shell and the problem is not statically determinate. However, for wall thicknesses which are small in comparison to the diameter, the variation in stresses through the thickness becomes very small and the compressive radial stress, which is equal to the internal pressure at the inner surface, is negligible in comparison to the axial and circumferential tensile stresses. The problem now becomes statically determinate. Since there is no bending or torsion, but only membrane stresses in a thin shell under pressure, the problem is analogous to the freely suspended cable.

An exact solution to a case of stress analysis is often very difficult or impossible and frequently gives only a marginally different result from a solution employing sensible approximations and simplifications to the problem. For the above reason it is often necessary and desirable to make simplifying assumptions at the start of a solution.

(1) *Cylinder with closed ends under internal pressure*

Assumptions

(a) The diameter to wall thickness ratio is large, and thus (b) there is a uniform distribution of stress in the material both in the axial and circumferential directions. (c) The radial stress is sufficiently small to be neglected, (d) that the solution to follow will only apply to cross sections at least one diameter's length away from the closed ends where there is non-uniform stress owing to local end restraint (see section 1.6.1).

Axial Equilibrium. The force acting on each closed end of the cylinder owing to the internal pressure P, Fig. 1.9(b), is obtained from the product of the pressure and the area on which it acts.

$$\text{Thus axial force} = P \times \pi r^2.$$

The part of the vessel shown in the free-body diagram in Fig. 1.9(b) is in axial equilibrium simply under the action of the applied force above and the axial stress, σ_x, in the material, the radial pressure shown having no axial resultant force. The cross-sectional area of material is approximately $2\pi rt$ and therefore the internal force $= \sigma_x \times 2\pi rt$, and for equilibrium

$$2\pi rt\sigma_x = \pi r^2 P$$
$$\text{or } \sigma_x = Pr/2t \tag{1.14}$$

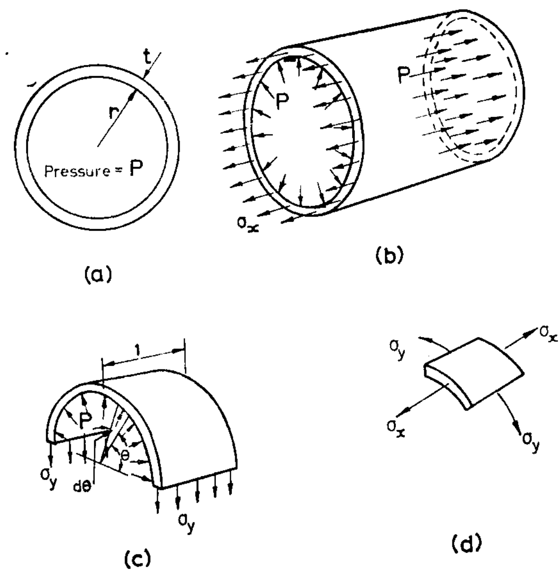


FIG. 1.9

Circumferential equilibrium. Considering the equilibrium of part of the vessel, if the cylinder is cut across a diameter as in the free-body diagram in Fig. 1.9(*c*) the internal pressure acting outwards must be in equilibrium with the circumferential stress, σ_y, as shown. Consider a unit length of cylinder, and a small length of shell subtending an angle $d\theta$ as shown in Fig. 1.9(*c*). The *radial* component of force on the element is $P \times 1 \times r d\theta$, hence the vertical component is $P \times 1 \times r d\theta \sin\theta$. Therefore the total vertical force due to pressure is

$$\int_0^{\pi} Pr \sin\theta \, d\theta = 2Pr.$$

It is useful to note that the vertical force is also given by considering the pressure acting on the *projected area* at the diameter. This fact also shows that the axial force is independent of the *shape* of the end closures.

The internal force required for equilibrium is obtained from the stress σ_y acting on the two ends of the strip of shell. Hence internal force $= \sigma_y \times 2 \times t \times 1$. For equilibrium $2t\sigma_y = 2rP$ or

$$\sigma_y = \frac{Pr}{t}. \tag{1.15}$$

Comparing equations (1.14) and (1.15) it is seen that the circumferential stress is twice the value of the axial stress. Figure 1.9(*d*) shows a small element of the shell subjected to the axial and circumferential (hoop) stresses.

(2) *Sphere under internal pressure*

The thin-walled sphere is a problem in which any diametral plane is a plane of symmetry and the internal pressure gives rise only to tension acting equally in all circumferential directions. The stress is obtained in terms of the pressure by considering equilibrium of part of the vessel, which for simplicity of analysis will be taken as a hemisphere with a horizontal diametral plane.

Vertical equilibrium. The vertical force owing to pressure is obtained by using the projected area at the diametral plane as shown previously and is thus

$$\text{Applied force} = P \times \pi r^2.$$

The reacting internal force from the hoop stress in the material is

$$\text{Internal force} = \sigma \times 2\pi r t.$$

For equilibrium of the part of the sphere shown in Fig.1.10(a)

$$2\pi rt\sigma = \pi r^2 P$$

or

$$\sigma = \frac{Pr}{2t} \tag{1.16}$$

this gives the tensile circumferential stress in any direction.

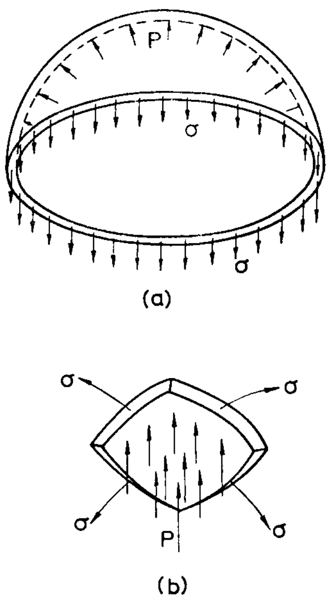


FIG. 1.10

The assumptions and analysis are similar to that for the first part of the cylinder.

It is interesting to note that the stress in a sphere is the same as the longitudinal stress in a cylinder having the same diameter and wall thickness—the equilibrium equation being essentially the same.

The stress system on an element in the sphere is shown in Fig. 1.10(b).

1.6.4. Thin Ring or Cylinder Rotating

Another interesting and slightly different type of problem involves forces set up in a thin ring or cylinder by centripetal action during rotation. If the angular velocity is constant, the force remains constant and the problem may be considered for statical equilibrium.

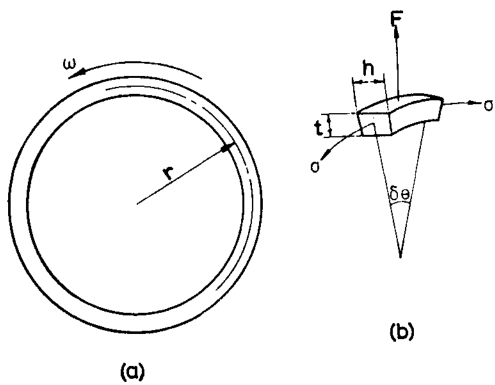


FIG. 1.11

The statical determinacy of the problem is a separate issue and in the case of the ring or cylinder for uniform stress the dimensions of the cross section must be small compared with the overall diameter. In addition the ring must not be restrained radially by, say, spokes.

The rotating ring shown in Fig. 1.11 may also be regarded as a slice through a cylinder. A small element of arc of the ring of cross-sectional area $A(=th)$, rotating at uniform angular velocity, ω, is shown in Fig. 1.11(b). The forces acting on the element are the radial inertia force, F, and the circumferential tensile stress,

σ, over the area A. It is assumed that radial and axial stresses are zero or negligible.

The resolved component of the two tensile forces, σA, radially inwards is $2\sigma A \sin(\delta\theta/2)$ which for small angle $\delta\theta$ may be written as $\sigma A \delta\theta$.

If the weight per unit volume of the material is w then the mass, m, of the element is $wAr\delta\theta/g$ where r is the radius of the centre line of the ring. The radial centrifugal force is given by

$$F = m\omega^2 r = (wAr\delta\theta/g)\,\omega^2 r$$

From D'Alembert's principle the radial inertia force will be in " equilibrium " with the resolved component of the internal force in the ring so that

$$\sigma A \delta\theta = \frac{wAr\delta\theta}{g}\,\omega^2 r$$

or

$$\sigma = \frac{w}{g}\,\omega^2 r^2. \tag{1.17}$$

It is interesting to note that in equation (1.17) the tensile stress is independent of shape and area of the cross section of the ring.

1.6.5. Friction on a Brakeband

A rotating shaft or wheel can be slowed down or stopped by the effect of friction between the surfaces of a brakeband and the rotating object. The frictional force is proportional to the reaction normal between the two surfaces and both vary along the arc of contact. The reaction can only be obtained by the application of tension at each end of the brakeband. The rotating body and the band are illustrated in Fig. 1.12(*a*) and the free-body diagram for a small segment of the band is shown in Fig. 1.12(*b*). The tension cannot be constant along the element owing to the friction force μR, where μ is the coefficient of friction and R is the reaction normal per unit length so there is a variation from T

at one end to $T + \delta T$ at the other. In this problem there are unknown forces, T and R, both functions of θ, and therefore two equations of equilibrium are needed for a complete solution.

Radial Equilibrium

$$T \sin \frac{\delta\theta}{2} + (T + \delta T) \sin \frac{\delta\theta}{2} = R\delta s \qquad (1.18)$$

Circumferential Equilibrium

$$(T + \delta T) \cos \frac{\delta\theta}{2} - T \cos \frac{\delta\theta}{2} = \mu R\delta s \qquad (1.19)$$

where δs is the length of the element.

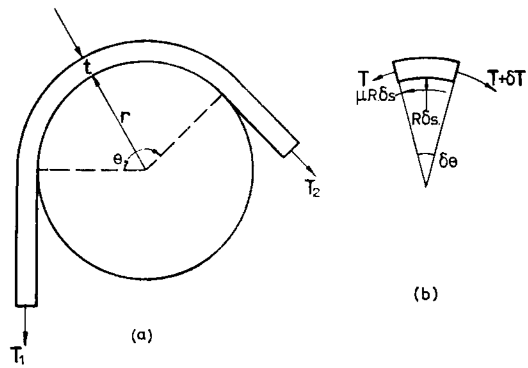


FIG. 1.12

For a small angle $\delta\theta$ the approximations are made that $\cos(\delta\theta/2) \to 1$ and $\sin(\delta\theta/2) \to (\delta\theta/2)$. Simplifying the above equations, and neglecting the product of small quantities, gives

$$T\delta\theta = R\delta s \text{ and } \delta T = \mu R\delta s:$$

eliminating the reaction R

$$\frac{\delta T}{T} = \mu\delta\theta.$$

In the limit as $\delta T/\delta\theta \to 0$,

$$\frac{dT}{T} = \mu d\theta:$$

integrating both sides

$$\int_{T_1}^{T_2} \frac{dT}{T} = \int_0^\theta \mu d\theta$$

or

$$\log_e(T_2/T_1) = \mu\theta,$$

hence

$$T_2 = T_1 e^{\mu\theta}. \tag{1.20}$$

The tensile stress in the band where it makes contact with the wheel is not uniform through the thickness, t, since the friction force is applied on one side of the band only. However, for practical purposes it would be sufficient to determine the *average* stress at the section carrying the maximum tensile force, i.e. maximum average stress

$$\sigma = \frac{T_2}{a} = \frac{T_1 e^{\mu\theta}}{a}, \tag{1.21}$$

where a is the cross-sectional area of the band.

1.6.6. Pure Torsion of a Thin-walled Circular Tube

The various examples of statically determinate stress systems studied so far, although involving different forms of external loading, have each resulted in direct stress. The present problem not only presents a further type of loading, that of pure torque, but also involves a different type of stress, namely, shear stress.

As in the case of thick- and thin-walled pressure vessels, the problem has to be restricted to a thin-walled tube in torsion if the solution is to be statically determinate. It will be demonstrated in Chapter 6 that for thick-walled and solid circular sections the distribution of stress is not uniform over the cross section. However, if the wall is thin enough the average stress can be considered since the variation is slight.

Referring to Fig. 1.13, a tube of mean radius r and wall thickness t is subjected to opposing torques at each end. The deformation action is that of twisting, and considering the equi-

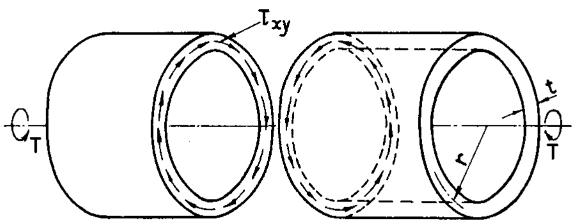


FIG. 1.13

librium of part of the tube by cutting it at some section away from the end effects and perpendicular to the axis, an internal reacting torque will be required for equilibrium and to maintain the twist. This reaction torque will take the form of a " uniform " shearing stress acting on the cut face as shown. The shear stress cannot vary around the tube or else this would imply a variation in the complementary longitudinal shear stress and a resultant axial force.

Internal torque $= \tau \times 2\pi rt \times r$.

External torque $= T$.

Equilibrium condition $2\pi r^2 r\tau = T$.

Hence

$$\tau = \frac{T}{2\pi r^2 t}. \tag{1.22}$$

1.7. Principle of Superposition: Forces and Stresses in a Statically Determinate System

The problems examined in previous sections have been extremely simple both in geometry and form of loading. It is often possible to break down a more complex loading case into several simple component parts. If this can be done, because stress is proportional to external loading, the solution for stresses (of the same type) for each of the components of loading can be simply added together to obtain the same result as solving for the complex loading in one operation. This is termed the principle of superposition and will be illustrated in several places in succeeding chapters.

EXAMPLE 1.1

A cylinder of internal diameter 6 in. and wall thickness $\frac{1}{8}$ in. is subjected to an internal pressure of 200 lb/in^2 and an axial tension of 1000 lb. Determine the axial stress by the method of superposition.

The axial stress due to internal pressure only is obtained from equation (1.14)

$$\sigma_x^P = \frac{Pr}{2t} = \frac{200 \times 3}{2 \times \frac{1}{8}} = 2400 \text{ lb/in}^2.$$

Axial stress due to axial tension only: for equilibrium

$$\sigma_x^T \times \pi \times 6 \times \tfrac{1}{8} = 1000$$

$$\sigma_x^T = 424 \text{ lb/in}^2.$$

Total axial stress $= \sigma_x^P + \sigma_x^T = 2400 + 424 =$ **2824 lb/in^2.**

The reader may verify this result by considering the total axial load directly.

Examples

1. A double-acting hydraulic cylinder has a piston 10 in. in diameter and a piston rod 3 in. in diameter. The water pressure is 1000 lb/in^2 on one side of the piston and 40 lb/in^2 on the other and on the return stroke the pressures are interchanged. Determine the maximum stress in the rod, allowing for the reduced piston area on the rod side.

2. A tie-bar $1\frac{1}{2}$ in. in diameter is subjected to tensile stress of 8000 lb/in^2. The bar is fastened to a cast-iron bracket which is held by four bolts. If the bolts are subjected to direct tension, determine the diameter at the root of the threads for a limiting stress of 5000 lb/in^2.

3. The maximum permissible tensile stress in the cables of a suspension bridge is 4 ton/in^2. The bridge is constructed of twin cables, and its weight, including the weight of its deck and including the maximum permissible distributed load, is 125 lb/ft length of the span. If the span is 90 ft and the height of the pier supporting one end is 15 ft greater than the height of the pier supporting the other, and the sag of the bridge below the lower pier is 5 ft, find the minimum possible cross-sectional area of each cable.

4. A cable of 160 ft span with each end at the same level carries a concentrated load of 600 lb at a distance of 120 ft from left support. The sag at the concentrated load is 12 ft. The weight of the cable may be taken as 4 lb/ft of horizontal span. Estimate the sag at mid span. (*Hint*: Determine distance x from left end of lowest point, which is not at the concentrated load, then treat as cable of span $2x$ subjected to distributed load only.)

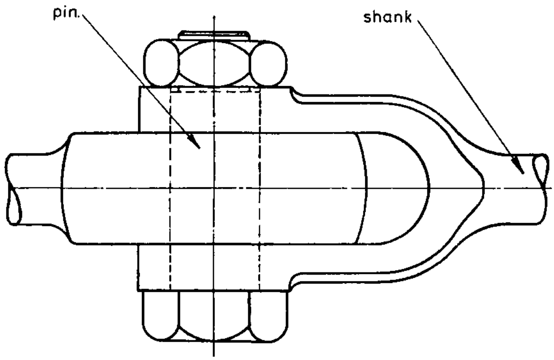


FIG. 1.14

5. A thin spherical steel vessel is made up of two hemispherical portions bolted together at flanges. The mean diameter of the sphere is 12 in. and wall thickness $\frac{1}{4}$ in. Assuming this vessel were a homogeneous sphere, what is the maximum working pressure for an allowable tensile stress in the shell of 10 ton/in^2?

If twenty bolts of $\frac{11}{16}$ in. diameter are used to hold the flanges together, what is the tensile stress in the bolts when the sphere is under full pressure?

6. A thin-walled steel cylinder of internal diameter 18 in. with closed ends is rotated about a longitudinal axis at a speed of 5000 rev/min. Whilst it is rotating it is subjected to an internal pressure of 600 lb/in^2. If the maximum allowable tensile stress in the material is 25,000 lb/in^2, calculate a suitable shell thickness. The method of superposition may be used. Steel weighs 0·28 lb/in^3; $g = 32{\cdot}2$ ft/sec^2.

7. A forked or pin joint as shown in Fig. 1.14 is required to transmit a load of 5000 lb. Determine the minimum permissible diameter of the pin and shank if the maximum permissible shear and tensile stresses in the material are 4 ton/in^2 and 7 ton/in^2 respectively.

CHAPTER 2

CONCEPTS OF DEFORMATION AND STRAIN

2.1. Introduction

In the previous chapter internal forces and stresses were found in several problems simply by the use of the condition of equilibrium of internal and external forces. In many cases this is not possible and internal forces and stresses can only be found by considering the geometry of deformation as well as statical

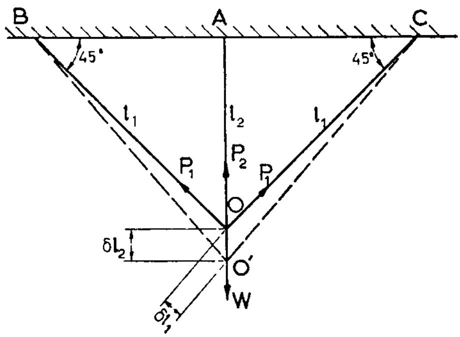


FIG. 2.1

equilibrium. For example, a problem is illustrated in Fig. 2.1 in which three pin-jointed members of the same material and cross-sectional area support a load W. It is desired to find the forces and hence the stresses in each member. The forces in OB and OC will be of the same magnitude, P_1, since the system is symmetrical, and if the force in OA is P_2, then for vertical equilibrium

$$2P_1 \cos 45° + P_2 = W.$$

It is seen that there are two unknowns but only one equation is obtainable, and thus although equilibrium is a *necessary* condition it is not a *sufficient* condition even to find the internal forces. Returning to the physical picture in Fig. 2.1 it is clear that each of the bars will stretch a small amount under the action of the load and the joint O will move to O' where OO' is δl_2. For the three members to remain joined at O', a condition of *geometrical compatibility*, there will be a simple relation for the *geometry of deformation* between $O'B$, $O'C$ and $O'A$, as shown by the dashed lines. Thus

$$OB = OC = l_1 = \sqrt{2} \times l_2$$

and the extension of OB and OC

$$\delta l_1 = \delta l_2 \cos 45° = \delta l_2/\sqrt{2}.$$

This step appears to have complicated the solution rather than assisted it, since there is now a further unknown quantity δl_2. However, if a *relationship between force and deformation* was known or could be found for the material from which the bars were made, there would then be sufficient equations for a complete solution of the unknown quantities. The problem just discussed is termed *statically indeterminate*.

The solution depends upon the following three conditions:

(1) Equilibrium of forces
(2) Geometry of deformation
(3) Force–deformation relationships.

This chapter is concerned with a study of deformation and strain and the application of the above three steps in the solution of several statically indeterminate systems. It also deals with the deformations in statically determinate problems, viz. Chapter 1, for the reasons given below.

2.2. Deformation

Deformations may occur in a material for a number of reasons, such as external applied loads, change in temperature, tightening of bolts, irradiation effects, etc. Bending, twisting, compression, tension and shear or combinations of these are common modes of deformation. In some materials, viz. rubber, plastics, wood, the deformations are quite large for relatively small loads, and readily observable by eye. In metals, however, the same loads would produce very small deformations which require the use of sensitive instruments for measurement.

Stress values do not always provide the limiting factor in design, for although a component may be safe and employ material economically with regard to stress, the deformations accompanying that stress might be dangerous or inconvenient. For example, too high a deflection of an aeroplane wing can result, among other things, in a detrimental change in aerodynamic characteristics. A lathe bed which was not sufficiently rigid would not permit of the required tolerances in machining. A perfectly safe " sag " in a dance hall floor might upset the balance of the dancers.

In this and succeeding chapters there will be many problems in which the analysis of displacements will be considered specifically in addition to the determination of stress magnitude.

2.3. Strain

As explained in Chapter 1 the effect of a force applied to bodies of different size can be compared in terms of stress, i.e. the force per unit area. Likewise the deformation of different bodies subjected to a particular load is a function of size and therefore comparisons are made by expressing deformation as a non-dimensional quantity given by the change in dimension per unit of original dimension, or in the case of shear as a change in angle between two initially perpendicular planes. The non-dimensional expression of deformation is termed *strain*.

Direct or Normal Strain

Consider the bar shown in Fig. 2.2 subjected to axial tensile loading if the resulting extension of the bar is δl and its original length is l_0, then the *direct tensile strain* is

$$\varepsilon = \frac{\delta l}{l_0}$$

If two bars identical in material, length l_0, and area were each subjected to a tensile force P then the extension in each would be the same, say δl, and the strain $\delta l/l_0$. The bars are now joined end on end and the same tensile force, P, applied. The overall extension of the combined bar will be $2\delta l$ but since the original length is now $2l_0$ the *strain* is $2\delta l/2l_0$ or $\delta l/l_0$, i.e. the same as for the separate bars. Hence strain rather than elongation is an appropriate measure of deformation.

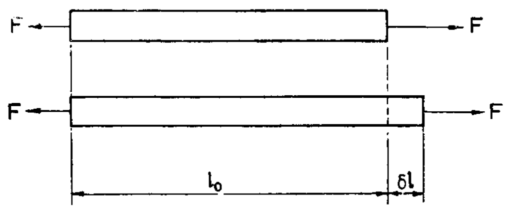


FIG. 2.2

Similarly if the bar had been compressed by forces an amount δl then the *compressive strain* would be

$$\varepsilon = -\delta l/l_0.$$

Strain is positive for increase in dimension and negative for a reduction in dimension.

The same suffix notation is used for strains as for stresses. ε_x is the strain of a line measured in the x direction and ε_y the strain of a line in the y direction.

Shear Strain

An element which is subjected to shear stress experiences deformation as shown in Fig. 2.3. The tangent of the angle through which two adjacent sides rotate relative to their initial position is termed *shear strain*. In many cases the angle is very small and the angle itself is used, expressed in *radians*, instead of the tangent, so that,

$$\gamma = A\hat{O}B - A'\hat{O}B' = \phi$$

when $A'OB' < AOB$ then γ is defined as a positive shear strain and $A'OB' > AOB$ is termed a negative shear strain.

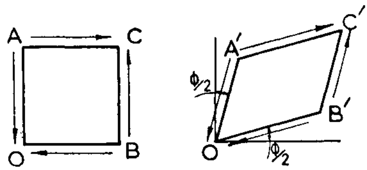


FIG. 2.3

Volumetric Strain

The term hydrostatic stress was used in Chapter 1 to describe a state of tensile or compressive stress equal in all directions within or external to a body. Hydrostatic stress causes a change in volume of the material which, if expressed per unit of original volume, gives a volumetric strain denoted by ε_v.

2.4. Elastic Load–Deformation Behaviour of Materials

Studies of material behaviour made by Robert Hooke in 1678 showed that up to a certain limit the extension δl of a bar subjected to an axial tensile loading P was often directly proportional to P, as in Fig. 2.4(*a*). This behaviour in which $\delta l \propto P$ is known as *Hooke's Law*. It is similarly found that for many materials uniaxial compressive load and compressive deformation are proportional up to a certain limit of load. A cylindrical bar which is twisted about its axis by opposing torques applied at each end is also found to have a linear torque–twist relationship up to a certain point. The maximum load up to which Hooke's law is applied is termed the *limit of proportionality*. If in each of the cases above at any particular load the same deformation existed both when increasing and decreasing load on the material and if after completely unloading the body it returned to exactly its original size before application of load, then it is said to have exhibited the property of *elasticity*. This behaviour exists only over a certain range of load and deformation, the end point being termed the *elastic limit*. The limit of proportionality is generally a shade lower than the elastic limit. If the elastic limit is exceeded then deformation is no longer proportional to load and it is found that there is some permanent deformation left after removal of the load, as illustrated in Fig. 2.4(*b*).

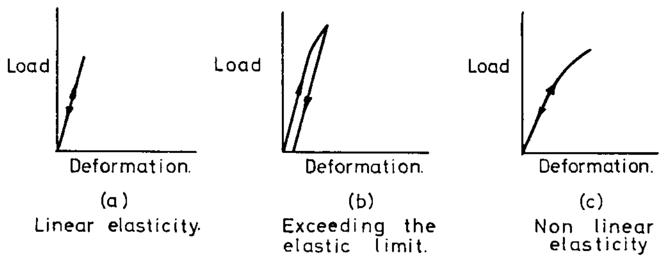


FIG. 2.4

Metals generally obey a linear load–deformation law up to their elastic limits which is termed *linear elasticity*: however, there are some materials, principally non-metallic, which have an elastic range as defined above, but exhibit a non-linear load–deformation relationship, Fig. 2.4(*c*).

2.5. Elastic Stress–Strain Behaviour of Materials

If the load in Fig. 2.4(*a*) is divided by the cross-sectional area of the bar, A, and the extensions on the abscissa are divided by the original length of the bar, l_0, a graph of stress against strain is obtained. Since A and l_0 are constants the stress–strain behaviour is also linear in the elastic range.

The slope of the line is constant and may be expressed as

$$\left(\frac{W}{A}\bigg/\frac{\delta l}{l_0}\right) = \frac{\sigma}{\varepsilon} = E$$

where E is a constant for the material, and is called *Young's Modulus of Elasticity*. Since ε is non-dimensional, E has dimensions of stress, i.e. force per unit area. Some typical *approximate* values of E are given for a few materials in Table 2.1.

A relationship between shear stress and shear strain may be derived from a torsion test on a cylindrical bar in which applied torque and angular twist are measured. The connections between torque and shear stress and twist and shear strain will be derived in Chapter 6. It is sufficient to state here that shear stress is proportional to shear strain within the elastic limit. Hence τ/γ = constant = G.

TABLE 2.1

Material	E lb/in²	G lb/in²
Steels	$28\text{–}30 \times 10^6$	$11{\cdot}5\text{–}12 \times 10^6$
Copper	$17\text{–}18 \times 10^6$	$5{\cdot}5\text{–}6{\cdot}5 \times 10^6$
Aluminium	$10\text{–}10{\cdot}5 \times 10^6$	$3{\cdot}6\text{–}3{\cdot}9 \times 10^6$
Glass	$7\text{–}11 \times 10^6$	$3{\cdot}8\text{–}4{\cdot}7 \times 10^6$
Plastic (Epoxy)	$0{\cdot}4 \times 10^6$	—

The constant of proportionality, G, is known as the *Modulus of Rigidity* or *Shear Modulus* and has dimensions of force per unit area. Approximate values of G for various materials are given in Table 2.1.

It can also be demonstrated experimentally that volumetric strain is proportional to hydrostatic stress within the elastic range. The constant relating those two quantities is termed the *Bulk Modulus* and is denoted by the symbol, K. Thus

$$\frac{\sigma}{\varepsilon_v} = K.$$

It will be shown in Chapter 3 that the elastic constants E, G and K are related to one another.

2.6. Plastic Stress–Strain Behaviour of Materials

Although this book is concerned with the theoretical solution of problems in elasticity, it is important to understand what effect there would be on a material or component if for some reason stress and strain went beyond the elastic range.

Let us again consider the most simple situation, that of a bar of, say, aluminium subjected to uniaxial tensile or compressive loading. Firstly then in Fig. 2.5(*a*) there is the elastic range which terminates at point *A*, the *elastic limit.* Just *below* at *B*

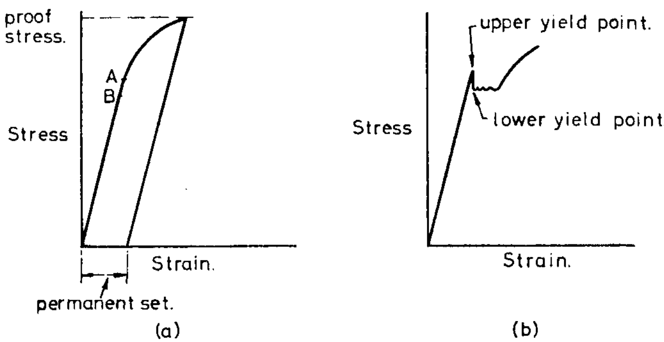


FIG. 2.5

but almost coinciding with *A* is what is termed the *proportional limit* which is the end of linear stress–strain behaviour, i.e. Hooke's law. Beyond the point, *A*, *yielding* commences and strain increases more rapidly than stress. If the load is removed, stress and strain reduce along a line which is nominally parallel to the initial elastic line and on complete unloading there is a permanent deformation remaining which is often called a *permanent set.* Because it is difficult to determine the proportional or elastic limits accurately, it is usual to determine a stress, either for calculation or specification, based on a very small permanent set of the *gauge length.* This is a parallel length of the bar from which measurements of extension are made by extensometer or some other arrangement. A permanent strain of 0·001 or 0·002 is generally adopted, i.e. one or two thousandths of an inch on an original gauge length of one inch, and they are usually spoken of as percentage strains, 0·1 or 0·2 per cent. Having obtained a stress–strain curve, a line is drawn from a strain on the abscissa of 0·001 or 0·002, parallel to the initial elastic line, and where this line intersects with the stress–strain curve gives the required value of the yield stress. This is normally termed a *proof stress* and the permanent set is attached to it thus: 0·1 per cent proof stress or 0·2 per cent proof stress.

Some steels exhibit a very sharply defined change from the elastic to the plastic range as shown in Fig. 2.5(*b*) which is termed the *yield point*. This may be defined as the first point at which there is an increase in strain without an increase in stress. Low carbon steels show an *upper yield point* at the end of the elastic range from which there is a sudden drop in stress to a *lower yield point*. The former is a rather insecure point which is influenced by a number of factors during testing.

After the initial yielding region continuing plastic strain or flow is only achieved by increasing the stress and when $d\sigma/d\varepsilon$ is again positive *strain hardening* is said to occur.

So far stress has been obtained by dividing the load applied to the material at any instant during stretching by the *original* cross-sectional area and this gives what is termed a *nominal stress* curve. In a simple tension test, area decreases with extension and hence the *actual* stress defined as the load divided by the actual area would be different as the test proceeded. If load is divided by actual area then a *true stress* curve is obtained. However, these two expressions of stress differ only slightly up to plastic strains of about 0·1 and for convenience in testing it is usual to quote nominal stress. Nominal and true stress–strain curves for uni-axial tension and compression tests are shown in Fig. 2.6(*a*). The point of maximum stress on the nominal tensile stress–strain curve is termed the *tensile strength* (previously the ultimate tensile strength or U.T.S.). Up to this point strain has been quite uniformly distributed along the length of the bar, but then at one cross section which is weaker than the rest, the rate of strengthening by work hardening becomes less than the increase in stress due to reduction in area. A more rapid reduction in cross section

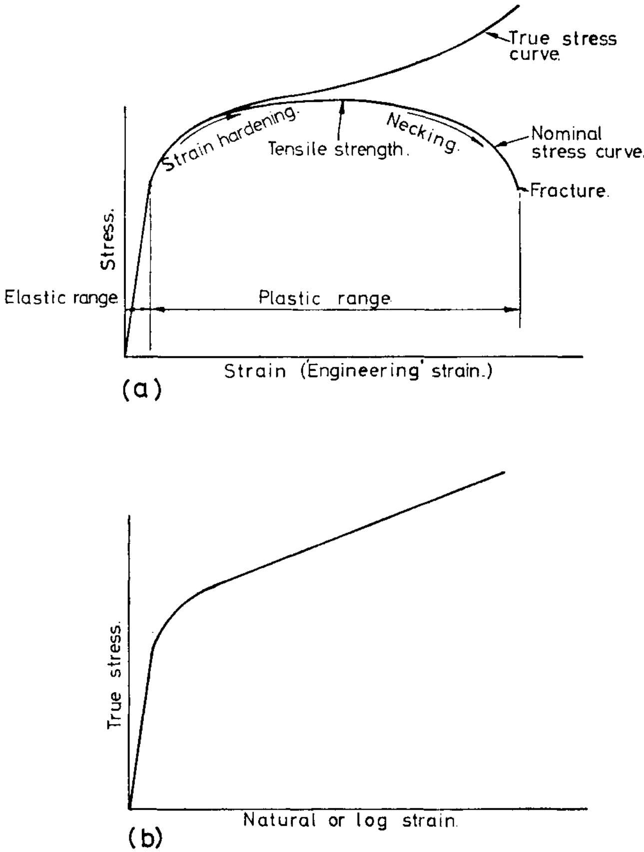


FIG. 2.6

occurs and a neck is formed. Once this happens, *less* load is required to keep straining the bar and the load–extension curve, and hence the nominal stress–strain curve, falls until fracture occurs in the neck. If stresses are determined on the actual area in the neck the true stress curve continues to rise after the tensile strength point.

In the same way that true stress was defined in terms of the current and not the original area, strain can also be defined in terms of the sum of a number of increments of strain in relation to the current length of the bar. This is termed true or natural strain, denoted by $\bar{\varepsilon}$.

$$\bar{\varepsilon} = \sum \frac{\delta l}{l},$$

where δl is the increment of strain on the current length l. The total strain between l_0 and l_f is

$$\bar{\varepsilon} = \int_{l_0}^{l_f} \frac{\mathrm{d}l}{l} = \log_e \left(\frac{l_f}{l_0}\right).$$

What has been used previously is called simply strain or often engineering strain. Hence engineering strain

$$\varepsilon = \delta l / l_0.$$

A true stress–true strain curve which is virtually the same whether obtained by a tension test or a compression test is illustrated in Fig. 2.6(*b*). A true stress–true strain curve is important for a fundamental understanding of material behaviour, but for normal engineering and design purposes a nominal stress–engineering strain curve is all that is required.

An important property is that of ductility, which is the ability of a material to undergo plastic deformation. Ductility is expressed principally by two quantities measured after a tensile test to fracture. The percentage elongation of the original gauge length, which is

$$\text{Percentage elongation} = \frac{l_f - l_0}{l_0} \times 100 \text{ per cent},$$

and the reduction in cross-sectional area at the neck, which is

$$\text{Percentage reduction in area} = \frac{A_0 - A_f}{A_0} \times 100 \text{ per cent}.$$

2.7. Solution of Some Statically Indeterminate Problems

(a) Frame

Now that a uniaxial stress–strain relationship has been established, the problem discussed in the introduction to this chapter can be completed.

Equilibrium

This condition was established as

$$\sqrt{2} \times P_1 + P_2 = W \tag{2.1}$$

Geometry of Deformation

For compatibility $$\delta l_1 = (1/\sqrt{2})\, \delta l_2 \tag{2.2}$$

It should also be noted that elastic deformations, particularly for metals, are extremely small and thus the overall geometry of the system is effectively unaltered by the deformations.

Stress–Strain Relationships

Let the area of each member be A and the Young's modulus of the material be E. Then for OB and OC

$$\frac{(P_1/A)}{(\delta l_1/l_1)} = E, \tag{2.3}$$

and for OA

$$\frac{(P_2/A)}{\delta l_2/l_2} = E. \tag{2.4}$$

Solution

Substituting for P_1 and P_2 in equation (2.1),

$$\frac{\sqrt{2}\delta l_1}{l_1} AE + \frac{\delta l_2}{l_2} AE = W.$$

Substituting for δl_1 and l_1 in terms of δl_2 and l_2 from equation (2.2)

$$\frac{\delta l_2}{l_2} AE = \frac{\sqrt{2}\,.\,W}{1 + \sqrt{2}},$$

so that from equation (2.4)

$$P_2 = \frac{\sqrt{2}\,.\,W}{1 + \sqrt{2}} \tag{2.5}$$

and from equation (2.1)

$$P_1 = \frac{W}{\sqrt{2}(1 + \sqrt{2})}. \tag{2.6}$$

From the above it is seen that the force in OA is twice that in OB and OC.

EXAMPLE 2.1

The frame in Fig. 2.1 is made of mild steel bars each of cross-sectional area 1 in^2 and the length of OA is 10 in. If the frame carries a load of 2 ton, determine the stress and deformation of the members. $E = 30 \times 10^6$ lb/in^2.

From equations (2.5) and (2.6)

$$P_1 = \frac{2 \times 2240}{1{\cdot}414 \times 2{\cdot}414} = 1313 \text{ lb},$$

$$P_2 = \frac{1{\cdot}414 \times 2 \times 2240}{2{\cdot}414} = 2626 \text{ lb}.$$

Stress in OB and OC = **1313 lb/in^2**

Stress in OA = **2626 lb/in^2**.

$$\text{Extension of } OB \text{ and } OC = \frac{\sigma_1 l_1}{E} = \frac{1313 \times 1{\cdot}414 \times 10}{30 \times 10^6}$$

$$= \mathbf{0{\cdot}00062 \text{ in.}}$$

$$\text{Extension of } OA = \frac{\sigma_2 l_2}{E} = \frac{2626 \times 10}{30 \times 10^6}$$

$$= \mathbf{0{\cdot}000875 \text{ in.}}$$

The smallness of the above values confirm the statement made earlier that the overall geometry may be regarded as being unaltered by the deformations.

(b) Composite Members

Some components of structures incorporate two or more different materials which have different load–deformation characteristics. The following example illustrates the application of the three basic steps proposed in section 2.1 to this type of problem.

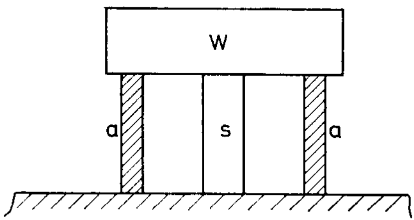


FIG. 2.7

A compression member consists of a solid cylindrical steel bar placed concentrically inside an aluminium tube as shown in Fig. 2.7. The bar and tube are of the same length, rest on a rigid base and support a load W. Subscripts a and s will be used to denote quantities relating to the aluminium tube and steel bar respectively. It is required to find the stresses and strains in the assembly.

Equilibrium

If the forces in the members are P_s and P_a then

$$P_s + P_a = -W. \tag{2.7}$$

Geometry of Deformation

The compression of the bar and tube must be the same, therefore

$$\delta l_a = \delta l_s. \tag{2.8}$$

Stress–Strain Relationship

$$\frac{\sigma_s}{\varepsilon_s} = E_s \tag{2.9}$$

and

$$\frac{\sigma_a}{\varepsilon_a} = E_a. \tag{2.10}$$

Solution

The cross-sectional areas of the bar and tube are A_s and A_a respectively and therefore from equations (2.9) and (2.10)

$$P_s = A_s\varepsilon_s E_s,$$

$$P_a = A_a\varepsilon_a E_a.$$

Substituting for P_s and P_a into the equilibrium equation (2.7)

$$A_s\varepsilon_s E_s + A_a\varepsilon_a E_a = -W \tag{2.11}$$

and since the bar and tube are the same initial length and have the same compression, the strain must be equal in each, therefore

$$\varepsilon_s = \varepsilon_a = \varepsilon$$

so that from equation (2.11)

$$\varepsilon = -\frac{W}{A_s E_s + A_a E_a}.$$

The stress in the steel bar is thus

$$\sigma_s = E_s\varepsilon = -\frac{WE_s}{A_s E_s + A_a E_a} \tag{2.12}$$

and in the aluminium tube is

$$\sigma_a = E_a\varepsilon = -\frac{WE_a}{A_s E_s + A_a E_a}. \tag{2.13}$$

(c) Thermal Strain

The effect of a change of temperature on a piece of material is a small change in size accompanied by alterations to the mechanical properties such as strength, ductility, modulus, etc. A temperature change of, say, 100°C is not very significant in relation to properties, but the change in size and hence strain although small can, in some circumstances, induce considerable stresses and it is this aspect which provides a further interesting statically indeterminate problem. The dependence of size on temperature variation is measured in terms of the basic quantity known as the *coefficient of linear thermal expansion* per unit temperature per unit length.

Consider a rod of length l_0 at temperature T_0 and a change in temperature to a new value T. Then the alteration in length is given by

$$\delta l = \alpha l_0 (T - T_0).$$

This may be expressed in terms of thermal strain as

$$\varepsilon_T = \frac{\delta l}{l_0} = \alpha(T - T_0). \tag{2.14}$$

It is seen that an increase in temperature will cause expansion and thus positive strain, while a decrease in temperature will result in contraction and negative strain. The interesting feature about this behaviour is that if there is no restraint on the material we can have strain unaccompanied by stress. However, if there is any restriction on free change in size then a stress will result termed *thermal stress*. The total strain in a body experiencing thermal stress may be divided into two components, the strain associated with the stress and the strain resulting from temperature change. Thus

$$\varepsilon = \varepsilon_\sigma + \varepsilon_T \tag{2.15}$$

or

$$\varepsilon = \sigma/E + \alpha(T - T_0),$$

which is a more general form of the uniaxial stress–strain law.

If there is unrestrained expansion then there will be no stress and $\varepsilon_\sigma = 0$. Hence the total strain will be simply the free thermal strain. If there were complete restraint then $\varepsilon = 0$ and $\varepsilon_\sigma = -\varepsilon_T$.

The above principle will now be applied to the case of a bimetallic component. A copper sleeve loosely surrounds a steel bar, they are of the same length and stress free at temperature T_0, and their ends are rigidly fixed together. Determine the stress in the copper and steel at temperature T.

Superscripts c and s will be used to denote the two materials; A = cross-sectional area; E = modulus of elasticity; α = coefficient of expansion.

Equilibrium

The change in temperature will cause a change in length of the assembly, but because of the different values for α, each material will put a restraint on the other. Since there is no external force, the sum of the internal forces in the copper and steel must be zero. Therefore

$$P^c + P^s = 0$$

or

$$\sigma^c A^c + \sigma^s A^s = 0 \tag{2.16}$$

for equilibrium.

Geometry of Deformation

Since the two materials are initially stress free and their ends are tied together the total strain must be the same for each. Therefore

$$\varepsilon^c = \varepsilon^s \quad \text{or} \quad \varepsilon^c_\sigma + \varepsilon^c_T = \varepsilon^s_\sigma + \varepsilon^s_T. \tag{2.17}$$

Stress–Strain Relation

$$\varepsilon^c = \frac{\sigma^c}{E^c} + \alpha^c(T - T_0) \tag{2.18}$$

and

$$\varepsilon^s = \frac{\sigma^s}{E^s} + \alpha^s(T - T_0). \tag{2.19}$$

Solution

Equating ε^c and ε^s from equations (2.18) and (2.19) gives

$$\frac{\sigma^c}{E^c} + \alpha^c(T - T_0) = \frac{\sigma^s}{E^s} + \alpha^s(T - T_0).$$

Substituting for, say, σ^s from equation (2.16)

$$\frac{\sigma^c}{E^c} + \alpha^c(T - T_0) = -\frac{1}{E^s}\cdot\frac{\sigma^c A^c}{A^s} + \alpha^s(T - T_0)$$

$$\sigma^c\left\{\frac{1}{E^c} + \frac{A^c}{E^s A^s}\right\} = (T - T_0)(\alpha^s - \alpha^c),$$

or

$$\sigma^c = \frac{A^s E^s E^c(T - T_0)(\alpha^s - \alpha^c)}{(A^s E^s + A^c E^c)} \tag{2.20}$$

and

$$\sigma^s = -\frac{A^c E^c E^s(T - T_0)(\alpha^s - \alpha^c)}{(A^s E^s + A^c E^c)}. \tag{2.21}$$

The negative sign for σ^s does not indicate a compressive stress but simply that it is opposite in sign to σ^c. The type of stress in each material is determined by the *numerical* values of the quantities, T_0, T, α^s and α^c. For example, $E^s = 30 \times 10^6$ lb/in^2, $E^c = 17 \times 10^6$ lb/in^2, $\alpha^s = 11 \times 10^{-6}$/°C, $\alpha^c = 16 \times 10^{-6}$/°C and let $A^s = 1$ in^2, $A^c = 2$ in^2, $T_0 = 20$°C and $T = 100$°C. Then

$$\sigma^c = \frac{1 \times 30 \times 10^6 \times 17 \times 10^6\,(100 - 20)\,(11 - 16) \times 10^{-6}}{(1 \times 30 \times 10^6 + 2 \times 17 \times 10^6)}$$

$$= -\frac{30 \times 17 \times 80 \times 5}{30 + 34}$$

$$= -3190 \text{ lb/in}^2,$$

and since

$$\sigma^s = -\frac{\sigma^c A^c}{A^s}$$

$$\sigma^s = -\frac{-3190 \times 2}{1}$$

$$= +6380 \text{ lb/in}^2.$$

Thus for an *increase* in temperature, α^c being greater than α^s, the copper is prevented from expanding as much as if it was free and is therefore put into compression. The steel is forced to expand more than it would if free and is therefore in tension.

(d) A Flexible Mounting

A flexible mounting for a machine of weight W consists of a *rigid* rectangular plate, supported on four coil springs of stiffness†

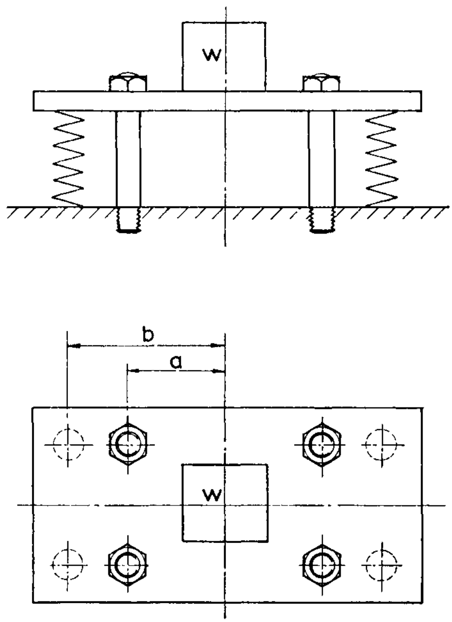


FIG. 2.8

K, placed symmetrically with respect to the corners of the plate as shown in Fig. 2.8. After the machine has been located centrally on the plate, the springs are further loaded by means of four bolts having their lower ends fixed firmly into concrete and their upper

†The term stiffness is defined as the load per unit deformation of an elastic member.

ends passing freely through the plate. The nuts are tightened until the bolts are stretched by an amount, y. The springs have then been compressed by an amount y' due to the combined action of the load W and the tension in the bolts. Determine how far the machine can be moved along a central line parallel to the long edge of the plate without the change in length of the bolts exceeding half the original extension, y.

The system is symmetrical about a vertical central plane parallel to the long edge of the plate and it is only necessary to consider the effect of W acting on a pair of springs and bolts.

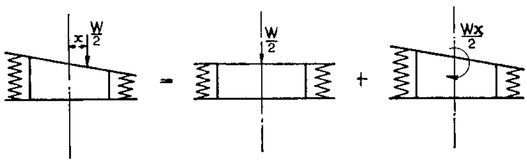


FIG. 2.9

The first step is to consider a likely mode of deformation and the associated equilibrium condition. When the load is moved a distance x away from the centre as in Fig. 2.9, one end of the plate will sink further and the other end will rise. The statical equivalent to the offset load is a central load $W/2$ and a couple about the centre of the plate $Wx/2$. The latter will cause simply a small rotation of the plate and the fall at one end will equal the rise at the other.

Let T = initial tension in bolts due to extension y;

F = force exerted by spring due to the compression y';

ΔT = *change* in force in bolt for offset load;

ΔF = *change* in force in spring for offset load;

k = stiffness of a bolt;

K = stiffness of a spring;

δ = *change* in length of bolt for offset load;

δ' = *change* in length of spring for offset load.

Equilibrium

As stated above the statical equivalent of the offset load is a vertical force and a moment applied at the centre of the plate. Hence for vertical equilibrium

$$\frac{W}{2} + 2T - 2F = 0. \tag{2.22}$$

For rotational equilibrium about the centre

$$\frac{W}{2}x + 2\Delta Ta + 2\Delta Fb = 0, \tag{2.23}$$

where a and b are as shown in Fig. 2.8.

Load Deformation

For vertical central load

$$\frac{T}{y} = k; \quad \frac{F}{y'} = K. \tag{2.24}$$

For rotation due to the offset load

$$\frac{\Delta T}{\delta} = k; \quad \frac{\Delta F}{\delta'} = K. \tag{2.25}$$

Geometry

Since the plate is rigid, from similar triangles

$$\frac{\delta}{\delta'} = \frac{a}{b}. \tag{2.26}$$

The stipulated condition is that

$$\delta \ngtr \tfrac{1}{2}y$$

or for max x,

$$\delta = \tfrac{1}{2}y \tag{2.27}$$

Solution

There are thus eight equations above to solve for the eight unknown quantities.

Substituting for ΔT and ΔF from equations (2.25) in the moment equation (2.23),

$$-\frac{Wx}{2} + 2k\delta a + 2K\delta' b = 0 \tag{2.28}$$

From equation (2.26)

$$\delta' = \frac{b}{a}\delta$$

and for max x,

$$\delta = \tfrac{1}{2}y.$$

Substituting these values into equation (2.28)

$$-\frac{Wx}{2} + kya + \frac{Kyb^2}{a} = 0. \tag{2.29}$$

Eliminating T and F between equations (2.22) and (2.24) gives

$$k = \frac{Ky' - (W/4)}{y}.$$

Substituting for K in equation (2.29) and simplifying

$$x = \frac{2K}{Wa}(a^2y' - b^2y) - \frac{a}{2}$$

or

$$\frac{x}{a} = \frac{2K}{W}[y' - (b/a)^2y] - \tfrac{1}{2} \tag{2.30}$$

2.8. General Stress–Strain Relationships

A bar which is extended under tensile stress will also undergo a contraction in the transverse direction. The strain which occurs perpendicular to the direction of a stress is termed *lateral* or *transverse strain* and is found to be of opposite sign and proportional, in the elastic range, to the strain in the direction of the stress. The ratio of lateral to longitudinal strain is called Poisson's ratio and will be given the symbol ν. It is important to remember that lateral strain does not imply that there is also stress in the lateral direction. A typical value of ν for metals is 0·3.

Consider the cubical element in Fig. 2.10(a) subjected to a uniaxial stress, σ_x, then the corresponding strain system is shown in Fig. 2.10(b). In the x direction there is ε_x and in the y and z

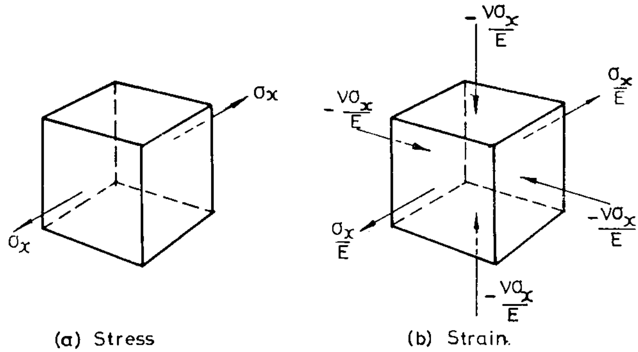


FIG. 2.10

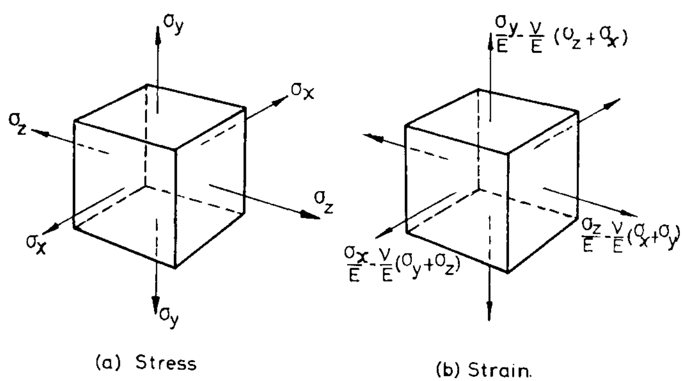


FIG. 2.11

directions $-\nu\varepsilon_x$ and $-\nu\varepsilon_x$ respectively. The above strains may be written in terms of stress as $\varepsilon_x = \sigma_x/E$; $\varepsilon_y = \varepsilon_z = -\nu\sigma_x/E$, the negative sign indicating contraction.

The element in Fig. 2.11 is subjected to triaxial stresses σ_x, σ_y and σ_z. The *total* strain in the x direction is therefore composed of a strain due to σ_x, a lateral strain due to σ_y and further lateral strain due to σ_z.

In Chapter 1 it was shown that the resultant effect of several forces could also be obtained as the sum of the individual effects of the forces by the principle of superposition. Likewise the separate strains in this case can be added to get the resultant strain, hence

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

or

$$\left.\begin{aligned}
\varepsilon_x &= \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z). \\
&\text{Similarly} \\
\varepsilon_y &= \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_z + \sigma_x) \\
&\text{and} \\
\varepsilon_z &= \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_x + \sigma_y).
\end{aligned}\right\} \quad (2.31)$$

There is no lateral strain effect associated with shear strain, hence the shear stress–shear strain relationship is the same for both uniaxial and complex strain systems. If in addition to strain due to stress there is also thermal strain due to change in temperature, then the most general form of the six stress–strain relationships is obtained as,

$$\left.\begin{aligned}
\varepsilon_x &= \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z) + \alpha(T - T_0) \\
\varepsilon_y &= \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_z + \sigma_x) + \alpha(T - T_0) \\
\varepsilon_z &= \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_x + \sigma_y) + \alpha(T - T_0).
\end{aligned}\right\} \quad (2.32)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}; \quad \gamma_{yz} = \frac{\tau_{yz}}{G}; \quad \gamma_{zx} = \frac{\tau_{zx}}{G}. \quad (2.33)$$

2.9. Material Under Constraint

A cube of material just fits between two rigid blocks and hence displacement is completely prevented in one direction. In a perpendicular direction a compressive force F is applied, and along the third axis there is no force. Determine the stresses and strains set up in the cube.

Equilibrium

Assuming a cube face area of A then
$\sigma_x = -F/A$, $\sigma_y A = F'$ (the force exerted on the rigid blocks) and $\sigma_z = 0$ (2.34)

Geometry of Deformation

Because of the rigid blocks,

$$\varepsilon_y = 0. \tag{2.35}$$

Stress–Strain Relationships

$$\left.\begin{aligned} \varepsilon_x &= \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z) \\ \varepsilon_y &= \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_z + \sigma_x) \\ \varepsilon_z &= \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_x + \sigma_y) \end{aligned}\right\} \tag{2.36}$$

Solution

Inserting the equilibrium and geometry conditions into the above three equations (2.36) gives

$$\varepsilon_x = -\frac{F}{AE} - \frac{\nu\sigma_y}{E} \tag{2.37}$$

$$0 = \frac{\sigma_y}{E} + \frac{\nu F}{AE} \tag{2.38}$$

and

$$\varepsilon_z = +\frac{\nu F}{AE} - \frac{\nu\sigma_y}{E}. \tag{2.39}$$

From equation (2.38)

$$\sigma_y = -\frac{\nu F}{A}$$

and hence

$$F' = -\nu F.$$

Substituting for σ_y in equations (2.37) and (2.39)

$$\varepsilon_x = -\frac{F}{AE} + \frac{\nu^2 F}{AE} = -\frac{F}{AE}(1 - \nu^2)$$

$$\varepsilon_y = +\frac{\nu F}{AE} + \frac{\nu^2 F}{AE} = +\frac{\nu F}{AE}(1 + \nu).$$

2.10. Strains in a Statically Determinate Problem

The stresses in a thin-walled cylinder under internal pressure were found in Chapter 1 as a statically determinate problem and now that stress–strain relationships have been developed the strains in the cylinder can be found.

From equations (1.14) and (1.15) from equilibrium

$$\sigma_x = \frac{Pr}{2t}; \ \sigma_y = \frac{Pr}{t} \text{ and } \sigma_z \text{ is negligible in comparison,}$$

therefore, from the stress–strain equations (2.31) the axial strain is

$$\varepsilon_x = \frac{Pr}{2tE} - \frac{\nu Pr}{tE} = \frac{Pr}{2tE}(1 - 2\nu)$$

and the circumferential or hoop strain is

$$\varepsilon_y = \frac{Pr}{tE} - \frac{\nu Pr}{2tE} = \frac{Pr}{2tE}(2 - \nu).$$

Taking a value for ν of 0·3 it is found that the ratio of the hoop to axial strains is

$$\frac{\varepsilon_y}{\varepsilon_x} = \frac{1\cdot7}{0\cdot4} = 4\cdot25,$$

whereas the ratio for hoop to axial stresses was only 2·0.

In the thin sphere there is only circumferential stress and strain,

$$\varepsilon = \frac{\sigma}{E} - \frac{\nu\sigma}{E}$$

and since

$$\sigma = \frac{Pr}{2t}$$

$$\varepsilon = \frac{Pr}{2tE}(1 - \nu).$$

2.11. Volume Changes

The following problem analyses the change in volume of a vessel subjected to pressure and makes use of the relationship between hydrostatic stress and volume strain.

A thin spherical steel shell has a mean diameter of 10 ft and wall thickness 0·25 in., and is just filled with water at 20°C and atmospheric pressure. The problem is to find the rise in gauge pressure if the temperature of the water and shell rises to 50°C, and then to determine the volume of water that would escape if a small leak developed at the top of the vessel.

Steel: Young's modulus $E = 30 \times 10^6$ lb/in^2,
Coefficient of linear expansion: 11×10^{-6} per °C,
Poisson's ratio $= 0{\cdot}3$.

Water: Bulk modulus $K = 46 \times 10^6$ lb/ft^2,
Coefficient of volumetric expansion: $0{\cdot}207 \times 10^{-3}$ per °C.

Equilibrium

Let the gauge pressure in the sphere after rise in temperature be P, then from Chapter 1 the equilibrium condition is

$$\sigma = \frac{Pr}{2t} = 120P \tag{2.40}$$

Geometry of Deformation

If there is to be a pressure at all then the water and sphere must remain in overall contact and hence

change in volume of sphere = change in volume of water

or

$$\varepsilon_{v_{\text{sphere}}} = \varepsilon_{v_{\text{water}}}, \tag{2.41}$$

since original volume is the same for each.

Stress–Strain Relationships

For the water the total volumetric strain is the sum of that due to pressure and that due to thermal strain

$$\varepsilon_{v_{\text{water}}} = -(P/K) + \alpha_v(T - T_0). \tag{2.42}$$

For the sphere the total volumetric strain is a function of strain due to stress (from pressure) and thermal strain,

$$\varepsilon_{v_{\text{sphere}}} = \varepsilon_{v_{\text{stress}}} + \varepsilon_{v_{\text{thermal}}}. \tag{2.43}$$

Solution

From equation (2.42)

$$\begin{aligned}\varepsilon_{v_{\text{water}}} &= -\frac{144P}{46 \times 10^6} + 0{\cdot}207 \times 10^{-3} \times 30 \\ &= -\,3{\cdot}13P \times 10^{-6} + 6210 \times 10^{-6}.\end{aligned}$$

Change in internal capacity or volume of the sphere may be written as

$$\tfrac{4}{3}\pi(r + \delta r)^3 - \tfrac{4}{3}\pi r^3$$

which gives, neglecting products of the small quantity δr,

$$\tfrac{4}{3}\pi \times 3r^2\delta r$$

expressing this as a volumetric strain

$$\frac{\tfrac{4}{3}\pi \times 3r^2\delta r}{\tfrac{4}{3}\pi r^3} = 3\,\frac{\delta r}{r}$$

It will now be shown that $\delta r/r$ is the linear or hoop strain in the material of the sphere.

$$\text{Change in circumference} = 2\pi(r + \delta r) - 2\pi r$$
$$= 2\pi\delta r.$$

$$\therefore \text{Hoop strain} = \frac{2\pi\delta r}{2\pi r} = \frac{\delta r}{r}.$$

$\therefore$ Volumetric strain of the vessel $= 3 \times$ hoop strain.

Now the hoop strain is given by equation (2.32)

$$\varepsilon = \frac{\sigma}{E} - \frac{\nu\sigma}{E} + \alpha(T - T_0)$$
$$= \frac{120P}{30 \times 10^6}(1 - 0{\cdot}3) + 11 \times 10^{-6} \times 30.$$

Therefore, total volumetric strain of vessel is

$$\varepsilon_{v_{\text{sphere}}} = 3\left\{\frac{120 \times 0{\cdot}7P}{30 \times 10^6} + 330 \times 10^{-6}\right\}.$$

Hence using equation (2.41)

$$-3{\cdot}13P \times 10^{-6} + 6210 \times 10^{-6} =$$
$$3\left\{\frac{120 \times 0{\cdot}7P}{30 \times 10^6} + 330 \times 10^{-6}\right\},$$

from which

$$P = \mathbf{452{\cdot}7\ lb/in^2}.$$

The volume of water which escapes through the leak is simply the difference of the *free* thermal expansions of the water and the vessel since obviously there is no pressure present to affect the issue.

Volume of water escaping

$$= \{6210 \times 10^{-6} - 3 \times 330 \times 10^{-6}\} \times (4/3)\pi \times 5^3$$
$$= 2{\cdot}733\ \text{ft}^3$$
$$= \mathbf{4723\ in^3}.$$

Examples

1. The following load–extension data results from a tension test on a mild steel test piece. The initial gauge length is 2 in., the original diameter is 0·505 in. and the final diameter is 0·391 in. Calculate the nominal stresses and strains from the data and plot on one sheet the nominal stress–strain curves (*a*) for all the values, (*b*) for strains below 0·005 to a larger scale. Also plot a true stress–true strain curve for the results beyond a strain of 0·025. To determine the actual area for true stresses assume constancy of volume, i.e.

$$A_1 = \frac{A_0 L_0}{L_1}.$$

From the curves determine the following:

(1) Limit of proportionality stress;
(2) Upper yield stress;
(3) Lower yield stress;
(4) 0·2 per cent proof stress;
(5) 0·5 per cent proof stress;
(6) Tensile strength;
(7) Nominal stress at fracture;
(8) Actual stress at fracture;
(9) Elongation on 2 in. gauge length;
(10) Reduction in area.

Load:	1200	1800	2400	3000	3600	4200	4800
Extension:	0·0002	0·0004	0·0006	0·0008	0·0010	0·0012	0·0014
Load:	5400	6000	6500	7000	7400	8000	8300
Extension:	0·0016	0·0018	0·002	0·0023	0·0026	0·0032	0·0038
Load:	8500	8500	8400	8200	8200	8400	9400
Extension:	0·0058	0·0068	0·008	0·0098	0·02	0·03	0·05
Load:	10600	11400	12300	12800	13500	13700	13100
Extension:	0·075	0·100	0·150	0·200	0·300	0·380	0·450
Load:	11500						
Extension:	0·500 (Fracture).						

2. A mild steel bolt has a minimum diameter of 0·25 in. and there are 20 threads to the inch. The bolt is used to fasten together parts of a machine which can be considered to be rigid. When the parts are just brought together, the distance between the head of the bolt and the nut is 10 in. Neglecting end effects in the bolts, estimate the tension which will be induced in the bolt if the nut is then tightened by one-sixth of a turn. $E = 30 \times 10^6$ lb/in².

3. A uniform steel rope 1000 ft long hangs down a shaft. Find the elongation of the first 500 ft at the top if the density of steel is 480 lb/ft³, and $E = 30 \times 10^6$ lb/in². (Cambridge)

4. A pull of 1000 lb is resisted jointly by three parallel taut wires with their ends so fixed that each stretches equally. The first wire is steel, 0·02 in² cross section; the second is copper, 0·05 in² cross section; the third wire is such that 5000 lbf tension in it produces a stretch of 1 in. $E = 30{,}000{,}000$ lb/in^2 for steel and 15,000,000 lb/in^2 for copper.

Taking the effective length of each wire as 100 in., find the force in each wire and the stretch.

5. Two vertical rods are each rigidly fastened at the upper end at a distance of 24 in. apart. Each rod is 10 ft long and of $\frac{1}{2}$ in. diameter. A horizontal cross-bar connects the lower ends of the rods and on it is placed a load of 1000 lb so that the cross-bar remains horizontal. Find the position of the load on the cross-bar and estimate the stress in each rod. One rod is of wrought-iron for which $E = 28 \times 10^6$ lb/in^2, and the other of bronze for which $E = 9 \times 10^6$ lb/in^2.

6. A weight of 20 ton is supported by three short pillars each of 1 in^2 section. The centre pillar is of steel and the two outer ones of copper. The pillars are so adjusted that at a temperature of 15°C each carries one-third of the total load. The temperature is then raised to 115°C. Estimate the stress in each pillar at 15°C and at 115°C. $E_s = 30 \times 10^6$ lb/in^2, $E_c = 12 \times 10^6$ lb/in^2, $\alpha_s = 12 \times 10^{-6}$/°C, $\alpha_c = 18{\cdot}5 \times 10^{-6}$/°C.

7. A copper ring of radial thickness $\frac{1}{16}$ in. is to be shrunk on to the outside of a steel ring of thickness $\frac{1}{4}$ in.

The copper ring is 0·002 in. less in inner diameter than the outer diameter of the steel ring. What change in temperature is required of the former so that it will just slide on to the latter? The nominal interface diameter is 6 in.

What will be the hoop stress in each ring and also the interface pressure when assembled and back at room temperature? Assume there is no axial stress in the rings. $E_s = 30 \times 10^6$ lb/in^2, $E_c = 15 \times 10^6$ lb/in^2, $\alpha_c = 18 \times 10^{-6}$/°C.

8. An elastic packing piece is bolted between a rigid rectangular plate and a rigid foundation by two bolts pitched 10 in. apart and symmetrically placed on the long centre line of the plate which is 15 in. long. The tension in each bolt is initially 4000 lb, the extension of each bolt is 0·0005 in. and the compression of the packing piece is 0·02 in. If one bolt is further tightened to a tension of 5000 lb, determine the tension in the other bolt. (London)

9. What pressure will increase the capacity of a mild steel cylinder having a diameter/thickness ratio of 30 by 1/2000 of its original value? Also find the proportionate increase in volume of fluid to achieve that pressure. $E = 30 \times 10^6$ lb/in^2, Poisson's ratio = 0·3, bulk modulus of fluid = $3{\cdot}2 \times 10^5$ lb/in^2.

10. A solid cylindrical bar is compressed along its axis and lateral expansion is restrained to one-third of the value when unrestrained. Show that in these circumstances the modulus of elasticity in the axial direction is modified to

$$\frac{3(1 - \nu)}{(3 - 3\nu - 4\nu^2)}$$

times the normal value for E. Determine also the lateral stress as a function of the axial stress.

11. A thin-walled steel cylinder of internal radius 5 in. and wall thickness $\frac{1}{4}$ in. is cooled through 50°C. Determine values for the internal pressure and axial load which, if applied, would restore the cylinder to its original dimensions. $\alpha = 11 \times 10^{-6}$/°C; $E = 30 \times 10^6$ lb/in^2; $\nu = 0{\cdot}3$.

CHAPTER 3

ANALYSIS OF STRESS AND STRAIN

3.1. Introduction

In the problems discussed in the previous chapters the stress planes analysed had only one component of stress acting on them. If, instead of the particular planes chosen, more arbitrary planes had been selected, it would have been found that both normal

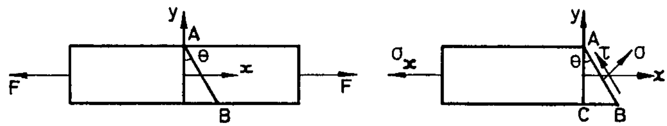


FIG. 3.1

and shear stresses existed on those planes simultaneously. Consider for example the simple case of a bar of rectangular cross section, thickness t as shown in Fig. 3.1, which is subjected to an axial tensile force, F. Now each part of the bar must in itself be in equilibrium, so if the bar is cut along a plane AB, inclined at an angle θ to the y-axis to provide a free-body, the stress conditions on AB can be found from equilibrium of *forces* on the left-hand part of the bar.

Resolving perpendicular to AB:

$$\sigma \times AB \times t = \sigma_x \times AC \times t \times \cos\theta,$$

hence

$$\sigma = \sigma_x \cos^2\theta. \tag{3.1}$$

Resolving parallel to AB:

$$\tau \times AB \times t = -\sigma_x \times AC \times t \times \sin\theta,$$

$$\tau = -\sigma_x \sin\theta\cos\theta, \tag{3.2}$$

the negative sign showing that the shear stress actually occurs in the direction opposite to that shown in the figure.

From this simplified example the analysis is extended in the following sections for a general two-dimensional stress system.

3.2. Plane Stress

The element shown in Fig. 3.2 is subjected to a set of stress components, in which there are no stresses on the two faces of

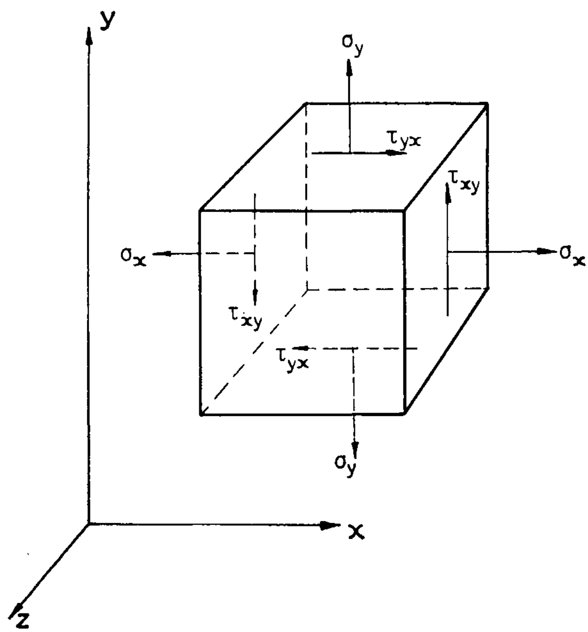


FIG. 3.2

the element perpendicular to the z direction. This is termed a condition of *plane stress* and this situation is found, or approximately so, in a number of engineering problems.

Another important feature is that because a very small element is considered there is no variation of stress over the faces of the

C

element and when the element is reduced to an infinitesimal size, the stress components give the *state of stress* at a point in the material. All the stresses have been drawn in the positive sense. The sign convention is as follows:

> A face of the element is defined as a positive or negative face depending on whether the outward normal to the face is in a direction positive or negative in relation to the coordinate axes. The direction of the arrow representing the stress is also similarly defined as positive or negative. The actual sign of the stress, be it a normal or shear stress, is then obtained as the product of the signs of the directions of the arrow and the normal to the face.

For example τ_{xy} is the shear stress acting on the plane whose normal is in the x direction. The second suffix y denotes that the shear stress direction is parallel to the y-axis. In Fig. 3.2 the shear stresses are all positive because the x, y suffices are related to directions which are either both positive or both negative and when taken together represent a positive shear. If the directions of the shear stress arrows were all reversed they would become negative, since one suffix would indicate a negative direction and the other positive. Pairs of complementary shear stress components are therefore either both positive or both negative. It is also seen that the order of the suffices is irrelevant as far as the value of the shear stress is concerned since $\tau_{xy} = \tau_{yx}$, being complementary, and therefore in analysis either one or the other may be written.

3.3. Stress Components on any Plane in Terms of Coordinate Stresses

The tension bar studied in section 3.1 was a problem in plane stress with only one known stress component, σ_x. In the general case of a body subjected to plane stress in which there are known stress components σ_x, σ_y and τ_{xy}, it is possible to determine the stress components on any plane as a function of the known stresses and the relationships will now be derived. The element in Fig. 3.3 is cut to give the resulting wedge-shaped element of thickness, t, with stress components as shown in Fig. 3.3. The directions of the normal and shear stresses on AC are such that when $\theta = 0°$ they are consistent in sign with σ_x and τ_{xy}. If θ is made 90°, σ corresponds with σ_y but $-\tau$ corresponds with τ_{yx}. In order to find the normal stress σ and shear stress τ on the inclined plane in terms of the coordinate stresses, the equilibrium of the element is studied by considering *forces* normal and parallel to AC.

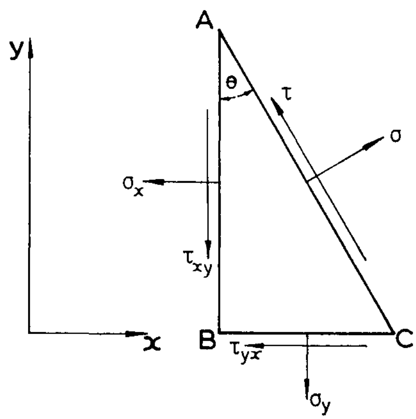


FIG. 3.3

Equilibrium perpendicular to AC:

$$AC \times t \times \sigma = AB \times t \times \sigma_x \cos\theta + BC \times t \times \sigma_y \sin\theta + AB \times t \times \tau_{xy} \sin\theta + BC \times t \times \tau_{yx} \cos\theta,$$

so that

$$\sigma = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy} \sin\theta\cos\theta + \tau_{yx} \sin\theta\cos\theta$$

or

$$\sigma = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy} \sin 2\theta, \qquad (3.3)$$

making use of the fact that $\tau_{xy} = \tau_{yx}$.

Equilibrium parallel to AC:

$$AC \times t \times \tau = -AB \times t \times \sigma_x \sin\theta + BC \times t \times \sigma_y \cos\theta + AB \times t \times \tau_{xy} \cos\theta - BC \times t \times \tau_{yx} \sin\theta,$$

from which

$$\tau = -\left(\frac{\sigma_x - \sigma_y}{2}\right) \sin 2\theta + \tau_{xy} \cos 2\theta. \qquad (3.4)$$

The normal and shear stress on any plane can be found in terms of normal and shear stresses in the coordinate directions from equations (3.3) and (3.4).

3.4. Mohr's Circle for Stress

The equations above can be solved analytically. Mohr's graphical construction is quick and simple, however, and one diagram can give the stresses on any desired plane.

If equations (3.3) and (3.4) are written in the form

$$\sigma - \tfrac{1}{2}(\sigma_x + \sigma_y) = \tfrac{1}{2}(\sigma_x - \sigma_y) \cos 2\theta + \tau_{xy} \sin 2\theta,$$

$$-\tau = \tfrac{1}{2}(\sigma_x - \sigma_y) \sin 2\theta - \tau_{xy} \cos 2\theta,$$

then squaring and adding the two equations gives

$$[\sigma - \tfrac{1}{2}(\sigma_x + \sigma_y)]^2 + \tau^2 = \tfrac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{yx}^2, \qquad (3.5)$$

which is the equation for a circle of radius $\{\tfrac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2\}^{1/2}$ with centre at the point $[\tfrac{1}{2}(\sigma_x + \sigma_y), 0]$.

This is the basis for the geometrical representation of the state of stress at a point devised by Mohr. The circle is plotted with reference to axes of normal stress σ as abscissa and shear stress τ as ordinate. The sign convention used on the circle will be normal stress positive to the right and negative to the left of the origin. Shear stresses which might be described as tending to cause a clockwise rotation of an element are plotted above the abscissa axis and shear stresses appearing as counterclockwise rotation are plotted below the axis.

It is important to remember that shear stress plotted below the σ-axis, although being regarded as negative in the circle construction, may be either positive or negative on the physical

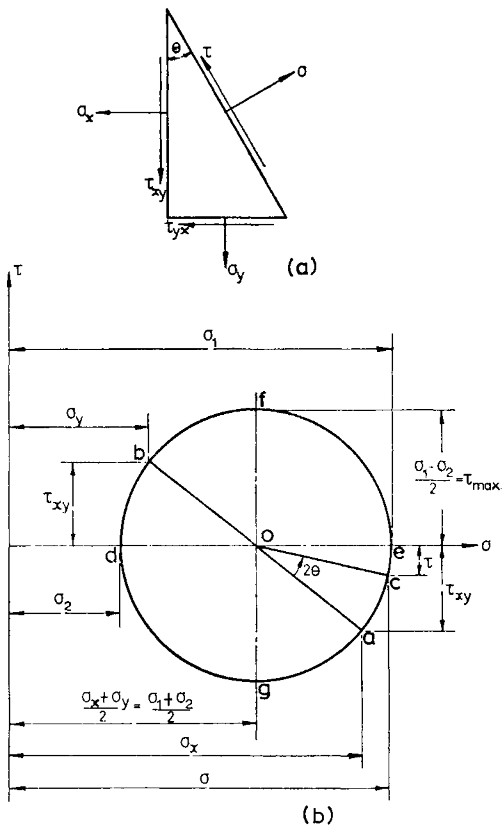


FIG. 3.4

element according to the shear stress convention defined previously. Likewise, positive shear stresses on the circle may be either positive or negative on the element.

To construct the Mohr's circle for the state of stress shown in Fig. 3.4(a), the coordinates (σ_x, $-\tau_{xy}$) are plotted at a (the shear stress is here counterclockwise and therefore below the σ-axis) and (σ_y, τ_{yx}) at b (shear stress clockwise and above the σ-axis). The line joining the above points cuts the σ-axis at O which is the centre of the circle and hence the circle can now be drawn as in Fig. 3.4(b). It should be noted that the planes perpendicular to the Ox and Oy axes which are located 90° apart in the element appear on the circle separated by 180°. The stress components on the plane inclined at θ to the plane perpendicular to Ox are found by setting off 2θ counterclockwise from Oa and drawing a radius to cut the circle at c. The coordinates (σ, τ) of the point c are the stress components in the required direction and it can be verified that these are consistent with equations (3.3) and (3.4).

There are four important planes represented on the circle by the points d, e, f and g. The first two represent perpendicular planes on which the shear stress is zero; the normal stress is a maximum at e and minimum at d. These normal stresses are termed *principal stresses* and the planes oe and od are *principal planes*. The mutually perpendicular planes of and og bisect the principal planes and are subjected to the maximum value of shear stress in the material.

EXAMPLE 3.1

Construct the Mohr's circle for the state of stress shown in Fig. 3.5. Find the stress components on planes at 70° to the x-axis and determine the principal stresses and planes, and maximum shear stress.

The construction of the Mohr's circle is shown in Fig. 3.5 based on the coordinate stresses $\sigma_x = 6$, $\sigma_y = -1$ and $\tau_{xy} = -4$. The required stress planes are such that $\theta = 20°$ and 160° to the y direction and are obtained on the circle by plotting the doubled angles in the counterclockwise direction to give the points a and b. The principal stresses are determined from the ends of the horizontal diameter at c and d. The stress components for the various cases are shown in Fig. 3.5(c) and (d). The maximum shear stresses are found from the top and bottom points on the circle as $\tau_{max} = 5{\cdot}35$ ton/in^2.

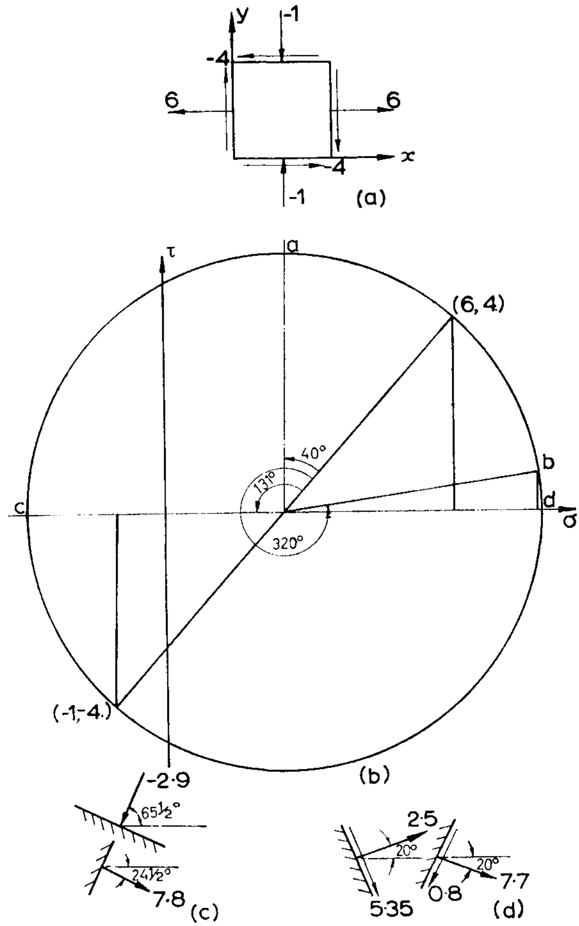


FIG. 3.5

Further examples of Mohr's circle solutions to states of stress may be found in Chapters 4 and 6 on bending and torsion.

To summarise the analytical relationships which are represented on the Mohr's circle; the principal stresses are given by

$$\sigma_1 = \frac{(\sigma_x + \sigma_y)}{2} + \tfrac{1}{2}\{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2\}^{1/2},$$

$$\sigma_2 = \frac{(\sigma_x + \sigma_y)}{2} - \tfrac{1}{2}\{(\sigma_x - \sigma_y)^2 + 4\tau_{yx}^2\}^{1/2}. \qquad (3.6)$$

The maximum shear stress in the plane of σ_1 and σ_2 is given by

$$\tau_{\max} = \frac{(\sigma_1 - \sigma_2)}{2}. \qquad (3.7)$$

However as will be seen on page 89 the maximum shear stress in the material without specifying the plane, is also a function of a third principal stress σ_3, even when it is zero.

The principal planes are defined by

$$\tan 2\theta = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

and the planes of maximum shear stress by

$$\cot 2\theta = \frac{-2\tau_{xy}}{(\sigma_x - \sigma_y)}.$$

3.5. Strain Components in Any Direction in Terms of Coordinate Strains

The conditions of strain in a material are directly related through the elastic constants to the state of stress. It is therefore important to be able to understand and analyse states of strain, and knowing the strains, perhaps from direct measurement, related to one set of axes to transform to unknown strains in other directions.

Consider the rectangular element of material *ABCD*, Fig. 3.6(*a*), which is in an unstrained condition. When subjected to a general system of normal and shear stresses it is deformed as in

Fig. 3.6(*b*) to $A'B'C'D'$. Normal strain is defined as the fractional change in original length of a line and is positive when the line elongates and negative when it contracts. Shear strain is defined as the tangent of the change in angle between two initially perpendicular lines. Since elastic deformations are small it is usual to use the change in angle itself (in radians) rather than the tangent. The sign convention to be used for shear strain is such

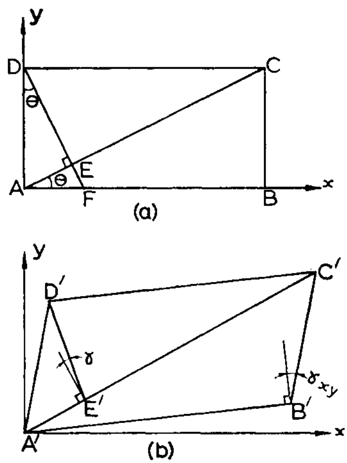


FIG. 3.6

that if the sides of the deformed element as in Fig. 3.6(*b*) have positive slopes in relation to the coordinate axes then the shear strain is positive, and conversely if the sides have negative slopes then the shear strain angle is negative.

Since the shear strain at the four corners of the element is equal and is an angular rotation between x and y planes it is immaterial whether γ_{xy} or γ_{yx} is used to denote shear strain.

The tensile strain is required along a line AC inclined at θ to the x-axis and all deformations are presumed to be small.

In the deformed element

$$A'C'^2 = A'B'^2 + B'C'^2 - 2A'B' \,.\, B'C' \,.\, \cos A'B'C' \quad (3.8)$$

C*

and if ε is the strain along $A'C'$ and ε_x and ε_y are the coordinate strains then

$$A'C' = AC(1 + \varepsilon);$$

$$A'B' = AB(1 + \varepsilon_x) = AC \cos\theta\,(1 + \varepsilon_x)$$

and $$B'C' = BC(1 + \varepsilon_y) = AC \sin\theta\,(1 + \varepsilon_y);$$

$$A'B'C' = (\pi/2) + \gamma_{xy}.$$

Substituting the above values in the equation (3.8) and simplifying:

$$(1 + \varepsilon)^2 = (1 + \varepsilon_x)^2 \cos^2\theta + (1 + \varepsilon_y)^2 \sin^2\theta - 2(1 + \varepsilon_x)(1 + \varepsilon_y) \sin\theta \cos\theta \cos(90 + \gamma_{xy}).$$

Omitting the second and higher powers of small quantities, and writing $\cos(90 + \gamma_{xy}) = -\sin\gamma_{xy} \simeq -\gamma_{xy}$

$$2\varepsilon = 2\varepsilon_x \cos^2\theta + 2\varepsilon_y \sin^2\theta + 2\gamma_{xy} \sin\theta \cos\theta$$

or

$$\varepsilon = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta \qquad (3.9)$$

The strain in a direction perpendicular to $A'C'$ is given by substituting $90° + \theta$ for θ, thus

$$\varepsilon' = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta - \frac{\gamma_{xy}}{2} \sin 2\theta \qquad (3.10)$$

To determine the shear strain between the arbitrary planes AC and DF it is necessary to consider the change in the angle AED, Fig. 3.6(a). In the deformed element AED becomes $A'E'D'$ and the shear strain is γ as shown in Fig. 3.6(b), thus

$$A'D'^2 = A'E'^2 + D'E'^2 - 2A'E' \,.\, D'E' \cos(90 + \gamma). \qquad (3.11)$$

Now the *strain* in $A'E'$ is the same as the strain in $A'C'$ and the strain in $D'E'$ denoted by ε' is obtained from equation (3.10). Therefore

$$A'D' = AD(1 + \varepsilon_y); \quad A'E' = AE(1 + \varepsilon); \quad D'E' = DE(1 + \varepsilon')$$

Substituting the above values into equation (3.11)

$$AD^2(1 + \varepsilon_y)^2 = AE^2(1 + \varepsilon)^2 + DE^2(1 + \varepsilon')^2 + 2AE \,.\, DE(1 + \varepsilon)(1 + \varepsilon')\gamma$$

Simplifying as previously

$$\varepsilon_y = \frac{\varepsilon + \varepsilon'}{2} + \frac{\varepsilon' - \varepsilon}{2} \cos 2\theta + \frac{\gamma}{2} \sin 2\theta.$$

Therefore

$$-\gamma = \frac{\varepsilon + \varepsilon'}{\sin 2\theta} + \frac{\varepsilon' - \varepsilon}{\sin 2\theta} \cos 2\theta - \frac{2\varepsilon_y}{\sin 2\theta}. \tag{3.12}$$

Adding and subtracting equations (3.9) and (3.10) gives

$$\varepsilon' + \varepsilon = \varepsilon_x + \varepsilon_y$$

and

$$\varepsilon' - \varepsilon = -(\varepsilon_x - \varepsilon_y) \cos 2\theta - \gamma_{xy} \sin 2\theta.$$

Substituting for $\varepsilon' + \varepsilon$ and $\varepsilon' - \varepsilon$ in equation (3.12) and simplifying

$$\gamma = -(\varepsilon_x - \varepsilon_y) \sin 2\theta + \gamma_{xy} \cos 2\theta. \tag{3.13}$$

Equations (3.9) and (3.13) should be compared in form with equations (3.3) and (3.4) for normal and shear stresses on any plane.

3.6. Mohr's Circle for Strain

The similarity in form between the stress equations (3.3) and (3.4) and the strain equations (3.9) and (3.13) indicates that a circle construction for the state of strain at a point can be derived in a similar manner to that for stress.

Equations (3.9) and (3.13) can be combined to give:

$$[\varepsilon - \tfrac{1}{2}(\varepsilon_x + \varepsilon_y)]^2 + \left(\frac{\gamma}{2}\right)^2 = \tfrac{1}{4}(\varepsilon_x - \varepsilon_y)^2 + \tfrac{1}{4}\gamma_{xy}^2 \tag{3.14}$$

which is the equation for a circle of radius $\{\frac{1}{4}(\varepsilon_x - \varepsilon_y)^2 + \frac{1}{4}\gamma_{xy}^2\}^{1/2}$ with centre at the point $[\frac{1}{2}(\varepsilon_x + \varepsilon_y), 0]$ in relation to axes of ε and

$\gamma/2$. It is seen that the circle above is of the same form as the stress circle, as also is the method of construction and the various particular points. For example, referring to Fig. 3.7, at the points e and d the shear strain is zero and the planes od and oe

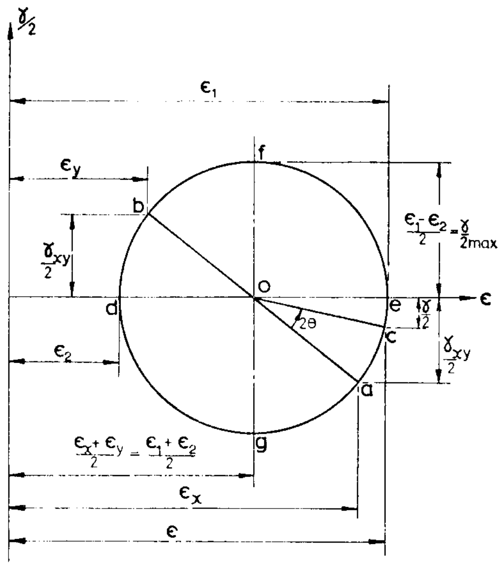


FIG. 3.7

are principal planes and the strains referred to these planes are principal strains ε_1 and ε_2, where

$$\left.\begin{aligned}\varepsilon_1 &= \frac{(\varepsilon_x + \varepsilon_y)}{2} + \tfrac{1}{2}\{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2\}^{1/2}\\ \varepsilon_2 &= \frac{(\varepsilon_x + \varepsilon_y)}{2} - \tfrac{1}{2}\{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2\}^{1/2}\end{aligned}\right\} \quad (3.15)$$

It has been stated that on the planes of principal strain the shear strain is zero. But from the elastic shear stress–shear strain relationship, if the shear strain is zero so must also be the shear stress and planes on which the shear stress is zero are planes of principal stress. Therefore the planes of principal stress and strain coincide.

The angles between the coordinate and principal planes in terms of strain are obtained from the relationship

$$\tan 2\theta = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}. \tag{3.16}$$

Shear strain is a maximum at the top and bottom points on the circle f and g which are planes at 45° to the principal planes. Hence,

$$\frac{\gamma_{max}}{2} = \pm \tfrac{1}{2}\{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2\}^{1/2}$$

$$\frac{\gamma_{max}}{2} = \frac{\varepsilon_1 - \varepsilon_2}{2}.$$

Therefore

$$\gamma_{max} = \varepsilon_1 - \varepsilon_2. \tag{3.17}$$

The correct position for plotting shear strain on the circle, i.e. above or below the ε-axis, may be found either by relating the deformation of the element to the corresponding shear stress system or by the convention that a shear strain angle which is measured clockwise from the undeformed plane is plotted below the ε-axis and a counterclockwise angle is plotted above the ε-axis.

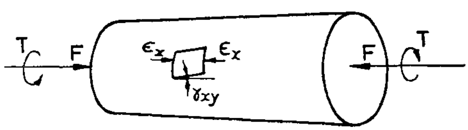


FIG. 3.8

EXAMPLE 3.2

A marine propeller shaft is subjected to thrust and torque as shown in Fig. 3.8 which for an element on the surface give rise to a compressive strain of 0·0002, and a shear strain of 0·0003. Calculate the principal strains, the maximum shear strain at a point, and construct the Mohr strain circle.

$$\varepsilon_x = -0{\cdot}0002,\ \varepsilon_y = 0,\ \gamma_{xy} = 0{\cdot}0003,$$

$$\varepsilon_1, \varepsilon_2 = -0{\cdot}0002 \pm \tfrac{1}{2}\{(-0{\cdot}0002)^2 + (0{\cdot}0003)^2\}^{1/2}$$

$$= -0{\cdot}0001 \pm \tfrac{1}{2}(13 \times 10^{-8})^{1/2}.$$

$$\varepsilon_1 = +\mathbf{0{\cdot}00008},$$

$$\varepsilon_2 = -\mathbf{0{\cdot}00028}.$$

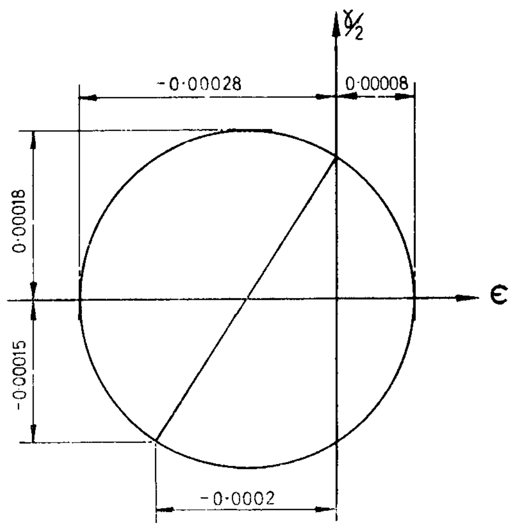


FIG. 3.9

$$\text{Maximum shear strain} = \varepsilon_1 - \varepsilon_2 = 0{\cdot}00008 - (-0{\cdot}00028)$$

$$= \mathbf{0{\cdot}00036}.$$

The Mohr's circle is shown in Fig. 3.9.

3.7. Strain Rosettes

Referring to equation (3.9) it is seen that if it is required to determine the coordinate strains, ε_x, ε_y and γ_{xy}, then these three unknowns can be found from three values for ε at known angles to the coordinate directions.

Surface strains can be measured on a component by means of an electrical wire resistance strain gauge. This consists essentially of a grid of fine wire or metal foil stuck to the metal surface from which the strain is transmitted. Longitudinal strain in the wire causes a change in resistance of the wire which can be measured on a Wheatstone bridge. The percentage change in resistance is proportional to strain. If three strain gauges are

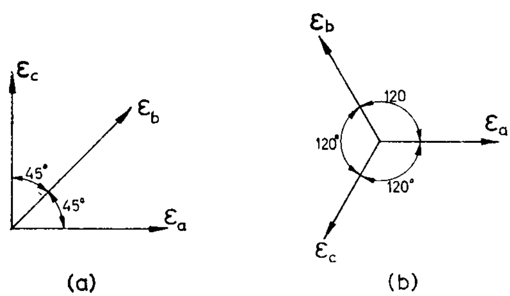


FIG. 3.10

fixed to a surface at different known angles to the coordinate directions, the three measured strains, ε', ε'', ε''', enable the coordinate strains to be determined, using equation (3.9) or Mohr's strain circle.

The two most common arrangements for a strain rosette are illustrated in Fig. 3.10(*a*) and (*b*), known as the rectangular or 45° rosette, and the equiangular or 120° rosette respectively.

The 45° rosette is a very simple one to use and analyse. For example, in Fig. 3.10(*a*), the shear strain associated with the axes *a*, *c*, can be found using equation (3.9) and putting $\theta = 45°$, thus

$$\varepsilon_b = \frac{(\varepsilon_a + \varepsilon_c)}{2} + \frac{\gamma_{ac}}{2},$$

therefore

$$\gamma_{ac} = 2\varepsilon_b - \varepsilon_a - \varepsilon_c.$$

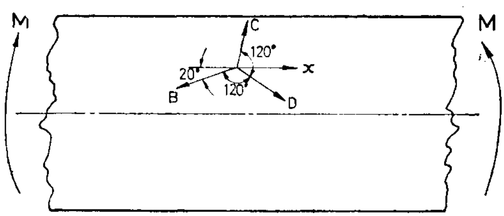


FIG. 3.11(*a*)

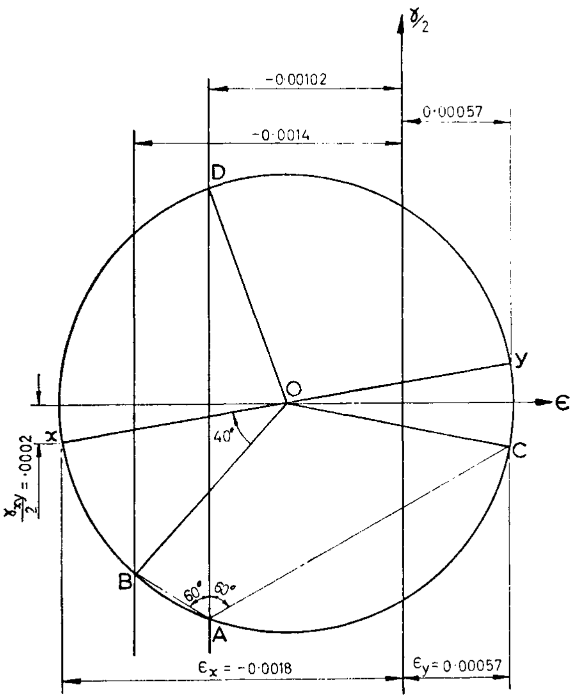


FIG. 3.11(*b*)

Knowing ε_a, ε_c and γ_{ac} for the perpendicular axes a, c, the principal strains ε_1 and ε_2 could then be determined from equations (3.15) and their directions from equation (3.16).

The analysis of the 120° rosette is arithmetically more involved than that above owing to the different angle. The following example illustrates the Mohr's circle construction for the 120° rosette, after which the construction for a 45° rosette is self evident.

EXAMPLE 3.3

A strain gauge rosette fixed at a point on an aluminium alloy beam, Fig. 3.10, gives three measured strains at 20°, 140° and 260° counterclockwise to the beam axis of values $-0{\cdot}0014$, $-0{\cdot}00102$ and $+0{\cdot}00057$ respectively. Construct a Mohr's strain circle and hence find normal and shear stresses in the coordinate directions. Young's modulus $= 10 \times 10^6$ lb/in^2. Shear modulus $= 3{\cdot}8 \times 10^6$ lb/in^2. Poisson's ratio $= 0{\cdot}315$.

Firstly, draw a vertical line representing the shear strain axis. Then three more vertical lines, Fig. 3.11, are drawn at the appropriate perpendicular distances from the shear strain axis representing the three abscissa normal strains $-0{\cdot}0014$, $-0{\cdot}00102$ and $+0{\cdot}00057$. From an arbitrary point A on the middle line of the three draw lines at 60° to cut the two outer lines at B and C. The perpendicular bisectors of AB and AC intersect at O to give the centre of the strain circle. With centre O draw a circle through B and C. Then OB, OC and OD represent the three strain gauge directions. An angle of 40° (2 × 20°) is set off clockwise (320° counterclockwise) from OB, and the line OX represents the x-axis and OY the y-axis. The required strains ε_x, ε_y and $\gamma_{xy}/2$ are now read off the circle as $-0{\cdot}0018$, $+0{\cdot}00057$ and $0{\cdot}0002$.

The normal stresses are obtained from the stress–strain relationships

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E}$$

and

$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu\sigma_x}{E}.$$

From which

$$\left.\begin{aligned} \sigma_x &= E(\varepsilon_x + \nu\varepsilon_y)(1 - \nu^2) \\ \text{and} \quad \sigma_y &= E(\varepsilon_y + \nu\varepsilon_x)(1 - \nu^2) \end{aligned}\right\} \quad (3.18)$$

Hence

$$\sigma_x = 10 \times 10^6 \times (-0{\cdot}0018 + 0{\cdot}315 \times 0{\cdot}00057)(1 - 0{\cdot}315^2)$$
$$= \mathbf{14{,}600\ lb/in^2.}$$
$$\sigma_y = 10 \times 10^6 \times (0{\cdot}00057 - 0{\cdot}315 \times 0{\cdot}0018)(1 - 0{\cdot}315^2)$$
$$= \mathbf{0.}$$

Since there is no lateral strain influence on shear strain the shear stress–shear strain relation gives

$$\tau_{xy} = G\gamma_{xy}$$

or

$$\tau_{xy} = 3{\cdot}8 \times 10^6 \times (2 \times 0{\cdot}0002) = \mathbf{1520\ lb/in^2.}$$

3.8. Relationships Between the Elastic Constants

A material which has mechanical properties which are the same in all directions is said to be *isotropic* and an isotropic material has only four elastic constants, E, G, K and ν. These constants have been used on a number of occasions in previous sections when writing down stress–strain relationships. In this section it will be shown that there are two independent relations between the elastic constants.

Firstly consider the element of material in Fig. 3.12 having sides of length x, y and z and thus a volume $V_0 = xyz$. When the element is subjected to a stress system such as that shown, the lengths of the sides become $x + \delta x$, $y + \delta y$ and $z + \delta z$. The new volume V is therefore

$$V = (x + \delta x)(y + \delta y)(z + \delta z)$$

which simplifies to

$$V = xyz + zy\delta x + xz\delta y + xy\delta z$$

neglecting products of small quantities.

Therefore change in volume

$$\delta V = V - V_0 = +zy\delta x + xz\delta y + xy\delta z$$

and the volumetric strain is

$$\varepsilon_V = \frac{\delta V}{V} = +\frac{\delta x}{x} + \frac{\delta y}{y} + \frac{\delta z}{z}$$

$$= \varepsilon_x + \varepsilon_y + \varepsilon_z, \tag{3.19}$$

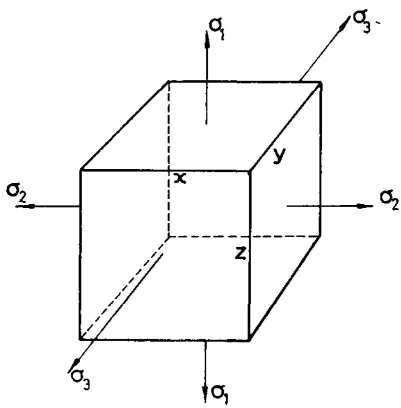


FIG. 3.12

which is the sum of the three linear strains in the coordinate directions.

Let the stress system in Fig. 3.12 become hydrostatic compression, then

$$\sigma_x = \sigma_y = \sigma_z = -\sigma.$$

The stress–strain relationships for the element are

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = -\frac{\sigma}{E} + \frac{\nu\sigma}{E} + \frac{\nu\sigma}{E} = -\frac{\sigma}{E}(1 - 2\nu).$$

Thus from equation (3.19)

$$\varepsilon_V = -\frac{3\sigma}{E}(1 - 2\nu).$$

But volumetric strain:

$$\frac{\delta V}{V_0} = \varepsilon_V = -\frac{\sigma}{K} \text{ (where } K = \text{ bulk modulus)}$$

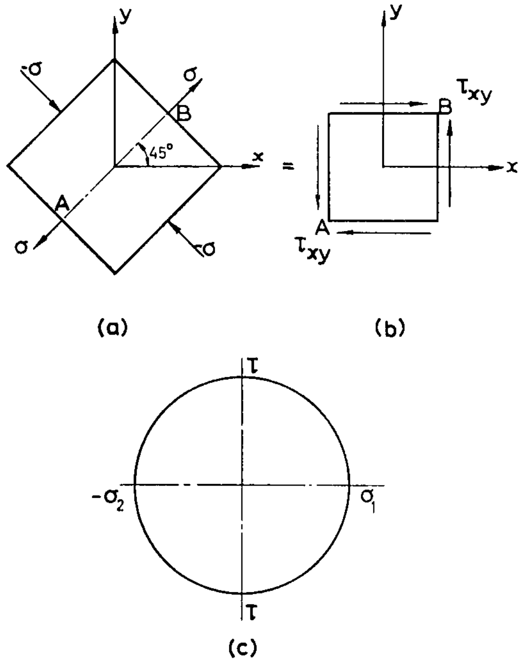


FIG. 3.13

therefore

$$-\frac{\sigma}{K} = -\frac{3\sigma}{E}(1 - 2\nu),$$

or

$$\underline{E = 3K(1 - 2\nu).} \tag{3.20}$$

The element of material of unit thickness shown in Fig. 3.13(*a*) is subjected to tensile stress, σ, in a direction at 45° to the co-ordinate axes, and an equal compressive stress, $-\sigma$, in the perpendicular direction. By considering the equilibrium of the element it is seen that the stress system in Fig. 3.13(*a*) is equivalent to that for pure shearing stress in Fig. 3.13(*b*). The Mohr's circle is shown in Fig. 3.13(*c*) where

$$\sigma = \sigma_1 = -\sigma_2 = \tau_{max}.$$

The strain in the direction *AB* in Fig. 3.13(*a*) is given by

$$\varepsilon_{AB} = \frac{\sigma}{E} - \frac{\nu}{E}(-\sigma)$$
$$= (\sigma/E)(1 + \nu)$$

But the strain in *AB* in Fig. 3.13(*b*) is obtained from equation (3.9), putting ε_x and $\varepsilon_y = 0$ since there are no normal strains in this case and hence $\varepsilon_{AB} = \frac{1}{2}\gamma_{xy}\sin 2\theta$, and since $\theta = 45°$ and $\gamma_{xy} = \tau_{xy}/G$

$$\varepsilon_{AB} = \frac{\tau_{xy}}{2G}.$$

Equating the two values for ε_{AB},

$$\frac{\tau_{xy}}{2G} = \frac{\sigma}{E}(1 + \nu).$$

But as stated earlier $\tau_{xy} = \sigma$, therefore

$$\frac{1}{2G} = \frac{1}{E}(1 + \nu)$$

or

$$\underline{E = 2G(1 + \nu).} \tag{3.21}$$

Thus from the expressions (3.20) and (3.21) if any two of the elastic constants are known or can be measured the remaining two can be determined.

3.9. Further Two-dimensional Stress–Strain Relationships

Now that relationships between the elastic constants have been established, it is possible to derive for an element of material two stress–strain equations which are independent of orientation of the element and give a measure of its change in volume and shape.

Consider the element under plane stress σ_x, σ_y and τ_{xy} shown in Fig. 3.14, then the normal and shear stresses on planes inclined at θ and $\theta + 90$ to the x, y system will be

$$\left.\begin{aligned} \sigma &= \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta \\ \sigma' &= \tfrac{1}{2}(\sigma_x + \sigma_y) - \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta - \tau_{xy}\sin 2\theta \\ \tau &= -\tfrac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta \end{aligned}\right\} \quad (3.22)$$

The strains on the inclined planes are ε, ε' and γ given by

$$\left.\begin{aligned} \varepsilon &= \tfrac{1}{2}(\varepsilon_x + \varepsilon_y) + \tfrac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta \\ \varepsilon' &= \tfrac{1}{2}(\varepsilon_x + \varepsilon_y) - \tfrac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta \\ \gamma &= -(\varepsilon_x - \varepsilon_y)\sin 2\theta + \gamma_{xy}\cos 2\theta \end{aligned}\right\} \quad (3.23)$$

The stress–strain relationships derived earlier are

$$\varepsilon = \frac{\sigma}{E} - \frac{\nu\sigma'}{E} \quad (3.24)$$

$$\varepsilon' = \frac{\sigma'}{E} - \frac{\nu\sigma}{E} \quad (3.25)$$

and

$$\gamma = \frac{\tau}{G}. \quad (3.26)$$

So that it would appear that there are three stress–strain relationships associated with a two-dimensional stress–strain field. However it will now be shown that only two of these are independent.

Adding and subtracting equations (3.24) and (3.25) gives

$$\varepsilon + \varepsilon' = (\sigma + \sigma')\frac{(1 - \nu)}{E} \quad (3.27)$$

$$\varepsilon - \varepsilon' = (\sigma - \sigma')\frac{(1 + \nu)}{E}. \quad (3.28)$$

Substituting for the strains, equations (3.23), and stresses, equations (3.22), into equations (3.27), (3.28) and (3.26) and putting $\tau_{xy} = G\gamma_{xy}$ and simplifying gives

$$\varepsilon_x + \varepsilon_y = (\sigma_x + \sigma_y)\frac{(1 - \nu)}{E} \tag{3.29}$$

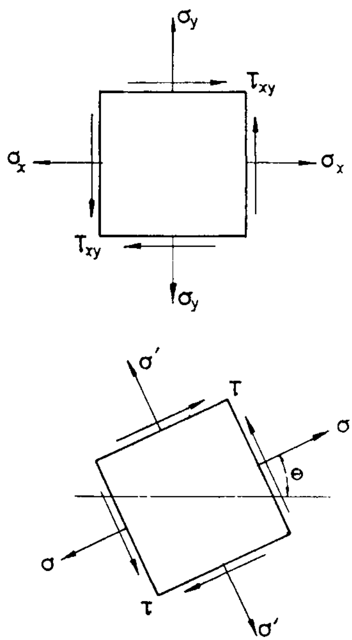


FIG. 3.14

$$(\varepsilon_x - \varepsilon_y) = (\sigma_x - \sigma_y)\frac{(1 + \nu)}{E} + \left\{\frac{2G}{E}(1 + \nu)\,\gamma_{xy} - \gamma_{xy}\right\}\tan 2\theta \tag{3.30}$$

and

$$(\varepsilon_x - \varepsilon_y) = (1/2G)(\sigma_x - \sigma_y). \tag{3.31}$$

Equation (3.30) appears to be dependent on θ: however, if using the relationship $E = 2G(1 + \nu)$ then the term inside the curly bracket becomes zero, and the equation reduces to

$$\varepsilon_x - \varepsilon_y = (\sigma_x - \sigma_y)\frac{(1 + \nu)}{E}. \tag{3.32}$$

By the further substitution

$$\frac{(1 + \nu)}{E} = \frac{1}{2G}$$

it is seen that equation (3.31) is identical with equation (3.32). It should be noted that equations (3.29) and (3.32) are in exactly the same form as equations (3.27) and (3.28).

There are therefore two stress–strain relations in a plane stress condition, of which the first is concerned with change in volume only and the second with shear distortion or change in shape.

$$\varepsilon_x + \varepsilon_y = (\sigma_x + \sigma_y)\frac{(1 - \nu)}{E},$$

$$\varepsilon_x - \varepsilon_y = (\sigma_x - \sigma_y)\frac{(1 + \nu)}{E}.$$

3.10. Strain Energy

During a tension test in the elastic range the work done on the specimen is completely recoverable on unloading, since by definition of elasticity the load–deformation relationship is reversible.

The work done on the specimen is equal to the energy stored in the material. Within the elastic range this energy which is stored when under load and is released on unloading is termed *strain energy*. Consider the load–extension diagram in Fig. 3.15(a), the work done during a small increment of extension, dx, is wdx and the total area under the curve up to the elastic limit at point A is $\frac{1}{2}W\delta l$.

Dividing by the area a and length of specimen l_0 gives stored strain energy,

$$U = \tfrac{1}{2}\frac{W}{a}\frac{\delta l}{l_0} = \tfrac{1}{2}\sigma\varepsilon \text{ per unit volume}$$

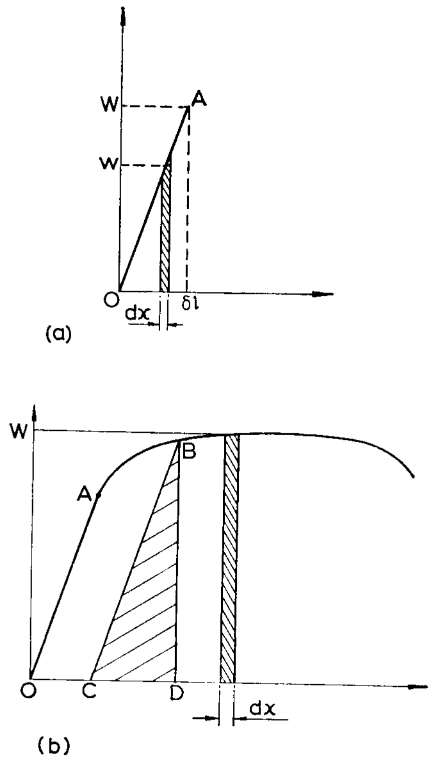


FIG. 3.15

or

$$U = \frac{\sigma^2}{2E} \text{ per unit volume}$$

for a uniaxial stress system.

If straining is continued beyond the elastic range then on unloading along BC, Fig. 3.15(b), only part of the work done is recoverable, the area BCD. The area $OABC$ is energy dissipated mostly in the form of heat.

The total work to fracture is represented by the area under the complete load–extension curve.

3.11. Maximum Stress Due to Suddenly Applied Load

In the discussion of the tensile test the load was applied gradually to the test bar and the work done in the elastic range was the *average* load, $\frac{1}{2}W$ times the distance moved, δl.

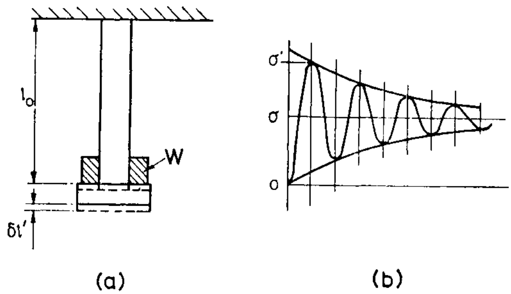


FIG. 3.16

Now suppose that a bar fixed vertically at the top, with a flange at the lower end, Fig. 3.16, has a load W suddenly released on to the flange. Let the momentary maximum extension, strain and stress in the bar be $\delta l'$, ε' and σ' respectively. In addition, if the masses of the bar and flange are small compared with the load W, then a reasonable approximation to the behaviour is made by neglecting the effect of the former. Because the *full* load moves through the extension $\delta l'$, then

$$\text{Work done} = W\delta l'$$

The strain energy per unit volume stored in the bar is $\frac{1}{2}\sigma'\varepsilon'$ so that the total strain energy U is

$$\tfrac{1}{2}\sigma'\varepsilon' a l_0 = \tfrac{1}{2}\sigma' a \delta l'.$$

Now the work done = strain energy stored

$$W\delta l' = \tfrac{1}{2}\sigma' a\delta l'$$

so that $\sigma' = 2W/a$ and $W/a = \sigma$, the stress due to a *gradually* applied load, therefore

$$\sigma' = 2\sigma$$

or the momentary maximum stress due to a suddenly applied load is twice the value of stress for a gradually applied load.

The bar will subsequently oscillate about the statical equilibrium position, while the stresses and deformations rapidly die away as shown in Fig. 3.16(*b*) to the values obtained for a gradually applied load. However, the momentary stress intensification by a factor of 2 might have very serious consequences on a component or structure.

3.12. Maximum Stress Due to Impact

An extension of the above problem would be the case where the load W was dropped on to the flange from a height h, in which case there would be a momentary extension of the bar, $\delta l'$. Then the total potential energy $= W(h + \delta l')$ and the momentarily stored strain energy is

$$U' = \tfrac{1}{2}\sigma' a\delta l'.$$

Then neglecting the mass of the bar and flange and assuming no losses of energy during impact

$$\tfrac{1}{2}\sigma' a\delta l' = W(h + \delta l')$$

or

$$\tfrac{1}{2}\sigma'\delta l' = \frac{Wh}{a} + \frac{W\delta l'}{a}.$$

Now $\delta l' = (\sigma'/E)\,.\,l_0$ and $W/a = \sigma$, the final steady stress. Therefore

$$\sigma'^2 - 2\sigma\sigma' - \frac{2\sigma Eh}{l_0} = 0$$

and

$$\sigma' = \sigma + \left(\sigma^2 + \frac{2\sigma Eh}{l_0}\right)^{1/2}. \qquad (3.33)$$

The momentary maximum stress σ' can be expressed in terms of the velocity of impact by substituting $h = v^2/2g$.

It is seen that in equation (3.33) if $h = 0$ then $\sigma' = 2\sigma$ being the result obtained in the previous section.

It is also interesting to note the magnitude of the ratio h/l_0 in relation to the momentary stress σ'. For example, if σ' is not to exceed 4σ and the bar is steel so that $E = 30 \times 10^6$ lb/in^2, then by substitution into equation (3.33)

$$\frac{h}{l_0} = \frac{\sigma'}{30 \times 10^6}.$$

Taking a limiting stress for σ' of, say, 40,000 lb/in^2, then $h = l_0/750$, i.e. for a bar 30 in. in length, the limiting drop of the weight for this particular case would be $h = 0{\cdot}04$ in.

The true situation for the stress during the impact of one mass on another is rather more complex than as indicated in the approximate analysis above. In practice the deformation and stress imposed on the bar at the point of impact take time to propagate along the length of the bar. The *stress wave*, as it is called, on reaching the fixed end of the bar will be reflected towards the point of initiation and thus a complex state of stress can arise.

3.13. Strain Energy in Three-dimensional Stress System

In section (3.15) dealing with yielding under complex stress, two theories are based on the concept that yielding is a function of the strain energy level in the material. The derivation of strain energy in a complex stress system is dealt with as follows.

If an element of material is subjected to principal stresses σ_1, σ_2, σ_3 then the strain energy stored in the system by superposition for the three principal directions will be

$$U = \tfrac{1}{2}\sigma_1\varepsilon_1 + \tfrac{1}{2}\sigma_2\varepsilon_2 + \tfrac{1}{2}\sigma_3\varepsilon_3 \text{ per unit volume.}$$

Using the stress–strain relationships

$$\varepsilon_1 = \frac{\sigma_1}{E} - \left(\frac{\nu}{E}\right)(\sigma_2 + \sigma_3), \text{ etc.}$$

$$U = \tfrac{1}{2}\sigma_1\left\{\frac{\sigma_1}{E} - \frac{\nu}{E}(\sigma_2 + \sigma_3)\right\} + \tfrac{1}{2}\sigma_2\left\{\frac{\sigma_2}{E} - \frac{\nu}{E}(\sigma_3 + \sigma_1)\right\} + \tfrac{1}{2}\sigma_3\left\{\frac{\sigma_3}{E} - \frac{\nu}{E}(\sigma_1 + \sigma_2)\right\}.$$

After simplifying

$$U = \frac{1}{2E}[(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)], \qquad (3.34)$$

which in the case of the two-dimensional stress system when $\sigma_3 = 0$ gives

$$U = \frac{1}{2E}(\sigma_1^2 + \sigma_2^2 - 2\nu\sigma_1\sigma_2). \qquad (3.35)$$

3.14. Strain Energy Owing to Distortion or Shear

Another way of considering the stress system acting on the element of material in the previous section is as follows:

Let there be a stress $\bar{\sigma}$ acting on every face of magnitude $\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$, i.e. a mean value of the three principal stresses. Then this would be a condition of *hydrostatic* stress.

In order to have the correct value of principal stress on each face there would have to be components

$$\left.\begin{aligned} \sigma_1' &= \tfrac{1}{3}(2\sigma_1 - \sigma_2 - \sigma_3), \\ \sigma_2' &= \tfrac{1}{3}(2\sigma_2 - \sigma_3 - \sigma_1), \\ \sigma_3' &= \tfrac{1}{3}(2\sigma_3 - \sigma_1 - \sigma_2), \end{aligned}\right\} \qquad (3.36)$$

which are termed *deviatoric* stresses, acting in conjunction with the above hydrostatic stress so that

$$\bar{\sigma} + \sigma_1' = \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) + \tfrac{1}{3}(2\sigma - \sigma_2 - \sigma_3) = \sigma_1, \text{ etc.}$$

Now the hydrostatic stress $\bar{\sigma}$ causes change in volume without distortional or shear strain hence distortion is due to the deviatoric stress components σ_1', σ_2' and σ_3' for which the strains are given by

$$\varepsilon_1' = \frac{1}{3E}(2\sigma_1 - \sigma_2 - \sigma_3) - \frac{\nu}{3E}(2\sigma_2 - \sigma_3 - \sigma_1) - \frac{\nu}{3E}(2\sigma_3 - \sigma_1 - \sigma_2)$$

$$\left.\begin{aligned} &= \frac{1+\nu}{3E}(2\sigma_1 - \sigma_2 - \sigma_3), \\ &\text{and similarly} \\ \varepsilon_2' &= \frac{1+\nu}{3E}(2\sigma_2 - \sigma_3 - \sigma_1), \\ \varepsilon_3' &= \frac{1+\nu}{3E}(2\sigma_3 - \sigma_1 - \sigma_2). \end{aligned}\right\} \tag{3.37}$$

The distortion or shear strain energy is therefore

$$\begin{aligned} U_s &= \tfrac{1}{2}\sigma_1'\varepsilon_1' + \tfrac{1}{2}\sigma_2'\varepsilon_2' + \tfrac{1}{2}\sigma_3'\varepsilon_3' \\ &= \frac{1+\nu}{18E}(2\sigma_1 - \sigma_2 - \sigma_3)^2 + \frac{1+\nu}{18E}(2\sigma_2 - \sigma_3 - \sigma_1)^2 \\ &\qquad + \frac{1+\nu}{18E}(2\sigma_3 - \sigma_1 - \sigma_2)^2, \end{aligned}$$

which on simplifying gives

$$U_s = \frac{1+\nu}{6E}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2].$$

Putting

$$\frac{1+\nu}{E} = \frac{1}{2G},$$

$$U_s = \frac{1}{12G}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]. \tag{3.38}$$

In the two-dimensional case when $\sigma_3 = 0$

$$U_s = \frac{1}{6G}(\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2). \tag{3.39}$$

The importance of the strain energy expressions of this and the previous section will become apparent in the next section.

3.15. Criteria for Yielding Under Complex Stress

In the simple uniaxial tension test discussed in Chapter 2, at the end of the elastic range yielding and plastic deformation were said to occur. A pertinent question that may then be asked is what property of the material governs the onset of yielding in uniaxial and complex stress systems? The obvious explanation that yielding occurs at some critical value of tensile stress is not necessarily true. However, it is reasonable to assume that a critical value of a quantity such as normal stress, shear stress, strain or strain energy has to be attained before yielding commences. Experimental observations on *ductile* materials under complex stress conditions have supported the hypotheses that a critical value of either the maximum shearing stress (Tresca, 1864) or the shear (distortion) strain energy (Von Mises, 1913) give the best prediction for yielding in the majority of cases. The critical value mentioned above is obtained from the observation of yielding in a simple tensile test.

Physically, yielding takes the form of bands of the atomic structure sliding over one another, which effect can be seen on the polished surface of a tensile specimen as Lüders lines or bands inclined to the longitudinal axis at about 45°. These are planes of maximum shearing stress indicating that the two hypotheses above involving shear are appropriate. More confirmation is obtained for the foregoing when it is found that hydrostatic stress does not cause yielding.

Further details about the mechanism of yield and plastic deformation can be found in *Mechanical Properties of Engineering Materials* by W. D. Biggs, Pergamon, 1965.

3.15.1. Maximum Shear Stress Theory

This theory postulates that, in a complex stress system, yielding occurs when the maximum shear stress attains the value of the maximum shear stress at yielding in simple tension.

In a two- or three-dimensional stress system the maximum shear stress is given by half the difference of the maximum and minimum principal stresses (see equation (3.7) for plane stress). Therefore in the case of simple tension where the principal stresses are σ_1, 0 and 0 the maximum shear stress at yielding is

$$\frac{(\sigma_Y - 0)}{2} = \frac{\sigma_Y}{2}$$

where σ_Y is the yield or proof stress in tension.

In a general two-dimensional stress system there are principal stresses σ_1, σ_2 and 0.

If $\sigma_1 > \sigma_2 > 0$, i.e. σ_1 and σ_2 both tensile, then the maximum shear stress would be

$$\tau_{\max} = \frac{(\sigma_1 - 0)}{2} = \frac{\sigma_1}{2}$$

and yielding occurs when $\sigma_1/2 = \sigma_Y/2$ or $\sigma_1 = \sigma_Y$. Alternatively if $\sigma_2 > \sigma_1 > 0$ then

$$\tau_{\max} = \frac{\sigma_2}{2}.$$

and $\sigma_2/2 = \sigma_Y/2$ or $\sigma_2 = \sigma_Y$.

These conditions may be represented on a diagram of σ_1 plotted against σ_2 as in Fig. 3.17, in which *AB* and *BC* form a boundary between elastic and plastic behaviour. *DE* and *EF* are the corresponding boundaries for compressive stresses (usually σ_Y is taken as equal in tension or compression).

When one of the principal stresses is positive and the other negative, then the maximum shear stress is

$$\tau_{max} = \frac{(\sigma_1 - \sigma_2)}{2}$$

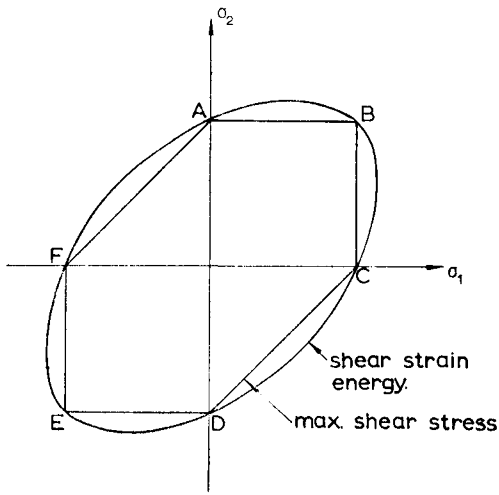


FIG. 3.17

for $\sigma_1 > 0 > \sigma_2$ and yielding occurs when

$$\frac{(\sigma_1 - \sigma_2)}{2} = \frac{\sigma_Y}{2} \text{ or } \sigma_1 - \sigma_2 = \sigma_Y.$$

Alternatively

$$\tau_{max} = \frac{(\sigma_2 - \sigma_1)}{2}$$

for $\sigma_2 > 0 > \sigma_1$, so that for yielding $\sigma_2 - \sigma_1 = \sigma_Y$.

These conditions are represented by the lines *CD* and *FA* respectively. The hexagon *ABCDEF* is termed a *yield locus* and all points within the hexagon represent elastic stress conditions and points outside are in the plastic range.

In a *three-dimensional* stress system where there are principal stresses σ_1, σ_2, σ_3, the conditions for yield are that

$$\sigma_1 - \sigma_3 = \sigma_Y \text{ when } \sigma_1 > \sigma_2 > \sigma_3$$

$$\sigma_1 - \sigma_2 = \sigma_Y \text{ when } \sigma_1 > \sigma_3 > \sigma_2$$

$$\sigma_2 - \sigma_3 = \sigma_Y \text{ when } \sigma_2 > \sigma_1 > \sigma_3$$

$$\sigma_2 - \sigma_1 = \sigma_Y \text{ when } \sigma_2 > \sigma_3 > \sigma_1$$

$$\sigma_3 - \sigma_2 = \sigma_Y \text{ when } \sigma_3 > \sigma_1 > \sigma_2$$

$$\sigma_3 - \sigma_1 = \sigma_Y \text{ when } \sigma_3 > \sigma_2 > \sigma_1,$$

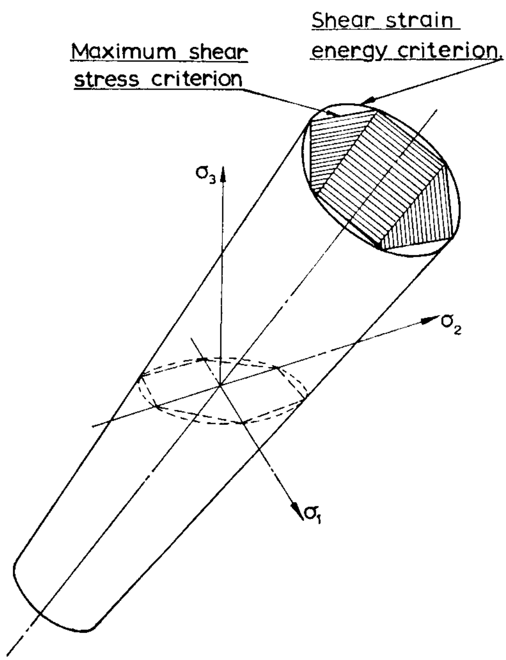


FIG. 3.18

and the *yield envelope* is a hexagonal cylinder with its longitudinal axis equally inclined to the $\sigma_1, \sigma_2, \sigma_3$ coordinate directions as shown in Fig. 3.18.

3.15.2. Shear Strain Energy Theory

An expression for shear strain energy was derived in section 3.14, and for the case of *two-dimensional* stress

$$U_s = (1/6G)(\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2),$$

therefore at yield in simple tension $\sigma_1 = \sigma_Y$ and $\sigma_2 = 0$, hence

$$U_s = \frac{\sigma_Y^2}{6G}$$

It is postulated that yielding will occur in a complex stress system when the shear strain energy is equal to that in simple tension (or compression) at yielding. Therefore

$$\frac{1}{6G}(\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2) = \frac{\sigma_Y^2}{6G}$$

or

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_Y^2.$$

This is an equation for an ellipse which is shown plotted in Fig. 3.17. The ellipse, which it is noticed coincides with the maximum shear stress theory at *A*, *B*, *C*, *D*, *E* and *F*, is the yield locus, stress conditions inside being elastic and outside plastic.

In the *three-dimensional* stress system the shear strain energy is

$$U_s = \frac{1}{12G}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2],$$

hence the condition for yielding is

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_Y^2.$$

For the case of three principal stresses the *yield envelope* is a cylinder of circular cross section with its axis inclined equally to the σ_1, σ_2, σ_3 coordinate directions as shown in Fig. 3.18. It is seen that the plane $\sigma_3 = 0$ cuts the cylinder to form the ellipse discussed above.

There have been a number of other yield theories put forward but all have shown less satisfactory correlation with experimental behaviour than for the two shear theories above.

One such example was the theory proposed by Haigh in which the total strain energy (section 3.13) was used to define the onset of yielding. Thus

$$\sigma_1^2 + \sigma_2^2 - 2\nu\sigma_1\sigma_2 = \sigma_Y^2.$$

The yield locus again takes the form of an ellipse.

3.16. Failure of Brittle Materials

Brittle materials such as flake cast iron, concrete, glass and ceramics exhibit little or no plastic deformation before fracture, hence a yield criterion is not applicable.

A criterion of fracture, based on physical arguments and observations on brittle materials, which is often adopted, is that failure will occur when a critical value of the maximum principal tensile stress is reached.

Ductile materials can occasionally fail in a brittle manner owing to a number of factors, perhaps the most important of which is a stress condition approaching hydrostatic tension in which case shear and hence yielding is prevented. (Discussion of Brittle Fracture in *Mechanical Properties of Engineering Materials* by W. D. Biggs.)

EXAMPLE 3.4

A circular steel cylinder of wall thickness $\frac{1}{2}$ in. and internal diameter 10 in., is subjected to an internal pressure of 2000 lb/in^2. If the yield stress of the material in simple tension is 35,000 lb/in^2, determine how much axial tension or thrust can be applied before yielding occurs, according to (a) the maximum shear stress (b) the shear strain energy theories.

Internal pressure:

Firstly, it is desirable to establish that yielding is not occurring under the action of internal pressure alone.

$$\text{Hoop stress } \sigma_y = \frac{Pr}{t} = \frac{2000 \times 5}{0{\cdot}5} = 20{,}000 \text{ lb/in}^2.$$

$$\text{Axial stress } \sigma_x = \frac{Pr}{2t} = 10{,}000 \text{ lb/in}^2.$$

$$\text{Radial stress } \sigma_z = 0$$

(actually varies from 2000 lb/in^2 internal to zero external).

	Simple tension	*Cylinder*
Max. shear stress		
	$35{,}000/2 = 17{,}500$	$(20{,}000 - 0)/2 = 10{,}000$
Shear strain energy		
	$\frac{1}{6G}(35{,}000)^2$	$\frac{1}{6G}\{(20{,}000)^2 + (10{,}000)^2 - (20{,}000 \times 10{,}000)\}$
	$= \frac{1}{6G} \times 12{\cdot}2 \times 10^8$	$= \frac{1}{6G} \times 3 \times 10^8$

It is evident from the above figures that yielding has not occurred according to either theory.

Axial Tension

Maximum shear stress criterion. It was shown above that the hoop stress alone was not sufficient to cause yielding and, as σ_{hoop} and σ_{axial} are both tensile, the former being fixed in value and $\sigma_z \simeq 0$, therefore the maximum shear stress and yielding will be a function of the axial stress due to the addition of axial tensile force.

Therefore for yielding

$$\frac{\sigma_{\text{axial}} - 0}{2} = \frac{\sigma_Y}{2}.$$

If F is the added axial force then

$$\frac{F}{\pi \times 10 \times \frac{1}{2}} + 10{,}000 = 35{,}000,$$

$$F = \frac{25{,}000 \times 5\pi}{2240} = \mathbf{175\ ton.}$$

Shear strain energy criterion. Using the shear strain energy theory, let the total axial tensile stress due to pressure plus axial force be σ_a. Then

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_Y^2,$$

therefore

$$\sigma_a^2 + (20{,}000)^2 - 20{,}000\sigma_a = (35{,}000)^2. \qquad (3.40)$$

The solution of this quadratic equation gives the two roots

$$\sigma_a = +40{,}400 \text{ lb/in}^2 \text{ and } -20{,}400 \text{ lb/in}^2.$$

In this case the positive root is required, therefore

$$F = (40{,}400 - 10{,}000)\frac{\pi \times 10 \times \frac{1}{2}}{2240}$$

$$= \mathbf{213\ ton.}$$

Axial Compression

Maximum shear stress criterion. Since yielding does not occur due to internal pressure alone, when the axial compression is applied it will first nullify the axial tensile stress due to pressure, and will then cause axial compressive stress.

Hence maximum shear stress will be

$$\frac{(\sigma_{\text{hoop}} - \sigma_{\text{axial}})}{2} \text{ since } \sigma_z \simeq 0.$$

Therefore for yielding $\sigma_{\text{hoop}} - \sigma_{\text{axial}} = \sigma_Y = 35{,}000$. Thus

$$\sigma_{\text{axial}} = 20{,}000 - 35{,}000$$
$$= -15{,}000 \text{ lb/in}^2.$$

But

$$\sigma_{\text{axial}} = 10{,}000 + \frac{F}{5\pi},$$

therefore

$$10{,}000 + \frac{F}{5\pi} = -15{,}000,$$

$$F = \frac{25{,}000 \times 5\pi}{2240} = \mathbf{-175\ ton.}$$

Shear strain energy criterion. For yielding according to the shear strain energy criterion, the same expression applies as

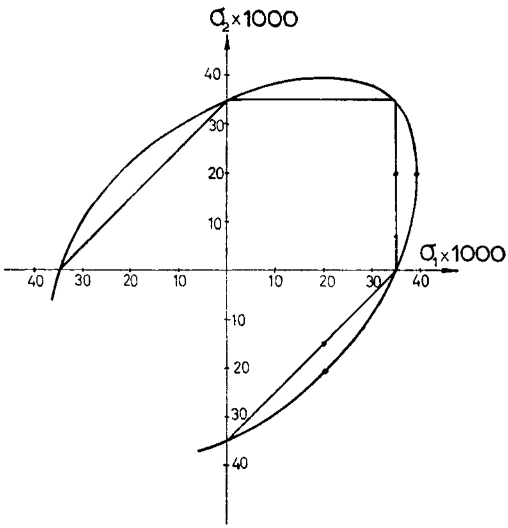


FIG. 3.19

derived above, equation (3.40), but on this occasion the negative root is used which is

$$\sigma_a = -20{,}400$$

therefore

$$\frac{F}{5\pi} + 10{,}000 = -20{,}400$$

$$F = \mathbf{-213\ ton.}$$

Thus yielding is caused by an *additional* amount of axial stress which has the same value for both tension and compression. The location on the yield locus of the stress systems discussed above is shown in Fig. 3.19.

The case of a thin cylinder subjected to internal pressure and axial load is one that has been used on a number of occasions for experimental studies of complex yield behaviour.

Examples

1. At a point in a complex stress field $\sigma_x = 4$ ton/in², $\sigma_y = 8$ ton/in² and $\tau_{xy} = -2$ ton/in². Find the normal and shear stresses on a plane at 45° to the y-axis.

2. At a point in a boiler rivet, the material of the rivet is undergoing the action of a shear stress of 5 ton/in² while resisting movement between the boiler plates and a tensile stress of 4 ton/in², due to the extension of the rivet.

Find the magnitude of the tensile stresses at the same point acting on the two planes making an angle of 80° to the axis of the rivet.

3. Show that the sum of the normal components of stress on any two planes at right angles is constant in a material subjected to a two-dimensional stress system.

At a point in a material there are normal stresses $\sigma_x = 2$ ton/in² and $\sigma_y = 4$ ton/in² tensile, together with a shear stress $\tau_{xy} = 1{\cdot}5$ ton/in². Find the value of the principal stresses and the inclination of the principal planes to the y-direction.

4. At a point in a body the principal stresses are tensile stresses of 350 and 200 lb/in² respectively. Find the resultant stress in magnitude and direction on a plane whose normal makes an angle of 30° with the normal to the plane of maximum principal stress.

5. Construct Mohr's circle for the following point stresses: $\sigma_x = 6$ ton/in²; $\sigma_y = 1$ ton/in²; $\tau_{xy} = -2$ ton/in², and hence determine the stress components and planes on which the shear stress is a maximum.

6. Explain the Mohr stress circle method for obtaining the stresses on any interface. Draw the diagram for principal stresses of 8 ton/in² (tensile) and 5 ton/in² (compr.) and calculate the magnitude and direction of the resultant stresses on planes making angles of 20° and 65° with the plane of the first principal stress. Find also the normal and tangential stresses on these planes.

7. The loads applied to a piece of material cause a shear stress of 4 ton/in², together with a normal tensile stress on a certain plane. Find the value of this tensile stress if it makes an angle of 30° with the major principal stress. What are the values of the principal stresses? (London)

8. At a point in a material the resultant stress on a plane A is 4 ton/in² inclined at 30° to the normal and on a plane B is 1 ton/in² inclined at 45° to the normal. Find the principal stresses and show the position of the two planes A and B relative to the principal planes. Use Mohr's circle only. Planes A and B are not necessarily at right angles to each other.

9. In a state of plane strain the coordinate strains are $\epsilon_x = 0{\cdot}0008$, $\epsilon_y = 0{\cdot}0001$, $\gamma_{xy} = -0{\cdot}0008$. Determine the magnitude and direction of the principal strains.

10. For coordinate strain components $\epsilon_x = -0{\cdot}0008$, $\epsilon_y = -0{\cdot}0002$ and $\gamma_{xy} = -0{\cdot}0006$, use Mohr's construction to find the planes for which the shear strain is a maximum. Also determine the normal strain related to the above planes.

11. A 60° rosette strain gauge measures strains of 0·00046, 0·0002 and −0·00016. Use Mohr's strain circle to determine the magnitude and direction of the principal strains and hence principal stresses. $E = 30 \times 10^6$ lb/in²; $v = 0{\cdot}3$.

12. The strains recorded from a rectangular rosette gauge are $\epsilon_0 = 0{\cdot}000328$, $\epsilon_{45°} = 0{\cdot}00046$ and $\epsilon_{90°} = -0{\cdot}000415$. Determine the magnitude of the principal strains and stresses, either graphically or analytically. $E = 30 \times 10^6$ lb/in²; $v = 0{\cdot}3$.

13. A rectangular block sustains stresses in three directions at right angles to each other of 5 tensile, 4 compressive and 6 tensile ton/in² respectively. Assuming the value of Poisson's ratio for this material is 1/3·5 and that the value of E is 13,000 ton/in², determine the strain in each of the three directions and the values of the volume and torsion moduli. (London)

14. Derive the formula for the bulk modulus of a material in terms of Young's modulus and Poisson's ratio. In a tensile test on a steel tube of external diameter 0·75 in. and bore 0·5 in., an axial load of 0·2 ton produced a stretch on a length of 2 in. of $1{\cdot}2 \times 10^{-4}$ in. and a lateral contraction of the outer diameter of $1{\cdot}31 \times 10^{-5}$ in. Calculate Young's modulus, Poisson's ratio, and the bulk modulus for the material. (London)

15. A rod, 12 in. long, of 2 in. diameter at the top and 1 in. at the bottom, tapering uniformly, is stretched by a gradually applied load of 8 ton. Find the work done. $E = 13{,}000$ ton/in².

16. A sliding weight of 4000 lb is dropped down a vertical rod which is suspended from the top, and is provided with a collar at the bottom end. The length of the rod is 12 ft and the diameter is 2 in. In order to reduce the shock, a helical buffer spring is placed on the collar; the spring will compress 1 in. per 1000 lb dead load. Taking account of the work done in compressing the spring and in stretching the bar, find approximately the height, measured from the top of the uncompressed spring, from which the weight must be dropped in order to produce a momentary stress of 10,000 lb/in² in the bar. Young's modulus = 30,000,000 lb/in². (London)

17. The state of stress at a point in a component is $\sigma_x = 20{,}000$ lb/in², $\sigma_y = -10{,}000$ lb/in², $\tau_{xy} = 20{,}000$ lb/in². If the yield stress of the material in simple tension is 48,000 lb/in², determine whether yielding has occurred at the point according to (*a*) maximum shear stress criterion, (*b*) shear strain energy criterion.

18. A thin-walled steel cylinder with flat ends is 3 ft internal diameter, and has a wall thickness of ½ in. Find the internal pressure to cause yielding according to the shear strain energy and maximum shear stress criteria of yielding if the yield stress of the steel in simple tension is 22 ton/in^2. Assume that the radial stress is zero.

If the cylinder is also subjected to a torque of 85,000 lb/ft (assume constant shear stress in the wall), determine the maximum allowable internal pressure according to the above two theories.

CHAPTER 4

THEORY OF BENDING

4.1. Introduction

One of the most common modes of deformation of a material due to the action of externally applied loads is bending. Joists supporting a floor, railway wagon axles, motor car leaf springs and the wings of an aeroplane are a few examples of components subjected to bending, or flexure as it is often termed. The design of elements such as these requires a knowledge of the internal forces and moments, stresses and deflections. An exact solution for all different types of problems in bending, slender and thick members, plates and shells is beyond the scope of this book. However, the problem of the slender elastic member in which the cross-sectional dimensions are small compared with the length is one which is amenable to a relatively simple treatment based on the principles of equilibrium, compatibility and load–deformation relations. The slender member, subjected to transverse loading which causes bending, is generally termed a beam and the present chapter is concerned with the analysis of forces, moments and stresses in elastic beams. The deflection of elastic beams will be discussed in the next chapter.

The first part of this chapter deals with the analysis of bending and shear stresses in beams during symmetrical bending. This is followed by the problems of unsymmetrical bending and thin-walled open sections. The final stages of the chapter are devoted to combined bending and direct stress, composite members, and curved bars.

4.2. Forms of Loading and Support

The transverse externally applied load on a beam can take one of two forms, *concentrated* or *distributed*. The former is illustrated in Fig. 4.1(*a*) in which the load acts on the surface of the beam along a line perpendicular to the longitudinal axis. This, of course, is an idealisation, and in practice a concentrated load will cover a very short length of the beam.

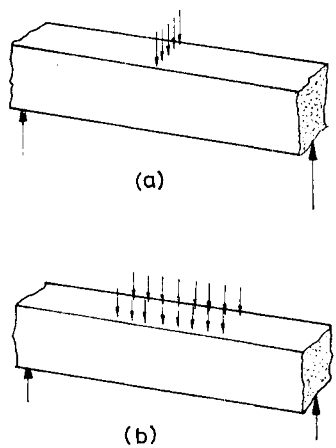
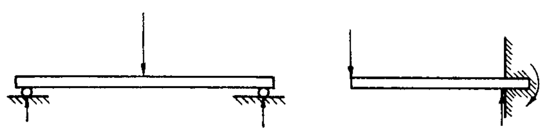


FIG. 4.1

A load which is distributed is shown in Fig. 4.1(*b*) and occupies a length of the beam surface. In future analyses the load intensity is always taken as constant across the beam thickness, but may be uniformly or non-uniformly distributed along part or the whole length of the beam. In practice the particular conditions of force and displacement at beam supports may vary considerably. Theoretical solutions of beam problems generally employ two simplified forms of support. These are termed *simply supported* and *built-in* (fixed). The former is illustrated in Fig. 4.2(*a*) in which the beam rests on knife-edges or rollers so that when the beam is bent under load the support reaction is only a transverse force and there is no restraining couple, hence the deflection is zero and the beam is free to adopt a slope at the support dictated by the applied load. The built-in support shown in Fig. 4.2(*b*) reacts with a transverse force *and* a couple, and both deflection and slope are fully restrained. The particular example illustrated, of a beam built-in at one end and free at the other, is termed a *cantilever*.

The number and type of support also has a further important bearing on a beam solution by making it either *statically determinate* or *statically indeterminate*. In the former, the support reactions can be found simply from force and moment equilibrium equations, and this applies for example to beams on two simple supports or one built-in end and no other support. The



(a) (b)

FIG. 4.2

two equilibrium equations are insufficient to find the reactions at the supports of a statically indeterminate beam owing to the presence of redundant forces.

In general if n is the number of reaction forces and moments and r is the number of redundant forces then

$$r = n - 2.$$

In order to provide equations additional to the two from equilibrium to solve for the redundancies, it is necessary to consider conditions of compatibility in terms of the displacements of the beam, as, for example, was done in the frame problem in Chapter 2, section 7(*a*).

4.3. Shear Force and Bending Moment

The first step towards the analysis of stress distribution within the beam is to consider the internal reactions caused by the external loading.

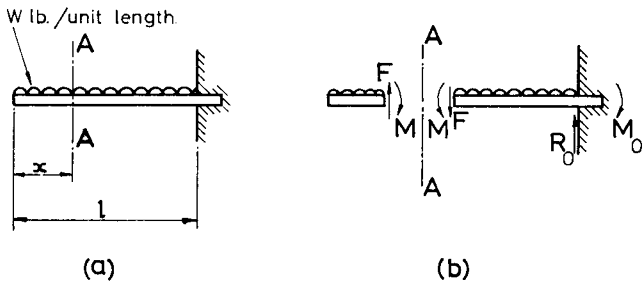


FIG. 4.3

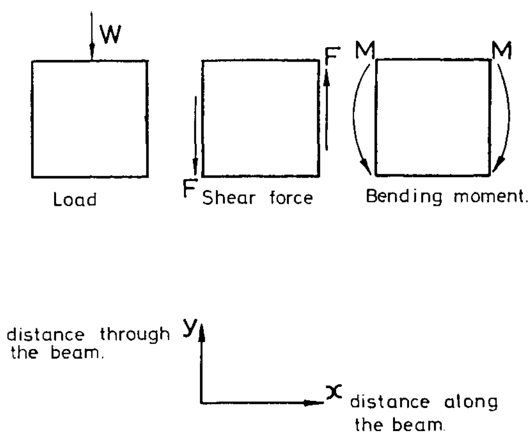


FIG. 4.4

The statically determinate cantilever shown in Fig. 4.3(*a*) is subjected to loading, *w*/unit length, uniformly distributed along the length, *l*. If the beam is imagined to be cut at an arbitrary section, *AA*, it is seen that in order to maintain equilibrium a vertical force and a moment would be required at that section as shown in Fig. 4.3(*b*). These are termed the *shear force* and *bending moment*. These internal reactions may easily be determined at any section along the beam by writing down the equations for vertical and moment equilibrium for either part of the beam.

The sign conventions† for positive forces, moments and distances measured on the beam are illustrated on Fig. 4.4.

EXAMPLE 4.1

In the cantilever beam shown in Fig. 4.3 the intensity of loading is w lb/unit length and the length of the beam is l. Then considering equilibrium of the portion of the beam to the left of section AA:

Vertical equilibrium

$$\text{Shear force} = F_{AA} = wx.$$

Moment equilibrium

$$\text{Bending moment} = M_{AA} = wx \times \frac{x}{2} = \frac{wx^2}{2}.$$

This is obtained by considering the total load on the length x, which is wx, as acting as a concentrated load at the mid-point of the length x.

At the left end of the beam $x = 0$.

Therefore $F = 0$ and $M = 0$ at the support $x = l$. Therefore

$$F = wl = R_0$$

and

$$\mathrm{M} = \frac{wl^2}{2} = M_0,$$

where R_0 and M_0 are the reactions exerted by the support on the right end of the beam.

† See appendix for alternative sign convention.

If the shear forces and bending moments are computed for a number of sections along a beam a graph can be plotted showing the distribution along the beam giving what is termed a shear force diagram and bending moment diagram.

These are illustrated in Fig. 4.5 for this problem.

The same results as above may be obtained by considering equilibrium on the right-hand part of the beam, as follows:

$$F_{AA} + w(l - x) = R_0$$

and

$$M_{AA} + \frac{w}{2}(l - x)^2 = M_0,$$

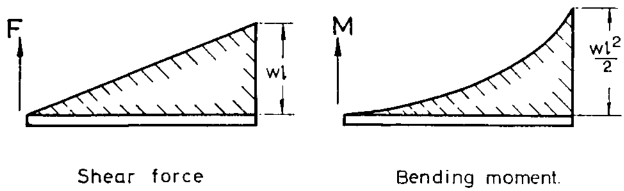


FIG. 4.5

but for equilibrium of the *whole* beam $R_0 = wl$ and $M_0 = wl^2/2$, therefore

$$F_{AA} + w(l - x) = wl,$$

or

$$F_{AA} = wx$$

and

$$M_{AA} + \frac{w}{2}(l - x)^2 = \frac{wl^2}{2}$$

or

$$M_{AA} = \frac{wl^2}{2} - \frac{w}{2}(l - x)^2.$$

EXAMPLE 4.2

The statically determinate beam shown in Fig. 4.6 carries two concentrated loads W_1 and W_2 and, because the loading is not continuous along the length of the beam, one cannot write down one quilibrium equation that applies for all cross sections. Therefore, expressions have to be written for AC, CD and DB separately. As the beam is simply supported, there are only vertical reactions R_A and R_B at the supports.

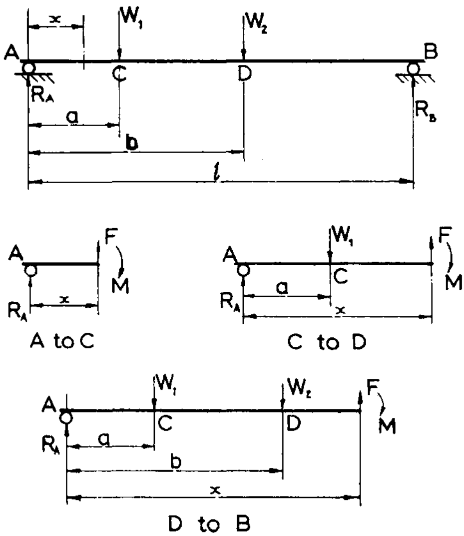


FIG. 4.6

Vertical Equilibrium

$$AC.\ 0 < x < a \qquad F = -R_A$$
$$CD.\ a < x < b \qquad F = -R_A + W_1$$
$$DB.\ b < x < l \qquad F = -R_A + W_1 + W_2.$$

Moment Equilibrium

In each case moments are taken about the right-hand end of the length x.

$$AC.\ 0 < x < a \qquad M = -R_A x$$
$$CD.\ a < x < b \qquad M = -R_A x + W_1(x - a)$$
$$DB.\ b < x < l \qquad M = -R_A x + W_1(x - a) + W_2(x - b).$$

The reactions can be found from vertical and moment equilibrium for the whole beam hence

$$R_A + R_B = W_1 + W_2$$

and

$$R_A l = W_1(l - a) + W_2(l - b)$$

therefore

$$R_A = [W_1(l - a) + W_2(l - b)]\,(1/l)$$

$$R_B = [W_1 a + W_2 b]\,(1/l)$$

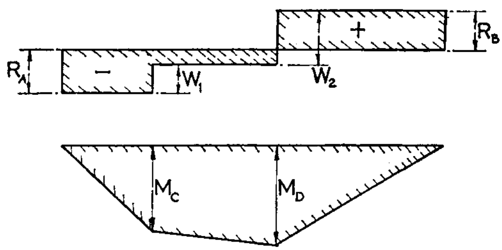


FIG. 4.7

The shear force and bending moment diagrams are shown in Fig. 4.7. A few points should be noted about these diagrams. The shear force at each end of the beams is equal to the vertical support reaction. There appears to be an abrupt change in shear force at those sections where concentrated loads are applied. This would not be the case in practice, however, since the load would not be ideally concentrated. Finally, bending moment is zero where simple supports occur.

A more detailed treatment of shear force and bending moment in statically determinate and indeterminate beams can be found in the companion book in this series, *Beams and Framed Structures* by J. Heyman (Pergamon, 1964).

4.4. Load, Shear and Moment Equilibrium Relationships

Consider a small length of beam dx supporting a distributed load of intensity, w, as shown in Fig. 4.8. The shear force and bending moment will vary from F and M at the left end to $F + (\mathrm{d}F/\mathrm{d}x) \,.\, \mathrm{d}x$ and $M + (\mathrm{d}M/\mathrm{d}x) \,.\, \mathrm{d}x$ at the right end.

Vertical Equilibrium

$$F + w\mathrm{d}x - F - \left(\frac{\mathrm{d}F}{\mathrm{d}x}\right) \mathrm{d}x = 0$$

so that

$$w = \frac{\mathrm{d}F}{\mathrm{d}x}. \tag{4.1}$$

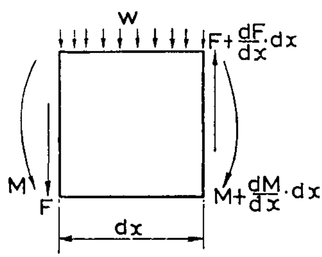


FIG. 4.8

Thus the slope of the shear force diagram is equal to the intensity of loading at any section.

It follows that between any two sections denoted by 1 and 2

$$\int_1^2 \mathrm{d}F = \int_1^2 w\mathrm{d}x$$

or

$$F_2 - F_1 = \int_1^2 w\mathrm{d}x.$$

Thus the *change* in shear force between any two cross sections may be obtained from the area under the load distribution curve between those sections.

Moment Equilibrium

Taking moments about the right-hand section

$$M + F\mathrm{d}x + w\frac{\mathrm{d}x^2}{2} - M - \frac{\mathrm{d}M}{\mathrm{d}x}\mathrm{d}x = 0.$$

Neglecting the second order term in dx gives

$$F = \frac{\mathrm{d}M}{\mathrm{d}x} \tag{4.2}$$

Thus the slope of the bending moment diagram is equal to the shear force at any section, from which

$$\int_1^2 \mathrm{d}M = \int_1^2 F\mathrm{d}x$$

therefore

$$M_2 - M_1 = \int_1^2 F\mathrm{d}x.$$

Thus the *change* in bending moment between any two sections is found from the area under the shear force diagram between those sections.

The bending moment can be related to load distribution from equations (4.1) and (4.2), thus

$$w = \frac{\mathrm{d}^2M}{\mathrm{d}x^2}. \tag{4.3}$$

4.5. Stresses and Deformations in Pure Bending

In the preceding sections, by considering equilibrium of the beam, the externally applied load has been related to the shear force and bending moment within the beam. The next step is to establish the distributions of normal and shear stress in the beam in terms of bending moment and shear force.

The fundamental principles were laid down in Chapter 2, section 1. However, those steps will be considered in a different order, since at this stage there is insufficient information about the stress distribution to enable an equilibrium equation to be formulated. The approach will be to consider a prismatic beam of symmetrical cross section subjected to a simple loading condition from which the *geometry of deformation* can be studied and strain distribution determined. The *stress–strain relations* then give the stress distribution which can be related to forces and moments through an *equilibrium condition.*

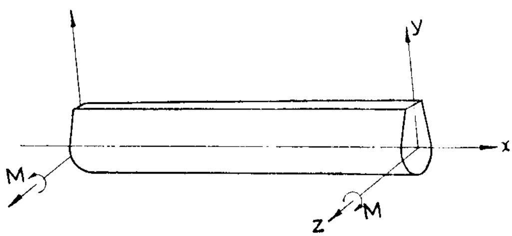


FIG. 4.9

The problem chosen for analysis is a beam of symmetrical cross section subjected to an external moment at each end causing bending in the plane of symmetry, Fig. 4.9. The beam incurs a bending moment which is constant along the length and is therefore said to be in pure bending.

Geometry of Deformation

First consider the three cross sections of the beam, AB, CD and EF, in Fig. 4.10(a) which are plane before bending occurs. Owing to the axial and transverse symmetry of the problem, these cross sections of the beam, Fig. 4.10(b), must remain plane during bending otherwise the material could not retain continuity throughout the beam.

For example, if section $C'D'$ were to bow out to the right, by symmetry section $E'F'$ would have to bow out to the left, but this is obviously impossible or there would be a gap in the material at $A'B'$.

Furthermore, since the mode of deformation must be identical for all three sections, it is implied that the beam takes up a curvature which is an arc of circle and the projection of the

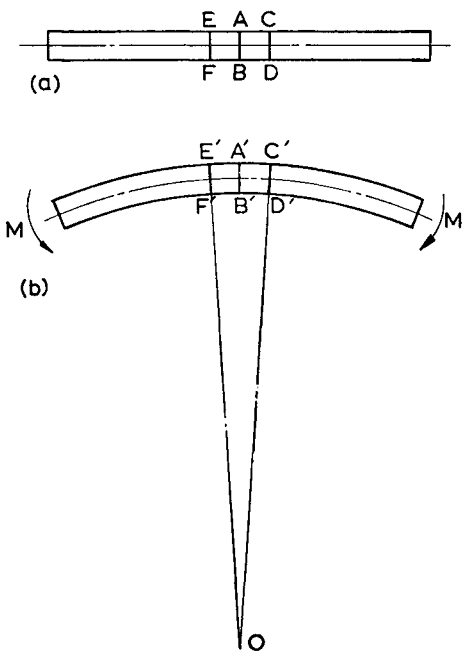


FIG. 4.10

three planes pass through a common centre, O. It is evident from Fig. 4.10(b) that the upper surface of the beam will increase in length and the lower surface will decrease in length. It follows therefore that there must be a xz-plane somewhere between the upper and lower surfaces which does not change in length. This plane is termed the *neutral surface* and a transverse axis of the beam lying in the neutral surface is termed the *neutral axis*, as illustrated in Fig. 4.11. Let us now consider the deformations between two sections AB and CD of an initially straight beam, Fig. 4.12, a distance δx apart. A longitudinal fibre PQ at a distance y above the neutral axis will have the same length as the fibre RS at the neutral axis initially. It is now assumed that all deformations are small and that during bending y is still the distance of $P'Q'$ from the neutral axis. During bending PQ stretches to become $P'Q'$ but RS being at the neutral axis is unstrained when it becomes $R'S'$.

Therefore,

$$R'S' = RS = \delta x = R\delta\theta$$

$$P'Q' = (R + y)\,\delta\theta$$

and the longitudinal strain in fibre $P'Q'$ is

$$\varepsilon_x = \frac{(P'Q' - PQ)}{PQ}$$

but

$$PQ = RS = R'S' = R\delta\theta$$

therefore

$$\varepsilon_x = \frac{\{(R + y)\delta\theta - R\delta\theta\}}{R\delta\theta}$$

or

$$\varepsilon_x = \frac{y}{R}. \tag{4.4}$$

Also $$R = \frac{\mathrm{d}x}{\mathrm{d}\theta},$$

therefore

$$\varepsilon_x = \frac{y}{R} = y\frac{\mathrm{d}\theta}{\mathrm{d}x}. \tag{4.5}$$

From the above it is seen that strain is distributed linearly across the section being zero at the neutral surface and having

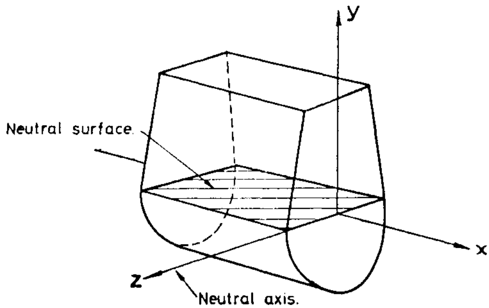


FIG. 4.11

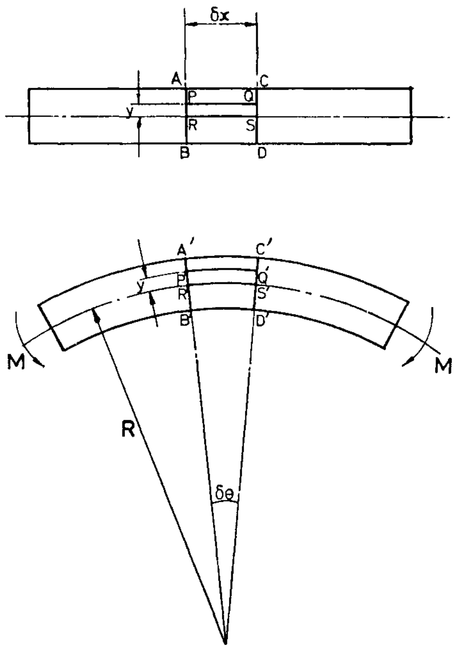


FIG. 4.12

maximum values at the outer surfaces. It is important to note here that equation (4.4) is entirely independent of the type of material, whether it is elastic or plastic and linear or non-linear in stress–strain. However, the analysis to follow is only concerned with linear elastic materials.

Stress–Strain Relationship

Using the relationship derived earlier, we can say that

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z) = \frac{y}{R}. \tag{4.6}$$

There is no information regarding the stress, or for that matter the strains, in the y and z directions. However, the only external load is acting in the xy-plane and that bending moment can be balanced entirely by axial stresses σ_x. As there is no load in the y and z directions on the external faces of the beam the normal (and shear) stresses will be zero. In view of the slenderness of the cross section compared with the length it will therefore be assumed that σ_y and σ_z are in fact zero throughout the depth and thickness. Hence equation (4.6) reduces to

$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{y}{R}$$

or

$$\frac{\sigma_x}{y} = \frac{E}{R}. \tag{4.7}$$

From the symmetry of loading and geometry it can also safely be assumed that

$$\tau_{xy} = \tau_{xz} = 0.$$

Equilibrium

In the previous section, the longitudinal bending stress was obtained in terms of the geometry of deformation and it is seen from equation (4.7) that the distribution of σ_x is linear across the section, Fig. 4.13, being zero at the neutral axis, when $y = 0$, and a maximum at the outer surfaces of the beam. The next step in the analysis is to relate the bending stress to the external applied moment. This may be done by considering the equilibrium of axial forces and moments over a cross section.

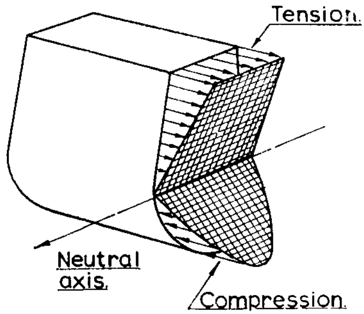


FIG. 4.13

Consider a small element of area, δA, at a distance y from the neutral surface, Fig. 4.14. The force on this element in the x direction is

$$\delta F_x = \sigma_x \delta A,$$

therefore the total axial force along the beam is

$$F_x = \int_A \sigma_x \mathrm{d}A,$$

where A is the area of the cross section.

But since there is no external axial force

$$F_x = \int_A \sigma_x \mathrm{d}A = 0.$$

Substituting for σ_x from equation (4.7)

$$\frac{E}{R} \int_A y \mathrm{d}A = 0.$$

Since E/R is not zero, the integral $\int_A y\mathrm{d}A$ must be zero, and as this is the first moment of area of the cross section about the neutral axis it is evident that the neutral axis must pass through the centroid of the cross section.

Returning to the element once more, the moment of the axial force on the element about the neutral surface is $\delta F_x y$. Therefore the total *internal resisting moment* is

$$\int_A y\mathrm{d}F_x = \int_A y\sigma_x\mathrm{d}A.$$

This must balance the external applied moment M_z, so that for equilibrium

$$\int_A y\sigma_x\mathrm{d}A = M_z.$$

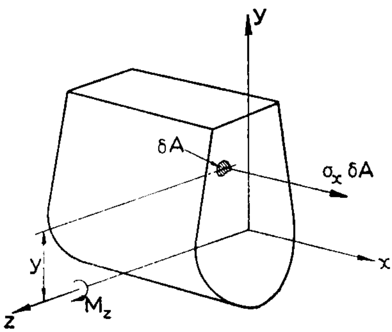


FIG. 4.14

Substituting for σ_x from equation (4.7) gives

$$M_z = \frac{E}{R}\int_A y^2\mathrm{d}A.$$

Now $\int_A y^2\mathrm{d}A$ is the *second moment of area* of the cross section about the neutral axis, zz, and will be denoted by I_z. Therefore

$$M_z = \frac{EI_z}{R}$$

or

$$\frac{M_z}{I_z} = \frac{E}{R}. \tag{4.8}$$

The fundamental relationships for pure bending of an initially straight elastic beam which relate the bending stress, bending moment, and geometry of deformation, are therefore

$$\frac{M_z}{I_z} = \frac{\sigma_x}{y} = \frac{E}{R}. \tag{4.9}$$

It is usual to find bending stresses from a knowledge of the applied load and hence bending moment rather than the radius of curvature, hence from equation (4.9) above

$$\sigma_x = \frac{M_z}{I_z} y. \tag{4.10}$$

The greatest bending stress occurs where y is greatest; if the largest value of y without regard to sign is h, then the greatest bending stress, again without regard to sign is given by

$$\sigma_{x_{\max}} = \frac{M_z h}{I_z}.$$

The quantity I_z/h is a function only of geometry and is termed the *section modulus* denoted by Z_z so that

$$Z_z = I_z/h$$

and

$$\sigma_{x_{\max}} = \frac{M_z}{Z_z}.$$

4.6. Transverse Deformation of a Beam Cross Section

Regarding deformations in the y and z directions, it is apparent that changes in length of the beam will require changes in the plane of the cross section. For example, the fibres in compression will result in an increase in the thickness of the beam owing to the Poisson's ratio effect, whereas the region which is in tension will have a decrease in beam thickness. The transverse strains will be

$$\varepsilon_y = -\frac{\nu\sigma_x}{E} \quad \text{and} \quad \varepsilon_z = -\frac{\nu\sigma_x}{E},$$

and the cross section of the beam will take up the shape shown (exaggerated) in Fig. 4.15.

The neutral surface instead of being plane will become a curved surface. This effect is known as *anticlastic curvature*. It does not upset the solution for longitudinal stresses since the deformations are so small that coordinates in the deformed cross section can be regarded as being the same as in the undeformed section.

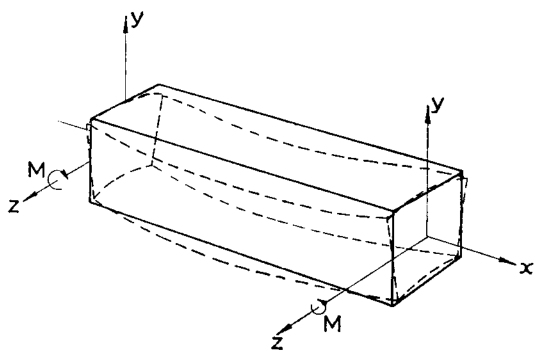


FIG. 4.15

4.7. Stresses in Symmetrical Beams Under General Loading Conditions

The solution obtained in the previous section for stresses set up by pure bending is an exact solution for cross sections whose distance from the point of application of the external moment is more than the depth of the beam.† In practice, however, it is seldom that a beam is subjected to pure bending; in most cases there will be transverse loads which will give rise to shear forces as well as bending moments (as in section 4.3 previously).

† It is generally not convenient for the external moment to be applied to the beam section in such a way as to produce the linear variation of bending stress at that section. However, it has been shown (St. Venant) that the manner in which forces and moments are applied to a material only affects the stress distribution locally.

4.7.1. Bending Stresses

If shear forces are present they are accompanied by a variation in bending moment along the beam and an exact solution for the stress distribution is no longer possible except by the *theory of elasticity* for certain special forms of load variation.

It is generally accepted and has been verified theoretically and experimentally that in such cases the linear distribution of bending stress given in equation (4.7) is quite acceptable in accuracy. The main difference from the pure bending case is that there is variation of the magnitude of bending stress along the *length* of the beam corresponding to the variation in bending moment. Equation (4.9) will therefore be assumed to hold true in such cases.

4.7.2. Shear Stresses

The presence of shear force indicates that there must be shear stresses on transverse planes in the beam. It is not possible to make use of the conditions of geometry of deformation and the stress–strain relationships, except in the development of an exact solution. However, from the assumptions about the bending stress distribution just stated it is possible to make an estimate of the transverse and longitudinal shear stress distributions in the beam, by using only the condition of equilibrium.

Firstly consider the bending stress distribution in the short section of beam of length dx shown in Fig. 4.16. Owing to the fact that bending moment increases from M_z at the left end to $M_z + (\mathrm{d}M/\mathrm{d}x) \,.\, \mathrm{d}x$ at the right end, the bending stress on any arbitrary fibre must increase from $\sigma_x = (y/I_z)M_z$ to $\sigma_x + (\mathrm{d}\sigma_x/\mathrm{d}x) \,.\, \mathrm{d}x = (y/I_z)[M_z + (\mathrm{d}M/\mathrm{d}x) \,.\, \mathrm{d}x]$ for constant y.

Now consider the strip of beam above an xz-plane, $ABCD$, Fig. 4.17, at a distance y_1 from the neutral surface. Each fibre in this strip will have an increase of bending stress dσ_x along the length dx, so that taken over the area A' there will be a resultant axial force, P_x. The only way of satisfying the condition of equilibrium of the strip is by means of a shear force F_{xy} acting on the underside of the plane $ABCD$. Therefore

$$F_{xy} = P_x = \int_{A'} \mathrm{d}\sigma_x \mathrm{d}A$$

but

$$\mathrm{d}\sigma_x = \mathrm{d}M \, . \frac{y}{I}.$$

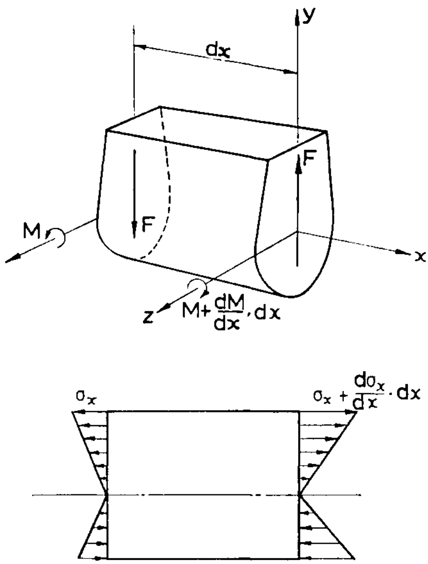


FIG. 4.16

Therefore

$$F_{xy} = \int_{A'} \left(\mathrm{d}M \, . \frac{y}{I}\right) \mathrm{d}A.$$

If the width of the strip at the plane $y = y_1$ is b and this is small compared with the depth of the beam, then the shear stress on $ABCD$ is nearly uniformly distributed and

$$\tau_{xy} = \frac{F_{xy}}{b\mathrm{d}x}$$

or

$$\tau_{xy} = \frac{dM}{dx} \cdot \frac{1}{bI} \int_{A'} y\,dA,$$

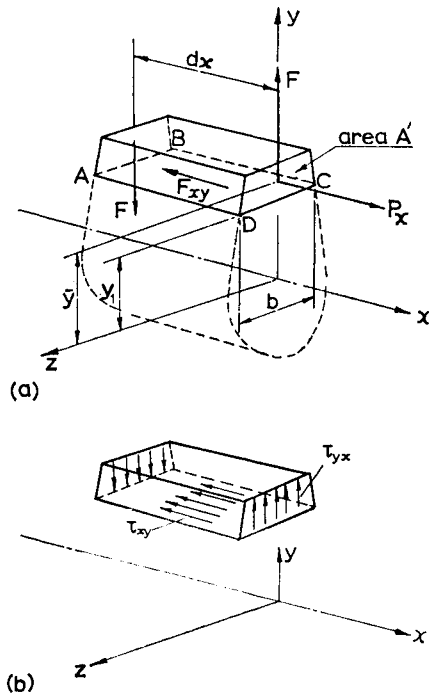


FIG. 4.17

but $dM/dx = F$, the vertical shear force on the section. Therefore

$$\tau_{xy} = \frac{F}{bI} \int_{A'} y\,dA. \tag{4.11}$$

The integral is the first moment of area of A' about the neutral surface, therefore

$$\int_{A'} y\mathrm{d}A = A'\bar{y}$$

where $\bar{y}$ is the distance of the centroid of A' from the neutral surface.

Hence

$$\tau_{xy} = \frac{FA'\bar{y}}{bI}. \tag{4.12}$$

The shear stress τ_{yx} acting vertically in the plane of the cross section has the same value as in equation (4.12) since it is the complementary shear stress.

These shear stresses are illustrated diagrammatically in the correct sense in Fig. 4.17(*b*).

The solution above was based on a *constant* shear force along the beam and this is the one case, as can be shown by the theory of elasticity, in which the engineering solution and equations (4.9) and (4.11) are exact. If the shear force varies along the beam then the above equations are no longer exact: however, for practical purposes the error obtained by using those equations is small if the beam is long compared with the cross section.

Equation (4.12) has a wider application than just as an aid to determining transverse shear stress. For example, let us suppose that the strip in Fig. 4.17(*a*) was not an integral part of a beam, but was either welded along the long edges or riveted to the beam. The net longitudinal force p_x would have to be in equilibrium with the axial force in the weld runs, or the force tending to shear the rivets, at the joint between the strip and the main part of the beam.

Let the axial force per unit length of beam be q_x, then

$$q_x = b\tau_{xy}$$

or

$$q_x = \frac{FA'\bar{y}}{I}.$$

EXAMPLE 4.3

A beam of rectangular cross section, depth d, thickness b, is simply supported over a span of length l, and carries a concentrated load W at mid-span. Determine the maximum value of the bending stress, and the distribution and maximum value of the transverse shear stress.

As the cross section is symmetrical about zz, the neutral axis and centroid occur at mid-depth and the second moment of area of the section is given by

$$I = \int_{-d/2}^{+d/2} by^2 \mathrm{d}y = \frac{bd^3}{12}.$$

The greatest value of the bending moment occurs at mid-span, Fig. 4.18, and is

$$M_{\max} = \frac{W}{2} \times \frac{l}{2} = \frac{Wl}{4}.$$

The maximum bending stress occurs at the outer fibres where $y = \pm d/2$.

Therefore using equation (4.10)

$$\sigma_{x\,\max} = M_{\max} y_{\max}/I = \pm \frac{Wl}{4} \times \frac{12}{bd^3} \times \frac{d}{2}$$
$$= \pm \frac{3}{2} \frac{Wl}{bd^2}.$$

Although changing sign at the centre of the span, the shear force is constant in magnitude along the whole span and is equal to $W/2$.

Considering the cross section shown in Fig. 4.19, the transverse shear stress on some arbitrary line AB a distance y from the neutral surface is given by equation (4.12), where A' is the shaded area above AB and $\bar{y}$ is the distance of the centroid of A' from the neutral surface. Therefore

$$A'\bar{y} = b\left(\frac{d}{2} - y\right) \times \tfrac{1}{2}\left(\frac{d}{2} + y\right) = \frac{b}{2}\left[\left(\frac{d}{2}\right)^2 - (y)^2\right].$$

The vertical shear stress is therefore

$$(\tau_{xy})_{AB} = \frac{F}{bI} \times \frac{b}{2}\left[\left(\frac{d}{2}\right)^2 - y^2\right]$$

$$= \frac{W}{4I}\left[\left(\frac{d}{2}\right)^2 - y^2\right]. \tag{4.13}$$

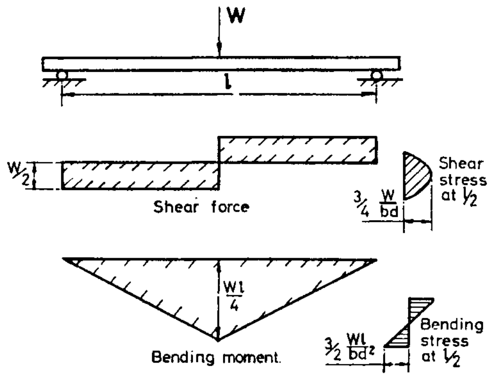


FIG. 4.18

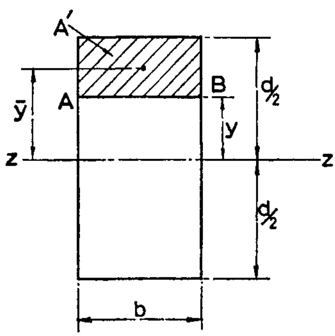


FIG. 4.19

The above expression shows that the distribution of vertical shear stress down the depth of the section is parabolic. The shear stress is zero at the outer fibres when $y = \pm d/2$, as it must be since the complementary shear stress in the longitudinal direction must be zero at a free surface. The maximum value is at the neutral surface when $y = 0$, therefore

$$\tau_{xy_{\max}} = \frac{Wd^2}{16I} = \frac{3W}{4bd}.$$

If uniformly distributed the shear stress would be given by the shear force divided by the area or

$$\tau_{xy_{\text{mean}}} = \frac{W}{2bd}.$$

Hence the maximum shear stress is 1·5 times the mean value.

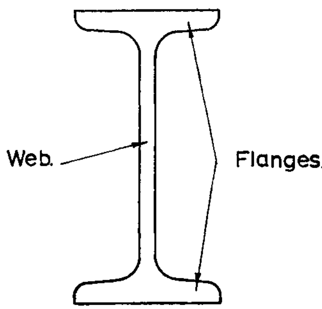


FIG. 4.20

4.8. Bending and Shear Stresses in I-section Beams

Beams having a cross section shaped in the form of an I, as in Fig. 4.20, are widely used in the construction of buildings, bridges, etc.

The I is an efficient shape to resist bending and shear. Material is concentrated, by means of the flanges, where bending stress and leverage are both high, and as will be seen the web carries almost all the shear.

As the beam section is symmetrical about the zz-axis the neutral axis lies at mid-depth of the web.

The practical I-section of Fig. 4.20 is usually idealized for simplicity of calculation into the rectangular shapes shown in Fig. 4.21. The second moment of area of the web about the neutral axis is

$$I_{\text{web}} = \frac{t_2 d^3}{12}.$$

For the flanges the parallel axis theorem is used. It states that the second moment of area about zz is equal to the second moment about the centroid of the flange plus the area of the

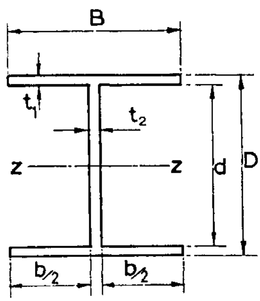


FIG. 4.21

flange times the square of the distance of its centroid from the zz-axis. Therefore

$$I_{\text{flange}} = \frac{Bt_1^3}{12} + Bt_1\left(\frac{d}{2} + \frac{t_1}{2}\right)^2.$$

Hence

$$I_{\text{total}} = \frac{t_2 d^3}{12} + 2\left\{\frac{Bt_1^3}{12} + Bt_1\left(\frac{d}{2} + \frac{t_1}{2}\right)^2\right\}.$$

In most cases the first term inside the bracket is small compared with the second term and can therefore be neglected.

An alternative way of expressing the I of the section is to consider it as a solid rectangle $B \times D$ and then subtract the seeond moment of the rectangles $b/2 \times$ d.

So that

$$I = \frac{BD^3}{12} - 2 \cdot \frac{b}{2} \cdot \frac{d^3}{12}$$

$$= \frac{BD^3 - bd^3}{12}.$$

The maximum value of y for the section is $D/2$ and from equation (4.10) the maximum bending stress is

$$\sigma_x = \frac{6MD}{(BD^3 - bd^3)}.$$

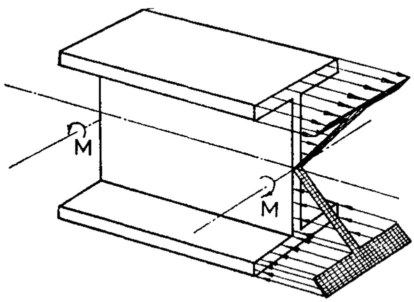


FIG. 4.22

The distribution of bending stress is illustrated in Fig. 4.22. A calculation of the bending moment taken by the flanges and by the whole section gives a ratio

$$\frac{M_{\text{flanges}}}{M_{\text{total}}} = \frac{(D^3 - d^3)B}{BD^3 - bd^3}.$$

For an I-section of proportions $D = 5$ in., $d = 4\frac{1}{2}$ in., $B = 3$ in., $b = 2\frac{3}{4}$ in.

$$\frac{M_f}{M_t} \simeq 0{\cdot}812,$$

thus 81 per cent of the total resisting moment is supplied by the flanges. While discussing the use of the flanges it should be noted that failure of an I-section is not necessarily a function of the strength of the tension flange, since care has to be exercised that elastic instability (lateral bending and twisting) does not occur in those parts of the I-section which are subjected to compression.

The shear stress distributions in an I-beam are rather more complex than in a rectangular section. Firstly consider the distribution of vertical shear stress τ_{xy} parallel to the axis yy.

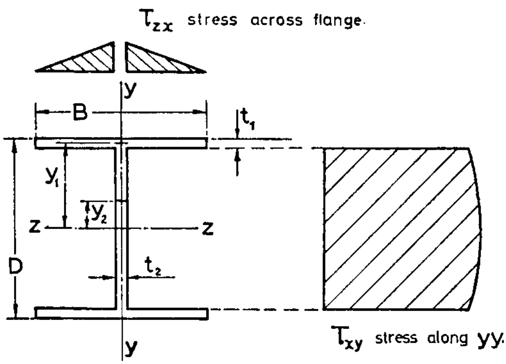


FIG. 4.23

In the web

$$\tau_{xy} = \frac{F}{t_2 I}\left[\int_{(D/2-t_1)}^{D/2} yB\mathrm{d}y + \int_{y_2}^{(D/2-t_1)} yt_2\mathrm{d}y\right]$$

$$= \frac{t_1 DBF}{2t_2 I} + \frac{F}{2I}\left[\left(\frac{D}{2} - t_1\right)^2 - y_2^2\right]. \qquad (4.14)$$

The above expression is parabolic and gives a distribution as shown in Fig. 4.23.

It would appear that where the flange joins the web and above, from equation (4.11)

$$\tau_{xy} = \frac{F}{BI}\int_{y_1}^{D/2} yB\mathrm{d}y = \frac{F}{2I}\left[\left(\frac{D}{2}\right)^2 - y_1^2\right]. \qquad (4.15)$$

However this cannot be correct, since at $y = d/2$, i.e. the underside of the flange, the longitudinal complementary shear stress must be zero at the free surface. This implies that τ_{xy} must also be zero when $y = d/2$, which is seen not to be the case in equation (4.15). Since τ_{xy} must be zero at both the upper and lower surfaces of the flange, it is evident that this shear stress contributes little towards balancing the variation of longitudinal bending stress in the flange. In fact it will be shown that it is the horizontal shear stress τ_{zx} and its complementary stress τ_{xz} which are of importance in the flanges.

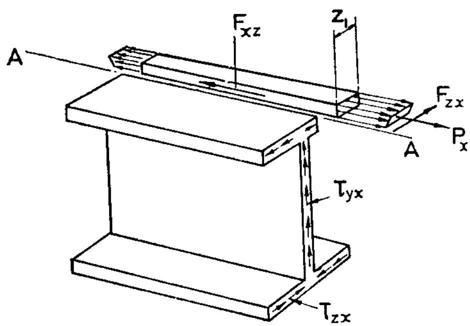


FIG. 4.24

The horizontal shear stress distribution in the flanges may be found by cutting off a portion along the line AA, Fig. 4.24. Then the net end load P_x, due to the increase in σ_x along the length must be balanced by the shear force F_{xz}, shown in the xy-plane of the flange. By a similar analysis to that used previously

$$\tau_{xz} = \tau_{zx} = \frac{F}{t_1 I} \times A\bar{y}$$

$$= \frac{F z_1 t_1 (D - t_1)}{2 t_1 I}.$$

Therefore

$$\tau_{zx} = \frac{z_1(D - t_1)}{2I} \,.\, F. \tag{4.16}$$

This is a linear distribution of shear stress in the z direction being zero at the outer edges of the flanges and maximum where they join the web. The distribution is shown diagrammatically in Fig. 4.23. The maximum value is given for $z_1 = b/2$.

4.9. Unsymmetrical Pure Bending

The previous analysis has been concerned with bending about an axis of symmetry. However, many occasions arise in practice where bending will occur either of a section which does not have any axes of symmetry or for a symmetrical section about an unsymmetrical axis.

4.9.1. Properties of Areas

In order to express the conditions of equilibrium in unsymmetrical bending a knowledge of first and second moments of area about arbitrary axes through the centroid of the section is required. It is therefore necessary to study certain properties of areas before embarking on the analysis of stress distribution in unsymmetrical bending.

Consider the section shown in Fig. 4.25 having its centroid at O and axes Oy and Oz.

The first and second moments of area of the section about Oz are respectively

$$\int_A y\mathrm{d}A = 0 \text{ and } \int_A y^2\mathrm{d}A = I_z.$$

About the axis Oy the moments are

$$\int_A z\mathrm{d}A = 0 \text{ and } \int_A z^2\mathrm{d}A = I_y.$$

If the coordinates y and z of an element of area are multiplied to give $yz\mathrm{d}A$ a quantity is obtained which has units of second moment of area, and for the whole section may be written as

$$\int_A yz\mathrm{d}A = I_{yz}.$$

This is called the *product moment of area*.

It is always possible to find two mutually perpendicular centroidal axes for a section such that $I_{yz} = 0$; such axes are then termed the *principal axes* for the section. The second moments of area about the principal axes are termed principal second moments of area, denoted by I_1 and I_2.

The principal axes can be found by considering the relationships between moments and products of area for two arbitrary sets of axes Oy, Oz and Oy', Oz' inclined to each other at an angle θ.

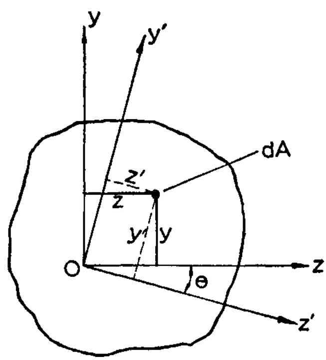


FIG. 4.25

From Fig. 4.25:

$$y' = y \cos \theta + z \sin \theta$$

and

$$z' = z \cos \theta - y \sin \theta.$$

Now

$$I_{z'} = \int_A (y \cos \theta + z \sin \theta)^2 \, dA$$

$$= \cos^2 \theta \int_A y^2 dA + 2 \sin \theta \cos \theta \int_A yz dA + \sin^2 \theta \int_A z^2 dA$$

$$= I_z \cos^2 \theta + 2I_{zy} \sin \theta \cos \theta + I_y \sin^2 \theta$$

or

$$I_{z'} = \tfrac{1}{2}(I_z + I_y) + \tfrac{1}{2}(I_z - I_y) \cos 2\theta + I_{zy} \sin 2\theta. \qquad (4.17)$$

Similarly

$$I_{y'} = \tfrac{1}{2}(I_z + I_y) - \tfrac{1}{2}(I_z - I_y)\cos 2\theta - I_{zy}\sin 2\theta. \qquad (4.18)$$

The product moment of area about Oz', Oy' is

$$I_{z'y'} = \int_A (y\cos\theta + z\sin\theta)(z\cos\theta - y\sin\theta)\mathrm{d}A,$$

which simplifies to give

$$I_{z'y'} = -\tfrac{1}{2}(I_z - I_y)\sin 2\theta + I_{zy}\cos 2\theta. \qquad (4.19)$$

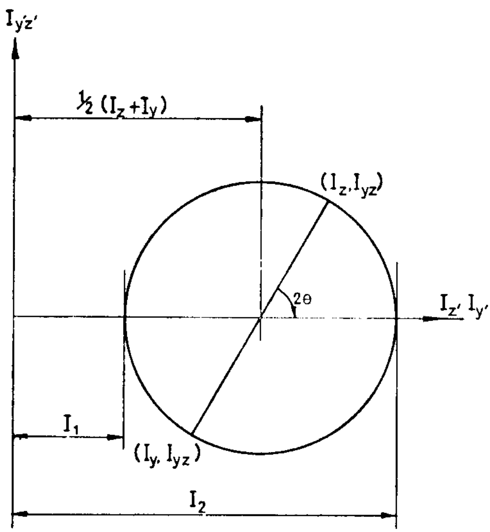


FIG. 4.26

Equations (4.17) and (4.19) are in a form which is exactly parallel to equations (3.3) and (3.4) in Chapter 3 for the normal and shear stresses on an inclined plane. This shows that a *circle construction* for second moments of area about sets of axes can be employed in a similar manner to Mohr's circle for stress and strain.

Writing equations (4.17) and (4.19) in the following form,

$$I_{z'} - \tfrac{1}{2}(I_z + I_y) = \tfrac{1}{2}(I_z - I_y)\cos 2\theta + I_{zy}\sin 2\theta,$$

$$-I_{z'y'} = \tfrac{1}{2}(I_z - I_y)\sin 2\theta - I_{zy}\cos 2\theta,$$

and then squaring and adding gives

$$[I_{z'} - \tfrac{1}{2}(I_z + I_y)]^2 + (I_{z'y'})^2 = \tfrac{1}{2}(I_z - I_y)^2 + (I_{zy})^2 \quad (4.20)$$

which is the equation for a circle relating $I_{z'}$ and $I_{z'y'}$, of radius $\{\frac{1}{2}(I_z - I_y)^2 + (I_{zy})^2\}^{1/2}$ with centre at the point $[\frac{1}{2}(I_z + I_y), 0]$.

The circle, Fig. 4.26, is constructed in a similar manner to the stress circle discussed previously.† From Fig. 4.26 it is seen that the principal second moments of area I_1 and I_2 about the principal axes are given by

$$I_1, I_2 = \tfrac{1}{2}(I_z + I_y) \pm \tfrac{1}{2}\{(I_z - I_y)^2 + 4(I_{zy})^2\}^{1/2} \quad (4.21)$$

and the inclination of the principal axes to the z, y system is expressed as

$$\tan 2\theta = \frac{2I_{zy}}{(I_z - I_y)}. \quad (4.22)$$

4.9.2. Bending Moment and Stresses

Unsymmetrical pure bending of a beam is illustrated in Fig. 4.27 for *positive* external moments M_y and M_z applied about an arbitrary set of centroidal axes. Considering first bending in the xy-plane, the equilibrium equations are:

$$\int_A \sigma_x \mathrm{d}A = 0 \text{ (neutral surface through the centroid);}$$

$$\int_A \sigma_x y \mathrm{d}A = M_z; \quad \int_A \sigma_x z \mathrm{d}A = M_y$$

† The convention for plotting in this case is that if I_{yz} is positive then the coordinates (I_z, I_{yz}) are plotted above the horizontal axis, and conversely $I_z, I_{yz})$ are plotted below the axis if I_{yz} is negative. Angles between axes on the circle are *twice* those on the element (Fig. 4.25) and are plotted in the *clockwise* sense from the z or y directions, the same as in Fig. 4.25.

but $\sigma_x = Ey/R_y$ where R_y is the radius of curvature in the xy-plane, hence

$$M_z = \frac{EI_z}{R_y} \quad \text{and} \quad M_y = \frac{EI_{yz}}{R_y}.$$

For bending in the xz-plane the equilibrium requirements are

$$\int_A \sigma_x \mathrm{d}A = 0; \quad \int_A \sigma_x z \mathrm{d}A = M_y; \quad \int_A \sigma_x y \mathrm{d}A = M_z,$$

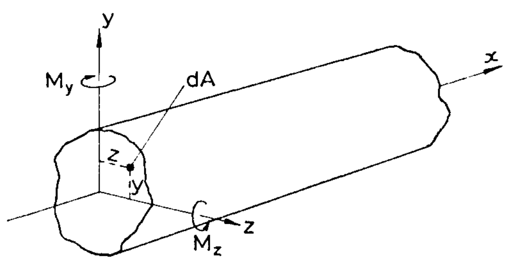


FIG. 4.27

but $\sigma_x = Ez/R_z$ where R_z is the radius of curvature in the xz-plane, hence

$$M_y = \frac{EI_y}{R_z} \quad \text{and} \quad M_z = \frac{EI_{yz}}{R_z}.$$

For bending simultaneously in the resultant of the xy- and xz-plane the separate effects above may be superposed so that

$$\left.\begin{aligned} M_y &= \frac{EI_y}{R_z} + \frac{EI_{yz}}{R_y} \\ \text{and} \qquad M_z &= \frac{EI_z}{R_y} + \frac{EI_{yz}}{R_z}. \end{aligned}\right\} \tag{4.23}$$

The radii of curvature are obtained from the above equations,

$$\frac{1}{R_y} = \frac{M_z I_y - M_y I_{yz}}{E(I_y I_z - I_{yz}^2)}$$

$$\frac{1}{R_z} = \frac{M_y I_z - M_z I_{yz}}{E(I_y I_z - I_{yz}^2)}.$$

The resultant bending stress is therefore

$$\sigma_x = \frac{yE}{R_y} + \frac{zE}{R_z}$$

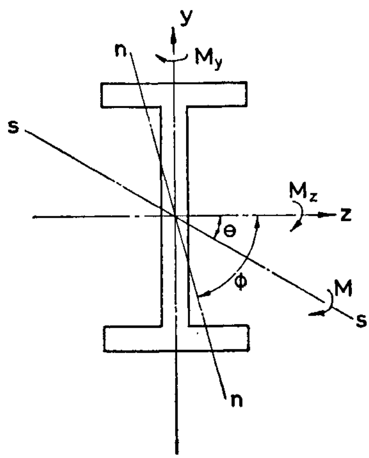


FIG. 4.28

or

$$\sigma_x = \frac{y(M_z I_y - M_y I_{yz}) + z(M_y I_z - M_z I_{yz})}{(I_y I_z - I_{yz}^2)}. \qquad (4.24)$$

If either M_z or M_y is in the opposite sense, i.e. negative to that shown in Fig. 4.27 then the appropriate signs must be changed in equation (4.24) and elsewhere.

The neutral surface, where $\sigma_x = 0$, is defined by the plane

$$y(M_z I_y - M_y I_{yz}) + z(M_y I_z - M_z I_{yz}) = 0. \qquad (4.25)$$

For principal axes, $I_{yz} = 0$, and

$$\sigma_x = \frac{M_z y}{I_z} + \frac{M_y z}{I_y}. \qquad (4.26)$$

If there had been only a single external applied moment M about an axis ss inclined at θ to one of the principal axes as in Fig. 4.28, then the moment vector M can be resolved into components M_y and M_z about the y- and z-axes so that

$$M_y = M \sin\theta \quad \text{and} \quad M_z = M \cos\theta,$$

and substituting in equation (4.26) above,

$$\sigma_x = \frac{Mz \sin\theta}{I_y} + \frac{My \cos\theta}{I_z}. \qquad (4.27)$$

The neutral plane will no longer be perpendicular to the plane of bending, as in the symmetrical problem, but the neutral plane can be determined by putting $\sigma_x = 0$ above, then

$$\frac{z \sin\theta}{I_y} + \frac{y \cos\theta}{I_z} = 0$$

and

$$\frac{y}{z} = -\frac{I_z}{I_y}\tan\theta = -\tan\phi, \qquad (4.28)$$

where ϕ is the inclination of nn, the neutral surface, to the z-axis.

The neutral surface is perpendicular to the plane of bending if either $I_z = I_y$ or $\theta = 0$.

Example 4.4

The equal angle section shown in Fig. 4.29(a) has a bending moment of 2000 lb in. applied about a centroidal axis (ss) inclined to the angle as indicated. Determine the principal second moments of area, the neutral plane and the maximum bending stress.

It is firstly necessary to locate the centroid by considering first

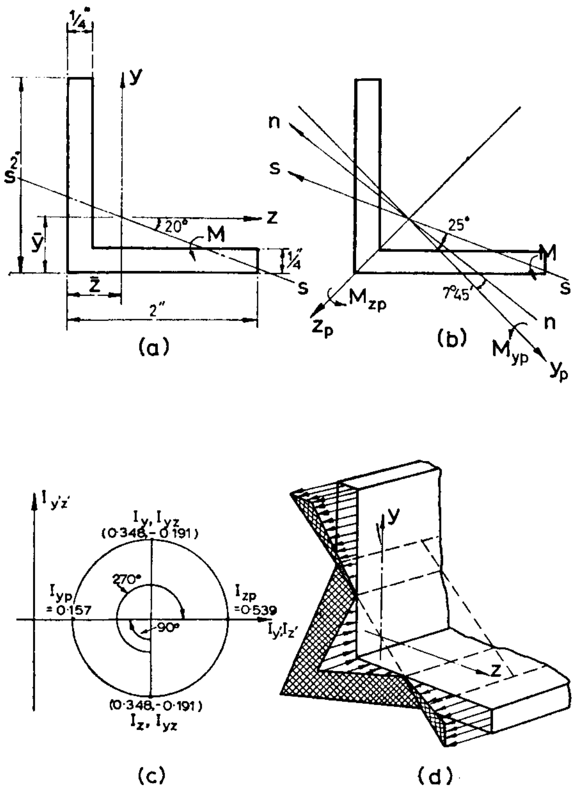


FIG. 4.29

moments of area. Let $\bar{z}$ and $\bar{y}$ be the distances of the centroid from the outer edges of the angle then,

$$(4 - \tfrac{1}{4}) \times \tfrac{1}{4}\bar{z} = \tfrac{1}{4} \times 2 \times (2/2) + (2 - \tfrac{1}{4}) \times \tfrac{1}{4} \times \tfrac{1}{8},$$

$$\bar{z} = 0{\cdot}590.$$

From symmetry $\bar{y} = 0{\cdot}59$.

Also from symmetry

$$I_y = I_z = \frac{1 \cdot 75}{12} \times (\tfrac{1}{4})^3 + 1 \cdot 75 \times \tfrac{1}{4} \times 0 \cdot 465^2 + \tfrac{1}{12} \times \tfrac{1}{4} \times 2^3$$
$$+ 2 \times \tfrac{1}{4} \times 0 \cdot 41^2$$
$$= 0 \cdot 0023 + 0 \cdot 0946 + 0 \cdot 1667 + 0 \cdot 084$$
$$= \mathbf{0 \cdot 348\ in^4}.$$

$$I_{yz} = \int_{-0\cdot 59}^{-0\cdot 34} z dz \times \int_{-0\cdot 59}^{+1\cdot 41} y dy + \int_{-0\cdot 59}^{+1\cdot 41} z dz \times \int_{-0\cdot 59}^{-0\cdot 34} y dy$$
$$= 2 \times \tfrac{1}{2} \times 0 \cdot 25 \times (-0 \cdot 93) \times \tfrac{1}{2} \times 2 \times 0 \cdot 82$$
$$= \mathbf{-0 \cdot 191\ in^4}.$$

The inclination of the principal axes is given by equation (4.22)

$$\tan 2\theta = \frac{2 \times -0 \cdot 191}{0 \cdot 348 - 0 \cdot 348} = \infty.$$

Therefore $\theta = 45°$ and $135°$, which is readily observed from the symmetry of the section.

From equation (4.21) the principal second moments of area are

$$I_{zp}, I_{yp} = \tfrac{1}{2}(0 \cdot 348 + 0 \cdot 348) \pm \tfrac{1}{2}\{0 + 4 \times (-0 \cdot 191)^2\}^{1/2}$$
$$= 0 \cdot 348 \pm 0 \cdot 191$$
$$= 0 \cdot 539 \text{ and } 0 \cdot 157.$$

Alternatively, the above information could be obtained from the circle construction shown in Fig. 4.29(*c*). It is seen that the major principal axis *zp* is 135° (270° on the circle) clockwise from the *z*-axis and the minor principal axis *yp* is 45° clockwise. The *positive* directions of z_p and y_p are shown in Fig. 4.29(*b*).

The axis of bending *ss* is inclined to the principal axis y_p at 25°. Therefore the resolved components of the bending moment are

$$-M_{yp} = M \cos 25° \quad \text{and} \quad -M_{zp} = M \sin 25°.$$

The negative signs are necessary because M_{yp} and M_{zp} are in the opposite sense to that shown in Fig. 4.27.

The neutral surface is therefore defined by

$$\sigma_x = \frac{-y_p M \sin 25°}{I_{zp}} + \frac{-z_p M \cos 25°}{I_{yp}} = 0,$$

or

$$\frac{z_p}{y_p} = -\tan 25° \times \frac{0{\cdot}157}{0{\cdot}539} = -0{\cdot}136.$$

The neutral plane *nn* lies at 7°45′ anticlockwise from the y_p-axis.

On the tensile stress side the furthest point from the neutral axis is the top right-hand corner of the vertical leg whose co-ordinates are

$$y_p = -\frac{(2-\frac{1}{4})}{\sqrt{2}} = -1{\cdot}236$$

and

$$z_p = -1{\cdot}236 + \frac{0{\cdot}34}{\cos 45} = -0{\cdot}755.$$

Therefore

$$\sigma_x = 2000 \times \left\{-1{\cdot}236 \times \frac{-\sin 25}{0{\cdot}539} - 0{\cdot}755 \times \frac{-\cos 25}{0{\cdot}157}\right\}$$

$$= \mathbf{10{,}650\ lb/in^2}.$$

The maximum compressive stress occurs at the corner of the angle whose coordinates are $y_p = 0$ and $z_p = +0{\cdot}59/\cos 45 = +0{\cdot}835$ in.

$$\sigma_x = 2000\left(0 - \frac{0{\cdot}835 \times 0{\cdot}9063}{0{\cdot}157}\right) \simeq \mathbf{-9640\ lb/in^2}.$$

The distribution of bending stress is shown diagrammatically in Fig. 4.29(*d*).

4.10. Shear Stress in Thin-walled Open Sections; Shear Centre

Beams of unsymmetrical cross section which sustain a bending moment varying along the length are subjected to shearing stresses as discussed previously for symmetrical sections. There are a number of beam sections widely used particularly in aircraft construction in which the thickness of material is small compared

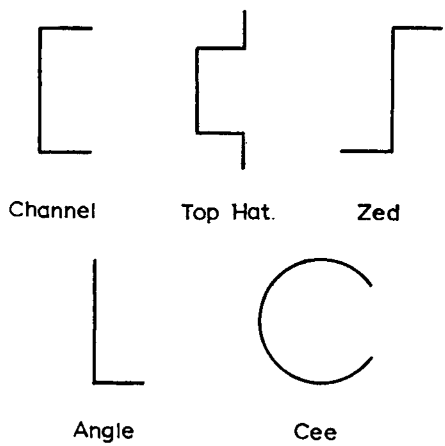


FIG. 4.30

with the overall geometry and there is only one or no axis of symmetry. These members are termed *thin-walled open* sections and some common shapes are shown in Fig. 4.30.

The arguments applied to the shear stress distribution in the flanges of the I-section (page 128) may also be applied in the above cases; however, there is one important difference owing to the lack of symmetry in the latter.

If the external applied forces, which set up bending moments and shear forces, act through the centroid of the section, then in addition to bending, *twisting* of the beam will generally occur. To avoid twisting, and cause only bending, it is necessary for the forces to act through a particular point, which may not coincide with the centroid. The position of this point is a function only of the geometry of the beam section and it is termed the *shear centre*.

4.10.1. Bending of a Channel Section without Twist

Before deriving a general theory for the bending of open-walled sections a simple example will be studied and the existence of a

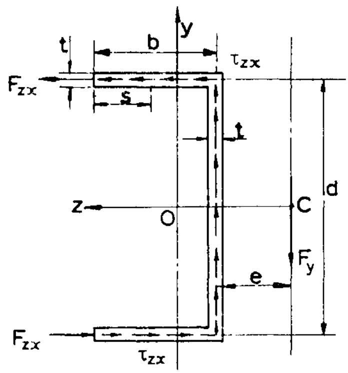


FIG. 4.31

shear centre established. Referring to Fig. 4.31 in which the channel section is loaded by a vertical force F_y, the y- and z-axes are principal axes and hence $I_{yz} = 0$. For the shear stress in the flanges the analysis is similar to that in section 4.8, equation (4.16) for the I-section.

$$\tau_{zx} = \frac{F_y}{tI_z}\int_A y\mathrm{d}A$$

and for a length s of the flange

$$\tau_{zx} = \frac{F_y}{tI_z}\int_0^s yt\mathrm{d}s = \frac{F_y}{tI_z} \times \frac{tsd}{2}.$$

The shear stress varies linearly with s from zero at the left to the maximum at the centre line of the web,

$$\max \tau_{zx} = \frac{F_y bd}{2I_z}.$$

The average shear stress is $F_y bd/4I_z$ and therefore the horizontal shear force in the top and bottom flange is

$$F_{zx} = \frac{F_y b^2 td}{4I_z}.$$

The couple about the x-axis of these shear forces is

$$F_{zx} d = \frac{F_y b^2 d^2 t}{4I_z}.$$

Let the vertical force F_y act through a point C, the shear centre, at a distance e from the middle of the web. Then twisting of the section is avoided if the moment eF_y balances the moment $F_{zx}d$ due to the horizontal shear forces. So that for equilibrium

$$eF_y = \frac{F_y b^2 d^2 t}{4I_z}$$

or

$$e = \frac{b^2 d^2 t}{4I_z} \tag{4.29}$$

which locates the position of the shear centre. The vertical shear stress τ_{xy} in the web may be found in the same way as for the I-section earlier.

4.10.2. Shear Stresses and Shear Centre for a Curved Open Section

The curved open wall section, illustrated in Fig. 4.32(a) is fixed at one end and subjected to a transverse force F through the centroid of the section at the other end. The force F can be replaced by the components F_y and F_z acting through the centroid in the coordinate directions. These forces will give rise to bending moments M_y and M_z and hence bending stresses, and also shear stresses in the cross section and along the length of the member.

It is assumed that shear stress through the wall thickness is negligible and that it is the shear parallel to the wall boundary which is of importance.

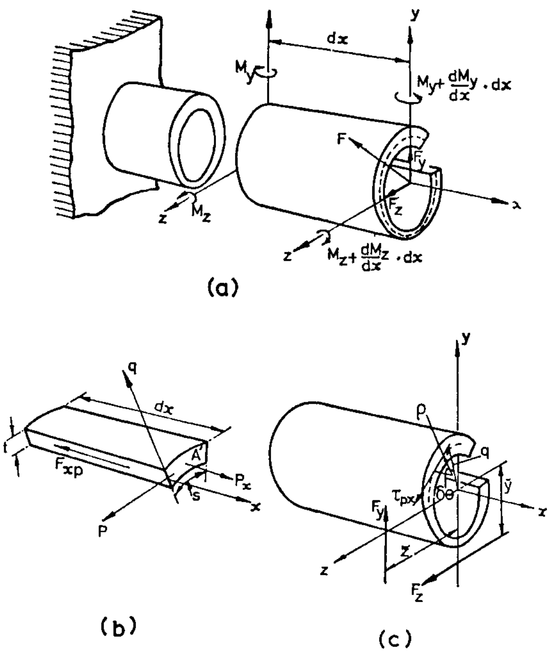


FIG. 4.32

The equilibrium condition in the x direction for the strip, Fig. 4.32(b), cut from the section subjected to bending in the xy plane alone, due to M_z, is

$$F_{xp} = P_x \int_{A'} \mathrm{d}\sigma_x \mathrm{d}A,$$

where $\mathrm{d}\sigma_x$ is the increase in bending stress along the strip and

F_{xp} is the shear force on the edge of the strip. Now from equation (4.24), putting $M_y = 0$

$$d\sigma_x = dM_z \frac{(yI_y - zI_{yz})}{(I_yI_z - I_{yz}^2)}.$$

Therefore

$$F_{xp} = \frac{dM_z}{(I_yI_z - I_{yz}^2)} \int_{A'} (yI_y - zI_{yz})dA$$

or

$$\tau_{xp} = \tau_{px} = \frac{F_{xp}}{tdx} = \frac{dM_z}{dx} \cdot \frac{1}{t(I_yI_z - I_{yz}^2)} [I_y\int_{A'} ydA - I_{yz}\int_{A'} zdA],$$

$$\tau_{px} = \frac{F_y}{t(I_yI_z - I_{yz}^2)} [I_y\int_{A'} ydA - I_{yz}\int_{A'} zdA]. \tag{4.30}$$

For bending in the xz-plane alone, due to M_y

$$\tau'_{px} = \frac{F_z}{t(I_yI_z - I_{yz}^2)} [I_z\int_{A'} zdA - I_{yz}\int_{A'} ydA]. \tag{4.31}$$

The integrals in the above expressions are the first moments of the area A' about the y- and z-axes. The resultant value of the shear stress at a particular point is given by superposition of the above equations.

The shear stress τ_{px} will give rise to a torque about the x-axis and hence twisting of the beam. It is therefore necessary for bending only and no twist to arrange that F_y and F_z act through the shear centre having coordinates $\bar{y}$ and $\bar{z}$ as in Fig. 4.32(c).

If the components F_y and F_z and hence the original force F are made to act through the shear centre, then we can find the coordinates $\bar{y}$ and $\bar{z}$ of the shear centre by considering torsional equilibrium of the section so that

$$F_y\bar{z} = \oint q\tau_{px}t\rho d\theta \tag{4.32}$$

and

$$F_z\bar{y} = \oint q\tau'_{px}t\rho d\theta, \tag{4.33}$$

where ρ is the radius of curvature of the element and q is the perpendicular distance of τ_{px} or τ'_{px} from the x-axis. (τ_{px} and

τ'_{px} are given by equations (4.30) and (4.31).) Since F_y and F_z appear on each side of the above equations and therefore vanish it is seen that the position of the shear centre is only a function of the geometry of the cross section.

4.11. Principal Stresses and Yield Criteria in Bending

Once the distributions of normal stress, σ_x, and shear stress, τ_{yx}, have been determined for a beam cross-section, equations (3.6) can be applied to determine the principal stresses and planes at any point in the beam. Remembering that σ_y is zero, the principal stresses are

$$\sigma_1, \sigma_2 = \sigma_x/2 \pm \tfrac{1}{2}(\sigma_x^2 + 4\tau_{yx}^2)^{1/2}.$$

At the outer fibres $\tau_{yx} = 0$ and $\sigma_1 = \sigma_x, \sigma_2 = 0$.

At the neutral axis $\sigma_x = 0$ and $\sigma_1 = -\sigma_2 = \tau_{yx}$, which is a state of pure shear.

There are few cases in which the maximum principal stress is greater than the maximum bending stress σ_x. It can occur where the web joins the flange in an I-section beam, however, because the bending stress is near its maximum value and there is a sharp increase in shear stress (see Fig. 4.23).

In most cases a beam will yield initially owing to local stresses at the point, or regions of load application, and plastic deformation will spread from these points. However, yield may also occur at other parts of a more complex cross section such as I or top hat, in which case for the maximum shear stress criteria

$$\sigma_1 - \sigma_2 = \sigma_Y,$$

which, in terms of the bending and shear stresses, gives

$$(\sigma_x^2 + 4\tau_{xy}^2)^{1/2} = \sigma_Y. \tag{4.34}$$

In the case of the shear strain energy criterion

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_Y^2$$

or

$$(\sigma_1 - \sigma_2)^2 + \sigma_1\sigma_2 = \sigma_Y^2,$$

which on substituting for σ_1 and σ_2 in terms of σ_x and τ_{xy} gives

$$(\sigma_x^2 + 3\tau_{xy}^2)^{1/2} = \sigma_Y. \tag{4.35}$$

4.12. Combined Bending and Direct Stress

A number of loading situations arise in service where a member is subjected to a combination of bending and end load. If the latter is compressive then the succeeding discussion only relates to short members where instability (buckling) cannot occur. (For a discussion of elastic stability see *The Stability of Structural Members* by M. R. Horne.)

Problems involving combined bending and axial load can be dealt with by superposition of the individual components of stress.

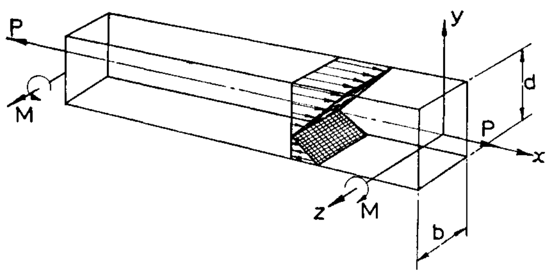


FIG. 4.33

Consider the rectangular section beam shown in Fig. 4.33 subjected to bending moments M about the z-axis, and an axial load P in the x direction.

If the end load acted alone, there would be a longitudinal stress

$$\sigma_x = \frac{P}{A}.$$

If the moment acted alone, the axial stress would be

$$\sigma_x = \pm \frac{My}{I}.$$

By superposition the resultant stress due to P and M is

$$\sigma_x = \left(\frac{P}{A}\right) \pm \left(\frac{My}{I}\right)$$

$$= \left(\frac{P}{bd}\right) \pm \left(\frac{12My}{bd^3}\right). \qquad (4.36)$$

The distribution of σ_x over the cross section is shown in Figs. 4.33 and 4.34 by the shaded portions. An interesting feature is

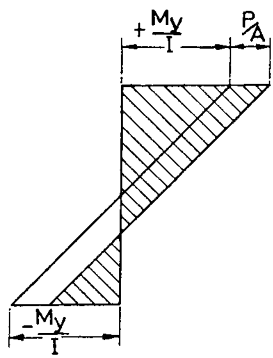


FIG. 4.34

that the neutral surface no longer passes through the centroid of the cross section since

$$\int_A \sigma_x \mathrm{d}A \neq 0,$$

as in the case of simple bending.

Note that y in equation (4.36) is a distance from the centroid of the section and *not* from the new neutral axis.

4.12.1. Eccentric End Load

The short beam illustrated in Fig. 4.35 is subjected to a compressive load P parallel to the x centroidal axis, but at an eccentricity e. The beam is in equilibrium under the equivalent system of axial loads P and couples Pe. The distribution of axial stress σ_x is therefore

$$\sigma_x = -\frac{P}{A} \pm \frac{Pey}{I},$$

which may be written as

$$\sigma_x = -\frac{P}{A}\left(1 \pm \frac{ey}{k^2}\right), \tag{4.37}$$

where k is the radius of gyration of the cross section about its centroid.

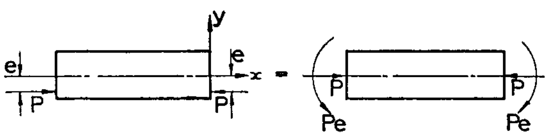


FIG. 4.35

In some materials which are strong in compression but weak in tension, such as concrete, it is necessary to limit the eccentricity (e) so that no tensile stress is set up. The condition for no tension in the cross section is that the compressive stress due to P is greater than or equal to the tensile bending stress set up by the moment Pe. Hence from equation (4.37) above

$$1 \geqslant \frac{ey_{\max}}{k^2}$$

or

$$e \leqslant \frac{k^2}{y_{\max}}.$$

For a rectangular section $k^2 = I/A = bd^3/12bd = d^2/12$, and $y_{\max} = d/2$. Therefore

$$e \leqslant \frac{d}{6}.$$

That is, for no tension in the section an end compressive load in the xy-plane must lie within a distance $d/6$ from either side of the centroid, i.e. in the *middle third* of the section. The distributions of σ_x for various amounts of eccentricity are shown in Fig. 4.36.

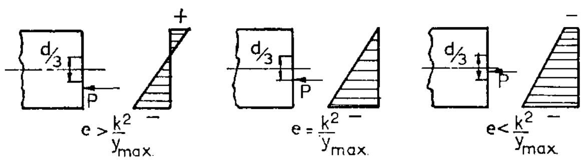


FIG. 4.36

EXAMPLE 4.5

In a tensile test within the elastic range on a specimen of circular cross section an extensometer is being used which will only measure deformation on one side of the specimen. Determine how much eccentricity of loading will give rise to a 5 per cent difference between the surface stress derived from the extensometer and the average stress over the cross section.

Let the average stress be σ, then for a 5 per cent error due to non-axial loading the resultant stress on one edge of the specimen will be $0{\cdot}95\sigma$ and at the opposite end of the diameter $1{\cdot}05\sigma$. From equation (4.37)

$$0{\cdot}95\sigma = \sigma\left(1 - \frac{ey_{\max}}{k^2}\right).$$

Therefore

$$e = \frac{0{\cdot}05\,k^2}{y_{\max}}.$$

For a circular cross section $I = Ak^2 = \pi d^4/64$ and hence $k^2 = d^2/16$.

Therefore

$$e = \frac{0{\cdot}05d^2/16}{d/2}$$
$$= 0{\cdot}00625d.$$

Thus in a tensile test on a specimen of $\frac{1}{2}$ in. diameter, the eccentricity of loading must be *less* than 0·003 in. to avoid surface stresses being more than 5 per cent greater than the average direct stress.

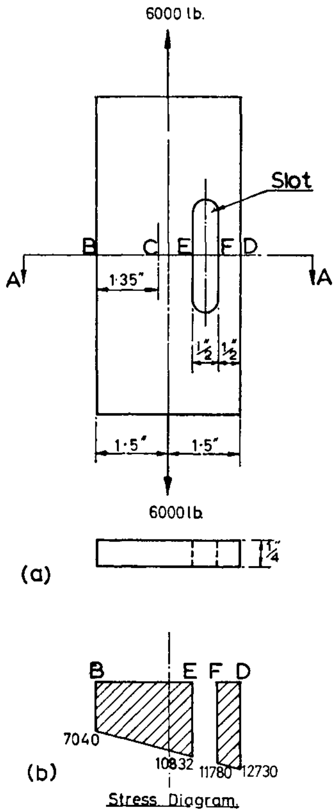


FIG. 4.37

EXAMPLE 4.6

A slotted machine link $\frac{1}{4}$ in. thick, illustrated in Fig. 4.37(*a*), is subjected to a tensile load of 6000 lb acting along the centre line of the end faces. Find the stress distribution for a section through the slot such as AA.

We must first find the centroid C of the section AA, and by taking the first moment of area about the left side

$$(2 \times \tfrac{1}{4} + \tfrac{1}{2} \times \tfrac{1}{4})\,\bar{x} = 2 \times \tfrac{1}{4} \times 1 + \tfrac{1}{2} \times \tfrac{1}{4} \times 2\tfrac{3}{4}$$

$$\bar{x} = 1{\cdot}35 \text{ in.}$$

Hence the applied load is acting eccentrically with respect to the centroid of the slot cross section AA by an amount

$$e = 1{\cdot}5 - 1{\cdot}35 = 0{\cdot}15 \text{ in.}$$

This eccentricity gives rise to a bending moment of

$$M = 6000 \times 0{\cdot}15 = 900 \text{ lb in.}$$

For *bending* only the neutral axis passes through C, and the greatest bending stresses occur at B and D, therefore $y_{max} = 1{\cdot}35$ in. and $1{\cdot}65$ in. and a calculation of the second moment of area gives $I = 0{\cdot}4746 \text{ in}^4$.

Therefore

$$\sigma_B = -\frac{900 \times 1{\cdot}35}{0{\cdot}4746} = -2560 \text{ lb/in}^2$$

and

$$\sigma_D = +\frac{900 \times 1{\cdot}65}{0{\cdot}4746} = +3130 \text{ lb/in}^2.$$

The direct stress $= \dfrac{6000}{2{\cdot}50 \times 0{\cdot}25} = +\ 9600 \text{ lb/in}^2.$

Resultant stress at B = $+9600 - 2560 = +\ 7040 \text{ lb/in}^2$.

Resultant stress at D = $+9600 + 3130 = +12{,}730 \text{ lb/in}^2$.

Resultant stress at E = $+9600 + 1232 = +10{,}832 \text{ lb/in}^2$.

Resultant stress at F = $+9600 + 2180 = +11{,}780 \text{ lb/in}^2$.

The distribution of stress is shown in Fig. 4.37(*b*).

4.13. Composite Beams

In some circumstances it may be necessary or desirable to construct a beam such that the cross section contains two different materials. Usually the object is for one material to act as a reinforcement to the other, perhaps weaker, material. There would be a number of reasons (cost, weight, size, etc.) why the whole beam could not be made from the stronger material. The positioning of the reinforcement material might not be symmetrical with respect to the centroid of the cross section and it could be embedded within, or fixed in some manner to the outside of the main bulk material.

The arguments which were applied to the analysis of simple bending of a homogeneous beam also apply to the composite beam since the two materials constrain each other to deform in the same manner, e.g. to an arc of a circle for pure bending.

The three basic conditions of Chapter 2, section 1, are used again.

(1) Equilibrium of moments and forces with bending and shear stresses.
(2) Geometry of deformation which is a linear distribution of strain over the cross section.
(3) Load–deformation relationship.

Equilibrium

Since there must be no net end load, longitudinal equilibrium gives

$$\int_{A_m} \sigma_m \mathrm{d}A_m + \int_{A_r} \sigma_r \mathrm{d}A_r = 0, \tag{4.38}$$

where subscripts m and r refer to main and reinforcing materials respectively.

Equilibrium of internal and external bending moments gives

$$M_m + M_r = M,$$

hence

$$\int_{A_m} \sigma_m y_m \mathrm{d}A_m + \int_{A_r} \sigma_r y_r \mathrm{d}A_r = M. \tag{4.39}$$

The above equations are independent of whether the material is in an elastic or plastic condition.

Geometry of Deformation

For a linear elastic strain distribution

$$\frac{1}{R} = \frac{\varepsilon_m}{y_m} = \frac{\varepsilon_r}{y_r}. \tag{4.40}$$

These relationships are purely a function of geometry and therefore are independent of the material and its properties.

Stress–Strain Relations

$$\left.\begin{aligned} \sigma_m &= E_m \varepsilon_m \\ \text{and} \qquad \sigma_r &= E_r \varepsilon_r. \end{aligned}\right\} \tag{4.41}$$

4.13.1. Reinforced Concrete Beam

Perhaps the most common example of a composite beam is the use of steel bars to reinforce concrete. The steel is always embedded in the concrete on the tension side of the beam owing to the weakness of concrete in tension, but reinforcement may also be included on the compression side to keep the overall beam section to a reasonable size.

Consider the case illustrated in Fig. 4.38 and make the conventional assumption that the concrete takes all the compression and the reinforcing bars all the tension.

All the required relationships have been derived above and it is only necessary now to solve for the unknown quantities as required.

Let the distance of the neutral axis from the outer surface in compression be h, Fig. 4.38(b), and the ratio of the elastic moduli

$$\frac{(E_{\text{steel}})}{(E_{\text{concrete}})} = m.$$

From equations (4.40) and (4.41)

$$\sigma_c = \frac{y_c E_c}{R} \quad \text{and} \quad \sigma_s = \frac{y_s E_s}{R}.$$

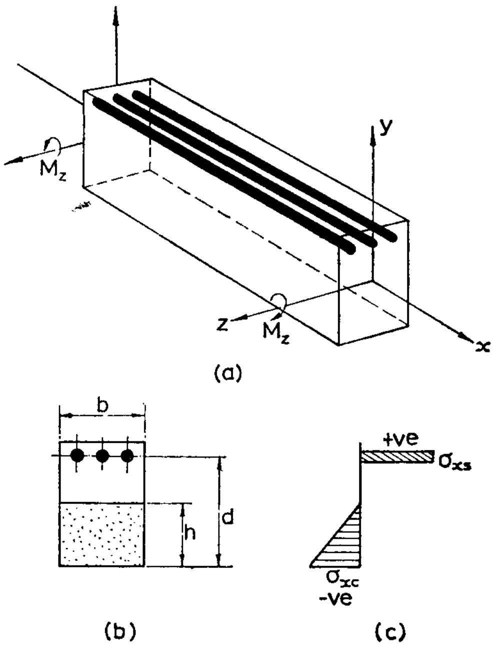


FIG. 4.38

Substituting the stresses into the equation (4.38) for longitudinal equilibrium and noting that the stresses in the steel and concrete are of opposite sign, gives

$$-\frac{E_c}{R}\int_{A_c} y_c \mathrm{d}A_c + \frac{E_s}{R}\int_{A_s} y_s \mathrm{d}A_s = 0. \tag{4.42}$$

F

For the steel reinforcement the tensile stress is considered constant over the cross-sectional area A_s and concentrated at $y = (d - h)$ as in Fig. 4.38(c), so that equation (4.42) becomes

$$-E_c\frac{bh^2}{2} + E_s(d-h)A_s = 0,$$

and thus

$$h = \left\{\left(\frac{mA_s}{b}\right)^2 + \frac{2mA_cd}{b}\right\}^{1/2} - \frac{mA_s}{b}, \qquad (4.43)$$

which gives the position of the neutral axis.

Substituting for σ_c and σ_s in equation (4.39) for equilibrium of moments

$$\frac{E_c}{R}\int_{A_c} y_c^2 \mathrm{d}A_c + \frac{E_s}{R}\int_{A_s} y_s^2 \mathrm{d}A_s = M$$

or

$$\frac{E_c}{R}\cdot\frac{bh^3}{3} + \frac{E_s}{R}A_s(d-h)^2 = M.$$

Now

$$\frac{1}{R} = \frac{\sigma_c}{y_cE_c} = \frac{\sigma_s}{y_sE_s}$$

so that substituting for $1/R$ gives

$$\left.\begin{aligned}\sigma_c &= \frac{My_c}{[\frac{1}{3}bh^3 + mA_s(d-h)^2]}\\ \sigma_s &= \frac{M(d-h)m}{[\frac{1}{3}bh^3 + mA_s(d-h)^2]}.\end{aligned}\right\} \qquad (4.44)$$

4.13.2. Reinforcing Plates and Sandwich Beams

Consider a beam cross section consisting of a central part of, say, plastic or timber with reinforcing plates firmly bonded (no sliding) to the upper and lower surfaces along the length of the beam as shown in Fig. 4.39. The section is symmetrical about the centroid and neutral surface.

From equations (4.40) and (4.41) $\sigma_m = y_m E_m/R$ and $\sigma_r = y_r E_r/R$.

Substitution in the equilibrium equation (4.39) gives

$$\left(\frac{E_m}{R}\right)\int_{A_m} y_m^2 \mathrm{d}A_m + \left(\frac{E_r}{R}\right)\int_{A_r} y_r^2 \mathrm{d}A_r = M,$$

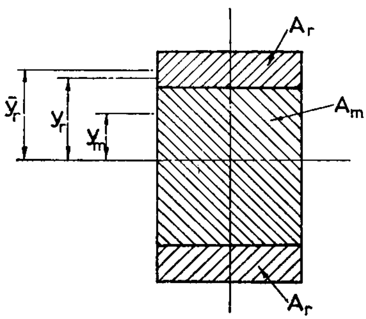


FIG. 4.39

but the integrals are the second moments of area of the core and cover plates, I_m and I_r respectively, about the neutral surface.

Therefore

$$(E_m I_m + E_r I_r)/R = M.$$

Substituting for $1/R$ gives

$$\frac{\sigma_m}{E_m y_m} = \frac{\sigma_r}{E_r y_r} = \frac{M}{(E_m I_m + E_r I_r)},$$

or

$$\left.\begin{aligned} \sigma_m &= \frac{M E_m y_m}{(E_m I_m + E_r I_r)} \\ \text{and} \qquad & \\ \sigma_r &= \frac{M E_r y_r}{(E_m I_m + E_r I_r)}. \end{aligned}\right\} \tag{4.45}$$

The longitudinal shear stress or bond stress τ_{xy} between the core and cover plates may be found by using equation (4.11) for longitudinal equilibrium. The longitudinal shear force at the bond is therefore

$$F_{xy} = \int_{A_r} d\sigma_r dA_r,$$

but from equation (4.45)

$$d\sigma_r = \frac{dM \,.\, E_r y_r}{(E_m I_m + E_r I_r)}.$$

The bond shear stress is therefore

$$\tau_{xy} = \frac{F_{xy}}{b \,.\, dx} = \frac{dM}{dx} \cdot \frac{E_r}{b(E_m I_m + E_r I_r)} \int_{A_r} y_r dA_r,$$

or

$$\tau_{xy} = \frac{F A_r \bar{y}_r}{b} \cdot \frac{E_r}{(E_m I_m + E_r I_r)} \tag{4.46}$$

where F is the vertical shear force on the section.

Example 4.7

An interesting problem which involves thermal strain, combined bending and direct stress and two different materials is the bimetallic strip frequently used to control temperature in a thermostat.

If the two materials were not bonded together then on changing their temperature each would undergo a free change in length proportional to their different coefficients of linear expansion. When bonded together each material exerts a restraint on the other since their change in length must now be the same at the junction. The axial tensile force set up in the one half will be in equilibrium with axial compressive force in the other. In addition the longitudinal forces above are acting eccentrically with respect to the centroids of each part of the section, and hence bending moments cause a curvature of the strip.

In the example shown, Fig. 4.40(*a*) and (*b*), because brass has the larger coefficient of linear expansion, the centre of curvature will be below the strip, and forces and moments as shown will be set up.

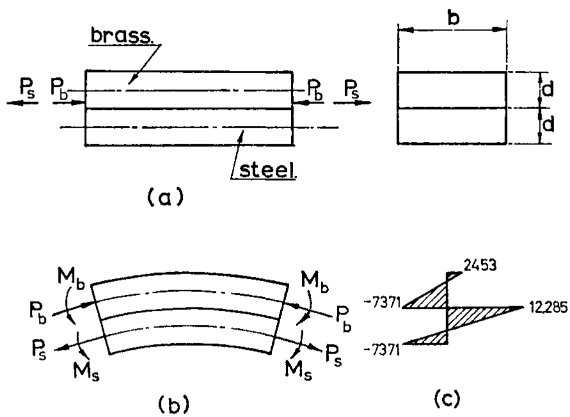


FIG. 4.40

Equilibrium

Denoting brass and steel by the subscripts b and s, then for longitudinal equilibrium

$$P_b = P_s = P, \tag{4.47}$$

and for equilibrium of moments

$$M_b + M_s = M = Pd. \tag{4.48}$$

Geometry of Deformation

For *increase* in temperature T and coefficient of linear thermal expansion α_b and α_s, the brass and steel undergo the following longitudinal strains at the common face.

	Brass	*Steel*
Thermal strain	$\alpha_b T$	$\alpha_s T$
Direct strain	$-\dfrac{P_b}{E_b bd}$	$+\dfrac{P_s}{E_s bd}$
Bending strain	$-\dfrac{d}{2R}$	$+\dfrac{d}{2R}$

(Where R is the radius of curvature of the interface.)

By superposition the total strains in each material are obtained and these must be equal at the common surface. Therefore

$$\alpha_b T - \frac{P_b}{E_b bd} - \frac{d}{2R} = \alpha_s T + \frac{P_s}{E_s bd} + \frac{d}{2R}. \tag{4.49}$$

Load–Deformation Relationships

$$M_b = \frac{E_b I_b}{R} \quad \text{and} \quad M_s = \frac{E_s I_s}{R}. \tag{4.50}$$

Putting $P_b = P_s = P$ in equation (4.49) and eliminating R, M_b and M_s using equations (4.48), (4.49) and (4.50) gives

$$(\alpha_b - \alpha_s)T - \left(\frac{P}{bd}\right)\left(\frac{1}{E_b} + \frac{1}{E_s}\right) - \frac{Pd^2}{(E_b I_b + E_s I_s)} = 0.$$

Direct stress due to P is therefore

$$\sigma = \frac{P}{bd} = (\alpha_b - \alpha_s)T \Bigg/ \left[\frac{E_b + E_s}{E_b \times E_s} + \frac{bd^3}{E_b I_b + E_s I_s}\right]. \tag{4.51}$$

For $I_b = I_s = bd^3/12$

$$\sigma = K(\alpha_b - \alpha_s)T$$

where $K = \dfrac{(E_b + E_s)E_b E_s}{(E_b + E_s)^2 + 12E_b E_s}$

Now solving for $1/R$ from equations (4.48), (4.50) and (4.51) gives

$$\frac{1}{R} = \left\{\frac{12K}{(E_b + E_s)}\right\}(\alpha_b - \alpha_s)\frac{T}{d} \tag{4.52}$$

but for bending

$$\sigma_b = \frac{E_b y}{R} \quad \text{and} \quad \sigma_s = \frac{E_s y}{R}.$$

Therefore

$$\sigma_b = \frac{12KT}{d} \frac{(\alpha_b - \alpha_s)}{E_b + E_s} E_b y \tag{4.53}$$

and

$$\sigma_s = \frac{12KT}{d} \frac{(\alpha_b - \alpha_s)}{E_b + E_s} E_s y. \tag{4.54}$$

Therefore, total stress by superposition is

$$\textit{Brass}: \quad \sigma = -KT(\alpha_b - \alpha_s) \pm \frac{12KT}{d} \frac{(\alpha_b - \alpha_s)}{E_b + E_s} E_b y$$

$$= -KT(\alpha_b - \alpha_s)\left[1 \pm \frac{12E_b y}{d(E_b + E_s)}\right]. \tag{4.55}$$

$$\textit{Steel}: \quad \sigma = KT(\alpha_b - \alpha_s)\left[1 \pm \frac{12E_s y}{d(E_b + E_s)}\right]. \tag{4.56}$$

For $\alpha_b = 20 \times 10^{-6}/°C$; $\alpha_s = 11 \times 10^{-6}/°C$; $E_b = 15 \times 10^6$ lb/in^2; $E_s = 30 \times 10^6$ lb/in^2 and $T = 100°C$ the maximum tensile and compressive stresses are found to be:

Brass: $\sigma_{max} = +\ 2453$ lb/in^2 and -7371 lb/in^2.

Steel: $\sigma_{max} = +12{,}285$ lb/in^2 and -7371 lb/in^2.

The distribution of stress is shown in Fig. 4.40(*b*).

4.14. Curved Bars

The theory of bending derived so far has been applied to initially straight bars and beams. The analysis will now be extended to cover beams which are initially curved. The geometry of curved bars has an important bearing on the bending stress distribution. If the depth of the cross section is small compared with the radius of curvature, then the stress distribution is linear as for straight beams. On the other hand if the depth of section is of the same order as the radius of curvature then a non-linear stress distribution occurs during bending.

Similar assumptions are made for curved beams as for straight beams, plane cross sections remain plane, etc., although a few of the assumptions are not strictly accurate for the case of a bar with a small radius of curvature.

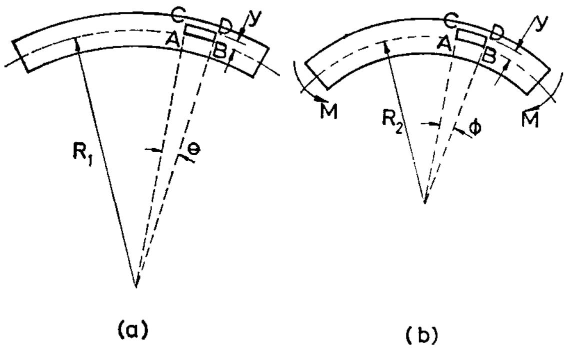


FIG. 4.41

Consider the curved bar shown unloaded in Fig. 4.41(*a*) and subjected to pure bending M (Fig. 4.41(*b*)) with initial and final radii of the neutral axis R_1 and R_2 respectively. The strain in a small element CD at a distance y from the neutral axis is derived as for the straight beam and is

$$\varepsilon_{CD} = \frac{(R_2 + y)\phi - (R_1 + y)\theta}{(R_1 + y)\theta},$$

but for an element AB at the neutral surface there is no change in length, so that $R_1\theta = R_2\phi$. Therefore

$$\varepsilon_{CD} = \frac{y(\phi - \theta)}{(R_1 + y)\theta}.$$

Making the substitution $\phi = R_1\theta/R_2$ gives

$$\varepsilon_{CD} = \frac{y\{(R_1/R_2) - 1\}}{R_1 + y} = \frac{y(R_1 - R_2)}{R_2(R_1 + y)}. \tag{4.57}$$

For the slender beam y can be neglected compared with R_1 and

$$\varepsilon = y\left(\frac{1}{R_2} - \frac{1}{R_1}\right). \tag{4.58}$$

For R_1 infinite, i.e. a straight beam, the expression reduces to that found previously.

By using the same concept as for the straight beam it could be shown that for no end load the centroidal axis and the neutral axis coincide, and for equilibrium of moments

$$\frac{M}{I} = \frac{\sigma}{y} = E\left(\frac{1}{R_2} - \frac{1}{R_1}\right).$$

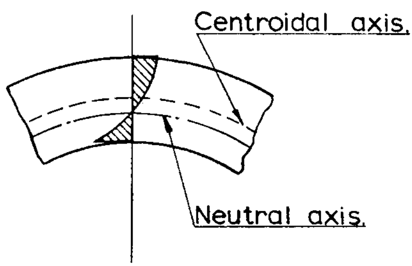


FIG. 4.42

4.14.1. Bars Having a Small Radius of Curvature

For the beam in which y is not negligible compared with R_1, the strain at distance y from the neutral axis is given by equation (4.57).

This is no longer a linear distribution of strain across the section as for the slender beam and hence the distribution of stress is non-linear, as indicated in Fig. 4.42.

F*

The resultant axial load is still zero, so that

$$\int_A \sigma \mathrm{d}A = 0.$$

But from equation (4.57)

$$\sigma = E\varepsilon = \frac{Ey(R_1 - R_2)}{R_2(R_1 + y)}. \tag{4.59}$$

Therefore

$$\frac{E(R_1 - R_2)}{R_2} \int \frac{y}{A(R_1 + y)} \mathrm{d}A = 0. \tag{4.60}$$

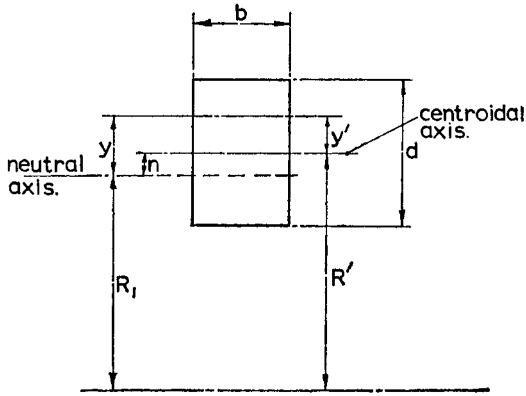


FIG. 4.43

Now the integral must be zero, but it is no longer the first moment of area about the centroid as for the slender curved or straight beam hence the *neutral axis and the centroid of the section do not coincide.*

Let the distance of the neutral axis from the centroidal axis be n, and the radius of curvature of the centroidal axis and distances measured from it be R' and y' respectively, where $y' = y - n$ and $R' = R_1 + n$, as shown in Fig. 4.43.

Put

$$\int_A \frac{y'}{R' + y'} \, dA = -mA,$$

where m is a number and A is the cross-sectional area, then

$$\int_A \frac{y'}{R' + y'} \, dA = \int_A \frac{y - n}{R_1 + y} \, dA$$

or

$$-mA = \left(1 + \frac{n}{R_1}\right) \int_A \frac{y}{R_1 + y} \, dA - \int_A \frac{n}{R_1} \, dA.$$

But

$$\int_A \frac{y}{R_1 + y} \, dA = 0$$

from equation (4.60). Therefore

$$-mA = -\frac{nA}{R_1}$$

or

$$n = mR_1. \tag{4.61}$$

For equilibrium of the internal and external moments

$$\int_A \sigma y \, dA = M$$

or

$$\frac{E(R_1 - R_2)}{R_2} \int_A \frac{y^2}{(R_1 + y)} \, dA = M. \tag{4.62}$$

The integral in equation (4.62) can be written as

$$\int_A \frac{y^2}{R_1 + y} \, dA = \int_A y \, dA - \int_A \frac{R_1 y}{R_1 + y} \, dA$$

$$= \int_A y \, dA = \int_A (y' + n) \, dA = + \int_A n \, dA.$$

Thus

$$\int_A \left(\frac{y^2}{R_1 + y}\right) \mathrm{d}A = nA$$

$$= mAR_1. \tag{4.63}$$

Therefore from equations (4.62) and (4.63)

$$\left\{\frac{E(R_1 - R_2)}{R_2}\right\} mAR_1 = M.$$

Now from equation (4.59),

$$\frac{E(R_1 - R_2)}{R_2} = \frac{\sigma(R_1 + y)}{y},$$

so that

$$\sigma = \frac{My}{mAR_1(R_1 + y)}. \tag{4.64}$$

In order to determine the magnitude of bending stress it is first necessary to determine the values of m and n for the particular shape of cross section.

Rectangular Section

For a section of width b and depth d

$$-mA = \int_A \frac{y'}{R' + y'} \mathrm{d}A = \int \mathrm{d}A - \int_{-d/2}^{+d/2} \frac{bR'\mathrm{d}y'}{R' + y'}$$

$$= bd - bR' \log_e \frac{R' + (d/2)}{R' - (d/2)}.$$

From which

$$m = \frac{R'}{d} \log_e \frac{R' + (d/2)}{R' - (d/2)} - 1.$$

The distance of the neutral axis from the centroid is

$$n = mR_1 \quad = m(R' - n)$$

$$n = \frac{mR'}{m + 1} = R' - \left\{d/\log_e\left[\frac{R' + (d/2)}{R' - (d/2)}\right]\right\}.$$

To obtain an accurate value for m or n it is necessary to have a very accurate value for

$$\log_e\left[\frac{R' + (d/2)}{R' - (d/2)}\right].$$

This can best be obtained by expressing as a series thus

$$\log_e\left[\frac{R' + (d/2)}{R' - (d/2)}\right] = \frac{d}{R'}\left\{1 + \tfrac{1}{3}\left(\frac{d}{2R'}\right)^2 + \tfrac{1}{5}\left(\frac{d}{2R'}\right)^4 + \dots\right\}.$$

So that

$$m = \tfrac{1}{3}\left(\frac{d}{2R'}\right)^2 + \tfrac{1}{5}\left(\frac{d}{2R'}\right)^4 + \dots$$

Sufficient number of terms are then taken for the required accuracy.

Taking only the first term of the series

$$m \simeq \frac{d^2}{12R'^2} \quad \text{and} \quad n \simeq \frac{d^2R'}{d^2 + 12R'^2}$$

The radius of curvature of the neutral axis is

$$R_1 = R' - n = R' - \frac{d^2R'}{d^2 + 12R'^2}$$

$$= \frac{12R'^3}{d^2 + 12R'^2}.$$

For the slender beam where d is small compared with R' the above expression gives $R_1 = R'$, i.e. the neutral and centroidal axes coincide.

EXAMPLE 4.8

Find the maximum tensile and compressive stresses at the section AA of the curved bar shown in Fig. 4.44.

Taking two terms of the series for m

$$m = \tfrac{1}{3}\left(\frac{2}{2 \times 4}\right)^2 + \tfrac{1}{5}\left(\frac{2}{2 \times 4}\right)^4 = 0{\cdot}0216$$

and for bending only

$$n = \frac{mR'}{m+1} = \frac{4 \times 0{\cdot}0216}{1{\cdot}0216} = 0{\cdot}0846.$$

The bending moment acting on AA is $M = 1000(3 + 4) = 7000$ in. lb.

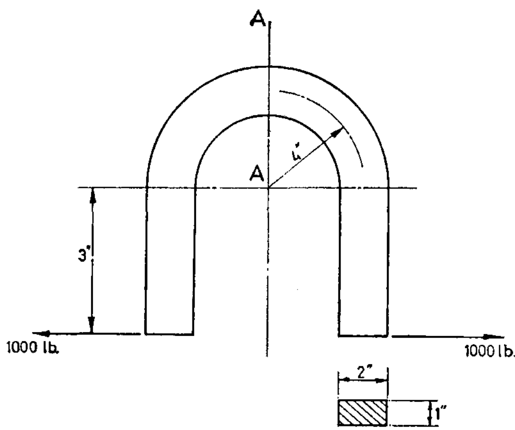


FIG. 4.44

From equation (4.64) the maximum stresses due to bending are

$$\sigma_1 = -\frac{7000 \times 1{\cdot}0846}{0{\cdot}0846 \times 2 \times 5} = -\ 9000 \text{ lb/in}^2.$$

$$\sigma_2 = -\frac{7000 \times -0{\cdot}9154}{0{\cdot}0846 \times 2 \times 3} = +\ 12{,}600 \text{ lb/in}^2.$$

The axial stress on AA due to the load is $1000/(2 \times 1) = +500$ lb/in^2.

Hence by superposition the resultant maximum stresses are

$$\sigma_1 = -\mathbf{8500\ lb/in^2} \quad \text{and} \quad \sigma_2 = +\mathbf{13{,}100\ lb/in^2}.$$

For an initially straight beam having the same cross section subjected to the same bending moment the maximum bending stresses would be $\sigma = \pm 10{,}500$ lb/in^2.

Examples

1. A balcony projects 6 ft and is supported by several beams spaced 8 ft apart. Find the greatest bending moment on each beam when the balcony is loaded with 100 lb/ft^2.

If the end of each beam is supported by a pillar which carries one-third of the whole load on the beam, what is then the greatest bending moment in each beam?

With the pillars (carrying one-third of the whole load) placed 1 ft from the end of the beam resketch the shearing force and bending moment diagrams.

2. A beam ABC is simply supported at B and C and AB is a cantilevered portion ($AB = 5$ ft; $BC = 15$ ft). The loading consists of 2 ton concentrated at A, 3 ton concentrated at D, 11 ft from C, and 4 ton concentrated at 5 ft from C. In addition, the beam carries a uniformly distributed load of 2 ton/ft over the length DC. Draw dimensioned sketches of the shearing force and bending moment diagrams. (London)

3. A beam 60 ft long is supported horizontally by vertical reactions at one end A, and a point B, 40 ft from A. There is a uniformly distributed load of 0·5 ton/ft over the whole beam, a concentrated load of 12 ton at the middle point of AB, and another concentrated load of 8 ton at the end of the overhanging portion. Draw accurately the bending moment and shearing force diagrams for the beam. (Cambridge)

4. A horizontal beam is simply supported at the ends and carries a uniformly distributed load of 1 ton/ft between the supports placed 30 ft apart. Counterclockwise moments of 40 and 32 ton ft respectively are applied to the two ends of the beam at the supports. Draw, approximately to scale, the bending moment diagram for the beam and find:

(i) the reactions at the supports,

(ii) the position and magnitude of the greatest bending moment. (London)

5. A vertical flagstaff standing 30 ft above the ground is of square section throughout, the dimensions being 6 in. × 6 in. at the ground, tapering uniformly to 3 in. × 3 in. at the top. A horizontal pull of 70 lb is applied at the top, the direction of loading being along a diagonal of the section. Calculate the maximum stress due to bending. (London)

6. In a small gantry for unloading goods from a railway wagon, it is proposed to carry the lifting tackle on a rolled steel joist, 9 in. × 4 in., of weight 21 lb/ft, supported at the ends, and of effective length 14 ft. The equivalent dead load on the joist due to the load to be raised is 3 ton, and this may act at any point of the middle 12 ft. By considering the fibre stress and the shear, examine whether the joist is suitable. The flanges are 4 in. × 0·46 in. and the web is 0·3 in. thick. The allowable fibre stress is 7·5 ton/in², and the allowable shearing stress 5 ton/in². (Cambridge)

7. A steel beam of I-section 30 ft long is supported horizontally at its extremities and carries a load of 20 ton concentrated at a point 10 ft from one end. The web is 20 in. × $\frac{5}{8}$ in. and each flange is 8 in. × 1 in. Calculate for a section 15 ft from the end: (*a*) the maximum longitudinal stress, (*b*) the maximum shearing stress in the web, (*c*) the principal stresses at the top and bottom of the web. (Cambridge)

8. The section of an open rectangular cast-iron channel for carrying water is 12 in. wide and 8 in. deep measured externally, the metal being 1 in. thick. The weight of the cast-iron and the water it holds is 125 lb per ft run. The channel is 20 ft long and is supported at its ends.

Calculate the intensity of the shearing stress in the vertical sides at a point 7 in. from the top in a section 5 ft from the end. (Cambridge)

9. A bar of rectangular cross section, 3 in. × $\frac{1}{4}$ in. thick, has a slot 1 in. long and $\frac{1}{2}$ in. deep cut into one side of the bar. When in use, a tensile load of 4 ton acts along the axis through the centres of area of the end cross sections of the bar. Determine the stress distribution for a transverse section part way along the slot. Also calculate the greatest possible depth of the slot if the maximum stress across a section as above is not to exceed 12 ton/in². Neglect any end effects at the slot. (London)

10. A 6 in. × 4 in. × $\frac{1}{2}$ in. unequal angle-bar is placed with the long leg vertical and the short leg at the top and is used as a beam supported at each end, the span being 10 ft. What vertical central load can be placed on the angle-bar in order that the maximum stress due to bending may not exceed 7 ton/in²?

11. A cantilever beam of length, L, has a ⅂-shaped section with the web vertical. The depth of section between flange centres is $2a$ and the width of each flange to the web centre line is a. The thickness of material is t throughout. Determine the maximum bending stress for a vertical load W applied at the free end along the centre line of the web.

$$I_{zz} = \tfrac{8}{3}\, ta^3, \; I_{yy} = \tfrac{2}{3}\, ta^3, \; I_{yz} = -ta^3.$$

12. A cantilevered beam is of H-section, the web being horizontal. The depth of one flange is $3a$ and the other is $2a$ and the distance between the centre lines of the flanges is $2a$. The material is of thickness t ($\ll a$) throughout. Determine the distance of the shear centre from the centre line of the larger flange for a vertically applied load W.

13. A thin-walled cylindrical tube of mean radius R and thickness t is cut longitudinally to make a semi-cylinder which is used as a cantilever beam. The load acts parallel to the cut section. Demonstrate that the shear centre is $4R/\pi$ from the cut section.

14. A reinforced concrete T-beam has a flange 60 in. wide and 4 in. deep. The reinforcement is placed in the rib 15 in. from the upper edge of the flange. The beam is designed so that the neutral axis coincides with the lower edge of the flange. The limits of stress are for steel 16,000 lb/in² and for concrete

600 lb/in². The ratio ($E_{\text{steel}}/E_{\text{concrete}}$) is 15. Calculate (*a*) the area of the reinforcement, (*b*) the moment of resistance of the beam, (*c*) the actual maximum stress in the steel and in the concrete. (London)

15. Two rectangular bars, one of brass and the other of steel, each $1\frac{1}{2}$ in. $\times$ $\frac{3}{8}$ in., are placed together to form a beam $1\frac{1}{2}$ in. wide and $\frac{3}{4}$ in. deep, on two supports 30 in. apart. The brass bar is on top of the steel.

Determine the maximum central load which can be applied to the beam if the bars are (*a*) separate and can bend independently, (*b*) firmly secured to each other throughout their length. For brass $E = 12{\cdot}5 \times 10^6$ lb/in², for steel $E = 30 \times 10^6$ lb/in².

Maximum allowable stress in brass = 10,000 lb/in², and in steel = 15,000 lb/in². (London)

16. The proportions of a proving ring for calibrating a testing machine of 50 ton capacity are to be determined. The ratio of the mean diameter to the depth of section of the ring is to be 30 : 1 and its width 3 in. If a bending stress of 20,000 lb/in² is not to be exceeded at full load, determine the relevant proportions of the ring.

17. A chain link is made of $\frac{1}{2}$ in. round steel and is semicircular at each end, the mean diameter of which is $1\frac{1}{4}$ in. The straight sides of the link are each $\frac{3}{4}$ in. long. If the link carries a load of $\frac{1}{2}$ ton, estimate the greatest tensile and compressive stresses in the link.

CHAPTER 5

DEFLECTIONS DUE TO BENDING

5.1. Introduction

The stresses set up in a beam during bending are not necessarily the critical and only factor to be considered in design. In a number of cases as, for example, the problems mentioned on page 27, in addition to keeping a stress criterion within the elastic range it is also necessary to limit the deflections during bending.

The present chapter is concerned therefore with the determination of deflections of straight and curved beams due to bending and shearing action. The analyses only relate to cases of bending about principal axes of the section.

5.2. Curvature, Slope and Deflection Relationships

For long slender beams the contribution to the deflection made by shearing force is almost negligible compared with the deflection due to bending moment.

In order to determine slope and deflection due to bending only, it is necessary to find the *shape* of the elastic curve of the neutral axis. This curve is dependent on the bending moment distribution along the length.

It was shown in the previous chapter that for an initially straight beam subjected to pure bending, the neutral axis bends into an arc of circle of radius R, and the applied bending moment is related to the curvature as

$$\frac{M_z}{EI_z} = \frac{1}{R}. \tag{5.1}$$

For a small length of beam δs subtending an angle $\delta\theta$ at the centre of curvature, Fig. 5.1, the positive curvature for a hogging bending moment is given by

$$\frac{1}{R} = \frac{\delta\theta}{\delta s},$$

and in the limit

$$= \frac{d\theta}{ds}. \tag{5.2}$$

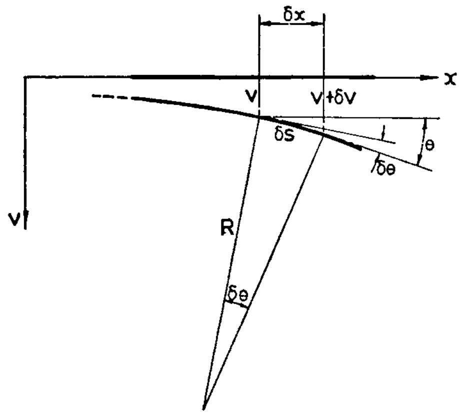


FIG. 5.1

The projection of the element on the x-axis is of length δx (Fig. 5.1) and for *small deflections* of an initially straight beam $\delta s = \delta x$.

If displacements of the neutral axis from the original unloaded position of the beam are v measured perpendicular to the x-axis and *positive downwards*,† then the deflections of each end of the element shown in Fig. 5.1 are v and $v + \delta v$. Again for small displacements

$$\tan\theta = \frac{dv}{dx}.$$

† See appendix for alternative sign convention.

Differentiating with respect to s

$$\sec^2\theta\frac{d\theta}{ds} = \frac{d}{ds}\left(\frac{dv}{dx}\right) = \frac{d^2v}{dx^2}\cdot\frac{dx}{ds}$$

$$= \frac{d^2v}{dx^2}\cos\theta,$$

or

$$\sec^3\theta\frac{d\theta}{ds} = \frac{d^2v}{dx^2}.$$

Therefore

$$\frac{1}{R} = \frac{d\theta}{ds} = \frac{d^2v}{dx^2}\Big/ \pm\{1+\tan^2\theta\}^{3/2}$$

so that

$$\frac{1}{R} = \frac{d^2v}{dx^2}\Big/ \pm\left\{1+\left(\frac{dv}{dx}\right)^2\right\}^{3/2}.$$

For small displacements $(dv/dx)^2$ can be neglected with respect to unity in the bracket and

$$\frac{1}{R} = \pm\frac{d^2v}{dx^2}$$

$$= +\frac{d^2v}{dx^2}, \tag{5.3}$$

the positive sign being the correct one to take for the sign convention used.

Combining equations (5.1) and (5.3) and omitting the z subscript, as problems are only being considered for bending in the xy-plane, gives

$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$

or

$$EI\frac{d^2v}{dx^2} = M. \tag{5.4}$$

This is a most important expression relating the curvature with bending moment. The slope, dv/dx, and the deflection v of the beam at any point can be derived from the above expression by integration, once M has been expressed as a function of x. It is occasionally useful to remember that slope and deflection are also related through bending moment to shear force and load distribution.

The distribution of bending moment along a beam in terms of the loading and x was discussed in the previous chapter and since the flexural rigidity EI is constant for most beams, the solution of the differential equation is relatively simple, thus

$$\frac{d^2v}{dx^2} = \frac{M}{EI},$$

$$\text{slope} = \frac{dv}{dx} = \frac{1}{EI}\int_0^l M dx + A$$

$$\text{and deflection} = v = \frac{1}{EI}\int_0^l\int_0^l M dx dx + [Ax]_0^l + B.$$

The constants of integration A and B can be obtained from the *boundary conditions*, that is, the known conditions of slope and deflection at the supports.

If the expression for bending moment cannot be integrated mathematically then recourse must be made to graphical integration. It was found in the previous chapter that shear force could be obtained from the area under the load curve and bending moment from the area under the shear force curve. Likewise here the slope and deflection are found from the areas under the bending moment and slope curves respectively.

EXAMPLE 5.1

A uniform simply supported beam of flexural rigidity EI and length l carries between the supports a uniformly distributed load w per unit length. Determine the shape of the deflection curve and the maximum value of the deflection.

The bending moment distribution as shown in Fig. 5.2 is

$$M = -\frac{wlx}{2} + \frac{wx^2}{2}.$$

Therefore

$$EI\frac{d^2v}{dx^2} = -\frac{wlx}{2} + \frac{wx^2}{2}.$$

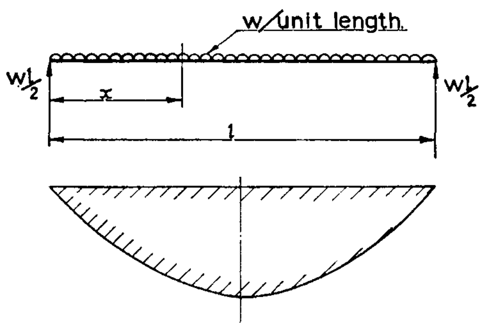


FIG. 5.2

Integrating both sides twice in succession gives

$$EI\frac{dv}{dx} = -\frac{wlx^2}{4} + \frac{wx^3}{6} + A$$

and

$$EIv = -\frac{wlx^3}{12} + \frac{wx^4}{24} + Ax + B.$$

Boundary Conditions

When $x = 0,\ v = 0$
and when $x = l,\ v = 0,$

from which $B = 0$ and $A = \frac{wl^3}{12} - \frac{wl^3}{24} = \frac{wl^3}{24}.$

The deflection curve is therefore

$$v = \left(-\frac{wlx^3}{12} + \frac{wx^4}{24} + \frac{wl^3x}{24}\right)\frac{l}{EI}. \tag{5.5}$$

For the maximum deflection

$$v = v_{max} \text{ when } x = l/2 \text{ (by symmetry).}$$

(In general $v = v_{max}$ when $dv/dx = 0$.)

Therefore

$$v_{max} = \frac{5wl^4}{384EI}. \tag{5.6}$$

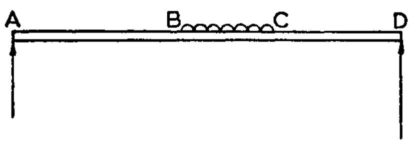


FIG. 5.3

5.3. Discontinuous Loading: Macaulay's Method

It was seen in the previous chapter in example 4.2, when considering the bending moment distribution for a beam with discontinuous loading, that a bending moment expression has to be written for each part of the beam. This means that in deriving slope and deflection a double integration would have to be performed on each bending moment expression and two constants would result for *each* section of the beam. A further example of discontinuous loading is shown in Fig. 5.3 and there would be three bending moment equations and thus six constants of integration in this case. There are apparently only two boundary conditions, those of zero deflection at each end. However, at the points of discontinuity, B and C, both slope and deflection must be continuous from one section to the next, so that:

$$\text{At } B \qquad \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)_{AB} = \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)_{BC} \quad \text{and } v_{AB} = v_{BC}$$

$$\text{and at } C \qquad \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)_{BC} = \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)_{CD} \quad \text{and } v_{BC} = v_{CD}.$$

The above four plus the two end conditions enable the six constants of integration to be determined. The derivation of the

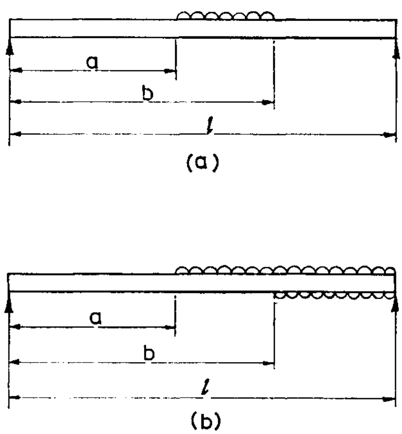


FIG. 5.4

deflection curve by the above approach is rather tedious; it is therefore an advantage to use the mathematical technique known as a *step function*, commonly known as Macaulay's method† when applied to beam solutions. This approach requires only *one* expression to be written down to cover the bending moment conditions for the whole length of beam and hence, on integration, only two unknown constants have to be determined.

† W. H. Macaulay, Note on the Deflection of Beams, *Messenger of Mathematics*, **48**, 129–130, 1919.

Briefly, the step function is a function of x of the form $f_n(x) = [x - a]^n$ such that for $x < a$, $f_n(x) = 0$ and for $x > a$ $f_n(x) = (x - a)^n$. *Note*: the change in the form of brackets used: the square brackets are particularly chosen to indicate the use of a step function, the curved brackets representing normal mathematical procedure. The important features when using the step function in analysis are that if on substitution of a value for x the quantity *inside* the square bracket becomes negative it

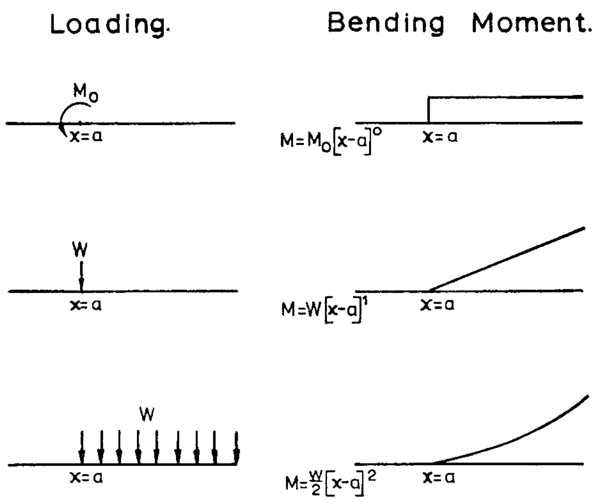


FIG. 5.5

is omitted from further analysis. Square bracket terms must be integrated in such a way as to preserve the identity of the bracket, i.e.

$$\int [x - a]^2 \, d(x - a) = (\tfrac{1}{3})[x - a]^3.$$

Finally for mathematical continuity distributed loading, as in Fig. 5.4(*a*), must be *arranged* to continue to $x = l$ whether starting from $x = 0$ or $x = a$. This may be effected by the superposition of loadings which cancel each other out in the required portions of the beam as shown in Fig. 5.4(*b*).

The three common step functions for bending moment are shown in Fig. 5.5.

EXAMPLE 5.2

Find the deflection curve and the maximum deflection for the beam shown in Fig. 5.6 for which the flexural rigidity is 2000×10^6 lb in^2.

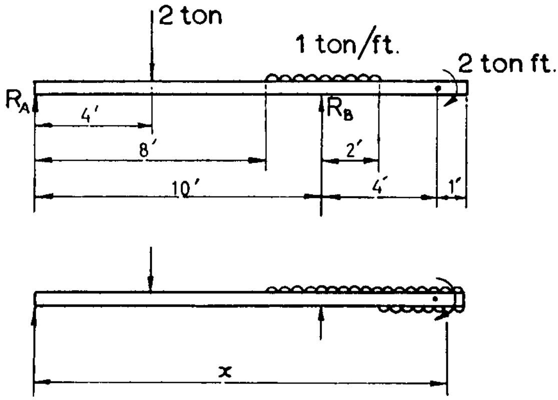


FIG. 5.6

As the problem is statically determinate, the reactions can first be found from equilibrium of forces and moments.

Thus $$R_A + R_B = 2 + 1 \times 4 = 6 \text{ ton}$$

and $$2 + 4 \times 10 + 2 \times 4 - 10R_B = 0,$$

whence $R_B = 5$ ton and $R_A = 1$ ton.

To apply the step function solution it is necessary to extend the distributed load to the right-hand end and balance out with an equal upward load beneath the beam (Fig. 5.4(*b*)). Then the general expression for bending moment for a section at the right-hand end of the beam is

$$M = EI\frac{d^2v}{dx^2} = -1 \times x + 2[x-4]^1 + (\tfrac{1}{2}) \times 1 \times [x-8]^2$$
$$-5[x-10]^1 - (\tfrac{1}{2}) \times 1 \times [x-12]^2 - 2[x-14]^0 \text{ ton ft.} \tag{5.7}$$

Therefore

$$EI\frac{dv}{dx} = -\frac{x^2}{2} + \tfrac{2}{2}[x-4]^2 + \tfrac{1}{6}[x-8]^3 - \tfrac{5}{2}[x-10]^2$$
$$- \tfrac{1}{6}[x-12]^3 - 2[x-14]^1 + A \text{ ton ft}^2. \tag{5.8}$$

$$EIv = -\frac{x^3}{6} + \tfrac{2}{6}[x-4]^3 + \tfrac{1}{24}[x-8]^4 - \tfrac{5}{6}[x-10]^3$$
$$- \tfrac{1}{24}[x-12]^4 + \tfrac{2}{2}[x-14]^2 + Ax + B \text{ ton ft}^3. \tag{5.9}$$

Boundary Conditions

When $x = 0$, $v = 0$

and when $x = 10$, $v = 0$.

The first condition makes all the terms zero or negative hence $B = 0$.

For the second condition

$0 = -\dfrac{10^3}{6} + \tfrac{2}{6}(10-4)^3 + \tfrac{1}{24}(10-8)^4 + 10A$ in which the terms that have been omitted are, $\tfrac{5}{6}(10-10)^3$, $\tfrac{1}{24}[10-12]^4$ and $\tfrac{2}{2}[10-14]^2$ because they are zero or negative inside the brackets.

Hence $A = 9{\cdot}4$ ton ft^2.

The deflection curve is therefore

$$v = \frac{1}{EI}\left\{-\frac{x^3}{6} + \tfrac{1}{3}[x-4]^3 + \tfrac{1}{24}[x-8]^4 - \tfrac{5}{6}[x-10]^3\right.$$
$$\left. - \tfrac{1}{24}[x-12]^4 - [x-14]^2 + 9{\cdot}4x\right\}.$$

To find the maximum deflection in this problem it would be incorrect to put $dv/dx = 0$ in equation (5.8), since that is a general expression for which some of the terms might not apply, depending on where the true position of maximum deflection occurred. Inspection and trial and error may be used on the various sections of the beam until a correct solution is obtained. Alternatively, if dv/dx is found at the end of each segment, then the maximum deflection occurs in the segment for which the slope is positive at one end and negative at the other end. In this problem a correct position of zero slope is found to occur in the second section where $x = 4{\cdot}37$ ft. However, this might not be the position of maximum deflection, which could occur at the right-hand end, where of course the slope would not be zero.

When $x = 4{\cdot}37$ ft

$$v = \frac{1}{EI}\left\{-\frac{4{\cdot}37^3}{6} + \tfrac{1}{3} \times 0{\cdot}37^3 + 9{\cdot}4 \times 4{\cdot}37\right\}$$

$$= \frac{27{\cdot}2}{EI}.$$

When $x = 15$ ft

$$v = \frac{1}{EI}\left\{-\frac{15^3}{6} + \tfrac{1}{3} \times 11^3 + \tfrac{1}{24} \times 7^4 - \tfrac{5}{6} \times 5^3 \right.$$

$$\left. - \tfrac{1}{24} \times 3^4 - 1^2 + 9{\cdot}4 \times 15\right\}$$

$$= \frac{13{\cdot}6}{EI}.$$

Therefore

$$v_{\max} = \frac{27{\cdot}2 \times 2240 \times 1728}{2000 \times 10^6}\ \text{in.}$$

$$= 0{\cdot}0527\ \text{in.}$$

Problems involving complex conditions of loading as above can frequently be solved more easily by using the principle of superposition. The deflection at any point is determined for each of the load components separately and the resultant deflection due to the total load is then found by summation of the increments of deflection.

Statically indeterminate beam problems can be solved for slope and deflection by exactly the same principles as for statically determinate beams except that the support reactions cannot be found at the start of the solution by consideration of statics. The reactions have to be left as symbols until the slope and deflection equations are obtained. However, there are *always* sufficient known boundary conditions at that stage to solve for the two constants of integration and the unknown redundancies.

5.4. Deflections by Geometry of Deformation

An alternative method of obtaining deflections due to bending, other than by graphical solutions, is to consider the geometry of the deformed beam.

If one end of a length x of a beam rotates through an angle $\mathrm{d}\phi$ with respect to the other end, then the deflection of one end relative to the other is

$$v = x\mathrm{d}\phi,$$

but

$$\mathrm{d}\phi = \left(\frac{M}{EI}\right)\mathrm{d}x.$$

Therefore

$$v = \left(\frac{M}{EI}\right)x\mathrm{d}x$$

and for a beam of length, l, the deflection is

$$v = \int_0^l \left(\frac{M}{EI}\right)x\mathrm{d}x.$$

In the case of a cantilever with a concentrated load W at the free end

$$M = Wx.$$

Hence the deflection under the load is

$$v = \int_0^l \left(\frac{Wx}{EI}\right) x\mathrm{d}x$$

$$= \frac{Wl^3}{3EI}.$$

For a simply supported beam carrying a distributed load w per unit length

$$M = -\left(\frac{wlx}{2}\right) + \left(\frac{wx^2}{2}\right)$$

and the deflection at the centre is

$$v = \int_0^{l/2} \left(-\frac{wlx^2}{2} + \frac{wx^3}{2}\right) \mathrm{d}x = -\frac{5}{384}\frac{wl^4}{EI}.$$

The negative sign occurs simply because the end of the beam, from which x is measured, has an upward " deflection " relative to the centre.

For a more detailed treatment of bending deflections the reader is referred to *Beams and Framed Structures* by Jacques Heyman.

5.5. Deflection of Beams Due to Shear Force

The previous sections have only considered the deflections caused by bending moments, but if shearing force is also present then the distortions of the beam due to shear stresses will also contribute to the total deflection. However, it will now be shown as stated earlier that the proportion of the deflection due to shear is very small compared with that due to bending when the cross section is small in relation to the length.

In simple bending theory it was assumed that plane cross sections remain plane, but this is not strictly true if shearing distortions occur, and cross sections take up the shape shown in

Fig. 5.7. The element of the beam at the neutral axis remains perpendicular to that axis and sliding occurs of one section over the next. The slope of the shear deflection curve *alone* can then be expressed in terms of the shear strain at the neutral axis so that, in the limit

$$\frac{dv_s}{dx} = \gamma_o, \tag{5.10}$$

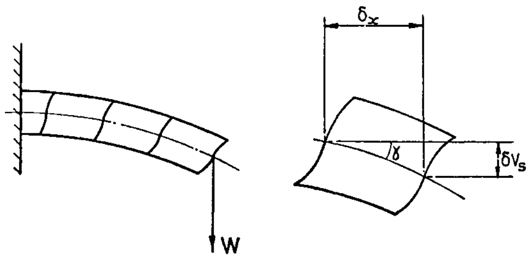


FIG. 5.7

where dv_s is the shear deflection over a length dx and γ_o is the shear strain at the neutral axis.

Now $\gamma_o = \tau_o/G$ at the neutral axis. Therefore

$$dv_s = \frac{\tau_o}{G}\,dx$$

or

$$v_s = \int \left(\frac{\tau_o}{G}\right) dx + A. \tag{5.11}$$

For a rectangular section from example 4.3

$$\tau_o = \tfrac{3}{2}\frac{F}{bd}.$$

Therefore

$$v_s = \int \tfrac{3}{2}\frac{F}{bdG}\,dx + A.$$

For a cantilever with a concentrated load at the free end $F = W$. Therefore

$$v_s = \tfrac{3}{2}\frac{Wx}{bdG} + A.$$

When $x = 0$, $v_s = 0$ therefore $A = 0$. At the free end $x = l$ and

$$v_s = \tfrac{3}{2}\frac{Wl}{bdG}. \tag{5.12}$$

The bending deflection at the free end is

$$v_b = \frac{Wl^3}{3EI}.$$

Total deflection is then

$$v_b + v_s = \frac{Wl^3}{3EI} + \frac{3Wl}{2bdG} = \frac{Wl^3}{3EI}\left[1 + \tfrac{3}{8}\frac{E}{G}\left(\frac{d}{l}\right)^2\right].$$

For $v = 0{\cdot}3$ $E/G = 2{\cdot}6$, and the deflection is

$$v_t = \frac{Wl^3}{3EI}\left[1 + 0{\cdot}98\left(\frac{d}{l}\right)^2\right]. \tag{5.13}$$

A more rigorous solution obtained by the theory of elasticity in which the built-in cross section is considered as not being able to distort gives

$$v_t = \frac{Wl^3}{3EI}\left[1 + 0{\cdot}71\left(\frac{d}{l}\right)^2 - 0{\cdot}1\left(\frac{d}{l}\right)^3\right]. \tag{5.14}$$

In either solution above it is seen that unless l is of the same order as the depth of section, v_s can be neglected with respect to v_b.

The deflection of a simply supported beam carrying a central point load can be considered as two cantilevers back to back. The deflection is therefore obtained by substituting $W/2$ and $l/2$ in equation (5.14), hence

$$v_t = \frac{Wl^3}{48EI}\left[1 + 2{\cdot}84\left(\frac{d}{l}\right)^2 - 0{\cdot}8\left(\frac{d}{l}\right)^3\right]. \tag{5.15}$$

For an I-section it was shown in the previous chapter that almost all the shear is taken in the web and that the shear stress at the neutral axis is close to the average shear stress in the web. Denoting web area by A_w the shear deflection is therefore

$$v_s = \int \left(\frac{F}{A_w G}\right) \mathrm{d}x + A. \tag{5.16}$$

If shear force is not constant along the beam but varies owing to distributed loading, then the appropriate expression for F in terms of x must be used in the integral above.

EXAMPLE 5.3.

A standard I-section R.S.J. 18 × 7 is used as a cantilever 10 ft in length supporting a load of 40 ton at the free end. The effective web area is 8·8 in^2 and the second moment of area is 1149 in^4. Determine and compare the shear and bending deflections at the free end.

$$E = 30 \times 10^6 \text{ lb/in}^2; \; G = 12 \times 10^6 \text{ lb/in}^2.$$

From equation (5.16) the shear deflection at the free end is

$$v_s = \frac{Wl}{A_w G} = \frac{40 \times 10 \times 12 \times 2240}{8{\cdot}8 \times 12 \times 10^6} = \mathbf{0{\cdot}102 \text{ in.}}$$

The bending deflection is

$$v_b = \frac{Wl^3}{3EI} = \frac{40 \times 2240 \times 120^3}{3 \times 30 \times 10^6 \times 1149} = \mathbf{1{\cdot}49 \text{ in.}}$$

The ratio of shear to bending deflection is (0·102/1·49) × 100 = 6·85 per cent.

5.6. Strain Energy Solution for Beam Deflection

The deflections of beams and frames can also be determined by indirect methods using the Principle of Virtual Work, or alternatively the First Theorem of Complementary Energy. In the special case when the behaviour is elastic and deflections are caused solely by the loads, this latter theorem reduces to

Castigliano's theorem Part II. The use of this and the former theorems is discussed in detail in *Structural Theorems and Their Applications* by B. G. Neal, and in this chapter attention will be confined to problems which can be treated by Castigliano's Theorem (Part II). A necessary preliminary is to derive a relationship for the elastic strain energy stored during bending.

It was shown in Chapter 4 that the angular rotation of one end of an element of length ds relative to the other for either initially straight or curved beams when subjected to a bending moment M is

$$d\theta = \frac{M\,ds}{EI}$$

and, since in the elastic range rotation is proportional to bending moment, the strain energy stored in the element is

$$\begin{aligned} dU &= \tfrac{1}{2}M\,d\theta \\ &= \frac{M^2\,ds}{2EI}. \end{aligned}$$

The total strain energy† stored in a beam of length l is therefore

$$U = \int_0^l \frac{M^2\,ds}{2EI}. \tag{5.17}$$

---

† The complementary energy for an element of beam of length ds, for an incremental change in bending moment dM, is

$$C = \int_0^M \theta\,dM = \int_0^l \int_0^M \left(\frac{dM}{R}\right) ds.$$

In the special case of a *linear-elastic* beam $1/R = M/EI$, hence the complementary energy

$$\begin{aligned} C &= \int_0^l \int_0^M \left(\frac{M}{EI}\right) dM\,ds = \int_0^l \left(\frac{M^2}{2EI}\right) ds \\ &= U \text{ (the strain energy).} \end{aligned}$$

The equivalence of elastic strain energy stored and external work done can be used in the case of a single concentrated load to determine the deflection under the load.

Consider a cantilever of length l carrying a load W at the free end, then if the deflection under the load is δ the work done $=$ $W\delta/2$.

Now the bending moment M at any point distance x from the load is Wx therefore the total strain energy

$$= \int_0^l \left(\frac{W^2x^2}{2EI}\right) \mathrm{d}x$$

$$= \frac{W^2l^3}{6EI}$$

Equating work done to strain energy gives

$$\frac{W\delta}{2} = \frac{W^2l^3}{6EI}$$

or

$$\delta = \frac{Wl^3}{3EI}.$$

5.7. Application of Castigliano's Theorem (Part II) to Beams

Castigliano's theorem (Part II) is applicable to linear elastic systems, where displacements are small, and in which deflections are due solely to applied loads, so that lack of fit, temperature stresses, etc. are excluded. For such systems the deflection of the point of application of a load in the direction of the load is given by the partial derivative of the strain energy of the system with respect to the load, or

$$\frac{\partial U}{\partial W} = \Delta, \tag{5.18}$$

where Δ is the deflection in the direction of the load W.

In the simple case of the cantilever discussed above

$$U = \frac{W^2 l^3}{6EI}$$

and

$$\Delta = \frac{\partial U}{\partial W} = \frac{2Wl^3}{6EI} = \frac{Wl^3}{3EI}.$$

The application of this theorem to a problem with distributed loading only requires a particular approach since there is no concentrated load at which the deflection can be obtained. Consider the case of a simply supported beam carrying a uniformly distributed load w for which the deflection at the centre is required.

On to the distributed loading a fictitious concentrated load W is superimposed at the centre and equation (5.18) is used to calculate the deflection due to both w and W. The required deflection is then obtained by putting $W = 0$. The bending moment distribution is given by

$$M = -\frac{Wx}{2} - \frac{wl}{2}x + \frac{wx^2}{2}.$$

The strain energy is then

$$U = \frac{1}{2EI}\int_0^l M^2 \mathrm{d}x = \frac{1}{EI}\int_0^{l/2} M^2 \mathrm{d}x$$

using the condition of symmetry.

The deflection under the concentrated load W is

$$\delta = \frac{\partial U}{\partial W} = \int 2\frac{M}{EI}\cdot\frac{\partial M}{\partial W}\,.\,\mathrm{d}x.$$

Now

$$\frac{\partial M}{\partial W} = -\frac{x}{2}$$

so that

$$\Delta = \frac{2}{EI}\int_0^{l/2} \frac{x}{2}\left(\frac{Wx}{2} + \frac{wlx}{2} - \frac{wx^2}{2}\right)\mathrm{d}x.$$

If we now put $W = 0$, the deflection due to w is found to be

$$\Delta = \frac{2}{EI}\int_0^{l/2}\left(+\frac{wlx}{2} - \frac{wx^2}{2}\right)\frac{x}{2}\,dx$$

$$= \frac{5}{384}\frac{wl^4}{EI},$$

which is the result found earlier in equation (5.6).

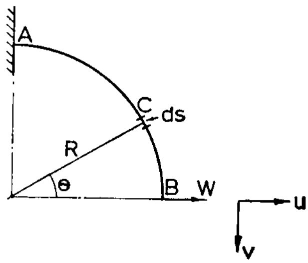


FIG. 5.8

5.8. Deflection of Slender Curved Bars

There are two approaches to this problem; firstly by a direct analysis of the geometry of displacement of a small element and then integrating along the length of the bar to obtain the total deflection. The alternative is to use the energy method described above.

(a) *Solution by Geometry of Deformation*

As an example of both these methods of solution, consider the curved bar AB in Fig. 5.8 which forms a quadrant of a circle, built-in at A. A horizontal load W is applied at B and the vertical and horizontal components of deflection of B, v and u are to be determined.

On a small element of length ds at C the bending moment will be

$$M = -WR \sin \theta,$$

the negative sign being used for a bending moment tending to decrease curvature.

The bending moment will cause a rotation of one end of the element relative to the other of $d\phi$, where $d\phi = (M/EI)ds = (M/EI)Rd\theta$. If AC and CB are considered as rigid, then the horizontal displacement of B due to the rotation of the element will be

$$du = -R \sin \theta \, d\phi$$

and substituting for $d\phi$ and M from above,

$$du = +\frac{WR^3}{EI} \sin^2 \theta d\theta.$$

Thus the total horizontal deflection of B for rotation of all small elements between A and B is

$$u = \int_0^{\pi/2} + \frac{WR^3}{EI} \sin^2 \theta d\theta$$

whence

$$u = +\frac{\pi}{4}\frac{WR^3}{EI}. \tag{5.18}$$

The vertical displacement of B due to rotation of the element will be

$$dv = R(1 - \cos \theta)d\phi$$

the total deflection of B vertically is

$$v = \int_0^{\pi/2} - \frac{WR^3 \sin \theta \, (1 - \cos \theta)d\theta}{EI},$$

$$v = -\frac{WR^3}{2EI}. \tag{5.19}$$

(*b*) *Solution by Castigliano's Theorem* (*Part II*)

Considering first the displacement in the direction of the load: The strain energy

$$U = \int_0^{\pi/2} \left(\frac{M^2}{2EI}\right) R d\theta$$

so that

$$u = \frac{\partial U}{\partial W} = \int_0^{\pi/2} \frac{M}{EI} \cdot \frac{\partial M}{\partial W} R d\theta,$$

$$\frac{\partial M}{\partial W} = -\, R \sin \theta.$$

$$u = \int_0^{\pi/2} - \frac{WR \sin \theta}{EI} (-R \sin \theta) \,.\, R d\theta$$

$$= \int_0^{\pi/2} \frac{WR^3}{EI} \sin^2 \theta d\theta$$

which is the same as in the analysis of the geometry of displacement.

Therefore

$$u = \frac{\pi}{4} \frac{WR^3}{EI}.$$

To determine the vertical displacement of B, an imaginary vertical force W_0 is applied. Then the bending moment on C will be

$$M_c = -\, WR \sin \theta + W_0 R(1 - \cos \theta)$$

and

$$\frac{\partial M_c}{\partial W_0} = R(1 - \cos \theta).$$

At this stage put $W_0 = 0$ in the expression for M_c, so that

$$v = \frac{\partial U}{\partial W_0} = \int_0^{\pi/2} -\frac{WR}{EI} \sin\theta \,.\, R(1 - \cos\theta) R d\theta$$

$$= \int_0^{\pi/2} -\frac{WR^3}{EI} \sin\theta (1 - \cos\theta) d\theta,$$

which is again precisely the same as in the previous analysis so that the Castigliano theorem arrives at the same point in the solution as the direct geometrical attack but by a different process.

Therefore

$$v = -\frac{WR^3}{2EI}.$$

5.9. Deflections in a Statically Indeterminate Problem: A Slender Ring

The calibration of a testing machine is frequently carried out using what is known as a *proving ring*. This is a slender ring of rectangular cross section with a dial gauge so mounted as to measure the change in one diameter when forces are applied to the ring along that diameter. Within the elastic range the change in diameter is proportional to the diametral load. The ring itself, of course, has to be calibrated initially and this is often done on special apparatus at the National Physical Laboratory.

(*a*) *Solution by Geometry of Deformation*

Consider the ring shown in Fig. 5.9(*a*) which is subjected to diametral compression. Because of the double symmetry of the system it is only necessary to consider the equilibrium and displacements of one quadrant as shown in Fig. 5.9(*b*). When the ring is cut at B, since this is a section of symmetry there is no shear force and equilibrium is satisfied by a vertical force $W/2$

and a moment, M_0, which is statically indeterminate. For convenience, point A is considered fixed and the vertical and horizontal relative displacements of B will be found, the former being equal to the vertical displacement at A in the full ring. There is no vertical displacement at B in the full ring.

The single redundancy M_0 is found from the geometrical condition that the slopes at A and B do not change under load.

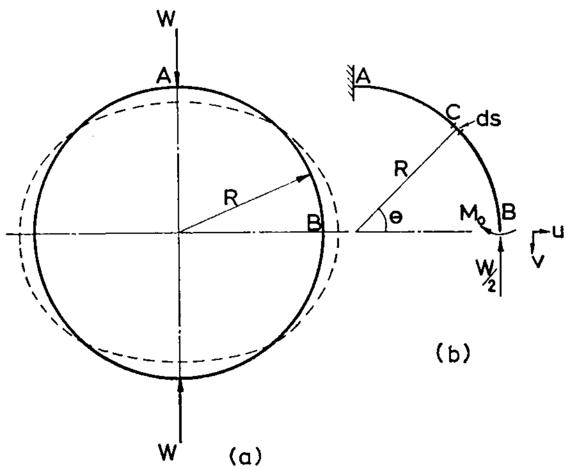


FIG. 5.9

If the rotation of a small element ds at C is $d\phi$, then the equation of compatibility is

$$\int_A^B d\phi = 0$$

or

$$\int_A^B (M/EI)ds = 0.$$

Now

$$M = M_0 - \frac{WR}{2}(1 - \cos\theta)$$

therefore

$$\int_0^{\pi/2}\left[M_0 - \frac{WR}{2}(1 - \cos\theta)\right]Rd\theta = 0$$

from which

$$M_0 = WR\left(\tfrac{1}{2} - \frac{1}{\pi}\right) \tag{5.20}$$

and

$$M = WR\left[\tfrac{1}{2}\cos\theta - \frac{1}{\pi}\right]. \tag{5.21}$$

From this point the analysis is similar to that for the previous example of the quadrant.

The horizontal displacement is therefore

$$u = \int_0^{\pi/2} -\frac{WR^3}{EI}\left[\left(\frac{\cos\theta}{2}\right) - \left(\frac{1}{\pi}\right)\right]\sin\theta\, d\theta,$$

from which

$$u = \frac{WR^3}{EI}\left(\frac{1}{\pi} - \tfrac{1}{4}\right) \tag{5.22}$$

or the increase in horizontal *diameter* of the ring is

$$2u = \frac{WR^3}{EI}\left(\frac{2}{\pi} - \tfrac{1}{2}\right) = 0{\cdot}136\frac{WR^3}{EI}.$$

The vertical displacement of B, actually A in the full ring, is

$$v = \int_0^{\pi/2}\frac{WR^3}{EI}\left(\frac{\cos\theta}{2} - \frac{1}{\pi}\right)\left(1 - \cos\theta\right)d\theta$$

from which

$$v = -\frac{WR^3}{EI}\left(\frac{\pi}{8} - \frac{1}{\pi}\right) \tag{5.23}$$

or the decrease in vertical *diameter* of the ring is

$$2v = -\frac{WR^3}{EI}\left(\frac{\pi}{4} - \frac{2}{\pi}\right) = -0{\cdot}149\frac{WR^3}{EI}.$$

(*b*) *Solution by Castigliano's Theorem of Compatibility*

In linear elastic problems if a beam or frame is free from stress when unloaded, and stresses are set up solely by applied loading, then Castigliano's theorem of compatibility reduces to the condition that $\partial U/\partial R = 0$ where U is the strain energy in the system and R is a redundancy.

In the case of the ring analysed in the previous section

$$U = \int_0^{\pi/2} \frac{1}{2EI}\left[M_0 - \frac{WR}{2}(1 - \cos\theta)\right]^2 R d\theta \qquad (5.24)$$

so that

$$\frac{\partial U}{\partial M_0} = \int_0^{\pi/2} \frac{1}{EI}\left[M_0 - \frac{WR}{2}(1 - \cos\theta)\right] R d\theta = 0,$$

whence

$$M_0 = WR\left(\tfrac{1}{2} - \frac{1}{\pi}\right),$$

as shown in the previous section by the direct geometrical attack.

Examples

1. A pole made of mild steel tube, 10 ft long, 6 in. external diameter and $\frac{1}{2}$ in. thick, is firmly fixed in the ground. A horizontal pull is applied at a point 6 ft from the ground, the magnitude of the pull being 2000 lb. Find the deflection at the top of the pole. $E = 13{,}000$ ton/in^2.

2. A steel beam AB, 15 ft long, is simply supported at its ends and carries concentrated vertical loads of 4 ton and 5 ton at points 5 ft and 12 ft respectively from A. Calculate the deflection of the beam under the 4 ton load. $I = 484$ in^4, $E = 30 \times 10^6$ lb/in^2.

3. A uniform beam of length $l + 2a$ is supported at two points l ft apart and overhung at each end a length a. It is loaded with concentrated loads W at each extremity and $2W$ at the centre of the span. Determine expressions for the deflection at each load. (London)

4. A horizontal beam 10 ft in length is simply supported at each end. The loading as measured from the left-hand end is: anticlockwise couple of 1 ton ft at 5 ft; uniformly distributed load of $\frac{1}{2}$ ton/ft between 5 ft and 8 ft; concentrated load of 2 ton at 8 ft. Calculate the deflection at a section 6 ft from the left-hand end. $I = 20$ in^4; $E = 10 \times 10^6$ lb/in^2.

5. A beam 10 ft long is simply supported at its ends and carries a varying distributed load over the whole span. The equation to the loading curve is $w = ax^2 + bx + c$, where w is the load intensity in ton/ft run, at a distance x along the beam, measured from an origin at the left-hand support, and a, b and c are constants. The load intensity is zero at each end of the beam and reaches a maximum value of 5 ton/ft at the centre of the span. Calculate the slope of the beam at each support and the deflection at the centre. $E = 30 \times 10^6$ lb/in^2, $I = 975$ in^4.

6. A horizontal beam of length L, freely supported at each end, carries a load which increases at a uniform rate, from zero at one end to an intensity wL at the other end. Determine the position of, and the value of, the maximum deflection.

7. A beam 6 ft long is fixed at one end and simply supported at the same level at the other end by a rigid prop. A load of 3 ton is uniformly distributed over 3 ft of the span from the fixed end. Determine the position and magnitude of the maximum deflection. $I = 13{\cdot}9$ in^4; $E = 30 \times 10^6$ lb/in^2. (Belfast)

8. By considering the strain energy in shear stored in an elemental volume of a beam, show that, for a cantilever of length l and cross section $b \times d$ carrying a load W at the free end, the shear deflection is $6Wl/(5bdG)$. Why is this value different from that obtained in equation (5.12)?

9. Compare the strain energy of a beam, simply supported at the ends and loaded with a uniformly distributed load, with that of the same beam centrally loaded and having the same value of the maximum bending stress. (London)

10. A 20 ft length of girder, having a second moment of area of 81 in^4, is used horizontally in a structure and may be considered simply supported at its ends. A second girder 10 ft long and weighing 21 lb/ft length is accidentally dropped and falls vertically through a height of 2 ft before striking the first girder at a point on the upper surface 5 ft from one support.

If the full weight of the falling girder is transmitted to the simply supported beam, calculate the maximum instantaneous deflection of the beam at the point of impact. (London)

11. A beam of length l, simply supported at each end, carries a concentrated load W at a distance a from the left end, and a uniformly distributed load w per unit length over the whole span. Use the Castigliano theorem to find the deflection under the load.

12. Determine the vertical and horizontal displacements of the free end of the slender curved bar of uniform cross section shown in Fig. 5.10.

13. A slender U-shaped member has a centre line consisting of a semi-circle of radius r and straight portions of length l. Determine the increase in distance between the ends due to forces W applied at each end acting in a direction perpendicular to the axis of symmetry of the U.

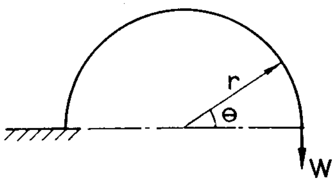


FIG. 5.10

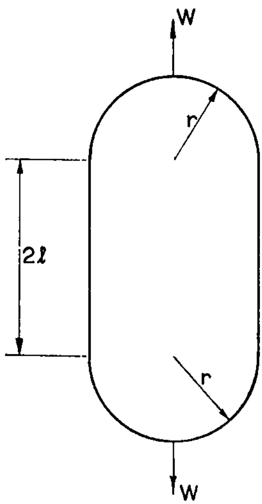


FIG. 5.11

14. The link shown in Fig. 5.11 has cross-sectional dimensions which are small in comparison with r. Determine the maximum bending moment for the loading shown.

CHAPTER 6

ELEMENTARY THEORY OF TORSION

6.1. Introduction

After axial loading and bending, the next of the common modes of deformation is torsion. In engineering, one frequent example of a member being subjected to torsion is that of the shaft which transmits power from one component to another. In pure torsion interest is centred on the stress distribution set up by externally applied torque about the longitudinal axis, Fig. 6.1, and deformations which take the form of twisting or rotation of one end of the shaft relative to the other.

This chapter covers the elastic torsion of various thin-walled closed and open sections as well as the shaft of solid circular cross section. Non-circular solid sections subjected to torsion are beyond the scope of this book.

6.2. Pure Torsion of a Thin-walled Circular Tube

This problem was studied briefly in Chapter 1, and was shown to be statically determinate, i.e. stresses could be found simply from the condition of equilibrium of forces. The conditions of stress and strain in the tube will now be studied in more detail, using the cylindrical coordinate system r, θ, z. The tube shown in Fig. 6.1 is of mean radius r with a wall thickness t and is subjected to a pure torque T about the longitudinal z-axis. The cross section at one end rotates through an angle θ with respect to the other end.

The longitudinal and transverse symmetry of the tube leads to the conclusion that out-of-plane distortion of an initially plane cross section cannot occur. For example, if the ends either dished or bulged, as in Fig. 6.2, a series of similarly deformed slices could not be put together to form a continuous length of tube. Hence deformation takes the form of rotation of one plane relative to the next and so on, the planes remaining normal to the

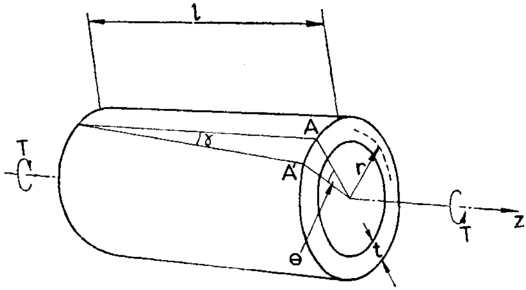


FIG. 6.1

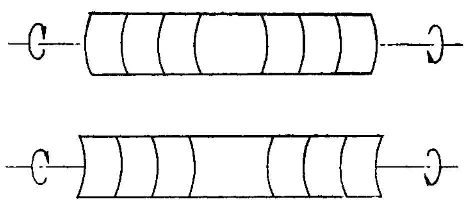


FIG. 6.2

axis of the tube. There is no apparent reason why the tube should not change in length or expand or contract circumferentially. There are no externally applied axial or radial forces, however, so if there are normal stresses in the r, θ and z directions their resultant must be zero. It was shown in Chapter 1 that equilibrium with the applied torque was satisfied by the existence of the shear stress $\tau_{\theta z}$ and of course the complementary shear stress $\tau_{z\theta}$. It will therefore be assumed that normal stresses do not play a significant part in pure torsion and that

$$\sigma_r = \sigma_\theta = \sigma_z = 0.$$

It can be shown in an exact solution by the mathematical theory of elasticity that the above assumptions are correct.

Owing to the rotation of one plane relative to the next an element of the tube wall deforms as shown in Fig. 6.3, the edges *AD* and *BC* remain radial and the angles *DAF* and *DAB* remain at right angles and so

$$\tau_{r\theta} = \tau_{rz} = 0.$$

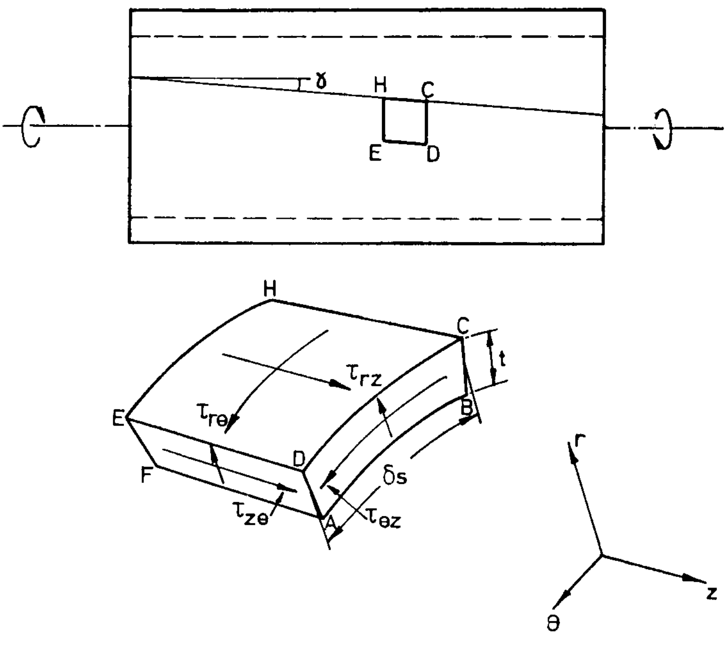


FIG. 6.3

Equilibrium

The tangential shear force on the element of length δs, Fig. 6.3, is

$$\tau_{\theta z}\delta st$$

and this has a moment about the z-axis of an amount

$$\tau_{\theta z}r\delta st.$$

For the complete circumference $\delta s = 2\pi r$ and therefore for equilibrium with the external torque

$$T = 2\pi r^2 \tau_{\theta z} t, \tag{6.1}$$

where $\tau_{\theta z}$ is assumed to be constant over the thickness t.

The above result was that obtained in Chapter 1.

Geometry of Deformation

For an angle of twist θ over a length l of the tube shown in Fig. 6.1, a point initially at A has moved to A' such that

$$AA' = \gamma_{\theta z} l = r\theta$$

or

$$\gamma_{\theta z} = \frac{r\theta}{l}. \tag{6.2}$$

Stress–Strain Relationship

The shear stress and shear strain are related as

$$\tau_{\theta z} = \gamma_{\theta z} G. \tag{6.3}$$

Equations (6.1), (6.2) and (6.3) are interdependent and may be written as

$$\tau_{\theta z} = \frac{Gr\theta}{l} = \frac{1}{2\pi r^2} \cdot \frac{T}{t}$$

or

$$\frac{\tau_{\theta z}}{r} = \frac{T}{2\pi r^3 t} = \frac{G\theta}{l}. \tag{6.4}$$

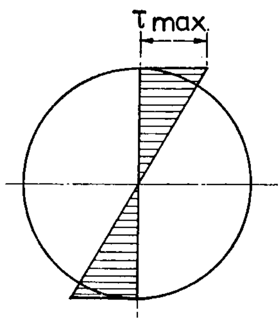
6.3. Pure Torsion of a Solid Circular Shaft

The analysis for the thin-walled tube can be used in the case of a solid circular shaft, by imagining the latter to be built up of an infinite number of tubes just fitting inside each other and all experiencing the same angle of twist, θ. It is apparent that the conditions that plane cross sections remain plane and normal to the axis and radial lines remain straight and radial during twisting, which were deduced for the tube, must also apply to the solid section.

Applying equation (6.2) to any two of the tubes of radius r_p and r_q which form the solid section, since θ and l are constant,

$$\frac{\theta}{l} = \frac{\gamma_p}{r_p} = \frac{\gamma_q}{r_q}$$

and hence the shear strain is proportional to the radius, being zero at the centre of the shaft and a maximum at the outer surface.



Distribution of shear stress τ

FIG. 6.4

From the shear stress–shear strain relation it is seen that shear stress also varies linearly with radius over the cross section as shown in Fig. 6.4.

The sum of all the torques on the separate tubes must be equal to the total torque T applied to the shaft so that using equation (6.1) and writing dr for the thickness t

$$T = \int_0^{r_0} 2\pi r^2 \tau \mathrm{d}r.$$

But from equation (6.4)

$$\tau = \frac{Gr\theta}{l} \tag{6.5}$$

Therefore

$$T = \frac{G\theta}{l}\int_0^{r_0} 2\pi r^3 \mathrm{d}r. \tag{6.6}$$

Now

$$\int_0^{r_0} 2\pi r^3 \mathrm{d}r$$

is the polar second moment of area of the cross section about the z-axis and will be denoted by I_p. Evaluating the integral gives $I_p = \pi r_0^4/2$. The constant term $G\theta/l$ in equation (6.6) is equal to τ/r and therefore

$$T = \frac{\tau}{r} . I_p \quad \text{or} \quad \frac{T}{I_p} = \frac{\tau}{r}, \tag{6.7}$$

which is the equilibrium relationship for a solid circular shaft under torsion. Combining equations (6.5) and (6.7) gives the expression

$$\frac{T}{I_p} = \frac{\tau}{r} = \frac{G\theta}{l}. \tag{6.8}$$

It is interesting to note the similarity between the above relationship and equation (4.9) for bending. However, whereas the latter applies to many shapes of cross section, equation (6.8) is valid only for circular cross sections.

6.4. Hollow Circular Shafts

The above analysis for the solid shaft is similarly applicable to the hollow shaft. Thus the torsion relationship equation (6.8) also expresses the conditions of equilibrium and compatibility for a hollow circular shaft. However, the radial boundaries are now $r = r_i$ and $r = r_o$, the inner and outer radii respectively, and thus the polar second moment of area is

$$\begin{aligned} I_p &= \int_{r_i}^{r_o} 2\pi r^3 \mathrm{d}r \\ &= \frac{\pi}{2}(r_o^4 - r_i^4). \end{aligned}$$

The shear stress varies linearly from Tr_i/I_p at the bore to Tr_o/I_p at the outer surface as shown in Fig. 6.5.

The hollow shaft is more efficient in its use of stressed material than the solid shaft because the core of a solid shaft has relatively low stresses as compared with the outer layers, cf. Chapter 4: I-beams. However, hollow shafts are not used widely in practice, owing to the cost of machining, unless saving of weight is at a premium, or it is necessary to pass services down the centre of the shaft.

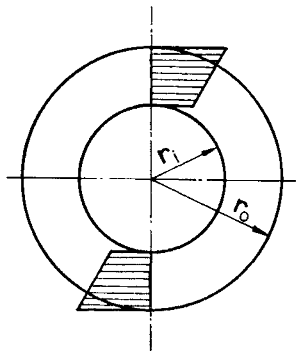


FIG. 6.5

EXAMPLE 6.1

A hollow circular shaft having a rate of external to internal diameter of 2:1 is to be used to replace a solid circular shaft of 4 in. diameter. If the maximum shear stress is to be the same for each shaft determine the dimensions and saving in weight for the hollow shaft.

Let the inner and outer radii of the hollow shaft be r and $2r$ respectively.

$$\text{Hollow shaft:} \quad I_p = \frac{\pi}{2}(16r^4 - r^4) = 7{\cdot}5\pi r^4 \text{ in}^4.$$

$$\text{Solid shaft:} \quad I_p = \frac{\pi}{2}\,.\,2^4 = 8\pi \text{ in}^4.$$

If the torque transmitted by each shaft is T then for equilibrium:

Hollow shaft: $T = \dfrac{\tau_{max}}{2r} 7{\cdot}5\pi r^4.$

Solid shaft: $T = \dfrac{\tau_{max}}{2} 8\pi.$

Therefore

$$\frac{\tau_{max}}{2} 8\pi = \frac{\tau_{max}}{2r} 7{\cdot}5\pi r^4,$$

$$r^3 = \frac{8}{7{\cdot}5},$$

$$r = 1{\cdot}02 \text{ in.}$$

The internal diameter is therefore 2·04 in. and the external diameter is 4·08 in.

The saving in weight will be

$$\left\{1 - \frac{\rho l\pi(2{\cdot}04^2 - 1{\cdot}02^2)}{\rho l\pi \times 2^2}\right\} \times 100\% = 22\%$$

where ρ is the weight per unit volume and l is the length of the shaft.

Example 6.2

A solid circular shaft has a diameter of 6 in. for a length of 45 in. and 3 in. for the remaining 20 in. If the total twist is limited to 2°, what horsepower can be transmitted at 600 rev/min? Determine the maximum shear stress in each part of the shaft. Shear modulus $G = 12 \times 10^6$ lb/in^2.

The shaft is illustrated in Fig. 6.6, and in subsequent analysis the larger part is denoted by A and the smaller by B.

Each part of the shaft must be in equilibrium and therefore

$$T_A = T_B = T.$$

If the angles of twist of A and B are θ_A and θ_B respectively then the total twist is

$$\theta_A + \theta_B = \frac{2}{57{\cdot}3} = 0{\cdot}035 \text{ rad.} \tag{6.9}$$

From the torsion equation (6.8)

$$\theta_A = \frac{Tl_A}{I_{pA}G} = \frac{45T}{(\pi/2)3^4 \times 12 \times 10^6} = \frac{T}{\pi \times 108 \times 10^5},$$

$$\theta_B = \frac{Tl_B}{I_{pB}G} = \frac{20T}{(\pi/2) \times 1{\cdot}5^4 \times 12 \times 10^6} = \frac{T}{\pi \times 15{\cdot}2 \times 10^5}.$$

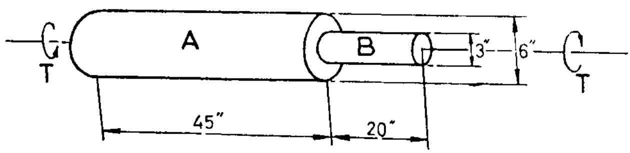


FIG. 6.6

From equation (6.9)

$$0{\cdot}00294 \times 10^{-5}T + 0{\cdot}0209 \times 10^{-5}T = 0{\cdot}035,$$

$$T = 147{,}000 \text{ lb in.}$$

The horsepower which can be transmitted is

$$\text{h.p.} = \frac{2\pi \times 600 \times 147{,}000}{33{,}000 \times 12}$$

$$= 1400.$$

To find the maximum shear stresses in each part

$$(\tau_A)_{\max} = \frac{Tr_A}{I_{pA}} = \frac{147{,}000 \times 3}{(\pi/2) \times 3^4}$$

$$= 3{,}470 \text{ lb/in}^2$$

and

$$(\tau_B)_{\max} = \frac{Tr_B}{I_{pB}} = \frac{147{,}000 \times 1{\cdot}5}{(\pi/2) \times 1{\cdot}5^4}$$

$$= 27{,}700 \text{ lb/in}^2.$$

6.5. Combined Torsion and Axial Loading of a Shaft

The state of stress for an element on the surface of a shaft in pure torsion is shown in Fig. 6.7(*a*) and the corresponding Mohr's circle in Fig. 6.7(*b*). σ_θ and σ_z are zero and so the circle has its

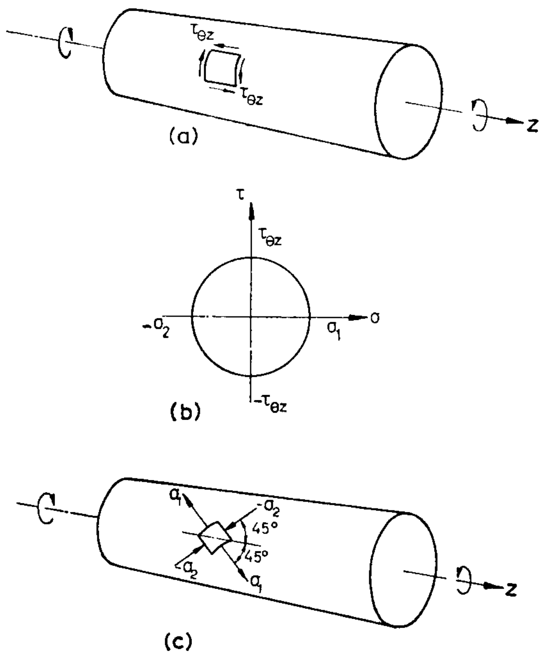


FIG. 6.7

centre at the origin and the principal normal stresses are $\sigma_1 = -\sigma_2 = \pm\tau_{\theta z}$ inclined at 45° directions to the z-axis as shown in Fig 6.7(c).

If axial loading is applied to the shaft, a direct stress σ_z is added to the element to give the state of stress illustrated in Fig. 6.8.

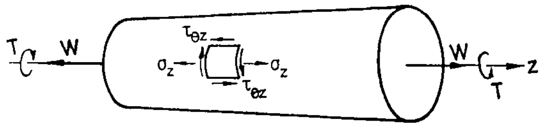

T
W
$\tau_{\theta z}$
σ_z
σ_z
$\tau_{\theta z}$
W
z
T

FIG. 6.8

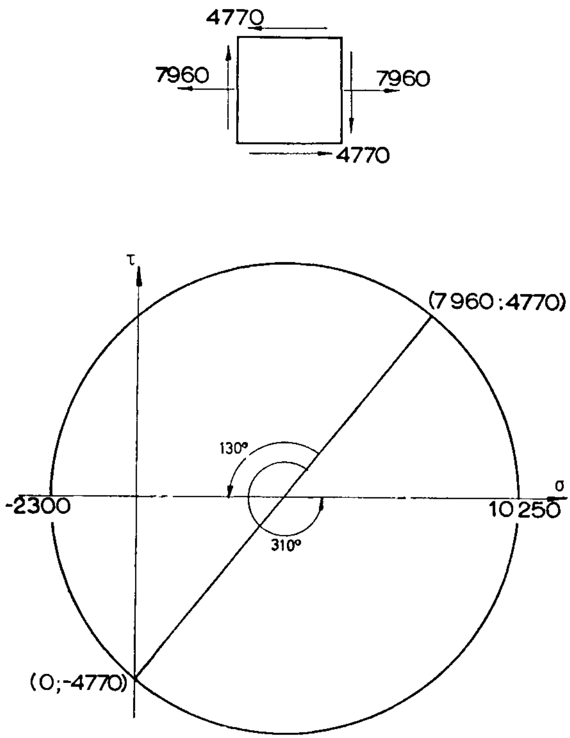

4770
7960
7960
4770
τ
(7 960 ; 4770)
130°
σ
-2300
10 250
310°
(0;-4770)

FIG. 6.9

EXAMPLE 6.3

Determine the magnitude and direction of principal stresses at the surface of a solid circular shaft of 4 in. diameter subjected to a torque of 60,000 lb in. and an axial tensile force of 100,000 lb.

Cross-sectional area of shaft $= \frac{\pi}{4} \times 4^2 = 4\pi$ in^2.

Therefore

$$\sigma_z = \frac{100{,}000}{4\pi} = 7960 \text{ lb/in}^2.$$

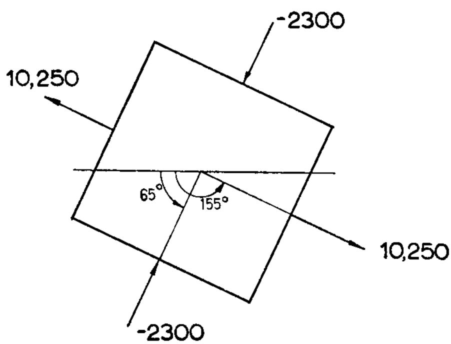


FIG. 6.10

Polar second moment of area $I_p = \frac{\pi}{32} \times 4^4 = 8\pi$ in^4,

hence

$$\tau_{\theta z} = \frac{60{,}000 \times 2}{8\pi} = 4770 \text{ lb/in}^2.$$

The circumferential stress σ_θ is zero.

The Mohr's circle is shown constructed in Fig. 6.9, from which the principal stresses are found to be

$$\sigma_1 = 10{,}250 \text{ lb/in}^2$$

and

$$\sigma_2 = -2300 \text{ lb/in}^2.$$

The directions of principal stresses are given in Fig. 6.10.

6.6. Strain Energy in Simple Shear and in Torsion

Strain energy relationships were shown in Chapter 5 to be useful for determining beam deflections, and in a similar manner, a strain energy equation can be used to find an angle of twist in torsion.

For the case of an element of material under simple uniform shear stress, Fig. 6.11(*a*), the stored strain energy is

$$U_s = \tfrac{1}{2}\tau\gamma \text{ per unit volume}$$

$$= \frac{\tau^2}{2G} \text{ per unit volume,} \qquad (6.10)$$

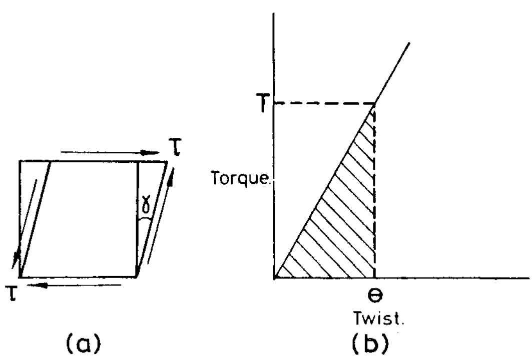


FIG. 6.11

which is similar in form to the strain energy U stored in an element under direct stress.

From relationship equation (6.8) the angle of twist θ is proportional to the applied torque T within the elastic range. Referring to Fig. 6.11(*b*) the work done in applying a torque T which is equal to the stored strain energy, is

$$U_s = \tfrac{1}{2}T\theta.$$

Substituting for θ in terms of T gives

$$U_s = \tfrac{1}{2}T \cdot \frac{Tl}{I_p G}$$

$$= \tfrac{1}{2}\frac{T^2 l}{G I_p}, \qquad (6.11)$$

which is similar in form to the expression for strain energy stored in bending, equation (5.17), $U = M^2 l/2EI$.

Now

$$T = \frac{\tau_{max} I_p}{r_o},$$

and so the shear strain energy in terms of the *maximum* shear stress at the surface of a solid shaft is

$$U_s = \frac{\tau^2{}_{max} I_p l}{2 G r_o^2}$$

$$= \frac{\tau^2{}_{max}}{4G} \text{ per unit volume.} \qquad (6.12)$$

6.7. Thin-walled Closed Tubes of Non-circular Cross Section

Thin-walled tubular members, which are not necessarily of circular section, are sometimes used in structures, for example, elliptical or box type sections in aircraft construction. An exact solution for the stresses and displacements set up by torsion in such members is beyond the scope of the present text. An *engineering* solution, however, based on the requirements of equilibrium together with certain simplifying assumptions, enables the shear stress distribution to be determined with sufficient accuracy for most purposes.

The tubular prismatic member shown in Fig. 6.12(a) is subjected to torsion about the z-axis without end restraint. It has a cross-sectional shape and wall thickness which is irregular around the periphery, but shape and dimensions are constant along the length. The coordinate system *n*, *s*, *z* will be used for directions normal, tangential and axial at any point in the tube wall. Since the inner and outer surfaces of the tube are free from loading and the wall is thin compared with the dimensions of the cross section, it is not unreasonable to assume that throughout the thickness

$$\sigma_n = \tau_{nz} = \tau_{ns} = 0.$$

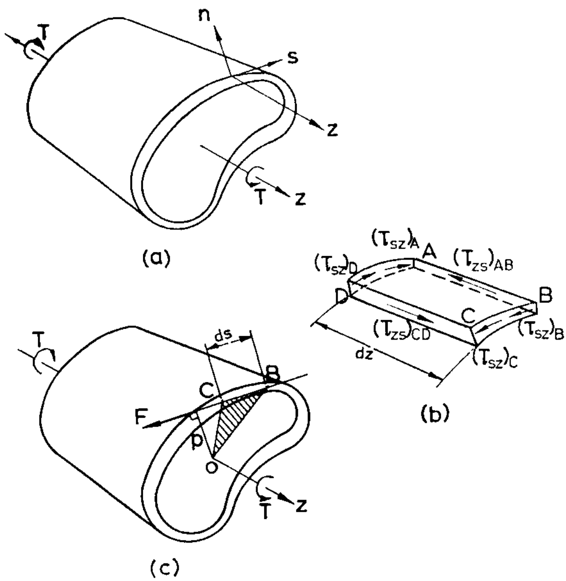


FIG. 6.12

Previous experience of torsion would indicate that there must be a shear stress τ_{sz}, and for radial equilibrium of an element such as in Fig. 6.12(*b*) it is evident that there is no tangential stress σ_s. If there is a stress distribution of σ_z, the axial *resultant* must be zero since there is no axial force, and it will therefore be assumed that $\sigma_z = 0$.

Consider the arbitrary element $ABCD$ cut from the tube. The tangential shear stress τ_{sz} will be assumed constant through the wall thickness although it may vary from point B to C and A to D. Now the tube is constant in dimensions along the length, therefore each cross section will experience the same shear stress distribution and τ_{sz} at A will equal τ_{sz} at B similarly $(\tau_{sz})_C = (\tau_{sz})_D$. Therefore the complementary shear stresses τ_{zs} along AB and CD will be constant although not necessarily equal. Then for longitudinal equilibrium of the element

$$\tau_{AB} t_{AB} \mathrm{d}z = \tau_{CD} t_{CD} \mathrm{d}z, \tag{6.13}$$

where t_{AB} and t_{CD} are the thicknesses of the element along AB and CD respectively.

Also in the transverse direction for the complementary shear stresses

$$\tau_{CD} t_{CD} = (\tau_{sz} t)_C \text{ and } \tau_{AB} t_{AB} = (\tau_{sz} t)_B,$$

whence from equation (6.13)

$$(\tau_{sz} t)_C = (\tau_{sz} t)_B$$

and similarly at A and D.

It is evident that the product of shear stress and thickness is constant for all parts of the tube. This constant is termed the *shear flow* and is denoted by q, the shear force per unit length of circumference.

Therefore

$$q = \tau t = \text{constant}.$$

The magnitude of the shear stress may be found in terms of the applied torque by considering the equilibrium of the tube. The element of wall of length ds shown in Fig. 6.12(c) is subjected to a shear force $F = q\mathrm{d}s$.

The moment of this force about any arbitrary point O within the section is

$$\mathrm{d}T = Fp = qp\mathrm{d}s,$$

where p is the perpendicular distance from the element to the point O.

Hence the torque for the whole section is

$$T = \oint qp\mathrm{d}s$$
$$= 2q \oint \mathrm{d}A,$$

where $\mathrm{d}A$ is the area of the shaded triangle OCB. Therefore $\oint \mathrm{d}A$ is twice the area of cross section (A) contained within the median line of the wall or

$$T = 2Aq.$$

Since $q = \tau t$ the shear stress at a point in the wall is given by

$$\tau = \frac{T}{2At}, \tag{6.14}$$

where t is the wall thickness *at that point.*

Owing to the variation of shear stress around the circumference of the tube it is not possible to predict the deformations and hence the angle of twist in the simple manner used for the circular shaft or tube. The angle of twist (θ) can be determined, however, from the strain energy stored in the tube. Considering an axial strip along which the shear stress is constant, then, from equation (6.10), the shear strain energy per unit volume is

$$U_s = \frac{\tau^2}{2G}.$$

If the strip is of length l, thickness t and width $\mathrm{d}s$, then the energy stored in the strip is

$$U_s = \frac{\tau^2}{2G} \times lt\mathrm{d}s.$$

Therefore the energy stored in the complete tube is

$$U_s = \oint \frac{\tau^2}{2G} lt\mathrm{d}s.$$

Substituting for τ from equation (6.14) gives

$$U_s = \oint \frac{T^2}{8A^2t^2G} lt\,\mathrm{d}s$$

$$= \frac{T^2 l}{8A^2G} \oint \frac{\mathrm{d}s}{t}. \tag{6.15}$$

But the stored energy is equal to the work, $\frac{1}{2}T\theta$, since in the elastic range the torque is proportional to the angle of twist θ, therefore

$$\tfrac{1}{2}T\theta = \frac{T^2 l}{8A^2G} \oint \frac{\mathrm{d}s}{t},$$

so that

$$\theta = \frac{Tl}{4A^2G} \oint \frac{\mathrm{d}s}{t}. \tag{6.16}$$

If the tube is of *constant* thickness t around the circumference s, then

$$\theta = \frac{Tls}{4A^2Gt},$$

or substituting for T from equation (6.14),

$$\theta = \frac{\tau l s}{2AG}. \tag{6.17}$$

EXAMPLE 6.4

The light alloy stabilising strut of a high-wing monoplane is 6 ft long, and has the cross section shown in Fig. 6.13. Determine the torque that can be sustained and the angle of twist if the maximum shear stress is limited to 4000 lb/in^2. $G = 3{\cdot}9 \times 10^6$ lb/in^2.

The area enclosed by the median line of the wall thickness is

$$A = \pi \times 1^2 + 2 \times 2 = 7{\cdot}142 \text{ in}^2.$$

The total length of the median line is

$$s = 2 \times 2 + \pi \times 2 = 10{\cdot}28 \text{ in.}$$

From equation (6.14) the allowable torque is

$$T = 2At\tau.$$

Therefore

$$T = 2 \times 7{\cdot}142 \times 0{\cdot}05 \times 4000$$
$$= 2860 \text{ lb in.}$$

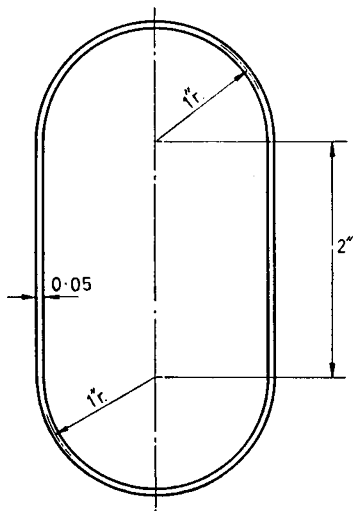


FIG. 6.13

The angle of twist is obtained from equation (6.17)

$$\theta = \frac{\tau l s}{2AG}.$$

Therefore

$$\theta = \frac{4000 \times 72 \times 10{\cdot}28}{2 \times 7{\cdot}142 \times 3{\cdot}9 \times 10^6}$$
$$= 0{\cdot}0533 \text{ rad} = 3{\cdot}05°.$$

6.8. Torsion of a Thin Rectangular Strip

An approximate solution for the torsion of a prismatic rectangular strip, whose thickness is small compared with the width, Fig. 6.14(a), can be obtained by considering the strip to be built up of a series of thin-walled concentric tubes which all twist by the same amount. One of these tubes is shown in Fig. 6.14(b) and the area enclosed by the median line is

$$A = (b - 2h)2h + \pi h^2.$$

If b is large compared with h then

$$A \approx 2bh.$$

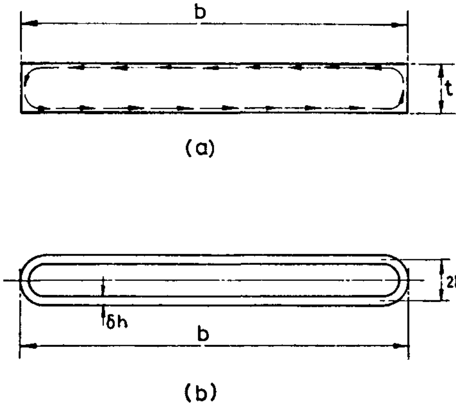


FIG. 6.14

If the tube is subjected to a torque δT then the shear stress from equation (6.14) is

$$\tau = \frac{\delta T}{4bh\delta h}. \tag{6.18}$$

From equation (6.17) the angle of twist is given by

$$\theta = \frac{\tau l 2b}{4bhG}$$

or

$$\theta = \frac{\tau l}{2hG}. \tag{6.19}$$

H

Therefore

$$\frac{\tau}{2h} = \frac{G\theta}{l}.$$

In the rectangular strip therefore the shear stress parallel to the long edge is proportional to the distance h from the central axis. The maximum shear stress occurs at the outer surface and is

$$\tau_{\max} = \frac{tG\theta}{l}. \tag{6.20}$$

From equation (6.18), as the tube becomes infinitely thin

$$\frac{\mathrm{d}T}{\mathrm{d}h} = 4bh\tau$$

and the torque carried by the strip is

$$T = \int_0^{t/2} 4bh\tau \mathrm{d}h.$$

Substituting for τ from equation (6.19) gives

$$T = \int_0^{t/2} \left(\frac{8bG\theta}{l}\right) h^2 \mathrm{d}h$$

$$= (\tfrac{1}{3})bt^3 \frac{G\theta}{l}. \tag{6.21}$$

The quantity $bt^3/3$ is termed the *torsion constant*, denoted by J, but it is *not* the polar second moment of area I_p for the section. Thus $T/J = G\theta/l$.

EXAMPLE 6.5

Determine the angle of twist per unit length and the maximum shear stress in the aluminium channel section shown in Fig. 6.15 when subjected to a pure torque of 200 lb in., and no warping restraint. Shear modulus $= 3{\cdot}9 \times 10^6$ lb/in^2.

The channel is treated as consisting of three rectangular strips, the flanges and the web, and the analysis above will be used on each part.

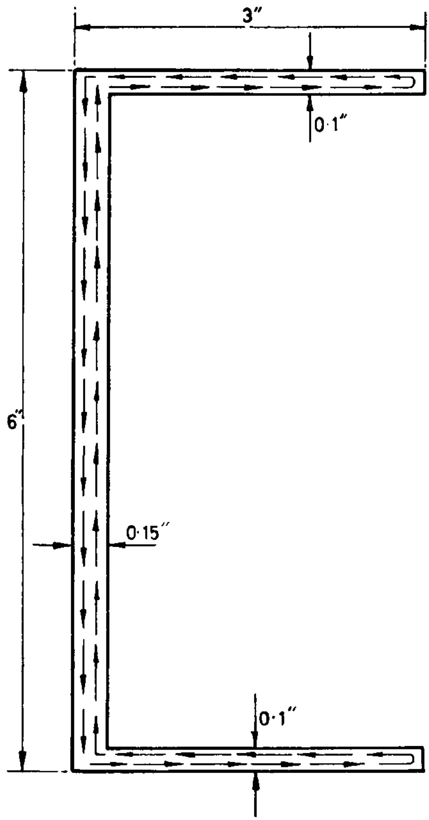


FIG. 6.15

Let the proportion of the torque carried by the flanges and web be T_1 and T_2 respectively, then from equation (6.21)

$$T_1 = \tfrac{1}{3} \times 3 \times 0{\cdot}1^3\, G\frac{\theta}{l} = 0{\cdot}001\, G\frac{\theta}{l}$$

and

$$T_2 = \tfrac{1}{3} \times 6 \times 0{\cdot}15^3\, G\,\frac{\theta}{l} = 0{\cdot}00675\, G\,\frac{\theta}{l}$$

but

$$2T_1 + T_2 = 200.$$

Therefore

$$0{\cdot}00875\, G\,\frac{\theta}{l} = 200$$

or

$$\frac{\theta}{l} = \frac{200}{0{\cdot}00875 \times 3{\cdot}9 \times 10^6} = 0{\cdot}00587 \text{ rad/in.}$$

Therefore angle of twist/ft length $= 0{\cdot}00587 \times 57{\cdot}3 \times 12$

$$= 3{\cdot}18°/\text{ft}.$$

From equation (6.20) the maximum shear stress in the flanges is

$$\tau_{max} = 0{\cdot}1 \times 3{\cdot}9 \times 10^6 \times 0{\cdot}00587$$
$$= 2900 \text{ lb/in}^2,$$

and in the web

$$\tau_{max} = 0{\cdot}15 \times 3{\cdot}9 \times 10^6 \times 0{\cdot}00587$$
$$= 4350 \text{ lb/in}^2.$$

6.9. The Effect of Warping on Torsion of Open Sections

In the previous section and the above example dealing with a thin-walled open section subjected to torsion, the pure torque was supposed to be applied to each end of the member in such a way that there was no axial restraint. Owing to the variation in transverse shear stress, for example in the flanges of the channel above, there is also a variation in longitudinal complementary shear stress which results in axial movement of one flange with respect to the other. Therefore cross sections which were initially plane do not remain so during torsion and there is warping of any cross section. If one or more sections of a member are constrained in some manner to remain plane during torsion then warping is restrained. Resisting torque in the section is supplied in two ways, by simple torsion and also by torque set up through the restraint of warping. Thus an applied torque will cause a smaller angle of twist than when the section is free to warp and torsional stiffness may be considerably increased if warping is restrained. Non-circular closed and solid section members also exhibit warping but the effect is much smaller in comparison with the open section. A detailed treatment of warping behaviour is beyond the scope of this book, but one problem that can be tackled by a particular but relatively simple approach is that of the I-beam.

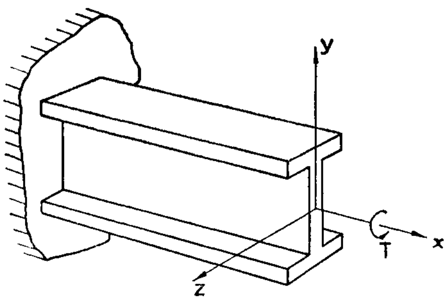


FIG. 6.16

6.10. Torsion of an I-beam with Warping Restrained

Consider an I-beam which is built-in at one end and has a pure torque applied about the longitudinal centroidal axis at the other as in Fig. 6.16. The applied torque is resisted partly by shear stress due to twisting of the flanges and web, which can be found by the three-rectangle approach of example 6.5, and also by the shear stress set up by *bending* of the flanges about the y-axis. It is this latter feature and simple twisting of the web about its centroidal axis which simplifies the analysis of warping in an I-beam.

Now considering a portion of beam of length dx, whose angle of twist is dθ, then from equation (6.21)

$$\frac{T_o}{J} = G \cdot \frac{d\theta}{dx}$$

where T_o is that part of the torque which is absorbed in simple twisting of the beam.

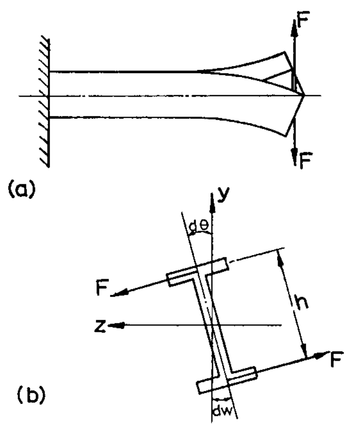


FIG. 6.17

The shear force F across the width of the flanges, Fig. 6.17(a), is obtained from the rate of change of bending moment about the y-axis, so that

$$F = \frac{dM_y}{dx},$$

but

$$M_y = -EI_y \frac{d^2w}{dx^2},$$

where w is the displacement in the z direction. Therefore

$$F = -EI_y \frac{d^3 w}{dx^3}.$$

If the distance between the centroids of the flanges is h then the resisting torque due to warping restraint is

$$\begin{aligned} T_w &= Fh \\ &= -EI_y h \frac{d^3 w}{dx^3}, \end{aligned} \tag{6.22}$$

but from Fig. 6.17

$$dw = \frac{h}{2} d\theta$$

or

$$\frac{d^3 w}{dx^3} = \frac{h}{2} \frac{d^3 \theta}{dx^3}.$$

Substituting the above in equation (6.22) gives

$$T_w = -EI_y \frac{h^2}{2} \frac{d^3 \theta}{dx^3}. \tag{6.23}$$

But the total torque applied to the section is

$$T = T_o + T_w,$$

therefore

$$T = GJ \frac{d\theta}{dx} - EI_y \frac{h^2}{2} \frac{d^3 \theta}{dx^3}.$$

The above equation can be written in a general form as

$$T = C \frac{d\theta}{dx} - C' \frac{d^3 \theta}{dx^3} \tag{6.24}$$

where C and C' are constants which are only a function of the geometry of the cross section and the elastic moduli of the material. The constant C' is often referred to as the *warping constant*.

EXAMPLE

An I-beam has a cross section as shown in Fig. 6.18. It is 20 ft long and is built-in to a support at one end. A torque of 2000 lb in. is applied at the free end about the longitudinal axis

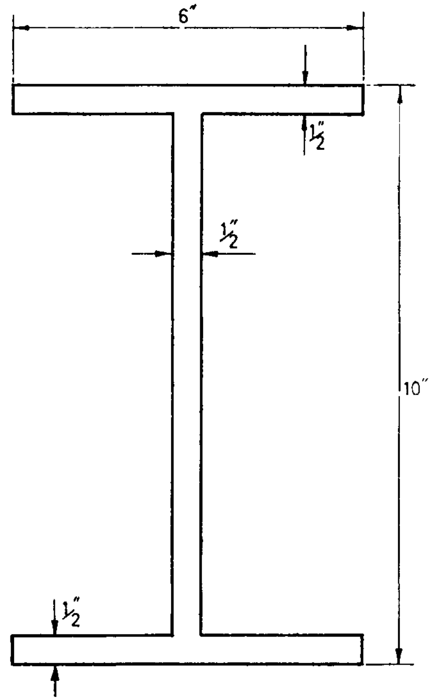


FIG. 6.18

and warping is restrained. Determine the angle of twist at the free end.

$$G = 12 \times 10^6 \text{ lb/in}^2 \text{ and } E = 30 \times 10^6 \text{ lb/in}^2.$$

The first step is to calculate the constants C and C' for which values of J and I_y are required.

The I-section can be treated as three rectangular strips such that $J = \Sigma \frac{1}{3} bh^3$ (see equation (6.21)). Therefore

$$J = \tfrac{1}{3} \times 9 \times 0{\cdot}5^3 + 2 \times \tfrac{1}{3} \times 6 \times 0{\cdot}5^3$$
$$= 0{\cdot}875 \text{ in}^4,$$
$$C = GJ = 12 \times 10^6 \times 0{\cdot}875$$
$$= 10{\cdot}5 \times 10^6 \text{ lb in}^2,$$
$$I_y = 2 \times \frac{0{\cdot}5 \times 6^3}{12} + \frac{9 \times 0{\cdot}5^3}{12}$$
$$= 18{\cdot}1 \text{ in}^4.$$

The warping constant $C' = EI_y \dfrac{h^2}{2}$

$$= 30 \times 10^6 \times 18{\cdot}1 \times \frac{9{\cdot}5^2}{2}$$
$$= 24{,}500 \times 10^6 \text{ lb in}^4.$$

If x is measured from the support then the boundary conditions are:

At $x = 0$, $\dfrac{\mathrm{d}w}{\mathrm{d}x} = 0$ therefore $\dfrac{\mathrm{d}\theta}{\mathrm{d}x} = 0$.

At $x = l$, $M_y = 0$ therefore $\dfrac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 0$ and hence $\dfrac{\mathrm{d}^2\theta}{\mathrm{d}x^2} = 0$.

Then the solution to equation (6.24) is

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{T}{C}\left[1 - \frac{\cosh\{(l - x)/\sqrt{(C'/C)}\}}{\cosh\{l/\sqrt{(C'/C)}\}}\right]$$

The angle of twist is obtained by integrating the above and using the condition $\theta = 0$ at $x = 0$ to determine the integration constant, giving

$$\theta = \frac{T}{C}\left[x + \frac{\sqrt{(C'/C)}\sinh\{(l - x)/\sqrt{(C'/C)}\}}{\cosh\{l/\sqrt{(C'/C)}\}} - \sqrt{(C'/C)}\tanh\{l/\sqrt{(C'/C)}\}\right]. \tag{6.25}$$

At the free end $x = l$ and

$$\theta = \frac{T}{C}\left[l - \sqrt{(C'/C)} \tan h\{l/\sqrt{]C'/C)\}\right] \text{rad},$$

therefore

$$\theta = \frac{2000}{10{\cdot}5 \times 10^6}(240 - 48{\cdot}3 \tanh 4{\cdot}97)$$

$$= 0{\cdot}0365 \text{ rad} = 2{\cdot}09^\circ.$$

In the absence of warping restraint the angle of twist is simply

$$\theta = \frac{Tl}{C} = 0{\cdot}0457 \text{ rad} = 2{\cdot}62^\circ.$$

Examples

1. A hollow shaft, of diameter ratio $\frac{3}{8}$, is required to transmit 800 h.p. at 110 rev/min, the maximum torque being 12 per cent greater than the mean. The shearing stress is not to exceed 4 ton/in^2 and the twist in a length of 10 ft is not to exceed 1°. Calculate the minimum external diameter of the shaft satisfying these conditions. Take $G = 5300$ ton/in^2. (London)

2. A composite shaft of circular cross section and $19\frac{1}{2}$ in. long is rigidly fixed at each end. A 12 in. length of this shaft is of 2 in. diameter and is made of bronze to which is joined the remaining $7\frac{1}{2}$ in. length of 1 in. diameter made of steel. If the limiting shear stress in the steel is 8000 lb/in^2, determine the maximum torque that can be applied at the joint. What is then the maximum shear stress in the bronze? Rigidity modulus of steel $= 12 \times 10^6$ lb/in^2, rigidity modulus of bronze $= 6 \times 10^6$ lb/in^2. (London)

3. A solid alloy shaft of 2 in. diameter is to be coupled in series with a hollow steel shaft of the same external diameter. Find the internal diameter of the steel shaft if the angle of twist per unit length is to be 75 per cent of that of the alloy shaft.

Determine the speed at which the shafts are to be driven to transmit 250 h.p., if the limits of shearing stress are to be 3·5 and 5 ton/in^2 in the alloy and steel respectively. $G_{\text{steel}} = 2{\cdot}2 \times G_{\text{alloy}}$. (London)

4. A shaft is 5 ft long, of 3 in. diameter at one end, and tapers at a uniform rate to 4 in. diameter at the other end. The larger end is firmly fixed and a torque of 2500 lb is applied to the smaller end. Find the total angle of twist and the maximum shear stress. Modulus of torsion $= 11{\cdot}5 \times 10^6$ lb/in^2. (London)

5. A thin tube of mean diameter 1 in. and thickness 1/20 in. is subjected to a pull of $\frac{3}{4}$ ton, and an axial twisting moment of $\frac{3}{8}$ ton in. Find the magnitude and direction of the principal stresses. (Cambridge)

6. A shaft is fitted with a strain gauge, the axis of which is inclined at 45° to the axis of the shaft. When calibrated with the shaft in pure torsion the gauge records a strain of 0·0008.

In service there is an end load which causes the shaft to extend by 0·0090 in. and to contract 0·0005 in. in diameter. What would the strain gauge record in service if subjected to the same torque as before? Shaft length 6 in., diameter 1 in. Assume that the strain gauge only responds to the component of tensile or compressive strain along its axis.

7. Two shafts, one of steel and the other of phosphor-bronze, are of the same length and are subjected to equal torques. If the steel shaft is of 1 in. diameter, find the diameter of the phosphor-bronze shaft so that it will store the same amount of strain energy per unit volume as the steel shaft. Also determine the ratio of the maximum stresses induced in the two shafts. Phosphor-bronze, $G = 7 \times 10^6$ lb/in²; steel, $G = 11{\cdot}7 \times 10^6$ lb/in².

(I.Mech.E.)

8. A 16 in. wide by 0·10 in. thick steel sheet is to be formed into a hollow section by bending through 360° and welding the long edges together (butt joint). The shape may be (1) circular, (2) square, or (3) a 6 in. × 2 in. rectangle. Assume a median length of 16 in. (no stretching of the sheet due to bending), and square corners for non-circular sections. The allowable shearing stress is 12,000 lb/in². For each of the shapes listed, determine the magnitude of the maximum permissible torque.

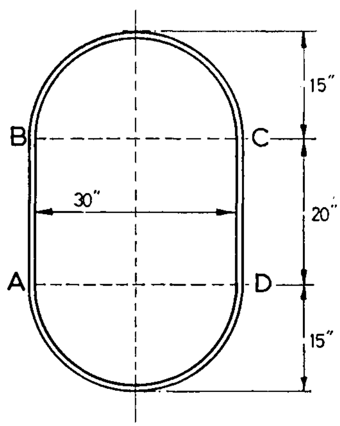


FIG. 6.19

9. Figure 6.19 represents the cross section of an airplane fuselage made of aluminium alloy. The sheet thicknesses are 0·050 in. from A to B and C to D, 0·040 in. from B to C, and 0·032 in. from D to A. For a maximum torque of 669,000 in. lb, determine the magnitude of the maximum shear stress, and the angle of twist per unit length. $G = 4 \times 10^6$ lb/in^2.

10. If a longitudinal slit were made in one side of the hollow sections (2) and (3) in Problem 8, what would then be the maximum permissible torque? Note the difference in magnitude of the torques in the two problems.

Determine the ratio of the angle of twist for the open to the closed section in the two cases.

CHAPTER 7

PROBLEMS IN BENDING AND TORSION

7.1. Introduction

In previous chapters the engineering theories of bending and torsion have been developed and applied separately to appropriate problems. This chapter is concerned with components which are subjected to the combined effects of bending and torsion, such as shafts, coil springs or curved members. These provide useful examples of complex stress systems. The bending and torsion relationships derived in earlier chapters will be used frequently in this chapter and are quoted below for ease of reference

$$\frac{\sigma_x}{y} = \frac{M_z}{I} = \frac{E}{R} \quad \text{or} \quad E\left(\frac{1}{R} - \frac{1}{R_o}\right), \tag{7.1}$$

$$\frac{\tau_{xz}}{r} = \frac{T}{I_p} = \frac{G\theta}{l}. \tag{7.2}$$

7.2 Combined Bending and Torsion of a Circular Shaft

The first example is illustrated in Fig. 7.1 in which a solid circular propeller shaft is supported in bearings at A and B and transmits a torque T. The weight of the propeller W causes bending in the shaft, but any bending due to the weight of the shaft will be considered as negligible. The torque is constant along the length of the shaft and hence from equation (7.2) the maximum shear stress at the outer surface is

$$\tau_{xz_{\max}} = \frac{16T}{\pi d^3}$$

where d is the diameter. The transverse shear stress due to bending is zero where the maximum bending and torsional shear stresses occur and therefore need not be considered.

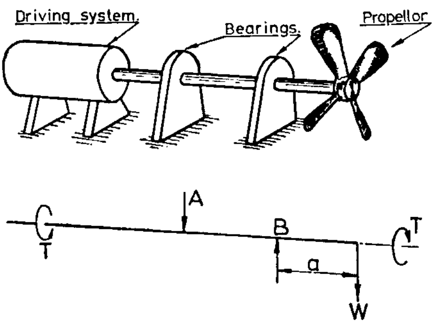


FIG. 7.1

The maximum bending moment and hence maximum bending stress occurs at B, where $M_{\max} = Wa$, neglecting the weight of shaft, and from equation (7.1)

$$\sigma_{x_{\max}} = \frac{32M}{\pi d^3}$$

The principal stresses at the surface of the shafts from equation (3.6), putting $\sigma_y = 0$, will be

$$\sigma_1, \sigma_2 = \frac{\sigma_x}{2} \pm \tfrac{1}{2}\{\sigma_x^2 + 4\tau_{xz}^2\}^{1/2},$$

which may be expressed in terms of the bending moment and torque as

$$\sigma_1, \sigma_2 = \frac{16}{\pi d^3}\left[M \pm \{M^2 + T^2\}^{1/2}. \right] \tag{7.3}$$

Since the third principal stress is zero and σ_2 is negative, the maximum shear stress is

$$\frac{\sigma_1 - \sigma_2}{2} = \frac{16}{\pi d^3}\{M^2 + T^2\}^{1/2}.$$

The above expression may be used when considering the criterion of yielding of ductile shafts, and if σ_Y is the yield stress in simple tension, then according to the Tresca theory at yield

$$\frac{\sigma_Y}{2} = \frac{16}{\pi d^3}(M^2 + T^2)^{1/2}$$

or

$$M^2 + T^2 = \left\{\frac{\pi d^3}{32}\sigma_Y\right\}^2. \tag{7.4}$$

Alternatively if the von Mises criterion, section 3.15.2, is used, it can be shown that

$$M^2 + \tfrac{3}{4}T^2 = \left\{\frac{\pi d^3}{32}\sigma_Y\right\}^2. \tag{7.5}$$

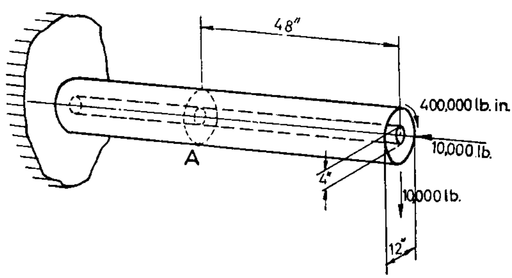


FIG. 7.2

EXAMPLE 7.1

The hollow circular shaft shown in Fig. 7.2 is subjected to the combined action of axial thrust, bending and torsion. Determine the principal stresses and maximum shear stress at point A on the surface of the shaft.

For the cross section shown

$$\text{Area} = \frac{\pi}{4}(12^2 - 4^2) = 100{\cdot}5 \text{ in}^2,$$

$$I = \frac{\pi}{64}(12^4 - 4^4) = 1005 \text{ in}^4,$$

$$I_p = \frac{\pi}{32}(12^4 - 4^4) = 2010 \text{ in}^4,$$

$$\text{Axial stress} = -\frac{10{,}000}{100{\cdot}5} = -99{\cdot}5 \text{ lb/in}^2,$$

$$\text{Bending stress at } A = \frac{My}{I} = -\frac{10{,}000 \times 48 \times 6}{1005}$$
$$= -2865 \text{ lb/in}^2,$$

$$\text{Torsional shear stress at } A = \frac{Tr}{I_p} = \frac{400{,}000 \times 6}{2010}$$
$$= 1194 \text{ lb/in}^2.$$

Transverse shear stress at A due to bending is zero since the complementary longitudinal shear is zero at the free surface

$$\text{Resultant direct stress at } A = -2865 - 99{\cdot}5$$
$$= -2965 \text{ lb/in}^2.$$

Therefore, principal stresses at A are

$$\sigma_{1,2} = -\frac{2965}{2} \pm \tfrac{1}{2}\{(-2965)^2 + 4 \times (1194)^2\}^{1/2}$$
$$= -1483 \pm 1905,$$
$$\sigma_1 = +422 \text{ lb/in}^2,$$
$$\sigma_2 = -3388 \text{ lb/in}^2.$$

With the third principal stress zero the maximum shear stress is

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = 422 - (-3388)$$

$$= 1905 \text{ lb/in}^2.$$

7.3. Bending and Torsion of a Member Curved in Plan

If a curved bar is subjected to loading out of the plane of curvature of the bar, then both bending and twisting will occur.

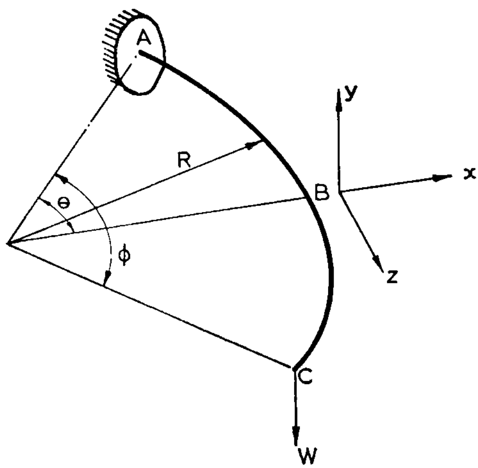


FIG. 7.3

The centre-line of the slender curved member of circular cross section shown in Fig. 7.3 is an arc of a circle. It is fixed at one end and carries a concentrated load acting perpendicular to the plane of curvature at the free end. The moments on a section such as B referred to the coordinate axes are

$$M_x = WR \sin(\phi - \theta) \text{ (Bending)}, \tag{7.6}$$

$$M_y = 0, \tag{7.7}$$

$$M_z = T = WR\{1 - \cos(\phi - \theta)\} \text{ (Torsion)}. \tag{7.8}$$

It is seen from the above equations that both bending moment and torque are functions of the distance along the bar. The direct and shear stresses are determined from the bending and torsion relationships, equations (7.1) and (7.2). The transverse shear stress due to bending is zero at the outside of the bar and where it is a maximum, the bending stress and torsional shear stress are zero.

The deflection of the bar can be determined using Castigliano's Theorem (Part II) discussed earlier. Thus

$$U = \int_0^{\phi} \left(\frac{M^2}{2EI} + \frac{T^2}{2GI_p}\right) R d\theta. \tag{7.9}$$

From equation (5.18) the deflection Δ is given as

$$\Delta = \frac{\partial U}{\partial W},$$

and

$$\frac{\partial M}{\partial W} = R \sin(\phi - \theta)$$

$$\frac{\partial T}{\partial W} = R\{1 - \cos(\phi - \theta)\}.$$

Therefore

$$\Delta = \int_0^{\phi} \left[\frac{WR^3}{EI} \sin^2(\phi - \theta) + \frac{WR^3}{GJ}\{1 - \cos(\phi - \theta)\}^2\right] d\theta. \tag{7.10}$$

EXAMPLE 7.2

A solid circular bar of 1 in. diameter is fixed at one end and is curved in a 60° arc of circle of radius 30 in. The free end of the bar is subjected to a load of 200 lb in a direction perpendicular to the plane of initial curvature of the bar. Determine the maximum values for principal stresses and deflection. If the bar had formed a semi-circle, where would the greatest value of the maximum principal stress occur?

$$E = 30 \times 10^6 \text{ lb/in}^2 \quad \text{and} \quad G = 12 \times 10^6 \text{ lb/in}^2.$$

In this case the bending moment and torque are both maximum at the support and from equations (7.6) and (7.8)

$$M = 200 \times 30 \sin 60 = 5196 \text{ lb in.}$$

$$T = 200 \times 30\,(1 - \cos 60) = 3000 \text{ lb in.}$$

From equation (7.3) the maximum principal stresses are

$$\sigma_1, \sigma_2 = \frac{16}{\pi \times 1^3}\,[5196 \pm \{(5196)^2 + (3000)^2\}^{1/2}],$$

therefore

$$\sigma_1 = 57{,}000 \text{ lb/in}^2,$$

$$\sigma_2 = -4100 \text{ lb/in}^2.$$

After integration of equation (7.10)

$$\Delta = \frac{WR^3}{2EI}\left[\theta + \tfrac{1}{2}\sin 2(\phi - \theta)\right]_0^{\pi/3} + \frac{WR^3}{GI_p}\left[\frac{3\theta}{2} + 2\sin(\phi - \theta) - \tfrac{1}{4}\sin 2(\phi - \theta)\right]_0^{\pi/3},$$

where $\phi = \pi/3$.

Hence

$$\Delta = \frac{WR^3}{2EI}\left(\pi/3 - \frac{\sqrt{3}}{4}\right) + \frac{WR^3}{GI_p}\left[\pi/2 - \sqrt{3} + \frac{\sqrt{3}}{8}\right]$$

$$= \frac{0{\cdot}614 \times 200 \times (30)^3 \times 64}{2 \times 30 \times 10^6 \times \pi \times 1^4} + \frac{0{\cdot}0555 \times 200 \times (30)^3 \times 32}{12 \times 10^6 \times \pi \times 1^4}$$

$$= 1{\cdot}379 \text{ in.}$$

Since the bending moment and torque are functions of ϕ and θ it does not necessarily follow that the maximum values occur at the support.

The maximum principal stress from equation (7.3) is

$$\sigma_1 = \frac{16}{\pi d^3}\{M + (M^2 + T^2)^{1/2}\},$$

and substituting for the values of M and T

$$\sigma_1 = \frac{16WR}{\pi d^3}\left[\sin(\phi-\theta) + \left\{\sin^2(\phi-\theta) + \{1-\cos(\phi-\theta)\}^2\right\}^{1/2}\right]$$

or

$$\sigma_1 = \frac{16WR}{\pi d^3}\left[\sin(\phi-\theta) + \left\{2-2\cos(\phi-\theta)\right\}^{1/2}\right]. \quad (7.11)$$

The greatest value of σ_1, from equation (7.11), is found to occur when $\phi - \theta = 2\pi/3$ and $4\pi/3$.

For the case of the bar forming a semi-circle $\phi = \pi$, and therefore σ_1 has its greatest value at $\theta = \pi - 2\pi/3$ or $\theta = 60°$.

7.4. Helical Springs

An important application of the bending and torsion theories derived in earlier chapters occurs in the case of the helical spring. The geometry of a spring is shown in Fig. 7.4. The centre-line of the wire forming the spring is a helix on a cylindrical surface, such that the helix angle is β. A portion of the wire is shown, and on the cross section at O mutually perpendicular axes Ox, Oy and Oz are set up.

Perhaps the most common form of loading on a spring is a force W along the central axis UV. Since this force acts at a distance R, the coil radius, from the axis of the wire there will be moments set up as follows

$$M_z = WR\cos\beta \text{ (causing torsion in the wire)}, \quad (7.12)$$

$$M_y = WR \sin \beta \text{ (causing bending about the } y\text{-axis and a change in } R), \tag{7.13}$$
$$M_x = 0.$$

In addition to the above, cross sections of the wire are subjected to a transverse shear force $W \cos \beta$ and an axial force $W \sin \beta$.

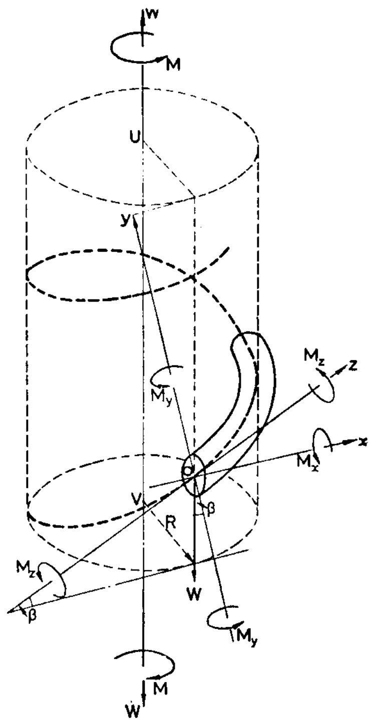


FIG. 7.4

Helical springs are designed and manufactured in two categories, (1) closed coiled (2) open coiled. In the former the helix angle β is very small and the coils almost touch each other. In the second case the helix angle is larger and the coils are spaced further apart.

7.5. Close Coiled Spring Subjected to Axial Force

As stated above, in this spring the angle β is small, and hence $\sin \beta \to 0$ and $\cos \beta \to 1$, therefore bending stresses and stresses due to axial force $W \sin \beta$ can be neglected. The wire is therefore subjected to axial torque $M_z = T = WR$ and a transverse shear force W. The shear stress due to torsion at the outer surface of the wire (Fig. 7.5) is

$$\tau_{\max} = \frac{Td}{2I_p} = \frac{16WR}{\pi d^3}, \tag{7.14}$$

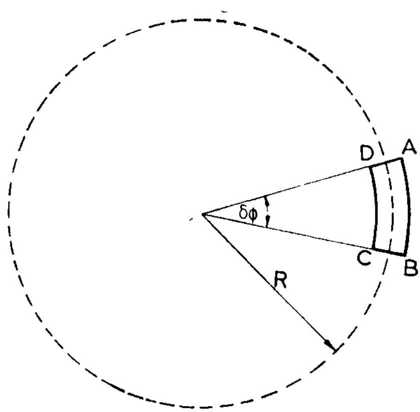


FIG. 7.5

where d is the wire diameter. This shear stress is generally taken as constant around the periphery of the wire, although in fact it is slightly higher on the inside of the coil at CD than on the outside at AB. This is because, for an angle of twist of section BC relative to AD, the shear strain at CD is slightly larger than at AB owing to CD being shorter in length than AB.

A very rough estimate of the mean transverse shear stress due to W is obtained as

$$\tau' = \frac{4W}{\pi d^2}.$$

On the outside of the coil this shear stress is opposite to the torsional shear stress, but on the inside the two shear stresses add together so that the resultant shear stress is

$$\frac{16WR}{\pi d^3}\left(1 + \frac{d}{4R}\right). \tag{7.15}$$

Except in the case of heavy coil springs, d is small compared with $4R$ and the second term in the bracket above can be ignored.

An important property of a spring is its *stiffness*. This is the constant of proportionality k between the load W applied to a spring and the resulting deflection δ, so that

$$W = k\delta,$$

where k is the stiffness.

The next step is therefore to obtain a load deflection relationship. Let the axial deflection of the spring be δ, then the work done by the load W gradually applied is

$$\tfrac{1}{2}W\delta.$$

Consider the small element of wire $ABCD$ in Fig. 7.5. The cross section at BC rotates about its axis relative to AD an amount $\delta\theta$ under the action of the torque $T = WR$. Then the work done by the element of wire in twisting is $\frac{1}{2}T\delta\theta$. Hence, as shown in equation (6.11), the total energy stored in the spring is

$$U = \int_0^{2\pi n} \frac{T^2 R}{2GI_p}\, d\phi,$$

where n is the number of complete coils and thus $2\pi n$ is approximately the length of the wire in the spring.

Substituting for $T = WR$ and integrating,

$$U = \frac{\pi n W^2 R^3}{GI_p}. \tag{7.16}$$

But the total energy stored is equal to the work done by the load. Therefore

$$\tfrac{1}{2}W\delta = \frac{\pi n W^2 R^3}{GI_p} \text{ or } \delta = \frac{2\pi n W R^3}{GI_p},$$

and

$$\delta = \frac{64 n W R^3}{G d^4}. \tag{7.17}$$

The stiffness of the spring is

$$\frac{W}{\delta} = \frac{Gd^4}{64 n R^3}, \tag{7.18}$$

which is a constant.

The stored energy or resilience in terms of the maximum shear stress is obtained by eliminating W between equations (7.16) and (7.14), hence

$$\begin{aligned} U &= \frac{\pi n R^3}{GI_p} \times \left(\frac{\pi d^3 \tau}{16R}\right)^2 \\ &= \pi^2 n R d^2 \tau^2 / 8G \\ &= \frac{\tau^2}{4G} \times \text{volume of wire.} \end{aligned} \tag{7.19}$$

Example 7.3

A vertical copper wire 50 in. long and 0·0125 in. diameter is fixed at its upper end and is attached at the lower end to the top of a steel close-coiled helical spring whose axis is also vertical. The bottom of the spring is fixed and initially the system is just taut. If the temperature of the copper wire is decreased by 50°C and there is no change in temperature of the spring find the force in the wire and the total strain energy stored in the wire and spring.

Diameter of the spring wire = 0·125 in.; coil diameter = 1·25 in.; number of coils = 12; G (steel) = 12×10^6 lb/in^2; E (copper) = 15×10^6 lb/in^2; α (copper) = $17{\cdot}5 \times 10^{-6}$ per °C.

Let the final deflection of the spring be δ. Then from equation (7.18) the force on the spring is

$$W = \frac{\delta G d^4}{64 n R^3} = \frac{12 \times 10^6 \times 0{\cdot}125^4 \delta}{64 \times 12 \times 0{\cdot}613^3} = 15{\cdot}6\delta \text{ lb.}$$

Strain in the copper wire $= \alpha T - \dfrac{\delta}{L}$.

Therefore, force on the wire

$$= \left(\alpha T - \frac{\delta}{L}\right) EA$$

$$= \left(17{\cdot}5 \times 10^{-6} \times 50 - \frac{\delta}{50}\right) \frac{\pi}{4} \times 0{\cdot}0125^2 \times 15 \times 10^6$$

$$= 1{\cdot}61 - 36{\cdot}8\delta.$$

For equilibrium, force in wire = force on the spring, therefore

$$15{\cdot}6\delta = 1{\cdot}61 - 36{\cdot}8\delta$$

$$\delta = \frac{1{\cdot}61}{52{\cdot}4} = 0{\cdot}0307 \text{ in.}$$

Therefore the force in the system $= 15{\cdot}6 \times 0{\cdot}0307$

$= 0{\cdot}479$ lb.

Total strain energy $= 2 \times \frac{1}{2} W\delta$

$= 0{\cdot}479 \times 0{\cdot}0307$ in. lb

$= 0{\cdot}0147$ in. lb.

7.6. Close Coiled Helical Spring Subjected to Axial Couple

If a helical spring is subjected to a couple M about its axis UV, then referring to the geometry of the spring in Fig. 7.4, the resolved components of the couple about the coordinate axes at section O are

$M_z = M \sin\beta$ (causing torsion of the wire), (7.20)

$M_y = M \cos\beta$ (causing bending of the coils to a smaller or larger radius), (7.21)

$M_x = 0.$

Since in the case of the close-coiled spring β is small,

$$M_z \simeq 0, \; M_y \simeq M,$$

so that the maximum bending stress in the wire is

$$\sigma = \frac{(M_y)_{max}}{I} = \frac{32M}{\pi d^3}.$$

The stiffness of the spring when subjected to an axial couple will now be considered using the same approach as in section 7.5 for

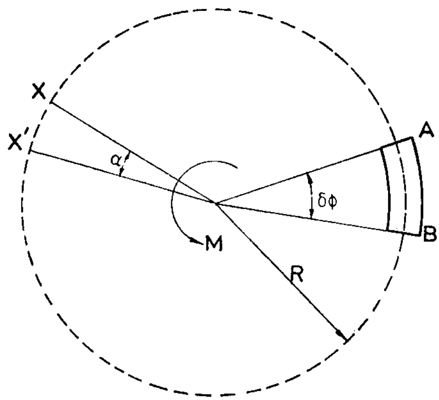


FIG. 7.6

the stiffness under axial force. Let one end of the spring move around the longitudinal axis by an amount α relative to the other end of the spring as in Fig. 7.6; then the work done by the couple M is $\frac{1}{2}M\alpha$.

Consider a small element of wire AB of initial mean radius R, the angle subtended at the centre of the coil being $\delta\phi$. Then from equation (5.17) the stored energy in the element due to bending is

$$\delta U = \tfrac{1}{2}\frac{M^2}{EI}R\delta\phi.$$

Total strain energy

$$U = \int_0^{2\pi n} \frac{M^2}{2EI} R d\phi.$$

Therefore

$$U = \frac{\pi n R M^2}{EI}. \tag{7.22}$$

Equating the work done to the stored energy

$$\tfrac{1}{2}M\alpha = \frac{\pi n R M^2}{EI},$$

$$\alpha = \frac{128 n R M}{E d^4} \tag{7.23}$$

and the stiffness is

$$\frac{M}{\alpha} = \frac{E d^4}{128 n R}. \tag{7.24}$$

7.7. Open Coiled Helical Spring

(*a*) *Axial Force W*

In the case of the open-coiled spring sin β cannot be treated as zero and cos β is no longer nearly unity. Therefore the effects of both bending and torsion on the wire must be included in the analysis. However, the assumption will still be made that the transverse shear stress and axial force in the wire can be neglected in comparison with the torsional and bending stresses.

On an arbitrary cross section of the wire from section 7.4 the torque is $WR \cos \beta$ and the bending moment is $WR \sin \beta$. The total strain energy stored is therefore

$$U = 2\pi n R \left(\frac{(WR \cos \beta)^2}{2GI_p} + \frac{(WR \sin \beta)^2}{2EI} \right) \sec \beta$$

$$= \pi n R^3 W^2 \left(\frac{\cos^2 \beta}{GI_p} + \frac{\sin^2 \beta}{EI} \right) \sec \beta, \tag{7.25}$$

where the total length of the wire is $2\pi n R \sec \beta$.

The work done by the load W is $\frac{1}{2} W\delta$, therefore

$$\tfrac{1}{2} W\delta = \pi n R^3 W^2 \left(\frac{\cos^2 \beta}{GI_p} + \frac{\sin^2 \beta}{EI} \right) \sec \beta$$

or

$$\delta = \frac{64 n R^3 W \sec \beta}{d^4} \left(\frac{\cos^2 \beta}{G} + \frac{2 \sin^2 \beta}{E} \right). \tag{7.26}$$

(*b*) *Axial Couple M*

As stated in section 7.6, the resolved components of the axial couple M about any cross section are a torque $M \sin \beta$ and a bending moment $M \cos \beta$. The total strain energy stored is therefore

$$U = \pi n R \sec \beta \left(\frac{(M \sin \beta)^2}{2GI_p} + \frac{(M \cos \beta)^2}{2EI} \right)$$

$$= M^2 \pi n R \sec \beta \left(\frac{\sin^2 \beta}{GI_p} + \frac{\cos^2 \beta}{EI} \right) \tag{7.27}$$

The work done by the applied couple M is $\frac{1}{2} M\alpha$, where α is the angle one end of the spring moves around the longitudinal axis relative to the other end.

Equating the work done to the stored energy

$$\tfrac{1}{2} M\alpha = M^2 \pi n R \sec \beta \left(\frac{\sin^2 \beta}{GI_p} + \frac{\cos^2 \beta}{EI} \right),$$

hence

$$\alpha = \frac{128nRM\sec\beta}{d^4}\left(\frac{\sin^2\beta}{2G} + \frac{\cos^2\beta}{E}\right). \tag{7.28}$$

(*c*) *Principal Stresses*

In the open-coiled helical spring, since both bending and torsional shear stresses are present, principal stresses are of interest. From equation (7.3), section 7.2, at the outside of the wire

$$\sigma_1, \sigma_2 = \frac{16}{\pi d^3}\{M \pm (M^2 + T^2)^{1/2}\}.$$

Axial force W case (section 7.7(*a*))

$$M = WR\sin\beta; \quad T = WR\cos\beta$$

Therefore

$$\begin{aligned}\sigma_1, \sigma_2 &= \frac{16}{\pi d^3}[WR\sin\beta \pm \{W^2R^2(\sin^2\beta + \cos^2\beta)\}^{1/2}]\\ &= \frac{16}{\pi d^3}(WR\sin\beta \pm WR)\\ &= \frac{16WR}{\pi d^3}(\sin\beta \pm 1).\end{aligned} \tag{7.29}$$

For the close-coiled spring where $\beta \to 0$,

$$\sigma_1, \sigma_2 = \pm\frac{16WR}{\pi d^3}.$$

Axial couple M case (section 7.7(*b*))

$$M = M\cos\beta; \quad T = M\sin\beta.$$

Therefore

$$\begin{aligned}\sigma_1, \sigma_2 &= \frac{16}{\pi d^3}[M\cos\beta \pm \{M^2(\cos^2\beta + \sin^2\beta)\}^{1/2}]\\ &= \frac{16M}{\pi d^3}(\cos\beta \pm 1).\end{aligned} \tag{7.30}$$

For the close-coiled spring where $\beta \to 0$,

$$\sigma_1, \sigma_2 = +\frac{32M}{\pi d^3} \quad \text{and} \quad 0.$$

$$\text{Maximum shear stress} = \frac{\sigma_1 - \sigma_2}{2} = \tau_{\max}$$

since $\sigma_1 > 0 > \sigma_2$.

$$\left.\begin{aligned} &\text{Open-coiled axial force } \tau_{\max} = \frac{16WR}{\pi d^3}, \\ &\text{Open-coiled axial couple } \tau_{\max} = \frac{16M}{\pi d^3}. \end{aligned}\right\} \quad (7.31)$$

It is interesting to note that in each case above, the maximum shearing stress is independent of the coil angle β.

Example 7.4

A flexible coupling between two shafts in-line consists of a 6-coil spring of helix angle 30°. The mean coil radius is 1 in. and wire diameter 0·2 in. The yield stress of the spring wire in simple tension is 90,000 lb/in^2 and G and E are 12×10^6 lb/in^2 and 30×10^6 lb/in^2 respectively. Determine the maximum shaft torque that can be transmitted through the coupling and the take-up angle of rotation of one shaft relative to the other using the maximum shear stress yield criterion.

The limiting shear stress for the wire according to the Tresca yield criterion = 90,000/2 = 45,000 lb/in^2.

Hence from equation (7.31) maximum shaft torque that can be transmitted without yielding is

$$M = \frac{45{,}000 \times \pi \times 0{\cdot}2^3}{16} \text{ lb in.}$$

$$= 70{\cdot}68 \text{ lb in.}$$

The angle of rotation of one shaft relative to the other on initial transmission of torque is obtained from equation (7.28) where

$$\alpha = \frac{128 \times 6 \times 1 \times 70{\cdot}68 \sec 30}{0{\cdot}2^4} \left\{ \frac{\sin^2 30}{2 \times 12 \times 10^6} + \frac{\cos^2 30}{30 \times 10^6} \right\}$$

$$= 39{\cdot}2 \times 10^6 \left\{ \frac{0{\cdot}25}{24 \times 10^6} + \frac{0{\cdot}75}{30 \times 10^6} \right\}$$

$$= 1{\cdot}39 \text{ rad} = 79{\cdot}5°.$$

7.8. Approximate Theory for Leaf Springs

The spring shown in Fig. 7.7(*a*) consisting of a series of strips of equal width, thickness and curvature, but varying length is

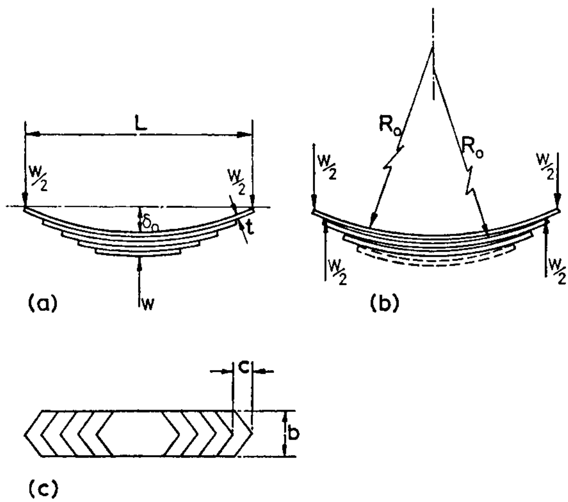


FIG. 7.7

well known as the type used in automobiles, locomotives and railway carriages. If the strips all have the same radius of curvature R_0, contact between the strips is made only at the ends, Fig. 7.7(*b*). Hence the parallel length of the leaves is subjected to pure bending and also the triangular end portions, Fig. 7.7(c), are loaded as cantilevers for which the linearly changing width provides a constant bending stress along the length c.

For a central load W the maximum bending moment for each leaf is given by

$$M = \frac{Wc}{2}$$

and if the width and thickness for all leaves are b and t respectively, then for each leaf

$$I = \frac{bt^3}{12}.$$

Therefore the maximum bending stress is

$$\sigma = \frac{Wc}{2} \times \frac{12}{bt^3} \times \frac{t}{2}.$$

For pure bending along the parallel length of all leaves, the load must be transmitted at the tip and base of each triangular portion of each leaf and it is seen in Fig. 7.7(c) therefore that

$$c = \frac{L}{2n},$$

so that

$$\sigma = \frac{3}{2}\frac{WL}{nbt^2}.$$

The central deflection of a leaf subjected to a pure bending moment M is given by

$$\begin{aligned} \delta &= \frac{ML^2}{8EI} \\ &= \frac{Wc}{2} \times \frac{L^2}{8} \times \frac{12}{Ebt^3} \\ &= \frac{3}{8}\frac{WL^3}{Ebt^3 n} \qquad (7.32) \end{aligned}$$

If R_0 and R are the unloaded and loaded radii of curvature respectively, then

$$\frac{1}{R_o} - \frac{1}{R} = \frac{M}{EI} = \frac{6Wc}{Ebt^3}$$

$$= \frac{3WL}{nEbt^3}. \qquad (7.33)$$

The load necessary to straighten the spring is termed the proof load W_o and is found by putting $R = \infty$ in equation (7.33) then

$$\frac{1}{R_o} = \frac{3W_oL}{nEbt^3}.$$

Now from the circular arc

$$R_o = \frac{L^2}{8\delta_o}$$

where δ_o is the initial dip of the spring below the horizontal, Fig. 7.7(*a*). Therefore

$$W_o = \frac{8nEbt^3\delta_o}{3L^3}. \qquad (7.34)$$

This expression is similarly obtained by putting $W = W_o$ and $\delta = \delta_o$ in equation (7.32).

The maximum strain energy stored in the spring on application of the proof load is

$$U_0 = \tfrac{1}{2}W_o\delta_o$$

$$= \frac{3}{16}\frac{W_o^2L^3}{nEbt^3}. \qquad (7.35)$$

EXAMPLE 7.5

A motor car leaf spring is to have a length of 40 in. and a leaf width of 2 in. When the car is stationary the dip of the spring below the horizontal will be 2 in. It is estimated that the most severe load on each spring when the laden car is moving over rough ground will be 380 lb. If the maximum bending stress is not to exceed 40,000 lb/in^2 and Young's Modulus for the steel is 30×10^6 lb/in^2 determine the required number and thickness of the leaves. Assume that the most severe load is also the proof load for straightening the spring.

The radius of curvature of the spring is given by

$$R_o = \frac{L^2}{8\delta_o} = \frac{40^2}{8 \times 2} = 100 \text{ in.}$$

Now $2\sigma_o/t = E/R_o$ where t is a leaf thickness. Therefore

$$t = \frac{2 \times 40{,}000 \times 100}{30 \times 10^6} = 0{\cdot}267 \text{ in.}$$

It is more convenient to make the thickness $t = \frac{1}{4}$ in.

From equation (7.34) the number of leaves may be obtained, therefore

$$n = \frac{3L^3 W_o}{8Ebt^3\delta_o}$$

$$= \frac{3 \times 40^3 \times 380}{8 \times 30 \times 10^6 \times 2 \times 0{\cdot}25^3 \times 2}$$

$$= 4{\cdot}86.$$

Therefore five leaves will be required.

The actual maximum bending stress on straightening will then be

$$\sigma = \frac{3 \times 380 \times 40}{2 \times 5 \times 2 \times 0{\cdot}25^2}$$

$$= 36{,}500 \text{ lb/in}^2,$$

which is within the allowable limit.

Examples

1. A solid shaft of 4 in. diameter is subjected to a bending moment of 25 ton in. Find the maximum torque which the shaft can transmit in addition to carrying the bending moment if the allowable direct stress is not to exceed 5 ton/in^2 and the maximum shear stress $3\frac{1}{2}$ ton/in^2. Which of these two limiting stresses is reached? Give the maximum value of the other stress.

2. A mild steel shaft of 2 in. diameter is subjected to a bending moment of 17,280 in. lb. If the yield point of the steel in simple tension is 30,000 lb/in^2, find the maximum torque that can also be applied according to (*a*) the maximum principal stress, (*b*) the maximum shear stress, (*c*) the shear strain energy theories of yielding.

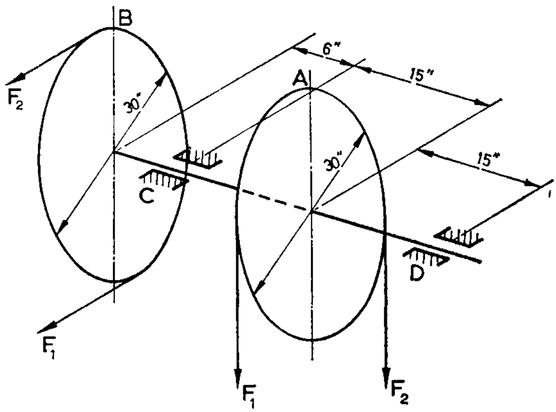


FIG. 7.8

3. A pulley shaft is supported in bearings as shown in Fig. 7.8. Pulley *A* receives 60 h.p. at 250 rev/min through a vertical belt and power is transmitted from pulley *B* by a horizontal belt. The ratio of the belt tensions is $F_1/F_2 = 3$. Determine a minimum diameter for the shaft using the shear strain energy criterion of yielding. Yield stress in simple tension for the shaft material is 18 ton/in^2.

4. A pipe is formed into a U-shape, the straight portions being 3 ft long and the semi-circular end being of 1 ft radius. One end of the pipe is clamped so that the U lies in a horizontal plane. The free end of the pipe is loaded vertically with 200 lb. Plot the distribution of bending moment and torque from one end of the pipe to the other.

5. Two strain gauges are fitted at $\pm 45°$ to the axis of a 3 in. diameter shaft. The shaft is rotating and, in addition to transmitting power, it is subjected to an unknown bending moment and a direct thrust. The readings of the gauges are recorded, and it is found that the maxima or minima values for each gauge occur at 180° intervals of shaft rotation and are $-0{\cdot}00060$ and $+0{\cdot}00030$ for the two gauges at one instant and $-0{\cdot}00050$ and $+0{\cdot}00040$ for the same gauges 180° of rotation later. Determine the transmitted torque and applied bending moments and thrust. Assume all the forces and moments are steady, i.e. do not vary during each rotation of the shaft. $E = 30 \times 10^6$ lb/in², $\nu = 0{\cdot}29$, $G = 11{\cdot}6 \times 10^6$ lb/in².

6. The principal strains e_1 and e_2 are measured at a point on the surface of a shaft which is subjected to bending and torsion. The values are $e_1 = +0{\cdot}0011$, $e_2 = -0{\cdot}0006$ and e_1 is inclined at 20° to the axis of the shaft.

If the diameter of the shaft is 2 in. and the rigidity modulus for the material is 12×10^6 lb/in², determine analytically the applied torque and the maximum shear stress in the material at the point concerned, and check graphically.

7. A hollow shaft, of 1 in. external and 0·5 in. internal diameter and 8 in. long, is fixed horizontal as a cantilever. Across the free end, perpendicular to the axis of the shaft, is rigidly fixed a horizontal lever arm, 6 in. long, from the centre of the shaft. A load of 50 lb falls 1 in. on to the end of the lever. Determine the momentary maximum deflection at the lever end due to bending and twisting of the shaft.

8. A close-coiled helical spring is made from a steel wire of 0·25 in. diameter, and there are ten free coils having a mean diameter of 3 in. The spring is subjected to an axial load of 20 lbf. Determine the maximum intensity of shear stress in the steel and the total deflection under the above load. What is the stiffness of the spring in pounds per foot of deflection? $G = 12 \times 10^6$ lb/in². (I.Mech.E.)

9. In an open-coil helical spring of ten coils, the stresses due to bending and twisting are 14,000 and 15,000 lb/in² respectively when the spring is axially loaded. Assuming the mean diameter of the coils to be eight times the diameter of the wire, find the maximum permissible axial load and the diameter of the wire for a maximum extension of 0·7 in. $E = 30 \times 10^6$ lb/in², $G = 11 \times 10^6$ lb/in². (London)

10. Two shafts in line, which are prevented from moving axially, are connected by a helical spring, the spring fitting loosely on the shafts and having its ends fixed to the shafts. Show that, if the coils of the spring are of circular cross section, and are inclined at 45° to the axis, the couple per unit angle of twist is given by

$$\frac{r^4}{8\sqrt{2}\,.\,nR}\left(\frac{E}{2} + G\right),$$

where r is the radius of the cross section, R is the mean coil radius and n the number of coils. (Cambridge)

11. A laminated steel carriage spring, 30 in. long, is built up of 9 leaves of equal thickness and 2 in. wide. Neglecting friction, find the thickness of the leaves if the stress in the material is to be limited to 25 ton/in² when the spring is loaded with 750 lb in the centre.

To what radius should the leaves be initially bent for the spring to be flat when under this load and what will be its central deflection? $E = 30 \times 10^6$ lb/in². (London)

APPENDIX

Sign Conventions in Bending Theory

THE importance of choosing and adhering to consistent sign conventions in Mechanics of Solids was emphasised in various chapters. In most parts of the subject there is a considerable measure of uniformity amongst text books, teachers and engineers on sign conventions; for example, tension is invariably regarded as positive and compression negative in sign. However, in the case of bending theory there are at least six quantities, distance, bending moment, shear force, etc., which are interrelated, and as there is no standard sign convention established, there is these days a number of different conventions in regular use. The user of one particular convention will often strongly advocate the advantages of that one compared to another. Although it is sometimes argued that a student should readily be able to switch from using one convention to an alternative, for various reasons this is not found to be easy or satisfactory in practice.

The bending convention used in this book is the same as in other books in this series, e.g. *Beams and Framed Structures* by J. Heyman, and its merits or otherwise will not be discussed here. However, in order to cater partially for those not liking or accustomed to that convention, two alternatives are outlined briefly below:

(*a*) The first convention is based on mathematical procedure in which distance y and deflection v are measured positive upwards, distance x is positive from left to right, and rotation is positive in the anticlockwise direction.

Figure A.1 illustrates the convention for positive load, bending moment and shear force, and Fig. A.2 shows a small length of beam in equilibrium under the above forces and moments.

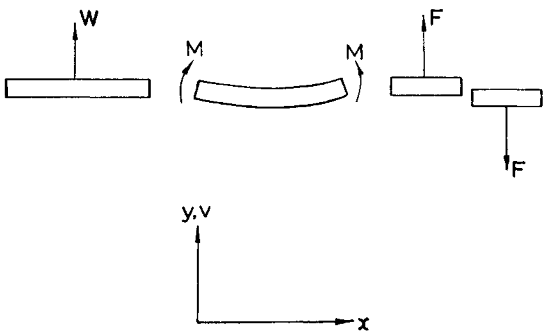


FIG. A.1

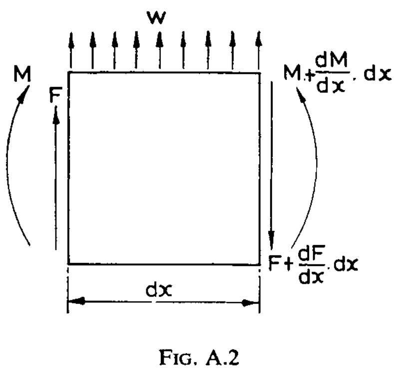


FIG. A.2

The merit of this convention is its conformity with that used in mathematics. On the other hand, an exception may be taken to beams thus being generally subjected to negative loads which produce negative deflections. The mathematical convention is preferable when studying vibrating beam theory.

The more important relationships derived in Chapters 4 and 5 are given below with the appropriate signs for this convention.

$$\frac{dF}{dx} = w \qquad (4.1A) \qquad\qquad \frac{dM}{dx} = F \qquad (4.2A)$$

$$\sigma_x = -\frac{M_z y}{I_z} \qquad (4.10A)$$

(For a positive bending moment, positive tensile bending stress is obtained where y is negative and compressive stress where y is positive.)

$$\frac{d^2v}{dx^2} = \frac{M}{EI} \qquad (5.4A)$$

If it is preferred to have downward loading and deflection positive, the mathematical convention can still be employed in a consistent manner by rotating all the diagrams above through 180°. The equations above will then remain the same.

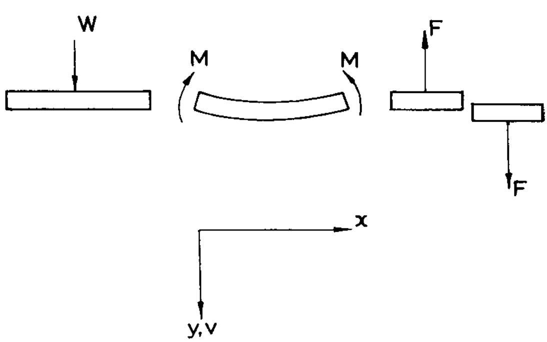


FIG. A.3

(*b*) Another fairly common convention in which loads and deflections are positive downwards is illustrated in Fig. A.3. For this convention, the equations above become

$$\frac{\mathrm{d}F}{\mathrm{d}x} = -w \qquad (4.1\text{B}) \qquad\qquad \frac{\mathrm{d}M}{\mathrm{d}x} = F \qquad (4.2\text{B})$$

$$\sigma_x = \frac{M_z y}{I_z} \qquad (4.10\text{B}) \qquad\qquad \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = \frac{M}{EI} \qquad (5.4\text{B})$$

ANSWERS TO EXAMPLES

Chapter 1

1. 10,700 lb/in^2. **2.** 0·948 in. **3.** 0·754 in^2. **4.** 10·78 ft.
5. 1865 lb/in^2; 12·7 ton/in^2. **6.** 0·604 in. **7.** 0·595 in.; 0·637 in.

Chapter 2

2. 1228 lb. **3.** 0·51 in.
4. 325 lb; 405 lb; 270 lb; 0·054 in.
5. 3855 lb/in^2 (iron); 1239 lb/in^2 (bronze); 5·8 in. from the iron rod.
6. 6·66 ton/in^2; 8·6 ton/in^2 (copper); 2·8 ton/in^2 (steel).
7. 4440 lb/in^2 (copper); 1110 lb/in^2 (steel); 92·5 lb/in^2 interface pressure.
8. 4140 lb.
9. 526·3 lb/in^2; 2·15 × 10^{-3} times original volume.
10. $\sigma_y = \frac{2}{3}\dfrac{v\sigma_x}{(1 - v)}$.
11. 1180 lb/in^2; 92,600 lb.

Chapter 3

1. $\sigma = 4$ ton/in^2; $\tau = -2$ ton/in^2.
2. 5·58 and 2·17 ton/in^2.
3. 4·8 and 1·2 ton/in^2; 61° 48′, 151° 48′.
4. 319 lb/in^2; 11° 45′.
5. $\sigma = 3{\cdot}5$ ton/in^2; $\tau_{max} = 3{\cdot}2$ ton/in^2; 64½°, 154½°.
6. Resultant 7·71 and −5·66 ton/in^2; normal 6·48 and −2·68 ton/in^2; shear 4·18 and 4·98 ton/in^2.
7. 4·63 ton/in^2; 6·93 and −2·31 ton/in^2.
8. 4·9 and 0·6 ton/in^2; 34° 30′, 80°.
9. 0·00098; 0·00008; 24·4°, 65·6°.
10. $\gamma_{max} = 0{\cdot}00085$; $\varepsilon = -0{\cdot}0005$.
11. 15,400 lb/in^2, −1245 lb/in^2; 17° and 107°.
12. 5·815 × 10^{-4}; −6·685 × 10^{-4}; 12,450 and −16,320 lb/in^2.
13. 0·000341, 0·000548, 0·00044; 10,111 and 5055 ton/in^2.
14. 13,600 ton/in^2; 0·292; 10,900 ton/in^2.
15. 0·0188 ton in.
16. 92 in.
17. (*a*) Yes. (*b*) No.
18. 1580 and 1369 lb/in^2 respectively; 1585 and 1369 lb/in^2.

Chapter 4

1. 14,400 lb ft; 4800 lb ft.
2. S.F. ton: A to B, -2; B to D, $+12{\cdot}29$; $+9{\cdot}29$ at D to $-2{\cdot}71$ at 4 ton load (straight line); $-6{\cdot}71$ at 4 ton load to $-16{\cdot}71$ at C (straight line). B.M. (ton ft): zero at A to -10 at B (straight line); -10 at B to $+39{\cdot}2$ at D (straight line); $+39{\cdot}2$ at D to $+58{\cdot}6$ at 4 ton load with max. of $+61{\cdot}4$ at 6·36 ft from C (parabolic); 58·6 at 4 ton load to zero at C (parabolic).
3. 90 ton ft at centre of AB, -260 ton ft at B; $+9{\cdot}5$ ton at A, $-22{\cdot}5$ ton at left of B, $+18$ ton at right of B.
4. (i) 17·4 and 12·6 ton; (ii) 111·4 ton ft at 17·4 ft from the end with the 40 ton ft moment.
5. 1173 lb/in^2.
6. 7·3 ton/in^2 bending; 1·2 ton/in^2 shear.
7. (*a*) 8·2 ton/in^2; (*b*) 1·14 ton/in^2; (*c*) 7·54 ton/in^2, $-0{\cdot}08$ ton/in^2.
8. 49·8 lb/in^2.
9. Max. stresses; $+2{\cdot}56$ and $+10{\cdot}24$ ton/in^2; 0·617 in.
10. 0·74 ton.
11. $(6/7)(wl/ta^2)$.
12. $(16/35)a$.
14. 2·9 in^2; 636,000 lb in.; 16,000 lb/in^2; 388 lb/in^2.
15. 219 lb.
16. Depth of section = 0·6 in.; diameter = 18 in.
17. $+6{\cdot}375$ and $-11{\cdot}8$ ton/in^2.

Chapter 5

1. 0·518 in.
2. 0·1015 in.
3. $\dfrac{Wal}{EI}\left(\dfrac{a^2}{3}+\dfrac{2}{a}-\dfrac{l}{8}\right)$ at ends; $\dfrac{Wl^2}{8EI}\left(\dfrac{l}{3}-a\right)$ at mid-span.
4. 0·08 in.
5. $\pm 0{\cdot}00184$ rad; 0·00585 in.
6. $0{\cdot}52L$ from left end; $\dfrac{2{\cdot}345}{360}\dfrac{wL^5}{EI}$.
7. 2·93 ft from free end; 0·0254 in.
8. Neglects the work done by the stress distribution at the fixed end.
9. Energy concentrated load = $\frac{5}{8}$ energy distributed load.
10. 0·835 in.
11. $\dfrac{Wa^2b^2}{3EI(a+b)}+\dfrac{wab}{24EI}(a^2+b^2+3ab)$.
12. $\dfrac{3\pi}{2}\dfrac{Wr^3}{EI}$ vertical; $\dfrac{2Wr^3}{EI}$ horizontal.

13. $\dfrac{2W}{EI}\left\{\dfrac{l^3}{3} + r\left(\dfrac{\pi}{2}l^2 + \dfrac{\pi}{4}r^2 + 2lr\right)\right\}$.

14. $\dfrac{Wr}{2}\left\{\dfrac{r(\pi - 2)}{2l + \pi r} - 1\right\}$.

Chapter 6

1. 7·68 in.
2. 9420 lb in; 5000 lb/in².
3. 1·584 in.; 1480 rev/min.
4. 0·652°; 5650 lb/in².
5. 11·5 ton/in², −1·98 ton/in²; 31·75°, 121·75°.
6. 0·0013.
7. 1·089 in.; $\sigma_s/\sigma_b = 1{\cdot}293$.
8. 48,900; 38,400; 28,800 in. lb.
9. 8000 lb/in²; 0·0582°/ft.
10. (2) 640 in. lb; 20 (3) 640 in. lb; 15.

Chapter 7

1. 28·41 ton in.; 3·01 ton/in² shear stress.
2. 24,300 lb in.; 16,000 lb in.; 18,500 lb in.
3. 1·74 in.
5. Torque 55,300 lb in.; Moment 11,150 lb in.; Thrust 59,600 lb.
6. 20,400 lb/in²; 20,550 lb in.
7. 0·216 in.
8. 9780 lb/in²; 0·9215 in.; 260 lb.
9. 39 lb; 0·22 in.
11. 0·183 in.; 49 in.; 2·3 in.

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