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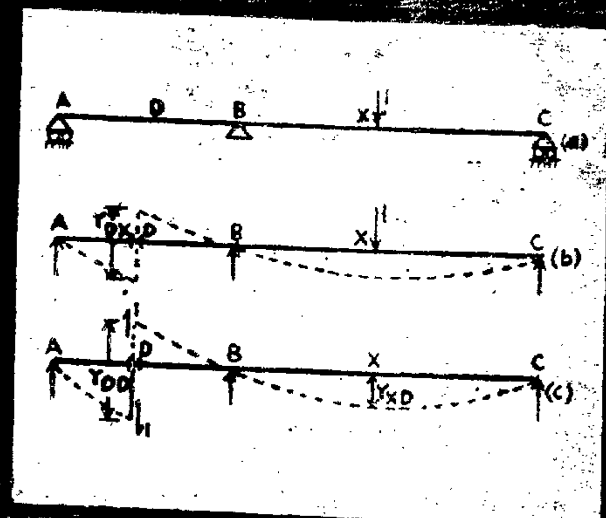

B.C. PUNMIA
ASHOK JAIN
ARUN JAIN

S.I. UNITS

SMTS II

Theory

of

Structures

[CONTAINING 28 CHAPTERS]

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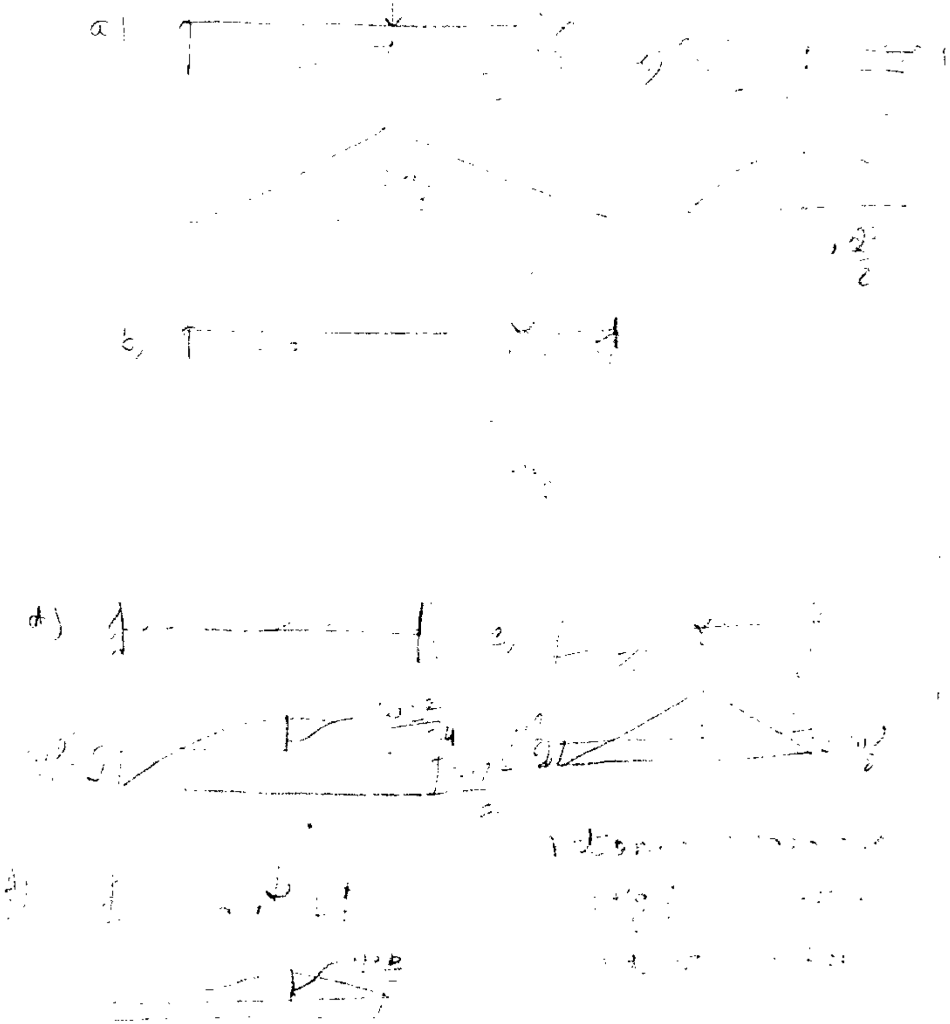

STRENGTH OF MATERIALS
AND
THEORY OF STRUCTURES

(VOLUME II)

By

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Preface to the Third Edition

In this volume, the author has attempted to present more advanced topics in strength of materials and mechanics of structures in a rigorous and coherent manner. Both volumes I and II cover the entire course for degree, diploma and AMIE examinations in this subject. Volume II has been divided into four parts. Part I contains four chapters on moving loads on statically determinate beams and frames, as well as on statically indeterminate beams. Part II deals with statically indeterminate structures and contains ten chapters, including a chapter on deflection of perfect frames. Part III contains five chapters on advanced topics on strength of materials. These chapters are mostly useful for mechanical engineering students. Part IV has three chapters on miscellaneous topics, including one on elementary theory of elasticity. The contents of the book are so designed that the book is equally useful to civil as well as mechanical and electrical engineering students.

The book is written entirely in metric units. Each topic introduced is thoroughly described, the theory is rigorously developed, and a large number of numerical examples are included to illustrate its application. General statements of important principles and methods are almost invariably given by practical illustrations. A large number of problems are available at the end of each chapter, to enable the student to test his reading at different stages of his studies.

The author is highly thankful to Prof. S.C. Goyal and Shri O.P. Kalani for their permission to reproduce some chapters and examples from the book 'Strength of Materials and Theory of Structures Vol. II' written by the author in their collaboration. Thanks are also due to the senate of the London University and to the Secretary of the Institution of Structural Engineers, London, for their kind permission to reproduce their examination questions. The London University is in no way responsible for the accuracy of the answers. In preparing this text, the author has freely consulted many excellent books on the subject and the help is gratefully acknowledged.

Every effort was made to eliminate errors in the book, but should the reader discover some, the author would appreciate having these brought to his attention. Suggestions from the readers for improvement in the book will be most gratefully acknowledged.

Jodhpur,
January 17, 1971 **B.C. PUNMIA**

Preface to the Fourth Edition

In the fourth edition, the subject matter of the book has been updated. An appendix, containing questions from the AMIE section B examinations in "Theory of Structures", has been added. The author is thankful to Shri J.N. Shrivastava for his valuable suggestions for improvements in the book.

Jodhpur,
January 1, 1974 **B.C. PUNMIA**

Preface to the Sixth Edition

In the Sixth Edition, the subject matter has been revised and updated. A large number of Examples in SI units have been added at the end of the book.

Jodhpur,
15th June, 1982 **B.C. PUNMIA**

Preface to the Seventh Edition

In the Seventh Edition, the entire book has been rewritten using SI units. The old diagrams have been replaced by new ones.

Jodhpur
Deepawali **B.C. PUNMIA**
18.11.85

Preface to the Eighth Edition

In the Eighth Edition, the subject matter has been revised and updated. A new chapter on Building Frames has been added at the end of the book.

Jodhpur
1st Sept., 1988 **B.C. PUNMIA**

Preface to the Ninth Edition

In the Ninth Edition, the book has been completely **rewritten.** The book has been divided into **four sections. Section 1,** containing **five chapters,** is on 'Moving Loads'. **Section 2** on 'Statically Indeterminates Structures', contain **eleven chapters. Section three** is devoted to 'Advanced Topics in Strength of Materials' and has **six chapters.** Lastly, **section 4** has six chapters on 'Miscellaneous Topics'. **Thus in the Ninth Edition of the book, which has 28 chapters, five new chapters have been added.**

In each chapter, the subject matter has been rearranged and new articles have been added. Many new **advanced problems** have been added which will be useful for competitive examinations.

Jodhpur **B.C. PUNMIA**
Mahaveer Jayantee **ASHOK KUMAR JAIN**
15-4-92 **ARUN KUMAR JAIN**

Contents

PART I : MOVING LOADS

PART II
STATICALLY INDETERMINATE STRUCTURES

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ADVANCED TOPICS IN STRENGTH OF MATERIALS

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MISCELLANEOUS TOPICS

SECTION 1

MOVING LOADS

Chapter

1. **ROLLING LOADS**
2. **INFLUENCE LINES FOR BEAMS**
3. **INFLUENCE LINES FOR GIRDERS WITH FLOOR BEAMS**
4. **INFLUENCE LINES FOR FRAMES**
5. **THE MULLER-BRESLAU PRINCIPLE**

1

Rolling Loads

1·1. INTRODUCTION

In the case of static or fixed load positions, the B.M. and S.F. diagrams can be plotted for a girder, by the simple principles of statics. In the case of rolling loads, however, the B.M. and S.F. at a section of the girder change as the loads move from one position to the other. The problem is, therefore, two-fold :

(*i*) to determine the load positions for maximum bending moment or shear force for a given section of a girder and to compute its value, and (*ii*) to determine the load positions so as to cause absolute maximum bending moment or shear force *anywhere* on the girder.

For every cross-section of girder, the maximum B.M. and S.F. can be worked out by placing the loads in appropriate positions. When these are plotted for all the sections of the girder, we get the *maximum B.M. and maximum S.F. diagrams*. The ordinate of a maximum B.M. or S.F. diagram at a section gives the *maximum B.M.* (or S.F.) at that section, due to a given train of loads.

We shall consider the following cases of loadings :

1. Single concentrated load.
2. Uniformly distributed load longer than the span of the girder.
3. Uniformly distributed load shorter than the span of the girder.
4. Two loads with a specified distance between them.
5. Multiple concentrated loads (train of wheel loads).

Sign Conventions

The following sign conventions will be followed for B.M. and S.F. at a given section (Fig. 1·1).

(1) A shear force having an upward direction to the right hand side of a section or downwards to the left of the section will be taken positive. Similarly, a negative S.F. will be one that has a downward direction to the right of the section or upward direction to the left of the section [Fig. 1·1(*a*, *b*)].

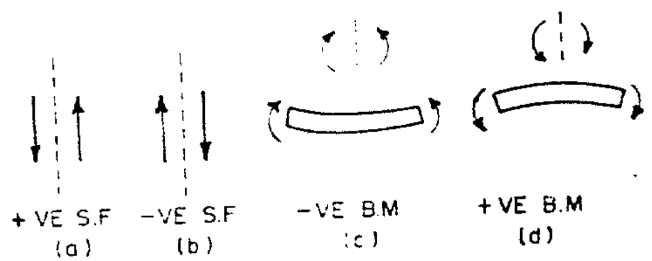


Fig. 1·1.

(2) A B.M. causing concavity upwards will be taken as negative and will be called sagging B.M. Similarly a B.M. causing convexity upwards will be taken as positive, and will be called hogging bending moment [Fig. 1·1(*c*, *d*)].

1·2. SINGLE CONCENTRATED LOAD

Let us now consider a single concentrated load W travelling or rolling along a simply supported beam or girder AB, of span L, from left to right.

(*a*) MAXIMUM SHEAR FORCE DIAGRAMS

Consider a point C, distant x from the left support A. Let the distance of load W be y from A. For any load position, the reaction $R_B=\dfrac{Wy}{L}$ and $R_A=\dfrac{W(L-y)}{L}$.

(*ai*) Load in AC (y<x)

Let the load W be in AC, such that y is lesser than x.

Then F_x (at C)$=+R_B=+\dfrac{Wy}{L}$ (1)

Thus, the shear force F_x increases as y increases till $y=x$, in which case,

$$F_{max}=+\frac{Wx}{L} \qquad (1\cdot1)$$

This happens when the load is on the section (C) itself, thus

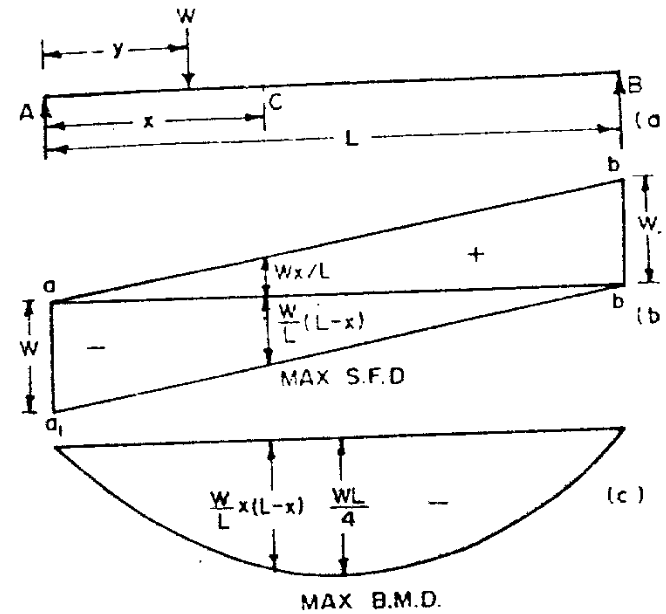


Fig. 1·2.

For different values of x (*i.e.* for different position of the section C), the maximum positive shear force given by equation 1·1 will vary *linearly* with x.

Thus, at $x=0$, $F_{max.}\,(+)=0$

at $x=L$, $F_{max.}\,(+)=+\dfrac{WL}{L}=+W=F_{max.\ max.}$

The absolute maximum positive S.F. therefore occurs at the right hand support, its value being equal to $+W$.

The maximum +ve shear force diagram is represented by abb_1 of Fig. 1·2(*b*).

(*aii*) Load in CB (y>x)

We have seen that when the wheel load W reaches the section C, maximum +ve S.F. of value $+\dfrac{Wx}{L}$ occurs at the section. When the load moves further (*i.e.* when y becomes greater than x), we have

F_x (at C)$=-R_A=-\dfrac{W(L-y)}{L}$ (2)

Thus, the shear force changes sign immediately when the load crosses the section. The maximum negative S.F. occurs evidently when $y=x$ (*i.e.* when y is least and the load is in CB).

Thus, $F_{max.} = -\frac{W(L-x)}{L}$ (1·2)

For different values of x (*i.e.* for different positions of the section C), the maximum negative shear force, given by equation 1·2 will vary *linearly* with x.

Thus, at $x=0$, $F_{max.}(-) = -\frac{WL}{L} = -W = F_{max.\ max.}$

at $x=L$, $F_{max.}(-) = -\frac{W(L-L)}{L} = 0.$

The absolute maximum —ve S.F. therefore occurs at the left hand support, its value being $-W$.

The maximum negative S.F.D. is represented by aa_1b of Fig. 1·2(b).

(b) MAXIMUM BENDING MOMENT DIAGRAM

Let us now draw the maximum bending moment diagram for the beam AB. It must be noted that a simply supported beam, under downward loads, bends causing concavity to the upper side. *Hence the bending moment is always negative for all sections of the beam.* Therefore, the maximum bending moment diagram will also be negative.

(bi) Load in AC ($y<x$)

When the load is in AC,

$$M_x = -R_B(L-x)$$
$$= -\frac{Wy}{L}(L-x) \quad (1)$$

This increases as y increases. When the load reaches the section C, $y=x$, and the section has the maximum bending moment.

$$M_{max} = -\frac{Wx}{L}(L-x) \quad (1·3)$$

(bii) Load in CB ($y>x$)

When the load W crosses the section C,

$$M_x = -R_A.x = -\frac{W(L-y)}{L}x \quad (2)$$

This increases as y *decreases*. When the load is on the section C, $y=x$, and the section has the maximum bending moment,

$$M_{max.} = -\frac{W(L-x)x}{L}$$

which is the same as equation 1·3.

Thus, the maximum bending moment at a section occurs when the load is on the section itself.

For different values of x (*i.e.* for different positions of section C) the maximum bending moment given by equation 1·3 will vary *parabolically* with x. Fig. 1·2(c) shows the maximum bending moment diagram. For absolute maximum bending moment

$$\frac{dM_{max.}}{dx} = 0$$

$$\therefore \quad -\frac{W}{L}(L-2x) = 0$$

or $$x = \frac{L}{2}.$$

Thus, the absolute maximum bending moment occurs at the centre of the span, and its value is given by

$$M_{max.\ max.} = -\frac{W}{L}\left(L-\frac{L}{2}\right)\frac{L}{2} = -\frac{WL}{4}.$$

1·3. UNIFORMLY DISTRIBUTED LOAD LONGER THAN THE SPAN OF THE GIRDER

Let us now study the case of the uniformly distributed load w per unit length, longer than the span, and moving from left to right.

(a) MAXIMUM S.F. DIAGRAM

Let us consider a section C distant x from left support A, as shown in Fig. 1·3(a). Let the head of the load be distant y from A.

Reaction $\quad R_B = \frac{wy^2}{2L}.$

When the load is in AC (*i.e.* $y<x$),

$$F_x = +R_B = +\frac{wy^2}{2L} \quad (1)$$

This evidently increases as y increases, until the head of the load reaches the section C (*i.e.*, when $y=x$).

$$\therefore \quad F_{max.} = +\frac{wx^2}{2L} \quad (1·4)$$

When the load still moves further, it can be proved that the value of F_x given by equation 1·4 decreases. To prove this statement, let the head of the load move by a distance δx from C towards B, and let δR_B be the corresponding increase in the reaction at B.

Then $\quad R_B + \delta R_B = \frac{w(x+\delta x)^2}{2L}$

and $$F_x = +(R_B+\delta R_B) - w.\,\delta x$$
$$= +\frac{w(x+\delta x)^2}{2L} - w.\,\delta x$$

$$=+\frac{wx^2}{2L}+\left\{wx\,.\,\frac{\delta x}{L}-w\delta x\right\} \qquad (2)$$

(neglecting the small quantities of second order).

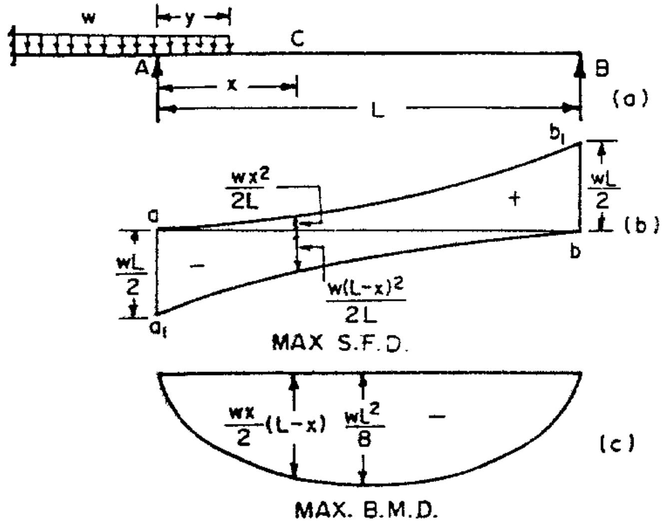


Fig. 1·3.

Since $\frac{x}{L}$ is less than 1, the expression inside the bracket is negative. Hence Fx given by (2) is less than Fx given by equation 1·4. *Thus, the maximum positive shear at a section occurs when the head of the load reaches the section,* (*i.e.* *when the left-portion AC is loaded and the right portion CB is empty*).

The maximum positive S.F. diagram can be plotted by giving different values of x in equation 1·4.

Thus, at $x=0$, $F_{max.}=0$

at $x=L$, $F_{max.}\ (+)=+\frac{wL^2}{2L}=\frac{wL}{2}$.

The absolute maximum positive S.F. occurs at the right hand support. The maximum +ve S.F.D. is shown by abb_1 in Fig. 1·3(*b*).

The S.F. at section C will continue to decrease as the load advances further. When the load covers the entire span,

$$F_X=-R_A+w.x=-\frac{wL}{2}+wx \qquad (3)$$

This is negative for the section C to be in the left half of the portion.

Let the load still move on so that the portion CB is fully loaded and portion AC is partially loaded, and we have

$$F_X=+R_B-w(L-x) \qquad (4)$$

In the above expression, the quantity $w(L-x)$ remains constant as the load still moves further, while R_B diminishes. Thus, with the onward movement of the load, the negative value of F_X increases. When the tail of the load reaches the section C, we have

$$F_X=-R_A=-\frac{w(L-x)^2}{2L}$$

This is the maximum value of negative shear force at C. As the load moves further, R_A decreases, and hence F_X decreases.

Thus, $$F_{max.}=-\frac{w(L-x)^2}{2L} \qquad (1.5)$$

The maximum negative shear force thus occurs when AC is empty and CB is fully loaded. To plot the maximum negative S.F. diagram vary x from 0 to L.

At $x=0$, $F_{max.}=-\frac{wL^2}{2L}=-\frac{wL}{2}=F_{max.max.}(-\text{ve})$

At $x=L$, $F_{max.}=0$

The maximum −ve S.F.D. is shown by aa_1b in Fig. 1·3 (*b*).

(*b*) MAXIMUM B.M. DIAGRAM

Let the head of the load be in AC, such that $y<x$.

$$R_B=\frac{wy^2}{2L}$$

$$\therefore \qquad M_X=-R_B(L-x)=-\frac{wy^2}{2L}\,(L-x) \qquad (1)$$

The value of Mx goes on increasing as y increases till the head of the load reaches the section C, and

$$Mx=-\frac{wx^2}{2L}\,(L-x) \qquad (2)$$

Let the load now advance further by a small amount δx, and let δB_B be the corresponding increase in the reaction at B, such that

$$R+\delta R_B=\frac{w}{2L}\,(x+\delta x)^2$$

Hence $$Mx=-(R_B+\delta R_B)(L-x)+w.\delta x\,.\,\frac{\delta x}{2}$$

$$=-\frac{w}{2L}(x+\delta x)^2\,(L-x)+\frac{w\delta x^2}{2} \qquad (3)$$

This is evidently more than that given by (2). Hence the B.M. at the section C continues to increase as the load moves further, till it occupies the whole span. In that case,

$$M_x = -\frac{wL}{2}x - \frac{wx^2}{2} = -\frac{wx(L-x)}{2} \qquad (4)$$

As the load still moves further, so that portion AC is partially loaded, and portion CB is fully loaded, we have

$$M_x = -R_B(L-x) + \frac{w(L-x)^2}{2} \qquad (5)$$

In the above expression, the quantity $\frac{w(L-x)^2}{2}$ is constant, while R_B diminishes as the load moves further. Hence M_x decreases till the tail of the load reaches the section C.

When the load is in the portion CB only, AC is empty.

$$M_x = -R_A \cdot x \qquad (6)$$

Since R_A decreases as the load moves further, M_x also decreases. It can thus be concluded that the values of M_x given by (5) and (6) are less than that given by (4). Thus, *the maximum B.M. at the section occurs when the whole span is loaded*, and its value is given by

$$M_{max} = -\frac{wx(L-x)}{2} \qquad ...(1\cdot5)$$

The maximum bending moment diagram is evidently a parabola as shown in Fig 1·3 (*c*). The absolute maximum bending moment evidently occurs at the centre of the span ($x = L/2$).

Thus, $\qquad M_{max \cdot max} = -\frac{w}{2}\frac{L}{2}\left(L - \frac{L}{2}\right) = -\frac{wL^2}{8}$

1·4. UNIFORMLY DISTRIBUTED LOAD SHORTER THAN THE SPAN OF THE GIRDER

Let the uniformly distributed load w/unit length extend over a length a such that $a < L$.

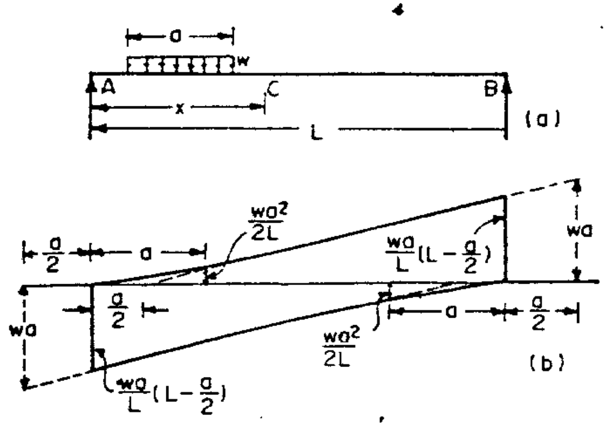
(*a*) MAXIMUM S.F. DIAGRAMS

(*ai*) Maximum Positive S.F.

(1) *Let the position of the section C be such that $x < a$.*

When the head of load reaches the section C, the portion AC is fully loaded (since $a > x$).

$$\therefore \qquad F_{max} = +R_B = +\frac{wx^2}{2L} \qquad ...[1\cdot6(a)]$$



MAX. S.F.D.

Fig. 1·4

This is parabolic relation and is valid for values of x between 0 and a. At $x=0$, $F_{max.}=0$.

and $\qquad$ at $x=a$, $F_{max} = +\frac{wa^2}{2L}$.

(2) *Let the position of the section C be such that $x > a$.*

When the load is in AC, the portion AC is partially loaded, and $F_x = +R_B$, which goes on increasing as the head of the load approaches C. When the head of the load reaches C, we have

$$F_{max} = +R_B = +\frac{wa}{L}\left(x - \frac{a}{2}\right) \qquad ...(1\cdot6)$$

This is a straight line relation.

At $\quad x = \frac{a}{2}$, $F_{max.} = 0$

At $x=a$, $F_{max.} = +\frac{wa}{L}\left(a - \frac{a}{2}\right) = +\frac{wa^2}{2L}$

At $x=L$, $F_{max} = +\frac{wa}{L}\left(L - \frac{a}{2}\right)$

The maximum +ve S.F.D. thus consists of a parabola upto a distance of a from A, and then straight line upto B.

(*ai*) Maximum Negative S.F.

As the load moves, further the S.F. decreases. For a particular load position, it becomes zero, and then changes sign and becomes negative. As the load still moves further, the negative S.F., at C increases. For maximum negative shear force at C, the span AC

should be empty and the reaction at A should be a maximum. In other words, the tail of the load should be at C, and the load should extend from C towards B.

When the tail of the load is at C,

$$F_{max.}=-R_A=-\frac{wa}{L}\left(L-x-\frac{a}{2}\right) \qquad (1\cdot7)$$

This is the equation of a straight line, and is valid for all values of x between 0 to $(L-a)$.

Thus, at $x=0$, $F_{max.}=-\frac{wa}{L}\left(L-\frac{a}{2}\right)$

at $x=(L-a)$, $F_{max.}=-\frac{wa}{L}\left(L-L+a-\frac{a}{2}\right)=-\frac{wa^2}{2L}$.

When the position of the section C is such that $x>L-a$

[*i.e.* when x varies from $(L-a)$ to L]

$$F_{max.}=-R_A=-\frac{w}{2L}(L-x)^2 \qquad (1\cdot8)$$

Thus, $F_{max.}$ is independent of a, and varies parabolically.

At $x=L-a$, $F_{max.}=-\frac{w}{2L}(L-L+a)^2=-\frac{wa^2}{2L}$, as before

At $x=L$, $F_{max}=-\frac{w}{2L}(L-L)^2=0$.

Thus, the maximum negative shear force diagram is a straight line from $x=0$ to $x=L-a$, and a parabola between $x=L-a$ to $x=L$.

The absolute maximum positive S.F. occurs at support B, when the head of the load is at B, and the absolute maximum negative S.F. is at A when the tail of the load is at A.

The maximum positive and negative S.F. diagrams have been shown in Fig. 1·4 (*b*).

(*b*) MAXIMUM B.M. DIAGRAM

Let the length of the U.D.L. be a. When the load is in the portion AC, the B.M. at the section C is given by

$$M_X=-R_B(L-x)$$

This goes on increasing as the head of the load approaches the section C. When the head of the load crosses the section C, the B.M. still goes on increasing, till it attains maximum value at a specific load position. On further movement, the B.M. at the section C decreases.

For the maximum B.M. at the section C the load is to be so arranged that its C.G. is at a distance y from A, as shown in Fig. 1·5(*a*).

In this load position,

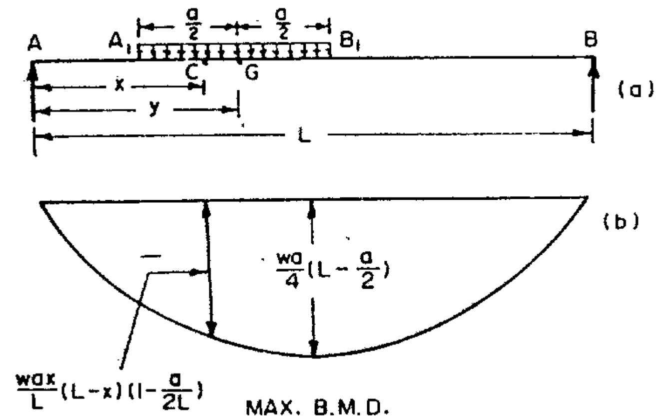


Fig. 1·5

$$R_B=wa\frac{y}{L}$$

$$\text{Distance } CB_1=\left(y-x+\frac{a}{2}\right)$$

$$M_X=R_B(L-x)+w\frac{(CB_1)^2}{2}$$

$$=-\frac{way}{L}(L-x)+\frac{w}{2}\left(y-x+\frac{a}{2}\right)^2. \qquad (2)$$

For M_X to be maximum, differentiate it with respect to y and equate to zero. Thus, we have

$$\frac{dM_X}{dy}=0=-\frac{wa}{L}(L-x)+w\left(y-x+\frac{a}{2}\right)$$

or $$\frac{a}{L}(L-x)=\left(y-x+\frac{a}{2}\right) \qquad (1\cdot9)$$

In the above equation, $a=A_1B_1$; $L=AB$; $L-x=CB$

and $$y-x+\frac{a}{2}=CB_1$$

Hence equation 1·9 can be expressed geometrically as

$$\frac{A_1B_1}{AB}\cdot CB=CB_1$$

or $$\frac{CB}{CB_1}=\frac{AB}{A_1B_1}=\frac{AB-CB}{A_1B_1-CB_1}=\frac{AC}{A_1C}$$

or $$\frac{A_1C}{CB_1}=\frac{AC}{CB} \qquad (1\cdot10)$$

Thus, for maximum bending moment at a section, the load position is such that the section divides the load in the same ratio as it divides the span. This relation will be found to hold good generally, both for the point loads as well as the uniformly distributed loads.

Equation 1·9 is directly useful for the location of the U.D.L. for the maximum B.M.

For the maximum B.M. at C, we get, from equation 1·9,

$$y=\frac{a}{L}(L-x)+x-\frac{a}{2}=\frac{a}{2}+x-\frac{ax}{L}$$

and $$y-x+\frac{a}{2}=\frac{a}{L}(L-x)$$

Substituting the values in (2), we get

$$M_{max}=-\frac{wa}{L}(L-x)\left(\frac{a}{2}+x-\frac{ax}{L}\right)+\frac{w}{2}\left\{\frac{a}{L}(L-x)\right\}^2$$

$$=-\frac{wax}{L}(L-x)\left(1-\frac{a}{2L}\right) \qquad (1\cdot11)$$

The maximum B.M. diagram can now be plotted by giving different values to x in equation 1·11. Absolute maximum B.M. occurs evidently at the centre, when $x=L/2$.

Thus, from equation 1·11,

$$M_{max\cdot max}=-\frac{wa}{L}\cdot\frac{L}{2}\left(L-\frac{L}{2}\right)\left(1-\frac{a}{2L}\right)=-\frac{wa}{4}\left(L-\frac{a}{2}\right)$$

The above value can also be obtained by considering Fig. 1·5, and applying the deduction of equation 1·10 independently.

Thus, for maximum bending moment at the centre of the span,

$$\frac{A_1C}{CB_1}=\frac{AC}{CB}$$

where $$AC=CB=\frac{L}{2}$$

$$\therefore \quad \frac{A_1C}{CB_1}=\frac{L/2}{L/2}=1$$

or $$A_1C=CB_1=a/2$$

In this position. $R_A=R_B=\dfrac{wa}{2}$

and $M_{max.}$ (at centre) $=M_{max\cdot max}=-\dfrac{wa}{2}\cdot\dfrac{L}{2}+\dfrac{w}{2}\left(\dfrac{a}{2}\right)^2$

$$=-\frac{wa}{4}\left(L-\frac{a}{2}\right)$$

which is same as before.

1·5. TWO POINT LOADS WITH A FIXED DISTANCE BETWEEN THEM

Let us now consider two point loads W_1 and W_2 at a fixed distance d apart, moving from left to right with W_1 leading. Let the leading load W_1 be smaller than W_2.

(*a*) MAXIMUM POSITIVE SHEAR FORCE

For positive shear force at the section C, we have to consider the three load positions :

(1) Both loads to the left of the section C.

(2) Load W_1 to the right of C and W_2 to the left of it.

(3) Both loads to the right of C : For this load position, there will be no positive shear (since $F_X=-R_A$), and hence we will consider only the first two load positions.

(*ai*) Both Loads to the Left of C

For this load position,

$$F_X=+R_B \qquad (1)$$

This increases as the leading load reaches near the section C, and is maximum when W_1 is *just to the left* of C.

(1) *When $x<d$*, only W_1 will be on the girder and W_2 will be off the span, with W_1 at C. Hence,

$$^{a1}F_{max.}=+R_B=+\frac{W_1x}{L} \qquad \text{(I)}(1\cdot12)$$

(2) *When $x>d$*, both W_1 and W_2 will be on the girder with W_1 at C, Hence

$$^{1}F_{max.}=+R_B=+\frac{W_1x+W_2(x-d)}{L} \qquad \text{(II)}(1\cdot13)$$

(*aii*) W_1 to the Right of C and W_2 to the Left of C

For this load position,

$$F_X=+R_B-W_1 \qquad (2)$$

Since R_B increases as W_1 and W_2 reach near B, the maximum S.F. occurs when the load is *just to the left of C*.

(3) *When* $(L-x)>d$, both W_1 and W_2 will be on the beam with W_2 at C, Hence

$${}^{2}F_{max.}=+R_B-W_1=\frac{W_2x+W_1(x+d)}{L}-W_1 \qquad \text{(III)(1·14)}$$

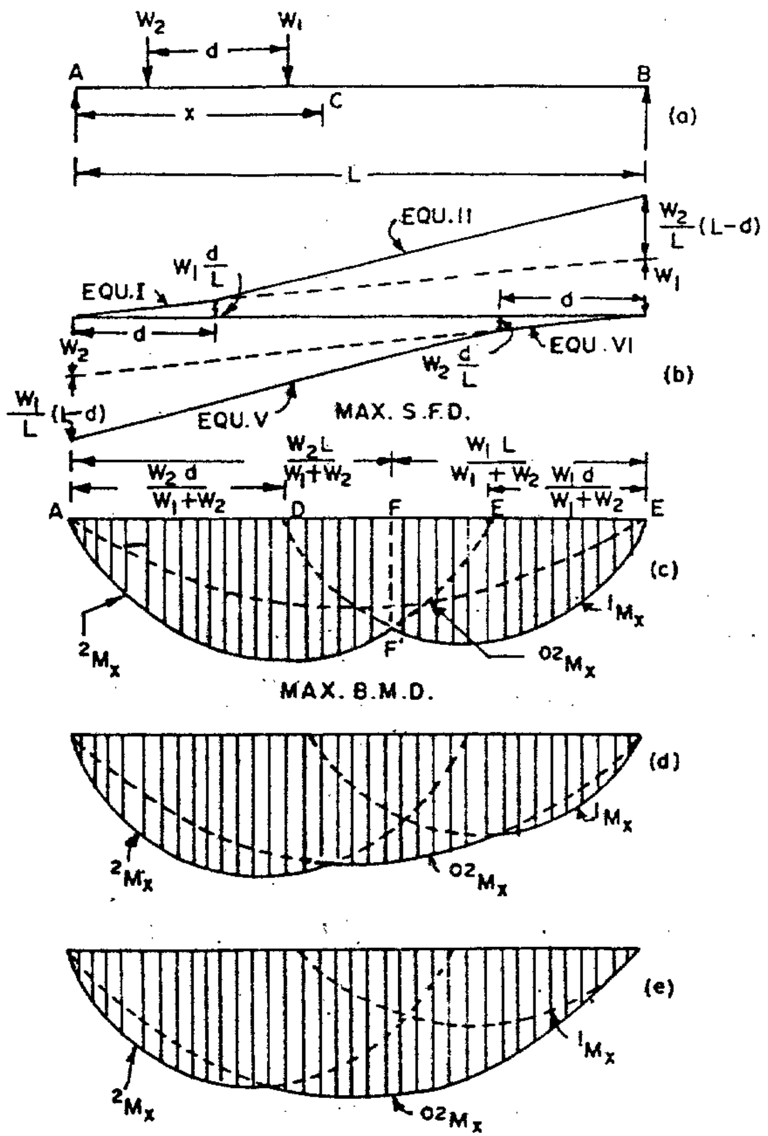


Fig. 1·6.

(4) *When* $(L-x)<d$, load W_1 will be off the girder while W_2 is at C. Hence

$${}^{02}F_{max.}=+R_B=+\frac{W_2x}{L} \qquad \text{(IV)(1·15)}$$

Thus, we have four equations for Fx (equations, I, II, III and IV) and one or the other of these equations will give maximum positive shear force depending upon the relative magnitudes of x and d.

To find which of these four equations will give F_{max}, we shall divide the beam in three zones :

(*i*) *Zone* (1) : $x=0$ to $x=d$
(*ii*) *Zone* (2) : $x=d$ to $x=(L-d)$
(*iii*) *Zone* (3) : $x=(L-d)$ to $x=L$

(*i*) **Zone** (1) : $x=0$ to $x=d$.

The first zone under consideration is from $x=0$ to $x=d$, and for this, both equations I as well as III will be applicable. For equation (I), W_1 is at C while W_2 is off the girder. For equation III, W_2 is at C and W_1 is to the right of it. Out of the two, equation I will give the larger value if

$$\frac{W_1x}{L}>\frac{W_2x+W_1(x+d)}{L}-W_1$$

or

$$x<\frac{W_1(L-d)}{W_2} \qquad (1·16)$$

Thus, when $x<\dfrac{W_1(L-d)}{W_2}$, equation I will give greater F_{max}. Beyond this value $\left(\text{i.e., } x=\dfrac{W_1(L-d)}{W_2} \text{ to } x=d\right)$ equation III will give greater F_{max}.

(*ii*) **Zone** (2) $[x=d$ to $x=L-d]$

The second zone under consideration is from $x=d$ to $x=L-d$, and for this both equations II as well as III will be applicable. For Eq. (II), W_1 is at C and W_2 is to the left of it. For Eq. (III), W_2 is at C and W_1 is to the right of it. Out of the two, equation (II) will give larger value if

$${}^{1}F_{max.}>{}^{2}F_{max.}$$

i.e.

$$\frac{W_1x+W_2(x-d)}{L}>\frac{W_2x+W_1(x-d)}{L}-W_1$$

or

$$(W_1+W_2)d<W_1L$$

or

$$d<\frac{W_1L}{W_1+W_2} \qquad (1·17)$$

Thus when the value of d is less than $\frac{W_1L}{W_1+W_2}$, max. +ve S.F. will occur when the leading load reaches the section. *This is the standard case for which max. S.F.D. has been drawn in Fig. 1·6(b).*

Thus, when $d < \frac{W_1L}{W_1+W_2}$, equation II will give F_{max}.

When $d > \frac{W_1L}{W_1+W_2}$, equation III will give F_{max}.

and maximum +ve S.F. will occur when the rear wheel load reaches the section.

Thus from Eq. I, when $x=0$, $F_{max.}=0$

when $x=d$, $F_{max.}=\frac{W_1d}{L}$

From Eq. II when $x=d$, $F_{max.}=\frac{W_1d}{L}$

when $x=L$, $F_{max.}=W_1+\frac{W_2(L-d)}{L}$

(b) MAXIMUM NEGATIVE SHEAR FORCE

In this case also, we will consider the three load positions for maximum negative shear force at the section (C):

(1) Both loads to the right of C.

(2) W_1 to the right of C and W_2 to the left of C.

(3) Both loads to the left of C: For this position, there will be no negative S.F. (Since $F_A=+R_B$) and hence we will consider only the first two load positions.

(i) Both loads to the right of C

For this load position,

$$F_X=-R_A \tag{1}$$

This increases when R_A increases. Hence the maximum value occurs when W_2 is *just to the right of C.*

(1) *When* $(L-x)>d$, both W_2 and W_1 will be on the girder, with W_2 at C. Hence

$$^{2}F_{max.}=-R_A=-\frac{W_2(L-x)+W_1(L-x-d)}{L} \qquad \text{(V)(1·18)}$$

(2) *When* $(L-x)<d$, only W_2 will be at C *and* W_1 will be off the girder. Hence

$$^{02}F_{max.}=-R_A=-\frac{W_2(L-x)}{L} \qquad \text{(VI)(1·19)}$$

***(bii)* W₁ to the right of C and W₂ to the left**

In this case, maximum negative S.F. occurs when W_1 *is just to the right of C.*

(3) *When* $x<d$, the load W_2 will be off the girder with load W_1 *just to the right* of C. Hence

$$^{01}F_{max.}=-R_A=-\frac{W_1(L-x)}{L} \qquad \text{(VII)(1·20)}$$

(4) When $x>d$, both W_1 and W_2 will be on the girder, with W_1 just to the right of C. Hence

$$^{1}F_{max.}=-\frac{W_1(L-x)+W_2(L-x+d)}{L}+W_2 \qquad \text{(VIII)(1·21)}$$

Equations V, VI, VII and VIII are valid for appropriate range of x.

For the case when $W_1<W_2$, equation V will give the maximum negative shear force, and is valid for all values of x between 0 to $(L-d)$. Beyond this, W_1 is off the girder, and equation VI will be valid.

Thus, at $x=0$, $^{2}F_{max}=-\left\{W_2+W_1\frac{(L-d)}{L}\right\}$

at $x=(L-d)$, $^{2}F_{max.}=-\frac{W_2d}{L}$

The complete maximum S.F.D. has been shown in Fig. 1·6 (b). Eqs. VII and VIII will give maximum value only when $W_2<W_1$. *i.e.*, when the loads move in the reverse order.

Summary

(1) The maximum positive S.F. occurs only when both the loads are to the left of the section with W_1 just approaching it. This is valid if $d<\frac{W_1L}{W_1+W_2}$. The maximum +ve S.F.D. is governed by Eqs. I and II.

(2) If $d>\frac{W_1L}{W_1+W_2}$, maximum positive S.F. occurs when W_2 is just to the left of C, and W_1 is to the right of C. The maximum +ve S.F.D. is governed by Eqs. I and III.

(3) The maximum negative S.F. occurs only when both the loads are to the right of the section. The maximum negative S.F.D. is governed by Eqs. V and VI.

(4) If W_2 is greater than W_1 (*i.e.*, when the loads travel in reverse order), Eqs. VII and VIII are valid for maximum negative S.F.D.

See examples 1·2 and 1·3 for complete illustration.

(c) MAXIMUM BENDING MOMENT DIAGRAM

When the two loads W_1 and W_2 are to the left of section C.

$$M_X=-R_B(L-x) \qquad (1)$$

This goes on increasing, as R_B increases, till W_1 reaches the section C. Let ${}^1M_X$ be the bending moment at C when W_1 is on the section, and W_2 is to the left of it.

Then
$$^1M_X=-R_B(L-x)$$
$$=-\frac{[W_1x+W_2(x-d)]}{L}(L-x) \qquad \text{(I)(1·22)}$$

When both the loads are to the right of section C,

$$M_X=-R_A.x$$

This is evidently maximum when W_2 is at C and W_1 is ahead of it. Let ${}^2M_X$ be the bending moment at C when W_2 is on it and W_1 is to the right of it.

Then,
$$^2M_X=-R_A.x$$
$$=-\frac{x}{L}\left\{W_1(L-x-d)+W_2(L-x)\right\} \qquad \text{(II)(1·23)}$$

As the loads still move further, R_A decreases, and hence ${}^2M_X$ decreases.

As a third possibility of getting maximum bending moment at C, let W_1 be to the right of C, and W_2 to the left of it at a distance y from C. Let ${}^3M_X$ be the bending moment at C, for this loading. Then,

$$^3M_X=-R_B(L-x)+W_1(d-y)$$
$$=-\left\{\frac{W_2(x-y)+W_1(x-y+d)}{L}\right\}(L-x)+W_1(d-y)$$

The above equation may be rewritten in terms of ${}^1M_X$ and ${}^2M_X$ as under :

$$^3M_X={}^1M_X-\frac{d-y}{L}\left\{W_1.x-W_2(L-x)\right\}$$

and
$$^3M_X={}^2M_X+\frac{y}{L}\left\{W_1x-W_2(L-x)\right\}$$

If $W_1.x>W_2(L-x)$, ${}^1M_X>{}^3M_X>{}^2M_X$

If $W_1x<W_2(L-x)$, ${}^2M_X>{}^3M_X>{}^1M_X$.

It is clear, therefore, that in either case, ${}^3M_X$ will not be maximum.

Hence maximum B.M. at the section is either ${}^1M_X$ or ${}^2M_X$, whichever is larger. Fig. 1·6 (c) shows both the parabolas giving ${}^1M_X$ and ${}^2M_X$ governed by equations I and II respectively.

Now ${}^1M_X$ is greater than ${}^2M_X$ if

$$\frac{W_1x+W_2(x-d)}{L}(L-x)>\frac{W_1(L-x-d)+W_2(L-x)}{L}x$$

i.e.
$$x>\frac{W_2L}{W_1+W_2} \qquad \ldots(1·24)$$

For $x<\dfrac{W_2L}{W_1+W_2}$, ${}^2M_X$ is maximum

For $x>\dfrac{W_2L}{W_1+W_2}$, ${}^1M_X$ is a maximum.

Now, ${}^1M_X$ is zero at $x=\dfrac{W_2d}{W_1+W_2}$, and at $x=L$.

${}^2M_X$ is zero at $x=0$, and at $(L-x)=\dfrac{W_1d}{W_1+W_2}$.

Both the parabolas cross each other at F', where ${}^1M_X={}^2M_X$. To find the position of this section, put ${}^1M_X={}^2M_X$.

Thus
$$\frac{W_1.x+W_2.(x-d)}{L}(L-x)=\frac{W_1(L-x-d)+W_2(L-x)}{L}x$$

From which x (or AF)$=\dfrac{W_2L}{W_1+W_2}$.

Thus, F divides AB in the ratio of $W_1 : W_2$. For all sections from A to F, maximum B.M. is given by ${}^2M_X$, and for all sections from F to B, the maximum B.M. is given by ${}^1M_X$.

The maximum value of ${}^2M_X$ in equation II will occur at

$$x=\frac{AE}{2}=\frac{1}{2}\left\{L-\frac{W_1d}{W_1+W_2}\right\}$$

When $W_2>W_1$ then absolutely maximum B.M. anywhere in the girder occurs in the ${}^2M_X$ range at $x=\frac{1}{2}AE$.

In case $(L-x)<d$, W_1 is off the girder when W_2 is on the section. Let ${}^{02}M_X$ be the bending moment at C when W_2 is on it and W_1 is off the girder.

Then
$$^{02}M_X=-R_B(L-x)=-\frac{W_2.x}{L}(L-x) \quad \text{(III)} \qquad \ldots(1·25)$$

Similarly, when $x<d$, W_2 is off the girder when W_1 is on the section. Let ${}^{01}M_x$ be the bending moment at C when M_1 is on it and W_2 is off the girder.

Then $${}^{01}M_X=-R_B(L-x)=-\frac{W_1.x}{L}(L-x) \qquad \text{(IV)} \qquad ...(1{\cdot}26)$$

For some sections over a portion of the girder, ${}^{02}M_X$ may sometimes be greater than ${}^{1}M_X$. Fig. 1·6(c) and (d) show the B.M.D. for two such possibilities. See also examples 1·3 and 1·4 for such possibilities.

Example 1·1. *A uniformly distributed load of 1 kN per metre run, 6 m long crosses a girder of 16 m span. Construct the maximum S.F. and B.M. diagram and calculate the values at sections at 3 m, 5 m and 8 m from the left hand support.*

Solution.

(*a*) *Maximum positive S.F. diagram.*

The maximum positive and negative S.F. diagrams are plotted xactly in the same manner as explained in § 1·4.

For x upto 6 m (=a)

$$F_{max.}=+\frac{wx^2}{2L}=\frac{1\times x^2}{2\times 16}=\frac{x^2}{32}\text{ kN} \qquad ...[1{\cdot}6(a)]$$

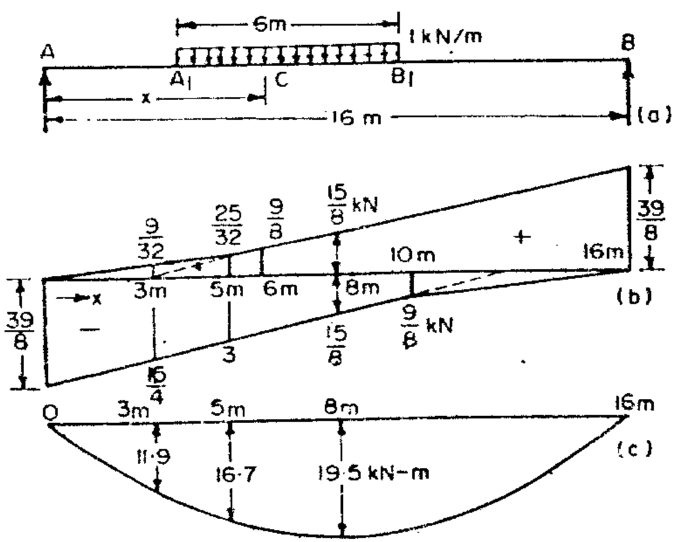


Fig. 1·7

At $x=0,\ F_0=0$

$x=3$ m, $F_3=+\dfrac{9}{32}$ kN

$x=5$ m, $F_5=+\dfrac{25}{32}$ kN

$x=6$ m$(=a)$, $F_6=+\dfrac{36}{32}=+\dfrac{9}{8}$ kN

For x>6

$$F_{max.}=+\frac{wa}{L}\left(x-\frac{a}{2}\right)=\frac{1\times 6}{16}\left(x-\frac{6}{2}\right)=+\frac{3}{8}(x-3) \qquad (1{\cdot}6)$$

At $x=6$. $F_6=+\dfrac{3}{8}(6-3)=+\dfrac{9}{8}$, as before.

$x=8$, $F_8=+\dfrac{3}{8}(8-3)=+\dfrac{15}{8}$ kN

$x=16$, $F_{16}=F_{max.max}=+\dfrac{3}{8}(16-3)=+\dfrac{39}{8}$ kN

(*b*) *Maximum negative S.F. diagram*

(1) For x between 0 to $(L-a)=16-6=10$ m ;

$$F_{max}=-R_A=-\frac{wa}{L}\left(L-x-\frac{a}{2}\right) \qquad ...(1{\cdot}7)$$

$$=-\frac{1\times 6}{16}\left(16-x-\frac{6}{2}\right)=-\frac{3}{8}(13-x)$$

At $x=0$, $F_0=-\dfrac{3}{8}(13-0)=-\dfrac{39}{8}$ kN

$x=3$, $F_3=-\dfrac{3}{8}(13-3)=-\dfrac{15}{4}$ kN

$x=5$, $F_5=-\dfrac{3}{8}(13-5)=-3$ kN

$x=8$, $F_8=-\dfrac{3}{8}(13-8)=-\dfrac{15}{8}$ kN

$x=10$ m, $F_{10}=-\dfrac{3}{8}(13-10)=-\dfrac{9}{8}$ kN

(5) For x between 10 m to 16 m :

$$F_{max}=-R_A=-\frac{w}{2L}(L-x)^2 \qquad ...(1{\cdot}8)$$

$$=-\frac{1}{2\times 16}(16-x)^2=-\frac{1}{32}(16-x)^2$$

At $x=10$, $F_{10}=-\dfrac{1}{32}(16-10)^2=-\dfrac{36}{32}=-\dfrac{9}{8}$, as before

$x=16$, $F_{16}=-\dfrac{1}{32}(16-16)^2=0.$

(c) *Maximum bending moment*

For getting maximum bending moment at a section, the load should be so arranged that the section divides it in the same ratio as it divides the span.

Thus, from equation 1·10,

$$\frac{A_1C}{CB_1}=\frac{AC}{CB}.$$

Hence $AC=x$; $CB=L-x$.

$$\therefore \quad \frac{A_1C}{CB_1}=\frac{x}{L-x}$$

or $$\frac{A_1C+CB_1}{CB_1}=\frac{A_1B_1}{CB_1}=\frac{x+L-x}{L-x}=\frac{L}{L-x}$$

or $$CB_1=\frac{L-x}{L}\times A_1B_1=\frac{a(L-x)}{L}$$

(1) *For* $x=3$ *m*

$$CB_1=\frac{a}{L}(L-x)=\frac{6}{16}(16-3)=\frac{39}{8}\text{ m}$$

$$\therefore \quad A_1C=6-\frac{39}{8}=\frac{9}{8}\text{ m}$$

$$BB_1=16-3-\frac{39}{8}=\frac{65}{8}\text{ m}$$

$$\therefore \quad R_A=(6\times 1)\left(\frac{65}{8}+3\right)\frac{1}{16}=\frac{267}{64}\text{ kN}$$

$$\therefore \quad M_3=-\frac{267}{64}\times 3+\frac{1}{2}\times\left(\frac{9}{8}\right)^2=-11{\cdot}9\text{ kN-m}$$

Alternatively, from equation 1·11, we have

$$\boxed{M_{max.}=-\frac{wax}{L}(L-x)\left(1-\frac{a}{2L}\right)} \quad \ldots(1{\cdot}11)$$

$$\therefore \quad M_3=-\frac{1\times 6\times 3}{16}(16-3)\left(1-\frac{6}{2\times 16}\right)$$

$$=-11{\cdot}9\text{ kN-m}$$

which is the same as before.

(2) *For* $x=5$ *and* $x=8$

For these sections also, equation 1·10 can be used to first locate the position of the U.D.L. and the maximum B.M. can then be calculated, or else equation 1·11 can be used to compute maximum B.M. directly. Thus, from equation 1·11,

$$M_5=-\frac{1\times 6\times 5}{16}(16-5)\left(1-\frac{6}{2\times 16}\right)$$

$$=-16{\cdot}7\text{ kN-m}$$

and $$M_8=-\frac{1\times 6\times 8}{16}(16-8)\left(1-\frac{6}{2\times 16}\right)=-19{\cdot}5\text{ kN-m}$$

$$=M_{max\cdot max}.$$

The results can now be summarised as below :

Section x	*Max. +ve S.F.*	*Max. −ve S.F.*	*Max. B.M.*
3 m	$+\frac{9}{32}$ kN	$-\frac{15}{4}$ kN	−11·9 kN-m
5 m	$+\frac{25}{32}$ kN	−3 kN	−16·7 kN-m
8 m	$+\frac{15}{8}$ kN	$-\frac{15}{8}$ kN	−19·5 kN-m

Example 1·2. *Two point loads of 4 kN and 6 kN spaced 6 m apart cross a girder of 16 m span, the 4 kN load leading from left to right. Construct the maximum S.F. and B.M. diagrams, stating the absolute maximum values.*

Solution

$W_1=4$ kN ; $W_2=6$ kN ; $d=6$ m ; $L=16$ m

$$\boxed{\frac{W_1L}{W_1+W_2}}=\frac{4\times 16}{4+6}=6{\cdot}4$$

$$\therefore \quad d=6<6{\cdot}4.$$

Thus the data is for the standard case, and the S.F.D. will be as that shown in Fig. 1·6(*a*).

(*a*) *Maximum +ve S.F. diagram*

Since $d<\frac{W_1L}{W_1+W_2}$, the maximum +ve S.F. at any section will occur under the leading load of 4 kN.

For x upto 6 m (=d), $F_{max.}=+\frac{W_1x}{L}=+\frac{4x}{16}=+\frac{x}{4}$ kN

$$\therefore \quad F_6=+\frac{6}{4}=+1{\cdot}5\text{ kN}$$

For x more than 6 m (=d), ${}^1F_{max.}$ will be greater than ${}^2F_{max.}$ and is given by equation 1·13, *i.e.*

$${}^1F_{max}=+R_B=+\frac{W_1x+W_2(x-d)}{L}=+\frac{(W_1+W_2)x-W_2d}{L}$$

$$=\frac{(4+6)x-6\times 6}{16}=\frac{5x-18}{8}$$

At $$x=L=16,\ F_{max\cdot max}=\frac{(5\times 16)-18}{8}=+\frac{31}{4}$$

(*b*) *Maximum* $-$*ve S.F. Diagram*

For all sections between $x=0$ and $x=L-d=16-6=10$ m, the maximum negative S.F. will be when the load $W_2(=6$ kN$)$ is at the section, with W_1 $(=4$ kN$)$ ahead of it. The variation is given by equation 1·18., *i.e.*,

$$^2F_{max.}=-R_A=-\frac{W_2(L-x)+W_1(L-x-d)}{L} \qquad \ldots(1{\cdot}18)$$

At $x=0$, $F_A=F_{max\cdot max}=-\dfrac{6(16)+4(16-0-6)}{16}=-8{\cdot}5$ kN

At $x=10$ m, $F_{10}=-\dfrac{6\times6+4(0)}{16}=-\dfrac{9}{4}$ kN

For all sections between $x=10$ to $x=16$, W_1 will be off the girder when W_2 is on it. Hence, max. $-$ve S.F. is given by equation 1·19, *i.e.*

$$^{02}F_{max.}=-R_A=-\frac{W_2(L-x)}{L} \qquad \ldots(1{\cdot}19)$$

At $x=10$ m, $F_{10}=-\dfrac{6(16-10)}{16}=-\dfrac{9}{4}$, as before.

At $x=16$, $F_B=0$.

The max. $+$ve and $-$ve S.F.D. have been shown in Fig. 1·8(*b*).

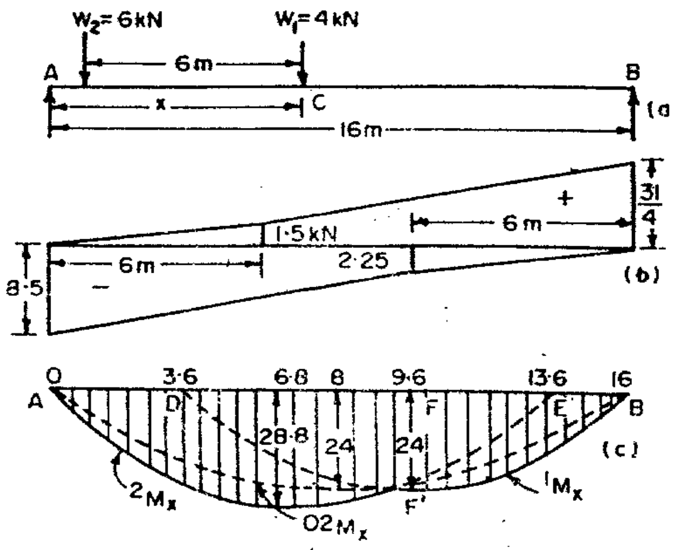


Fig. 1·8

(*c*) *Maximum bending moment diagram*

As discussed earlier, the maximum B.M. at a section may occur under any one of the following conditions :

(*i*) Max. B.M. under W_1 and W_2 behind it (^1Mx).
(*ii*) Max. B.M. under W_2 with W_1 ahead of it (^2Mx).
(*iii*) Max. B.M. under W_2 with W_1 off the girder ($^{02}M_X$).

We shall investigate all the three possibilities.

(*a*) *Max. B.M. under* W_1 (^1M_X) :

From equation 1·22, we have

$$^1M_X=-\frac{W_1x+W_2(x-d)}{L}(L-x) \qquad \ldots(1{\cdot}22)$$

$$=-\frac{4x+6(x-6)}{16}(16-x)$$

$$=(36-10x)\left(1-\frac{x}{16}\right)=-(10x-36)\left(1-\frac{x}{16}\right)$$

This is zero at $x=3{\cdot}6$ m and $x=16$ m.

Thus, $DB=16-3{\cdot}6=12{\cdot}4$ m.

$\therefore$ $^1M_{max}$ will occur at $x=3{\cdot}6+\dfrac{12{\cdot}4}{2}=9{\cdot}8$ m, its value being.

$$^1M_{max.}=(36-98)\left(1-\frac{9{\cdot}8}{16}\right)=-24{\cdot}25 \text{ kN-m.}$$

(*b*) *Max. B.M. under* W_2 (^2Mx) :

From equation 1·23, we have

$$^2Mx=-\frac{x}{L}\left\{W_1(L-x-d)+W_2(L-x)\right\}$$

$$=-\frac{x}{16}\left\{4(16-x-6)+6(16-x)\right\}$$

$$=-\frac{x}{16}\left[136-10x\right]$$

This is zero at $x=0$, and $x=13{\cdot}6$ m

Thus $AE=13{\cdot}6$ m

$^2M_{max}$. occurs at $x=\dfrac{AE}{2}=6{\cdot}8$ m, its value being

$$^2M_{max.}=-\frac{6{\cdot}8}{16}\left\{136-68\right\}=28{\cdot}8 \text{ kN-m}$$

Since $\dfrac{W_2L}{W_1+W_2}=\dfrac{6\times16}{6+4}=9{\cdot}6$ m, for $x<9{\cdot}6$. ^2Mx is greater than ^1Mx, and for $x>9{\cdot}6$ m, ^1Mx is greater than ^2Mx.

$$^{2}Mx = -\frac{9\cdot6}{16}(136-96) = -24 \text{ kN-m}$$

$$\left(1-\frac{9\cdot6}{16}\right) = -24 \text{ kN-m}$$

...us ^{2}Mx and ^{1}Mx are equal at $x=9\cdot6$ m (This is a check).

(c) *Max. B.M. under W_2 when W_1 is off the girder* (^{02}Mx)

From equation 1·25,

$$^{02}Mx = -\frac{W_2 x}{L}(L-x) = -\frac{6x}{16}(16-x)$$

and this is valid from $x=L-d=16-6=10$ m, to $x=L=16$ m. In this range ^{02}Mx is greater than ^{1}Mx if

$$-\frac{6x}{16}(16-x) > \left(36-10x\right)\left(\frac{16-x}{16}\right)$$

or if $\quad 6x > 10x-36$

or $\quad 36 > 4x$

or $\quad x < 9$ m.

But since the equation of ^{02}Mx is valid for x greater than 10 m, the above condition cannot be fulfilled, and hence ^{02}Mx is less than ^{1}Mx between $x=10$ to $x=16$ m. ^{02}Mx has its maximum value at $x=\frac{L}{2}=8$ m, its value being equal to

$$^{02}Mx = -\frac{6\times8}{16}(16-8)$$

$$= -24 \text{ kN-m}$$

The maximum B.M.D. has been drawn in Fig. 1·8(*c*).

Example 1·3. *Solve example 1·2 if $W_1=4$ kN; $W_2=6$ kN; $d=6$ m and the span $L=12$ m.*

Solution.

For the present case $\frac{W_1}{W_1+W_2}L = \frac{4\times12}{4+6} = 4\cdot8$ m.

$$\therefore \qquad d = 6 > \frac{W_1 L}{W_1+W_2}.$$

Thus the case is not the standard one, and the S.F.D. will not be similar to that of Fig. 1·6(*b*).

(*a*) *Maximum +ve S.F. Diagram*

(*i*) For $x=0$ to $x=d=6$ m, when only W_1 is on the span, and W_2 is off the span the maximum S.F., from equation 1·12, is given by

$$^{1}F_{max} = +R_B = +\frac{W_1 x}{L} = +\frac{4x}{12} = +\frac{x}{3} \text{ kN} \qquad ...(1)$$

(*ii*) For $x=0$ to $x=d=6$ m, when W_2 is just to the left of the section and W_1 ahead of it.

$$^{2}F_{max} = +R_B - W_1 = \frac{W_2 x}{L} + \frac{W_1(x+d)}{L} - W_1$$

$$= \frac{6x}{12} + \frac{4(x+6)}{12} - 4 = \left(\frac{5}{6}x - 2\right) \qquad ...(2)$$

Thus, for $x=0$ to $x=6$, F_{max} is given by both equation (1) and (2). $^{2}F_{max}$ will be greater than F_{max} only if

$$\left(\frac{5}{6}x-2\right) > \frac{x}{3}$$

or $\quad$ if $5x-12 > 2x$

or $\quad$ if $3x > 12$

or $\quad$ if $x > 4$

Thus, for $x=0$ to $x=4$, $^{1}F_{max}$ (given by equation 1) gives the maximum S.F., while for $x=4$ to $x=6$, $^{2}F_{max}$ gives the maximum values.

Thus, at $x=4$, $F_4 = +\frac{4}{3}$ kN

at $x=6$, $F_6 = +3$ kN

(*iii*) For $x=d=6$ m (or $L-d=6$ m) to $x=L=12$; both W_1 and W_2 on the span:

The maximum S.F. is given by equation 1·13

$$^{1}F_{max} = +R_B = +\frac{W_1 x + W_2(x-d)}{L} \qquad (1\cdot13)$$

$$= \frac{4x+6(x-6)}{12} = \frac{5}{6}(x-3\cdot6) \qquad (3)$$

(*iv*) For $x=d=6$ m to $x=L=12$, with W_2 on the section and W_1 off the girder:

The maximum S.F. is given by equation 1·5.

$$^{02}F_{max} = +R_B = +\frac{W_2 x}{L} = +\frac{6x}{12} = +\frac{x}{2} \text{ kN} \qquad ...(4)$$

Thus, for $x=6$ to $x=12$, the maximum S.F. is given by equation (3) and (4). Evidently $^{02}F_{max}$ will be greater than $^{1}F_{max}$, if

$$\frac{x}{2} > \frac{5}{6}(x-3\cdot6)$$

or $\quad x < 9$

Thus, from $x=6$ to $x=9$, maximum S.F. will be governed by $^{02}F_{max}$ (equation 4), while from $x=9$ to $x=12$, maximum S.F. will be governed by $^{1}F_{max}$.

At $x=9$, $F_9=+4{\cdot}5$ kN
$x=12$, $F_{12}=+7$ kN

(b) *Maximum negative S.F. diagram*

(i) When both W_1 and W_2 are on the girder :

Max. −ve S.F. at C will occur when W_2 is just to the right of C and W_1 is ahead of it by 6 m.

Thus, from equation 1·18,

$$^2F_{max.}=-R_A=-\frac{W_2(L-x)+W_1(L-x-d)}{L}$$

$$=-\frac{6(12-x)+4(12-x-6)}{12}=-\left(8-\frac{5}{6}x\right) \quad ...(1)$$

This is valid from $x=0$ to $x=L-d=12-6=6$ m.

At $x=0$, $F_{max.\ max.}=-8$ kN ; At $x=6$, $F_6=-(8-5)=-3$ kN.

(ii) *When W_2 is on the section, and W_1 off the girder (for $x=6$ to $x=12$)*

The max. −ve S.F. given by equation 1·19

$$^{02}F_{max.}=-R_A=-\frac{W_2(L-x)}{L}=-\frac{6}{12}(12-x) \quad ...(ii)$$

At $x=6$, $F_6=-\frac{1}{2}(6)=-3$ kN, as before,

$x=12$, $F_{12}=0$.

The complete S.F.D. is shown in Fig. 1·9(b).

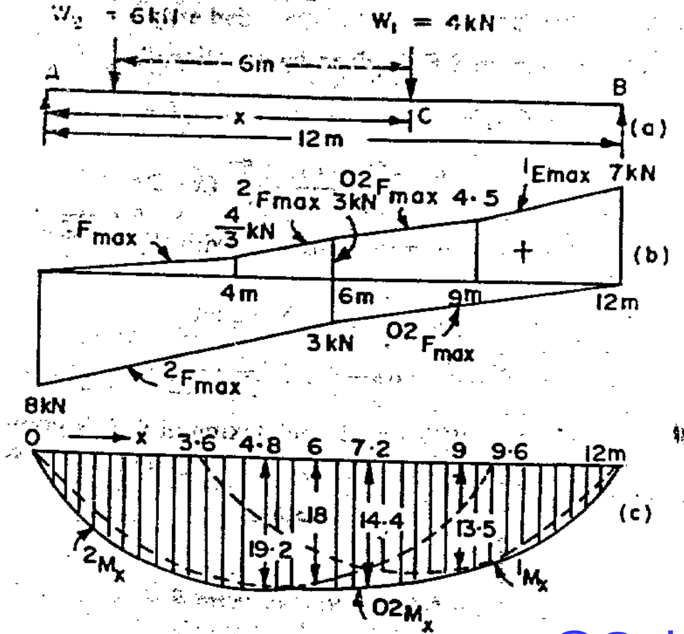


Fig. 1·9.

(c) *Maximum bending moment diagram*

The maximum bending moment may occur under any one of the following three conditions :

(i) Maximum bending moment under W_2 with W_1 ahead of it (^2Mx).

(ii) Maximum bending moment under W_1 and W_2 behind it (^1Mx).

(iii) Maximum bending moment under W_2 with W_1 off the span, (^{02}Mx).

Let us investigate all the three possibilities :

(i) *Maximum bending moment under W_2 with W_1 ahead of it* (^2Mx)

From Equation 1·23,

$$^2Mx=-R_A.x=-\frac{x}{L}\left\{W_1(L-x-d)+W_2(L-x)\right\} \quad (1{\cdot}23)$$

$$=-\frac{x}{12}\left\{4(12-x-6)+6(12-x)\right\}=-x\left(8-\frac{5}{6}x\right) \quad \text{(I)}$$

This is zero at $x=0$ and at $x=\frac{48}{5}=9{\cdot}6$ m

^2Mx will have its max. value at $x=\frac{9{\cdot}6}{2}=4{\cdot}8$ m

$$\therefore\ ^2M_{max}=-4{\cdot}8\left(8-\frac{5}{6}\times4{\cdot}8\right)=-19{\cdot}2 \text{ kN-m}$$

(ii) *Max. bending moment under W_1 with W_2 behind it (^1Mx)*

From equation 1·22, we have

$$^1Mx=-\frac{W_1x+W_2(x-d)}{L}(L-x) \quad (1{\cdot}22)$$

$$=-\frac{4x+6(x-6)}{12}(12-x)=-\left(\frac{5}{6}x-3\right)(12-x) \quad \text{(II)}$$

This is zero at $x=3{\cdot}6$ m, and at $x=12$ m

^1Mx is maximum at $x=3{\cdot}6+\frac{12-3{\cdot}6}{2}=7{\cdot}8$ m

$$\therefore\quad ^1M_{max}=-\left(\frac{5}{6}\times7{\cdot}8-3\right)(12-7{\cdot}8)=-14{\cdot}7 \text{ kN-m}$$

Now $^1M_{max.}$ will be greater than $^2M_{max.}$ for

$$x>\frac{W_2L}{W_1+W_2}>\frac{6\times12}{4+6}>7{\cdot}2 \text{ m.}$$

Thus for $x=0$ to $x=7{\cdot}2$, max. B.M. will be governed by $^2M_{max}$ and from $x>7{\cdot}2$, max. B.M. may be governed by $^1M_{max}$.

At $x=7{\cdot}2$, $\quad {}^1M_{7\cdot2}=-\left(\frac{5}{6}\times7{\cdot}2-3\right)(12-7{\cdot}2)=14{\cdot}4$ kN-m

$${}^2M_{7\cdot2}=-7{\cdot}2\left(8-\frac{5}{6}\times7{\cdot}2\right)=-14{\cdot}4 \text{ kN-m}$$

(check)

(*iii*) *Max. bending moment under W_1 with W_2 off the girder*

From equation 1·25,

$${}^{02}Mx=-\frac{W_2x}{L}(L-x)=-\frac{6x}{12}(12-x)$$
$$=-x(6-0{\cdot}5x) \qquad \text{(III)}$$

To get the section where ${}^2Mx$, is equal to ${}^{02}Mx$, we have

$$x\left(8-\frac{5}{6}x\right)=x(6-0{\cdot}5x)$$

or $$x=6 \text{ m}$$

The common value of B.M. is given by

$${}^2M_6={}^{02}M_6=-6(6-0{\cdot}5\times6)=-18 \text{ kN-m}$$

To get the section where ${}^{02}Mx$ and ${}^1Mx$ are equal, we have

$$\left(\frac{5}{6}x-3\right)(12-x)=x(6-0{\cdot}5x)$$

which gives $x=9$ m.

The common value of the max. B.M. is given by

$${}^{02}M_9={}^1M_9=-9(6-0{\cdot}5\times9)=-13{\cdot}5 \text{ kN-m}$$

The maximum value of ${}^{02}M_6$ evidently occurs at $x=\frac{L}{2}=6$ m

its value being equal to ${}^{02}M_6=-6(6-0{\cdot}5\times6)=-18$ kN-m

Hence, to Summarise :

(*i*) For $x=0$ to $x=6$, Max. B.M. is governed by ${}^2Mx$.

(*ii*) For $x=6$ to $x=9$, Max. B.M. is governed by ${}^{02}Mx$.

(*iii*) For $x=9$ to $x=12$, Max. B.M. is governed by ${}^1Mx$.

The complete max. B.M. diagram is shown in Fig. 1·9(*c*). The absolute max. B.M. will be under W_2, at $x=4{\cdot}8$ m, its value being equal to **19·2 kN-m.**

Example 1·4. *Two point loads W_1 and W_2 $(W_2>W_1)$ spaced at a distance 'd' travel from left to right across a simply supported girder, with W_1 leading. Prove that the limiting span below which the greatest bending moment anywhere in the girder will occur when the load W_1 has gone off the girder, is equal to $\left(1\pm\sqrt{\frac{W_2}{W_1+W_2}}\right)d$.*

Hence, draw the max. B.M. diagram if $W_1=4$ kN ; $W_2=6$ kN ; $d=6$ m and the span $L=10$ m.

Solution. (Refer Fig. 1·6 *a, e*).

Since W_2 is greater than W_1, the maximum va[lue of] ${}^2Mx$ at a section occurs when the load W_2 is at the [section] ahead of it. The maximum value of ${}^2M_{max}$ occurs at

$x=\frac{1}{2}\left\{L-\frac{W_1d}{W_1+W_2}\right\}$. The maximum value of ${}^2M_{max}$ is obtained by substituting the value of x in equation 1·23.

Thus $${}^2M_{max}=-\frac{x}{L}\{W_1(L-x-d)+W_2(L-x)\}$$

$$=-\frac{1}{2L}\left\{L-\frac{W_1d}{W_1+W_2}\right\}\left[(W_1+W_2)\left\{\left(L-\frac{1}{2}\left(L-\frac{W_1d}{W_1+W_2}\right)\right\}-W_1d\right)\right]$$

$$=-\frac{1}{4L(W_1+W_2)}\left[\{(W_1+W_2)L-W_1d\}\{(W_1+W_2)L-W_1d\}\right]$$

$$=-\frac{\{(W_1+W_2)L-W_1d\}^2}{4L(W_1+W_2)} \qquad \text{(I) (1·27)}$$

Also, when W_1 is off the girder, the maximum B.M. under W_2 at the section is given by equation 1·25,

$${}^{02}M_x=-\frac{W_2x}{L}(L-x) \qquad (1{\cdot}25)$$

Evidently, its maximum value occurs at $x=\frac{L}{2}$

and $${}^{02}M_{max}=-\frac{W_2L}{L\times2}\left(L-\frac{L}{2}\right)=-\frac{W_2L}{4} \qquad \text{(II)}$$

To have the *greatest* B.M. governed by ${}^{02}m_{max}$, we have

$$\frac{W_2L}{4}>\frac{\{(W_1+W_2)L-W_1d\}^2}{4L(W_1+W_2)}$$

or $$W_2L^2(W_1+W_2)>(W_1+W_2)^2L^2+W_1^2d^2-2W_1dL(W_1+W_2)$$

or $$L^2(W_1+W_2)\{(W_1+W_2)-W_2\}+W_1^2d^2-2W_1dL(W_1+W_2)<0$$

or $$L^2(W_1+W_2)-2dL(W_1+W_2)+W_1d^2<0$$

which gives $$L<\left[1\pm\sqrt{1-\frac{W_1}{W_1+W_2}}\right]d$$

or $$L<\left[1\pm\sqrt{\frac{W_2}{W_1+W_2}}\right]d \qquad (1{\cdot}28)$$

Numerical part :

To ascertain whether the span is less than that given by equation 1·28, substitute the values in equation 1·28.

Thus, $L_{lim} = \left[\ 1 \pm \sqrt{\dfrac{6}{4+6}}\ \right] 6$

$= (1 \pm 0.775) 6 = 1{\cdot}35$ m or $10{\cdot}65$ m

$= 10.65$ (Taking the greater limiting value)

Since our span is even *lesser* than the greater permissible value, ${}^2M_x$ will be greater than ${}^2M_{max}$.

Now $\quad {}^{01}M_x = -\dfrac{W_2 x}{L}(L-x) = -\dfrac{6x}{10}(10-x)$

This is maximum at $x = \dfrac{L}{2} = 5$ m

$\because \quad M_{max.\ max.} = {}^{02}M_5 = -\dfrac{6 \times 5}{10} \times 5 = -15$ kN-m.

To plot the max. B.M. diagram, let us again investigate all the three possibilities :

(*a*) *Max. B.M. under W_2 with W_1 ahead of it* (${}^2M_x$)

From Eq. 1·23.

$${}^2M_x = -\frac{x}{L}\{W_1(L-x-d) + W_2(L-x)\}$$

$$= -\frac{x}{10}\{4(10-x-6) + 6(10-x)\}$$

$$= -x(7{\cdot}6 - x) \qquad (i)$$

This is zero at $x = 0$ and $x = 7{\cdot}6$ m

(*b*) *Max B.M. under W_1 with W_2 behind it* (${}^1M_x$)

From Eq. 1·23,

$${}^1M_x = -\frac{W_1 x + W_2(x-d)}{L}(L-x)$$

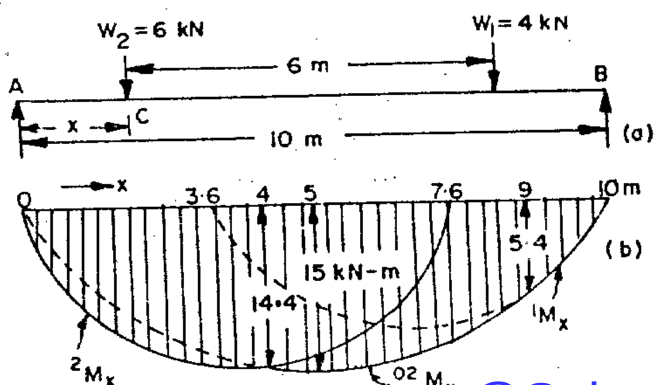


Fig. 1·10

$$= -\frac{4x + 6(x-6)}{10}(10-x)$$

$$= -(x - 3{\cdot}6)(10 - x) \qquad (ii)$$

This is zero at $x = 3{\cdot}6$ m and $x = 10$ m.

(*e*) *Max. B.M. under W_2 with W_1 off the girder* (${}^{02}M_x$).

From equation 1·25,

$${}^{02}M_x = -\frac{W_2 x}{L}(L-x) = -\frac{6x}{10}(10-x)$$

$$= -0{\cdot}6x(10-x) \qquad (iii)$$

To find the section where ${}^2M_x$ and ${}^{02}M_x$ are equal, we have

$0{\cdot}6x(10-x) = x(7{\cdot}6 - x)$

This gives $\quad x = 4$m.

Thus, $\quad {}^2M_4 = {}^{02}M_4 = -4(7{\cdot}6 - 4) = -14{\cdot}4$ kN-m.

To find the section where ${}^1M_x$ and ${}^{02}M_x$ will be equal, we have

$0{\cdot}6x(10-x) = (x - 3{\cdot}6)(10 - x)$

which gives $\quad x = 9$ m

Thus $\quad {}^1M_9 = {}^{02}M_9 = -0{\cdot}6 \times 9(10-9) = -5{\cdot}4$ kN.

Hence, we get

(*i*) For $x = 0$ to $x = 4$ m, Max. B.M. is governed by ${}^2M_x$

(*ii*) For $x = 4$ m to $x = 9$ m, Max. B.M. is governed by ${}^{02}M_x$.

(*iii*) For $x = 9$ m to $x = 10$ m, Max. B.M. is governed by ${}^1M_x$.

The max. B.M.D. is shown in Fig. 1·10.

Example 1·5. *Plot the maximum bending moment diagram for a simply-supported girder with the following data :*

$W_1 = 3$ kN (leading)
$W_2 = 6$ kN
$d = 6$ m
$L = 10$ m.

Prove that maximum B.M. occurs under W_2 when W_1 is off the span.

Solution

$$L_{limit} = \left(1 \pm \sqrt{\frac{W_2}{W_1 + W_2}}\right) d$$

$$= \left(1 \pm \sqrt{\frac{6}{3+6}}\right) 6 = 10{\cdot}89 \text{ m}.$$

Since $L = 10$ m, maximum B.M. will occur under W_2 when W_1 is off the girder, *i.e.* ${}^{02}M_{max}$. will be greater than ${}^2M_{max}$.

To plot the maximum B.M. diagram, we will investigate all the possibilities.

(a) *Maximum B.M. under W_2 with W_1 ahead of it* (^2Mx)

From equation 1·23, we have

$$^2Mx=-\frac{x}{L}\left\{W_1(L-x-d)+W_2(L-x)\right\} \quad (1\cdot23)$$

$$=-\frac{x}{L}\left\{3(10-x-6)+6(10-x)\right\}$$

$$=-x(7\cdot2-0\cdot9\,x) \quad (1)$$

It will be zero at $x=0$, and $x=8$ m.

(b) *Maximum B.M. under W_1 with W_2 behind it* (^1Mx)

From equation 1·22.

$$^1Mx=\frac{W_1x+W_2(x-d)}{L}(L-x) \quad (1\cdot22)$$

$$=-\frac{3x+6(x-6)}{10}(10-x)$$

$$=-(0\cdot9x-3\cdot6)\,(10-x) \quad (2)$$

It will be zero at $x=4$ m and $x=10$ m.

(c) *Maximum B.M. under W_2 with W_1 off the girder* (^{02}Mx)

From equation 1·25,

$$^{02}Mx=-\frac{W_2x}{L}(L-x)$$

$$=-\frac{6x}{10}(10-x)=-0\cdot6x(10-x) \quad (3)$$

To find the section where ^2Mx and ^{02}Mx are equal, we have

$$x(7\cdot2-0\cdot9x)=0\cdot6x(10-x)$$

which gives $x=4$ m.

$\because \quad {}^2M_4={}^{02}M_4=-0\cdot6\times4(10-4)=-14\cdot4$ kN-m.

For the rest of the span, ^{02}Mx will be greater than ^1Mx only if

$$0\cdot6x(10-x)>(0\cdot9x-3\cdot6)\,(10-x)$$

or if $\quad 0\cdot6x>0\cdot9x-3\cdot6$

or if $\quad x<12$ m.

However, since maximum value of $x=L=10$ m, ^{02}Mx will always be greater than ^1Mx.

The absolute maximum B.M. any where in the girder will evidently be governed by ^{02}Mx. It will occur at $x=\frac{L}{2}=5$ m and its value is

$$M_{max\cdot max}={}^{02}M_5=-0\cdot6\times5\times5=-15 \text{ kN-m}.$$

The maximum B.M.D. is shown in Fig. 1·11.

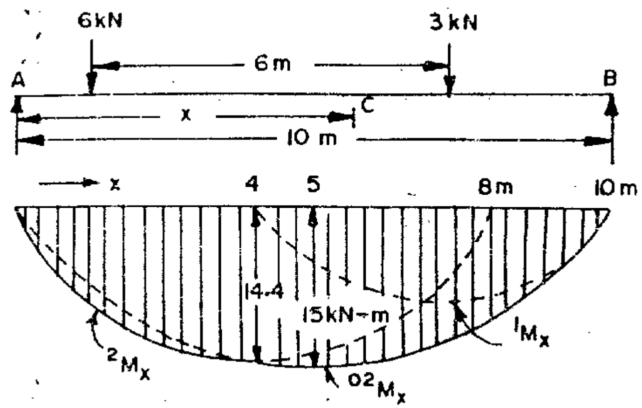


Fig. 1·11

1·6. SEVERAL POINT LOADS : MAXIMUM B.M.

Let us now take the case of a train of wheel loads W_1, W_2...W_n crossing a simply supported girder. For getting the position and amount of maximum bending moment, we shall discuss the following two propositions.

PROPOSITION 1

When a series of wheel-loads cross a girder, simply supported at the ends, the maximum bending moment under any given wheel load occurs when the centre of the span is mid way between the C.G. of the load system and the wheel load under consideration.

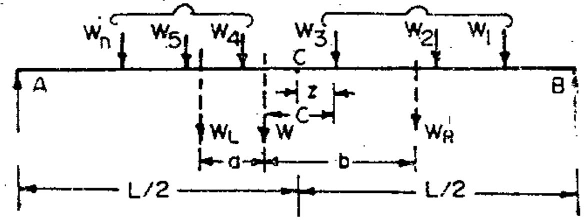


Fig. 1·12

Thus, in Fig. 1·12, let us find the maximum bending moment under the wheel load W_3, of the train of wheel loads W_1, W_2...W_n. Let W_L be the resultant of all loads to the left of W_3 and W_R be the resultant of all loads to the right of W_3 and inclusive of W_3. Let W be the resultant of the load system, situated at a from W_L, b from W_R and c from W_3. For given load system a, b and c are constants.

To get the maximum B.M. under W_2, let the load W_2 be placed at a distance z from the centre C of the span. It is required to find the value of the variable z.

$$\text{Reaction } R_A=\frac{W}{L}\left[\frac{L}{2}+(c-z)\right]$$

$$\text{B.M. under } W_2 \text{ is } M=-R_A\left(\frac{L}{2}+z\right)+W_L(a+c)$$

$$=-\frac{W}{L}\left[\frac{L}{2}+(c-z)\right]\left(\frac{L}{2}+z\right)+W_L(a+c)$$

$$=-\frac{W}{L}\left(\frac{L^2}{4}+cz-z^2+\frac{cL}{2}\right)+W_L(a+c)$$

For maximum M,

$$\frac{dM}{dz}=-\frac{W}{L}(c-2z)=0$$

or

$$z=\frac{c}{2} \qquad \ldots(1{\cdot}29)$$

Hence the centre of the span is midway between W and W_2. This proves the proposition.

The above proposition can be used to find the maximum B.M. under desired wheel load. However, to get absolute maximum B.M, any where on the girder, several trials are to be made. Any one load must first be chosen and arranged according to the condition of equation 1·29 derived above, and the maximum B.M. is calculated. Another wheel load can then be chosen and the procedure repeated to get another value of maximum B.M. Two or three such trials may sometimes be needed, and the absolute maximum B.M. will be the greatest of these. However, to reduce the number of trials to a minimum, the following points must always be kept in mind :

1. The maximum B.M. always occurs under a wheel load, and *not* any where between two wheel loads.
2. Absolute maximum B.M. always occurs at a section *near* the centre of the span. (It never occurs at the centre unless the C.G. of the resultant load coincides with the centre line of some heavy wheel load).
3. The wheel load should be so selected that the centre of the span is midway between the C.G. of the load system and wheel load under consideration.
4. The absolute maximum B.M. generally occurs under the heavier wheel load—specially that which is very near to the C.G. of the load system.

PROPOSITION 2

The maximum bending moment at any given section of a simply supported beam, due to given system of point loads crossing the beam occurs when the average loading on the portion to the left of it is equal to the average loading to the right of it, i.e. when the section divides the load in the same ratio as it divides the span.

The proposition is very useful for locating the load position for maximum B.M. at a *given* section, and has already been proved for uniformly distributed load in § 1·4.

Let it be required to find the load position for maximum B.M. at a point C, distant x from A. Let W be the resultant load located at y from A, for maximum B.M. at C. Let W_L be the resultant of the loads to the left of C and W_R the resultant to the right of C.

$$R_A=\frac{W(L-y)}{L}$$

$$\therefore \quad M_X=-R_A.x+W_L[x-(y-a)]$$

$$=-\frac{W(L-y)}{L}x+W_L(x-y+a).$$

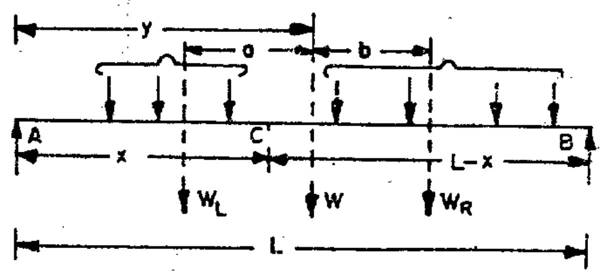


Fig. 1·13

In the above expression, y is the only variable. For maximum M_x,

$$\frac{dM_x}{dy}=+\frac{W.x}{L}-W_L=0 \text{ (or should change sign)}$$

or

$$\frac{W_L}{x}=\frac{W}{L}=\frac{W-W_L}{L-x}=\frac{W_R}{L-x} \qquad \ldots(1{\cdot}30)$$

In other words, the average load on the portion to the left of C is equal to the average load on the portion to the right of C.

Actually, in isolated load systems, $\frac{dM_x}{dy}$ *cannot be equal to zero but will have sudden steps changing from a positive value to a negative one.* Generally, the loading will be such that either AC is heavier and CB lighter, or *vice versa*. *Hence the maximum* B.M. *at* C *will occur when* $\left(\frac{W_L}{x}-\frac{W_R}{L-x}\right)$ *changes sign*. The value $\left(\frac{W_L}{x}-\frac{W_R}{L-x}\right)$ can change sign only when a load crosses C from left to right, thus increasing W_R and decreasing W_L. *Hence to get the value of maximum* B.M. *at a section, one of the wheel loads should be placed at the section, so that if that load is considered as a part of* W_L, *the expression* $\left(\frac{W_L}{x}-\frac{W_R}{L-x}\right)$ *is positive, but if considered as part of* W_R, *the expression* $\left(\frac{W_L}{x}-\frac{W_R}{L-x}\right)$ *becomes negative*. If on rolling the loads from left to right, $\left(\frac{W_L}{x}-\frac{W_R}{L-x}\right)$ does not change the sign from +ve to −ve, but instead, increases or remains positive, the loads should be rolled to the right so that next load comes over to the section. With this new load at the section, $\left(\frac{W_L}{x}-\frac{W_R}{L-x}\right)$ should again be investigated for the two positions, as described above, till it changes sign. In the passage of a series of wheel loads, two or more positions of the load system may occur satisfying the above condition of change of sign from +ve to −ve. In such a case, the value of M_x at the section for each of these load positions must be calculated, and the greatest of these taken as the maximum B.M. at the section.

It must always be remembered that maximum B.M. at any section occurs when the wheel load is over it.

1·7. SEVERAL POINT LOADS : MAX. S.F. AT A SECTION

Let us now investigate the load position for getting maximum S.F. at a section due to several point loads $W_1, W_2, \ldots, W_n$. The process of locating the load position for maxima is that of trial and error. However, the max. S.F. at the section occurs when one of the loads is on the section.

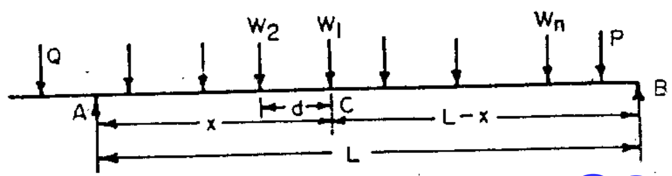


Fig. 1·14

To get the max. +ve S.F. at C let the load W_1 be at the section C, and let another load W_2 be at d behind it. If the loads are rolled to the right by a distance d, so that W_2 comes at C, the S.F. at the section C will be changed. This change (δB) consists of two components :

(*i*) Increase δR_B (gradually as the loads roll)

$\delta R_B=\frac{Wd}{L}$, where W=resultant of all loads on the span.

(*ii*) Sudden decrease or drop equal to W_1.

Hence $\boxed{\delta F=\delta R_B-W_1=\frac{Wd}{L}-W_1}$ ...(1·31)

If this change is positive, rolling will increase +ve S.F. In such a case, the rolling must be contained till equation 1·31 becomes negative.

The above discussion is true only if no load either enters or leaves the span when the system is rolled by the specified distance d.

To discuss the most general case, let the load Q enter the span a distance a, and load P move beyond B a distance b, due to rolling. If W is the resultant load before the advance, we have

$$\delta R_B=\frac{Wd}{L}+\frac{Qa}{L}-P\left(1+\frac{b}{L}\right)$$

Hence $\delta F=\delta R_B-W_1$

$$=\frac{Wd}{L}+\frac{Qa}{L}-P\left(1+\frac{b}{L}\right)-W_1$$

$$=\frac{Wd}{L}-(P+W_1)-\frac{Pb}{L}+\frac{Qa}{L} \quad \text{...(1·32)}$$

Since $\frac{b}{L}$ and $\frac{a}{L}$ are usually small compared with unity, the last two terms of the above expression may be neglected for *approximation*. Hence, we get

$$\delta F\simeq\frac{Wd}{L}-(W_1+P) \quad \text{...(1·33)}$$

If this is +ve, rolling will increase the S.F. From the above, it is evident that the load entering the span does not change the S.F. *appreciably* while the load leaving the span does.

If all the loads are equal and equally spaced $\left(\frac{Wd}{L}-W_1\right)$ *will always be negative and, hence, maximum S.F. at the section will occur when the first load reaches the section.*

The absolute maximum +ve S.F. evidently occurs at the right support, for which the criterion of equation 1·33 must be tried.

Equation 1·31 (or 1·33) can also be used for getting maximum −ve S.F. at the section. However, in this case, the advance (or rolling) must be to the left till δF is increased (or becomes +ve). The first chosen wheel load W_1 is considered just to the right of the section before such movement. If the whole system is now moved to the left by distance, say c, negative S.F. will increase only if $\left(\frac{Wc}{L}-W_1\right)$ is positive. If it becomes negative, the load position before such movement gives maximum −ve S.F. If it becomes positive, movement must be permitted till the expression becomes negative.

Example 1·6. *The system of concentrated loads shown in Fig. 1·15 (a) rolls from left to right across a beam simply supported over a span of 40 m, the 4 kN load leading. For a section 15 m from the left hand support, determine :*

(a) The maximum bending moment.

(b) The maximum shearing force.

Solution.

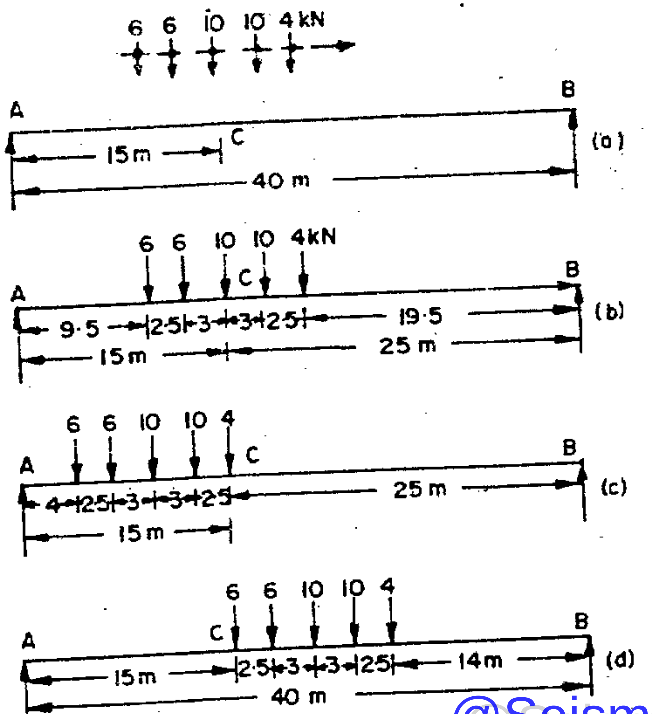


Fig. 1·15

(a) *Maximum B.M.*

By inspection, it is clear that the maximum B.M. at C will occur when the central 10 kN load is over the section, so that when the loads are rolled across the section, the condition of loading in $AC-CB$ will alter from heavier-lighter to lighter-heavier. The loads are arranged as shown in Fig. 1·15 (*b*).

Give small movement to the left,

$$\frac{W_L}{x}-\frac{W_R}{L-x}=\frac{6+6+10}{15}-\frac{10+4}{40-15}=1·47-0·56=+0·91.$$

Giving small movement to the right,

$$\frac{W_L}{x}-\frac{W_R}{L-x}=\frac{6+6}{15}-\frac{10+10+4}{40-15}=0·8-0·96=-0·16.$$

Since $\frac{W_L}{x}-\frac{W_R}{L-x}$ changes sign, the bending moment will decrease if the central 10 kN load is displaced from C. Hence the arrangement of the load shown in Fig. 1·15 (*b*) gives the maximum B.M.

Now $R_B=\frac{1}{40}\Big[(9·5\times6)+(12\times6)+(10\times15)+(10+18)+(4\times20·5)\Big]$

$=13·525$ kN

$\therefore\ M_c=-(13·525\times25)+(4\times5·5)+(10\times3)=-286$ **kN-m**

(b) ***Maximum S.F.***

For maximum +ve shear force, let us try with the first 4 kN load at the section C, with the load arranged as shown in Fig. 1·15 (*c*). Since the next load (*i.e.* 10 kN load) is at a distance $d=2·5$ m, let us roll the loads to the right by 2·5 m.

Here W=total load$=(6+6+10+10+4)=36$ kN

$W_1=4$ kN

$d=2·5$

$$\therefore\quad \delta F=\frac{Wd}{L}-W_1=\frac{36\times2·5}{40}-4=2·25-4=-1·75$$

Since it is negative, the S.F. decreases. Hence the maximum positive S.F. occurs when the 4 kN load is just to the left of the section C, as shown in Fig. 1·15 (*c*).

$\therefore\ R_B=\frac{1}{40}\Big[(6\times4)+(6\times6·5)+(10\times9·5)+(10\times12·5)+(4\times15)\Big]$

$=8·575$ kN

$\therefore\ F_c=R_B=+8·575$ **kN**

Similarly, for negative S.F. at C, let us try with the last 6 kN load at C, as shown in Fig. 1·15(d). Since the next 6 kN load is at a distance $d=2{\cdot}5$ m, let us roll the loads to the left by 2·5 m.

Here $W=36$ kN

$W_1=6$ kN ; $d=2{\cdot}5$ m

$$\therefore \quad \delta F=\frac{Wd}{L}-W_1=\frac{36\times 2{\cdot}5}{40}-6=22{\cdot}5-6=-3{\cdot}75$$

This shows that the $-$ve S.F. will be decreased. Hence the maximum negative S.F. occurs when the 6 kN load is just to the right of the section C, as shown in Fig. 1·15 (d).

$$\therefore R_A=\frac{1}{40}\Big[(4\times 14)+(10\times 16{\cdot}5)+(10\times 19{\cdot}5)+(6\times 22{\cdot}5)+(6\times 25)\Big]$$

$$=17{\cdot}525 \text{ kN}$$

$$\therefore F_c=-R_A=\mathbf{-17{\cdot}525\ kN.}$$

Example 1·7. *The following system of the wheel loads crosses a span of 25 m.*

Wheel load	*16*	*16*	*20*	*20*	*20 kN*
Distance between centre	*3*	*3*	*4*	*4 metres.*	

Find the maximum value of bending moment and shearing force in the span.

Solution

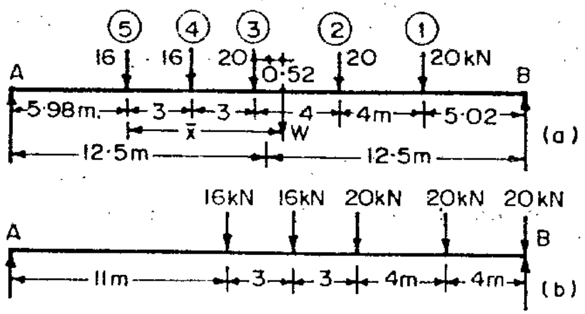


Fig. 1·16.

(a) **Maximum B.M.**

Let us number the loads as 1, 2, 3, etc.

Then $W=20+20+20+16+16=92$ kN

Taking moment of all loads about load no. 5, we get

$92\bar{x}=(16\times 3)+(20\times 6)+(20\times 10)+(20\times 14)$

$\therefore \bar{x}=7{\cdot}04$ m

Let us try with the third load ($i.e.$, 20 kN load). Maximum M under it will occur when the centre of the span is equidistant from load no. 3 and the C.G. of the loads.

Distance of load no. 3 and $W=7{\cdot}04-6=1{\cdot}04$ m.

Distance of load 3 from centre of span $=\dfrac{1{\cdot}04}{2}=0{\cdot}52$ m

$i.e.$ load no. 3 is at a distance of 0·52 m from the centre of the span, as shown in Fig. 1·16 (a).

In this position,

$$R_B=\frac{1}{25}\Big[(16\times 5{\cdot}98)+(16\times 8{\cdot}98)+(20\times 11{\cdot}98)+(20\times 15{\cdot}98)+(20\times 19{\cdot}98)\Big]=48 \text{ kN}$$

$$\therefore \quad M_{max}=-R_B(12{\cdot}5+0{\cdot}52)+(20\times 8)+(20\times 4)$$

$$=-48\times 13{\cdot}02+160+80=\mathbf{-384{\cdot}96\ kN\text{-}m.}$$

(b) **Maximum S.F.**

Maximum S.F. values is either R_A or R_B. As the C.G. of the load can approach nearer to B than to A, $R_B>R_A$ for limiting load position.

Keep the first load ($i.e.$, 20 kN) just to the left of B. Since next load is at 4 m distance, give a movement of 4 m

Thus, $W=92$ kN

$W_1=20$ kN

$d=4$ m

$P=$ load leaving the span $=20$ kN

From equation 1·33,

$$\boxed{\delta F=\frac{Wd}{L}-(W_1+P)}=\frac{92\times 4}{15}-(20+20)$$

$$=24{\cdot}5-40=-15{\cdot}5.$$

The negative sign shows that the shear force will decrease if the loads are moved. Hence the arrangement of the loads for maximum S.F. will be as shown in Fig. 1·16(b).

Considering the first 20 kN load just to the left of B, we have

$$R_B=\frac{1}{25}\Big[(16\times 11)+(16\times 14)+(20\times 17)+(20\times 21)+(20\times 25)\Big]$$

$$=66{\cdot}4 \text{ kN}$$

$$\therefore \quad F_{max.}=+R_B=\mathbf{+66{\cdot}4\ kN.}$$

Example 1·8. *A girder, simply supported over a span 20 m, is traversed by a moving load as shown in Fig. 1·17. Determine the maximum B.M. at 8 m from the left hand support.*

Solution.

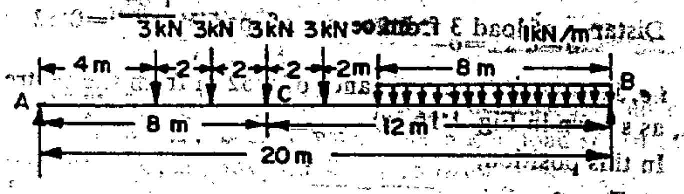


Fig. 1·17.

Let us try with the second 3 kN load at the section *C*, as shown.

Giving slight motion to the left :

$$\frac{W_L}{x}-\frac{W_R}{L-x}=\frac{3+3+3}{8}-\frac{3+8}{12}=1·125-0·91=+0·215$$

Give slight motion to the right :

$$\frac{W_L}{x}-\frac{W_R}{L-x}=\frac{3+3}{8}-\frac{3+3+8}{12}=0·75-1·17=-0·42$$

Since $\frac{W_L}{x}-\frac{W_R}{L-x}$ changes sign from +ve to −ve, the maximum will occur when the loads are arranged as shown.

$$R_A=\frac{1}{20}\Big[(8\times1\times4)+(3\times10)+(3\times12)+(3\times14)+(3\times16)\Big]$$

$$=9·4 \text{ kN}$$

$$\therefore \quad M_{max}=-(9·4\times8)+(3\times4)+(3\times2)=-\mathbf{57·2\ kN\text{-}m.}$$

1·8. EQUIVALENT UNIFORMLY DISTRIBUTED LOAD

A given system of loading crossing a girder can always be replaced by uniformly distributed load, longer than the span, such that bending moment or S.F., due to this equivalent static load, every where is atleast equal to that caused by the actual system of moving loads. Such a static load is known as *equivalent uniformly distributed load* (*E.U.D.L.*). The E.U.D.L. will be different for B.M. and S.F. The bending moment diagram for E.U.D.L. will be a parabola symmetrical about the base and must completely envelope the maximum bending moment diagram for the moving loads.

Let us now find the E.U.D.L. for the following cases, for B.M. purposes :

(*a*) Single point load.

(*b*) U.D.L. shorter than the span.

(*c*) Two point loads W_1 and W_2 at distance *d* apart.

(*a*) **E.U.D.L. for Single Point Load :**

The maximum B.M. at the section *C*, distant *x* from left support to single point load is given by equation 1·3,

$$M_{max}=-\frac{Wx}{L}(L-x) \qquad (1)$$

If *w'* is E.U.D.L. over the whole span, B.M. at the section *C* is given by

$$M=\frac{w'L}{2}x+\frac{w'x^2}{2}=-\frac{w'x}{2}(L-x) \qquad (2)$$

Equating (1) and (2), we get

$$\frac{w'x}{2}(L-x)=\frac{Wx}{L}(L-x)$$

or

$$w'=\frac{2W}{L} \qquad (1·34)$$

The same result could be obtained by equating the bending moment at the centre, *i.e.*

$$\frac{w'L^2}{8}=\frac{WL}{4}$$

or $\boxed{w'=\frac{2W}{L}}$ which is the same as above.

(*b*) **E.U.D.L. for U.D.L. shorter than the span :**

The max. B.M. at the centre of the span, due to U.D.L. shorter than the span, is given by

$$M_{max}=-\frac{wa}{4}\left(L-\frac{a}{2}\right)$$

where *a* is the length of the U.D.L.

The B.M. at the centre of span, due to E.U.D.L. *w'* is

$$M=-\frac{w'L^2}{8}$$

Equating the two, we get

$$\frac{w'L^2}{8}=\frac{wa}{4}\left(L-\frac{a}{2}\right)$$

or

$$\boxed{w'=\frac{2wa}{L^2}\left(L-\frac{a}{2}\right)} \qquad (1·35)$$

(*c*) **E.U.D.L. for the point loads W_1 and W_2 at a distance *d* apart :**

The E.U.D.L. for this must be such that the B.M.D. due to this completely envelops $^{2}M_x$, $^{02}M_x$, and $^{1}M_x$ diagrams. This can be there if the tangent to the curve of B.M. due to E.U.D.L. at the support is equal to the greater of the tangents to ${}^2Mx$ and ${}^1Mx$ (or ${}^{02}Mx$ diagaams at their corresponding ends).

Thus, in example 1·2, the equation of ${}^2Mx$ is given by

$${}^2Mx=y=\frac{x}{16}\,(136-10x)$$

$$\therefore \quad \frac{dy}{dx}(\text{at } x=0)=\frac{1}{16}\,(136)=8{\cdot}5 \qquad (1)$$

The equation of ${}^1Mx$ is given by

$${}^1Mx=(10x-36)\left(1-\frac{x}{16}\right)=(12{\cdot}25x-0{\cdot}625x^2-36)$$

$$\therefore \quad \frac{dy}{dx}\ (\text{at } x=16)=12{\cdot}25-1{\cdot}25\times16=-7{\cdot}75$$

The minus sign simply shows that the inclination of the tangent is in anticlockwise direction.

$$\therefore \quad \text{Greater } \frac{dy}{dx} \text{ due to the actual loading}=8{\cdot}5$$

The equation of B.M. at any point, due to E.U.D.L. w' is

$$Mx=y=\frac{w'x(L-x)L}{2L}=\frac{w'x(16-x)}{2}$$

$$\therefore \quad \frac{dy}{dx}\ (\text{at } x=0)=8w'$$

Equating this to the greater of (1) and (2), we get

$$8w'=8{\cdot}5$$

$$\therefore \quad w'=\frac{8{\cdot}5}{8}=1{\cdot}06 \text{ kN/m}$$

This will give max. B.M. $=\frac{w'L^2}{8}=\frac{8{\cdot}5}{8}\times\frac{16\times16}{8}=34$ kN-m.

The actual absolute Max. B.M.=28·8 kN-m, as found in example 1·2.

Similarly, the E.U.D.L. on the considerations of max. shear can also be computed.

1·9 COMBINED DEAD AND MOVING LOAD S.F. DIAGRAMS : FOCAL LENGTH

Let a girder AB, simply supported over a span L, carry a uniformly distributed dead load w/unit length. Also due to certain system of moving loads, let w'' be the E.U.D.L., based on shear *considerations.*

Fig. 1·18(a) shows the S.F.D. due to dead load.

Fig. 1·18(b) shows the S.F.D. due to E.U.D.L. At any distance x, S.F. due to E.U.D.L. is given by

$$Fx(+\text{ve})=\frac{w''x^2}{2L}$$

and
$$Fx(-\text{ve})=\frac{w''(L-x)^2}{2L}$$

Fig. 1·18(c) shows the combined S.F.D., obtained after superimposing the two diagrams.

Thus, by combining $-$ve S.F. of (a) with $+$ve S.F. of (b), we get final shear=ordinate C_1C_2. Similarly by combining $-$ve S.F. of (a) with $-$ve S.F. of (b), we get final shear=ordinate C_1C_3. Hence in the combined diagram, the final shear at any point is given by vertical intercepts between dead load S.F. and the curves of E.U.D.L.

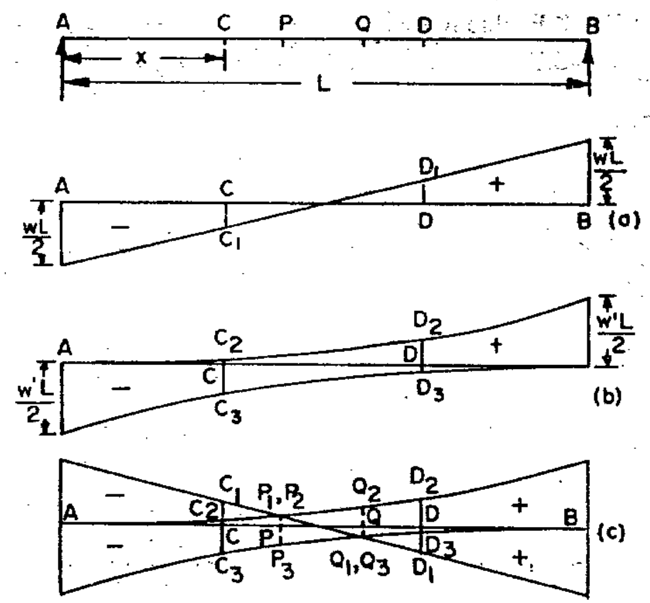


Fig. 1·18

From Fig. 1·18(c), we make the following observations :

At point C, S.F.$=C_1C_2$ and C_1C_3 (both negative)

At point P, S.F.$=P_1P_2(=0)$ and P_1P_3 (negative)

At point Q, S.F.$=Q_1Q_2$ (positive) and $Q_1Q_3(=0)$.

At point, D S.F.$=D_1D_2$ and D_1D_3 (both positive).

From the above, we make the following conclusions :

(*a*) For all sections to the left of P, the final S.F. is always negative.

(*b*) For all sections to the right of Q, the final S.F. is always positive.

(*c*) For all sections between P and Q, the final S.F. is both positive and negative. That is, the S.F. changes sign as the load moves over the portion PQ only. Such a portion of the girder, over which the final S.F. changes sign, is called the *focal length*. If such a girder is of lattice type, counter bracing is needed for this portion. In Fig. 1·18 (*c*), thus, PQ is the focal length of the girder.

Example 1·9. *Calculate the focal length of a girder of 16 m span carrying a dead load of 3 kN/m and E.U.D.L. of 6 kN/m for shear.*

Solution : (Fig. 1·18)

Let F_d=S.F. due to dead load, at any section.

F_l=S.F. due to E.U.D.L. at any section.

Then, $$F_d=-\frac{wL}{2}+wx=-\frac{3\times16}{2}+3x=-24+3x \qquad (1)$$

$$F_l(+\text{ve})=+\frac{w'x^2}{2L}=+\frac{6x^2}{2\times16}=+\frac{3x^2}{16} \qquad (2)$$

At the point P [Fig. 1·18(*c*)], F_d+F_l (+ve)$=0$

$$\therefore \qquad -24+3x+\frac{3x^2}{16}=0$$

which gives $x=AP=5·85$ m

By symmetry $BQ=AP=5·85$ m

$\therefore$ Focal length$=PQ=AB-2AP=16-2\times5·85=$**4·3 m.**

Example 1·10. *Calculate the focal length of a girder of 16 m span, carrying a dead load of 3 kN/m and a uniform live load of 2 kN/m, 4 m long, travelling from left to right.*

Solution. (Fig. 1·18)

$$F_d=-\frac{wL}{2}+wx=-\frac{3\times16}{2}+3x=-24+3x \qquad (1)$$

For $x>5$ m, $F_l(+\text{ve})=+\frac{wa}{L}\left(x-\frac{a}{2}\right)$ ...(see equation 1·6)

$$=+\frac{2\times4}{16}\left(x-\frac{4}{2}\right)=0·5(x-2)$$

At the point P [Fig. 1·18 (*c*)], F_d+F_l (+ve)$=0$

$$\therefore \qquad -24+3x+0·5(x-2)=0$$

which gives $x=AP=7·14$ m

By symmetry, $QB=AP=7·14$ m

$\therefore$ Focal length $PQ=16-2\times7·14=$**1·72 m**

Example 1·11. *Calculate the focal length of the girder of example 1·2 if it also carries a dead load of intensity 3 kN/m over the whole span.*

Solution

For the given girder : $L=16$ m ; $W_1=4$ kN

$W_2=6$ kN ; $d=6$ m

For any section distant x from A,

$$F_d=-\frac{wL}{2}+wx=-\frac{3\times16}{2}+3x=-24+3x \qquad (1)$$

(*a*) For $x>6$, max. +ve S.F. due to live load is given by

$$F_l(+\text{ve})={}^1F_x=+\frac{W_1x+W_2(x-d)}{16}=+\frac{(W_1+W_2)x-W_2d}{L}$$

$$=+\frac{(4+6)x-6\times6}{16}=+\frac{5x-18}{8}=\frac{5}{8}x-2·25 \qquad (2)$$

At the point P [Fig. 1·18(*c*)], we have

$$F_d+F_l(+\text{ve})=0$$

or $$-24+3x+\frac{5}{8}x-2·25=0$$

which gives $x=AP=7·24$ m.

(*b*) Again, for $x>8<10$, we have

$$F_l(-\text{ve})={}^2F_x=-\frac{W_2(L-x)+W_1(L-x-d)}{L} \qquad (1·18)$$

$$=-\left[\frac{6(16-x)+4(16-x-6)}{16}\right]$$

$$=-8·5+\frac{5}{8}x \qquad (3)$$

For the point O, we have

$$F_d+F_l(-\text{ve})=0$$

$$\therefore \qquad -24+3x-8·5+\frac{5}{8}x=0$$

which gives $x=AQ=8·96$ m.

Hence focal length$=AQ-AP=8·96-7·24$

$=$**1·72 m**

PROBLEMS

1. A single rolling load of 10 kN rolls along a girder of 20 m span. Draw the diagrams of maximum B.M. and maximum S.F. positive and negative. What will be the absolute maximum (+) S.F. and B.M. ?

2. A uniform load of 1 kN/m, 4 m long cross a girder of 16 m span. Construct the maximum S.F. and B.M. diagrams and calculate values at section 6 m and 8 m from left hand support.

3. Two concentrated rolling loads of 12 and 6 kN, placed 4·5 m apart, travel along a freely supported girder of 16 m span. Sketch the graphs of maximum shearing force and maximum bending moment and indicate the position and magnitudes of the greater value.

4. A simply supported girder has a span of 40 m. A moving load consisting of a uniformly distributed load of 1 kN/m over a length of 8 m preceded by a concentrated load of 6 kN moving at a fixed distance of 2 m in front of the distributed load, crosses the beam.

Find (*a*) the point of the beam at which the greatest bending moment occurs, (*b*) the position of the load where it occurs, (*c*) the value of the greatest B.M.

5. A simply-supported beam is traversed by a train of wheel loads of irregular spacing and unequal weights. State and prove (*a*) a rule giving the train position for the bending moment under a particular load to have its maximum value, and (*b*) a rule giving the train position for the bending moment at a given point on the beam to its maximum value.

6. A freely supported gantry girder of effective span L carries a travelling crane with two wheel loads, each W at spacing a, this spacing being less than $\frac{L}{2}$. Find, from first principles, the maximum bending moment induced by the loads.

If the spacing a is increased, find the maximum value of a (in terms of L) for which the maximum bending moment will occur at the centre of the span with only one wheel on the girder. (*U.L.*)

7. A system of moving loads cross a girder of 36 m span which is simply supported at its ends. The loads and their distances are as follows :

Wheel loads (kN)	10	10	20	20	16
Distance between centres	3	4·5	4	3·5	

Determine

(*a*) The maximum bending moment at the quarter span.

(*b*) The maximum bending moment in the girder.

For each case, make a sketch of the girder showing clearly the section where the bending moment occurs and the corresponding position of the loads.

8. The following system of concentrated loads roll from left to right on a span of 15 m, 4 kN load leading :

Load	2	6	6	5	4	kN
Distance	1·5	1·5	2	1		metres

For a section 4 m from the left hand support, determine (*a*) the maximum bending moment, (*b*) maximum S.F.

9. The following system of wheel loads crosses a plate girder of 30 m span :

Wheel load	8	18	18	15	kN
Distance between centres	4·5	3·5	4		m

Determine the maximum value of the shearing force which may be produced at the middle point of the span. Also, find the equivalent uniformly distributed load which could produce the same maximum bending moment at midspan. (*U.L.*)

10. A simply supported beam of span L is crossed by a uniformly distributed load of length m and of total weight W. If L is greater than m, obtain from first principles an expression for the maximum bending moment at any point at distance a from one support. Hence show that a single point load of $W\left(1-\frac{m}{2L}\right)$, travelling across the span will give the same maximum moment everywhere along the beam as the above uniformly distributed load. (*U.L.*)

11. The following arrangement of axle load is carried by a single bridge girder across a clear span of 30 m.

Axle loads	5	5	10	10	10	kN
Spacing	2·5	2·5	2·5	2·5		m

Determine the maximum bending moment and maximum S.F. at section distant 10 m from left hand abutment. The 5 kN load leads, and the system may pass over the bridge from either side.

12. A beam, simply supported over a span L is traversed by a uniformly distributed load of intensity w and length $\frac{L}{5}$. If the beam also carries a dead load, uniformly distributed over the span, of intensity $\frac{w}{2}$ indicate on the diagram the length of the beam for which there is reversal of shear force.

Answers

1. S.F. : ± 10 kN ; B.M. : -50 kN-m.
2. 13·12 kN-m ; 14·10 kN-m.
3. $F_{max \cdot max}$ (+ve)=14·62 kN at right hand support.
   $F_{max \cdot max}$ (−ve)=16·31 kN at left support.
   $M_{max \cdot max}$=59·1 kN at 7·25 m from left support.
4. (*a*) 20·26 m from left support.
   (*b*) Tail of load at distance of 13·17 m from support.
   (*c*) M_{max}=118·5 kN-m.
5. $M_{max}=\frac{W}{8L}(2L-a)^2$; a=0·5862
6. M_{max} at left quarter span=373·8 kN-m.
   M_{max} at right quarter span=396·8 kN-m.
   $M_{max \cdot max}$=519·3 kN-m with central wheel load placed 0·48 m off the centre.
8. 51·9 kN-m ; −12·1 kN when 4th load is on the section.
9. (*a*) 19·4 kN ; (*b*) w=3·11 kN/m.
10. 216·7 kN-m ; −21·25 kN.
12. 0·2286 L.

2

Influence Lines

2·1. DEFINITION

An influence line for any *given point* or section of a structure is a curve whose ordinates represent to scale the variation of a function, such as shear force, bending moment, deflection, etc. at the point or section as unit load moves across the structure. In other words, an influence line for any given point C on a structure is such a curve that its ordinate at any point D gives the bending moment, shear force or similar quantity at C when a unit load is placed at D. For statically determine structures, the influence lines for B.M., shear or stress are composed of straight lines, while they are curvilinear for statically indeterminate structures. The influence lines are very useful in the speedy determination of the value of a function at *the given section* under any complex system of loading. These also help to determine, in an easy manner, the disposition of the load system so as to cause the maximum value of the function at the section.

The difference between a curve of B.M. or S.F. (as discussed in the previous chapter) and an influence line of B.M. or S.F. must be clearly understood at this stage. The ordinate of a curve of B.M. or S.F. gives the value of the B.M. or S.F. at the section where the ordinate has been drawn, while in the case of an influence line, the ordinate at any point gives the value of the B.M. or S.F. only at the given section (for which the influence line has been drawn) and *not* at the point at which the ordinate has been drawn. Also, there is one single B.M. or S.F. curve of the *whole beam* under the action of a given set or train of loads, while there are infinite number of influence lines, one for each section of the beam, drawn for a unit rolling load.

2·2. INFLUENCE LINE FOR SHEAR FORCE

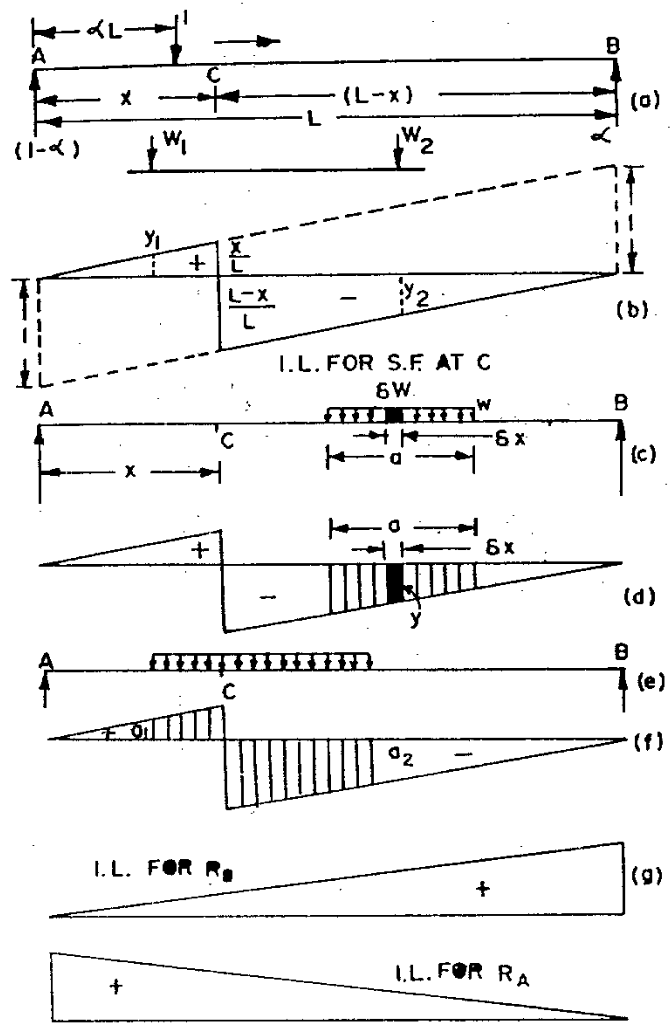


Fig. 2·1

Let us consider a simply supported beam AB of span L, and construct the influence line for S.F. at a section C distant x from the left support. *The position of the section is fixed, while the unit load moves from left to right.* The problem is to plot the *variation* of S.F. at the given section C, as the unit load moves along the beam.

At any instant, let the unit load be at a distance αL from the support A. Then, $R_B=\alpha$, and $R_A=(1-\alpha)$.

$\therefore$ Shear force at $C=F_C=+R_B=+\alpha$ (1)

The variation is linear, and is valid for all positions of load between 0 to x from A.

When the load is at A, $\alpha L=0$, $\therefore$ $F_C=0$

When the load is at C, $\alpha L=x$

$$\therefore \quad F_C=+\alpha=+\frac{x}{L}$$

When the unit load crosses the section C, $\alpha L>x$, and hence

$$F_C=-R_A=-(1-\alpha) \quad (2)$$

Thus, the S.F. changes sign as the unit load crosses the section. The variation is linear, and is valid for all load positions between x to L from A.

When the unit load is slightly to the right of C, $\alpha L=x$

$$\therefore \quad F_C=-(1-\alpha)=-\left(1-\frac{x}{L}\right)=-\frac{L-x}{L}$$

When the unit load is at support B, $\alpha L=L$ or $\alpha=1$

$$\therefore \quad F_C=-(1-\alpha)=-(1-1)=0$$

The complete influence line diagram for the S.F. at C is given in Fig. 2·1 (b).

As per definition, the ordinate $+y_1$ at a point gives the S.F. at C, due to unit load at the point where the ordinate y_1 is measured. Hence if a load W_1 is acting at that point, and y_1 is the ordinate of *I.L.* under it, the S.F. at $C=+W_1y_1$. Similarly, if a load W_2 is acting at a certain point, and $-y_2$ is the ordinate of the influence line under the point of application of the load, the S.F. at C will be $-W_2y_2$. If W_1 and W_2 are acting simultaneously, the S.F. at $C=W_1y_1-W_2y_2$. Hence if the beam is being acted upon by loads W_1, W_2, W_3..., W_n, and $y_1, y_2, \ldots y_n$ are the corresponding influence line ordinates under them, the S.F. at C is

$$F_C=W_1y_1+W_2y_2+W_3y_3+\ldots W_ny_n=\Sigma Wy \quad (2{\cdot}1)$$

In the above equation, the numerical value of the ordinate y is to be substituted with its proper algebraic sign, *i.e.* +ve if it is of positive diagram, and −ve if of negative portion of the influence line diagram.

Let us now take the case of U.D.L. (w) of a length a, placed in the position shown in Fig. 2·1(c). Let us consider a length δx of the load, and the corresponding elementary load $\delta W=w.\delta x$. Hence S.F. at C, due to the elementary load δW is

$$\delta F_C=\delta W.y$$

(where y is the influence line ordinate under δW)

$$\delta F_C=w\delta x.y \quad (3)$$

$=w\times$ area of the elementary strip of the I.L. diagram [shown in Fig. 2·1(d)]

Therefore, the shear force at C, due to total U.D.L. of length a given by

$$F_C=\Sigma w(\delta x.y)=w\Sigma\delta x.y \quad (2{\cdot}2)$$

$=w\times$ area of I.L. diagram under the U.D.L. [shown shaded in Fig. 2·1(d)]

Hence the S.F. at C, due to U.D.L. of length a is equal to the area of the I.L. diagram under the U.D.L. multiplied by the intensity of the load.

Fig. 2·1(e) shows the U.D.L. extending to both the sides of the section C. In this case, the S.F. at C is obtained by multiplying the *net area* by the intensity of the load.

Thus, $F_C=w(+a_1-a_2)$

where a_1=area of the positive S.F. diagram under the U.D.L.

a_2=area of the negative S.F. diagram under the load.

If $a_1=a_2$, $F_C=0$.

Influence Line for the Reactions

If the section C is located at the support B, the value of $x=L$ and hence the ordinate of +ve I.L. diagram under $B=\frac{x}{L}=\frac{L}{L}=1$.

Thus, the I.L. for reaction at B=I.L. for shear at C when $x=L$, and is a triangle having a maximum ordinate of unity under B. However the I.L. for reactions at A and B can be plotted independently as under :

When the load is at a distance αL from A,

$R_B=+\alpha$ and $R_A=+(1-\alpha)$

When the load is at A, $\alpha=0$

$\therefore$ $R_B=0$; $R_A=+1$

When the load is at B, $\alpha L=L$; or $\alpha=1$

$\therefore$ $R_A=+\alpha=+1$: $R_A=+(1-\alpha)=0$.

Hence the I.L. for R_B consists of a triangle having zero ordinate at A and unit ordinate at B. Similarly, the I.L. for R_A consists of a triangle having unit ordinate at A and zero ordinate at B as shown in Fig. 2·1(g) and (h) respectively.

2·3. INFLUENCE LINE FOR BENDING MOMENT

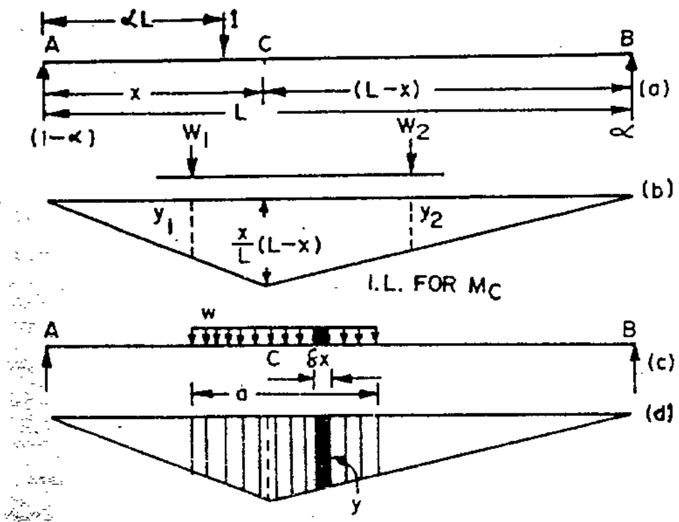


Fig. 2·2.

Let us now construct the I.L. for B.M. at C.

When the unit load is at a distance αL from A, such that $\alpha L < x$, we have $R_B = \alpha$ and $R_A = (1-\alpha)$.

$$\therefore \quad M_C = -R_B(L-x) = -\alpha(L-x)$$

The variation is linear, and is valid for load position distant 0 to x from A.

When the unit load is at A, $\alpha L = 0$.

$$\therefore \quad M_C = 0$$

When the unit load is at C, $\alpha L = x$

$$\therefore \quad M_C = -\frac{x}{L}(L-x) \tag{2·3}$$

When the unit load is at C, $\alpha L > x$

$$\therefore \quad M_C = -R_A \cdot x = -(1-\alpha)x \tag{1}$$

The variation is linear, and is valid for load position distant x to L from A.

When the load is at C, $\alpha L = x$

$$\therefore \quad M_C = -\left(1-\frac{x}{L}\right)x = -\frac{L-x}{L}\,x$$

which is the same as equation 2·3.

Thus, the I.L. diagram for M_C is a triangle having a maximum ordinate of $\frac{x}{L}(L-x)$ *under the section* as shown in Fig. 2·2(*b*).

If there are two loads W_1 and W_2 acting, and if y_1 and y_2 are the influence line ordinates under these loads, we have by definition

$$M_C = -(W_1y_1 + W_2y_2)$$

Hence if there are number of point loads $W_1, W_2, \ldots W_n$ and the corresponding I.L. ordinates under them are $y_1, y_2, \ldots y_n$ we have

$$M_C = (W_1y_1 + W_2y_2 + \ldots\ldots W_ny_n)$$
$$= -\Sigma Wy \tag{2·4}$$

Let there be an U.D.L. of intensity w, and length a, as shown in Fig. 2·2(*c*). Consider an elementry length δx of the load, such that the elementary load $\delta W = w\delta x$. Let y be the average ordinate under the elementary load. Then the B.M. at C due to this elementary load is given by

$$\delta W_C = \delta W \,.\, y = -w\delta x . y$$
$$= -w \times \text{area of the elementary strip of the I.L. diagram [shown thick in Fig. 2·2(d)].}$$

Hence the B.M. at C, due to the total U.D.L. of length a is

$$M_C = -\Sigma w(\delta x . y)$$
$$= -w \times \text{area of I.L. diagram under U.D.L. [shown shaded in Fig. 2·2 (d)]} \tag{2·5}$$

Thus, the B.M. at C, due to U.D.L. of length a is equal to the intensity of load multiplied by the area of I.L. diagram under the uniformly distributed load.

2·4. LOAD POSITION FOR MAXIMUM S.F. AT A SECTION

In chapter 1 on rolling loads, we have derived the load positions for maximum S.F. at a given section. We will now use the influence line for determination of the position of loads for maximum S.F. at the section C. We shall take different loading conditions.

1. Single point load

Let a single point load of magnitude W roll from left to right. Referring to I.L. of S.F. at the section C distant x from A [Fig. 2·1 (*b*)] maximum positive S.F. will occur when the load is *just to the left* of C, and maximum −ve S.F. will occur when the load is *just to the right* of C.

$$\text{Thus, } F_C(\text{max. +ve}) = +\frac{Wx}{L}$$

and

$$F_C(\text{max. −ve}) = -\frac{W(L-x)}{L}.$$

2. U.D.L. longer than the span

From the I.L. for S.F. at C, Fig. 2·1(b), it is clear that the max. +ve S.F. will occur when the span AC is loaded and CB is empty and max. −ve S.F. will occur when the span CB is loaded and AC is empty.

Thus, $Fc\ (\text{max}\ +\text{ve})=w\cdot\frac{1}{2}\ x\cdot\frac{x}{L}=\frac{wx^2}{2L}$

and $Fc(\text{max.}\ -\text{ve})=w\cdot\frac{1}{2}\left(L-x\right)\frac{(L-x)}{L}=\frac{w(L-x)^2}{2L}$

3. U.D.L. shorter than the span

Let the U.D.L. of length a travel from left to right. From Fig. 2·1 (b), maximum +ve S.F. at C will occur when the head of the load reaches C, while maximum −ve S.F. will occur when the tail of the load is at C.

4. Several Point Loads

For several point loads, we may use the same criterion, as discussed in the previous chapter. Thus, if a load W_1 is at the section C, with other loads in appropriate position, and the loads are moved by a distance d such that next load comes over C, the change δFc is given by

$$\delta Fc=\frac{Wd}{L}-W_1.$$

If the above expression is positive, it indicates an increase in S.F. and the loads must be permitted to roll to get greater S.F. The procedure must be repeated till the above expression changes sign, which indicates that greatest peak has been passed.

2·5. LOAD POSITION FOR MAXIMUM B.M. AT A SECTION

Here also, we shall consider all the loading conditions :

1. Single Point Load

Let a single point load W roll from left to right. Since the I.L. diagram for B.M. at C has the maximum ordinate under C itself [see Fig. 2·2(b)], maximum B.M. will occur when the load is at C itself.

Thus $Mc\ (\text{max})=W\ \frac{x}{L}\ (L-x)$

2. U.D.L. greater than the span

Refer to Fig. 2·2 (b) maximum B.M. will occur when the U.D.L. occupies the whole span.

Thus $Mc(\text{max.})=w\times\text{area of I.L. diagram}$

$$=w\times\frac{1}{2}\times L\,\frac{x}{L}\left(L-x\right)$$

$$=\frac{wx\,(L-x)}{2}$$

3. U.D.L. Shorter than the span

Let the uniformly distributed load be of length a. The load has to be arranged, with respect to section C, in such a way that the area of the I.L. diagram under the load is maximum.

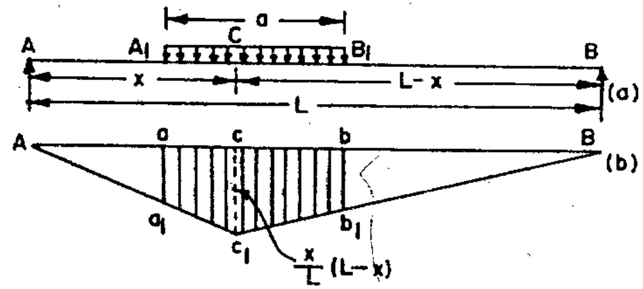


Fig. 2·3.

Let the load be arranged in the position as shown in Fig, 2·3, so that the shaded area of I.L. diagram is maximum. That is, a small movement of the loading to the left or right will decrease the area of the I.L. diagram. If a movement is given to the left, ordinate aa_1 will be decreased while bb_1 increased, and the net result will be the decrease in the area of I.L. diagram. Similary, if a movement is given to the right, ordinate bb_1 will be decreased while ordinate aa_1 will be increased and the net result will be the decrease in the area of the I.L. diagram. Evidently, maximum area will be obtained only if the ordinate aa_1 is equal to ordinate bb_1.

$$aa_1=\frac{x}{L}\left(L-x\right)\cdot\frac{AA_1}{x}=\frac{L-x}{L}\cdot AA_1$$

and $$bb_1=\frac{x}{L}\left(L-x\right)\cdot\frac{BB_1}{L-x}=\frac{x}{L}\cdot BB_1$$

Since aa_1 must be equal to bb_1 for maximum area,

$$\frac{L-x}{L}\cdot AA_1=\frac{x}{L}\cdot BB_1$$

or $$\frac{x}{L-x}=\frac{AA_1}{BB_1}=\frac{x-AA_1}{L-x-BB_1}=\frac{A_1C}{CB_1}$$

or $$\frac{AC}{CB}=\frac{A_1C}{CB_1} \tag{2·6}$$

which is the same as that derived in chapter 1.

Hence the maximum bending moment at a section occurs when the section divides the U.D.L. in the same ratio as it divides the span.

$\therefore$ M_C(max.) $= w \times$ area of I.L. diagram under the load

$$= w\left[(aa_1+cc_1)\frac{ac}{2}+(cc_1+bb_1)\frac{cb}{2}\right]$$

$$= w\left(aa_1+cc_1\right)\frac{a}{2}, \text{ since } aa_1=bb_1$$

$$= \frac{wa}{2}\left\{\frac{x(L-x)}{L}\cdot\frac{L-a}{L}+\frac{x(L-x)}{L}\right\}$$

$$= \frac{wax\,(L-x)}{2L^2}\cdot(2L+a) \tag{2·7}$$

4. Several Point Loads

Let the loads be so arranged that W_L is the resultant of the loads to the left of the section C, and W_R is the resultant of the loads to the right of C. Let y_1 and y_2 be the ordinates under W_L and W_R respectively.

The B.M. at C, for this arrangement is given by

$$M_C = W_1 \cdot y_1 + W_R \cdot y_2$$

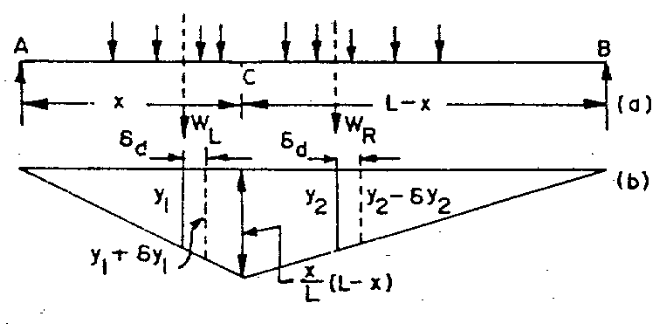


Fig. 2·4.

This will be maximum only if a small movement δd of the loads either to the left or to the right, will decrease its value. Let the loads be given a movement δd to the right, and let the new ordinate under W_L and W_R be $(y_1+\delta y_1)$ and $(y_2-\delta y_2)$ respectively. The corresponding change δM_C is given by

$$\delta M_C=[W_L(y_1+\delta y_1)+W_R(y_2-\delta y_2)]-[W_L.y_1+W_R.y_2]$$
$$=W_L.\delta y_1-W_R.\delta y_2$$

$$=W_L.\frac{L-x}{L}\,\delta d-W_R.\frac{x}{L}.\,\delta d$$

$$=\frac{x(L-x)}{L}\cdot\delta d\left(\frac{W_L}{L}-\frac{W_R}{L-x}\right) \tag{2·9}$$

Thus, δM_C is negative when $\left(\frac{W_L}{x}-\frac{W_R}{L-x}\right)$ becomes negative (or changes sign). The value $\frac{W_L}{x}-\frac{W_R}{L-x}$ can change sign only when a wheel load passes the section C, thus increasing W_R and decreasing W_L. Thus, to get maximum bending moment at a section, one of the loads should be placed at the section, so that if that load is considered as a part of W_L, the expression $\left(\frac{W_L}{x}-\frac{W_R}{L-x}\right)$ is positive, but if considered as a part of W_L, the expression $\left(\frac{W_L}{x}-\frac{W_R}{L-x}\right)$ is negative.

Example 2·1. *Two wheel loads of 16 and 8 kN, at a fixed distance apart of 2 m, cross a beam of 10 m span. Draw the influence line for bending moment and shear force for a point 4 m from the left abutment, and find the maximum bending moment and shear force at that point.*

Solution. (Fig. 2·5).

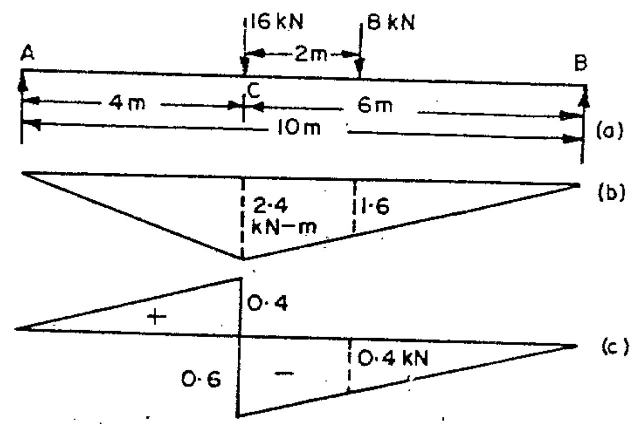


Fig. 2·5.

(*a*) *Max. B.M. at C*

The I.L. for B.M. at C distant 4 m from A is shown in Fig. 2·5(*b*).

The maximum ordinat under $C=\frac{x(L-x)}{L}=\frac{4\times 6}{10}=2·4$.

The B.M. at C is maximum when δWy is maximum. By inspection, M_{max} occurs when the loads are as in the position shown.

Ordinate under 16 kN load=2·4.

Ordinate under 8 kN load$=\frac{2{\cdot}4\times4}{6}=1{\cdot}6$

$\therefore \quad M_C=(16\times2{\cdot}4)+(8\times1{\cdot}6)=\mathbf{51{\cdot}2\ kN\text{-}m}$

(*b*) *Max, S.F. at C*

The I.L. for S.F. at a section *C* distant 4 m from *A* is shown in Fig. 2·5 (*c*).

The ordinate under *C* are $+\frac{x}{L}=+\frac{4}{10}=+0{\cdot}4$

and $-\frac{L-x}{x}=-\frac{6}{10}=-0{\cdot}6.$

By inspection of the I.L., max. S.F. occurs when the 16 kN load is just to the right of *C*, and the 8 kN load is ahead of it. Ordinate under 16 kN load$=-0{\cdot}6$. Ordinate under 8 kN load $=-\frac{0{\cdot}6}{6}\times4=-0{\cdot}4.$

$\therefore \quad F_C=-[16\times0{\cdot}6+8\times0{\cdot}4]=\mathbf{12{\cdot}8\ kN}$

It can be shown that the max. +ve S.F. at *C* will be lesser than 12·1 kN.

Example 2·2. *Make neat diagram of the influence lines for shearing force and B.M. at a section 3 m from one end of a simply supported beam, 12 m long. Use the diagram to calculate the maximum shearing force and the maximum bending moment at this section due to a uniformly distributed rolling load, 5 m long and of 2 kN per meter intensity.*

Solution. (Fig. 2·6)

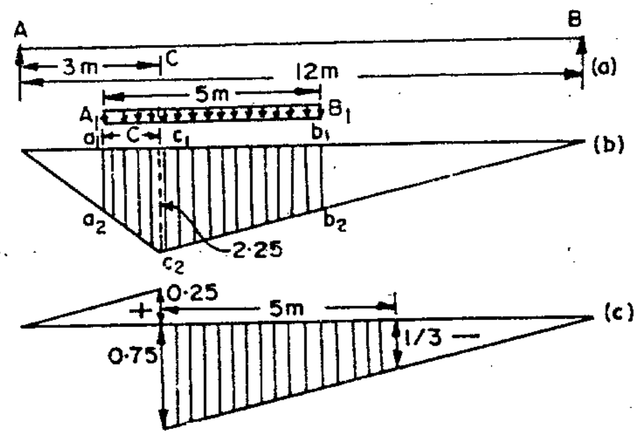


Fig. 2·6

66

(*a*) *I.L. for B.M.*

The ordinate c_1c_2 of the I.L. diagram for B.M. ε

=2·25. I.L. for B.M. at *C* is shown in Fig. 2.6(*b*) U.D.L. is shorter than the span, the load is to be so arranged that area of I.L. diagram under the load is maximum. For this condition $a_1a_2=b_1b_2$.

Let the tail of the load be at a distance *c* from *C*.

From equation 2·6, $\frac{AC}{CB}=\frac{A_1C}{CB_1}$

or $\quad \frac{3}{9}=\frac{c}{5-c}$

or $\quad 15-3c=9c$

From which $\quad c=1{\cdot}25$ m

The end ordinates are :

$$a_1a_2=\frac{2{\cdot}25}{3}\times1{\cdot}75=1{\cdot}3125$$

$$b_1b_2=\frac{2{\cdot}25}{9}\times5{\cdot}25=1{\cdot}3125$$

$$\therefore \quad M_C=\frac{1{\cdot}3125+2{\cdot}25}{2}\times(1{\cdot}25+3{\cdot}75)\times2$$

$$=\mathbf{17{\cdot}81\ kN\text{-}m}$$

(*b*) *I.L. for S.F.*

The I.L. for S.F. at *C* is shown in Fig. 2·6(*c*). The ordinates under *C* are$=+\frac{3}{12}=+0{\cdot}25$, and $-\frac{9}{12}=-0{\cdot}75$. By inspection, maximum S.F. at *C* will occur when the tail of the load is at *C*. The ordinate under the head of load$=\frac{0{\cdot}75\times4}{9}=\frac{1}{3}$.

Then $F_C=w\times$(Shaded area of I.L. under U.D.L.)

$$=2\times\frac{5}{2}\left(0{\cdot}75+\frac{1}{3}\right)=\mathbf{5{\cdot}42\ kN.}$$

Example 2·3. *A simply supported girder has a span of 25 m. Draw on squared paper the influence line for shearing force at a section 10 m from one end, and using the diagram determine the maximum shearing force due to the passage of a knife-edge load of 5 kN, followed immediately by a uniformly distributed load of 2·4 kN per metre extending over a length of 5m. The loads may cross in either direction.*

Solution. (Fig. 2·7)

The I.L. ordinate $cc_1=+\frac{10}{25}=+\frac{2}{5}$

$$cc_2=-\frac{15}{25}=-\frac{3}{5}$$

For maximum +ve S.F., the 5 kN load will be just to the left of C, and the U.D.L. behind or to the left of it. In this position, the ordinate aa_1 under the tail of the U.D.L. is

$$aa_1=\frac{2}{5}\times\frac{5}{10}=\frac{1}{5}$$

$$Fc(+\text{ve})=\left(5\times\frac{2}{5}\right)+2{\cdot}4\left(\frac{2}{5}+\frac{1}{5}\right)\frac{5}{2}=+5{\cdot}6\ \textbf{kN}$$

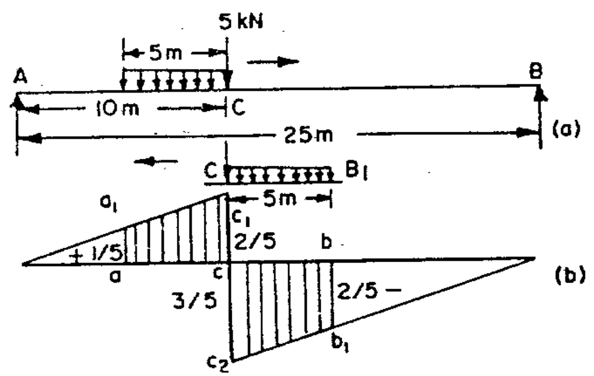


Fig. 2·7

For maximum +ve S.F. at C, the 5 kN load will be just to the right of C, with U.D.L. to the right of it. In this position, the ordinate bb_1 under the tail of the load is

$$bb_1=\frac{3}{5}\times\frac{10}{15}=\frac{2}{5}$$

$$\therefore\quad Fc\ (-\text{ve})=\left(5\times\frac{3}{5}\right)+2{\cdot}4\left(\frac{3}{5}+\frac{2}{5}\right)\frac{5}{2}=9\ \textbf{kN}$$

Hence the maximum S.F. at the section is the greater of the two. Its value is, therefore, 9 **kN.**

Example 2·4. *Four wheel loads of 6, 4, 8 and 5 kN cross a girder of 20 m span, from left to right followed by U.D.L. of 4 kN/m and 4 m long with the 6 kN load leading. The spacing between the loads in the same order are 3 m, 2m and 2 m. The head of the U.D.L. is at 2 m from the last 5 kN load. Using influence lines, calculate the S.F. and B.M. at a section 8 m from the left support when the 4 kN load is at centre of the span.*

Solution.

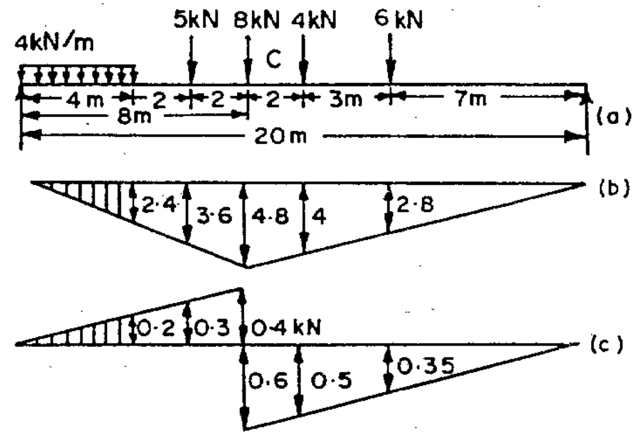


Fig. 2·8

(*a*) *Bending Moment*

The ordinate of I.L. for B.M. at $C=\frac{8\times12}{20}=4{\cdot}8$

When the 4 kN load is at the centre of the beam, the arrangement of the other loads will be as shown in Fig. 2·8 (*a*).

$\therefore$ Ordinate under 6 kN load$=\frac{4{\cdot}8}{12}\times7=2{\cdot}8$

Ordinate under 4 kN load$=\frac{4{\cdot}8}{12}\times10=4$

Ordinate under 5 kN load$=\frac{4{\cdot}8}{8}\times6=3{\cdot}6$

Ordinate under head of U.D.L.$=\frac{4{\cdot}8}{8}\times4=2{\cdot}4$

$$\therefore\quad Mc=-\Sigma My=-[(6\times2{\cdot}8)+(4\times4)+(8\times4{\cdot}8)+(5\times3{\cdot}6)+(\tfrac{1}{2}\times2{\cdot}4\times4\times4)]$$

$$=-108{\cdot}4\text{ kN-m} \qquad (1)$$

(*b*) *Shear Force*

The ordinate of I.L. for S.F. at C are $+\frac{8}{20}=+\frac{2}{5}=+0{\cdot}4$ and $-\frac{12}{20}=-\frac{3}{5}=-0{\cdot}6$

Hence ordinate under 6 kN load $=\frac{0{\cdot}6}{12}\times7=0{\cdot}35$

ordinate under 4 kN load$=\frac{0{\cdot}6}{12}\times10=0{\cdot}5$

ordinate under 5 kN load $=\frac{0.4}{8}\times 6=0.3$

ordinate under head of U.D.L. $=\frac{0.4}{8}\times 4=0.2$

(*i*) *maximum −ve S.F.*

For maximum −ve S.F., consider the 8 kN load just to the left of *C*.

Then $F_C=+\left\{(8\times 0.4)+(5\times 0.3)+\left(\frac{1}{2}\times 4\times 0.2\times 4\right)\right\}-\{(4\times 0.5)+(6\times 0.35)\}$

$=+2.2$ kN

(*ii*) *Maximum −ve S.F.*

For maximum −ve S.F., consider the 8 kN load just to the right of *C*.

Then $F_C=-\{(6\times 0.35)+(4\times 0.5)+(8\times 0.6)\}+\{(5\times 0.3)+(\frac{1}{2}\times 4\times 0.2\times 4)\}$

$=-5.8$ **kN** (2)

Hence the maximum S.F. at *C*, under the given load positions, will be the greater of the two, *i.e.* **5·8 kN.**

Example 2·5. *A horizontal beam ABC is hinged at A and simply supported at B. The span is 15 m. The cantilevered portion BC is 6 m long. Draw the influence line for bending moment for the points D and E respectively 12 m from A and 4 m from C. Hence find the maximum ± bending moments at D and the maximum bending moment at E due to a load of 1 kN/m of a length 3 m. State the corresponding position of the load.*

Solution

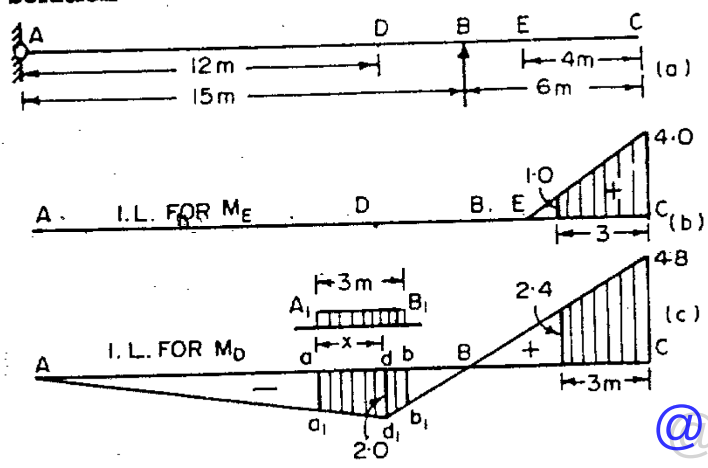


Fig. 2·9

Since the beam is hinged at *A*, it is statically determinate.

(*a*) *I.L. for B.M. at E*

Let the unit load roll from left to right. When the unit load is between *A* to *E*, the *B.M.* at *E* is equal to zero (since there is no load to the right of *E*).

When the unit load is in *EC*, distant *x* from *E*,

$$M_E=+1\times x=+x.$$

The variation is linear,

when $x=0$, $M_E=0$.

when $x=4$ m, $M_E=+4$.

Hence the ordinate of I.L. diagram is zero under *E* and +4 units under *C*. The I.L. for B.M. at *E* is shown in Fig. 2·9(*b*).

For maximum value of M_E, the head of the load should be at *C*. In this position, ordinate under the tail of the U.D.L. $=\frac{4}{4}\times 1$ $=1$.

Hence $M_E=+1\times\frac{3}{2}(1+4)=+7.5$ kN-m

(*b*) *I.L. for B.M. at D*

Let the unit load travel from left to right. When the load is at a distance *x* from *A*, such that $x<15$.

$$R_A=\frac{15-x}{15}\uparrow \text{ and } R_B=\frac{x}{15}\uparrow$$

When $x<12$ m, $M_D=-R_B\times 3=-\frac{x}{15}\times 3=-\frac{x}{5}$

when the load is at *A*. $x=0$, $\therefore$ $M_D=0$

when the load is at *D*, $x=12$, $\therefore$ $M_D=-\frac{12}{5}=-2.4$

When $12<x<15$ m, $M_D=-R_A\times 12=-\frac{15-x}{15}\times 12$

when the load is at *D*, $x=12$, $\therefore$ $M_D=-2.4$

when the load is at *B*, $x=15$, $\therefore$ $M_D=0$

when the load is in *BC*, distance *x* from *A*, $R_B=\frac{x}{15}\uparrow$

$\therefore$ $M_D=+1\,(x-12)-R_B\times 3=(x-12)-\frac{x}{15}\times 3=(0.8x-12)$

when the load is at *B*, $x=15$, $\therefore$ $M_D=0$

when the load is at *C*, $x=21$, $\therefore$ $M_D=+4.8$ KN-m.

The complete I.L. for B.M., at *D* is shown in Fig. 2·9(*c*).

Now, *for maximum positive B.M.*, the head of the load should be at C. In this position, the ordinate under the tail of the load $=\frac{4\cdot8}{6}\times3=2\cdot4$.

Hence M_D (+ve max.) $=+\frac{1\times3}{2}(2\cdot4+4\cdot8)=+$**10·8 kN-m.**

For maximum negative B.M., the load should be partially to the left and partially to the right of D such that the ordinate $aa_1=bb_1$. Let the tail of the load be a distance x from D. Then, from equation 2·6

$$\frac{AD}{DB}=\frac{A_1D}{DB_1}$$

or
$$\frac{12}{3}=\frac{x}{3-x}$$

From which $x=2\cdot4$ m.

$\therefore$ Ordinate $aa_1=bb_1=\frac{2\cdot4}{12}\times9\cdot6=1\cdot92$

$\therefore$ M_D (max. $-$ve) $=-\frac{1\times1}{2}[(aa_1+dd_1)\times ad+(dd_1+bb_1)db]$

$$=-\frac{1}{2}(aa_1+dd_1)\times ab=-\frac{1}{2}(1\cdot92+2\cdot4)\times3$$

$=-$**6·48 kN-m.**

Example 2·6. *A beam ABC is supported at A, B and C, and has a hinge at D distant 3 m from A. AB=7 m and BC=10 m. Draw the influence lines for:*

(*i*) *reactions at A, B and C.*
(*ii*) *S.F. at a point just to the right of B.*
(*iii*) *B.M. at a section 1 m to the right of B.*

Hence, if a U.D.L. of intensity 2 kN/m, and length 3 m, travels from left to right, calculate above quantities from which I.L. are drawn.

Solution

The direction of any reaction will be +ve if it is acting in upward direction (↑).

(*a*) *I.L. for reaction at R* (R_A)

Let the unit load roll from left to right. When the load is in AD, distant x from A,

$$R_A=\frac{3-x}{3}\uparrow \text{ (since } M_D)=0 \quad (1)$$

When the load is at A, $x=0$, $\therefore$ $R_A=1$
When the load is at D, $x=3$, $\therefore$ $R_A=0$.

As the load enters DB, the reaction R_A is always zero, since M_D has to be zero due to the hinge. Hence the I.L. for R_A consists of a triangle having a maximum ordinate of unity under A, and zero under D, as shown in Fig. 2·10(b).

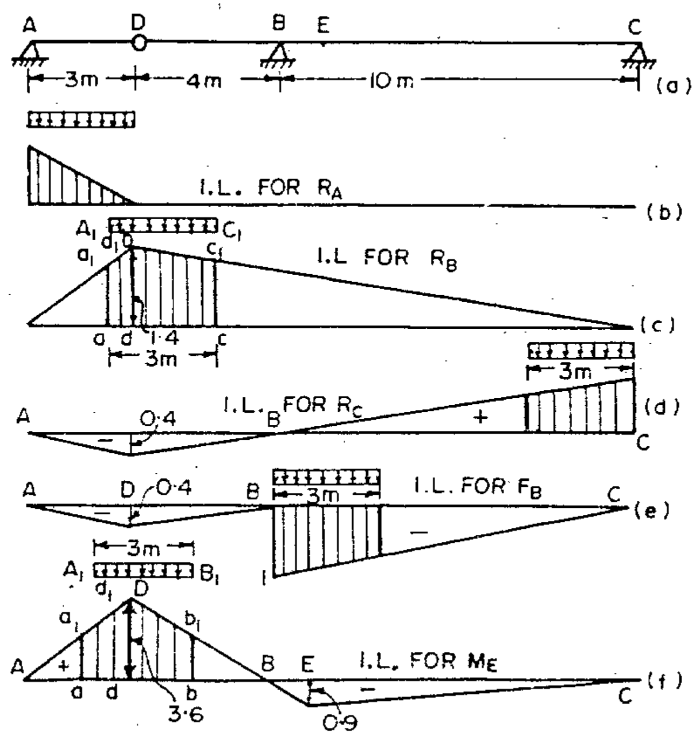


Fig. 2·10.

For maximum R_A, the U.D.L. of length 3 m occupy the whole portion AD.

$$\therefore \quad R_A=\frac{1}{2}(1\times3)\times2=3 \text{ kN}\uparrow$$

(*b*) *I.L. for reaction at B*(R_B)

When the unit load is in AD, distant x from A, $R=\left(1-\frac{x}{3}\right)\uparrow$

and hence pressure on DBC at $D=\frac{x}{3}\downarrow$

Taking moments about C,

$R_B\times10=$ pressure at $D\times14$

$$\therefore \quad R_B=\frac{x}{3}\times\frac{14}{10}=\frac{1\cdot4x}{3}\uparrow \quad (2)$$

When the unit load is at A, $x=0$, $\therefore R_B=0$
When the unit load is at D, $x=3$, $\therefore R_B=+1{\cdot}4$.

Now, let the load be in DBC, distant x from A. R_A will be zero for this range of load position. Hence, taking moments about C, we have

$$1\times(17-x)=R_B\times 10$$

From which $$R_B=1{\cdot}7-\frac{x}{10}\uparrow \tag{3}$$

When the load is at D, $x=3$ m, $\therefore R_B=+1{\cdot}4$
When the load is at B, $x=7$ m, $\therefore R_B=+1$
When the load is at C, $x=17$ $\therefore R_B=0$.

Thus the I.L. for R_B is triangle as shown in Fig. 2·10(c). For maximum R_B, let the tail of the U.D.L. be at x from D, so that shaded area is maximum. The criterion, given by Eq. 2·6 is

$$\frac{AD}{DC}=\frac{A_1D}{DC_1}$$

or $$\frac{3}{14}=\frac{x}{3-x},\text{ from which } x=\frac{9}{17}\text{ m}$$

$$\therefore \quad \text{ordinate } aa_1=cc_1=\frac{1{\cdot}4}{3}\left(3-\frac{9}{17}\right)=1{\cdot}15$$

$$R_B=2\times\frac{1}{2}\,[1{\cdot}15+1{\cdot}4]\times 3=\mathbf{7{\cdot}65\ kN}\uparrow$$

(c) *I.L. for Reaction at C* (R_C)

When the unit load is AD, distant x from A, $R_A=\left(1-\frac{x}{3}\right)\uparrow$

and hence pressure on DBC at $D=\frac{x}{3}\downarrow$

Taking moment about B,

$$\Sigma M_B=0=\frac{x}{3}\times 4+R_C\times 10$$

$$\therefore \quad R_C=-\frac{4x}{30}=-\frac{2x}{15}\left(\text{i.e. } R_C=\frac{2x}{15}\downarrow\right) \tag{4}$$

When the load is at A, $x=0$ $\therefore R_C=0$
When the load is at D, $x=3$ m, $\therefore R_C=-0{\cdot}4$

Now, let the load be in DBC, distant x from A. R_A will be zero for the range of load position. Hence taking moments about B, we have

$$\Sigma M_B=0=1\times(7-x)+R_C\times 10$$

or $$R_C=-\left(0{\cdot}7-\frac{x}{10}\right) \tag{5}$$

When the load is at D, $x=3$ m, $\therefore R_C=-0{\cdot}4$
When the load is at B, $x=7$ m $\therefore R_C=0$
When the load is at C, $x=17$ m, $\therefore R_C=+1$
The I.L. for R_C is shown in Fig. 2·10 (d).

By inspection, maximum R_C will occur when the head of the U.D.L. is at C. In this position, ordinate, under the tail of the U.D.L. $=+\frac{1}{10}\times 7=+0{\cdot}7$

$$\therefore \quad R_C=\frac{2}{2}\,(1+0{\cdot}7)3=+\mathbf{5{\cdot}1\ kN}\ (i.e.\ 5{\cdot}1\text{ kN}\uparrow)$$

(d) *I.L. for S.F. at a section just to the right of B*

When the load is between A to B.

$F_B=+R_C$, and hence the variation of F_B will be the same as that of R_C. Hence I.L. for F_B will have zero ordinate under A and B, and ordinate of $-0{\cdot}4$ under D.

When the load is in BC, at distance x from A, $R_A=0$ and $R_B=1{\cdot}7-\frac{x}{10}$ (from Eq. 3 above).

$$\text{Hence } F_B=-R_B=-\left(1{\cdot}7-\frac{x}{10}\right) \tag{6}$$

When the load is just to the right of B, $x=7$ m.

$$\therefore \quad F_B=-\left(1{\cdot}7-\frac{7}{10}\right)=-1.$$

When the load is at C, $x=17$ m

$$F_B=-\left(1{\cdot}7-\frac{17}{10}\right)=0.$$

The complete I.L. diagram is shown in Fig. 2·10 (e). It must be noted that the S.F. is always *negative* at this section. By inspection, maximum S.F. will occur when the tail of the load is at B. In this position, the ordinate under the head of the U.D.L. $=\frac{1}{10}\times 7=0{\cdot}7$.

$$\therefore \quad F_B(\text{max.})=-\frac{2}{2}\,(1+0{\cdot}7)\times 3=-\mathbf{5{\cdot}1\ kN.}$$

(e) *I.L. for B.M. at E, 1 m to the right of B*

When the unit load is in AD, distant x from A.

$$R_C=-\frac{2x}{15}\ \text{(from equation 4 above)}$$

$$\because \quad M_E=-R_C\times 9=-\left(-\frac{2x}{15}\times 9\right)=+1{\cdot}2x \tag{7}$$

when the load is at A, $x=0$, $\therefore$ $M_E=0$
when the load is at D, $x=3$ m $\therefore$ $M_E=+3{\cdot}6$ kN-m

When the load is in DE, distant x from A,

$$R_C=-\left(0{\cdot}7-\frac{x}{10}\right) \qquad \text{(from equation 5 above)}$$

$$\therefore \quad M_E=-R_C\times 9=+\left(0{\cdot}7-\frac{x}{10}\right)9=+6{\cdot}3-0{\cdot}9x \qquad (8)$$

when the load is at D, $x=3$ m, $\therefore$ $M_B=+3{\cdot}6$ kN-m
when the load is at B, $x=7$ m, $\therefore$ $M_E=0$
when the load is at E, $x=8$ m, $\therefore$ $M_E=-0{\cdot}9$ kN-m

When the load is in EC, distant x from A, $R_A=0$ and

$$R_B=1{\cdot}7-\frac{x}{10} \text{ (from equation 3 above)}$$

$$\therefore \quad M_E=-R_B\times 1=-\left(1{\cdot}7-\frac{x}{10}\right)$$

when the load is at E, $x=8$ m, $\therefore$ $M_E=-0{\cdot}9$ kN-m.
when the load is at C, $x=17$ m, $\therefore$ $M_E=0$

The complete I.L. for M_E is shown in Fig. 2·10 (f).

For maximum M_E, let the tail of the U.D.L. be at x from D. The corresponding area of I.L. diagram is shown shaded. Using criterion of equation 2·6, we have

$$\frac{AD}{DB}=\frac{A_1D}{DB_1}$$

or
$$\frac{3}{4}=\frac{x}{3-x}, \text{ from which } x=\frac{9}{7} \text{ m}$$

$$\therefore \quad \text{Ordinate } aa_1=bb_1=\frac{3{\cdot}6}{3}\left(3-\frac{9}{7}\right)=2{\cdot}06$$

$$\therefore \quad M_E=\frac{2}{2}(2{\cdot}06+3{\cdot}6)(3)=+16{\cdot}98 \textbf{ kN-m}$$

Example 2·7. *Draw dimensioned influence lines for the reactions at A and C and for the bending moment at E, the mid-point of the lower beam CF of the simply supported beam system shown in Fig. 2·11. By the use of these influence lines, calculate the greatest value of R_A, R_C and M_E due to the passage of two 10 kN rolling loads, 2 m apart which travel across the upper beam AB.*

Solution

(*a*) *I.L. for reaction at A*

Let the unit load be in AD, distant x from A.

Then $$R_A=\frac{6-x}{6}=1-\frac{x}{6} \qquad (1)$$

When the load is at A, $x=0$ $\therefore$ $R_A=+1$
When the load is at D, $x=6$ m, $\therefore$ $R_A=0$.

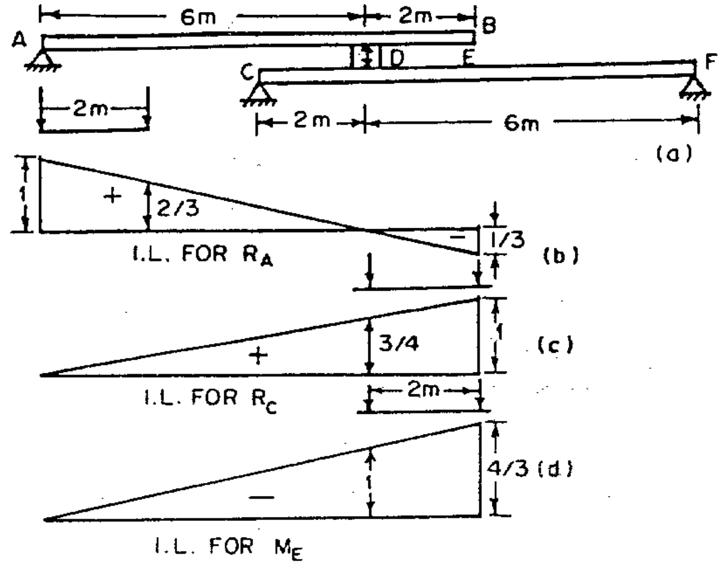


Fig. 2·11.

Let the load be in DB, distant x from A,

$$\text{Then} \quad R_A=-\frac{x-6}{6}=-\left(\frac{x}{6}-1\right) \qquad (2)$$

When the load is at D, $x=6$, $\therefore$ $R_A=0$

When the load is at B, $x=8$ m $\therefore R_A=-\left(\frac{8}{6}-1\right)=-\frac{1}{3}$

The I.L. for R_A is shown in Fig. 2·11(b).

By inspection, maximum R_A will be obtained when one 10 kN load is at A and other ahead of it at 2 m.

$$\therefore \quad \text{Ordinate under next 10 load}=\frac{1}{6}\times 4=\frac{2}{3}$$

$$\therefore \quad R_A=(10\times 1)+\left(10\times\frac{2}{3}\right)=+\frac{50}{3}=+16{\cdot}67 \textbf{ kN}$$

(*b*) *I.L. for reaction at C*

When the load is at a distance x from A

$$R_D=\frac{x}{6} \qquad (3)$$

Thus, for the lower beam CF, the downward load at $D=\frac{x}{6}$.

$$\therefore \quad R_C=\frac{6R_D}{8}=\frac{6}{8}\times\frac{x}{6}=\frac{x}{8} \qquad (4)$$

When the load is at A, $x=0$, $\quad\therefore\ R_C=0$

When the load is at B, $x=8$ m, $\quad\therefore\ R_C=+1$.

The I.L. for R_C is shown in Fig. 2·11(*c*).

By inspection, maximum R_C will be obtained when one 10 kN load is at B, and other behind it by 2 m (*i.e.* at D).

The ordinate under $D=\frac{1}{8}\times 6=+\frac{3}{4}$

$$\therefore \quad R_C=(10\times 1)+\left(10\times\frac{3}{4}\right)=+17\cdot 5 \text{ kN.}$$

(*c*) *I.L. for B.M. at E*

When the load is at a distance x from A,

$$R_D=\frac{x}{6} \qquad (3)$$

Thus, for the lower beam CF, the downward load at $D=\frac{x}{6}$.

$$\therefore \quad R_F=\frac{2R_D}{8}=\frac{2}{8}\times\frac{x}{6}=+\frac{x}{24}.$$

$$\therefore \quad M_E=R_F\times 4=-\frac{4x}{24}=-\frac{x}{6} \qquad (4)$$

when the load is at A, $x=0$, $\quad\therefore\ M_E=0$

when the load is at B, $x=8$ m, $\quad\therefore\ M_E=-\frac{8}{6}=-\frac{4}{3}$

The I.L. for M_E is shown in Fig. 2·11(*d*).

By inspection, maximum B.M. at E will occur when one point load is at B, and other 2 m behind it (*i.e.* at D). In this position, ordinate under $D=\frac{4}{3}\times\frac{1}{8}\times 6=1$

$$\therefore \quad M_E=-\left\{\left(10\times\frac{4}{3}\right)+(10+1)\right\}=-23\cdot 33 \text{ kN-m.}$$

PROBLEMS

1. Draw the influence lines for S.F. and B.M. at a section 5 m from one end of a simply supported beam, 25 m long. Hence calculate the maximum B.M. and S.F. at this section due to a uniformly rolling load of 8 m long and of intensity 1 kN/m.

2. A beam has a span of 20 m. Draw the I.L. for B.M. and S.F. for a section 8 m from the left hand support and determine the maximum B.M. and S.F. for this section due to two point loads of 8 and 4 kN at a fixed distance of 2 m apart rolling from left to right with either of the loads leading.

3. What is an influence line ?

Determine the position of maximum bending moment ... 12 and 4 kN, at a fixed distance of 4·5 m apart, cross a girde... Calculate the magnitude of the maximum bending moment, and ... position of the point at which the bending moment is the same in the two cases :

(*a*) When the 12 kN load is over the point. and

(*b*) When the 4 kN load is over the point ; the ends of the girder may be considered to be simply supported.

4. Draw the I.L. for B.M. at a point 10 m distant from the left hand abutment of a bridge girder of 25 m span, and find the maximum B.M. at that point due to a series of wheel loads 10, 20, 20, 20 and 20 tonnes at centre 4, 2·5, 2·5 and 2·5 metres. The loads can cross in either direction, the 10 kN wheel load leading in each case.

5. The following arrangement of axle loads is carried by a single bridge girder across a clear span of 30 m.

Axle load	5	5	10	10	10	kN
Spacings	2·5	2·5	2·5	2·5		metres

Draw dimensioned free hand sketches of the influence lines for shearing force and bending moment at a point 10 m from the left hand abutment, and determine the maximum bending moment and maximum shearing force at this point. The 5 kN load leads, and the system may pass over the bridge from either side.

6. A girder AB of length 30 m is simply-supported at C and D which are 5 and 20 m respectively from A. Draw the influence lines for B.M. and S.F. for the mid point of the girder and obtain the maxiumm B.M. and S.F. at this point when the girder is crossed by a uniformly distributed load w kN per metre which can occupy the whole or any part or parts of the span.

7. Draw the influence lines for the reactions at A, B and C, and for the bending moment at B in the structure shown in Fig. 2·12. Calculate the maximum value of each reaction and of the bending moment at B when a long uniformly distributed load of intensity w per/m crosses from A to C. The beam is hinged at mid-point of BC.

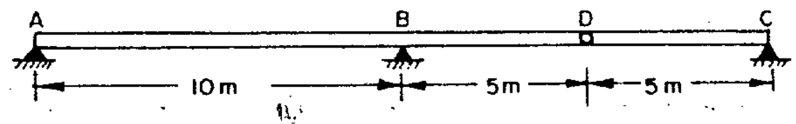


Fig. 2·12.

8. One span of a road bridge $ABCD$ consists of two cantilevers projecting from abutments A and D, and carrying a suspended span BC between them. $AB=CD=3L$; $BC=4L$. Draw the influence lines for :

(*a*) B.M. at A and at centre of BC.

(*b*) S.F. at B and at D.

9. Draw the general type of influence line for shear force at a point on a simply supported beam of span L, and deduce therefrom, giving a figured sketch, the diagram of maximum shear force of both signs due to the passage

over the span of a uniformly distributed load of intensity w and length $\frac{L}{5}$. If the beam also carries a dead load, uniformly distributed over the span of intensity $\frac{w}{2}$, indicate on the diagram the length of beam for which there is reversal of shearing force. (*U.L.*)

Answers

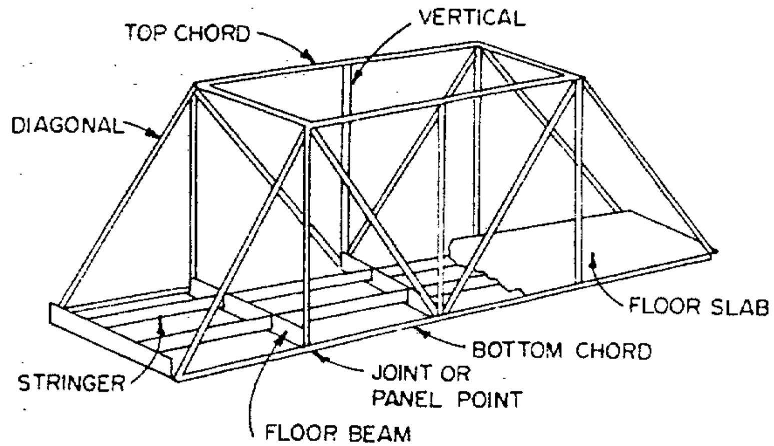
1. S.F. 5·12 kN ; B.M. 26·9 kN.
2. (*a*) 6·8 kN (*b*) 54·4 kN m.
3. (*a*) 91·25 kN-m at 11·94 m from left.
   (*b*) 74·75 kN-m at 14·19 m from the left.
   (c) Both moments equal at 18·75 m from left ; value=61·5 kN-m.
4. 424 kN-m.
5. 216·7 kN-m ; −21·25 kN.
6. $F_{max.}=+\frac{20}{3}w$; $-\frac{5w}{3}$ kN.
   $M_{max.}=37\cdot5\ w$; $-25\ w$ kN-m
7. $P_{AD}=46\cdot1$ kN (tension).
8. $R_A=5w\uparrow$; $R_B=15\ w\uparrow$; $R_C=2\cdot5\ w\uparrow$; $M_B=25\ w$ (hogging).
9. Focal length $=0\cdot2286\ L$.

3

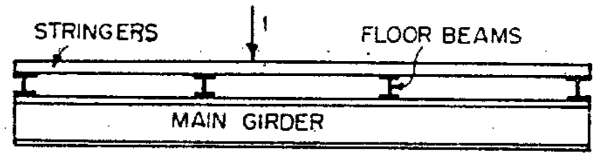
Influence Lines for Girders with Floor Beams

3·1. INTRODUCTION

We have seen in the previous chapter that an influence line for any *given point* or section of a structure is a curve whose ordinates



(a) TRUSS WITH FLOOR BEAMS



(b) GIRDER WITH FLOOR BEAM

Fig. 3·1.

represent to scale the variation of a function, such as shear force, bending moment, deflection etc. at a point or section, as a unit load moves across the structure. In the case of a beam, the unit load actually crosses each and every section and hence the influence line ordinate changes from point to point, as the unit load moves. However, in nearly all framed structures and girders with *floor beams*, the loads (whether knife edge load or uniformly distributed load) are applied at the *nodes* or *joints* only, so that we have the case of the loads applied at definite points [Fig. 3·1(*a*), (*b*)]. Due to this, the S.F. and B.M. for a panel, between the nodal points, are constant. *Hence influence lines for S.F. (or B.M.) are plotted for a panel and not for a particular* section of a girder.

3·2. Influence Line of S.F. for Girder with floor beams

Fig. 3·2(*a*), (*b*) shows such a system in which the loads are transmitted to the girder at definite points. For a given position of

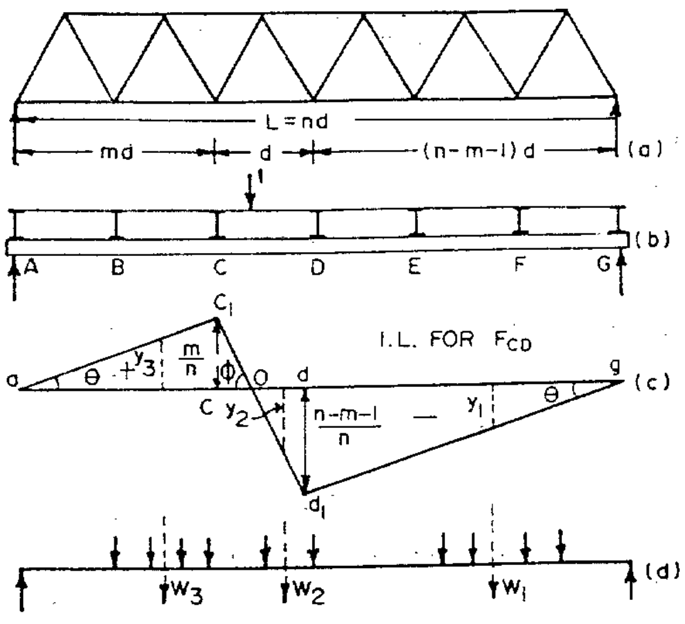


Fig. 3·2.

unit load, the S.F. for the whole of the *panel*, situated between the nodal points, is constant. Hence I.L. is plotted for S.F. of a panel and *not* for S.F. at a particular section of the girder.

Let such a girder or frame consist of n panels, each of length d such that the total length $L=nd$.

Let us plot the I.L. for S.F. in a panel CD [Fig. 3·2(*b*)]. Let there be m panels to the left of CD, and $(n-m-1)$ panels to its right. Thus CD is $(m+1)$th panel from the left support.

Let the unit load roll from A to G.

(*a*) *Load in AC*

When the load is at A, $R_G=0$, $\therefore\ F_{CD}=0$

When the load is at C, $R_G=\dfrac{md}{nd}=\dfrac{m}{n}$

$$F_{CD}=+R_G=+\frac{m}{n}$$

(*a*) *Load in DG*

Now, let the load be in portion DG.

when the load is at D, $R_A=\dfrac{(n-m-1)d}{nd}=\dfrac{n-m-1}{n}$

$$F_{CD}=-R_A=-\frac{n-m-1}{n}$$

when the load is at G, $R_A=0$

$$\therefore\qquad F_{CD}=0.$$

(*c*) *Load in CD*

When the load is at a distance x from C, load transmitted at the panel point $C=\dfrac{d-x}{d}$, and load transmitted at the panel point $D=\dfrac{x}{d}$.

$$\therefore\qquad F_{CD}=+R_G-\frac{x}{d}=\frac{md+x}{d}-\frac{x}{d}$$

The variation is linear.

When the load is at C, $x=0$. $\therefore\ F_{CD}=+\dfrac{m}{n}$, as before

When the load is at D, $x=d$.

$$\therefore\qquad F_{CD}=\frac{md+d}{nd}-\frac{d}{d}=-\frac{n-m-1}{n}$$

Thus the ordinate $CC_1=+\dfrac{m}{n}$

$$\text{ordinate } dd_1=-\frac{n-m-1}{n}$$

The ordinate is zero at some point O between C and D. The I.L. for F_{CD} is shown in Fig. 3·2(*c*).

LOAD POSITIONS FOR MAXIMUM S.F.

Let us now determine the load positions for maximum S.F. in panel CD.

(1) Single Point Load

Maximum +ve shear will occur when the point load is at C and maximum −ve shear will occur when the load is at D.

$$\therefore \quad F_{CD}\ (+\text{ve max.}) = \boxed{\frac{Wm}{n}} \qquad [3\cdot1(a)]$$

and $$F_{CD}\ (-\text{ve max.}) = \boxed{\frac{W}{n}(n-m-1)} \qquad [3\cdot1(b)]$$

(2) U.D.L. greater than the Span

Maximum +ve S.F. will occur when ao is fully loaded and og is empty, and maximum −ve S.F. will occur when og is fully loaded and ao is empty, o being the point of zero ordinate of the I.L. for F_{CD}. The position of the point o can very easily be located by the consideration of the triangles cc_1o and dd_1o. Thus,

$$\frac{cc_1}{dd_1} = \frac{co}{do} = \frac{co}{cd-co}$$

Since cc_1, dd_1, and cd are known, co can be calculated, and hence o can be located.

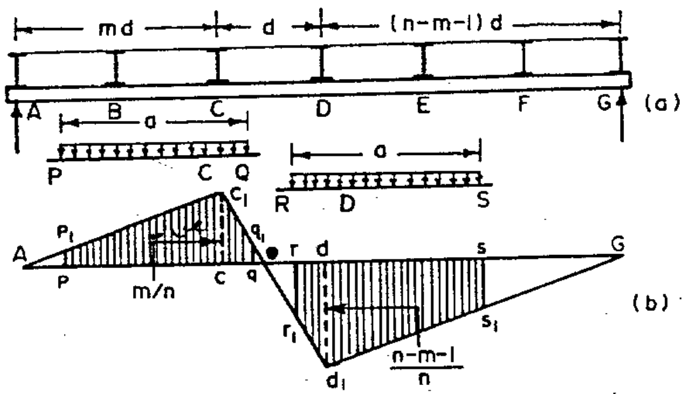


Fig. 3·3.

(3) U.D.L. shorter than the span

Let a U.D.L. of length a travel from left to right such that $a < Ao$, and also $a < oG$. For getting maximum positive shear, the shaded area of the +ve portion of the I.L. should be maximum. For this, the ordinate pp_1 should be equal to ordinate qq_1. Hence applying the criterion of equation 2·6,

$$\frac{AC}{CO} = \frac{PC}{CQ} = \frac{PC}{a-PC} \qquad [3\cdot2(a)]$$

Since AC, CO and a are known, PC can be computed, and then the shaded area can be known.

Similarly, for maximum negative shear, the ordinate rr_1 should be equal to ss_1. Hence, from criterion of equation 2·6,

$$\frac{OD}{DG} = \frac{RD}{DS} = \frac{RD}{a-RD} \qquad [3\cdot2(b)]$$

Knowing OD, DG and a, RD can be computed, and the shaded area can be known.

(3) Irregular Load System

Let a train of wheel loads travel from left to right. Let the arrangement of the load [Fig. 3·2(d)] be such that W_1 is the resultant of the load to the right of CD, W_3 is the resultant of the load to the left of CD, and W_2 is the resultant of the load on the panel CD itself. This arrangement will give maximum F_{CD} only if a small movement of the load system decreases the S.F.

Inclination θ of cc_1 or gd_1 is given by

$$\tan\theta = \frac{m/n}{md} = \frac{1}{nd} = \frac{1}{L}$$

Inclination ϕ of c_1d_1 is given by

$$\tan\phi = \frac{cc_1+dd_1}{cd} = \frac{(ac+dg)\tan\theta}{cd} = \frac{L-d}{d}\cdot\frac{1}{L}$$
$$= \frac{L-d}{d.L}$$

By giving a small movement δx to the right, the ordinates y_3 and y_2 are increased and ordinate y_1 is decreased. However, the decrease of y_1 increases the +ve S.F. Hence, the change in S.F. is given by

$$\delta F_{CD} = W_3\,\delta x \tan\theta - W_2\,\delta x \tan\phi + W_1 \delta x \tan\theta.$$

$$\therefore \quad \frac{\delta F_{CD}}{dx} = (W_3+W_1)\tan\theta - W_2\tan\phi$$

$$= \frac{W_3+W_1}{L} - \frac{W_2(L-d)}{d.L} = \frac{W_1+W_2+W_3}{L} - \frac{W_2}{d}.$$

$$= \boxed{\frac{W}{L} - \frac{W_2}{d}} \qquad (3\cdot3)$$

(where W is the total load).

Hence, the maximum, $\frac{W}{L} - \frac{W_2}{d}\left(\text{ or } \frac{W}{n} - W_2\right)$ should change sign. In the limiting case, when the loads are very near, $W_2 = \frac{W}{n}$

Hence the maximum S.F. in a panel occurs when the load in that panel is equal to the load divided by the number of panels.

3·4. INFLUENCE LINE OF B.M. FOR GIRDER WITH FLOOR BEAMS

Let us now draw the I.L. for bending moment at a point P, distant x from A, in the panel CD.

When the unit load is in AC, $\qquad M_P = -R_G(L-x)$ (1)

Again when the unit load is in DG, $\qquad M_P = -R_A . x$ (2)

Both these variations are same as for a girder without floor beam. It must be remembered that x is a fixed quantity in the above equation.

When the load is at C, $R_G = \dfrac{md}{nd} = \dfrac{m}{n}$

$$\therefore \quad M_P = \text{Ordinate } cc_1 = \frac{m}{n}(L-x) \qquad (3)$$

When the load is at D, $R_A = \dfrac{(n-m-1)d}{nd} = \dfrac{n-m-1}{n}$ (4)

$$\therefore \quad M_P = \text{Ordinate } dd_1 = \frac{n-m-1}{n}x.$$

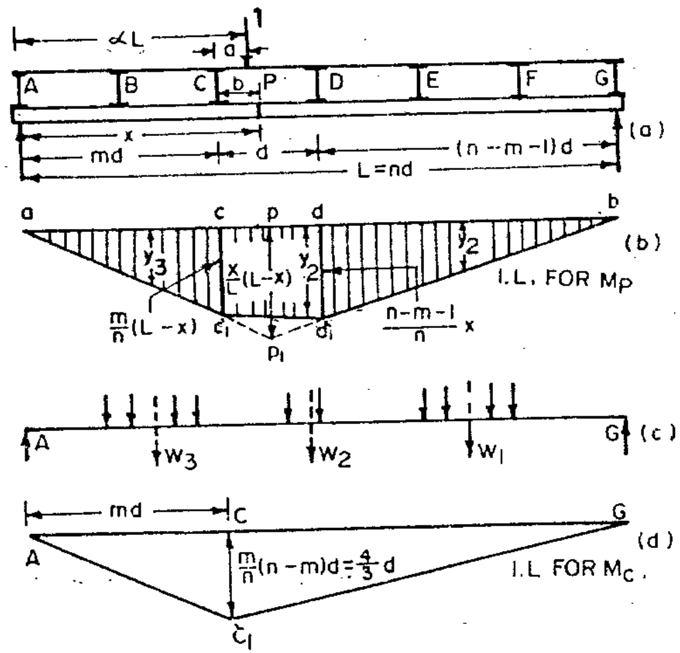


Fig. 3·4

If the girder were without the floor beam, the ordinate pp_1 under the section would have been $\dfrac{x}{L}(L-x)$, and the corresponding

ordinates cc_1 and dd_1 would have been

$$cc_1 = \frac{x}{L}(L-x) \times \frac{1}{x} \times md = \frac{m}{n}(L-x)$$

(which is the same as found above in Eq. 3)

and $\qquad dd_1 = \dfrac{x}{L}(L-x)\dfrac{1}{(L-x)} \times (n-m-1)d = \dfrac{n-m-1}{n}x$

(which is the same as found above in Eq. 4).

Hence the portion ac_1 and bd_1 of I.L. for a girder with floor beams can be obtained by constructing the I.L. for the beam assuming it to be without floor beam, making the central ordinate $pp_1 = \dfrac{x}{L}(L-x)$ joining p_1 to a and b, as shown in Fig. 3·4 (b).

To plot the portion of I.L. diagram under the bay CD, consider the unit load at a distance a from C. The panel point load transferred to C and D will be $\dfrac{d-a}{d}$ and $\dfrac{a}{d}$ respectively.

$$\text{Here } M_P = cc_1\left(\frac{d-a}{d}\right) + dd_1\left(\frac{a}{d}\right).$$

This is a linear function of a. When the load is at C, $a=0$, and hence $M_P = cc_1\left(\dfrac{d}{d}\right) = cc_1$. Similarly, when the load is at D, $a=d$, and $M_P = dd_1\left(\dfrac{d}{d}\right) = dd_1$. Hence the I.L. portion under panel CD is obtained by joining c_1 and d_1 by a straight line. The figure ac_1d_1b is thus the complete I.L. diagram for B.M. at point P is the panel CD.

The I.L. for the point P in any other panel can also be found in a similar manner. However, when the point P coincides with some panel or node point, such as C, the I.L. diagram will be a triangle. Fig. 3·4 (d) shows the I.L. for B.M. at C. The ordinate cc_1 under C [Fig. 3·4 (d)] $= \dfrac{md(n-m)d}{nd} = \dfrac{m}{n}(n-m)d$. When $n=6$ and $m=2$,

$$cc_1 = \frac{2}{6}(6-2)d = \frac{4}{3}d.$$

3·5. LOAD POSITIONS FOR MAXIMUM B.M.

1. Single Point Load

From the inspection of the I.L. for M_P, it is clear that $M_{P\,max.}$ will be obtained by putting the load either at C or at D, depending upon whether ordinate cc_1 is bigger or dd_1 is bigger.

$$\text{Thus} \quad M_{P\,max.} = -\frac{Wm}{n}(L-x) \qquad 3·4\ (a)$$

or $$=-\frac{W(n-m-1)}{n}x. \qquad 3\cdot4\ (b)$$

2. U.D.L. longer than the Span

Maximum B.M. will evidently occur when the load occupies the whole span. In that case, $M_P=w\times$ shaded area of I.L. diagram.

3. Irregular Load System

Let W_1 be the resultant of the loads to the right of D, W_3 the resultant of the loads to the left of C, and W_2 the resultant of the loads on the panel CD. Let the section P be at a distance b from C (such that $b=x-md$). Then, it can be proved that the maximum B.M. at P occurs when the expression

$$\frac{W(L-x)}{L}-\left\{W_2\left(\frac{d-b}{d}\right)+W_1\right\} \textit{ changes sign.}$$

As a rule, $b=\frac{d}{2}$, so that the above criterion reduces to

$$\frac{W(L-x)}{L}-\left(\frac{W_2}{2}+W_1\right) \textit{ changes sign} \qquad ...(3\cdot5)$$

Hence, to get the maximum value of M_P, the procedure is as follows: Place the load on the span such that the span is fully covered (if the load system is long) and with one load at D. First consider this load as part of W_2 and then as part of W_1. If this causes the expression of equation 3·5 to change sign, the position is the maximum required. If not, move the load on until another load comes at C or D, and apply the above criterion again.

Example 3·1. *A pratt girder shown in Fig. 3·5 consists of eight panels each 3·5 m square, the loading being on the lower boom. Draw the influence line for the force in the member EC and determine the maximum tension and maximum compression in EC due to*

(*a*) *a concentrated rolling load of 20 kN*,

(*b*) *a uniform live load of 10 kN/m and 10 m long. Indicate clearly for each of the four required values the corresponding load positions.*

Solution.

If we pass a section 1-1, it is clear that force in EC is equal to shear in $BC\times\sec 45°=F_{BC}\times\sqrt{2}$.

Hence $\qquad P_{EC}=F_{BC}\sqrt{2}. \qquad (1)$

Also, when the load is in ao, S.F. in panel BC will be positive, and hence force in EC will be compressive. Similarly, when the load is in od, S.F. in panel BC will be negative, and hence force in EC will be tensile.

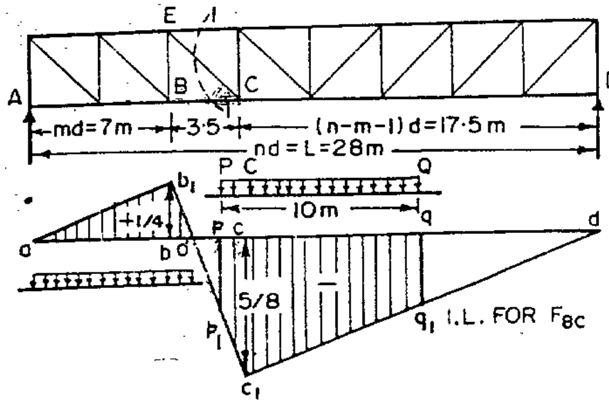


Fig. 3·5.

Let us first plot the I.L. for F_{BC}.

Hence $m=2$, $n=8$; $d=3\cdot5$

Ordinate $bb_1=\frac{m}{n}=\frac{2}{8}=\frac{1}{4}$

Ordinate $cc_1=\frac{n-m-1}{n}=\frac{8-2-1}{8}=\frac{5}{8}$

If o is the point of zero S.F., we have

$$\frac{bo}{bb_1}=\frac{co}{cc_1}=\frac{bo+co}{bb_1+cc_1}=\frac{3\cdot5}{\frac{1}{4}+\frac{5}{8}}=4 \text{ m}$$

$$\therefore \qquad bo=4\times bb_1=4\times\frac{1}{4}=1 \text{ m}$$

and $\qquad co=3\cdot5-1=2\cdot5$ m.

(*a*) *Concentrated load of 20 kN*

Maximum +ve S.F. in BC will occur when the point load is at B, while maximum −ve S.F. will occur when the point load is at C.

Hence F_{BC} (+ve max.) $=\frac{1}{4}\times20=5$ kN

$\therefore \qquad P_{EC}=F_{BC}\sqrt{2}=$ **$5\sqrt{2}$ kN** (compressive)

and $\qquad F_{BC}$ (−ve max.) $=\frac{5}{8}\times20=12\cdot5$ kN

$\therefore \qquad P_{EC}=F_{BC}\sqrt{2}=$ **$12\cdot5\sqrt{2}$ kN** (tensile)

(*b*) *U.D.L. of 10 kN/m, 10 m long*

$bo=1$ m, (found above)

$\therefore \qquad ao=7+1=8$ m and $od=28-8=20$ m.

Since the length of U.D.L. is more than ao, max. +ve S.F. will occur when the load occupies the whole of the portion ao.

Then $\quad F_{BC}$ (+ve max.)$=w\times$Area ob_1a

$$=10\times\frac{1}{2}\times8\times\frac{1}{4}=10 \text{ kN}$$

$\therefore \quad P_{EC}=F_{BC}\sqrt{2}=10\sqrt{2}$ **kN** (compressive)

For maximum negative shear, the load should be so arranged that the area of the I.L. diagram under it (shown dotted) is maximum. This will happen when ordinate $pp_1=qq_1$ (*i.e.* point c divides the load pq in the same ratio as it divides the base od of the triangle oc_1d). Hence applying the criterion of equation 2·6, we have

$$\frac{oc}{cd}=\frac{pc}{cq}=\frac{pc}{10-pc}$$

or $$\frac{2\cdot5}{20-2\cdot5}=\frac{pc}{10-pc}$$

From which $\quad pc=1\cdot25$ m

Also, $\quad op=oc-pc=2\cdot5-1\cdot25=1\cdot25$ m

Hence $$qq_1=pp_1=\frac{5}{8}\times\frac{1}{2\cdot5}\times1\cdot25=\frac{5}{16}$$

Hence $$F_{BC}\ (-\text{ve max})=\frac{1}{2}\left(\frac{5}{8}+\frac{5}{16}\right)\times10\times10=46\cdot88 \text{ kN}$$

$\therefore \quad P_{EC}=F_{BC}\,.\,\sqrt{2}=46\cdot88\sqrt{2}=66\cdot2$ kN (tensile).

PROBLEMS

1. A N-girder bridge (Fig. 3·6) has cross-girders at the lower panel points. The diagonals are at 45°. A live load of 6 kN/m (per girder), longer than the span, cross the bridge. Find the maximum forces in the three members *AB*, *AD* and *CD*.

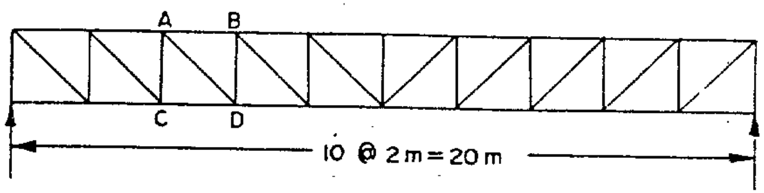


Fig. 3·6.

Answers

1. $P_{AD}=126$ kN (com.) ; $P_{CD}=96$ kN (tensile)
$P_{AD}=46\cdot1$ kN (tension).

4

Influence Lines for Stresses in Frames

4·1. PRATT TRUSS WITH PARALLEL CHORDS

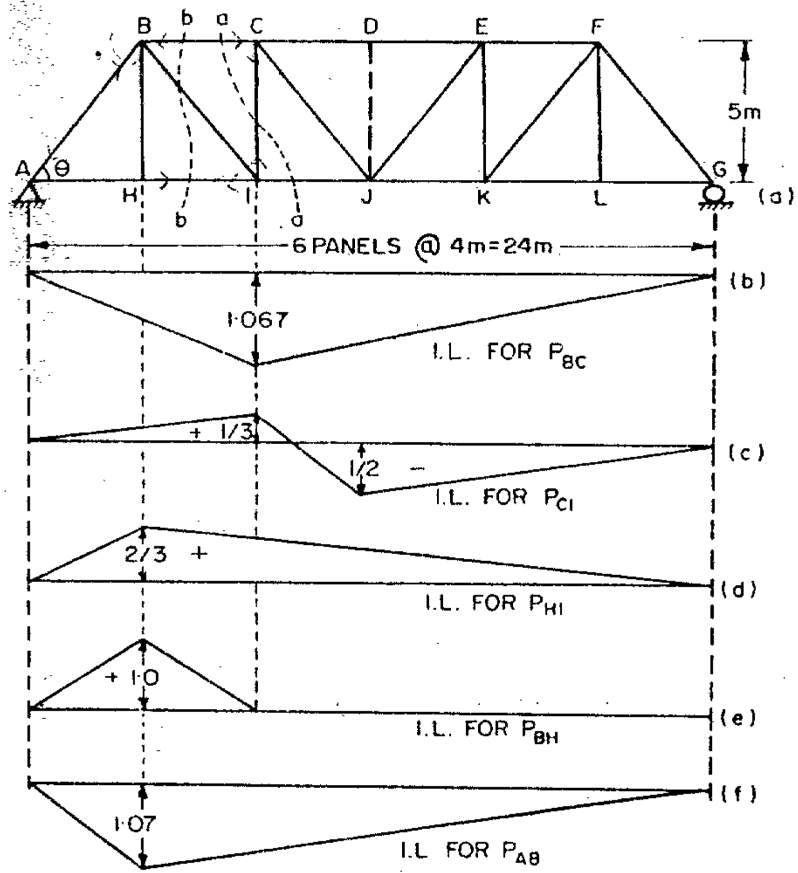


Fig. 4·1
Pratt truss with parallel chords.

Fig. 4·1 shows a pratt truss with 6 panels, each of length 4 m and of height 5 m. Let us draw the influence lines for stresses in members of panels *HI* and *IJ*. The truss is statically determinate.

$$\sin\theta=\frac{5}{\sqrt{4^2+5^2}}=\frac{5}{\sqrt{41}}=0{\cdot}78$$

$$\cos\theta=\frac{4}{\sqrt{41}}=0{\cdot}625\ ;\ \text{cosec}\ \theta=1{\cdot}28$$

(1) Influence line for P_{BC}

In order to find stress P_{BC} in member *BC*, pass a section *aa* as shown. Evidently ;

$$P_{BC}=\frac{M_I}{5}\ \text{(compression)}$$

where M_I = bending moment at joint *I*.

The influence line for bending moment at chord point *I* will be a triangle having a maximum ordinate equal to $\frac{8\times16}{24}=\frac{16}{3}$. Hence influence line for P_{BC} will also be a triangle having a maximum ordinate of $\frac{1}{5}\times\frac{16}{3}=\frac{16}{15}=1{\cdot}067$ under *I* as shown in Fig. 4·1 (*b*). The minus sign indicates compression.

(2) Influence line for P_{CI}

P_{CI} = Shear in panel *IJ*

Thus the I.L. for P_{CI} is the same as the I.L. for shear in the panel *IJ*. From § 3·2, the ordinates of the I.L. diagram are as under :

$$\text{Under point } I,\ \text{ordinate}=+\frac{8}{24}=+\frac{1}{3}$$

$$\text{Under point } J,\ \text{ordinate}=-\frac{12}{24}=-\frac{1}{2}$$

The I.L. diagram for P_{CI} is shown in Fig. 4·1 (*c*). When the load traverses the panel *IJ* there is reversal of stress in member *CI*. When the load is to the left of *I*, the S.F. is positive and the force in *CI* is tensile, while when the load is to the right of *J*, the S.F. is negative and the force in *CI* is compressive. *Thus a positive S.F. gives tension while a negative S.F. gives campression*

(3) Influence line for P_{HI}

Pass a section *bb*,

$$P_{HI}=\frac{M_B}{5}\ \text{(tension)}$$

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Hence the I.L. for P_{HI} will be a triangle, havin ordinate of $\frac{4\times20}{24}\times\frac{1}{5}=\frac{2}{3}$ under H, as shown in

(4) Influence line for P_{BH}

When the unit load is at *A*, $P_{BH}=0$

When the unit load is at *H*, $P_{BH}=1$ (tension)

When the unit load is at *I* or to the right of *I*, $P_{BH}=0$

The influence line for P_{BH} will therefore be a triangle having a maximum ordinate of unity under H, as shown in Fig. 4·1 (*e*).

(5) Influence line for P_{AB}

The force in *AB* can be found by resolution of forces at *A* in the vertical direction.

When the unit load is at *A*, $R_A=1$, and hence $P_{AB}=0$. When the unit is at *H*, $R_A=\frac{20}{24}=\frac{5}{6}$, and

$$P_{AB}=R_A\ \text{cosec}\ \theta=\frac{5}{6}\times1{\cdot}28=1{\cdot}07\ \text{(comp.)}$$

When the load is at *G*, $R_A=0$. $\therefore\ P_{AB}=0$

The I.L. for P_{AB} is shown in Fig. 4·1 (*f*).

4·2. PRATT TRUSS WITH INCLINED CHORDS

Fig. 4·2 (*a*) shows Pratt truss with inclined chords, consisting of 6 panels each of 4 m length.

(1) Influence line for P_{HI}

Pass a section *aa* cutting three members

$$P_{HI}=\frac{M_B}{BH}=\frac{1}{3}M_B\ \text{(tension)}$$

The influence line diagram will therefore be a triangle having a maximum ordinate $=\frac{1}{3}\left(\frac{4\times20}{24}\right)=\frac{10}{9}=1{\cdot}111$ under *H* as shown in Fig. 4·2 (*b*).

(2) Influence line for P_{BC}

$$P_{BC}=\frac{M_I}{x}\ \text{(compression)}$$

where *x* is the perpendicular distance between point *I* and *BC*. In order to find *x*, prolong *CB* back to meet *IA* produced, in *O*.

Now $\tan\alpha=\frac{5-3}{4}=\frac{1}{2}=\frac{5-3}{4}$ $\therefore\ \alpha=26°34'$

$\therefore$ $\sin\alpha=0{\cdot}447\ ;\ \cos\alpha=0{\cdot}894$

$$OH=\frac{3}{\tan\alpha}=\frac{3}{\frac{1}{2}}=6\ \text{m}$$

$$OI=\frac{5}{\tan\alpha}=\frac{5}{\frac{1}{2}}=10 \text{ m} \; ; \; OA=6-4=2 \text{ m}$$

Now $\quad x=OI\sin\alpha=10\times0{\cdot}447=4{\cdot}47$ m

$$\therefore \quad P_{BC}=\frac{M_I}{4{\cdot}47}$$

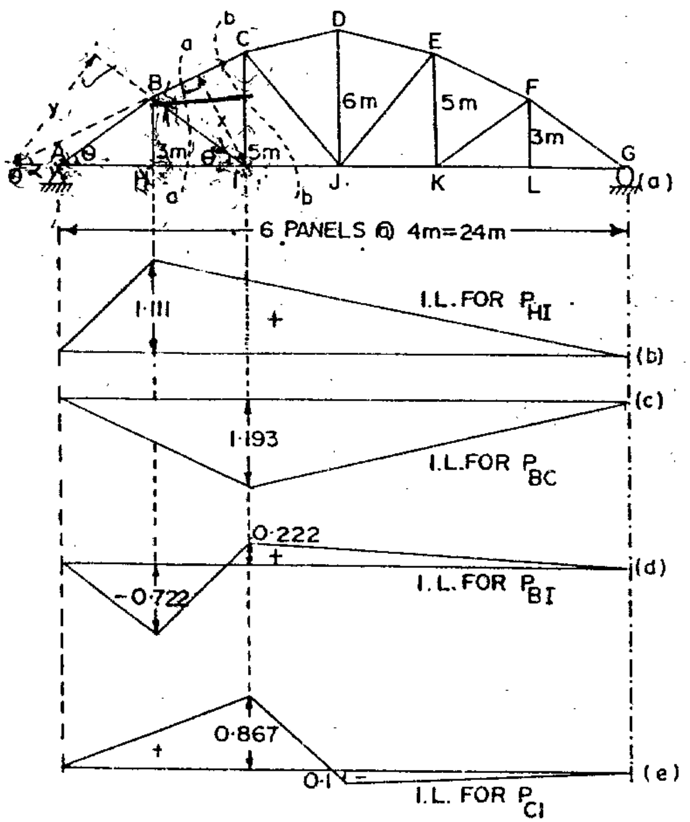


Fig. 4·2

Pratt truss with inclined chords.

The influence line diagram will be a triangle having a maximum ordinate$=\frac{1}{4{\cdot}47}\left(\frac{8\times16}{24}\right)=1{\cdot}193$, as shown in Fig. 4·2 (*c*).

(3) Influence line for P_{BI}

$$P_{BI}=\frac{M_0}{y}.$$

where y=perpendicular distance of point O from BI

$$=OI\sin\theta=10\times\frac{3}{5}=6 \text{ m}$$

$$\therefore \quad P_{BI}=\frac{M_0}{6}$$

where M_0 is the moment about O, of the forces to the left of section *aa*.

When the unit load is at A, $R_A=1$. Hence considering the forces to the left of the section *aa*, $M_0=0$. Hence $P_{BI}=0$. When the unit load is at H, $R_A=\frac{20}{24}=\frac{5}{6}$.

$$\therefore \quad P_{BI}=\frac{M_0}{6}=\frac{1}{6}\,[(1\times OH)-(R_A\times AO)] \text{ (comp.)}$$

$$=\frac{1}{6}\left\{(1\times6)-\left(\frac{5}{6}\times2\right)\right\}=0{\cdot}722 \text{ (comp.)}$$

When the unit load is at I, $R_A=\frac{16}{24}=\frac{2}{3}$

$$\therefore \quad P_{BI}=\frac{M_0}{6}=\frac{1}{6}\left\{R_A\times OA\right\}=\text{(tension)}$$

$$=\frac{1}{6}\left(\frac{2}{3}\times2\right)=\frac{2}{9}=0{\cdot}222 \text{ (tension)}$$

Thus, there is reversal of stress in BI as the load traverses the panel HI.

When the load is G, $R_A=0$; Hence M_0 and P_{BI} are zero.

The complete I.L. diagram for P_{BI} is shown in Fig. 4·2(*d*).

(4) Influence line for P_{CI}

Pass a section *bb* to cut the three members BC, CI and IJ. Since BC and IJ meet at O, when produced, we have

$$P_{CI}=\frac{M_0}{OI}=\frac{M_0}{10}$$

where M_0 is the moment, about O, of all forces to the left of section *bb*.

When the unit load is at A, $R_A=1$.

$\therefore \quad M_0=0$ and hence $P_{CI}=0$

When the unit load is between A and I, R_A is less than unity, and hence the net moment M_0 is clockwise. Hence P_{CI} will give an anti-clockwise moment, giving tensile force in it.

When the load is at I, $R_A=\frac{16}{24}=\frac{2}{3}$

$$\therefore \quad P_{CI}=\frac{M_0}{10}=\frac{1}{10}\left\{(1\times OI)-(R_A\times OA)\right\} \text{ (tension)}$$

$$=\frac{1}{10}\left\{(1\times10)-\left(\frac{2}{3}\times2\right)\right\}=0{\cdot}867 \text{ (tension)}$$

When the unit load is at J, $R_A=\frac{1}{2}$.

$$\therefore\ P_{CI}=\frac{M_0}{10}=\frac{1}{10}\left\{R_A\times OA\right\}\ \text{(compression)}$$

$$=\frac{1}{10}\times\tfrac{1}{2}\times2=0{\cdot}1\ \text{(compression)}$$

Thus, there is reversal of stress in CI as the unit load transverses the span IJ.

When the unit load is at G, $R_A=0$ and hence M_0 and P_{CI} are zero. The I.L. for P_{CI} is shown in Fig. 4·2(e).

4·3. WARREN TRUSS WITH INCLINED CHORDS

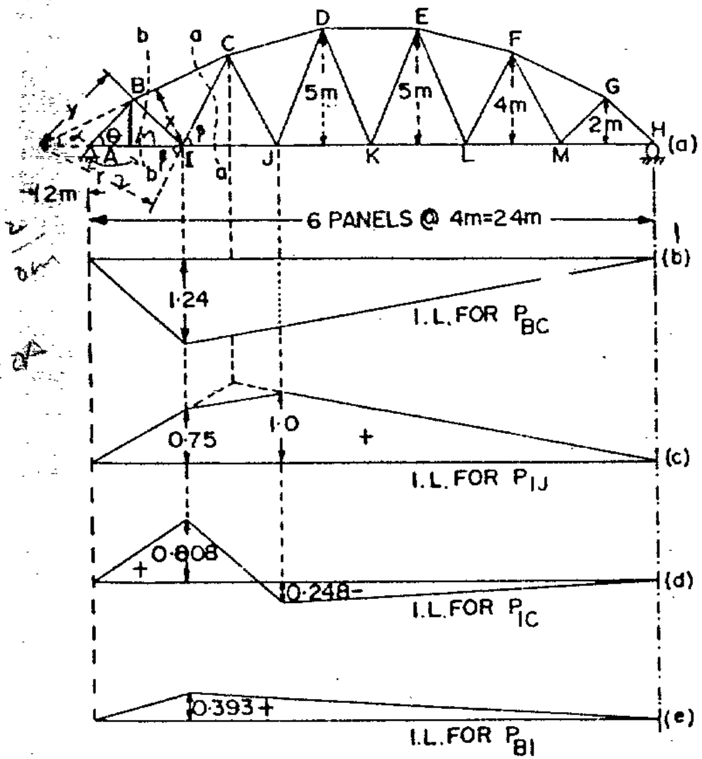


Fig. 4·3

Warren girder with inclined chords.

Fig. 4·3 shows a Warren girder with inclined chords. There are six panels each of 4 m span.

(1) **Influence line for P_{BC}**

Pass a section aa to cut members BC, CI and IJ

$$P_{BC}=\frac{M_I}{x}\ \text{(compression)}$$

where x=perpendicular distance of I from BC

$=OI\sin\alpha$

But $\tan\alpha=\dfrac{4-2}{2}=\dfrac{1}{2}$, $\quad\therefore\ \alpha=26°34'$; $\sin\alpha=0{\cdot}447$

$\therefore\quad OA=\dfrac{2}{\tan\alpha}-2=2$ m

$\therefore\quad OI=2+4=6$ m.

Hence $x=OI\sin\alpha=6\times0{\cdot}447=2{\cdot}68$ m

$\therefore\quad P_{BC}=\dfrac{M_I}{2{\cdot}68}$ (compression)

The influence line for P_{BC} will be a triangle having a maximum ordinate of $\dfrac{4\times20}{24}\times\dfrac{1}{2{\cdot}68}=1{\cdot}24$ under I, as shown in Fig 4·3 (a).

(2) **Influence line for P_{IJ}**

$$P_{IJ}=\frac{M_C}{4}\ \text{(tension)}$$

When the load is at A, $R_H=0$; hence M_C and P_{IJ} are zero.

When the load is at I, $R_H=\dfrac{4\times1}{24}=\dfrac{1}{6}$

$$\therefore\ P_{IJ}=\frac{M_C}{4}=\frac{1}{4}\left(\frac{1}{6}\times18\right)=0{\cdot}75.\ \text{(tension)}$$

When the load is at J, $R_A=\dfrac{1\times16}{24}=\dfrac{2}{3}$

$$\therefore\ P_{IJ}=\frac{M_C}{4}=\frac{1}{4}\left(\frac{2}{3}\times6\right)=1\ \text{(tension)}$$

When the load is at H, $R_B=1$ and $R_A=0$

Hence M_C and P_{IJ} are zero.

The influence line diagram for P_{IJ} has zero ordinates under A and H and ordinates of 0·75 and 1·0 under I and J, as shown in Fig. 4·3 (c).

(3) **Influence line for P_{IC}**

$$P_{IC}=\frac{M_0}{r}$$

where r=perpendicular distance of O from CI

$=OI\sin\beta$

But $OI=6$ m ; $\sin\beta=\dfrac{4}{\sqrt{2^2+4^2}}=0{\cdot}894$

$\therefore\quad r=6\times0{\cdot}894=5{\cdot}36$ m

$\therefore\quad P_{IC}=\dfrac{M_0}{5{\cdot}36}$

When the unit load is at A, $R_A=1$; Hence M_0 and P_{IC} are zero.

When the unit load is at I, $R_A=\frac{1\times 20}{24}=0{\cdot}833$

$$\therefore \quad P_{IC}=\frac{1}{5{\cdot}36}\left\{(1\times 6)-(0{\cdot}833\times 2)\right\}$$
$$=0{\cdot}808 \text{ (tension)}$$

When the unit load is J, $R_A=\frac{1\times 16}{24}=0{\cdot}667$

$$\therefore \quad P_{IC}=\frac{1}{5{\cdot}36}(0{\cdot}667\times 2)=0{\cdot}248 \text{ (compression)}$$

Thus, there is reversal of stress IC when the unit load crosses the panel IC. The I.L. for P_{IC} is shown in Fig. 4·3(d).

(4) **Influence line for** P_{BI}

Pass a section bb to cut members BC, BI and IA.

$$P_{BI}=\frac{M_0}{y}$$

where $\quad y$=perpendicular distance of O from BI
$$=OI\sin\theta$$

$$OI=6 \text{ m}; \sin\theta=\frac{2}{\sqrt{2^2+2^2}}=\frac{2}{\sqrt{8}}=0{\cdot}707$$

$$\because \quad y=6\times 0{\cdot}707=4{\cdot}24 \text{ m}$$

Hence $\quad P_{BI}=\frac{M_0}{4{\cdot}24}$.

When the unit load is at A, $R_A=1$. Hence M_0 and P_{BI} are zero.

When the unit load is I, $R_A=\frac{20}{24}=0{\cdot}833$

$$\therefore \quad P_{BI}=\frac{1}{4{\cdot}24}\left\{0{\cdot}833-2\right\}=0{\cdot}393 \text{ (tension)}$$

When the unit load is at H, $R_A=0$. Hence M_0 and P_{BI} are zero.

The I.L. for P_{BI} is shown in Fig 4·3(e).

4·4. K-TRUSS

(1) **Influence line for** P_{CD}

Pass a section aq as shown in Fig. 4·4 (a). Considering the equilibrium of the portion to the left of section aa and taking moments about O, we get

$$P_{CD}=\frac{M_0}{6} \text{ (compression)}$$

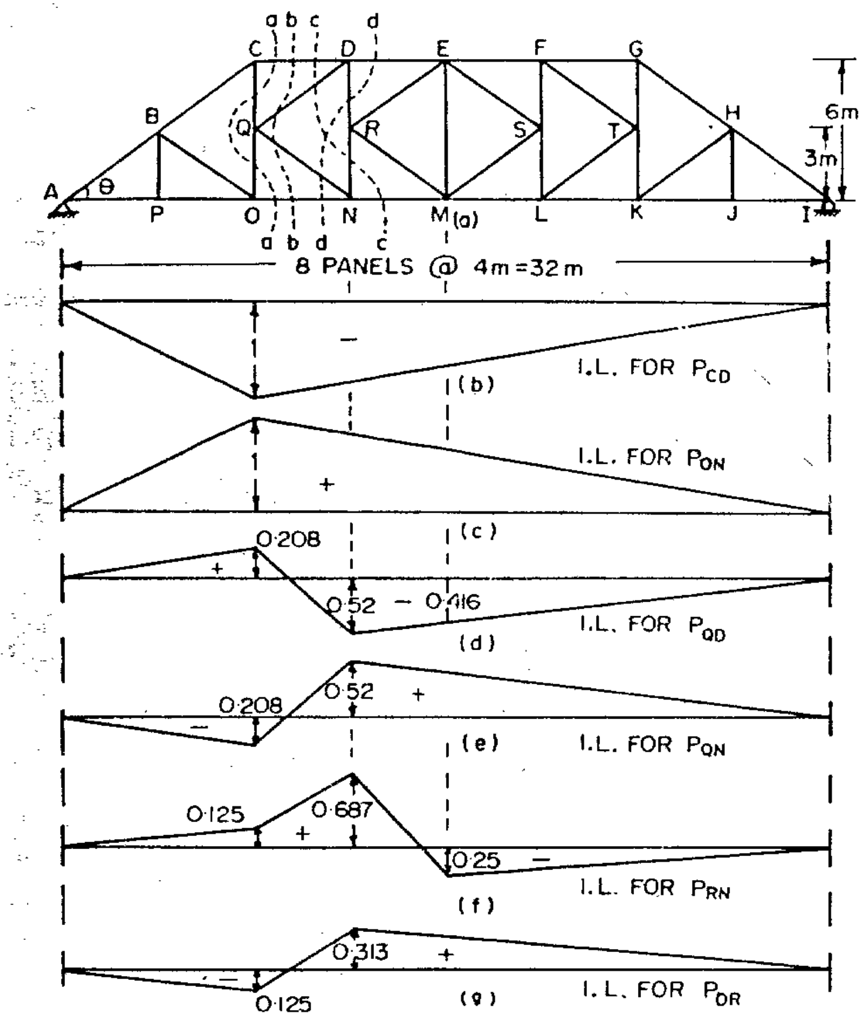


Fig. 4·4
K-Truss

The I.L. for P_{CD} will be triangle having a maximum ordinate of $\frac{8\times 24}{32}\times\frac{1}{6}=1$ under O, as shown in Fig. 4·4(b).

(2) **Influence line for** P_{ON}

$$P_{ON}=\frac{M_C}{6} \text{ (tension)}$$

The I.L. for P_{ON} will be a triangle having a maximum ordinate of $\frac{8\times24}{32}\times\frac{1}{6}=1$ under O, as shown in Fig. 4·4(c).

(3) Influence line for P_{QD} and P_{QN}

QD and QN have the same inclination with the vertical. Hence they will carry equal but opposite stresses. Thus, numerically,

$$P_{QD}=P_{QN}$$

Pass a section bb and consider the equilibrium of the left portion. Resolving vertically,

$$P_{QD}\sin\theta+P_{QN}\sin\theta=\text{shear in panel } ON=F_{ON}$$

When the unit load is at A, $R_A=1$, and hence shear in panel ON is zero. Therefore, P_{QD} and P_{QN} are zero.

When the unit load is at O,

$$R_A=\frac{1\times24}{32}=0{\cdot}75$$

$$\therefore\quad 2P_{QD}\sin\theta=F_{ON}=1-0{\cdot}75=0{\cdot}25 \text{ (tension)}$$

or $$P_{QD}=\frac{0{\cdot}25}{2\sin\theta}\qquad\left[\text{But }\sin\theta=\frac{3}{5}=0{\cdot}6\right]$$

$$=\frac{0{\cdot}25}{2\times0{\cdot}6}=0{\cdot}208 \text{ (tension)}$$

and $$P_{QN}=0{\cdot}208 \text{ (compression)}$$

When the unit load is at N,

$$R_A=\frac{1\times20}{32}=\frac{5}{8}=0{\cdot}625=F_{ON}$$

$$\therefore\quad P_{QD}=\frac{F_{ON}}{2\sin\theta}=\frac{0{\cdot}625}{2\times0{\cdot}6}=0{\cdot}52 \text{ (comp.)}$$

and $$P_{QN}=0{\cdot}52 \text{ (tension)}$$

When the unit load is at I, $R_A=0$. Hence F_{ON} is zero. Therefore, P_{QD} and P_{QN} are zero.

The I.L. for P_{QD} and P_{QN} are shown in Fig. 4·4 (d) and (e) respectively.

(4) Influence line for P_{RN}

Pass a section cc, cutting members CD, QD, RN and NM, and consider the equilibrium of the left portion.

(i) When the unit load is at O,

$$R_A=\frac{1\times24}{32}=0{\cdot}75 \text{ and } P_{QD}=0{\cdot}208 \text{ (tension)}$$

$$\therefore\quad P_{QD}\sin\theta+P_{RN}=1-R_A=1-0{\cdot}75=0{\cdot}25$$

$$\therefore\quad P_{RN}=0{\cdot}25-P_{QD}\sin\theta=0{\cdot}25-(0{\cdot}208\times0{\cdot}6)$$
$$=0{\cdot}125 \text{ (tension)}$$

(ii) When the unit load is at N,

$$R_A=\frac{1\times20}{32}=0{\cdot}625 \text{ and } P_{QD}=0{\cdot}52 \text{ (comp.)}$$

$$\therefore\quad -P_{QD}\sin\theta+P_{RN}=1-R_A=1-0{\cdot}625=0{\cdot}375$$

or $$P_{RN}=0{\cdot}375+(0{\cdot}52\times0{\cdot}6)=0{\cdot}687 \text{ (tension)}$$

(iii) When the unit load is at M,

$$R_A=\frac{1\times16}{32}=0{\cdot}5 \text{ and } P_{QD}=0{\cdot}416 \text{ (comp.)}$$

$$P_{RN}=R_A-P_{QD}\sin\theta=0{\cdot}5-(0{\cdot}416\times0{\cdot}6)$$
$$=0{\cdot}25 \text{ (comp.)}$$

The I.L. for P_{RN} is shown in Fig. 4·4 (f).

(5) Influence line for P_{DR}

Pass a section dd, cutting members DE, DR, QN and ON. Out of these four, stresses in members QN and ON are known. Consider the equilibrium of the portion to the left of section dd.

(i) When the unit load is at O,

$$R_A=0{\cdot}75 \text{ and } P_{QN}=0{\cdot}208 \text{ (comp.)}$$

$$P_{DR}=1-R_A-P_{QN}\sin\theta$$
$$=1-0{\cdot}75-(0{\cdot}208\times0{\cdot}6)$$
$$=0{\cdot}125 \text{ (comp.)}$$

(ii) When the unit load is at N

$$R_A=0{\cdot}625 \text{ and } P_{QN}=0{\cdot}52 \text{ (tension)}$$

$$\therefore\quad P_{QN}=R_A-P_{QN}\sin\theta=0{\cdot}625-(0{\cdot}52\times0{\cdot}5)$$
$$=0{\cdot}313 \text{ (tension)}$$

The I.L. for P_{QN} is shown in Fig. 4·4(g). The influence lines for stresses in other members can similarly be plotted.

4·5. BALTIMORE TRUSS WITH SUB-TIES : THROUGH TYPE

Simple trusses become uneconomical when the span exceeds 80 to 100 m. Earlier, multiple web systems were used in long span bridges. However, they are expensive and highly indeterminate, and are no longer used. The modern trend is to use some form of *sub-divided trusses* or K-truss. A sub-divided truss is obtained by placing in every panel of the truss some *secondary members* or diagonals. In contrast with primary members, which are stressed with all positions of the loads, secondary members are stressed only by the loads in certain limited positions.

Fig. 4·5 (a) shows a Baltimore truss (through type), with sub-ties. The load moves on the lower chords.

(1) Influence line for P_{HT} and P_{TI}

Pass a section aa cutting members BC, BN and HT

$$P_{HT}=P_{TI}=\frac{M_B}{15} \text{ (tension)}$$

The influence line will be a triangle having a maximum ordinate of $\frac{12\times60}{72}\times\frac{1}{15}=\frac{2}{3}$ under H, as shown in Fig. 4·5 (b).

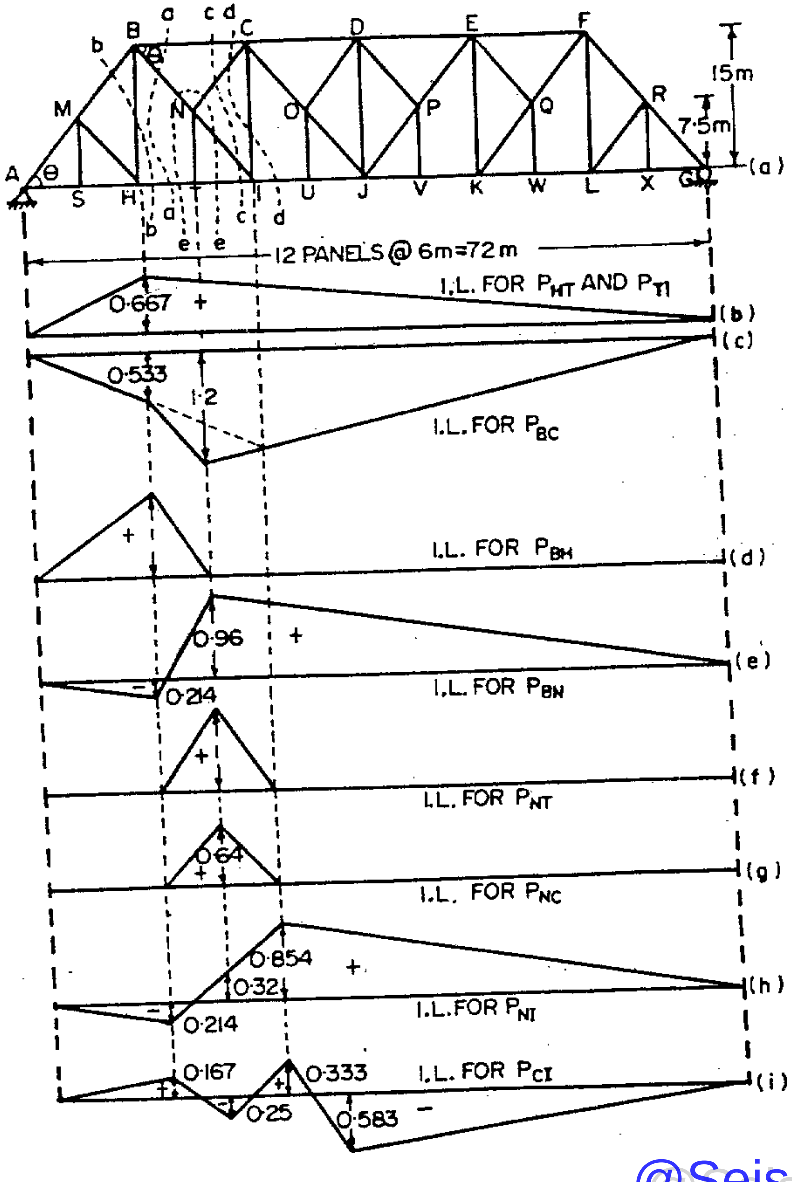


Fig. 4·5.
Baltimore truss with sub-ties. Through type.

(2) **Influence line for P_{BC}**

$$P_{BC}=\frac{M_I}{15}\text{ (compression)}$$

When the unit load is at H,

$$R_A=\frac{1\times60}{72}=\frac{5}{6}$$

$$\therefore \quad P_{BC}=\frac{1}{15}\left(\frac{5}{6}\times24-1\times12\right)$$

$$=\frac{8}{15}=0{\cdot}533\text{ (comp)}$$

When the unit load is at T,

$$R_A=\frac{1\times54}{72}=0{\cdot}75$$

$$\therefore \quad P_{BC}=\frac{1}{15}\left(0{\cdot}75\times24\right)=1{\cdot}2\text{ (comp.)}$$

When the load is at G, $R_A=0$

$$\therefore \quad P_{BC}=0$$

The influence line for P_{BC} is shown in Fig. 4·5 (c).

(3) **Influence line for P_{BH}**

Pass a section bb. Consider equilibrium of the left portion

$$P_{BH}=\frac{M_A}{12}\text{ (tension)}$$

M_A and hence P_{BH} are zero when the unit load is at A.
When the unit load is H,

$$P_{BH}=\frac{1}{12}(1\times12)=1\text{ (tension)}$$

When the unit load is at T or beyond T on right side there is no external force to the left of section bb except R_A. Hence $M_A=0$. Therefore, P_{BH} is zero. The I.L. for P_{BH} is, therefore, a triangle having zero ordinates under A and T, and a maximum ordinate of unity under H, shown in Fig. 4·5 (d).

(4) **Influence line for P_{BN}**

Considering the equilibrium to the left of section aa,

$$P_{BN}\sin\theta=\text{shear in panel } HT=F_{HT}$$

$$\sin\theta=\frac{15}{\sqrt{15^2+12^2}}=0{\cdot}781\ ;\ \cos\theta=\frac{12}{\sqrt{15^2+12^2}}=0{\cdot}625$$

When the unit load is at H, $R_A=\frac{1\times60}{72}=0{\cdot}833$

$$F_{HT}=1-R_A=1-0{\cdot}833=0{\cdot}167$$

$$P_{BN}=\frac{F_{HT}}{\sin\theta}=\frac{0{\cdot}167}{0{\cdot}781}=0{\cdot}214\text{ (compression)}$$

When the unit load is at T, $R_A=\dfrac{1\times 54}{72}=0{\cdot}75=F_{HT}$

$$\therefore \qquad P_{BN}=\frac{F_{HT}}{\sin\theta}=\frac{0{\cdot}75}{0{\cdot}781}=0{\cdot}96 \text{ (tension)}$$

When the unit load is at A or G, F_{HT} and P_{BN} are zero. The I.L. for P_{BN} is shown in Fig. 4·5 (*e*).

(5) **Influence line for P_{NT}**

NT is secondary member.

When the unit load is at H or to the left of H, $P_{NT}=0$

When the unit load is at T, $P_{NT}=1$ (tension)

When the unit load is at I or to the right of I, $P_{NT}=0$

The I.L. for P_{NT} is shown in Fig. 4·5 (*f*).

(6) **Influence line for P_{NC}**

NC is a sub-tie, and is thus a secondary member. Pass a horse shoe section *ee* cutting five members. Out of these, four members pass through I when produced. Hence take the moments about I. Consider the equilibrium of the portion enclosed by the horse shoe section *ee*.

When the unit load is at H or I, $M_I=0$, and hence $P_{NC}=0$.

When the unit load is at T, we get, by taking moment about point I:

$$P_{NC}\times(15\cos\theta)=1\times 6 \text{ (tension)}$$

$$\therefore \qquad P_{NC}=\frac{6}{15\cos\theta}=\frac{6}{15\times 0{\cdot}625}=0{\cdot}64 \text{ (tension)}$$

Thus, I.L. for P_{NC} is a triangle, as shown in Fig. 4·5 (*g*), and is similar to I.L. for P_{NT}. The vertical component of P_{NC} is equal to $0{\cdot}64\sin\theta=0{\cdot}64\times 0{\cdot}781=0{\cdot}5$. *Hence it is very interesting to note that in general, for both parallel and non-parallel chord trusses, where the secondary has the same slope as the main diagonal, the vertical components of stress in the secondary diagonal will be equal to one half of the load applied at the joint.*

(6) **Influence line for P_{NI}**

Pass a section *cc* cutting four members. Consider the equilibrium of the portion to the left of section *cc*.

Resolving forces vertically.

$$P_{NC}\sin\theta+P_{NI}\sin\theta=\text{S.F. in panel } TI=F_{TI}$$

$$P_{NC}+P_{NI}=F_{TI}\,.\,\text{cosec}\,\theta$$

The member *NC* will have stress only when the load is in span *HI*.

(*i*) When the unit load is at P, $P_{NC}=0$ and $R_A=0{\cdot}833$.

$$\therefore\ F_{TI}=1-0{\cdot}883=0{\cdot}167\ ;\ \text{cosec}\,\theta=\frac{1}{0{\cdot}781}$$

$$\therefore\ P_{NI}=\frac{0{\cdot}167}{0{\cdot}781}=0{\cdot}214 \text{ (compression)}$$

(*ii*) When the unit load is at T, $P_{NC}=0{\cdot}64$ (tension)

and $\qquad R_A=0{\cdot}75$

$$\therefore\ P_{NI}=(0{\cdot}74-1)\ \text{cosec}\,\theta+P_{NC}=-\frac{0{\cdot}25}{0{\cdot}781}\times 0{\cdot}64$$

$$=0{\cdot}32 \text{ (tension)}$$

(*iii*) When the unit load is at I, $P_{NC}=0$ and $R_A=0{\cdot}667$

$$P_{NI}=0{\cdot}667\ \text{cosec}\,\theta=\frac{0{\cdot}667}{0{\cdot}781}=0{\cdot}854 \text{ (tension)}$$

The I.L. for P_{NI} is shown in Fig. 4·5 (*h*).

(7) **Influence line for P_{CI}**

Pass a section *dd* cutting four sections. Consider equilibrium of left portion. Resolving the forces vertically,

$$P_{CI}+P_{NC}\sin\theta=\text{shear in panel } IU=F_{IU}$$

Member *NC* has stress when the load is in panel *HI* only.

(*i*) When the unit load is at H, $R_A=0{\cdot}833$ and $P_{NC}=0$

$\therefore\ P_{CI}=F_{IU}=(1-0{\cdot}833)=0{\cdot}167$ (tension)

(*ii*) When the unit load is at T, $R_A=0{\cdot}75$

and $\qquad P_{NC}=0{\cdot}64$ (tension)

$$P_{CI}=(0{\cdot}75-1)+P_{NC}\,.\,\sin\theta=-0{\cdot}25+(0{\cdot}64\times 0{\cdot}781)$$

$$=0{\cdot}25 \text{ (compression)}$$

(*iii*) When the unit load is at I, $R_A=0{\cdot}667$ and $P_{NC}=0$

$\therefore\ P_{CI}=(1-0{\cdot}667)=0{\cdot}333$ (tension)

(*iv*) When the unit load is at U, $R_A=0{\cdot}583$ and $P_{NC}=0$

$\therefore\ P_{CI}=0{\cdot}583$ (compression)

The I.L. for P_{CI} is shown in Fig. 4·5 (*i*).

4·6. BALTIMORE TRUSS WITH SUB-TIES : DECK TYPE

Fig. 4·6 (*a*) show a deck type Baltimore truss with sub-ties. The load moves on the top chord members.

(1) **Influence line for P_{NM}**

Pass a section *aa* to cut three members.

$$P_{NM}=\frac{M_C}{15} \text{ (tension)}$$

The influence line will be triangle having a maximum ordinate of $\dfrac{1}{15}\left(\dfrac{12\times 60}{72}\right)=\dfrac{2}{3}=0{\cdot}667$ under C, as shown in Fig. 4·6 (*b*).

(2) Influence line for P_{CD}

$$P_{CD}=\frac{M_M}{15}\text{(compression)}$$

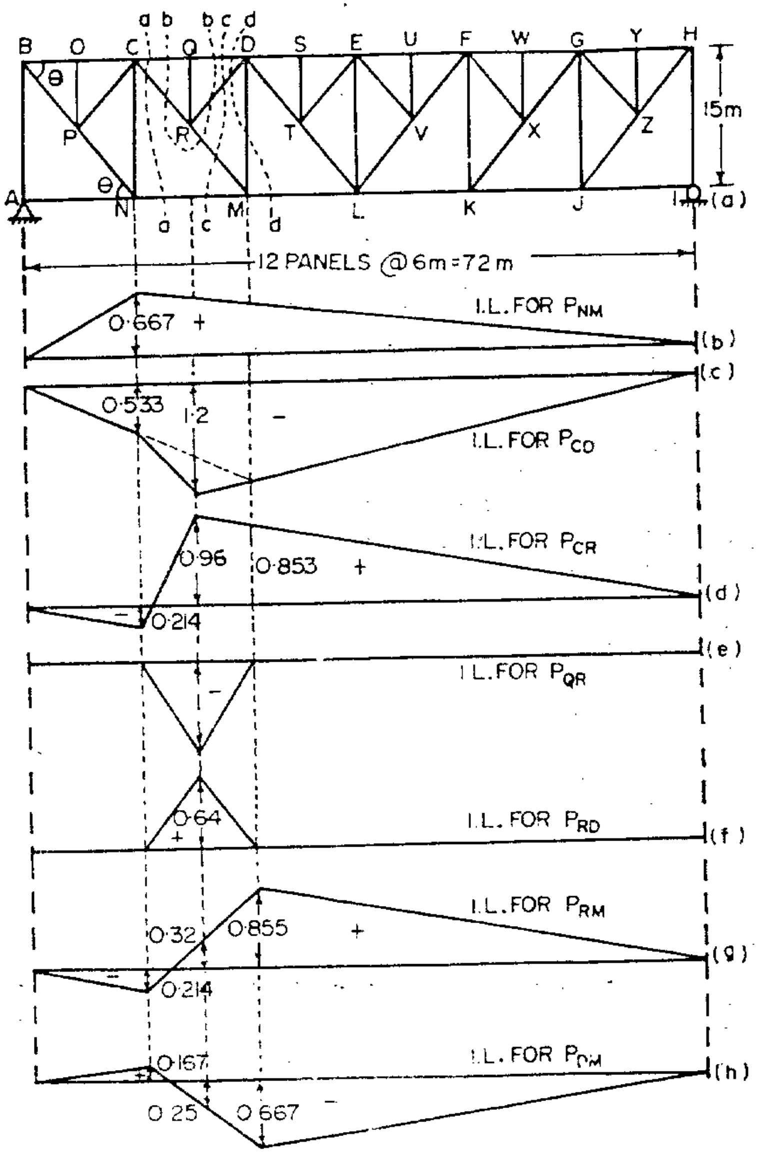


Fig. 4·6.
Baltimore truss with sub-ties (Deck type).

When the unit load it at C, $R_A=\frac{1\times60}{72}=\frac{5}{6}=0{\cdot}833$

$$\therefore\ P_{CD}=\frac{1}{15}\left(\frac{5}{6}\times24-1\times12\right)=\frac{8}{15}=0{\cdot}533\text{ (compression)}$$

When the unit load is at Q, $R_A=\frac{1\times54}{72}=\frac{3}{4}$.

$$\therefore\ P_{CD}=\frac{1}{15}\left(\frac{3}{4}\times24\right)=\frac{6}{5}=1{\cdot}2\text{ (compression)}$$

The I.L. for P_{CD} is shown in Fig. 4·6(c).

(3) Influence line for P_{CR}

Consider equilibrium of the portion to the left of aa and resolve the forces vertically. Then

$P_{CR}\sin\theta=$ shear in panel $CQ=F_{CQ}$

$\therefore\quad P_{CR}=F_{CQ}.\ \text{cosec}\ \theta.$

But $\sin\theta=\frac{15}{\sqrt{15^2+12^2}}=0{\cdot}781$; $\cos\theta=0{\cdot}625$

$\text{cosec}\ \theta=1{\cdot}28$

$P_{CR}=1{\cdot}28\ F_{CQ}$

(i) When the unit load is at C, $R_A=5/6=0{\cdot}833$

$\because\quad F_{CQ}=1-0{\cdot}833=0{\cdot}167$

$\therefore\quad P_{CR}=1{\cdot}28\ F_{CQ}=1{\cdot}28\times0{\cdot}167=0{\cdot}214$ (compression)

(ii) When the unit load is at Q, $R_A=\frac{3}{4}=0{\cdot}75$ and $F_{CQ}=0{\cdot}75$

$\therefore\ P_{CR}=1{\cdot}28\times0{\cdot}75=0{\cdot}96$ (tension)

Let I.L. for P_{CR} is shown in Fig. 4·6(d).

(4) Influence line for P_{QR}

QR is secondary member and hence it will be stressed when the load is in panel CD. The influence line for P_{QR} is shown in Fig. 4·6(e), having zero ordinate under C and D, and an ordinate of unity under Q.

(5) Influence line for P_{RD}

RD is a secondary member, and hence it will be stressed only when the load is in panel CD. Pass a horse shoe section bb cutting four members. Out of these, the line of action of forces in three members pass through point C. Hence take the moment of forces about point C. Consider the equilibrium of the portion enclosed by the section bb.

When the unit load is at C or D, M_C is zero, and hence P_{RD} is zero.

When the unit load is at Q, we get

$$M_C=1\times6=P_{RD}\times(12\sin\theta)$$

$$P_{RD}=\frac{1}{12}\text{ cosec }\theta=\tfrac{1}{2}\text{ cosec }\theta=\tfrac{1}{2}\times 1{\cdot}28=0{\cdot}64 \text{ (tension)}$$

The I.L. for P_{RD} is shown in Fig. 4·6 (*f*).

(6) **Influence line for P_{RM}**

Pass a section *cc* cutting four members *QD*, *RD*, *RM* and *NM*. The stress in *RD* is known. Consider the equilibrium of the left portion. Resolving the forces vertically, we get

$$P_{RD}\sin\theta+P_{RM}\sin\theta=\text{shear in panel } QD=F_{QD}$$

In the above relation, member *RD* will have stress only when the load is in *CD*.

(*i*) When the unit load is at *C*, $P_{RD}=0$ and $R_A=5/6=0{\cdot}833$

$\therefore$ $P_{RM}=\text{cosec }\theta.\ F_{QD}=1{\cdot}28(1-0{\cdot}833)=0{\cdot}214$ (comp.)

(*ii*) When the unit load is at *Q*, $P_{RD}=0{\cdot}64$ (tension) and $R_A=0{\cdot}75$

$$\because \quad P_{RM}=P_{RD}+(0{\cdot}75-1)\text{ cosec }\theta$$
$$=0{\cdot}64-0{\cdot}25\times 1{\cdot}28=0{\cdot}32 \text{ (tension).}$$

(*iii*) When the unit load is at *D*, $P_{RD}=0$ and $R_A=2/3$

$$\because \quad P_{RM}=\tfrac{2}{3}\text{ cosec }\theta=\tfrac{2}{3}\times 1{\cdot}28=0{\cdot}853 \text{ (tension).}$$

The I.L. for P_{RM} is shown in Fig. 4·6(*g*). It will be seen that the influence lines for P_{CR} and P_{RM} are exactly the same for load positions between *B* to *C* and between *D* to *H*.

(7) **Influence line for P_{DM}**

Pass a section *dd*, cutting four members. Considering the equilibrium of the left portion and resolving the forces vertically we get, in general :

$$P_{DM}+P_{RD}\sin\theta=\text{shear in panel } QD=F_{QD}$$

Member *RD* will have stress only when the load is in *CD*.

(*i*) When the unit load is at *C*, $R_A=0{\cdot}833$ and $P_{RD}=0$

$$\therefore \quad P_{DM}=F_{QD}=1-0{\cdot}833=0{\cdot}167 \text{ (tension)}$$

(*ii*) When the unit load is at *Q*, $R_A=0{\cdot}75$

and $\quad P_{RD}=0{\cdot}64$ (tension)

$$P_{DM}=(0{\cdot}75-1)+(0{\cdot}64\times 0{\cdot}781) \text{ (compression)}$$
$$=-0{\cdot}25+0{\cdot}5=0{\cdot}25 \text{ (compression)}$$

(*iii*) When the unit load is at *D*, $P_{RD}=0$ and $R_A=\tfrac{2}{3}$.

$$\therefore \quad P_{DM}=\tfrac{2}{3}=0{\cdot}667 \text{ (compression)}$$

The I.L. for P_{DM} is shown in Fig. 4·6(*h*).

4·7 BALTIMORE TRUSS WITH SUB-STRUTS : THROUGH TYPE

(1) **Influence line for P_{BC}**

Pass a section *aa* cutting three members. Considering equilibrium of the left portion,

$$P_{BC}=\frac{M_K}{15}\text{(compression)}$$

The influence line diagram for P_{BC} is thus a triangle having a maximum ordinate of $\dfrac{24\times 48}{72}\times\dfrac{1}{15}=1{\cdot}067$ (comp.) under *K* as in Fig. 4·7(*b*).

(2) **Influence line for P_{LK}**

$$P_{LK}=\frac{M_B}{15}\text{(tension)}$$

When the unit load is at *L*, $R_A=\dfrac{60}{72}=0{\cdot}833$

$$\therefore \quad P_{LK}=\frac{1}{15}(0{\cdot}833\times 12)=0{\cdot}667 \text{ (tension)}$$

When the unit load is at *T*, $R_A=\dfrac{54}{72}=0{\cdot}75$

$$\therefore \quad P_{LK}=\frac{1}{15}(0{\cdot}75\times 12+1\times 6)=1 \text{ (tension)}$$

When the unit load is at *K*, $R_A=\dfrac{48}{72}=0{\cdot}667$

$$\therefore \quad P_{LK}=\frac{1}{15}(0{\cdot}667\times 12)=0{\cdot}533 \text{ (tension)}$$

The I.L. diagram for P_{LK} is shown in Fig. 4·7(*c*).

(3) **Influence line for P_{NK}**

Resolving the forces vertically,

$$P_{NK}\sin\theta=\text{shear in panel } TK=F_{TK}$$

or $\quad P_{NK}=F_{TK}\text{ cosec }\theta$

$$\sin\theta=\frac{15}{\sqrt{15^2+12^2}}=0{\cdot}781\ ;\ \text{cosec }\theta=1{\cdot}28\ ;\ \cos\theta=0{\cdot}625$$

When the unit load is at *T*, $R_A=0{\cdot}75$

$$\therefore \quad P_{NK}=(1-0{\cdot}75)\times 1{\cdot}28=0{\cdot}32 \text{ (comp.)}$$

When the unit load is at *K*, $R_A=0{\cdot}667$

$$\therefore \quad P_{NK}=0{\cdot}667\times 1{\cdot}28=0{\cdot}853 \text{ (tension).}$$

The I.L. diagram for P_{NK} is shown in Fig. 4·7 (*d*).

(4) **Influence line for P_{NT}**

NT is a secondary member, and hence it will be stressed only when the load is in panel *LK*. Evidently, I.L. for P_{NT} will be a triangle having zero ordinates under *L* and *K* and an ordinate of unity under *T*, as shown in Fig. 4·7(*e*).

(5) Influence line for P_{LN}

Pass a horse shoe section *bb*, cutting five members *BN*, *NK* *LN*, *LT* and *TK*. Out of these, line of action of forces in four

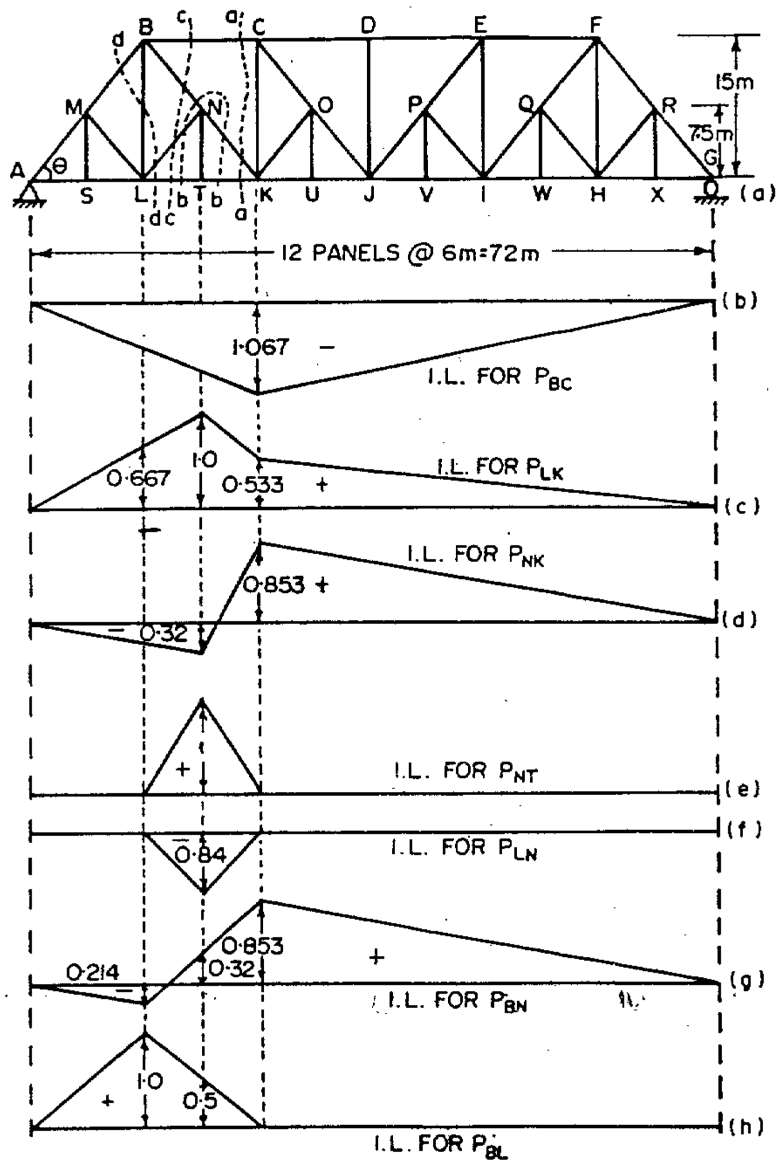


Fig. 4·7

Baltimore truss with sub-struts : Through type.

members pass through point *K*. Hence take the moment of the forces, about *K* and consider the equilibrium of the portion enclosed by the horse shoe section. Note that *LN* is a secondary member, and hence it will be stressed only when the load is in panel *LK*.

When the unit load is at *L*, $M_K=0$ and hence $P_{LN}=0$

When the unit load is at *T*,

$$M_K=1\times6=P_{LN}\times(12\sin\theta)$$

$$\therefore\quad P_{LN}=\frac{6}{12\sin\theta}=\tfrac{1}{2}\operatorname{cosec}\theta=\tfrac{1}{2}\times1{\cdot}28=0{\cdot}64 \text{ (comp.)}$$

When the unit load is at *K*, $M_K=0$, and hence $P_{LN}=0$

The I.L. diagram for P_{LN} is shown in Fig. 3·7 (*f*).

(6) Influence line for P_{BN}

Pass a section *cc*, cutting four members. Considering the equilibrium of the left portion, we get, in general,

$$P_{BN}\sin\theta+P_{LN}\sin\theta=\text{shear in } LT=F_{LT}$$

or $$P_{BN}+P_{LN}=F_{LT}\operatorname{cosec}\theta=1{\cdot}28\ F_{LT}$$

The member *LN* will be stressed only when the load is in *LK*.

When the unit load is at *L*, $R_A=0{\cdot}833$ and $P_{LN}=0$

$$\therefore\quad P_{BN}=(1-0{\cdot}833)\times1{\cdot}28=0{\cdot}214 \text{ (comp.)}$$

When the unit load is at *T*, $R_A=0{\cdot}75$ and $P_{LN}=0{\cdot}64$ (comp.)

$$\therefore\quad P_{BN}+0{\cdot}64=0{\cdot}75\times1{\cdot}28$$

or $$P_{BN}=0{\cdot}75\times1{\cdot}28-0{\cdot}64=0{\cdot}32 \text{ (tension)}$$

When the unit load is at *K*, $R_A=0{\cdot}667$ and $P_{LN}=0$

$$\therefore\quad P_{BN}=0{\cdot}667\times1{\cdot}28=0{\cdot}853 \text{ (tension)}$$

The I.L. diagram for P_{BN} is shown in Fig. 4·7 (*g*).

(7) Influence line for P_{BL}

Pass a section *dd*, cutting four members *BM*, *BL*, *LN* and *LT*. Out of these, the lines of action of forces in two members pass through *A*. Hence take the moment of all the forces, about *A* and consider the equilibrium of the left portion.

When the unit load is at *A*, P_{BL} is evidently zero.

When the unit load is at *L*, $P_{LN}=0$.

$$\therefore\quad P_{BL}\times12=1\times12$$

or $$P_{BL}=1 \text{ (tension).}$$

When the unit load is at *T*, $P_{LN}=0{\cdot}64$ (comp.)

$$\therefore\quad P_{BL}\times12=(P_{LN}\sin\theta)\times12$$

$$\therefore\quad P_{BL}=0{\cdot}64\times0{\cdot}781=0{\cdot}5 \text{ (tension)}$$

When the unit load is at *K*, $P_{LN}=0$

$$\therefore\quad P_{BL}\times12=\text{zero.}\quad\text{or}\quad P_{BL}\text{ is zero}$$

The I.L. for P_{BL} is shown in Fig. 4·7 (*h*).

4·8. PENNSYLVANIA OR PETTIT TRUSS WITH SUB-TIES

(1) Influence line for P_{CD}

Pass a section *aa* cutting three members

$$P_{CD}=\frac{M_J}{r}\text{(compression)}$$

where r=Perpendicular distance of J from CD.

Prolong DC backward to meet GA produced in O_1. Let θ_1 be the inclination of CD, as shown in Fig. 3·8 (*a*).

$$\tan\theta_1=\frac{15-13}{12}=\frac{1}{6}=0{\cdot}1667\ ;\ \theta_1=9°\ 28'$$

$$\therefore\quad \sin\theta_1=0{\cdot}165\ :\ \cos\theta_1=0{\cdot}986$$

$$O_1J=\frac{DJ}{\tan\theta_1}=15\times6=90\text{ m}\ ;\ O_1A=90-36=54\text{ m}$$

Now $\quad r=O_1J\sin\theta_1=90\times0{\cdot}165=14{\cdot}85$ m

$$\therefore\quad P_{CD}=\frac{M_J}{14{\cdot}85}\text{(compression)}$$

When the unit load is at K, $R_A=\frac{48}{72}=0{\cdot}667$

$$\therefore\quad P_{CD}=\frac{1}{14{\cdot}85}[1{\cdot}667\times36-1\times12]=0{\cdot}808\text{ (comp.)}$$

When the unit load is at U, $R_A=\frac{42}{72}=0{\cdot}583$

$$\therefore\quad P_{CD}=\frac{1}{14{\cdot}85}[0{\cdot}583\times36]=1{\cdot}412\text{ (comp.)}$$

The I.L. diagram for P_{CD} is shown in Fig. 4·8 (*b*).

(2) Influence line for P_{KJ}

$$P_{KJ}=\frac{M_C}{13}\text{(tension)}$$

The I.L. diagram for P_{KJ} will, therefore, be a triangle having a maximum ordinate of $\frac{24\times48}{72}\times\frac{1}{13}=1{\cdot}23$ under K as shown in Fig. 4·8 (*c*).

(3) Influence line for P_{CO}

Consider the equilibrium of the portion to the left of section *aa*. Take moment about O_1 where two members CD and KJ meet when produced.

$$P_{CO}=\frac{M_{OI}}{\bar{X}}$$

where $\quad \bar{X}$=perpendicular distance of CO from O_1

$=O_1J\sin\theta$

θ=inclination of CJ with O_1J

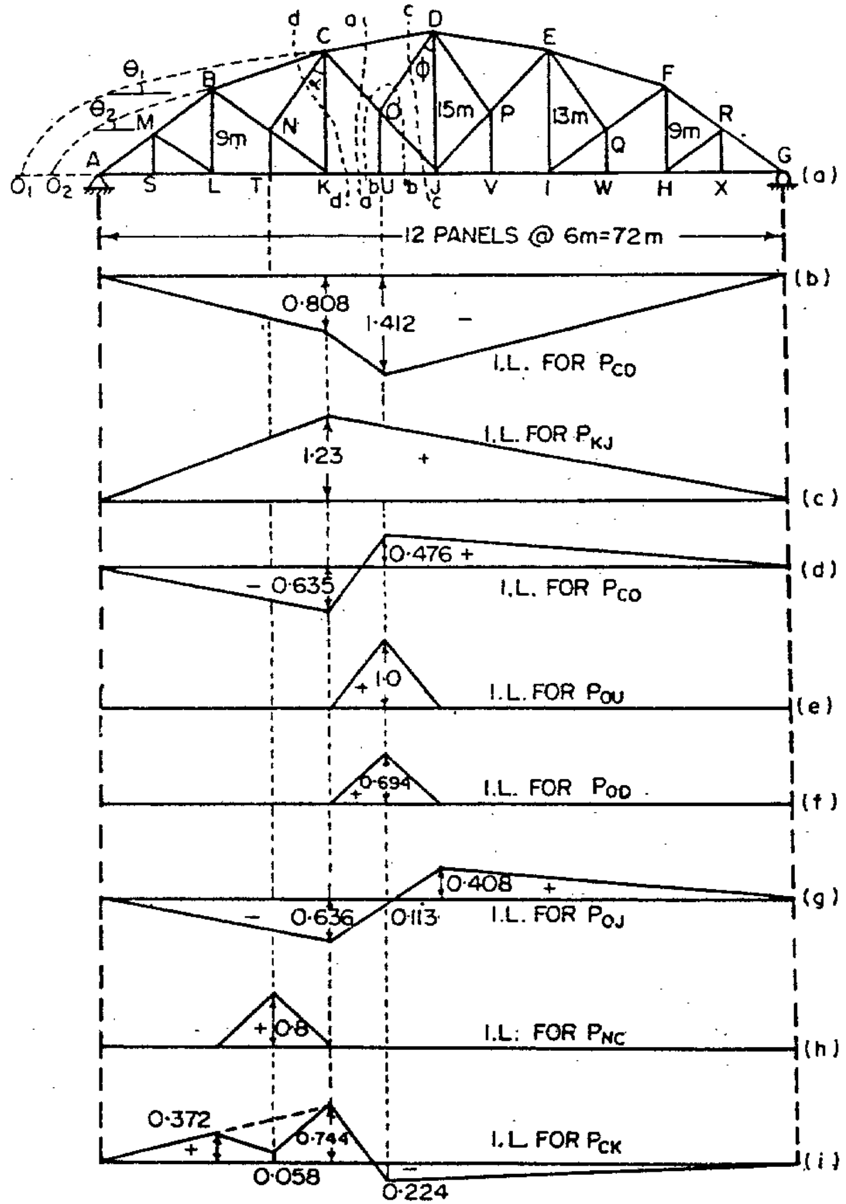


Fig. 4·8.
Pettit truss with sub-ties

Now $\sin\theta=\dfrac{13}{\sqrt{13^2+12^2}}=\dfrac{13}{17\cdot7}=0\cdot734$; $\operatorname{cosec}\theta=1\cdot364$

$$\cos\theta=\frac{12}{17\cdot7}=0\cdot678$$

$$\therefore \quad \bar{X}=90\times0\cdot734=66\cdot1 \text{ m}$$

$$\therefore \quad P_{CO}=\frac{M_{O_1}}{66\cdot1}$$

When the unit load is at K, $R_A=0\cdot667$

$$\therefore \quad P_{CO}=\frac{1}{66\cdot1}[1\times78-0\cdot667\times54]=0\cdot635 \text{ (comp.)}$$

When the unit load is at U, $R_A=0\cdot583$

$$\therefore \quad P_{CO}=\frac{1}{66\cdot1}(0\cdot583\times54)=0\cdot476 \text{ (tension)}$$

The I.L. diagram for P_{CO} is shown in Fig. 4·8 (*a*).

(4) **Influence line for** P_{OU}

OU is a secondary member, and hence will be stressed only when the unit load is in the panel *KJ*. The I.L. diagram will be a triangle having zero ordinates under *K* and *J*, and unit ordinate under *U* as shown in Fig. 4·8 (*e*).

(5) **Influence line for** P_{OD}

OD is also a secondary member, and will be stressed only when the unit load is in the panel *KJ*. Pass a horse shoe section *bb* cutting five members. Out of these, the lines of action of four forces pass through *J*. Hence take moments about *J* and consider the equilibrium of portion enclosed by the horse shoe section.

Thus

$$P_{OD}\times(15\sin\phi)=M_J$$

$$\sin\phi=\frac{6}{\sqrt{(8\cdot5)^2+6^2}}=\frac{6}{10\cdot4}=0\cdot577\ ;\ \cos\phi=\frac{8\cdot5}{10\cdot4}=0\cdot818$$

$$\therefore \quad P_{OD}=\frac{M_J}{15\sin\phi}=\frac{M_J}{15\times0\cdot577}=\frac{M_J}{8\cdot65}$$

When the unit load is at K, $M_J=0$ and hence $P_{OD}=0$.

When the unit load is at U, $M_J=1\times6=6$

$$\therefore \quad P_{OD}=\frac{M_J}{8\cdot65}=\frac{6}{8\cdot65}=0\cdot694 \text{ (tension)}$$

When the unit load is at J, $M_J=0$ and hence $P_{OD}=0$.

The influence line diagram for P_{OD} is shown in Fig. 4·8 (*f*).

(6) **Influence line for** P_{OJ}

Pass a section *cc* cutting four members. Out of these, the line of action of forces in two members pass through O_1 Hence consider

the equilibrium of the portion to the left of section *cc* and take moments about O_1.

(*i*) When the unit load is at K, $R_A=0\cdot677$ and $P_{OD}=0$

$$\therefore \quad (P_{OJ}\sin\theta)\times O_1J=1\times O_1K-0\cdot667\times O_1A$$

$$P_{OJ}=\frac{1}{0\cdot734\times90}[1\times78-0\cdot667\times54]=0\cdot636 \text{ (comp.)}$$

(*ii*) When the unit load is at U, $R_A=0\cdot583$ and $P_{OD}=0\cdot694$ (tension).

Taking moments about O_1, we get

$$(P_{OJ}\sin\theta)\ O_1J=(1\times O_1U)-(R_A\times O_1A)+(P_{OD}\sin\phi\times6\cdot5)-(P_{OD}\cos\phi\times O_1U)$$

$$\therefore \quad P_{OJ}=\frac{1}{0\cdot734\times90}[(1\times84)-(0\cdot583\times54)+(0\cdot694\times0\cdot577\times6\cdot5)-(0\cdot694\times0.818\times84)]$$

$$=0\cdot113 \text{ (compression)}.$$

When the unit load is at J, $R_A=0\cdot5$ and $P_{OD}=0$

$$\therefore \quad (P_{OJ}\sin\theta)\ O_1J=R_A\times O_1A$$

$$\therefore \quad P_{OJ}=\frac{0\cdot5\times54}{0\cdot734\times90}=0\cdot408 \text{ (tension)}$$

The I.L. diagram for P_{OJ} is shown in Fig. 4·8 (*g*).

(7) **Influence line for** P_{NC}

The influence line for P_{NC} can be drawn exactly in the same manner as that for P_{OD}. The I.L. diagram is a triangle with a central ordinate of 0·8 as shown in Fig. 4·8 (*h*). Reader is advised to compute this ordinate.

(8) **Influence line for** P_{CK}

Pass a section *dd* cutting four members.

If α is the inclination of member *NC* with the vertical

$$\sin\alpha=\frac{6}{\sqrt{8\cdot5^2+6^2}}=\frac{6}{10\cdot4}=0\cdot577\ ;\ \cos\alpha=\frac{8\cdot5}{10\cdot4}=0\cdot818$$

Prolong *CB* backwards to meet *GA* produced in O_2. Let θ_2 be the inclination of *CB* with horizontal.

Then, $\tan\theta_2=\dfrac{13-9}{12}=\dfrac{4}{12}=\dfrac{1}{3}$; $\theta_2=18°\ 26'$

$$\therefore \quad \sin\theta_2=0\cdot316 \text{ and } \cos\theta_2=0\cdot949$$

$$O_2K=\frac{CK}{\tan\theta_2}=13\times3=39 \text{ m}$$

$$\therefore \quad O_2A=39-24=15 \text{ m.}$$

Take the moments about O_2, of all unbalanced forces to the left of the section *dd*.

(*i*) When the unit load is at L, $R_A=\frac{60}{72}=0\cdot833$ and $P_{NC}=0$

$$\therefore \quad P_{CK}=\frac{M_{O_2}}{O_2K}=\frac{M_{O_2}}{39}=\frac{1}{39}\{(1\times27)-(0\cdot833\times15)\}$$

$$=0\cdot372 \text{ (tension).}$$

(*ii*) When the unit load is at T, $R_A=0\cdot75$ and $P_{NC}=0\cdot8$ (tension).

$$\therefore P_{CK}\times O_2K=(P_{NC}\cos\alpha\times O_2K)-(P_{NC}\sin\alpha\times13)+(R_A\times O_2A(-(1\times O_2T)$$

or
$$P_{CK}=\frac{1}{39}\{(0\cdot8\times0\cdot818\times39)-(0\cdot8\times0\cdot577\times13)+(0\cdot75\times15))-(1\times33)\}$$

$$=0\cdot058 \text{ (tension).}$$

(*iii*) When the unit load is at K, $R_A=0\cdot667$ and $P_{NC}=0$

$$\therefore P_{CK}=\frac{1}{39}\{(1\times39)-(0\cdot667\times15)\}=0\cdot744 \text{ (tension).}$$

(*iv*) When the unit load is at U, $R_A=0\cdot583$ and $P_{NC}=0$

$$\therefore P_{CK}=\frac{1}{39}\{0\cdot583\times15\}=0\cdot224 \text{ (comp.)}$$

The influence line diagram for P_{CK} is shown in Fig. 4·8 (*i*).

5

The Müller-Breslau Principle

5·1. INTRODUCTION

The Müller-Breslau principle or Müller-Breslau influence theorem is the most important tool in obtaining influence lines for statically determinate as well as statically indeterminate structures. The method is based on the concept of the influence line as a deflection curve. While developing the method about tewenty years after the influence line was first introduced by Winkler (1867), Müller-Breslau became aware of the great values of Maxwell's theorem of reciprocal displacement. In fact, Müller-Breslau principle is the straight application of Maxwell's reciprocal theorem. For a detailed study of the reciprocal theorem, the reader is advised to read articles 7·3 and 7·4 of chapter 7.

5·2. THE MÜLLER-BRESLAU PRINCIPLE

The Müller-Breslau principle may be stated as follows :

"*If an internal stress component, or a reaction component is considered to act through some small distance and thereby to deflect or displace a structure, the curve of the deflected or displaced structure will be, to some scale, the influence line for the stress or reaction component*".

To prove the validity of the above statement, let us consider a two span continuous beam ABC, freely supported at A and C, and continuous over support B and plot the influence line for reaction R_B at B.

Let a unit load act at a point X distant x from end A. If the support at B is removed, the beam will deflect as shown in Fig. 5·1. (*b*). Remove the unit load as well as the redundant reaction R_B, and place a downward unit load at B. The beam will deflect under the unit load, as shown in Fig. 5·1 (*c*).

Let y_{BB}=deflection at B due to unit load at B
y_{XB}=deflection at X due to unit load at B
y_{BX}=deflection at B due to unit load at X.

(The first suffix to y denotes the point where the deflection is reckoned and the second suffix denotes the position of the unit point load).

Thus, when the unit load is acting at X, the deflection of point B, in the absence of R_B, will be equal to y_{BX}. However, as the support at B is at the same level as A and C, the upward deflection at B due to R_B is to neutralize this downward deflection y_{BX}. Hence we get, from consistent deformation (chapter 7) :

$$R_B . y_{BB} = y_{BX}$$

By Maxwell's reciprocal theorem (7·3)

$$y_{BX} = y_{XB}$$

Hence $$R_B = \frac{y_{XB}}{y_{BB}} \tag{5·1}$$

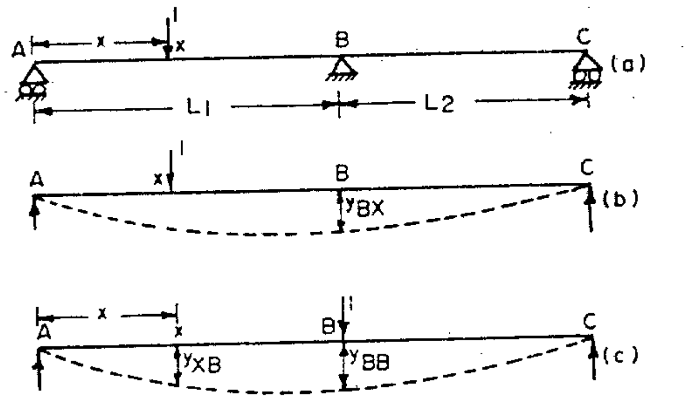


Fig. 5·1

Thus, the reaction at B, due to unit load at any point X is proportional to the deflection at the point X due to the unit load acting at B. In other words, the deflection curve shown in Fig. 5·1 (c) represents, to some scale, the influence line for R_B.

If the deflection y_{BB}, in the direction of unit load at B, is selected as unity, the deflection curve will directly give influence line for R_B.

5·3 INFLUENCE LINES FOR STATICALLY DETERMINATE BEAMS

The Müller-Breslau Principle is applicable both for statically determinate beams as well as for statically indeterminate beams. Let us first take statically determinate beam.

The Müller-Breslau *influence theorem* for statically determinate beams may be stated as follows :

"The influence line for an assigned function of a statically determinate beam may be obtained by removing the restraint offered by that function and introducing a directly related generalised unit displacement at the location and in the direction of the function.

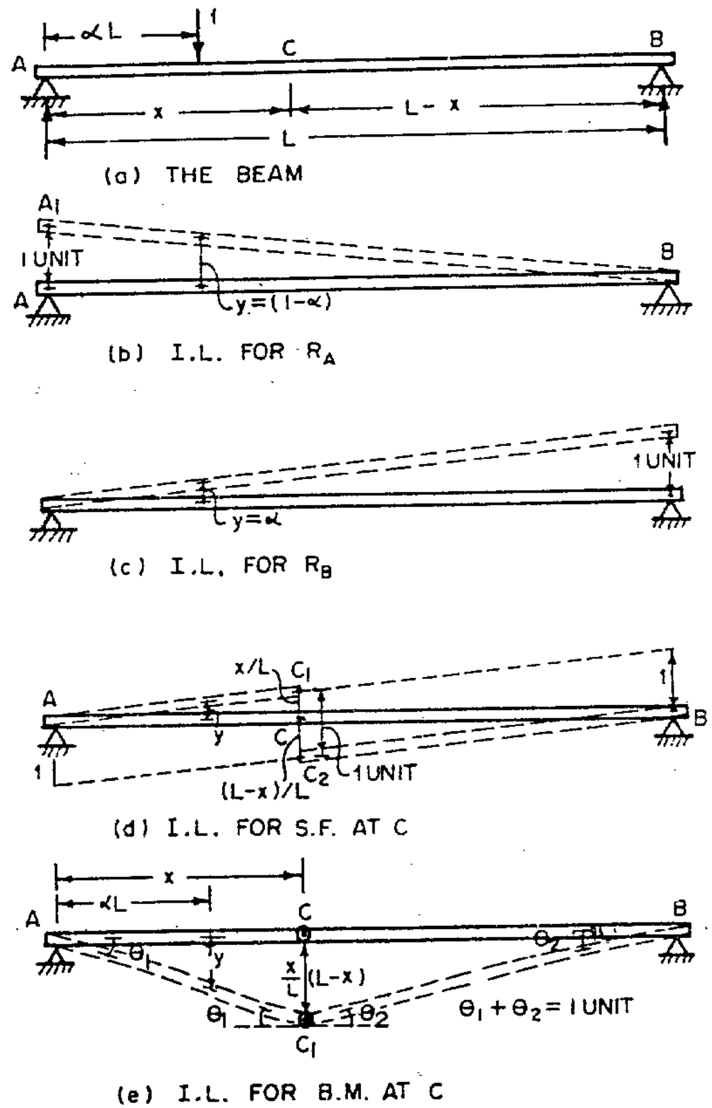


Fig. 5·2.

Fig. 5·2 shows a supply supported beam AB of span L.

1. I.L. For reaction R_A and R_B

The I.L. for reaction (R_A) at A can be found by lifting the beam off the support A by a unit distance, as shown in Fig 5·2 (*b*). The deflected shape gives the I.L. for R_A. This can be easily proved by applying the principle of *virtual work* to the rigid body motion of the beam shown in Fig. 5·2 (*b*). The total virtual work (δW) must be equal to zero since the resultant of the force system is zero. Thus, if the ordinate under the unit load is y, we have

$$\delta W = R_A(1\cdot0) - 1\cdot0(y) = 0$$

which gives $\quad y = R_A$

Which proves the proposition.

When the unit load is at a distance αL from A, the magnitude of y is given by the relaiion

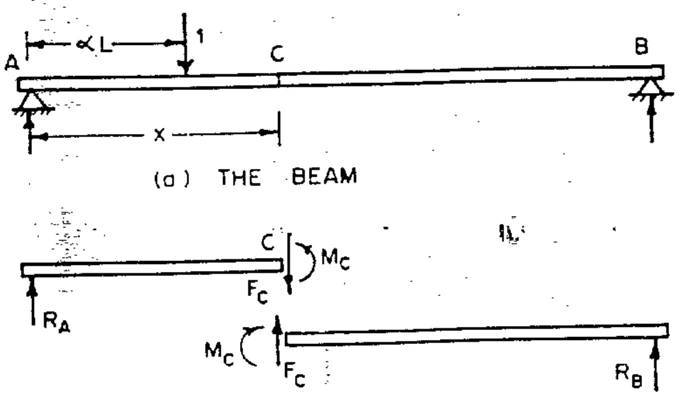
$$\frac{1}{L} = \frac{y}{L-\alpha L}$$

$$\therefore \quad y = (1-\alpha)$$

Similarly, the I.L. for reaction R_B can be found, as shown in Fig. 5·2 (*c*).

2. I.L. for S.F. at C

Let us find the I.L. for S.F. (F_C) at C. We know that S.F. (F_C) acts to both the sides of the section and is represented by (↓ : ↑). Hence cut the beam at C in two parts AC and CB. The free body diagram of the two parts is shown in Fig 5·3 (*b*). Let the beam go through rigid body motions of parts AC and CB, as shown in Fig. 5·2 (*d*), So that the total movement C_1C_2=unity. The deflected shape will then give the influence line for F_C. This can be very easily



(a) THE BEAM

(b) FREE BODY DIAGRAMS OF THE TWO PARTS

Fig. 5.3

proved by applying the principle of virtual work. Thus, if y is the ordinate of the I.L. under the unit load, we have

$$\delta W = R_A(0) - 1\cdot0(y) - M_C(\theta) + F_C(CC_1) + F_C(CC_2) + M_C(\theta) + R_B(0) = 0$$

or $\quad y = F_C(CC_1 + CC_2) = F_C(C_1C_2)$, where $C_1C_2 = 1$

$\therefore \quad y = F_C$

which proves the proposition.

Now, if the section C is at a distance x from A. CC_1 will be equal to x/L. and CC_2 will be equal to $(L-x)/L$. Similarly, the ordinate y under the unit load is given by

$$y = \left(\frac{x}{L}\right) \times \frac{\alpha L}{x} = \alpha$$

3. I.L. for B.M. (M_C) at C

For obtaining I.L. for M_C, introduce a hinge at C, and let the system go through rigid body motions of AC and CB as shown in Fig. 5·2 (*e*). Then

$$\delta W = R_A(0) - 1\cdot0(y) - F_C(CC_1) + M_C(\theta_1) + F_C(CC_1) + M_C(\theta_2) + R_B(0) = 0$$

or $\quad y = M_C(\theta_1 + \theta_2)$, where $\theta_1 + \theta_2 = 1$

$\therefore \quad y = M_C$

Now $\quad CC_1 = x\theta_1 = (L-x)\theta_2$

Thus, $\quad \theta_2 = \dfrac{x}{L-x}\theta_1$

But $\quad \theta_1 + \theta_2 = 1$

$\therefore \quad \theta_1 + \dfrac{x}{L-x}\theta_1 = 1$

or $\quad \theta_1\left(\dfrac{L}{L-x}\right) = 1 \quad \therefore \quad \theta_1 = \dfrac{L-x}{L}$

Hence $\quad CC_1 = x\theta_1 = \dfrac{x}{L}(L-x)$

Also, ordinate y is given by

$$\frac{y}{\alpha L} = \frac{CC_1}{x}$$

or

$$y = \alpha L \times \frac{1}{x} \times \frac{x}{L}(L-x) = \alpha(L-x)$$

When the unit load is at C, $\alpha L = x$, or $\alpha = x/L$

$$y = CC_1 = \frac{x}{L}(L-x)$$

Example 5·1. *A two span beam ABC has internal hinges at D and E. Using Muller-Breslau influence theorem, sketeh I.L. for (i) R_A (ii) R_B (iii) R_C and (iv) M_C.*

Solution. If there were no hinges at D and E, the beam would be *statically indeterminate* to second degree. However, provision of hinges at D and E makes the beam statically determinate.

(*i*) **I.L. For R_A** : According to Müller Breslau Theorem, in order to find I.L. for R_A, lift the beam off the support A by unity in the direction of R_A. The deflected shape of the beam, shown in Fig. 5·4 (*b*) gives the I.L. for R_A. It should be noted that because

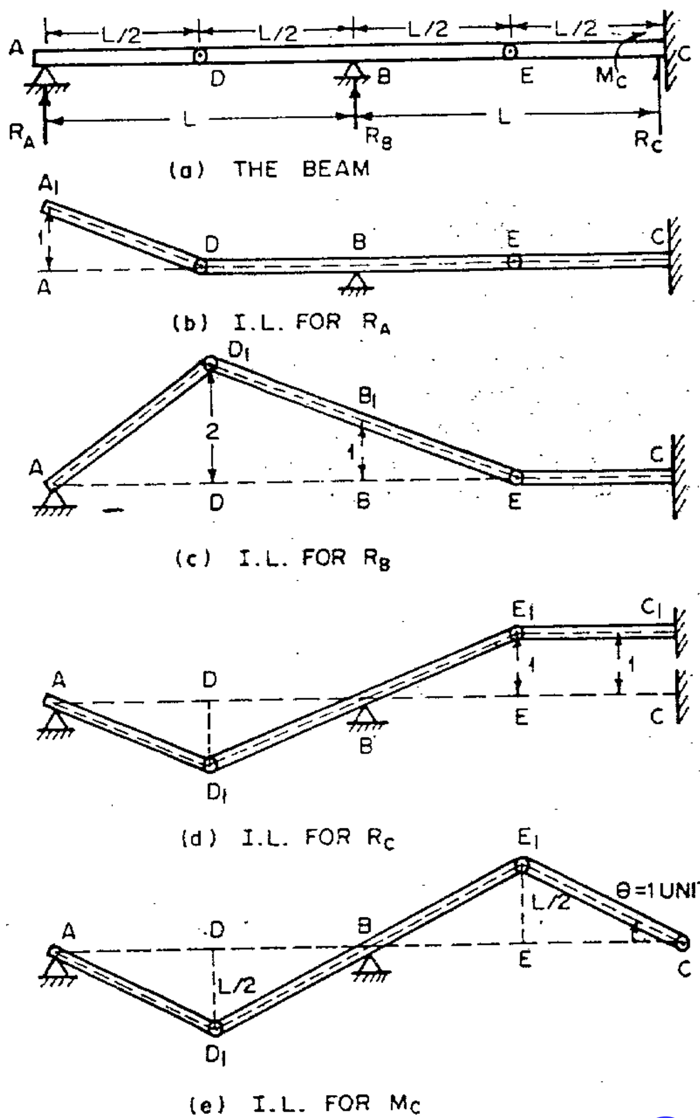


Fig. 5·4

of a hinge at D, only the portion AD will be lifted up, and the remaining portion will remain horizontal. This suggests that when the unit load crosses D, R_A will be zero and will continue to remain zero as the unit load moves along $DBEC$

(*ii*) **I.L. for R_B** : For R_B, lift the beam off the support B by unity. The beam will deflect as shown in Fig. 5·4(*c*), which will be I.L. for R_B. Ordinate DD_1 will evidently be equal $\frac{1}{L/2}\times L=2$. When the unit load crosses E, R_B will be zero.

(*iii*) **I.L. for R_C**. Lift the beam off the support C by unity. The beam will deflect as shown in Fig. 5·4 (*d*) which will be the I.L. for R_C, according to the Müller Breslau Principle.

Since $CC_1=1$, E will move to E_1 such that $EE_1=1$.

Hence, from geometry $DD_1=1$ in the negative direction. This suggests that when the unit load is between A to B, the reaction R_C will be negative *i.e.* R_C will act down wards).

(*iv*) **I.L. for M_C**. Let us assume M_C to be in clockwise direction. Hence in order to find I.L. for M_C, introduce a hinge at C and rotate the beam, at C, by $\theta=1$ unit. The beam will deflect as shown in Fig. 5·4 (*e*) which will evidently be the I.L. for M_C.

$$\text{Ordinate } EE_1=\frac{EC}{\theta}=\frac{L/2}{1}=\frac{L}{2}$$

Hence by geometry, $DD_1=\frac{L}{2}$.

When the unit load is between A to B, M_C will be negative, *i.e.* it will act in the counterclockwise direction. For the unit load positions between B and C, M_C will act in the clockwise direction, as marked in Fig 5·4 (*a*).

Example 5·2. *A two span beam ACDE has internal hinge at B and an overhang DE. Using Müller-Breslau influence theorem, draw influence lines for R_A, R_C, R_D, M_C, M_G, F_G.*

Solution.

(*i*) **I.L. for R_A**. For getting I.L. for R_A, lift the beam off the support A by unity. Due to internal hinge at B, only portion AB will be deflected, as shown in Fig. 5·5 (*b*) which is the I.L. for R_A. The reaction R_A remains zero for load positions between B to E.

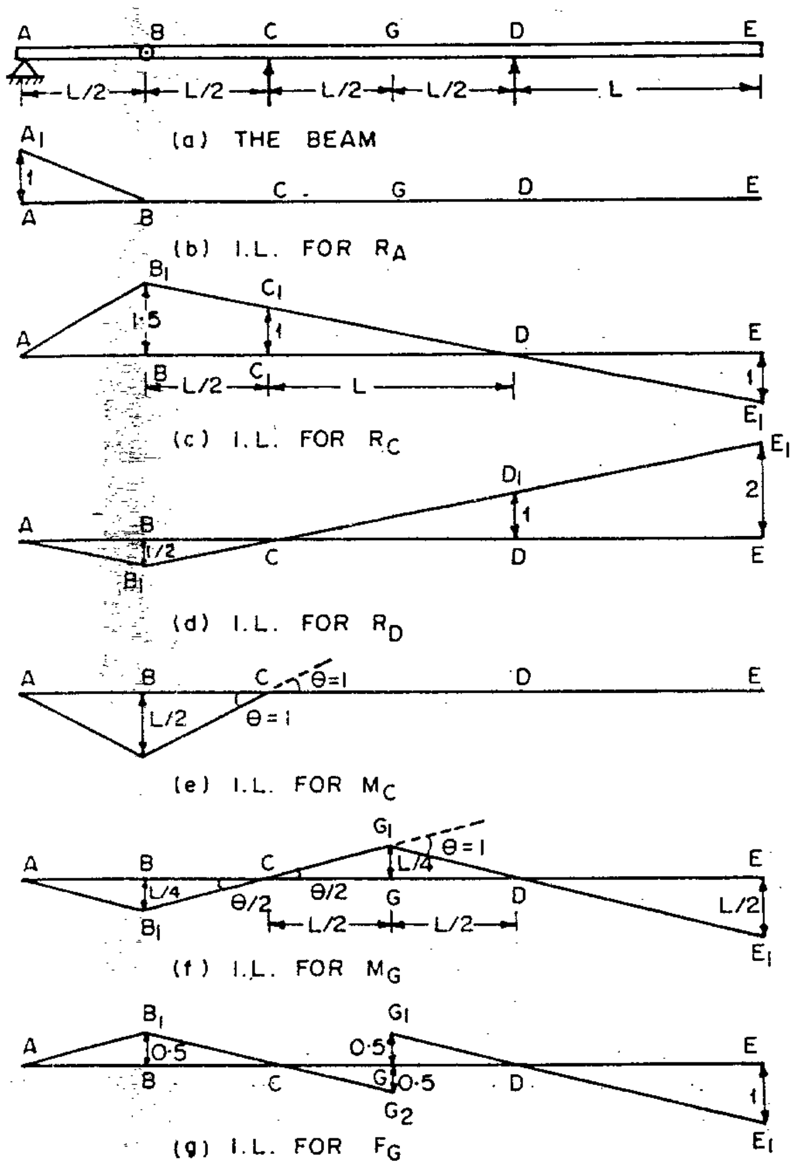


Fig. 5·5

(*ii*) **I.L. for** $\mathbf{R_C}$, Lift the beam off the support C by unity. The deflected shape, shown in Fig. 5·5(*c*), will be the I.L. for R_C as per Muller Breslau principle. The ordinate BB_1, is given by

$$BB_1 = \frac{CC_1}{CD} \times BD = \frac{1}{L} \times \frac{3}{2} L = 1{\cdot}5 \text{ (positive).}$$

Similarly $\quad EE_1 = CC_1 = 1$ (negative)

(*iii*) **I.L. for** $\mathbf{R_D}$. For getting I.L. for R_D, lift the beam off the support D by unity. The deflected shape of the beam, shown in Fig. 5·5 (*d*) will be the I.L. for R_D, where in

$$BB_1 = \frac{DD_1}{CD} \times BC = \frac{1}{L} \times \frac{L}{2} = \frac{1}{2} \text{ (negative)}$$

and
$$EE_1 = \frac{DD_1}{CD} \times CE = \frac{1}{L} \times 2L = 2 \text{ (positive)}$$

(*iv*) **I.L. for** $\mathbf{M_C}$. In order to get I.L. for M_C, introduce a hinge at C and rotate the beam by θ=unity, in the anticlockwise direction, assuming that M_C acts in the counter-clockwise direction. The deflected shape, shown in Fig. 5·5 (*e*) will be the I.L. for M_C, wherein, ordinate $BB_1 = BC(\theta) = \frac{L}{2}$ (negative). This means that when the load is in AC, the moment M_C will be in the *clockwise* direction. For unit load positions between C and E, M_C will *remain zero*.

(*v*) **I.L. for** $\mathbf{M_G}$. Introduce a hinge at G and permit unit relative rotation of the parts on opposite sides of G. The deflected shape of the beam, shown in Fig. 5·5 (*f*) will be I.L. for M_G, in which ordinate $BB_1 = \frac{L}{4}$ (negative) and ordinate $EE_1 = \frac{L}{2}$ (negative) while ordinate $GG_1 = \frac{L}{4}$.

(*vi*) For I.L. for F_G, apply a cut in the beam at G and give relative displacements of the two parts by unity. The deflected shape of the beam, shown in Fig. 5·5 (*g*) will give I.L. for F_G. Since G is situated midway between C and D, $GG_1 = GG_2 = 0{\cdot}5$. Evidently, BB_1 will be $+0{\cdot}5$ and EE_1 will be -1.

5·4. PROPPED CANTILEVERS

1. I.L. for prop reaction

In § 5·2, we have seen the application of Muller-Breslau principle for constructing influence line for vertical reaction component of a continuous beam. We shall now apply the principle for drawing I.L. for reaction R_B at the prop.

Let the unit load be at section X (Fig. 5·6*a*). According to the Muller-Breslau principle, remove the prop at B. The beam will deflect

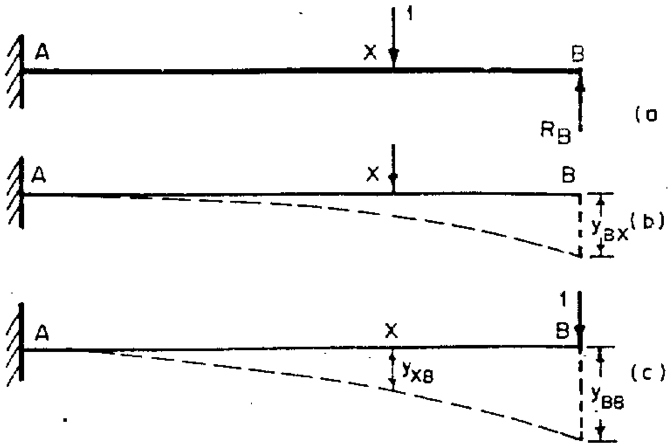


Fig. 5·6

as shown in Fig. 5·6 (*b*), in which y_{BX} is the deflection of *B* due to unit load at *X*. Now remove the unit load from *X* and place it at *B*. The beam will deflect as shown in Fig. 5·6 (*c*), in which y_{XB} is the deflection at *X* due to unit load at *B*, and y_{BB} is the deflection at *B* due to unit load at *B*. However, since the support at *B* is at the same level as *A*, the upward deflection at *B* due to R_B is to neutralise the downward deflection y_{BX}. Hence, we get from consistent deformation (Chapter 7).

$$R_B \, . \, y_{BB} = y_{BX}$$

By Maxwell's reciprocal theorem (§ 7·3)

$$y_{BX} = y_{XB}$$

$$\therefore \qquad R_B = \frac{y_{XB}}{y_{BB}} \qquad \ldots(5{\cdot}2)$$

Thus, the reaction at *B*, due to unit load at any point *X* is proportional to the deflection at point *X* due to the unit load acting at *B*. In other words, the deflection curve shown in Fig. 5·6 (*c*), represents, to some scale, the influence line for R_B.

If the deflection y_{BB} in the direction of unit load at *B* is selected as unity, the deflection curve will directly give the I.L. for R_B,

2. I.L. for M_A

Let the unit load be at *X*. Remove the fixed support at *A* and introduce a hinge. The beam will deflect as shown in Fig. 5·7 (*b*), when unit load is applied at *X*. Let ϕ'_{AX} be the rotation at *A* due to unit load at *X*. Now, remove the unit load at *X* and apply unit

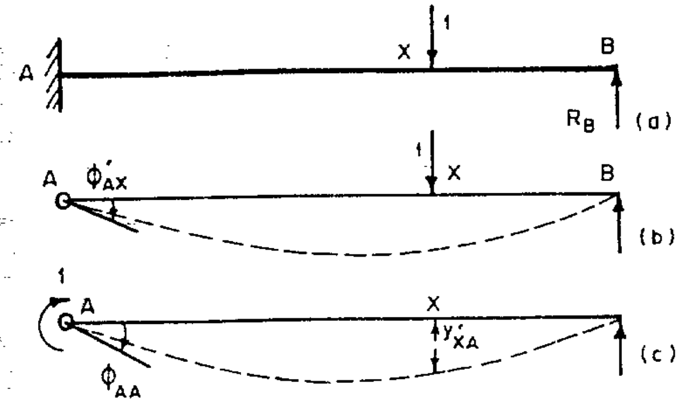


Fig. 5·7

moment at *A*. The beam will deflect as shown in Fig. 5·7 (*c*), where y'_{XA} = deflection at section *X* due to unit moment at *A* and ϕ_{AA} = rotation at *A* due to unit moment at *A*.

From method of consistent deformation,

$$M_A \, . \, \phi_{AA} = \phi'_{AX}$$

But from reciprocal theorem,

$$\phi'_{AX} = y'_{XA}$$

$$\therefore \qquad M_A \, . \, \phi_{AA} = y'_{XA}$$

$$\text{or} \qquad M_A = \frac{y'_{XA}}{\phi_{AA}} \qquad (5{\cdot}3)$$

The above relation suggests that M_A is proportional to y'_{XA}. In other words, the deflected curve of Fig. 5·7 (*c*) gives, to some scale, the influence line for M_A. If ϕ_{AA} is selected as unity, the deflection curve will directly give the I.L. for M_A.

3. I L. for M_D (Fig. 5·8 *a*)

Let the unit load be at *X*. We want to plot the influence line for bending moment M_D at the point *D*. According to Muller-Breslau principle, the internal stress component, for which the influence line is to be plotted is first removed. For the present case, this is accomplished by inserting a pin at *D*. The beam will then deflect under the unit load at *X*, as shown in Fig. 5·8 (*b*).

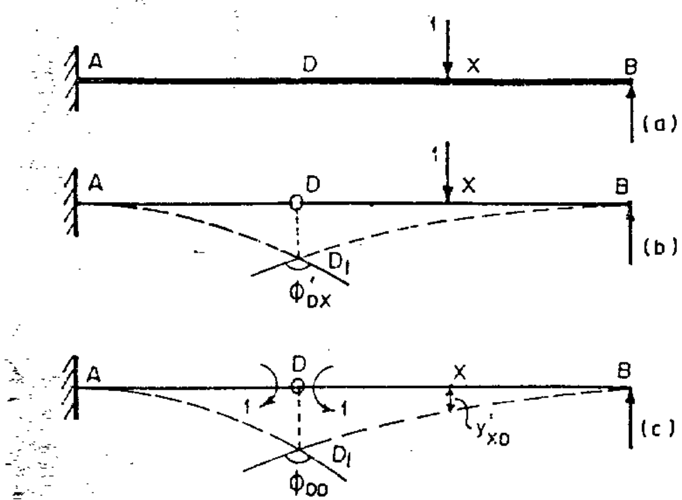


Fig. 5·8

Let ϕ'_{DX} be the rotation at D, due to unit load at X. Now remove the unit load from X, and apply a pair of unit moments at D, as shown in Fig. 5·8 (*c*). Let ϕ_{DD} be the resulting rotation at D and Y'_{XD} be the resulting deflection at X. Then, from method of consistent deformation,

$$M_D \cdot \phi_{DD} = \phi'_{DX}$$

But from reciprocal theorem,

$$\phi'_{DX} = Y'_{XD}$$

$$\therefore \quad M_D \cdot \phi_{DD} = Y'_{XD}$$

or
$$M_D = \frac{Y'_{XD}}{\phi_{DD}} \qquad (5·4)$$

The above relation suggests that M_D is proportional to Y'_{XD}. In other words, the deflected curve of Fig. 5·8 (*c*) gives to some scale, the influence line for M_D. If ϕ_{DD} is selected as unity, the deflection curve will directly give the I.L. for M_D.

4. I.L. for F_D

Let us now plot the I.L. for shear at D. Let the unit load be at X (Fig. 5·9 *a*). In order to remove the internal stress component, *i.e.* shear F_D at D, assume that the beam is cut at D and that a slide device is inserted in such a way that it permits relative transverse deflection between the two parts of the cut, as shown in Fig. 5·9(*b*), but which at the same time, maintains a common slope at both the ends of the cut. Let Y_{DX} be the relative linear deflection at D due to unit load at X.

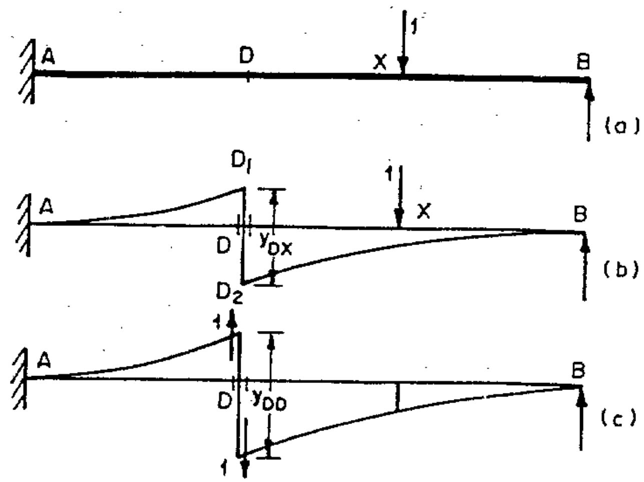


Fig. 5·9.

Now remove the unit load from X, and apply a pair of unit loads (shear) at D. The beam will deflect as shown in Fig. 5·9 (*c*), where Y_{DD} is the relative linear deflection at D due to pair of unit loads (shear) at D, and Y_{XD} be the corresponding deflection at X.

Now from consistent deformation,

$$F_D \cdot Y_{DD} = Y_{DX}$$

But from reciprocal theorem,

$$Y_{DX} = Y_{XD}$$

$$\therefore \quad F_D \cdot Y_{DD} = Y_{XD}$$

or
$$F_D = \frac{Y_{XD}}{Y_{DD}} \qquad ...(5·5)$$

Thus, F_D is proportional to Y_{XD}. In other words, the deflection curve of Fig. 5·9 (*c*) gives the I.L. for F_D, to some scale. If, however, Y_{DD} is taken as unity, the deflection at any point X of Fig. 5·9(*c*) gives the shear at D due to unit vertical load at X.

5·5. CONTINUOUS BEAM : INFLUENCE LINE FOR BENDING MOMENT

Article 5·2 illustrates the application of Müller-Breslau principle for constructing influence line for a vertical reaction component of a continuous beam. We shall now apply the principle for drawing the influence line for bending moment at any point D of a continuous beam shown in Fig. 5·10 (*a*).

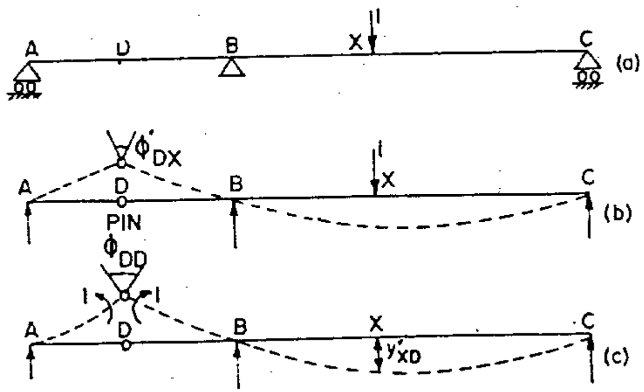


Fig. 5·10
Influence line for M_D

Let the unit load roll on the span ABC. At any instant, let it be at a point X. We want to plot the influence line for bending moment M_D at the point D. According to Müller-Breslau principle the internal stress component, for which the influence line is to be plotted, is first removed. For the present case, this is accomplished by inserting a pin at D. The beam will then deflect, under the unit load at X, as shown in Fig. 5·10(b). Let ϕ'_{DX} be the rotation at D, due to unit load at X. Now remove the unit load from X, and apply a pair of unit moments at D, as shown in Fig. 5·10 (c).

Let ϕ_{DD}=rotation at D, due to unit couple at D

y'_{XD}=deflection at X due to unit couple at D.

From method of consistent deformation (Chapter 7), we have

$$M_D \cdot \phi_{DD}=\phi'_{DX}$$

where M_D=bending moment at D due to unit load at X.

$$\therefore \quad M_D=\frac{\phi'_{XD}}{\phi_{DD}}$$

But from reciprocal theorem,

$$\phi'_{DX}=y'_{XD} \qquad \text{(see Eq. 7·7)}$$

$$\therefore \quad M_D=\frac{y'_{XD}}{\phi_{DD}} \qquad \ldots(5\cdot6)$$

Eq. 5·6 suggests that M_D is proportional to y'_{XD}. In other words, the deflection curve of Fig. 5·10 (c) gives, to some scale, the influence line for M_D. If however, ϕ_{DD} is selected to be unity, the deflection at any point X will give the bending moment at D. Eq. 5·6, therefore, further proves the validity of the Muller-Breslau principle applied to the influence line for bending moment.

5·6. CONTINUOUS BEAM : INFLUENCE LINE FOR SHEAR FORCE

Let us now study the applicability of the Müller-Breslau principle for plotting the influence line for the shear force at any point D of a continuous beam ABC shown in Fig. 5·11. Let the unit load be at any point X. In order to remove the internal stress components, *i.e.* shear F_D at D, assume that the beam is cut at D and that a slide device inserted in such a way that it permits relative transverse deflection between the two parts of the cut, as shown in Fig. 5·11 (b), but which at the same time, maintains a common slope at both the ends of the cut.

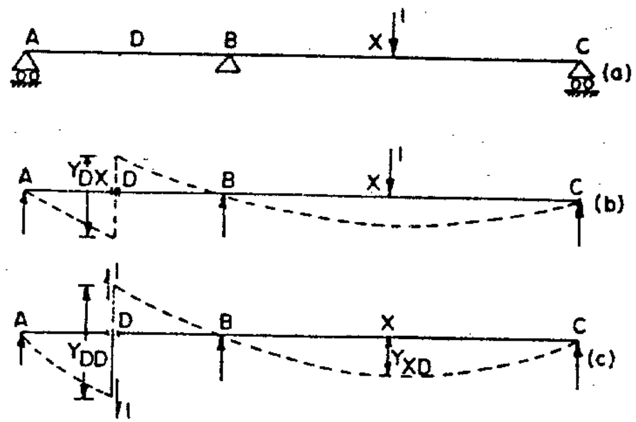


Fig. 5·11
Influence line for S.F. at D

Now remove the unit load from X, and apply a pair of unit loads (shear) at D. The beam will deflect as shown in 5·11 (c).

Let F_D=shear force at D.

Y_{DX}=relative linear deflection at D due to unit load at X.

Y_{DD}=relative linear deflection at D due to pair of unit load (shear) at D.

Y_{XD}=deflection at X due to pair of unit load (shear) at D.

Then, from compatibility of deformation at D, we have

$$F_D \cdot Y_{DD}=Y_{DX}$$

$$F_D=\frac{Y_{DX}}{Y_{DD}}$$

But from the reciprocal theorem,

$$Y_{DX}=Y_{XD}$$

$$\therefore \qquad F_D=\frac{Y_{XD}}{Y_{DD}} \qquad (5\cdot7)$$

Thus, F_D is proportional to Y_{XD}. In other words, the deflection curve of Fig. 5·11 (*c*) gives the influence line for shear at *D*, to some scale. If, however, Y_{DD} is taken as unity, the deflection at any point *X* of Fig. 5·11 (*c*) give the shear at *D* due to unit vertical load at *X*.

5·7. INFLUENCE LINE FOR HORIZONTAL REACTION

Let us now study the influence line for horizontal reaction at the hinged end *A* of a frame shown in Fig. 5·12(*a*). The unit vertical load can travel on *BC*, or unit horizontal load can travel on *AB*. Let us first take the case when the unit vertical load travel on *BC*.

According to the Muller-Breslau principle, the reaction component at *Q* is first removed. This is accomplished by supporting end *A* on rollers. The end *A* will deflect horizontally by Δ_{AX} due to unit vertical load at *X*, as shown in Fig. 5·12 (*b*). The unit vertical load at *X* is then removed and a unit horizontal load is applied at *A*. The frame will deflect as shown in Fig. 5·12(*c*).

Let $\quad \Delta_{AA}$=Horizontal deflection of *A* due to horizontal unit load at *A*.

Δ_{XA}=Vertical deflection at *X* due to unit horizontal load at *A*

H_A=Horizontal reaction at *A*

Then $\qquad H_A\Delta_{AA}=\Delta_{AX}$

or $$H_A=\frac{\Delta_{AX}}{\Delta_{AA}}$$

But $\qquad \Delta_{AX}=\Delta_{XA}$, from reciprocal theorem

Hence $$H_A=\frac{\Delta_{XA}}{\Delta_{AA}} \qquad (5\cdot8)$$

Eq. 5·8 shows that the deflection curve of *BC* [Fig. 5·12 (*c*)] gives the influence line, to some scale for H_A when a unit vertical load moves on *BC*. Similarly, it can be shown that the deflection curve of *AB* gives the influence line for H_A when a unit horizontal load moves on *AB*.

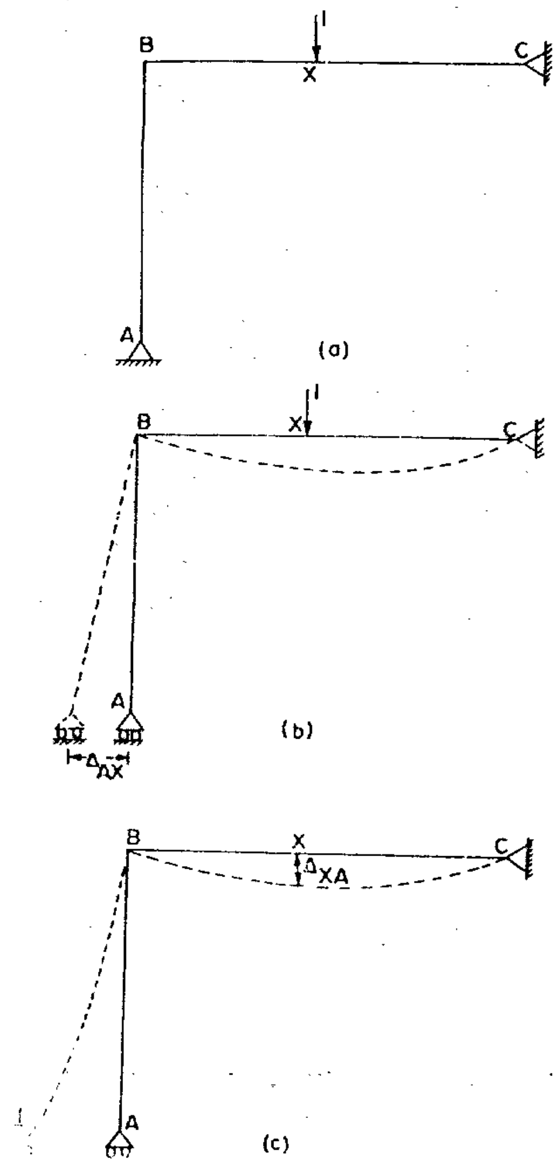


Fig. 5·12
Influ ce line for horizontal reaction at A.

Example 5·3. *Draw the influence lines for (i) reaction at B and (ii) moment at A for the propped cantilever shown in Fig. 5·13 (a). Compute the ordinates at intervals of 1·25 m.*

Solution

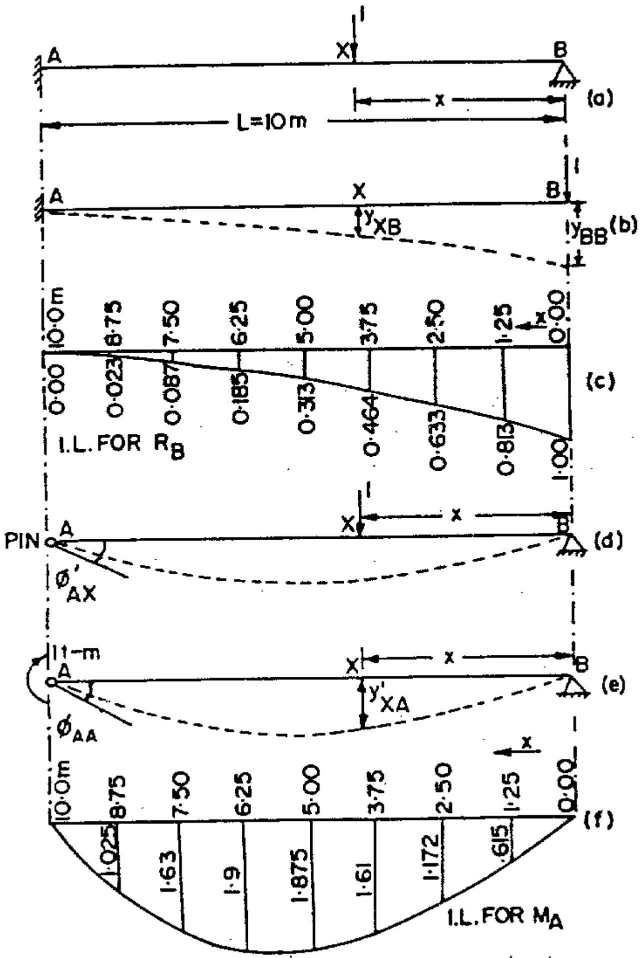


Fig. 5·13

(a) *I.L. for R_B*

From Eq. 5·2, $$R_B=\frac{y_{XB}}{y_{BB}} \quad (1)$$

To compute y_{XB} and y_{BB}, apply a unit vertical load at B as shown in Fig. 5·13 (b).

At any section X distant x from B, we have

$$EI\frac{d^2y}{dx^2}=1\,.\,x$$

Integrating, $$EI\frac{dy}{dx}=\frac{x^2}{2}+C_1$$

At $x=L$, $\frac{dy}{dx}=0$; $\therefore\ C_1=-\frac{L^2}{2}$

Hence $$EI\frac{dy}{dx}=\frac{x^2}{2}-\frac{L^2}{2}$$

Integrating further,

$$EIy=\frac{x^3}{6}-\frac{L^2}{2}x+C_2$$

At $x=L$, $y=0$, $\therefore\ C_2=\frac{L^3}{2}-\frac{L^3}{6}=\frac{L^3}{3}$

Hence $$EIy=\frac{x^3}{6}-\frac{L^2}{2}x+\frac{L^3}{3} \quad (2)$$

At $x=0$, $y=y_{BB}=\frac{L^3}{3EI}$

At $x=x$, $y=y_{XB}=\frac{1}{EI}\left(\frac{x^3}{6}-\frac{L^2}{2}x+\frac{L^3}{3}\right)$

Substituting these in (1), we get

$$R_B=\left(\frac{x^3}{6}-\frac{L^2}{2}x+\frac{L^3}{3}\right)\frac{3}{L^3}$$

or $$R_B=\frac{1}{2}\left(\frac{x^3}{L^3}-\frac{3x}{L}+2\right)$$

or $$R_B=\frac{1}{2}\left(n^3-3n+2\right),\ \text{where } \frac{x}{L}=n.$$

The ordinates of I.L. for R_B are computed in Table 5·1.

TABLE 5·1

x (m)	0	1·25	2·50	3·75	5	6·25	7·5	8·75	10
$n=\frac{x}{L}$	0	0·125	0·25	0·375	0·5	0·625	0·75	0·875	1
R_B	1	0·813	0·633	0·464	0·313	0·185	0·087	0·023	0

The I.L. for R_B is shown in Fig. 5·9(c)

(b) *I.L. for* M_A

In order to draw the I.L. for M_A, replace the fixed support at A by a pin, as shown in Fig. 5·13(d). Remove the external unit load and apply a unit couple at A, as shown in Fig. 5·13(e). Then from Eq. 5·3.

$$M_A=\frac{y'_{XA}}{\phi_{AA}} \qquad (3)$$

where y'_{XA}=vertical deflection at X due to unit couple at A

ϕ_{AA}=slope at A due to unit couple at A,

Let R'_B=Reaction at B, when unit moment is acting at A

$$=\frac{1}{L}\uparrow$$

$$\therefore \quad EI\frac{d^2y}{dx^2}=-R'_B\,.\,x=-\frac{x}{L}$$

$$EI\frac{dy}{dx}=-\frac{x^2}{2L}+C_1$$

and
$$EIy=-\frac{x^3}{6L}+C_1x+C_2$$

At
$$x=0,\ y=0. \quad \therefore \quad C_2=0$$

$$x=L,\ y=0,\ \therefore\ C_1=\frac{L}{6}$$

Hence
$$EI\frac{dy}{dx}=-\frac{x^2}{2L}+\frac{L}{6} \qquad (i)$$

and
$$EIy=-\frac{x^3}{6L}+\frac{Lx}{6} \qquad (ii)$$

At $x=L$,
$$\frac{dy}{dx}=\phi_{AA}=\frac{1}{EI}\left(-\frac{L^2}{2L}+\frac{L}{6}\right)=-\frac{L}{3} \qquad (4)$$

At $x=x$,
$$y=y'_{XA}=\frac{1}{EI}\left(-\frac{x^3}{6L}+\frac{Lx}{6}\right) \qquad (5)$$

Substituting these values in (3), we get

$$M_A=\left(\frac{x^3}{6L}-\frac{Lx}{6}\right)\times\frac{3}{L}=\frac{1}{2}\left(\frac{x^3}{L^2}-x\right)$$

This is thus the equation of the influence line for M_A. The ordinates are calculated in the tabular form in Table 5·2.

The minus sign shows that the direction of M_A is in reverse direction to that of the unit moment applied at A, *i.e.*, M_A act in anti-clockwise direction. The I.L. for M_A is shown in Fig. 5·13(f).

TABLE 5·2

x (m)	0	1·25	2·5	3·75	5	6·25	7·5	8·75	10
$\frac{x^3}{L^2}$	0	0·0195	0·156	0·53	1·25	2·45	4·24	6·7	10·0
M_A	0	−0·615	−1·172	−1·61	−1·875	−1·9	−1·63	−1·025	0

Example 5·4. *Determine the influence line for* R_A *for the continuous beam shown in Fig. 5·14. Compute the ordinates at every 1 m interval.*

Solution

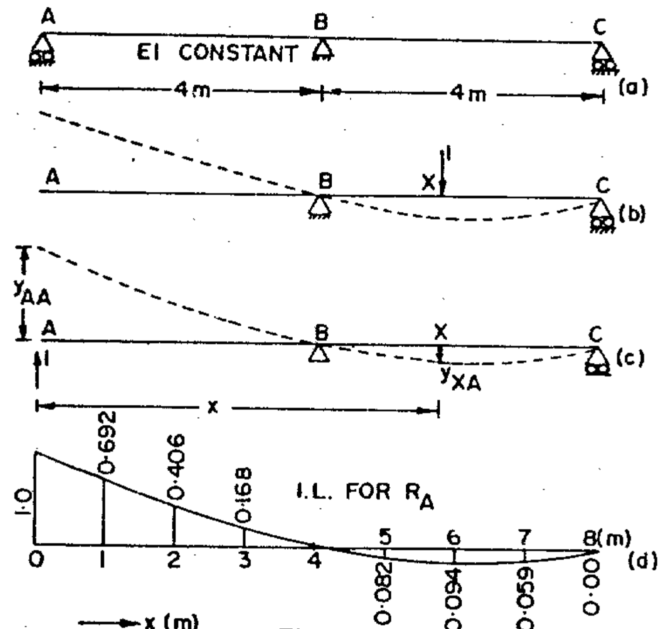


Fig. 5·14.

Apply a unit vertical load at A, as shown in Fig. 5·14(c). Then

$$R_A=\frac{y_{XA}}{y_{AA}}$$

From Fig. 5·10(c), $R_B=2\downarrow$ and $R_C=1\uparrow$

$$EI\frac{d^2y}{dx^2}=-1\times x \ \vdots\ +R_B(x-4)=-x\ \vdots\ +2(x-4)$$

$$\therefore \quad EI\frac{dy}{dx}=-\frac{x^2}{2}+C_1\ \vdots\ +(x-4)^2$$

and $$EIy=-\frac{x^3}{6}+C_1x+C_2 \;\vdots\; +\frac{(x-4)^3}{3}$$

At $x=4$, $y=0$, $\therefore\; 4C_1+C_2=\frac{64}{6}$

At $x=8$, $y=0$, $\therefore\; 8C_1+C_2=64$

$\therefore\; C_1=\frac{40}{3}$ and $C_2=-\frac{128}{3}$

$$\therefore\; EI\,y=-\frac{x^3}{6}+\frac{40}{3}x-\frac{128}{3} \;\vdots\; +\frac{(x-4)^3}{3}$$

At $x=0$, $y=y_{AA}=\frac{1}{EI}\left(-\frac{128}{3}\right)=-\frac{128}{3EI}$

At $x=x$, $y=y_{XA}=\left[-\frac{x^3}{6}+\frac{40}{3}x-\frac{128}{3} \;\vdots\; +\frac{(x-4)^3}{3}\right]\frac{1}{EI}$

The values of y_{XA}, for various values of x are given in table 5·3. The term $\frac{1}{EI}$ has been omitted for convenience.

TABLE 5·3

x (m)	y_{XA}	$R_A=\frac{y_{XA}}{y_{AA}}$
0	$0+0-\frac{128}{3}$	1
1	$-\frac{1}{6}+\frac{40}{3}-\frac{128}{3}=-\frac{177}{6}$	+0·692
2	$-\frac{8}{6}+\frac{80}{3}-\frac{128}{3}=-\frac{104}{6}$	+0·406
3	$-\frac{27}{6}+\frac{120}{3}-\frac{128}{3}=-\frac{43}{6}$	+0·168
4	$-\frac{64}{6}+\frac{160}{3}-\frac{128}{3}=0$	+0·00
5	$-\frac{125}{6}+\frac{200}{3}-\frac{128}{3}+\frac{1}{3}=+\frac{21}{6}$	−0·082
6	$-\frac{216}{6}+\frac{240}{3}-\frac{128}{3}+\frac{8}{3}=\frac{24}{6}$	−0·094
7	$-\frac{343}{6}+\frac{280}{3}-\frac{128}{3}+\frac{27}{3}=+\frac{15}{6}$	−0·059
8	$-\frac{512}{6}+\frac{320}{3}-\frac{128}{3}+\frac{64}{3}=0$	0

The I.L. for R_A is shown in Fig. 5·14(d). By inspection, the plus sign shows that R_A acts downward (↓) while minus sign shows that R_A acts upwards (↑).

Example 5·5. *Determine the influence line for the bending moment at D, the middle point of span BC, of a continuous beam shown in Fig. 5·15(a). Compute the ordinates at 1 m interval.*

Solution

In order to draw the I.L. for M_D, consider a pin at D. The beam will deflect under the unit load as shown in Fig. 5·15(b). Remove the external unit load and apply a pair of unit couples at

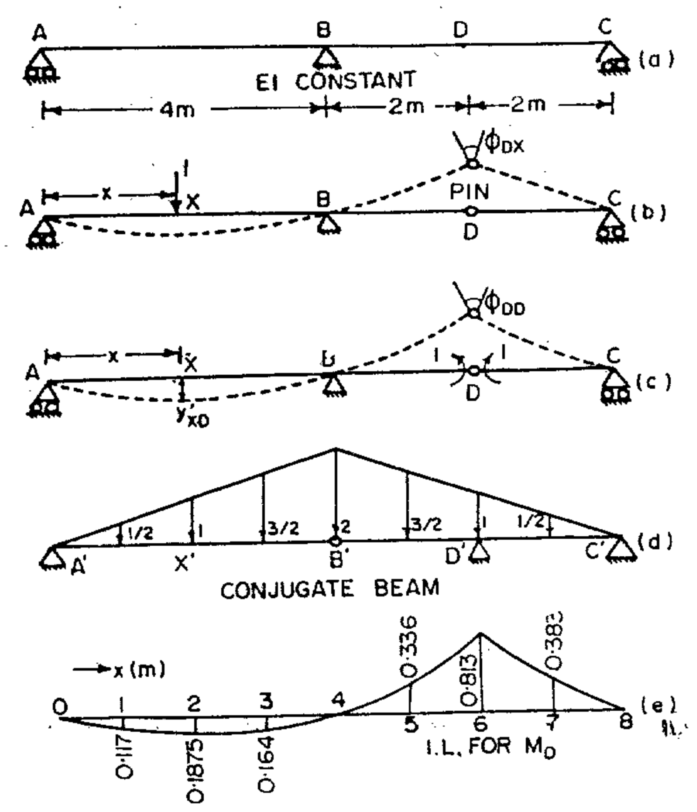


Fig. 5·15.

D as shown in Fig. 5·15(c). The bending moment M_D is then given by

$$M_D=\frac{y_{XD}}{\phi_{DD}} \qquad (i)$$

where y'_{XD} = Deflection at any section X due to unit couple at D.

ϕ_{DD} = Rotation at D, due to unit couple at D.

From Fig. 5·15(*c*) considering the equilibrium of the portion to the right of the pin,

$$M_D = 0 = (R_C \times 2) - 1 \circlearrowleft$$

or $$R_C = \tfrac{1}{2} \uparrow$$

Similarly, considering the equilibrium of the left portion,

$$M_D = 0 = R_A \times 6 \uparrow - R_B \times 2 - 1 \circlearrowright \qquad (i)$$

or $$6R_A = 2R_B + 1$$

Also, for the whole beam,

$$R_A \uparrow + R_C \uparrow = R_B \downarrow$$

$$\therefore \quad R_A + \tfrac{1}{2} = R_B$$

From (*i*) and (*ii*) $R_A = \frac{1}{2} \uparrow$ and $R_B = 1 \downarrow$ (*ii*)

Knowing all the three reactions, the bending moment at any section of the beam can be determined. The B.M.D. is a triangle having a maximum ordinate of -2 kN-m at B. In order to find y'_{XD} (*i.e.* to determine the deflection curve), we shall use the *conjugate beam method*. Fig. 5·15(*d*) shows a corresponding conjugate beam loaded with $-M/EI$ diagram. Thus, since $M_{max} = -2$ kN-m, the loading on the conjugate beam will be triangular, having a maximum intensity of $+2$ (*i.e.*, acting downwards) at B, EI being omitted. Since the real beam [Fig. 5·15(*c*)] has a zero deflection at B, the conjugate beam will have a pin at the corresponding point B' so that the B.M. there, representing deflection at B, is zero. Let us first determine the reactions R_A', R_D' and R_C' of the conjugate beam. Taking moments about the pin B' and considering the equilibrium of the left portion, we get

$$R_A' = \frac{1}{4}\left(\frac{1}{2} \times 4 \times 2 \times \frac{4}{3}\right) = \frac{4}{3} \uparrow$$

Considering the equilibrium of the right portion, we have

$$2R_D' + 4R_C' = \frac{1}{2} \times 4 \times 2 \times \frac{4}{3}$$

or $$R_D' + 2R_C' = \frac{8}{3} \qquad (i)$$

Also, $$R_A' + R_D' + R_C' = \frac{1}{2} \times 8 \times 2 = 8$$

or $$R_D' + R_C' = 8 - R_A' = 8 - \frac{4}{3} = \frac{20}{3} \qquad (ii)$$

From (*i*) and (*ii*), we get

$$R_D' = \frac{32}{3} \uparrow \text{ and } R_C' = -4, \textit{ i.e. } \text{acting} \downarrow$$

Now y'_{XD} of real beam $= M_X$ of conjugate beam

and ϕ_{DD} = relative change in angle at D

= sum of shears in conjugate beam to the right and left of the support D

$$= R_D' = \frac{32}{3}.$$

The calculation of M_X (and hence y'_{XD}) and M_D are done in the tabular form below. (Table 5·4)

TABLE 5·4

x (*m*)	$M_X = y'_{XD}$	$M_D = \dfrac{y'_{XD}}{\phi_{DD}}$
1	$\left(-\frac{4}{3} \times 1\right) + \left(\frac{1}{2} \times 1 \times \frac{1}{2} \times \frac{1}{3}\right) = -\frac{15}{12}$	$-\frac{15}{12} \times \frac{3}{32} = -0.177$
2	$\left(-\frac{4}{3} \times 2\right) + \left(\frac{1}{2} \times 2 \times 1 \times \frac{2}{3}\right) = -2$	$-\frac{2 \times 3}{32} = -0.1875$
3	$\left(-\frac{4}{3} \times 3\right) + \left(\frac{1}{2} \times 3 \times \frac{3}{2} \times \frac{3}{3}\right) = -\frac{7}{4}$	$-\frac{7}{4} \times \frac{3}{32} = -0.164$
4	$\left(-\frac{4}{3} \times 4\right) + \left(\frac{1}{2} \times 4 \times 2 \times \frac{4}{3}\right) = 0$	0
5	$(4 \times 3) - \left(\frac{32}{3} \times 1\right) + \left(\frac{1}{2} \times 3 \times \frac{3}{2} \times \frac{3}{3}\right) = \frac{43}{12}$	$\frac{43}{12} \times \frac{3}{32} = 0.336$
6	$(4 \times 2) + \left(\frac{1}{2} \times 2 \times 1 \times \frac{2}{3}\right) = \frac{26}{3}$	$\frac{26}{3} \times \frac{3}{32} = 0.813$
7	$(4 \times 1) + \left(\frac{1}{2} \times 1 \times \frac{1}{2} \times \frac{1}{3}\right) = \frac{49}{12}$	$\frac{49}{12} \times \frac{3}{22} = 0.383$
8	0	0

The influence line diagram for M_D is shown plotted in Fig. 5·15(*d*).

Alternative Solution

The deflection curve or the value of y'_{XD} can also be determined by the conventional double integration method used in example 5·3.

Refer Fig. 5·15(*c*), where $R_A = \frac{1}{2} \uparrow$, $R_B = 1 \downarrow$ and $R_C = \frac{1}{2} \uparrow$. Since there is discontinuity in the beam at the pin at D, we will treat the spans AD and DC separately.

(*i*) *For the span AD*,

Measuring x from the L.H. support A, we get

$$EI\frac{d^2y}{dx^2}=-\frac{x}{2}\ \vdots\ +(x-4)$$

$$\therefore\quad EI\frac{dy}{dx}=-\frac{x^2}{4}+C_1\ \vdots\ +\frac{(x-4)^2}{2}$$

and $$EIy=-\frac{x^3}{12}+C_1x+C_2\ \vdots\ +\frac{(x-4)^3}{6}$$

At $x=0$, $y=0$ $\therefore C_2=0$

At $x=4$, $y=0=-\dfrac{64}{12}+4C_1$

$$\therefore\ C_1=\frac{64}{12\times4}=\frac{4}{3}$$

Hence the slope and deflection equations for span AD are :

$$EI\frac{dy}{dx}=-\frac{x^2}{4}+\frac{4}{3}\ \vdots\ +\frac{(x-4)^2}{2} \qquad \text{(I)}$$

and $$EIy=-\frac{x^3}{12}+\frac{4}{3}x\ \vdots\ +\frac{(x-4)^3}{3} \qquad \text{(II)}$$

At $x=6$ m. $EIy=-\dfrac{216}{12}+\dfrac{24}{3}+\dfrac{8}{6}=-\dfrac{26}{3}$

and $$\left(EI\frac{dy}{dx}\right)_{DA}=-9+\frac{4}{3}+2=-\frac{17}{3}.$$

(*ii*) *For the span DC*

Shear at pin D, just to its right, is equal to $\frac{1}{2}\downarrow$ (*i.e.* shear at D is equal and opposite to Rc).

Measuring x from D, to right

$$EI\frac{d^2y}{dx^2}=\frac{1}{2}\,.\,x-1$$

$$EI\frac{dy}{dx}=\frac{x^2}{4}-x+C_1$$

and $$EIy=\frac{x^3}{12}-\frac{x^2}{2}+C_1x+C_2$$

At $x=0$, $EIy=-\dfrac{26}{3}$

$$\therefore\quad C_2=-\frac{26}{3}.$$

At $x=2$ m, $EIy=0=\dfrac{8}{12}-\dfrac{4}{2}+2C_1-\dfrac{26}{3}$

$$\therefore\quad C_1=+5.$$

Hence the slope and deflection equations for span DC are :

$$EI\frac{dy}{dx}=\frac{x^2}{4}-x+5 \qquad \text{(III)}$$

and $$EIy=\frac{x^3}{12}-\frac{x^2}{2}+5x-\frac{26}{3} \qquad \text{(IV)}$$

At $x=0$, $\left(EI\dfrac{dy}{dx}\right)_{DC}=+5$

Now, ϕ_{DD}=relative change in this angle at D

$$=\left(\frac{dy}{dx}\right)_{DA}-\left(\frac{dy}{dx}\right)_{DC}$$

$$=\frac{1}{EI}\left[-\frac{17}{3}-5\right]=-\frac{1}{EI}\ \frac{32}{3}.$$

The calculation of y'_{XD} is done in table 5·5. The term $\dfrac{1}{EI}$ has been omitted for convenience.

TABLE 5·5

Distance from A (m)	Eq. No.	y'_{XD}	$M_D=\dfrac{y'_{XD}}{\phi_{DD}}$
0	II ($x=0$)	0	0
1	II ($x=1$)	$-\frac{1}{12}+\frac{4}{3}=\frac{15}{12}$	$-\frac{15}{12}\times\frac{3}{12}=-0{\cdot}117$
2	II ($x=2$)	$-\frac{8}{12}+\frac{8}{3}=2$	$-2\times\frac{3}{32}=-0{\cdot}1875$
3	II ($x=3$)	$-\frac{27}{12}+\frac{12}{3}=\frac{7}{4}$	$-\frac{7}{4}\times\frac{3}{32}=-0{\cdot}164$
4	II ($x=4$)	$-\frac{64}{12}+\frac{16}{3}=0$	0
5	II ($x=5$)	$-\frac{125}{12}+\frac{20}{3}+\frac{1}{6}=-\frac{43}{12}$	$\frac{43}{12}\times\frac{3}{32}=0{\cdot}336$
6	IV ($x=6$)	$-\frac{216}{12}+\frac{24}{3}+\frac{8}{6}=-\frac{26}{3}$	$\frac{26}{3}\times\frac{3}{32}=0{\cdot}813$
7	IV ($x=1$)	$\frac{1}{12}-\frac{1}{2}+5-\frac{26}{3}=-\frac{49}{12}$	$\frac{49}{12}\times\frac{3}{32}=0{\cdot}383$
8	IV ($x=2$)	$\frac{8}{12}-\frac{4}{2}+10-\frac{26}{3}=0$	0

Example 5·6. *Determine the influence line for the shear force at D, the middle point of span BC, of a continuous beam shown in Fig. 5·16 (a). Compute the ordinates at 1 m interval.*

Solution

In order to plot the influence line for shear force F_D, the shearing resistance of the beam is first removed by inserting a sliding device which permits the relative movement between the two parts but does not impair the moment resistance. Thus the slide device is such that it maintains the same slope in the distorted beam to either side of the device. The unit load (external) is removed and a pair of the unit loads (unit shear) is applied at *D*. The beam will then distort as shown in Fig. 5·16(*c*). The S.F. at *D* is given by

$$F_D=\frac{Y_{XD}}{Y_{DD}} \qquad (1)$$

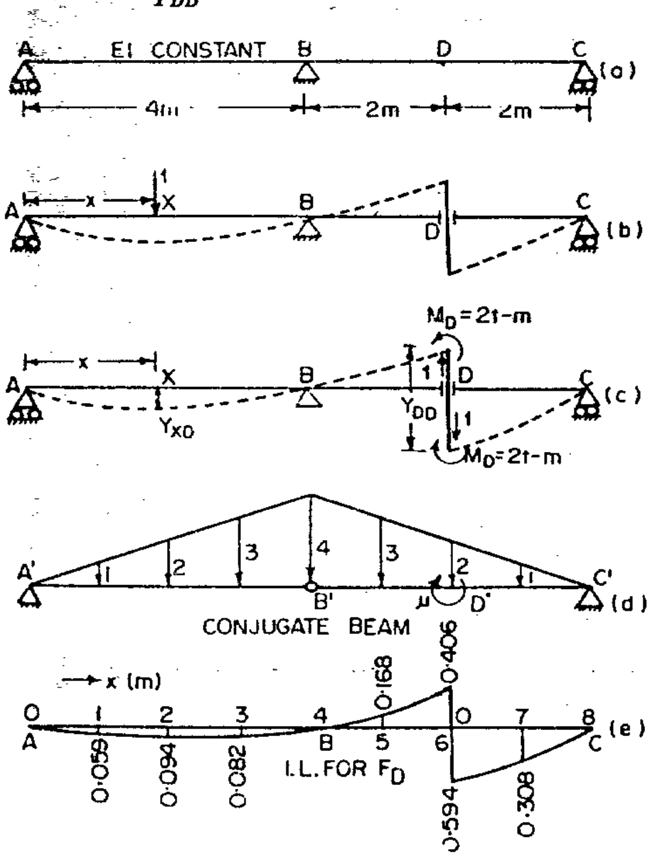


Fig. 5·16.

where Y_{XD}=deflection of beam at X due to unit shear at D.

Y_{DD}=total relative movement at D due to unit shear at D.

Considering the equilibrium of the portion DC to the right of the cut, we have

$$R_C=1\uparrow \text{ and } M_D=1\times2=2 \text{ kN-m}$$

Similarly, considering the equilibrium of the portion to the left of the cut, and taking moments about A, we get

$$4R_B=M_D+1\times6=2+6=8$$

$$R_B=2\downarrow$$

Hence $\quad R_A=2-1=1\uparrow$

The bending moment diagram will be a triangle having a maximum ordinate of -4 kN-m at B.

(*a*) Solution by conjugate beam method

Fig. 5·16 (*d*) shows the conjugate beam loaded with $-M/EI$ diagram. Omitting *EI* for convenience, the loading diagram will also be triangle having maximum ordinate of 4 at *B*, the load acting downwards. In addition to this loading, an unknown moment load μ will also act at D' of the conjugate beam, to satisfy the condition that the slope at both the sides of the real beam is the same. Since the slope at the real beam is represented by the shear of the conjugate beam, the shear just to the right of D' must be equal to the shear to the left of D'. This condition is satisfied by the moment μ acting at D' of the conjugate beam. There is a pin B' corresponding to the support B of the real beam.

Consider the equilibrium of the portion to the left of hinge B'.

$$\therefore \quad R_A'=\frac{1}{4}\left(\frac{1}{2}\times4\times4\times\frac{3}{4}\right)=\frac{8}{3}\uparrow$$

$$\therefore \quad R_C'=\text{total load}-R_A'=\left(\frac{1}{2}\times8\times4\right)-\frac{8}{3}=\frac{40}{3}\uparrow$$

Again, considering the equilibrium of the portion to the right of pin B', and taking moment about B', we get

$$\mu+\left(\frac{1}{2}\times4\times4\times\frac{4}{3}\right)=\frac{40}{3}\times4$$

$$\therefore \quad \mu=\frac{160}{3}-\frac{32}{3}=\frac{128}{3}$$

Since the bending moment of the conjugate beam represent the deflection of the corresponding point of the real beam, the moment μ represents the relative movement of the two parts of the slide. Hence

$$Y_{DD}=\mu=\frac{128}{3}$$

The calculations of M_X (and hence Y_{XD}) and F_D are done in the tabular form below (Table 5·6).

TABLE 5·6

x (m)	$M_X=Y_{XD}$	$F_D=\frac{Y_{XD}}{Y_{DD}}$
0	0	0
1	$\left(-\frac{8}{3}\times1\right)+\left(\frac{1}{2}\times1\times1\times\frac{1}{3}\right)=-\frac{5}{2}$	$\frac{-5\times3}{2\times128}=-0{\cdot}059$
2	$\left(-\frac{8}{3}\times2\right)+\left(\frac{1}{2}\times2\times2\times\frac{2}{3}\right)=-4$	$-\frac{4\times3}{128}=-0{\cdot}094$
3	$\left(-\frac{8}{3}\times3\right)+\left(\frac{1}{2}\times3\times3\times\frac{3}{3}\right)=-3{\cdot}5$	$-\frac{3{\cdot}5\times3}{128}=-0{\cdot}082$
4	$\left(-\frac{8}{3}\times4\right)+\left(\frac{1}{2}\times4\times4\times\frac{4}{3}\right)=0$	0
5	$\left(-\frac{40}{3}\times3\right)+\left(\frac{1}{2}\times3\times3\times\frac{3}{3}\right)+\frac{128}{3}=-\frac{43}{6}$	$\frac{43}{6}\times\frac{3}{128}=+0{\cdot}168$
6 (left)	$\left(-\frac{40}{3}\times2\right)+\left(\frac{1}{2}\times2\times2\times\frac{2}{3}\right)+\frac{128}{3}=+\frac{52}{3}$	$\frac{52}{3}\times\frac{3}{128}=+0{\cdot}406$
6(right)	$\left(-\frac{40}{3}\times2\right)+\left(\frac{1}{2}\times2\times2\times\frac{2}{3}\right)=-\frac{76}{3}$	$-\frac{76}{3}\times\frac{3}{128}=-0{\cdot}594$
7	$\left(-\frac{40}{3}\times1\right)+\left(\frac{1}{2}\times1\times1\times\frac{1}{3}\right)=-\frac{79}{6}$	$-\frac{79}{6}\times\frac{3}{120}=-0{\cdot}308$
0	0	0

The I.L. for F_D is shown in Fig. 5·16(*d*).

(*b*) Alternative Solution

We shall now determine the values of Y_{XD} for various values of x, by the double integration method. Since the beam is discontinuous at D, we will write the differential equations for both the portions separately. Refer Fig. 5·16 (*c*). The reactions, calculated earlier, are as follows :

$$R_A=1\uparrow\ :\ R_B=2\downarrow \text{ and } R_C=1\uparrow$$

(*i*) *For portion AD*

Measuring x from A, towards right,

$$EI\frac{d^2y}{dx^2}=-x\ \vdots\ +2(x-4)$$

$$\therefore\qquad EI\frac{dy}{dx}=-\frac{x^2}{2}+C_1\ \vdots\ +(x-4)^2$$

and $$EI\,y=-\frac{x^3}{6}+C_1x+C_2\ \vdots\ +\frac{(x-4)^3}{3}$$

At $x=0,\ y=0\ \therefore\ C_2=0$

At $x=4,\quad y=0=-\frac{64}{6}+4C_1\ ;\ \therefore\ C_1=\frac{64}{24}=\frac{8}{3}$

Hence the slope and deflection equations for portion AD are

$$EI\frac{dy}{dx}=-\frac{x^2}{2}+\frac{8}{3}\ \vdots\ +(x-4)^2 \qquad \text{(I)}$$

and $$EI\,y=-\frac{x^3}{6}+\frac{8}{3}x\ \vdots\ +\frac{(x-4)^3}{3} \qquad \text{(II)}$$

At $x=6$, $$\left(EI\frac{dy}{dx}\right)_{DA}=-\frac{36}{2}+\frac{8}{3}+4=-\frac{34}{3}$$

$$(EIy)_{DA}=-\frac{216}{6}+16+\frac{8}{3}=-\frac{52}{3}.$$

(*ii*) *For portion DC*

Measuring x from D, towards right, we get

$$EI\frac{d^2y}{dx^2}=-2+(1\times x)$$

$$EI\frac{dy}{dx}=-2x+\frac{x^2}{2}+C_1$$

$$EIy=-x^2+\frac{x^3}{6}+C_1x+C_2$$

$$x=0,\ \left(EI\frac{dy}{dx}\right)_{DC}=\left(EI\frac{dy}{dx}\right)_{DA}=-\frac{34}{3}=C_1$$

$$x=2,\ EIy=0=-4+\frac{8}{6}-\frac{34}{3}\times2+C_2$$

$$C_2=\frac{76}{3}$$

Hence the slope and deflection for portion DC are

$$EI\frac{dy}{dx}=-2x+\frac{x^2}{2}-\frac{34}{3} \qquad \text{(III)}$$

$$EIy=-x^2+\frac{x^3}{6}-\frac{34}{3}x+\frac{76}{3} \qquad \text{(IV)}$$

At $x=0,\ (EIy)_{DC}=\frac{76}{3}$

Hence $y_{DD}=(y_{DA})-(y_{DC})=\dfrac{1}{EI}\left(-\dfrac{52}{3}-\dfrac{76}{3}\right)=-\dfrac{128}{3EI}$

The calculations of Y_{XD} and F_D are done in the tabular form below (Table 5·7). The term EI has been omitted for convenience.

TABLE 5·7

Dist. from A (m)	*Eq. No.*	Y_{XD}	$F_D=\dfrac{Y_{XD}}{Y_{DD}}$
0	II ($x=0$)	0	0
1	II ($x=1$)	$-\frac{1}{6}+\frac{8}{3}=+\frac{5}{2}$	$-\frac{5}{2}\times\frac{3}{128}=-0{\cdot}059$
2	II ($x=2$)	$-\frac{8}{6}+\frac{16}{3}=+4$	$-\frac{4\times3}{28}=-0{\cdot}094$
3	II ($x=3$)	$-\frac{27}{6}+\frac{24}{3}=+\frac{7}{2}$	$-\frac{7\times3}{2\times128}=-0{\cdot}082$
4	II ($x=4$)	$-\frac{64}{6}+\frac{32}{3}=0$	0
5	II ($x=5$)	$-\frac{125}{6}+\frac{40}{3}+\frac{1}{3}=-\frac{43}{6}$	$+\frac{43}{6}\times\frac{3}{128}=+0{\cdot}168$
6 (left)	II ($x=6$)	$-\frac{216}{6}+\frac{48}{3}+\frac{8}{3}=-\frac{52}{3}$	$+\frac{52}{3}\times\frac{3}{128}=+0{\cdot}406$
6(Right)	IV ($x=0$)	$+\frac{76}{3}$	$-\frac{76}{3}\times\frac{3}{128}=-0{\cdot}594$
7	IV ($x=1$)	$-1+\frac{1}{6}-\frac{34}{3}+\frac{76}{3}=+\frac{79}{6}$	$-\frac{76}{6}\times\frac{3}{128}=-0{\cdot}308$
8	IV ($x=2$)	$-4+\frac{8}{6}-\frac{68}{3}+\frac{76}{3}=0$	0

5·8. FIXED BEAMS

1. I.L. for support moment

Let us now take a fixed beam AB, and plot the I.L. for support moment M_A at A. Let the unit load be at section X, distant x from A [Fig. 5·17(a)]. As per Müller-Breslau principle, introduce a hinge at A, thus getting a basic determinate structure, which will deflect under the unit load at X, as shown in Fig. 5·17(b). Let ϕ_{AX} be the resulting rotation at end A. Now remove the unit load and apply a unit moment at A, due to which the beam will deflect as shown in

Fig. 5·17 (c). Let ϕ_{AA} be the resulting rotation at A and y'_{XA} be the

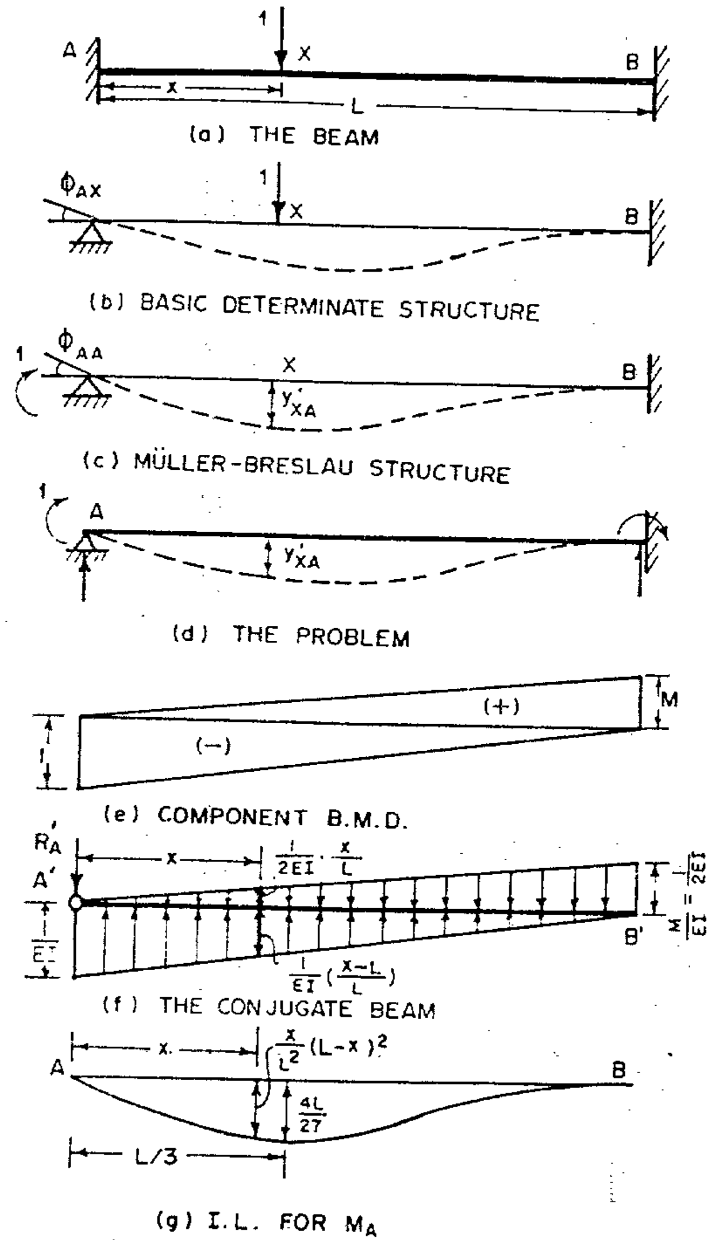


Fig. 5·17

deflection at x, due to unit moment applied at A. Then

$$M_A . \phi_{AA} = \phi_{AX}$$

But $\phi_{AX} = y'_{XA}$, by reciprocal theorem.

Hence $M_A . \phi_{AA} = y'_{XA}$

or
$$M_A = \frac{y'_{XA}}{\phi_{AA}} \tag{5·9}$$

Thus, the deflection of Fig. 5·17 (c), to some scale, gives the I.L. for M_A. If ϕ_{AA} is taken as unity, the deflection at any point X gives the I.L. for M_A.

Thus, the basic problem is shown in Fig. 5·17 (d), wherein we have to find the value of deflection y'_{XA} at X, due to unit moment applied at A. We will solve the problem by the conjugate beam method. Let the reactive moment at end B be M, due to unit moment applied at end A. The component B.M.D. is shown in Fig. 5·17 (e). The conjugate beam $A'B'$ along with the elastic loading (equal to M/EI diagram) is shown in Fig 5·17 (f).

For the conjugate beam, $M_A'=0$, because of the hinge at A'. Hence

$$M'_A = 0 = \frac{1}{2} M . L \left[\frac{2}{3} L\right] - \frac{1}{2} \frac{L}{EI} \times \frac{1}{3} L$$

which gives $M = \frac{1}{2}$.

For reaction R'_A at A, take moments at B' and equate to zero, since end B' of the conjugate beam is free.

$$\therefore\ R'_A \times L + \frac{1}{2} \times \frac{L}{EI} \times \frac{2}{3} L - \frac{1}{2} \times \frac{L}{2EI} \times \frac{1}{3} L = 0$$

which gives $R'_A = -\frac{L}{4EI}$

i.e. $$R_A' = \frac{L}{4EI} (\downarrow)$$

Now Mx' of the conjugate beam will give y'_{XA} of the real beam.

$$\therefore\ M'x = \frac{L}{4EI} x + \frac{1}{2}\left(\frac{1}{2EI} \frac{x}{L}\right) x . \frac{x}{3} - \frac{x^2}{6}\left[\frac{1}{EI}\left(\frac{x-L}{L}\right) + \frac{2}{EI}\right]$$

or
$$M'x = \frac{1}{12\ EIL} (3L^2x - 6x^2L - 3x^3)$$

$$\therefore\ y'_{XA} = M'x = \frac{1}{12\ EIL} (3L^2x - 6x^2L - 3x^3)$$

Also, ϕ_{AA} of real beam $= R'_A = -\frac{L}{4EI}$

Hence from Fig. 5·9,

$$M_A = \frac{y'_{XA}}{\phi_{AA}} = \frac{1}{12\ EIL}\left[3L^2x - 6x^2L - 3x^3\right]\left[-\frac{4\ EI}{L}\right]$$

or
$$M_A = -\frac{x}{L^2}(L-x)^2$$

The minus sign shows that M_A is opposite to the direction of unit moment applied at A.

$$\therefore\ M_A = \frac{x}{L^2} (L-x)^2 \tag{5·10}$$

Let us check this result by taking $x=a$ and $(L-x)=b$ and by taking a point load W in place of unit load. In that case,

$$M_A = \frac{W . a\, b^2}{L^2}$$

which matches with the well known result.

The I.L. for M_A is shown in Fig. 5·17 (g). For finding the maximum value of M_A, we have

$$\frac{dM_A}{dx} = 0 = \frac{1}{L^2} [L^2 + 3x^2 - 4Lx]$$

or
$$(L-x)(L-3x) = 0$$

From which, we get $x = L/3$.

$$\therefore\ M_A = \frac{L/3}{L^2}\left[L - \frac{L}{3}\right]^2 = \frac{4}{27} L$$

2. I.L. for Support reaction

Let us now plot I.L. for support reaction R_A, for a fixed beam shown in Fig. 5·18 (a). Fig. 5·18 (b) shows the basic determinate structure by removing the support reaction R_A, but by keeping fixidity intact at end A through an induced moment. The end A will deflect by an amount y_{AX}, due to unit load placed at X.

Now remove the unit load from X, and place it at end A, due to which the beam will deflect by y_{AA} at A and y_{XA} at X, as shown in Fig. 5·18 (c). From the method of consistent deformation,

$$R_A . y_{AA} = y_{AX}$$

But $y_{AX} = y_{XA}$, by reciprocal theorem

$$\therefore\ R_A . y_{AA} = y_{XA}$$

or
$$R_A = \frac{y_{XA}}{y_{AA}} \tag{5·11}$$

Thus, the deflection curve of Fig 5·18 (c). gives, to some scale, the I.L. for R_A. If y_{AA} is selected as unity, the deflection curve gives the I.L. for R_A, in which the ordinate y_{XA} at any point X, due to unit load placed at A, gives the ordinate of I.L. diagram. Thus the basic problem, shown in Fig. 5·18 (d) lies in finding the value of deflection y_{XA}, due to unit load and a reactive moment M at end A.

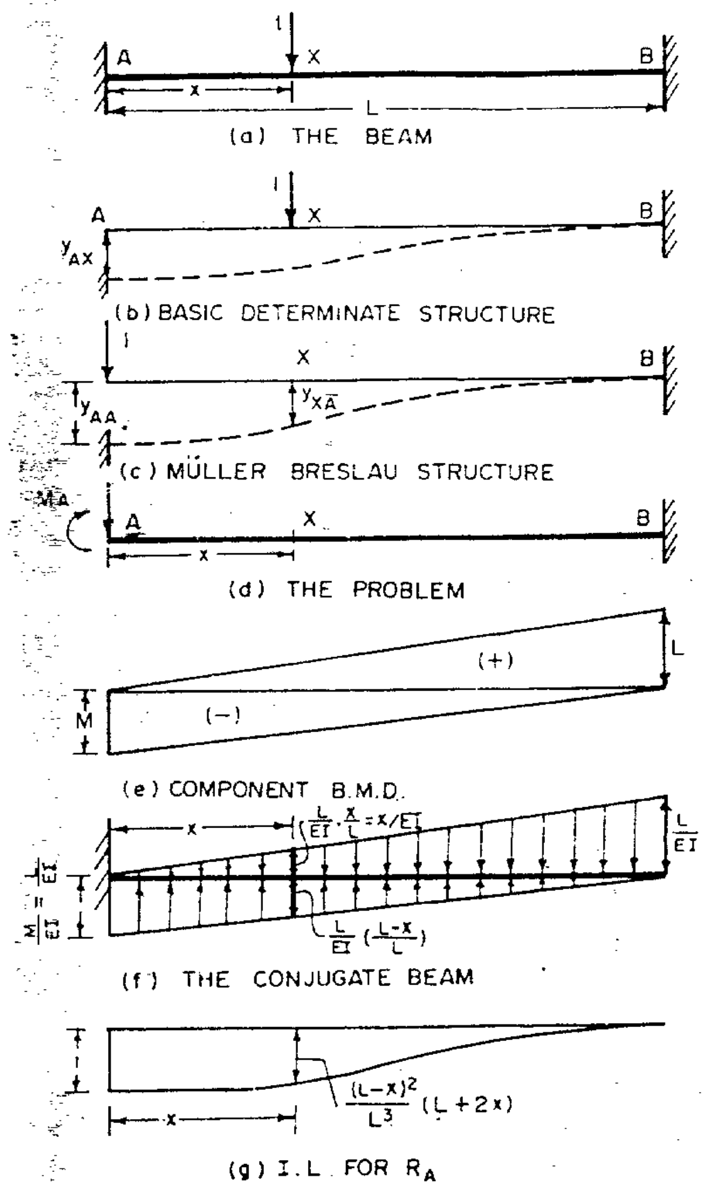


Fig. 5.18.

Fig. 5·18 (*e*) shows the component B.M. diagram for the beam problem of Fig. 5·18 (*d*), the corresponding conjugate beam $A'B'$ is shown in Fig 5·18 (*f*), with the elastic loading (M/EI), in which end A' is fixed while end B' is free. Since the slope at A of

the real beam, represented by shear at A' of the conjugate beam, is zero, we have $R_A'=0$ for the conjugate beam.

$$\therefore \quad R_A'=0=\frac{1}{2}\cdot\frac{L}{EI}\cdot L-\frac{1}{2}\,\frac{M}{EI}\cdot L$$

From which $M=L$.

Again deflection y_{xA} of the real beam is given by B.M. $M'x$ of the conjugate beam.

$$\therefore \quad y_{xA}=M'x=\frac{(L-x)^2}{6}\left(\frac{x}{EI}+\frac{2L}{EI}\right)-\frac{1}{2}\left\{\frac{L}{EI}\left(\frac{L-x}{L}\right)\right\}(L-x)\times\frac{(L-x)}{3}$$

or
$$y_{xA}=\frac{(L-x)^2}{6EI}(x+2L)-\frac{1}{6EI}(L-x)^3$$

or
$$y_{xA}=\frac{(L-x)^2}{6EI}(L+2x) \qquad (i)$$

$$\text{At } x=0,\ y_{AA}=M'_A=\frac{L^3}{6EI} \qquad (ii)$$

$$\text{Now } R_A=\frac{y_{xA}}{y_{AA}}=\frac{(L-x)^2}{6EI}\,(L+2x)\times\frac{6EI}{L^3}$$

or
$$R_A=\frac{(L-x)^2}{L^3}(L+2x) \qquad (5{\cdot}12)$$

which gives the equation of I.L. for R_A.

At $x=0$, $R_A=1$ (as expected).

Check. For a single point load W acting at a from A and b from B, we have the well known expression

$$R_A=\frac{Wb^2}{L^3}(3a+b)$$

Putting $W=1$, $a=x$ and $b=L-x$, we get

$$R_A=\frac{(L-x)^2}{L^3}(3x+L-x)=\frac{(L-x)^2}{L^3}(L+2x)$$

which is the same as Eq. 5·12.

Fig. 5·18 (*g*) shows the I.L. for R_A, which is a third degree curve.

PROBLEMS

1. A beam ABC of uniform section, length $2L$, is hinged at the collinear supports at its centre and ends. Derive the equation to the influence lines for bending moment at the central support. Taking $L=4$ m, plot the influence line to scale indicating values at every quarter of each span.

2. A continuous beam ABC is shown in Fig. 5·19. Compute the ordinate of the influence line for the reaction at C, at every quarter point of each span.

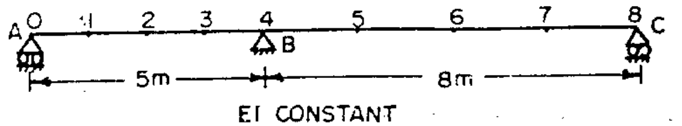


Fig. 5·19.

3. For the continuous beam shown in Fig. 5·20, draw the influence lines for reaction at A, B and C. Indicate the values at every quarter of each span.

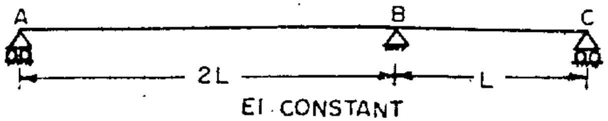


Fig. 5·20.

Answers

1. $O_0=0$; $O_1=+0{\cdot}236$; $O_2=+0{\cdot}376$; $O_3=+0{\cdot}328$; $O_4=0$
   $O_5=+0{\cdot}328$; $O_6=+0{\cdot}376$; $O_7=+0{\cdot}236$; $O_8=0$.
2. $O_0=0$; $O_1=-0{\cdot}028$; $O_2=-0{\cdot}045$; $O_3=-0{\cdot}039$; $O_4=0$
   $O_5=+0{\cdot}149$; $O_6=+0{\cdot}386$; $O_7=+0{\cdot}680$; $O_8=+1$.
3.

Ordinate	O_0	O_1	O_2	O_3	O_4	O_5	O_6	O_7	O_8
R_A	+1·00	+0·6718	+0·375	+0·1406	0	−0·0273	−0·0312	−0·0196	0
R_B	0·00	+0·4844	+0·8750	+1·0781	+1·00	+0·832	+0·5938	+0·3086	0
R_C	0·00	−0·1563	−0·2500	−0·2188	0	+0·1954	+0·4375	+0·7109	1·00

SECTION 2

STATICALLY INDETERMINATE STRUCTURES

Chapters

6

Statically Indeterminate Beams and Frames

6·1. INTRODUCTION

A *structural system*, generally called a *structure*, may be defined as an assembly of members such as bars, cables, arches, etc., with the purpose of transmitting external loads of the surrounding environment to the foundation. A structural system is in state of *equilibrium* if the constraints permit no rigid-body movement upon application of loads. The displacements or deformations are negligible in comparison with the dimensions of the structure. A structure is said to be *stable* when it deforms *elastically* and immediate elastic restraint is developed under the action of externally applied loads. The stability of a structure depends upon the number and arrangement of internal members and external reaction components. If a system does not have a sufficient number of internal or external constraints it will undergo a rigid-body movement upon the application of a small displacement. Such a system is said to be *statically unstable*, and is usually referred to as a *mechanism*.

If a structure is stable under the action of forces acting in a plane, *three conditions of equilibrium* must be satisfied : $\Sigma H=0$; $\Sigma V=0$ and $\Sigma M=0$, where ΣH is the algebraic sum of forces in horizontal or x-direction, ΣV is the algebraic sum of forces in vertical or y-direction and ΣM is the algebraic sum of moments of all the forces about a point. If the unknown forces (reactions and stress components) in the system can be determined by the equation of equilibrium alone, the system is said to be *statically determinate structure*. When the number of unknown reaction or stress (force) components exceeds the number of conditions of equilibrium, the system is said to be *statically indeterminate* or *redundant structure* and the excess restraints or members are described as *redundants*. In such cases, the equations of static equilibrium alone cannot provide solution ; they must be supplemented by the equations of compatibility of deformation. The *degree of indeterminateness or redundancy* is the number of unknown reactive restraints or stresses over and above the number of condition equations available for solution.

A redundant structure can further be classified into two categories : *externally redundant system* and *internally redundant system.* Externally redundant structures are those which have redundant reactive restraints. Internally redundant structures are those which have redundant members and are overstiff. However, a redundant structure may have both external as well as internal redundancies.

The main difference between the redundant structures and the statically determinate ones resides in the fact that the stress distribution depends for the first ones not only on the loading but also on the relative dimensions of their members and on the properties of materials of which the members are made. Statically indeterminate structures are very sensitive to such factors as the settlement of their supports, temperature variation, lack of fitness of members which give rise to additional stresses, while the same factors would have no influence whatsoever on statically determinate structures. However, statically indeterminate structures are most widely used in various engineering activities.

6·2. TYPES OF SUPPORTS : REACTION COMPONENTS

There are three types of supports which may be encountered in plane structures (*i*) *a roller support*, (*ii*) *a hinge support* and (*iii*) *a built-in or fixed supports.* These are shown in Fig. 6·1 (*a*), (*b*) and (*c*) respectively.

A *roller support* consists of two rockers—the upper rocker and the lower rocker, with a pin in between permitting the rotation of the upper rocker with respect to the lower one. Both the rockers can move together on rollers along the bearing plate. Such a support supplies a reactive force which acts normal to the surface of rolling and is directed through the centre of the hinged pin. Thus, only one parameter of the reaction, *i.e.* its magnitude, has to be known in order to determine the reactions completely. Such support is also known as *free end support* or *simple support.*

A *hinged support* [Fig. 6·1 (*b*)] differs from the roller support by the fact that the lower rocker is fixed and cannot move. The reaction passes through the centre of the pin but its magnitude and direction is unknown. In other words, this support has two reaction components—the horizontal and the vertical. Schematically, a hinged

support is represented by two bars connected by pin, or sometimes simply by a pin.

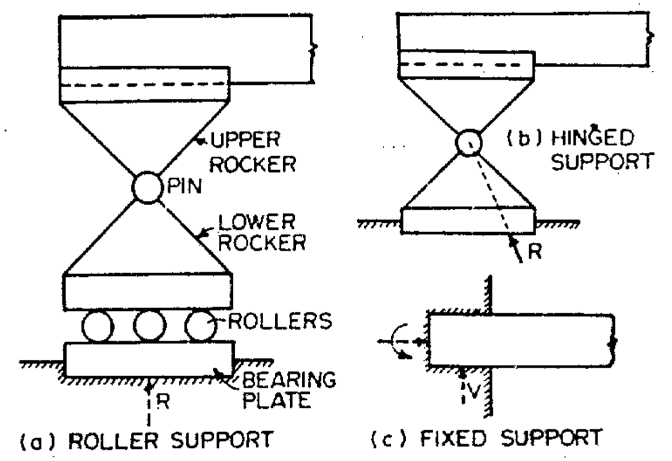


Fig. 6·1.
Type of Supports.

The *built-in-support*, shown in Fig. 6·1 (*c*), has zero degree of freedom. The determination of the reactions developed by this support requires the knowledge of three parameters—the direction and magnitude of a force passing through any chosen point (or its horizontal and vertical components) and the magnitude of the moment about the same point. Thus, a fixed support provides three reaction components.

6·3. EXTERNAL REDUNDANCY

For any structure, supported on external supports, the total reaction components can be easily found. The stability of a structure depends on the number and arrangement of the reaction components and component parts, rather than on the strength of the supports and parts of the structure. In general, three reaction components are necessary for the external stability of plane structures. This condition of three reaction components is *necessary* but not always *sufficient.* The arrangement of the three reaction components is very important from stability point of view. For example, if the lines of action of the three components are concurrent, the structure is externally unstable because the point of concurrency becomes the instantaneous centre of rotation giving a critical configuration. Similarly, a structure will also the unstable if the three reaction components have parallel lines of action, since the structure does not have any resistance to horizontal motion.

For a plane structure, three equations of static equilibrium are available. In addition to this, extra condition equation may sometimes be available by special features of construction, such as internal pins or links. A pin [Fig. 6·2 (*d*)] provided anywhere in the structure cannot transmit moment from one part of the structure to the other part and thus provides one additional condition equation : $\Sigma M=0$ at pin. Similarly, a link [consisting of a short bar with a pin at each end, as shown in [Fig. 6·2 (*e*)] provided anywhere in the structure is incapable of transmitting moment as well as horizontal force from one part to the other and thus provides two additional condition equations : $\Sigma M=0$ and $\Sigma H=0$ at the link. *Thus total number of condition equations of statical equilibrium for any structure are equal to the three equations of statical equilibrium plus additional condition equations because of a pin or a link anywhere in the structure.*

A structure is unstable if the total number of reaction components (*R*) are less than the total number of condition equations available. If the number of reaction components are equal to the condition equations, the structure is externally determinate. If, however, the number of reaction components are more than the condition equations the structure is *statically indeterminate externally*, the degree of indeterminacy or redundancy being equal to the number by which the reaction components exceed the condition equations, and is represented by the equation

$$E=R-r \tag{6·1}$$

where E=Degree of external redundancy

r=total number of condition equations available.

6·4. STATICALLY INDETERMINATE BEAMS

A continuous beam is a typical example of externally indeterminate structure. Fig. 6·2 shows some statically indeterminate beams.

Fig. 6·2 (*a*) shows a propped cantilever. For a *general loading* the total reaction components (*R*) are equal to $(3+2)=5$, while the total number of condition equations (*r*) are equal to 3. Hence the beam is statically indeterminate, externally, to second degree. However, for *vertical loading*, only two reaction components (M and V) are available at the fixed end and one reaction component (V) available at the propped end, making the total reaction components R equal to 3, while the number of condition equations of static equilibrium are only two ($\Sigma M=0$ and $\Sigma V=0$), and the beam is statically indeterminate to single degree.

Fig. 6·2(*b*) shows a fixed beam with 6 reaction components, and three condition equations. For the general system of loading,

therefore, a fixed beam is statically indeterminate to third degree. However, for vertical loading on the beam, the total reaction components are four only (two at each joint), and only two condition equations are availabe making the beam statically indeterminate to second degree only.

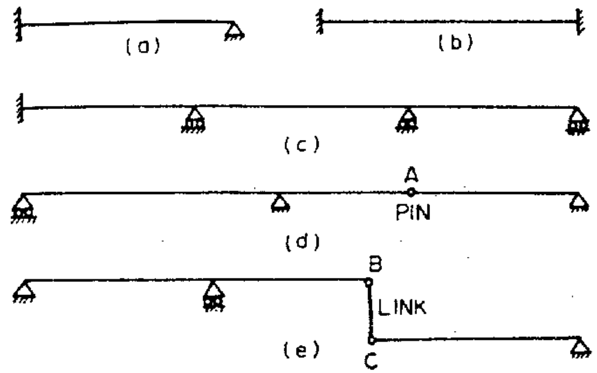


Fig. 6·2.
Statically indeterminate beams.

In Fig. 6·2(*c*), the total reaction components are equal to $(3+1+1+1)=6$ while the condition equations are three. Hence the beam is statically indeterminate to third degree, for the general system of loading.

In Fig. 6·2(*d*), there is a pin at *A*. The total number of condition equations are equal to three equations of static equilibrium plus one condition equation (*i.e* $\Sigma M=0$) at the pin at *A*, making a total of 4. The reaction components are equal to $(1+2+2)=5$. The beam is statically indeterminate to single degree only.

Similarly, the beam of Fig. 6·2(*e*) has a link *BC*, giving two additional condition equations. The total number of condition equations are, therefore, equal to $3+2=5$, while the reaction components are equal to $2+1+2=5$. The beam is, therefore, statically determinate.

It should be noted that in the case of continuous beams (or statically indeterminate beams), the shear and moment at any point in the beam are readily known once the reaction components are determined. Thus, these beams are statically determinate *internally*. The degree of indeterminacy of a beam is therefore equal to its external redundancy.

6·5. DEGREE OF REDUNDANCY OF ARTICULATED STRUCTURES

A pin jointed frame or articulated structure is composed of a number of bars or straight members connected by frictionless pins, forming geometrical figures which are usually triangles. A stable and determinate frame can be built up as an assemblage of triangles. The first triangle is made up of three joints and three members, and each successive triangle require two additional members and one additional joint. If j is the total number of joints and m is the total number of members, we get

$$m=3+2(j-3)$$

or
$$m=2j-3 \qquad (6{\cdot}2)$$

If, however, the total number of reaction components absolutely necessary for stability (and hence the total number of condition equations available) are r and not 3, the above equation can be written as

$$m=2j-r \qquad (6{\cdot}3)$$

The above equation gives the criterion for finding the degree of *internal indeterminacy* or internal redundancy. A truss or frame is said to be statically determinate *internally* if it has members given by Eq. 6·3. If, however, the number of members in a frame are more than given by Eq. 6·3, the frame is said to have internal redundancy. If it has fewer members, it is *unstable*. The degree of internal redundancy I is therefore given by

$$I=m-(2j-r) \qquad (6{\cdot}4)$$

The degree of external indeterminacy E is given by Eq. 6·1. Hence the total redundancy or indeterminateness which is equal to the sum of external indeterminateness and internal indeterminateness, is given by

$$T=E+I=\{(R-r)\}+\{m-(2j-r)\}$$

or
$$T=m+R-2j \qquad (6{\cdot}5)$$

where T=degree of redundancy (total)

R=total number of reaction components.

Fig. 6·3 shows some typical articulated structures.

(*i*) *In Fig. 6·3(a),*

$j=6$; $m=9$;

$R=2+1=3$; $r=3$.

$\therefore \quad E=3-3=0$

$I=m-(2j-r)=9-(2\times6-3)=0$

$\therefore \quad T=E+I=0$

Thus the frame is statically determinate both externally as well as internally and is stable.

(*ii*) *In Fig. 6·3(b).*

$j=8$; $m=15$

$R=2+2=4$; $r=3$

$\therefore \quad E=R-r=4-3=1$

$I=m-(2j-r)=15-(2\times8-3)=2$

$T=E+I=1+2=3$

Alternatively, $T=m+R-2j=15+4-16=3$.

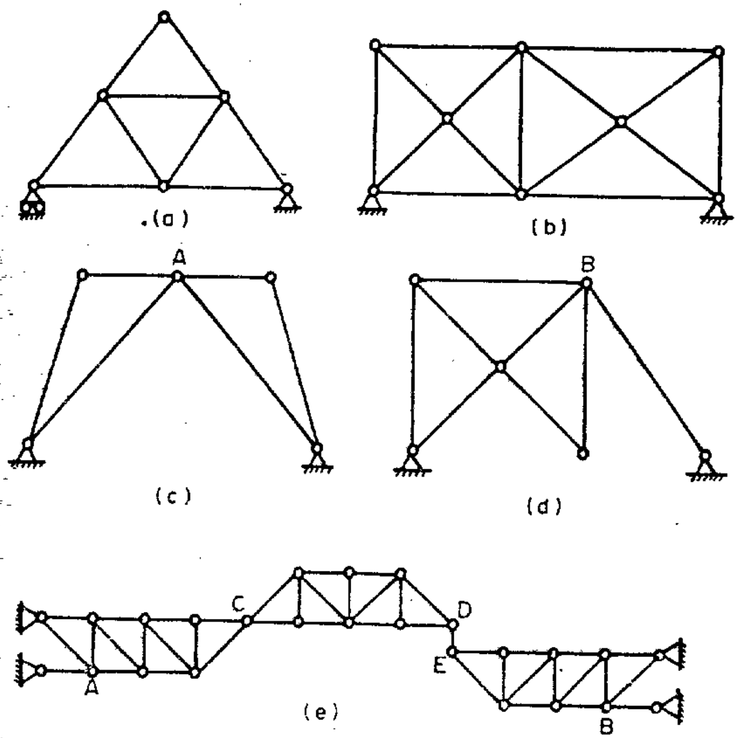


Fig. 6·3.
Articulated structures.

Hence the frame is indeterminate externally to first degree, internally to second degree and thus has a total degree of redundancy equal to 3.

(*iii*) *In Fig. 6·3(c),*

$j=5$; $m=6$

$R=4$; $r=3+1$ (due to hinge at A)$=4$

$\therefore \quad E=4-4=0$

$I=m-(2j-r)=6-(2\times5-4)=0$

$T=E+I=0$

(Also, $T=m+R-2j=6+4-10=0$).

The frame is thus statically determinate, both externally as well as internally.

(iv) *In Fig.* 6·3(*d*),

$$j=6\,;\ m=8$$
$$R=4\,;\ r=3+1=4$$
$$\therefore\quad E=4-4=0$$
$$I=m-(2j-r)=8-(2\times6-4)=0.$$

The frame is thus statically determinate.

(v) *In Fig.* 6·3(*e*),

$j=25\,;\ m=42$

$R=8\,;\ r=3+3$ (one for each hinge at A, B, C)+2 (link CD)=8

$\therefore\ I=42-(2\times25-8)=0.$

The frame is thus statically determinate.

6·6. DEGREE OF REDUNDANCY OF RIGIDLY JOINTED FRAMES

In the case of pin jointed frames, the members carry axial forces only and hence two equations are available at each joint. The members of stiff jointed frames, on the other hand, resist thrust, shear and bending moment. There are, therefore, three equations available at each joint.

Fig. 6·4 shows a stiff jointed frame. Assuming that the reaction components are known for the purposes of determining internal indeterminacy, and treating column AB as free body, the thrust, shear and moment in AB are known. Consider joint B at which there are total nine unknowns (*i.e.* thrust, shear and moment each for BA, BC and BE), out of which three unknowns of BA are known and three conditions of static equilibrium at joint B can be arbitrarily assigned to the three unknowns of BC. Thus, there remain three unknowns in the member BE at joint B. Once the internal stresses in BE are determined, the total number of knowns at joint E are six (three for ED and three for EG), out of which three equations of statical equilibrium of joint E can be arbitrarily assigned to the unknowns of the member ED. Thus there remain three unknowns in the member EG at the joint E. Knowing the stress components in BC, ED and GF, and using the equations of statical equilibrium at joints C, D and F, the unknown

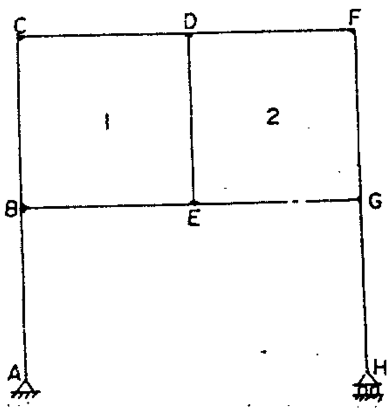


Fig. 6·4.

stress components in CD and FD can be easily determined. Thus, on the whole, there are 6 unknowns (3 for BE and 3 for EG) to be determined and the frame is internally indeterminate to 6th degree. In general, therefore, the degree of internal indeterminacy I can be represented by the formula :

$$I=3a \tag{6·6}$$

where a is the number of areas *completely enclosed* by members of the frame. In Fig. 6·4, $a=2$, and hence $I=3\times2=6$ as determined above. The above formula is also applicable to continuous beams (Fig. 6·2), where $a=0$ and hence $I=0$. *i.e.*, a continuous beam is statically determinate *internally* since the moment and shear at any point on the beam can be readily determined once the external redundant reactions are determined.

The external indeterminateness is given by Eq. 6·1.

$$E=R-r.$$

In the case of stiff jointed frame, $r=3$, since no hinge or link is provided in between the members. Hence

$$E=R-3 \tag{6·7}$$

Hence the total indeterminateness or redundancy is given by :

$$T=E+I=(R-r)+3a=(R-3)+3a \tag{6·8}$$

Fig. 6·5 shows some stiff-jointed structures.

(i) *In Fig.* 6·5(*a*),

$$R=3\times3=9\,;\ a=2$$
$$\therefore\quad E=R-3=9-3=6$$
$$I=3a=3\times2=6$$
$$\therefore\quad T=E+I=6+6=12$$

Thus the structure is statically indeterminate to twelfth degree.

(ii) *In Fig.* 6·5(*b*),

$$R=2+2+2=6\,;\ a=3$$
$$\therefore\quad E=R-3=6-3=3$$
$$I=3a=3\times3=9$$
$$\therefore\quad T=E+I=3+9=12$$

(iii) *In Fig.* 6·5(*c*),

$$R=2+2=4\,;\ a=0$$
$$\therefore\quad E=R-3=4-3=1$$

and $I=3\times0=0$ (*i.e.*, the structure is statically determinate internally)

$$\therefore\quad T=1.$$

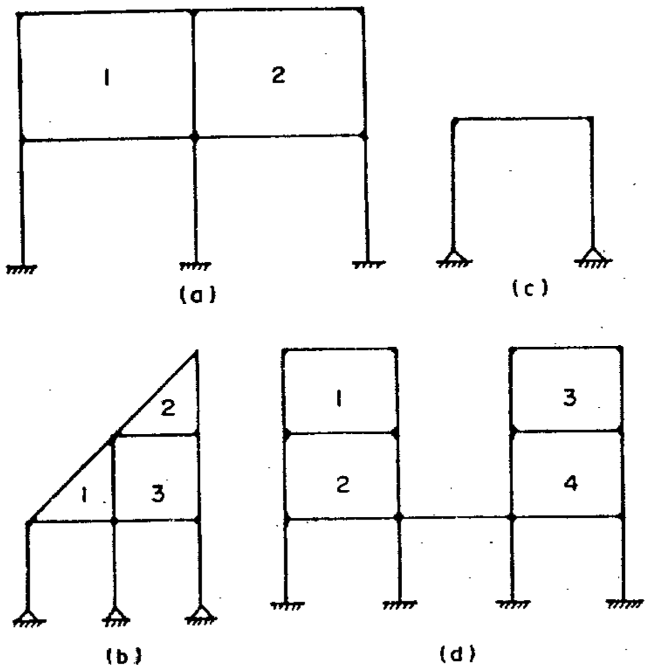


Fig. 6·5.
Rigid-jointed Structures.

(*iv*) *In Fig.* 6·5 (*d*),

$$R=3\times4=12\ ;\ a=4$$

$$\therefore \quad E=R-3=12-3=9$$

$$I=3a=3\times4=12$$

$$\therefore \quad T=E+I=9+12=21$$

6·7. METHOD OF ANALYSIS

There are two *basic methods* available for analysing statically indeterminate structures : (*i*) compatibility method, and (*ii*) Equilibrium method.

Compatibility method. This method is also sometimes known as *flexibility coefficient method* or *force method*. In this method the redundant forces are chosen as unknowns and additional equations are obtained by considering the geometrical conditions imposed on the deformations of the structures. The common methods that fall under this category are : the method of consistent deformation (or the general method), three moment theorem, column analogy

method, elastic centre method, Maxwell-Mohr equations, Castigliano's theorem of minimum strain energy, etc.

Equilibrium method. This method is also known as *deformation method* or *stiffness coefficient method*. In this method, displacements of joints are taken as unknowns. The equilibrium equations are expressed in terms of these displacements and the external loads are solved to give the actual joints displacements from which redundant forces can be computed. The common methods that fall under this category are : Slope-deflection method, moment distribution method, minimum potential energy method, etc.

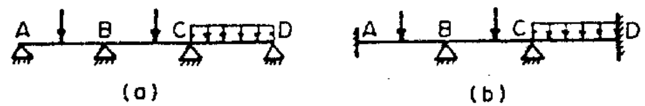


Fig. 6·6

The choice between compatibility method and equilibrium method largely depends upon the type of structure and the manner in which it is supported. For example, consider a continuous beam loaded as shown in Fig. 6·6(*a*). For the case of vertical loads shown, the reactions at *A*, *B*, *C* and *D* will be vertical—making a total of 4 reaction components while the only two equations (*i.e.*, $\Sigma V=0$ and $\Sigma M=0$) are available from statical equilibrium. Hence the structure is statically indeterminate to second degree and any two reactions can be taken as unknowns for the compatibility method, and only two compatibility equations will be required. On the other hand, if equilibrium methods were to be used, there are four unknown joint rotations (*i.e.*, θ_A, θ_B, θ_C and θ_D) and four equilibrium equations will have to be formulated and solved. The compatibility method will therefore be preferred for this beam. Now take the case of the same beam but fixed at *A* and *D* as shown in Fig. 6·6 (*b*). There will be two additional moments (unknowns), making total reaction components equal to 4+2=6. Since only two equations ($\Sigma V=0$ and $\Sigma M=0$) are available from statical equilibrium, the beam is statically indeterminate to the fourth degree. In the compatibility method four equations will have to be formulated and solved for any four unknown reactions. On the other hand, if equilibrium method were to be used, only two equations in terms of joints rotations θ_B and θ_C need be solved since θ_A and θ_D are each zero. Though the fixidity of joints *A* and *D* have increased the redundants to four, the joint displacements are reduced by two. The equilibrium method will be more suitable to this case.

PROBLEMS

1. Find the degree of indeterminateness of the beams shown in Fig. 6·7 for the general case of loading.

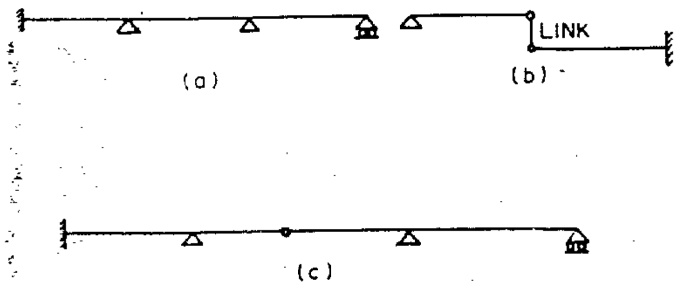


Fig. 6·7

2. Find the degree of internal, external and total indeterminateness of the pin-pointed frames shown in Fig. 6·8.

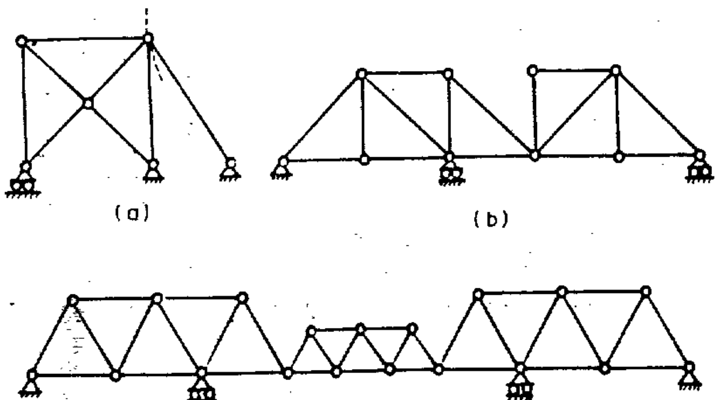


Fig. 6·8

3. Find the degree of redundancy of the stiff-jointed frames shown in Fig. 6·9.

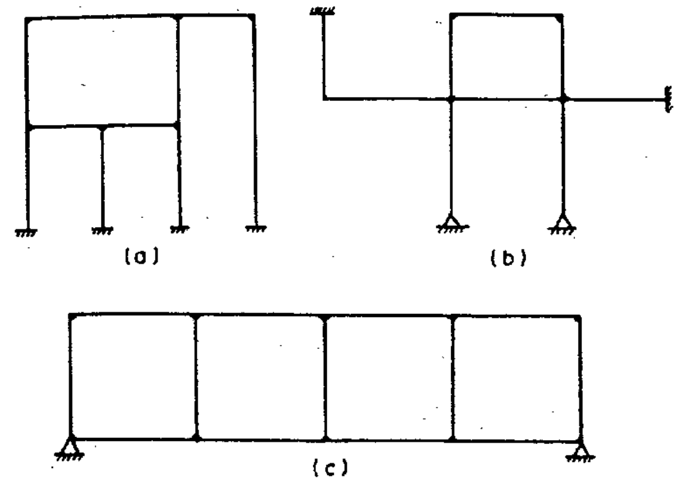


Fig. 6·9

3a $E = R - 3a$
12 − 3
= 9

Answers

1. (*a*) $E=5$ (*b*) $E=0$ (*c*) $E=4$.
2. (*a*) $E=0$; $I=0$; $T=1$
   (*b*) $E=0$; $I=0$; $T=0$
   (*c*) $E=1$; $I=0$; $T=1$.
3. (*a*) $T=12$ (*b*) $T=0$ (*c*) $T=13$.

I = 3a
− 3 × 1

T = 12

b) E = 10 − 3
= 7

I = 3
T = 10

16
m = 2j − 3
20 − 3
= 17

3 c) E = 4 − 3 = 1
I = 3 × 4 = 12
j = 10
T = 13

7

The General Method

(METHOD OF CONSISTENT DEFORMATION)

7·1. INTRODUCTION

The general method, or the *method of consistent deformation*, as is sometimes known, is credited to Clerk Maxwell (1864), Otto Mohr (1874) and Muller-Breslau (1886). This chapter, however deals with the Muller-Breslau version of the general method in which the condition equations of geometrical coherence of a structure are obtained by superposition of displacements as caused by the applied loads and individual redundant stresses and reactions. The degree of redundancy of a structure is equal to the number of excess reaction components over those required for statical equilibrium. The method essentially consist in replacing the redundant reaction components by unknown force or moment reactions and then writing the condition equations for geometrical coherence of the structure by the superpositions of the displacements as caused by the applied loads and the unknown redundant force and/or moments. The structure obtained by replacing the redundant reactions by unknown forces or moments is known as a *basic determinate structure*. A number of such basic determinate structures can be obtained out of the original redundant structure, depending upon the choice of the redundant reaction component(s).to be replaced.

7·2. STATICALLY INDETERMINATE BEAMS AND FRAMES

The statically indeterminate beam or frame is analysed by the method of consistent deformation by first obtaining a basic determinate structure. The condition equations for geometrical coherence of a structure with various types of supports are as follows :

1. If a roller support is removed, the deflection in the direction perpendicular to the plane of rolling must be zero.
2. If a hinged support is removed, the deflection in the vertical and horizontal directions at the point must be zero.

3. If a fixed support is removed, the rotation as well as the vertical and horizontal deflections at the point must be zero.

There are, thus as many physical conditions of geometry as there are redundant reaction components.

Example 7·1. *A cantilever of uniform flexural stiffness is proposed at the remote end. Find the load on the prop when a force W is applied at the centre of the cantilever.*

Solution

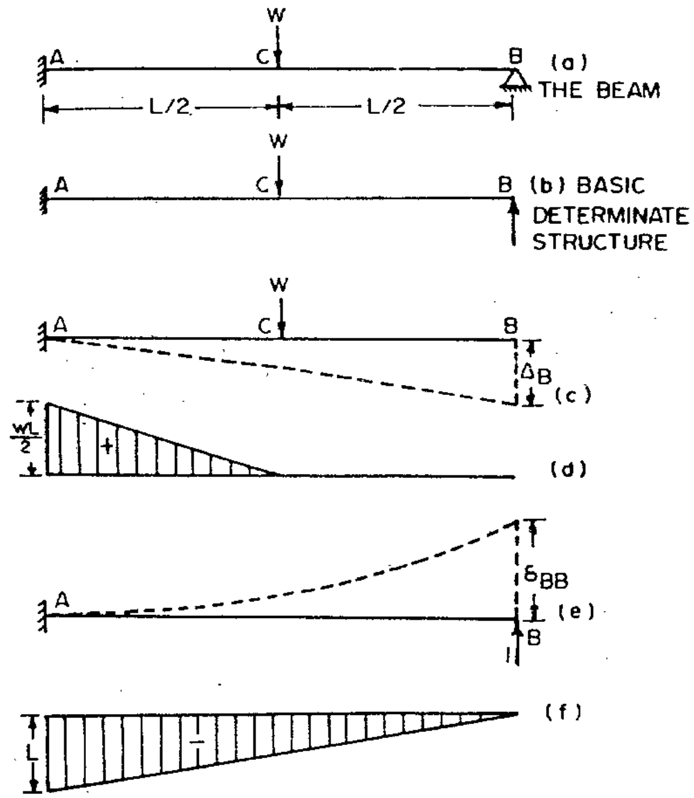


Fig. 7·1

Let V_B = Redundant reaction (vertical) at B.

The basic determinate structure is obtained by replacing the prop at B by an unknown vertical reaction $V_B\uparrow$ as shown in Fig. 7·1 (*b*).

Let Δ_B = Deflection of the end $B(\downarrow)$ of the basic determinate structure due to external loading [Fig. 7·1(*c*)].

δ_{BB}=Deflection of the end B of the basic determinate structure due to unit load at B (The first suffix denotes the point where the deflection is reckoned and the second suffix denotes the position of the unit point load) [Fig. 7·1(*e*)].

From conditions of geometry at B, we get

$$\Delta_B + V_B \cdot \delta_{BB} = 0 \tag{7·1}$$

The deflection Δ_B and δ_{BB} can be obtained either by the area moment method or the conjugate beam method. Fig. 7·1(*c*) shows the load W acting on the beam, with Δ_B as the deflection of the end B; the corresponding bending moment diagram is shown in Fig. 7·1(*d*).

$$\Delta_B = \frac{1}{EI}\sum_B^A A\bar{x} = \frac{1}{EI}\left(\frac{1}{2}\cdot\frac{WL}{2}\cdot\frac{L}{2}\right)\left(\frac{L}{2}+\frac{2}{3}\cdot\frac{L}{2}\right)$$

$$= +\frac{5}{48}\,\frac{WL^3}{EI}$$

Fig. 7·1(*e*) and (*f*) shows the unit load acting at B and the corresponding B.M.D. respectively.

$$\therefore\ \delta_{BB} = \frac{1}{EI}\sum_B^A A\bar{x} = \frac{1}{EI}\left(-\frac{1}{2}\cdot L\cdot L\right)\left(\frac{2}{3}\,L\right) = -\frac{L^3}{3EI}$$

Substituting in Eq. 7·1, we get

$$\frac{5}{48}\,\frac{WL^3}{EI} - V_B\cdot\frac{L^3}{3EI} = 0$$

or $$V_B = \frac{5}{16}W$$

Alternative Solution

An alternative basic determinate structure can be obtained by treating moment at A as redundant. The basic determinate structure, thus, has end A as simply supported with an unknown moment M_A acting, as shown in Fig. 7·2 (*b*).

Let θ_A=slope at end A of the basic determinate structure, due to the external loading as shown in Fig. 7·2(*c*).

ϕ_{AA}=slope at end A, of the basic determinate structure, due to unit moment acting at A as shown in Fig. 7·2(*e*).

Then by conditions of geometry at A,

$$\theta_A + M_A\cdot\phi_{AA} = 0 \tag{7·2}$$

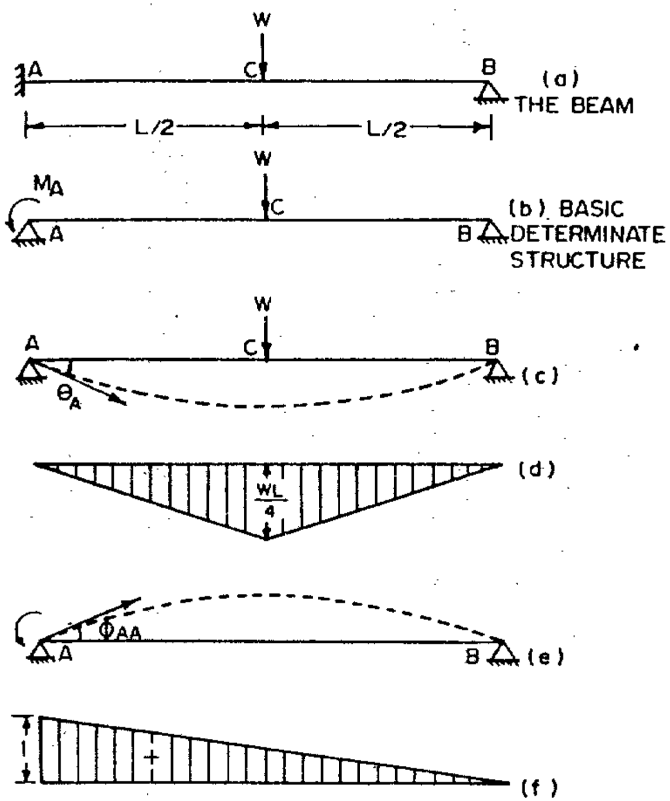


Fig. 7·2

Fig. 7·2(*d*) shows the B.M. diagram when the weight W is acting. Fig. 7·2 (*f*) shows the B.M. diagram for the unit moment acting at A. From the conjugate beam method, we get

$$\theta_A = +\frac{1}{2}\left[\frac{1}{2}L\cdot\frac{WL}{4}\right]\frac{1}{EI} = +\frac{WL^2}{16\,EI}$$

$$\phi_{AA} = -\frac{2}{3}\left[\frac{1}{2}\cdot L\cdot 1\right]\frac{1}{EI} = -\frac{L}{3EI}$$

Substituting in Eq. 7·2, we get

$$+\frac{WL^2}{16\,EI} - M_A\cdot\frac{L}{3EI} = 0$$

$$\therefore\quad M_A = +\frac{3WL}{16}.$$

Knowing M_A, V_A can be found by taking moments at B

∴ $$W\cdot\frac{L}{2}+M_A-V_A.L=0$$

∴ $$V_A=\frac{W}{2}+\frac{M_A}{L}=\frac{W}{2}+\frac{3W}{16}=\frac{11}{16}W\uparrow$$

∴ $$V_B=W-V_A=W-\frac{11}{16}W=\frac{5}{16}W\uparrow$$

Example 7·2. *A beam AB of span 4 m is fixed at A and B and carries a point load of 5 kN at a distance of 1 m from end A. Calculate the support moments by the method of consistent deformation.*

Solution

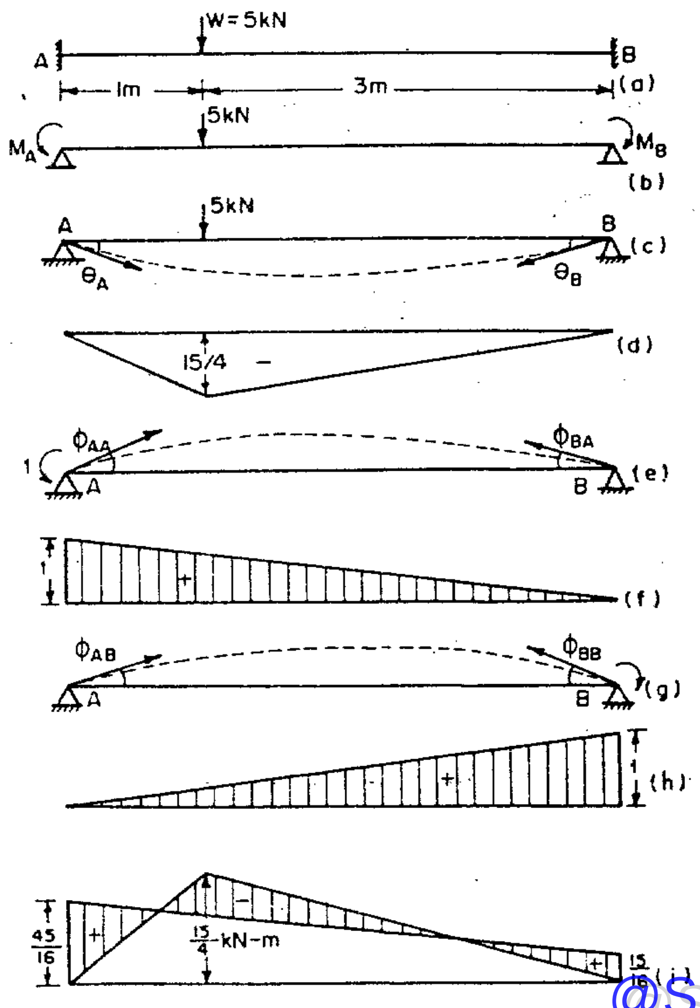


Fig. 7·3

For vertical loading, there are four reaction components: M_A, V_A, M_B and V_B while there are two equations ($\Sigma M=0$ and $\Sigma V=0$), available from statical equilibrium. The beam is, therefore, statically indeterminate to second degree. Let us choose M_A and M_B as the redundants. The basic determinate structure is shown in Fig. 7·3(*b*).

Let θ_A and θ_B=slope at A and B for the basic determinate beam, due to the external load.

ϕ_{AA}=slope at A due to unit moment at A

ϕ_{AB}=slope at A due to unit moment at B

ϕ_{BB}=slope at B due to unit moment at B.

ϕ_{BA}=slope at B due to unit moment at A.

Then, from condition of geometry at A and B, we get

$$\theta_A+M_A\,.\,\phi_{AA}+M_B\,.\,\phi_{AB}=0 \qquad [7·3(a)]$$

$$\theta_B+M_A\,.\,\phi_{BA}+M_B\,.\,\phi_{BB}=0 \qquad [7·3(b)]$$

Let us use conjugate beam method for the calculations of θ_A, θ_B, ϕ_{AA}, ϕ_{AB}, ϕ_{BB}, ϕ_{BA}. Fig. 7·3 (*d*), (*f*) and (*h*) show the bending moment diagrams for the external load, unit couple at A and unit couple at B respectively. These B.M. diagrams become the loading for the conjugate beam. The slope at any point of real beam is equal to the shear force at the corresponding point of the conjugate beam. The sign convention for positive and negative shear force is shown in Fig. 7·4.

Fig. 7·4

From Fig. 7·3(*d*), we get

θ_A=shear at end A of the conjugate beam

$$=\frac{1}{4}\left(\frac{1}{2}\times\frac{15}{4}\times 4\right)\left(\frac{4+3}{3}\right)\frac{1}{EI}=\frac{35}{8}\;\frac{1}{EI}$$

θ_B=shear at end B of the conjugate beam

$$=-\frac{1}{4}\left(\frac{1}{2}\times\frac{15}{4}\times 4\right)\left(\frac{4+1}{3}\right)\frac{1}{EI}=-\frac{25}{8}\;\frac{1}{EI}$$

Similarly from Fig. 7·3 (*f*)

$$\phi_{AA}=-\frac{1}{4}\left(\frac{1}{2}\times 4\times 1\right)\frac{2\times 4}{3}\cdot\frac{1}{EI}=-\frac{4}{3}\;\frac{1}{EI}$$

$$\phi_{BA}=+\frac{1}{4}\left(\frac{1}{4}\times 4\times 1\right)\frac{4}{3}\;\frac{1}{EI}=+\frac{2}{3}\;\frac{1}{EI}$$

And, from Fig. 7·3(*b*)

$$\phi_{AB}=-\frac{1}{4}\left(\frac{1}{2}\times4\times1\right)\frac{4}{3}\frac{1}{EI}=-\frac{2}{3}\frac{1}{EI}$$

$$\phi_{BB}=+\frac{1}{4}\left(\frac{1}{2}\times4\times1\right)\frac{2\times4}{3}\frac{1}{EI}=+\frac{4}{3}\frac{1}{EI}$$

Substituting these values in Eq. 7·3(*a*) and 7·3(*b*), we get

$$\frac{35}{8}\frac{1}{EI}-M_A\left(\frac{4}{3}\frac{1}{EI}\right)-M_B\left(\frac{2}{3}\frac{1}{EI}\right)=0 \qquad (1)$$

and

$$-\frac{25}{8}\frac{1}{EI}+M_A\left(\frac{2}{3}\frac{1}{EI}\right)+M_B\left(\frac{4}{3}\frac{1}{EI}\right)=0 \qquad (2)$$

Solving Eqs. (1) and (2), we get

$$M_A=+\frac{45}{16}\text{ kN-m and } M_B=+\frac{15}{16}\text{ kN-m.}$$

The final B.M.D. for the beam is shown in Fig. 7·3(*i*)(*a*)].

7·3. MAXWELL'S LAW OF RECIPROCAL DEFLECTION

As applied to beam deflections and rotations, Maxwell's theorem of reciprocal deflections has the following three versions :

(1) The deflection at A due to unit force at B is equal to deflection at B due to unit force at A [Fig. 7·5(*a*)].

Thus, $\delta_{AB}=\delta_{BA}$ (7·4)

(2) The slope at A due to unit couple at B is equal to the slope at B due to unit couple A [Fig. 7·5(*b*).

Thus, $\phi_{AB}=\phi_{BA}$ (7·5)

(3) The slope at A due to unit load at B is equal to deflection at B due to unit couple at A [Fig. 7·5(*c*)].

Thus, $\phi_{AB}'=\delta_{BA}'$ (7·6)

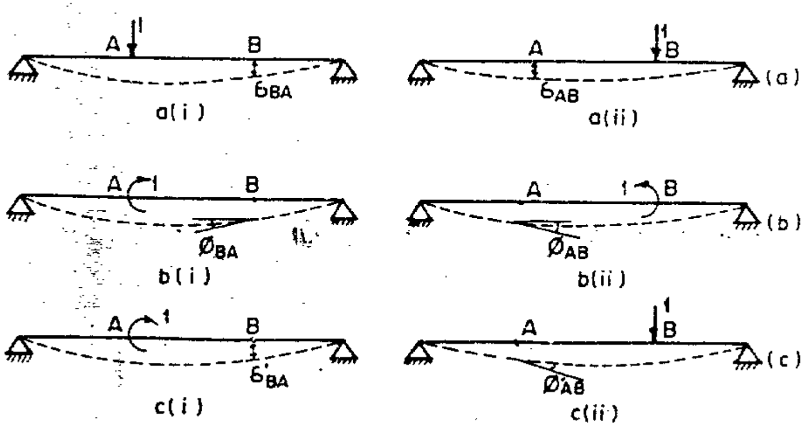


Fig. 7·5

Proof

By unit load method, $\delta=\int\frac{Mmdx}{EI}$ in general (See Vol. 1).

where M=bending moment at any point X due to external load.

m=bending moment at any point X due to unit load applied at the point where deflection is required.

Let mx_A=bending moment at any point X due to unit load at A.

mx_B=bending moment at any point X due to unit load at B.

When unit load (external load) is applied at A, $M=mx_A$.

To find deflection at B due to unit load at A, apply unit load at B. Then $m=mx_B$.

Hence $$\delta_{BA}=\int\frac{Mmdx}{EI}=\int\frac{mx_A\cdot mx_B}{EI}dx \qquad (1)$$

Similarly, when unit load (external load) is applied at B.

$$M=mx_B$$

To find the deflection at A due to unit load B, apply unit load at A. Then $m=mx_A$.

Hence $$\delta_{AB}=\int\frac{Mmdx}{EI}=\int\frac{mx_B\cdot mx_A}{EI} \qquad (2)$$

Comparing (1) and (2), we get

$$\delta_{AB}=\delta_{BA}.$$

Similarly, other versions of the reciprocal theorem can also be proved.

7·4. GENERALISED MAXWELL'S THEOREM : BETTI'S RECIPROCAL THEOREM

Generalised Statement. If an elastic system is in equilibrium under one set of forces with their corresponding displacements and if the same system is also in in equilibrium under second set of forces acting through the same points with their corresponding displacements then the product of first group of forces and the corresponding displacements caused by second group is equal to the product of the second group of forces and the corresponding displacements caused by the first group.

i.e $$P_A\Delta_A'+P_B\Delta_B'=P'_A\cdot\Delta_A+P_B'\Delta_B \qquad (7·7)$$

where P and Δ constitute first group of forces and their corresponding displacements, and P' and Δ' constitute second group of forces and displacements.

That is, the virtual work done by the first set of forces acting through the second set of displacements is equal to the virtual work done by the second set of forces acting through the first set of displacements.

In Betti's theorem, the symbols P and Δ can also denote couples and rotations respectively, as well as forces and linear deflections, *i.e.* $M_A.\theta_A'+M_A.\theta_B'=M_A'.\theta_A+M_B'.\theta_B$ (7·8)

Thus, according to Betti's law, we have, in general

$$\Sigma P.\Delta'=\Sigma M.\theta'=\Sigma P'\Delta+\Sigma M'.\theta \quad (7·9)$$

Example 7·3. *A continuous beam ABC is loaded as shown in Fig. 7·6 (a). Determine all reactions and draw B.M. and S.F. diagrams.*

Solution

A basic determinate structure is obtained by replacing the central support by an upward force V_B [Fig. 7·6 (*b*)].

Since there are three unknowns, *i e.* V_A, V_B and V_C, the beam is indeterminate to the first degree and the following condition equation will be used.

$$\Delta_B=V_B.\delta_{BB} \quad \text{(Numerically)} \quad (1)$$

Now $\Delta_B=W.\delta_{BD}$

But from reciprocal deflections, $\delta_{BD}=\delta_{DB}$

Hence $\Delta_B=W.\delta_{DB}$

Substituting in (1), we get the modified condition equation

$$W.\delta_{DB}=V_B.\delta_{BB} \quad (2)$$

When the unit load acts at B [Fig. 7·6 (*e*)], the B.M. diagram will be a triangle having a maximum ordinate of $+\frac{L}{2}$ at B. Hence the conjugate beam [Fig. 7·6 (*f*) loaded with $-\frac{M}{EI}$ diagram will be acted upon by a triangular load acting upwards.

From conjugate beam method, [Fig. 7·6 (*e*)] and [Fig. 7·6 (*f*)],

$EI\delta_{BB}$=B.M. at B due to loading of Fig. 7·6 (*f*)

$$=\left(\frac{L^2}{4}\times L\right)-\left(\frac{1}{2}\times L\times\frac{L}{2}\right)\left(\frac{1}{3}\;L\;\right)=\frac{L^3}{6}$$

and $EI\delta_{DB}$=B.M. at D due to loading of Fig. 7·6 (*f*)

$$=\left(\frac{L^2}{4}\times\frac{L}{2}\right)-\left(\frac{1}{2}\times\frac{L}{2}\times\frac{L}{4}\right)\left(\frac{1.L}{3\times 2}\right)=\frac{11L^3}{96}$$

Substituting in (2), we get

$$W.\;\frac{11L^3}{96}=V_B.\;\frac{L^3}{6}$$

or $$V_B=\frac{11}{16}\,W\uparrow$$

Taking moment about C, we get

$$V_A.2L+\frac{11}{16}\,WL=W.\;\frac{3}{2}\,L$$

or $$V_A=\frac{WL}{2L}\left(\frac{3}{2}-\frac{11}{16}\right)=\frac{13}{32}\,W\uparrow$$

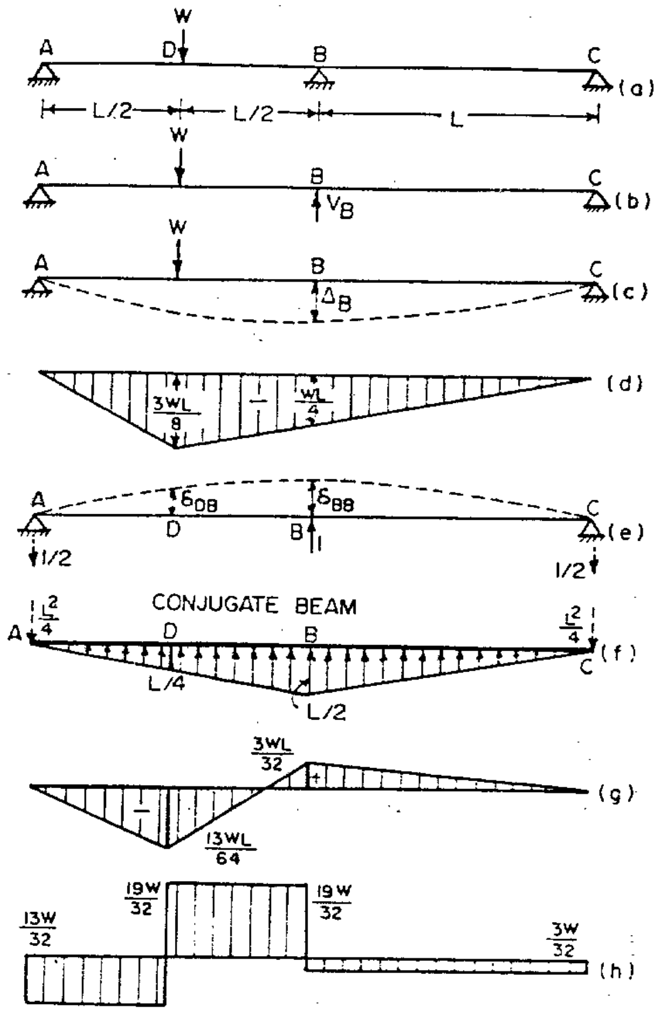


Fig. 7·6.

$$\therefore \quad V_C=W-\left(\frac{11}{16}W+\frac{13}{32}W\right)=\frac{32-22-13}{32}W$$

$$=-\frac{3}{32}W=\frac{3}{32}W\downarrow$$

B.M.D.

$$M_A=0,\ M_D=-\frac{13}{32}W.\frac{L}{2}=-\frac{13}{64}WL$$

$$M_B=+\frac{3}{32}WL.$$

The B.M.D. and S.F.D. have been shown in Fig. 7.6 (*g*) and 7·6 (*h*) respectively.

PROBLEMS

1. A beam is fixed at both the ends and carries a central point load. Find the support moments.

2. A cantilever of span L is propped at the free end. Calculate the prop reaction if it carries a uniformly distributed load of w per unit length.

3. A fixed beam of length L is loaded at third points by two point loads of W each. Calculate the fixing moments and plot the B.M. and S.F. diagrams.

4. A beam AB, of flexural rigidity EI and span L carries a uniformly distributed load of intensity w per unit length. It is encastre at A and B but support B settled during the application of the load by an amount δ. Show that if $\delta=\frac{wL^4}{72EI}$, there is no fixing moment at B.

Answers

1. $M_A=M_B=\frac{WL}{8}$.

2. $V_B=\frac{3}{8}wL$.

3. $\frac{2}{9}WL$.

8

Three Moment Equation Method

8·1. CLAPEYRON'S THEOREM OF THREE MOMENTS

General Statement

Let us consider two consecutive spans AB and BC of a continuous beam, loaded with *any system* of loading. The μ and μ' diagrams can be constructed as usual.

Fig. 8·1 (*a*) shows two consecutive spans AB-BC of a continuous beam with any type of loading. Let suffix 1 (*i.e.* L_1, E_1, I_1, etc.) stand for span AB, and suffix 2 (*i.e.* L_2, E_2, I_2, etc.) stand for span BC.

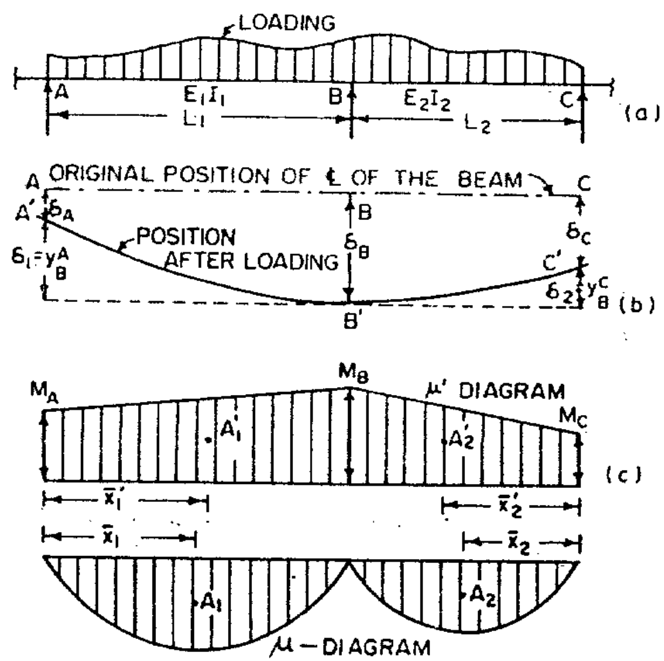


Fig. 8·1.

Fig. 8·1 (*b*) shows the deflected shape of the two spans after loading, in which the three supports A, B and C have settled to position A', B' and C' by the amounts δ_A, δ_B, δ_C respectively, below the original centre line.

Then $y_B^A = \delta_B - \delta_A = \delta_1$ (say)

where y_B^A = deflection of B with respect to A

Similarly, $y_B^C = \delta_B - \delta_C = \delta_2$ (say)

where y_B^C is the deflection of B with respect to C.

Fig. 8·1 (*c*) shows the fixing moment diagram, and Fig. 8·1 (*d*) shows the free bending moment diagram for the two spans.

Taking span AB first, and measuring x positive to the right, we have

$$E_1 I_1 \frac{d^2y}{dx^2} = \mu_x + \mu_x', \text{ with usual notations.}$$

Multiplying both sides by x and integrating over the range $x=0$ to $x=L_1$, we get

$$E_1 I_1 \left[x\frac{dy}{dx} - y \right]_0^{L_1} = \int_0^{L_1} \mu_x x\,dx + \int_0^{L_1} \mu_x' x\,dx$$

At $x = L_1$, $\frac{dy}{dx} = i_B$ and $y_B^A = \delta_1$

$$\therefore \quad E_1 I_1 (L_1 i_B - \delta_1) = A_1 \bar{x}_1 + A_1' \bar{x}_1'$$

From which $i_B = \frac{1}{E_1 I_1 L_1}(A_1\bar{x}_1 + A_1'\bar{x}') + \frac{\delta_1}{L_1}$ (1)

Similarly, considering span BC, taking C as origin, and x positive to the left, we can obtain,

$$i_B' = \frac{1}{E_2 I_2 L_2}(A_2\bar{x}_2 + A_2'\bar{x}_2') + \frac{\delta_2}{L_2} \qquad (2)$$

Due to the continuity of the beam, $i_B = -i_B'$

Hence adding (1) and (2), we get

$$\frac{1}{E_1 I_1 L_1}(A_1\bar{x}_1 + A_1'\bar{x}_1') + \frac{\delta_1}{L_1} + \frac{1}{E_2 I_2 L_2}(A_2\bar{x}_2 + A_2'\bar{x}_2') + \frac{\delta_2}{L_2} = 0$$

Substituting $A_1'\bar{x}_1' = (M_A + 2M_B)\frac{L_1^2}{6}$, and

$$A_2'\bar{x}_2' = (M_C + 2M_B)\frac{L_2^2}{6}, \text{ we get}$$

$$\frac{A_1\bar{x}_1}{E_1 I_1 L_1} + \frac{L_1}{6E_1 I_1}(M_A + 2M_B) + \frac{\delta_1}{L_1} + \frac{A_2\bar{x}_2}{E_2 I_2 L_2} + \frac{L_2}{6E_2 I_2}(M_C + 2M_B) + \frac{\delta_2}{L_2} = 0$$

Multiplying by 6 and rearranging, we get,

$$M_A \frac{L_1}{E_1 I_1} + 2M_B\left(\frac{L_1}{E_1 I_1} + \frac{L_2}{E_2 I_2}\right) + M_C \frac{L_2}{E_2 I_2} + \left(\frac{6A_1\bar{x}_1}{E_1 I_1 L_1} + \frac{6A_2\bar{x}_2}{E_2 I_2 L_2}\right) + 6\left(\frac{\delta_1}{L_1} + \frac{\delta_2}{L_2}\right) = 0 \qquad (8\cdot1)$$

This is the *generalised theorem of three moments*. While substituting the *numerical values* of A_1 and A_2 for a given loading system, proper care of the *sign* must be taken. For usual downward loading, A_1 and A_2 will be negative. Let us use the above equation for some special cases.

8·2. EI CONSTANT : GENERAL LOADING

If $E_1 I_1 = E_2 I_2 = EI$, then, from Eq. 8·1, we get

$$M_A L_1 + 2M_B(L_1 + L_2) + M_C L_2 + \frac{6A_1\bar{x}_1}{L_1} + \frac{6A_2\bar{x}_2}{L_2} + 6EI\left(\frac{\delta_1}{L_1} + \frac{\delta_2}{L_2}\right) = 0 \qquad (8\cdot2)$$

8·3. EI CONSTANT : NO SETTLEMENT

If $E_1 I_1 = E_2 I_2 = EI$

and $\delta_1 = 0$; $\delta_2 = 0$, we have

$$M_A L_1 + 2M_B(L_1 + L_2) + M_C L_2 + \frac{6A_1\bar{x}_1}{L_1} + \frac{6A_2\bar{x}_2}{L_2} = 0 \qquad (8.3)$$

8·4. EI CONSTANT : U.D.L. ON BOTH SPANS

If the beams have constant EI, and if the supports do not yield Eq. 8·3 will be applicable. Let w_1 and w_2 be the U.D.L. on the two spans respectively.

Then $A_1 = -\frac{2}{3}\left(\frac{w_1 L_1^2}{8}\right)L_1 = -\frac{w_1 L_1^3}{12}$

$$A_1\bar{x}_1 = -\frac{w_1 L_1^3}{12} \times \frac{L_1}{2} = -\frac{w_1 L_1^4}{24}$$

Similarly, $A_2\bar{x}_2 = -\frac{w_2 L_2^4}{24}$

Substituting these in Eq. 8·3, we get the special form

$$M_A L_1 + 2M_B(L_1 + L_2) + M_C L_2 + \frac{6}{L_1}\left(-\frac{w_1 L_1^4}{24}\right) + \frac{6}{L_2}\left(-\frac{w_2 L_2^4}{24}\right) = 0$$

$$M_A L_1 + 2M_B(L_1 + L_2) + M_C L_2 = \frac{w_1 L_1^3}{4} + \frac{w_2 L_2^3}{4} \qquad (8\cdot4)$$

If however, the supports settle, the above equation can be modified as follows :

$$M_AL_1+2M_B(L_1+L_2)+McL_1+6EI\left(\frac{\delta_1}{L_1}+\frac{\delta_2}{L_2}\right)=\frac{w_1L_1^3}{4}+\frac{w_2L_2^3}{4} \quad (8\cdot5)$$

8·5. FIXED BEAM

If the beam is fixed at both the ends, the three moment theorem can be used for finding out the support moments by imagining a zero span to the left of A and zero span to the right of B as shown in Fig. 8·2.

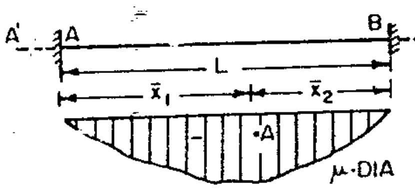


Fig. 8·2.

Thus, applying three moments theorem for the spans $A'A$-AB, we get

$$0+2M_A(0+L)+M_BL+0+\frac{6A\bar{x}_2}{L}=0$$

or
$$2M_AL+M_BL+\frac{6A\bar{x}_2}{L}=0$$

or
$$2M_A+M_B+\frac{6A\bar{x}_2}{L^2}=0 \quad (1)$$

Similarly, for span AB-BB'

$$M_AL+2M_B(L+0)+0+\frac{6A\bar{x}_1}{L}=0$$

or
$$M_A+2M_B+\frac{6A\bar{x}_1}{L^2}=0 \quad (2)$$

Solving (1) and (2), M_A and M_B can be easily found.

Example 8·1. *A beam ABC of length 2L rests on three supports equally spaced and is loaded with U.D.L. w/unit length throughout the length of the beam as shown in Fig. 8·3. Plot the B.M. and S.F. diagrams.*

Solution.

Applying the three moment theorem for U.D.L. (Eq. 8·4), we get

$$M_AL+2M_B(L+L)+McL=\frac{wL^3}{4}+\frac{wL^3}{4}$$

But $\quad M_A=0$ and $Mc=0$

$\therefore \quad 4M_BL=\frac{wL^3}{2}$ or $M_B=\frac{wL^2}{8}$

For the reaction R_A, write the equation of B.M. at B. Thus,

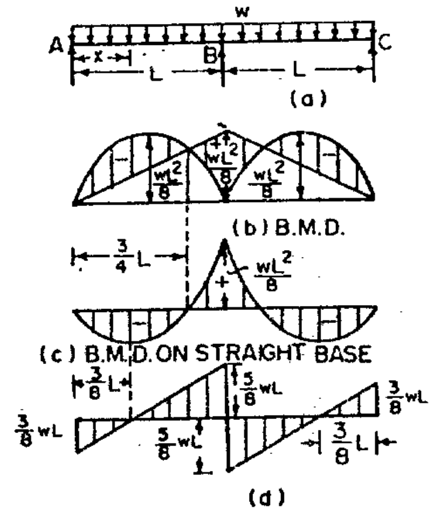


Fig. 8·3

$$-R_AL+wL\,\frac{L}{2}=M_B=+\frac{wL^2}{8}$$

$$\therefore \quad R_A=\frac{wL}{2}-\frac{wL}{8}=\frac{3}{8}\,wL$$

$$Rc=\frac{3}{8}\,wL \text{ by symmetry.}$$

$$\therefore \quad R_B=2wL-(R_A+Rc)=2wL-\frac{3}{4}\,wL=\frac{5}{4}\,wL$$

The equation for B.M. in span AB is

$$M_x=-\frac{3}{8}\,wLx+\frac{wx^2}{2}$$

At $\quad x=L,\ M_B=-\frac{3}{8}\,wL^2+\frac{wL^2}{2}=\frac{wL^2}{8}$

For point of inflexion,

$$-\frac{3}{8}\,wLx+\frac{wx^2}{2}=0 \text{ which gives } x=\frac{3}{4}L.$$

The B.M.D. can be drawn by superimposing μ-diagram over μ-diagram as shown in Fig. 8·3 (b).

Example 8·2. *A cantilever beam ABCD covers three spans, AB=6 m, BC=12 m and CD=4 m. It carries uniformly spread loads of 2 kN, 1 kN and 3 kN per metre run on AB, BC and CD respectively. If the girder is of same cross-section throughout, find the bending moment at the supports B and C and the pressure on each support. Plot the B.M. and S.F. diagrams.*

Solution

The free B.M. diagrams for *AB*, *BC* and *CD* can be constructed as usual.

For *AB*, $M_{max.}=\frac{wL^2}{8}=\frac{2\times6\times6}{8}=9$ kN·m

For *BC*, $M_{max.}=\frac{1\times12\times12}{8}=18$ kN-m

For *CD*, $M_{max.}=\frac{3\times4\times4}{8}=6$ kN-m

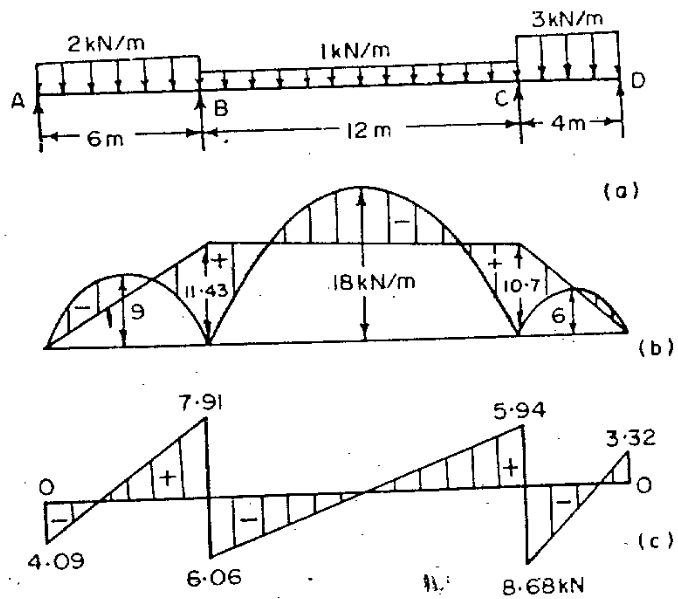


Fig. 8·4.

Applying the three moment theorem (Eq. 8·4) for spans *AB* and *BC*,

$$M_A\times6+2M_B(6+12)+M_C12=\frac{2\times6^3}{4}+\frac{1\times12^3}{4}$$

or $\quad 6M_A+36M_B+12M_C=108+432$

or $\quad 36M_B+12M_C=540$, since $M_A=0$

or $\quad M_B+0{\cdot}33\ M_C=15 \qquad (1)$

Similarly, for spans *BC-CD* :

$$M_B\times12+2M_C(12+4)+M_D\times4=\frac{1\times12^3}{4}+\frac{3\times4^3}{4}$$

$$12M_B+32M_C=432+48=480, \text{ since } M_D=0$$

or $\quad M_B+2{\cdot}667\ M_C=40 \qquad (2)$

From (1) and (2), we get

$$M_C=+\mathbf{10{\cdot}71} \text{ and } M_B=+\mathbf{11{\cdot}43\ kN\text{-}m}$$

For reaction at *A*, write expression for B.M. at *B*.

Thus $\quad -(R_A\times6)+(6\times2\times3)=M_B=+11{\cdot}43$

or $\quad R_A=\frac{36-11{\cdot}43}{6}=\frac{24{\cdot}57}{6}=\mathbf{4{\cdot}09\ kN}$

For reaction at *B*, write expression for B.M. at *C*.

Thus, $\quad -R_A\times18-R_B\times12+6\times2(12+3)+1\times12\times6$
$=M_C=+10{\cdot}71$

or $\quad 12R_B=-(4{\cdot}09\times18)+(12\times15)+72-10{\cdot}71$

or $\quad R_B=\frac{167{\cdot}66}{12}=\mathbf{13{\cdot}97\ kN}$

Similarly for R_D, write equation for B.M. at *C* :

$$-R_D\times4+(3\times4\times2)=M_C=+10{\cdot}71$$

or $\quad R_D=\frac{24-10{\cdot}71}{4}=3{\cdot}32$ kN

Here, $\quad R_C=(2\times6)+(1\times12)+(3\times4)-(R_A+R_B+R_D)$
$=36-(4{\cdot}09+13{\cdot}97+3{\cdot}32)=36-21{\cdot}38=\mathbf{14{\cdot}62\ kN}$

The B.M. and S.F. diagrams are shown in Fig. 8·4.

Example 8·3. *A continuous beam ABCD, 20 m long is carried on supports at its end and is propped at the same level at points 5 m and 12 m from left end A. It carries two concentrated loads of 80 kN and 50 kN at 2 m and 9 m respectively from A and uniformly distributed load of 10 kN/m run over the span CD. Find the B.M. at the reactions at the four supports.*

Solution

The free B.M. diagrams for three spans can be drawn as usual.

For span AB

$$M_{max.}=\frac{Wab}{L}=\frac{80\times2\times3}{5}=96 \text{ kN-m}$$

$$A=-\frac{1}{2}\times 5\times 96=-240\ ;\ \bar{x}=\frac{1}{3}(5+2)=\frac{7}{3}$$

$$\therefore\quad A\bar{x}=-240\times\frac{7}{3}=-560, \text{ with } A \text{ as origin.}$$

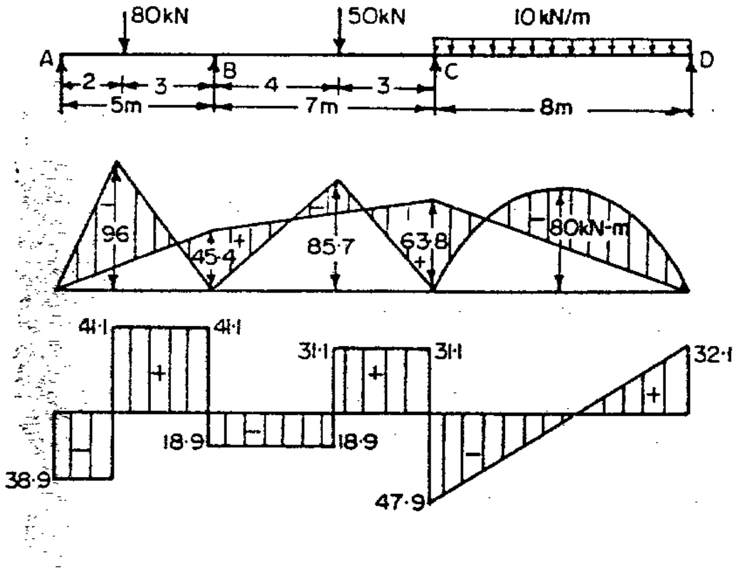


Fig. 8·5

For span BC

$$M_{max.}=\frac{50\times 4\times 3}{7}=85\cdot 7 \text{ kN-m}$$

with C as origin, $A\bar{x}=-(\frac{1}{2}\times 7\times 85\cdot 7)\frac{1}{3}(7+3)=-1000$

with B as origin, $A\bar{x}=-(\frac{1}{2}\times 7\times 85\cdot 7)\frac{1}{3}(7+4)=-1100$

For span CD

$$M_{max}=\frac{10\times 8^2}{8}=80 \text{ kN-m}$$

$$A\bar{x}=-\frac{2}{3}\times 80\times 8\times\frac{8}{2}=-1706\cdot 7, \text{ with } D \text{ as origin.}$$

$$M_D=0$$

Applying three moments theorem for span AB-BC,

$$5M_A+2M_B(5+7)+7M_C+\frac{6A_1\bar{x}_1}{5}+\frac{6A_2\bar{x}_2}{7}=0$$

or $\quad 5\times 0+24M_B+7M_C=\dfrac{6\times 560}{5}+\dfrac{6\times 1000}{7}=673+857=1530$

or $\quad M_B+0\cdot 29M_C=63\cdot 8$ (since $M_A=0$) $\quad$ (1)

Similarly, for span BC-CD

$$7M_B+2M_C(7+8)+8M_D+\frac{6A_1\bar{x}_1}{7}+\frac{6A_1\bar{x}_2}{8}=0$$

or $\quad 7M_B+30M_C+0=\dfrac{6\times 1100}{6}+\dfrac{6\times 1706\cdot 7}{8}=943+1280=2223$

or $\quad M_B+4\cdot 28M_C=317\cdot 6$ (since $M_D=0$) $\quad$ (2)

Solving (1) and (2), we get

$$M_C=\frac{317\cdot 6-63\cdot 8}{4\cdot 28-0\cdot 29}=63\cdot 5 \text{ kN-m}$$

and $\quad M_B=63\cdot 8-0\cdot 29\times 63\cdot 5=45\cdot 5$ **kN-m**

For reaction at A, write equation for M_B :

$(-R_A\times 5)+(80\times 3)=M_B=+45\cdot 4$

or $\quad R_A=\dfrac{240-45\cdot 4}{5}=\dfrac{194\cdot 6}{5}=38\cdot 9$ **kN**

For reaction at B, write equation for M_C :

$-R_A(5+7)-R_B\times 7+80(3+7)+(50\times 3)=M_C=+63\cdot 5$

or $\quad -(38\cdot 9\times 12)-7R_B+800+150=63\cdot 5$

$\therefore\quad R_B=60$ **kN**

For reaction at D, write expression for M_C ;

$-R_D\times 8+(10\times 8\times 4)=M_C=+63\cdot 5$

or $\quad R_D=\dfrac{320-63\cdot 5}{8}=32\cdot 1$ kN

For reaction at C,

$R_C=(80+50+80)-(R_A+R_B+R_D)$

$=210-(38\cdot 9+60+32\cdot 1)=79$ **kN.**

The B.M. and S.F. diagrams are shown in Fig. 8·6.

Example 8·4. *Solve example 8·3 if the support B sinks by 10 mm below A and C. Moment of inertia for the whole beam* $=85\times 10^6$ mm^4 *and* $E=2\cdot 1\times 10^6$ N/mm^2.

Solution

Applying three moment theorem for span AB-BC,

$$24M_B+7M_C-1530+6EI\left(\frac{\delta_1}{L_1}+\frac{\delta_2}{L_2}\right)=0$$

in which $\dfrac{6A_1\bar{x}_1}{L_1}+\dfrac{6A_2\bar{x}_2}{L_2}=-1530$ from example 8·3.

While substituting the numerical values of E_1, I_1, δ_1 and δ_2 proper care of units must be taken.

$$E=2\cdot 1\times 10^5 \text{ N/mm}^2=210 \text{ kN/mm}^2$$

$$I=85\times 10^6 \text{ mm}^4$$

$$\therefore \quad EI = 210 \times 85 \times 10^6 = 17850 \times 10^6 \text{ kN-mm}^2 = 17850 \text{ kN-m}^2$$

$$\delta_1 = 10 \text{ mm} = \frac{10}{1000}\text{ m} = \frac{1}{100} \text{ m} = \delta_2$$

Thus, substituting all the values in kN and m units,

$$24M_B + 7M_C - 1530 + 6 \times 17850\left(\frac{1}{100 \times 5} + \frac{1}{100 \times 7}\right) = 0$$

or $$M_B + 0{\cdot}29\, M_C = 48{\cdot}5 \qquad (1)$$

For span *BC-CD*,

δ_1 = movement of C with respect to $B = \times 10$ mm

(the movement of C being upwards with respect to B)

δ_2 = movement of C with respect to $D = 0$

$$\therefore \quad 7M_B + 30M_C - 2223 + 6 \times 17850\left(-\frac{1}{100 \times 7} + 0\right)$$

or $$M_B + 4{\cdot}28 M_C = 339{\cdot}6 \qquad (2)$$

Solving (1) and (2), we get

$M_C = 73$ kN-m

and $M_B = 27{\cdot}4$ kN-m.

The reactions at various supports can now be found in the same manner as illustrated in the previous examples.

Example 8·5. *Solve example 8·3 if the end A is fixed and D is simply supported.*

Solution

Imagine a point A' to the left of A such that $AA' = 0$

$\therefore$ For spans $A'A$ and AB

$$0 + 2M_A(5+0) + M_B \times 5 - \left(\frac{1}{2} \times 5 \times 96\right)\left\{\frac{1}{3}(5+3)\right\} = 0$$

or $$10M_A + 5M_B - 640 = 0$$

or $$M_A + 0{\cdot}5M_B = 64 \qquad (1)$$

For span *AB-BC*, we have

$$5M_A + 24M_B + 7M_C = 1530 \text{ (as in example 8·3)}$$

or $$M_A + 4{\cdot}8M_B + 1{\cdot}4\, M_C = 306 \qquad (2)$$

For span *BC-CD*, we have

$$M_B + 4{\cdot}28M_C = 317{\cdot}6 \text{ (as in example 8·3)} \qquad (3)$$

From (1) and (2),

$$4{\cdot}3M_B + 1{\cdot}4\, M_C = 242$$

or $$M_B + 0{\cdot}325M_C = 56{\cdot}2 \qquad (4)$$

From (3) and (4), we have $M_C = \dfrac{261{\cdot}4}{3{\cdot}955} = 66$ **kN-m**

$\therefore$ $M_B = 317{\cdot}6 - 283 = 34{\cdot}6$ **kN-m**

$M_A = 64 - 17{\cdot}3 = 46{\cdot}7$ **kN-m**

For reaction at D,

$$-R_D \times 8 + (8 \times 10 \times 4) = M_C = 66$$

or $$R_D = \frac{320 - 66}{8} = \frac{254}{8} = 31{\cdot}8 \text{ kN}$$

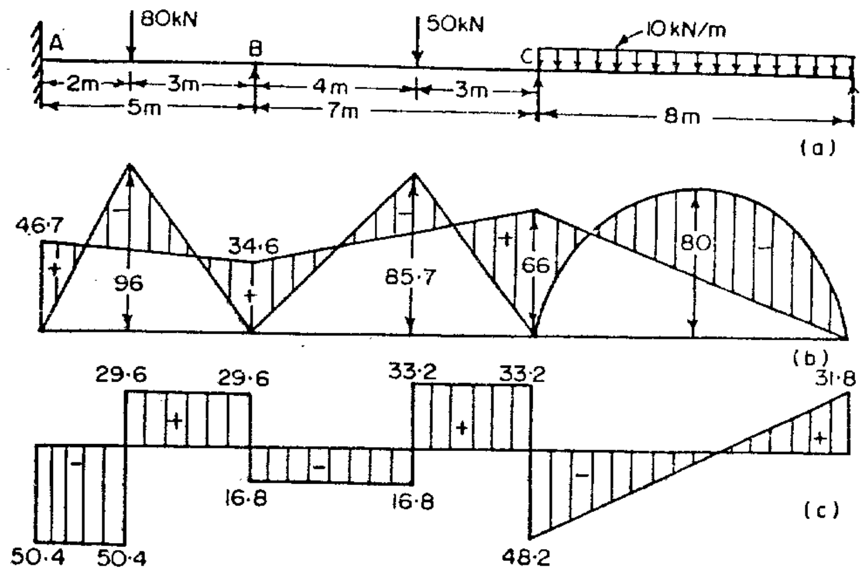


Fig. 8·6

For reaction at C,

$$(-R_D \times 15) - (R_C \times 7) + (80 \times 11) + (50 \times 4) = M_B = 34{\cdot}6$$

or $$7R_C = -476 + 880 + 200 - 34{\cdot}6$$

$$\therefore \quad R_C = 81{\cdot}4$$

For reaction at A,

or $$-R_A \times 5 + M_A + 80 \times 3 = M_B$$

or $$R_A = \frac{M_A - M_B + 240}{5} = \frac{46{\cdot}7 - 34{\cdot}6 + 240}{5} = 50{\cdot}4$$

For reaction at B,

$$R_B = (80 + 50 + 80) - (50{\cdot}4 + 81{\cdot}4 + 31{\cdot}8) = 46{\cdot}4$$

The B.M. and S.F. diagrams are shown in Fig. 8·6.

Example 8·6. *A straight elastic beam of uniform section rests on four similar elastic supports which are placed L metres apart. The supports are such that they are compressed by d for each unit of load* *upon them. Show that when a uniformly distributed load of total amount W comes on the beam, the reactions at central supports are each.*

$$\frac{W\left(\frac{11}{6}+\frac{3EId}{L^3}\right)}{\left(5+\frac{12EId}{L^3}\right)}.$$

Solution. (Fig. 8·7)

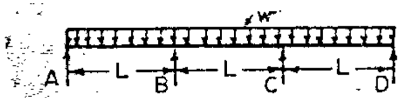


Fig. 8·7

U.D.L $\quad w=\frac{W}{3L}$ per unit length.

Sinking of support $A=R_A.d$; Sinking of support $B=R_B.d$.

Sinking of support $C=R_C.d$; Sinking of support $D=R_D.d$.

Applying the three moment theorem for the span AB-CD (Eq. 8·5)

$$M_A.L+2M_B.2L+M_C.L+6EI\left(\frac{\delta_1}{L}+\frac{\delta_2}{L}\right)=\frac{wL^3}{4}+\frac{wL^3}{4}=\frac{wL^3}{2} \qquad (1)$$

δ_1=deflection of B with respect to $A=\delta_B-\delta_A=(R_B-R_A)d$

δ_2=deflection of B with respect to $C=\delta_B-\delta_C=(R_B-R_C)d=0$

Since $R_B=R_C$, by symmetry,

$M_A=0$; $M_C=M_B$ by symmetry ; $w=\frac{W}{3L}$

Substituting the above values in (1), we get

$$4M_BL+M_BL+\frac{6EId}{L}(R_B-R_A)=\frac{W}{3L}\cdot\frac{L^3}{2}=\frac{WL^2}{6}$$

$$\therefore \quad 5M_BL+\frac{6EId}{L}(R_B-R_A)=\frac{WL^2}{6}$$

or $$M_B=\frac{1}{5}\left[\frac{WL}{6}-\frac{6EId}{L^2}(R_B-R_A)\right] \qquad (2)$$

Taking moments at B,

$$M_B=M_A-R_AL+\left(\frac{W}{3L}\right)\frac{L^2}{2}=0-R_AL+\frac{W}{6}L$$

$$=L\left(\frac{W}{6}-R_A\right) \qquad (3)$$

Also, $2R_B+2R_A=W$

or $$R_A=\left(\frac{W}{2}-R_B\right)$$

Substituting the values of M_B and R_A in (2), we get

$$L\left\{\frac{W}{6}-\left(\frac{W}{2}-R_B\right)\right\}=\frac{1}{5}\left[\frac{WL}{6}-\frac{6EId}{L^2}\left\{R_B-\left(\frac{W}{2}-R_B\right)\right\}\right]$$

or $$\frac{5WL}{6}-\frac{5}{2}WL-5R_B.L=\frac{WL}{6}-\frac{12EId}{L^2}R_B+\frac{3EIdW}{L^2}$$

or $$R_B\left(\frac{12EId}{L^2}+5L\right)=\frac{3EIdW}{L^2}+\frac{11}{6}WL$$

or $$R_B=\left[\frac{\frac{3EIWd}{L^2}+\frac{11}{6}WL}{\frac{12EId}{L^2}+5L}\right]=\frac{W\left(\frac{11}{6}+\frac{3EId}{L^3}\right)}{\left(5+\frac{12EId}{L^3}\right)}$$

Hence proved.

Example 8·7. *A bridge of uniform cross-section rests on rigid abutments at the ends and three equal pontoons as shown in Fig. 8·8 and has a concentrated load W, at the middle. When the bridge is unloaded the pontoons just touch it without exerting any force. With the load W at the middle and the two end pontoons removed the central deflection is one half what it would be with no pontoons.*

Find the reactions and draw the bending moment diagram for the bridge due to central load with three pontoons in position.

Solution. (Fig. 8·8)

Due to symmetry, $R_A=R_E$; $R_B=R_D$

Let the settlement of any pontoon be $=Rk$, where R is the reaction and k is a constant.

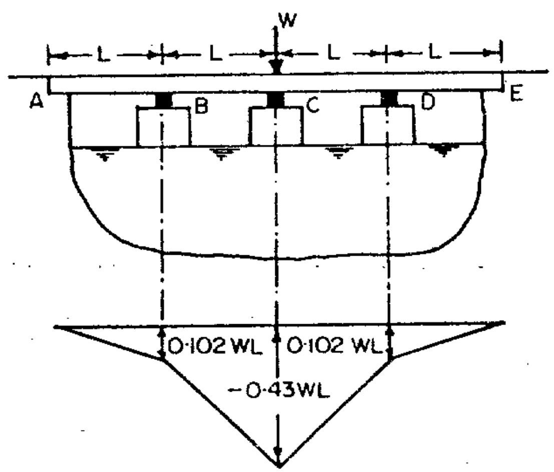


Fig. 8·8.

The value of k can be determined from the data given in the problem when there is only a central pontoon. In such a case, let P=reaction on the pontoon.

$\therefore$ Central deflection $=\frac{(W-P)(4L)^3}{48EI}=\frac{4(W-P)}{3EI}L^3$

with the pontoon.

With no pontoon, deflection $=\frac{W(4L)^3}{48\,EI}=\frac{4WL^3}{3EI}$

As per given condition. $\frac{4(W-P)L^3}{3EI}=\frac{1}{2}\,\frac{4WL^3}{3EI}$

from which, $P=\frac{W}{2}$.

Central deflection with the pontoon $=Pk=\frac{W}{2}k$.

But it is equal to $\frac{1}{2}\,\frac{4WL^3}{3\,EI}$

$\therefore \quad \frac{W}{2}k=\frac{1}{2}\,\frac{4WL^3}{3EI}$

or $\quad k=\frac{4L^3}{3EI}$; Also $\frac{6EIk}{L^2}=8L$.

Applying three moments theorem for span AB-BC.

$$M_A.L+2M_B(L+L)+M_CL+6EI\left(\frac{\delta_1}{L}+\frac{\delta_2}{L}\right)+\frac{6A_1\bar{x}_1}{L}+\frac{6A_2\bar{x}_2}{L}=0$$

Here, $M_A=0$, $A_1=0$, $A_2=0$ (since there is no loading on AB and BC)

$\delta_1=\delta_B-\delta_A=(R_Bk-0)=R_Bk$, since $\delta_A=0$

$\delta_2=\delta_B-\delta_C=(R_B-R_C)k$

$\therefore \quad 4M_BL+M_CL+\frac{6EIk}{L}(2R_B-R_C)=0$

or $\quad 4M_B+M_C+\frac{6EIk}{L^2}(2R_B-R_C)=0$

But $\quad \frac{6EIk}{L^2}=8L$

Hence $4M_B+M_C+8L(2R_B-R_C)=0$ (1)

For the span BC-CD,

$$M_BL+2M_C(2L)+M_DL+\frac{6EIk}{L}(R_C-R_B+R_C-R_D)=0$$

But $R_B=R_D$ and $M_B=M_D$

$\therefore \quad M_B+2M_C=\frac{6EIk}{L^2}(R_C-R_B)=0$ ...(2)

or $\quad M_B+2M_C+8L(R_C-R_B)0$ (2)

Now, $\quad -R_AL=M_B$ or $R_A=-\frac{M_B}{L}$

and $\quad -R_BL-R_A2L=M_C$

or $\quad R_B=-\frac{M_C}{L}-2R_A=-\frac{M_C}{L}+\frac{2M_B}{L}=\frac{2M_B-M_C}{L}$

Hence $\quad R_C=W-2R_A-2R_B=W+\frac{2M_B}{L}-\frac{4M_B-2M_C}{L}$

$=W+\frac{2(M_C-M_B)}{L}$

Substituting the above values in (1) and (2) we get

$4M_B+M_C+8(4M_B-2M_C-WL-2M_C+2M_B)=0$ (from 1)

or $\quad 52M_B-31M_C=8WL$ (4)

and $\quad M_B+2M_C+8(WL+2M_C-2M_B+M_C)=0$ (from 2)

or $\quad 31M_B+26M_C=8WL$ (5)

Solving (4) and (5), we get

$M_B=-0{\cdot}102\,WL=M_D$ and $M_C=-0{\cdot}43\,WL$

The B.M.D. has been shown in Fig. 8·8.

Example 8·8. *A continuous beam ABCD has a weight w per unit length and rests on four knife edge supports A, B, C and D. The middle span carries a load W as shown in Fig. 8·9. Find (a) bending moments at B and C, (b) reactions at A and D.*

Solution

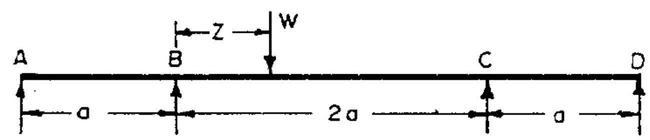


Fig. 8·9

For span AB,

$M_{max}=\frac{wa^2}{8}$

$A\bar{x}=-\frac{2}{3}\times\frac{wa^2}{8}\times a\times\frac{a}{2}=-\frac{wa^4}{24}$

For span BC,

$M_{max}=-\frac{w(2a)^2}{8}$ due to U.D.L. $=-\frac{wa^2}{2}$

$M_{max}=-\frac{Wz(2a-z)}{2a}$ due to point load.

$\therefore$ $A\bar{x}$ (with B as origin) $= -\left[\left\{\frac{2}{3}\times\frac{wa^2}{2}\times 2a\times a\right\} + \left\{\frac{1}{2}\,\frac{Wz\,(2a-z)}{2a}\times 2a\times\frac{1}{3}\,(2a+z)\right\}\right]$

$$= -\frac{2wa^4}{3} - \frac{Wz}{6}\,(2a-z)\,(2a+z)$$

$A\bar{x}$ (with C as origin) $= -\left[\left\{\frac{2}{3}\times\frac{wa^2}{2}\times 2a\times a\right\} + \left\{\frac{1}{2}\,\frac{Wz(2a-z)}{2a}\times 2a\times\frac{1}{3}\,(2a+2a-z)\right\}\right]$

$$= -\frac{2}{3}\,wa^4 - \frac{Wz}{6}\,(2a-z)(4a-z)$$

Applying three moment equation for span AB BC,

$$M_A.a + 2M_B\,(a+2a) + M_C.2a = \left[\frac{6}{a}\times\frac{Wa^4}{24}\right] + \frac{6}{2a}\left[\frac{2}{3}\,wa^4 + \frac{Wz}{6}\,(2a-z)(4a-z)\right]$$

or $\quad 6M_B + 2M_C = \frac{9}{4}\,wa^2 + \frac{Wz}{2a^2}\,(2a-z)(4a-z) \qquad (1)$

Similarly applying three moment equation for spans BC-CD

$$M_B.2a + 2M_C(2a+a) + M_D.a = \frac{6}{2a}\left[\frac{2}{3}\,wa^4 - \frac{Wz}{6}(2a-z)(2a+z)\right] + \left[\frac{6}{a}\times\frac{wa^4}{24}\right]$$

or $\quad 2M_B + 6M_C = \frac{9}{4}\,wa^2 + \frac{Wz}{2a^2}\,(2a-z)(2a+z) \qquad (2)$

From (1) and (2), we get

$$M_B = \frac{9}{32}\,wa^2 + \frac{Wz}{16a^2}\,(2a-z)(5a-2z)$$

and $\quad M_C = \frac{9}{32}\,wa^2 + \frac{Wz}{16a^2}\,(2a-z)(a+2z)$

For reaction at A, write equation for M_B

$$\therefore \quad -(R_A\times a) + \frac{wa^2}{2} = M_B$$

$$\therefore \quad R_A = \frac{1}{2}\,wa - \frac{M_B}{a} = \frac{1}{2}\,wa - \frac{9}{32}\,wa - \frac{Wz}{16a^3}\,(2a-z)(5a-2z)$$

or $\quad R_A = \frac{7}{32}\,wa - \frac{Wz}{16a^3}\,(2a-z)(5a-2z)$

Similarly,

$$R_D = \frac{1}{2}\,wa - \frac{M_C}{a} = \frac{7}{32}\,wa - \frac{Wz}{16a^3}(2a-z)(a+2z).$$

Example 8·9. *For a three span beam shown in Fig. 8·10(a) find the reactions and support moments, and draw the B.M. and S.F. diagrams.*

Solution

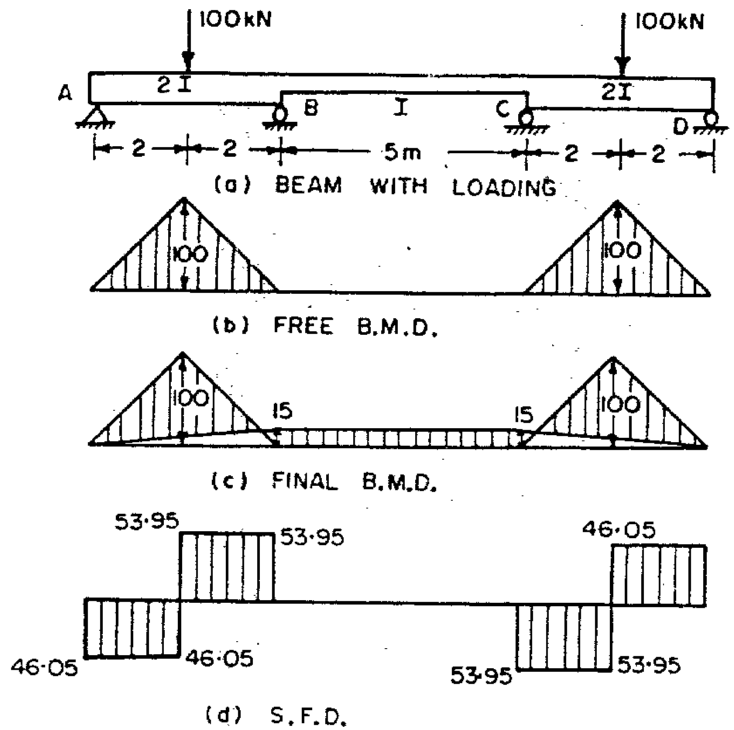


Fig. 8·10.

Here the moment of inertia is variable for each span, but E is the same. Hence from Eq. 8·1

$$M_A.\frac{L_1}{I_1} + 2M_B\left(\frac{L_1}{I_1}+\frac{L_2}{I_2}\right) + M_C\frac{L_2}{I_2} + \frac{6A_1\bar{x}_1}{I_1L_1} + \frac{6A_2\bar{x}_2}{I_2L_2} = 0$$

The free B.M.D. is shown in Fig. 8·10 (*b*).

For span AB

$$A\bar{x} = -\frac{1}{2}\times 4\times 100\times 2 = -400$$

For span BC

$A\bar{x}$ = zero, since there is no loading on BC.

Now $M_A\dfrac{L_1}{I_1}+2M_B\left(\dfrac{L_1}{I_1}+\dfrac{L_2}{I_2}\right)+M_C\dfrac{L_2}{I_2}+\dfrac{6A_1\bar{x}_1}{I_1L_1}+\dfrac{6A_2\bar{x}_2}{I_2L_2}=0$

Here $M_A=0$ and $M_B=M_C$, due to symmetry

$$\therefore \quad 2M_B\left(\frac{4}{2I}+\frac{5}{I}\right)+M_B\left(\frac{5}{I}\right)=\frac{6\times 400}{2I(4)}+0$$

or $\quad 14\,M_B+5M_B=300$

From which $M_B=15{\cdot}789$ kN-m $=M_C$

For reaction R_A, take moments about B,

$$-R_A(4)+100\,(2)=M_B$$

or $\quad R_A=\dfrac{200}{4}-\dfrac{M_B}{4}=50-\dfrac{15{\cdot}789}{4}=$ **46·05 kN** $=R_D$

Hence $R_B=100-R_A=100-46{\cdot}05=$ **53·95 kN** $=R_C$

The B.M. and S.F. diagrams are shown in Fig. 8·10 (*c*) and (*d*) respectively.

Example 8·10. *A two span continuous beam, fixed at the ends is loaded as shown in Fig. 8·11(a). Find the reactions and support moments and draw the B.M. and S.F. diagrams.*

Solution. The free B.M. diagrams for each span are shown in Fig. 8·11 (*b*).

For span AB

$$M_{max}=-\frac{6(10)^2}{8}=-75 \text{ kN-m}$$

$$A\bar{x}=-\frac{2}{3}\times 75\times 10(5)=-2500 \text{ (with } A \text{ or } B \text{ as origin)}$$

For span BC

$$A\bar{x}\text{ (with } B \text{ as origin)}=\left(\frac{1}{2}\times 4\times 68{\cdot}571\times\frac{8}{3}\right)-\left(\frac{1}{2}\times 3\times 51{\cdot}429\right)\times(4+1)=-20$$

$$A\bar{x}\text{ (with } C \text{ as origin)}=\left(\frac{1}{2}\times 4\times 68{\cdot}571\right)\times\left(3+\frac{4}{3}\right)-\left(\frac{1}{2}\times 3\times 57{\cdot}429\right)2$$

$$=+440$$

Imagine a point A' to the left of A, such that $AA'=0$

Hence for spans $A'A$ and AB

$$0+2M_A(0+10)+M_B(10)+0-\frac{6\times 2500}{10}=0$$

or $\quad 20M_A+10M_B=1500$

or $\quad 2M_A+M_B=150 \quad (1)$

For spans *AB-BC*, we have

$$M_A(10)+2M_B(10+7)+M_C(7)-\frac{6\times 2500}{10}+\frac{6\times 440}{7}=0$$

or $\quad 10M_A+34M_B+7M_C=1122{\cdot}86 \quad (2)$

Imagine a point C' to the right of C, such that $CC'=0$

Hence for spans $BC-CC'$,

$$M_B(7)+2M_C(7+0)+0-\frac{6\times 20}{7}+0=0$$

or $\quad 7M_B+14M_C=17{\cdot}142 \quad (3)$

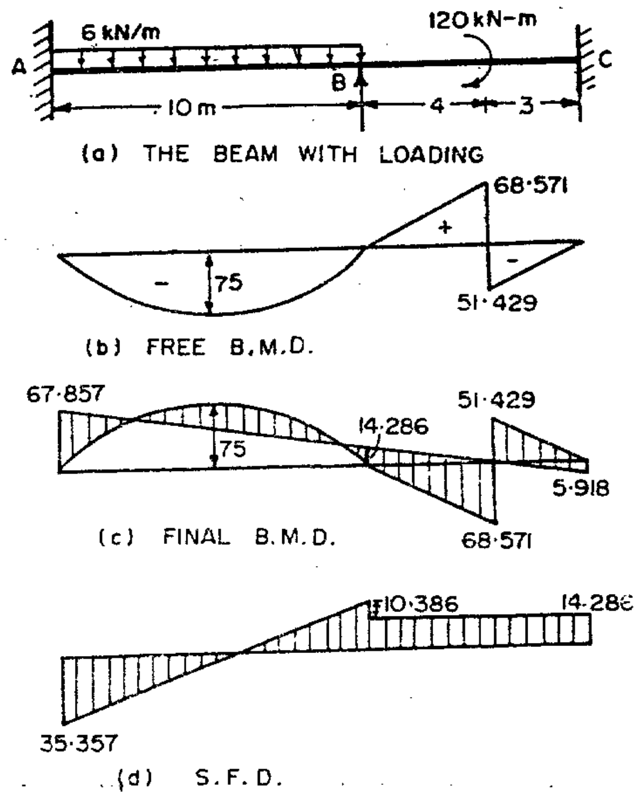


Fig. 8·11

From (1) and (2)

$$29M_B+7M_C=372{\cdot}86 \quad (4)$$

From (3) and (4), we get

$$M_B=14{\cdot}286 \text{ kN-m}$$

Hence from (1), $M_A=67{\cdot}857$ kN-m

and from (3) $\quad M_C=-5{\cdot}918$ (*i.e.* 5·918 ↺)

For reaction at A, take moments about B and equate to it to M_B

$$-R_A(10)+M_A+(10\times6\times5)=M_B$$

or

$$-10R_A+67{\cdot}857+300=14{\cdot}286$$

From which $R_A=35{\cdot}357$ kN(↑)

Similarly, for reaction at C, take moments about B and equate it to M_B.

$$\therefore \quad -R_C\times7-M_C+120=M_B$$

or

$$-7R_C-5{\cdot}918+120=14{\cdot}286$$

From which $R_C=14{\cdot}257$ kN(↑)

$$\therefore \quad R_B=(6\times10)-(R_A+R_C)=60-(35{\cdot}357+14{\cdot}257)$$
$$=10{\cdot}386(\uparrow)$$

The final B.M. and shear force diagrams are shown in Fig. 8·11 (*c*) and (*d*) respectively.

PROBLEMS

1. A fixed beam AB of span L carries a uniformly distributed load of w per unit length and is propped at a distance $\frac{L}{3}$ from A. If the deflection of the beam at this point is kR, where R=load on the prop, determine the magnitude of R.

2. A fixed beam carries a load which varies uniformly in intensity from zero at A to $2w$ at B. A prop is placed at mid-span which removes all the deflection at this point. Calculate the load carried by the prop.

3. A beam $ABCD$, 16 m long is continuous over three spans: AB=6 m ; BC=5 m ; and CD=5 m, the supports being at the same level. There is a uniformly distributed load of 20 kN/m over BC. On AB, is a point load of 80 kN at 2 m from A and CD, there is a point load of 60 kN at 3 m from D. Calculate the moments and reactions at the supports.

4. Solve question 3 if the support B sinks by 0·5 cm. I for the section is 9300 cm⁴ and $E=2{\cdot}10\times10^5$ N/mm².

5. Solve question 3 if the end A is fixed.

6. $ABCD$ is a straight uniform beam of length $4L$. It is freely supported at its ends A and D, and at two intermediate supports B and C distant L from either end. The supports at A and D are rigid but those at B and C are such that they deflect by an amount λ for each unit of load which is placed upon them. The beam carries a uniformly distributed load w per unit length along its entire length.

Show that the reactions at the supports are

$$\frac{wL}{8}\left[\frac{7L^3+48EI\lambda}{4L^3+3EI\lambda}\right] \text{and} \quad \frac{3wL}{8}\left[\frac{19L^3}{4L^3+3EI\lambda}\right]$$

(*Cambridge*)

7. A uniformly continuous girder ABC rests upon three similar floating supports, situated at each end and at the middle point B. The buoyancy of each float is such that every additional tonne of load increases its immersion by h. Initially, all the floats are equally immersed. If a load W tonnes is placed on the girder at B, show that the proportion carried by the central float is

$$\frac{W\left(1+\frac{3hEI}{a^3}\right)}{\left(1+\frac{9hEI}{a^3}\right)}, \text{ where } 2a \text{ is the length of the girder.}$$

8. A beam of length $2a$ and flexural rigidity EI carries a uniformly distributed load w/unit length and rests on three supports one at each end and one in the middle. Assuming that the beam was straight before loading, show that, for the greatest bending moment to be as small as possible, the central support must be $\frac{(8\sqrt{2}-11)wa^4}{24EI}$ lower than the end supports which are at the same level.

9. A beam rests on three supports A, B and C at the same level. The spans AB and BC are each of length L. The span AB is loaded with a load W concentrated at the middle, and the span BC has an equal load uniformly distributed. Find the reaction at the supports.

If the middle support sinks an amount $\frac{5}{96}\frac{WL^3}{EI}$ below the end supports, show that there will be no bending moment at B.

ANSWERS

1. $\frac{9wL^4}{4374\,EIk+16L^3}$
2. $\frac{wL}{2}$
3. $M_B=56{\cdot}8$ kN-m ; $M_C=45{\cdot}8$ kN-m ; R_A=43·[illegible] kN ; $R_B=88{\cdot}3$ kN ; $R_D=14{\cdot}8$ kN ; $R_C=93{\cdot}0$ kN
4. $M_B=44{\cdot}8$ kN-m ; $M_C=54{\cdot}9$ kN-m ; $R_C=44{\cdot}2$ kN ; $R_B=87{\cdot}8$ kN ; $R_C=95$ kN ; $R_D=13$ kN
5. $M_A=70{\cdot}6$ kN-m ; $M_B=36{\cdot}3$ kN-m ; $M_C=51{\cdot}0$ kN-m
6. $\frac{11}{32}W$; $\frac{21}{16}W$; $\frac{11}{32}W$.

9

The Slope Deflection Method

·1. INTRODUCTION : SIGN CONVENTIONS

The slope deflection method, in its present form, was first presented by Professor G.A. Maney (1915) of the University of Minnesota. In this method, the joints are considered to be rigid, *i.e.* the joints rotate as a whole and the angles between the tangents to the elastic curve meeting at the joint do not change due to deformation. The rotations of the joints are treated as unknowns. A series of simultaneous equations, each expressing the relation between the moments acting at the ends of the members are written in terms of slope and deflection. The solution of the slope-deflection equation along with the equilibrium equations, gives the values of the unknown rotations of the joints. Knowing these rotations, the end moments are calculated using the slope deflection equations During the decade just prior to the introduction of the moment distribution method, nearly all continuous frames were analyzed by the slopo deflection method.

The sign convention used in the case of bending of simple beams, etc., becomes clumsy if used for the case of more complex beams and frames where more than two members meet at a joint. In our earlier sign convention for simple beams, a moment is considered to be positive if it bends the beam convex upwards and negative if it bends the beam concave upwards. Thus, for the case of structure shown in Fig. 9·1, the three moments acting at the rigid joint B, where the three members BA, BC and BD meet are all positive according to the previous sign convention since all the three moments tend to bend the three corresponding beams convex upwards. Hence the equilibrium equation $\Sigma M_B=0$ at the joint B cannot be conveniently applied if the previous sign convention is used, though the joint B is in equilibrium.

However, the examination of joint B (Fig, 9·1) reveals that the moments M_{BA} and M_{BD} are clockwise while the moment M_{BC} is

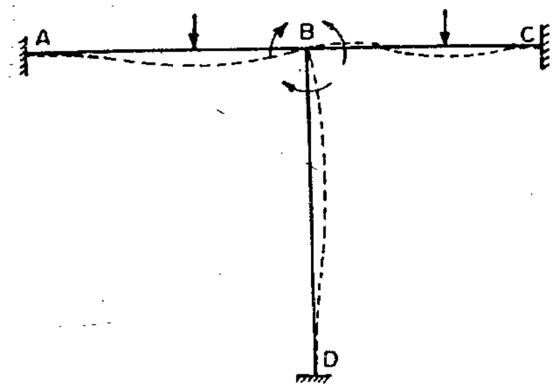


Fig. 9.1.

anti-clockwise. If the new sign convention is based on the direction of the moment, we get

$$M_{BA}+M_{BD}-M_{BC}=0.$$

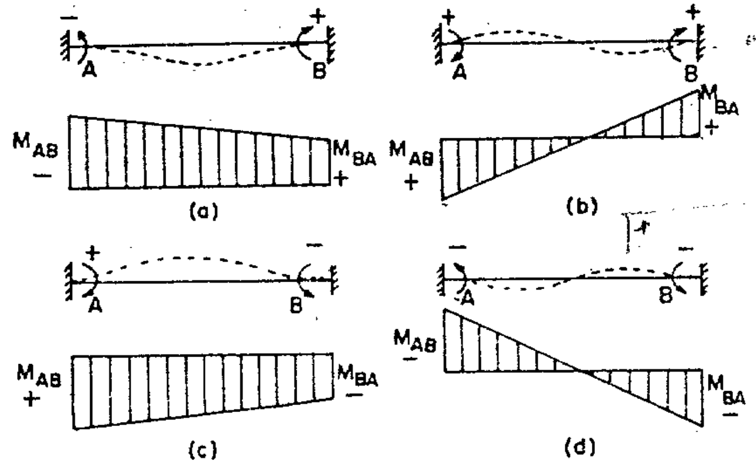


Fig. 9.2. Sign convention.

Hence in the new sign convention that will be used in this method, *a support moment acting in the clockwise direction will be taken as positive and that in the anti-clockwise direction as negative.* A corresponding change will have to be made while plotting the *support moment diagram.* For any span of a beam or member with rigid joints a positive support moment (or end moment) at the right hand end will be plotted above the base line and negative support moment below the base. Similarly, for the left hand end, the negative end moment is plotted above the base and positive end moment is plotted below the base line, as shown in Fig. 9·2.

[illegible] to the above sign convention for the end moments [illegible] convention for the rotation and settlement is adop- [illegible] rotation (or slope) will be taken as positive and [illegible] rotation as negative. (2) If one end of a beam settle, [illegible] deflection) will be taken as positive, if it rotates [illegible] whole in the clockwise direction, and negative if it [illegible] the beam as a whole in the anti-clockwise direction.

[illegible]MENTAL EQUATIONS

[illegible] the fundamental slope deflection equations, consider [illegible] at ends A and B, and subjected to external load- [illegible] in Fig. 9·3. Due to the external loading, the beam will [illegible] the ends will rotate. Let us now apply end moments [illegible] at ends A and B respectively, of such magnitude [illegible] at the ends A and B, due to the external [illegible] zero. In other words, the applied moments [illegible] of fixed end moments and therefore, a suffix [illegible] with these end moments. Such moment, which [illegible] to zero, will hereafter be called the fixed [illegible] easily calculated from the standard [illegible] given system of loading on the beam. [illegible] fixed moment at A for the beam AB. [illegible] fixing moment at B for the beam BA.

[illegible] rotate through θ_A and θ_B respectively, and [illegible] downwards by an amount δ. Thus the ends [illegible] movements, both rotational as well [illegible] beam takes the form shown by solid line in [illegible] be the additional moments resulting [illegible] beam from its initial position when the [illegible] same level. The rotation θ_A and [illegible] all positive. AB' is the tangent to the [illegible] to the horizontal, while B_1A' is [illegible] an angle θ_B to the horizontal line [illegible] moment m_{AB} and m_{BA}, causing rotations [illegible] can be easily calculated by the use of [illegible] (c) shows the component bending [illegible] moments.

[illegible] with reference to the tangent A

$$=-\frac{1}{EI}\sum_{B}^{A} A\bar{x}$$ (the minus sign is used because the deviation of B is in upward direction with respect to the tangent of A)

$$=-\frac{1}{EI}\left(\frac{1}{2}m_{BA}.L.\frac{L}{3}-\frac{1}{2}m_{AB}.L.\frac{2L}{3}\right)$$

$$=\frac{L^2}{6EI}\left(2m_{AB}-m_{BA}\right)$$

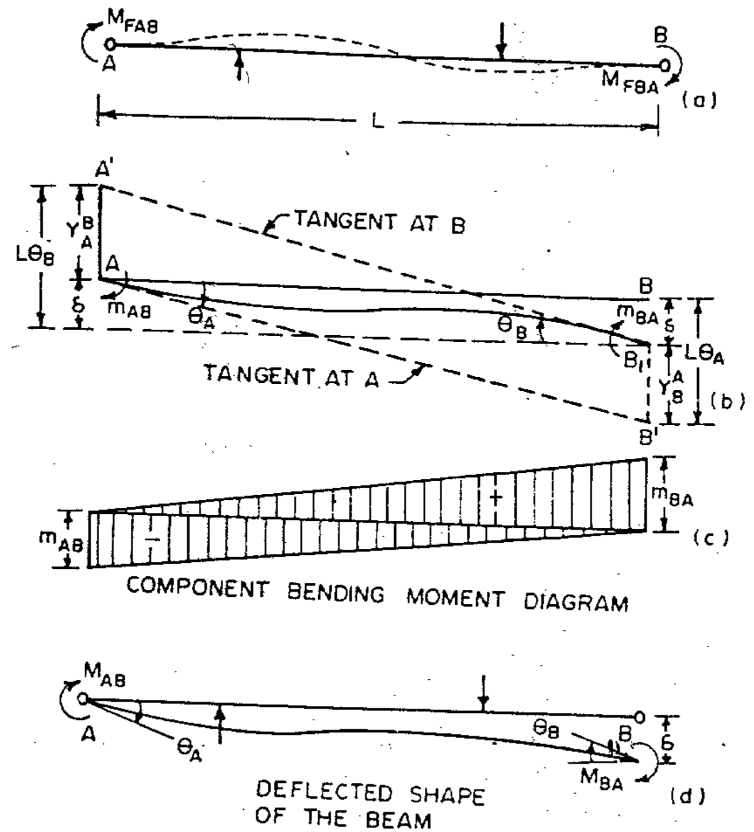


Fig. 9·3.
Derivation of slope-deflection equations.

But $Y_B^A = L\theta_A - \delta$ [From Fig. 9·3 (b)].

Hence $\frac{L^2}{6EI}\left(2m_{AB}-m_{BA}\right)=\left(L\theta_A-\delta\right)$

$$2m_{AB}-m_{BA}=\frac{6EI}{L}\left(\theta_A-\frac{\delta}{L}\right) \quad (1)$$

Similarly, Y_A^B = deviation of A with respect to the tangent at B

$$=\frac{1}{EI}\sum_A^B A\bar{x}$$

$$=\frac{1}{EI}\left(\frac{1}{2}m_{BA}.L.\ \frac{2L}{3}-\frac{1}{2}m_{AB}.L.\ \frac{L}{3}\right)$$

$$=\frac{L^2}{6EI}\left(2m_{BA}-m_{AB}\right)$$

But $Y_A^B=L\theta_B-\delta$ From Fig. 9·3 (*b*)

$$\therefore \quad \frac{L^2}{6EI}\left(2m_{BA}-m_{AB}\right)=L\theta_B-\delta$$

or
$$2m_{BA}-m_{AB}=\frac{6EI}{L}\left(\theta_B-\frac{\delta}{L}\right) \quad (2)$$

Solving (1) and (2) for m_{AB} and m_{BA}, we get

$$m_{AB}=\frac{2EI}{L}\left(2\theta_A+\theta_B-\frac{3\delta}{L}\right) \quad (i)$$

and
$$m_{BA}=\frac{2EI}{L}\left(2\theta_B+\theta_A-\frac{3\delta}{L}\right) \quad (ii)$$

Thus, the values of the additional moments m_{AB} and m_{BA} are known. Superimposing the effects of Fig. 9·3 (*a*) and 9·3 (*b*), we get the final deflected shape of the beam as shown in Fig. 9·3 (*d*), wherein θ_A and θ_B are the final rotations of the ends A and B, and δ is the deflection or settlement of the end B under the external loading. The final moments M_{AB} and M_{BA} at the ends A and B are respectively given by :

$$M_{AB}=m_{AB}+M_{FAB}=\frac{2EI}{L}\left(2\theta_A+\theta_B-\frac{3\delta}{L}\right)+M_{FAB} \quad (9·1)$$

$$M_{BA}=m_{BA}+M_{FBA}=\frac{2EI}{L}\left(2\theta_B+\theta_A-\frac{3\delta}{L}\right)+M_{FBA} \quad (9·2)$$

These are the fundamental slope deflection equations for the span AB. Writing $\frac{I}{L}=K$ and $\frac{\delta}{L}=R$, the slope-deflection equations are sometimes written in the following form :

$$M_{AB}=2EK(2\theta_A+\theta_B-3R)+M_{FAB} \quad (9·3)$$

$$M_{BA}=2EK(2\theta_B+\theta_A-3R)+M_{FBA} \quad (9·4)$$

9·3. CONTINUOUS BEAMS AND FRAMES WITHOUT JOINTS TRANSLATION

A continuous beam is essentially a statically indeterminate structure which must satisfy both the conditions of geometry as well as statical equilibrium. In the method of slope-deflection, the conditions of geometry are satisfied at the very outset of the slope deflection equations. In addition to this, the algebraic sum of the moments acting at a joint must be equal to zero. This condition furnishes as many equations as the number of joints. For example, for the continuous beam of Fig. 9·4, we get from the equilibrium of the joint B,

$$M_{BA}+M_{BC}=0 \quad (9·6)$$

Thus, we have two slope deflections equations (Eqs. 9·3 and 9·4), and one equilibrium equation (Eq. 9·5). The simultaneous solution of these equations give the unknown rotations. The end moments can then be calculated by substituting the values of these rotations in the slope-deflection equations. The procedure for the solution of a problem of continuous beam or frame without joint translation is summarised below :

1. Treat each span as a fixed beam and calculate the fixed end moments.
2. Write down slope deflection equations in terms of end moments, fixing moments, joint rotations and joint translation for each span.
3. Write down equilibrium equations for the individual joints.
4. Substitute the rotations back into the slope deflection equations and solve for the end moments.

Example 9·1. *A beam ABC, 10 m long, fixed at ends A and B is continuous over joint B and is loaded as shown in Fig. 9·4 (a). Using the slope deflection method, compute the end moments and plot the bending moment diagram. Also, sketch the deflected shape of the beam. The beam has constant EI for both the spans.*

Solution.

(*a*) *Fixed end moments*

Treating each span as a fixed beam, the fixed end moments are as follows :

$$M_{FAB}=-\frac{5\times3\times2^2}{5^2}=-2·4 \text{ kN-m}$$

$$M_{FBA}=+\frac{5\times2\times3^2}{5^2}=+3·6 \text{ kN-m}$$

$$M_{FBC}=-\frac{8\times5}{8}=-5{\cdot}0 \text{ kN-m}$$

$$M_{FCB}=+\frac{8\times5}{8}=+5{\cdot}0 \text{ kN-m}$$

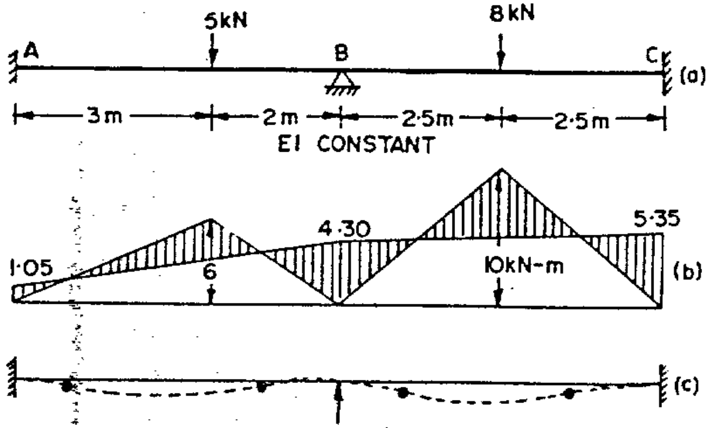


Fig. 9·4.

(*b*) *Slope deflection equations*

The end rotations θ_A and θ_C are zero since the beam is fixed at *A* and *C*. Hence there is only one unknown, θ_B. The ends do not settle and hence δ for each span is zero.

Let us assume θ_B to be positive. The result will indicate the correct sign. The slope deflection equations are as follows :

For span AB,

$$M_{AB}=\frac{2EI}{L}\left(2\theta_A+\theta_B-\frac{3\delta}{L}\right)+M_{FAB}$$

or $$M_{AB}=\frac{2EI}{5}(\theta_B)-2{\cdot}4=0{\cdot}4\ EI\ \theta_B-2{\cdot}4 \qquad (1)$$

and $$M_{BA}=\frac{2EI}{L}\left(2\theta_B+\theta_A-\frac{3\delta}{L}\right)+M_{FBA}$$

or $$M_{BA}=\frac{2EI}{5}(2\theta_B)+3{\cdot}6=0{\cdot}8\ EI\ \theta_B+3{\cdot}6 \qquad (2)$$

For span BC

$$M_{BC}=\frac{2EI}{5}(2\theta_B)-5{\cdot}0=0{\cdot}8\ EI\ \theta_B-5{\cdot}0 \qquad (3)$$

and $$M_{CB}=\frac{2EI}{5}(\theta_B)+5{\cdot}0=0{\cdot}4\ EI\ \theta_B+5{\cdot}0 \qquad (4)$$

(*c*) *Equilibrium equation*

Since there is only one unknown, *i.e.* θ_B, one equilibrium equation is sufficient. For the joint *B*, we have

$$M_{BA}+M_{BC}=0$$

$$\therefore \qquad (0{\cdot}8\ EI\ \theta_B+3{\cdot}6)+(0{\cdot}8\ EI\theta_B-5{\cdot}0)=0$$

or $$1{\cdot}6\ EI\ \theta_B=1{\cdot}4$$

or $$EI\ \theta_B=+\frac{1{\cdot}4}{1{\cdot}6}.$$

The plus sign indicates that θ_B is positive (*i.e.* rotation of tangent at *B* is clockwise).

(*d*) *Final moments*

Substituting the value of $EI\ \theta_B$ in Eqs. (1) to (4), we get

$$M_{AB}=0{\cdot}4\left(\frac{1{\cdot}4}{1{\cdot}6}\right)-2{\cdot}4=-1{\cdot}05 \text{ kN-m}$$

$$M_{BA}=0{\cdot}8\left(\frac{1{\cdot}4}{1{\cdot}6}\right)+3{\cdot}6=+4{\cdot}30 \text{ kN-m}$$

$$M_{BC}=0{\cdot}8\left(\frac{1{\cdot}4}{1{\cdot}6}\right)-5{\cdot}0=-4{\cdot}30 \text{ kN-m}$$

and $$M_{CB}=0{\cdot}4\left(\frac{1{\cdot}4}{1{\cdot}6}\right)+5{\cdot}0=+5{\cdot}35 \text{ kN-m}$$

Fig. 9·4 (*b*) shows the bending moment diagram. The deflected shape of the beam is shown in Fig. 9·4 (*c*).

Example 9·2. *Solve example 9·1 if ends A and C are simply supported (or hinged).*

Solution

(*a*) *Fixed end moment*

These are the same as calculated in the previous example :

$M_{FAB}=-2{\cdot}4$ kN-m ; $\quad M_{FBA}=+3{\cdot}6$ kN-m

$M_{FBC}=-5{\cdot}0$ kN-m ; $\quad M_{FCB}=+5{\cdot}0$ kN-m

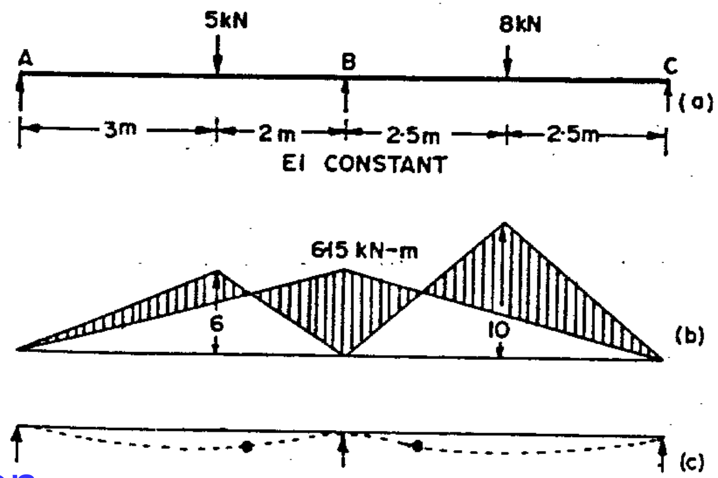


Fig. 9·5.

(*b*) *Slope deflection equations.*

For span AB.

$$M_{AB}=\frac{2EI}{5}(2\theta_A+\theta_B)-2{\cdot}4=0{\cdot}4\ EI(2\theta_A+\theta_B)-2{\cdot}4 \quad (1)$$

$$M_{BA}=\frac{2EI}{5}(2\theta_B+\theta_A)+3{\cdot}6=0{\cdot}4\ EI(2\theta_B+\theta_A)+3{\cdot}6 \quad (2)$$

For span BC,

$$M_{BC}=\frac{2EI}{5}(2\theta_B+\theta_C)-5{\cdot}0=0{\cdot}4\ EI(2\theta_B+\theta_C)-5{\cdot}0 \quad (3)$$

and $$M_{CB}=\frac{2EI}{5}(2\theta_C+\theta_B)+5{\cdot}0=0{\cdot}4\ EI(2\theta_C+\theta_B)+5{\cdot}0 \quad (4)$$

(*c*) *Equilibrium equations*

Since end *A* is freely supported, $M_{AB}=0$

$$\therefore \quad 0{\cdot}4\ EI(2\theta_A+\theta_B)-2{\cdot}4=0 \quad (I)$$

Also end *C* is freely supported, $M_{CB}=0$

$$\therefore \quad 0{\cdot}4\ EI(2\theta_C+\theta_B)+5{\cdot}0=0 \quad (II)$$

For the joint *B*, $M_{BA}+M_{BC}=0$

$$\therefore \quad [0{\cdot}4\ EI(2\theta_B+\theta_A)+3{\cdot}6]+[0{\cdot}4\ EI(2\theta_B+\theta_C)-5{\cdot}0]=0$$

or $$0{\cdot}4\ EI(4\theta_B+\theta_A+\theta_C)-1{\cdot}4=0 \quad (III)$$

Solving Eqs. I, II and III, we get

$$EI\ \theta_A=\frac{22{\cdot}5}{12} \quad (i)$$

$$EI\ \theta_B=\frac{27}{12} \quad (ii)$$

$$EI\ \theta_C=\frac{-88{\cdot}5}{12} \quad (iii)$$

(*d*) *Final moments*

Substituting the values of $EI\ \theta_A$ and $EI\ \theta_B$ in Eq. (2), we get

$$M_{BA}=0{\cdot}4\left[\frac{2\times27}{12}+\frac{22{\cdot}5}{12}\right]+3{\cdot}6=+6{\cdot}15\text{ kN-m}$$

As a check, substituting in Eq. (3)

$$M_{BC}=0{\cdot}4\left(\frac{2\times27}{12}-\frac{88{\cdot}5}{12}\right)-5{\cdot}0=-6{\cdot}15\text{ kN-m}$$

$$\therefore \quad M_{BA}+M_{BC}=+6{\cdot}15-6{\cdot}15=0.$$

The bending moment diagram and the deflected shape of the beam are shown in Fig. 9·5 (*b*) and (*c*) respectively.

Note. The beam is statically indeterminate to single degree only. This problem has also been solved by the moment distribution method (Example 10·2) treating the moment at *B* as unknown.

However, in the slope-deflection method, the slope or rotations are taken as unknowns and due to this the problem involves three unknown rotations θ_A, θ_B and θ_C. Hence the method of slope deflection is not recommended for such a problem.

Example 9·3. *A continuous beam ABCD consists of three span and is loaded as shown in Fig. 9·6 (a). Ends A and D are fixed. Determine the bending moments at the supports and plot the bending moment diagram.*

Solution

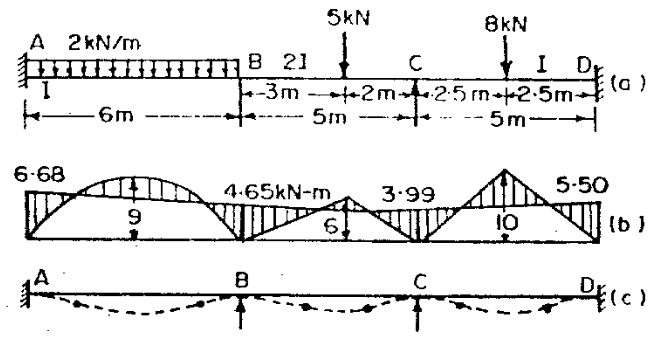


Fig. 9·6.

(*a*) *Fixed end moments*

$$M_{FAB}=-\frac{2\times6^2}{12}=-6\text{ kN-m}\ ;\ M_{FBA}=+\frac{2\times6^2}{12}=+6\text{ kN-m}$$

$$M_{FBC}=-\frac{5\times3\times2^2}{5^2}=-2{\cdot}4\text{ kN-m}$$

$$M_{FCB}=+\frac{5\times2\times3^2}{5^2}=+3{\cdot}6\text{ kN-m}$$

$$M_{FCD}=-\frac{8\times5}{8}=-5\text{ kN-m}\ ;\ M_{FDC}=+\frac{8\times5}{8}=+5\text{ kN-m}$$

(*b*) *Slope deflection equations*

θ_A and θ_D are zero since ends *A* and *D* are fixed.

$$M_{AB}=\frac{2E(I)}{6}[\theta_B]-6=\frac{EI}{3}\theta_B-6 \quad (1)$$

$$M_{BA}=\frac{2E(I)}{6}[2\theta_B]+6=\frac{2EI}{3}\theta_B+6 \quad (2)$$

$$M_{BC}=\frac{2E(2I)}{5}[2\theta_B+\theta_C]-2{\cdot}4=\frac{4EI}{5}(2\theta_B+\theta_C)-2{\cdot}4 \quad (3)$$

$$M_{CB}=\frac{2E(2I)}{5}[2\theta_C+\theta_B]+3{\cdot}6=\frac{4EI}{5}(\theta_B+2\theta_C)+3{\cdot}6 \quad (4)$$

$$M_{CD}=\frac{2E(I)}{5}[2\theta_C]-5=\frac{4EI}{5}\theta_C-5 \tag{5}$$

$$M_{DC}=\frac{2E(I)}{5}[\theta_C]+5=\frac{2EI}{5}\theta_C+5 \tag{6}$$

(*c*) *Equilibrium equations*

At joint B,

$$M_{BA}+M_{BC}=0$$

or
$$\left[\frac{2EI}{3}\theta_B+6\right]+\left[\frac{4EI}{5}(2\theta_B+\theta_C)-2{\cdot}4\right]=0$$

or
$$\frac{34EI}{3}\theta_B+\frac{4EI}{5}\theta_C+3{\cdot}6=0 \tag{I}$$

At joint C, $\quad M_{CB}+M_{CD}=0$

$$\therefore \quad \left[\frac{4EI}{5}(\theta_B+2\theta_C)+3{\cdot}6\right]+\left[\frac{4EI}{5}\theta_C-5\right]=0$$

or
$$\frac{4EI}{5}\theta_B+\frac{12EI}{5}\theta_C-1{\cdot}4=0 \tag{II}$$

From (I) and (II), we get

$$EI\theta_B=-2{\cdot}03 \tag{i}$$
$$EI\theta_C=+1{\cdot}26 \tag{ii}$$

(*d*) *Final moments*

Substituting these values in Eqs. (1) to (6), we get

$$M_{AB}=\frac{1}{3}(-2{\cdot}03)-6=-6{\cdot}68 \text{ kN-m}$$

$$M_{BA}=\frac{2}{3}(-2{\cdot}03)+6=+4{\cdot}65 \text{ kN-m}$$

$$M_{BC}=\frac{4}{5}[(-2\times2{\cdot}03)+1{\cdot}26]-2{\cdot}4=-4{\cdot}65 \text{ kN-m}$$

$$M_{CB}=\frac{4}{5}[(-2{\cdot}03)+(2\times1{\cdot}26)]\times3{\cdot}6=+3{\cdot}99 \text{ kN-m}$$

$$M_{CD}=\frac{4}{5}(1{\cdot}26)-5=-3{\cdot}99 \text{ kN-m}$$

$$M_{DC}=\frac{2}{5}(1{\cdot}26)+5=+5{\cdot}50 \text{ kN-m}$$

The bending moment diagram and the deflected shape are shown in Fig. 9·6 (*b*) and (*c*) respectively.

Example 9·4. *A continuous beam ABCD, 12 m long is fixed at A and D, and is loaded as shown in Fig. 9·7 (a). Analyse the beam completely if the following movements take place simultaneously :*

(*i*) *The end A yields, turning through* $\frac{1}{250}$ *radians in a clockwise direction.*

(*ii*) *End B sinks 30 mm in downward direction.*

(*iii*) *End C sinks 20 mm in downward direction.*

The beam has constant $I=38{\cdot}20\times10^5$ *mm*^{4} *and* $E=2\times10^5$ *N/mm*^{2}.

Solution

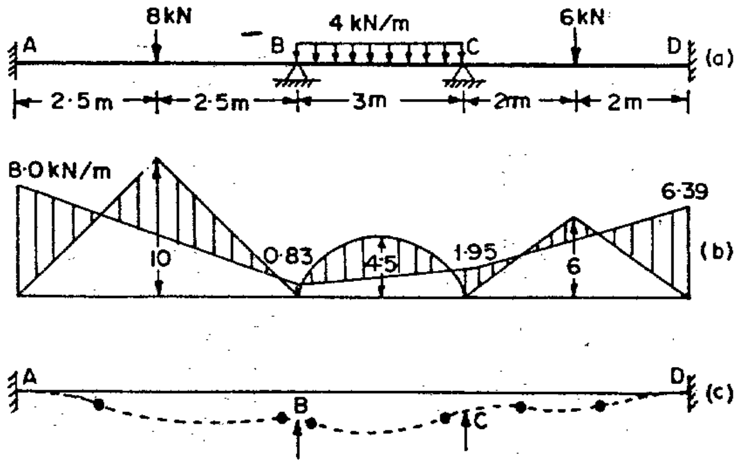


Fig. 9·7.

(*a*) *Fixed end moments*

$$M_{FAB}=-\frac{-8\times5}{8}=-5{\cdot}0 \text{ kN-m}$$

$$M_{FBA}=+\frac{8\times5}{8}=+5{\cdot}0 \text{ kN-m}$$

$$M_{FBC}=-\frac{4\times3^2}{12}=-3{\cdot}0 \text{ kN-m}$$

$$M_{FCB}=+\frac{4\times3^2}{12}=+3{\cdot}0 \text{ kN-m}$$

$$M_{FCD}=-\frac{6\times4}{8}=-3{\cdot}0 \text{ kN-m}$$

$$M_{FDC}=+\frac{6\times4}{8}=+3{\cdot}0 \text{ kN-m}$$

(*b*) *Slope deflection equations*

All the unknowns are assumed to be positive.

For AB,

$$K=\frac{I}{5}\ ;\ R=+\frac{30}{5000}=\frac{3}{500}\ ;\ \theta_A=+\frac{1}{250}$$

$$\therefore \quad M_{AB}=\frac{2EI}{5}\left(\frac{2}{250}+\theta_B-\frac{9}{500}\right)-5{\cdot}0 \tag{1}$$

$$M_{BA}=\frac{2EI}{5}\left(2\theta_B+\frac{1}{250}-\frac{9}{500}\right)+5\cdot0 \qquad (2)$$

For BC,

$$K=\frac{I}{3}\ ;\ R=-\frac{30-20}{3000}=-\frac{1}{300}$$

$$\therefore\ M_{BC}=\frac{2EI}{3}\left(2\theta_B+\theta_C+\frac{3\times1}{300}\right)-3\cdot0 \qquad (3)$$

$$M_{CB}=\frac{2EI}{3}\left(2\theta_C+\theta_B+\frac{3\times1}{300}\right)+3\cdot0 \qquad (4)$$

For CB,

$$K=\frac{I}{4}\ ;\ R=-\frac{20}{4000}=\frac{2}{400} \text{ and } \theta_D=0$$

$$\therefore\ M_{CD}=\frac{2EI}{4}\left(2\theta_C+\frac{3\times2}{400}\right)-3\cdot0 \qquad (5)$$

$$M_{DC}=\frac{2EI}{4}\left(\theta_C+\frac{3\times2}{400}\right)+3\cdot0 \qquad (6)$$

(c) *Equilibrium equations* :

There are two unknowns θ_B and θ_C. Thus two simultaneous equations will be required which will be provided by the conditions of equilibrium at joints B and C.

At joint B,

$$M_{BA}+M_{BC}=0$$

$$\therefore\ \frac{2EI}{5}\left(2\theta_B+\frac{1}{250}-\frac{9}{500}\right)+5+\frac{2EI}{3}\left(2\theta_B+\theta_C+\frac{3}{300}\right)-3\cdot0=0$$

or $$\frac{32}{15}EI\,\theta_B+\frac{2}{3}EI\,\theta_C+\frac{4}{3750}EI+2\cdot0=0$$

or $$EI\,\theta_B+\frac{5}{16}EI\,\theta_C+\frac{EI}{2000}+\frac{15}{16}=0 \qquad (7)$$

At joint C,

$$M_{CB}+M_{CD}=0$$

$$\therefore\ \frac{2EI}{3}\left(2\theta_C+\theta_B+\frac{3}{300}\right)+3+\frac{2EI}{4}\left(2\theta_C+\frac{3\times2}{400}\right)-3\cdot0=0$$

or $$\frac{7}{3}EI\theta_C+\frac{2}{3}EI\theta_B+\frac{17}{1200}EI=0$$

or $$EI\,\theta_B+\frac{7}{2}EI\theta_C+\frac{17}{800}EI=0 \qquad (8)$$

Subtracting Eq. (8) from Eq. (7),

$$\frac{5}{16}EI\theta_C+\frac{EI}{2000}+\frac{15}{16}-\frac{7}{2}EI\,\theta_C-\frac{17}{800}EI=0$$

or $$-\frac{51}{16}EI\,\theta_C=\frac{83}{4000}EI-\frac{15}{16}$$

But $EI=\frac{2\times10^5}{1000}\times(1000)^2\times\frac{38\cdot2\times10^5}{(1000)^4}=764$ kN-m²

$$\therefore\ -\frac{51}{16}\times764\,\theta_C=\frac{83}{4000}\times764-\frac{15}{16} \qquad (i)$$

or $$\theta_C=-6\cdot124\times10^{-3} \text{ radians}$$

Substituting the value of θ_C in Eq. (8),

$$\therefore\ EI\,\theta_B-\frac{7}{2}EI\,(6\cdot124\times10^{-3})+\frac{17}{800}EI=0$$

or $$\theta_B=0\cdot184\times10^{-3} \text{ radians} \qquad (ii)$$

(d) *Final moments*

Substituting the values of θ_B and θ_C in Eqs. (1) to (6), we get the values of moments at the supports :

$$M_{AB}=\frac{2\times764}{5}\left(\frac{2}{250}+0\cdot184\times10^{-3}-\frac{9}{500}\right)-5\cdot0=-8\cdot0 \text{ kN-m}$$

$$M_{BA}=\frac{2\times764}{5}\left(2\times0\cdot184\times10^{-3}+\frac{1}{250}-\frac{9}{500}\right)+5\cdot0$$
$$=+0\cdot83 \text{ kN-m}$$

$$M_{BC}=\frac{2\times764}{3}\left(2\times0\cdot184\times10^{-3}-6\cdot124\times10^{-3}+\frac{3}{300}\right)-3\cdot0$$
$$=-0\cdot83 \text{ kN-m}$$

$$M_{CB}=\frac{2\times764}{3}\left(-2\times6\cdot124\times10^{-3}+0\cdot184\times10^{-3}+\frac{3}{300}\right)+3\cdot0$$
$$=+1\cdot95 \text{ kN-m}$$

$$M_{CD}=\frac{2\times764}{4}\left(-2\times6\cdot124\times10^{-3}+\frac{6}{400}\right)-3\cdot0=-1\cdot95 \text{ kN-m}$$

$$M_{DC}=\frac{2\times764}{4}\left(-6\cdot124\times10^{-3}+\frac{6}{400}\right)+3\cdot00=+6\cdot39 \text{ kN-m}$$

The bending moment diagram and the deflected shape of the beam are shown in Fig. 9·7 (*b*) and (*c*) respectively.

Example 9·5. *A continuous beam ABC is supported on an elastic column BD and is loaded as shown in Fig. 9·8. Treating joint B as rigid, analyse the frame and plot the bending moment diagram and the deflected shape of the structure.*

Solution.

(a) *Fixed end moments*

$$M_{FAB}=-\frac{10\times2\times3^2}{5^2}=-7\cdot2 \text{ kN-m},\quad M_{FBA}=+\frac{10\times3\times2^2}{5^2}=+4\cdot8 \text{ kN-m}$$

$M_{FBC} = -\dfrac{2\times 3^2}{12} = -1{\cdot}5$ kN-m ; $M_{FCB} = +\dfrac{2\times 3^2}{12} = +1{\cdot}5$ kN-m

$M_{FBD} = M_{FDB} = 0$

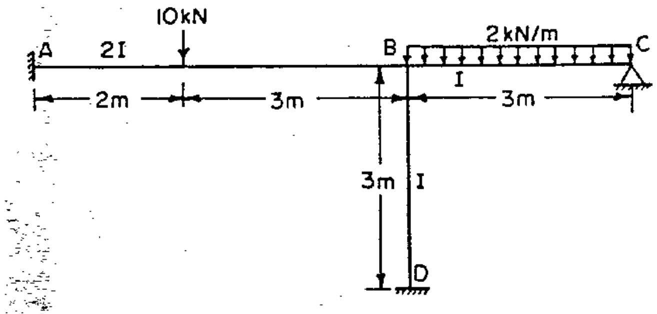


Fig. 9·8.

Slope deflection equations

The slopes θ_A and θ_D are zero since ends A and D are fixed.

For span AB :

$$M_{AB} = \frac{2E(2I)}{5}[\theta_B] - 7{\cdot}2 = \frac{4}{5}EI\theta_B - 7{\cdot}2 \qquad (1)$$

$$M_{BA} = \frac{2E(2I)}{5}[2\theta_B] + 4{\cdot}8 = \frac{8}{5}EI\theta_B + 4{\cdot}8 \qquad (2)$$

For span BC :

$$M_{BC} = \frac{2EI}{3}[2\theta_B + \theta_C] - 1{\cdot}5 = \frac{4EI}{3}\theta_B + \frac{2}{3}EI\theta_C - 1{\cdot}5 \qquad (3)$$

$$M_{CB} = \frac{2EI}{3}[2\theta_C + \theta_B] + 1{\cdot}5 = \frac{4}{3}EI\theta_C + \frac{2}{3}EI\theta_B + 1{\cdot}5 \qquad (4)$$

For span BD :

$$M_{BD} = \frac{2EI}{3}[2\theta_B] = \frac{4}{3}EI\theta_B \qquad (5)$$

$$M_{DB} = \frac{2EI}{3}[\theta_B] = \frac{2}{3}EI\theta_B \qquad (6)$$

(*c*) *Equilibrium equations*

At joint B, $M_{BA} + M_{BC} + M_{BD} = 0$

$$\therefore \left(\frac{8}{5}EI\theta_B + 4{\cdot}8\right) + \left(\frac{4EI}{3}\theta_B + \frac{2}{3}EI\theta_C - 1{\cdot}5\right) + \left(\frac{4}{3}EI\theta_B\right) = 0$$

or
$$\frac{64}{15}EI\theta_B + \frac{2}{3}EI\theta_C + 3{\cdot}3 = 0 \qquad \text{(I)}$$

At joint C, $M_{CB} = 0$

$$\therefore \quad \frac{4}{3}EI\theta_B + \frac{2}{3}EI\theta_B + 1{\cdot}5 = 0 \qquad \text{(II)}$$

Solving Eq. (I) and (II) for θ_B and θ_C, we get

$$EI\theta_B = -0{\cdot}648 \qquad (i)$$

and
$$EI\theta_C = -0{\cdot}801 \qquad (ii)$$

(*d*) *Final moments*

Substituting these values in Eq. (1) to (6), we get

$$M_{AB} = \frac{4}{5}(-0{\cdot}648) - 7{\cdot}2 = -7{\cdot}72 \text{ kN-m}$$

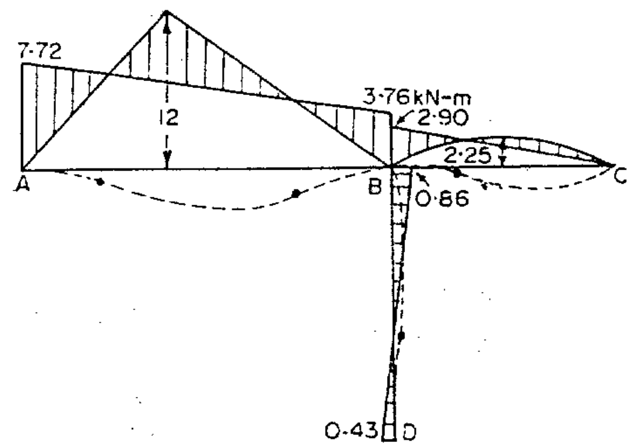


Fig. 9·9.

$$M_{BA} = \frac{8}{5}(-0{\cdot}648) + 4{\cdot}8 = +3{\cdot}76 \text{ kN-m}$$

$$M_{BC} = \frac{4}{5}(-0{\cdot}648) + \frac{2}{3}(-0{\cdot}801) - 1{\cdot}5 = -2{\cdot}90 \text{ kN-m}$$

$$M_{CB} = \frac{4}{3}(-0{\cdot}801) + \frac{2}{3}(-0{\cdot}648) + 1{\cdot}5 = 0$$

$$M_{BD} = \frac{4}{3}(-0{\cdot}648) = -0{\cdot}86 \text{ kN-m}$$

$$M_{DB} = \frac{2}{3}(-0{\cdot}648) = -0{\cdot}43 \text{ kN-m}$$

The bending moment diagram and the deflected shape of the structure are shown in Fig. 9·9.

Example 9·6. *Analyse the rigid frame shown in Fig. 9·10.*

Solution.

(*a*) *Fixed end moments*

$M_{FAB} = -\dfrac{2\times 4^2}{12} = -2{\cdot}67$ kN-m ; $M_{FBA} = +\dfrac{2\times 4^2}{12} = +2{\cdot}67$ kN-m

$$M_{FBD}=-\frac{4\times4}{8}=-2 \text{ kN-m} \; ; \; M_{FDB}=+\frac{4\times4}{8}=+2 \text{ kN-m}$$

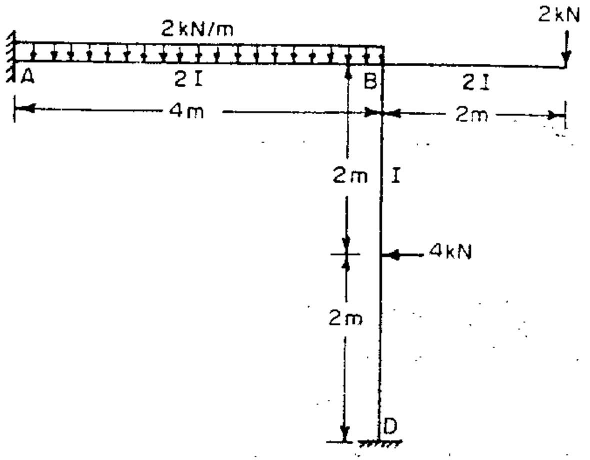


Fig. 9·10

(*b*) *Slope deflection equations*

θ_A and θ_D are zero.

$$M_{AB}=\frac{2E(2I)}{4}[\theta_B]-2{\cdot}67=EI\theta_B-2{\cdot}67 \quad (1)$$

$$M_{BA}=\frac{2EI(2I)}{4}[2\theta_B]+2{\cdot}67=2EI\theta_B+2{\cdot}67 \quad (2)$$

$$M_{BD}=\frac{2\,EI}{4}[2\theta_B]-2=EI\theta_B-2 \quad (3)$$

$$M_{DB}=\frac{2\,EI}{4}[\theta_B]+2=\frac{1}{2}EI\theta+2 \quad (4)$$

$M_{BC}=-2\times2=-4$ kN-m.

(*b*) *Equilibrium equation*

For the equilibrium of the joint *B*.

$$M_{BA}+M_{BD}+M_{BC}=0$$

$$\therefore \quad (2EI\theta_B+2{\cdot}67)+(EI\theta_B-2)+(-4)=0$$

or $\quad 3EI\theta_B=3{\cdot}33$

or $\quad EI\theta_B=1{\cdot}11$

(*d*) *Final moments*

Substituting the value of $EI\theta_B$ in Eqs. (1) to (4), we get

$M_{AB}=1{\cdot}11-2{\cdot}67=-1{\cdot}56$ kN-m

$M_{BA}=2(1{\cdot}11)+2{\cdot}67=+4{\cdot}89$ kN-m

$M_{BD}=1{\cdot}11-2=-0{\cdot}89$ kN-m

$M_{DB}=\frac{1}{2}(1{\cdot}11)+2=+2{\cdot}56$ kN-m

The bending moment diagram and the deflected shapes of the structure are shown in Fig. 9·11.

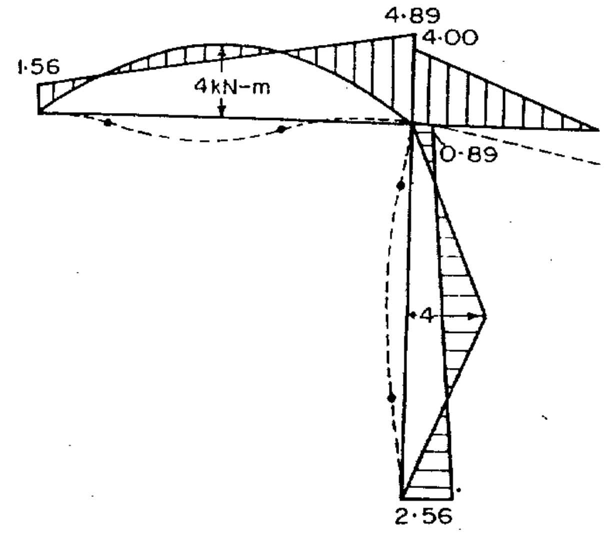


Fig. 9·11.

Example 9·7. *A portal frame ABCD is fixed at A and D, and is loaded as shown in Fig 9·12. Treating joints B and C as rigid, calculate the moments at A, B, C and D. Draw the bending moment diagram and sketch the deflected shape of the frame.*

Solution

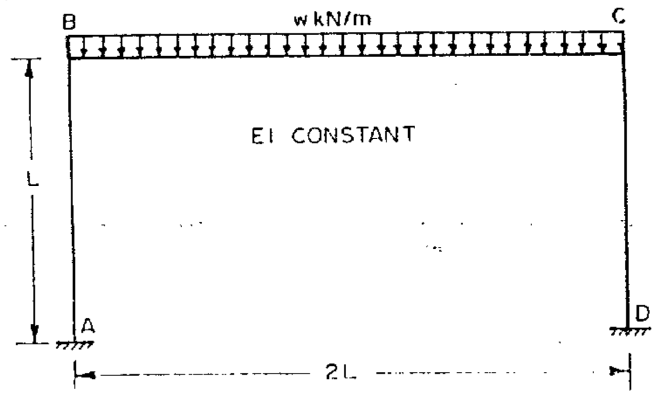


Fig 9·12.

(a) Fixed end moments

$$M_{FBC}=-\frac{w(2L)^2}{12}=-\frac{wL^2}{3};\quad M_{FCB}=+\frac{w(2L)^2}{12}=+\frac{wL^2}{3}$$

(b) Slope deflection equations

θ_A and θ_D are zero since ends A and D are fixed. Also, since the frame is symmetrical and the loading is also symmetrical, there is no side-sway or deflection of the frame. The unknowns are, therefore, θ_B and θ_C.

For AB,

$$M_{AB}=\frac{2EI}{L}(\theta_B)=\frac{2}{L}EI\theta_B \tag{1}$$

$$M_{BA}=\frac{2EI}{L}(2\theta_B)=\frac{4}{L}EI\theta_B \tag{2}$$

For BC,

$$M_{BC}=\frac{2EI}{2L}\left[2\theta_B+\theta_C\right]-\frac{wL^2}{3}=\frac{2}{L}EI\theta_B+\frac{1}{L}EI\theta_C-\frac{wL^2}{3} \tag{3}$$

$$M_{CB}=\frac{2EI}{2L}\left[2\theta_C+\theta_B\right]+\frac{wL^2}{3}=\frac{2}{L}EI\theta_C+\frac{1}{L}EI\theta_B+\frac{wL^2}{3} \tag{4}$$

For CD,

$$M_{CD}=\frac{2EI}{L}\left[2\theta_C\right]=\frac{4}{L}EI\theta_C \tag{5}$$

$$M_{DC}=\frac{2EI}{L}\left[\theta_C\right]=\frac{2}{L}EI\theta_C \tag{6}$$

(c) Equilibrium equations

At the joint B,

$$M_{AB}+M_{BC}=0$$

$$\therefore\quad \frac{4}{L}EI\theta_B+\frac{2}{L}EI\theta_B+\frac{1}{L}EI\theta_C-\frac{wL^2}{3}=0$$

or

$$\frac{6}{L}EI\theta_B+\frac{1}{L}EI\theta_C-\frac{wL^2}{3}=0$$

But by symmetry, $\theta_C=-\theta_B$

$$\therefore\quad \frac{5}{L}EI\theta_B=\frac{wL^2}{3}$$

or

$$\frac{EI}{L}\theta_B=\frac{wL^2}{3}\times\frac{1}{5}=\frac{wL^2}{15} \tag{i}$$

and

$$\frac{EI}{L}\theta_C=-\frac{EI}{L}\theta_B=-\frac{wL^2}{15} \tag{ii}$$

(d) Final moments

Substituting these values in Eqs. (1) to (6), we get

$$M_{AB}=2\left(\frac{wL^2}{15}\right)=+\frac{2}{15}wL^2$$

$$M_{BA}=4\left(\frac{wL^2}{15}\right)=+\frac{4}{15}wL^2$$

$$M_{BC}=2\left(\frac{wL^2}{15}\right)-\frac{wL^2}{15}-\frac{wL^2}{3}=-\frac{4}{15}wL^2$$

$$M_{CB}=2\left(-\frac{wL^2}{15}\right)+\frac{wL^2}{15}+\frac{wL^2}{3}=+\frac{4}{15}wL^2$$

$$M_{CD}=4\left(-\frac{wL^2}{15}\right)=-\frac{4}{15}wL^2$$

$$M_{DC}=2\left(-\frac{wL^2}{15}\right)=-\frac{2}{15}wL^2.$$

The bending moment diagram and the deflected shape of the frame are shown in Fig. 9·13.

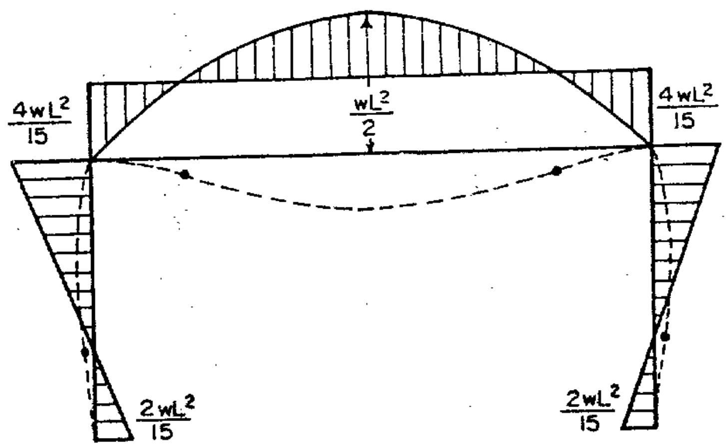


Fig. 9·13.

9·4 PORTAL FRAMES WITH SIDE SWAY

In the case of continuous beams, etc., the effect of yielding or settlement of support is taken into account by introducing initial fixed end moments. In the case of portal frames, however, the joint moment or 'sway' is not known and form an amount of the additional unknown. The portal frames may sway due to one of the following reasons :

1. Eccentric or unsymmetrical loading on the portal frame.
2. Unsymmetrical outline of portal frame.
3. Different end conditions of the columns of the portal frame.
4. Non-uniform section of the members of the frame.
5. Horizontal loading on the columns of the frame.
6. Settlement of the supports of the frame.
7. A combination of the above.

In such cases, the joint translations become additional unknown quantities. Some additional conditions will, therefore, be required for analysing the frame. The additional conditions of equilibrium are obtained from the consideration of the shear force exerted on the structure by the external loading. The horizontal shear exerted by a member is equal to the algebraic sum of the moments at the ends divided by the length of the member. Thus the horizontal shear resistance of all such members can be found and the algebraic sum of all such forces must balance the external horizontal loading, if any.

In Fig. 9.14, the horizontal reactions are given by,

$$H_A=\frac{M_{AB}+M_{BA}-Ph}{L_1}\rightarrow \tag{1}$$

and

$$H_D=\frac{M_{CD}+M_{DC}+\frac{1}{2}wL_2^2}{L_2}\rightarrow$$

The above reactions have been calculated on the assumption that all the end moments are clockwise.

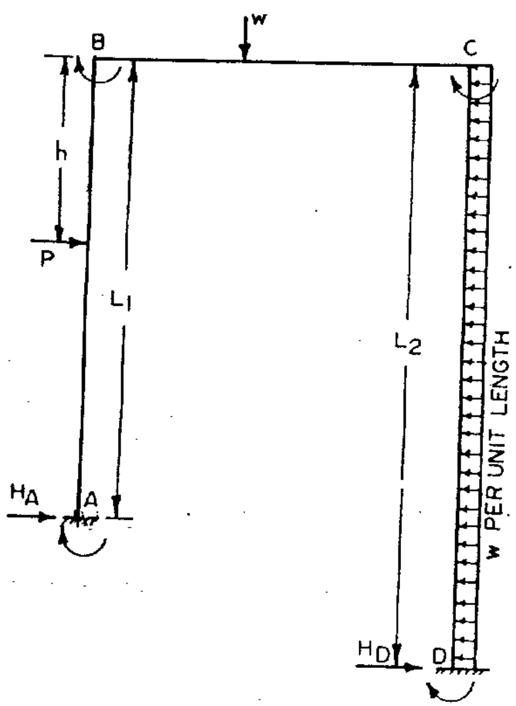


Fig. 9·14.

For the equilibrium of the frame, $\Sigma H=0$ (3)

$\because \quad H_A+H_D+P-wL_2=0$

The above equation is known as the *shear equation*. Substituting the values, we get

$$\frac{M_{AB}+M_{BA}-Ph}{L_1}+\frac{M_{CD}+M_{DC}+\frac{1}{2}wL_2^2}{L_2}+P-wL_2=0 \tag{4}$$

Equation (4) gives the general expression of shear equation. If however, $P=0$, we get

$$\frac{M_{AB}+M_{BA}}{L_1}+\frac{M_{CD}+M_{DC}+\frac{1}{2}wL_2^2}{L_2}-wL_2=0$$

and

$$\frac{M_{AB}+M_{BA}}{L_1}+\frac{M_{CD}+M_{DC}}{L_2}-\frac{wL_2}{2}=0 \tag{4(a)}$$

If w is zero, we get from Eq. (4)

$$\frac{M_{AB}+M_{BA}-Ph}{L_1}+\frac{M_{CD}+M_{DC}}{L_2}+P=0 \tag{4(b)}$$

If both P and w are zero ; we get

$$\frac{M_{AB}+M_{BA}}{L_1}+\frac{M_{CD}+M_{DC}}{L_2}=0 \tag{4(c)}$$

Example 9·8. *Analyse the portal frame shown in Fig. 9·15. Also sketch the deflected shape of the frame. The end A is fixed and end D is hinged.*

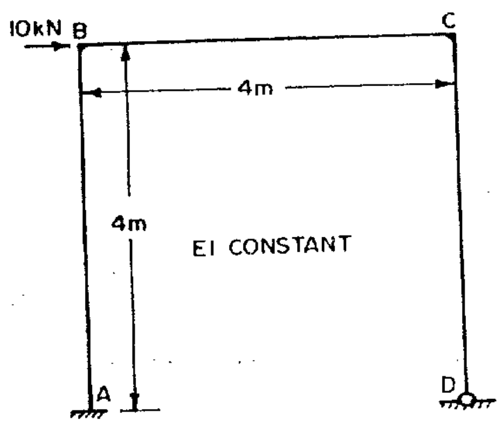


Fig. 9·15.

Solution

(*a*) *Fixed end moments* t :

There will be no fixed end moments for any of the spans of the frame as 10 kN load is acting on the joint.

(*b*) *Slope deflection equations* :

The unknowns in this case are θ_B, θ_C, θ_D and the joint translation δ. Because E, I and L are same for all members, K is constant. Assuming no change in the length of BC, the horizontal movement of B and C will equal to δ. For AB and CD, $R=\dfrac{\delta}{4}$.

$$M_{AB}=2EK(\theta_B-3R) \quad (1)$$
$$M_{BA}=2EK(2\theta_B-3R) \quad (2)$$
$$M_{BC}=2EK(2\theta_B+\theta_C) \quad (3)$$
$$M_{CB}=2EK(2\theta_C+\theta_B) \quad (4)$$
$$M_{CD}=2EK(2\theta_C+\theta_D-3R) \quad (5)$$
$$M_{DC}=2EK(2\theta_D+\theta_C-3R) \quad (6)$$

Since M_{DC} is zero, θ_D can be expressed in terms of θ_C, thereby reducing the number of unknowns to three.

$$M_{DC}=0=2\theta_D+\theta_C-3R$$

or $$\theta_D=\frac{3R-\theta_C}{2}.$$

(*c*) *Equilibrium equations* :

At joint B,

$$M_{BA}+M_{BC}=0$$

So, $$2EK(2\theta_B-3R)+2EK(2\theta_B+\theta_C)=0$$

or $$8EK\theta_B-6EKR+2EK\theta_C=0$$

or $$4\theta_B-3R+\theta_C=0 \quad (7)$$

At joint C,

$$M_{CB}+M_{CD}=0$$

So, $$2EK(2\theta_C+\theta_B)+2EK(2\theta_C+\theta_D-3R)=0$$

or $$4\theta_C+\theta_B+\theta_D-3R=0.$$

Substituting the values of $\theta_D=\dfrac{3R-\theta_C}{2}$, we get

$$4\theta_C+\theta_B+\frac{3R-\theta_C}{2}-3R=0$$

or $$7\theta_C+2\theta_B-3R=0 \quad (8)$$

(*d*) *Shear equations* :

$$\frac{M_{BA}+M_{AB}}{4}+\frac{M_{CD}}{4}+P=0$$

or $$\frac{2EK(2\theta_B-3R)+2EK(\theta_B-3R)}{4}+\frac{2EK(2\theta_C+\theta_D-3R)}{4}+10=0$$

or $$3\theta_B-6R+\frac{20}{EK}+2\theta_C+\frac{3R-\theta_C}{2}-3R=0$$

or $$6\theta_B-15R+\frac{40}{ER}+3\theta_C=0 \quad (9)$$

From equation 7, $\theta_C=3R-4\theta_B$

Substituting in equation (8)

$$21R-28\theta_B+2\theta_B-3R=0$$

or $$18R=26\theta_B$$

or $$R=\frac{13}{9}\theta_B \quad (i)$$

Hence $$\theta_C=\frac{3\times13}{9}\theta_B-4\theta_B=\frac{\theta_B}{3} \quad (ii)$$

and $$\theta_D=\frac{\frac{3\times13}{9}\times\theta_B-\frac{\theta_B}{3}}{2}=2\theta_B \quad (iii)$$

Substituting in equation (9), we get

$$6\theta_B-15\times\frac{13}{9}\theta_B+\frac{40}{EK}+\theta_B=0$$

or $$\frac{44}{3}\theta_B=\frac{40}{EK}$$

or $$\theta_B=\frac{30}{11\,EK}$$

and hence $$R=\frac{130}{33\,EK},\quad \theta_C=\frac{10}{11\,EK} \text{ and } \theta_D=\frac{60}{11\,EK}$$

(*e*) *Final moments* :

Substituting in equations 1 to 6, we get

$$M_{AB}=2EK\left(\frac{30}{11\,EK}-\frac{3\times130}{33\,EK}\right)=-18{\cdot}18 \text{ kN-m}$$

$$M_{BA}=2EK\left(\frac{2\times30}{11\,EK}-\frac{3\times130}{33\,EK}\right)=-12{\cdot}73 \text{ kN-m}$$

$$M_{BC}=2EK\left(\frac{2\times30}{11\,EK}+\frac{10}{11\,EK}\right)=+12{\cdot}73 \text{ kN-m}$$

$$M_{CB}=2EK\left(\frac{20}{11\,EK}+\frac{30}{11\,EK}\right)=+9{\cdot}09 \text{ kN-m}$$

and $$M_{CD}=2EK\left(\frac{20}{11\,EK}+\frac{60}{11\,EK}-\frac{3\times130}{33\,EK}\right)=-9{\cdot}09 \text{ kN-m}$$

The bending moment diagram and the deflected shape of the beam has been shown in Fig. 9·16.

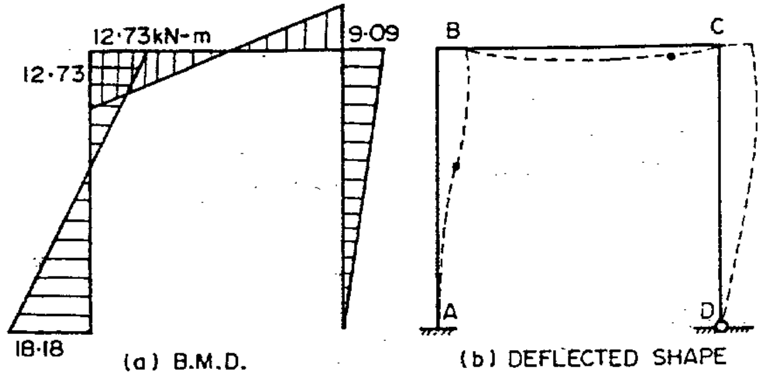


Fig. 9·16.

Example 9·16. *A portal frame ABCD is fixed at A and D, and has rigid joints at B and C. The column AB is 3 m long and column CD, 2 m long. The beam BC is 2 m long and is loaded with uniformly distributed load of intensity 6 kN/m. The moment of inertia of AB is 2 I and that of BC and CD is I (Fig. 9·17). Plot B.M. diagram and sketch the deflected shape of the frame.*

Solution

(*a*) *Fixed end moments*

$$M_{FBC}=+\frac{6\times 2^2}{12}=-2 \text{ kN-m}$$

$$M_{FCB}=+2 \text{ kN-m}$$

$$M_{FAB}=M_{FBA}=M_{FCD}=M_{FDC}=0.$$

Let the joints B and C move horizontally by δ.

(*b*) *Slope deflection equations*

$$M_{AB}=\frac{2.E.2I}{3}\left(\theta_B-\frac{3\delta}{3}\right)=\frac{4}{3}EI(\theta_B-\delta) \quad (1)$$

$$M_{BA}=\frac{2.E.2I}{3}\left(2\theta_B-\frac{3\delta}{3}\right)=\frac{4}{3}EI(2\theta_B-\delta) \quad (2)$$

$$M_{BC}=\frac{2.E.I}{2}\left(2\theta_B+\theta_C\right)-2 \quad (3)$$

$$M_{CB}=\frac{2.E.I}{2}\left(2\theta_C+\theta_B\right)+2 \quad (4)$$

$$M_{CD}=\frac{2.E.I}{2}\left(2\theta_C-\frac{3\delta}{2}\right)=EI(2\theta_C-1\cdot 5\,\delta) \quad (5)$$

and $M_{DC}=EI(\theta_C-1\cdot 5\,\delta)$ (6)

(*c*) *Equilibrium equations*

At joint B,

$$M_{BA}+M_{BC}=0$$

or $\frac{4}{3}EI(2\theta_B-\delta)+EI(2\theta_B+\theta_C)-2=0$

or $\frac{8}{3}\theta_B-\frac{4}{3}\delta+2\theta_B+\theta_C=\frac{2}{EI}$

or $14\theta_B+3\theta_C-4\delta=\frac{6}{EI}$ (7)

At joint C,

$$M_{CB}+M_{CD}=0$$

or $EI(2\theta_C+\theta_B)+2+EI(2\theta_C-1\cdot 5\delta)=0$

or $4\theta_C+\theta_B-1\cdot 5\delta+\frac{2}{EI}=0$ (8)

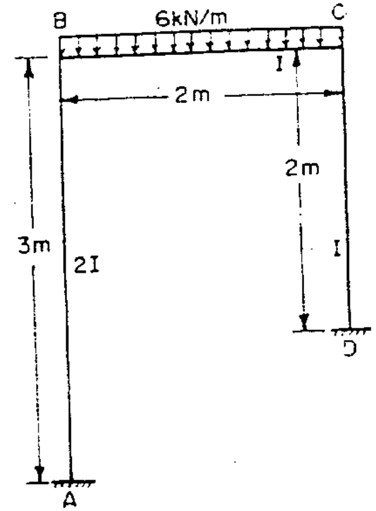


Fig. 9·17.

(*d*) *Shear equation*

$$\frac{M_{AB}+M_{BA}}{3}+\frac{M_{DC}+M_{CD}}{2}=0$$

$$\frac{\frac{4}{3}EI(\theta_B-\delta)+\frac{4}{3}EI(2\theta-\delta)}{3}+\frac{EI(\theta_C-1\cdot 5\delta)+EI(2\theta_C-1\cdot 5\delta)}{2}=0$$

or $8\theta_B-8\delta+16\theta_B-8\delta+9\theta_C-13\cdot 5\delta+18\theta_C-13\cdot 5\delta=0$

or $24\theta_B+27\theta_C=43\delta$ (9)

From equation 7, $\theta_C=\frac{2}{EI}+\frac{4\delta}{3}-\frac{14}{3}\theta_B$

Substituting the value of θ_C in equation 8,

$$\frac{8}{EI}+\frac{16}{3}\delta-\frac{56}{3}\theta_B+\theta_B-1\cdot 5\delta+\frac{2}{EI}=0$$

or $\frac{53}{3}\theta_B=\frac{10}{EI}+\frac{23}{6}\delta$

or $\theta_B=\frac{30}{53EI}+\frac{23}{106}\delta$ (10)

Substituting the value of θ_C in equation (9), we get

$$24\theta_B+\frac{54}{EI}+36\delta-126\theta_B=43\delta$$

or $102\theta_B=\frac{54}{EI}-7\delta$

or $\theta_B=\frac{54}{102EI}-\frac{7}{102}\delta$ (11)

Equating the value of θ_B given by equations 10 and 11,

$$\frac{30}{53EI}+\frac{23}{106}\delta=\frac{54}{102EI}-\frac{7}{102}\delta$$

$$\left(\frac{23}{106}+\frac{7}{102}\right)\delta=\frac{54}{102EI}-\frac{30}{53EI}$$

or
$$0{\cdot}286\delta=\frac{-0{\cdot}035}{EI}$$

or
$$\delta=\frac{-0{\cdot}123}{EI}$$

Substituting in equation 10,

$$\theta_B=\frac{30}{53EI}-\frac{23}{106}\times\frac{0{\cdot}123}{EI}=\frac{0{\cdot}538}{EI}$$

Similarly,
$$\theta_C=\frac{2}{EI}-\frac{4}{3}\times\frac{0{\cdot}123}{EI}-\frac{14}{3}\times\frac{0{\cdot}538}{EI}$$
$$=\frac{2}{EI}-\frac{0{\cdot}164}{EI}-\frac{2{\cdot}51}{EI}=\frac{-0{\cdot}674}{EI}.$$

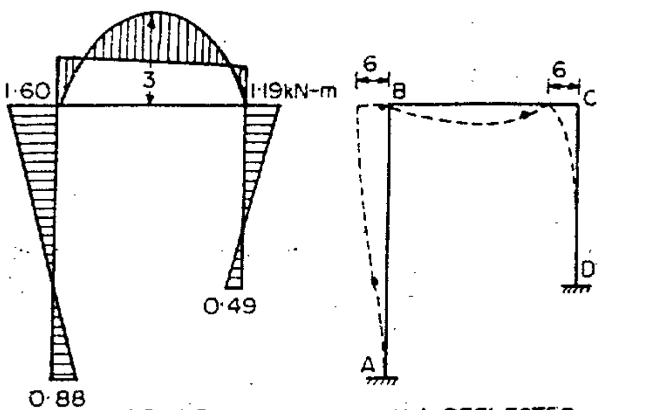
(e) *Final moments*

Substituting in equations 1 to 6, we get the values of end moments. Thus,

$$M_{AB}=\frac{4}{3}EI\left(\frac{0{\cdot}538}{EI}+\frac{0{\cdot}123}{EI}\right)=+0{\cdot}88 \text{ kN-m}$$

$$M_{BA}=\frac{4}{3}EI\left(\frac{2\times0{\cdot}538}{EI}+\frac{0{\cdot}123}{EI}\right)=+1{\cdot}60 \text{ kN-m}$$

$$M_{BC}=EI\left(\frac{2\times0{\cdot}538}{EI}-\frac{0{\cdot}674}{EI}\right)-2=-1{\cdot}60 \text{ kN-m}$$



(a) B.M.D. (b) DEFLECTED SHAPE

Fig. 9·18

$$M_{CB}=EI\left(\frac{-2\times0{\cdot}674}{EI}+\frac{0{\cdot}538}{EI}\right)+2=+1{\cdot}19 \text{ kN-m}$$

$$M_{CD}=EI\left(\frac{-2\times0{\cdot}674}{EI}+\frac{1{\cdot}5\times0{\cdot}123}{EI}\right)=-1{\cdot}19 \text{ kN-m}$$

and
$$M_{DC}=EI\left(\frac{-0{\cdot}674}{EI}+1{\cdot}5\times\frac{0{\cdot}123}{EI}\right)=-0{\cdot}49 \text{ kN-m}$$

The bending moment diagram and the deflected shape of the frame have been shown in Fig. 9·18.

Example 9·10. *A column AB fixed at the ends carries a load of 8 kN on the bracket as shown in Fig. 9·19. Plot the bending moment diagram and the deflected shape of the column.*

Solution

The load of 8 kN will give rise to a clockwise couple of 4 kN-m at *C*. The point *C* will be displaced by an amount δ.

(a) *Slope deflection equations*

There are two unknowns θ_C and δ. We shall assume θ_C to be positive. δ for *CA* is assumed positive and that for *CB*, is assumed negative.

$$M_{AC}=\frac{2EI}{3}\left(\theta_C-\frac{3\delta}{3}\right) \quad (1)$$

$$M_{CA}=\frac{2EI}{3}\left(2\theta_C-\frac{3\delta}{3}\right) \quad (2)$$

$$M_{CB}=\frac{2EI}{2}\left(2\theta_C+\frac{3\delta}{2}\right) \quad (3)$$

$$M_{BC}=\frac{2EI}{2}\left(\theta_C+\frac{3\delta}{2}\right) \quad (4)$$

(b) *Equilibrium equation*

As there are two unknowns, two equations will be required for finding out the values of unknowns. One equation will be provided by the fact that the clockwise couple at *C* causes clockwise moments in *CA* and *CB*.

$$\therefore \quad M_{CA}+M_{CB}=4$$

$$\frac{2EI}{3}\left(2\theta_C-\frac{3\delta}{3}\right)+\frac{2EI}{2}\left(2\theta_C+\frac{3}{2}\delta\right)=4$$

or
$$\frac{4}{3}EI\theta_C-\frac{2EI\delta}{3}+2EI\theta_C+\frac{3}{2}EI\delta=4$$

or
$$20EI\theta_C+5EI\delta=24$$

or
$$20\theta_C+5\delta=\frac{24}{EI} \quad (5)$$

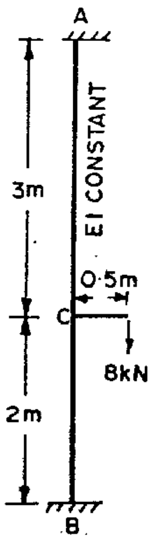


Fig. 9·19.

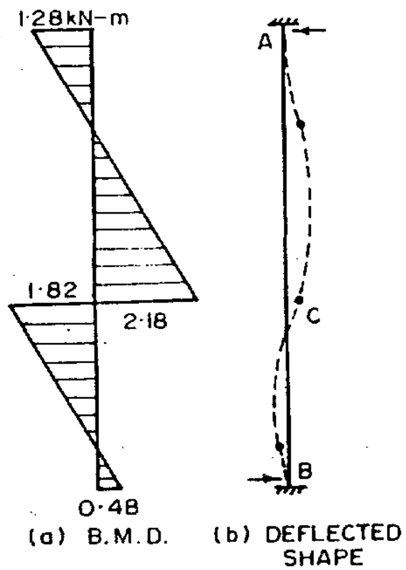


Fig. 9·20.

(c) Shear equation

The couple acting at C also gives rise to horizontal reaction at A and B, the two being equal in magnitude but opposite in direction.

Now horizontal reaction at $A=\dfrac{M_{AC}+M_{CA}}{3}$ and, horizontal reaction at $B=\dfrac{M_{CB}+M_{BC}}{2}$

As the two are equal so, $\dfrac{M_{AC}+M_{CA}}{2}=\dfrac{M_{CB}+M_{BC}}{2}$

$$\frac{\frac{2EI}{3}(\theta_C-\delta)+\frac{2EI}{3}(2\theta_C-\delta)}{3}=\frac{\frac{2EI}{2}\left(2\theta_C+\frac{3}{2}\delta\right)+\frac{2EI}{2}\left(\theta_C+\frac{3}{2}\delta\right)}{2}$$

or $$\frac{4}{3}EI\theta_C-\frac{4}{3}EI\delta+\frac{8}{3}EI\theta_C-\frac{4}{3}EI\delta.$$

$$=6EI\theta_C+\frac{9}{2}EI\delta+3EI\theta_C+\frac{9}{2}EI\delta$$

or $$30EI\theta_C=-70EI\delta$$

or $$\theta_C=-\frac{7}{3}\delta$$

Substituting in equation 5, we get

$$-\frac{140}{3}\delta+5\delta=\frac{24}{EI}$$

or $$-\frac{125}{3}\delta=\frac{24}{EI}$$

or $$\delta=\frac{-0{\cdot}576}{EI} \qquad (i)$$

and $$\theta_C=\frac{7}{3}\times\frac{0{\cdot}576}{EI}=\frac{1{\cdot}344}{EI} \qquad (ii)$$

(d) Final moments

The values of moments may now be found out by substituting the values of θ_C and δ in equations 1 to 4.

Thus, $$M_{AC}=\frac{2EI}{3}\left(\frac{1{\cdot}344}{EI}+\frac{0{\cdot}576}{EI}\right)=1{\cdot}28\text{ kN-m}$$

$$M_{CA}=\frac{2EI}{3}\left(\frac{2\times1{\cdot}344}{EI}+\frac{0{\cdot}576}{EI}\right)=2{\cdot}18\text{ kN-m}$$

$$M_{CB}=\frac{2EI}{2}\left(\frac{2\times1{\cdot}344}{EI}-\frac{3\times0{\cdot}576}{2EI}\right)=1{\cdot}82\text{ kN-m}$$

and $$M_{BC}=\frac{2EI}{2}\left(\frac{1{\cdot}344}{EI}-\frac{3}{2}\times\frac{0{\cdot}576}{EI}\right)=0{\cdot}48\text{ kN-m}$$

The B.M. diagram and the deflected shape have been shown in Fig. 9·20.

Example 9·11. *A portal frame ABCD is hinged at A and fixed at D and has stiff joints at B and C. The loading is as shown in Fig. 9·21. Draw the bending moment diagram and deflected shape of the frame.*

Solution

(a) Fixed end moments

$$M_{FBC}=-\frac{6\times2}{8}=-1{\cdot}5\text{ kN-m}$$

$$M_{FCB}=+1{\cdot}5\text{ kN-m}$$

$$M_{FCD}=-\frac{2\times4^2}{12}=-\frac{8}{3}\text{ kN-m}$$

$$M_{FDC}=+\frac{8}{3}\text{ kN-m}$$

(b) Slope deflection equations

Let joints B and C move horizontally by δ. There are four unknowns : θ_A, θ_B, θ_C and δ.

Assume all unknowns to be positive,

$$M_{AB}=\frac{2E\times 3I}{2\times 3}\left(2\theta_A+\theta_B-\frac{3\times\delta}{3}\right)=EI(2\theta_A+\theta_B-\delta) \quad (1)$$

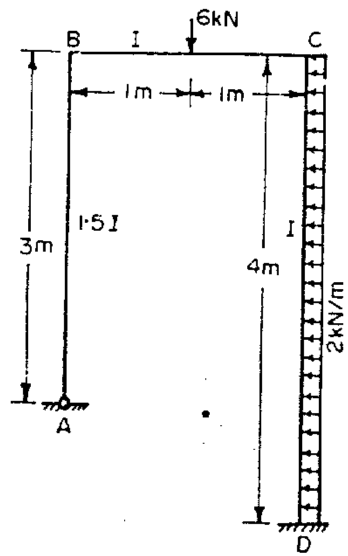


Fig. 9·21.

$$M_{BA}=\frac{2E\times 3I}{2\times 3}\left(2\theta_B+\theta_A-\frac{3\delta}{3}\right)=EI(2\theta_B+\theta_A-\delta) \quad (2)$$

$$M_{BC}=\frac{2EI}{2}(2\theta_B+\theta_C)-1{\cdot}5=EI(2\theta_B+\theta_C)-1{\cdot}5 \quad (3)$$

$$M_{CB}=\frac{2EI}{2}(2\theta_C+\theta_B)+1{\cdot}3=EI(2\theta_C+\theta_B)+1{\cdot}5 \quad (4)$$

$$M_{CD}=\frac{2EI}{4}\left(2\theta_C-\frac{3\delta}{4}\right)-\frac{8}{3}=\frac{EI}{2}\left(2\theta_C-\frac{3\delta}{4}\right)-\frac{8}{3} \quad (5)$$

$$M_{DC}=\frac{2EI}{4}\left(\theta_C-\frac{3\delta}{4}\right)+\frac{8}{3}=\frac{EI}{2}\left(\theta_C-\frac{3\delta}{2}\right)+\frac{8}{3} \quad (6)$$

(c) *Equilibrium equations*

At joint B,

$$M_{BA}+M_{BC}=0$$

or
$$EI(2\theta_B+\theta_A-\delta)+EI(2\theta_B+\theta_C)-\frac{3}{2}=0$$

or
$$4\theta_B+\theta_A+\theta_C-\delta-\frac{3}{2EI}=0 \quad (7)$$

At joint C,

$$M_{CB}+M_{CD}=0$$

or
$$EI(2\theta_C+\theta_B)+\frac{3}{2}+\frac{EI}{2}\left(2\theta_C-\frac{3\delta}{4}\right)-\frac{8}{3}=0$$

or
$$3\theta_C+\theta_B-\frac{3}{8}\delta-\frac{7}{6EI}=0 \quad (8)$$

(d) *Shear equation*

$$\frac{M_{AB}+M_{BA}}{L_1}+\frac{M_{CD}+M_{DC}}{L_2}=\frac{wL_2}{2}$$

or
$$\frac{EI(2\theta_A+\theta_B-\delta)+EI(2\theta_B+\theta_A-\delta)}{3}+\frac{\frac{EI}{2}\left(2\theta_C-\frac{3\delta}{4}\right)-\frac{8}{3}+\frac{EI}{2}\left(\theta_C-\frac{3\delta}{4}\right)+\frac{8}{3}}{4}=\frac{2\times 4}{2}$$

or
$$8\theta_A+4\theta_B-4\delta+8\theta_B+4\theta_A-4\delta+3\theta_C-\frac{9}{8}\delta+\frac{3\theta_C}{2}-\frac{9}{8}\delta=\frac{48}{EI}$$

or
$$12\theta_A+12\theta_B+\frac{9}{2}\theta_C-\frac{41}{4}\delta=\frac{48}{EI} \quad (9)$$

The end A is hinged. So $M_{AB}=0$

i.e.
$$EI(2\theta_A+\theta_B-\delta)=0$$

or
$$\theta_B=\delta-2\theta_A \quad (10)$$

Substituting the value of θ_B in equation 7,

$$4\delta-8\theta_A+\theta_A+\theta_C-\delta-\frac{3}{2EI}=0$$

or
$$\theta_C-7\theta_A+3\delta-\frac{3}{2EI}=0$$

or
$$\theta_C=\frac{3}{2EI}+7\theta_A-3\delta \quad (11)$$

Substituting the value of θ_B in equation 8,

$$3\theta_C+\delta-2\theta_A-\frac{3}{8}\delta-\frac{7}{6EI}=0 \quad (12)$$

or
$$3\theta_C-2\theta_A+\frac{5}{8}\delta-\frac{7}{6EI}=0$$

Substituting the values of θ_B and θ_C in equation 9, we get

$$12\theta_A+12\delta-24\theta_A+\frac{27}{4EI}+\frac{63}{2}\theta_A-\frac{27}{2}\delta-\frac{41}{4}\delta=\frac{48}{EI}$$

or
$$\theta_A=\frac{330}{156EI}+\frac{47}{78}\delta$$

Substituting the values of θ_C and θ_A in equation 12, we get

$$\frac{9}{2EI}+\frac{19\times330}{156EI}+\frac{47\times19}{78}\delta-98+\frac{5}{8}\delta-\frac{7}{6EI}=0$$

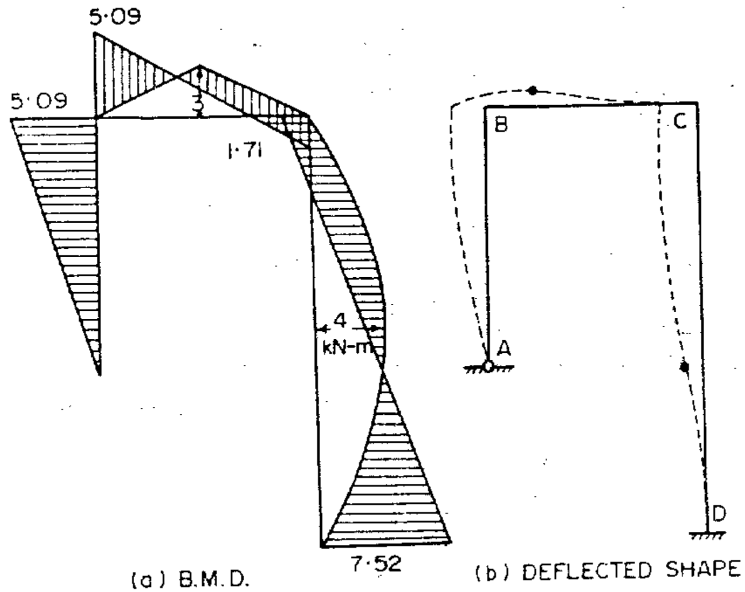


Fig. 9·22.

or $$\frac{959}{312}\delta=-\frac{3395}{78EI}$$

or $$\delta=-\frac{3395}{78EI}\times\frac{312}{959}=-\frac{13580}{959EI} \qquad (i)$$

Hence $$\theta_A=\frac{330}{156EI}-\frac{47}{78}\times\frac{13580}{959EI}=-\frac{6\cdot42}{EI} \qquad (ii)$$

$$\theta_B=\frac{3}{2EI}-\frac{7\times6\cdot42}{EI}+\frac{3\times13580}{959EI}=-\frac{0\cdot94}{EI} \qquad (iii)$$

and $$\theta_B=\frac{-13580}{959EI}+\frac{2\times6\cdot42}{EI}=-\frac{1\cdot33}{EI} \qquad (iv)$$

(e) *Final moments*

Substituting the values of θ_A, θ_B, θ_C and δ in equations 2 to 6, we get

$$M_{BA}=EI\left(\frac{-2\times1\cdot33}{EI}-\frac{6\cdot42}{EI}+\frac{13580}{959EI}\right)=+5\cdot09\text{ kN-m}$$

$$M_{BC}=EI\left(\frac{-2\times1\cdot33}{EI}-\frac{0\cdot94}{EI}\right)-1\cdot5=-5\cdot09\text{ kN-m}$$

$$M_{CB}=EI\left(\frac{-2\times0\cdot94}{EI}+\frac{1\cdot33}{EI}\right)+1\cdot5=-1\cdot71\text{ kN·m}$$

$$M_{CD}=\frac{EI}{2}\cdot\left(\frac{-2\times0\cdot94}{EI}+\frac{3}{4}\times\frac{13580}{959EI}\right)-\frac{8}{3}$$

$$=+0\cdot171\text{ kN-cm}$$

and $$M_{DC}=EI\left(\frac{-0\cdot94}{EI}+\frac{3\times13580}{4\times959EI}\right)+2\cdot67=+7\cdot52\text{ kN-m}$$

The bending moment diagram and the deflected shape of the frame have been shown in Fig. 9·22.

Example 9·12. *Analyse the frame shown in Fig. 9·23. EK is constant for the whole frame.*

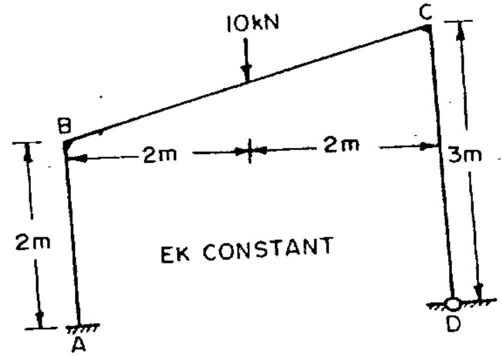


Fig. 9·23

Solution

(a) *Fixed end moments :*

$$M_{FBC}=\frac{-10\times4}{8}=-5\text{ kN·m}\ ;\ M_{FCB}=+5\text{ kN-m}$$

(b) *Slope deflection equations :*

The unknown quantities are θ_B, θ_C, θ_D and R. But as M_{DC} is zero, θ_D can be expressed in terms of θ_C thus leaving only three unknowns. Treating all these unknowns as positive, we get

$$M_{AB}=2EK\left(\theta_B-\frac{3\delta}{2}\right) \qquad (1)$$

$$M_{BA}=2EK\left(2\theta_B-\frac{3\delta}{2}\right) \qquad (2)$$

$$M_{BC}=2EK(2\theta_B+\theta_C)-5 \qquad (3)$$

$$M_{CB}=2EK\,(2\theta_C+\theta_B)+5 \qquad (4)$$

$$M_{CD}=2EK\left(2\theta_C+\theta_D-\frac{3\delta}{3}\right) \qquad (5)$$

and $$M_{DC}=2EK\left(2\theta_D+\theta_C-\frac{3\delta}{3}\right)=0$$

and $$\theta_D=\frac{\delta-\theta_C}{2} \qquad (6)$$

(c) *Equilibrium equations :*

At joint *B*,

$$M_{BA}+M_{BC}=0$$

$$2EK\left(2\theta_B-\frac{3}{2}\delta\right)+2EK(2\theta_B+\theta_C)-5=0$$

or $$8\theta_B+2\theta_C-3\delta=\frac{5}{EK} \qquad (7)$$

At joint *C*,

$$M_{CB}+M_{CD}=0$$

$$2EK(2\theta_C+\theta_B)+5+2EK\left(2\theta_C+\theta_D-\frac{3\delta}{3}\right)=0$$

or $$8\theta_C+2\theta_B+2\theta_D-2\delta+\frac{5}{EK}=0 \qquad (8)$$

(d) *Shear equations :*

$$\frac{M_{AB}+M_{BA}}{2}+\frac{M_{CD}}{3}=0$$

$$\frac{2EK\left(\theta_B-\frac{3}{2}\delta\right)+2EK\left(2\theta_B-\frac{3}{2}\delta\right)}{2}+\frac{2EK(2\theta_C+\theta_D-\delta)}{3}=0$$

or $$9\theta_B+4\theta_C+2\theta_D-11\delta=0 \qquad (9)$$

From equation 7, $\theta_B=\frac{5}{8EK}+\frac{3}{8}\delta-\frac{\theta_C}{4}$

Substituting the values of θ_D and θ_B in equation 8, we get

$$\frac{13}{2}\theta_C-\frac{\delta}{4}+\frac{25}{4EK}=0$$

or $$\delta=26\,\theta_C+\frac{25}{EK} \qquad (10)$$

Substituting the values of θ_B and θ_D in equation 9, we get

$$-\frac{53}{8}\delta+\frac{3}{4}\theta_C+\frac{45}{8EK}=0$$

or $$\delta=\frac{6}{53}\theta_C+\frac{45}{53EK} \qquad (11)$$

Equating the values of δ in equations 10 and 11, we get

$$26\,\theta_C+\frac{25}{EK}=\frac{6}{53}\theta_C+\frac{45}{53EK}$$

or $$\frac{1372}{53}\theta_C=\frac{-1280}{53EK} \quad \text{or} \quad \theta_C=\frac{-0.933}{EK} \qquad (i)$$

Hence $$\delta=\frac{-26\times0.933}{EK}+\frac{25}{EK}=\frac{0.75}{EK} \qquad (ii)$$

$$\theta_B=\frac{5}{8EK}+\frac{3\times0.75}{8\times EK}+\frac{0.933}{4EK}+\frac{1.139}{EK} \qquad (iii)$$

and $$\theta_D=\frac{\frac{0.75}{EK}+\frac{0.933}{EK}}{2}=\frac{0.842}{EK} \qquad (iv)$$

(e) *Final moments*

Substituting the values of θ_B, θ_C, θ_D and δ in equations 1 to 5, we get

$$M_{AB}=2EK\left(\frac{1.139}{EK}-1.5\times\frac{0.75}{EK}\right)=+0.028 \text{ kN-m}$$

$$M_{BA}=2EK\left(\frac{2\times1.139}{EK}-\frac{3}{2}\times\frac{0.75}{EK}\right)=+2.31 \text{ kN-m}$$

$$M_{BC}=2EK\left(\frac{2\times1.139}{EK}-\frac{0.933}{EK}\right)-5=-2.31 \text{ kN-m}$$

$$M_{CB}=2EK\left(\frac{-2\times0.933}{EK}+\frac{1.139}{EK}\right)+5=+3.55 \text{ kN-m}$$

and $$M_{CD}=2EK\left(\frac{-2\times0.933}{EK}+\frac{0.842}{EK}-\frac{0.75}{EK}\right)=-3.55 \text{ kN-m}$$

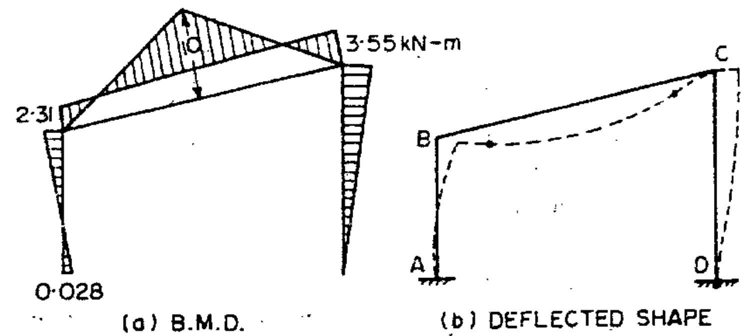


Fig. 9·24.

The bending moment diagram and the deflected shape of the frame have been shown in Fig. 9·24.

Example 9·13. *The frame shown in Fig. 9·25 has fixed ends at A and D. The end A rotates clockwise through* $\frac{0.20}{EK}$ *radians and the* *end D slips to the right through $\frac{0.4}{EK}$ units. Find the moments induced in the members of the frame and sketch the deflected shape. Take EK constant.*

Solution.

Since there is no external loading, there will be no fixed end moments.

When D moves to the right through a known distance Δ the joint B and C will move to the right through some unknown distance δ. The movement δ causes rotation of AB and DC and Δ causes negative rotation of CD. So, the net rotation of DC with respect to AB is the algebraic sum of rotations caused by δ and Δ. There are thus three unknowns : θ_B, θ_C and δ.

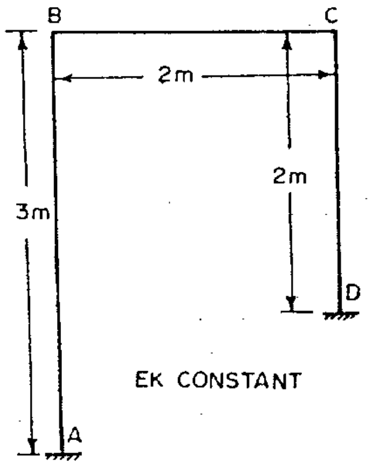


Fig. 9·25.

(a) *Slope deflection equations :*

$$M_{AB}=2EK\left(\frac{2\times0.2}{EK}+\theta_B-\frac{3\delta}{3}\right) \quad (1)$$

$$M_{BA}=2EK\left(2\theta_B+\frac{0.2}{EK}-\frac{3\delta}{3}\right) \quad (2)$$

$$M_{BC}=2EK\,(2\theta_B+\theta_C) \quad (3)$$

$$M_{CB}=2EK\,(2\theta_C+\theta_B) \quad (4)$$

$$M_{CD}=2EK\left\{2\theta_C+3\,\frac{\left(\frac{0.4}{EK}-\delta\right)}{2}\right\} \quad (5)$$

$$M_{DC}=2EK\left\{\theta_C+3\,\frac{\left(\frac{0.4}{EK}-\delta\right)}{2}\right\} \quad (6)$$

(b) *Equilibrium equations :*

At joint B.

$$M_{BA}+M_{BC}=0$$

or
$$2EK\left(2\theta_B+\frac{0.2}{EK}-\delta\right)+2EK\,(2\theta_B+\theta_C)=0$$

or
$$4\theta_B+\theta_C-\delta+\frac{0.2}{EK}=0$$

or
$$\theta_C=\delta-\frac{0.2}{EK}-4\theta_B \quad (7)$$

At joint C,

$$M_{CB}+M_{CD}=0$$

$$2EK\,(2\theta_C+\theta_B)+2EK\left\{2\theta_C+3\,\frac{\left(\frac{0.4}{EK}-\delta\right)}{2}\right\}=0$$

or
$$8\theta_C+2\theta_B+\frac{1.2}{EK}-3\delta=0 \quad (8)$$

(c) *Shear equation :*

$$\frac{M_{AB}+M_{BA}}{3}+\frac{M_{CD}+M_{DC}}{2}=0$$

or
$$\frac{2EK\left(\frac{0.4}{EK}+\theta_B-\delta\right)+2EK\left(2\theta_B+\frac{0.2}{EK}-\delta\right)}{3}+\frac{2EK\left\{2\theta_C+\frac{3}{2}\left(\frac{0.4}{EK}-\delta\right)\right\}}{2}+\frac{2EK\left\{\theta_C+\frac{3}{2}\left(\frac{0.4}{EK}-\delta\right)\right\}}{2}=0$$

This reduces to,

$$\frac{4.8}{EK}+6\theta_B+9\theta_C-13\,\delta=0 \quad (9)$$

Substituting the value of θ_C from equation 7 in equation 8, we get

$$8\,\delta-\frac{1.6}{EK}-32\,\theta_B+2\theta_B+\frac{1.2}{EK}-3\delta=0$$

or
$$5\,\delta-\frac{0.4}{EK}=30\,\theta_B$$

or
$$\theta_B=\frac{\delta}{6}-\frac{0.4}{30\,EK}$$

Substituting the values of θ_C and θ_B in equation 9, we get

$$\frac{4.8}{EK}+\delta-\frac{0.4}{5EK}+9\delta-\frac{1.8}{EK}-6\delta+\frac{0.48}{EK}-13\delta=0$$

or
$$\delta=\frac{3.4}{9EK} \quad \ldots(i)$$

Hence $\qquad \theta_B = \dfrac{3\cdot4}{9\times6EK} - \dfrac{0\cdot4}{30\,EK} = \dfrac{0\cdot0497}{EK}$ radians $\qquad$ ...(*ii*)

and $\qquad \theta_C = \dfrac{3\cdot4}{9EK} - \dfrac{0\cdot2}{EK} - \dfrac{4\times0\cdot0497}{EK} = -\dfrac{0\cdot021}{EK}$ radians ...(*iii*)

(*d*) *Final moments :*

Substituting in equations 1 to 6, we get the value of moments as follows :

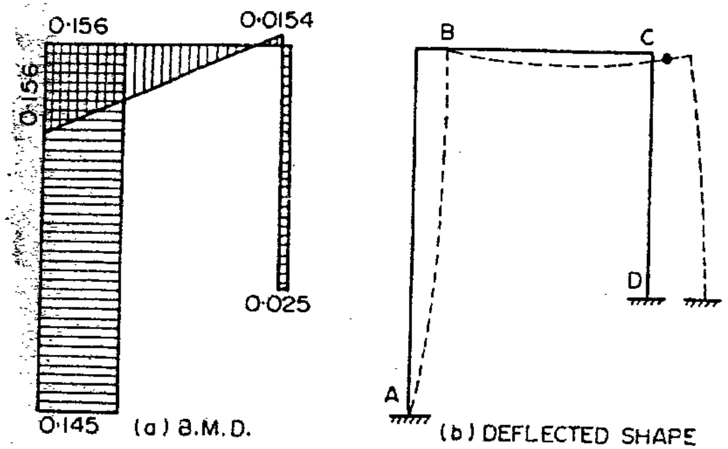


Fig. 9·26.

$$M_{AB} = +0\cdot145 \text{ units} \; ; \; M_{BA} = -0\cdot156 \text{ units}$$

$$M_{BC} = +0\cdot156 \text{ units} \; ; \; M_{CB} = +0\cdot0154 \text{ units}$$

$$M_{CD} = -0\cdot0154 \text{ units} \; ; \; M_{DC} = +0\cdot025 \text{ units}$$

It is to be noted that the quantity EK has the units of moment. The units of the above moments will, therefore, be the same as the units of EK.

The bending moment diagram and the deflected shape of the frame have been shown in Fig. 9·26.

Example 9·14. *The portal frame shown in Fig. 9·27 has fixed ends. If the end D sinks by* Δ, *find the moment induced in the frame. The members have the same uniform cross-section.*

Solution

When the end D sinks by Δ, the joints C will also sink by Δ and BC will rotate in clockwise direction. There will be side sway also on the right side. Let the movement of B and C, perpendicular to the axis of AB and DC be δ. There are thus three unknowns ; θ_B, θ_C and δ. Since there is no external loading, there will be no fixed end moments.

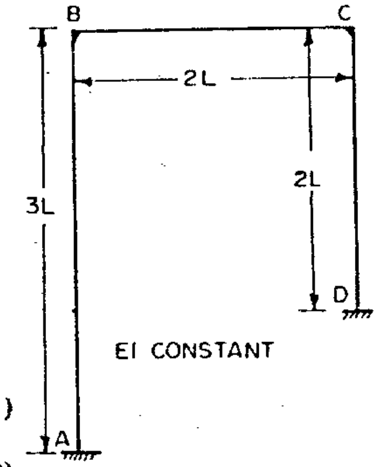


Fig. 9·27.

(*a*) *Slope deflection equations :*

$$M_{AB} = \frac{2EI}{3L}\left(\theta_B - \frac{3\delta}{3L}\right) \quad (1)$$

$$M_{BA} = \frac{2EI}{3L}\left(2\theta_B - \frac{3\delta}{3L}\right) \quad (2)$$

$$M_{BC} = \frac{2EI}{2L}\left(2\theta_B + \theta_C - \frac{3\Delta}{2L}\right) \quad (3)$$

$$M_{CB} = \frac{2EI}{2L}\left(2\theta_C + \theta_B - \frac{3\Delta}{2L}\right) \quad (4)$$

$$M_{CD} = \frac{2EI}{2L}\left(2\theta_C - \frac{3\delta}{2L}\right) \quad (5)$$

and $$M_{DC} = \frac{2EI}{2L}\left(\theta_C - \frac{3\delta}{2L}\right) \quad (6)$$

(*b*) *Equilibrium equations :*

At joint B,

$$M_{BA} + M_{BC} = 0$$

$$\frac{4EI\theta_B}{3L} - \frac{2EI\delta}{3L^2} + \frac{2EI\theta_B}{L} + \frac{EI\theta_C}{L} - \frac{3EI\Delta}{2L^2} = 0$$

or $$\frac{10}{3}\theta_B + \theta_C - \frac{2}{3L}\delta - \frac{3}{2L}\Delta = 0 \quad (7)$$

At joint C,

$$M_{CB} + M_{CD} = 0$$

$$\frac{2EI\theta_C}{L} + \frac{EI\theta_B}{L} - \frac{3EI\Delta}{2L^2} + \frac{2EI\theta_C}{L} - \frac{3EI\delta}{2L^2} = 0$$

or $$4\theta_C + \theta_B - \frac{3}{2L}\delta - \frac{3}{2L}\Delta = 0$$

or $$\theta_B = \frac{3}{2L}\delta + \frac{3}{2L}\Delta - 4\theta_C \quad (8)$$

(c) *Shear equation*

$$\frac{M_{AB}+M_{BA}}{3L}+\frac{M_{CD}+M_{DC}}{2L}=0$$

$$\frac{\frac{2EI\theta_B}{3L}-\frac{2EI\delta}{3L^2}+\frac{4EI\theta_B}{3L}-\frac{2EI\delta}{3L^2}}{3L}+\frac{\frac{2EI\theta_C}{L}-\frac{3EI\delta}{2L^2}+\frac{EI\theta_C}{L}-\frac{3EI\delta}{2L^2}}{2L}=0$$

It reduces to,

$$12\theta_B+27\theta_C-35\frac{\delta}{L}=0 \qquad (9)$$

Substituting the values of θ_B in equation 7,

$$\frac{5\delta}{L}+\frac{5\Delta}{L}-\frac{40}{3}\theta_C+\theta_C-\frac{2}{3}\frac{\delta}{L}-\frac{3\Delta}{2L}=0$$

or

$$\theta_C=\frac{13}{37}\frac{\delta}{L}+\frac{21}{74}\frac{\Delta}{L}$$

Substituting the value of θ_B in equation 9,

$$\frac{18\delta}{L}+\frac{18\Delta}{L}-48\theta_C+27\theta_C-\frac{35\delta}{L}=0$$

or

$$\theta_C=\frac{6\Delta}{7L}-\frac{17}{21}\frac{\delta}{L}$$

Equating the two values of θ_C,

$$\frac{6}{7L}\Delta-\frac{17}{21L}\delta=\frac{13}{37L}\delta+\frac{21}{74L}\Delta$$

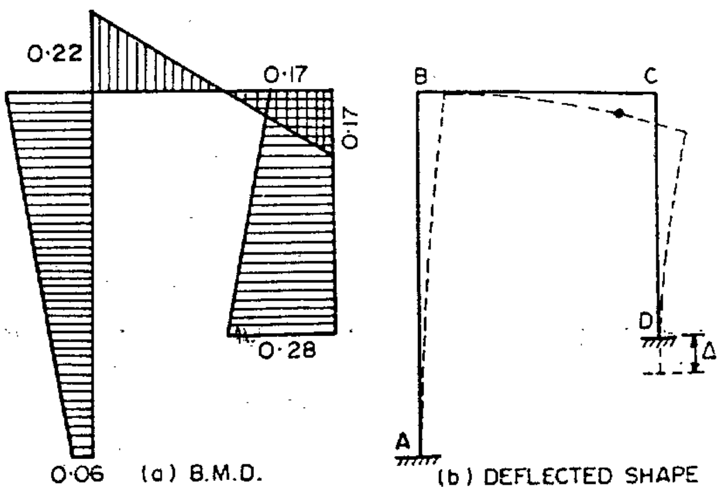


Fig. 9·28

or

$$\frac{13}{37L}\delta+\frac{17}{21L}\delta=\frac{6}{7L}\Delta-\frac{21}{74L}\Delta$$

or

$$1\cdot16\delta=0\cdot573\ \Delta$$

or

$$\delta=0\cdot4935\ \Delta \qquad (i)$$

Hence

$$\theta_C=\frac{13}{37L}\times0\cdot4935\ \Delta+\frac{21}{74L}\ \Delta=0\cdot457\frac{\Delta}{L} \qquad (ii)$$

and

$$\theta_B=\frac{3}{2L}\times0\cdot4935\ \Delta+\frac{3}{2L}\ \Delta-4\times0\cdot457\ \frac{\Delta}{L}$$

$$=0\cdot412\frac{\Delta}{L} \qquad (iii)$$

(d) *Final moments* :

By substituting the values of θ_B, θ_C and δ in equations 1 to 6 we get the end moments as follows :

$$M_{AB}=-\frac{0\cdot06EI\Delta}{L^2}\ ; \quad M_{BA}=+\frac{0\cdot22EI\Delta}{L^2}$$

$$M_{BC}=-\frac{0\cdot22EI\Delta}{L^2}\ ; \quad M_{CB}=-\frac{0\cdot17EI\Delta}{L^2}$$

$$M_{CD}=+\frac{0\cdot17EI\Delta}{L^2}\ ; \quad M_{DC}=-\frac{0\cdot28EI\Delta}{L^2}$$

The bending moment diagram and the deflected shape have been given in Fig 9·28. The values marked in Fig 9·28 (*a*) are to be multiplied by the factor $\frac{EI\Delta}{L^2}$.

PROBLEMS

1. A beam *ABC*, 12 m long, fixed at *A* and *C* and continuous over support *B*, is loaded as shown in Fig. 9·29. Calculate the end moments and plot the bending moment diagram.

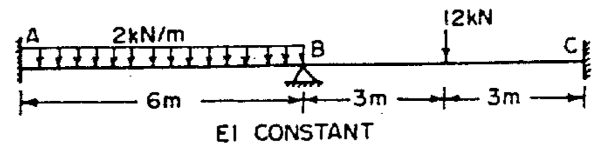


Fig. 9·29.

2. A continuous beam *ABCD* is fixed at ends *A* and *D*, and is loaded as shown in Fig. 9·30. Spans *AB*, *BC* and *CD* have moments of inertia of *I*, 1·51 and *I* respectively and are of the same material. Determine the moments at the supports and plot the bending moment diagram.

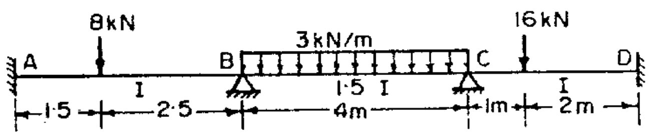


Fig. 9·30.

3. Solve problem 2 if there is no support at D.

4. Using the slope deflection method, calculate the moments at the support of the beam loaded as shown in Fig. 9·31.

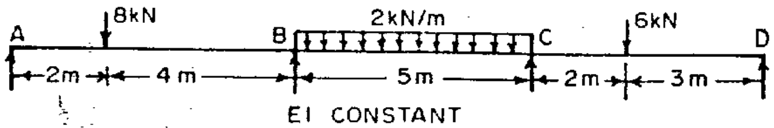


Fig. 9·31.

5. Draw the bending moment diagram and sketch the deflected shape of the frame shown in Fig. 9·32. All members a are of the same material.

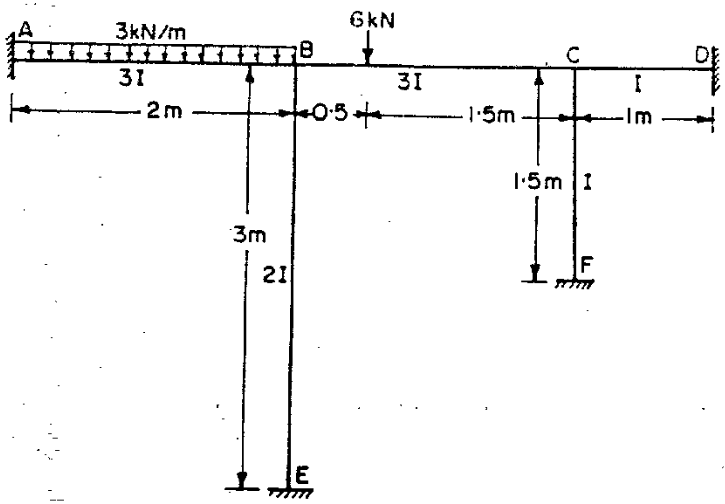


Fig. 9·32.

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6. Draw the bending moment diagram and sketch the deflected shape of the frame shown in Fig. 9·33.

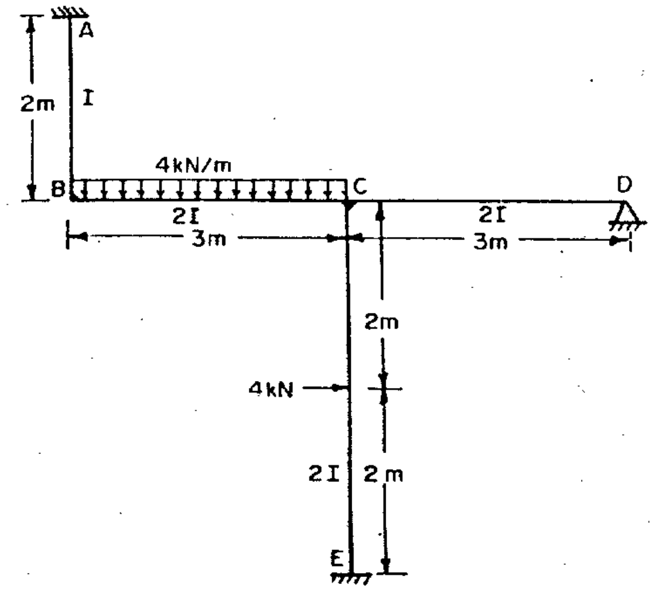


Fig. 9·33.

7. The frame *ABCDEF* shown in Fig. 9·34 has rigid joints throughout and is rigidly held at *A*, *E* and *F*. It carries a uniformly distributed load of w per unit length along *BD*. The stiffness ratios of the members are shown in the diagram and all the members are of equal length. Determine the bending moment throughout the frame and sketch the bending moment diagram.

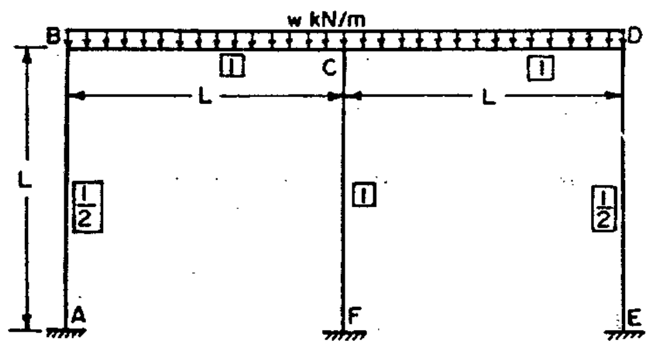


Fig. 9·34.

8. A portal frame *ABCD*, fixed at ends *A* and *D* carries a point load 2·5 kN as shown in Fig. 9·35. Draw the bending moment diagram and sketch the deflected shape of the beam.

9. Analyse completely the portal frame shown in Fig. 9.36.

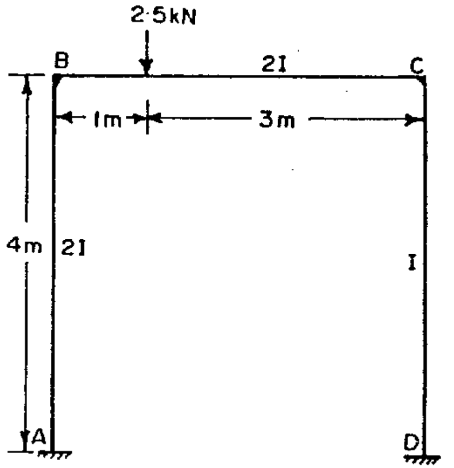


Fig. 9·35.

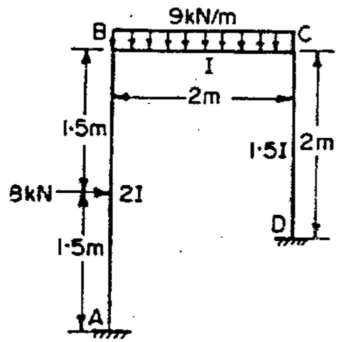


Fig. 9·36.

Answers :

1. $M_{AB}=-5{\cdot}25$; $M_{BA}=+7{\cdot}5$; $M_{BC}=-7{\cdot}5$; $M_{CB}=+9{\cdot}75$ kN-m.
2. $M_{AB}=-4{\cdot}6$ kN-m ; $M_{BA}=+2{\cdot}98$: $M_{BC}=-2{\cdot}98$; $M_{CB}=+5{\cdot}7$ $M_{CD}=-5{\cdot}7$; $M_{DC}=+4{\cdot}27$ kN-m.
3. $M_{AB}=-5{\cdot}83$; $M_{BA}=+0{\cdot}55$; $M_{BC}=-0{\cdot}55$; $M_{CB}=+16$ kN-m.
4. $M_{AB}=-7{\cdot}06$ kN-m ; $M_{BA}=+3{\cdot}63$; $M_{BC}=-3{\cdot}63$; $M_{CB}=+5{\cdot}14$; $M_{CD}=-5{\cdot}10$ kN-m.
5. $M_{AB}=-0{\cdot}82$ kN-m ; $M_{BA}=+1{\cdot}35$; $M_{BC}=-1{\cdot}51$; $M_{CB}=+0{\cdot}59$ $M_{CD}=-0{\cdot}23$; $M_{DC}=-0{\cdot}11$; $M_{BE}=+0{\cdot}15$; $M_{EB}=+0{\cdot}078$; $M_{CF}=-0{\cdot}15$; $M_{FC}=-0{\cdot}08$ kN-m.
6. $M_{AB}=+0{\cdot}91$; $M_{BA}=+1{\cdot}82$; $M_{BC}=-1{\cdot}82$; $M_{CB}=+1{\cdot}71$; $M_{CE}=+0{\cdot}14$; $M_{CB}=-1{\cdot}85$; $M_{EC}=-2{\cdot}93$.
7. $M_{AB}=+\frac{wL^2}{12}$; $M_{ED}=-\frac{wL^2}{72}$; $M_{BC}=-\frac{wL^2}{36}$; $M_{BA}=+\frac{wL^2}{36}$;

   $M_{CB}=+\frac{wL^2}{9}$; $M_{CD}=-\frac{wL^2}{9}$; $M_{DC}=+\frac{wL^2}{36}$; $M_{DE}=+\frac{wL^2}{36}$;

   $M_{CF}=M_{FC}=0$.
8. $M_{AB}=+0{\cdot}137$ kN-m ; $M_{BA}=+0{\cdot}647$; $M_{BC}=-0{\cdot}647$, $M_{CB}=+0{\cdot}461$; $M_{CD}=-0{\cdot}461$; $M_{DC}=-0{\cdot}325$.
9. $M_{AB}=-4{\cdot}34$ kN-m ; $M_{BA}=+2{\cdot}22$; $M_{BC}=-2{\cdot}22$; $M_{CB}=+3{\cdot}33$ $M_{CD}=-3{\cdot}33$; $M_{DC}=-3{\cdot}26$; $H_A=4{\cdot}703\leftarrow$; $H_D=3{\cdot}30\leftarrow$; $V_A=8{\cdot}45\uparrow$; $V_D=9{\cdot}55\uparrow$.

10

Moment Distribution Method

10·1. INTRODUCTION : SIGN CONVENTIONS

The method of moment distribution belongs to the group of approximate methods. Essentially, it consists of solving the simultaneous equations in the slope deflection method by successive approximations using a series of cycles, each converging towards ihe precise final result. The series may, therefore, be terminated whenever one reaches the degree of precision required by the particular problem under consideration. It leads to a very substantial reduction in the number of equations and in the case of structures, where joints can sustain angular twists alone but cannot be deflected, the method permits to avoid completely the solution of simultaneous equations with several unknowns. The method was first introduced by Prof. Hardy Cross in 1930. The moment distribution method could be used for the analysis of all types of statically indeterminate beams or rigid frames.

The sign convention used in the case of bending of simple beams, etc. becomes clumsy if used for the case of more complex

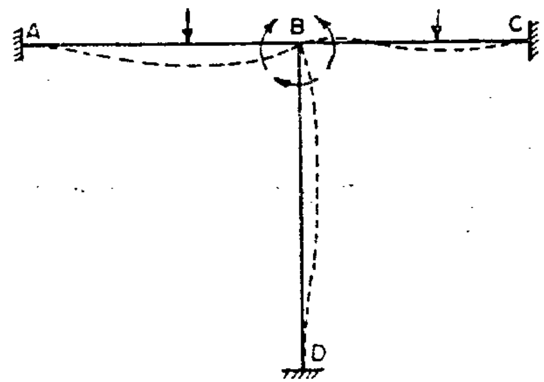


Fig. 10·1

beams and frames where more than two members meet at a joint. In our earlier sign convention for simple beams, a moment is considered to be positive if it bends the beam convex upwards, and negative if it bends the beam concave upwards. Thus, for the case of structure shown in Fig. 10·1, the three moments acting at the rigid joint B, where the three members BA, BC and BD meet, are all positive according to the previous convention since all the three moments tend to bend the three corresponding beams convex upwards. Hence the equilibrium equation $\Sigma M_B=0$ at the joint B cannot be conveniently applied if the previous sign convention is used, though the joint B is in equilibrium.

However, examination of joint E (Fig. 10·1) reveals that the moments, M_{BA} and M_{BD} are clockwise while the moment M_{BC} is anticlockwise. If the new sign convention is based on the direction of the moment, we get

$$M_{BA}+M_{BD}-M_{BC}=0.$$

Hence in the new sign convention that will be used in this method, *a support moment acting in the clockwise direction will be taken as positive and that in the anti-clockwise direction as negative.* A corresponding change will have to be made while plotting the *support moment diagram.* For any span of a beam or member with rigid joints, a positive support moment (or end moment) at the right

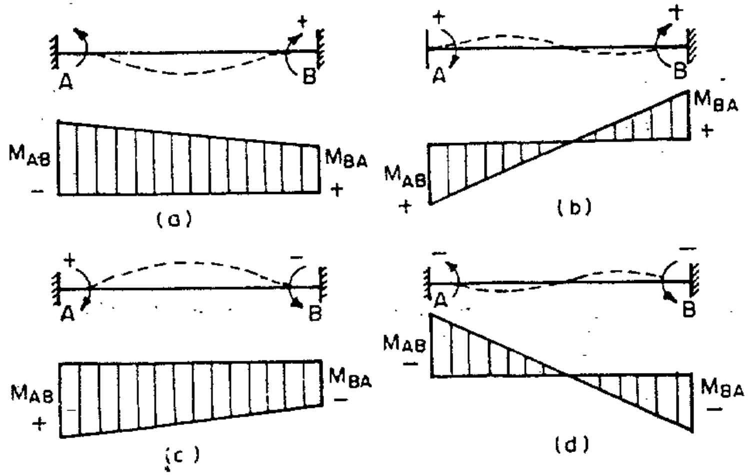


Fig. 10·2
Sign convention.

hand end will be plotted above the base line and negative support moment below the base. Similarly, for the left hand end, the negative end moment is plotted above the base line and positive end moment is plotted below the base line, as shown in Fig. 10·2.

In addition to the above sign convention for the end moments, the following sign convention for the rotation and settlement is adopted : (1) A clockwise rotation will be taken as positive and anti-clockwise rotation as negative. (2) If one end of the beam settles, the settlemet will be taken as positive if it rotates the beam as a whole in the clockwise direction, and negative if it rotates the beam as a whole in the anticlockwise direction.

10·2. FUNDAMENTAL PROPOSITIONS

For the better understanding of the theory and the mechanism of moment distribution, the following fundamental relations and deductions for prismatic beams are useful.

1. **Beam hinged at both ends**

Fig. 10·3 (a) shows a beam hinged at both ends, and subjected to a moment μ at the end A. Due to this, the rotations at the ends A

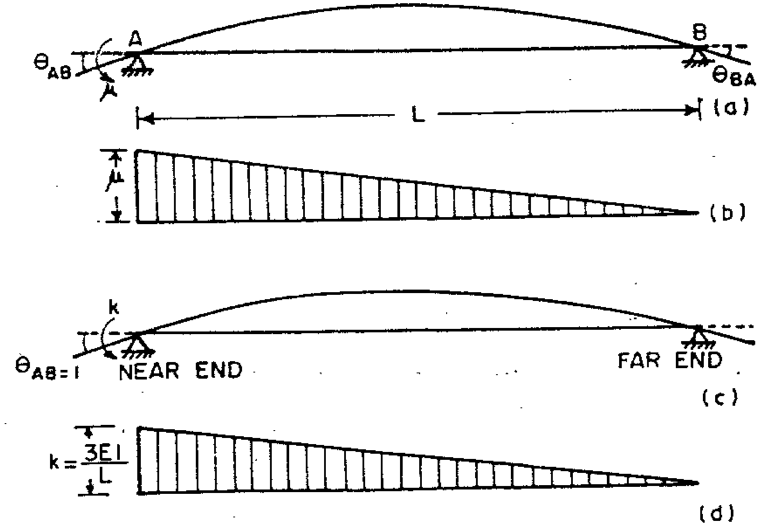


Fig. 10·3
Beam with both ends hinged.

and B are θ_{AB} and θ_{BA} respectively. Fig. 10·3 (*b*) shows the corresponding bending moment diagram. Since the beam is hinged at both the ends, the fixed end moments M_{FAB} and M_{FBA} are zero. Hence the slope-deflection equations for the span AB are :

$$M_{AB}=\frac{2EI}{L}[2\theta_{AB}+\theta_{BA}] \qquad (1)$$

and

$$M_{BA}=\frac{2EI}{L}[\theta_{AB}+2\theta_{BA}] \qquad (2)$$

For the joint B, the equilibrium equation is

$$M_{BA}=0$$

Hence, from (2), we get

$$\theta_{BA}=-\tfrac{1}{2}\theta_{AB}$$

Substituting this in (1) and noting that $M_{AB}=\mu$, we get

$$\mu=\frac{2EI}{L}\left[2\theta_{AB}-\frac{\theta_{AB}}{2}\right]=\frac{3EI}{L}\theta_{AB} \qquad [10·1\ (a)]$$

If θ_{AB}=unity, we have

$$\mu=k=\frac{3EI}{L} \qquad (10·1)$$

Eq. 10·1 gives the following important proposition :

Proposition 1

The moment k required to rotate the near end of a prismatic beam through a unit angle, without translation, the far end being freely supported, is given by

$$k=\frac{3EI}{L}$$

This moment k is known as *absolute stiffness* or *simply, stiffness. The stiffness of a member is the moment required to rotate the end under consideration through unit angle.*

2. **Beam hinged at one end and fixed at the other end**

Fig. 10·14 (*a*) shows a prismatic beam hinged at A and fixed at B, and subjected to a moment μ at the hinged end (called the near end). Due to moment μ, the end A rotates through angle θ_{AB} while the rotation of end B is zero since it is fixed. Let the induced moment at the end B be μ'. Since there is no external loading on the beam except the moment μ, the fixed end moments M_{FAB} and M_{FBA} are zero. Hence the slope deflection equation are :

$$M_{AB}=\mu=\frac{2EI}{L}[2\theta_{AB}+0] \qquad (1)$$

$$M_{BC}=\mu'=\frac{2EI}{L}[0+\theta_{AB}] \qquad (2)$$

From (1), we get

$$\mu=\frac{4EI}{L}\theta_{AB} \qquad (3)\ (10·2)$$

or

$$\theta_{AB}=\frac{\mu L}{4EI} \qquad (4)$$

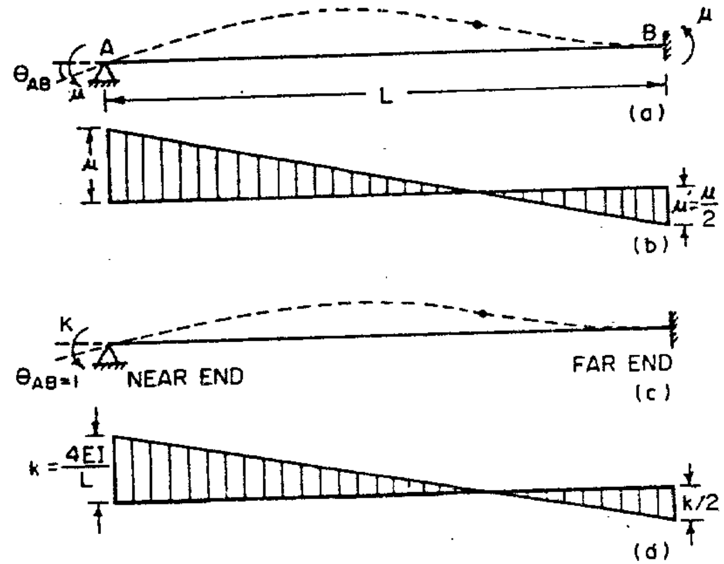


Fig. 10·4
Beam with far end fixed.

Substituting in (2) the value of θ_{AB}, we get

$$\mu'=\frac{\mu}{2} \qquad (10·3)$$

Eqs. 10·2 and 10·3 give the following two important propositions :

Proposition 2

A moment k required to rotate the near end of a prismatic beam through unit angle, without translation, the far end being fixed, is given by

$$k=\frac{4EI}{L} \qquad [10·2(a)]$$

Proposition 3

A moment which rotates the near end of prismatic beam without translation, the far end being fixed, induces at the far end a moment of one half its magnitude and in the same direction.

3. **Several members meeting at a joint**

Fig. 10·5 (*a*) shows members *AO*, *BO*, *CO* and *DO* meeting at a rigid joint *O*. Let L_1, L_2, L_3 and L_4 be the lengths and I_1, I_2, I_3 and I_4 be the moments of inertia of the respective members *AO*, *BO*, *CO* and *DO*. When an external moment μ is applied at the rigid joint *O*, the joint will rotate through an angle θ. Due to the rigidity of the joint, all the members will be rotated through the same angle θ. The applied moment μ will be resisted collectively by all the four members. Let μ_1, μ_2, μ_3 and μ_4 be the shares of the applied moment μ resisted by the members *OA*, *OB*, *OC* and *OD* respectively, as shown in Fig. 10·5 (*b*), so that $\mu=\mu_1+\mu_2+\mu_3+\mu_4$. The magnitudes of these moments depend upon the stiffness of the members, and are given by Eqs. 10·1 (*a*) and 10 2.

$$\mu_1=\frac{3E_1I_1}{L_1}\theta=k_1\theta \qquad (1)\ [\text{Eq. } 10{\cdot}1\ (a)]$$

$$\mu_2=\frac{4E_2I_2}{L_2}\theta=k_2\theta \qquad (2)\ [\text{Eq. } 10{\cdot}2]$$

$$\mu_3=\frac{3E_3I_3}{L_3}\theta=k_3\theta \qquad (3)\ [\text{Eq. } 10{\cdot}1(a)]$$

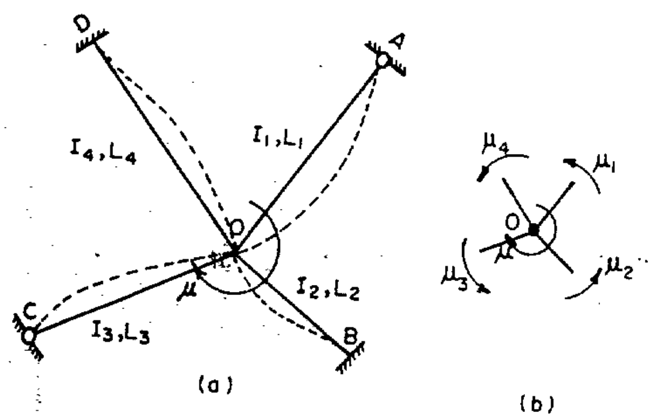


Fig. 10·5
Several members meeting at a joint.

$$\mu_4=\frac{4E_4I_4}{L_4}\theta=k_4\theta \qquad (4)\ (\text{Eq. } 10{\cdot}2)$$

where k_1, k_2, k_3 and k_4 are the stiffness of the members *OA*, *OB*, *OC* and *OD* respectively. From (1) to (4), we get

$$\mu_1:\mu_2:\mu_3:\mu_4::k_1:k_2:k_3:k_4 \qquad (I) \qquad (10{\cdot}4)$$

Also, from statics

$$\mu=\mu_1+\mu_2+\mu_3+\mu_4 \qquad (II)$$

From (I) and (II), we get the following expressions for the moments μ_1, μ_2, μ_3 and μ_4 :

$$\mu_1=\frac{k_1}{k_1+k_2+k_3+k_4}.\ \mu=\frac{k_1}{\Sigma k}\mu \qquad (a)$$

$$\mu_2=\frac{k_2}{k_1+k_2+k_3+k_4}.\ \mu=\frac{k_2}{\Sigma k}\mu \qquad (b)$$

$$\mu_3=\frac{k_3}{k_1+k_2+k_3+k_4}\ \mu=\frac{k_3}{\Sigma k}\mu \qquad (c)$$

and

$$\mu_4=\frac{k_4}{k_1+k_2+k_3+k_4}.\ \mu=\frac{k_4}{\Sigma k}\mu \qquad (d)$$

Eq. 10·4 gives the following proposition :

Proposition 4

A moment which tends to rotate a joint without translation, will be divided amongst the connecting members at the joint in proportion to their "stiffness".

The quantities $\frac{k_1}{\Sigma k}$, $\frac{k_2}{\Sigma k}$, $\frac{k_3}{\Sigma k}$ and $\frac{k_4}{\Sigma k}$ are called *distribution factors* for the members *OA*, *OB*, *OC* and *OD* respectively. The moments μ_1, μ_2, μ_3 and μ_4 are called the *distributed moments* and are in a direction opposite to that of μ since they finally restore the equilibrium at the joint.

Relative stiffness (*K*) : When several members meeting at a joint have different conditions of support at their other ends, it is always convenient to express the stiffness of the members in terms of *relative stiffness*. We have seen that the absolute stiffness (*k*) of a prismatic member with far end fixed is $\frac{4EI}{L}$. If all the prismatic members meeting at the joint are of the same material and are fixed at the far end, the stiffness of each member relative to the others may be represented by $\frac{I}{L}$. However, if some of the prismatic members meeting at the joints are freely supported at the other end, their relative stiffness may be taken equal to $\frac{3}{4}\cdot\frac{I}{L}$, since the absolute stiffness of such members is $\frac{3EI}{L}$ and is $\frac{3}{4}$ of that of the members fixed at the far end.

Example 10·1. *Calculate the distributed moments for the members OA, OB, OC and OD (Fig. 10·5), if their lengths are 150, 200, 100 and 200 cm and moments of inertia are 300, 400, 300 and 200 cm⁴ units respectively. The applied moment at joint O is 8100 N-cm.*

Solution

The calculations are arranged in Table 10·1.

TABLE 10·1

(Example 10·1)

Member	*Length (cm)*	*I cm⁴*	*Absolute stiffness (k)*	*Distribution factor* $\frac{k}{\Sigma k}$	*Distributed moment (N-cm)*
OA	150	300	$\frac{3E\times300}{150}=6E$	$\frac{6}{27}$	1800
OB	200	400	$\frac{4E\times400}{200}=8E$	$\frac{8}{27}$	2400
OC	100	300	$\frac{3E\times300}{100}=9E$	$\frac{9}{27}$	2700
OD	200	200	$\frac{4E\times200}{200}=4E$	$\frac{4}{27}$	1200
		Sum	$\Sigma k=27E$		8100

10·3. THE MOMENT DISTRIBUTION METHOD

The basic mechanism of moment distribution can be best understood with reference to Fig. 10·6 which shows a two span continuous beam *ABC* with ends *A* and *C* fixed. The method essentially consists of first locking all the joints (which are not fixed) so that each span of the structure behaves like a fixed beam. For the present

case of Fig. 10·6 (*a*), joint *B* is locked against rotation. The fixed end moments of the two beams *AB* and *BC* are:

$$M_{FAB}=-\frac{Wab^2}{L^2}=-\frac{5\times3\times2^2}{5^2}=-2{\cdot}40\text{ kN-m}$$

$$M_{FBA}=+\frac{Wa^2b}{L^2}=+\frac{5\times3^2\times2}{5^2}=+3{\cdot}60\text{ kN-m}$$

$$M_{FBC}=-\frac{WL}{8}=-\frac{8\times5}{8}=-5{\cdot}00\text{ kN-m}$$

$$M_{FCB}=+\frac{WL}{8}=+\frac{8\times5}{8}=+5{\cdot}00\text{ kN-m}$$

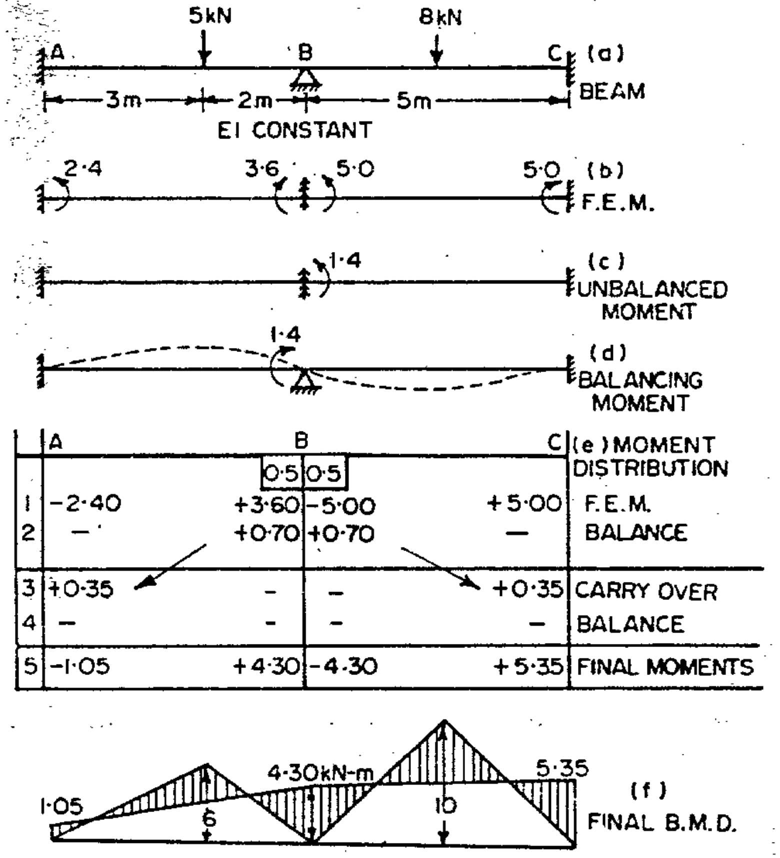


Fig. 10·6
Mechanism of moment distribution

The fixed end moment M_{FAB} and M_{FBC} are negative since they act in the anti-clockwise direction, while M_{FBA} and M_{FCB} act in the clockwise direction and hence positive according to our new sign convention. Fig. 10·6(*b*) shows the fixed end moments marked in the appropriate directions. At the joint *B*, there is an unbalanced moment of 1·4 kN-m (*i.e.* 5·00−3·60=1·40) acting in the anti-clockwise direction. It is this unbalanced moment which keeps the joint *B* locked against rotation. Actually, joint *B* is free to rotate. In order to permit it to rotate. or to unlock it, a balancing moment of +1·4 kN-m (*i.e.* clockwise) is applied at *B* as shown in Fig. 10·6 (*d*). Now according to proposition 4, end moment applied at a joint is to be resisted by the members meeting at the joint in proportion to their stiffnesses. Here, both the beams have equal span of 5m, and their *EI* is also the same. Hence their stiffnesses are equal. Due to this the *distribution factors* at *B* will be 0·5 for each of the beams *BA* and *BC* as marked in Fig. 10·6(*e*) showing the moment distribution in the tabular form. Thus, the balancing moment of +1·4 kN-m is distributed equally to both the beams *BA* and *BC*, each one getting a balancing moment of +0·70 kN-m, as indicated in step 2 of Fig. 10·6(*e*). Since the far ends of the beams *BA* and *BC* are fixed, the *carry over moments* at the far ends *A* and *C* are each +0·35 kN-m as indicated in step 3. Thus, all the three steps (step 1 : calculation of *F.E.M.*, step 2 : balancing the joint, and step 3 : carry over to the far ends) constitute *one cycle* of moment distribution.

At the third step. Fig. 10·6(*e*). when the moments are carried over from the near ends to the far ends there is no *carry over moments* from joint *A* to *B*, or from joint *C* to *B*, since joints *A* and *C* are originally fixed. *It should be remembered that a fixed joint does not need any balancing and it absorbs all the moments carried over to it from the other end.* Thus, in the third step, joint *B* does not have any unbalanced moment. The second cycle consists of the balancing of the joint *B*, which, in the present case, is not required. Hence there are no balancing moments against step 4 and the moment distribution is over, The last step consists of finding the final moments at each joint by taking the algebraic sum of the moments in each of the vertical columns for *A*, *B* and *C*. The final moments are :

At *A*, $M_{AB}=-1{\cdot}05$ kN-m

At *B*, $\left.\begin{array}{l} M_{BA}=+4{\cdot}30 \text{ kN-m} \\ M_{BC}=-4{\cdot}30 \text{ kN-m} \end{array}\right]$ joint perfectly balanced

At *C*, $M_{CB}=+5{\cdot}35$ kN-m

Fig. 10·6 (*f*) shows the final bending moment diagram for the beam.

In general, the complete moment distribution process consists of a number of cycles. The carried over moments (step 3) may constitute the unbalanced moments for the next cycle. The following examples explain the procedure for other cases when the far ends of the beams may or may not be fixed. For example, if joints *A* and *C* are not initially fixed, but are hinged, they are locked first, and then balanced in step 2. Then, in step 3 joint *B* receives carry over moments from both joints *A* as well as *C*. These moments become the unbalanced moments for the second cycle, as indicated in Example 10·2.

Example 10·2. *Solve problem of Fig. 10·6 (a) if ends A and C are simply supported (or hinged).*

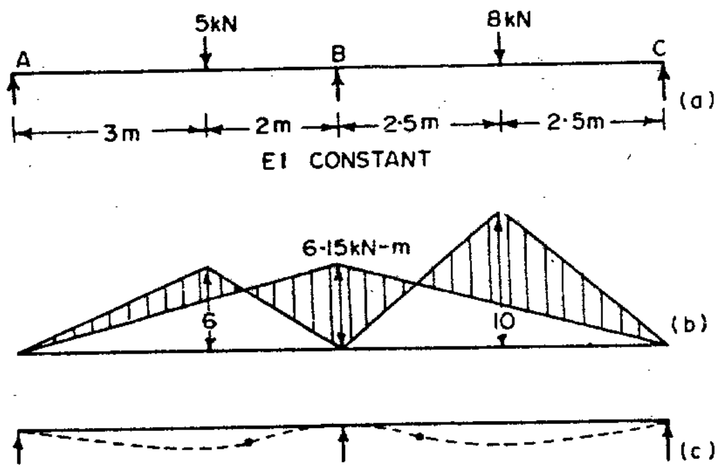


Fig. 10·7.

Solution

Step 1. Lock the joints *A*, *B* and *C*, and calculate the fixed end moments by treating *AB* and *BC* as two fixed beams. The moments are.

$M_{FAB}=-2{\cdot}40$ kN-m ; $M_{FBA}=+3{\cdot}60$ kN-m ;

$M_{FBC}=-5{\cdot}00$ kN-m ; $M_{FCB}=+5{\cdot}00$ kN-m

Since both the beams have the same values of E, I and L, and have the same end conditions, their relative stiffness will be equal. Thus the distribution factors will be 0·5 each at joint B for members BA and BC.

Step 2. Unlock the joints A, B and C in succession, and balance them by applying balanoing moments in a direction opposite to the unbalanced moment at that joint. Thus, at joint A, the unbalanced moment is $-2{\cdot}40$ kN-m, and hence the balancing moment will be $+2{\cdot}40$ kN-m. At joint C, the balancing moment will be $-5{\cdot}00$ kNm. At joint B, the unbalanced moment is $-1{\cdot}40$ and hence the balancing moment will be $+1{\cdot}40$ kN-m, which will be applied equally (*i.e.* $+0{\cdot}70$ kN-m) to each of the spans BA and BC at joint B. This is indicated in step 2 of Table 10·2.

TABLE 10·2

(Example 10·2)

Cycle		*A*	*B*		*C*	
			0·5	0·5		
	1	−2·40	+3·60	−5·00	+5 00	F.E.M.
	2	+2·40	+0·70	+0·70	−5·00	Balance
I	3	+0·35	+1·20	−2·50	+0·35	Carry over
	4	−0·35	+0·65	+0·65	−0·35	Balance
II	5	+0·32	−0·17	−0·17	+0·32	Carry over
	6	−0·32	+0·17	+0·17	−0·32	Balance
IV	7	+0·09	−0·16	−0·16	+0·09	Carry over
	8	−0·09	+0·16	+0·16	−0·09	Balance
V	9	+0·08	−0·04	−0·04	+0·08	Carry over
	10	−0·08	+0·04	+0·04	−0·08	Balance
		0	+6·15	−6·15	0	Final

Step 3. Half of the balancing moment, with the same sign (see proposition 3) is carried over to the opposits joint. These carried

over moments constitute unbalanced moment for the second cycle. Hence a line is drawn after step 2. The complete moment distribution is shown in Table 10·2. The moment distribution procedure can be stopped at the end of any cycle when the needed accuracy is achieved. In the present case, the carried over moments at joint B from joints A and C are equal in the third cycle. Hence the procedure can be terminated at the end of the third cycle.

Fig. 10·8 shows the final bending moment diagram and the deflected shape of the beam.

Example 10·3. *A continuous beam ABCD consists of three span, and is loaded as shown in Fig. 10·8(a). Ends A and D are fixed. Determine the bending moments at the supports and plot the bending moment diagram.*

Solution

(*a*) *Fixed end moments* (kN-m units)

$$M_{FAB}=-\frac{2\times6^2}{12}=-6\ ;\ M_{FBA}=+\frac{2\times6^2}{12}=+6$$

$$M_{FBC}=-\frac{5\times3\times2^2}{5^2}=-2{\cdot}4\ ;\ M_{FCB}=+\frac{5\times2\times^23}{5^2}=+3{\cdot}6$$

$$M_{FCD}=-\frac{8\times5}{8}=-5\ ;\ M_{FDC}=+\frac{8\times5}{8}=+5$$

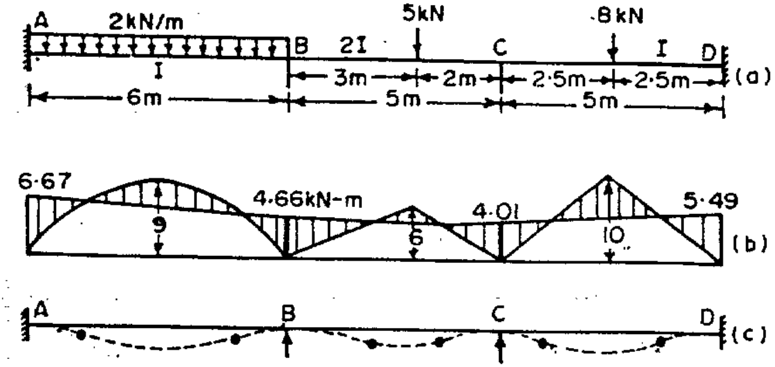


Fig. 10·8

(*b*) *Distribution factors*

The relative stiffness and distribution factors are calculated in Table 10·3.

TABLE 10·3

Joint	Member	Relative stiffness	Sum	Distribution factor
B	BA	$\frac{I}{6}$	$\frac{17I}{30}$	$\frac{5}{17}$
	BC	$\frac{2I}{5}$		$\frac{12}{17}$
C	CB	$\frac{2I}{5}$	$\frac{3I}{5}$	$\frac{2}{3}$
	CD	$\frac{I}{5}$		$\frac{1}{3}$

(c) *Moment distribution*

The moment distribution is carried out in Table 10·4. The final bending moment diagram and the deflected shape of the beam are shown in Fig. 10·8 (*b*) and (*c*) respectively.

TABLE 10·4

A	B		C		D	
	$\frac{5}{17}$	$\frac{12}{17}$	$\frac{2}{3}$	$\frac{1}{3}$		
−6·00	+6·00	−2·40	+3·60	−5·00	+5·00	F.E.M.
—	−1·06	−2·54	+0·93	+0·47	—	Balance
−0·53	—	+0·46	−1·27	—	+0·23	Carry over
—	−0·14	−0·32	+0·85	+0·42	—	Balance
−0·07	—	+0·42	−0·16	—	+0·21	C
—	−0·12	−0·30	+0·11	+0·05	—	B
−0·06	—	+0·05	−0·15	—	+0·03	C
−0·01 ←	−0·02	−0·03	+0·10	+0·05 →	+0·02	Balance & carry over to fixed ends
−6·67 ←	+4·66	−4·66	+4·01	−4·01	+5·49	Final moments

Note. It should be noted that the last cycle ends with the balancing of hinged or continuous joints and with carry over to the fixed joints.

Example 10·4. *Solve example 10·3 if the ends A and D are simply supported (or hinged).*

Solution :

(a) Fixed end moments

Lock all the joints against rotation. The fixed end moments will be the same as found in the previous example.

(b) Distribution factors

Considering joints *A* and *D* locked when the other joints are balanced, the same distribution factors, as calculated in the previous example will be applicable.

(c) *Moment distribution* (Table 10·5)

TABLE 10·5

A	B		C		D	
	$\frac{5}{17}$	$\frac{12}{17}$	$\frac{2}{3}$	$\frac{1}{3}$		
−6·00	+6·00	−2·40	+3·60	−5·00	+5·00	F.E.M.
+6·00	−1·06	−2·54	+0·93	+0·47	−5·00	Balance
−0·53	+3·00	+0·46	−1·27	−2·50	+0·23	Carry over
+0·53	−1·02	−2·44	+2·51	+1.26	−0·23	Balance
−0·51	+0·26	+1·26	−1·22	−0·11	+0·63	C
+0·51	−0·45	−1·07	+0·89	+0·44	−0·63	B
−0·22	+0·26	+0·44	−0·53	−0·32	+0·22	C
+0·22	−0·21	−0·49	+0·57	+0·28	−0·22	B
−0·11	+0·11	+0·29	−0·25	−0·11	+0·14	C
+0·11	−0·12	−0·28	+0·24	+0·12	−0·14	B
−0·06	+0·06	+0·12	−0·14	−0·07	+0·06	C
+0·06	−0·05	−0·13	+0·14	+0·07	+0·06	B
−0·02	+0·03	+0·07	−0·06	−0·03	+0·03	C
+0·02	−0·03	−0·07	+0·06	+0·03	−0·03	B
0·00	+6·78	−6·78	+5·47	−5·47	0·00	Final moments

Alternative Solution

An alternative method of solving the problem is to keep ends *A* and *D* *free* throughout the operation, except in the beginning when the fixed end moments are calculated. In that case, the relative stiffness of *BA* and *CD* will be $\frac{3}{4}$th of what have been taken in the previous example. The revised distribution factors are calculated in Table 10·6.

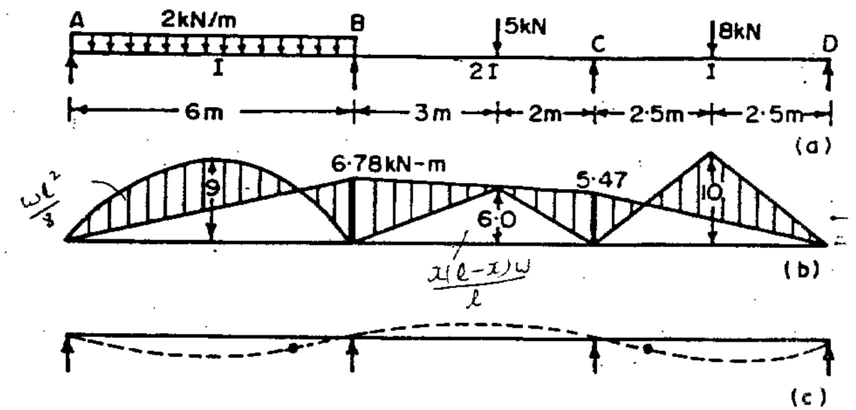


Fig. 10·9

TABLE 10·6

Joint	*Member*	*Relative stiffness*	*Sum*	*D.F.*
B	*BA*	$\frac{3}{4}\cdot\frac{I}{6}$	$\frac{63I}{120}$	$\frac{15}{63}=\frac{5}{21}$
	BC	$\frac{2I}{5}$		$\frac{48}{63}=\frac{16}{21}$
C	*CB*	$\frac{2I}{5}$	$\frac{11I}{20}$	$\frac{8}{11}$
	CD	$\frac{3}{4}\cdot\frac{I}{5}$		$\frac{3}{11}$

In the first cycle of the moment distribution (Table 10·7), only joint *A* and *D* are released and balanced, and half of the balanced moments are carried over to *B* and *C* respectively. A line is then drawn and the initial moments are found by taking the sum of the

first two lines of the first cycle. In process of the balancing and carry over that follows in the subsequent cycles, ends *B* and *D* are kept permanently free so that they neither need balancing nor is any moment carried over to them. The moment distribution is shown in Table 10·7

TABLE 10·7

A	*B*		*C*		*D*	
	$\frac{5}{21}$	$\frac{16}{21}$	$\frac{8}{11}$	$\frac{3}{11}$		
−6·00	+6·00	−2·40	+3·60	−5·00	+5·00	F.E.M.
+6·00	+3·00	—	—	−2·50	−5·00	Release *A* & *B* and carry over
0	+9·00	−2·40	+3·60	−7·50	0·00	Initial moments
—	−1·56	−5·04	+2·84	+1·06	—	Balance
—	—	+1·42	−2·52	—	—	Carry over
—	−0·34	−1·08	+1·83	+0·69	—	Balance
—	—	+0·92	−0·54	—	—	C
—	−0·22	−0·70	+0·40	+0·15	—	B
—	—	+0·20	−0·35	—	—	C
—	−0.05	−0·15	+0·25	+0·10	—	B
—	—	+0·12	−0·07	—	—	C
—	−0·03	−0·09	+0·05	+0·02	—	B
—	—	+0·02	−0·05	—	—	C
—	−0·01	−0·01	+0·03	+0·02	—	B
0·00	+6·79	−6·79	+5·46	−5·46	0·00	Final moments

Example 10·5. *Solve example 10·3 if there is no support at the end.*

Solution.

(*a*) *Fixed end moments*

Ends B and C are clamped and AB and BC are considered as fixed beams. The overhanging portion CD becomes a cantilever fixed at C. Hence

$$M_{FAB}=-6{\cdot}00 \text{ kN-m}\,; \qquad M_{FBA}=+6{\cdot}00 \text{ kN-m}$$
$$M_{FBC}=-2{\cdot}40 \text{ kN-m}: \qquad M_{FCB}=+3{\cdot}60 \text{ kN-m}$$
$$M_{FCD}=-8\times 2{\cdot}5=-20 \text{ kN-m}.$$

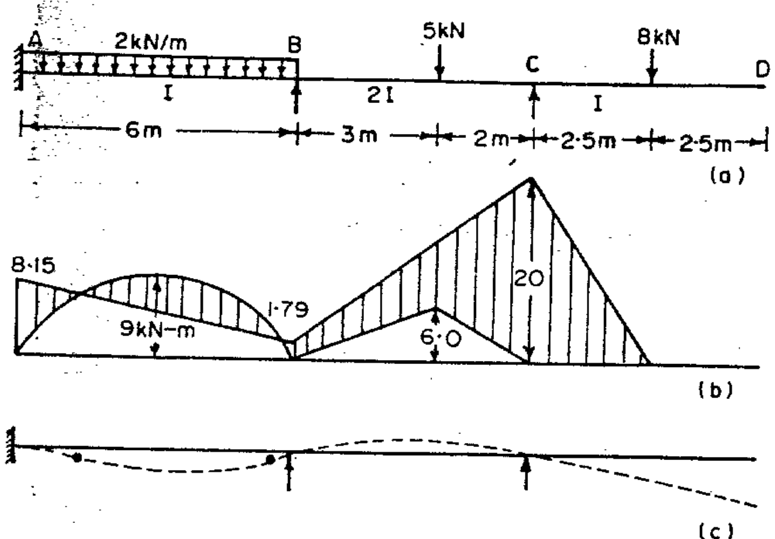


Fig. 10·10

(*b*) *Distribution factors*

The stiffness of the cantilever CD is zero since it has no resistance to rotation if an external moment is applied at the freely supported end C. The distribution factors at B are calculated on the premise that end C will be kept free throughout the process of moment distribution.

(*c*) *Moment distribution*

In the first cycle of the moment distribution, end C is balanced and half the balancing moments are carried over to end B. A line is then drawn and the initial unbalanced moments are found by taking the sum of the moments in the first two lines of the first cycle. In

TABLE 10·8

Joint	Member	Relative Stiffness	Sum	D.F.
B	BA	$\frac{I}{6}$	$\frac{7I}{15}$	$\frac{5}{14}$
	BC	$\frac{3}{4}\cdot\frac{2I}{5}$		$\frac{9}{14}$
C	CB	$\frac{2I}{5}$	$\frac{2I}{5}$	1
	CD	0		0

the process of balancing and carry over, end C is kept permanently free so that it neither needs balancing nor is any moment carried over to it.

TABLE 10·9.

A	B		C		D	
	$\frac{5}{14}$	$\frac{9}{14}$	1	0		
$-6{\cdot}00$	$+6{\cdot}00$	$-2{\cdot}40$	$+3{\cdot}60$	$-20{\cdot}00$	$0{\cdot}00$	F.E.M.
—	—	$+8{\cdot}20$ ←	$+16{\cdot}40$	—	—	Balance C and carry over to B
$-6{\cdot}00$	$+6{\cdot}00$	$+5{\cdot}80$	$+20{\cdot}00$	$-20{\cdot}00$	—	Initial
$-2{\cdot}11$ ←	$-4{\cdot}21$	$-7{\cdot}59$	—	—	—	Balance B and carry over to A
$-8{\cdot}11$	$+1{\cdot}79$	$-1{\cdot}79$	$+20{\cdot}00$	$-20{\cdot}00$	—	Final moments

10·4. SINKING OF SUPPORTS

(*a*) Beam fixed at both the ends

In the previous treatment, we have considered continuous beams resting on rigid supports which do not yield or sink under loads. However, if one of the ends of the beam sinks, additional moments will be induced at both the ends of the beam. If the sinking of the support is such as to rotate the beam *as a whole* in the clockwise direction, the moments at both the ends will be in the *anti-clockwise direction* and of equal magnitude. Similarly, if the sinking rotates the beam in anti-clockwise direction, the induced moments at both the ends will be in clockwise direction.

Fig. 10·11[*a*)(*i*)] shows a fixed beam of span L, with right hand support sunk by an amount δ. Fig. 10·11 [*a*(*ii*)] shows the component bending moment diagram due to the moments M induced at each end, while Fig. 10·12 [*a*(*iii*)] shows the net bending moment diagram. The beam has evidently, a point of contraflexure at its middle point.

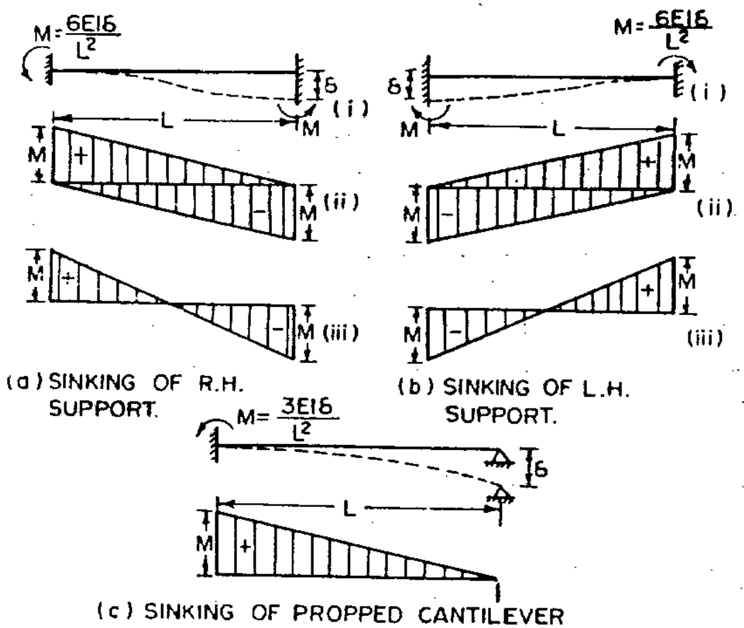


Fig. 10·11
Sinking of Supports

From the conjugate beam method, we get

$$\delta=\frac{1}{EI}\left[\frac{ML}{2}\times\frac{2}{3}L-\frac{ML}{2}\cdot\frac{1}{3}L\right)=\frac{ML^2}{6EI}$$

Hence $$\boxed{M=\frac{6EI\delta}{L^2}} \qquad (10\cdot6)$$

This is, thus, the expression for the moment induced at both the ends. According to our new sign convention since this moment acts in anti-clockwise direction at each end, it is *negative*.

Similarly, if the left hand of the beam sinks by an amount δ [Fig. 10·11 *b* (*i*)], the moment induced at each end is given by Eq. 10·6 and will be positive at each end.

(*b*) **Beam fixed at one end and freely supported at the other end** [Fig. 10·11 (*c*)].

If the beam is freely supported at the other end, the moment induced at the fixed end will be in the anti-clockwise direction if the rotation of the beam as a whole is clockwise and *vice versa*. The bending moment diagram will be a triangle having an ordinate M at the fixed end. From conjugate beam method, we get

$$\therefore \qquad \delta=\frac{1}{EI}\left(\frac{ML}{2}\times\frac{2}{3}L\right)=\frac{ML^2}{3EI}$$

or $$\boxed{M=\frac{3EI\delta}{L^2}} \qquad \ldots(10\cdot7)$$

10·5. CONTINUOUS BEAM ON ELASTIC PROPS

In the case of a continuous beam supported on elastic props, there are usually more than two members meeting at a joint. In such a case, proposition 4 is used. The unbalanced moment at any joint is distributed amongst all the members meeting at the joint in the ratio of their relative stiffness. See examples 10·8 to 10·11 for illustration.

10·6. PORTAL FRAMES WITH NO SIDE SWAY

A simple portal frame consists of a beam resting on two columns. The junction of the beam with the columns consist of rigid joints. The analysis of such a frame, when the loading conditions and the geometry of the frame is such that there is no joint translation or sway, is similar to that of continuous beam on elastic props. A portal frame may, in general, have more than one span and more than one storey. Examples 10·12, 10·13 and 10·14 illustrate the procedure of analysis of such frames when they do not have any joint translation.

Example 10·6. *A continuous beam ABC is shown in Fig. 10·12 (a). Calculate the moments induced at the ends if support B settle by 30 mm. Draw the bending moment diagram and the deflected shape of the beam. Take $E=2\times10^5$ N/mm² and $I=3\times10^6$ mm⁴ constant for the whole beam.*

Solution

(*a*) *Fixed end moments*

Clamp the supports B and C against rotation so that each span behaves as the fixed beam. Due to downward settlement of support

B, beam *AB* rotates clockwise as a whole. Hence fixed end moments at *A* and *B* will be anti-clockwise (*i.e.* negative) :

$$M_{FAB}=M_{FBA}=\frac{-6EI\delta}{L^2}=\frac{-6\times2\times10^5\times3\times10^6\times30}{(3000)^2}\text{ N-mm}$$

$$=-12\times10^6\text{ N-mm}=-12\text{ kN-m.}$$

Similarly, for the fixed beam *BC*,

$$M_{FBC}=M_{FCB}=+\frac{6EI\delta}{L^2}=+\frac{6\times2\times10^5\times3\times10^6\times30}{(2000)^2}\text{ N-mm}$$

$$=+27\times10^6\text{ N-mm}=+27\text{ kN-m.}$$

For the span *CD*, there will be no fixed end moments because neither of the ends sinks.

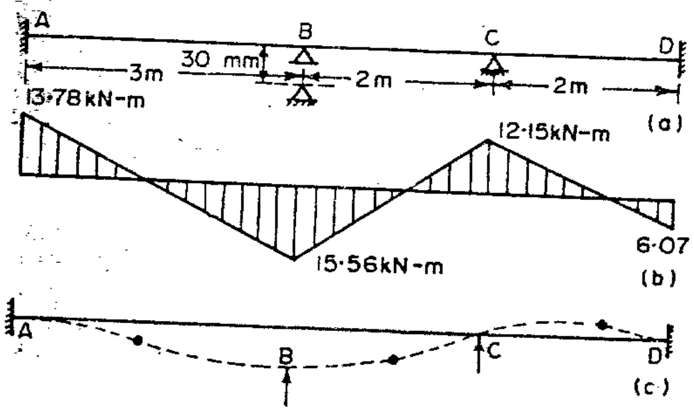


Fig. 10·12.

(*b*) *Distribution factors* (Table 10·10).

TABLE 10·10.

Joint	*Members*	*Relative-stiffness*	*Sum*	*D.F.*
B	*BA*	$\frac{I}{3}$	$\frac{5I}{6}$	$\frac{2}{5}=0{\cdot}4$
	BC	$\frac{I}{2}$		$\frac{3}{5}=0{\cdot}6$
C	*CB*	$\frac{I}{2}$	*I*	$\frac{1}{2}=0{\cdot}5$
	CD	$\frac{I}{2}$		$\frac{1}{2}=0{\cdot}5$

(*c*) *Moment distribution* (Table 10·11)

TABLE 10·11

A	*B*		*C*		*D*	
	0·4	0·6	0·5	0·5		
−12·00	−12·00	+27·00	+27·00	—	—	F.E.M.
	−6·00	−9·00	−13·50	−13·50	—	Balance
−3·00	—	−6·75	−4·50	—	−6·75	Carry over
—	+2·70	+4·05	+2·25	+2·25	—	Balance
+1·35	—	+1·13	+2·03	—	+1·13	Carry over
—	−0·45	−0·68	−1·01	−1·02	—	Balance
−0·22	—	−0·50	−0·34	—	−0·51	Carry over
—	+0·20	+0·30	+0·17	+0·17	—	Balance
+0·10	—	+0·08	+0·15	—	+0·08	Carry over
—	−0·03	−0·05	−0·08	−0·07	—	Balance
−0·02	—	−0·04	−0·03	—	−0·03	Carry over
+0·01	← −0·02	+0·02	+0·01	+0·02 →	+0·01	Balance and carry over to *A* and *D*
−13·78	−15·56	+15·56	+12·15	−12·15	−6·07	Final moments

The bending moment diagram and the deflected shape of the beam are shown in Fig. 10·12 (*b*) and (*c*) respectively.

Example 10·7. *A horizontal beam ABCD is carried on hinged supports and is continuous over three equal spans each of 3 m. All the supports are initially at the same level. The beam is loaded as shown in Fig. 10·13 (a). Plot the bending moment diagram and sketch the deflected shape of the beam if the support A settles by 10 mm, B settles by 30 mm and C settles by 20 mm. The moment of inertia of the whole beam is* $2{\cdot}4\times10^6$ *mm*^{4} *units. Take* $E=2\times10^5$ *N/mm*^{2}.

Solution.

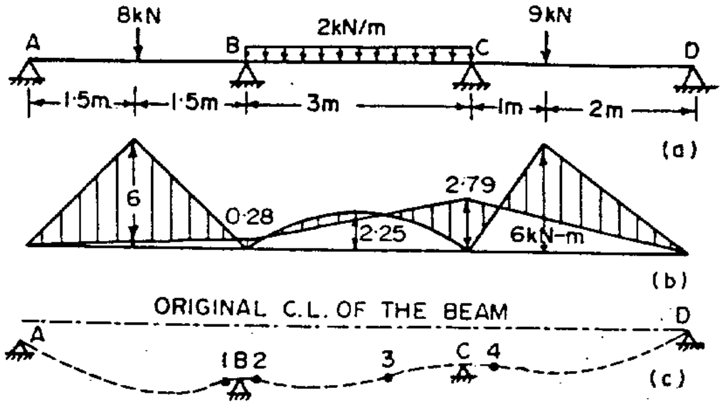


Fig. 10·13.

(*a*) *Fixed end moments*

Clamp all the joints against rotation so that each span behaves as a separate fixed beam. The fixed end moment at each end, will be the algebraic sum of the fixed end moments caused by the external loading and the settlement of supports.

For the span AB, end B sinks by $30-10=20$ mm ↓ relative to end A, and hence the F.E.M. due to this settlement will be of negative sign.

$$\therefore\ M_{FAB}=-\frac{WL}{8}-\frac{6EI\delta}{L^2}=-\frac{8\times3}{8}-\frac{6\times2\times10^5\times2{\cdot}4\times10^6\times20}{(3000)^2\times10^6}$$

$$=-3-6{\cdot}4=-9{\cdot}4 \text{ kN-m}$$

$$M_{FBA}=+\frac{WL}{8}-\frac{6EI\delta}{L^2}=+\frac{8\times3}{8}-\frac{6\times2\times10^5\times2{\cdot}4\times10^6\times20}{(3000)^2\times10^6}$$

$$=+3-6{\cdot}4=-3{\cdot}4 \text{ kN m.}$$

For the span BC, end B sinks $30-20=10$ mm ↓ relative to end C. The F.E.M. due to this settlement will be clockwise (*i.e.* positive) since the sinking of the supports rotate the beam as a whole in the anticlockwise direction.

$$\therefore\ M_{FBC}=-\frac{wL^2}{12}+\frac{6EI\delta}{L^2}=-\frac{2\times3^2}{12}+\frac{6\times2\times10^5\times2{\cdot}4\times10^6\times10}{(3000)^2\times10^6}$$

$$=-1{\cdot}5+3{\cdot}2=+1{\cdot}7 \text{ kN-m}$$

$$M_{FCB}=+\frac{wL^2}{12}+\frac{6EI\delta}{L^2}=1{\cdot}5+3{\cdot}2=+4{\cdot}7 \text{ kN-m.}$$

For the span CD, end C moves 20 mm ↓ relative to D, and hence the F.E.M. will be positive.

$$\therefore\ M_{FCD}=-\frac{Wab^2}{L^2}+\frac{6EI\delta}{L^2}$$

$$=-\frac{9\times1\times2^2}{3^2}+\frac{6\times2\times10^5\times2{\cdot}4\times10^6\times20}{(3000)^2\times10^6}$$

$$=-4+6{\cdot}4=+2{\cdot}4 \text{ kN-m}$$

$$M_{FDC}=+\frac{Wba^2}{L^2}+\frac{6EI\delta}{L^2}$$

$$=+\frac{9\times2\times1^2}{3^2}+\frac{6\times2\times10^5\times2{\cdot}4\times10^6\times20}{(3000)^2\times10^6}$$

$$=+2+6{\cdot}4=+8{\cdot}4 \text{ kN-m}$$

Distribution factors (Table 10·12)

TABLE 10·12

Joint	*Member*	*Relative stiffness*	*Sum*	*D.F.*
B	BA	$\frac{3}{4}\cdot\frac{I}{L}$	$\frac{7I}{4}$	$\frac{3}{7}$
	BC	$\frac{I}{L}$		$\frac{4}{7}$
C	CB	$\frac{I}{L}$	$\frac{7I}{4L}$	$\frac{4}{7}$
	CD	$\frac{3}{4}\ \frac{I}{L}$		$\frac{3}{7}$

(c) *Moment distribution* (Table 10·13)

TABLE 10·13.

A	B		C		D	
	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{4}{7}$	$\frac{3}{7}$		
−9·40	−3·40	+1·70	+4·70	+2·40	+8·40	F.E.M.
+9·40→	+4·70			−4·20 ←	−8·40	Balance *A* & *D* and carry over to *B* and *C*
0·00	+1·30	+1·70	+4·70	−1·80	0·00	Initial moments
—	+1·29	−1·71	−1·66	−1·24	—	Balance
—	—	−0·83	−0·85	—	—	Carry over
—	+0·36	+0·47	+0·49	+0·36	—	Balance
—	—	+0·25	+0·24	—	—	Carry over
—	−0·11	−0·14	−0·10	−0·14	—	Balance
—	—	−0·05	−0·07	—	—	Carry over
—	+0·02	+0·03	+0·04	+0·03	—	Balance
0·00	+0·28	−0·28	+2·79	−2·79	0·00	Final moments

The B.M.D. and the deflected shape are shown in Fig. 10·13 (*b*) and (*c*) respectively. The beam has four points of contraflexure.

Example 10·8. *A continuous beam ABC is supported on an elastic column BD, and is loaded as shown in Fig. 10·14. Treating joint B as rigid, analyse the frame and plot the B.M.D. and sketch the deflected shape of the structure.*

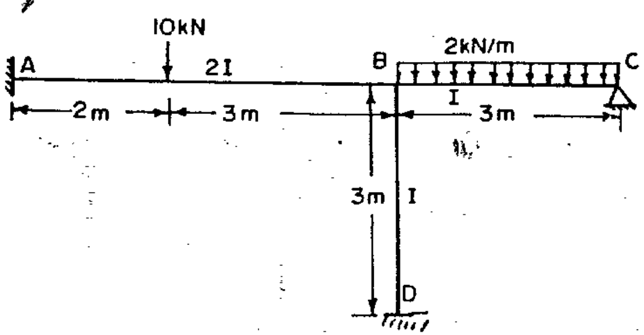


Fig. 10·14.

Solution

(*a*) *Fixed end moments*

$$M_{FAB} = -\frac{Wab^2}{L^2} = -\frac{10\times2\times3^2}{5^2} = -7·2 \text{ kN-m}$$

$$M_{FBA} = -\frac{Wba^2}{L^2} = +\frac{10\times3\times2^2}{5^2} = +4·8 \text{ kN-m}$$

$$M_{FBC} = -\frac{wL^2}{12} = -\frac{2\times3^2}{12} = -1·5 \text{ kN-m}$$

$M_{FCB} = +1·5$ kN-m ; $M_{FBD} = M_{FDB} = 0$

(*b*) *Distribution factors* (Table 10·14)

TABLE 10·14

Joint	*Member*	*Relative stiffness*	*Sum*	*D.F.*
	BA	$\frac{2I}{5}$		$\frac{24}{59}=0·406$
B	*BC*	$\frac{3}{4}\cdot\frac{I}{3}=\frac{I}{4}$	$\frac{59I}{60}$	$\frac{15}{59}=0·254$
	BD	$\frac{I}{3}=\frac{I}{3}$		$\frac{20}{59}=0·340$

(*c*) *Moment Distribution* (Table 10·15)

TABLE 10·15

A	B			C	
	BA 0·406	*BD* 0·340	*BC* 0·254		
—7·20	+4·80	0·00	—1·50	+1·50	F.E.M.
			—0·75 ←	—1·50	Balance *C* and carry over to *B*
—7·20	+4·80	0·00	—2·25	0·00	Initial moments
—	—1·03	—0·87	—0·65		Balance
—0·52	—	—	—	—	Carry over
—	—	—	—	—	Balance
—7·72	+3·77	—0·87	—2·90	—	Final moments

From proposition 3,

$M_{DB}=\frac{1}{2}M_{BD}=\frac{1}{2}(-0\cdot87)=-0\cdot435\simeq-0\cdot44$ kN-m

The B.M.D. is shown in Fig. 10·15. The dotted lines show the deflected shape of the beam.

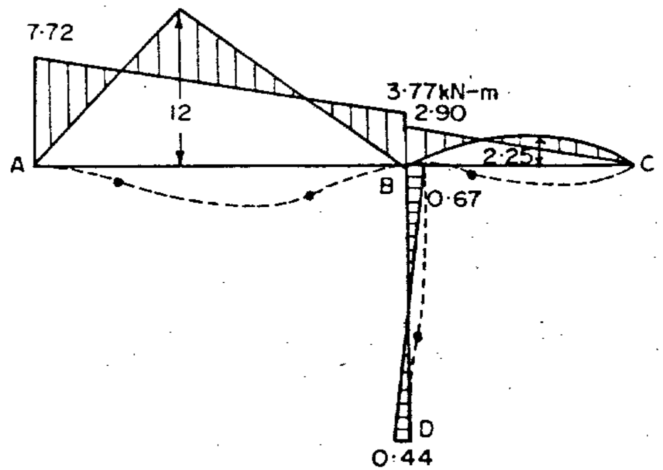


Fig. 10·15

Example 10·9. *The continuous beam shown in Fig. 10·16 has rigidly fixed ends at C and D, is pinned at E and has rigid joints at A and B. The members are of uniform section and material throughout. Sketch the bending moment diagram for the frame, showing all important values. Also, find the values of the horizontal and vertical reactions at D and E.*

Solution

(*a*) *Fixed end moments* :

$$M_{FAB}=-\frac{12\times1\times2^2}{3^2}-\frac{12\times2\times1^2}{3^2}=-\frac{16}{3}-\frac{8}{3}$$

$$=-8 \text{ kN-m}$$

$$M_{FBA}=+\frac{12\times2\times1^2}{3^2}+\frac{12\times1\times2^2}{3^2}=+\frac{8}{3}+\frac{16}{3}$$

$$=+8 \text{ kN-m.}$$

$$M_{FBC}=-\frac{4\times4^2}{12}=-5\cdot33 \text{ kN-m}$$

$$M_{FCB}=+\frac{4\times4^2}{12}=+5\cdot33 \text{ kN-m.}$$

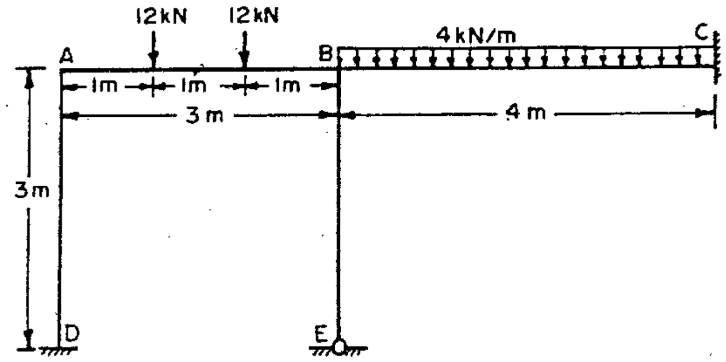


Fig. 10·16

(*b*) *Distribution factors* (Table 10·16)

TABLE 10·16

Joint	*Member*	*Relative stiffness*	*Sum*	*D.F.*
A	*AD*	$\frac{I}{3}$	$\frac{2I}{3}$	$\frac{1}{2}=0\cdot5$
	AB	$\frac{I}{3}$		$\frac{1}{2}=0\cdot5$
B	*BA*	$\frac{I}{3}$	$\frac{10I}{12}$	0·4
	BE	$\frac{3}{4}\cdot\frac{I}{3}=\frac{I}{4}$		0·3
	BC	$\frac{I}{4}$		0·3

(c) *Moment distribution* (Table 10·17)

8 - 5.33 = 2.67

2.67 × .4 = -1.07

TABLE 10·17

D	A		B			C	
DA	AD 0·5	AB 0·5	BA 0·4	BE 0·3	BC 0·3	CB	
0·00	0·00	−8·00	+8·00	0·00	−5·33	+5·33	F.E.M.
—	+4·00	+4·00	−1·07	−0·80	−0·80	—	B
+2·00	—	−0·53	+2·00	—	—	−0·40	C
—	+0·26	+0·27	−0·80	−0·60	−0·60	—	B
+0·13	—	−0·40	+0·13	—	—	−0·30	C
—	+0·20	+0·20	−0·05	−0·04	−0·04	—	B
+0·10	—	−0·02	+0·10	—	—	−0·02	C
+0·01	← +0·01	+0·01	−0·04	−0·03	−0·03 →	−0·01	B & C
+2·24	+4·47	−4·47	+8·27	−1·47	−6·80	+4·60	Final moment

The bending moment diagram is shown in Fig. 10·17

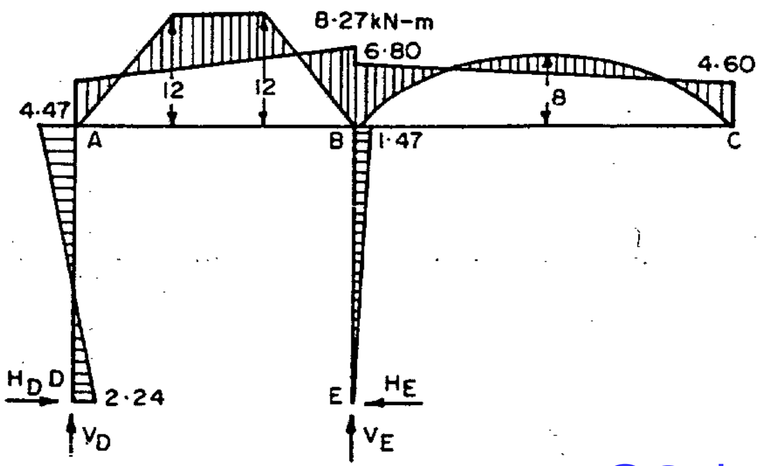


Fig. 10·17

(d) *Calculation of reactions*

Considering the equilibrium of *AD* and taking moments about *A*.

$$H_D = \frac{M_{DA}+M_{AD}}{3} = \frac{2{\cdot}24+4{\cdot}47}{3} = \mathbf{2{\cdot}24\ kN} \rightarrow.$$

Similarly, taking moments about *B*, of all forces below *B*, we get,

$$-1{\cdot}47+3H_E=0$$

or

$$H_E = \frac{1{\cdot}47}{3} = \mathbf{0{\cdot}49\ kN} \leftarrow.$$

Taking moments about *B*, of all forces to the right to *B*,

$$-6{\cdot}80+4{\cdot}60-4V_C+4\times4\times2=0$$

or

$$V_C = \mathbf{7{\cdot}45\ kN} \uparrow$$

Taking moments about *B*, of all forces to the left of *B*,

$$8{\cdot}27+2{\cdot}24+3V_D-(3\times2{\cdot}24)-(12\times1)-(12\times2)=0$$

$$\therefore \quad V_D = \mathbf{10{\cdot}74\ kN} \uparrow.$$

Considering the vertical equilibrium of the whole frame

$$V_E+7{\cdot}45+10{\cdot}74-12-12\ (4\times4)$$

$$\therefore \quad V_E = \mathbf{21{\cdot}81\ kN} \uparrow.$$

Example 10·10. *Analyse the rigid frame shown in Fig. 10·18.*

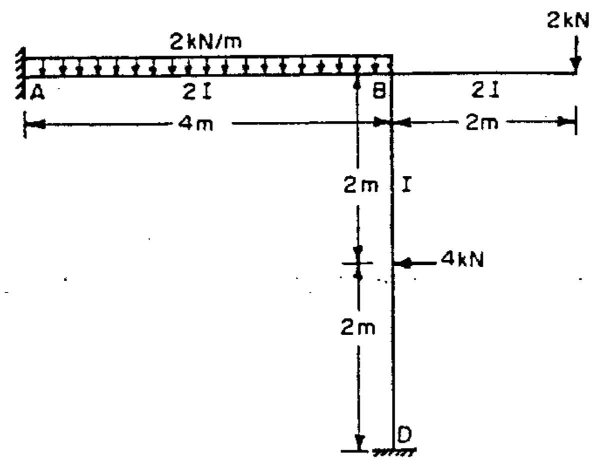


Fig. 10·18.

Solution

(*a*) *Fixed end moments*

Clamp all joints.

$$M_{FAB}=-\frac{2\times4^2}{12}=-2{\cdot}67\text{ kN-m}$$
$$M_{FBA}=+2{\cdot}67\text{ kN-m}$$
$$M_{FBC}=-2\times2=-4\text{ kN-m}$$
$$M_{FBD}=-\frac{4\times4}{8}=-2\text{ kN-m}$$
$$M_{FDB}=+2\text{ kN-m}$$

(*b*) *Distribution factors* (Table 10·18)

TABLE 10·18

Joint	*Member*	*Relative stiffness*	*Sum*	*D.F.*
	BA	$\frac{2I}{4}$		$\frac{2}{3}$
B	*BC*	0	$\frac{3I}{4}$	0
	BD	$\frac{I}{4}$		$\frac{1}{3}$

(*c*) *Moment distribution* (Table 10·19)

TABLE 10·19

A	*B*			*D*	*Joint*
AB	*BA*	*BC*	*BD*	*DB*	*Member*
	2/3	0	1/3		*Distribution factor*
−2·67	+2·67	−4·00	−2·00	+2·00	F.E.M.
—	+2·22	0·00	+1·11	—	Balance
+1·11	—	—	—	+0·56	Carry over
					Balance
−1·56	+4·89	−4·00	−0·89	+2·56	Final moments

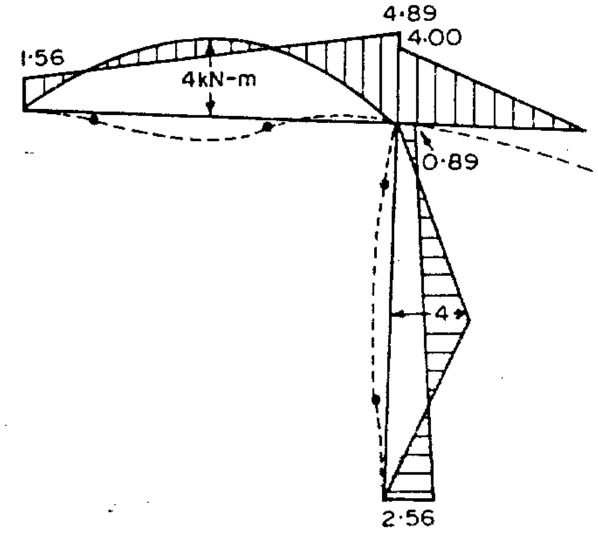


Fig. 10·19.

Fig. 10·19 shows the bending moment diagram and the deflected shape of the structure.

Example 10·11. *Draw the bending moment diagram and sketch the deflected shape of the frame shown in Fig. 10·20.*

Solution

(*a*) *Fixed end moments,*

$$M_{FBC}=-\frac{4\times3^2}{12}=-3\text{ kN-m}\ ;\ M_{FCB}=+\frac{4\times3^2}{12}=+3\text{ kN-m}$$

$$M_{FCE}=+\frac{4\times4}{8}=+2\text{ kN-m}\ ;\ M_{FEC}=-\frac{4\times4}{8}=-2\text{ kN-m}$$

(*b*) *Distribution factors* (Table 10·20)

(*c*) *Moment distribution* (Table 10·21)

The bending moment diagram and the deflected shape of the frame are shown in Fig. 10·21.

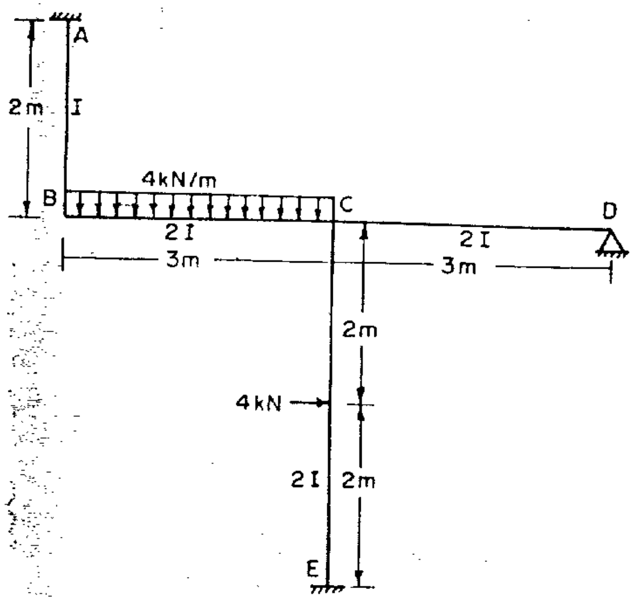


Fig. 10·20.

Table 10·20

Joint	Member	Relative stiffness	Sum	D.F.
B	BA	$\frac{I}{2}=\frac{3I}{6}$	$\frac{7I}{6}$	$\frac{3}{7}=0{\cdot}43$
	BC	$\frac{2I}{3}=\frac{4I}{6}$		$\frac{4}{7}=0{\cdot}67$
C	CB	$\frac{2I}{3}=\frac{4I}{6}$	$\frac{10I}{6}$	0·4
	CE	$\frac{2I}{4}=\frac{3I}{6}$		0·3
	CD	$\frac{3}{4}\cdot\frac{2I}{4}=\frac{3I}{6}$		0·3

Table 10·21

A	B		C			D	E	Joint
AB	BA	BC	CB	CE	CD	DC	EC	Member
	0·43	0·57	0·4	0·3	0·3			D.F.
—	—	−3·00	+3·00	+2·00	—	—	−2·00	F.E.M.
—	+1·29	−1·71	−2·00	−1·50	−1·50	—	—	Balance
+0·65	—	−1·00	+0·86	—	—	—	−0·75	Carry over
—	+0·43	+0·57	−0·34	−0·26	−0·26	—	—	Balance
+0·21	—	−0·17	+0·28	—	—	—	−0·13	Carry over
—	+0·07	+0·10	−0·12	−0·08	−0·08	—	—	Balance
+0·04	—	−0·06	+0·05	—	—	—	−0·04	Carry over
+0·01	+0·03	+0·03	−0·02	−0·02	−0·01	—	−0·01	Balance and Carry over to Fixed ends
+0·91	+1·82	−1·82	+1·71	+0·14	−1·85	—	−2·93	Final moments

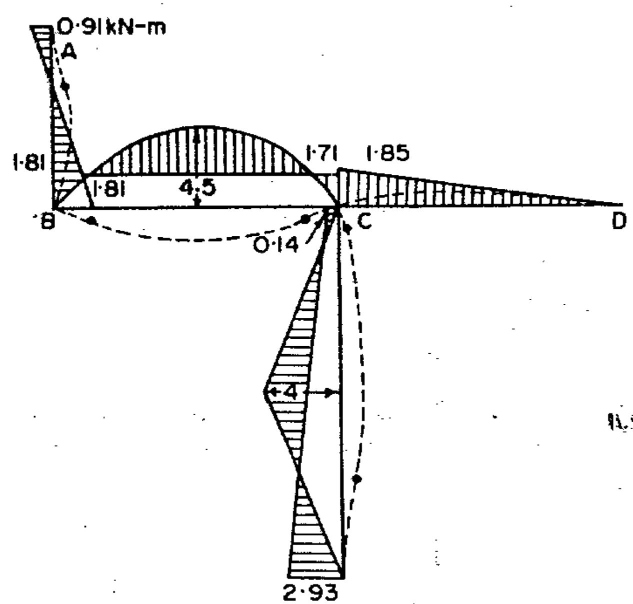


Fig. 10·21.

Example 10·12. *Analyse the portal frame shown in Fig. 10·22 by moment distribution method. The frame is fixed at A and D and has rigid joints at B and C. Draw the bending moment diagram and sketch the deflected shape of the structure.*

Solution

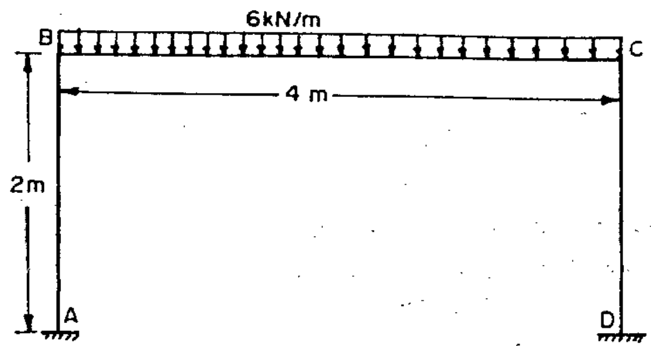


Fig. 10·22.

(*a*) *Fixed end moments*

$$M_{FBC}=-\frac{6\times4^2}{12}=-8\text{ kN-m}\ ;\ M_{FCB}=+\frac{6\times4^2}{12}=+8\text{ kN-m}$$

(*b*) *Distribution factors* (Table 10·22)

TABLE 10·22

Joint	*Member*	*Relative stiffness*	*Sum*	*D.F*
(*B* or *C*)	*BA* (or *CD*)	$\frac{I}{2}=\frac{2I}{4}$	$\frac{3I}{4}$	$\frac{2}{3}$
	BC (or *CB*)	$\frac{I}{4}$		$\frac{1}{3}$

(*b*) *Moment distribution* (Table 10·23)

TABLE 10·23

A	*B*		*C*		*D*	
	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$		
—	—	−8·0	+8·0	—	—	F.E.M.
—	+5·33	+2·67	−2·67	−5·33	—	Balance
+2·67	—	−1·34	+1·34	—	−2·67	Carry over
—	+0·89	+0·45	−0·45	−0·89	—	Balance
+0·44	—	−0·22	+0·22	—	−0·04	Carry over
—	+0·15	+0·07	−0·07	−0·15	—	Balance
+0·07	—	−0·03	+0·03	—	−0·07	Carry over
+0·01 ←	+0·02	+0·01	−0·01	−0·02 →	−0·01	B, and C.O
+3·19	+6·39	−6·39	+6·39	−6·39	−3·19	Final moments

The bending moment diagram and the deflected shape of the structure have been shown in Fig. 10·23(*a*) and (*b*) respectively.

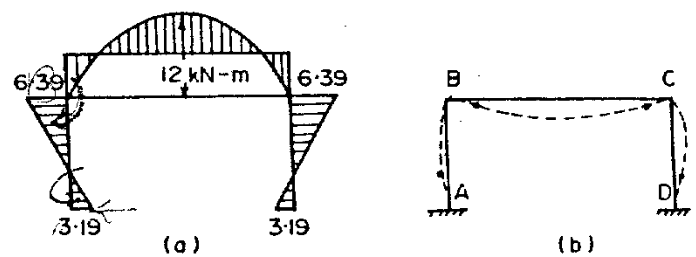


Fig. 10·23.

For finding out the horizontal reaction at *B*, consider the equilibrium of *AB*. Taking moments about *B*, we get

$$H_A=+\frac{3\cdot19+6\cdot39}{2}=+4\cdot79\text{ kN}=4\cdot79\text{ kN}\rightarrow$$

Similarly, $$H_D=\frac{-3\cdot19-6\cdot39}{2}=-4\cdot79\text{ kN}=4\cdot79\text{ kN}\leftarrow$$

$$V_A=V_B=12\text{ kN}\uparrow$$

Example 10·13. *Analyse the portal frome shown in Fig. 10·24. Draw the bending moment diagram and the deflected shape of the structure.*

Solution

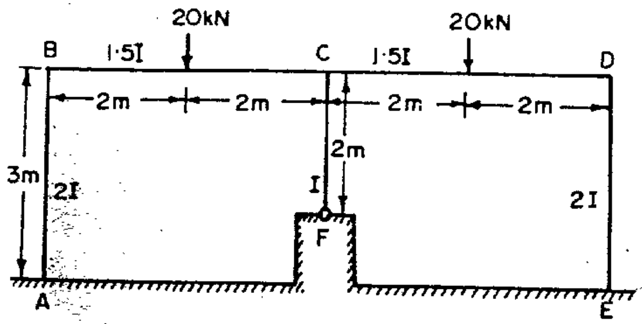


Fig. 10·24.

(*a*) *Fixed end moments*

$$M_{FBC}=M_{FCD}=-\frac{20\times 4}{8}=-10 \text{ kN-m}$$

$$M_{FCB}=M_{FDC}=+\frac{20\times 4}{8}=+10 \text{ kN-m.}$$

(*b*) *Distribution factors* (Table 10·24)

TABLE 10·24

Joint	*Member*	*Relative stiffness*	*Sum*	*D.F.*
B (or *D*)	*BA* (or *DE*)	$\frac{2I}{3}=\frac{8I}{12}$	$\frac{12\cdot 5I}{12}$	0·64
	BC (or *DC*)	$\frac{1\cdot 5I}{4}=\frac{4\cdot 5I}{12}$		0·36
C	*CB*	$\frac{1\cdot 5I}{4}=\frac{3I}{8}$	$\frac{9}{8}I$	0·333
	CF	$\frac{3}{4}\times\frac{I}{2}=\frac{3I}{8}$		0·333
	CD	$\frac{1\cdot 5I}{4}=\frac{3I}{8}$		0·333

(*c*) *Moment distribution* (Table 10·25)

TABLE 10·25

A	*B*		*C*			*D*		*E*	
	BA 0·64	*BC* 0·36	*CB* 0·333	*CF* 0·333	*CD* 0·333	*DC* 0·36	*DE* 0·64		
—	—	−10·00	+10·0	—	−10·0	+10·0	—	—	F.E.M.
—	+6·40	+3·60	—	—	—	−3·60	−6·40	—	B.
+3·20	—	—	+1·80	—	−1·80	—	—	−3·20	C.O.
—	—	—	—	—	—	—	—	—	B.
+3·20	+6·40	−6·40	+11·8	0	−11·8	+6·40	−6·40	−3·20	Final moments

Moment at column base = 0

The bending moment diagram and the deflected shape of the structure have been shown in Fig. 10·25.

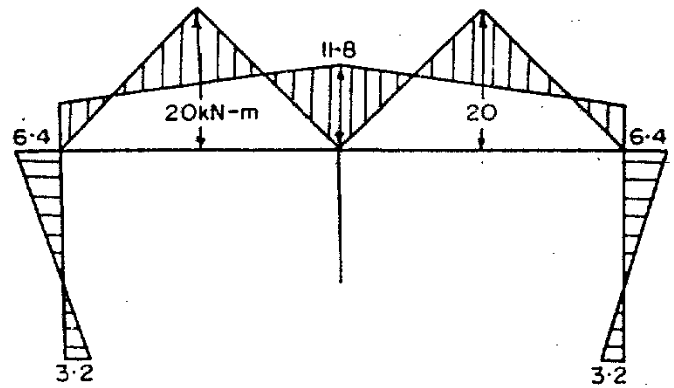


Fig. 10·25

Example 10·14. *Analyse the single span double storey portal frame shown in Fig. 10·26. The ends A and F are fixed.*

Solution

(*a*) *Fixed end moments*

$$M_{FBE}=-\frac{3\times 9^2}{12}=-20\cdot 25 \text{ kN-m}$$

$$M_{FEB}=+20\cdot 25 \text{ kN-m}$$

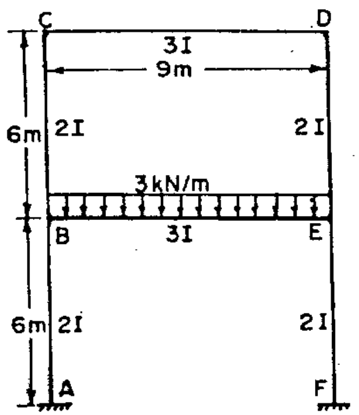


Fig. 10·26

(*b*) *Distribution factors* (Table 10·26)

TABLE 10·26

Joint	*Member*	*k*	*Sum*	*D.F.*
B	*BA*	$\frac{2I}{6}=\frac{I}{3}$	I	$\frac{1}{3}$
	BE	$\frac{3I}{9}=\frac{I}{3}$		$\frac{1}{3}$
	BC	$\frac{2I}{6}=\frac{I}{3}$		$\frac{1}{3}$
C	*CB*	$\frac{2I}{6}=\frac{I}{3}$	$\frac{2I}{3}$	$\frac{1}{2}$
	CD	$\frac{3I}{9}=\frac{I}{3}$		$\frac{1}{2}$

(*c*) *Moment distribution* (Table 10·27) on page 285.

The bending moment diagram and the deflected shape of the frame have been shown in Fig. 10·27.

TABLE 10·27

Joint	*A*	*B*			*C*		*D*		*E*			*F*
Member	*AB*	*BA*	*BE*	*BC*	*CB*	*CD*	*DC*	*DE*	*ED*	*EB*	*EF*	
D.F.		1/3	1/3	1/3	1/2	1/2	1/2	1/2	1/3	1/3	1/3	
F.E.M.	—	—	−20·25	—	—	—	—	—	—	+20·25	—	—
Balance	—	+6·75	+6·75	+6·75	—	—	—	—	−6·75	−6·75	−6·75	—
Carry over	+3·37		−3·37	—	+3·37	—	—	−3·37	—	+3·37	—	−3·37
Balance	—	+1·12	+1·13	+1·12	−1·68	−1·69	+1·69	+1·68	−1·12	−1·13	−1·12	—
Carry over	+0·56	—	−0·56	−0·84	+0·56	+0·84	−0·84	−0·56	+0·83	+0·56	—	−0·56
Balance	—	+0·47	+0·46	+0·47	−0·70	−0·70	+0·70	+0·70	−0·47	−0·46	−0·47	—
Carry over	+0·23	—	−0·23	−0·35	+0·23	+0·35	−0·35	−0·23	+0·35	+0·23	—	−0·23
Balance	—	+0·19	+0·20	+0·19	−0·29	−0·29	+0·29	+0·29	−0·19	−0·20	−0·19	—
Carry over	+0·10	—	−0·10	−0·14	+0·10	+0·14	−0·14	−0·10	+0·14	+0·10	—	−0·10
Balance		+0·08	+0·08	+0·08	−0·12	−0·12	+0·12	+0·12	−0·08	+0·08	−0·08	—
Carry over	+0·04	—	−0·04	−0·06	+0·04	+0·06	−0·06	−0·04	+0·06	+0·04	—	−0·04
Balance & Carry over	+0·02	+0·03	+0·04	+0·03	−0·05	−0·05	+0·05	+0·05	−0·03	−0·04	−0·03	−0·02
Final Moments	+4·32	+8·64	−15·89	+7·25	+1·46	−1·46	+1·46	−1·46	−7·25	15·89	−8·64	−4·32

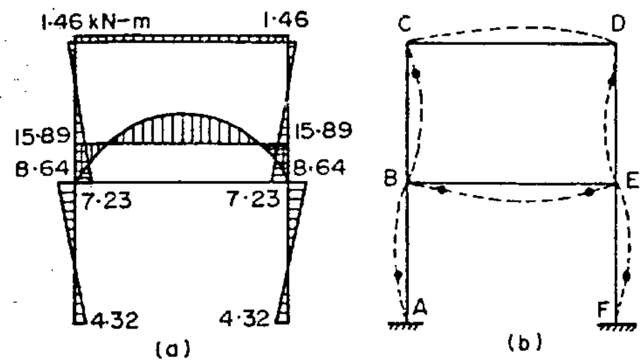


Fig. 10·27.

Example 10·15. *Analyse the box culvert shown in Fig. 10·28. All the joints A, B, C and D are rigid. Plot the bending moment diagram and the deflected shape of the frame.*

Solution

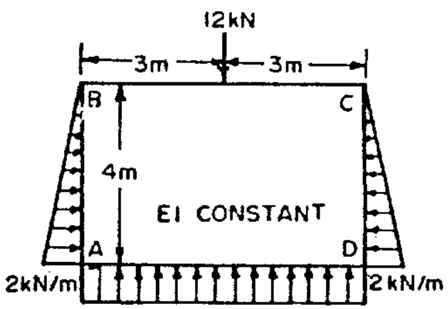


Fig. 10·28

(*a*) *Fixed end moments*

$$M_{FAB}=-\frac{wL^2}{20}=-\frac{2\times4^2}{20}=-1\cdot6 \text{ kN-m}$$

$$M_{FBA}=+\frac{wL^2}{30}=+\frac{2\times4^2}{30}=+1\cdot07 \text{ kN-m}$$

$$M_{FBC}=-\frac{12\times6}{8}=-9 \text{ kN-m} ; \; M_{FCB}=+9 \text{ kN-m}$$

$$M_{FCD}=-1\cdot07 \text{ kN-m and } M_{FDC}=+1\cdot6 \text{ kN-m}$$

$$M_{FAD}=+\frac{2\times6^2}{12}=+6 \text{ kN-m and } M_{FDA}=-6 \text{ kN-m}$$

(*b*) *Distribution factors* (Table 10·29)

TABLE 10·29

Joint	*Member*	*Relative stiffness*	*Sum*	*D.F.*
A	*AD*	$\frac{I}{6}=\frac{2I}{12}$	$\frac{5I}{12}$	0·4
	AB	$\frac{I}{4}=\frac{3I}{12}$		0·6
B	*BA*	$\frac{I}{4}=\frac{3I}{12}$	$\frac{5I}{12}$	0·6
	BC	$\frac{I}{6}=\frac{2I}{12}$		0·4

(*c*) *Moment distribution*

The frame may be cut in *AD* and opened out. The moment distribution will be carried out as usual as shown in Table 10·29.

TABLE 19·29

(*D*)	*A*	*B*		*C*		*D*	(*A*)	
0·4	0·6	0·6	0·4	0·4	0·6	0·6	0·4	
+6·00	—1·60	+1·07	—9·00	+9·00	—1·07	+1·60	—6·00	F.E.M.
—1·76	—2·64	+4·76	+3·17	—3·17	—4·76	+2·64	+1·76	B.
+0·88	+2·38	—1·32	—1·59	+1·59	+1·32	—2·38	—0·88	C.O.
—1·30	—1·96	+1·75	+1·16	—1·16	—1·75	+1·96	+1·30	B.
+0·65	+0·87	—0 98	—0·58	+0·58	+0·98	—0·87	—0·65	C.O.
—0·61	—0·91	+0·94	+0·62	—0·62	—0·94	+0·91	+0·61	B.
+0·30	+0·47	—0·45	—0·31	+0·31	+0·45	—0·47	—0·30	C.O.
—0·31	—0·46	+0·45	+0·30	—0·30	—0·46	+0·45	+0·31	B.
+3·85	—3·85	+6·23	—0·23	+6·23	—6·23	+3·85	—3·85	

The B.M.D. and the deflected shape have been shown in Fig. 10·29.

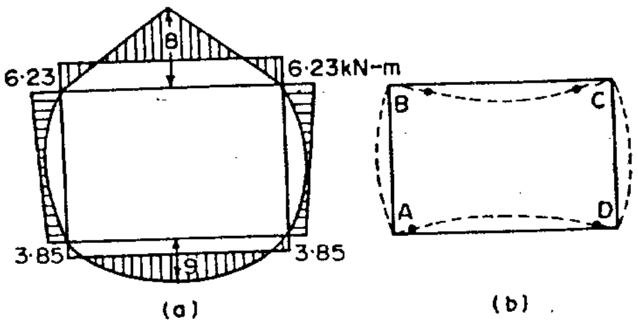


Fig. 10·29

10·7. PORTAL FRAMES WITH SIDE SWAY

In the case of continuous beams, etc., the effect of yielding or settlement of supports was taken into account by introducing initial fixed end moments. In the case of portal frames, however, the amount of 'sway' or joint movement is not known and the analysis is done by assuming some arbitrary fixed moments. These assumed fixed moments due to side sway are then distributed and the reactions (horizontal as well as vertical) are found. The algebraic sum of the horizontal reactions due to the assumed sway moments must be equal to the sway force. If not, the assumed sway moments are reduced proportionately as discussed below.

Causes of Side Sway

The portal frames sway due to one of the following reasons :

(1) Eccentric or unsymmetrical loading on the portal frame.
(2) Unsymmetrical out-line of portal frame
(3) Different end conditions of the columns of the portal frame.
(4) Non-uniform section of the members of the frame.
(5) Horizontal loading on the columns of the frame.
(6) Settlement of the supports of the frame.
(7) A combination of the above.

Method of Analysis

The analysis of portal frames with side sway is done in the following steps :

Step 1. (*a*) Hold the joints against side sway by applying a force P (Fig. 10·35). Calculate the fixed end moments due to external loads and distribute the moments.

(*b*) Calculate the horizontal and vertical reactions. The algebraic sum of the two horizontal reactions at the column bases will give the value of the restraining or holding force P. The sway force S will then be in the opposite direction and of the magnitude of P.

Step 2. (*a*) Remove the holding force P and permit the joints to sway. This will cause a set of fixed end moments. To start with, assume suitable sway moments at the four joints A, B, C and D of the frame, in proportion given by Eq. 10·8 or 10·9 or 10·10 (see below), as the case may be. Distribute these arbitrary sway moments.

(*b*) Calculate the horizontal and vertical reactions due to the assumed sway moments. The algebraic sum of the horizontal reactions of the two column bases must be equal to the sway force S calculated in step (1 *b*). If not, reduce the assumed sway moments proportionately as discussed below. The sway moments must be of such magnitude that the algebraic sum of the horizontal reactions due to sway is equal to the sway force S.

Let H_1 and H_2 be the horizontal reactions.

Let $c(H_1+H_2)=S$.

Then, Actual Sway Moments $=c\times$ Assumed Sway Moments.

Thus the actual sway moments are known.

Step 3. (*a*) The final moments at each joint will be equal to the algebraic sum of the moments due to initial moments (as obtained in step 1 (*a*) and the moments due to actual sway (as obtained in step 2).

(*b*) The final reactions will be equal to the algebraic sum of those found in 1 (*b*) and 2 (*b*).

Ratio of Sway Moments at Column Heads

When the joints sway, a set of moments are introduced at the two column heads (and bases) of a portal frame. The *ratio* of the sway moments at the two column heads (*i.e.* $M_{BA} : M_{CD}$) will depend upon the end conditions. Let us now take different end conditions to derive the standard expressions for the *ratio* of the sway moments.

Case I. Both ends hinged (Fig. 10·30)

Consider a portal frame with dimensions as shown in Fig. 10·30. Let a force P cause the frame to sway, so that the joint B moves to B' through a horizontal distance δ. Considering no change in the length of BC, joint C will move to C' through distance δ.

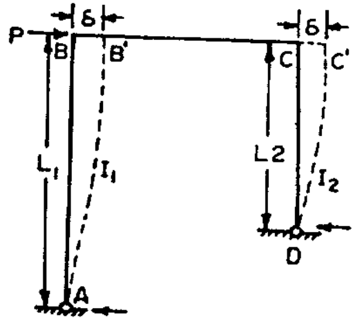


Fig. 10·30.

From 10·4, for a beam hinged at one end and fixed at the other, the fixed end moment due to movement or settlement of the support is given by Eq. 10·7.

$$M_{BA}=\frac{3EI_1\delta}{L_1^2} \qquad (i)$$

Similarly, the fixed end moment induced at C due to movement, is

$$M_{CD}=\frac{3EI_2\delta}{L_2^2} \qquad (ii)$$

Dividing (*i*) by (*ii*), we get

$$\frac{M_{BA}}{M_{CD}}=\frac{I_1/L_1^2}{I_2/L_2^2} \qquad (10\cdot8)$$

Since both the columns rotate in the same direction, the moments M_{BA} and M_{CD} will be of the *same sign* (either positive or negative, as the case may be). For the case of Fig. 10·30, both the moments will be of the *negative* sign since the columns tend to rotate in the *clockwise* direction.

Case II. Both ends fixed

Fig. 10·31 shows a portal frame fixed at both the ends. The movement $BB'=CC'=\delta$.

For the beam AB, fixed at ends A and B, the fixing moments due to the movement of the joint B by δ is given by

$$M_{BA}=M_{AB}=\frac{6EI_1\delta}{L_1^2} \qquad (i)$$

For the beam CD, fixed at ends C and D, the fixing moments due to the movement of the joint C by δ is given by

$$M_{CD}=M_{DC}=\frac{6EI_2\delta}{L_2^2} \qquad (ii)$$

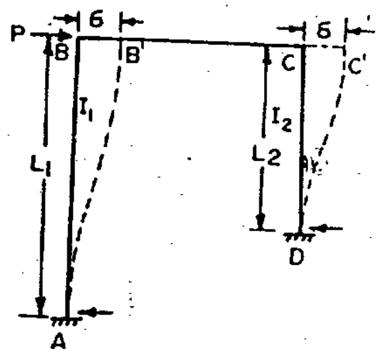


Fig. 10·31

From (*i*) and (*ii*), we get

$$\frac{M_{BA}}{M_{CD}}=\frac{I_1/L_1^2}{I_2/L_2^2} \qquad (10\cdot9)$$

Both the moments, M_{BA} and M_{CD}, will always be of the same sign. For the clockwise rotation (or for the sways to the right), the sway moments will be negative. For the anticlockwise rotation (or for the sway to the left), the sway moments will be positive.

Case III. One end fixed and other end hinged

Let us now take the case of a portal frame fixed at one end and hinged at the other end, as shown in Fig. 10·32. The movement $BB'=CC'=\delta$.

For the beam AB, fixed at A and B, the fixing moments due to the movement of the joint B by δ is given by

$$M_{BA}=M_{AB}=\frac{6EI_1\delta}{L_1^2} \qquad (i)$$

For the beam CD, fixed at C and hinged at D, the fixing moment due to the movement of the joint C by δ is given by

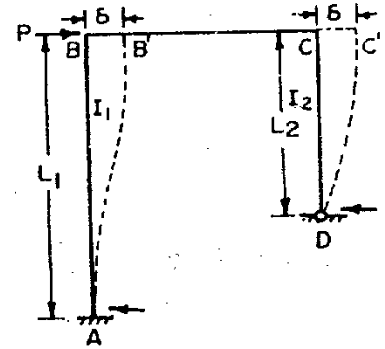


Fig. 10·32

$$M_{CD}=\frac{3EI_2\delta}{L_2^2} \qquad (ii)$$

From (*i*) and (*ii*), we get

$$\frac{M_{BA}}{M_{CD}}=\frac{2I_1/L_1^2}{I_2/L_2^2} \qquad (10\cdot10)$$

Thus equations 10·8, 10·9 and 10·10 give the *ratio* of the moments induced at the column heads due to side sway, for various end conditions.

Example 10·16. *Analyse the portal frame shown in Fig. 10·33. The end A is fixed and D is hinged. The joints B and C are rigid. Draw the bending moment diagram and sketch the deflected shape of the frame.*

Solution

As the load is acting on the joint, there will be no fixed end moments. However, due to side sway, moments will be induced at joint A, B and C.

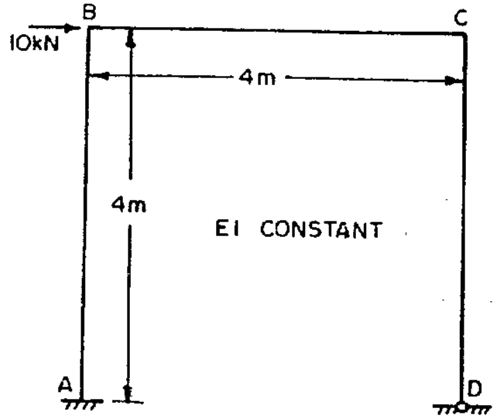


Fig. 10·33

Distribution factors (Table 10·30).

TABLE 10·30

Joint	*Member*	*Relative stiffness*	*Sum*	*D.F.*
B	*BA*	$\frac{I}{4}$	$\frac{2I}{4}$	0·5
	BC	$\frac{I}{4}$		0·5
C	*CB*	$\frac{I}{4}=\frac{4I}{16}$	$\frac{7I}{16}$	0·57
	CD	$\frac{3}{4}\times\frac{I}{4}=\frac{3I}{16}$		0·43

Side Sway

Under the action of the 10 kN load, there will be side sway to the right and the columns *AB* and *CD* will rotate in a clockwise direction. Thus negative moments will be induced at *A*, *B* and *C* in these columns. As the end *A* is fixed and *D* is hinged, the ratio of moments will be :

$$\frac{M_{BA}}{M_{CD}}=\frac{2I_1/L_1^2}{I_2/L_2^2}=\frac{2I/4^2}{I/4^2}=\frac{2}{1}$$

Also, $M_{BA}=M_{AB}$

Let us, first of all, assume arbitrary values of these moments and find out the corresponding sway force.

Let $M_{CD}=-5$ kN-m

$\therefore$ $M_{BA}=M_{AB}=-10$ kN-m

Moment distribution (Table 10·31)

TABLE 10·31

A	B		C		D	
	0·5	0·5	0·57	0·43		
−10·0	−10·0	—	—	−5·0	0	F.E.M.
—	+5·0	+5·0	+2·86	+2·14	—	Balance
+2·50	—	+1·43	+2·50	—	—	Carry over
—	−0·72	−0·71	−1·43	−1·07	—	Balance
−0·36	—	−0·72	−0·36	—	—	Carry over
—	+0·36	+0·36	+0·21	+0·15	—	Balance
+0·18	—	+0·10	+0·18	—	—	Carry over
—	−0·05	−0·05	−0·10	−0·08	—	Balance
−0·03	—	−0·05	−0·03	—	—	Carry over
+0·01 ←	+0·03	+0·02	+0·02	+0·01	—	Balance and Carry over
−7·70	−5·38	+5·38	+3·85	−3·85	0	Final moment

Horizontal reaction at $A=\frac{-7\cdot70-5\cdot38}{4}=-\frac{13\cdot08}{4}=3\cdot27$ kN←

Horizontal reaction at $D=-\frac{3\cdot85}{4}=0\cdot963$ kN←

The sway force causing the assumed moments$=3\cdot27+0\cdot963$
$=4\cdot233$ kN→

But actual sway force is 10 kN : hence the moments will be increased proportionately in the ratio of $\frac{10}{4\cdot203}$, as shown in Table 10·32.

TABLE 10·32

	A	B		C		D
Sway=4·233 kN	−7·70	−5·48	+5·38	+3·85	−3·85	0
Sway=10 kN	−18·18	−12·73	+12·73	+9·09	−9·09	0

The horizontal reaction at $A=\frac{3{\cdot}27}{4{\cdot}233}\times 10=7{\cdot}72$ kN←

The horizontal reaction at $D=\frac{0{\cdot}963}{4{\cdot}233}\times 10=2{\cdot}28$ kN←

The bending moment diagram and the deflected shape have been shown in Fig. 10·34.

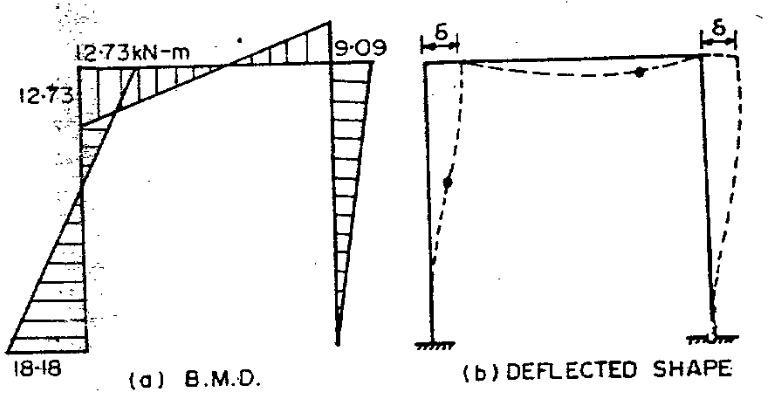


Fig. 10·34

Example 10·17. *Draw the bending moment diagram and sketch the deflected shape of the frame shown in Fig. 10·35. The ends A and D are fixed and BC is loaded with U.D.L. of 6 kN/m.*

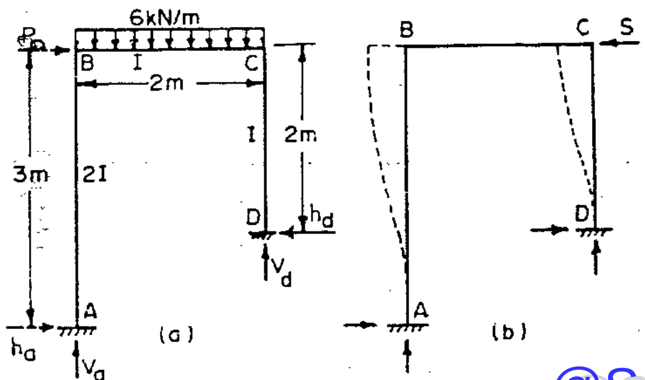


Fig. 10·35

Solution

(*a*) *Fixed end moments*

$$M_{FBC}=-\frac{6\times 2^2}{12}=-2 \text{ kN-m} ;\ M_{FCB}=+2 \text{ kN-m}$$

Let a horizontal force *P* be applied at *B* to hold it against translation. The moment distribution is then done as usual.

(*b*) *Distribution factors* (Table 10·33).

TABLE 10·33

Joint	Member	Relative stiffness	Sum	D.F.
B	BA	$\frac{2I}{3}=\frac{4I}{6}$	$\frac{7I}{6}$	0·57
	BC	$\frac{I}{2}=\frac{3I}{6}$		0·43
C	CB	$\frac{I}{2}$	I	0·50
	CD	$\frac{I}{2}$		0·50

(*c*) *Moment distribution* (Table 10·34)

TABLE 10·34

A	B		C		D	
	0·57	0·43	0·5	0·5		
0	0	−2·0	+2·0	0	0	F.E.M.
—	+1·14	+0·86	−1·0	−1·0	—	Balance
+0·57	—	−0·50	+0·43	—	−0·50	Carry over
—	+0·29	+0·21	−0·21	−0·22	—	Balance
+0·15	—	−0·11	+0·10	—	−0·11	Carry over
—	+0·06	+0·05	−0·05	−0·05	—	Balance
+0·03	—	−0·03	+0·03	—	−0·03	Carry over
+0·01 ←	+0·02	+0·01	−0·02	−0·01 →	—	B. & Carry over
+0·76	+1·51	−1·51	+1·28	−1·28	−0·64	Final moments

Horizontal reaction at A, $h_a=\dfrac{0{\cdot}76+1{\cdot}51}{3}=\dfrac{2{\cdot}27}{3}=0{\cdot}76$ kN$\rightarrow$

and horizontal reaction at D, $h_d=\dfrac{1{\cdot}28+0{\cdot}64}{2}=\dfrac{1{\cdot}92}{2}=0{\cdot}96$ kN$\leftarrow$

The value of P preventing side sway $=0{\cdot}96-0{\cdot}76$
$=0{\cdot}20$ kN$\rightarrow$

(*d*) *Side Sway*

Now let a sway force $S=0{\cdot}20$ kN $\leftarrow$ be applied at C. This will cause the columns AB and DC to rotate in anticlockwise direction and thus clockwise moments will be induced at column heads such that

$$\frac{M_{BA}}{M_{CD}}=\frac{I_1/L_1^2}{I_2/L_2^2}=\frac{2I/9}{I/4}=\frac{8}{9}$$

We shall assume arbitrary values of sway moments in the above proportion.

Let $M_{BA}=M_{AB}=+8$ kN-m and $M_{CD}=M_{DC}=+9$ kN-m.

(*e*) *Distribution of correcting moments* (Table 10·35)

TABLE 10·35

A	B	B	C	C	D	
	0·57	0·42	0·5	0·5		
+8·0	+8·0	—	—	+9·0	+9·0	F.E.M.
—	−4·57	−3·43	−4·5	−4·5	—	Balance
−2·29	—	−2·25	−1·72	—	−2·25	Carry over
—	+1·29	+0·96	+0·86	+0·86	—	Balance
+0·64	—	+0·43	+0·48	—	+0·43	Carry over
—	−0·25	−0·18	−0·24	−0·24	—	Balance
−0·13	—	−0·12	−0·09	—	−0·12	Carry over
—	+0·07	+0·05	+0·05	+0·04	—	Balance
+0·04	—	+0·02	+0·03	—	+0·02	Carry over
—	−0·01	−0·01	−0·02	−0·01	—	Balance
+6·26	+4·53	−4·53	−5·15	+5·15	+7·08	Final moments

Horizontal reaction at $A=\dfrac{6{\cdot}26+4{\cdot}53}{3}=\dfrac{10{\cdot}79}{3}=3{\cdot}60$ kN$\rightarrow$

Horizontal reaction at $D=\dfrac{5{\cdot}15+7{\cdot}08}{2}=\dfrac{12{\cdot}23}{2}=6{\cdot}12$ kN$\rightarrow$

So the sway force, which induces the assumed moments
$=3{\cdot}6+6{\cdot}12=9{\cdot}72$ kN$\leftarrow$

But the actual sway force is $=0{\cdot}20$ kN$\leftarrow$

The correction in the moments may now be carried out as shown in Table 10·36.

TABLE 10·36

	A	B	B	C	C	D
1. Sway=9·72 kN	+6·26	+4·53	−4·53	−5·15	+5·15	+7·08
2. Sway=0·20 kN	+0·12	+0·09	−0·09	−0·10	+0·10	+0·15
3. Non-sway	+0·76	+1·51	−1·51	+1·28	−1·28	−0·64
4. Final moments	+0·88	+1·60	−1·60	+1·18	−1 18	−0·49

The final reactions are as follows :

Horizontal reaction at $A=\dfrac{0{\cdot}88+1{\cdot}60}{3}=0{\cdot}83$ kN$\rightarrow$

Horizontal reaction at $D=\dfrac{-1{\cdot}18-0{\cdot}49}{2}=0{\cdot}83$ kN$\leftarrow$

For vertical reaction at A take moments about D,

$$2V_A-(0{\cdot}83\times1)-(12\times1)+0{\cdot}88-0{\cdot}49=0$$

or $\quad V_A=6{\cdot}22$ kN and $V_D=12-6{\cdot}22=5{\cdot}78$ kN

The bending moment diagram and the deflected shape have been shown in Fig. 10·36.

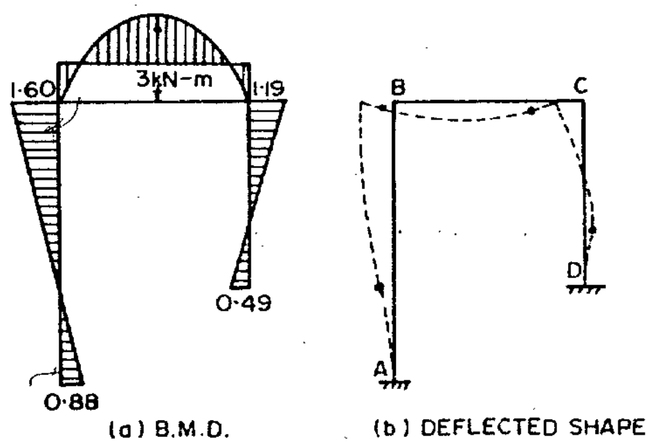


Fig. 10·36

Example 10·18. *The frame shown in Fig. 10·37 is hinged at A. The end D is fixed and the joints B and C are rigid. The column CD is subjected to a horizontal loading of 2 kN/m. A concentrated load of 6 kN acts on BC at 1 m from B. Analyse the frame completely and sketch its deflected shape.*

Solution

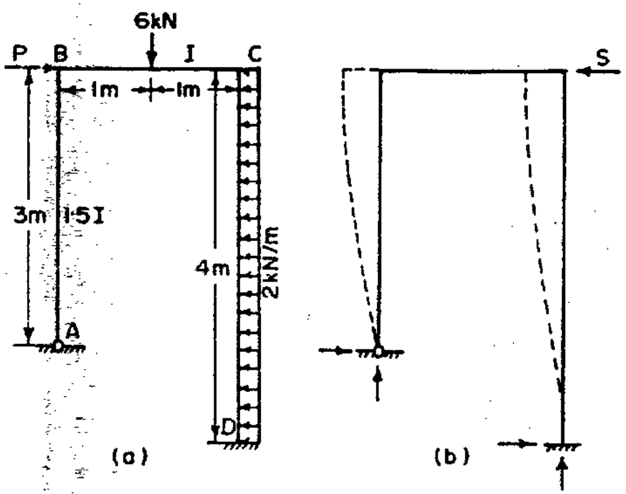


Fig. 10·37.

TABLE 10·37

Joint	*Member*	*Relative stiffness*	*Sum*	*D.F.*
B	BA	$\frac{3}{4}\times\frac{1\cdot5I}{3}=\frac{3I}{8}$	$\frac{7I}{8}$	$\frac{3}{7}$
	BC	$\frac{I}{2}=\frac{4I}{8}$		$\frac{4}{7}$
C	CB	$\frac{I}{2}=\frac{2I}{4}$	$\frac{3I}{4}$	$\frac{2}{3}$
	CD	$\frac{I}{4}$		$\frac{1}{3}$

(*a*) *Fixed end moments*

$$M_{FBC}=-\frac{6\times1\times1^2}{4}=-1\cdot5\text{ kN-m};\ M_{FCB}=+1\cdot5\text{ kN-m}$$

$$M_{FCD}=-\frac{2\times4^2}{12}=-2\cdot67\text{ kN-m};\ M_{FDC}=+2\cdot67\text{ kN-m}$$

(*b*) *Distribution factors* (Table 10·37)

(*c*) *Moment distribution*

Assume a horizonal force P to be applied at the joint B to prevent the side sway. The moment distribution, neglecting the sway will then be carried out as shown in Table 10·38.

TABLE 10·38

A	B		C		D	
	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{2}{3}$	$\frac{1}{3}$		
0	0	−1·50	+1·50	−2·67	+2·67	F.E.M.
—	+0·64	+0·86	+0·78	+0·39	—	Balance
—	—	+0·39	+0·43	—	+0·20	Carry over
—	−0·17	−0·22	−0·29	−1·40	—	Balance
—	—	−0·15	−0·11	—	−0·07	Carry over
—	+0·06	+0·09	+0·07	+0·04	—	Balance
—	—	+0·04	+0·05	—	+0·02	Carry over
—	−0 02	−0·02	−0·03	−0·02 →	−0·01	B & C.O.
—	+0·51	−0·51	+2·40	−2·40	+2·81	Final moments

Horizontal reaction at $A=\frac{0\cdot51}{3}=0\cdot17$ kN→

For horizontal reaction at D *i.e.*, h_d, taking moments about C for equilibrium of CD,

$$h_d\times4=4\times2\times2+2\cdot81-2\cdot40=16\cdot41$$

or $$h_d=4\cdot10\text{ kN}\rightarrow$$

So $$P=8-0\cdot17-4\cdot10=3\cdot73\text{ kN}\rightarrow$$

(d) Side Sway

As actually there is no force like P acting at joint B, so apply an equal opposite force S=3·73 kN ← at joint C to neutralise the effect of P.

This sway force S will rotate the column in anticlockwise direction and thus inducing clockwise moments at the end of the columns. The ratio of moments is as under :

$$\frac{M_{CD}}{M_{BA}}=\frac{2I/16}{3I/2\times 9}=\frac{6}{8}$$

So let $M_{CD}=M_{DC}=+6{\cdot}0$ kN

and $M_{BA}=+8{\cdot}0$ kN.

(e) Distribution of correcting moments (Table 10·39)

TABLE 10·39

	3/7	4/7	2/3	1/3		
—	+8·0	—	—	+6·0	+6·0	F.E.M.
—	−3·43	−4·57	−4·0	−2·0	—	Balance
—	—	−2·0	−2·28	—	−1·0	Carry over
—	+0·86	+1·14	+1·52	+0·76	—	Balance
—	—	+0·76	+0·57	—	+0·38	Carry over
—	−0·33	−0·43	+0·38	−0·19	—	Balance
—	—	−1·19	−0·21	—	−0·10	Carry over
—	+0·08	+0·11	+0·14	+0·07	—	Balance
—	—	+0·07	+0·06	—	+0·04	Carry over
—	−0·03	−0·04	−0·04	−0·02	—	Balance
—	—	−0·02	−0·02	—	−0·01	Carry over
—	+0·01	+0·01	+0·01	+0·01	—	Balance
0	+5·16	−5·16	−4·63	+4·63	+5·31	Final moments

Horizontal reaction at $A=\frac{5{\cdot}16}{3}=1{\cdot}72$ kN →

Horizontal reaction at $D=\frac{4{\cdot}63+5{\cdot}31}{4}=\frac{9{\cdot}94}{4}=2{\cdot}485$ kN

Corresponding sway force=1·72+2·485=4·205 kN←

But the actual sway force is 3·73 kN only. Hence the moments will be reduced proportionately and added algebraically to the non-sway moments as shown in Table 10·40.

TABLE 10·40

	A	B		C		D
1. Sway=4·205 kN	0	+5·16	−5·16	−4·63	+4·63	+5·31
2. Sway=3·73 kN	0	+4·58	−4·58	−4·11	+4·11	+4·71
3. Non-sway moments	0	+0 51	−0·51	+2·40	−2·40	+2·81
4. Final moments	0	+5·09	−5·09	−1·72	+1·71	+7·52

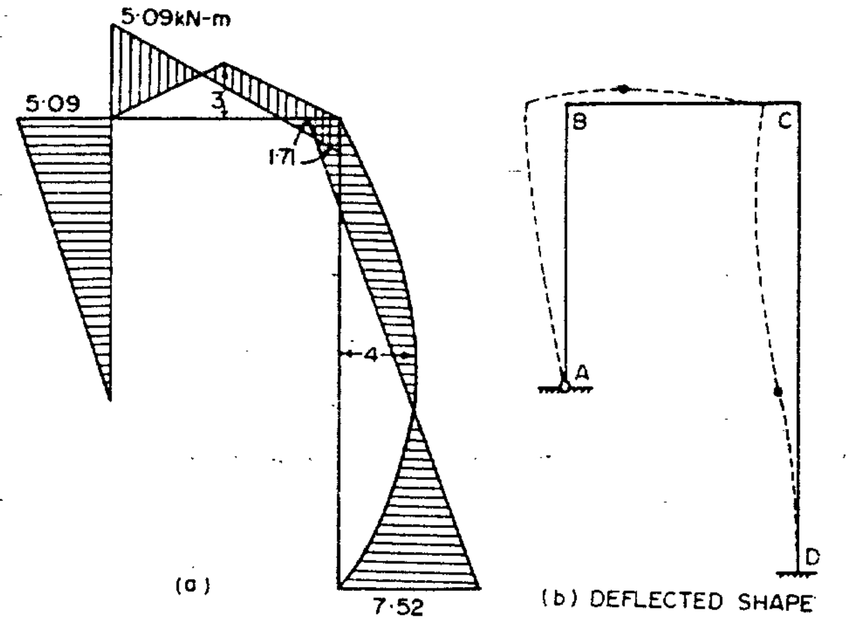


Fig. 10·38

Horizontal reaction at $A=\frac{5\cdot09}{3}=1\cdot70$ kN→

Horizontal reaction at $D=\frac{8\times2+1\cdot71+7\cdot52}{4}=6\cdot3$ kN→

For vertical reaction at A, taking moments about D,

$$V_a\times2=6\times1+8\times2-1\cdot70\times1-7\cdot52=12\cdot78$$

or $$V_a=6\cdot39\text{ kN}\uparrow$$

$$\therefore\quad V_d=0\cdot39\text{ kN}\downarrow$$

The bending moment diagram and the deflected shape of the frame have been shown in Fig. 10·38.

Example 10·19. *Use the method of moment distribution to analyse the portal frame shown in Fig. 10·39 if the hinged support D sinks by an amount* Δ. *The members have the same uniform cross-section.*

Solution

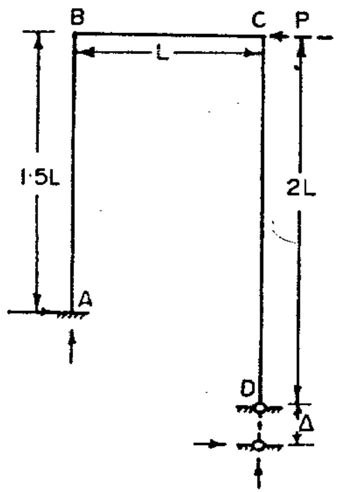


Fig. 10·39

(*a*) *Fixed end moments*

When the hinged end D sinks, the end C of the beam will also settle by the same amount. Due to settlement, there will be side sway in the right side. Let a force P be applied at C to prevent this side sway.

The settlement of end C will induce moments in BC in anti-clockwise direction.

$$M_{FBC}=M_{FCB}=-\frac{6EI\Delta}{L^2}=-6c\text{ (Say)}$$

where $$c=\frac{EI\Delta}{L^2}$$

(*b*) *Distribution factors* (Table 10·41).

TABLE 10·41

Joint	*Member*	*Relative stiffness*	*Sum*	*D.F.*
B	*BA*	$\frac{I\times2}{3L}=\frac{2I}{3L}$	$\frac{5I}{3L}$	0·4
	BC	$\frac{I}{L}=\frac{3I}{3L}$		0·6
C	*CB*	$\frac{I}{L}=\frac{8I}{8L}$	$\frac{11I}{8L}$	0·73
	CD	$\frac{3}{4}\times\frac{I}{2L}=\frac{3I}{8L}$		0·27

(*c*) *Moment distribution*

TABLE 10·42

A	*B*		*C*		*D*	
	0·4	0·6	0·73	0·27		
—	—	$-6\cdot0c$	$-6\cdot0c$	—	—	F.E.M.
—	$+2\cdot4c$	$+3\cdot6c$	$+4\cdot36c$	$+1\cdot64c$	—	Balance
$+1\cdot2c$	—	$+2\cdot18c$	$+1\cdot8c$	—	—	Carry over
—	$-0\cdot87c$	$-1\cdot31c$	$-1\cdot31c$	$-0\cdot49c$	—	Balance
$-0\cdot44c$	—	$-0\cdot66c$	$-0\cdot66c$	—	—	Carry over
—	$+0\cdot26c$	$+0\cdot40c$	$+0\cdot48c$	$+0\cdot18c$	—	Balance
$+0\cdot13c$	—	$+0\cdot24c$	$+0\cdot20c$	—	—	Carry over
—	$-0\cdot10c$	$-0\cdot14c$	$-0\cdot14c$	$-0\cdot06c$	—	Balance
$-0\cdot05c$	—	$-0\cdot07c$	$-0\cdot07c$	—	—	Carry over
—	$+0\cdot03c$	$+0\cdot04c$	$+0\cdot05c$	$+0\cdot02c$	—	Balance
$+0\cdot02c$	—	$+0\cdot03c$	$+0\cdot02c$	—	—	Carry over
$-0\cdot01c\leftarrow$	$+0\cdot01c$	$+0\cdot02c$	$-0\cdot02c$	$-0\cdot01c$	—	B & C.O.
$+0\cdot85c$	$+1\cdot71c$	$-1\cdot71c$	$-1\cdot28c$	$+1\cdot28c$	0	

The moment distribution, assuming absence of sway, will be as shown in Table 10·42.

Horizontal reaction at $A=\dfrac{0{\cdot}85c+1{\cdot}71c}{3/2L}=\dfrac{1{\cdot}713c}{L}$

$=\dfrac{1{\cdot}713EI\Delta}{L^3}\rightarrow$

Horizontal reaction at $D=\dfrac{1{\cdot}28c}{2L}=\dfrac{0{\cdot}64EI\Delta}{L^3}\rightarrow$

$P=(1{\cdot}713+0{\cdot}64)\dfrac{EI\Delta}{L^3}=\dfrac{2{\cdot}353EI\Delta}{3}$

(*d*) *Side Sway*

Now apply a force $S=\dfrac{2{\cdot}353EI\Delta}{L^3}$ at B in opposite direction to that of P. Side sway will induce anticlockwise moment A, B and C, such that

$$\frac{M_{BA}\text{ or }M_{AB}}{M_{CD}}=\frac{2I_1/L_1^2}{I_2/L_2^2}=\frac{8/9}{1/4}=\frac{32}{9}$$

Arbitrary values of sway moments in above proportion are assumed as given below :

$$M_{BA}=M_{AB}=-8{\cdot}0\ ;\ M_{CD}=-2{\cdot}25$$

(*e*) *Distribution of correcting moments :*

TABLE 10·43

A	B		C		D	
	0·4	0·6	0·73	0·27		
−8·0	−8·0	—	—	−2·25	—	F.E.M.
—	+3·20	+4·80	+1·64	+0·61	—	Balance
+1·60	—	+0·82	+2·40	—	—	Carry over
—	+0·33	−0·49	−1·75	−0·65	—	Balance
−0·16	—	−0·87	−0·24	—	—	Carry over
—	+0·35	+0·52	+0·17	+0·07	—	Balance
+0·18	—	+0·09	+0·26	—	—	Carry over
—	−0·04	−0·05	−0·19	−0·07	—	Balance
−0·02	—	−0·09	−0·03	—	—	Carry over
+0·02 ←	+0·04	+0·05	+0·02	+0·01	—	B. & C.O.
−6·38	−4·78	+4·78	+2·28	−2·28	—	

The distribution of these sway moments is carried out as shown in Table 10·43.

The horizontal reaction at $A=-\dfrac{6{\cdot}38+4{\cdot}78}{3/2L}=\dfrac{7{\cdot}44}{L}\leftarrow$

The horizontal reaction at $D=\dfrac{2{\cdot}28}{2L}=\dfrac{1{\cdot}14}{L}\leftarrow$

The sway force $=\dfrac{7{\cdot}44}{L}+\dfrac{1{\cdot}14}{L}=\dfrac{8{\cdot}58}{L}\rightarrow$

But the actual sway force is $\dfrac{2{\cdot}353\ EI\Delta}{L^3}$ and hence the sway moments will have to be corrected accordingly and added to the non-sway moments as illustrated in Table 10·44.

TABLE 10·44

	A	B		C	
1. Sway $=\dfrac{8{\cdot}58}{L}$	−6·38	−4·73	+4·78	+2·28	−2·28
2. Sway $=\dfrac{2{\cdot}353EI\Delta}{L^3}$	$-\dfrac{1{\cdot}75EI\Delta}{L^3}$	$-\dfrac{1{\cdot}31EI\Delta}{L^3}$	$+\dfrac{1{\cdot}31EI\Delta}{L^2}$	$+\dfrac{0{\cdot}63EI\Delta}{L^2}$	$-\dfrac{0{\cdot}63EI\Delta}{L^2}$
3. Non-sway moment	$+\dfrac{0{\cdot}85EI\Delta}{L^2}$	$+\dfrac{1{\cdot}71EI\Delta}{L^2}$	$-\dfrac{1{\cdot}71EI\Delta}{L^2}$	$-\dfrac{1{\cdot}28EI\Delta}{L^2}$	$+\dfrac{1{\cdot}28EI\Delta}{L^2}$
4. Final moment	$-\dfrac{0{\cdot}90EI\Delta}{L^2}$	$+\dfrac{0{\cdot}4EI\Delta}{L^2}$	$-\dfrac{0{\cdot}4EI\Delta}{L^2}$	$-\dfrac{0{\cdot}65EI\Delta}{L^2}$	$+\dfrac{0{\cdot}65EI\Delta}{L^2}$

The bending moment diagram and the deflected shape have been shown in Fig. 10·40. The values marked in the figure are to be multiplied by the factor $\dfrac{EI\Delta}{L^2}$.

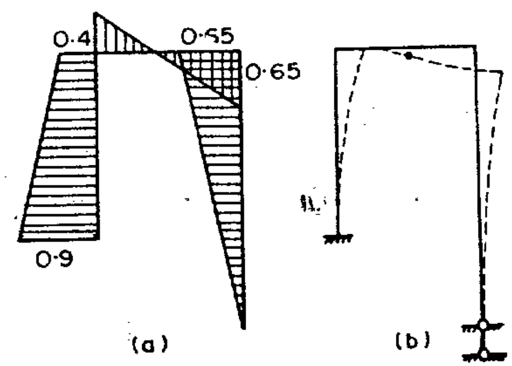


Fig. 10·40.

Example 10·20. *Analyse the double span single storey portal frame shown in Fig. 10·41. The beam BC is loaded with a U.D.L. of 3 kN/m run. The ends A, F and E are hinged and joints B, C and D are rigid. The moment of inertia of beams is double the same for the columns.*

Solution

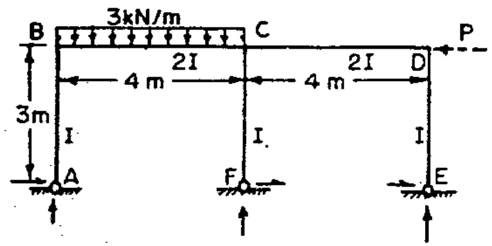


Fig. 10·41.

(*a*) *Fixed end moments*

$$M_{FBC}=-\frac{3\times4^2}{12}=-4\text{ kN-m}\ ;\ M_{FCB}=+4\cdot0\text{ kN-m}$$

(*b*) *Distribution factors* (Table 10·45)

TABLE 10·45

Joint	*Member*	*Relative stiffness*	*Sum*	*D.F.*
B	*BA*	$\frac{3}{4}\times\frac{I}{3}=\frac{I}{4}$	$\frac{3I}{4}$	0·33
	BC	$\frac{2I}{4}$		0·67
C	*CB*	$\frac{2I}{4}$	$\frac{5I}{4}$	0·4
	CF	$\frac{3}{4}\times\frac{I}{3}=\frac{I}{4}$		0·2
	CD	$\frac{2I}{4}$		0·4

(*a*) *Moment distribution*

Let a horizontal force *P* act at joint *D* to prevent the side sway. The moment distribution is carried out as shown in Table 10·46.

TABLE 10·46

A	*B* 0·33 *BA*	0·67 *BC*	*C* 0·4 *CB*	0·2 *CF*	0·4 *CD*	*D* 0·67 *DC*	0·33 *DE*	*E*	
—	—	−4·0	+4·0	—	—	—	—	—	F.E.M.
	+1·33	+2·67	−1·60	−0·80	−1·60	—	—	—	Balance
—	—	−0·80	+1·33	—	—	−0·80	—	—	Carry over
	+0·27	+0·53	−0·53	−0·27	−0·53	+0·53	+0·27	—	Balance
—	—	−0·27	+0·27	—	+0·27	−0·27	—	—	Carry over.
	+0·09	+0·18	−0·22	−0·10	−0·22	+0·18	+0·09	—	Balance
—	—	−0·11	+0·09	—	+0·09	−0·11	—	—	Carry over
	+0·04	+0·07	−0·07	−0·04	−0·07	+0·07	+0·04	—	Balance
		−0·03	+0·04	—	+0·03	−0·03	—	—	Carry over
	−0·01	−0·02	−0·03	−0·01	−0·03	+0·02	+0·01	—	Balance
—	+1·74	−1·74	+3·28	−1·22	−2·06	−0·41	+0·41	—	

Horizontal reaction at $A=\frac{1\cdot74}{3}=0\cdot58$ kN→

Horizontal reaction at $E=\frac{0\cdot41}{3}=0\cdot137$ kN→

Horizontal reaction at $F=\frac{1\cdot22}{3}=0\cdot407$ kN←

So $P=0\cdot58+0\cdot137-0\cdot407=0\cdot31$ kN←

(*d*) *Side sway*

Let a force $S=0\cdot31$ kN→ be made to act at joint *B* to neutralise the effect of *P*. The joints *B*, *C* and *D* each will move on right side by equal amount inducing anticlockwise moments in columns at *B*, *C* and *D*. As the end condition of all the three columns and *EI* is same, $M_{BA}=M_{CF}=M_{DE}$.

Let us assume an arbitrary value of 10 kN-m for these moments.

(*e*) *Distribution of sway moments* (Table 10·47)

TABLE 10·47

A	*B*		*C*			*D*		*E*	
	BA 0·33	*BC* 0·67	*CB* 0·4	*CF* 0·2	*CD* 0·4	*DC* 0·67	*DE* 0·33		
–	–10·0	–	–	–10·0	–	–	–10·0	–	F.E.M.
–	+3·33	+6·67	+4·0	+2·0	+4·0	+6·67	+3·33	–	B.
–	–	+2·0	+3·33	–	+3·33	+2·0	–	–	C.O.
–	–0·67	–1·33	–2·67	–1·32	–2·67	–1·33	–0·67	–	B.
–	–	–1·33	–0·67	–	–0·67	–1·33	–	–	C.O.
–	+0·44	+0·89	+0·54	+0·26	+0·54	+0·89	+0·44	–	B.
–	–	+0·27	+0·45	–	+0·45	+0·27	–	–	C.O.
–	–0·09	–0·18	–0·36	+0·18	–0·36	–0·18	–0·09	–	B.
–	–	–0·18	–0·09	–	–0·09	–0·18	–	–	C.O.
–	+0·06	+0·12	+0·07	+0·04	+0·07	+0·12	+0·06	–	B.
–	–	+0·03	+0·06	–	+0·06	+0·03	–	–	C.O.
–	–0·01	–0·02	–0·05	–0·02	–0·05	–0·02	–0·01	–	B.
–	–	–0·03	–	–	–	–0·03	–	–	C.O.
–	+0·01	+0·02	–	–	–	+0·02	+0·01	–	B.
–	–6·93	+6·93	+4·61	–9·22	+4·61	+6·93	–6·93	–	Final Moments

Horizontal reaction at $A=\frac{6{\cdot}93}{3}=2{\cdot}31$ kN←

Horizontal reaction at $E=\frac{6{\cdot}93}{3}=2{\cdot}31$ kN←

Horizontal reaction at $F=\frac{9{\cdot}22}{3}=3{\cdot}07$ kN←

Sway force corresponding to assumed moments

$$=2{\cdot}31+2{\cdot}31+3{\cdot}07=7{\cdot}69 \text{ kN}\rightarrow$$

But the actual sway force is 0·31 kN. Thus the sway moments will have to be reduced proportionately. These may then be added to non-sway moments to obtain the final moments as shown in Table 10·48.

TABLE 10·48

	A	*B*		*C*			*D*		*E*
				CB	*CF*	*CD*			
1. Sway=7·69 kN	0	–6·93	+6·93	+4·61	–9·22	+4·61	+6·93	–6·93	0
2. Sway=0·31 kN	0	–0·28	+0·28	+0·19	–0·38	+0·19	+0·28	–0·28	0
3. Non-sway moments	0	+1·74	–1·71	+3·28	–1·22	–2·06	–0·41	+0·41	0
4. Final moment	0	+1·46	–1·46	+3·47	–1·60	–1·87	–0·13	+0·13	0

Horizontal reaction at $A=\frac{1{\cdot}46}{3}=0{\cdot}485$ kN→

Horizontal reaction at $F=\frac{1{\cdot}60}{3}=0{\cdot}53$ kN←

Horizontal reaction at $E=\frac{0{\cdot}13}{3}=0{\cdot}045$ kN→

For vertical reactions taking moment about C for equilibrium of left side portion,

$$0{\cdot}485\times3+3\times4\times2-3{\cdot}47=V_a\times4$$

or $$4V_a=21{\cdot}99$$

or $$V_a=5{\cdot}5 \text{ kN}\uparrow$$

Taking moments about C for the equilibrium of frame on right hand side,

$$V_e\times4=-0{\cdot}045\times3-1{\cdot}87=-2{\cdot}005$$

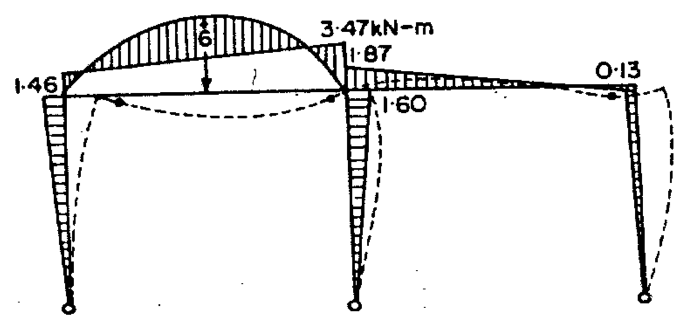


Fig. 10·42.

or $$V_e = 0.50 \downarrow$$

$$V_f = 12 + 0.5 - 5.5 = 7.0 \text{ kN} \uparrow$$

The bending moment diagram and the deflected shape have been shown in Fig. 10·42.

10·8. PORTAL FRAMES WITH INCLINED MEMBERS

In the previous article, we have considered the sway moments due to the moment of the column heads in a direction perpendicular to the centre line of the columns. Due to such movement, the beam *BC* has a motion of translation only. If, however, the columns are inclined to the vertical, the column head will move, due to side sway in a direction perpendicular to the centre line of the column thereby giving a motion of rotation to the beam *BC* in addition to the motion of translation. Due to motion of rotation, the beam *BC* will also have sway moments at joints *B* and *C*. In order to find the ratio of sway moments we shall consider the three cases with different end conditions.

Case I : Both Ends Hinged

Let us consider a portal frame *ABCD*, hinged at *A* and *D*. Let the column *AB* be of length L_1, moment of inertia I_1 and its inclination to the horizontal be θ_1. Similarly, let the column *CD* be of length L_2, moment of inertia I_2 and its inclination with horizontal be θ_2. Let the beam *BC* be of length *L* and its moment of inertia be *I*.

Let the frame be distorted as shown in Fig. 10·43 due to sway. Since the displacements are small, joint *B* will move to B_1 in direction BB_1 perpendicular to *AB*. Similarly, joint *C* will move to C_1 in direction CC_1 perpendicular to *CD*. Consequently, the beam *BC* will be distorted to B_1C_1.

Let $BB_1 = \delta_1$ and $CC_1 = \delta_2$. Draw lines B_1B' and C_1C' perpendicular to *BC* from B_1 and C_1 respectively.

Let δ = Vertical displacement of *C* with regard to *B* after distortion,

or $$\delta = B_1B' + C'C_1$$

Considering no change in the length of the beam *BC*, we have

$$BB' = CC'$$

or $$\delta_1 \sin \theta_1 = \delta_2 \sin \theta_2 \quad (1)$$

(2)

Equations (1) and (2) give the relationship between the displacement of various joints. From the above equations, δ_1 and δ_2 can be calculated in terms of δ for given values of θ_1 and θ_2.

Now considering column *AB* as a beam fixed at *B* and hinged at *A* with the joint displacement δ_1, we have

$$M_{BA} = \frac{3EI_1\delta_1}{L_1^2} \quad (3)$$

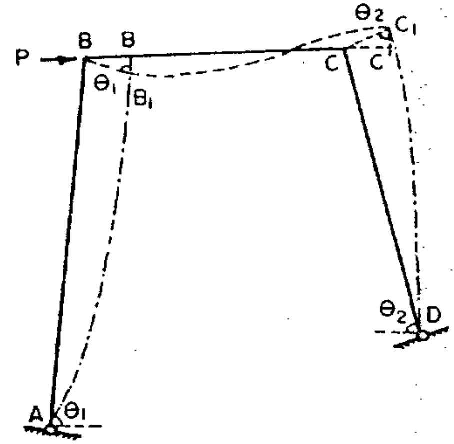


Fig. 10·43.

Similarly, for the column *CD*, we have,

$$M_{CD} = \frac{3EI_2\delta_2}{L_2^2} \quad (4)$$

Considering the beam *BC* as fixed at *B* and *C* and the joint *C* having displacement δ with respect to *B*, we have,

$$M_{BC} = M_{CB} = \frac{6EI\delta}{L^2} \quad (5)$$

From equations (3), (4) and (5), we get

$$M_{BC} : M_{BA} : M_{CD} :: \frac{2I\delta}{L^2} : \frac{I_1\delta_1}{L_1^2} : \frac{I_2\delta_2}{L_2^2} \quad (10·11)$$

It should be noted here that the moments M_{BA} and M_{CD} will be of the same sign and M_{BC} (or M_{CB}) will be of different signs. For the case illustrated in Fig. 10·43 M_{BA} and M_{CD} are of negative sign as the columns rotate in clockwise direction while M_{BC} (or M_{CB}) of positive sign since the beam *BC* rotates in the anticlockwise direction.

Case II : Both Ends Fixed (Fig. 10·44)

With the same notations as that of Fig. 10·43, we have

$$\delta_1 \sin \theta_1 = \delta_2 \sin \theta_2 \qquad (1)$$

and

$$\delta = \delta_1 \cos \theta_1 + \delta_2 \cos \delta_2 \qquad (2)$$

Considering column AB to be fixed at A and B, the movement of the joint B by δ_1 will cause moments in the column AB. Thus

$$M_{BA} = M_{AB} = \frac{6EI_1\delta_1}{L_1^2} \qquad (3)$$

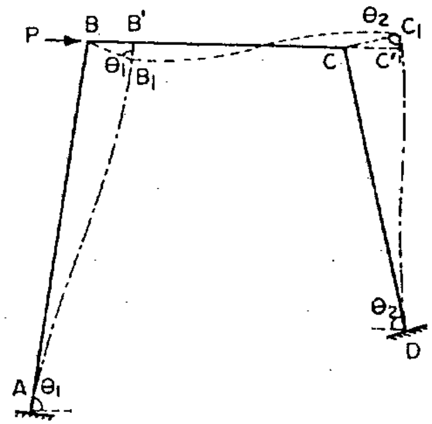


Fig. 10·44.

Similarly, for the column CD, we have,

$$M_{CD} = M_{DC} = \frac{6EI_2\delta_2}{L_2^2} \qquad (4)$$

and for the beam BC, we have

$$M_{BC} = M_{CB} = \frac{6EI\delta}{L^2} \qquad (5)$$

Hence from (3), (4) and (5), we get

$$M_{BC} : M_{BA} : M_{CD} :: \frac{I\delta}{L^2} : \frac{I_1\delta_1}{L_1^2} : \frac{I_2\delta_2}{L_2^2} \qquad (10·12)$$

For the case of sway illustrated in Fig. 10·44, M_{BC} will be of positive sign while M_{BA} and M_{CD} will be of negative sign.

Case II : One End Fixed, Other End Hinged (Fig. 10·45)

In this case also, we have

$$\delta_1 \sin \theta_1 = \delta_2 \sin \theta_2 \qquad (1)$$

and

$$\delta = \delta_1 \cos \theta_1 + \delta_2 \cos \theta_2 \qquad (2)$$

For the column AB, fixed at A and B, we have

$$M_{BA} = M_{AB} = \frac{6EI_1\delta_1}{L_1^2} \qquad (3)$$

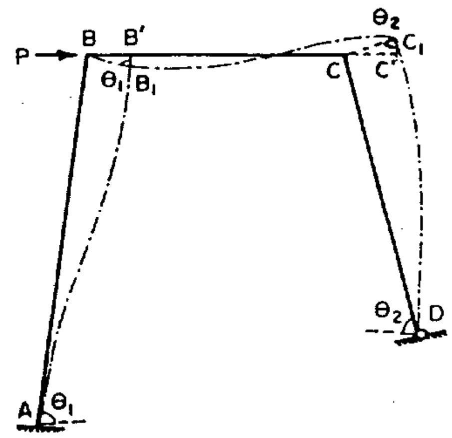


Fig. 10·45.

For the column CD, fixed at C and hinged at D, w have

$$M_{CD} = \frac{3EI_2\delta_2}{L_2^2} \qquad (4)$$

For the beam BC, fixed at B and C, we have

$$M_{BC} = M_{CB} = \frac{6EI\delta}{L^2} \qquad (5)$$

From (3), (4) and (5), we get,

$$M_{BC} : M_{BA} : M_{CD} :: \frac{2I\delta}{L^2} : \frac{2I_1\delta_1}{L_1^2} : \frac{I_2\delta_2}{L_2^2} \qquad (10·13)$$

Example 10·21. *Analyse the portal frame shown in Fig. 10·46. The end A is fixed and D is hinged. The beam BC is loaded with a U.D.L. of 5 kN/m.*

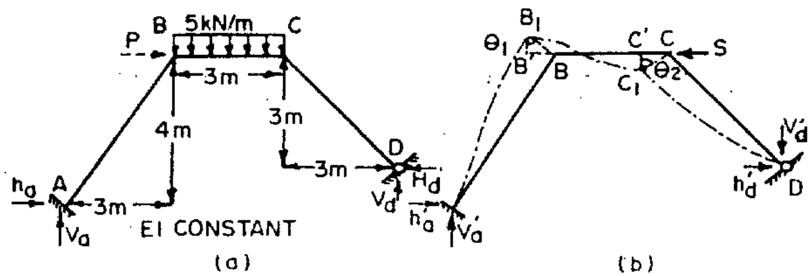


Fig. 10·46.

Solution

(*a*) *Fixed end moments*

$$M_{FBC}=-\frac{5\times3^2}{12}=-3{\cdot}75 \text{ kN-m} \; ; \; M_{FCB}=+3{\cdot}75 \text{ kN-m.}$$

(*b*) *Distribution factors* (Table 10·49)

TABLE 10·49

Joints	*Member*	*Relative stiffness*	*Sum*	*D.F.*
B	BA	$\frac{I}{5}=\frac{3I}{15}$	$\frac{8I}{15}$	0·38
	BC	$\frac{I}{3}=\frac{5I}{15}$		0·62
C	CB	$\frac{I}{3}=0{\cdot}33I$	0·51 *I*	0·65
		$\frac{3}{4}\times\frac{I}{3\sqrt{2}}=0{\cdot}18I$		0·35

(*c*) *Moment distribution*

TABLE 10·50

A	B		C		D	
	0·38	0·62	0·65	0·35		
—	—	−3·75	+3·75	—	—	F.E.M.
	+1·41	+2·34	−2·44	−1·31		Balance
+0·70	—	−1·22	+1·17	—	—	Carry over
	+0·46	+0·76	−0·76	−0·41	—	Balance
+0·23	—	−0·38	+0·38	—	—	Carry over
	+0·14	+0·24	−0·25	−0·13	—	Balance
+0·07	—	−0·13	+0·12	—	—	Carry over
—	+0·05	+0·08	−0·08	−0·04	—	Balance
+0·03	—	−0·04	+0·04	—	—	Carry over
+0·01 ←	+0·02	+0·02	−0·02	−0·02	—	B. and C.O.
+1·04	+2·08	−2·08	+1·91	−1·91	—	Final

Assuming a horizontal force P to be applied at joint C as shown in figure 10·46(*a*) to prevent the side sway of the frame, the moment distribution is carried out as shown in Table 10·50.

To find out the reactions at A and D proceed as follows :

Taking moments about B for equilibrium of AB,

$$V_a\times3+1{\cdot}04+2{\cdot}08=4\times h_a$$

$$4h_a=3V_a+3{\cdot}12 \quad (1)$$

Taking moments about C for equilibrium of CD

$$V_d\times3+1{\cdot}91=h_d\times3$$

or $$3V_d+1{\cdot}91=3h_d \quad (2)$$

Also, $$P=h_d-h_a \quad (3)$$

Taking moments about A for the equilibrium of the whole frame we get,

$$V_d\times9-P\times4=5\times3\times4{\cdot}5-h_d\times1+1{\cdot}04$$

or $$9V_d+h_d-4P=66{\cdot}46 \quad (4)$$

Also $$V_a+V_d=(5\times3)=15 \quad (5)$$

Substituting the value of P in (5), we get

$$9V_d+h_d+4h_d-4h_a=66{\cdot}46 \quad (6)$$

Now adding equation (1) and (6), we get,

$$9V_d+5h_a=66{\cdot}46+3V_a+3{\cdot}12$$

or $$9V_d+5h_d-3(15-V_d)=66{\cdot}46+3{\cdot}12$$

or $$12V_d+5h_d=114{\cdot}58$$

Substituting the value of h_d from equation (2),

$$12V_d+5(V_d+0{\cdot}637)=114{\cdot}58$$

or $$V_d=6{\cdot}93 \text{ kN}$$

Hence $$V_a=15-6{\cdot}93=8{\cdot}07 \text{ kN}$$

Substituting in equation (2), we get,

$$3h_d=3V_d+1{\cdot}91=3\times6{\cdot}93+1{\cdot}91=22{\cdot}7$$

or $$h_d=7{\cdot}57 \text{ kN}$$

Similarly from (1) we get $h_a=6{\cdot}83$ kN

From equation (3), we get,

$$P=7{\cdot}57-6{\cdot}83=0{\cdot}74 \text{ kN}\rightarrow$$

(*d*) *Side Sway*

Now apply a force $S=0{\cdot}74\leftarrow$ at C in opposite direction to that of P to neutralise its effect. The frame will now experience side sway as shown in Fig. 10·46(*b*).

Now, let $BB_1=\delta_1$; $CC_2=\delta_2$

Difference in level of B_1 and $C_1=\delta=\delta_1\cos\theta_1+\delta_2\cos\theta_2$

or $$\delta=\delta_1\times\frac{3}{5}+\delta_2\times\frac{1}{\sqrt2}$$

Also $\delta_1\sin\theta_1=\delta_2\sin\theta_2$

or $$\delta_1\times\frac{4}{5}=\delta_2\times\frac{1}{\sqrt2}\text{ or }\delta_2=\frac{4\sqrt2}{5}\delta_1$$

or $$\delta_1=\frac{5}{4\sqrt2}\delta_1$$

So $$\delta=\frac{3}{4\sqrt2}\delta_2+\frac{1}{\sqrt2}\delta_2=\frac{7}{4\sqrt2}\delta_2$$

$$=\frac{7}{4\sqrt2}\times\frac{4\sqrt2}{5}\delta_1=\frac{7}{5}\delta_1$$

Hence $$\delta_1=\frac{5}{7}\delta=0{\cdot}714\delta$$

and $$\delta_2=\frac{4\sqrt2}{7}\delta=0{\cdot}808\,\delta$$

Now $$M_{AB}=\frac{6EI\delta_1}{l_1^2}=\frac{6EI\times0{\cdot}714\delta}{25}=0{\cdot}171EI\delta=M_{BA}$$

$$M_{BC}=M_{CB}=-\frac{6EI\delta}{9}=-0{\cdot}667\,EI\delta$$

and $$M_{CD}=\frac{3EI\delta_2}{l_2^2}=\frac{3EI\times0{\cdot}808\,\delta}{18}=0{\cdot}135\,EI\delta$$

Keeping these relations between the fixed end moments we can put arbitrary value of each bending moment as follows :

$$M_{AB}=M_{BA}=+5{\cdot}00\text{ kN-m}$$
$$M_{BC}=M_{CB}=-19{\cdot}50\text{ kN-m}$$
$$M_{CD}=+3{\cdot}95\text{ kN-m}$$

(*e*) *Distribution of sway moments*

The moments may now be distributed as shown in Table 10·51.

TABLE 10·51

A	*B*		*C*		*D*	
	0·38	0·62	0·65	0·35		
+5·00	+5·00	−19·50	−19·50	+3·95	0	F.E.M.
—	+5·52	+8·98	+10·10	+5·45	—	Balance
−2·76	—	+5·05	+4·49	—	—	Carry over
—	−1·92	−3·13	−2·92	−1·57	—	Balance
−0·96	—	−1·46	−1·57	—	—	Carry over
—	+0·55	+0·91	+1·02	+0·55	—	Balance
+0·28	—	+0·51	+0·46	—	—	Carry over
—	−0·19	−0·32	−0·30	−0·16	—	Balance
−0·10	—	−0·15	−0·16	—	—	Carry over
+0·03	+0·06	+0·09	+0·10	+0·06	—	Balance
+7·01	+9·02	−9·02	−8·28	+8·28	—	Final

Taking moments about B for equilibrium of AB,

$$V'_a\times3+7{\cdot}01+9{\cdot}02=h'_a\times4$$

or $$4h'_a=3V'_a+16{\cdot}03 \qquad (1)$$

Taking moments about C for equilibrium of DC,

$$h'_d\times3=V'_d\times3+8{\cdot}28 \qquad (2)$$

Also $$S=h'_a+h'_d \qquad (3)$$

and $$V'_a=V'_d \qquad (4)$$

Taking moment about A for the equilibrium of the whole frame,

$$V'_d\times9+0{\cdot}71+h'_d\times1=S\times4=4h'_a+4h'_d \qquad (5)$$

Substracting (1) from (5), we get

$$9V'_d+7{\cdot}01+h'_d-3V'_d-16{\cdot}03=4h'_d$$

or $$6V'_d-9{\cdot}02=3h'_d$$

Substituting the value of $3h'_d$ from (2), we get

$$6V'_d-9{\cdot}02=3V'_d+8{\cdot}28$$

or $$3V'_d=17{\cdot}3$$

or $$V'_d=5{\cdot}77 \text{ kN} \downarrow$$

and $$V'_a=5{\cdot}77 \text{ kN} \uparrow$$

Also $$3h'_d=3\times V'_d+8{\cdot}28=17{\cdot}30+8{\cdot}28=25{\cdot}28$$

or $$h'_d=8{\cdot}53 \text{ kN} \rightarrow$$

From equation (1)

$$4h'_a=3\times5{\cdot}77+16{\cdot}03=33{\cdot}33$$

or $$h'_a=8{\cdot}33 \text{ kN} \rightarrow$$

So $$S=8{\cdot}33+8{\cdot}53=16{\cdot}86 \text{ kN}.$$

The actual sway force is 0·74 kN only and hence the moments will have to be reduced accordingly. The final moments will be found out as shown in Table 10·52.

TABLE 10·52

	A	B		C		D
1. Sway=16·86 kN	+7·01	+9·02	−9·02	−8·28	+8·28	0
2. Sway=0·74 kN	+0·31	+0·40	−0·40	−0·36	+0·36	0
3. Non-sway moment	+1·04	+2·08	−2·08	+1·91	−1·91	0
4. Final moments	+1·35	+2·48	−2·48	+1·55	−1·55	0

$$\text{Vertical reaction at } A=8{\cdot}07+\frac{5{\cdot}77}{16{\cdot}86}\times0{\cdot}74=8{\cdot}32 \text{ kN}\uparrow$$

$$\text{Vertical reaction at } D=6{\cdot}93-\frac{5{\cdot}77}{16{\cdot}86}\times0{\cdot}74=6{\cdot}86 \text{ kN}\uparrow$$

$$\text{Horizontal reaction at } A=6{\cdot}83+\frac{8{\cdot}33}{16{\cdot}86}\times0{\cdot}74=7{\cdot}20 \text{ kN}\rightarrow$$

$$\text{Horizontal reaction at } D=7{\cdot}57-\frac{8{\cdot}53}{16{\cdot}86}\times0{\cdot}74=7{\cdot}20 \text{ kN}\leftarrow$$

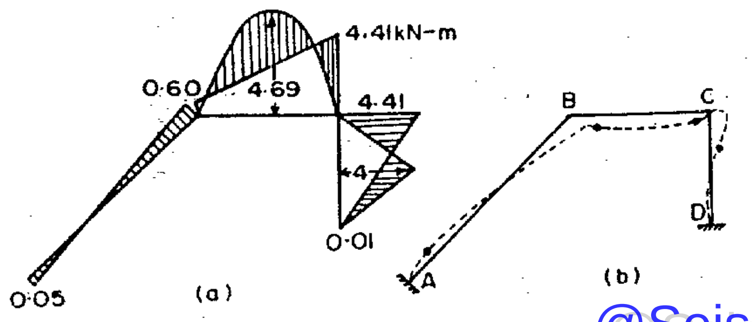


Fig. 10·47.

The bending moment diagram and the deflected shape have been shown in Fig. 10·47.

Example 10·22. *Analyse the inclined portal frame shown in Fig. 10·48 completely and draw bending moment diagram and deflected shape of the frame.*

Solution

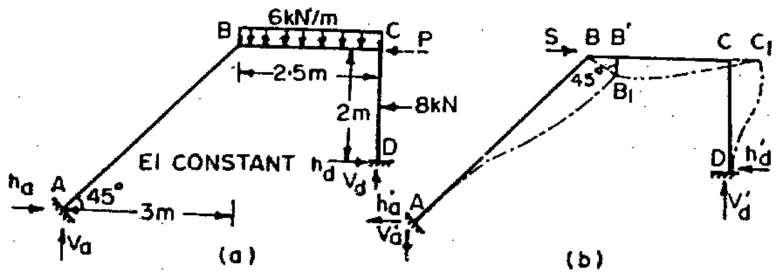


Fig. 10·48.

$$AB=3\sqrt{2}=4{\cdot}24 \text{ m}$$

(a) Fixed end moment

$$M_{FBC}=-\frac{6\times2{\cdot}5^2}{12}=-3{\cdot}13 \text{ kN-m}\ ;\ M_{FCB}=+3{\cdot}13 \text{ kN-m}$$

$$M_{FCD}=-\frac{8\times2}{8}=-2{\cdot}0 \text{ kN-m}\ ;\ M_{FDC}=+2{\cdot}0 \text{ kN-m}$$

(b) Distribution factors (Table 10·53)

TABLE 10·53

Joint	*Member*	*Relative stiffness*	*Sum*	*D.F.*
B	*BA*	$\frac{I}{4{\cdot}24}=0{\cdot}236I$	$0{\cdot}636I$	0·37
	BC	$\frac{I}{2{\cdot}5}=0{\cdot}4I$		0·63
C	*CB*	$\frac{I}{2{\cdot}5}=0{\cdot}4I$	$0{\cdot}9I$	0·44
	CD	$\frac{I}{2}=0{\cdot}5I$		0·56

(c) *Moment Distribution*

A force P is applied at joint C, as shown in Fig. 10·48(a) and moment distribution is carried out as indicated in Table 10·54.

TABLE 10·54

A	B		C		D	
	0·37	0·63	0·44	0·56		
–	–	–3·13	+3·13	–2·0	+2·0	F.E.M.
–	+1·16	+1·97	–0·50	–0·63	–	Balance
+0·58	–	–0·25	+0·99	–	–0·32	Carry over
–	+0·09	+0·16	–0·43	–0·56	–	Balance
+0·05	–	–0·22	+0·08	–	–0·28	Carry over
–	+0·08	+0·11	–0·04	–0·04	–	Balance
+0·04	–	–0·02	+0·07	–	–0·02	Carry over
–	+0·01	+0·01	–0·03	–0·04→	–0·02	Balance
+0·67	+1·34	–1·34	+3·27	–3·27	+1·36	

For equilibrium of AB, taking moments about B,

$$V_a \times 3 + 0{\cdot}67 + 1{\cdot}34 = 3h_a$$

$$3h_a = 3V_a + 2{\cdot}01 \qquad (1)$$

Taking moments about C for equilibrium of DC,

$$8 \times 1 + 1{\cdot}36 - 3{\cdot}27 = h_d \times 2$$

or $$h_d = 3{\cdot}05 \text{ kN} \rightarrow$$

Also $$P = h_a + h_d - 8 = h_a + 3{\cdot}05 - 8$$

$$= h_a - 4{\cdot}95 \qquad (2)$$

Now, taking moments about A for the equilibrium of the whole frame, we get

$$3{\cdot}05 \times 1 - P \times 3 + 6 \times 2{\cdot}5 \times 4{\cdot}25 + 1{\cdot}36 + 0{\cdot}67 = 8 \times 2 + V_a \times 5{\cdot}50$$

Putting value of P from (3) and solving, we get,

$$67{\cdot}68 - 3h_a = 5{\cdot}50\ V_d$$

Adding (1) and (3), we get

$$67{\cdot}68 = 5{\cdot}50\ V_a + 3V_a + 2{\cdot}01$$

Also $$V_a = 15 - V_d$$

So. $$67{\cdot}68 = 5{\cdot}5\ V_d + 3(15 - V_d) + 2{\cdot}01$$

$$= 2{\cdot}5\ V_d + 47{\cdot}01$$

or $$V_d = 8{\cdot}27 \text{ kN} \uparrow$$

and $$V_a = 15 - 8{\cdot}27 = 6{\cdot}73 \text{ kN} \uparrow$$

Substituting the value of V_a in equation (1), we get,

$$3h_a = 3 \times 6{\cdot}73 + 2{\cdot}01 = 19{\cdot}86$$

or $$h_a = 7{\cdot}40 \text{ kN} \rightarrow$$

From equation (2), we get

$$P = 7{\cdot}40 - 4{\cdot}95 = 2{\cdot}45 \text{ kN} \rightarrow$$

(d) *Side Sway*

Now apply a force $S = 2{\cdot}45$ kN→ at B so as to neutralise the effect of force P. The frame $ABCD$ will be deformed to AB_1C_1D as shown in Fig. 10·48 (b).

Now $$B_1B' = \delta\ ;\ CC_1 = \delta_2 \text{ and } BB_1 = \delta_1$$

So $$\delta = \delta_1 \cos 45° = \frac{\delta_1}{\sqrt{2}}$$

Also $\delta_1 \sin 45° = \delta_2$

or $$\frac{\delta_1}{\sqrt{2}} = \delta_2$$

So $$\delta = \delta_2 = \frac{\delta_1}{\sqrt{2}}$$

The relations between the sway moments may now be put as under :

$$M_{AB} = M_{BA} = -\frac{6EI\delta_1}{(3\sqrt{2})^2} = -\frac{6EI \times \delta\sqrt{2}}{18} = -0{\cdot}47EI\delta$$

$$M_{BC} = M_{CB} = +\frac{6EI\delta}{2{\cdot}5^2} = +0{\cdot}96\ EI\ \delta$$

$$M_{CD} = M_{DC} = -\frac{6EI\delta_2}{2^2} = -\frac{6EI\delta}{4} = -1{\cdot}5EI\delta$$

The sway moments are given arbitrary values as below, satisfying the above relations.

$$M_{AB} = M_{BA} = -10{\cdot}0 \text{ kN-m}$$

$$M_{BC} = M_{CB} = +20{\cdot}4 \text{ kN-m}$$

and $$M_{CD} = M_{DC} = -31{\cdot}9 \text{ kN-m}$$

(c) *Distribution of sway moments*

The moment distribution may now be carried out as shown in Table 10·55.

TABLE 10·55

A	B		C		D	
	0·37	0·63	0·44	0·56		
—10·0	—10·0	+20·4	+20·4	—31·9	—31·9	F.E.M.
—	—3·85	—6·55	+5·06	+6·44	—	Balance
—1·92	—	+2·53	—3·27	—	+3·22	Carry over
—	—0·94	—1·59	+1·44	+1·83	—	Balance
—0·47	—	+0·72	—0·80	—	+0·92	Carry over
—	—0·27	—0·45	+0·33	+0·47	—	Balance
—0·14	—	+0·16	—0·22	—	+0·23	Carry over
—	—0·66	—0·10	+0·10	+0·12	—	Balance
—0·03	—	+0·05	—0·05	—	+0·06	Carry over
—0·01 ←	—0·02	—0·03	—0·02	+0·03→	+0·02	B. & C.O.
—12·57	—15·14	+15·14	+23·01	—23·01	—27·45	Final moments

Taking moments about B for equilibrium of AB,

$$3h_a' = V_a' \times 3 + 12{\cdot}57 + 15{\cdot}14 = 3V_a' + 27{\cdot}71 \quad (1)$$

Taking moments about C for equilibrium of CD,

$$2h_d' = 23{\cdot}01 + 27{\cdot}45 = 50{\cdot}46$$

$$\therefore \quad h_d' = 25{\cdot}23 \text{ kN} \leftarrow \quad (2)$$

Also $\quad S = h'_a + h'_d = h'_a + 25{\cdot}23 \quad (3)$

and $\quad V'_a = V'_d$

Taking moments about A, for the equilibrium of the whole frame,

$$5 \times 3 = h'_d \times 1 + V'_d \times 5{\cdot}5 + 27{\cdot}45 + 12{\cdot}57$$

Substituting the values of P and h'_d from equations (2) and (3), we get,

$$3h'_a = 5{\cdot}5\, V'_d - 10{\cdot}46$$

Substracting (1) from (4), we get,

$$2{\cdot}5V'_a = 38{\cdot}17 \text{ kN}$$

or $\quad V'_a = 15{\cdot}27 \text{ kN} = V'_d$

Substituting in equation (1), we get,

$$3h'_a = 3 \times 15{\cdot}27 + 27{\cdot}71 = 73{\cdot}52$$

So. $\quad h'_a = 24{\cdot}51 \text{ kN} \leftarrow$

and $\quad S = h'_a + h'_d = 24{\cdot}51 + 25{\cdot}23 = 49{\cdot}74 \text{ kN} \rightarrow$

The final moments may now be got as shown in Table 10·56.

TABLE 10·56

	A	B		C		D
1. Sway=49·74 kN	—12·57	—15·14	+15·14	+23·01	—23·01	—27·45
2. Sway=2·45 kN	—0·62	—0·74	+0·74	+1·14	—1·14	—1·35
3. Non-sway moments	+0·67	+1·34	—1·34	+3·27	—3·27	+1·36
4. Final moments	+0·05	+0·60	—0·60	+4·41	—4·41	+0·01

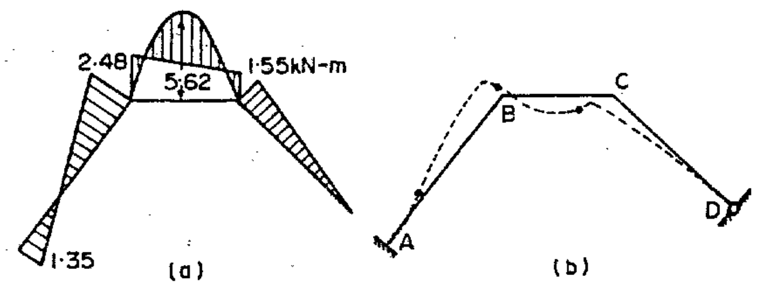


Fig. 10·49.

Horizontal reaction at $A = 7{\cdot}40 - \dfrac{24{\cdot}51 \times 2{\cdot}45}{49{\cdot}74} = 6{\cdot}20 \text{ kN} \rightarrow$

Horizontal reaction at $D = 3{\cdot}05 - \dfrac{25{\cdot}24 \times 2{\cdot}45}{49{\cdot}74} = 1{\cdot}80 \text{ kN} \rightarrow$

Vertical reaction at $A = 6{\cdot}73 - \dfrac{15{\cdot}27 \times 2{\cdot}45}{49{\cdot}74} = 5{\cdot}98 \text{ kN} \uparrow$

Vertical reaction at $D = 8{\cdot}27 + \dfrac{15{\cdot}27 \times 2{\cdot}45}{49{\cdot}74} = 9{\cdot}02 \text{ kN} \uparrow$

The bending moments diagram and the deflected shape have been plotted in Fig. 10·49.

PROBLEMS

1. A continuous beam *ABCD* is fixed at ends *A* and *D*, and is loaded as shown in Fig. 10·50. Spans *AB*, *BC* and *CD* have moments of inertia of *I*, 1·5 *I*

and I respectively and are of the same material. Determine the moments at the supports and plot the bending moment diagram.

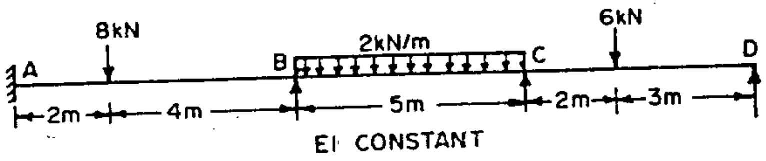


Fig. 10·50.

2. Solve problem 1 if both ends A and D are freely supported.

3. Solve problem 1 if there is no support at the end D.

4. A beam ABC, 12 m long, fixed at A and C and continuous over support B, is loaded as shown in Fig. 10·51. Calculate the end moments and plot the bending moment diagram.

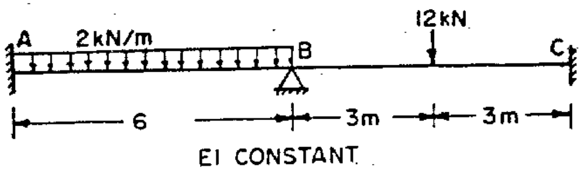


Fig. 10·51.

5. A beam $ABCD$, 16 m long is continuous over three spans and is loaded as shown in Fig. 10·52. Calculate the moments and reactions at the supports.

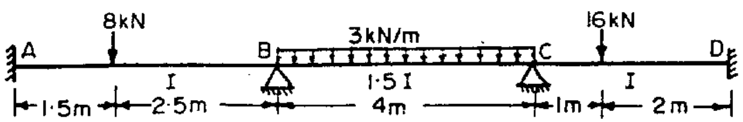


Fig. 10.52.

6. Solve Problem 5 if the support B sinks by 5 mm downwards. I for the beam is 93×10^6 mm^4 throughout. Take $E = 2{\cdot}1 \times 10^5$ N/mm^2.

7. A continuous beam $ABCD$, 20 m long is simply supported at its ends and is propped at the same level at points 5 m and 12 m from left hand A. It carries two concentrated loads of 8 kN and 5 kN at 2 m and 9 m respectively from A and a uniformly distributed load of 1 kN/m over the span CD. Find the B.M. at the props if the support B sinks by 10 mm below A and C. Moment of inertia for the whole beam $= 85 \times 10^6$ mm^4 and $E = 2{\cdot}1 \times 10^5$ N/mm^2.

8. Draw the bending moment diagram for the frame shown in Fig. 10·53. The frame has stiff joint at B and is fixed at A, C and D.

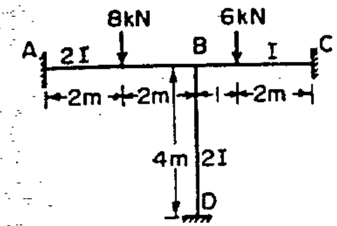


Fig. 10·53.

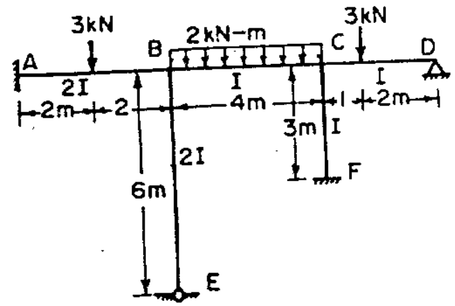


Fig. 10·54.

9. Analyse the continuous one storey frame shown in Fig. 10·54.

10. Fig. 10·55 shows a two span portal frame with the columns fixed at end A, E and F and carries uniformly distributed load of w kN/m along BD. The stiffness ratios of the members are shown in the diagram and all the members are of equal length.

(a) Determine the bending moments throughout the frame and sketch the bending moment diagram.

(b) If Young's modulus E is constant and the central column sinks by an amount $\triangle$, determine the changes in the moments at B, C and D in terms of E, $\triangle$ and L.

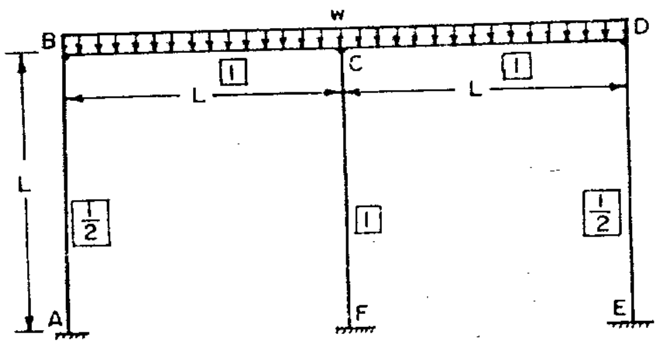


Fig. 10·55.

11. A portal frame $ABCD$ fixed at ends A and D carries a point load W as shown in Fig. 10·56. Draw the bending moments diagram and sketch the deflected shape of the beam.

12. A portal frame $ABCD$ is fixed at A and hinged at D. Draw the bending moment diagram due to a point load of 9 kN as shown in Fig. 10·57. Calculate the reactions and sketch the deflected shape of the frame.

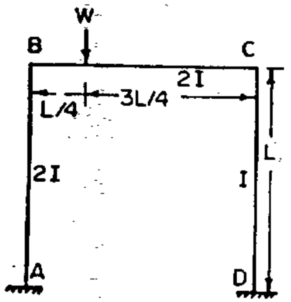


Fig. 10·56.

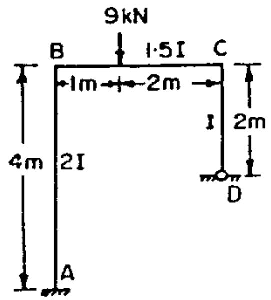


Fig. 10·57.

13. Analyse completely the portal frame shown in Fig. 10·58.

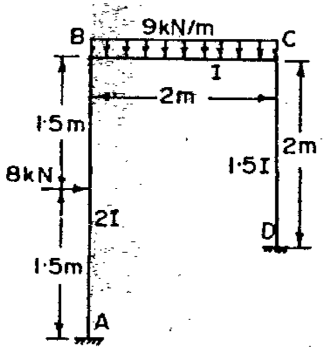


Fig. 10·58.

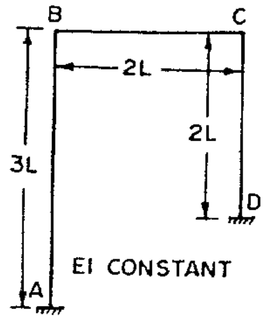


Fig. 10·59.

14. The portal frame shown in 10·59 has fixed ends. If D sinks by Δ, find the moments induced in the frame. All the members have the same uniform cross-section.

15. Determine the bending moments at the joints and draw the bending moment diagram for the frame shown in Fig. 10·60. The flexural rigidity of the members is as shown.

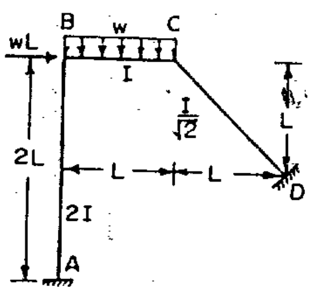


Fig. 10·60.

Answers

1. $M_{AB}=-4·6$ kN-m ; $M_{BA}=+2·98$; $M_{BC}=-2·98$ kN-m ; $M_{CB}=+5·7$; $M_{CD}=-5·7$; $M_{DC}=+4·27$.
2. $M_{BA}=+4·19$ kN-m ; $M_{BC}=-4·19$; $M_{CB}=+6·53$; $M_{CD}=+6·53$.
3. $M_{AB}=-5·83$; $M_{BA}=+0·55$; $M_{BC}=-0·55$; $M_{CB}=+16$ kN-m
4. $M_{AB}=-5·25$; $M_{BA}=+7·5$; $M_{BC}=-7·5$; $M_{CB}=+9·75$ kN-m.
5. $M_{BA}=+5·68$ kN-m ; $M_{BC}=-5·68$; $M_{CB}=+4·58$; $M_{CD}=-4·58$; $R_A=4·39$; $R_B=8·83$; $R_C=9·3$; $R_D=1·48$.
6. $M_{BA}=+4·48$; $M_{BC}=-4·48$; $M_{CB}=+5·49$; $M_{CD}=-5·49$ kN-m ; $R_A=4·42$; $R_B=8·78$; $R_C=9·50$; $R_D=1·30$ kN
7. $M_{BA}=+2·74$; $M_{BC}=-2·74$; $M_{CB}=+7·3$; $M_{CD}=-7·3$ kN-m
8. $M_{AB}=-4·25$; $M_{BA}=+3·5$; $M_{BC}=-3$; $M_{BD}=-0·5$; $M_{CB}=+1·17$
9. $M_{AB}=-1·84$; $M_{BA}=+2·06$; $M_{BC}=-2·45$; $M_{BE}=-0·39$; $M_{CB}=+2·51$; $M_{CD}=-2·01$; $M_{CF}=-0·48$; $M_{FC}=-0·24$
10. (a) $M_{AB}=+\dfrac{wL^2}{72}$; $M_{ED}=-\dfrac{wL^2}{72}$; $M_{BC}=-\dfrac{wL^2}{36}$

    $M_{BA}=+\dfrac{wL^2}{36}$; $M_{CB}=+\dfrac{wL^2}{9}$; $M_{CD}=-\dfrac{wL^2}{9}$;

    $M_{DC}=+\dfrac{wL^2}{36}$; $M_{DE}=-\dfrac{wL^2}{36}$; $M_{CF}=M_{FC}=0$

    (b) Moments at B increased by $\dfrac{2EI\Delta}{L^2}$

    Moment at C decreased by $\dfrac{4EI\Delta}{L^2}$
11. $M_{AB}=+0·0137\ WL$; $M_{BA}=+0·0647\ WL$
    $M_{BC}=-0·0647\ WL$; $M_{CB}=+0·0461\ WL$
    $M_{CD}=-0·0461\ WL$; $M_{DC}=-0·0325\ WL$
12. $M_{AB}=+0·96$; $M_{BA}=+2·25$
    $M_{BC}=-2·25$; $M_{CB}=+1·60$
    $M_{CD}=-1·60$; $M_{DC}=0$
    $H_A=0·80\rightarrow$; $H_D=0·80\leftarrow$
    $V_A=6·22\uparrow$; $V_D=2·78\uparrow$
13. $M_{AB}=-4·34$; $M_{BA}=+2·22$; $M_{BC}=-2·22$; $M_{CB}=+3·33$; $M_{CD}=-3·33$; $M_{DC}=-3·26$; $H_A=4·7\leftarrow$; $H_D=3·3\leftarrow$; $V_A=8·45\uparrow$; $V_D=9·55\uparrow$
14. $M_{AB}=-\dfrac{0·06EI\Delta}{L^2}$; $M_{BA}=+\dfrac{0·22EI\Delta}{L^2}$

    $M_{BC}=-\dfrac{0·22EI\Delta}{L^2}$; $M_{CB}=-\dfrac{0·17EI\Delta}{L^2}$

    $M_{CD}=+\dfrac{0·17EI\Delta}{L^2}$; $M_{DC}=-\dfrac{0·28EI\Delta}{L^2}$
15. $M_{AB}=-0·0676\ wL^2$; $M_{BA}=-0·0676\ wL^2$
    $M_{BC}=+0·0676\ wL^2$; $M_{CB}=+0·1436\ wL^2$
    $M_{CD}=-0·1436\ wL^2$; $M_{DC}=-0·113\ wL^2$.

11

The Column Analogy Method

11·1. THE COLUMN ANALOGY

The column analogy method was suggested by Prof. Hardy Cross in 1930, and is the most useful in the analysis of beams and curved members with two fixed supports and of rigid frames upto third degree of redundancy. He proved the mathematical similarity or analogy between the stresses created on a column section subjected to eccentric load and the moments imposed on a member due to fixidity of its supports.

To understand the analogy, consider a fixed beam AB with loads W_1, W_2, etc. (Fig. 11·1). The indeterminate bending moment diagram (or fixed bending moment diagram or M_i dia.) is shown in Fig. 11·1 (b) and the static bending moment diagram (or Ms dia.) is shown in Fig. 11·1(c). Fig. 11·1(d) shows a short height of a column having the length L equal to the span of the beam, and width equal to $\frac{1}{EI}$.

The ends A and B of the beam are fixed, and hence θ_A and θ_B are zero. It follows, therefore, that the area of the total bending moment diagram from A to B is zero.

Thus, we have

$$EI\theta_{B}^{A} = -\sum_{B}^{A} (\text{Area of B.M. diagram}) = 0$$

or
$$\int_0^L \frac{Msdx}{EI} + \int_0^L \frac{M_i dx}{EI} = 0$$

or
$$\int_0^L \frac{Msdx}{EI} = -\int_0^L \frac{M_i dx}{EI} \qquad (1)$$

Let us now load the column by loading diagram equal to $\frac{Ms}{EI}$ diagram [Fig. 11·1 (e)]. The pressure distribution diagram on the

column will be as shown in Fig. 11·1 (f). Studying Figs. 11·1 (b) and 11·1 (f), it is evident that both the diagrams are similar and this similarity can be utilised to find the fixed end moments M_A and M_B.

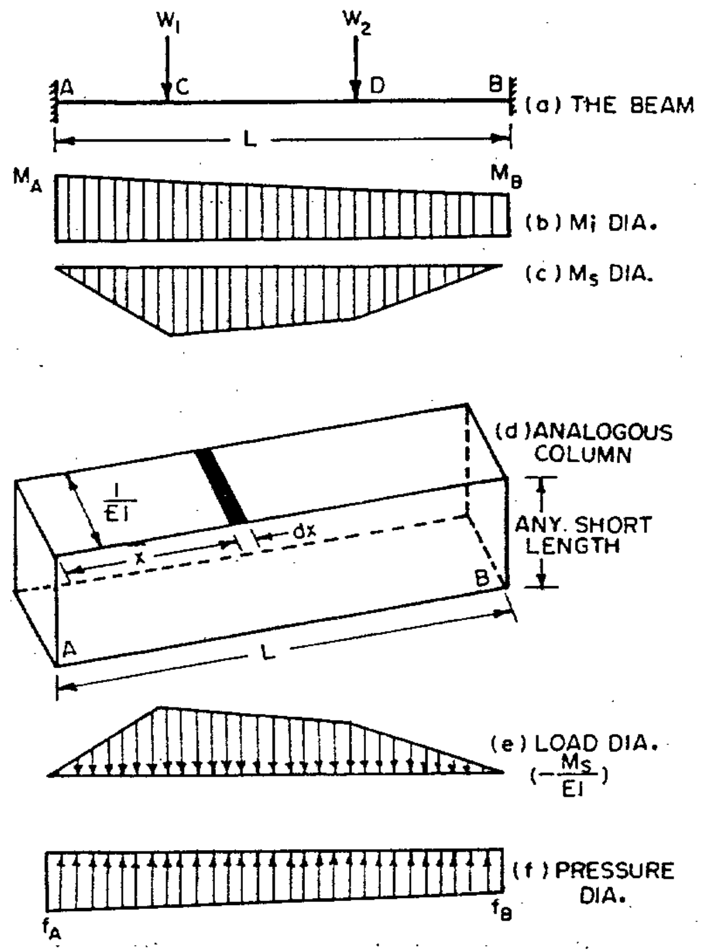


Fig. 11·1
The Column Analogy.

The pressure f_A and f_B on the end of the analogous column can be determined from the following relations :

(i) the total pressure ($\int f\,.dA$) at the base of the column is equal to the total pressure (P) on the top of the column, and (*ii*) the moment of the total pressure, at the base of the column, about A is equal to the moment of the total pressure at the top of the column, about A.

Let us consider a small section dx of the column, situated at a distance x from the end A. Let f be the intensity of pressure at the base at that section.

Thus we have

$$P=\int_0^L f\,.\,dA \tag{2}$$

where P=total load on the column

$$=\text{area of } \frac{Ms}{EI} \text{ diagram}$$

and dA=area of the small section of the column

But
$$P=-\int_0^L \frac{Msdx}{EI}=\int_0^L \frac{M_I dx}{EI} \tag{3}$$

(Thus minus sign has been used because Ms diagram is negative.)

Thus from (2) and (1),

$$\int_0^L \frac{M_I dx}{EI}=\int_0^L f\,dA \tag{11·1}$$

From equation 11·1, it is clear that the fixed end bending moment diagram is analogous to the pressure diagram of an eccentrically loaded column. Since the $\frac{M_I}{EI}$ diagram and the pressure diagram are trapeziums of equal base (L), it follows that

and
$$\left.\begin{matrix} M_A=f_A \\ M_B=f_B \end{matrix}\right\} \tag{11·2}$$

Thus, if the pressure diagram is known, the fixing moment can easily be determined. The column shown in Fig. 11·1 (*d*) is known as the *analogous column* having width of such a magnitude that the total pressure at its base is equal to the total load on the column.

$$\text{Total load on the column}=-\int_0^L \frac{Msdx}{EI}=\int \frac{M_I dx}{EI}$$

$$\text{Total pressure of the base}=\int_0^L f\,.\,dA$$

Comparing the two, it is evident that in order that M_I is equal to f, the area dA of the section must be equal to $\frac{dx}{EI}$. But dA= Length × width=x × width. *Hence the width of the analogous column is equal to* $\frac{1}{EI}$.

The above discussions can be summarised below :

(1) The analogous column has a length (L) equal to the span of the indeterminate beam.

(2) The width of the column is equal to $\frac{1}{EI}$. The height of the column is assumed to be small so that it acts as a short column.

(3) The loading on the analogous column is that represented by $-\frac{Ms}{EI}$ diagram.

(4) The trapezoidal pressure distribution diagram at the base of the column represents also the M_I diagram.

11·2. APPLICATION OF THE ANALOGY FOR FIXED BEAMS

Let us start with the analysis of a fixed beam which is statically indeterminate to second degree for vertical loading. The stress diagram of the analogous column will also give the M_I diagram of the fixed beam if the analogous column is loaded with the $\frac{Ms}{EI}$ diagram. The Ms diagram may be drawn by releasing statically indeterminate moments and forces until the beam becomes statically determinate. Any basic statically determinate structure may be chosen. The statically indeterminate moment at any point of the beam will be equal to the stress (f) of the column at the same point. Once the M_I diagram is known, the final bending moment at any point of the beam may be found from the relation

$$M=Ms+M_I$$

Sign Convention

The following sign convention will be adopted.

1. *A moment* bending the beam convex upwards will be taken as positive and that bending the beam concave upwards as negative.

2. A downward load on the column will be taken as positive and upward load as negative. Thus the positive downward load on the column will be the $-\frac{Ms}{EI}$ diagram.

3. An upward *pressure* at the base of the column will be taken as positive and downward pressure as negative.

11·3. PROPERTIES OF A SYMMETRICAL ANALOGOUS COLUMN

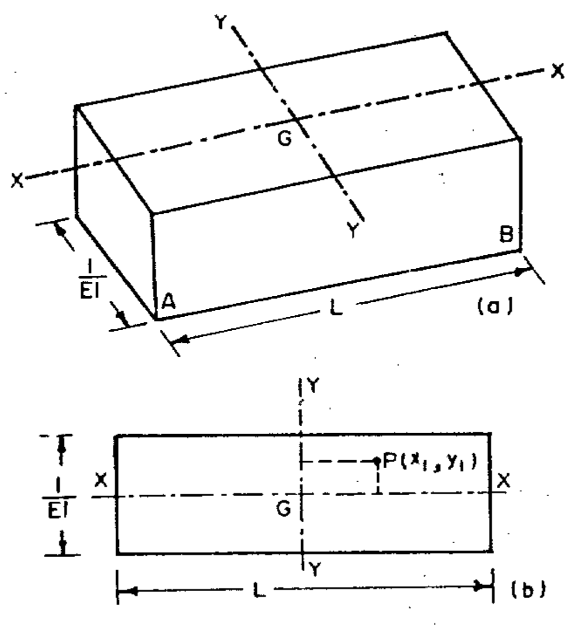


Fig. 11·2.

Fig 11·2 (*a*) shows an analogous column having width $\frac{1}{EI}$ and length L equal to the length or span of the statically indeterminate structure. The column is symmetrical, having the axis x-x along the span or length and the axis y-y perpendicular to it. Fig. 11·2 (*b*) shows the plan view. Let P be the point of application of the load on the column, having the co-ordinates (x_1, y_1).

For the analogous column,

$$A = L \cdot \frac{1}{EI} = \frac{L}{EI}$$

$$I_{YY} = \frac{1}{12} \cdot \frac{1}{EI} \cdot L^3 = \frac{L^3}{12EI}$$

$$I_{XX} = \frac{1}{12} L \cdot \frac{1}{(EI)^3} = \frac{L}{12(EI)^3} = \text{negligible}$$

The total stress (f) at any point having co-ordinates (x, y) referred to the principle axis, will be given by

$$f = f_0 \pm \frac{M_{XX} . y}{I_{XX}} \pm \frac{M_{YY} . x}{I_{YY}} \quad (11·3)$$

where f_0 = direct stress $= \frac{P}{A}$, if P is the total load on the column

M_{XX} = the moment $= P \cdot y_1$

M_{YY} = the moment $= P \cdot x_1$

If y_1 is zero, as is generally the case, $M_{XX}(=Py_1)$ will be zero and the stress f at any point (x, y) will be given by

$$f = \frac{P}{A} \pm \frac{M_{YY} . x}{I_{YY}} \quad (11·4)$$

Example 11·1. *A beam AB of span L is fixed at both the ends and carries a uniformly distributed load w per unit length. Using the column analogy method, compute the fixed end moments.*

Solution (A)

Fig. 11·3 (*a*) shows the loaded beam. If the simply-supported beam is chosen as the basic determinate structure, the Ms diagram will be a parabola having a central ordinate $= \frac{wL^2}{8}$.

Fig. 11·3 (*d*) shows the corresponding analogous column. Due to symmetry, f_A and f_B will be equal.

$$\text{Total load } P = -\int_0^L \frac{Msdx}{EI} = -\frac{1}{EI}\left[-\frac{2}{3} \cdot L \cdot \frac{wL^2}{8}\right] = \frac{wL^3}{12EI}$$

$$\text{Area of the column} = A = \frac{L}{EI} \cdot$$

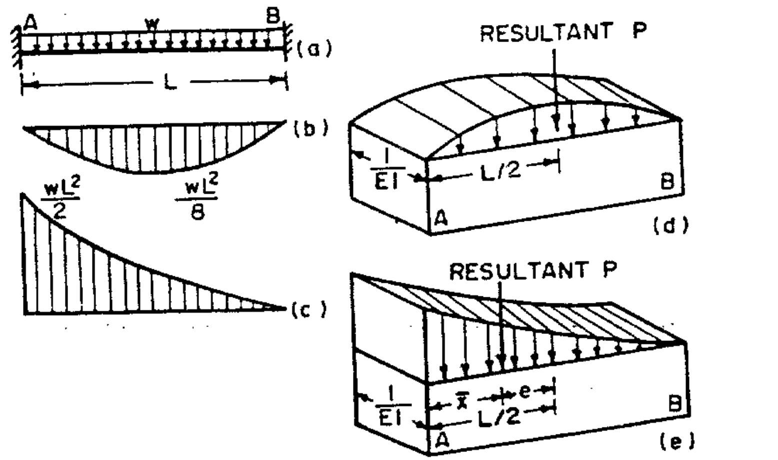


Fig. 11·3.

Since the loading is symmetrical about x-x and y-y axis, M_{XX} and M_{YY} are each zero, and the stress f at any point is given by

$$f=f_0=\frac{P}{A}$$

Hence $\quad f_A=M_A=f_0=\frac{P}{A}=\frac{wL^3}{12EI}\div\frac{L}{EI}=+\frac{wL^2}{12}$

and $\quad f_B=M_B=f_0=+\frac{wL^2}{12}.$

Alternative Solution

The problem can also be solved by taking the cantilever as the basic determinate structure. For that, the support B is completely removed, and the end A is kept fixed. The Ms diagram will be a parabola having maximum ordinate of $+\frac{wL^2}{2}$ at A, as shown in Fig. 11·3 (*c*). The corresponding analogous column is shown in Fig. 11·3 (*e*).

Resultant load $P=-\int_0^L \frac{Msdx}{EI}=-\frac{1}{EI}\left[\frac{1}{3}\times L\times\frac{wL^2}{2}\right]$

$$=-\frac{wL^3}{6EI}$$

$\bar{x}$ from $A=\frac{L}{4}$

$\therefore$ Eccentricity $e=\frac{L}{2}-\bar{x}=\frac{L}{2}-\frac{L}{4}=\frac{L}{4}$

Moment $M_{YY}=P\,.\,e=-\frac{wL^3}{6EI}\times\frac{L}{4}=-\frac{wL^4}{24EI}$

Moment $M_{XX}=0$

$$I_{YY}=\frac{L^3}{12EI}$$

Area $A=\frac{L}{EI}$

The stress (f) at any point (x, y) is given by

$$f=\frac{P}{A}\pm\frac{M_{YY}\,.\,x}{I_{YY}}$$

$$=\left(-\frac{wL^3}{6EI}\times\frac{EI}{L}\right)\pm\left(-\frac{wL^4}{24EI}\times\frac{12EI}{L^3}x\right)$$

$$=\left(-\frac{wL^2}{6}\right)\pm\left(-\frac{wL}{2}x\right) \quad \text{(i)}$$

At A, $\quad x=\frac{L}{2}$

Hence $\quad f_A=\left(-\frac{wL^2}{6}\right)+\left(-\frac{wL}{2}\cdot\frac{L}{2}\right)$

$$=-\frac{wL^2}{6}-\frac{wL^2}{4}=-\frac{5wL^2}{12}$$

At B, $\quad x=\frac{L}{2}.$

Hence $\quad f_B=\left(-\frac{wL^2}{6}\right)-\left(-\frac{wL}{2}\cdot\frac{L}{2}\right)$

$$=-\frac{wL^2}{6}+\frac{wL^2}{4}=+\frac{wL^2}{12}$$

Thus, $\quad M_A=M_I+M_S=-\frac{5wL^2}{12}+\frac{wL^2}{2}=+\frac{wL^2}{12}$

$$M_B=M_I+M_S=+\frac{wL^2}{12}+0=+\frac{wL^2}{12}.$$

Example 11·2. *A beam AB of span L is fixed at A and B, and carries a point load W at a distance a from A and b from B. Calculate the support moments.*

Solution

Let us choose the simply-supported beam as the basic determinate structure. The corresponding Ms diagram is shown in Fig. 11·4 (*b*). The analogous column is loaded with $-\frac{Ms}{EI}$ diagram, as shown in Fig. 11·4 (*e*). The resultant or total load on the column is

$$P=-\int_0^L \frac{Msdx}{EI}=-\frac{1}{EI}\left[-\frac{1}{2}.L.\frac{Wab}{L}\right]=\frac{Wab}{2EI}$$

Distance of C.G. from $A=\bar{x}=\frac{L+a}{3}$

Eccentricity $e=\frac{L}{2}-\left(\frac{L+a}{3}\right)=\frac{L-2a}{6}=\frac{b-a}{6}$

The stress at any point in the column is given by

$$f=\frac{P}{A}\pm\frac{M_{YY}}{I_{YY}}\,.\,x,\ \text{where } A=\frac{L}{EI}\ \text{and } I_{YY}=\frac{L^3}{12EI}$$

$$\therefore \quad f=\frac{Wab}{2EI}\cdot\frac{EI}{L}\pm\frac{Wab}{2EI}\cdot\frac{b-a}{6}\cdot\frac{12EI}{L^3}\,.\,x$$

$$=\frac{Wab}{2L}\pm\frac{Wab(b-a)}{L^3}\,.\,x$$

For $A, f=f_A$ and $x=\frac{L}{2}$

$$\therefore\ f_A=\frac{Wab}{2L}+\frac{Wab(b-a)}{L^3}\cdot\frac{L}{2}=\frac{Wab}{2L^2}(L+b-a)=\frac{Wab^2}{L^2}$$

For $B, f=f_B$ and $x=\frac{L}{2}$

$$\therefore\ f_B=\frac{Wab}{2L}-\frac{Wab(b-a)}{L^3}\cdot\frac{L}{2}=\frac{Wab}{2L^2}(L-b+a)=\frac{Wa^2b}{L^2}$$

Hence $$M_A=f_A=+\frac{Wab^2}{L^2}$$

and $$M_B=f_B=+\frac{Wba^2}{L^2}$$

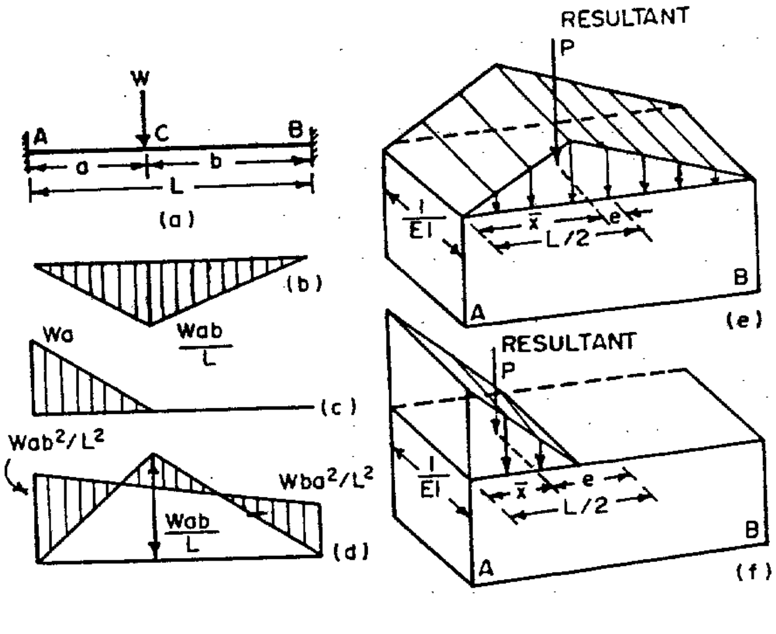


Fig. 11·4.

The final *B.M.* diagram is shown in Fig. 11·4 (*d*).

Alternative Solution

Alternatively, the problem can be solved by removing the support *B* completely, thus making the cantilever *AB* as the basic determinate struture. The corresponding *B.M.* diagram is shown in Fig. 11·4 (*c*) and the loaded analogous column in Fig. 11·4 (*f*).

The resultant or total load on the column is

$$P=-\int_0^L\frac{Msdx}{EI}=-\frac{1}{EI}\left[+\frac{1}{2}\ .\ a\ .\ Wa\right]=-\frac{Wa^2}{2EI}$$

Distance of C.G. from $A=\bar{x}=\frac{a}{3}$

Eccentricity $e=\frac{L}{2}-\frac{a}{3}=\frac{3L-2a}{6}=\frac{a+3b}{6}$

Area $A=\frac{L}{EI}$; $I_{YY}=\frac{L^3}{12\ EI}$

$$M_{YY}=P\ .\ e=-\frac{Wa^2}{2EI}\cdot\frac{a+3b}{6}$$

The stress at any point in the column is given by

$$f=\frac{P}{A}\pm\frac{M_{YY}}{I_{YY}}.\ x$$

$$=\left(-\frac{Wa^2}{2EI}\cdot\frac{EI}{L}\right)\pm\left(-\frac{Wa^2}{2EI}\cdot\frac{a+3b}{6}\cdot\frac{12EI}{L^3}.\ x\right)$$

$$=\left(-\frac{Wa^2}{2L}\right)\pm\left\{-\frac{Wa^2}{L^3}(a+3b)x\right\}$$

For A, $x=L/2$ and $f=f_A$

$$\therefore\ f_A=\left(-\frac{Wa^2}{2L}\right)+\left\{-\frac{Wa^2}{L^3}(a+3b).\ \frac{L}{2}\right\}$$

$$-\frac{Wa^2}{2L}-\frac{Wa^2}{2L^2}(a+3b)=-\frac{Wa^2}{2L^2}(L+a+3b)=-\frac{Wa^2}{L^2}(a+2b)$$

For B, $x=L/2$ and $f=f_B$

$$\therefore\ f_B=\left(-\frac{Wa^2}{2L}\right)-\left\{-\frac{Wa^2}{L^3}(a+3b)\ .\ \frac{L}{2}\right\}$$

or $$f_B=-\frac{Wa^2}{2L}+\frac{Wa^2}{2L^2}(a+3b)=\frac{Wa^2}{2L^2}(a+3b-L)=\frac{Wa^2b}{L^2}$$

Hence $M_A=M_I+M_S=f_A+M_S=-\frac{Wa^2}{L^2}(a+2b)+Wa$

$$=\frac{Wa}{L^2}(L^2-a^2-2ab)=\frac{Wab^2}{L^2}$$

and $M_B=M_I+M_S=f_B+M_S=\frac{Wa^2b}{L^2}+0=\frac{Wa^2b}{L^2}$.

Example 11·3. *A beam AB of span L is fixed at both the ends and carries a point load W at its centre. The moment of inertia of first half portion of the beam is 2I and that of the next half is I. Compute the fixed end moments.*

Solution

Let us first solve the problem by taking the simply-supported beam as the basic determinate structure. The corresponding B.M.

diagram is shown in Fig. 11·5 and the loaded analogous beam is shown in Fig. 11·5 (*d*).

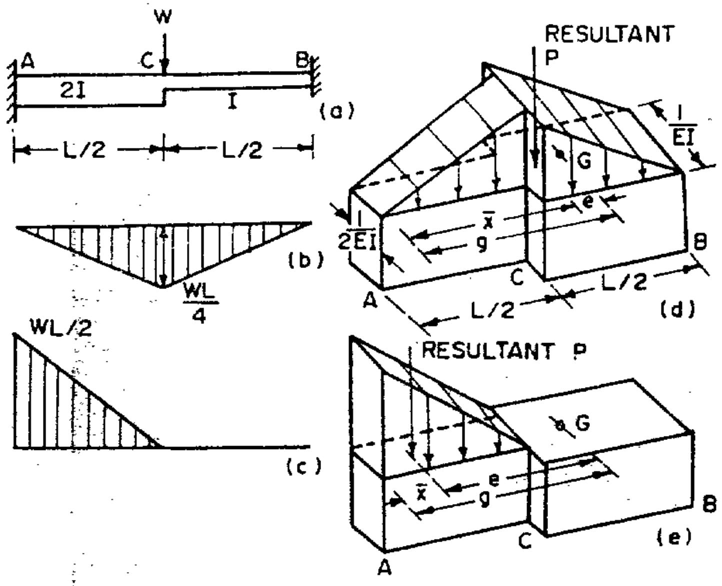


Fig. 11·5.

The total load $= -\int_0^L \frac{M_s dx}{EI}$

$$= -\left[\left\{\frac{1}{2EI}\cdot\frac{1}{2}\cdot\frac{L}{2}\left(-\frac{WL}{4}\right)\right\}+\left(\frac{1}{EI}\cdot\frac{1}{2}\cdot\frac{L}{2}\left(-\frac{WL}{4}\right)\right)\right]$$

$$=\frac{WL^2}{32EI}+\frac{WL^2}{16EI}=\frac{3WL^2}{32EI}$$

The C.G. $\bar{x}$ of the load, measured from *A*, is given by

$$\frac{3WL^2}{32EI}\,\bar{x}=\frac{WL^2}{32EI}\left(\frac{2}{3}\cdot\frac{L}{2}\right)+\frac{WL^2}{16EI}\left(\frac{L}{2}+\frac{1}{3}\cdot\frac{L}{2}\right)$$

From which $\bar{x}=\frac{5L}{9}$

Properties of the analogous column :

Area *A* of the analogous column

$$=\left(\frac{1}{2EI}\cdot\frac{L}{2}\right)+\left(\frac{1}{EI}\cdot\frac{L}{2}\right)=\frac{3L}{4EI}$$

The distance *g* of the centroid of the column section is given by

$$\frac{3L}{4EI}\cdot g=\left(\frac{1}{2EI}\cdot\frac{L}{2}\right)\left(\frac{L}{4}\right)+\left(\frac{1}{EI}\cdot\frac{L}{2}\right)\left(\frac{L}{2}+\frac{L}{4}\right)$$

From which $g=\frac{7L}{12}$

∴ Eccentricity *e* of the load $=\frac{7L}{12}-\frac{5L}{9}=\frac{L}{36}$

$$I_{AA}=\frac{1}{3}\cdot\frac{1}{EI}\cdot L^3-\frac{1}{3}\left(\frac{1}{2EI}\right)\left(\frac{L}{2}\right)^3=\frac{5}{16}\frac{L^3}{EI}$$

$$I_{YY}=I_{AA}-Ag^2=\frac{5}{16}\frac{L^3}{EI}-\frac{3L}{4EI}\left(\frac{7L}{12}\right)^2=\frac{11}{192}\frac{L^3}{EI}$$

$$M_{YY}=P\,.\,e=\frac{3WL^2}{32EI}\times\frac{L}{36}=\frac{WL^3}{384EI}$$

The stress *f* at any point is given by

$f=\frac{P}{A}\pm\frac{M_{YY}\,.\,x}{I_{YY}}$, where *x* is measured from *y-y* axis

$$=\frac{3WL^2}{32EI}\cdot\frac{4EI}{3L}\cdot\pm\frac{WL^3}{384EI}\cdot\frac{192}{11}\frac{EI}{L^3}\cdot x$$

$$=\frac{WL}{8}\pm\frac{W}{22}x$$

For *A*, $x=g=\frac{7}{12}L$

Hence $f_A=\frac{WL}{8}+\frac{W}{22}\cdot\frac{7}{12}L=\frac{5}{33}WL$

For *B*, $x=L-g=L-\frac{7}{12}L=\frac{5}{12}L$

∴ $f_B=\frac{WL}{8}-\frac{W}{22}\cdot\frac{5}{12}L=\frac{7}{66}WL$

Hence $M_A=M_i+M_s=f_A+M_s=\frac{5}{33}WL+0=+\frac{5}{33}WL$

and $M_B=M_i+M_s=f_B+M_s=\frac{7}{66}WL+0=+\frac{7}{66}WL.$

Alternative Solution

Let us now solve the problem by taking the cantilever as the basic determinate structure. The *Ms* diagram is shown in Fig. 11·5 (*c*) and the loaded column in Fig. 11·5 (*e*).

The total load $=P=-\int_0^L\frac{M_s dx}{EI}$

$$=-\left[\frac{1}{2EI}\cdot\frac{1}{2}\cdot\frac{L}{2}\left(+\frac{WL}{2}\right)\right]=-\frac{WL^2}{16EI}$$

The C.G. $\bar{x}$ of the load, measured from A, is given by

$$\bar{x}=\frac{1}{3}\left(\frac{L}{2}\right)=\frac{L}{6}$$

For the previous solution :

$$A=\frac{3L}{4EI}$$

$$g=\frac{7L}{12}$$

$$I_{YY}=\frac{11}{192}\ \frac{L^3}{EI}$$

The eccentricity $e=g-\bar{x}=\dfrac{7L}{12}-\dfrac{L}{6}=\dfrac{5}{12}L$

$$\therefore \quad M_{YY}=P\,.\,e=-\frac{WL^2}{16\,EI}\cdot\frac{5}{12}L=-\frac{5WL^3}{192EI}$$

The stress f at any point is given by

$f=\dfrac{P}{A}\pm\dfrac{M_{YY}\,.\,x}{I_{YY}}$, where x is measured from y-y axis.

$$=\left(-\frac{WL^2}{16EI}\cdot\frac{4EI}{3L}\right)\pm\left(-\frac{5WL^3}{192EI}\cdot\frac{192EI}{11L^3}\,.\,x\right)$$

$$=\left(-\frac{WL}{12}\right)\pm\left(-\frac{5Wx}{11}\right)$$

For A, $x=g=\dfrac{7L}{13}$

$$\therefore \quad f_A=\left(-\frac{WL}{12}\right)+\left(-\frac{5W}{12}\cdot\frac{7L}{12}\right)=-\frac{WL}{12}-\frac{35WL}{12\times 11}=\frac{23}{66}WL$$

For B, $x=L-g=L-\dfrac{7L}{12}=\dfrac{5L}{12}$

$$\therefore f_B=\left(-\frac{WL}{12}\right)=\left(-\frac{5W}{11}\cdot\frac{5L}{12}\right)=-\frac{WL}{12}+\frac{25WL}{12\times 11}=\frac{7WL}{66}$$

Hence $M_A=M_I+M_S=-\dfrac{23}{66}WL+\dfrac{WL}{2}=+\dfrac{5}{33}WL$

$$M_B=M_I+M_S=+\frac{7WL}{66}+0=+\frac{7WL}{66}$$

Example 11·4. *An encastre beam of span L carries a uniformly distributed load w. The second moment of area of the central half of the beam is I_1 and that of the end portion is I_2. Neglecting the weight of the beam itself find the ratio of I_2 to I_1 so that the magnitude of the bending moment at the centre is one-third of that of the fixing moments at the ends.* (U.L.)

Solution

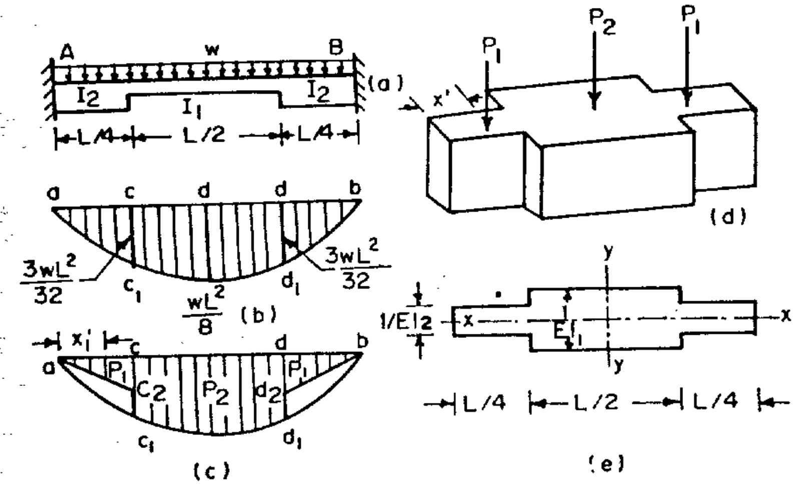


Fig. 11·6.

Taking simply-supported beam as the basic determinate structure, the Ms diagram is shown in Fig. 11·6(b). The ordinates of Ms diagram at C and D are

$$Ms \text{ at } C \text{ and } D=-\frac{wL}{4}\left(\frac{L}{4}\right)+\frac{w}{2}\left(\frac{L}{4}\right)^2=-\frac{3wL^2}{32}$$

$$Ms \text{ at the centre of the beam}=-\frac{wL^2}{8}$$

Fig. 11·6(c) shows the $\dfrac{Ms}{EI}$ diagram which also represents the load on the analogous column.

Let area $acc_2=bdd_2=$load P_1

and area $cc_1d_1d=$Load P_2

$$\therefore \quad P_1=-\int_0^{L/4}\frac{Ms\,dx}{EI_2}=-\frac{1}{EI_2}\int_0^{L/4}\left(-\frac{wL}{2}x+\frac{wx^2}{2}\right)dx$$

$$=\frac{1}{EI_2}\left[\frac{wL}{4}\left(\frac{L}{4}\right)^2-\frac{w}{6}\left(\frac{L}{4}\right)^3\right]=\frac{5wL^3}{384EI_2}$$

To get $\bar{x}$ of P_1 measured from a, we have

$$P_1\bar{x}_1=-\frac{1}{EI_2}\int_0^{L/4}\left(-\frac{wLx}{2}+\frac{wx^2}{2}\right)xdx$$

or $\dfrac{5}{384}\ \dfrac{wL^3}{EI_2}\ \bar{x}_1=\dfrac{1}{EI_2}\left[\dfrac{wL}{6}\left(\dfrac{L}{4}\right)^3-\dfrac{w}{8}\left(\dfrac{L}{4}\right)^4\right]=\dfrac{13}{16\times 384}\cdot\dfrac{wL^4}{EI_2}$

From which $\bar{x}_1=\frac{13}{80}L$.

Similarly, Load $P_2=-\frac{1}{EI_2}$ (area cc_1d_1d)

$$=-\frac{1}{EI_1}\text{ (area } ac_1d_1b-2acc_1)$$

$$=-\frac{1}{EI_1}\left[-\left\{\frac{2}{3}L\frac{wL^2}{8}-2\times\frac{5wL^3}{384}\right\}\right]=\frac{11}{192}\frac{wL^3}{EI_1}$$

And $\bar{x}_2$, measured from a is equal to $\frac{L}{2}$.

The net loadings P_1, P_2 and P_1 are shown on the analogous column in Fig. 11·5 (d).

Resultant load $P=2P_1+P_2=\frac{5wL^3}{192EI_2}+\frac{11}{192}\frac{wL^3}{EI_1}$

and $\bar{x}=\frac{L}{2}$, due to symmetry.

Area $A=\left(2\cdot\frac{1}{EI}\cdot\frac{L}{4}\right)+\left(\frac{1}{EI_1}\cdot\frac{L}{2}\right)=\frac{L}{2EI_2}+\frac{L}{2EI_1}$

Due to symmetry, $M_{YY}=0$.

$$\therefore\ f=f_A=f_B=\frac{\left(\frac{5wL^3}{192EI_2}+\frac{11}{192}\frac{wL^3}{EI_1}\right)}{\left(\frac{L}{2EI_2}+\frac{L}{2EI_1}\right)}$$

$$=\frac{wL^2}{96}\left\{\frac{\frac{5}{I_2}+\frac{11}{I_1}}{\frac{1}{I_2}+\frac{1}{I_1}}\right\}=\frac{wL^2}{96}\left(\frac{5I_1+11I_2}{I_1+I_2}\right)$$

Now M_S at the centre $=\frac{wL^2}{8}$

M_I at the centre $=f$

$\therefore$ M (net at the centre) $=M_S-M_I=\frac{wL^2}{8}-f$ (numerically)

M (at ends) $=M_I+M_S=M_I+0=f$ (numerically)

As per given condition

$$\frac{wL^2}{8}-f=\frac{f}{3}$$

$$\frac{wL^2}{8}=f+\frac{f}{3}=\frac{4}{3}f=\frac{4}{3}\cdot\frac{wL^2}{96}\left(\frac{5I_1+11I_2}{I_1+I_2}\right)$$

or $9I_2+9I_1=11I_2+5I_1$

From which $\frac{I_1}{I_2}=\frac{1}{2}$

11·4. PORTAL FRAMES

Portal frames having total indeterminacy upto third degree can also be solved by the column analogy method. The analogous column, in plan, has the same outline as that of the portal frame, the width equal to $\frac{1}{EI}$. If I is different for different members of the frame, the width $\frac{1}{EI}$ will also change correspondingly. The loading on the top of the analogous column will be equal to $-\frac{M_S}{EI}$ diagram. Any basic determinate structure can be derived by removing the redundants and the M_S diagram can be plotted. The loading

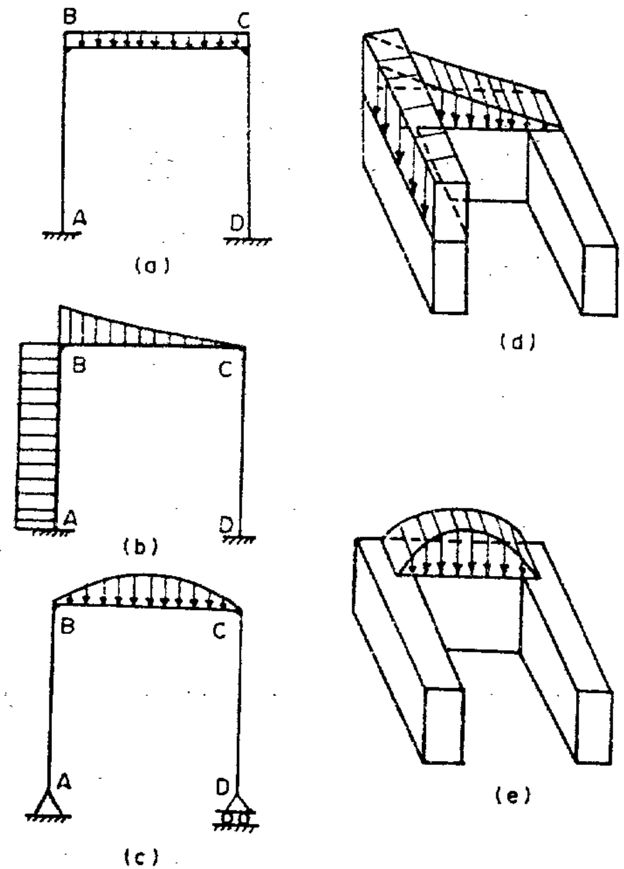


Fig. 11·7.

is *downward* if M_S causes compression on the *outside* of the frame. The pressure on the bottom of the analogous column at any point will then give the indeterminate moment M_I at the point. The final moment is then given by

$$M = M_S + M_I$$

Fig. 11·7(*a*) shows a portal frame fixed at ends A and D and loaded with uniformly distributed load on BC. Fig. 11·7(*b*) shows a basic determinate structure derived by removing the end D completely, the M_S diagram being drawn on the frame itself. Fig. 11·7(*d*) shows the corresponding load on the analogous column. Another basic determinate structure can be obtained by making end A hinged and supporting end D on rollers as shown in Fig. 11·7(*c*) along with the M_S diagram. Fig. 11·7(*e*) shows the corresponding loading on the analogous column. Fig. 11·7(*c*) and (*e*) also correspond to the basic determinate structure derived by treating the beam BC as hinged (or simply supported) at B and C. In a similar manner, basic determinate structure can be obtained for other end conditions of portal frames.

11·5. THE GENERALISED COLUMN FLEXURE FORMULA

Upto this stage, we have analysed a column which is symmetrical about both the principal centroidal axes, as in § 11·3. While analysing the portal frames, we come across such analogous columns which are not symmetrical about the principal centroidal axes. In a generalised case, the stress f at any point x, y in a column section can be expressed as

$$f = a + bx + cy \qquad (1)$$

where a, b, c are constants to be determined.

Hence load $P = \int f \,.\, dA = a\int dA + b\int x \,.\, dA + c\int y \,.\, dA.$

Since the principal axes are through the centroid, and x and y are measured with reference to these, $\int x \,.\, dA$ and $\int y \,.\, dA$ are each zero.

Hence $\qquad P = a\int dA,$

or

$$a = \frac{P}{\int dA} = \frac{P}{A} \qquad (2)$$

Also, $M_{XX} = \int (f \,.\, dA)y = \int (a + bx + cy)y \, dA$

$$= a\int y \, dA + b\int xy \, dA + c\int y^2 \, dA$$

$$= 0 + bI_{XY} + cI_{XX} \qquad (3)$$

Since $\int y \, dA = 0$, $\int xy \, dA = I_{XY}$ and $\int y^2 \, dA = I_{XX}$

Similarly, $M_{YY} = \int (f \,.\, dA)x = \int (a + bx + cy)x \, dA$

$$= a\int x \,.\, dA + b\int x^2 \, dA + c\int xy \, dA$$

$$= 0 + bI_{YY} + c \,.\, I_{XY} \qquad (4)$$

Since $\int x \, dA = 0$, $\int x^2 \, dA = I_{YY}$ and $\int xy \, dA = I_{XY}$

Solving equations (3) and (4) for b and c, we get

$$b = \frac{M_{YY} - M_{XX}\dfrac{I_{XY}}{I_{XX}}}{I_{YY} - \dfrac{I_{XY}^2}{I_{XX}}} = \frac{M'_{YY}}{I'_{YY}} \qquad (5)$$

$$c = \frac{M_{XX} - M_{YY}\dfrac{I_{XY}}{I_{YY}}}{I_{XX} - \dfrac{I_{XY}^2}{I_{YY}}} = \frac{M'_{XX}}{I'_{XX}} \qquad (6)$$

Substituting the values of a, b and c in (1), we get

$$f = \frac{P}{A} + \left[\frac{M_{YY}\, I_{XX} - M_{XX}\, I_{XY}}{I_{YY} \,.\, I_{XX} - I_{XY}^2}\right]x + \left[\frac{M_{XX}\, I_{YY} - M_{YY} \,.\, I_{XY}}{I_{XX} \,.\, I_{YY} - I_{XY}^2}\right]y \qquad (11·4)$$

If, however, the portal frame is such that one of the principal centroidal axis is the axis of symmetry, I_{XY} becomes zero. Equation 11·4 then reduces to the form

$$f = \frac{P}{A} + \frac{M_{YY}}{I_{YY}} \,.\, x + \frac{M_{XX}}{I_{XX}} \,.\, y \qquad (11·5)$$

Sign Convention

While applying equation 11·4 or 11·5 the following sign convention is to be followed rigidly : x is reckoned positive when measured to the right, and negative when measured to the left of the y-y axis. Similarly, y is reckoned positive when measured upwards and negative when measured downwards to the x-x axis. Both x and y will be measured with centroid of the column section as the origin. This is represented in Fig. 11·8.

The length of every member of the analogous column will be equal to the length of the corresponding member of the frame, and its width will be equal to $\frac{1}{EI}$ where I is the moment of inertia of the member. Since the width $\frac{1}{EI}$ is extremely small in comparison

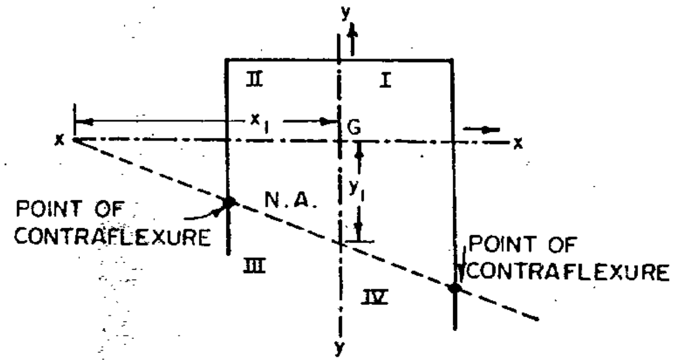


Fig. 11·8.

to its length, the column section may be looked upon as a line diagram when calculating the centroid of the section, or calculating I_{XX} and I_{YY}.

Neutral Axis of Analogous Column :

The neutral axis of the analogous column is that axis, at every point of which the stress f is zero. Hence the points of intersection of N.A. with the legs of the portal will be the points of contraflexure since f and hence M_t will be zero at these points. The position of the N.A. (Fig. 11·8) can very easily be located by calculating the co-ordinates x_1 and y_1 in which it intersects the x-axis and y-axis respectively. The co-ordinates x_1 and y_1 can be calculated on the premise that stress f on each of these points, *i.e.* $(x_1, 0)$ and $(0, y_1)$ is zero.

11·6. PORTAL FRAME WITH HINGED LEG(S)

A hinge represents the possibility of an indefinitely large rotation. Since it offers no resistance to rotation, the flexural rigidity EI at a hinge is zero. This results in the following :

(1) Since the width of the analogous column is $\frac{1}{EI}$, its area of cross-section corresponding to the hinge is infinite.

(2) If the portal frame has one end fixed and the other end hinged, both axes will pass through the hinge as shown in Fig.

11·9 (*a*). The area of the analogous column will be infinite, and is assumed to be concentrated at the hinge.

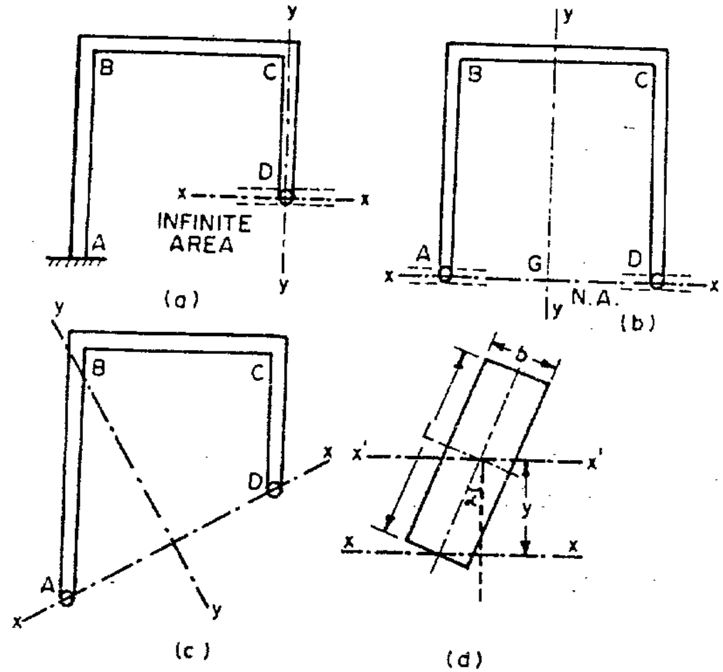


Fig. 11·9.

(3) If the portal frame has both legs hinged at the ends, the centroid of the column section lies midway between the hinges. The area A and I_{YY} of the analogous column becomes infinite. The N.A. of the column section will pass through the hinges, as shown in Fig. 11·9 (*b*).

The fibre stress at any point, for case of Fig. 11·9 (*b*) is then given by

$$f = \frac{P}{\infty} + \frac{M_{YY} \cdot x}{\infty} + \frac{M_{XX} \cdot y}{I_{XX}}$$

$$= \frac{M_{XX} \cdot y}{I_{XX}} \qquad (11·6)$$

(4) If both the legs are hinged at the base, but the hinges are not at the same level, the x-x axis, and hence the N.A., will pass through both the hinges as shown in Fig. 11·9 (*c*). The area A and I_{YY} of the analogous column becomes infinite and hence the stress

f at any point is given by equation 11·6, *i.e.*

$$f=\frac{Mxx\,.\,y}{Ixx}$$

Since the hinges are not at the same level, the axis x-x is inclined to all the three members, and Ixx can be calculated with reference to Fig. 11·9 (d).

Let the longitudinal axis of a member of length L and width b be inclined at angle α to the line perpendicular the axis x-x passing through its base. If the x'-x' axis, parallel to x-x axis, passes through the centroid of the section, we have

$$Ix'x'=\frac{bL^3}{12}\cos^2\alpha$$

and $\quad Ixx=Ix'x'+ay^2$

where $\quad a$ = area of cross-section of the member

and $\quad y$ = perpendicular distance between the two axes.

Example 11·5. *A portal frame ABCD is fixed at A and D and has stiff joints B and C. It carries a uniformly distributed load w per unit length on BC. Plot the bending moment diagram. EI is constant for the whole of the frame.*

Solution

The basic determinate structure is derived by treating BC hinged at B and C. The Ms diagram is parabola having maximum ordinate $=-\frac{w(2L)^2}{8}=-\frac{wL^2}{2}$ at the mid-span. The analogous column loaded with $-\frac{Ms}{EI}$ diagram is shown in Fig. 11·10 (b). The analogous column is shown in plan, and fully dimensioned in Fig. 11·10 (c).

Load $\quad P=-\int_0^L\frac{Msdx}{EI}=-\frac{1}{EI}\left[\frac{2}{3}\times(2L)\left(-\frac{wL^2}{2}\right)\right]$

$$=\frac{2}{3}\,\frac{wL^3}{EI}.$$

The point of the application of P, along with the principal centroidal axes, are shown in Fig. 11·10 (c). Since both the legs are having similar properties, *i.e.* same length, end conditions and constant EI, y-y axis is the axis of symmetry.

Area $\quad A=\frac{1}{EI}\,[L+L+2L]=\frac{4L}{EI}$

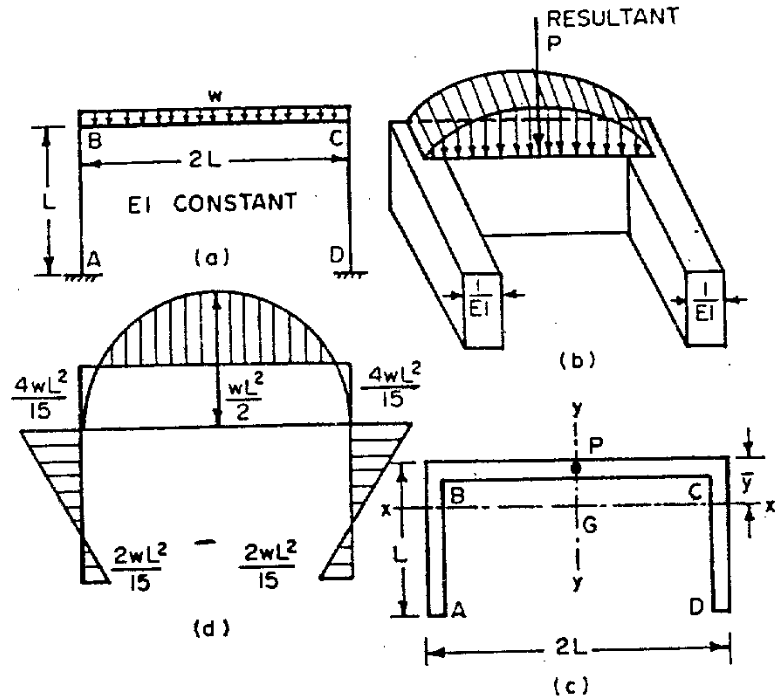


Fig. 11·10.

To find the position of x-x axis, take moments about BC

Thus $\frac{4L}{EI}\,.\,\bar{y}=\left(\frac{L}{EI}\times\frac{L}{2}\right)+\left(\frac{L}{EI}\times\frac{L}{2}\right)=\frac{L^2}{EI}$

(the contribution of the section BC being negligible)

$$\therefore\qquad \bar{y}=\frac{L}{4}$$

$$I_{BC}=\frac{1}{3}\,\frac{1}{EI}\,(L)^3+\frac{1}{3}\,\frac{1}{EI}\,(L)^3=\frac{2L^3}{3EI}$$

(the contribution of the section BC being negligible)

$$Ixx=I_{BC}-A\bar{y}^2=\frac{2L^3}{3EI}-\frac{4L}{EI}\left(\frac{L}{4}\right)^2=\frac{5}{12}\,\frac{L^3}{EI}$$

$$I_{YY}=\frac{1}{12}\times\frac{1}{EI}(2L)^3+2\times\frac{L}{EI}\,(L)^2=\frac{8}{3}\,\frac{L^3}{EI}$$

(the contribution of AB and CD about their own axis being negligible).

$Ixx=0$, since y-y axis is the axis of symmetry.

Hence the fibre stress at any point is given by

$$f=\frac{P}{A}+\frac{M_{YY}\,.\,x}{I_{YY}}+\frac{M_{XX}}{I_{XX}}\,.\,y$$

$$M_{XX}=P\bar{y}=\frac{2}{3}\;\frac{wL^3}{EI}\,.\,\frac{L}{4}=\frac{wL^4}{6EI}$$

$$M_{YY}=0$$

$$\therefore\quad f=\frac{2}{3}\;\frac{wL^3}{EI}\,.\,\frac{EI}{4L}+\frac{wL^4}{6EI}\,.\,\frac{12EI}{5L^3}\,.\,y$$

$$=\frac{wL^2}{6}+\frac{2wL}{5}\,y$$

At B and C, $y=+L/4$

$$\therefore\quad f_B=f_C=\frac{wL^2}{6}+\frac{2wL^2}{20}=+\frac{4}{15}wL^2$$

At A and D, $y=-\frac{3}{4}L$

$$\therefore\quad f_A=f_D=\frac{wL^2}{6}-\frac{2wL}{5}\times\frac{3}{4}\;L=-\frac{2}{15}wL^2$$

Since M_S is zero at each of the points A, B, C and D, we have

$$M_A=M_{IA}=f_A=-\frac{2}{15}wL^2$$

$$M_B=M_{IB}=f_B=+\frac{4}{15}wL^2$$

$$M_C=M_{IC}=f_C=+\frac{4}{15}wL^2$$

$$M_D=M_{ID}=f_D=-\frac{2}{15}wL^2$$

The final B.M.D. is shown in Fig. 11·10 (*d*).

Example 11·6. *A portal frame ABCD is hinged at A and D, and has rigid joints at B and C. The frame is loaded as shown in Fig. 11·11 (a) . Plot the bending moment diagram for the frame.*

Solution

The basic determinate structure is derived by treating BC hinged at B and C. The M_S diagram, also the load diagram, is shown in Fig. 11·11(*b*). The ordinate of M_S diagram and the load

$$=-\frac{4\times1{\cdot}5\times3}{1{\cdot}5}=-4\text{ kN-m}$$

The total load $=-\int\frac{M_S}{EI}\,dx=-\frac{1}{EI}\left[\frac{1}{2}\times4{\cdot}5(-4)\right]=\frac{9}{EI}$

acting at $\frac{1}{3}(1{\cdot}5+4{\cdot}5)=2$ m from A.

Since ends A and D are hinged, the analogous column will have infinite area at the hinges and the remaining area of legs and beam becomes negligible. Hence the x-x axis will pass through the hinges,

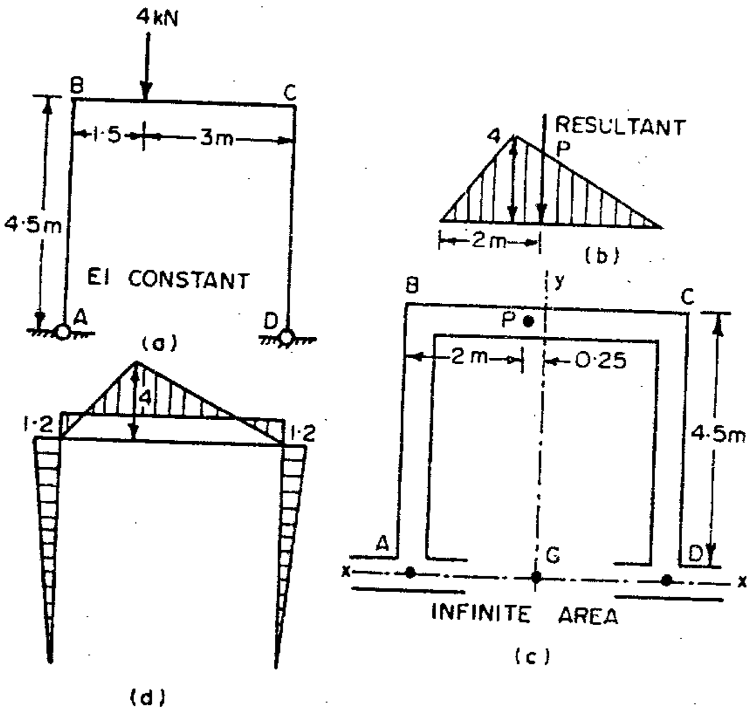


Fig. 11·11.

and the y-y axis is the axis of symmetry. The centroid G is mid-way between the hinges, as shown in Fig. 11·11 (*c*).

Hence $I_{YY}=\infty$, and the stress is given by equation 11·6

i.e.
$$f=\frac{M_{XX}}{I_{XX}}\,.\,y$$

$$M_{XX}=\frac{9}{EI}\times4{\cdot}5=\frac{40{\cdot}5}{EI}$$

$$I_{XX}=2\left\{\frac{1}{3}\times\frac{1}{EI}(4{\cdot}5)^3\right\}+\left\{\left(\frac{4{\cdot}5}{EI}\right)(4{\cdot}5)^2\right\}$$

$$=\frac{151{\cdot}9}{EI}$$

$$\therefore \qquad f = \frac{40{\cdot}5}{EI} \times \frac{EI}{151{\cdot}9}\, y = \frac{y}{3{\cdot}75}$$

For A and D, $y=0$

$$\therefore \qquad f_A = f_D = M_A = M_D = 0$$

For B and C, $y = 4{\cdot}5$ m

$$\therefore \qquad f_B = f_C = \frac{4{\cdot}5}{3{\cdot}75} = 1{\cdot}2$$

Hence $\qquad M_{IB} = M_{IC} = +1{\cdot}2$ kN-m

Since M_S is zero at B and C.

$$M_B = M_C = 1{\cdot}2 \text{ kN-m}$$

The B.M. diagram is shown in Fig. 11·11(d).

Example 11·7. *A portal frame ABCD is fixed at A and D, and has rigid joints at B and D, and is loaded as shown. Plot the bending moment diagram for the frame.*

Solution.

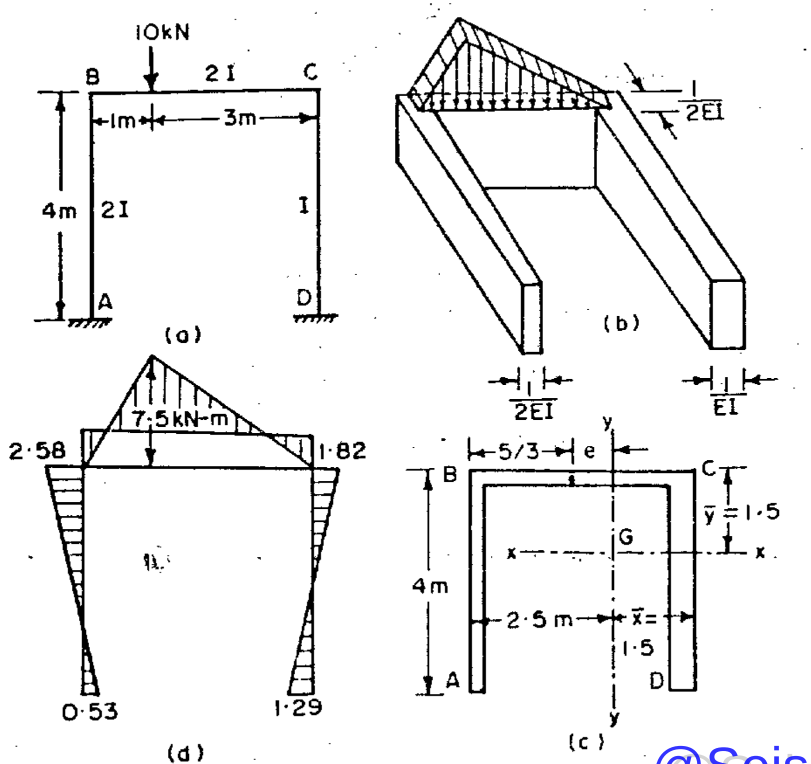


Fig. 11·12

The basic determinate structure is obtained by treating the beam BC hinged at B and C. The M_S diagram will be a triangle having a maximum ordinate $= -\dfrac{10\times1\times3}{4} = -7{\cdot}5$ kN-m under the load.

$$\therefore \qquad P = -\int \frac{M_S}{EI}\,dx = -\frac{1}{2EI}\left[\frac{1}{2}\times4\times(-7{\cdot}5)\right] = \frac{7{\cdot}5}{EI}$$

acting at $\dfrac{1}{3}(1+4) = \dfrac{5}{4}$ m from B. The point of application of the resultant load P is shown in Fig. 11·12 (c), where the analogous column section is shown fully dimensioned along with the position of the centroidal axes.

Since both the legs have different moment of inertia, the column will be unsymmetrical about both the axes, and the stress at any point is given by Eq. 11·4.

Area $\qquad A = \dfrac{4}{2EI} + \dfrac{4}{2EI} + \dfrac{4}{EI} = \dfrac{8}{EI}$

To locate the centroid G, or to determine $\bar{x}$ and, $\bar{y}$, take moment about CD and BC.

Thus
$$\bar{x} = \frac{\dfrac{4}{2EI}\times2 + \dfrac{4}{2EI}\times4}{\dfrac{8}{EI}} = 1{\cdot}5 \text{ m}$$

$$\bar{y} = \frac{\dfrac{4}{2EI}\times2 + \dfrac{4}{EI}\times2}{\dfrac{8}{EI}} = 1{\cdot}5 \text{ m}$$

$$M_{XX} = P\bar{y} = \frac{7{\cdot}5}{EI}\times1{\cdot}5 = \frac{11{\cdot}25}{EI}$$

$$e = 2{\cdot}5 - \frac{5}{3} = \frac{2{\cdot}5}{3}$$

$$M_{YY} = \frac{7{\cdot}5}{EI}\left(-\frac{2{\cdot}5}{3}\right) = -\frac{6{\cdot}25}{EI}$$

(the minus sign with e being used because the eccentricity is to the left of y-y axis).

The calculations of various moments of inertia etc., are done in the table below, I_{OX} and I_{OY} denote the moments of inertia of individual members about their own centroidal axes.

Member	AB	BC	CD	Sum
Length	4 m	4 m	4 m	
Area	$\frac{4}{2EI}=\frac{2}{EI}$	$\frac{4}{2EI}=\frac{2}{EI}$	$\frac{4}{EI}$	$\frac{8}{EI}$
$\bar{x}$	−2·5 m	−0·5 m	+1·5	
$\bar{y}$	−0·5 m	+1·5 m	−0·5 m	
I_{OX}	$\frac{1}{12}\cdot\frac{1}{2EI}(4)^3=\frac{8}{3EI}$	0	$\frac{1}{12}\cdot\frac{1}{EI}(4)^3=\frac{16}{3EI}$	$\frac{8}{EI}$
I_{OY}	0	$\frac{1}{12}\cdot\frac{1}{2EI}(4)^3=\frac{8}{3EI}$	0	$\frac{8}{3EI}$
$A\bar{x}^2$	$\frac{12\cdot5}{EI}$	$\frac{0\cdot5}{EI}$	$\frac{9}{EI}$	$\frac{22}{EI}$
$A\bar{y}^2$	$\frac{0\cdot5}{EI}$	$\frac{4\cdot5}{EI}$	$\frac{1}{EI}$	$\frac{6}{EI}$
$A\bar{x}\bar{y}$	$+\frac{2\cdot5}{EI}$	$-\frac{1\cdot5}{EI}$	$-\frac{3}{EI}$	$-\frac{2}{EI}$

Now $\quad I_{XX}=\Sigma(I_{OX}+A\bar{y}^2)=\frac{8}{EI}+\frac{6}{EI}=\frac{14}{EI}$

$$I_{YY}=\Sigma(I_{OY}+A\bar{x}^2)=\frac{8}{3EI}+\frac{22}{EI}=\frac{74}{3EI}$$

$$I_{XY}=\Sigma A\bar{x}\bar{y}=-\frac{2}{EI}$$

Hence we have

$$\frac{M_{YY}\,.\,I_{XX}-M_{XX}\,.\,I_{XY}}{I_{YY}\,.\,I_{XX}-I_{XY}^2}=\frac{\frac{-6.25}{EI}\times\frac{14}{EI}-\frac{11\cdot25}{EI}\times\left(-\frac{2}{EI}\right)}{\frac{74}{3EI}\times\frac{14}{EI}-\left(-\frac{2}{EI}\right)^2}$$

$$=\frac{-87\cdot5+22\cdot5}{345-4}=-0\cdot19$$

$$\frac{M_{XX}\,.\,I_{YY}-M_{YY}.I_{XY}}{I_{YY}\,.\,I_{XX}-I_{XY}^2}=\frac{\frac{11\cdot25}{EI}\left(\frac{74}{3EI}\right)-\left(-\frac{6\cdot25}{EI}\right)\left(-\frac{2}{EI}\right)}{\frac{74}{3EI}\times\frac{14}{EI}-\left(-\frac{2}{EI}\right)^2}$$

$$=\frac{277\cdot5-12\cdot5}{345-4}=+0\cdot777$$

and $\quad \frac{P}{A}=\frac{7\cdot5}{EI}\times\frac{EI}{8}=0\cdot938$

Substituting these values in equation 11·5, we get

$f=0\cdot938-0\cdot19x+0\cdot777y$

(1) For A, $\quad x=-2\cdot5$ m ; $y=-2\cdot5$ m

$\therefore f_B=M_{IA}=0\cdot938+(0\cdot19\times2\cdot5)-(0\cdot777\times2\cdot5)=-0\cdot53$ kN-m

(2) For B, $x=-2\cdot5$ m ; $y=+1\cdot5$ m

$\therefore f_B=M_{IB}=0\cdot938+(0\cdot19\times2\cdot5)\times(0\cdot777\times1\cdot5)=+2\cdot58$ kN-m

(3) For C, $x=+1\cdot5$ m ; $y=+1\cdot5$ m

$\therefore f_C=M_{IC}=0\cdot938-(0\cdot19\times1\cdot5)+(0\cdot777\times1\cdot5)=+1\cdot82$ kN-m

(4) For D, $x=+1\cdot5$ m ; $y=-2\cdot5$ m

$\therefore f_D=M_{ID}=0\cdot938-(0\cdot19\times15)-(0\cdot777\times2\cdot5)=-1\cdot29$ kN-m

Since M_S is zero at points A, B, C and D, we have

$$M_A=M_{IA}=-0\cdot53 \text{ kN-m}$$
$$M_B=M_{IB}=+2\cdot58 \text{ kN-m}$$
$$M_C=M_{IC}=+1\cdot82 \text{ kN-m}$$
$$M_D=M_{ID}=-1\cdot29 \text{ kN-m}$$

The B.M.D. is drawn in Fig. 11·12 *(d)*.

Example 11·8. *A portal frame ABCD has end A fixed and end D hinged, with rigid joints at B and C. Plot the bending moment diagram if the frame is loaded as shown in Fig. 11·13(a).*

Solution

The basic determinate structure is derived by making the beam BC simply supported. The $\frac{M_S}{EI}$ diagram and hence the load diagram is shown in Fig. 11·13 *(b)*. The ordinate of M_S under the load

$$=-\frac{6\times1\cdot5\times3}{4\cdot5}=-6 \text{ kN-m}$$

The total load $P=-\int\frac{M_S}{EI}dx=-\frac{1}{1\cdot5\;EI}\left\{\frac{1}{2}\times4\cdot5(-6)\right\}=\frac{9}{EI}$

Acting at $\frac{1}{3}(1\cdot5+4\cdot5)=2$ m from B.

Since the leg CD is hinged at D, the analogous column has infinite area concentrated at D, and the area of the remaining members is negligible. Hence both the centroidal axes will pass through the hinge D as shown in Fig. 11.13 (c) which illustrates the analogous

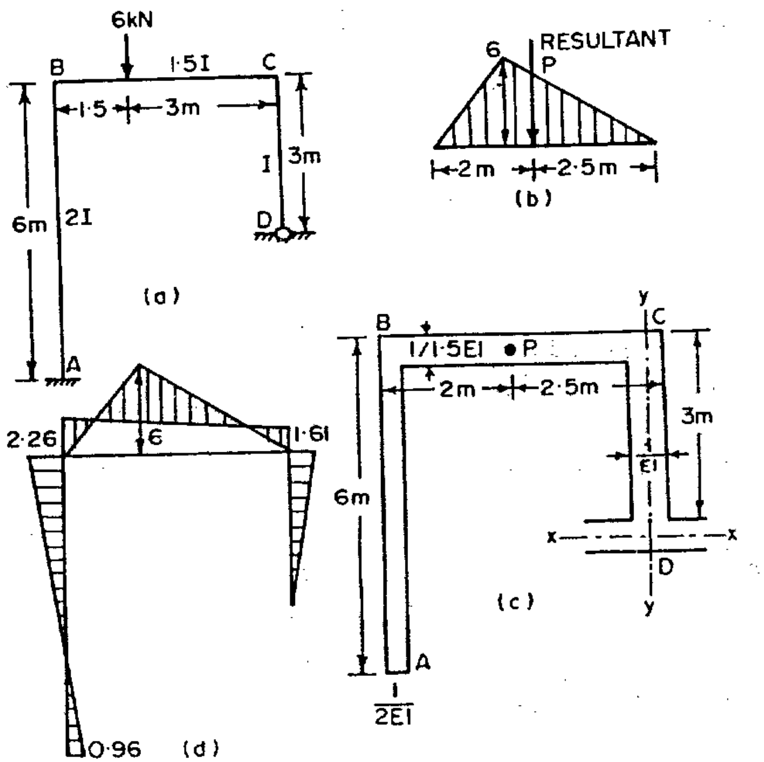


Fig. 11·13.

column fully dimensioned. P is the point of application of the resultant load on the analogous column.

Properties of the analogous column :

$A=\infty$

$$I_{XX}=\frac{1}{12}\cdot\frac{1}{2EI}(6^3)+\frac{1}{3}\cdot\frac{1}{EI}(3)^3+\frac{4.5}{1.5EI}(3)^2=\frac{45}{EI}$$

The contribution of BC about its own centroidal axis (being negligible).

$$I_{YY}=\frac{6}{2EI}(4.5)^2+\frac{1}{3}\,\frac{(4.5)^3}{1.5EI}=\frac{81}{EI}$$

$$I_{XY}=\Sigma A\bar{x}\bar{y}=\frac{6}{2EI}(-4.5)(0)+\frac{4.5}{1.5EI}(-2.25)(3)+\frac{3}{EI}(0)(1.5)$$

$$=-\frac{20.25}{EI}$$

$$M_{XX}=\frac{9}{EI}(3)=+\frac{27}{EI}$$

$$M_{YY}=\frac{9}{EI}(-2.5)=-\frac{22.5}{EI}$$

$$\frac{M_{YY}\,.\,I_{XX}-M_{XX}\,.\,I_{XY}}{I_{YY}\,.\,I_{XX}-I_{XY}^2}=\frac{\left(-\frac{22.5}{EI}\right)\left(\frac{45}{EI}\right)-\left(\frac{27}{EI}\right)\left(-\frac{20.25}{EI}\right)}{\left(\frac{81}{EI}\right)\left(\frac{45}{EI}\right)-\left(-\frac{20.25}{EI}\right)^2}$$

$$=-0.144$$

$$\frac{M_{XX}\,.\,I_{YY}-M_{YY}\,.\,I_{XY}}{I_{YY}\,.\,I_{XX}-I_{XY}^2}=\frac{\left(\frac{27}{EI}\right)\left(\frac{81}{EI}\right)-\left(\frac{22.5}{EI}\right)\left(-\frac{20.25}{EI}\right)}{\left(\frac{81}{EI}\right)\left(\frac{45}{EI}\right)-\left(-\frac{20.25}{EI}\right)^2}$$

$$=+0.535$$

Substituing the values in equation 11·5, we get

$$f=\frac{P}{\infty}+(-0.144)x+(0.535)y$$

or $$f=-0.144x+0.535y$$

For A, $x=-4.5$; $y=-3$

$\therefore$ $f_A=(-0.144)(-4.5)+(0.535)(-3)$

$=-0.96$ kN-m $=M_{IA}$

For B, $x=-4.5$; $y=+3$

$\because$ $f_B=(-0.144)(-4.5)+(0.535)(+3)$

$=+2.26$ kN-m $=M_{IB}$

For C, $x=0$; $y=+3$

$\therefore$ $f_C=0.535\times 3=+1.61$ kN-m $=M_{IC}$

For D, $x=0$; $y=0$

$\therefore$ $f_D=0=M_{ID}$

Since M_S is zero at the points A, B, C and D, we have

$M_A=M_{IA}=-0.96$ kN-m

$M_B=M_{IB}=+2.26$ kN-m

$M_C=M_{IC}=+1.61$ kN-m

$M_D=M_{ID}=$ zero.

The B.M.D. is shown in Fig. 11·13 (d).

Example 11·9. *A portal frame ABCD has legs hinged at A and D, and has stiff joints at B and C. Draw the B.M. for the loading shown in Fig. 11·14 (a).*

Solution

The basic determinate structure is derived by considering the beam BC simply supported. The M_S diagram will be a parabola having central ordinate equal to $-\dfrac{w(2L)^2}{8}=-\dfrac{wL^2}{2}$.

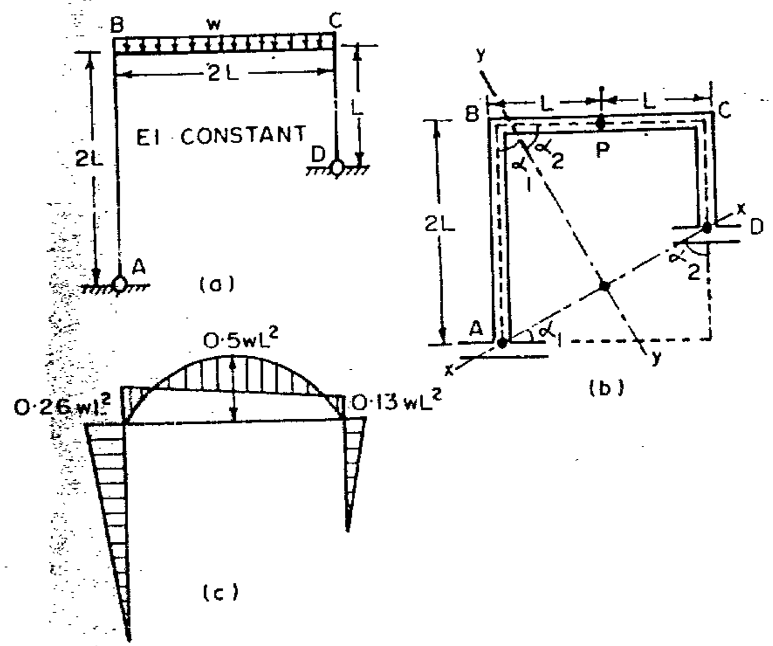


Fig. 11·14

The total load $P=-\int \dfrac{Ms}{EI}\,dx=-\dfrac{1}{EI}\left[\dfrac{2}{3}(2L)\left(-\dfrac{wL^2}{2}\right)\right]$

$$=\frac{2wL^3}{3EI}$$

Acting at a distance L from B and C.

The analogous column is shown in Fig. 11·14 (b). Since ends A and D are hinged, it has infinite area concentrated at A and D, and the axis x-x, therefore, passes through both the hinges. The axis y-y is perpendicular to x-x, and passes through a point midway between AD.

Properties of analogous column

$$A=\infty$$

$$I_{YY}=\infty$$

$I_{XY}=0$, since the reference axes are the principal axes of inertia.

M_{XX}=(Load)×(perpendicular distance between P and x-x)

$$=\frac{2wL^3}{3EI}\times(1\cdot5\,L\cos\alpha_1)=\frac{wL^4}{EI}\cos\alpha_1$$

$$=\frac{2}{\sqrt{5}}\,\frac{wL^4}{EI}$$

where α_1 is the angle made by the longitudinal axes of AB and CD with y-y axis, and α_2 is the angle made by the longitudinal axis of BC with y-y axis.

The moment of inertia of any inclined member [Fig. 11·9 (d)] is given by

$$I_{x'x'}=\frac{bL^3}{12}\cos^2\alpha$$

$$I_{XX}=I_{x'x'}+ay^2$$

The calculations of $I_{x'x'}$ and I_{XX} etc. for various members is done in tabular form below.

Member	*cos α*	*cos² α*	$I_{x'x'}$	*Area* (a)	y^2	I_{XX}
AB	$\frac{2}{\sqrt{5}}$	0·8	$\frac{1(2L)^3}{12EI}\times0\cdot8$	$\frac{2L}{EI}$	$(L\cos\alpha_1)^2=0\cdot8L^2$	$\frac{25\cdot6L}{12EI}$
BC	$\frac{1}{\sqrt{5}}$	0·2	$\frac{1(2L)^3}{12EI}\times0\cdot2$	$\frac{2L}{EI}$	$(3L\cos\alpha_2)^2=1\cdot8L^2$	$\frac{44\cdot8L^3}{12\,EI}$
CD	$\frac{2}{\sqrt{5}}$	0·8	$\frac{1\,(L)^3}{12EI}\times0\,8$	$\frac{L}{EI}$	$(\frac{1}{2}L\cos\alpha_1)^2=0.2L^2$	$\frac{3\cdot2L^3}{12EI}$

$$\Sigma I_{XX}=\frac{6\cdot14L^3}{EI}$$

The stress at any point is given by equation 11·5, *i.e.*

$$f=\frac{M_{XX}}{I_{XX}}y$$

$$=\frac{2wL^4}{\sqrt{5}EI}\times\frac{EI}{6\cdot14L^3}y=\frac{2wL}{6\cdot15\sqrt{5}}y$$

At A and D, $y=0$

$\therefore\quad f_A=f_D=0$

At B, $y=2L\cos\alpha_1=\dfrac{4L}{\sqrt{5}}$

$\therefore\quad f_B=\dfrac{2wL}{6\cdot14\sqrt{5}}\times\dfrac{4L}{\sqrt{5}}=0\cdot26\,wL^2$

At C, $\qquad y = L \cos \alpha_1 = \dfrac{2L}{\sqrt{5}}$

$$\therefore \qquad f_C = \frac{2wL}{6 \cdot 14\sqrt{5}} \times \frac{2L}{\sqrt{5}} = 0 \cdot 13\ wL^2$$

Since Ms is zero at each of the points A, B, C, and D, we have

$$M_A = M_{IA} = f_A = 0$$
$$M_B = M_{IB} = f_B = +0 \cdot 26\ wL^2$$
$$M_C = M_{IC} = f_C = +0 \cdot 13\ wL^2$$
$$M_D = M_{ID} = f_D = 0$$

Tne B.M.D. is shown in Fig. 11·14 (c).

Example 11·10. *A culvert shown in Fig. 11·15 (a) is of constant section throughout and carries a central load of 4 kN on BC. Determine the moments at the corners of the culvert and draw the B.M.D. Assume a uniformly distributed reactive force under the base.*

Solution

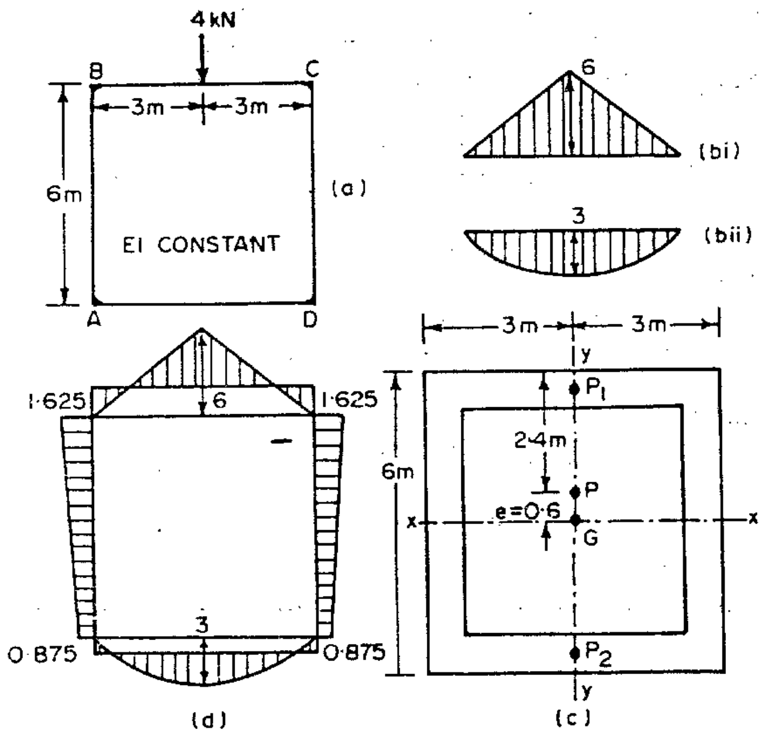


Fig. 11·15

The basic determinate structure is derived by treating the joints A, B, C and D, hinged. The Ms diagram for BC will be a triangle having maximum central ordinate of $-\dfrac{4\times 6}{4} = -6$ kN-m. The Ms diagram for AD will be a parabola having a maximum central ordinate of $-\dfrac{4\times 6}{8} = -3$ kN-m.

The total load P_1 on $BC = -\dfrac{1}{EI}\left[\dfrac{1}{2}\times 6(-6)\right] = \dfrac{18}{EI}$

The total load P_2 on $AD = -\dfrac{1}{EI}\left[\dfrac{2}{3}\times 6(-3)\right] = \dfrac{12}{EI}$.

The points of application of P_1 and P_2 are shown in Fig. 11·15 (c) along with the position of the centroidal axes of the analogous column.

The resultant load $P = P_1 + P_2 = \dfrac{18}{EI} + \dfrac{12}{EI} = \dfrac{30}{EI}$

Acting at $\left(\dfrac{12}{EI}\times 6\right)\dfrac{EI}{30} = 2 \cdot 4$ m from face BC.

$$e = 3 - 2 \cdot 4 = 0 \cdot 6 \text{ m}$$

$$M_{XX} = P.e = \frac{30}{EI}\times 0 \cdot 6 = \frac{18}{EI}$$

$$A = \frac{6+6+6+6}{EI} = \frac{24}{EI}$$

$$I_{XX} = 2\left(\frac{1}{12}\cdot\frac{1}{EI}\times 6^3\right) + 2\left(\frac{6}{EI}\times 3^2\right) = \frac{144}{EI}$$

Since y-y axis is the axis of symmetry, the stress f is given by

$$f = \frac{P}{A} + \frac{M_{XX}}{I_{XX}}\,y = \left(\frac{30}{EI}\times\frac{EI}{24}\right) + \left(\frac{18}{EI}\times\frac{EI}{144}\right)y = 1 \cdot 25 + \frac{y}{8}$$

At A and D, $y = -3$

$$\therefore \quad f_A = f_D = 1 \cdot 25 - \frac{3}{8} = +0 \cdot 875 \text{ kN-m} = M_A = M_D$$

At B and D, $y = +3$

$$\therefore \quad f_B = f_C = 1 \cdot 25 + \frac{3}{8} = +1 \cdot 625 \text{ kN-m} = M_B = M_D$$

The B.M.D. is shown in Fig. 11·15 (d).

Example 11·11. *A portal frame ABCD is fixed at A and hinged at D and carries a horizontal load of 10 kN at B as shown in Fig. 11·16. (a). Compute the moments at A, B and C.*

Solution

The basic determinate structure is derived by removing the end D completely. The Ms diagram will be a triangle having a maximum ordinate of $+10\times 4 = +40$ kN-m at A as shown in Fig. 11·16 (b).

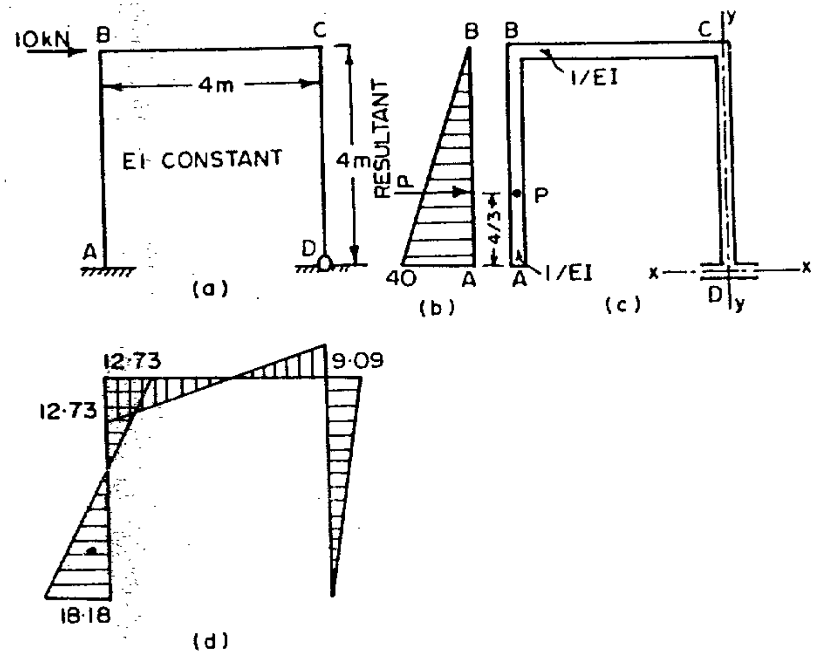


Fig. 11·16

$$\text{Resultant load} = -\int \frac{Ms}{EI} dx$$

$$= -\frac{1}{EI}\left\{\frac{1}{2} \times 4(+40)\right\} = -\frac{80}{EI}$$

Acting at $\frac{4}{3}$ m from A.

Since the frame is hinged at D, the analogous column will have infinite area concentrated at D. Both the axes will pass through D as shown in Fig. 11·16 (c).

Properties of analogous column,

$A = \infty$

$$I_{XX} = \frac{1}{3} \cdot \frac{1}{EI}(4)^3 + \frac{1}{3} \cdot \frac{1}{EI}(4)^3 + \frac{4}{EI}(4)^2 = \frac{320}{3EI}$$

$$I_{YY} = \frac{4}{EI}(4)^2 + \frac{1}{3} \cdot \frac{1}{EI}(4)^3 = \frac{256}{3EI}$$

$$I_{XY} = \Sigma A\bar{x}\bar{y} = \frac{4}{EI}(-4)(+2) + \frac{4}{EI}(-2)(+4) + \frac{4}{EI}(0)(+2)$$

$$= -\frac{64}{EI}$$

$$M_{XX} = \left(-\frac{80}{EI}\right)\left(\frac{4}{3}\right) = -\frac{320}{3EI}$$

$$M_{YY} = \left(-\frac{80}{EI}\right)(-4) = +\frac{320}{EI}$$

$$\frac{M_{YY} \cdot I_{XX} - M_{XX} \cdot I_{XY}}{I_{YY} \cdot I_{XX} - I_{XY}^2} = \frac{\left(\frac{320}{EI}\right)\left(\frac{320}{3EI}\right) - \left(-\frac{320}{EI}\right)\left(-\frac{64}{EI}\right)}{\left(\frac{256}{3EI}\right)\left(\frac{320}{3EI}\right) - \left(-\frac{64}{EI}\right)^2} = +\frac{60}{11}$$

$$\frac{M_{XX} \cdot I_{YY} - M_{YY} \cdot I_{XY}}{I_{YY} \cdot I_{XX} - I_{XY}^2} = \frac{\left(-\frac{320}{3EI}\right)\left(\frac{256}{3EI}\right) - \left(\frac{320}{EI}\right)\left(-\frac{64}{EI}\right)}{\left(\frac{256}{3EI}\right)\left(\frac{320}{3EI}\right) - \left(-\frac{64}{EI}\right)^2} = +\frac{25}{11}$$

Hence the stress f at any point is given by

$$f = +\frac{60}{11}x + \frac{25}{11}y$$

At A, $x = -4$; $y = 0$

$\therefore$ $f_A = M_{IA} = -21{\cdot}82$

At B, $x = -4$; $y = +4$

$\therefore$ $f_B = M_{IB} = -12{\cdot}73$

At C, $x = 0$; $y = +4$

$\therefore$ $f_C = M_{IC} = +9{\cdot}09$

At D, $x = 0$; $y = 0$

$\therefore$ $f_D = M_{ID} = 0$

The final moments are as follows :

$M_A = M_{IA} + M_{SA} = -21{\cdot}82 + 40 = -18{\cdot}18$ kN-m

$M_B = M_{IB} + M_{SB} = -12{\cdot}73 + 0 = -12{\cdot}73$ kN-m

$M_C = M_{IC} + M_{SC} = +9{\cdot}09 + 0 = +9{\cdot}09$ kN-m

$M_D = M_{ID} + M_{SD} = 0$

The final B.M.D. is shown in Fig. 11·16 (d).

PROBLEMS

1. Find the support moments of a built-in beam loaded at third point by two point loads W each. EI is constant throughout.

2. A girder of 36 ft, span is fixed horizontally at the end. A downward vertical load of 12 tons acts on the girder at a distance of 12 ft. from the left hand end and an upward vertical force of 8 tons acts at a distance of 18 ft. from the right hand end. Determine the end reactions and fixing couples and draw the bending moment and shearing force diagram for the girder. (*U.L.*)

3. An encastre beam of span L carries a load wL uniformly distributed over the span. The second moment of area of the beam section is not the same throughout ; for a length $L/4$ at each end the value is $2I$ and for the middle length $L/2$, it is I.

Determine the bending moment at the end of the beam and sketch the bending moment diagram, showing on it the values at the ends and at midspan. (*U.L.*)

4. A beam of 20 m span is fixed at both the ends. A couple of 12 kN-m is applied to the beam at a distance 8 m from the left hand support, about a horizontal axis at right angles to the beam. Find the fixing couple at each end and plot the B.M. diagrams.

5. A beam *AB* of span 3 m is fixed at both the ends and carries a point load of 10 kN at *C*, distant 1 m from *A*. The moment of inertia of the portion *AC* of the beam is $2I$ and that of portion *CB* is I. Calculate the fixed end moments.

6. A portal frame *ABCD* is fixed at *A* and *D*, and has rigid joints at *B* and *C* and is loaded as shown in Fig. 11·17. Plot the bending moment diagram for the frame.

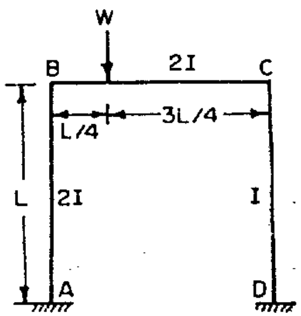


Fig. 11·17.

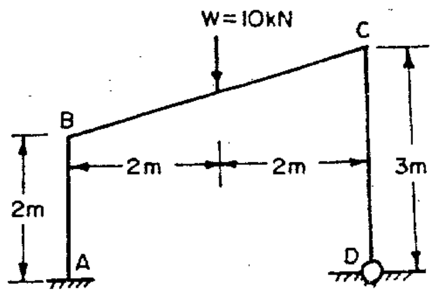


Fig. 11·18.

7. Analyse the portal frame shown in Fig. 11·18. *EI* is constant for the whole frame.

8. Draw the bending moment diagram and the deflected shape of the frame shown in Fig. 11·19. The ends *A* and *D* are fixed and *BC* is loaded with U.D.L. of 10 kN/m.

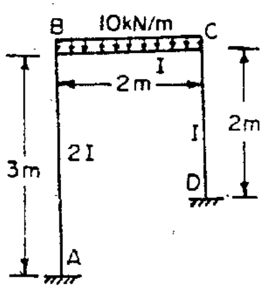


Fig. 11·19.

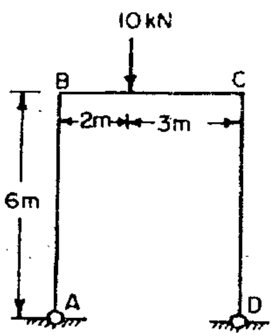


Fig. 11·20.

9. A rectangular box frame is 10 ft. wide and 6 ft. deep. The second moments of area of the horizontal members are twice those of the vertical members. The frame carries an inward uniformly distributed load of 20 tons

per foot run along the top and bottom horizontal members only. Calculate the bending moments at the corners of the frame. (*A.M.I.C.E.*)

10. A portal *ABCD* is hinged at *A* and *D*, and has stiff joints at *B* and *C*. Draw the bending moment diagram due to a point load of 10 kN as shown in Fig. 11·20. *EI*=const.

11. A portal frame *ABCD* has ends *A* and *D* hinged, and carries U.D.L. of 3 kN/m on *AB* as shown in Fig. 11·21. Plot the B.M. diagram.

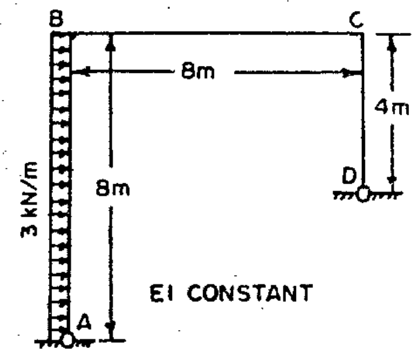


Fig. 11·21.

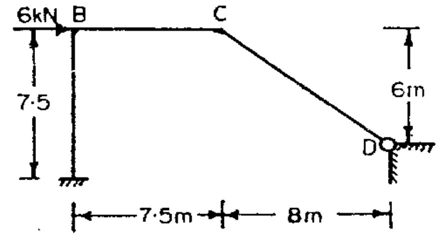


Fig. 11·22.

12. Fig. 11·22 gives the dimensions of a continuous frame *ABCD* in which *EI* is constant. The end *A* is fixed and the end *D* is hinged. Draw the bending moment diagram for this frame, marking on it all important values, when the frame is subjected to a horizontal load of 6 kN applied at *B*.

Answers

1. $\frac{2}{9}WL$.
2. $R_1=4\cdot89\uparrow$; $R_2=0\cdot89\downarrow$
   $M_1=28$ t-ft ; $M_2=4$ t-ft.
3. $M_1=M_2=+\frac{3}{32}WL^2$.
4. M (Left)$=1\cdot44$; M (Right) $3\cdot84$ kN-m both in the same directions as the external moment ; $R=0\cdot864$ kN.
5. $M_A=5\cdot40$ kN-m ; $M_B=1\cdot76$ kN-m.
6. $M_A=+0\cdot0137\ WL$; $M_B=+0\cdot0647\ WL$.
   $M_C=+0\cdot0461\ WL$; $M_D=-0.0325\ WL$.
7. $M_A=+0\cdot028$ kN-m; $M_B=+2\cdot31$ kN-m.
   $M_C=+3\cdot55$ kN-m ; $M_D=0$.
8. $M_A=-1\cdot47$ kN-m ; $M_B=+2\cdot67$ kN-m
   $M_C=+1\cdot98$ kN-m ; $M_D=-0\cdot81$ kN-m.
9. 7·58 t-ft.
10. $M_B=+3\cdot32$ kN-m ; $M_C=+3\cdot32$ kN-m.
11. $M_B=-12\cdot8$ kN-m ; $M_C=+41\cdot6$ kN-m.
12. $M_A=+8\cdot38$ kN-m ; $M_B=-8\cdot5$ kN-m
    $M_C=+6\cdot5$ kN-m ; $M_D=0$.

12

Method of Strain Energy

12·1. GENERAL PRINCIPLES

When external force (*i.e.* axial load or moment) acts on an elastic body, it deforms. If the elastic limit is not exceeded, the work done in straining the material is stored in it in the form of resilience of internal strain energy. By equating the external work done by applied loads as they deform the elastic body to the internal strain energy stored in the body, we obtain a method of determining deflections that is based on the principle of conservation of energy. The energy principles presented in this chapter have the broad scope of their application to the analysis of redundant systems also.

In pin jointed structures, where the members are in tension or compression, the energy stored depends on direct forces only. However, in beams and frames having rigid joints, shear stress and bending stress may also occur at any section, and the total strain energy stored depends on the magnitudes of direct force, shear and moment. While analysing statically indeterminate structures, the work done by direct and shear forces is *neglected* since it is very small in comparison to that done by bending. Before discussing the various strain energy theorems, let us first derive standard expressions for strain energy stored in linear elastic systems under various loadings.

12·2. STRAIN ENERGY IN LINEAR ELASTIC SYSTEMS

(*i*) Axial Loading

Let us consider a straight bar of length L, having uniform cross-sectional area A. If an axial load P is applied gradually, and if the bar undergoes a deformation Δ, the work done, stored as strain energy (U) in the body, will be equal to average force ($\frac{1}{2}P$) multiplied by the deformation Δ.

Thus $$U=\frac{1}{2}P\cdot\Delta$$

But $$\Delta=\frac{PL}{AE} \text{ from Hooke's Law.}$$

$\therefore$ $$U=\frac{1}{2}P\cdot\frac{PL}{AE}=\frac{P^2L}{2AE} \qquad (12\cdot1)$$

If, however, the bar has variable area of cross section, consider a small section of length dx and area of cross-section Ax. The strain energy dU stored in this small element of length dx will be, from Eq. 12·1,

$$dU=\frac{P^2dx}{2A_xE}.$$

The total strain energy U can be obtained by integrating the above expression over the length of the bar

Thus $$U=\int_0^L \frac{P^2dx}{2A_xE} \qquad (12\cdot2)$$

(*ii*) Flexural Loading (moment or Couple) :

Let us now consider a member of length L subjected to uniform bending moment M. Consider an element of length dx, and let d be the change in the slope of the element due to applied moment M. If M is applied gradually, the strain energy stored in the small element will be,

$$dU=\frac{1}{2}M\cdot di$$

But $$\frac{di}{dx}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}=\frac{M}{EI}$$

or $$di=\frac{M}{EI}\cdot dx$$

Hence $$dU=\frac{1}{2}M\cdot\frac{M}{EI}\cdot dx=\frac{M^2}{2EI}dx$$

Integrating this over the entire length, we get the total strain energy stored in the member. Thus,

$$U=\int_0^L \frac{M^2dx}{2EI} \qquad (12\cdot3)$$

12·3. CASTIGLIANO'S FIRST THEOREM

The concept of elastic strain energy can be very useful in the study of deflections of various points of structure under load. Instead of directly equating the external work to the internal strain energy, considerable simplification is obtained by *Castigliano's first theorem* which states that the deflection caused by any external force is equal to the partial derivative of the strain energy with respect to that force.

A generalised statement of the theorem is as follows :

"If there is any elastic system in equilibrium under the action of a set of forces W_1, W_2, W_3, ..., W_n and corresponding displacements δ_1, δ_2, δ_3.... δ_n, and a set of moments M_1, M_2 M_3,..., M_n, and corresponding rotations ϕ_1, ϕ_2, ϕ_3,..., ϕ_n, then the partial derivative of the total strain energy U with respect to any one of the forces or moments taken individually would yield its corresponding displacement in its direction of action."

Expressed mathematically,

$$\frac{\partial U}{\partial W_1}=\delta_1 \tag{12·4}$$

and

$$\frac{\partial U}{\partial M_1}=\phi_1 \tag{12·5}$$

Proof :

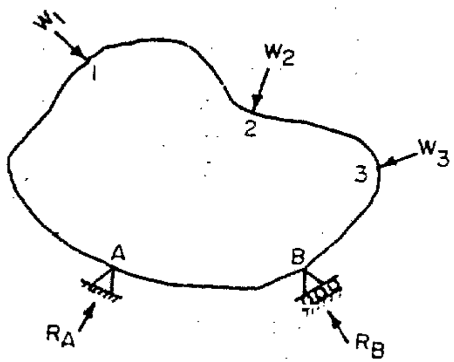


Fig. 12·1.

Consider an elastic body (Fig. 12·1) subjected to loads W_1, W_2, W_3, ... etc. each applied independently. Let the body be supported at *A*, *B* etc. The *reactions* R_A, R_B *etc. do not do work while the body deforms* because the hinge reaction is fixed and cannot move (and therefore the work done is zero) and the roller reaction is perpendicular to the displacements of the roller. Assume that the material follows Hooke's law, the displacements of the points of loading will be linear functions of the load and the principle of superposition will hold.

Let δ_1, δ_2, δ_3, ..., etc. be the deflections of points 1, 2, 3, etc. *in the direction of the loads at these points.* The total strain energy *U* is then given by

$$U=\frac{1}{2}(W_1\delta_1+W_2\delta_2+W_3\delta_3+...) \tag{1}$$

Let the load W_1 be increased by an amount dW_1, after the loads have been applied. Due to this, there will be small change in the deformation of the body, and the strain energy will be increased slightly by an amount *dU*. Expressing this small increase as the rate of change of *U* with respect to W_1 times dW_1, the new strain energy will be

$$U+\frac{\partial U}{\partial W_1}.dW_1 \tag{2}$$

On the assumption that principle of superposition applies, the final strain energy does not depend upon the order in which the forces are applied. Hence assuming that dW_1 is acting on the body, prior to the application of W_1, W_2, W_3 etc. the deflections will be infinitely small and the corresponding strain energy of the second order can be neglected. Now when W_1, W_2, W_3 ..etc. are applied (with dW_1 still acting initially), the points 1, 2, 3 etc. will move through δ_1, δ_2, δ_3 etc. in the direction of these forces and the strain energy *U* will be as given by (1) above. However, in doing so, the small load dW_1, which is acting prior to the application of W_1, rides through a distance δ_1 and produces the external work increment $dU=dW_1 \cdot \delta_1$. Hence the new strain energy, when the loads are applied in this order, is

$$U=dW_1 \cdot \delta_1 \tag{3}$$

Since the final strain energy does not depend upon the order in which the forces are applied, we get, by equating (2) and (3)

$$U+dW_1.\delta_1=U+\frac{\partial U}{\partial W_1}.dW_1$$

or

$$\delta_1=\frac{\partial U}{\partial W_1}$$

which proves the proposition.

Similarly, it can be proved that $\phi_1=\dfrac{\partial U}{\partial M_1}$.

12·4 DEFLECTION OF BEAMS ETC. BY CASTIGLIANO'S FIRST THEOREM

In this chapter, we shall compute the deflections of beams and other members connected by rigid joints. The case of joint deflection of an articulated structure has been dealt with separately in chapter 13.

Castigliano's first theorem (Eq. 12·4) can be used for computing the deflection of beams and frames with rigid joint.

If a member carries an axial force, the energy stored is given by

$$U=\int_0^L \frac{P^2dx}{2AE} \text{ (from Eq. 12·2).}$$

In the above expression, P is the axial force in the member, and is the function of external load W_1, W_2 etc. If it is required to compute the deflection δ_1 in the direction of W_1, we have, from Castingliano's first theorem,

$$\delta_1=\frac{\partial U}{\partial W_1}=\int_0^L \frac{P}{AE}\cdot\frac{\partial P}{\partial W_1}\cdot dx \quad (12·6)$$

In the above expression, $\frac{\partial P}{\partial W_1}$ *is best evaluated by differentiating inside the integral sign before integrating*. This is permissible because W_1 is not a function of x.

If, however, the strain energy is due to bending, and not due to axial load,

$$U=\int_0^L \frac{M^2dx}{2EI}, \text{ (from equation 12·3)}$$

(where M is a function of the load W_1)

and $$\delta_1=\frac{\partial U}{\partial W_1}=\int_0^L M\left(\frac{\partial M}{\partial W_1}\right)\frac{dx}{EI} \quad (12·7)$$

In the above expression also, $\frac{\partial M}{\partial W_1}$ is evaluated by differentiating inside the integral sign before integrating.

If no load is acting at a point where the deflection is desired, fictitious load W is applied at the point, in the direction the deflection is required. Then, *after* differentiating but *before* integrating, the fictitious load is set to zero. The method is sometimes known as the *fictitious load method*.

If, however, the rotation ϕ_1 is required in the direction of M_1 equation 10·7 is modified as follows :

$$\phi_1=\frac{\partial U}{\partial M_1}=\int_0^L M\left(\frac{\partial M}{\partial M_1}\right)\frac{dx}{EI} \quad (12·8)$$

where M is a function of M_1.

The procedure will now be illustrated with the help of few worked examples.

Example 12·1. *Calculate the central deflection, and the slope at ends of a simply supported beam carrying a U.D.L. w per unit length over the whole span.*

Solution

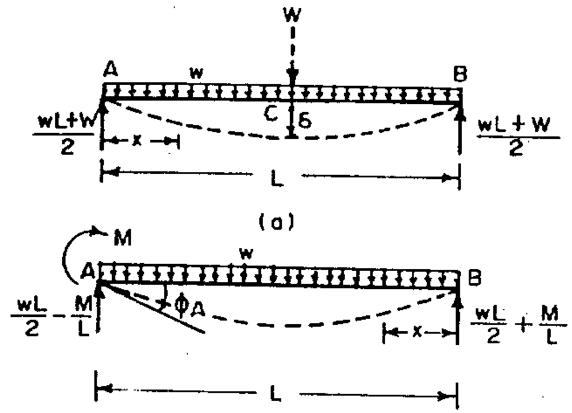


Fig. 12·2.

(*a*) *Central deflection*

Since no point load is acting at the centre where the deflection is required, apply a fictitious load W there, as shown in Fig. 12·2(*a*). The reactions at A and B will $\left(\frac{wL}{2}+\frac{W}{2}\right)\uparrow$ each.

Then

$$\delta c=\frac{\partial U}{\partial W}=\frac{1}{EI}\int_0^L Mx\frac{\partial Mx}{\partial W}\cdot dx \quad (1)$$

where Mx is the bending moment at any section distant x from A.

$$Mx=-\left(\frac{wL}{2}+\frac{W}{2}\right)x+\frac{wx^2}{2}$$

$$\frac{\partial Mx}{\partial W}=-\frac{x}{2}$$

Substituting in (1), we get

$$\delta c=\frac{2}{EI}\int_0^{L/2}\left[\left(\frac{wL}{2}+\frac{W}{2}\right)x-\frac{wx^2}{2}\right]\frac{x}{2}dx$$

Putting $W=0$,

$$\delta c=\frac{2}{EI}\int_0^{L/2}\left(\frac{wL}{2}x-\frac{wx^2}{2}\right)\frac{x}{2}dx$$

$$=\frac{2}{EI}\left[\frac{wLx^3}{12}-\frac{wx^4}{16}\right]_0^{L/2}=\frac{5}{384}\frac{wL^4}{EI}$$

(*b*) *Slope at ends*

To obtain the slope at the end A, say, apply a fictitious moment M as shown in Fig. 12·2(*b*). The reations at A and B will be respectively $\left(\frac{wL}{2}-\frac{M}{L}\right)$ and $\left(\frac{wL}{2}+\frac{M}{L}\right)$.

Measuring x from B, we have, Eq. 12·8,

$$\phi_A=\frac{\partial U}{\partial M}=\frac{1}{EI}\int_0^L Mx\cdot\frac{\partial Mx}{\partial M}\cdot dx \qquad (2)$$

where Mx is the moment at a point distant x from the origin (*i.e.* B) and is a function of M.

$$Mx=-\left(\frac{wL}{2}+\frac{M}{L}\right)x+\frac{wx^2}{2}$$

$$\therefore\quad \frac{dMx}{dM}=-\frac{x}{L}$$

Substituting the values in (2), we get

$$\phi_A=\frac{1}{EI}\int_0^L\left[\left(\frac{wL}{2}+\frac{M}{L}\right)x-\frac{wx^2}{2}\right]\frac{x}{L}\,dx$$

Putting $M=0$

$$\phi_A=\frac{1}{EI}\int_0^L\left(\frac{wL}{2}x-\frac{wx^2}{2}\right)\frac{x}{L}\,dx$$

$$=\frac{1}{EI}\left[\frac{wx^3}{6}-\frac{wx^4}{8L}\right]_0^L=+\frac{wL^3}{24EI}.$$

The plus sign signifies that the rotation is in the direction of the moment M, *i.e.* clockwise.

Example 12·2. *Using Castigliano's first theorem, determine the deflection and rotation of the overhanging end A of the beam loaded as shown in Fig. 12·3(a).*

Solution

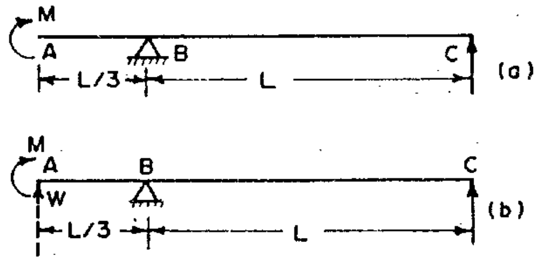


Fig. 12·3.

(*a*) *Rotation of A*

For the loading of Fig. 12·3(*a*), the reaction $R_B=\frac{M}{L}\downarrow$ and

$R_C=\frac{M}{L}\uparrow$

Now $\phi_A=\dfrac{\partial U}{\partial M}=\dfrac{1}{EI}\displaystyle\int_A^B Mx\cdot\frac{\partial Mx}{\partial M}\,dx+\frac{1}{EI}\int_C^B Mx\cdot\frac{\partial Mx}{\partial M}\,dx$

For any point distant x from A, between A and B, $\left(i.e.\ x=0 \text{ to } x=\frac{L}{3}\right)$

$$Mx=-M\ ;\ \frac{\partial Mx}{\partial M}=-1.$$

For any point distant x from C, between C and B, (*i.e.* $x=0$ to $x=L$)

$$Mx=-\frac{M}{L}x$$

and
$$\frac{\partial Mx}{\partial M}=-\frac{x}{L}$$

Substituting the values, we get

$$\phi_A=\frac{\partial U}{\partial M}=\frac{1}{EI}\int_0^{L/3}(-M)(-1)dx+\frac{1}{EI}\int_0^L\left(-\frac{M}{L}x\right)\left(-\frac{x}{L}\right)dx$$

$$=\frac{M}{EI}\Big[x\Big]_0^{L/3}+\frac{M}{L^2EI}\left[\frac{x^3}{3}\right]_0^L$$

$$=\frac{ML}{3EI}+\frac{ML}{3EI}$$

$$=\frac{2ML}{3EI}\ \text{(clockwise)}$$

(*b*) *Deflection of A*

To find the deflection at P, apply a fictitious load W at A, in upward direction as shown in Fig. 12·2(*b*).

The reaction $R_B=\left(M+\frac{4}{3}WL\right)\frac{1}{L}\downarrow$

The reaction $R_C=\left(M+\frac{1}{3}WL\right)\frac{1}{L}\uparrow$

Then

$$\delta_A=\frac{\partial U}{\partial W}=\frac{1}{EI}\int_A^B Mx\frac{\partial Mx}{\partial W}\cdot dx+\frac{1}{EI}\int_C^B Mx\frac{\partial Mx}{\partial W}\cdot dx$$

For the portion AB, $x=0$ at A to $x=\frac{L}{3}$ at B,

$$M_x=-M-Wx$$

$$\frac{\partial M_x}{\partial W}=-x$$

For the portion CB, $x=0$ at to C to $x=L$ at B.

$$M_x=-\left(M+\frac{1}{3}WL\right)\frac{1}{L}\cdot x$$

$$\frac{\partial Mx}{\partial W}=-\frac{x}{3}$$

Substituting the values, we get

$$\delta_A=\frac{1}{EI}\int_0^{L/3}(M+Wx)x\,.\,dx+\frac{1}{EI}\int_0^L\left(M+\frac{1}{3}WL\right)\frac{x}{L}\cdot\frac{x}{3}\,dx$$

Putting $W=0$

$$\delta_A=\frac{1}{EI}\int_0^{L/3} M.x.dx+\frac{1}{EI}\int_0^L\frac{Mx^2}{3L}\,dx$$

$$=\frac{M}{EI}\left[\frac{x^2}{2}\right]_0^{L/3}+\frac{M}{3EIL}\left[\frac{x^3}{3}\right]_0^L$$

$$=\frac{ML^2}{18EI}+\frac{ML^2}{9EI}=\frac{ML^2}{6EI}$$

Example 12·3. *A freely supported beam of span L carries a central load W. The sectional area of the beam is so designed that the moment of inertia of the section increases uniformly from I at ends to 1·5 I at the middle. Calculate the central deflection.*

Solution

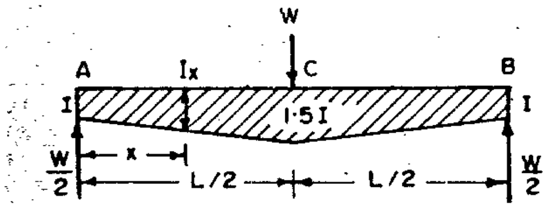


Fig. 12·4

The deflection is given by

$$\delta_C=\frac{\partial U}{\partial W}=\frac{1}{E}\int\frac{M_x}{I_x}\cdot\frac{\partial M_x}{\partial W}\cdot dx$$

In the above integral, $I_x=I+\frac{I}{2}\cdot\frac{2}{L}\cdot x=I\left(1+\frac{x}{L}\right)$ and is a function of x.

$$M_x=-\frac{W}{2}x\text{, from } x=0 \text{ to } x=L/2$$

$$\frac{\partial M_x}{\partial W}=-\frac{x}{2}$$

Substituting the value, we get

$$\delta_C=\frac{2}{E}\int_0^{L/2}\frac{1}{I\left(1+\frac{x}{L}\right)}\frac{W}{2}x\cdot\frac{x}{2}dx$$

$$=\frac{W}{2EI}\int_0^{L/2}\frac{x}{\left(1+\frac{x}{L}\right)}\,dx$$

Substituting $x+L=t$ in the above integral, and simplifying, we get

$$\delta_C=\frac{WL}{2EI}\int_L^{3L/2}\left(t-2L+\frac{L^2}{t}\right)dt$$

$$=\frac{WL}{2EI}\left[\frac{t^2}{2}-2Lt+L^2\log_e t\right]_L^{3L/2}$$

$$=0{\cdot}015\,\frac{WL^3}{EI}.$$

Example 12·4. *A beam of uniform section and of length 2L is freely supported by rigid supports at its ends and by an elastic prop at its centre. If the prop deflects by an amount λ times the load it carries and if the beam carries a total distributed load of W, show that the load carried by the prop is* $\dfrac{5W}{8\left(1+\dfrac{6EI\lambda}{L^3}\right)}$ (*U.L.*)

Solution

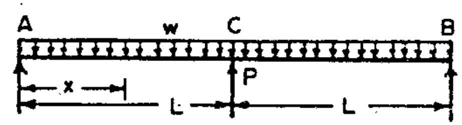


Fig. 12·5

Let the prop reaction be $=P$

Load on beam $=W$

$\therefore$ U.D.L. $=\dfrac{W}{2L}$ per unit length

Reaction at ends $=R=\dfrac{1}{2}(W-P)$

The deflection at the prop is given by

$$\delta_C=\frac{\partial U}{\partial P}=\frac{1}{EI}\int M_x\frac{\partial M_x}{\partial P}dx$$

Since the prop deflects by an amount λ times the load it carries, we have $\delta_C=-P\,.\,\lambda$.

(Minus sign has been used since deflection is in a direction opposite to the line of action of P).

Hence $$-P\lambda=\frac{1}{EI}\int M_x\frac{\partial M_x}{\partial P}\,dx$$

or $$-P=\frac{1}{\lambda\,.\,EI}\int M_x\frac{\partial M_x}{\partial P}\,dx \qquad (1)$$

For any section distant x from A,

$$Mx=-\frac{1}{2}(W-P)x+\frac{W}{2L}\frac{x^2}{2}$$

$$\frac{\partial Mx}{\partial P}=+\frac{x}{2}$$

Substituting the value in (1), we get

or $$-P=\frac{2}{\lambda\,.\,EI}\int_0^L\left\{\frac{-1}{2}(W-P)x+\frac{W}{2L}\frac{x^2}{2}\right\}\frac{x}{2}\,dx$$

or $$P.\frac{\lambda EI}{2}=\left[\frac{1}{12}(W-P)x^3-\frac{W}{32}\frac{x^4}{L}\right]_0^L$$

or $$P\frac{\lambda EI}{2}=\frac{WL^3}{12}-\frac{PL^3}{12}-\frac{WL^3}{32}=\frac{5WL^3}{96}-\frac{PL^3}{12}$$

or $$P\left(\frac{6EI\lambda}{L^3}+1\right)=\frac{5W}{8}$$

From which $$P=\frac{5W}{8\left(1+\frac{6EI\lambda}{L^3}\right)}$$

Example 12·5. *A vertical load W is applied to the rigid cantilever frame shown in Fig. 12·6. Assuming EI to be constant throughout the frame, determine the horizontal and vertical displacements of the point C. Neglect axial deformations.*

Solution

Vertical deflection of C

The vertical deflection of C is given by

$$\delta_{CV}=\frac{\partial U}{\partial W}=\frac{1}{EI}\int Mx\frac{\partial Mx}{\partial W}\,dx \qquad (1)$$

For BC, measuring x from C,

$$Mx=+Wx$$

$$\frac{\partial Mx}{\partial W}=+x$$

For BA, measuring x from B

$$Mx=+\frac{WL}{2}\ \text{(constant)}$$

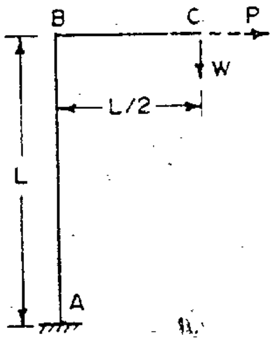


Fig. 12·6

$$\frac{\partial Mx}{\partial W}=+\frac{L}{2}$$

Substituting the values in (1), we get

$$\delta_{CV}=\frac{1}{EI}\left[\int_0^{L/2}Wx\,.\,xdx+\int_0^L\frac{WL}{2}\,.\,\frac{L}{2}dx\right]$$

$$=\frac{1}{EI}\left[\left\{\frac{Wx^3}{3}\right\}_0^{L/2}+\left\{\frac{WL^2}{4}x\right\}_0^{l}\right]$$

$$=\frac{1}{EI}\left[\frac{WL^3}{24}+\frac{WL^3}{4}\right]=\frac{7WL^3}{24EI}\ \text{(Answer).}$$

(*b*) *Horizontal deflection of C* :

To compute the horizontal deflection, apply fictitious horizontal load P at C, as shown in Fig. 12·6. Then

$$\delta_{CH}=\frac{\partial U}{\partial P}=\frac{1}{EI}\int Mx\frac{\partial Mx}{\partial P}\,.\,dx \qquad (2)$$

For BC, measuring x from C,

$$Mx=+Wx$$

$$\frac{\partial M}{\partial P}=0$$

For BA, measuring x from B,

$$Mx=+\frac{WL}{2}+Px$$

$$\frac{\partial Mx}{\partial P}=+x$$

Substituting the values in (2), we get

$$\delta_{CH}=\frac{1}{EI}\left[\int_0^{L/2}Wx(0)\,dx+\int_0^L\left(\frac{WL}{2}+P\,.\,x\right)x\,dx\right]$$

$$=\frac{1}{EI}\left[\frac{WLx^2}{4}+\frac{Px^3}{3}\right]_0^L$$

Applying the limits and putting $P=0$, we get

$$\delta_{CH}=\frac{1}{EI}\,.\,\frac{WL^3}{4}=\frac{WL^3}{4EI}\ \text{(Answer)}$$

Example 12·6. *Obtain an expression for the vertical displacement of point A in the bent cantilever shown in Fig. 12·7(a).*

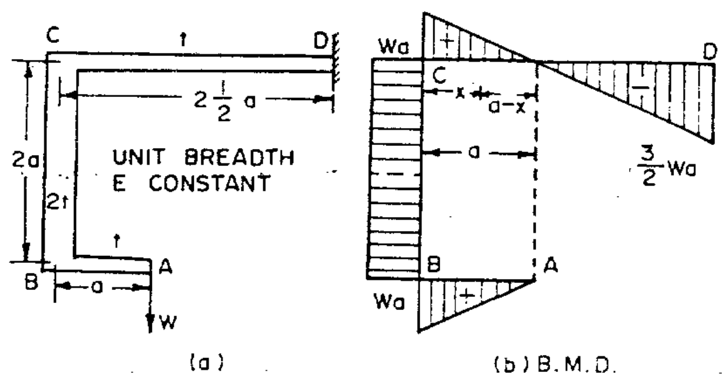


Fig. 12·7

Solution

The vertical displacement of A is given by

$$\delta_A = \frac{\partial U}{\partial W} = \frac{1}{E}\int \frac{M_X}{I_X} \frac{\partial M_X}{\partial W} \cdot dx \qquad (1)$$

where the integration is carried over the whole frame. A bending moment will be designated positive if it produces convexity to the *inside* of the frame.

The B.M.D. for the whole frame is shown in Fig. 12·7(*b*).

(*i*) *For AB* :

Width = unity

$$I = \frac{1}{12} \times 1 \times t^3 = \frac{t^3}{12}$$

$M_X = +W_X$ (measuring x from A)

$$\frac{\partial M_X}{\partial W} = +x$$

Limits of x from 0 at A to a at B.

(*ii*) *For BC* :

Width = unity

$$I = \frac{1}{12} \times 1(2t^2) = \frac{2t^3}{3}$$

$$M_X = +Wa$$

$$\frac{\partial M_X}{\partial W} = +a$$

Limits of x from 0 at B to $2a$ at C.

(*iii*) *For CD* :

Width = unity

$$I = \frac{1}{12} \times 1\ (t)^3 = \frac{t^3}{12}$$

$Mx = +W(a-x)$, x being measured from C.

(Evidently, $Mx=0$ when $x=a$ where the line of action of W cuts the member CD).

$$\frac{\partial Mx}{\partial W} = a - x$$

Limits of x from 0 at $5a/2$ at D

Substituting the values in (1), we get

$$\delta_A = \frac{1}{E}\left[\int_0^a Wx(x)\frac{12}{t^3}dx + \int_0^{2a} Wa(a)\frac{3}{2t^3}dx + \int_0^{\frac{5a}{2}} W(a-x)(a-x)\frac{12}{t^3}dx\right]$$

$$= \frac{1}{Et^3}\left[\left(4Wx^3\right)_0^a + \left(\frac{3}{2}Wa^2x\right)_0^{2a} + 12W\left(a^2x + \frac{x^3}{3} - \frac{2ax^2}{2}\right)_0^{\frac{5a}{2}}\right]$$

$$= \frac{1}{Et^3}\left[4Wa^3 + 3Wa^3 + 12W\left(\frac{5}{2}a^3 + \frac{125}{24}a^3 - \frac{25}{4}a^3\right)\right]$$

$$= \frac{24{\cdot}5\ Wa^3}{Et^3} \text{ (Answer).}$$

Example 12·7. *A steel tube having outside and inside diameters of 10 cm and 6 cm respectively is bent into the form of a quadrant of 2 m radius. One end is rigidly attached to a horizontal base plate to which a tangent to that end is perpendicular, and the free end supports a load of 1000 N. Determine the vertical and horizontal deflections of the free end under this load. The dimensions of the cross-section may be considered as small relative to the radius of curvature.*

$$E = 2 \times 10^5\ N/mm^2$$

Solution.

(*a*) *Vertical deflection of end A*

$$\delta_{AV} = \frac{\partial U}{\partial W} = \frac{1}{EI}\int M \frac{\partial M}{\partial W}\ ds \qquad (1)$$

where the integration is carried over the whole frame along the *curved* surface. Hence ds has been used in place of dx.

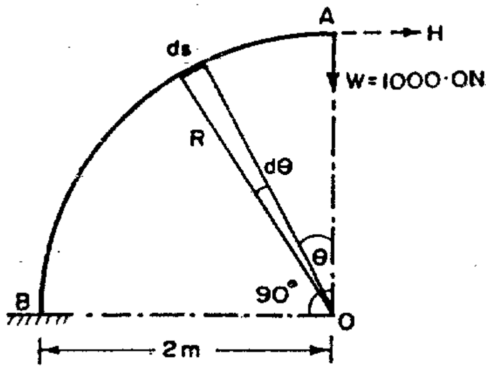


Fig. 12·8.

Let us consider a small element of curved length ds subtending $d\theta$ at the centre, and being at an angle θ from the line OA, O being the centre of the quadrant of the circle.

$$M = WR \sin\theta$$

$$\frac{\partial M}{\partial W} = R \sin\theta$$

$$ds = Rd\theta$$

Limits of θ from 0 at A to $\pi/2$ at B.

Substituting in (1), we get

$$\delta_{AV} = \frac{1}{EI}\int_0^{\pi/2} WR\sin\theta \cdot R\sin\theta \cdot Rd\theta$$

$$= \frac{WR^3}{EI}\int_0^{\pi/2}\sin^2\theta\ d\theta = \frac{WR^3}{EI} \cdot \frac{\pi}{4}$$

Now $\quad I = \frac{\pi}{64}(10^4 - 6^4) = 427\ cm^4 = 427 \times 10^4\ mm^4$

$\therefore \quad EI = 2 \times 10^5 \times 427 \times 10^4 = 854 \times 10^9$ N-mm²

$R = 2m = 2000$ mm

$$W=1000 \text{ N}$$

$$\therefore \quad \delta_{AV}=\frac{1000\times(2000)^3}{854\times10^9}\cdot\frac{\pi}{4}=7{\cdot}35 \text{ mm}$$

(d) *Horizontal deflection of A*

Apply a fictitious horizontal load H as shown in Fig. 12·8. Then

$$\delta_{AH}=\frac{\partial U}{\partial H}=\frac{1}{EI}\int M\,.\,\frac{\partial M}{\partial H}\,ds \tag{2}$$

$$M=WR\sin\theta+HR\,(1-\cos\theta)$$

$$\frac{\partial M}{\partial H}=R(1-\cos\theta)$$

$$ds=R\,.\,d\theta$$

$$\delta_{HA}=\frac{1}{EI}\int_0^{\pi/2}\Big\{WR\sin\theta+HR(1-\cos\theta)\Big\}R(1-\cos\theta)\,.\,Rd\theta$$

Putting $H=0$

$$\delta_{AH}=\frac{WR^3}{EI}\int_0^{\pi/2}\sin\theta(1-\cos\theta)\,d\theta=\frac{WR^3}{EI}\int_0^{\pi/2}\left(\sin\theta-\frac{\sin 2\theta}{2}\right)d\theta$$

$$=\frac{WR^3}{EI}\left[-\cos\theta+\frac{\cos\theta}{4}\right]_0^{\pi/2}=\frac{WR^3}{2EI}$$

Substituting the numerical values, we get

$$\delta_{AH}=\frac{1000\,(2000)^3}{2\times854\times10^9}=4{\cdot}68 \text{ mm}$$

Example 12·8. *A steel bar bent to the shape shown in Fig. 12.9 is fixed at A and carries a vertical load W at C. Calculate the vertical deflecton of C. EI is constant throughout.*

Solution

$$\delta_C=\frac{\partial U}{\partial M}=\frac{1}{EI}\int_C^B M\,\frac{\partial M}{\partial W}\,ds+\frac{1}{EI}\int_B^A M\,\frac{\partial M}{\partial W}\,dx \tag{1}$$

(*i*) *For CB,*

For any radius vector subtending an angle θ with OC, we have (see example 12·7) :

$$M=WR\sin\theta$$

$$\frac{\partial M}{\partial W}=R\sin\theta$$

$$ds=R\,.\,d\theta$$

$$\therefore \quad \int_C^B M\,\frac{\partial M}{\partial W}\,ds=\int_0^{\pi/2} WR^3\sin\theta$$

$$=\frac{\pi}{4}WR^3$$

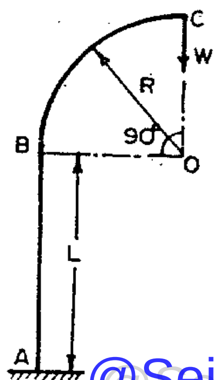


Fig. 12·9

(*ii*) *For BA,*

Measuring x from B,

$$M=WR$$

$$\frac{\partial M}{\partial W}=R$$

$$\therefore \quad \int_B^A M\,\frac{\partial M}{\partial W}\,dx=\int_0^L WR^2\,dx=WR^2 L$$

Substituting the values in (1), we get

$$\delta_C=\frac{1}{EI}\left[\frac{\pi}{4}WR^3+WR^2 L\right]$$

$$=\frac{WR^2}{4EI}\left(\pi R+4L\right) \text{ Answer.}$$

Example 12·9. *A circular arch rib of constant flexural rigidity is encastre at A as shown in Fig. 12·10. The end B is tied horizontally with a force H such that it can only move vertically when a load W is hung at B. Find the ratio H/W.*

Solution

The horizonal deflection of B is given by

$$\delta_{BH}=\frac{\partial U}{\partial H}=\int_A^B M\,\frac{\partial M}{\partial H}\,ds$$

As per condition of the problem, $\delta_{BH}=0$

$$\therefore \int_A^B M\,\frac{\partial M}{\delta H}\,.\,ds=0 \quad (1)$$

Consider an element ds, the angular distance of its radius vector being at θ from OA.

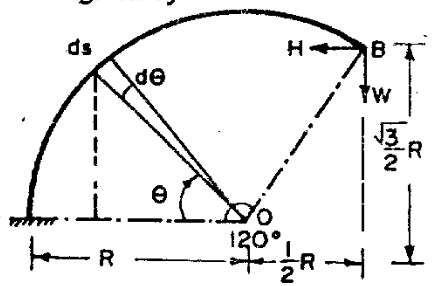


Fig. 12·10.

$$M=-H\left(\frac{\sqrt3}{2}R-R\sin\theta\right)+W\left(\frac{1}{2}R+R\cos\theta\right)$$

$$\frac{\partial M}{\partial W}=-R\left(\frac{\sqrt3}{2}-\sin\theta\right);\ ds=Rd\theta$$

Substituting in (1), we get

$$\int_0^{2\pi/3}\left\{HR\left(\frac{\sqrt3}{2}-\sin\theta\right)-WR\left(\frac{1}{2}+\cos\theta\right)\right\}R\left(\frac{\sqrt3}{2}-\sin\theta\right)Rd\theta=0$$

$$\text{or } HR^3\int_0^{2\pi/3}\left(\frac{\sqrt3}{2}-\sin\theta\right)^2 d\theta-WR^3\int_0^{2\pi/3}\left(\frac{1}{2}+\cos\theta\right)\left(\frac{\sqrt3}{2}-\sin\theta\right)d\theta$$

$$\therefore \quad \frac{H}{W}=\frac{\int_0^{2\pi/3}\left(\frac{1}{2}+\cos\theta\right)\left(\frac{\sqrt3}{2}-\sin\theta\right)d\theta}{\int_0^{2\pi/3}\left(\frac{\sqrt3}{2}-\sin\theta\right)^2 d\theta}$$

The numerator of the right hand side

$$=\frac{1}{2}\int_0^{2\pi/3}(\sqrt{3}+2\sqrt{3}\cos\theta-2\sin\theta-2\sin 2\theta)d\theta$$

$$=\frac{1}{4}\left[\frac{2\pi}{3}\sqrt{3}+2\sqrt{3}\,.\,\frac{\sqrt{3}}{2}+2\left(-\frac{1}{2}\right)+\left(-\frac{1}{2}\right)\right]=1\cdot28$$

The denominator of the right hand side

$$=\int_0^{2\pi/3}\left(\frac{3}{4}-\sqrt{3}\sin\theta+\sin^2\theta\right)d\theta$$

$$=\frac{3}{4}\cdot\frac{2\pi}{3}-\sqrt{3}\left(-\frac{1}{2}\right)+\frac{\pi}{3}-\frac{1}{4}\left(-\frac{\sqrt{3}}{2}\right)=1\cdot97$$

Hence $\frac{H}{W}=\frac{1\cdot28}{1\cdot97}=0\cdot65.$

12·5. MINIMUM STRAIN ENERGY AND CASTIGLIANO'S SECOND THEOREM

1. Minimum Strain Energy

In the sixth chapter, we have discussed two methods of analysing the statically indeterminate structures : the *displacement method* (equilibrium method or stiffness coefficient method) and the *force method* (compatibility method or flexibility coefficient method). For the *displacement method* of analysis, Castigliano's first theorem may be used to express the conditions of equilibrium. For the *force method* of analysis, Castigliano's second theorem may be used to express the conditions of compatibility. Castigliano stated that, *among all the statically possible states of stress in a structure subjected to a variation of stress during which the conditions of equilibrium are maintained, the correct one is that which makes the strain energy of the system a minimum.*

Thus, if U is the strain energy stored in a elastic body, and if R_1 and R_2 etc. are the redundant reactions or forces, then if there are no support movements and no change in the temperature, the reduntants R_1, R_2, etc., must be such as to make the strain energy a minimum.

Expressed mathematically,

$$\left.\begin{aligned}\frac{\partial U}{\partial R_1}&=0 \qquad (1)\\ \frac{\partial U}{\partial R_2}&=0 \qquad (2)\end{aligned}\right] \qquad (12\cdot8)$$

This set of equations is interpreted as follows : of all possible set of values that redundant forces in the system may assume, the correct set of values is that which makes the strain energy a minimum. There will be one equation for each redundant force and a set of equations corresponding to the conditions of compatibility will be obtained.

2. Castigliano's Second Theorem

Castigliano extended the principle of least work to the *self straining* systems. For example, if λ is a small strain, or displacement, within the elastic limit, in the direction of the redundant force T, we have

$$\frac{\partial U}{\partial T}=\lambda \qquad (12\cdot9)$$

The self-straining may be caused by the settlement of the support of a redundant reaction by an amount λ, or by the initial misfit of a member by an amount λ too short.

Actually, equation 12·9 represents a theorem imposing the conditions of compatibility and need not be associated with a minimum of strain energy. In this text, equation 12·2 will be called as *Castigliano's second theorem*, while equation 12·8 will be called *Castigliano's theorem of minimum strain energy*, which is a particular case of a Castigliano's second theorem when $\lambda=0$ (*i.e.* when the redundant supports do not yield or when there is no initial lack of fit in the redundant members).

3. Proof

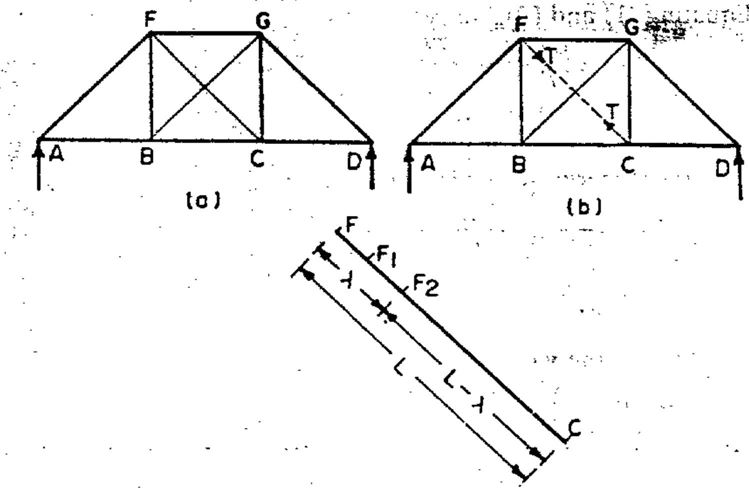


Fig. 12·11.

Consider a redundant frame shown in Fig. 12·11 (*a*), in which FC is a redundant member of geometrical length L. Let the actual length of the member FC be $(L-\lambda)$, λ being the initial lack of fit. The member FC is shown in Fig. 12·11 (*c*), in which the lack of fit λ has been shown exaggerated. F_2C represents, thus, the actual length $(L-\lambda)$ of the member. When it is fitted to the truss, the member will have to be pulled such that F_2 and F coincide. In doing so, a tensile force will be induced in the member and this tensile force will pull the joint F towards F_2. Let F_1 be the final position of the end (and of the joint F), such that the end F_2 has moved to F_1 and the member has been *extended* by an amount F_2F_1 in the fixing operation. According to Hooke's Law

$$F_2F_1 = \text{deformation} = \frac{T(L-\lambda)}{AE} = \frac{TL}{AE} \text{ (approx.)}$$

where T is the force (tensile) induced in the member.

$$\text{Hence } FF_1 = FF_2 - F_1F_2 = \lambda - \frac{TL}{AE} \quad \ldots(1)$$

Let the member FC be removed, and consider a tensile force T applied at the corners F and C, as shown in Fig. 12·11 (b), so that the basic system is not changed.

Now, FF_1 = relative deflection of F and C

$$= \frac{\partial U^1}{\partial T} \quad \ldots(2),$$

according to Castigliano's first theorem, where U^1 is the strain energy of the whole frame *except* that of the member FC.

Equating (1) and (2), we get

$$\frac{\partial U^1}{\partial T} = \lambda - \frac{TL}{AE}$$

or
$$\frac{\partial U^1}{\partial T} + \frac{TL}{AE} = \lambda$$

The strain energy stored in the member FC due to a force T is

$$U_{FC} = \frac{1}{2} T \cdot \frac{TL}{AE} = \frac{T^2L}{2\,AE}$$

$$\therefore \quad \frac{\partial U_{FC}}{\partial T} = \frac{TL}{AE}$$

Substituting the value of $\frac{TL}{AE}$ in (3), we get

$$\frac{\partial U^1}{\partial T} + \frac{\partial U_{FC}}{\partial T} = \lambda$$

or
$$\frac{\partial U}{\partial T} = \lambda$$

where $U = U^1 + U_{FC}$.

Equation 12·9 represents Castigliano's second theorem.

If, however, there is no initial lack of fit, $\lambda = 0$, and hence

$$\frac{\partial U}{\partial T} = 0 \quad \ldots(12·10)$$

This equation represents Castigliano's theorem of minimum strain energy.

(i) *Southwell suggested that Castigliano's theorem of least work in effect, means that the strain energy of a linearly elastic system that is initially unstrained will have less strain energy stored in it when subjected to a total load system than it would have if it were self-strained.*

12·6. ANALYSIS OF STATICALLY INDETERMINATE BEAMS AND PORTAL FRAMES BY MINIMUM STRAIN ENERGY

We shall now discuss the application of Castigliano's theorem of minimum strain energy for the analysis of statically indeterminate beams and portal frames. The analysis of redundant pin jointed beams and portal frames. The analysis of redundant pin jointed frames have been discussed separately in chapter 14 and the analysis of redundant arches have been dealt with in chapter 16.

The redundant reaction of the beam or portal frame are removed and a basic determinate structure is derived. If U is the *total* strain energy due to the external loads *and* the redundant reaction R, we have from Castigliano's theorem of minimum strain energy,

$$\frac{\partial U}{\partial R} = 0$$

If the beam or frame has two or more than two redundant reactions, the partial derivative of the total strain energy U, due to the external loads and the redundant reaction R_1, R_2 etc. with respect to the redundants is then equated to zero.

Thus,
$$\frac{\partial U}{\partial R_1} = 0$$
$$\frac{\partial U}{\partial R_2} = 0 \text{ etc.}$$

If the degree of redundancy is *more* than two, the slope deflection method or the moment distribution method is more convenient.

Example 12·10. *A continuous beam of two equal spans L is uniformly loaded over its entire length. Find the magnitude R of the middle reaction by using the Castigliano's theorem.*

Solution

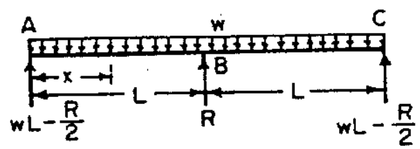


Fig. 12·12.

Let R be the redundant reaction at B,

Then $$\frac{\partial U_{AC}}{\partial R}=\frac{1}{EI}\int_A^C M\,\frac{\partial M}{\partial R}\,dx=0 \quad (1)$$

The reactions at A and $C=\left(wL-\frac{R}{2}\right)$ each.

At any point distant x from A

$$M=-\left(wL-\frac{R}{2}\right)x+\frac{wx^2}{2}$$

$$\therefore \quad \frac{\partial M}{\partial R}=+\frac{x}{2}$$

Substituting the values in (1), we get

$$\frac{2}{EI}\int_0^L\left\{-\left(wL-\frac{R}{2}\right)x+\frac{wx^2}{2}\right\}\frac{x}{2}\,dx=0$$

or $$\left[-\left(wL-\frac{R}{2}\right)\frac{x^3}{6}+\frac{w}{4}\,\frac{x^4}{4}\right]_0^L=0$$

or $$-\frac{wL^4}{6}+\frac{RL^3}{12}+\frac{wL^4}{16}=0$$

From which $R=\frac{5}{4}wL.$

Example 12·11. *Two wood beams of indentical cross-section are supported at their ends and cross at their mid-points as shown in Fig. 12·13. What interactive force R will exist between the two beams at C when a vertical load W is applied to the upper beam as shown ?*

Solution

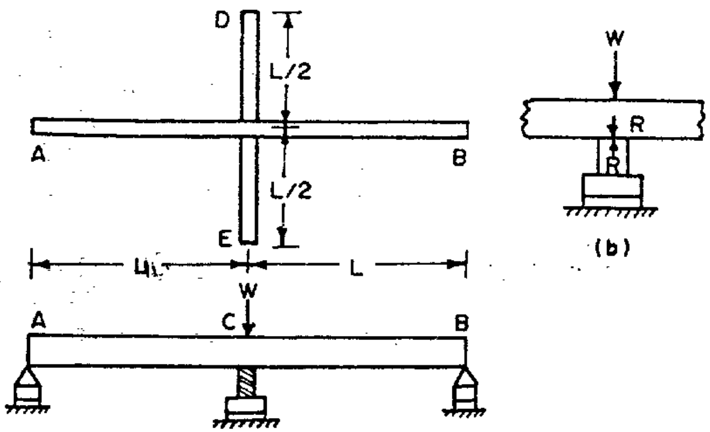


Fig. 12·13

Let us treat the interactive forces R as a generalised force. The corresponding displacement in the direction of each interactive force R as shown in Fig. 12·13 (*b*) is the relative displacement between the midpoints of the two beams. Since both the beams remain in contact, the relative displacement is zero. Hence if U is the total strain energy stored in both the beams AB and DE, we have

$$\frac{\partial U}{\partial R}=0=\frac{\partial U_{AB}}{\partial R}+\frac{\partial U_{DE}}{\partial R} \quad (1)$$

For any simply supported beam of span l and loaded with a central point load P, we have

$$M_x=-\frac{P}{2}x, \text{ (for } x=0 \text{ to } x=l/2)$$

$$\therefore \quad U=\int_0^l \frac{M^2}{2EI}dx$$

$$=2\int_0^{l/2}\frac{P^2}{8EI}x^2dx$$

$$=\frac{P^2}{4EI}\left(\frac{x^3}{3}\right)_0^{l/2}=\frac{P^2l^3}{96EI} \quad (2)$$

Hence, for the beam AB

Span $l=2L$

The net downward load $P=(W-R)$

$$\therefore \quad U_{AB}=\frac{(W-R)^2 8L^3}{96EI}=\frac{(W-R)^2L^3}{12EI}$$

$$\therefore \quad \frac{\partial U_{AB}}{\partial R}=-\frac{(W-R)L^3}{6EI}$$

Similarly, for the span DE

Span $l=D$

The net downward load $P=R$

$$\therefore \quad U_{DE}=\frac{R^2L^3}{96\,EI}$$

$$\therefore \quad \frac{\partial U_{DE}}{\partial R}=\frac{RL^3}{48EI}$$

Substituting the values in (1), we get

$$\frac{\partial U}{\partial R}=-\frac{(W-R)L^3}{6EI}+\frac{RL^3}{48EI}=0$$

or $$-8(W-R)+R=0$$

from which $$R=\frac{8}{9}W.$$

Example 12·12. *A uniform continuous bar ABCD is built-in at A and laterally supported at B as shown in Fig. 12·14 (a). Find the* *relative force R at B due to the action of a vertical load W at D as shown. Neglect the effect of direct compression in the vertical portions of the bar. Joint C is stiff. Sketch the B.M.D. for the frame.*

Solution

Let the reactive force at B be R as shown in Fig. 12·14(a). Since the support B does not yield, we have

$$\frac{\partial U_{DA}}{\partial R}=0=\frac{\partial U_{DC}}{\partial R}+\frac{\partial U_{CB}}{\partial R}+\frac{\partial U_{BA}}{\partial R} \quad (1)$$

(i) *For member DC*

$Mx=Wx$, x being measured from D

$$\frac{\partial Mx}{\partial R}=0$$

$$\therefore \quad \frac{\partial U_{DC}}{\partial R}=\frac{1}{EI}\int_0^L Mx\frac{\partial Mx}{\partial R}\,dx=0 \quad (2)$$

(ii) *For member CB*

Measuring x from B, towards B,

$$Mx=+WL$$

$$\therefore \quad \frac{\partial Mx}{\partial R}=0$$

$$\therefore \quad \frac{\partial U_{CB}}{\partial R}=\frac{1}{EI}\int_0^L Mx\frac{\partial Mx}{\partial R}\,dx=0 \quad (3)$$

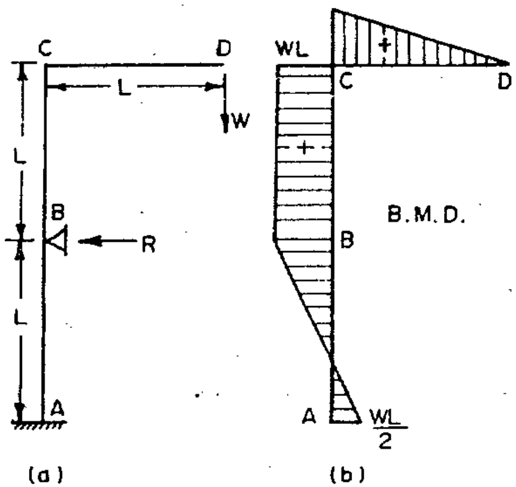


Fig. 12·14

(ii) *For member BA*

Measuring x from B towards A.

$$Mx=+WL-Rx$$

$$\frac{\partial Mx}{\partial R}=-x$$

$$\therefore \quad \frac{\partial U_{BA}}{\partial R}=\frac{1}{EI}\int_0^L (WL-Rx)(-x)dx=\frac{1}{EI}\left[\frac{Rx^3}{3}-\frac{WLx^2}{2}\right]_0^L$$

$$=\frac{1}{EI}\left(\frac{RL^3}{3}-\frac{WL^3}{2}\right) \quad (4)$$

Substituting the values in (1), we get

$$\frac{\partial U_{DA}}{\partial R}=\frac{1}{EI}\left(\frac{RL^3}{3}-\frac{WL^3}{2}\right)=0$$

From which $R=\dfrac{3W}{2}$

The B.M.D. has been shown in Fig. 12·14(b).

Example 12·13. *A portal frame ABCD is hinged at A and D, and has rigid joints B and C. The frame is loaded as shown in Fig. 12·15. Using the method of minimum strain energy, analyse the frame and plot the B.M. diagram.*

Solution

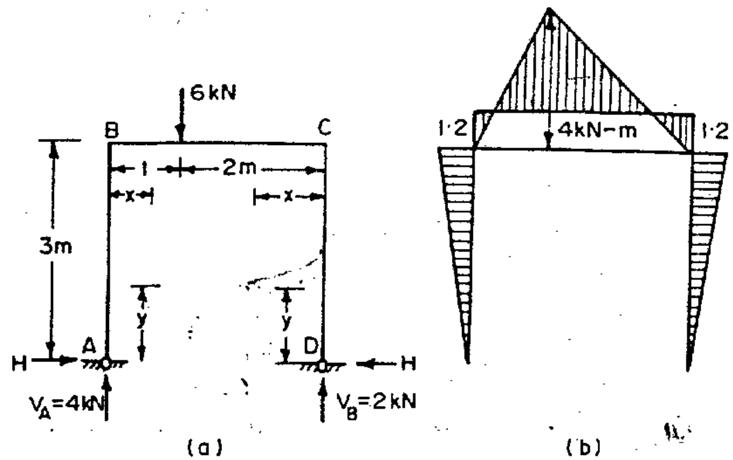


Fig. 12·15.

The frame is statically indeterminate to first degree.

Let the horizontal reaction at A be H. Since there is no external horizontal force acting, the horizontal reaction at D will also be H. Let us treat H as the redundant reaction. Since the support A does not move,

$$\frac{\partial U}{\partial H}=0=\frac{\partial U_{AB}}{\partial H}+\frac{\partial U_{BC}}{\partial H}+\frac{\partial U_{DC}}{\partial H} \qquad (1)$$

Taking moments at D, $V_A=\frac{6\times 2}{3}=4$ kN

Hence $V_D=6-4=2$ kN.

The work is most conveniently done in the tabular form below:

Member	*M*	$\partial M/\partial H$	*Limit*
AB	$+Hy$	$+y$	0 to 3
DC	$+Hy$	$+y$	0 to 3
BE	$-4x+3H$	$+3$	0 to 1
CE	$-2x+3H$	$+3$	0 to 2

For AB, $\frac{\partial U_{AB}}{\partial H}=\frac{1}{EI}\int_0^3 (Hy)(y)\,dy=\frac{H}{3EI}(3)^3=\frac{9H}{EI}$

For DC, $\frac{\partial U_{DC}}{\partial H}=\frac{1}{EI}\int_0^3 (Hy)\ (y)\ dx=\frac{H}{3EI}(3)^3=\frac{9H}{EI}$

For BC, $\frac{\partial U_{BC}}{\partial H}=\frac{1}{EI}\left[\int_0^1 (-4x+3H)(+3)dx\right.$

$$\left.+\int_0^2 (-2x+3H)(+3)dx\right]$$

$$=\frac{1}{EI}\left[(-6+9H)+(-12+18H)\right]=\frac{9}{EI}(-2+3H)$$

Substituting the values in (1), we get

$$\frac{9H}{EI}+\frac{9H}{EI}+\frac{9}{EI}(-2+3H)=0$$

From which $H=0{\cdot}4$ kN

Hence $M_A=0$

$M_B=3H=0{\cdot}4\times 3=1{\cdot}2$ kN-m

$M_C=3H=1{\cdot}2$ kN-m

$M_D=0.$

The B.M.D. is shown in Fig. 12·15(*b*).

Example 12·14. *Using the principle of least work, analyse the portal frame shown in Fig. 12·16(a). Also, plot the B.M. diagram.*

Solution

The structure is statically indeterminate to first degree. Let us treat the horizontal reaction $H(\leftarrow)$ at A as redundant. The horizontal reaction at D will evidently be $=(3-H)\leftarrow$.

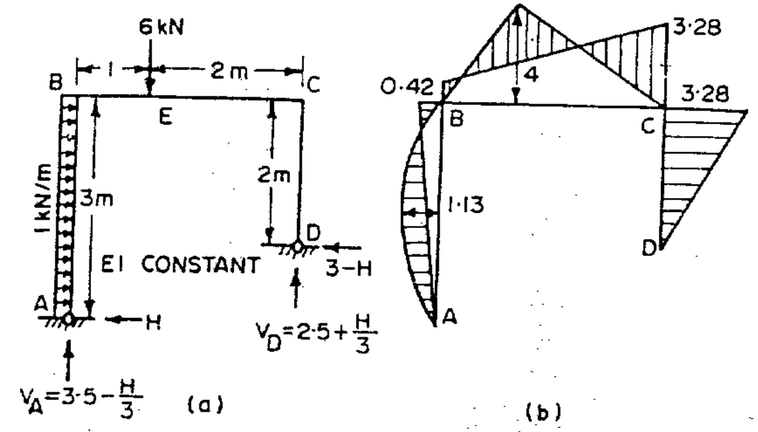


Fig. 12·16.

By taking the moments at D, we get

$$(V_A\times 3)+H(3-2)+(3\times 1)(2\times 1{\cdot}5)-(6\times 2)=0.$$

$$V_A=3{\cdot}5-\frac{H}{3}$$

and hence $V_D=6-V_A=2{\cdot}5+\frac{H}{3}$

By the theorem of minimum strain energy,

$$\frac{\partial U}{\partial H}=0=\frac{\partial U_{AB}}{\partial H}+\frac{\partial U_{BE}}{\partial H}+\frac{\partial U_{CE}}{\partial H}+\frac{\partial U_{DC}}{\partial H}$$

(1) *For member AB*

Taking A as the origin,

$$M=+\frac{1\times x^2}{2}-H\,.\,x$$

$$\frac{\partial M}{\partial H}=-x$$

$$\frac{\partial U_{AB}}{\partial H}=\frac{1}{EI}\int_0^3 M\,\frac{\partial M}{\partial H}dx=\frac{1}{EI}\int_0^3\left(\frac{x^2}{2}-Hx\right)\left(-x\right)dx$$

$$=\left[\frac{1}{EI}\left(\frac{Hx^3}{3}-\frac{x^4}{8}\right)\right]_0^3=\frac{1}{EI}(9H-10{\cdot}12) \qquad (1)$$

(2) *For the member BE*

Taking B as the origin

$$M=(-H\times3)+(3\times1{\cdot}5)-\left(3{\cdot}5-\frac{H}{3}\right)x$$

$$=-3H+4{\cdot}5-3{\cdot}5x+\frac{Hx}{3}$$

$$\frac{\partial M}{\partial H}=-3+\frac{x}{3}$$

$$\therefore\quad \frac{\partial M_{BE}}{\partial H}=\frac{1}{EI}\int_0^1 M\frac{\partial M}{\partial H}dx$$

$$=\frac{1}{EI}\int_0^1\left(-3H+4{\cdot}5-3{\cdot}5x+\frac{Hx}{3}\right)\left(-3+\frac{x}{3}\right)dx$$

$$=\frac{1}{EI}(8{\cdot}04H-7{\cdot}9) \qquad (2)$$

(3) *For the member CE*

Taking C as the origin,

$$M=+(3-H)\times2-\left(2{\cdot}5+\frac{H}{3}\right)x$$

$$\frac{\partial M}{\partial H}=-2-\frac{x}{3}$$

$$\therefore\quad \frac{\partial U_{CE}}{\partial H}=\frac{1}{EI}\int_0^2 M\frac{\partial M}{\partial H}dx$$

$$=\frac{1}{EI}\int_0^2\left\{(3-H)\times2-\left(2{\cdot}5+\frac{H}{3}\right)x\right\}\left(-2-\frac{x}{3}\right)dx$$

$$=\frac{1}{EI}(10{\cdot}96H-15{\cdot}78) \qquad (3)$$

(4) *For the member DC*

Taking D as the origin

$$M=+(3-H)x$$

$$\frac{\partial M}{\partial H}=-x$$

$$\therefore\quad \frac{\partial U_{DC}}{\partial H}=\frac{1}{EI}\int_0^2 M\frac{\partial M}{\partial H}dx=\frac{1}{EI}\int_0^2(3-H)x(-x)dx$$

$$=\frac{1}{EI}(2{\cdot}67H-8) \qquad (4)$$

Substituting the values, we get

$$\frac{\partial U}{\partial H}=0=\frac{1}{EI}[(9H-10{\cdot}12)+(8{\cdot}04H-7{\cdot}9)+(10{\cdot}96H-15{\cdot}78)$$
$$+(-8+2{\cdot}67H)]$$

or $\qquad 30{\cdot}67\,H=41{\cdot}80$

From which **H=1·36 kN.**

Hence $V_A=3{\cdot}5-\frac{H}{3}=3{\cdot}5-\frac{1{\cdot}36}{3}=3{\cdot}05$ kN

$V_D=6-3{\cdot}05=2{\cdot}95$ kN

$M_A=0$

$M_B=\frac{1\times3^2}{2}-(1{\cdot}36\times3)=+0{\cdot}42$ kN-m

$M_C=(3-H)2=(3-1{\cdot}36)2=+3{\cdot}28$ kN-m

$M_D=0$

The B.M. diagram is shown in Fig. 12·16(*b*).

Example 12·15. *A frame ABC is hinged at A and C, and has stiff joints at B, as shown in Fig. 12·17(a). Analyse the frame completely, and draw the B.M. diagram.*

Solution

The structure is statically indeterminate to single degree. Horizontal reactions at A and C will be equal, of magnitude H, say. Treating H as redundant.

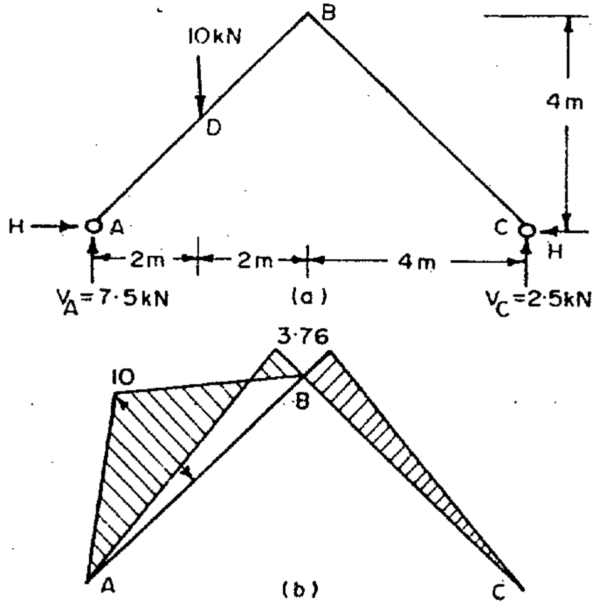


Fig. 12·17.

$$\frac{\partial U}{\partial H}=0=\frac{\partial U_{AD}}{\partial H}+\frac{\partial U_{DB}}{\partial H}+\frac{\partial U_{CB}}{\partial H} \qquad (1)$$

Taking moments about C, $V_A=7{\cdot}5$; Hence $V_C=2{\cdot}5$ kN.

It must always be remembered that in the case of curved or inclined members, the integrations must be taken along the members of frame.

If (x, y) are the co-ordinates of any point, and s is its distance along the member, from the origin, we have

$x=y=0{\cdot}707\ s$ (from the geometry of the frame).

(1) *For the member AD :*

Length $AD=2{\cdot}83$ m.

Taking A as the origin,

$$M=-7{\cdot}5x+Hy$$

Writing x and y in terms of s,

$$M=-5{\cdot}3s+0{\cdot}707\ H.s.$$

$$\frac{\partial M}{\partial H}=+0{\cdot}707s$$

$$\frac{\partial U_{AD}}{\partial H}=\frac{1}{EI}\int_0^{2\cdot83}(-5{\cdot}3s-0{\cdot}707\ Hs)(+0{\cdot}707s)\ ds$$

$$=\frac{1}{EI}(3{\cdot}75\ H-28) \qquad (2)$$

(2) *For the member DB*

Taking A as the origin,

$$M=-7{\cdot}5x+Hy+10(x-2)$$
$$=-5{\cdot}3s+0{\cdot}707\ Hs+10(0{\cdot}707s-2)$$
$$=1{\cdot}77s+0{\cdot}707\ Hs-20$$

$\dfrac{\partial M}{\partial H}=0{\cdot}707\ s$; Length $AB=5{\cdot}66$ m.

$$\therefore\ \frac{\partial U_{DB}}{\partial H}=\frac{1}{EI}\int_{2\cdot83}^{5\cdot66}(1{\cdot}77s+0{\cdot}707\ Hs-20)(0{\cdot}707s)\ ds$$

$$=\frac{1}{EI}(26{\cdot}6\ H-103) \qquad (3)$$

(3) *For the member BC*

Length $BC=5{\cdot}66$ m

Taking C as the origin,

$$M=-2{\cdot}5x+Hy$$
$$=-1{\cdot}77s+0{\cdot}707\ Hs$$

$$\frac{\partial M}{\partial H}=0{\cdot}707\ s$$

$$\therefore\ \frac{\partial U_{CB}}{\partial H}=\frac{1}{EI}\int_{0}^{5\cdot66}(-1{\cdot}77s+0{\cdot}707\ H.s)(0{\cdot}707\ s)\ ds$$

$$=\frac{1}{EI}(30{\cdot}4H-76) \qquad (4)$$

Substituting the values in (1), we get

$$\frac{\partial U}{\partial H}=0=\frac{1}{EI}\left[(3{\cdot}75H-28)+(26{\cdot}6H-103)+(30{\cdot}4H-76)\right]$$

or $\quad 60{\cdot}75H-207=0$

from which $\quad H=3{\cdot}44$ kN

Hence, we have

$$M_B=+(H\times4)-(V_B\times4)$$
$$=(3{\cdot}44\times4)-(2{\cdot}5\times4)$$
$$=\mathbf{3{\cdot}76\ kN\text{-}m}$$

The B.M. diagram is shown in Fig. 12·17(b).

Example 12·16. *A beam AB of span 3 m is fixed at both the ends and carries a point load of 9 kN at C distant 1 m from A. The moment of inertia of the portion AC of the beam is 2I and that of portion CB is I. Calculate the fixed end moments and reactions.*

Solution

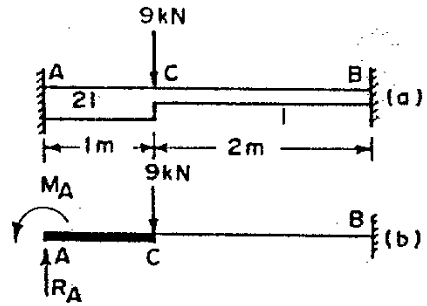


Fig. 12·18.

This is a problem of *second degree* indeterminacy. There are four unknowns M_A, R_A, M_B and R_B. Only two equations of statics are available, *i.e.* $\Sigma V=0$ and $\Sigma M=0$. Let us Choose M_A and R_A as redundants, as shown in Fig. 12·18(b).

Since the end A does not settle, we have

$$\delta_A=\frac{\partial U_{AB}}{\partial R_A}=0=\int_A^B\frac{Mx}{EI}\ \frac{\partial Mx}{\partial R_A}dx \qquad (1)$$

Again, since the end A does not rotate, we have

$$\phi_A=\frac{\partial U_{AB}}{\partial M_A}=0=\int_A^B\frac{Mx}{EI}\ \frac{\partial Mx}{\partial M_A}dx \qquad (2)$$

Thus, there are two unknowns : M_A and R_A, and we have two equations from the theorem of minimum strain energy.

(1) *For portion AC*

Taking A as the origin, we have

$$M_x = +M_A - R_A.x$$

$$\frac{\partial M_x}{\partial R_A} = -x$$

$$\frac{\partial M_x}{\partial M_A} = +1$$

Moment of inertia$=2I$

Limits of x : 0 to 1 m

Hence $$\int_A^C \frac{M_x}{EI}\frac{\partial M_x}{\partial R_A}\,dx = \int_0^1 \frac{(M_A - R_A.x)(-x)}{2EI}\,dx$$

$$= \frac{1}{2EI}\left[\frac{-M_A(1)^2}{2} + \frac{R_A(1)^3}{3}\right] = \frac{1}{2EI}\left[\frac{R_A}{3} - \frac{M_A}{2}\right]$$

and $$\int_A^C \frac{M_x}{EI}\frac{\partial M_x}{\partial M_A}\,dx = \int_0^1 \frac{(M_A - R_A.x)(+1)}{2EI}\,dx$$

$$= \frac{1}{2EI}\left[M_A(1) - \frac{R_A(1)^2}{2}\right] = \frac{1}{2EI}\left[M_A - \frac{R_A}{2}\right]$$

(2) *For portion CB*

Taking A as the origin, we have

$$M_x = +M_A - R_A.x + 9(x-1)$$

$$\frac{\partial M_x}{\partial R_A} = -x$$

$$\frac{\partial M_x}{\partial M_A} = +1$$

Moment of inertia$=I$

Limits of x : 1 to 3 m

Hence $$\int_C^B \frac{M_x}{EI}\frac{\partial M_x}{\partial R_A}\,dx = \frac{1}{EI}\int_1^3 \{M_A - R_A x + 9(x-1)\}(-x)\,dx$$

$$= \frac{1}{EI}\left[-4M_A + \frac{26}{3}R_A - 42\right]$$

and $$\int_C^B \frac{M_x}{EI}\frac{\partial M_x}{\partial M_A}\,dx = \frac{1}{EI}\int_1^3 \{M_A - R_A.x + 9(x-1)\}\,dx$$

$$= \frac{1}{EI}(2M_A - 4R_A + 18)$$

Substituting these values in Eqs. (1) and (3), we get

$$\frac{\partial U_{AB}}{\partial R_A} = 0 = \frac{1}{2EI}\left[\frac{R_A}{3} - \frac{M_A}{2}\right] + \frac{1}{EI}\left[-4M_A + \frac{26}{3}R_A - 42\right]$$

or $$2{\cdot}08\,R_A - M_A = 9{\cdot}88 \qquad \ldots(3)$$

and $$\frac{\partial U_{AB}}{\partial M_A} = 0 = \frac{1}{2EI}\left[M_A - \frac{R_A}{2}\right] + \frac{1}{EI}\left[2M_A - 4R_A + 18\right]$$

or $$M_A - 1{\cdot}7\,R_A = -7{\cdot}2 \qquad \ldots(4)$$

Solving (3) and (4), we get

$M_A = 4{\cdot}8$ kN-m (*i.e.* assumed direction is correct)

$R_A = +7{\cdot}05$ **kN-m** (*i.e.* the assumed direction is correct)

To find M_B, take moments at B, and apply the condition $\Sigma M=0$ there. Take clockwise moment as positive and anti-clockwise moment as negative. Taking M_B clockwise, we have

$\therefore \quad M_B - M_A + R_A(3) - 9\times 2 = 0$

or $\quad M_B - 4{\cdot}8 + (7{\cdot}05\times 3) - 18 = 0$

$\therefore \quad M_B = +1{\cdot}65$ **kN-m** (*i.e.* assumed direction is correct)

To find R_B, apply $\Sigma V=0$ for the whole frame

$\therefore \quad R_B = 9 - R_A = 9 - 7{\cdot}05 = 1{\cdot}95$ **kN.**

Example 12·17. *Using Castigliano's theorem of minimum strain energy, analyse the frame shown in Fig. 12·19 (a). EI is constant for the whole frame.*

Solution

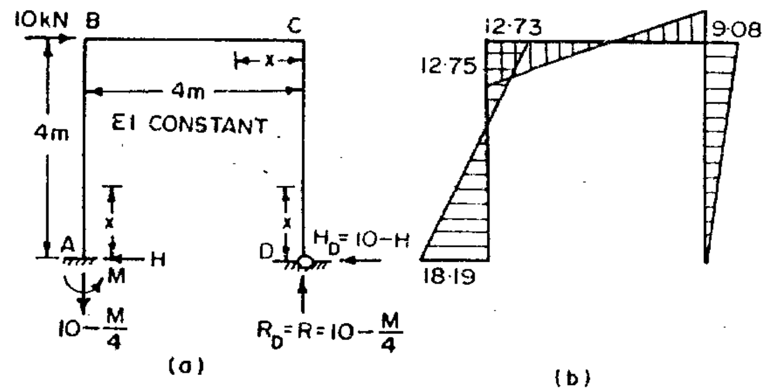


Fig. 12·19.

There are five unknowns : M, R and H at A, H_D and R_D at D. Let all these reactions act in the directions as shown in Fig. 12·19 (*a*). Since three equations (*i.e.* $\Sigma H=0$, $\Sigma V=0$, $\Sigma M=0$) are available from statical equilibrium, the frame is statically indeterminate to *second degree*. Let M and H (both at A) be the redundants.

(*i*) Since ΣH is zero for the whole frame :

$$H_D + H - 10 = 0$$

or $$H_D = 10 - H \qquad (1)$$

(*ii*) Since ΣV is zero for the whole frame :

$$R_D = R \text{ (numerically)} \qquad (2)$$

where R is the vertical reaction at A, acting downwards.

(*iii*) To find the value of R in terms of M and H, apply the third equation of statical equilibrium, *i.e.*, $\Sigma M=0$. Taking moments at D, treating clockwise as positive, we get

$$-M-4R+40=0$$

or

$$R=10-\frac{M}{4}=R_D \qquad (3)$$

Thus the other three reactions (*i.e.* R, R_D and H_D) are known in terms of the redundants H and M.

From the theorem of minimum strain energy,

$$\frac{\partial U}{\partial M}=0=\int \frac{M_x}{EI}\cdot\frac{\partial M_x}{\partial M}\,dx \qquad \text{(I)}$$

and

$$\frac{\partial U}{\partial H}=0=\int \frac{M_x}{EI}\cdot\frac{\partial M_x}{\partial H}\,dx \qquad \text{(II)}$$

The calculations for various quantities are performed in the tabular form below :

Member	M_x	$\partial M_x/\partial M$	$\partial M_x/\partial H$	*Limits of x*
AB	$+M-Hx$	$+1$	$-x$	0 to 4
DC	$+(10-H)x$	0	$-x$	0 to 4
CB	$+(10-H)4-\left(10-\frac{M}{4}\right)x$	$+\frac{x}{4}$	-4	0 to 4

$$\frac{\partial U}{\partial M}=0=\int_0^4 (M-Hx)(1)dx+\int_0^4 (10-H)x(0)dx+$$

$$\int_0^4 \left\{\left(10-H\right)4-\left(10-\frac{M}{4}\right)x\right\}\left(+\frac{x}{4}\right)dx$$

$$=\left[Mx-\frac{Hx^2}{2}\right]_0^4+\left[\text{ zero }\right]+\left[\frac{(10-H)x^2}{2}-\left(10-\frac{M}{4}\right)\frac{x^3}{12}\right]_0^4$$

which gives $M-3H+5=0$

Similarly,

$$\frac{\partial U}{\partial H}=0=\int_0^4 (M-Hx)(-x)\,dx+\int_0^4 (10-H)x(-x)+$$

$$\int_0^4 \left\{(10-H)\,4-\left(10-\frac{M}{4}\right)x\right\}(-4)dx$$

$$=\left[-\frac{Mx^2}{2}+\frac{Hx^3}{3}\right]_0^4+\left[-\frac{10x^3}{3}+\frac{Hx^3}{3}\right]_0^4$$

$$+\left[-160x+16Hx+\frac{40x^2}{2}-\frac{Mx^2}{2}\right]_0^4$$

which gives $\quad -M+\frac{20}{3}H-\frac{100}{3}=0 \qquad$ (II)

Solving (I) and (II), we get

$$H=+7{\cdot}73 \text{ kN}$$

and

$$M=+18{\cdot}19 \text{ kN-m}$$

Plus sign with the numerical values of H and M indicates that the assumed directions are correct.

The other reactions are

$$R=10-\frac{M}{4}=+5{\cdot}45 \text{ kN (}i.e. \text{ downwards)}$$

$$R_D=+5{\cdot}45 \text{ kN}$$

$$H_D=10-H=+2{\cdot}27 \text{ kN}$$

$$M_B=+M-4H=18{\cdot}19-30{\cdot}92=-12{\cdot}73 \text{ kN-m}$$

$$M_C=+4H_D=+9{\cdot}08 \text{ kN-m}$$

The B.M. diagram has been shown in Fig. 12·19(*b*).

PROBLEMS

1. A simply supported beam with overhang is loaded as shown in Fig. 12·20. Using the theorem of Castigliano find the vertical deflection of point *C*.

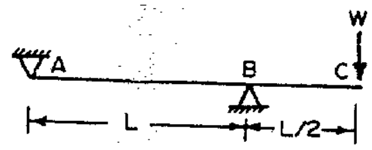


Fig. 12·20.

2. A cantilever of length L is loaded with uniformly distributed load w per unit length over the whole span. It is propped at the free end. Calculate the prop reaction if (*a*) the prop is rigid (*b*) prop yields by an amount λ under unit load.

3. For a uniformly loaded beam *AB* with built-in ends, determine the end moments by using theorem of Castigliano.

4. A cantilever *AB*, loaded at the end *B*, is supported by a short cantilever *CD* of the cross-section as cantilever *AB* as shown in Fig. 12·21. Prove that the pressure X between the two beams at C is given by $X=\frac{3P}{4}\left(\frac{L}{L_1}-\frac{1}{3}\right)$ (*A.M.I.E.*)

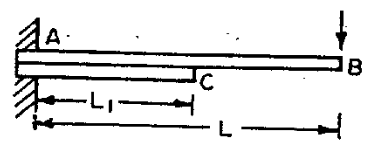


Fig. 12·21.

5. A frame shown in Fig. 12·22 consists of a column *AB*, fixed at *A* and having rigid connection at *B* with a double cantilever. The frame carries a point load W at *C*. Calculate the vertical deflection of *C* and *D*. *EI* is constant for the whole stucture.

6. A structure shown in Fig. 12·23 consists of an upright cantilever AB of length $3R$ and a semicircular portion BC of radius R. The flexual rigidity is constant throughout. It carries a load W, acting vertically at C. Determine the vertical deflection of C and the horizontal deflection of B.

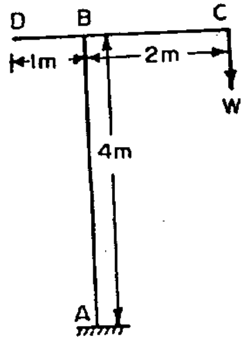


Fig. 12·22.

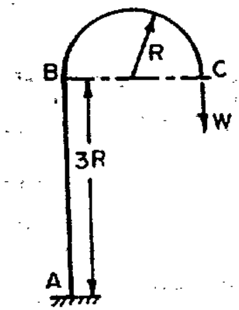


Fig. 12·23.

7. A circular bar is bent into the shape of a half-ring and supported in a vertical plane as shown in Fig. 12·24. Determine the horizontal movement of point C and the vertical movement of point B.

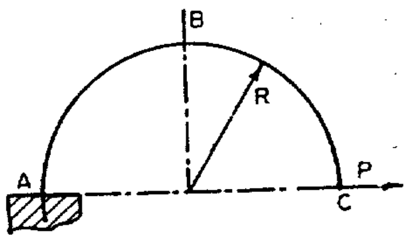


Fig. 12·24.

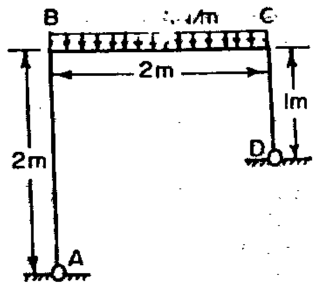


Fig. 12·25.

8. Using the method of minimum strain energy, analyse the portal frame shown in Fig. 12·25. Plot the B.M. diagram. EI is constant.

9. Solve problem 9 (Fig. 11·21) of chapter 11 by minimum strain energy.

Answers

1. $\frac{WL^3}{8EI}$.

2. (a) $\frac{3}{8}wL$. (b) $\frac{3}{8}\ \frac{wL^4}{L^3+3EI\lambda}$.

3. $M_A=M_B=\frac{wL^2}{12}$.

5. $\delta_C=\frac{56W}{3EI}\downarrow$; $\delta_D=\frac{8W}{EI}\uparrow$.

6. $\delta_{CV}=16{\cdot}71\ \frac{WR^3}{EI}$; $\delta_{BH}=\frac{9WR^3}{EI}$.

7. $\delta_{CH}=\frac{PR^3\pi}{2EI}$; $\delta_{BV}=\frac{PR^3}{2EI}$.

8. $M_B=+0{\cdot}78$; $M_C=+0{\cdot}39$; $H_A=0{\cdot}39\rightarrow$
$H_D=0{\cdot}39\leftarrow$; $R_A=3{\cdot}2$; $R_D=2{\cdot}8$.

13

Deflection of Perfect Frames

13·1. GENERAL

An articulated structure or a truss is composed of a number of bars or members connected by frictionless pins, forming geometrical figures which are usually triangles. A truss is said to be statically determinate internally if it has the members given by the equation

$$m=2j-r$$

where m=Total number of members
j=Total number of joints.
r=Total number of condition equations available.

In the above equation, the value of r is usually 3 but if there is an additional hinge separating the structure in two parts, $r=3+1=4$, and if there is a link somewhere in the structure, $r=3+2=5$. The frame is said to be perfect if the number m is equal to right hand side of the equation.

When external loads are applied on the truss, at the joints, the members carry internal forces, usually called the stresses which may be either tensile or compressive. According to Hooke's law, if any axial member of length L carries a force P, it will be deformed by an amount $\frac{PL}{AE}$ where A is the area of cross-section of the members. In the truss, therefore, the members carrying tension will be elongated while those carrying compression will be shortened. Thus, all the joints will move from their initial position, and will occupy their final equilibrium position. The axial deformation in the members of the truss may also be due to temperature changes or due to errors in fabrication or lack of fit of some members, and the joints may move from their original position. *This movement of each joint is defined as the deflection of the joint in the direction of movement.* The resolved part of the movement in the vertical direction is called *vertical deflection* of the joint, and that in the horizontal direction is known as the *horizontal deflection* of the joint. In general, each joint has both vertical as well as horizontal deflection, unless constrained to move in a given direction.

There are various methods of computing the joint deflection of a perfect frame. We shall, however, discuss the following methods :

1. The unit load method.
2. Deflection by Castigliano's first theorem.
3. Graphical method : By Williot-Mohr Diagram.

The first two methods are analytical, and require the full knowledge of the methods of finding the stress in plane frames under static loading (see Author's Vol. I).

13·2. THE UNIT LOAD METHOD

To develop the method, let us consider a perfect frame as shown in Fig. 13.1, where W_1, W_2, W_3 ..W_n etc., are external loads.

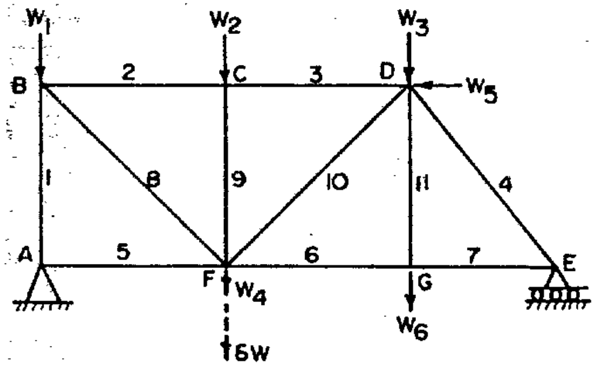


Fig. 13.1.

Let P_1, P_2, P_3,...P_n etc., be the forces or stresses in the members 1, 2, 3, ...n etc., due to the external loading.

Let us find the vertical displacement (δv) of the joint F due to the external system of loads shown.

Apply gradually an infinitely small load δW at the joint F in the vertical direction, and let δv be the deflection of the joint. The work by δW will be $\frac{1}{2}\ \delta W \,.\, \delta v$ (2)

If u_1, u_2, u_3..., u_n are the forces in the various members due to *unit vertical load* at the joint F, the forces in these members due to a load δW at F will be $u_1.\delta W$, $u_2.\delta W$, $u_3.\delta W$,...$u_n\delta W$, respectively.

If any member of length L has the force $u.\delta W$, its extra deformation due to δW at F will be $=\frac{(u\,.\,\delta W)\,L}{AE}$.

$$\therefore \text{ Work stored in the member} = \frac{1}{2}\text{ (force) (deformation)}$$

$$= \frac{1}{2}(P+u\,.\,\delta W)\times\frac{u.\delta W.L}{AE}$$

$$= \frac{1}{2}\ \frac{P.u\,L.\delta W}{AE}$$

(neglecting the product of small quantities)

Total work stored in all the members

$$= \sum_1^n \frac{1}{2}\,.\,\frac{P.u\,L.\delta W}{AE}$$

$$= \frac{1}{2}\delta W \sum_1^n \frac{PuL}{AE} \qquad (2)$$

Equating the work supplied to the work stored, we get

$$\frac{1}{2}\delta W.\delta v = \frac{1}{2}\delta W.\sum_1^n \frac{PuL}{AE}$$

$$\delta v = \sum_1^n \frac{PuL}{AE} = \sum_1^n \frac{puL}{E} \qquad (13\cdot1)$$

where n is the total number of members.

If, however, horizontal deflection is required, it can similarly be proved that

$$\delta_H = \sum_1^n \frac{Pu'L}{AE} \qquad (13\cdot2)$$

where P=Force in any number due to external loads.

p=Intensity of stress in any member due to external loads.

u=Force in any member due to unit vertical load applied at the joint where deflection is required.

u'=Force in any number due to unit horizontal load applied at the joint where deflection is required.

Steps :

The method of computing the deflection of a joint can be summarised below :

1. Find the forces P_1, P_2,..., P_n in all the members due to external loads.

2. Remove the external loads and apply the unit vertical load at the joint if the vertical deflection of the joint is required, and find the stress $u_1, u_2, \ldots, u_n$ in all the members. (If horizontal deflection is required, apply unit horizontal load there and find $u'_1, u'_2, \ldots, u'_n$).

3. Apply equation 13·1 for vertical deflection and 13·2 for horizontal deflection of the joint.

13·3. JOINT DEFLECTION IF LINEAR DEFORMATION OF ALL THE MEMBERS ARE KNOWN

If, in the place of external loads, the deformations $\Delta_1, \Delta_2 \ldots, \Delta_n$ etc. of all the members are known, the deflection δ can be calculated as follows :

Equation 13·1 can be rewritten as

$$\delta_V = \sum_1^n u . \frac{PL}{AE}$$

But $\frac{PL}{AE}$ = deformation of the member $= \Delta$ (according to Hooke's law).

$$\therefore \quad \delta_V = \sum_1^n u\Delta \qquad (13·4)$$

Similarly,

$$\delta_H = \sum_1^n u'\Delta \qquad (13·5)$$

Hence, in order to find the deflection δ in such cases apply unit load at the joint, in the direction the deflection is required, and apply Eq. (13·4) or (13·5). See example 13·9 for illustration.

DEFLECTION OF A JOINT DUE TO TEMPERATURE VARIATION

Let $\Delta_1, \Delta_2, \Delta_3 \ldots, \Delta_n$ be the changes in the lengths of various members of a perfect frame due to temperature variation. To find the deflection of an unloaded frame due to temperature variations apply the unit load at the joint and calculate $u_1, u_2 \ldots$ and apply equation 13·4 (or 13·5) for the joint deflection.

$$\text{Thus,} \qquad \delta = \sum_1^n u\Delta_1 = u_1\Delta_1 + u_2\Delta_2 \ldots\ldots u_n\Delta_n \qquad (13·6)$$

If the change in length (Δ) of certain members is zero, the product $u.\Delta$ for those members will be substituted as zero in the above equation. If, for example, there is only one member in which

there is change in length Δ_1, the deflection of a particular joint will be equal to $u_1\Delta_1$, where u_1 is the stress in that member due to the unit load at the joint under consideration. See example 13·10.

DEFLECTION OF A JOINT DUE TO LACK OF FIT OF CERTAIN MEMBERS

Let $\Delta_1, \Delta_2 \ldots\ldots \Delta_n$ be the lack of fit in the members. The joint deflection can be found by equation 13·4 or 13·5, *i.e.*

$$\delta = \sum_1^n u\Delta = u_1\Delta_1 + u_2\Delta_2 + \ldots u_n\Delta_n$$

If there is only one member having lack of fit Δ_1, the deflection of a particular joint will be equal to $u_1\Delta_1$, where u_1 is the stress in that member due to unit load at the joint under consideration.

Example 13·1. *Determine the vertical and horizontal displacements of the point C of the pin-joined frame shown in Fig. 13·12(a).*

The cross-sectional area of AB is 100 sq. mm and of AC and BC 150 sq mm each. $E = 2 \times 10^5$ N/mm².

Solution

The vertical and horizontal deflections of the joint C are given by,

$$\delta_V = \Sigma \frac{PuL}{AE} \qquad (1)$$

and

$$\delta_H = \Sigma \frac{Pu'L}{AE} \qquad (2)$$

Let us now find P, u and u' in each member.

(a) *Stresses due to external loading*

$$AC = \sqrt{3^2 + 4^2} = 5 \text{ m}$$

$$\sin\theta = \frac{3}{5} = 0·6 \; ; \quad \cos\theta = \frac{4}{5} = 0·8$$

Resolving at the joint C, we get

$$6 = P_{AC} \sin\theta + P_{BC} \sin\theta$$

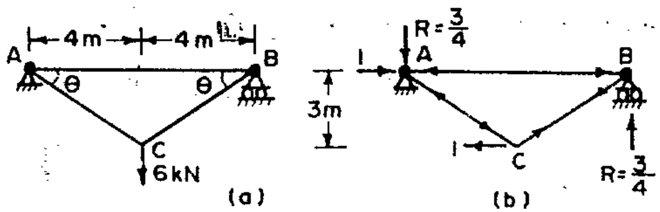


Fig. 13·2.

Resolving horizontally, $P_{AC}=P_{BC}$

$\therefore$ $2P_{AC}\sin\theta=6$, from which

$$P_{AC}=P_{BC}=\frac{6}{2\sin\theta}=\frac{6}{2\times0{\cdot}6}=+5\text{ kN (tension)}$$

(Use + sign for tension and − sign for compression)

Resolving horizontally at A,

$$P_{AB}=P_{AC}\cos\theta=5\times0{\cdot}8=4\text{ kN (comp.)}$$
$$=-4\text{ kN.}$$

(*b*) *Stresses due to unit vertical load at C*

Apply unit vertical load at C.

The stresses in each member will be $\frac{1}{6}$th of those obtained above.

Thus $$u_{AC}=u_{BC}=+\frac{5}{6}$$

and $$u_{AB}=-\frac{4}{6}=-\frac{2}{3}$$

(*c*) *Stresses due to unit horizontal load at C*

Assuming that the horizontal movement of joint C is to the left, apply a unit horizontal load at C as shown in Fig. 13·2(*b*), along with the reactions.

Resolving vertically at joint C, we get

$$u'_{CA}=u'_{CB}\text{ (numerically)}$$

Resolving horizontally,

$$u'_{CB}\cos\theta+u'_{CA}\cos\theta=1$$

or $$u'_{CB}=u'_{CA}=\frac{1}{2\cos\theta}=\frac{1}{2\times0{\cdot}8}=\frac{5}{8}\text{ kN}$$

$\therefore$ $$u'_{CB}=+\frac{5}{8}\text{ kN};\; u'_{CA}=-\frac{5}{8}\text{ kN}$$

Resolving horizontally at B, we get

$$u'_{AB}=u'_{BC}\cos\theta=\frac{5}{8}\times0{\cdot}8=0{\cdot}5\text{ kN (comp.)}$$

To calculate $\frac{PuL}{A}$ and $\frac{Pu'L}{A}$, the results are tabulated below

Member	Length L (mm)	Area (mm)²	P (kN)	u (kN)	u′ (kN)	$\frac{PuL}{A}$	$\frac{Pu'L}{A}$
AB	8000	100	−4	−2/3	−1/2	+640/3	+160
BC	5000	150	+5	+5/6	+5/8	+2500/18	+2500/24
CA	5000	150	+5	+5/6	−5/8	+2500/18	−2500/24
					Sum	+491	+160

$$E=2\times10^5\text{ N/mm}^2=200\text{ kN/mm}^2$$

$$\delta_V=\Sigma\frac{PuL}{AE}=+\frac{491}{200}=+2{\cdot}45\text{ mm}$$

$$\delta_H=\Sigma\frac{Pu'L}{AE}=+\frac{160}{200}=+0{\cdot}8\text{ mm.}$$

(The plus signs indicate that the assumed directions are correct).

Check

Since the end B is supported on roller, the movement of B is horizontal only, and is equal to the deformation of the bar AB. Since the structure is symmetrical about C, and loading is also central, it is evident that horizontal movement of $C=\frac{1}{2}\times$ deformation of AB.

$\therefore$ $$\delta_H=\Sigma\frac{1}{2}\frac{P_{AB}\,.\,L_{AB}}{A_{AB}\,.\,E}$$
$$=\frac{1}{2}\frac{4\times8000}{100\times200}=0{\cdot}8\text{ mm}$$

Example 13·2. *A frame ABCD consists of two equilateral triangles and is hinged at A and supported on rollers at D as shown in Fig. 13·3. Determine the vertical deflection of C and horizontal movement of D due to a load W applied vertically at C. All the members are of Length L. All the tension member are of area a and compression members of area 2a.*

Solution

The vertical deflection of C is given by

$$\delta_{CV}=\sum_{1}^{n}\frac{PuL}{AE}\qquad(1)$$

where u is the stress due to unit vertical load at C.

The horizontal deflection of D is given by

$$\delta_{DH}=\sum_{1}^{n}\frac{Pu'L}{AE} \qquad (2)$$

where u' is the stress due to unit horizontal load at D.

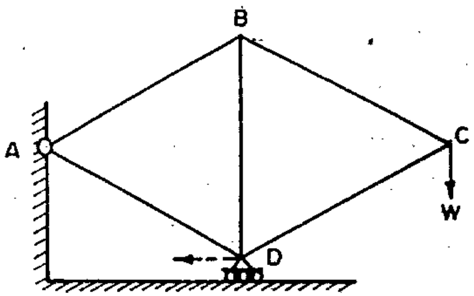


Fig. 13·3.

(a) *Stresses due to external loading*

$$\cos 60°=\frac{1}{2};\ \cos 30°=\frac{\sqrt{3}}{2}$$

Resolving horizontally at joint C,

$$P_{CB}=P_{CD}$$

Resolving vertically at joint C,

$$2P_{CB}\cos 60°=W$$

$\therefore$ $P_{BC}=W$ (tension)

and $P_{CD}=W$ (comp.)

Resolving horizontally at B,

$$P_{BA}=P_{BC}=W \text{ (tension)}$$

Resolving vertically at B,

$$P_{BD}=2P_{AB}\cos 60°=W \text{ (comp.)}$$

Resolving horizontally at D,

$$P_{DA}=P_{DC}=W \text{ (comp.)}$$

(b) *Stresses due to unit vertical load at C*

To find u in all members, put $W=1$ in the expressions found above.

(c) *Stresses due to unit vertical load at D*

The find the horizontal deflection of D, apply unit horizontal load at D, in the direction shown in Fig. 13·3.

By resolution at joint C,

$$u'_{BC}=u'_{CD}=0 \quad \text{(Since there is no load at } B)$$

Resolving horizontally at B,

$$u'_{BA}=u'_{BC}=0$$

Hence $u'_{BD}=0$

Resolving horizontally at D,

$$u_{DA}\cos 30°=1$$

or $$u_{DA}=\frac{1}{\cos 30°}=\frac{2}{\sqrt{3}} \text{ (comp.)}$$

The results are tabulated below :

Member	Length	Area	P	u	u'	$\frac{PuL}{AE}$	$\frac{pu'L}{AE}$
AB	L	a	$+W$	$+1$	0	$+\frac{WL}{aE}$	0
BC	L	a	$+W$	$+1$	0	$+\frac{WL}{aE}$	0
CD	L	$2a$	$-W$	-1	0	$+\frac{WL}{2aE}$	0
AD	L	$2a$	$-W$	-1	$-\frac{2}{\sqrt{3}}$	$+\frac{WL}{2aE}$	$+\frac{WL}{\sqrt{3}aE}$
BD	L	$2a$	$-W$	-1	0	$+\frac{WL}{2aE}$	0
					Sum	$+\frac{7WL}{2aE}$	$+\frac{WL}{\sqrt{3}aE}$

Hence $$\delta_{CV}=\Sigma\frac{PuL}{AE}=\frac{7WL}{2aE}\downarrow$$

and $$\delta_{DH}=\Sigma\frac{Pu'L}{AE}=\frac{WL}{\sqrt{3}aE}\leftarrow$$

Example 13·3. *Fig. 13·4 represents a crane structure attached to a vertical wall and carrying a vertical load of 20 kN at C.*

All tension members are stressed to 80 N/mm² and all compression members to 50 N/mm². Determine the horizontal and vertical deflection of the end C. Take E=2×10⁵ N/mm². All members, except CD, have a length of 2 m. AE=2 m.

Solution

The horizontal and vertical deflections of C are given by

$$\delta_V=\sum_1^n \frac{PuL}{E}=\sum_1^n \frac{PuL}{E} \qquad (1)$$

and $$\delta_H=\sum_1^n \frac{Pu'L}{E}=\sum_1^n \frac{Pu'L}{AE} \qquad (2)$$

where p=intensity of stress in each member, and is known.

(*a*) *Calculation of stresses due to unit vertical load at C.*

To calculate u in all members, apply a unit vertical load at C.

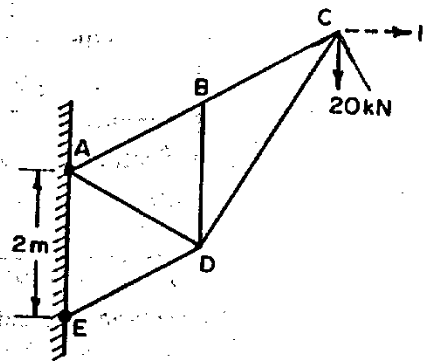


Fig. 13·4.

Resolving perpendicular to BC at C,

$$u_{CD}\sin 30°=1\times\sin 60°$$

$$\therefore \quad u_{CD}=\frac{\sin 60°}{\sin 30°}=\sqrt{3}\text{ (comp.)}$$

$$u_{CB}=u_{CD}\cos 30°-1\cos 60°$$

$$=\sqrt{3}\times\frac{\sqrt{3}}{2}-\frac{1}{2}=1{\cdot}0\text{ (tension)}$$

$u_{AB}=u_{BC}=1{\cdot}0$ (tension)

$u_{BD}=0$ (by resolving perpendicular to AB at B)

Resolving perpendicular to ED at D,

$$u_{AD}\sin 60°=u_{DC}\sin 30°$$

or $$u_{AD}=\sqrt{3}\times\frac{1}{2}\times\frac{2}{\sqrt{3}}=1{\cdot}0\text{ (tension)}$$

Resolving along ED,

$$u_{ED}=u_{AD}\cos 60°+u_{DC}\cos 30°=2{\cdot}0\text{ (comp.)}$$

(*b*) *Calculation of stresses due to unit horizontal load at C*,

Apply unit horizontal load at C, as shown.

Resolving perpendicular to BC, at C,

$$u'_{CD}\sin 30°=1{\cdot}0\sin 30°$$

$$u'_{CD}=1\text{ (comp.)}$$

Resolving along BC, at C,

$$u'_{CB}=u'_{CD}\cos 30°+1{\cdot}0\cos 30°$$

$$=\left(1{\cdot}0\times\frac{\sqrt{3}}{2}\right)+\left(1{\cdot}0\times\frac{\sqrt{3}}{2}\right)=\sqrt{3}\text{ (tension)}$$

$$\therefore \quad u'_{BA}=u'_{CB}=\sqrt{3} \quad \text{(tension)}$$

$$u'_{BD}=0$$

Resolving perpendicular to ED, at D,

$$u'_{AD}\sin 60°=u'_{CD}\sin 30°$$

$$\therefore \quad u'_{AD}=\frac{1}{2}\times\frac{2}{\sqrt{3}}=\frac{1}{\sqrt{3}}\text{ (tension)}$$

Resolving along ED, at D,

$$u'_{ED}=u'_{AD}\cos 60°+u'_{DC}\cos 30°$$

$$=\left(\frac{1}{\sqrt{3}}\times\frac{1}{2}\right)+\left(1\times\frac{\sqrt{3}}{2}\right)=\frac{2}{\sqrt{3}}\text{ (comp.)}$$

The results are tabulated below :

Member	*L* (*mm*)	*p* (*N/mm*²)	*u*	*u'*	*puL*	*pu'L*
AB	2000	+80	+1·0	$+\sqrt{3}$	$+16\times10^4$	$+16\sqrt{3}\times10^4$
BC	2000	+80	+1·0	$+\sqrt{3}$	$+16\times10^4$	$+16\sqrt{3}\times10^4$
AD	2000	+80	+1·0	$+\frac{1}{\sqrt{3}}$	$+16\times10^4$	$+\frac{16}{\sqrt{3}}\times10^4$
BD	2000	0	0	0	0	0
ED	2000	−50	−2·0	$-\frac{2}{\sqrt{3}}$	$+20\times10^4$	$+\frac{20}{\sqrt{3}}\times10^4$
DC	$2000\sqrt{3}$	−50	$-\sqrt{3}$	−1·0	$+30\times10^4$	$+10\sqrt{3}\times10^4$
				Sum	$+98\times10^4$	$+93{\cdot}5\times10^4$

$$\therefore \quad \delta_V=\sum_1^n\frac{puL}{E}=\frac{98\times10^4}{2\times10^5}=4{\cdot}9\text{ mm }(\downarrow)$$

and $$\delta_H=\sum_1^n\frac{pu'L}{E}=\frac{93{\cdot}5\times10^4}{2\times10^5}=4{\cdot}67\text{ cm. }(\rightarrow)$$

Example 13·4. *The steel truss shown in Fig. 13·5 is anchored at A and supported on rollers at B. If the truss is so designed that, under the given loading, all tension members are stressed to 100 N/mm² and all compression members to 80 N/mm², find the vertical deflection of the point C. Take E=2×10⁵ N/mm².*

Find also the lateral displacement of the end B.

Solution

The vertical deflection is given by

$$\delta_{CV}=\sum_1^n\frac{puL}{E}$$

where p is the stress due to external loading.

To find the values of u, apply a unit vertical load at C, and analyse the frame. The results are tabulated below. (The students are advised to work out the stresses themselves).

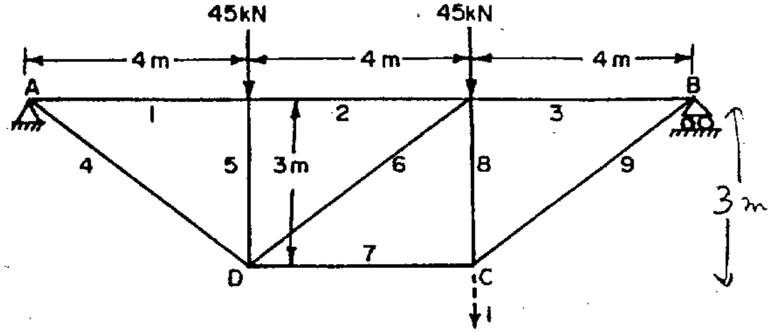


Fig. 13·5
(+for tension ; — for compression)

Member	Length (mm)	p (N/mm^2)	u	puL
1	4000	−80	−4/9	$+\frac{128}{9}\times 10^4$
2	4000	−80	−4/9	$+\frac{128}{9}\times 10^4$
3	4000	−80	−8/9	$+\frac{256}{9}\times 10^4$
4	5000	+100	+5/9	$+\frac{250}{9}\times 10^4$
5	3000	−80	0	0
6	5000	0	−5/9	0
7	4000	−100	+8/9	$+\frac{320}{9}\times 10^4$
8	3000	−80	+1/3	-8×10^4
9	5000	+100	+10/9	$+\frac{500}{9}\times 10^4$
			Sum	$+168\times 10^4$

$$\delta_{CV}=\sum_1^n \frac{puL}{E}=\frac{168\times 10^4}{2\times 10^5}=8{\cdot}4 \text{ mm}$$

(*b*) *Horizontal deflection of B*

Since the roller at B moves in the horizontal direction only, its movement will evidently be equal to the axial shortening of the members 1, 2 and 3.

Thus $\delta_{BH}=\Delta_1+\Delta_2+\Delta_3$

$$=\sum_1^3 \frac{pL}{E}=\frac{3(80\times 4000)}{2\times 10^5}=4{\cdot}8 \text{ mm } (\leftarrow).$$

Example 13·5. *The frame shown in Fig. 13·6 consists of four panels each 2·5 m wide, and the cross-sectional areas of the members are such that, when the frame carries equal loads at the panel points of the lower chord, the stress in all the tension members is f N/mm². and the stress in all the compression members of 0·8 f N/mm². Determine the value of f if the ratio of the maximum deflection to span is* $\frac{1}{900}$.

Take $E=2{\cdot}0\times 10^5$ N/mm^2.

Solution

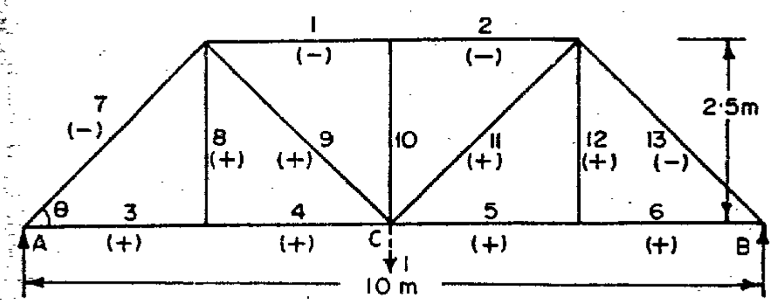


Fig. 13·6.

By inspection, it can be seen that for the loads at the panel points of the lower panel, the top chord members will be in compression, and the bottom chord members, verticals and diagonals will be in tension.

Due to symmetrical loading, the maximum deflection occurs at C. Apply unit load at C to find u in all the members. All the members have been numbered 1, 2... etc. in Fig. 13·6.

By inspection, $u_8=0$; $u_{10}=0$; $u_{12}=0$.

Reaction $R_A=R_B=\frac{1}{2}$; $\theta=45°$; $\cos\theta=\sin\theta=\frac{1}{\sqrt{2}}$

$$\therefore \quad u_7 = \frac{R_A}{\sin\theta} = \frac{\sqrt{2}}{2} \text{ (comp.)}$$

$$u_3 = u_7 \cos\theta = \frac{\sqrt{2}}{2} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2} \text{ (tension)} = u_4$$

$$u_9 = \frac{u_4}{\cos\theta} = \frac{\sqrt{2}}{2} \text{ (tension)}$$

Also, $\quad u_7 \cos\theta + u_9 \cos\theta = u_1$

$$\therefore \quad u_1 = \frac{\sqrt{2}}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{2} \times \frac{1}{\sqrt{2}} = 1{\cdot}0 \text{ (comp.)}$$

The results for *half* the truss are tabulated below :

Member	*Length L (mm)*	*p (N/mm²)*	*u*	*puL*
1	2500	$-0{\cdot}8f$	$-1{\cdot}0$	$+2000f$
3	2500	$+f$	$+\frac{1}{2}$	$+1250f$
4	2500	$+f$	$+\frac{1}{2}$	$+1250f$
7	$2500\sqrt{2}$	$-0{\cdot}8f$	$-\frac{\sqrt{2}}{2}$	$+2000f$
8	2500	$+f$	0	0
9	$2500\sqrt{2}$	$+f$	$+\frac{\sqrt{2}}{2}$	$+2500f$
			Sum	$+9000f$

$$\therefore \quad \delta_C = \sum_1^n \frac{puL}{E} = \frac{(9000f \times 2)}{2{\cdot}0 \times 10^5} = 0{\cdot}09f \text{ mm}$$

As per given condition, $\delta_C = \frac{1}{900} \times \text{span}$

$$= \frac{1}{900} \times 10000 = \frac{100}{9} \text{ mm}$$

Hence $\quad 0{\cdot}09f = \frac{100}{9}$

or $\quad f = \frac{100}{9 \times 0{\cdot}09} = 123{\cdot}5 \text{ N/mm}^2.$

Example 13·6. *Fig. 13·7(a) shows the outline of truss used for lifting a load which is disributed as 2 kN on each of the four points Q, R, S and T. The members PQ, PR, PS and PT, each have an area of 65 sq. mm and the members QR, RS and ST each have of 130 sq. mm. Determine the vertical deflection of Q and R relative to support P. Take E = 2×10⁵ N/mm².*

Solution

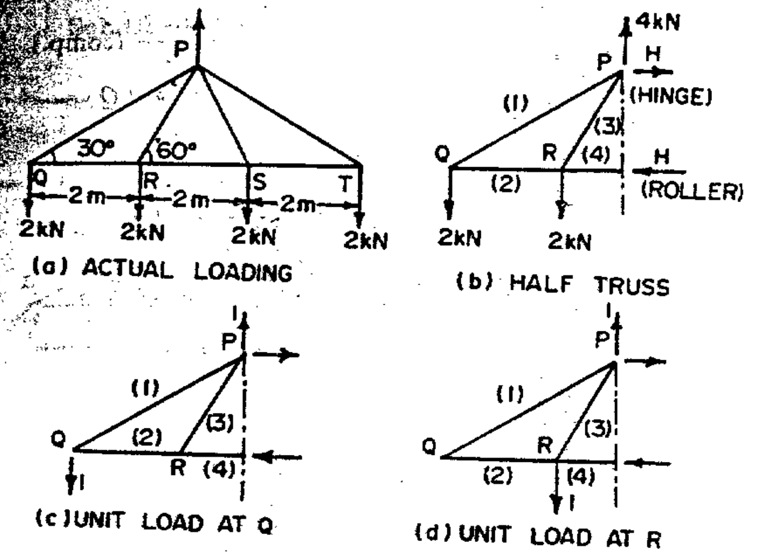


Fig. 13·7.

Since the loading and truss are symmetrical, consider half truss only. For the equilibrium of the half truss, the horizontal reaction (*H*) will be supplied by the other half of the truss as shown in Fig. 13·7 (*b*). Thus the half truss of Fig. 13·7 (*b*) is obtained by assuming the truss having cut by a vertical section through *P*, and the basic system of Fig. 13·7 (*a*) does not change. Fig. 13·7 (*c*) shows the half truss under unit vertical load at *Q*, and Fig. 13·7 (*d*) shows the half truss under unit vertical load at *R*.

The vertical deflection of *Q* and *R* are given by

$$\delta_{QV} = \Sigma \frac{PuL}{AE} \quad \text{(1)}$$

and

$$\delta_{RV} = \Sigma \frac{Pu'L}{AE} \quad \text{(2)}$$

The summation being made for half the truss only.

where u is the stress in any member due to unit vertical load at Q, and u' is the stress due to unit vertical load at R. The members have been numbered 1, 2, 3, etc.

(a) *Calculation of stresses due to external loading*

$$P_1 \sin 30° = 2 \; ; \qquad \therefore \quad P_1 = 4 \text{ (tension)}$$

$$P_2 = P_1 \cos 30° = \frac{4\sqrt{3}}{2} = 2\sqrt{2} \text{ (comp.)}$$

$$P_3 \sin 60° = 2 \; ; \qquad \therefore \quad P_3 = \frac{2\times 2}{\sqrt{3}} = \frac{4\sqrt{3}}{3} \text{ (tension)}$$

$$P_4 = P_3 \cos 60° + P_2 = \frac{4\sqrt{3}}{3} \times \frac{1}{2} + 2\sqrt{3} = \frac{8\sqrt{3}}{3} \text{ (comp.)}$$

(b) *Calculation of stresses due to unit vertical load at Q*

By inspection, $u_3 = 0$

$$u_1 \sin 30° = 1 : \qquad \therefore \quad u_1 = 2 \text{ (tension)}$$

$$u_2 = u_1 \cos 30° = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3} \text{ (comp.)}$$

$$u_4 = u_2 = \sqrt{3} \text{ (comp.)}$$

(c) *Calculation of stresses due to unit vertical load at R*

Since there is no load at Q.

$$u'_1 = 0$$

$$u'_2 = 0$$

$$u'_3 \sin 60° = 1, \qquad \therefore \quad u'_3 = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \text{ (tension)}$$

$$u'_4 = u'_3 \cos 60° = \frac{2\sqrt{3}}{3} \times \frac{1}{2} = \frac{\sqrt{3}}{3} \text{ (comp.)}$$

The results are tabulated below :

(+for tension ; −for comp.)

Member	*L* (*mm*)	*A* (*mm²*)	*p*	*u*	*u'*	$\frac{PuL}{A}$	$\frac{Pu'L}{A}$
1	$2000\sqrt{3}$	65	+4	+2	0	+426	0
2	2000	130	$-2\sqrt{3}$	$-\sqrt{3}$	0	+92·3	0
3	2000	65	$+\frac{4\sqrt{3}}{3}$	0	$+\frac{2\sqrt{3}}{3}$	0	+82
4	1000	130	$-\frac{8\sqrt{3}}{3}$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$	+61·5	+20·5
					Sum	+579·8	+102·5

$E = 2\times 10^5$ N/mm² $= 200$ kN/mm²

$$\therefore \qquad \delta_{QV} = \frac{579{\cdot}8}{200} = 2{\cdot}9 \text{ mm}$$

$$\delta_{RV} = \frac{102{\cdot}5}{200} = 0{\cdot}51 \text{ mm.}$$

Example 13·7. *The roof truss shown in Fig. 13·8 has members with cross-sectional areas such that when the loading is as shown, all members are subjected to the same intensity of stress either tensile or compressive. If the vertical deflection of joint C is 15 mm, determine the change in the span of the truss.*

Solution.

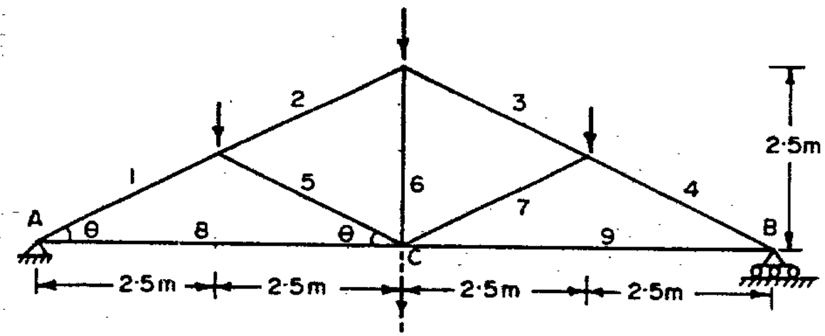


Fig. 13·8.

To find the vertical deflection of point C, apply a unit vertical load at C, as shown in Fig. 13·8. Let $\pm p$ be the intensity of stress in the members due external loading. The members have been numbered 1, 2, 3 etc.

$$L_1 = L_2 = L_3 = L_4 = L_5 = L_7 = \sqrt{(2{\cdot}5)^2 + \left(\frac{2{\cdot}5}{2}\right)^2} = \frac{2{\cdot}5}{2}\sqrt{5} \text{ m}$$

$$= 1250\sqrt{5} \text{ mm}$$

$$L_6 = 2{\cdot}5 \text{ m} \; ; \; L_8 = L_9 = 5 \text{ m}$$

$$\sin\theta = \frac{2{\cdot}5}{2{\cdot}5\sqrt{5}} = \frac{1}{\sqrt{5}}$$

$$\cos\theta = \frac{5}{2{\cdot}5\sqrt{5}} = \frac{2}{\sqrt{5}}$$

$$u_1 \sin\theta = \frac{1}{2} \qquad \text{or} \quad u_1 = \frac{1}{2}\sqrt{5} = \frac{\sqrt{5}}{2} = u_4 \text{ (comp.)}$$

$$u_8 = u_1 \cos\theta = \frac{\sqrt{5}}{2} \times \frac{2}{\sqrt{2}} = 1 = u_9 \text{ (tension)}$$

$u_6 = u_7 = 0$; $\quad \therefore \quad u_8 = 1$ (tension)

$u_2 = u_3 = u_1 = \dfrac{\sqrt{5}}{2}$ (comp.)

The results are tabulated below, giving correct signs to $\pm p$ obtained by inspection of Fig. 13·8 under the actual loads.

(+for tension ; —for compression)

Member	Length L(cm)	Stress p	u	puL
1	$1250\sqrt{5}$	$-p$	$-\frac{\sqrt{5}}{2}$	$+3125p$
2	$1250\sqrt{5}$	$-p$	$-\frac{\sqrt{5}}{2}$	$+3125p$
3	$1250\sqrt{5}$	$-p$	$-\frac{\sqrt{5}}{2}$	$+3125p$
4	$1250\sqrt{5}$	$-p$	$-\frac{\sqrt{5}}{2}$	$+3125p$
5	$1250\sqrt{5}$	$-p$	0	0
6	2500	$+p$	$+1$	$+2500p$
7	$1250\sqrt{5}$	$-p$	0	0
8	5000	$+p$	$+1$	$+5000p$
9	5000	$+p$	$+1$	$+5000p$
			Sum	$+25000p$

$$\therefore \quad \delta_C = \Sigma \frac{puL}{E} = \frac{25000p}{E} \text{ mm} \qquad (1)$$

But $\qquad \delta_C = 15$ mm (given) $\qquad (2)$

Equating (1) and (2),

$$25000 \frac{p}{E} = 15$$

$$\therefore \quad \frac{p}{E} = \frac{15}{25000} = \frac{3}{5000}$$

Now, change in the span $AB = \Delta_{AC} + \Delta_{CB}$

$$= 2\left\{\frac{p\,(5000)}{E}\right\}$$

$$= 10000\,\frac{p}{E}$$

Substituting the value of $\frac{p}{E}$, we get

$\therefore$ Change in the span $AB = \dfrac{10000 \times 3}{5000} = \dfrac{30}{5} = 6$ **mm.**

Example 13·8. *The frame shown in Fig. 13·9 consists of four panels each 2·5 m wide, and cross-sectional areas of the members are such that when the frame carries equal loads at the panel points of the ~~lower chord, the~~ stresses in all tension members is 100 N/mm² and the stress in all the compression members is 80 N/mm². Determine the relative movement between the joints C and K in the direction CK. Take E = 2 × 10⁵ N/mm².*

Solution

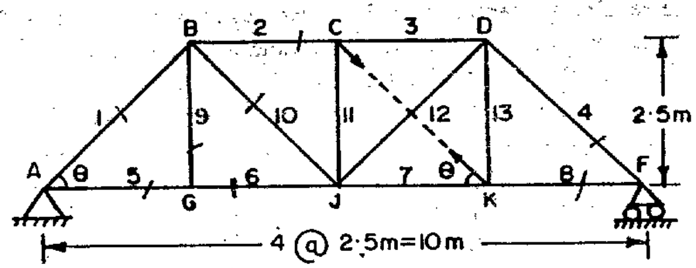


Fig. 13·9.

To find the relative movement between joints C and K, apply unit loads at C and K in the direction CK. The movement δ of the joints C and K towards each other is then given by

$$\delta = \sum_{1}^{u} \frac{puL}{E}$$

There will be no reaction at A and F due to the unit loads.

Hence

$u_1 = 0$; $u_5 = 0$; $u_6 = 0$; $u_9 = 0$; $u_{10} = 0$; $u_2 = 0$ and $u_4 = 0$; $u_8 = 0$

Resolving at joint C,

$$u_{11} = 1 \sin\theta = \frac{1}{\sqrt{2}} \text{ (comp.)}$$

$$u_3 = 1 \cos\theta = \frac{1}{\sqrt{2}} \text{ (comp.)}$$

Resolving at joint K

$$u_7 = 1 \cos\theta = \frac{1}{\sqrt{2}} \text{ (comp.)}$$

$$u_{13} = 1 \sin\theta = \frac{1}{\sqrt{2}} \text{ (comp.)}$$

Resolving at D,

$$u_{12}\cos\theta=u_{13}$$

$$\therefore \quad u_{12}=\frac{u_{13}}{\cos\theta}=\frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{1}=1 \text{ (tension)}$$

By inspection, it can be seen that for the loads at the panel points of the lower panel, the top chord members will be in compression, and the bottom chord members, verticals and diagonals will be in tension. Member CJ does not carry any stress.

Thus, $p_1=p_2=p_3=p_4=80$ N/mm² (comp.)

$p_5=p_9=p_6=p_{10}=100$ N/mm² (tension)

$p_1=p_8=p_{12}=p_{13}=100$ N/mm² (tension)

$p_{11}=0.$

The results are tabulated below. However, since δ is the function of the product of p and u, those members having $u=0$ or $p=0$ (and hence $p.u=0$) have *not* been included in the table.

(+tension ; −comp.)

Member	*Length (L mm)*	*p*	*u*	*puL*
u_3	2500	−80	$-\frac{1}{\sqrt{2}}$	$+14{\cdot}14\times10^4$
u_7	2500	+100	$-\frac{1}{\sqrt{2}}$	$-17{\cdot}66\times10^4$
u_{12}	$2500\sqrt{2}$	+100	+1	$+35{\cdot}32\times10^4$
u_{13}	2500	+100	$-\frac{1}{2}$	$-17{\cdot}66\times10^4$
			Sum	$+14{\cdot}14\times10^4$

$$E=20\times10^5 \text{ N/mm}^2$$

$$\therefore \quad \delta=\frac{14{\cdot}14\times10^4}{2{\cdot}0\times10^5}\text{ mm}=0{\cdot}706 \text{ mm}$$

Example 13·9. *Fig. 13·10(a) shows a pin-jointed frame which is hinged to rigid-supports A and D which are at the same level. All members have the same length and the span AD is the same as the length of the members. Due to a certain loading, the changes in length of the members are estimated as AB : +0·185 in, BC : +0·240 in., BD : −0·200 in., CD : −0·365 in (+denotes extension).* (U.L.)

Find the horizontal and vertical movements of C.

Solution

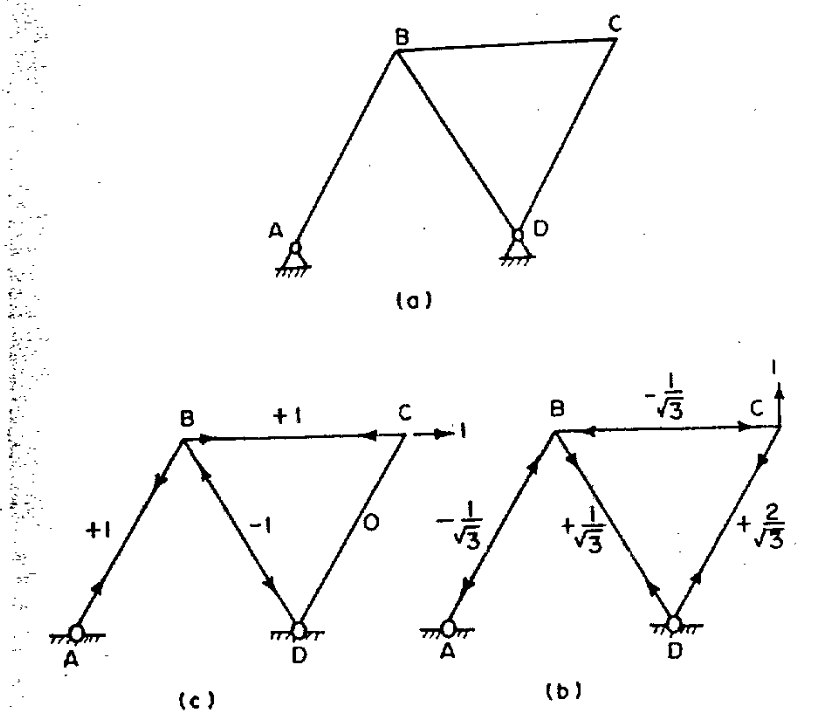


Fig. 13·10.

The horizontal and vertical deflections of C are given by

$$\delta_V=\sum_1^n\frac{PuL}{AE}=\sum_1^n u\Delta$$

$$\delta_H=\sum_1^n\frac{Pu'L}{AE}=\sum_1^n u'\Delta \qquad (2)$$

where u is the force in any member due to unit vertical load at C, and, u' is the force due to unit horizontal load at C.

To find the values of u, apply unit vertical load at C. Fig. 13·10(b) shows the induced stresses in the members. Similarly, Fig. 13·10(c) shows the stresses u' due to unit horizontal load at C. (Students are advised to work out the stresses independently).

The results are tabulated below.

(+ for tension and extension ; − for compression and contraction)

Member	Deformation Δ (in.)	u	u'	$u\Delta$	$u'\Delta$
AB	+0·185	$-\frac{1}{\sqrt{3}}$	+1	−0·107	−0·185
BC	+0·240	$-\frac{1}{\sqrt{3}}$	+1	−0·139	+0·240
CD	−0·365	$+\frac{2}{\sqrt{3}}$	0	−0·422	0
BD	−0·200	$+\frac{1}{\sqrt{3}}$	−1	−0·116	+0·200
			Sum	−0·784	+0·625

$$\therefore \quad \delta v=\sum_1^n u\Delta=-0{\cdot}748 \text{ in, i.e. } 0{\cdot}748 \text{ in.} \downarrow$$

(The minus sign indicates that δv is in the reverse direction of that assumed).

and $$\delta_H=\sum_1^n u'\Delta=+0{\cdot}625 \text{ in. } (\rightarrow).$$

Example 13·10. *Determine the vertical deflection of the joint C of the frame shown in Fig. 13·11, due to temperature rise of 60° F in the upper chords only. The coefficient of expansion=6·0×10⁻⁶ per 1°F and E=2×10⁶ kg/cm².*

Solution

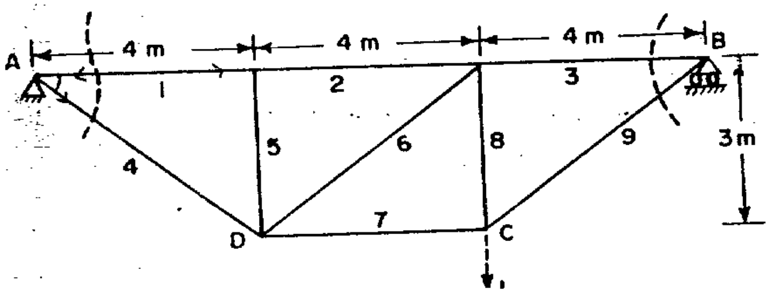


Fig. 13·11.

Increase in length of each member of the upper chord

$$=L\alpha t=400\times 6\times 10^{-6}\times 60=+0{\cdot}144 \text{ cm}$$

The vertical deflection of C is given by

$$\delta=\Sigma u\Delta$$

To find u, apply unit vertical load at C. Since the change in length (Δ) occurs only in the three top chord members, stresses in these members only need be found out.

$$\text{Reaction at } A=\frac{4}{12}=\frac{1}{3}$$

$$\text{Reaction at } B=\frac{8}{12}=\frac{2}{3}$$

Passing a section cutting members 1 and 4, and taking moments at D, we get

$$u_1=\left(\frac{1}{3}\cdot 4\right)\frac{1}{3}=\frac{4}{9}\text{ (comp.)}$$

Similarly, passing a section cutting members 3 and 9, and taking moments at C, we get

$$u_3=\left(\frac{2}{3}\times 4\right)\frac{1}{3}=\frac{8}{9}\text{ (comp.)}$$

Also $$u_2=u_1=\frac{4}{9}\text{ (comp.)}$$

$$\therefore \quad \delta_C=u_1\,\Delta_1+u_2\,\Delta_2+u_3\,\Delta_3$$

$$=\left\{\left(-\frac{4}{9}\right)+\left(-\frac{4}{9}\right)+\left(-\frac{8}{9}\right)\right\}\times(+0{\cdot}144)$$

$$=-0{\cdot}256 \text{ cm, i.e., } 0{\cdot}256 \text{ cm} \uparrow.$$

Example 13·11. *Determine the horizontal and vertical deflection of the joint C of the frame shown in Fig. 13·12, if the member DF has a lack of fit of 1 cm (long).*

Solution

Let u be the stress in member DF due to unit vertical load at C, [Fig. 13·12 (*a*)] and u' be the stress in it due to unit horizontal load at C, [Fig. 13·12 (*b*)]. Since only one member has lack of fit, we have

$$\delta_{CV}=u_6\Delta_6=u_6\times(+1)=u_6 \quad (1)$$

(Since $\Delta_6=+1$ cm)

and $$\delta_{CH}=u'_6\Delta_6=u'_6\times(+1)=u'_6 \quad (2)$$

Refer Fig. 13·12 (*a*). Reactions at A and B will be $\frac{1}{3}$ and $\frac{2}{3}$ respectively. Pass a section to cut the members 2, 6 and 7

Then force in member 6=(shear in the panel)×(cosec θ)

$$\text{cosec } \theta = \frac{5}{3}$$

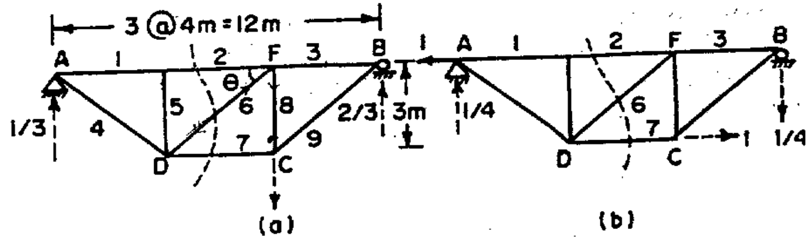


Fig. 13·12.

$$\therefore \quad u_6 = \frac{1}{3} \times \frac{5}{3} = \frac{5}{9} \text{(comp.)} = -\frac{5}{9}$$

(since compression has negative sign)

$$\therefore \quad \delta_{CV} = u_6 = -\frac{5}{9} \text{ cm, } i.e. \ \frac{5}{9} \text{ cm} \uparrow.$$

Similarly, when a unit horizontal load is applied at C, [Fig. 13·12 (b)], the horizontal reaction at A=1 (←). Since the unit horizontal load at C and the reaction at A produce a couple in the anticlockwise direction, equal and opposite vertical reactions of magnitude $\frac{1\times3}{12}=\frac{1}{4}$ will be induced at A and B as shown.

To find the force u'_6 in FD, pass a section to cut the members 2, 6 and 7.

$$\text{Then} \quad u'_6 = \text{(shear in the panel)} \times (\text{cosec } \theta)$$

$$= \frac{1}{4} \times \frac{5}{3} = \frac{5}{12} \text{(comp.)} = -\frac{5}{12}$$

$$\text{Hence} \quad \delta_{CH} = u'_6 = -\frac{5}{12} \text{ cm, } i.e. \ \frac{5}{12} \text{ cm} \leftarrow$$

13·4. DEFLECTION BY CASTIGLIANO'S FIRST THEOREM

In chapter 12, it has been proved that the partial derivative of total strain energy with respect to a force gives the deflection in the direction of the force. This is Castigliano's first theorem, and its application for finding the deflection of beams etc. has already been studied earlier.

In articulated structures, the loads are applied at panel points only, and hence the members carry axial forces, (either tension or

compression) only. Thus the strain energy will be due to direct forces and is given by

$$U = \sum_1^n \frac{P^2L}{2AE} \qquad (1)$$

where P is force in any member due to external loading. If W is an external load acting at a joint and it is required to find the deflection of the joint in the direction of the application of the load, we have, according to Castigliano's first theorem.

$$\boxed{\delta = \frac{\partial U}{\partial W} = \sum_1^n \frac{PL}{AE} \cdot \frac{\partial P}{\partial W}} \qquad (13·7)$$

This is the expression for the deflection of a joint. If, however, it is required to find the deflection of a joint where W is not acting, a fictitious load W is applied there in the required direction. The fictitious load W is then equated to zero. A similar method may be employed for the joint where external load is acting, but the deflection is required to be found in some other direction.

Procedure for Computing Deflection

1. Apply a fictitious load W at the joint in the direction in which the deflection is required, if no such external load is acting.

2. Find the force P in all members. The force P will be a function of W and the external load. Thus, in general,

$$P = a + bW$$

where a and b are the constants depending upon the geometry of the truss, position of the load, the position of the member and the system of external loading. In some cases, either a may be zero, or b may be zero, or both a and b may zero.

3. Find the value $\frac{\partial P}{\partial W}(=b)$.

4. Calculate $\frac{PL}{AE} \cdot \frac{\partial P}{\partial W}$ for each member. If W were a fictitious load, equate it to zero.

5. $\Sigma \frac{PL}{AE} \cdot \frac{\partial P}{\partial W}$ gives the required deflection.

Comparison with Unit Load Method

The deflection by the unit load method is given by

$$\delta = \sum_1^n \frac{PuL}{AE} = \sum_1^n \frac{PL}{AE} \cdot u = \sum_1^n \Delta \cdot u \qquad (1)$$

The deflection by Castigliano's theorem is given by

$$\delta=\sum_1^n \frac{PL}{AE}\cdot\frac{\partial P}{\partial W} \qquad (2)$$

If expressions (1) and (2) are compared, it is evident that $\frac{\partial P}{\partial W}=u$. Thus, both the expressions are the same. However, due to different form, the procedure for computation is different. In the unit load method, one has to analyse the frame twice for finding P and u in each member while in the latter method, only one analysis is needed. However, the expressions for P, by the Castigliano's method, are sometimes long and cumbersome.

Example 13·12. *Solve example 13·2 by Castigliano's first theorem.*

Solution. (Refer 13·3).

(*a*) *Vertical deflection of C*

An external load W is already acting vertically at C. Hence

$$\delta_{CV}=\sum_1^n \frac{PL}{AE}\,\frac{\partial P}{\partial W}$$

The value P in various members have already been calculated in example 13·2. The values of P and $\frac{\partial P}{\partial W}$ etc. have been entered, and computations done in a tabular form below :

(+ for tension ; — for compression)

Member	*Length L*	*Area A*	*P*	$\frac{\partial P}{\partial W}$	$\frac{PL}{A}\,\frac{\partial P}{\partial W}$
AB	*L*	*a*	$+W$	$+1$	$+\frac{WL}{a}$
BC	*L*	*a*	$+W$	$+1$	$+\frac{WL}{a}$
AD	*L*	$2a$	$-W$	-1	$+\frac{WL}{2a}$
CD	*L*	$2a$	$-W$	-1	$+\frac{WL}{2a}$
BD	*L*	$2a$	$-W$	-1	$+\frac{WL}{2a}$
					$\frac{7WL}{2a}$

Hence $\qquad \delta_{CV}=\sum_1^n \frac{PL}{AE}\cdot\frac{\partial P}{\partial W}=\frac{7WL}{2aE}$

(*b*) *Horizontal deflection of C*

Apply a horizontal load H at the joint D (in the dotted direction shown in Fig. 13·3), *with the external load W still acting on the frame.* The horizontal deflection of C is given by

$$\delta_{DH}=\sum_1^n \frac{PL}{AE}\cdot\frac{\partial P}{\partial H}$$

It will be seen that the forces in *all members* except AD will be the same as in the previous case (*i.e.*, when only W is acting and $H=0$). The force in AD can be found by resolving horizontally at A. As the horizontal reaction at A is H, we have

$$P_{AD}\cos 30°=H+P_{AB}\cos 30°=H+W\cos 30°.$$

$$\therefore \qquad P_{AD}=H\,\frac{2}{\sqrt{3}}+W$$

and $\qquad \frac{\partial P_{AD}}{\partial H}=\frac{2}{\sqrt{3}}$

The forces in the four members are as below :

$$P_{AB}=+W\,;\ \frac{\partial P_{AB}}{\partial H}=0$$

$$P_{BC}=+W\,;\ \frac{\partial P_{BC}}{\partial H}=0$$

$$P_{CD}=-W\,;\ \frac{\partial P_{CD}}{\partial H}=0$$

$$P_{BD}=-W\,;\ \frac{\partial P_{BD}}{\partial H}=0$$

Hence $\qquad \delta_{DH}=\sum_1^n \frac{PL}{AE}\cdot\frac{\partial P}{\partial H}$

$$=\left(\frac{2}{\sqrt{3}}\,H+W\right)\frac{2}{\sqrt{3}}\cdot\frac{L}{2aE}$$

$$=\frac{WL}{\sqrt{3}aE} \text{ by putting } H=0.$$

Example 13·13. *A pin joined frame shown in Fig. 13·13 is hinged to a rigid wall at A and is free to slide vertically at E. The frame carries a vertical load W at B. The area of each tension member is a and of each comperssion member 2a and the length AE is L. Obtain an expression for the vertical displacement of C.* (U.L.)

Solution

Due to external loading, member AB, AD and AE will carry tension, and the area of each of these members is therefore, a. Members BD and DE carry compression and hence their area is $2a$. Members BC and CD carry zero stress.

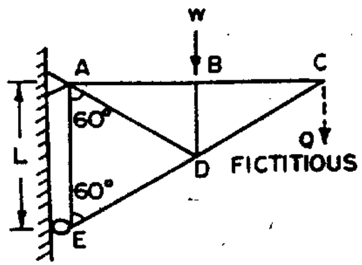


Fig. 13·13.

To find the vertical deflection of C, apply a fictitious vertical load Q there. Then

$$\delta_{CV}=\sum_1^n \frac{PL}{AE}\cdot\frac{\partial P}{\partial Q}$$

where P=stress in any member due to both external and fictitious load

Vertical Reaction at E=0 (roller)

Vertical Reaction at $A=(W+Q)$

The stresses in BC, CD and AB will be in terms of Q only, and hence $P.\dfrac{\partial P}{\partial Q}$ will be zero when the value of Q is put zero. Similarly stresses in BD and AD will be the function of W only and hence $\dfrac{\partial P}{\partial Q}$ and $P.\dfrac{\partial P}{\partial Q}$ will be zero for these two members. Thus $\Sigma\,\dfrac{PL}{AE}\cdot\dfrac{\partial P}{\partial Q}$ is required for members AE and DE only.

Pass a section to cut AB, AD and ED. Taking moments about A, we get

$$P_{DE}=\frac{(W\times AB)+(Q\times AC)}{L\sin 60°}$$

$$=\frac{WL\cos 30°+2QL\cos 30°}{L\sin 60°}=(W+2Q)\text{ (compression)}$$

Resolving vertically at E,

$$P_{AE}=P_{DE}\cos 60°=\frac{1}{2}(W+2Q)\text{ tension}$$

To make it more clear how $P.\dfrac{\partial P}{\partial Q}$ is zero for the five members, value of P and $\dfrac{\partial P}{\partial Q}$ are tabulated below for all the members.

Member	Length L	Area A	P	$\dfrac{\partial P}{\partial Q}$	$\dfrac{PL}{AE}\dfrac{\partial P}{\partial Q}$ after putting $Q=0$
AB	$\frac{\sqrt{3}}{2}L$	a	$+\frac{4Q}{\sqrt{3}}$	$+\frac{4}{\sqrt{3}}$	$\frac{16}{3}\times\frac{\sqrt{3}}{2}\frac{QL}{aE}=0$
BD	$\frac{L}{2}$	$2a$	$-W$	0	0
	$\frac{\sqrt{3}}{2}L$	a (say)	$+\frac{4Q}{\sqrt{3}}$	$+\frac{4}{\sqrt{3}}$	$\frac{16}{3}\times\frac{\sqrt{3}}{2}\frac{QL}{aE}=0$
CD	L	a (say)	$-2Q$	-2	$\frac{4L}{aE}Q=0$
AD	L	a	$+W$	0	0
AE	L	a	$+\frac{1}{2}(W+2Q)$	$+1$	$\frac{L(W+2Q)}{2aE}=\frac{WL}{2aE}$
DE	L	$2a$	$-(W+2Q)$	-2	$\frac{2(W+2Q)L}{2aE}=\frac{WL}{aE}$
				Sum	$\frac{3WL}{2aE}$

$$\therefore\quad \delta_{CV}=\sum_1^n\frac{PL}{AE}\cdot\frac{\partial P}{\partial Q}=\frac{3WL}{2aE}.$$

13·5. MAXWELL'S RECIPROCAL THEOREM APPLIED TO FRAMES

As applied to the deflection of articulated structures, Maxwell's theorem of reciprocal deflection has the following statement :

"In a perfect frame under equilibrium, the deflection of any joint A due to a load at the joint B is equal to the deflection of the joint B due to same load at the joint A".

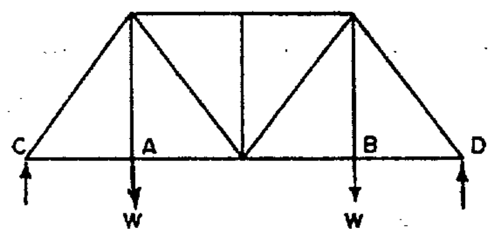


Fig. 13·14.

Expressed mathematically,

$$_B\delta_A = {}_A\delta_B$$

where $_B\delta_A$ = deflection of A due to load at B

$_A\delta_B$ = deflection of B due to load at A

Thus in Fig. 13·14, let W be the load at B. According to unit load method, the deflection of the joint A is given by

$$_B\delta_A = \sum_1^n \frac{PuL}{AE} \qquad \ldots(1)$$

where P = Stress in any member due to W at B.

u = Stress in any member due to unit load A.

Now remove the load from B, and apply it at the joint A. Then the deflection of joint B is given by

$$_A\delta_B = \sum_1^n \frac{P'u'L}{AE} \qquad \ldots(2)$$

where P' = Stress in any member due to W at A

$= uW$

and u' = Stress in any member due to unit load at B

$$= \frac{P}{W}$$

Substituting these values of P' and u' in (2), we get

$$_A\delta_B = \sum_1^n \frac{L}{AE}(uW)\left(\frac{P}{W}\right) = \sum_1^n \frac{PuL}{AE} \qquad \ldots(1)$$

Comparing (1) and (3), we get

$$_B\delta_A = {}_A\delta_B \qquad \ldots(13\cdot8)$$

which proves the statement.

Example 13·14. *Determine the vertical displacement of both lower points C and D for the pin jointed frame shown in Fig. 13·15. The cross-sectional area of all members is 130 sq. mm. and the modulus of elasticity is 200 kN/mm². Determine the magnitude of an additional vertical load W placed at D necessary to increase the deflection at C by 50%.*

Solution

From Castigliano's first theorem, the deflections of C and D due to external load is given by

$$\delta_C = \sum_1^n \frac{Pu_1L}{AE} \text{ and } \delta_D = \sum_1^n \frac{Pu_2L}{AE}$$

where P = force in any member due to the load of 9 kN acting at C.

u_1 = force in any member due to unit load at C.

u_2 = force in any member due to unit load at D.

The calculations of P, u_1 and u_2 are presented in the tabular form below :

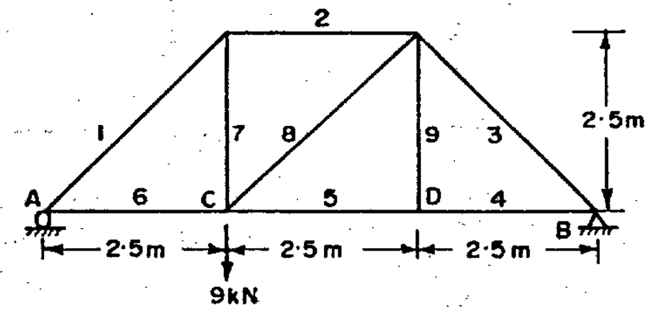


Fig. 13.15.

(+for tension ;—for compression)

Member	L (mm)	P	u_1	u_2	Pu_1L	Pu_2L
1	$2500\sqrt{2}$	$-6\sqrt{2}$	$-\frac{2}{3}\sqrt{2}$	$-\frac{\sqrt{2}}{4}$	$+2\cdot828\times10^4$	$+1\cdot414\times10^4$
2	2500	-6	$-\frac{2}{3}$	$-\frac{1}{3}$	$+1\ 0\times10^4$	$+0\cdot5\times10^4$
3	$2500\sqrt{2}$	$-3\sqrt{2}$	$-\frac{1}{3}\sqrt{3}$	$-\frac{2\sqrt{2}}{3}$	$+0\cdot707\times10^4$	$+1\cdot414\times10^4$
4	2500	$+3$	$+\frac{1}{3}$	$+\frac{2}{3}$	$+0\cdot250\times10^4$	$+0\cdot5\times10^4$
5	2500	$+3$	$+\frac{1}{3}$	$+\frac{2}{3}$	$+0\cdot250\times10^4$	$+0\cdot5\times10^4$
6	2500	$+6$	$+\frac{2}{3}$	$+\frac{1}{3}$	$+1\cdot0\times10^4$	$+0\cdot5\times10^4$
7	2500	$+6$	$+\frac{2}{3}$	$+\frac{1}{3}$	$+1\cdot0\times10^4$	$+0\cdot5\times10^4$
8	$2500\sqrt{2}$	$+3\sqrt{2}$	$+\frac{1}{3}\sqrt{2}$	$-\frac{\sqrt{2}}{3}$	$+0\cdot707\times10^4$	$-0\cdot707\times10^4$
9	2500	0	0	$+1$	0	0
				Sum	$7\cdot742\times10^4$	$4\cdot621\times10^4$

$$\therefore \quad \delta_C = \sum_1^n \frac{Pu_1L}{AE} = \frac{7{\cdot}742\times10^4}{130\times200} = 2{\cdot}98 \text{ mm} \downarrow \qquad (1)$$

$$\delta_D = \sum_1^n \frac{Pu_2L}{AE} = \frac{4{\cdot}621\times10^4}{130\times200} = 1{\cdot}78 \text{ mm} \downarrow \qquad (2)$$

Let the additional load at D be W.

We want $_D\delta_C = \frac{2{\cdot}98}{2} = 1{\cdot}49$ mm.

According to Maxwell's reciprocal theorem

$$_C\delta_D = {}_D\delta_C$$

Hence $_C\delta_D = 1{\cdot}49$ mm $\qquad (3)$

When a load of 9 kN is at C, $\delta_D = 1{\cdot}78$ mm

Hence when a load of W is at C, $_C\delta_D = \frac{1{\cdot}78}{9} W \qquad (4)$

Equating (3) and (4), we get

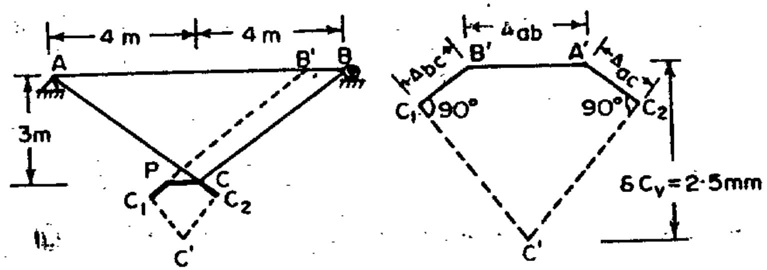
$$\frac{1{\cdot}78}{9} W = 1{\cdot}49$$

or $$W = \frac{1{\cdot}49\times9}{1{\cdot}78} = 7{\cdot}53 \text{ kN.}$$

13·6. GRAPHICAL METHOD

(a) Williot Diagram

In the graphical method, the extension or the contraction of each member is first calculated by Hooke's law $\left(i.e.\ \Delta = \frac{PL}{AE}\right)$. These stretching or shortening of the members are then plotted to get the position of the joints and the deflections.



(a) (b) WILLIOT DIAGRAM

Fig. 13.16.

To start with, let us take the simple triangular truss of example 13·1, which carries a point load of 6 tonnes at the point C. The members AC and BC carry a tensile force of 5 kN each, and AB

carries a compressive force of 4 kN. Hence the extensions or contractions of various members are :

$$\Delta_{AC} = \Delta_{BC} = \frac{5\times1000\times5000}{150\times2\times10^5} = +0{\cdot}834 \text{ mm } (i.e. \text{ extension})$$

and $$\Delta_{AB} = \frac{4\times1000\times8000}{100\times2\times10^5} = -1{\cdot}6 \text{ mm } (i.e. \text{ contraction})$$

The deformed shape of the frame can now be plotted by using the changed lengths as the sides and plotting the triangle ABC. However, since Δ_{AC} and Δ_{AB} are extremely small, the deformed shape of the frame will practically coincide with the original shape, and thus accurate measurements for the deflection of the joint C cannot be made. For this reason, bigger scale is used for plotting the *changes* in the lengths. For the truss shown in Fig. 13·16 (*a*), the point A is *position fixed*. Also the movement of the roller at B is horizontal, and hence the direction AB is also fixed. Thus, when AB carries compression under the given system of loading, B will move towards A, by an amount $BB' = \Delta_{AB}$. This contraction Δ_{AB} $(=BB')$ has been marked on BA, to the left of B, on a bigger scale than that of the original diagram. From B', a line $B'P$ is drawn parallel to BC and equal in length to BC.

Now on AC, CC_2 is made equal to Δ_{AC} (extension) on the enlarged scale. Similarly on $B'P$. PC_1 is made equal to Δ_{BC} (extension). Then by striking arcs from centres A and B' with radii AC_2 and $B'C_1$ respectively the intersection gives C', the new position of point C. For every small changes in lengths, the angle $\widehat{CC_2C'}$ and PC_1C' are right angles. Hence right angles are set off at C_1 and C_2, intersecting at C'. We thus get the figure $PCC_2C'C_1$, which is of much larger scale than that of the original diagram ABC. C' is the final position of the joint C, and the perpendicular distance of C' from PC gives the vertical deflection of the joint C.

In the case of bigger frames, it is convenient to draw the deflection diagram (such as $PCC_2C'C_1$ separately, as shown in Fig. 13·16 (*b*). Point A being fixed in position, it is chosen as the reference point, and the direction AB is the reference direction. Thus, in Fig. 13·16 (*b*), point A' is first chosen, and point B' is marked to the left of A' (since point B moves to the *left relative to* A), and by a magnitude $A'B' = \Delta_{AB}$. Line $A'B'$ is evidently parallel to the direction AB. Since AC extends, point C moves to the right relative to A, and in the direction AC. Hence the extension AC_2

($=\Delta_{AC}$) is marked parallel to AC, marking the point C_2 to the right of A'. Similarly, BC extends, and point C moves to the left relative to B, and in the direction BC. Hence the extension $B'C_1$ ($=\Delta_{BC}$) is marked parallel to BC, marking the point C_1 to the left of B'. Perpendiculars are then drawn at C_1 and C_2, thus meeting at C. The deflection of joint C', when scaled, comes out to be 2·5 mm. The diagram 13·16(b) is known as the *Williot diagram.*

(*b*) **Williot-Mohr Diagram**

In the case illustrated above, the reference point A was fixed in position and the reference direction AB was also fixed. In some cases however there may not be even a single member which is either fixed in direction or carries zero stress. In such a case, Williot diagram is first plotted with reference to the direction of any member. The necessary correction (known as Mohr's correction) is then made to the diagram. The final diagram so obtained is known as the *Williot-Mohr* diagram. The procedure for plotting the Williot-Mohr diagram is decribed in example 13·16.

Example 13.15. *Solve example 13·9 by graphical method.*

Solution.

In this frame, both points A and D are position fixed, and hence the direction AD is fixed. Thus there is no change in the length AD, and $\Delta_{AD}=0$.

To draw the Williot diagram [Fig. 13·17(b)]; points A' and D' coincide. To find the position B' (of the joint B), $A'B_1$ drawn parallel to AB and equal to Δ_{AB}. Since Δ_{AB} is positive (*i.e.* extension), B moves away from A, in the direction $A'B_1$. Similarly, since Δ_{BD} is negative (*i.e* contraction), B moves towards D. Hence $D'B_2$ is drawn parallel to BC, in the direction B to D, and equal to Δ_{BD}. Perpendiculars are drawn at B_1 and B_2 to meet at B'. Thus, B' is the location of the joint B relative to joints A and D.

Similarly, to locate the position C' (of the joint C), draw $B'C_1$ parallel to BC (since Δ_{BD} is extension and C moves away from B in the direction B to C), and equal to Δ_{BC}. Similarly draw $D'C_2$ parallel to CD (since Δ_{CD} is contraction and C moves towards D in the direction C to D), and equal to Δ_{CD}. Perpendiculars are then drawn at C_1 and C_2 to meet at C'. C' is then the location of the joint C relative to joints A and D.

From the Williot diagram [Fig. 13·17(b)], the vertical deflection of $C=\delta_{CV}$=vertical distance between A' and C'=0·78" (↓) and the horizontal deflection of $C=\delta_H$=horizontal distance between A' and C'=0·63"→.

In the Williot diagram, all the firm lines denote the changes in the lengths of the members, all the dotted lines are the perpendiculars, and all letters with 'dash' (i.e. A', B', C' etc.) are the deflected positions of joints relative to the fixed points and direction chosen. This convention in the drawing will be followed in the next example also.

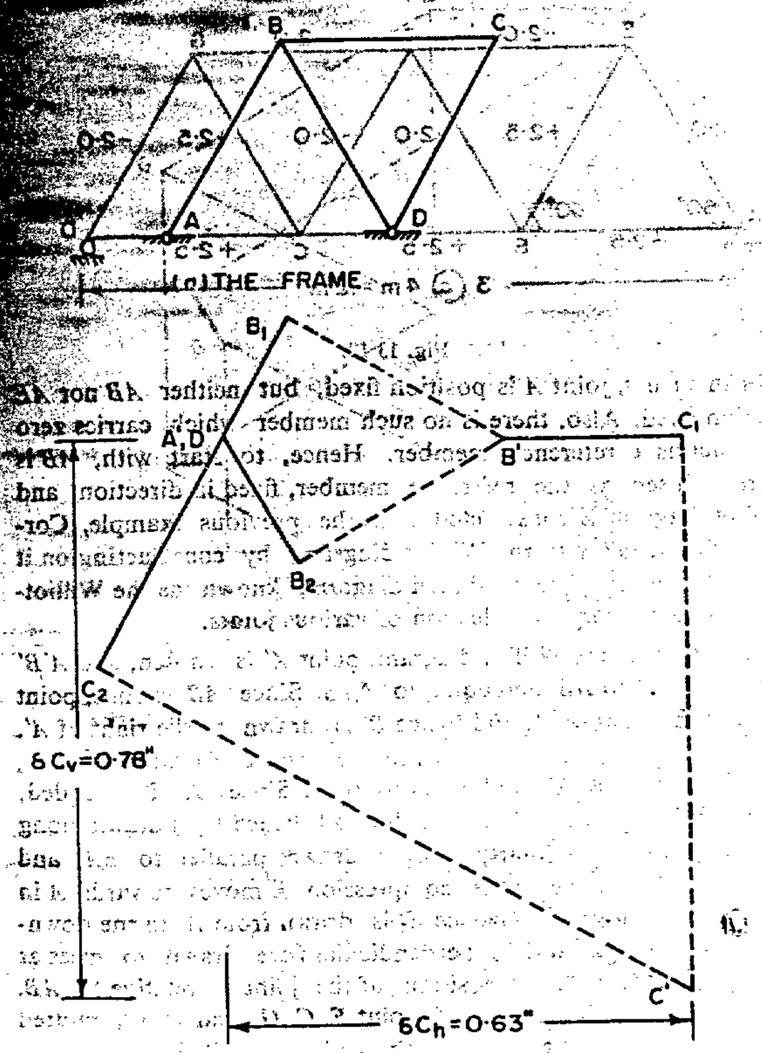


Fig. 13·17.

Example 13·16. *The members of a Warren truss are subjected to the changes (in mm) in length, shown against each members in Fig. 13·18 due to a certain loading.*

Draw the Williot-Mohr diagram, and find the vertical deflection of the joint C.

Solution.

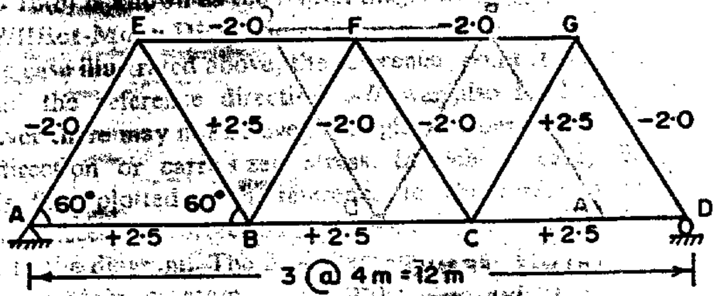


Fig. 13·18.

In this truss, joint *A* is position fixed, but neither *AB* nor *AE* is direction fixed. Also, there is no such member which carries zero stress, to act as a reference member. Hence, to start with, *AB* is arbitrarily chosen as the reference member, fixed in direction, and the Williot diagram is constructed as in the previous example. Correction is then applied to the Williot diagram, by constructing on it the Mohr's diagram. The combined diagram, known as the Williot-Mohr's diagram gives the deflection of various joints.

To construct the Williot diagram, point A' is chosen, and $A'B'$ is drawn parallel to *AB*, and equal to Δ_{AB}. Since *AB* extends, point *B* moves to the right of *A*, and hence B' is drawn to the right of A'. Now, to plot E', the deflected position of point *E* relative to $A'B'$, draw $B'E_2$ parallel to *BE*, and equal to Δ_{BC}. Since *BE* is extended, *E* moves away from *B*, in direction *BE*, and hence E_2 is drawn along the direction *B* to *E*. Similarly, $A'E_1$ is drawn parallel to *EA*, and equal to Δ_{EA}. Since *AE* carries compression, *E* moves towards *A* in downward direction, and hence E_1 is drawn from A' in the downward direction. At E_1 and E_2, perpendiculars are drawn to meet at E', thus giving the deflected position of the joint *E* relative to *AB*. Similarly, deflected positions of point *F*, *C*, *G* and *D* are plotted at F' C', G' and D' respectively, as shown in Fig. 13·19.

From the Williot diagram so obtained, the point D' is the deflected position of *D* relative to *AB*, and hence the vertical distance

between A' and D' gives vertical deflection of *D* relative to *A*. Actually, since *D* is supported on rollers, both *A* and *D* are at the *same level*, and the vertical deflection of *D* relative to *A* must be zero. This discrepancy has crept in due to the fact that the Williot dia-

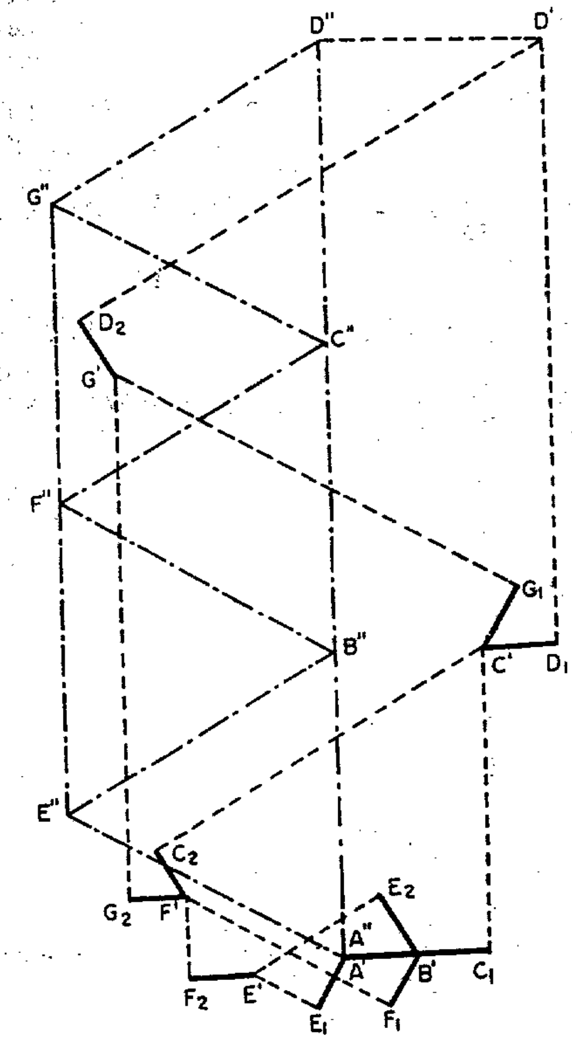


Fig. 13·19.

gram has been plotted with reference to the original axis of the member *AB*. Actually, *AB* is not fixed, and it rotates about the joint *A*.

The necessary correction is applied through the Mohr's diagram. The deformed truss should be rotated about the hinge A, in clockwise direction, in such a way that the joint D comes in level with the joint A. The vertical displacement given to the joint D is equal to the height of D' above A'. The horizontal component of the displacement represented by D' still remains. Due to the rotation of the truss, other joints will also be proportionately displaced.

To draw the correction diagram (*i.e.*, Mohr's diagram), draw $D'D''$ parallel to BA, to meet the vertical line through A' in D''. The line $A'D''$ represents the lower chord of the truss. The other joints of the lower chord, B and C, will lie on $A'D''$. Hence, on the base $A'D''$ construct the figure $A''\ B''\ C''\ D''\ G''\ F''\ E''$ similar to that of the truss $ABCDGFE$, as shown by chain-dotted lines in Fig. 13·19. The figure $A''\ B''\ C''\ D''\ G''\ F''\ E''$ is known as the Mohr's diagram, and must be constructed on such side of the line $A''D''$ that after rotating it by 90°, it becomes parallel and similar to the original truss diagram. In the combined diagram, the following points must be carefully noted.

(*i*) The firm lines represent the deformation of various members.

(*ii*) The dotted lines represent the perpendicular drawn to locate the deflected position of the joints.

(*iii*) The letters with 'dash' (*i.e.*, B', C' etc.) represent the deflected position of the corresponding joints, relative to A, taking AP as the fixed reference direction.

(*iv*) The displacement of various joints B, C, D, E, F, G are represented by the vectors $B'B''$, $C'C''$, $D'D''$, $E'E''$, $F'F''$, and $G'G''$. Thus the vertical displacement of the joint D is the vertical component of the vector $D'D''$, and is evidently zero. The horizontal displacement of D is equal to the horizontal component of the vector $D'D''$, and is evidently equal to the length $D'D''$.

From the Williot-Mohr diagram,

$$\delta_{CV} = \text{vertical component of vector } C'C''$$
$$= 10{\cdot}5 \text{ mm.}$$

PROBLEMS

1. Fig. 13·21 shows a pin jointed plane frame hinged to a rigid wall at C and D, and carrying a vertical load W at A. The area of each tension member is 'a' and that of each compression member is '$2a$'. The length AD is

L and all members have a modulus of elasticity E. Determine analytically or graphically, the vertical and horizontal displacements of joint A. (*U.L.*)

2. A Crane structure is shown in Fig. 13·21. The length of the member AD is $2L$ and all other members are of length L. The cross-sectional area of AD is '$2a$' and that of all other members is 'a'. Determine the horizontal and vertical deflection of the joint F due to a vertical load W at that joint. (*U.L.*)

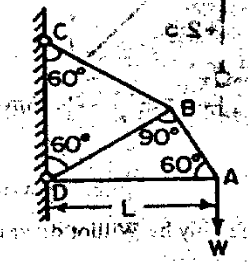


Fig. 13·20.

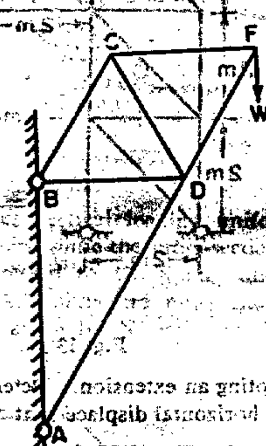


Fig. 13·21.

3. A pin jointed structure shown in Fig. 13·22 is pinned to an abutment at J and rests on rollers at G. A load W is applied horizontally at D. Determine the horizontal movement of D. if the area of all tension members is 'a' and that of all compression members '$2a$'. The panels are all equilateral and of side length L. (*U.L.*)

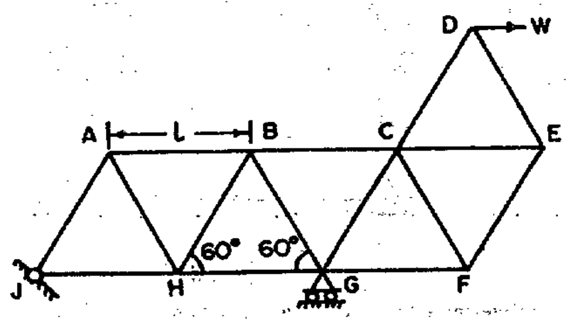
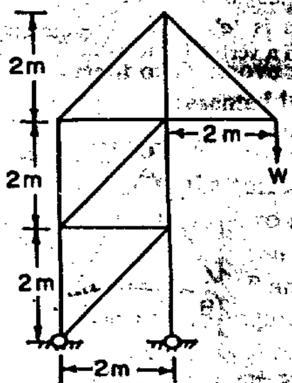


Fig. 13·22.

4. Determine the vertical deflection of the load in the structure shown in Fig. 13·23. The tension members are stressed to 124 N/mm² and the compression members are stressed to 93 N/mm². $E = 2{\cdot}01 \times 10^5$ N/mm².

5. The frame shown in Fig. 13·24 carries load which produce deformations (in mm) as shown against each of the members, a positive sign



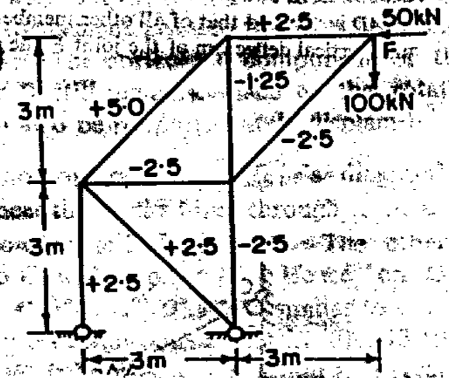


Fig. 13·23. Fig. 13·24.

denoting an extension. Determine, perferably by Williot diagram, the vertical and horizontal displacement of F.

6. Fig. 13·25 shows a small pin-jointed frame which is hinged at B and supported by a roller at C. Bar BC is horizontal and the lengths of the members are :

$$AB=30 \text{ cm} ; \; BC=50 \text{ cm} ; \; AC=45 \text{ cm}$$

The lengths of the bars are adjustable. Find the vertical and horizontal movements of A due to the following changes in lengths.

$$AB : +2{\cdot}5 \text{ mm} ; \; AC : -1{\cdot}0 \text{ mm} ; \; BC : -1{\cdot}5 \text{ mm}.$$

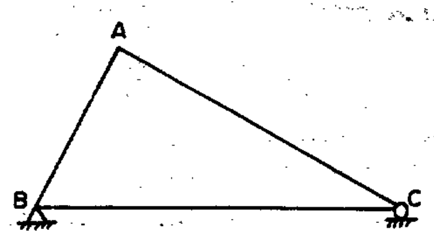


Fig. 13·25.

7. Determine the horizontal deflection of the roller support C of the frame shown in Fig. 13·26 due to applied load of 8 tonnes at B. Members AB, BC and BD are each of 800 sq. mm area and AD and CD are each of 1600 sq. mm area. $E=2{\cdot}06\times10^5$ N/mm².

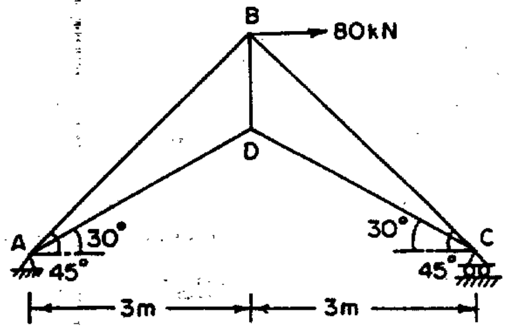


Fig. 13·26.

8. The truss shown in Fig. 13·27 carries vertical loading uniformly divided between the panel points of the lower chord while the cross-section of the members are such that all loaded ties are stressed to 135 N/mm² and all loaded struts to 90 N/mm² under the loading. All joints are pin-joints and the value of E for the material of the truss is $2{\cdot}02\times10^5$ N/mm². Find the vertical deflection of the point A.

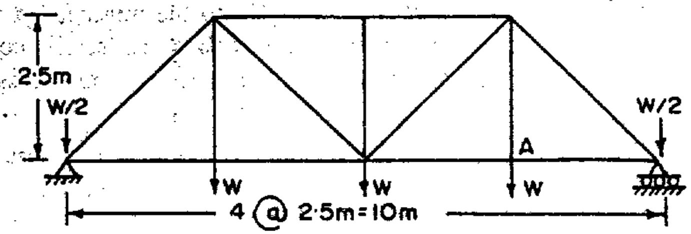


Fig. 13·27.

Answers

1. $\delta_V=2{\cdot}57\,\dfrac{WL}{aE}$; $\delta_H=0{\cdot}29\,\dfrac{WL}{aE}$.
2. $\delta_V=\dfrac{5{\cdot}67\,WL}{3aE}$; $\delta_H=\dfrac{4{\cdot}04\,WL}{aE}$.
3. $\Delta=12{\cdot}88\,\dfrac{WI}{aE}$.
4. 14·8 mm.
5. 24·2 mm ; 14·7 mm.
6. $\delta_V=0{\cdot}212\uparrow$; $\delta_H=0{\cdot}133\rightarrow$
7. $\delta_H=16{\cdot}35$ mm.
8. 8·33 mm.

14

Redundant Frames

14·1. DEGREE OF REDUNDANCY

A frame is said to be statically indeterminate when the number of unknown reactions or stress components exceed the total number of condition equations of equilibrium. If the number of unknowns are equal to the number of condition equations available, the frame is said to be a perfect frame. The *total* degree of indeterminacy or redundancy of a frame is therefore equal to the number by which the unknowns (*i.e.*, reaction components as well as stress components) exceed the condition equations of equilibrium. The excess restraints or members are described as *redundants.*

To find the total degree of indeterminacy, the structure may be, somewhat arbitrarily classified as statically indeterminate externally, internally or both. In case of externally redundant structures, there are redundant reactive restraints. The degree of external indeterminacy or redundancy is given by

$$E=R-r \tag{14·1}$$

where E=external redundancy,

R=total number of reaction components (one for a roller, two for a hinge, and three for a fixed support),

r=total number of condition equations available (*i.e.* minimum number of reaction components required for the stability of the frame).

The structure is said to be internally indeterminate if it has redundant members, and are therefore overstiff. The degree of internal redundancy (I) is given by

$$I=m-(2j-r) \tag{14·2}$$

where I=degree of internal redundancy

m=total number of members

j=total number of joints.

r=minimum number of reaction components required for the stability of the structure.

A structure may be redundant both internally as well as externally (*i.e.* it has redundant reaction components as well as redundant members). In this case, the total redundancy (T) is given by :

$$T=E+I$$
$$=(R-r)+m-2j+r=R+m-2j$$
$$=m-(2j-R) \tag{14·3}$$

Difference between equations 14·2 and 14·3 must be carefully noticed. In equation 14·2, r is the *minimum* number of reaction components required for the *stability* of the structure, while in equation 14·3, R is the *actual* number of the reaction components present in the structure.

14·2. APPLICATION OF CASTIGLIANO'S THEOREM OF MINIMUM STRAIN ENERGY

The principle of Least Work is a statement of the practical fact that if an elastic structure is in a state of stable equilibrium under any forces whatsoever, then the work stored is smallest amount possible. The theorem of minimum strain energy can, therefore, be used for analysing the redundant frames. To use this method, the redundant members are replaced by the unknown forces (T_1, T_2 etc.) acting at the joints. The statically determinate system which results from the removal of the redundant members is called the *base* or *principal* or *perfect system*. Then, by Castigliano's theorem of minimum strain energy, we get

$$\frac{\partial U}{\partial T_1}=0, \quad \frac{\partial U}{\partial T_2}=0, \text{ etc.}$$

where U is the *total strain energy* (*inclusive* of that in the redundant members) of the frame. The number of equations will be the same as the number of unknowns.

As an illustration, consider a frame shown in Fig. 14·1 (*a*). The total number of reaction components $R=(2+1)=3$: minimum reaction components required is $r=3$ (from statical equilibrium).

Hence $E=R-r=3-3=0$, and the frame is externally determinate.

Now $m=8$ and $j=5$; $r=3$

Hence $I=m-(2j-r)=8-(2\times5-3)=1$

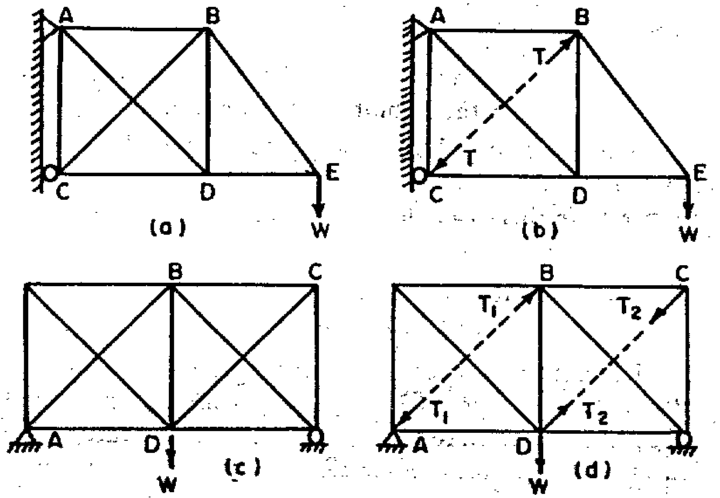


Fig. 14·1

Thus frame is internally indeterminate to *single degree i.e.*, it has one redundant member. Considering member AB as the redundant, replace it by a force T at the corners B and C, as shown in Fig. 14·1 (*b*). The *base system* is thus obtained, and stresses in various members can be calculated in terms of external loads and the redundant force T. If U is the total strain energy stored in the frame (inclusive of that in BC), we have

$$\frac{\partial U}{\partial T}=0$$

Now strain energy in any member carrying axial force is $=\dfrac{P^2L}{2AE}$

Hence $\quad U=\sum_1^n \dfrac{P^2L}{2AE}$

and $$\frac{\partial U}{\partial T}=\sum_1^n P.\frac{\partial P}{\partial T}\cdot\frac{L}{AE}=0 \qquad (14\cdot4)$$

The force P in any member will be a function of the external load W and the redundant force T, *i.e.*

$$P=aT+bW$$

$$\therefore \qquad \frac{\partial P}{\partial T}=a$$

where a may be zero for some remote members.

For the frame of Fig 14·1(*c*), $m=11$, and $j=6$, $r=3$

$$\therefore \qquad I=11-(6\times2-3)=2$$

Hence the frame is redundant to second degree, *i.e.* it has two redundant members. Choosing AB and CD as the redundant members they can be replaced by the forces T_1 and T_2 at the appropriate corners to obtain the *base system* as shown in Fig. 14·1(*d*). The forces (P) in the various members can now be calculated. In general, $P=aT_1+bT_2+cW$. If U is the total strain energy stored in the system (inclusive of that in the redundant members), we have, from the theorem of minimum strain energy :

$$\frac{\partial U}{\partial T_1}=\sum_1^n P\frac{\partial P}{\partial T_1}\cdot\frac{L}{AE}=0 \qquad [14\cdot5\,(a)]$$

and $$\frac{\partial U}{\partial T_2}=\sum_1^n \frac{\partial P}{\partial T_2}\cdot\frac{L}{AE}=0 \qquad [14\cdot5(b)]$$

The simultaneous solution of equations 14·5(*a*) and 14·5 (*b*) gives the values of the redundant forces T_1 and T_2 in the members AB and CD respectively. If the value of T(or T_1 or T_2) comes out to be *negative*, the actual force in the redundant member will be of the reverse sign; *i.e.* if the original assumption is a compressive force designated by ←...→ at the joints, the actual force will be tensile, and *vice versa.*

Examples will now follow to illustrate the application of the principle of minimum strain energy.

Example 14·1. *Find the force in the member BC of the frame loaded as shown in Fig. 14·2. All the members have the same cross-sectional area.*

Solution

The frame is redundant to single degree, since $I=m-(2j-r)=6-(2\times4-3)=1$. Treating BC as the redundant, and assuming that it carries a tensile force T, apply forces T at joints B and C as shown, and remove the member. The forces in various members can now be found as under :

$$\sin\theta=\frac{3}{5}=0\cdot6\ ;\ \cos\theta=\frac{4}{5}=0\cdot8$$

At B, resolving vertically,

$P_{BD}=T\sin\theta=0\cdot6\ T$ (comp.)

Resolving horizontally,

$P_{BA}=T\cos\theta=0\cdot8\ T$ (comp.)

At A, resolving horizontally

$$P_{AD}=\frac{1}{\cos\theta}(10-P_{AB})$$
$$=1\cdot25\ (10-0\cdot8\ T)\text{ comp.}$$
$$=12\cdot5-T\text{ comp.}$$

Resolving vertically,

$$P_{AC}=P_{AD}\sin\theta$$
$$=0\cdot6(12\cdot5-T)$$

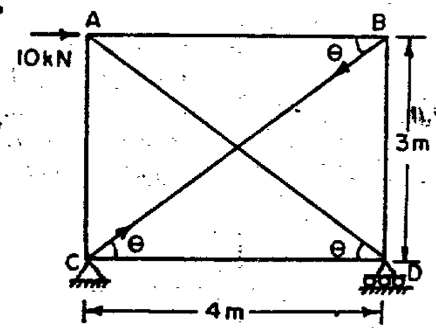


Fig. 14.2.

$$=7\cdot5-0\cdot6\ T \text{ (tension)}$$

Resolving horizontally at D, $P_{CD}=P_{AD}\cos\theta$

$$=10-0\cdot8\ T \text{ (tension)}$$

The values of P, $\frac{\partial P}{\partial R}$, etc., are entered in the tabular form below :

Member	Length m	P	$\frac{\partial P}{\partial E}$	$P\frac{\partial P}{\partial T}.L$
AB	4	$-0\cdot8\ T$	$-0\cdot8$	$+2\cdot56\ T$
BD	3	$-0\cdot6\ T$	$-0\cdot6$	$+1\cdot08\ T$
CD	4	$10-0\cdot8T$	$-0\cdot8$	$-32+2\cdot56\ T$
AC	3	$7\cdot5-0\cdot6T$	$-0\cdot6$	$-13\cdot5+1\cdot08\ T$
AD	5	$T-12\cdot5\ T$	$+1\cdot0$	$-62\cdot5+5\ T$
BC	5	T	$+1\cdot0$	$+5\ T$
				$-108+17\cdot28\ T$

The terms A and E have not been included in the above table since they are *same* for all the members and cancel out.

$$\therefore \quad \frac{\partial U}{\partial T}=0=\sum_{1}^{n}P.\ \frac{\partial P}{\partial T}.\ \frac{L}{AE}=0$$

or

$$-108+17\cdot28\ T=0$$

which gives $T=+6\cdot25$ kN. (tension)

The plus sign indicates that the assumed sign of T (*i.e.* tension) is correct.

Example 14·2. *A frame work consists of six bars of uniform cross-sectional area, and hinged together to form a square with two diagonals, is suspended from one end as shown in Fig. 14·3. At the opposite corner a load of 10 kN is suspended. Calculate the forces in all the members. The diagonals act independently.*

Solution

The number of reaction components (r) necessary for the equilibrium of the frame is 3, Also, $m=6$; $j=4$. Hence $I=6-(2\times4-3)=1$. Hence the frame is indeterminate to single degree.

Treating member AC to be redundant, replace it with tensile force T at the joints A and C as shown.

$$\text{Then}\quad \frac{\partial U}{\partial T}=\sum_{1}^{n}P\ \frac{\partial P}{\partial T}\cdot\frac{L}{AE}=0$$

Since A and E are the same for all the members, we have

$$\frac{\partial U}{\partial T}=\sum_{1}^{n}P\frac{\partial P}{\partial T}\ .\ L=0$$

Calculation of stresses in the members

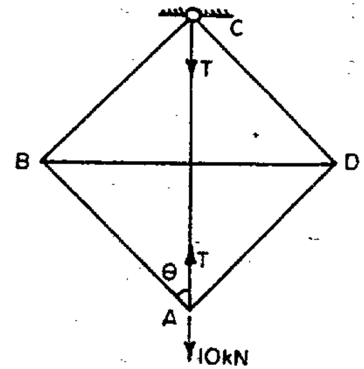


Fig. 14·3.

Let the length of the side of the square$=L$

$\therefore$ Lenth of the diagonal $=L\sqrt{2}$.

Resolving horizontally at A,

$$P_{AB}=P_{AD}$$

Resolving vertically,

$$2P_{AB}\cos\theta+T=10$$

or

$$P_{AB}=P_{AD}=\frac{\sqrt{2}}{2}\ (10-T)=\frac{1}{\sqrt{2}}\ (10-T),\quad \text{tension}$$

Reaction at $C=10$ kN ↑. Hence $P_{BC}=P_{CD}=\frac{1}{\sqrt{2}}(10-T)$, tension.

Resolving at B,

$$P_{BD}=P_{BC}\cos\theta+P_{AB}\cos\theta$$

$$=\frac{2}{\sqrt{2}}\left\{\frac{1}{\sqrt{2}}\ (10-T)\right\}$$

$$=10-T, \text{ compression}$$

The result may now be tabulated as shown on next page from the table we get

$$\therefore \quad \sum_{1}^{n}P\ \frac{\partial P}{\partial T}.\ L=(4\cdot828\ T-34\cdot14)L=0$$

which gives $T=+7\cdot07$ **kN**.

The $+$sign indicates that the sign of the assumed stress in AC is correct (*i.e.*, it carries tension). The value of T can now be substituted in column (3), and the stresses computed as shown in column (6).

(+ for tension ;—for compression)

Member (1)	Length (2)	P (3)	$\frac{\partial P}{\partial T}$ (4)	$P.\frac{\partial P}{\partial T}.L$ (5)	Final stress (P) (6)
AB	L	$\frac{1}{\sqrt{2}}(10-T)$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}(10-T)L$	+2·07
BC	L	$\frac{1}{\sqrt{2}}(10-T)$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}(10-T)L$	+2·07
CD	L	$\frac{1}{\sqrt{2}}(10-T)$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}(10-T)L$	+2·07
DA	L	$\frac{1}{\sqrt{2}}(10-T)$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}(10-T)L$	+2·07
BD	$L\sqrt{2}$	$-10+T$	+1	$\sqrt{2}(T-10)L$	−2·93
AC	$L\sqrt{2}$	T	+1	$\sqrt{2}TL$	+7·07
			Sum	$(4\cdot828T-34\cdot14)L$	

Example 14·3. *In the frame work shown in Fig. 14·4, the member AB, BC and CA have area of cross-section '2a' and the member DA, DB and DC have area of cross-section 'a'. Find the force in the member DA due to a load of 10 tonnes applied horizontally at A.*

Solution

$$m=6\ ;\ j=4\ ;\ r=3$$

$$I=6-(2\times4-3)=1$$

Hence the frame is indeterminate to single degree. Treating AD as the redundant, replace it by tensile force T at the joints A and D. Then

$$\frac{\partial U}{\partial T}=0=\sum_{1}^{n} P\frac{\partial P}{\partial T}\cdot\frac{L}{A}=0$$

(E being same for all members).

At the joint D, resolving horizontally, $P_{BD}=P_{DC}$

Resolving vertically,

$$P_{BD}\cos 45°+P_{DC}\cos 45°=T$$

$$\therefore\quad P_{BD}=P_{DC}=\frac{T}{\sqrt{2}}\ \text{(tension)}.$$

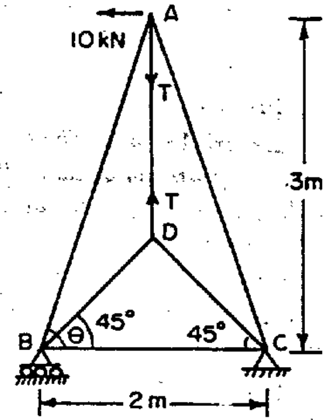


Fig. 14·4

At the joint A, resolving vertically,

$$P_{AB}\sin\theta=T+P_{AC}\sin\theta \quad (1)$$

Resolving horizontally,

$$P_{AB}\cos\theta+P_{AC}\cos\theta=10 \quad (2)$$

where

$$\sin\theta=\frac{3}{\sqrt{10}}=0\cdot948\ ;$$

$$\cos\theta=\frac{1}{\sqrt{10}}=0\cdot316$$

Solving (1) and (2), we get

$$P_{AB}=0\cdot527\ T+15\cdot85\ \text{(comp.)}$$

and

$$P_{AC}=15\cdot85-0\cdot527\ T\ \text{(tension)}$$

Resolving horizontally at B,

$$P_{BC}=P_{AB}\cos\theta-P_{BD}\cos 45°$$

$$=(0\cdot527\ T+15\cdot85)\,0\cdot316-\frac{T}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}$$

$$=-0\cdot334\ T+5\ \text{(tension)}$$

The values of P and $\frac{\partial P}{\partial T}$ etc. are tabulated below :

(+ for tension ; —for compression)

Member	Length	Area	P	$\frac{\partial P}{\partial T}$	$\left(P.\frac{\partial P}{\partial T}\frac{L}{A}\right)a$
AB	3·16	$2a$	$-(0\cdot527\ T+15\cdot85)$	−0·527	$0\cdot439\ T+13\cdot2$
AC	3·16	$2a$	$15\cdot85-0\cdot527\ T$	−0·527	$-13\cdot2+0\cdot439$
BC	2·00	$2a$	$-0\cdot334\ T+5$	−0·334	$0\cdot112\ T-1\cdot67$
BD	$\sqrt{2}$	a	$+\frac{T}{\sqrt{2}}$	$+\frac{1}{\sqrt{2}}$	$+0\cdot707\ T$
CD	$\sqrt{2}$	a	$+\frac{T}{\sqrt{2}}$	$+\frac{1}{\sqrt{2}}$	$+0\cdot707\ T$
AD	2·0	a	$+T$	+1	$+2\ T$
				Sum	$4\cdot404\ T-1\cdot67$

$$\sum_1^n P \frac{\partial P}{\partial T} \frac{L}{A} = 0 = \frac{1}{a}(4{\cdot}404\,T - 1{\cdot}67)$$

From which $T = 0{\cdot}379$ **kN** (tension).

Example 14·4. *The frame shown in Fig. 14·5 is pin-jointed to a rigid support at A and B and the joints C and D are also pinned. The diagonals AD and BC act independently and the members are all of the same cross-section and material. ABC and BCD are equilateral triangles.*

Initially there is no load in any of the members which may be assumed weightless.

If a load of 5 kN is hung at D, calculate the forces in all of the members.

Solution

Both the hinges are essential for the equilibrium of the frame, Hence $r = 2 + 2 = 4$.

Also, $m = 5$; $j = 4$

$\therefore \quad I = 5 - (2 \times 4 - 4) = 1$

Thus the frame is indeterminate to single degree.

Treating AD as the redundant member it may be replaced by compressive force R at the joints A and D as shown. The stresses in various members can now be calculated as under :

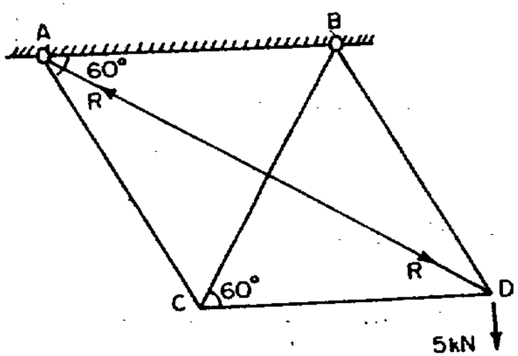


Fig. 14.5.

At the joint D

Resolving vertically, $P_{BD} \frac{\sqrt{3}}{2} - \frac{R}{2} = 5$

or $\qquad P_{BD} = \frac{10+R}{\sqrt{3}}$ (tension)

Resolving horizontally, $P_{CD} = P_{BD} \cos 60° - R \cos 30°$

$$= \frac{10+R}{\sqrt{3}} \times \frac{1}{2} - \frac{\sqrt{3}}{2} R$$

$$= \frac{5-R}{\sqrt{3}} \text{ (compression)}$$

At the joint C

Since members CA and CB are equally inclined to vertical,

$P_{CB} = P_{CA}$

Resolving horizontally, $2P_{AC} \cos 60° = P_{CD} = \frac{5-R}{\sqrt{3}}$

$\therefore \qquad P_{AC} = P_{CD} = \frac{5-R}{\sqrt{3}}$ (compression)

and $\qquad P_{CB} = \frac{5-R}{\sqrt{3}}$ (tension)

The calculation for $P \cdot \frac{\partial U}{\partial R} \cdot L$ is done in the tabular form below :

(+ for tension ; —for compression)

Member (1)	*Length* (2)	*P* (3)	$\frac{\partial P}{\partial R}$ (4)	$P \cdot \frac{\partial P}{\partial R} L$ (5)	*Final Force* (6)
AC	*L*	$-\frac{5-R}{\sqrt{3}}$	$+\frac{1}{\sqrt{3}}$	$\frac{L}{3}(R-5)$	$-2{\cdot}57$
BC	*L*	$+\frac{5-R}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$\frac{L}{3}(R-5)$	$+2{\cdot}57$
CD	*L*	$-\frac{5-R}{\sqrt{3}}$	$+\frac{1}{\sqrt{3}}$	$\frac{L}{3}(R-5)$	$-2{\cdot}57$
DA	$\sqrt{3}L$	$-R$	-1	$\sqrt{3}\,RL$	$-0{\cdot}545$
BD	*L*	$+\frac{10+R}{\sqrt{3}}$	$+\frac{1}{\sqrt{3}}$	$\frac{L}{\sqrt{3}}(10+R)$	$+6{\cdot}07$
			Sum	$\frac{L}{3}(9{\cdot}19\,R-5)$	

$$\therefore \qquad \sum_1^n P \frac{\partial P}{\partial R} \frac{L}{A} = 0 = \frac{L}{3}(9{\cdot}19\,R - 5)$$

which gives $R = 0{\cdot}545$ kN (compression)

Substituting the value of R in column 3, the force in each member can be calculated, and tabulated as shown in column (6).

Example 14·5. *The pin jointed frame shown in Fig. 14·6 is of the shape of a regular hexagon. All the members have the same area of cross-section. Calculate the force in the member CH.*

Solution

There are 3 reaction components necessary for the equilibrium of the whole frame, and hence $r=3$.

$m=12$; $j=7$

$I=12-(7\times2-3)=1$

Hence the frame is indeterminate to single degree. Treating member CH to be redundant, replace it by tensile force T at the joints C and H as shown.

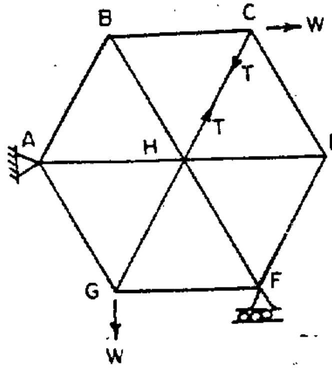


Fig. 14·6

At the joint C

Resolving vertically, $P_{CD}=T$ (compression)

Resolving horizontally, $P_{CB}=W-P_{CD}\cos 60°-T\cos 60°$

$=(W-T)$, tension

At the joint B

Resolving vertically, $P_{BA}=P_{BH}$

Resolving horizontally, $P=P_{BA}\cos 60°+P_{BH}\cos 60°=P_{BC}$

$=W-T$

$\therefore$ $P_{BA}=(W-T)$, tension

and $P_{BH}=(W-T)$, compression

At the joint D

Resolving vertically $P_{DF}=P_{CD}=T$ (comp.)

Resolving horizontally, $P_{DH}=2P_{CD}\cos 60°=T$ (tension)

At the joint F

Since there is roller at F, it can take only the vertical reaction V_F. Thus, at the hinge A, horizontal reaction is $H_A=W\leftarrow$. Let V_A and V_F be the vertical reactions at A and F respectively. Taking moments about A, we get

$$V_F\left(L+\frac{L}{2}\right)=\frac{WL}{2}+W\frac{\sqrt{3}}{2}L$$

From which $V_F=0·91\ W\uparrow$

$\therefore$ $V_A=W-0·91\ W=0·09\ W\uparrow$

Now resolving vertically at F,

$$P_{FH}\cos 30°+P_{FD}\cos 30°=V_F=0·91\ W$$

$\therefore$ $$P_{FH}=0·91\ W\times\frac{2}{\sqrt{3}}-T=(1·05\ W-T)\text{ compression.}$$

Resolving horizontally,

$$P_{FG}=P_{DF}\cos 60°-P_{FH}\cos 60°=\frac{T}{2}-\frac{1}{2}(1·05\ W-T)$$

$$=(T-0·525\ W),\text{ compression.}$$

At the joint G

Resolving horizontally,

$$P_{GA}\cos 60°+P_{GF}=P_{GH}\cos 60°$$

or $$\frac{1}{2}P_{GA}+(T-0·525\ W)=\frac{1}{2}(1·155\ W-P_{GA})$$

From which $P_{GA}=(1·103\ W-T)$, tension.

Resolving vertically.

$$P_{GH}+P_{GA}=W\sec 30°=1·155\ W$$

$\therefore$ $$P_{GH}=1·155\ W-(1·103\ W-T)$$

$$=(0·052\ W+T)\text{ tension.}$$

At the joint A

Resolving horizontally,

$$P_{AH}+W=P_{AB}\cos 60°+P_{AG}\cos 60°$$

$$P_{AH}=\frac{1}{2}(W-T)+\frac{1}{2}(1·103\ W-T)-W$$

$$=(0·052\ W-T),\text{ compression}$$

The summation may now be carried out in the tabular form below. $\frac{L}{AE}$ is same for all the members, and has not been included in the table.

$\therefore$ $$\frac{\partial U}{\partial T}=\Sigma P\frac{\partial P}{\partial T}\cdot\frac{L}{AE}=0$$

or $$12\ T-5·678\ W=0$$

or $$T=0·437\ \text{W (tension).}$$

(+ for tension :— for compression)

Member	P	$\frac{\partial P}{\partial T}$	$P\frac{\partial P}{\partial T}$
AB	$+(W-T)$	-1	$T-W$
BC	$+(W-T)$	-1	$T-W$
CD	$-T$	-1	T
DF	$-T$	-1	T
FG	$-(T-0.525\ W)$	-1	$T-0.525\ W$
GA	$+(1.103\ W-T)$	-1	$T-1.103\ W$
AH	$-(0.052\ W-T)$	$+1$	$T-0.052\ W$
BH	$-(W-T)$	$+1$	$T-W$
CH	$+T$	$+1$	T
DH	$+T$	$+1$	T
FH	$-(1.05\ W-T)$	$+1$	$T-1.05\ W$
GH	$+(0.052\ W+T)$	$+1$	$T+0.052\ W$
		Sum	$12\ T-5.678\ W$

Example 14·6. *Find the axial force in the member BC of the frame shown in Fig. 14·7. The figures in brackets indicate the cross-sectional area in cm². The members are all of the same material.*

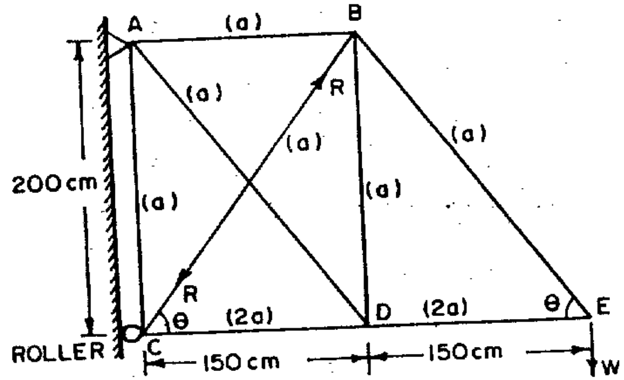


Fig. 14·7.

Solution

There are 3 reaction components necessary for the equilibrium of the whole frame, and hence $r=3$

$$m=8\ ; \quad j=5$$

$$\therefore \quad I=8-(5+2-3)=1$$

Hence the frame is indeterminate to first degree. Treating member BC to be redundant, replace it by a compressive force R at the joints B and C as shown.

$$\sin\theta=\frac{200}{\sqrt{(200)^2+(150)^2}}=\frac{200}{250}=0.8$$

$$\cos\theta=\frac{150}{250}=0.6$$

At the joint E

$$P_{BE}=W\ \text{cosec}\ \theta=1.25\ W\ \text{(tension)}$$

$$P_{DE}=P_{BE}\cos\theta=1.25\ W\,(0.6)=0.75\ W\ \text{(comp.)}$$

At the joint B

Resolving vertically,

$$P_{BD}=(1.25\ W\times0.8)-(R\times0.8)=W-0.8\ R\ \text{(comp.)}$$

Resolving horizontally

$$P_{AB}=(1.25\ W\times0.6)+(R\times0.6)=0.75\ W+0.6\ R\ \text{(tension)}$$

At the joint D

Resolving vertically,

$$P_{AD}=\frac{1}{0.8}\times P_{BD}=1.25\ W-R\ \text{(tension)}$$

Resolving horizontally.

$$P_{CD}=0.73\ W+0.6\ (1.25\ W-R)=1.5\ W-0.6\ R\ \text{(comp.)}$$

At the joint C

Since there is a roller at C, vertical reaction is zero. Hence, resolving vertically,

$$P_{AC}=R\times0.8=0.8\ R\ \text{(tension)}$$

The result may now be tabulated as below.

$$\text{Now} \quad \frac{\partial U}{\partial R}=\sum_{1}^{n}P\,\frac{\partial P}{\partial R}\cdot\frac{L}{AE}=0$$

$\therefore$ From the table

$$\frac{1}{a}\,(837\ R-472.5\ W)=0$$

From which $R=0.564\ W$ (compression).

(+ for tension ; — for compression)

Member	L (cm)	A (cm)^2	P	$\frac{\partial P}{\partial R}$	$P \cdot \frac{\partial P}{\partial R} \cdot \frac{L}{A} a$
AB	150	a	$0.75\,W+0.6\,R$	+0·6	$(67.5\,W+54\,R)$
AC	200	a	$0.8\,R$	+0·8	$(128\,R)$
CD	150	$2a$	$0.6\,R-1.5\,W$	+0·6	$(27\,R-67.5\,W)$
BD	200	a	$0.8\,R-W$	+0·8	$(128\,R-160\,W)$
BE	250	a	$1.25\,W$	0	0
DE	150	$2a$	$-0.75\,W$	0	0
AD	250	a	$1.25W-R$	−1	$(250\,R-312.5\,W)$
BC	250	a	$-R$	−1	$(250\,R)$
				Sum	$(837\,R-472.5\,W)$

Example 14·7. *Find the force in the member AC of the frame shown in Fig. 14·8. The quantity AE is constant for all the members.*

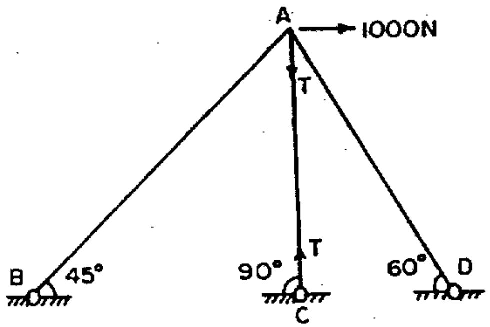


Fig. 14·8.

Solution

A close study of the frame will reveal that all the three hinged supports are essential for the stability of the frame. Thus the reaction component necessary for the equilibrium of the frame is $r=3\times2=6$

$$m=3\ ;\ j=4$$

$$\therefore \qquad I=3-(2\times4-6)=1$$

Hence the frame is indeterminate to single degree. Considering the member AC to be redundant, replace it by tensile forces (T) at the joints A and C, as shown.

Resolving horizontally at A,

$$P_{AB}\cos 45°+P_{AD}\cos 60°=1000 \qquad (1)$$

Resolving vertically at A,

$$P_{AB}\cos 45°+T=P_{AD}\sin 60°$$

From (1) and (2), we get

$$P_{AB}=896-0.516\,T \text{ (tension)}$$

$$P_{AD}=0.733\,T+733 \text{ (comp.)}$$

For the whole frame,

$$\frac{\partial U}{\partial T}=0=\sum_{1}^{n} P\,\frac{\partial P}{\partial R}\cdot\frac{L}{AE}$$

The summation may be carried in the tabular form below :

(+ for tension ; — for compression)

Member	L	P	$\frac{\partial P}{\partial R}$	$P\frac{\partial P}{\partial R}L$
AB	$2\sqrt{}L$	$896-0.516\,T$	−0·516	$(-656+0.376\,T)\,L$
AD	$\frac{2}{\sqrt{3}}L$	$-(0.733\,T+733)$	−0·733	$(0.620\,T+620)\,T$
AC	L	$+T$	+1	TL
			Sum	$(1.996T-36)\,L$

$$\therefore \quad 1.996\,T-36=0$$

or $$T=18 \text{ N (tension).}$$

Example 14·8. *The bars of the pin jointed frame shown in Fig. 14.9 are of the same material and have the same cross-sectional area. Show that the forces in AB and CD are compressive and tensile respectively of magnitude* $\frac{W}{2}\left(\frac{1+2\sqrt{2}}{3+4\sqrt{2}}\right)$. (U.L.)

Solution

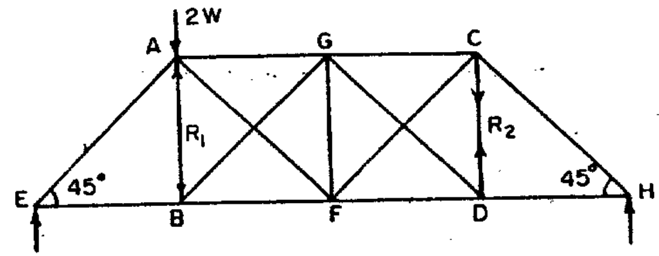


Fig. 14·9.

From the frame,

$$r=3$$
$$m=15$$
$$j=8$$
$$I=15-(2\times 8-3)=2.$$

Hence the frame is indeterminate to second degree. Treating AB and CD as redundants, replace them with the forces, R_1 and R_2 at the joints (A, B) and (C, D) respectively, as shown. It is assumed that AB carries a compressive force (R_1) and CD carries a tensile force (R_2).

$$\text{Reaction at } E=\frac{2W\times 3}{4}=\frac{3}{2}W\uparrow.$$

$$\text{Reaction at } H=\frac{2W\times 1}{4}=\frac{W}{2}\uparrow.$$

At the joint E

$$P_{EA}=\frac{3}{2}W \text{ cosec } 45^\circ=\frac{3}{\sqrt{2}}W \text{ (comp.)}$$

$$P_{EB}=P_{EA}\cos 45^\circ=\frac{3}{\sqrt{2}}W\times\frac{1}{\sqrt{2}}=\frac{3}{2}W \text{ (tension)}$$

At the joint A

Resolving vertically

$$P_{AF}\frac{1}{\sqrt{2}}=\frac{3}{\sqrt{2}}W\times\frac{1}{\sqrt{2}}+R_1-2W$$

$$P_{AF}=\frac{3}{\sqrt{2}}W+\sqrt{2}R_1-2\sqrt{2}W$$

$$=\left(R_1-\frac{W}{2}\right)\sqrt{2} \text{ (tension)}$$

Resolving horizontally,

$$P_{AG}=\frac{3}{\sqrt{2}}W\times\frac{1}{\sqrt{2}}+\left(R_1-\frac{W}{2}\right)\frac{\sqrt{2}}{\sqrt{2}}=(W+R_1) \text{ (comp.)}$$

At the joint B

$$P_{BG}=P_{AB}\text{ cosec } 45^\circ=\sqrt{2}R_1 \text{ (tension)}$$

and $P_{BF}=P_{EB}-P_{BG}\cos 45^\circ=\frac{3}{2}W-\sqrt{2}\,R_1\frac{1}{\sqrt{2}}=\left(\frac{3}{2}W-R_1\right)$ (tension)

At the joint H

$$P_{CH}=\frac{W}{2}\text{ cosec } 45^\circ=\frac{W}{2}\sqrt{2}=\frac{W}{\sqrt{2}} \text{ (compression)}$$

$$P_{DH}=P_{CH}\cos 45^\circ=\frac{W}{\sqrt{2}}\times\frac{1}{\sqrt{2}}=\frac{W}{2} \text{ (tension).}$$

At the joint D

$$P_{GD}=\sqrt{2}\,R_2 \text{ (comp.)}$$

$$P_{FD}=P_{DH}+P_{GD}\cos 45^\circ=\frac{W}{2}+\sqrt{2}R_2\times\frac{1}{\sqrt{2}}=\left(\frac{W}{2}+R_2\right) \text{ tension.}$$

At the joint C

Resolving vertically, $P_{CF}\times\frac{1}{\sqrt{2}}=\frac{W}{\sqrt{2}}\times\frac{1}{\sqrt{2}}-R_2$

$$P_{CF}=\left(\frac{W}{\sqrt{2}}-\sqrt{2}\,R_2\right) \text{ tension}$$

Resolving horizontally, $P_{CG}=\frac{W}{\sqrt{2}}\times\frac{1}{\sqrt{2}}+\left(\frac{W}{\sqrt{2}}-\sqrt{2}\,R_2\right)\frac{1}{\sqrt{2}}$

$$=\frac{W}{2}+\frac{W}{2}-R_2$$

$$=(W-R_2) \text{ compression.}$$

At the joint G

Resolving vertically, $P_{GF}=-P_{GB}\cos 45^\circ+P_{GD}\cos 45^\circ$

$$=-\sqrt{2}R_1\times\frac{1}{\sqrt{2}}+2R_2\times\frac{1}{\sqrt{2}}$$

$$=(R_2-R_1) \text{ tension.}$$

Now, according to Castigliano's theorem of minimum strain energy :

$$\frac{\partial U}{\partial R_1}=0=\sum_1^n P\frac{\partial P}{\partial R_1}\cdot\frac{L}{AE} \quad (1)$$

and

$$\frac{\partial U}{\partial R_2}=0=\sum_1^n P\frac{\partial P}{\partial R_2}\cdot\frac{L}{AE} \quad (2)$$

The summation is carried out in tabular form on next page.

From the Table,

$$\Sigma P\frac{\partial P}{\partial R_1}\cdot L=(4+4\sqrt{2})\,R_1-R_2-\frac{W}{2}(1+2\sqrt{2})=0 \quad (1)$$

and

$$\Sigma P=\frac{\partial P}{\partial R_2}\cdot L=(4+4\sqrt{2})\,R_2-R_1-\frac{W}{2}(1+2\sqrt{2})=0 \quad (2)$$

The simultaneous solution of (1) and (2) gives

$$R_1=\frac{W}{2}\left(\frac{1+2\sqrt{2}}{3+4\sqrt{2}}\right), \text{ compression}$$

and
$$R_2=\frac{W}{2}\left(\frac{1+2\sqrt{2}}{3+4\sqrt{2}}\right), \text{ tension.}$$

Member	L	P	$\frac{\partial P}{\partial R_1}$	$\frac{\partial P}{\partial R_2}$	$P\frac{\partial P}{\partial R_1}\cdot L$	$P\frac{\partial P}{\partial R_2}L$
EA	$\sqrt{2}$	$-\frac{3}{\sqrt{2}}W$	0	0	0	0
AG	1	$-(W+R_1)$	-1	0	$W+R_1$	0
GC	2	$-(W-R_2)$	0	$+1$	0	$-W+R_2$
CH	$\sqrt{2}$	$-\frac{W}{\sqrt{2}}$	0	0	0	0
EB	1	$\frac{3}{2}W$	0	0	0	0
BF	1	$\left(\frac{3}{2}W-R_1\right)$	-1	0	$-\frac{3}{2}W+R_1$	0
FD	1	$\left(\frac{W}{2}+R_2\right)$	0	$+1$	0	$\frac{W}{2}+R_2$
DH	1	$\frac{W}{2}$	0	0	0	0
AB	1	$-R_1$	-1	0	R_1	0
BG	$\sqrt{2}$	$\sqrt{2}R_1$	$+\sqrt{2}$	0	$2\sqrt{2}R_1$	0
AF	$\sqrt{2}$	$\left(R_1-\frac{W}{2}\right)\sqrt{2}$	$+\sqrt{2}$	0	$2\sqrt{2}R_1-\sqrt{2}W$	0
GF	1	(R_2-R_1)	-1	$+1$	R_1-R_2	R_2-R_1
GD	$\sqrt{2}$	$-\sqrt{2}R_2$	0	$-\sqrt{2}$	0	$2\sqrt{2}R_2$
FC	$\sqrt{2}$	$\frac{W}{\sqrt{2}}-\sqrt{2}R_2$	0	$-\sqrt{2}$	0	$-\sqrt{2}W+2\sqrt{2}R_2$
CD	1	R_2	0	$+1$	0	R_2

Example 14·9. *Find the forces in all the members of the frame shown in Fig. 14·10. All the bars are of same area of cross-section and are of same material.*

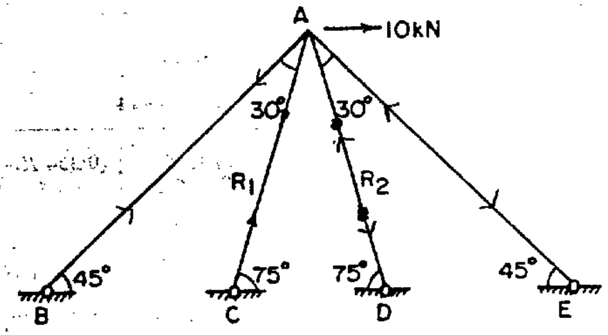


Fig. 14·10.

Solution

All the four hinged supports are necessary for the equilibrium of the frame. If any of the hinged support is replaced with a roller support, the frame becomes unstable.

Hence $r=4\times2=8$

$m=4$

$j=5$

$\therefore$ $I=4-(2\times5-8)=2$

Thus the frame is indeterminate to *second degree*. Considering AC and AD as redundants, replace them with forces R_1 and R_2 at A, C and D as shown. It is assumed that AC carries a tensile force R_1 and AD carries a compressive force R_2.

Consider the equilibrium of the joint A.

Resolving horizontally, we get

$P_{AB}\cos 45°+R_1\cos 75°+R_2\cos 75°+P_{AE}\cos 45°=10$

$0{\cdot}707\ P_{AB}+0{\cdot}259\ (R_1+R_2)+0{\cdot}707\ P_{AE}=10$ (1)

Resolving vertically, we get

$P_{AB}\sin 45°+R_1\sin 75°=P_{AE}\sin 45°+R_2\sin 75°$

$0{\cdot}707\ P_{AB}+0{\cdot}966\ (R_1-R_2)-0{\cdot}707\ P_{AE}=0$ (2)

Solving (1) and (2), we get

$P_{AB}=0{\cdot}5\ R_2-0{\cdot}866\ R_1+7{\cdot}07$ (tension)

$P_{AE}=0{\cdot}5\ R_2-0{\cdot}866\ R_2+7{\cdot}07$ (compression)

Let the height of the frame $=L$

$\therefore$ Length of AB and $AE=L$ cosec $45°=1{\cdot}414\ L$

Length of AC and $AD=L$ cosec $75°=1{\cdot}035\ L$

The result may now be tabulated.

Member	L	P	$\frac{\partial P}{\partial R_1}$	$\frac{\partial P}{\partial R_2}$	$P\frac{\partial P}{\partial R_1}L$	$P\frac{\partial P}{\partial R_2}L$
AB	$1{\cdot}414\ L$	$(0{\cdot}5\ R_2-0{\cdot}866\ R_1+7{\cdot}07)$	$-0{\cdot}866$	$+0{\cdot}5$	$-(0{\cdot}612\ R_2-1{\cdot}06R_1+8{\cdot}66)L$	$(0{\cdot}354\ R_2-0{\cdot}612R_1+5{\cdot}0)L$
AC	$1{\cdot}035\ L$	$+R_1$	$+1$	0	$1{\cdot}035\ R_1L$	0
AD	$1{\cdot}035\ L$	$-R_2$	0	-1	0	$1{\cdot}085\ R_2L$
AE	$1{\cdot}414\ L$	$0{\cdot}866\ R_2-0{\cdot}5R_1-7{\cdot}07$	$-0{\cdot}5$	$+0{\cdot}866$	$-(0{\cdot}622\ R_1-0{\cdot}354R_1-5{\cdot}0)L$	$(1{\cdot}06R_2-0{\cdot}612\ R_1-8{\cdot}66)L$

$$\therefore \frac{\partial U}{\partial R_1}=0=\sum_1^n P\frac{\partial P}{\partial R_1}L=(2{\cdot}449\ R_1-1{\cdot}224\ R_2-3{\cdot}66)\ L \quad (1)$$

$$\frac{\partial U}{\partial R_2}=0=\sum_1^n P\frac{\partial P}{\partial R_2}L=(2{\cdot}449\ R_2-1{\cdot}224\ R_1-3{\cdot}66)\ L \quad (2)$$

Solving (1) and (2), we get

$R_1=3$ **kN** (tension)

$R_2=3$ **kN** (comp.)

Hence $P_{AB}=6$ **kN** (tension)

$P_{AE}=9$ **kN** (comp.)

14·3. MAXWELL'S METHOD

The stresses in redundant frames can also be evaluated by the method given by Clerk Maxwell.

Fig. 14·11 (*a*) shows a redundant frame. Treating AC as the redundant, replace it by tensile forces T at A and C, as shown in Fig. 14·11 (*b*).

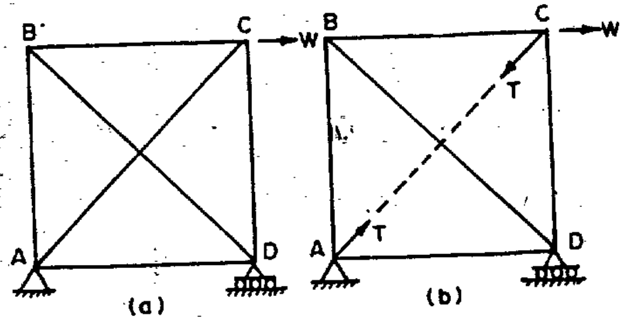


Fig. 14·11.

Due to the application of forces T, the joints A and C will move towards each other, thus shortening the length AC.

Let P=Force in any member due to external load acting on the perfect frame *obtained by removing the redundant member*.

u=Force in any member due to unit loads at A and C.

uT=Force in any member due to force T at A and C.

Hence the total force in any member, due to external loading as well as force T at A and C is $=(P+uT)$.

Hence from Eq. 13·1, the deflection of joints A and C towards each other $=\sum_1^{n-1}\frac{(P+uT)uL}{AE}$

$$\therefore \text{ Shortening of length } AC=\sum_1^{n-1}\frac{(P+uT)uL}{AE} \quad (1)$$

(The summation being done for all members except AC).

Since member AC carries tensile force (T), it is elongated by the amount $\frac{TL'}{A'E'}$ where L', A' and E' stand for AC.

Since the deformation of the member AC is consistent with the movement of joints A and C, we get

$$\sum_1^{n-1}\frac{(P+uT)uL}{AE}=-\frac{TL'}{A'E'} \quad (2)\ (14{\cdot}6)$$

Thus T can be found by the solution of the above equation.

Rewriting the above equation, we get

$$\sum_1^{n-1}\frac{PuL}{AE}+\sum_1^{n-1}\frac{u^2TL}{AE}=-\frac{TL'}{A'E'}$$

In the above expression, u is the force in any member due to unit load at the joints A and C. Hence, for the member AC itself, $u=1$. Therefore the term $\frac{TL'}{A'E'}$ can also be written as $\frac{u^2TL'}{A'E'}$.

Thus, we get

$$\sum_1^{n-1}\frac{PuL}{AE}+\sum_1^{n-1}\frac{u^2TL}{AE}=-\frac{u^2TL'}{A'E'}$$

or
$$\sum_1^{n-1}\frac{PuL}{AE}=-\sum_1^{n-1}\frac{u^2TL}{AE}-\frac{u^2TL'}{A'E'}$$

or
$$\sum_1^{n-1}\frac{PuL}{AE}=-\sum_1^{n}\frac{u^2TL}{AE}=-T\sum_1^{n}\frac{u^2L}{AE}$$

From which $T = -\dfrac{\sum_{1}^{n-1} \dfrac{PuL}{AE}}{\sum_{1}^{n} \dfrac{u^2L}{AE}}$ (14·7)

(where n is the total number of members)

Notes :

1. In the above equation, $\sum_{1}^{n-1}$ is the summation for the whole frame except the redundant member, while $\sum_{1}^{n}$ is the summation for all members including the redundant member.

(2) Comparing Maxwell's method with that of Castigliano's method, it will be seen that in the Maxwell's method the stress in each member has to be calculated twice, first due to the external loads (in the absence of the redundant member) and second by unit force applied at the ends of the redundant member.

(3) Equations 14·6 and 14·7 are *valid only for tensile force in the redundant members.* This is an important point to note. Thus to start with, the redundant member is assumed to carry tensile force.

If at the end of the solution, a positive sign is obtained with the numerical value of T, it will be tensile, as assumed earlier. If negative sign is obtained with the numerical value of T, it will be compressive.

Procedure :

(1) Remove the redundant member completely.

(2) Find the force (P) in each member due to external loading.

(3) Apply unit load (→...←) at the end joints of redundant member, and find unit force (u) in each member.

(4) Apply Eq. (14·6) or (14·7) to find the force T.

Frames with two or more redundant members

Let the frame be redundant to second degree. Remove the two redundant members, thus making the frame perfect.

Let P= Force in any member due to external loading, obtained after making the frame 'perfect' by removing the redundant members.

u_1= Force in any member due to unit pulls at the joints of the first redundant member.

u_2= Force in any member due to unit pulls at the joints of the redundant member

Hence the force in any member $= (P + u_1T_1 + u_2T_2)$

Movement of joints (towards each other)

$$= \sum_{1}^{n-2} \frac{(P + u_1T_1 + u_2T_2)u.L}{AE}$$

The extension of the first redundant bar $= \dfrac{T_1L'}{A'E}$

Since the movement of the joints is consistent with the deformation of the redundant member, we get

$$\sum_{1}^{n-2} \frac{(P + u_1T_1 + u_2T_2)u_1L}{AE} = -\frac{T_1L'}{A'E} \quad [14·8(a)]$$

(where L' and A' stand for the first redundant member)

Similarly, for the second redundant member.

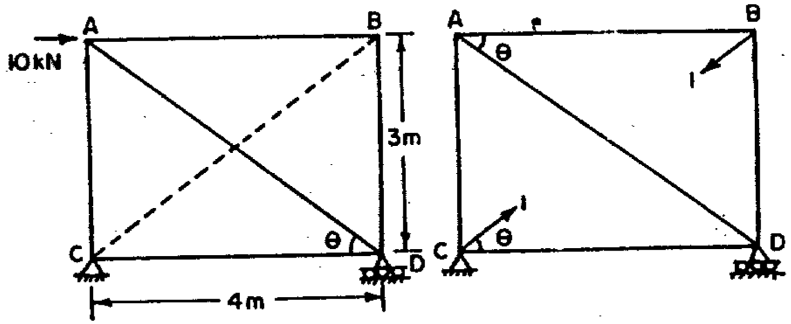
$$\sum_{1}^{n-2} \frac{(P + u_1T_1 + u_2T_2)u_2L}{AE} = -\frac{T_2L''}{A''E} \quad [14·8(b)]$$

(where L'' and A'' stand for the second redundant member)

The simultaneous solution of Eqs. 14·8(a) and 14·8(b) will give the values of the redundant forces T_1 and T_2.

Example 14·10. *Solve example 14·1 by Maxwell's method.*

Solution. (Ref. Fig. 14·12).



(a) BC REMOVED (b) UNIT LOAD

Fig. 14·12.

(a) *Calculation of P* : sin θ = 0·6 ; cos θ = 0·8

Remove the redundant member, as shown in Fig. 14·12(a).

Since there is no external load at the joint B,

$P_{AB} = 0$

$P_{BD} = 0$

Resolving horizontally at joint A,

$$P_{AB}\cos\theta=10$$

$$\therefore\quad P_{AD}=\frac{10}{0{\cdot}8}=12{\cdot}5 \text{ kN (comp.)}.$$

Resolving vertically at joint A,

$$P_{AC}=P_{AD}\sin\theta=12{\cdot}5\times0{\cdot}6=7{\cdot}5 \text{ kN (tension)}$$

$$P_{CD}=\text{horizontal reaction at } C=10 \text{ kN (tension)}$$

(Alternately, $P_{CD}=P_{AD}\cos\theta=12{\cdot}5\times0{\cdot}8=10$ kN)

(*b*) *Calculation of u* [Fig. 14·12 (*b*)]: Apply unit loads at C and D.

Reaction at C and D will be zero since unit loads are equal and opposite.

Resolving at joint C,

$$u_{AC}=1\sin\theta=0{\cdot}6 \text{ (comp.)}$$

$$u_{CD}=1\cos\theta=0{\cdot}8 \text{ (comp.)}$$

Resolving at the joint A,

$$u_{AD}\sin\theta=u_{AC}=0{\cdot}6$$

$$\therefore\quad u_{AD}=\frac{0{\cdot}6}{0{\cdot}6}=1 \text{ (tension)}$$

Resolving at the joint B,

$$u_{AB}=1\cos\theta=0{\cdot}8 \text{ (comp.)}$$

$$u_{BD}=1\sin\theta=0{\cdot}6 \text{ (comp.)}$$

The summation may be carried out in the tabular form below :

(+for tension ; —for compression)

Member	*L* (*m*)	*P*	*u*	*PuL*	u^2L
AB	4	0	−0·8	0	2·56
AC	3	+7·5	−0·6	−13·5	1·08
CD	4	+10·0	−0·8	−32·0	2·56
BD	3	0	−0·6	0	1·08
AD	5	−12·5	+1·0	−62·5	5·00
BC	5	—	+1·0	—	5·00
			Sum	−108·0	17·28

From equation 14·7,

$$T=\frac{\sum_{1}^{n-1}\frac{PuL}{AE}}{\sum_{1}^{n}\frac{u^2L}{AE}}$$

$$=-\frac{-108}{17{\cdot}28}=+6{\cdot}25 \text{ kN (tension)}$$

(Plus sign indicates that the redundant member carries tension).

Example 14·11. *In the plane braced frame work shown in Fig. 14·13 (a) all the members have the same cross sectional area and are made of the same material. Determine the force in all bars when loads W are applied as shown.* (*U.L.*)

Solution

Let us treat FE as the redundant member, carrying a tensile force T. Since the frame is indeterminate to single degree, we have

$$T=-\frac{\sum_{1}^{n-1}\frac{PuL}{AE}}{\sum_{1}^{n}\frac{u^2L}{AE}}=-\frac{\sum_{1}^{n-1}PuL}{\sum_{1}^{u}u^2.L} \qquad (1)$$

(Since AE is constant for all members.)

(*a*) *Calculation of P*

To calculate P in each member due to external loading, remove the redundent member FE, and make the frame statically determinate (or perfect), as shown in Fig. 14·13(*b*).

By inspection, all the inclined members and diagonals are inclined at 45° to the horizontal.

At the joint A : $P_{AF}=P_{AB}$

and $$2P_{AF}\frac{1}{\sqrt{2}}=W$$

$$\therefore\quad P_{AF}=P_{ED}=\frac{W\sqrt{2}}{2}=\frac{W}{\sqrt{2}} \text{(tension)}$$

and $$P_{AB}=P_{CD}=\frac{W}{\sqrt{2}} \text{(comp)}$$

At the joint F : $P_{FC}=P_{AF}=\frac{W}{\sqrt{2}}\text{(tension)}=P_{EB}$

$$P_{FB}=2P_{FA}\frac{1}{\sqrt{2}}=2.\frac{W}{\sqrt{2}}\times\frac{1}{\sqrt{2}}=W \text{ (comp.)}$$

$P_{EC}=W$ (comp.), by symmetry.

At the joint B.

$$P_{BC}=(P_{AB}+P_{BC})\frac{1}{\sqrt{2}}=\left(\frac{W}{\sqrt{2}}+\frac{W}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}=W\text{ (comp.)}$$

(*b*) *Calculation of u*

Remove the external load and apply unit pulls at F and E, as shown in Fig. 14·13(c). Since the applied unit pulls are equal and opposite, and act along the same direction, the reactions at B and C are each zero.

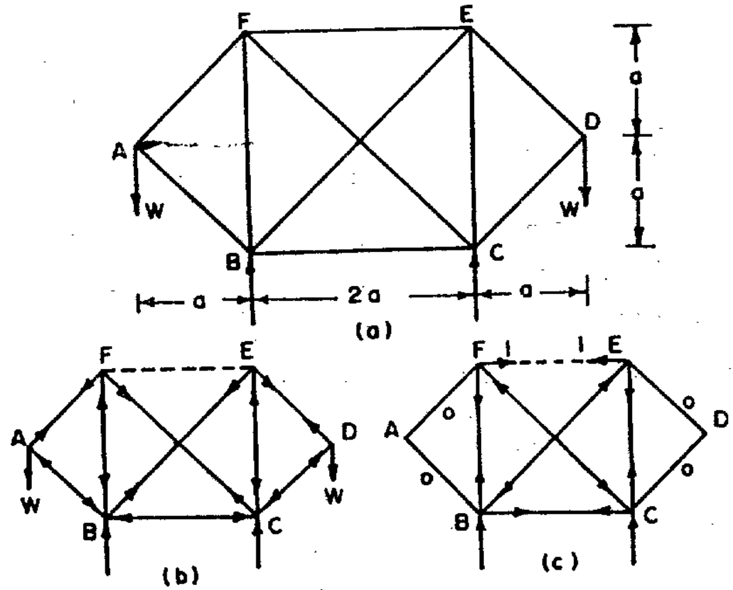


Fig. 14·13.

Since no external load is acting at joints A and D,

$$u_{AF}=0\ ;\ u_{AB}=0\ ;\ u_{ED}=0\ ;\ u_{CD}=0.$$

At the joint F, $u_{FC}=1\sqrt{2}=\sqrt{2}$ (comp.)$=u_{BE}$

$$u_{FB}=u_{FC}\times\frac{1}{\sqrt{2}}=1\text{ (tension)}=u_{CE}$$

At the joint R, $u_{BC}=u_{BE}\times\dfrac{1}{\sqrt{2}}=1$ (tension)

The result may now be tabulated below.

From the Table

$$T=-\frac{\sum\limits_{1}^{n-1} PuL}{\sum\limits_{1}^{n} u^2L}=\frac{-11\cdot636Wa}{19\cdot312a}=+0\cdot6035\ W$$

Hence $T=0\cdot6035\ W$ (tension).

The actual stress in any member is equal to ($P=uT$), and has been entered in the last column of the given table. The reader is advised to work out this example by Castigliano's method.

Member	*L*	*P*	*u*	*PuL*	u^2L	*Actual stress* $=(P+uT)$
AB	$\sqrt{2}a$	$-\dfrac{W}{\sqrt{2}}$	0	0	0	$-0\cdot707W$
BC	$2a$	$-W$	+1	$-2Wa$	$2a$	$-0\cdot3965W$
CD	$\sqrt{2}a$	$-\dfrac{W}{\sqrt{2}}$	0	0	0	$-0\cdot707W$
AF	$\sqrt{2}a$	$+\dfrac{W}{\sqrt{2}}$	0	0	0	$+0\cdot707W$
ED	$\sqrt{2}a$	$+\dfrac{W}{\sqrt{2}}$	0	0	0	$+0\cdot707W$
BF	$2a$	$-W$	+1	$-2Wa$	$2a$	$-0\cdot3965W$
CE	$2a$	$-W$	+1	$-2Wa$	$2a$	$-0\cdot3965W$
FC	$2\sqrt{2}a$	$+\dfrac{W}{\sqrt{2}}$	$-\sqrt{2}$	$-2\sqrt{2}Wa$	$4\sqrt{2}a$	$-0\cdot1465W$
BE	$2\sqrt{2}a$	$+\dfrac{W}{\sqrt{2}}$	$-\sqrt{2}$	$-2\sqrt{2}Wa$	$4\sqrt{2}a$	$-0\cdot1465W$
FE	$2a$	—	+1	—	$2a$	$+0\cdot6035W$
			Sum	$-11\cdot636Wa$	$+19\cdot312a$	

14·4. STRESSES DUE TO ERROR IN LENGTH

If any one member of a redundant frame has lack of fit, stresses will be induced in all the members of the redundant frame when that member is forced in position. Let a member of redundant frame be *short* in length by an amount λ. When this member is forced into position, it will exert pull T at the joints (or ends) of the member. Thus, the two joints will have a tendency to move towards each other while the member (having lack of fit) will be subjected to a tensile force T. In the equilibrium position, by compatibility of deformation, we have :

INWARD MOVEMENT OF JOINTS + EXTENSION OF THE MEMBER $=\lambda$ (1)

From chapter 13, the inward movement of the joints due to a force T acting at the joints $= \sum_{1}^{n-1} \frac{PuL}{AE}$

$$= \sum_{1}^{n-1} \frac{(uT)uL}{AE} \quad (2)$$

Where u = force in any member due to unit *pulls* at the two joints

P = force in any member due to pulls T at the two joints $= uT$

Also the extension of the member having lack of fit, due to the tensile force $= \frac{TL'}{A'E}$ (3)

(Where L' and A' stand for that member)

Substituting the values in (1), we get

$$\sum_{1}^{n-1} \frac{(uT)uL}{AE} + \frac{TL'}{A'E} = \lambda$$

As in the previous article, writing $\frac{TL'}{A'E} = \frac{u^2TL'}{A'E}$, we get

$$\sum_{1}^{n-1} \frac{(uT)uL}{AE} + \frac{u^2TL'}{A'E} = \lambda$$

or

$$\sum_{1}^{n} \frac{u^2TL}{AE} = \lambda$$

Hence

$$T = + \frac{\lambda}{\sum_{1}^{n} \frac{u^2L}{AE}} \quad (14\cdot9)$$

where λ is taken to be positive if the member is short in length (so as to exert pull T at the joints), and negative if the member is excess in length (so as to apply push at the joints).

Analysis by Castigliano's Theorem :

The stresses in members of a redundant frame due to lack of fit of a member can also be found by Castigliano's theorem as under :

Let the member carry a force T when forced in position. All the members of frame will be strained, and the partial derivative of the total strain energy with respect to the force T will, according to Castigliano's second theorem, be equal to the lack of fit.

Thus, $\frac{\partial U}{\partial T} = \sum_{1}^{n} P \frac{\partial P}{\partial T} \cdot \frac{L}{AE} = \lambda$ (4)

Now, when there is no external load,

$$P = uT$$

and

$$\frac{\partial P}{\partial T} = u$$

Substituting the values in (4), we get

$$\sum_{1}^{n} (uT)(u) \frac{L}{AE} = \lambda$$

from which

$$T = \frac{\lambda}{\sum_{1}^{n} \frac{u^2L}{AE}}$$

which is the same as equation 14·9 found by Maxwell's method.

Example 14·12. *Find the forces in all the members of the frame shown in Fig. 14·14(a), if the member BC is short in length by 10 mm and is forced into position. Take $E = 2 \times 10^5$ N/mm². All members have same area of cross-section of 100 mm².*

Solution

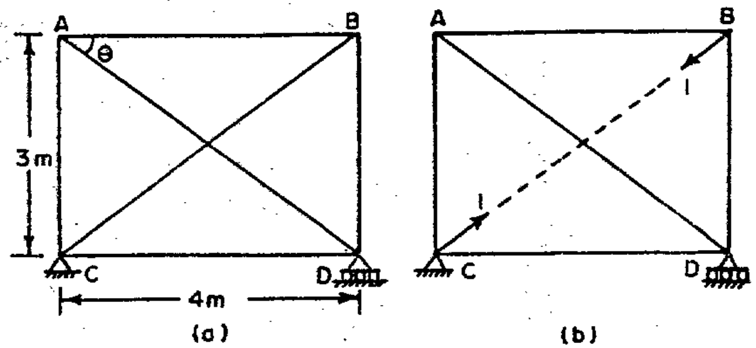


Fig. 14·14.

To find the tensile force T in the member BC by equation 14·9 remove the member BC and apply unit pulls at joints B and C as shown in Fig. 14·4(*b*). The stresses (*u*) in each member can be calculated, and these will be same as calculated in example 14·9.

The values may be arranged in the tabular form on next page.

$$\sum_{1}^{n} u^2L = 17280 \text{ kN·mm units}$$

$$\lambda = +10 \text{ mm}$$

$$E = 2 \times 10^5 \text{ N/mm}^2 = 200 \text{ kN/mm}^2$$

$$A = 100 \text{ mm}^2$$

(+for tension : — for compression)

Member	L (mm)	u	u^2L	Actual stress P=uT (kN)
AB	4000	−0·8	2560	−9·26
AC	3000	−0·6	1080	−6·94
CD	4000	−0·8	2560	−9·26
BD	3000	−0·6	1080	−6·94
AD	5000	+1·0	5000	+11·57
BC	5000	+1·0	5000	+11·57
		Sum	17280	

Now $$T=+\frac{\lambda}{\sum_1^n \frac{u^2L}{AE}}$$

$$=+\frac{10}{\frac{17280}{100\times 200}}=+11{\cdot}57 \text{ kN}$$

Hence stress in the member $BC=11{\cdot}57$ kN (tensile)

Stress in any other member is uT, and has been tabulated in the last column of the table above.

14·5. COMBINED STRESSES DUE TO EXTERNAL LOAD AND ERROR IN LENGTH

Let us now discuss the case of combined stresses in the members of a redundant frame having error in length of one member and subjected to external loads. The actual or final stress in any member can be found by analysing the two effects separately and then superimposing them. However, the labour can be very much reduced by computing the stresses in one operation by Castigliano's second theorem as discussed below.

Let a member of the redundant frame be *short* in length by an amount λ.

By Castigliano's second theorem, if U is total strain energy stored in the frame, both due to forcing the member in position, as well as due to external load, we have

$$\frac{\partial U}{\partial T}=\sum_1^n P\frac{\partial P}{\partial T}\cdot\frac{L}{AE}=\lambda \quad (1)$$

where T=*tensile* force in the member having lack of fit.
λ=amount by which the member is *short*.
P=actual force in any member=$(F+uT)$.
u=stress in any member due to unit pulls at the joints (or ends) of the member having lack of fit.
uT=stress in any member due to pull T at the joints (or ends) of the member having lack of fit.
F=stress in any member due to external loads alone when the member (having lack of fit) is not put in position (*i.e.*, when the frame is perfect one).

Now $P=F+uT$

$$\frac{\partial P}{\partial T}=u$$

Substituting in (1), we get

$$\sum_1^n (F+uT).u\frac{L}{AE}=\lambda$$

or $$\sum_1^n\frac{FuL}{AE}+\sum_1^n\frac{u^2TL}{AE}=\lambda \quad (14{\cdot}10)$$

The above equation can also be expressed in the form :

$$T=\frac{\lambda-\sum_1^n\frac{FuL}{AE}}{\sum_1^n\frac{u^2L}{AE}} \quad (14{\cdot}11)$$

Special Cases :

(1) If the member does not have and lack of fit, *i.e.*, when $\lambda=0$, we get

$$T=-\frac{\sum_1^n\frac{FuL}{AE}}{\sum_1^n\frac{u^2L}{AE}}$$

If should be noted that F is to be found when the redundant member is removed. Hence F for the redundant member is *nil* and $\sum_1^n\frac{FuL}{AE}$ is the same as $\sum_1^{n-1}\frac{FuL}{AE}$ Thus the above equation can also be written as

$$T=-\frac{\sum_{1}^{n-1}\frac{FuL}{AE}}{\sum_{1}^{u}\frac{u^2L}{AE}} \qquad [14\cdot11(a)]$$

The above equation is the same as equation 14·7 derived earlier.

(2) If there is no external load, but there is lack of fit λ (short), we have $F=0$ in each member, and hence $\sum_{1}^{n}\frac{FuL}{AE}=0$. Thus

$$T=+\frac{\lambda}{\sum_{1}^{n}\frac{u^2L}{AE}} \qquad [14\cdot11(b)]$$

The above equation is the same as equation 14·9 derived earlier.

Note : *In all these equations, λ is taken positive when the member is short in length and negative when it is excess in length.*

Procedure for Computation

(1) Remove the members having lack of fit, and calculate F_1, F_2. etc. in the member, due to external loading.

(2) Remove the external loads and apply unit *pulls* at the ends of the redundant member (*i.e.*, member having lack of fit), and calculate u_1 and u_2 etc. in members.

(3) Calculate T from equation 14·10 after substituting the various quantities with their proper algebraic sign, *i.e.*

(*i*) Tensile force is positive and compressive force negative.

(*ii*) λ is positive if the member is short in length and negative if excess in length.

Example 14·13. *In the pin-jointed framework shown in Fig. 14·15(a), all the members have the same cross sectional area=2 in², and E=1300 t/in². The support at A is hinged and it may be assumed that E is supported on a roller. During construction, member EB was made $\frac{1}{8}$ in. too long and was forced into place.*

Determine the resultant force in members EB and AD when the frame supports the 8 ton load as shown. (A.M.I. Struct. E)

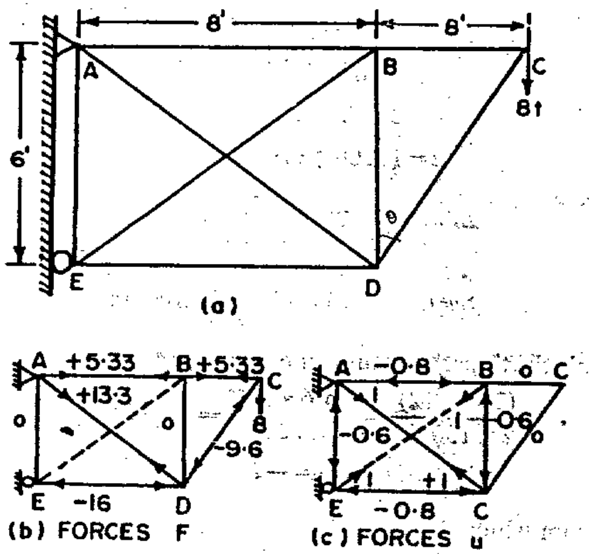


Fig. 14·15.

Solution

(1) To calculate F in the members, remove the member EB, as shown in Fig. 14·15(*b*). The values of F with their proper signs have been marked on the diagram.

(2) Remove the external load and apply unit pulls at the joints E and B, as shown in Fig. 14·15(*c*), wherein the values of u have been marked with their proper signs.

The result may now be tabulated as below :

Member	*F* (*tons*)	*n* (*tons*)	*L* (*ft*)	*Ful* (*ton²-ft*)	*u²L* (*ton²-ft*)
AB	+5·33	−0·8	8	−34	+5·14
BC	+5·33	0	4	0	0
ED	−16	−0·8	8	+102	+5·14
AE	0	−0·6	6	0	+2·16
BD	0	−0·6	6	0	+2·16
AD	+13·3	+1·0	10	+133	+10
CD	−9·6	0	$\sqrt{52}$	0	+0
BE	0	+1·0	10	0	+10
			Sum	+201	+34·6

$$\lambda = -\frac{1''}{8} = 1{\cdot}125'' \text{ (since } BE \text{ is excess in length).}$$

$$A = 2 \text{ in}^2.$$

$$E = 13000 \text{ t/in}^2.$$

$$\sum_1^n FuL = +(201 \times 12) \text{ ton}^2\text{-in.}$$

$$\sum_1^n u^2 LT = +(34{\cdot}6\, T \times 12) \text{ ton}^2\text{-in.}$$

Substituting the values in equation 14·10, we get

$$\frac{(201 \times 12)}{2 \times 13000} + \frac{34{\cdot}6 \times 12T}{2 \times 13000} = -0{\cdot}125$$

or $$(201 + 34{\cdot}6\, T) = -271$$

From which $T = -\frac{472}{34{\cdot}6} = -13{\cdot}6$ tons

Hence force in $EB = 13{\cdot}6$ tons (compression)

Force in $AD = (F + uT)$

$$= +13{\cdot}3 + (1{\cdot}0)(-13{\cdot}6) = -0{\cdot}3$$

$$= 0{\cdot}3 \text{ ton (compression).}$$

14·6. EXTERNALLY INDETERMINATE FRAMES

Structures may be somewhat arbitrarily classified as statically indeterminate *externally*, *internally* or *both*. Externally statically indeterminate structures are those which have redundant reaction restraints. The degree of external indeterminacy is given by the expression,

$$E = R - r$$

where E = degree of external indeterminacy.

R = total number of reaction-components (one for a roller, two for a hinge and three for fixed support).

r = total number of reaction components actually necessary for the stability of the structure (or total number of condition equations available).

Solution by Castigliano's Theorem :

If the structure has one redundant reaction component (say H) we have, from Castigliano's theorem of minimum strain energy,

$$\frac{\partial U}{\partial H} = 0 = \sum_1^n P \frac{\partial P}{\partial H} \cdot \frac{L}{AE} \qquad (14{\cdot}12)$$

If, however, the frame has two redundant reaction components H_1 and H_2, we have

$$\frac{\partial U}{\partial H_1} = 0 = \sum_1^n P \cdot \frac{\partial P}{\partial H_1} \cdot \frac{L}{AE} \qquad [14{\cdot}13(a)]$$

and $$\frac{\partial U}{\partial H_2} = 0 = \sum_1^n P \cdot \frac{\partial P}{\partial H_2} \cdot \frac{L}{AE} \qquad [14{\cdot}13(b)]$$

where n is the total number of members in the frame.

To calculate the value of P in the members the redundant reaction restraint is removed, and in its place an external force (H) is applied in the appropriate direction.

Solution by Maxwell's Method

The redundant reaction (H) is given by the equation

$$H = -\frac{\sum_1^n \frac{PuL}{AE}}{\sum_1^n \frac{u^2 L}{AE}} \qquad (12{\cdot}14)$$

where H = redundant reaction component

P = force in any member due to external loading, after removing the redundant reaction and making the structure statically determinate

u = force in any member due to unit force applied at the support in the direction of the redundant reaction.

The actual force in any member will be equal to $(P + uH)$ and can be calculated after knowing H.

Yielding of support :

If the support yields by an amount λ in the direction of H, we have, by Castigliano's theorem,

$$\frac{\partial U}{\partial H} = \sum_1^n P \cdot \frac{\partial P}{\partial H} \cdot \frac{L}{AE} = \lambda \qquad (14{\cdot}15)$$

and $$H = \frac{\lambda - \sum \frac{PuL}{AE}}{\sum \frac{u^2 L}{AE}} \quad \text{by Maxwell's method.} \qquad (14{\cdot}16)$$

Example 14·14. *Determine the reaction at B, taking it as a redundant, and forces in the members of the truss shown in Fig. 14·16(a). The value of* $\frac{L}{AE}$ *is constant for all members.*

Solution

The total number of reaction components $(R) = 2 + 1 + 1 = 4$.

Actual reaction components necessary for the stability of the frame=2+1=3 (*i.e.*, even if the roller at B is removed, the frame will be stable).

Thus, $E=R-r=4-3=1$

i.e. the frame is externally indeterminate to single degree.

To get the values of P in all the members, make the frame statically determinate by removing the redundant reaction at B, shown in Fig. 14·16(b).

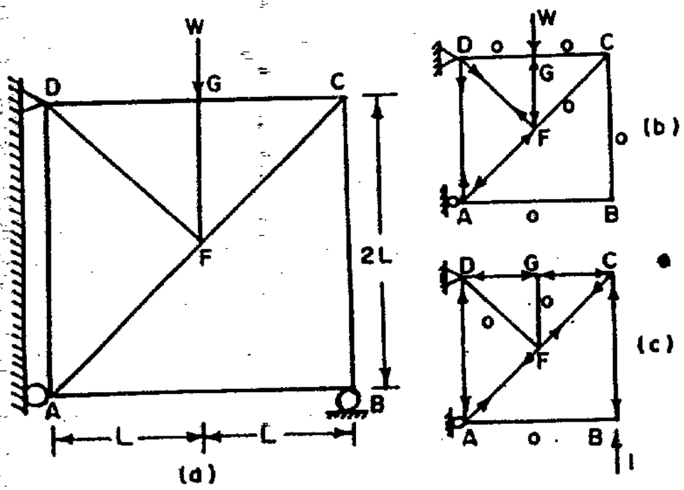


Fig. 14·16.

To get the values of u in all the members, apply unit load at B in the direction as the reaction as shown in Fig. 14·16(c).

(b) *Calculation of P* [*Fig. 14·16*(b)]

By inspection, all the inclined members are at 45° to the horizontal. Hence $\sin\theta=\cos\theta=\dfrac{1}{\sqrt{2}}$.

Since there is no vertical load at B,

$$P_{BC}=0\ ;\quad P_{AB}=0.$$

Hence $\quad P_{CF}=0\ ;\quad P_{GC}=0\ ;\quad P_{DG}=0$

At the joint G, $\quad P_{GF}=W$ (comp.)

At the joint F, $\quad P_{DF}=P_{GF}\dfrac{1}{\sqrt{2}}=\dfrac{W}{\sqrt{2}}=\dfrac{W\sqrt{2}}{2}$ (tension)

$$P_{FA}=P_{GF}\frac{1}{\sqrt{2}}=\frac{W}{\sqrt{2}}=\frac{W\sqrt{2}}{2}\ \text{(comp.)}$$

At the joint A, $\quad P_{AD}=P_{AF}\dfrac{1}{\sqrt{2}}=\dfrac{W}{\sqrt{2}}\cdot\dfrac{1}{\sqrt{2}}=\dfrac{W}{2}$ (tension)

(b) *Calculate of u* [*Fig. 12·16*(c)].

At the joint B, $\quad P_{BC}=1$ (comp.)

$\quad P_{AB}=0$

At the joint C, $\quad P_{CF}=P_{BC}\sec\theta=\sqrt{2}$ (tension)

$$P_{CG}=P_{CF}\cos\theta=\sqrt{2}\cdot\frac{1}{\sqrt{2}}=1\ \text{(tension.)}$$

At the joint G, $\quad P_{GD}=P_{CG}=1$ (comp.)

$\quad P_{GF}=0$

At the joint F, $\quad P_{DF}=0$

$\quad P_{FA}=P_{CF}=\sqrt{2}$ (tension)

At the joint A, $\quad P_{AD}=P_{AF}\cdot\dfrac{1}{\sqrt{2}}=\sqrt{2}\times\dfrac{1}{\sqrt{2}}=1$ (comp.)

The calculations may be done in the tabular form below :

(+for tension ; —for compression)

Member	*P*	*u*	*P.u*	u^2	*Actual stres* $(P+u\,V_B)$
AB	0	0	0	0	0
BC	0	-1	0	1	$-0{\cdot}1875\ W$
CG	0	-1	0	1	$-0{\cdot}1875\ W$
GD	0	-1	0	1	$-0{\cdot}1875\ W$
DA	$+W/2$	-1	$-W/2$	1	$+0{\cdot}3125\ W$
AF	$-W\sqrt{2}/2$	$+\sqrt{2}$	$-W$	2	$-0{\cdot}442\ W$
FC	0	$+\sqrt{2}$	0	2	$+0{\cdot}265\ W$
FG	$+W$	0	0	0	$-W$
FD	$+W\sqrt{2}/2$	0	0	0	$+0{\cdot}707\ W$
			$-\dfrac{3}{2}W$	8	

The vertical reaction (V_B) at the roller at B is given by

$$V_B=\frac{\sum_1^n \dfrac{PuL}{AE}}{\sum_1^n \dfrac{u^2L}{AE}}=\frac{\sum_1^n Pu}{\sum_1^n u^2}\quad\text{since }\frac{L}{AE}\text{ is constant.}$$

$$\text{Hence}\quad V_B=-\frac{\sum_1^n P\,.\,u}{\sum_1^n u^2}=-\frac{-\frac{3}{2}W}{8}=+\frac{3}{16}W$$

$$=0{\cdot}1875\,W\,(\uparrow).$$

The actual stress each member will be $(P+uV_B)$ and has been tabulated in the last column of the table above.

Example 14·15. *Fig. 14·17 (a) shows a pin joined frame of uniform material supported on a roller at B and hinged at A and C. The ratio of length to cross-sectional area is constant for each member of frame. Find the forces in all members.*

Solution

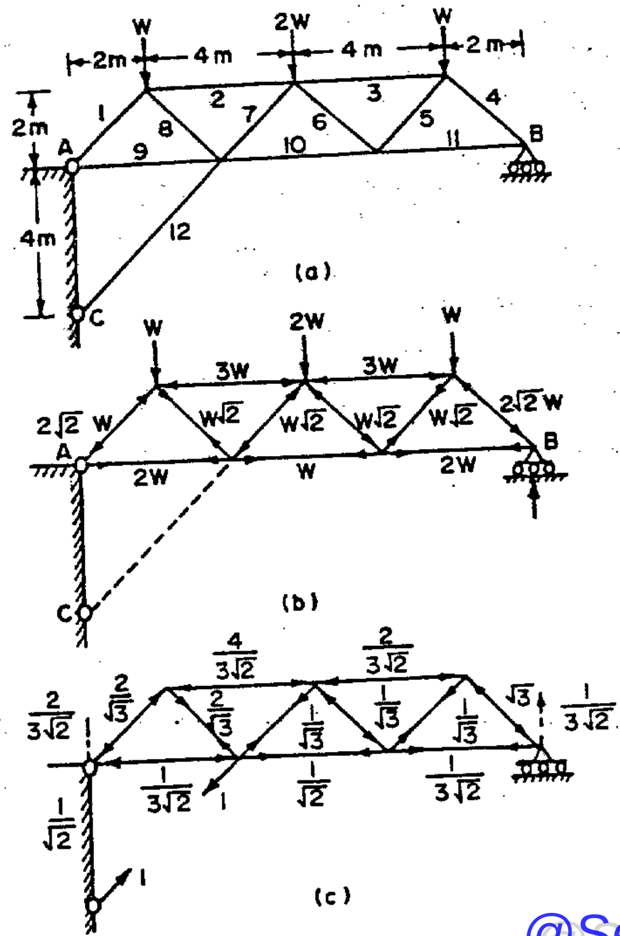


Fig. 14·17.

R=total reaction components$=2+2+1=5$

r=Actual reaction components necessary$=2+2=4$

$\therefore\quad E=R-r=5-4=1$

Thus the frame is indeterminate to single degree. We shall solve this example by treating member No. 12 as the redundant member carrying a force T assumed tensile to start with.

Then, by Maxwell's method,

$$T=-\frac{\sum_1^{n-1}\frac{PuL}{AE}}{\sum_1^{n}\frac{u^2L}{AE}}=-\frac{\sum_1^{n-1}Pu}{\sum_1^{n}u^2}$$

$$\left(\text{Since }\frac{L}{AE}\text{ is constant}\right)$$

(*a*) *Calculation of P*

To calculate P in all the members due to external loads, remove member No. 12, as shown in Fig. 14·17(*b*). The stresses P have been marked against each member in the diagram.

(*b*) *Calculation of u*

To calculate u, remove the external load, and apply unit pulls at the ends (or joints) of the redundant member, as shown in Fig. 14·17(*c*). The stresses u have been marked in the diagram.

The values may now be tabulated as under :

$$\text{Hence}\qquad T=-\frac{\sum_1^{n-1}Pu}{\sum_1^{n}u^2}=-\frac{\frac{16}{\sqrt{2}}W}{\frac{73}{18}}=-2{\cdot}78\,W$$

The minus sign indicates that member No. 12 carries a compressive force of 2·78 W.

The actual stress in any member will be $(P+Tu)$ and have been entered in the last column of the table.

(+for tension ; — for compression)

Member	P	u	Pu	u^2	Actual stress $(P+Tu)$
1	$-2\sqrt{2}W$	$-\frac{2}{3}$	$+\frac{4\sqrt{2}}{3}W$	$\frac{4}{9}$	$-0{\cdot}974\ W$
2	$-3W$	$-\frac{4}{3\sqrt{2}}$	$+\frac{4}{\sqrt{2}}W$	$\frac{16}{18}$	$-0{\cdot}380\ W$
3	$-3W$	$-\frac{2}{3\sqrt{2}}$	$+\frac{2}{\sqrt{2}}W$	$\frac{4}{18}$	$-1{\cdot}693\ W$
4	$-2\sqrt{2}W$	$-\frac{1}{3}$	$+\frac{2\sqrt{2}}{2}W$	$\frac{1}{9}$	$-1{\cdot}902\ W$
5	$+W\sqrt{2}$	$+\frac{1}{3}$	$+\frac{\sqrt{2}}{3}W$	$\frac{1}{9}$	$+0{\cdot}486\ W$
6	$-W\sqrt{2}$	$-\frac{1}{3}$	$+\frac{\sqrt{2}}{3}W$	$\frac{1}{9}$	$-0{\cdot}486\ W$
7	$-W\sqrt{2}$	$+\frac{1}{3}$	$-\frac{\sqrt{2}}{3}W$	$\frac{1}{9}$	$-2{\cdot}342\ W$
8	$+W\sqrt{2}$	$+\frac{2}{3}$	$+\frac{2\sqrt{2}}{3}W$	$\frac{4}{9}$	$-0{\cdot}440\ W$
9	$+2W$	$-\frac{1}{3\sqrt{2}}$	$-\frac{2}{3\sqrt{2}}W$	$\frac{1}{18}$	$+2{\cdot}656\ W$
10	$+4W$	$+\frac{1}{\sqrt{2}}$	$+\frac{4}{\sqrt{2}}W$	$\frac{1}{2}$	$+2{\cdot}034\ W$
11	$+2W$	$+\frac{1}{3\sqrt{2}}$	$+\frac{2}{3\sqrt{2}}W$	$\frac{1}{18}$	$-1{\cdot}344\ W$
12	—	$+1$	—	1	$-2{\cdot}780\ W$
			$+\frac{16}{\sqrt{2}}W$	$\frac{73}{18}$	

Example 14·16. *A pin jointed frame of uniform material has loads and dimensions as shown in Fig. 14·18(a). The cross-sectional area is 100 mm² for each member of the frame. The truss is hinged to rigid supports at A and B.*

Calculate (a) *Reactions at A and B.*

(b) *Horizontal reaction at B if it moves (or yields) by 10 mm horizontally to the right.*

(c) *The maximum horizontal yielding of the support B to have zero horizontal reaction there.*

Take $E=2\times10^5\ N/mm^2\ (200\ kN/mm^2)$.

Solution

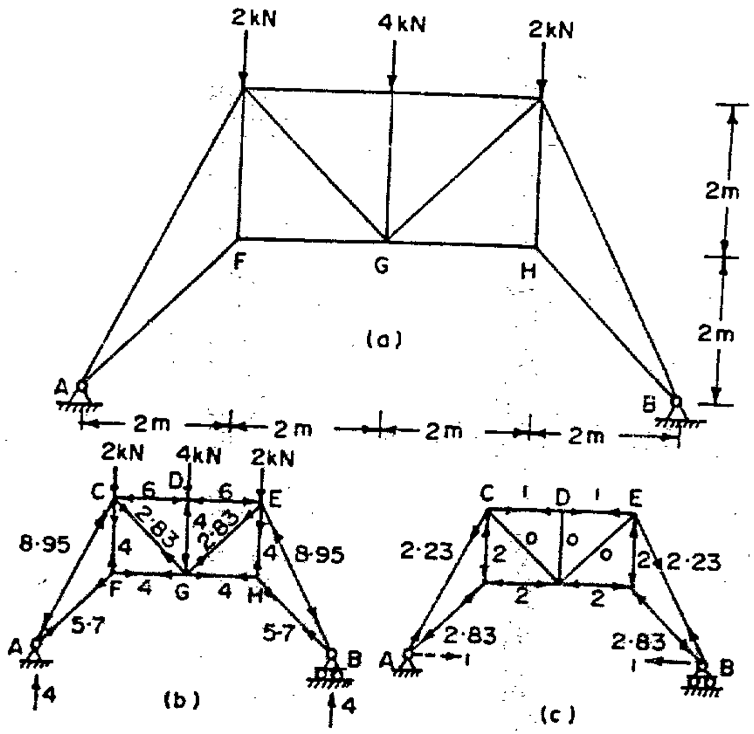


Fig. 14·18

(a) The horizontal reaction (H) at B is given by

$$H=-\frac{\Sigma\dfrac{PuL}{AE}}{\Sigma\dfrac{u^2L}{AE}} \tag{1}$$

To calculate P, make the structure statically determinate by providing a roller support at B, as shown in Fig. 14·18 (b), where the stresses due to external loading have been marked.

To calculate u, remove the external loads and apply unit pull at the joint B as shown in Fig. 14·18 (c). The stresses have been marked on the diagram.

The computations are done in the tabular form below :

Member	Length L (mm)	P (kN)	u	PuL	u²L
AC	4470	−8·95	+2·23	−89400	22210
CD	2000	−6·0	+1·00	−12000	2000
DE	2000	−6·0	+1·00	−12000	2000
EB	4470	−8·95	+2·23	−89400	22210
GH	2000	+4·0	−2·00	−16000	8000
FG	2000	+4·0	−2·00	−16000	8000
FA	2830	+5·7	−2·83	−45700	22640
HB	2830	+5·7	−2·83	−45700	22640
CF	2000	+4·0	−2·00	−16000	8000
CG	2830	+2·83	0	0	0
DG	2000	−4·0	0	0	0
EG	2830	+2·83	0	0	0
EH	2000	+4·0	−2·00	−16000	8000
			Sum	−358200	125700

Substituting the values in (1), we get

$$H=-\frac{-358200}{125700}=\mathbf{2{\cdot}85\ kN}\ (\leftarrow)$$

$$R_A=R_B=\sqrt{4^2+(2{\cdot}85)^2}=\mathbf{4{\cdot}91\ kN.}$$

(*b*) The general expression for the horizontal reaction at *B* is

$$H=\frac{\lambda-\Sigma\dfrac{PuL}{AE}}{\Sigma\dfrac{u^2L}{AE}} \qquad (14{\cdot}16)$$

where λ=horizontal movement of *B* towards *A*.

For the present case $\lambda=-10$ mm (since *B* moves away from *A* or opposite to *H*).

$$A=100\ \text{mm}^2$$

$$E=2\times10^5\ \text{N/mm}^2=200\ \text{kN/mm}^2$$

$$\therefore\quad H=\frac{-10-\left(\dfrac{-358200}{100\times200}\right)}{\left(\dfrac{125700}{100\times200}\right)}=\frac{-10+17{\cdot}91}{6{\cdot}285}$$

$$=\mathbf{1{\cdot}26\ kN}\ (\leftarrow).$$

(*c*) For this case $H=0$.

Substituting the values in equation 14·16, we have

$$H=0=\frac{\lambda-\Sigma\dfrac{PuL}{AE}}{\Sigma\dfrac{u^2L}{AE}}$$

or

$$\lambda=\Sigma\frac{PuL}{AE}$$

$$=\frac{-358200}{100\times200}=-17{\cdot}91\ \text{mm}$$

$\therefore$ Horizontal yielding of $B=\mathbf{17{\cdot}91\ mm}$ $(\rightarrow)$.

14·7. TRUSSED BEAMS

A beam strengthened by providing ties and struts is known as a trussed beam. In the case of an ordinary beam, the strain energy is mainly due to bending. In the case of a truss or framed structure, the strain energy stored is due to direct forces, since the members carry only axial forces. In the case of a trussed beam, however, the strain energy stored is due to both bending as well as direct force. The ties and struts of the trussed beam, however, carry only direct forces, and the strain energy due to direct forces only. Such a trussed beam is shown in Figs. 14·19 and 14·20.

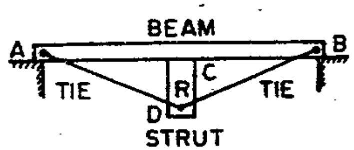


Fig. 14·19.

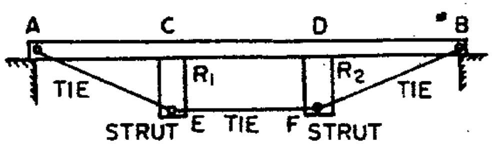


Fig. 14·20.

By the provision of ties and struts, a trussed beam is a statically indeterminate structure, and can be analysed by Castigliano's theorem of minimum strain energy. Thus, in Fig. 14·19 the beam is statically indeterminate to single degree. Let the thrust (R) in CD be redundant.

Strain energy in the beam due to bending $=\int_0^L \frac{M^2dx}{2EI}$

Strain energy in all the four members due to axial forces

$$=\sum_1^n \frac{P^2L}{2AE}$$

From Castigliano's theorem of minimum strain energy,

$$\frac{\partial U}{\partial R}=0=\int_0^L M\frac{\partial M}{\partial R}\cdot\frac{dx}{EI}+\sum_1^n P\,\frac{\partial P}{\partial R}\cdot\frac{L}{AE} \qquad (14·17)$$

where $\sum_1^n$ is the summation for all the members of the trussed beam.

If, however, there are two redundant forces R_1 and R_2 as in Fig. 14·20, we have

$$\frac{\partial U}{\partial R_1}=0=\int_0^L M\frac{\partial M}{\partial R_1}\cdot\frac{dx}{EI}+\sum_1^n P\,\frac{\partial P}{\partial R_1}\cdot\frac{L}{AE} \qquad [14·18\ (a)]$$

and

$$\frac{\partial U}{\partial R_2}=0=\int_0^L M\frac{\partial M}{\partial R_2}\cdot\frac{dx}{EI}+\sum_1^n P\,\frac{\partial P}{\partial R_2}\cdot\frac{L}{AE} \qquad [14·18\ (b)]$$

The solutions of these equations gives the values of the redundant forces. Afer having known the redundant forces, the forces in the ties bars etc. can be calculated, and the B.M. and S.F. diagram can be plotted for the main beam.

Example 14·17. *A trussed timber beam, 120 mm wide and 160 mm deep, is 4 m long and has a central C.I. strut 1 m long and 1000 mm² area of cross-section. The tie rods are of steel and 500 mm² area of cross-section. Calculate the thrust in the strut if the beam carries a uniformly distributed load of 10 kN/m. Take E for wood, C.I. and steel as $1·0\times10^4$ N/mm², 1×10^5 N/mm² and $2·0\times10^5$ N/mm² respectively.*

Solution

Let the redundant force in the strut CD be R.

Then $\frac{\partial U}{\partial R}=0$

Now $\cos\theta=\frac{2}{\sqrt5}=0·894$, $\sin\theta=\frac{1}{\sqrt5}=0·447$;

$\operatorname{cosec}\theta=\sqrt5=2·24$

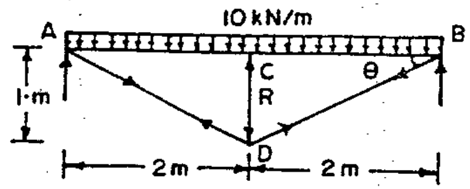


Fig. 14·21

$P_{AD}=P_{BD}=\frac{R}{2}\operatorname{cosec}\theta=1·12\,R$ (tension)

Reactions at A and $B=(10\times2)-P_{AD}\sin\theta$

$=20-\frac{1}{2}R$

(a) For thv beam AB

Strain energy due to bending $=U_{ABI}=2\int_0^2 \frac{M^2dx}{2EI}$

$$\therefore\quad \frac{\partial U_{ABI}}{\partial R}=2\int_0^2 M\,\frac{\partial M}{\partial R}\,\frac{dx}{EI}$$

At any section distant x from A,

$$M=-\left(20-\frac{1}{2}R\right)x+5x^2$$

$$\frac{\partial M}{\partial R}=+\frac{1}{2}x$$

$$\therefore\quad \frac{\partial U_{ABI}}{\partial R}=2\int_0^2\left\{-\left(20-\frac{1}{2}R\right)x+5x^2\right\}\frac{1}{2}x\,\frac{dx}{EI}$$

$$=\frac{1}{EI}\int_0^2\left(-20x^2+\frac{1}{2}Rx^2+5x^3\right)\frac{dx}{EI}$$

$$=\frac{1}{EI}\left[\frac{-20x^3}{3}+\frac{1}{2}R\frac{x^3}{3}+\frac{5x^4}{4}\right]_0^2$$

$=\frac{1}{EI}\left[\frac{4R-100}{3}\right]$, where EI is in kN-m² units.

E for timber $=1·0\times10^4$ N/mm² $=10$ kN/mm² $=1\times10^7$ kN/m²

$$\therefore\quad EI=(1\times10^7)\left[\frac{0·12(0·16)^3}{12}\right]=409·6 \text{ kN/m}^2 \text{ units.}$$

$$\therefore\quad \frac{\partial U_{ABI}}{\partial R}=\frac{4R-100}{3\times409·6}$$

(b) Strain energy due to direct forces

The tension in tie bars $=P_{AD}=P_{BD}=1{\cdot}12R$

Compression in the beam $=P_{AD}\cos\theta=(1{\cdot}12\,R)(0{\cdot}894)=R$

The calculations are arranged in the tabular form below :

Member	L (m)	A (mm^2)	E (kN/mm^2)	P (kN)	$\frac{\partial P}{\partial R}$	$P.\frac{\partial P}{\partial R}.\frac{L}{AE}$
CD	1	1000	100	$-R$	-1	$R\times1\times\frac{1}{1000\times100}$ $=R\times10^{-5}$
AD	2·24	500	200	$+1{\cdot}12R$	$+1{\cdot}12$	$(1{\cdot}12)^2R\,\frac{2{\cdot}24}{00\times200}$ $=2{\cdot}8R\times10^{-5}$
BD	2·24	500	200	$+1{\cdot}12R$	$+1{\cdot}12$	$2{\cdot}8R\times10^{-5}$
AB	4	19200	10	$-R$	-1	$R\times1\times\frac{4}{19200\times10}$ $=2{\cdot}08R\times10^{-5}$
						$8{\cdot}68R\times10^{-5}$

$$\therefore\quad \sum_1^n P.\frac{\partial P}{\partial R}.\frac{L}{AE}=8{\cdot}68R\times10^{-5}$$

$$\text{Now}\quad \frac{\partial U}{\partial R}=0=\int M\frac{\partial M}{\partial R}.\frac{dx}{EI}+\sum_1^n P\frac{\partial P}{\partial R}.\frac{L}{AE}$$

$$\therefore\quad \frac{4R-100}{3\times409{\cdot}6}+8{\cdot}68R\times10^{-5}=0$$

which gives $\quad R=24{\cdot}35$ **kN.**

(**Note.** If the support *CD* were *rigid*, the value of *R* would have been $=\frac{5}{8}wL=\frac{5}{8}\times10\times4=25$ kN).

PROBLEMS

1. The material and cross-sectional area of the bars of the frame shown in Fig. 14·22 are same. Show that force in *BC* is 0·707 *W* tensile. (*U.L.*)

2. Find the forces in the members of the frame work shown in Fig. 14·23. The quantity *AE* is constant for all the members.

3. Determine the axial force in the members of the frame shown in Fig. 14·24. The cross-sectional area of bars *AB* and *AC* is $2a$ and that of other member is a.

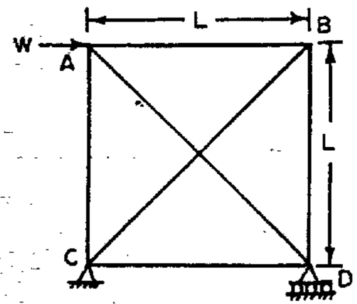


Fig. 14·22.

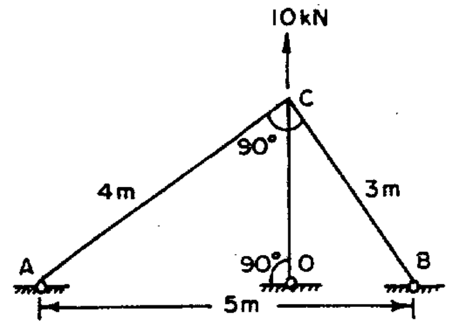


Fig. 14·23.

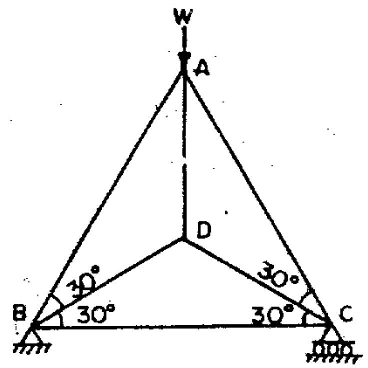


Fig. 14·24.

4. A braced cantilever is loaded as shown in Fig. 14·25. All the members are of the same cross-sectional area. Find the axial force in the member *BC*.

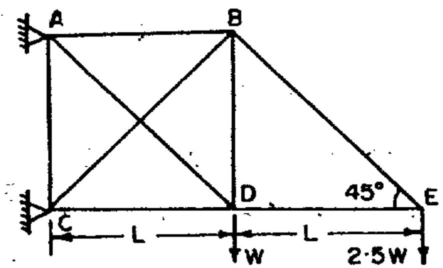


Fig. 14·25.

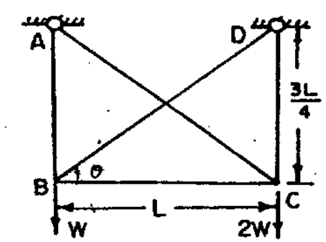


Fig. 14·26.

5. Determine the forces in the members of the frame, shown in Fig. 14·26 which is pinned to supports *A* and *D* and carries loads of *W* and *2W* at *B* and *C* respectively. The members *AB* and *CD* are $3a$ and the remainder a in cross-sectional area.

6. Find the force in the member AF of the pin-jointed frame-work shown in Fig. 14·27. All members have the same area of cross-section and are of the same material.

7. The three rods AD, BD and CD are pinned to each other at D and to a rigid member ABC at A, B and C respectively, as shown in Fig. 14·28. The rods AD and BD are each of 1000 sq. mm and CD of 1600 sq. mm. If a horizontal load of 100 kN is applied at D, find the loads in all the three members. Take $E=2{\cdot}1\times10^5$ N/mm² (210 kN/mm²)

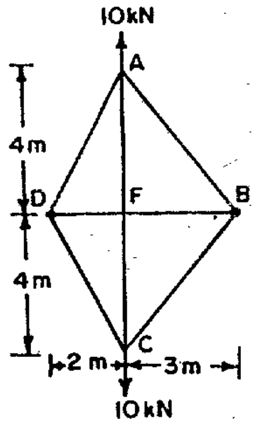


Fig. 14·27.

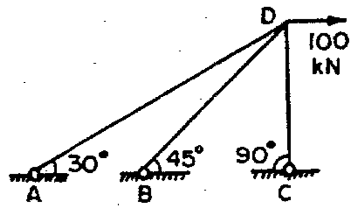


Fig. 14·28.

8. A pin-jointed frame is loaded as shown in Fig. 14·29. The frame is hinged at A and supported on rollers at B. The ratio $\dfrac{\text{Length}}{\text{Area}}$ for all the members is the same. Treating CF as the redundant member, obtain the forces in the central members CF and DE.

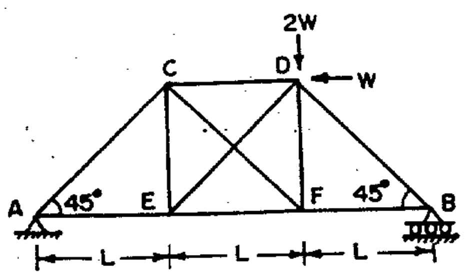


Fig. 14·29.

9. The frame work shown in Fig. 14·30 is made from bars all having the same extensibility AE. It is supported at B and C and carries loads at A

10. Fig. 14·31 shows a pin jointed frame work carrying a vertical load of 10 kN at E and supported by vertical reactions at A and B. The dimensions of the figure are such that if the line CE is omitted, all the angles are either 30°, 60° or 90°. The members are all of the same material and cross-section. Find the load in the member CE.

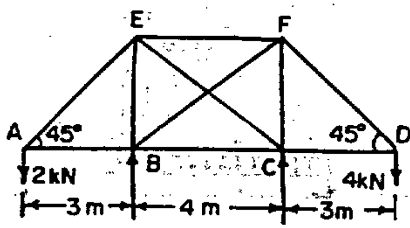


Fig. 14·30.

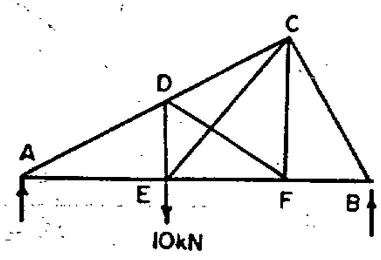


Fig. 14·31.

11. A pin-jointed rectangular frame with two diagonals is built up as shown in Fig. 14·32. The bar BC is the last to be added and is short by 5 mm. Find the force in the member BC when it is forced into position. The cross-section area of each side bar is 1000 mm² and of each diagonal 500 mm². Take $E=2{\cdot}1\times10^5$ N/mm².

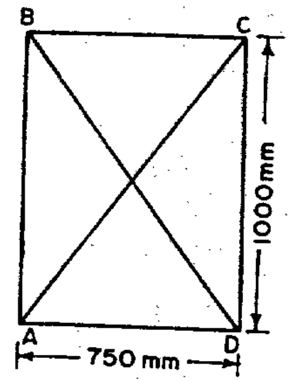


Fig. 14·32.

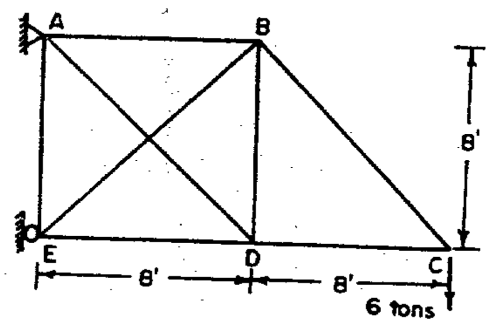
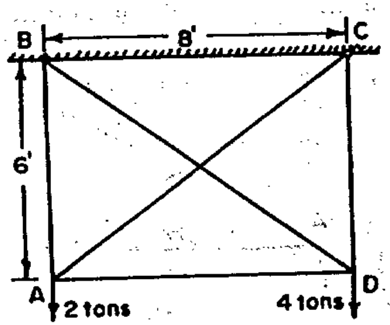


Fig. 14·33.

12. In a pin-jointed frame shown in Fig. 14·33, the cross-sectional area of each member is 2 in², and E=13000 ton/in². During construction the member EB was made 1/16 in. too long and was forced into plane. Determine the force developed in each of the diagonal members AD and EB when a vertical load of 6 tons is applid at C. *(A.M.I. Struct. E)*

13. The rectangular frame shown in Fig. 14·34 consists of five bars pinned together at A, B, C and D and suspended in a vertical plane from a beam. All the bars are of steel and 2 in² in cross-sectional area. Find the force in the member AD due to the two loads at A and D. If the temperature is raised 20°C, the distance BC remaining unchanged, find the force in each of

the bars. Take $E=13200$ tons/in². Coefficient of linear expansion $=11\times10^{-6}$ per °C. (L.U.)



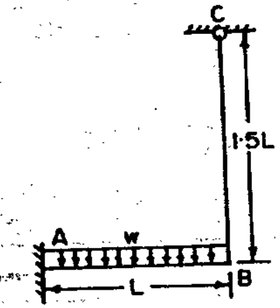


Fig. 14·34. Fig. 14·35.

14. A horizontal cantilever of length L has its free end attached to a vertical tie rod 1·5 L long, which is initially unstrained. The moment of inertia of the section of the cantilever is I and the area of the cross-section of the tie rod is a. Prove that the load (R) taken by the tie rod due to $U.D.L.$ of w/unit length on the cantilever is $R=\dfrac{3waL^3}{4(2aL^2+9I)}$.

E for both the rod and cantilever is the same.

15. A trussed timber beam, 200 mm wide and 300 mm deep, is 6 m long and has a central $C.I.$ strut 1 m long and 80 mm dia. The tie rods are of steel and 30 mm in diameter. Calculate the thrust in the strut if the beam carries a uniformly distributed load of 30 kN/m.

E for wood $=1\times10^4$ N/mm² (10 kN/mm²)
E for $C.I.=1\times10^5$ N/mm² (100 kN/mm²)
E for steel $=2\times10^5$ N/mm² (200 kN/mm²)

16. In the pin-joined frame work shown in Fig. 14·36, the three triangles

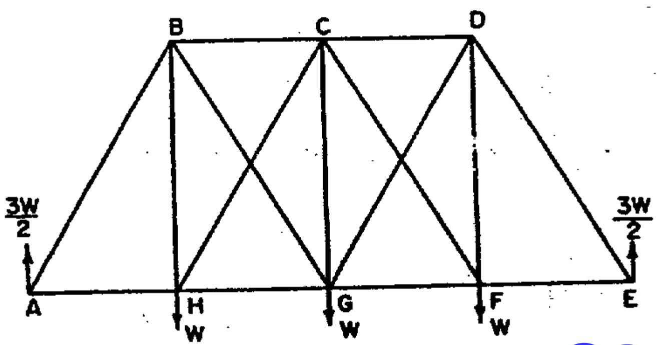


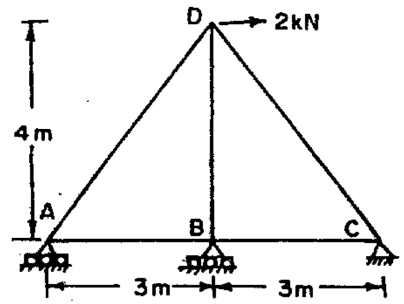
Fig. 14·36.

ABG, HCF and GDE are equilateral. The members AB, BC, CD, DE, BH, CG and DF has a cross-section $2a$, and the remaining members a. Treating the verticals BH and DF as redundant members show that they are subjected to tension of magnitude.

$$\frac{42+6\sqrt{3}}{35+9\sqrt{3}}W.$$

(Cambrige)

17. Compute the reaction and forces in the members of the truss shown in Fig 14·37. L/AE is constant for all members.



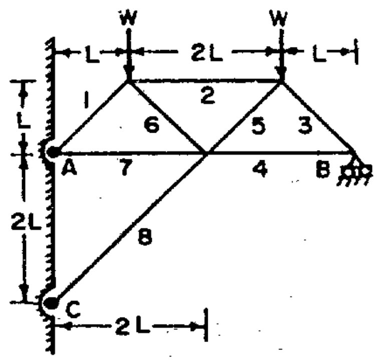
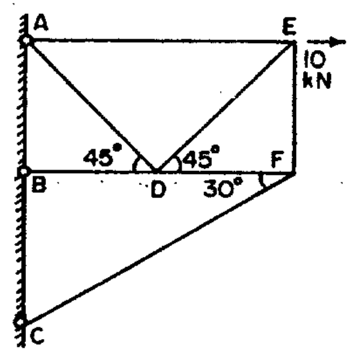


Fig. 14·37. Fig. 14·38.

18. A pin-jointed frame of uniform marerial has the loads and dimensions shown in Fig. 14·38. The ratio of length to cross-sectional area is constant for each member of the frame. The frame is hinged to foundations at A and C and is supported by a horizontal free-roller bearing at B. Taking member 8 as the redundant member, calculate the magnitude and kind of force it carries due to loads.

19. Calculate the force in the members of the frame shown in Fig. 14·39 which is supported on pins at A, B. and C and carries a load of 10 kN at E. All the bars have equal areas.



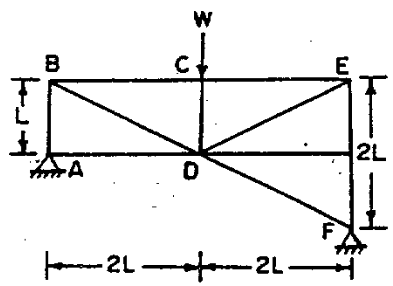


Fig. 14·39. Fig. 14·40.

20. The structure shown in Fig. 14·40 is hinged at A and F and subjected to a load W at C. Assuming the sectional area of all the members to be the same, determine the force in each of the members AB and DE.

Answers

2. $P_{AC}=+2\cdot5$ kN ; $P_{BC}=+3\cdot33$ kN ; $P_{CD}=+5\cdot83$ kN.
3. $P_{AB}=P_{AC}=-0\cdot535\ W$: $P_{BC}=+0\cdot33\ W$; $P_{BD}=P_{AD}=P_{CD}=-0\cdot07\ W$
4. $2\cdot6\ W$ (comp.)
5. $P_{AC}=P_{BD}=+0\cdot136\ W$; $P_{BC}=0\cdot108\ W$; $P_{DC}=+1\cdot919\ W$;
   $P_{AB}=+0\cdot919\ W$.
6. $P_{AF}=+5\cdot13$ kN
7. P_{A}[illegible]$=96\cdot5$ kN ; $P_{BD}=+23\cdot5$ kN : $P_{CD}=-65$ kN.
8. $P_{CF}=+0\cdot53\ W$; $P_{DE}=-0\cdot87\ W$.
9. $1\cdot876$ kN (comp.)
10. $5\cdot34$ kN.
11. $+55$ kN.
12. EB : $7\cdot74$ tons (comp.) ; AD : $0\cdot75$ ton (comp.)
13. (a) $-0\cdot588$ ton.
    (b) $P_{AB}=+2\cdot29$; $P_{AD}=+0\cdot42$; $P_{CD}=+4\cdot29$;
    $P_{AC}=-0\cdot52$; $P_{DB}=-0\cdot52$
15. $98\cdot7$ kN.
17. $P_{AB}=P_{BC}=-0\cdot86$ kN ; $P_{CD}=-1\cdot894$ kN.
    $P_{DA}=+1\cdot438$ kN ; $P_{DB}=+0\cdot364$ kN.
18. $0\cdot77\ W$ (comp.).
19. $P_{AE}=+9\cdot43$ kN ; $P_{BD}=+2\cdot14$ kN ; $P_{DF}=+0\cdot98$ kN ;
    $P_{EF}=-0\cdot58$ kN ; $P_{AD}=-0\cdot81$ kN ; $P_{DE}=+0\cdot81$ kN ;
    $P_{FC}=-1\cdot15$ kN.
20. $P_{AB}=0\cdot299\ W$; $P_{DE}=0\cdot669\ W$.

15

Cables and Suspension Bridges

15·1 INTRODUCTION

Suspension bridges are used for highways, where the span of a bridge is more than 200 m. Essentially, a suspension bridge (Fig. 15·1) consists of the following elements : (*i*) the cable, (*ii*) suspenders, (*iii*) decking, including the stiffening girder, (*iv*) supporting tower, and (*v*) anchorage.

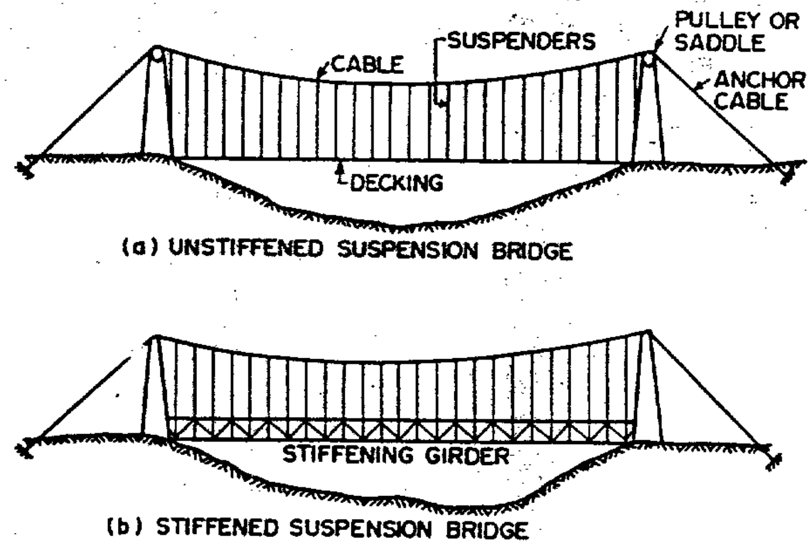


Fig. 15·1.
Elements of a suspension bridge.

The traffic load of the decking is transferred to main cable through the suspenders. Since the cable is the main load bearing member, the curvature of the cable of an unstiffened bridge changes as the load moves on the decking. To avoid this, the decking is

stiffened by provision of either a three hinged or two hinged stiffen-
ing girder. The stiffening girder transfers a uniformly distributed or
equal load to each suspender, irrespective of the load position on
the decking. The suspension cable is supported on either side. There
are two arrangements generally used. The suspension cable may
either pass over a smooth frictionless pulley and anchored to the
other side, or it may be attached to a saddle placed on rollers. In the
former case, the tension on the cable on the two sides of the pulley
are equal while in the latter case, the horizontal components of the
tension on the two sides are equal since the cable cannot have a
movement relative to the saddle. The cable consists of either wire
rope, parallel wires joined with clips or eye bar links. The cable can
carry direct tension only, and the bending moment at any point on
the cable is zero. The suspenders consist of round rods or ropes
with turn-buckles so that adjustment in their lengths may be done if
required. The anchorage consists of huge mass of concrete, designed
to resist the tension of the cable.

15·2. EQUILIBRIUM OF LIGHT CABLE : GENERAL CABLE THEOREM

Fig. 15·2 shows a light cord or cable, suspended from two points A and B and subjected to a number of point loads W_1, W_2......W_n. Let L be the horizontal span of the cable and α be the inclination of the line AB, with the horizontal. Evidently, the difference in elevation between the two supports A and B is equal to $L \tan \alpha$.

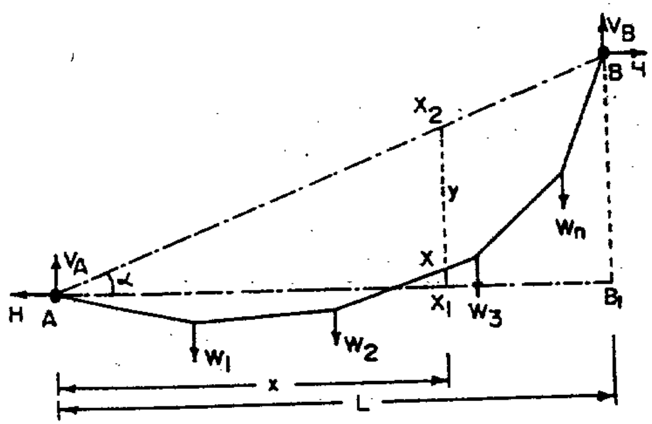


Fig. 15·2.
Equilibrium of a light cable.

Let V_A and V_B be the vertical components of reactions at A and B. Since there is no horizontal loading on the cable, the horizontal reaction (H) at the ends A and B will be equal in magnitude but opposite in direction. Since the cable is in equilibrium, it will take the shape of a funicular polygon for the load system, and will, therefore, deform as shown.

In order to find the vertical reaction V_A, take moments about B :

$$-V_A \,.\, L - H \,.\, L \tan \alpha + \Sigma M_B = 0$$

or
$$V_A = \frac{\Sigma M_B}{L} - H \tan \alpha \qquad (15·1)$$

where ΣM_B = sum of moments of all external loads about B.

Consider any point X at a horizontal distance x from A.

Evidently, $X_1X_2 = x \tan \alpha$

Assuming that the cable is perfectly flexible so that the bending moment at any point on the cable is zero, the sum of moments (ΣM_x) of all external forces to the left of point X is zero.

$$\therefore \quad -H(XX_1) - V_A \,.\, x + \Sigma M_x = 0$$

or
$$-H(x \tan \alpha - y) - V_A \,.\, x + \Sigma M_x = 0$$

where M_x = sum of moments of all forces to the left of X.
$y = XX_2$

Substituting the value of V_A from Eq. 15·1, we get

$$-H(x \tan \alpha - y) - \left\{\frac{\Sigma M_B}{L} - H \tan \alpha\right\} x + \Sigma M_x = 0$$

or
$$Hy - \frac{x}{L}\Sigma M_B + \Sigma M_X = 0$$

or
$$Hy = \frac{x}{L}\Sigma M_B - \Sigma M_x \qquad (15·2)$$

Eq. 15·2 is the *general cable theorem.*

15·3. UNIFORMLY LOADED CABLE

(a) EXPRESSION FOR HORIZONTAL REACTION

Fig. 15·3 shows a cable supporting a uniformly distributed load of intensity p per unit length. From the general cable theorem derived in the previous article we have

$$Hy = \frac{x}{L}\Sigma M_B - \Sigma M_x$$

where $y = XX_2$ = vertical ordinate between the line AB and chord at the point X.

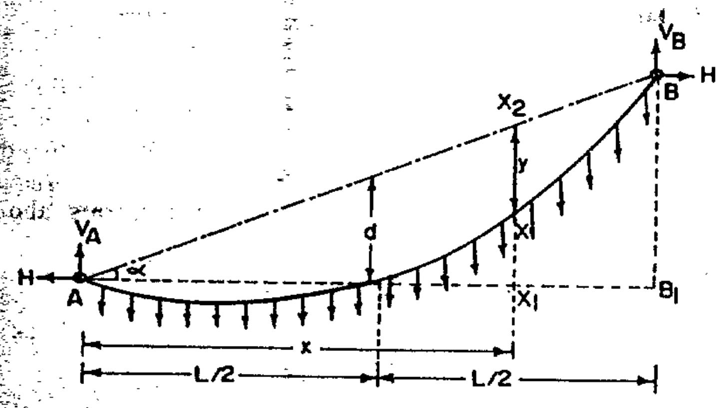


Fig. 15.3.

$$\Sigma M_B = pL \cdot \frac{L}{2} = p\,\frac{L^2}{2}$$

$$\Sigma M_x = px \cdot \frac{x}{2} = p\,\frac{x^2}{2}$$

$$\therefore \quad Hy = \frac{x}{L} \cdot p\frac{L^2}{2} - p\,\frac{x^2}{2} = p\,\frac{Lx}{2} - p\,\frac{x^2}{2} \qquad (15{\cdot}3)$$

At the mid-span, $x = L/2$ and $y = d = dip$ of the cable.

$$\therefore \quad Hd = p\,\frac{L}{2} \cdot \frac{L}{2} - \frac{p}{2}\left(\frac{L}{2}\right)^2 = p\,\frac{L^2}{8}$$

Hence $$H = p\,\frac{L^2}{8d} \qquad (15{\cdot}4)$$

Eq. 15·4 gives the expression for the horizontal reaction H and is valid whether the cable chord is inclined or horizontal.

Alternative method when the cable chord is horizontal.

Alternatively equation 15·4 can be derived in a slightly different manner for the special case when the cable chord is horizontal (Fig. 15·4).

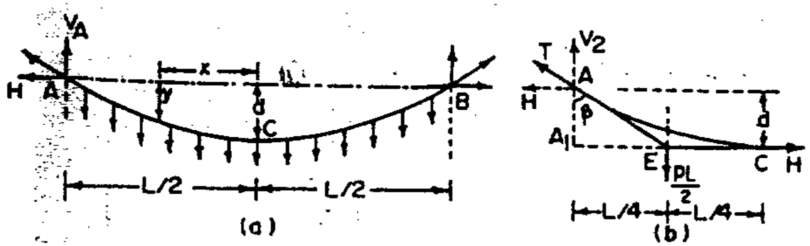


Fig. 15·4.
Cable chord horizontal.

Due to symmetry, $V_A = V_B = p \cdot \frac{L}{2}$.

Let C be the lowest point of the cable, at its middle, where the dip is equal to d. Consider the equilibrium of the portion CA to the left of C [Fig. 15·4 (*b*)]. This left portion is in equilibrium under there forces : (*i*) the cable tension T at A (*i.e.*, the resultant of H and V_A), (*ii*) the external load $pL/2$ acting at $L/4$ from C, and (*iii*) the horizontal cable tension H at C. All these forces must meet at a point E, distant $L/4$ from C. Thus triangle AA_1E becomes a triangle of forces from which

$$\frac{H}{A_1E} = \frac{T}{AE} = \frac{pL/2}{AA_1}$$

$$\therefore \quad H = \frac{pL}{2} \cdot \frac{A_1E}{AA_1} = \frac{pL}{2} \cdot \frac{L}{4} \cdot \frac{1}{d} = \frac{pL^2}{8d}$$

which is the same as obtained earlier.

(*b*) EXPRESSION FOR CABLE TENSION AT THE ENDS

The cable tension T at any end is the resultant of vertical and horizontal reactions at the end. Thus

$$T_A = \sqrt{V_A^2 + H^2}$$

$$T_B = \sqrt{V_B^2 + H^2} \qquad (15{\cdot}5)$$

Knowing H from Eq. 15·4 and V_A from Eq. 15·1, the cable tension T_A or T_B can be easily calculated. When the cable chord is horizontal, $V_A = V_B = \frac{pL}{2}$. Hence

$$T_A = T_B = T = \sqrt{\left(\frac{pL}{2}\right)^2 + \left(\frac{pL^2}{8d}\right)^2}$$

or $$T = \frac{pL}{2}\sqrt{1 + \frac{L^2}{16d^2}} \qquad (15{\cdot}6)$$

or $$T = H\sqrt{1 + \frac{16d^2}{L^2}} \qquad (15{\cdot}7)$$

The inclination β of T with the vertical is given by

$$\tan\beta = \frac{H}{V} = \frac{pL^2}{8d} \cdot \frac{2}{pL} = \frac{L}{4d} \qquad (15{\cdot}8)$$

It should be remembered that the horizontal component of cable tension at any point will be equal to H.

(c) SHAPE OF THE CABLE

Let us now determine the shape of the cable under the uniformly distributed load. Substituting the value of H (Eq. 15·4) in Eq. 15·3, we get

$$\left(\frac{pL^2}{8d}\right) y=\frac{pLx}{2}-\frac{px^2}{2}$$

or
$$y=\frac{4dx}{L^2}(L-x) \tag{13·9}$$

This is, thus, the equation of the curve with respect to the cable chord. The cable, thus, takes the form of a parabola when subjected to uniformly distributed load.

(d) LENGTH OF THE CABLE : BOTH ENDS AT THE SAME LEVEL

When both the ends of the cable are at the same level [Fig. 15·4 (a)], the equation of the parabola can be written, with C as the origin, as follows :

$$y=kx^2$$

At A, $\quad x=\frac{L}{2}$ and $y=d$

$$\therefore \quad k=\frac{y}{x^2}=\frac{d}{(L/2)^2}=\frac{4d}{L^2}$$

$$\therefore \quad y=\frac{4d}{L^2}x^2 \tag{15·10}$$

$$\therefore \quad \frac{dy}{dx}=\frac{8d}{L^2}x$$

Consider an element of length ds of the curve, having co-ordinates x and y. The total length s of the curve is given by

$$s=\int_0^L ds=2\int_2^{L/2}\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}dx$$

$$=2\int_0^{L/2}\left(1+\frac{64d^2}{L^4}x^2\right)^{1/2}dx$$

Expanding $\left(1+\frac{64d^2}{L^4}x^2\right)^{1/2}$ by Binomial theorem, and neglecting higher powers of $\frac{d^2}{L^4}x^2$, we get

$$s=2\int_0^{L/2}\left(1+\frac{1}{2}\;\frac{64d^2}{L^4}x^2+\ldots\ldots\right)dx$$

$$=2\left[x+\frac{32d^2}{3L^4}x^3\right]_0^{L/2}=2\left[\frac{L}{2}+\frac{4}{3}\;\frac{d^2L^3}{L^4}\right]$$

$$\therefore \quad s=L+\frac{8}{3}\;\frac{d^2}{L} \tag{15·11}$$

(e) LENGTH OF THE CABLE : ENDS AT DIFFERENT LEVELS

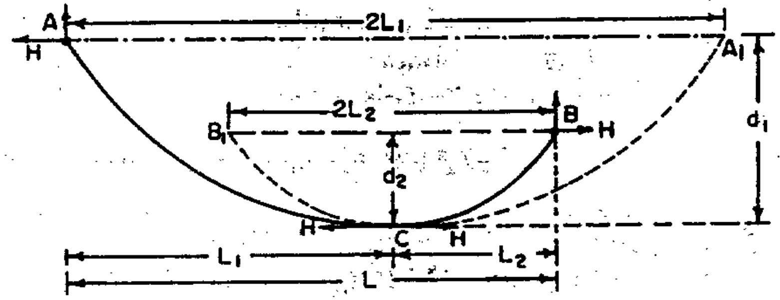


Fig. 15·5

Supports at different levels.

Consider a cable AB with the supports A and B at different levels. Let C be the lowest point of the cable, such that the horizontal equivalent of AC is L_1 and that of CB is L_2.

Evidently, $\quad L_1+L_2=L \quad$ (1)

Imagine the portion AC to be extended to a point A_1 such that A and A_1 are at the same level. Let d_1 be the dip of this hypothetical cable, below the chord AA_1. From Eq. 15.4, we have

$$H=\frac{pL^2}{8d}\text{, where } L=2L_1 \text{ and } d=d_1$$

$$\therefore \quad H=\frac{p}{8}\;\frac{(2L)^2}{d_1}=\frac{pL_1^2}{2d_1} \tag{i}$$

Similarly, imagine the portion BC to be extended to a point B_1 such that B and B_1 are at the same level. Let d_2 be the dip of this hypothetical cable, below the chord BB_1. From Eq. 15·4

$$H=\frac{pL^2}{8d}\text{, where } L=2L_2 \text{ and } d=d_2$$

$$\therefore \quad H=\frac{p}{8}\;\frac{(2L_2)^2}{d_2}=\frac{pL_2^2}{2d_2} \qquad (ii)\ [15·12(a)]$$

Since H is the same at C for both the portions of the cable, we get

$$\frac{pL_1^2}{2d_1}=\frac{pL_2^2}{2d_2}$$

or
$$\frac{L_1}{L_2}=\sqrt{\frac{d_1}{d_2}} \qquad (2)\ (15·12)$$

Solving (1) and (2), the values of L_1 and L_2 can be known in terms of L, d_1 and d_2.

In order to find the vertical reaction V_A at A, take moments about B. Then

$$V_A=\frac{1}{L}\left[\frac{pL^2}{2}+H(d_1-d_2)\right], \text{ where } H=\frac{pL_1^2}{2d_1}$$

$$=\frac{p}{2L}\left[L^2+\frac{L_1^2}{d_1}(d_1-d_2)\right]=\frac{p}{2L}\left[L^2+L_1^2-L_1^2\frac{d_2}{d_1}\right]$$

$$=\frac{p}{2L}\left[L^2+L_1^2-L_1^2\times\frac{L_2^2}{L_1^2}\right]$$

$$=\frac{p}{2L}\left[L_1^2+L_2^2+2L_1L_2+L_1^2-L_2^2\right]=\frac{p}{2L}[2L_1^2+2L_1L_2]$$

$$\therefore V_A=pL_1$$

Similarly, $V_B=pL_2$.

For the imaginary cable ACA_1, the length s_1 is given by

$$s_1=2L_1+\frac{8}{3}\frac{d_1^2}{2L_1}=2L_1+\frac{4}{3}\frac{d_1^2}{L_1}$$

Similarly, the length of the cable BCB_1 is given by

$$s_2=2L_2+\frac{8}{3}\frac{d_2^2}{2L_2}=2L_2+\frac{4}{3}\frac{d_2^2}{L_2}$$

Hence the total length of the actual cable ABC is

$$s=\frac{1}{2}(s_1+s_2)$$

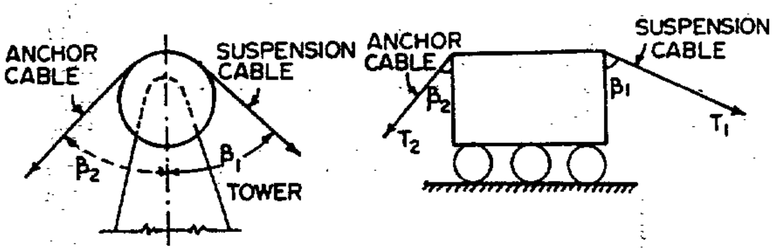
or $$s=\frac{1}{2}\left\{\left(2L_1+\frac{4}{3}\frac{d_1^2}{L_1}\right)+\left(2L_2+\frac{4}{3}\frac{d_2^2}{L_2}\right)\right\}$$

or $$s=L_1+\frac{2}{3}\frac{d_1^2}{L_1}+L_2+\frac{2}{3}\frac{d_2^2}{L_2}$$

or $$s=L_1+\frac{2}{3}\frac{d_1^2}{L_1}+\frac{2}{3}\frac{d_2^2}{L_2} \qquad (15{\cdot}13)$$

15·4. ANCHOR CABLES

The suspension cable is supported on either sides, on the supporting towers. The anchor cables transfer the tension of the suspension cable to the anchorage consisting of huge mass of concrete. There are generally two arrangements for this. The suspension cable can either be passed over the guide pulley for anchoring it to the other side or it can be attached to a saddle mounted on roller. Fig. 15·6 (*a*) and (*b*) show both the arrangements.



(a) GUIDE PULLEY SUPPORT (b) ROLLER SUPPORT

Fig. 15·6.

In the former case, when the suspension cable passes over the guide pulley and forms the part of the anchor cable to the other side, the tension T in the cable is the same on both the sides.

Let β_1=inclination of the suspension cable, with vertical.

β_2=inclination of the anchor cable, with vertical.

$\therefore$ Pressure on the top of pier$=V_P=T\cos\beta_1+T\cos\beta_2$

$$=T(\cos\beta_1+\cos\beta_2) \qquad (15{\cdot}14)$$

Horizontal force on the top of the pier

$$=T\sin\beta_1-T\sin\beta_2$$

$$=T(\sin\beta_1-\sin\beta_2) \qquad (15{\cdot}15)$$

This horizontal force will cause bending moment in the tower.

If the cable is supported on a saddle mounted on rollers, as shown in Fig. 15·6 (*b*), the horizontal components of the tensions in the suspension cable and the anchor cable will be equal since the rollers do not have any horizontal reaction.

$$\therefore T_1\sin\beta_1=T_2\sin\beta_2=H \qquad (15{\cdot}16)$$

The vertical pressure on the pier is given by

$$V_P=T_1\cos\beta_1+T_2\cos\beta_2 \qquad (15{\cdot}17)$$

15·5. TEMPERATURE STRESES IN SUSPENSION CABLE

Let s=length of the cable

δs=change in the length due to change in temperature

δd=corresponding change in the dip.

From Eq. 13·11,

$$s=L+\frac{8}{3}\frac{d^2}{L}$$

$$\delta s=\frac{16}{3}\frac{d}{L}\delta d$$

or $$\delta d=\frac{3}{16}\frac{L}{d}\delta s \qquad (1)$$

But $\delta s=s\,.\,\alpha\,.\,t$

where α=coefficient of thermal expansion of cable

t=change in the temperature

$$\therefore \qquad \delta s = \alpha t\left(L+\frac{8}{3}\,\frac{d^2}{L}\right)$$

or
$$\delta s = L\alpha t + \frac{8}{3}\,\frac{d^2}{L}\,\alpha t.$$

Neglecting $\frac{8}{3}\,\frac{d^2}{L}\,\alpha t$ in comparison to $L\,.\,\alpha\,.\,t$, we have

$$\delta s = L\,.\,\alpha\,.\,t \qquad (2)$$

Substituting this in (1), we get

$$\delta d = \frac{3}{16}\,\frac{L}{d}\left(L\,.\,\alpha\,.\,t\right) = \frac{3}{16}\,\frac{L^2}{d}\,\alpha\,.\,t \qquad (15{\cdot}18)$$

It is to be noted that when the temperature rises, L will increase, and hence δd will increase. Similarly, when the temperature falls, L will decrease, and hence, δd will decrease. Let us now find the corresponding change in the value of H due to this change in d.

$$H = \frac{pL^2}{8d}$$

or
$$H \propto \frac{1}{d}$$

$$\therefore \qquad \frac{\delta H}{H} = -\frac{\delta d}{d}$$

If f is the stress in the cable,

$$\therefore \qquad \frac{\delta f}{f} = \frac{\delta H}{H} = -\frac{\delta d}{d}$$

where δf = change in the cable stress.

$$\frac{\delta f}{f} = \frac{\delta H}{H} = -\frac{\delta d}{d} = -\frac{3}{16}\,\frac{L^2}{d^2}\,\alpha\,.\,t \qquad (15{\cdot}10)$$

Example 15·1. *A light cable, 18 m long, is supported at two ends at the same level. The supports are 16 m apart. The cable supports three loads of 8, 10, and 12 N dividing the 16 m distance in four equal parts. Find the shape of the string and the tension in various portions.*

Solution.

Let $CC_1 = y_C$; $DD_1 = y_D$ and $EE_1 = y_E$

Since both the supports are at the same level,

$$V_A = \frac{1}{16}\left\{(8\times12)+(10\times8)+(12\times4)\right\} = 14 \text{ N}$$

Since the cable is in equilibrium, the shape taken by it is that of a funicular polygon. The shape, thus, represents the bending moment diagram to some scale.

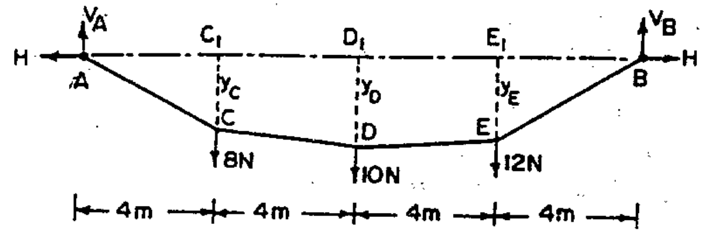


Fig. 15·7.

$$\therefore \quad y_C : y_D : y_E = (14\times4) : (14\times8-8\times4) : (14\times12-8\times8-10\times4)$$

or
$$y_C : y_D : y_E = 56 : 80 : 64$$

or
$$y_C : y_D : y_E = 1 : \frac{10}{7} : \frac{8}{7}$$

$$y_D = \frac{10}{7}\,y_C \text{ and } y_E = \frac{8}{7}\,y_C$$

Distance $AC = \sqrt{16+y_C^2} = 4\sqrt{1+0{\cdot}0625\,y_C^2}$

$$CD = \sqrt{16+(y_D-y_C)^2} = \sqrt{16+\left(\frac{10}{7}-1\right)^2 y_C^2}$$

$$= \sqrt{16+\frac{9}{49}y_C^2} = 4\sqrt{1+0{\cdot}0115\,y_C^2}$$

$$DE = \sqrt{16+(y_D-y_E)^2} = \sqrt{16+\left(\frac{10}{7}-\frac{8}{7}\right)^2 y_C^2}$$

$$= \sqrt{16+\frac{4}{49}\,y_C^2} = 4\sqrt{1+0{\cdot}0051\,y_C^2}$$

$$EB = \sqrt{16+y_E^2} = \sqrt{16+\left(\frac{8}{7}y_C\right)^2}$$

$$= \sqrt{16+\frac{64}{49}\,y_C^2} = \sqrt{1+0{\cdot}0816\,y_C^2}$$

$\therefore$ Total length $AB = AC+CD+DE+EB$

$$\therefore \quad 18 = 4[(1+0{\cdot}0625\,y_C^2)^{1/2}+(1+0{\cdot}0115\,y_C^2)^{1/2} +(1+0{\cdot}0051\,y_C^2)^{1/2}+(1+0{\cdot}0816\,y_C^2)^{1/2}]$$

or
$$4{\cdot}5 = \left[\left(1+\frac{0{\cdot}00625}{2}y_C^2\right)+\left(1+\frac{0{\cdot}0115}{2}y_C^2\right) +\left(1+\frac{0{\cdot}0051}{2}y_C^2\right)+\left(1+\frac{0{\cdot}0816}{2}y_C^2\right)\right]$$

or
$$4{\cdot}5 = [4+0{\cdot}08\,y_C^2]$$

$$\therefore \quad y_C = 2{\cdot}5 \text{ m}$$

$$\therefore \quad y_D = \frac{10}{7}\,y_C = 3{\cdot}57 \text{ m}$$

and $\qquad y_E=\frac{8}{7}\,y_C=2{\cdot}86$ m

Thus, with the known values of y_C, y_D and y_E, the shape of the cable is determined.

In order to find the horizontal reaction H, apply the general cable theorem (Eq. 15·2) at point C.

$$\therefore \quad Hy_C=\frac{4}{16}\,\Sigma M_B-\Sigma M_C$$

where ΣM_B=sum of moments of *external* loads, about B

$=(8\times 12)+(10\times 8)+(12\times 4)=224$

ΣM_C=sum of moments of *external* loads, about C

$=0.$

$$y_C=2{\cdot}50$$

$$\therefore \quad 2{\cdot}5H=\frac{4}{16}\,(224)=56$$

$$H=\frac{56}{2{\cdot}50}=22{\cdot}4$$

Hence tension in the part AC is given by

$$T_{AC}=\sqrt{V_A^2+H^2}=\sqrt{(14)^2+(22{\cdot}4)^2}=26{\cdot}4 \text{ N}$$

The tensions in parts CD, DE and EB will be such that their horizontal component is equal to $H=22{\cdot}4$.

Example 15·2. *A cable is used to support five equal and equidistant loads over a span of 39 metres. Find the length of the cable required and its sectional area if the safe tensile stress is 140 N/mm². The central dip of the cable is 2·5 m and loads are 5 kN each.*

Solution

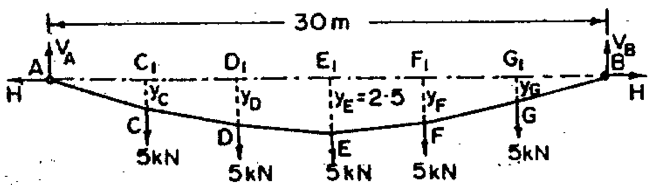


Fig. 15·8.

Since the cable is in equilibrium, the shape of the cable will correspond to the funicular polygon for the load system. The ordinates at C, D, E, F and G will be proportional to the bending moments at these points. Since both the cables are at the same level, we have

$$V_A=\tfrac{1}{2}\,(5+5+5+5+5)=12{\cdot}5 \text{ kN.}$$

$$\therefore \quad CC_1=GG_1=\mu_C=(12{\cdot}5\times 5)=62{\cdot}5$$

$$DD_1=FF_1=\mu_D=(12{\cdot}5\times 10)-(5\times 5)=100$$

$$EE_1=\mu_E=(12{\cdot}5\times 15)-(5\times 10)-(5\times 5)=112{\cdot}5$$

$$\therefore \quad EE_1 : DD_1 : CC_1 :: 112{\cdot}5 : 100 : 62{\cdot}5$$

$$y_E : y_D : y_C :: 1 : 0{\cdot}89 : 0{\cdot}556$$

But $\qquad y_E=2{\cdot}5$ m

$$\therefore \quad y_D=y_F=2{\cdot}5\times 0{\cdot}89=2{\cdot}22 \text{ m}$$

$$y_C=y_G=2{\cdot}5\times 0{\cdot}556=1{\cdot}39 \text{ m}$$

The length of the cable $=2(AC+CD+DE)$

$$=2\left[5\left\{1+\frac{1{\cdot}39^2}{25}\right\}^{\frac{1}{2}}+5\left\{1+\frac{(2{\cdot}22-1{\cdot}39)^2}{25}\right\}^{\frac{1}{2}}+5\left\{1+\frac{(2{\cdot}5-2{\cdot}22)^2}{25}\right\}^{\frac{1}{2}}\right]$$

$$=10\left[1+\frac{1{\cdot}93}{50}+1+\frac{0{\cdot}69}{50}+1+\frac{0{\cdot}08}{50}\right]$$

$$=30{\cdot}54 \text{ m}$$

The length of the cable can also be found approximately by treating the string as a parabola. In that case,

$$s=L+\frac{8}{3}\,\frac{d^2}{L}=30+\frac{8}{3}\,\frac{(2{\cdot}5)^2}{30}=30{\cdot}56 \text{ m}$$

To find the horizontal reaction H, take moment about C of all forces to the left of it and equate it to zero. Thus,

$$M_C=0=(H\times 1{\cdot}39)-V_A\times 5=1{\cdot}39\,H-5\times 12{\cdot}5$$

$$\therefore \quad H=\frac{5\times 1{\cdot}25}{1{\cdot}39}=45 \text{ kN}$$

The maximum tension in $AC=\sqrt{(45^2)+(12{\cdot}5)^2}=46{\cdot}6$ kN

$$\therefore \quad \text{Area required} \quad =\frac{46{\cdot}6\times 1000}{140}=333 \text{ mm}^2$$

Example 15·3. *A flexible rope weighing 1 N per metre span between two points 40 m apart and at the same level, 12 m above the ground. It is to carry a concentrated load of 300 N at a point P on the rope which is to be at a horizontal distance of 10 m from the left hand support. What is the maximum height above the ground to which the point P may be raised if the maximum tension in the rope is not to exceed 1000 N? Assume that the distances measured along the rope are equal to their horizontal projection.*

Solution

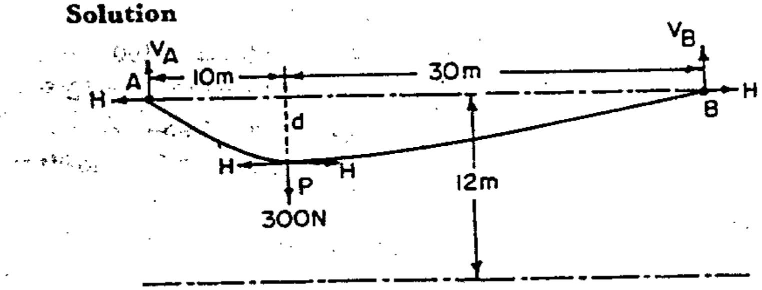


Fig. 15·9.

Let d=dip of the cable at P.

Taking moments about B, we get

$$(V_A \times 40)=(300\times 30)+(1\times 40\times 20)$$

or $$V_A=\frac{9800}{40}=245 \text{ N}$$

and $$V_A=(300+40\times 1)-245=95 \text{ N}$$

The maximum tension in the cable$=T_A=\sqrt{V_A^2+H^2}$

$\therefore$ $$1000=\sqrt{(245)^2+(H)^2}$$

or $$10^6=6\times 10^4+H^2$$

$\therefore$ $$H=970 \text{ N}$$

Taking moments about P, we get

$$M_P=0=Hd-V_A\times 10$$

or $$970d-245\times 10=0$$

$\therefore$ $$d=\frac{2450}{970}=2{\cdot}52 \text{ m}$$

$\therefore$ Height of P above ground$=12-2{\cdot}52=$**9·48 m.**

Example 15·4. *A wire of uniform material weighing 0·32 lb. per cu. in. hangs between two points 120 ft. apart horizontally. with one end 3 ft. above the other. The sag of the wire measured from the highest point is 5 ft. Calculate the maximum stress in the wire.* (U.L.)

Solution

Let p=weight of cable, per ft. length (in lbs).

From Eq. 15·12,

$$\frac{L_1}{L_2}=\sqrt{\frac{d_1}{d_2}}$$

where d_1=maximum dip=5′ ; $d_2=5-3=2'$

$\therefore$ $$\frac{L_1}{L_2}=\sqrt{\frac{5}{2}}=1{\cdot}58$$

$\therefore$ $$L_1=1{\cdot}58\ L_2 \quad (1)$$

Also, $$L_1+L_2=120' \quad (2)$$

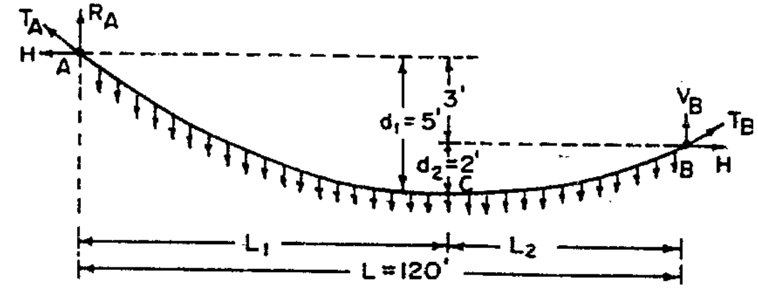


Fig. 15·10.

$\therefore$ From (1) and (2), $L_1=73{\cdot}5'$; $L_2=46{\cdot}5'$

Reaction $$R_A=pL_1=73{\cdot}5\ p$$

$$H=\frac{pL_1^2}{2d_1}=\frac{p(73{\cdot}5)^2}{10}=540{\cdot}25p$$

$\therefore$ Max. tension$=T_A=\sqrt{R_A^2+H^2}$

$$=p(73{\cdot}5^2+540{\cdot}25^2)^{1/2}=545\ p$$

But $$p=0{\cdot}32\times 12A$$

where A=area of cross-section

$\therefore$ $$T_A=545\times 0{\cdot}32\times 12A=2090\ A \text{ lb}$$

$\therefore$ Max. tensile stress at end A=**2090 lb/in².**

Example 15·5. *A flexible suspension cable of weight $\frac{3}{4}$ N/m hangs between two vertical walls 60 m apart, the left hand end being attached to the wall at point 10 m below the right hand end. A concentrated load of 100 N is attached to the cable in such a manner that the point of attachment of the load is 20 m horizontally from the left hand wall and 5 m below the left hand support. Show that the maximum resultant cable tension is at the right hand end and find its value. The cable weight may be taken as uniformly distributed horizontally.*

Solution

To find the reactions V_B and H, take moments about A :

$\because$ $$60\ V_B=20W+\frac{pL^2}{2}+10\ H$$

$\therefore$ $$V_B=\frac{1}{60}\left(20\times 100+\frac{3}{4}\times\frac{3600}{2}+10\ H\right)$$

or $$V_B=\frac{1}{6}(335+H) \quad (1)$$

Again, since the cable is flexible, $M_C=0$

$$40V_B - p\times 40\times 20\times 15\ H=0$$

or $$8V_B-\tfrac{3}{4}\times 160-3\ H=0$$

$$V_B=15+\frac{3}{8}H \qquad (2)$$

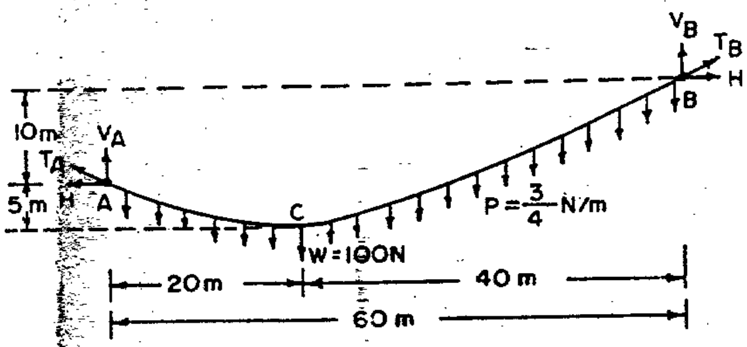


Fig. 15·11.

From (1) and (2), we get

$$H=196\ \text{N} \quad \text{and} \quad V_B=88{\cdot}5\ \text{N}$$

$$V_A=(\tfrac{3}{4}\times 60)+100-V_B=45+100-88{\cdot}5=56{\cdot}5$$

Since $V_B>V_A$, T_B will be greater than T_A.

$$T_B=\sqrt{V_B^2+H^2}$$

$$=\sqrt{(88{\cdot}5)^2+(196)^2}=215\ \text{N}.$$

Example 15·6. *A suspension cable 160 m span and 16 m central dip carries a load of ½ kN per lineal horizontal metre. Calculate the maximum and minimum tensions in the cable. Find horizontal and vertical forces in each pier under the following alternative conditions :*

(a) If the cable passes over frictionless rollers on the top of the piers.

(b) If the cable is firmly clamped to saddles carried on friction less roller on the top of the piers.

In each case the back stay is inclined at 30° to the horizontal.

Solution

The maximum tension (T) in the cable is always at its ends while the minimum tension in the cable is at its lowest point and is equal to H.

Now $$T_{min}=H=\frac{pL^2}{8d}=\frac{\frac{1}{2}(160)^2}{8\times 16}=100\ \text{kN}$$

From Eq. 15·7,

$$T=T_{max}=H\sqrt{1+16\frac{d^2}{L^2}}$$

$$=100\sqrt{1+\frac{16\times 16^2}{160^2}}=108\ \text{kN}$$

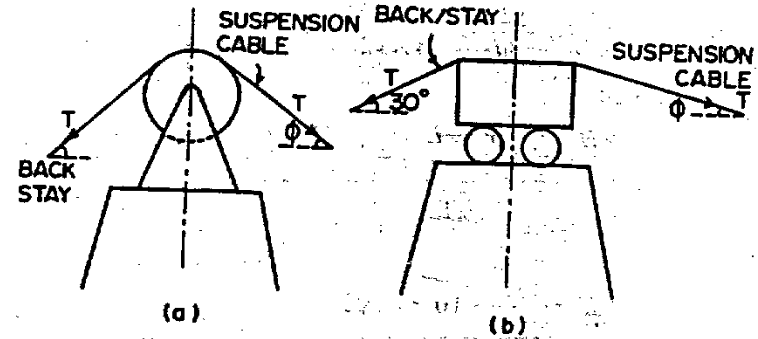


Fig. 15·12.

Forces in the pier

(*d*) When the cable passes over the frictionless pulley, the tension in the back stay is equal to the tension in the cable. Let the inclination of the cable be ϕ with horizontal.

Then the load on pier $=T\sin\phi+T\sin 30°$

$$=V+T\sin 30°$$

$$=40+108\times\frac{1}{2}=94\ \text{kN}$$

Horizontal shear $=T\cos\phi-T\cos 30°$

$$=\overrightarrow{H}-\overleftarrow{T\cos 30}$$

$$=100-(108\times 0{\cdot}866)=6{\cdot}47\ \text{kN}\rightarrow$$

(*a*) Rollers permit no horizontal shear on towers. Hence the horizontal components of tension balance. Let Ts = tension in the back stay.

$\therefore$ $$Ts\cos 30°=T\cos\phi=H=100$$

$\therefore$ $$Ts=\frac{100}{\cos 30°}=\frac{100}{0{\cdot}866}=115{\cdot}5\ \text{kN}$$

$\therefore$ Total compression on pier

$$=Ts\sin 30°+T\sin\phi$$

$$=Ts\sin 30°+V$$

$$=\frac{115{\cdot}5}{3}+40=97{\cdot}8\ \text{kN}.$$

Example 15·7. *A cable is swung between two points at the same level with a central dip of 12 m over a span of 120 m. The cable carries a uniformly distributed load of intensity 2 kN/m of horizontal length. Calculate the change in the horizontal tension if the temperature rises by 20°F from the original. Take $\alpha = 6\times10^{-6}$ per 1°F.*

Solution

$$H=\frac{pL^2}{8d}=\frac{2(120)^2}{8\times12}=300 \text{ kN}$$

From Eq. 3·19

$$\frac{\delta H}{H}=-\frac{3}{16}\left(\frac{L}{d}\right)^2\alpha\, t$$

$$=-\frac{3}{16}\left(\frac{120}{12}\right)^2\times6\times10^{-6}\times20=-\frac{9}{4}\times10^{-3}$$

$$\therefore\quad \delta H=-\frac{9}{4}\times10^{-3}\times300=-0.675 \text{ kN}$$

i.e. $\delta H = 0.675$ **kN** (decrease)

15·6. THREE HINGED STIFFENING GIRDER

Since the cable of the suspension bridge is the main load bearing member, the curvature of the cable of an unstiffened bridge changes as the load moves on the decking. To avoid this, the decking is stiffened by provision of either a three hinged stiffening girder or a two hinged stiffening girder. The stiffening girder transfers a uniform or equal load to each suspender, irrespective of the position of the load on the decking. Fig. 15·13 shows a suspension bridge with a stiffening girder hinged at the abutments D and F and also at the centre E. We shall now consider the effect of a unit point load rolling on the decking and plot (*i*) bending moment diagram for fixed load position, (*ii*) influence line for horizontal reaction H of the cable (*iii*) influence line for bending moment at a section, (*iv*) maximum bending moment diagram due to a point load W and (*v*) maximum bending moment diagram due to a uniformly distributed load of intensity w.

For the purposes of analysis of the above items, let us first consider the equilibrium of the cable as well as the stiffening girder separately.

(1) EQUILIBRIUM OF THE CABLE

By the provision of a stiffening girder the suspenders carry a uniformly distributed load of intensity p, irrespective of the nature

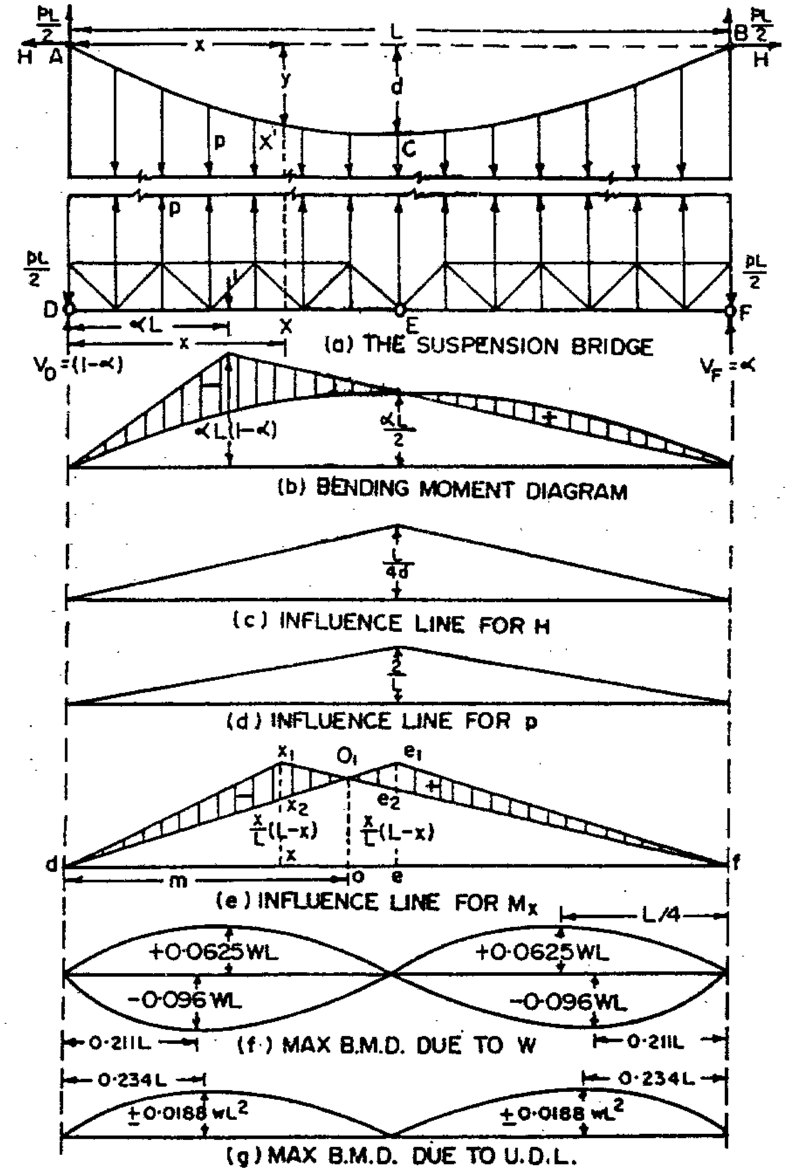


Fig. 15·13

Three hinged stiffening girder.

and position of the load on the decking. Thus the cable is subjected to a downward uniform load of intensity p per unit length, as shown in the upper part of Fig. 15·13(a). The reactions at the support A and B are equal, the horizontal reaction being equal the H and the vertical reaction equal to $\frac{pL}{2}$.

From Eq. 15·5

$$H=\frac{pL^2}{8d} \qquad (1)$$

Since the cable is flexible, the bending moment at any point on it is equal to zero. Consider any point X' distant x from left support A, Then

$$M_x=0=Hy-\frac{pL}{2}x+\frac{px^2}{2}$$

$$\therefore \quad Hy=\left(\frac{pL}{2}x-\frac{px^2}{2}\right) \qquad (2)$$

The equation of the parabola (cable), with A as the origin may be written as

$$y=kx(L-x)$$

$$\text{at} \quad x=\frac{L}{2},\ y=d$$

$$k=\frac{4d}{L^2}$$

$$\text{Hence} \quad y=\frac{4dx}{L^2}(L-x) \qquad (3)$$

(2) EQUILIBRIUM OF THE GIRDER

The lower part of Fig. 15·13(a) shows the equilibrium of the three hinged stiffening girder which is subjected to the following forces : (i) the external unit load acting at a distance of αL from left hand support, (ii) the simply supported reactions $V_D=(1-\alpha)$ and $V_F=\alpha$ respectively at points D and F, due to the external load, (iii) the uniformly distributed pull p exerted by the suspenders, and (iv) downward reactions $\frac{pL}{2}$ at D and F, due to the pull p of the hingers.

(3) BENDING MOMENT DIAGRAM

Consider a point X distant x from the left hand support. The bending moment at X may be expressed as follows :

$$M_X=\left[-V_D.x+1\,(x-\alpha L)\right]+\left[\frac{pL}{2}x-\frac{px^2}{2}\right] \qquad (4)$$

Thus the bending moment at X consists of two parts. The first part, $i.e.$ $[-V_D.x+1\,(x-\alpha L)]$ may be designated as μ_X, where μ_X is the bending moment at the point X treating the girder as simply supported beam. The second part, $i.e.$ $\left[\frac{pL}{2}x-\frac{px^2}{2}\right]$ is found to be equal to $H.y$, from Eq. (2). Hence Eq. (4) may be re-written as :

$$M_X=\mu_X+H.y \qquad (15\cdot20)$$

The bending moment diagram for the girder can thus be obtained by superimposing the μ_X diagram over the Hy diagram. The μ_X diagram for a simply supported beam is a triangle having an ordinate of $-\frac{Wab}{L}=-\frac{1\times\alpha L\,(1-\alpha)L}{L}=-\alpha L(1-\alpha)$ under the point load, while the Hy diagram will be a parabola, since y varies parabolically while H is constant for a fixed load position. For the load position at αL from left support, $H=\frac{\alpha L}{2d}$ (see sub-Section 4 below). The maximum value of y is equal to d at the centre. Hence the Hy diagram is a parabola having a maximum ordinate of $\frac{\alpha L}{2d}\cdot d=\frac{\alpha L}{2}$ under the centre of the cable. The final bending moment diagram is shown in Fig 15·13(b).

(4) INFLUENCE LINE FOR H

In order to find the value of H for the unit load position at a distance αL, we apply Eq. 15·20 at the central hinge E of the girder where the bending moment is zero.

$$\therefore \quad M_E=0=\mu_E=H.y$$

$$H=-\frac{\mu_E}{y} \qquad (15\cdot21)$$

$$\text{Now} \quad \mu_E=-\alpha\cdot\frac{L}{2},\ \text{and } y=d \text{ at } E$$

$$H=\frac{\alpha L}{2d} \qquad (15\cdot22)$$

Eq. 15·22 suggests that for a given load position, H is constant. However, when the unit load changes its position, H also changes linearly as per Eq. 15·22. However, Eq. 15·22 is valid for load position between D and E, $i.e.$ for the range $\alpha L=0$ to $\alpha L=L/2$.

When $\alpha L=0, \quad H=0$

$$\alpha L=\frac{L}{2}, \quad H=\frac{L}{4d}$$

By symmetry, when the unit load is at F, $H=O$. Thus, the influence line diagram for H is a triangle having a maximum ordinate of $\frac{L}{4d}$ under the central hinge, as shown in Fig. 15·13(c).

(5) INFLUENCE LINE FOR p

From Eq. 15·4

$$H=\frac{pL^2}{8d}$$

or

$$p=\frac{8d}{L^2}\cdot H \qquad (15.23)$$

But $H=\frac{\alpha L}{2d}$, from Eq. 15·22

$$\therefore \quad p=\frac{8d}{L^2}\cdot\frac{\alpha L}{2d}=\frac{4\alpha}{L} \qquad (15·24)$$

Thus the load carried by the suspenders vary with the load position in the case of a three hinged stiffening girder. Eq. 15·24 is valid for the range of $\alpha L=0$ to $\alpha L=\frac{L}{2}$.

When the load is at D, $\alpha=0$ $\quad\therefore\ p=0$.

When the load is at the central hinge E, $\alpha L=\frac{L}{2}$, or $\alpha=\frac{1}{2}$

$$\therefore \quad p=\frac{4}{L}\cdot\frac{1}{2}=\frac{2}{L}.$$

By symmetry, when the load is at F, $p=0$

The influence line for p is shown in Fig. 15·13(d).

It will be seen later that in the case of a two hinged stiffening girder, p is constant, and does not vary with the load position.

(6) INFLUENCE LINE FOR BENDING MOMENT

We have

$$Mx=\mu_x+Hy$$

Thus, the I.L. for Mx can be obtained by superimposing I.L. for μ_x on the I.L. for $H.y$.

The I.L. for μ_x is a triangle having a maximum ordinate of $\frac{x(L-x)}{L}$ under the section X.

The I.L. for $H.y$ will be a triangle. It should be noted that for the given section X, y is a fixed quantity equal to $\frac{4d}{L^2}\,x\,(L-x)$.

Thus, the I.L. for $H.y$ has a maximum ordinate of $\frac{L}{4d}y$

$$=\frac{L}{4d}\cdot\frac{4d}{L^2}x(L-x)=+\frac{x}{L}(L-x)$$

under the central hinge. Fig. 15·15(e) shows the I.L. for Mx in which the maximum ordinates of positive and negative bending moments are equal.

(7) MAXIMUM BENDING MOMENT DIAGRAM DUE TO A SINGLE POINT LOAD W

(a) Maximum negative bending moment diagram

From the I.L. for Mx it is clear that the maximum negative bending moment at section X occurs when the point load is at the section, its value being given by

$$Mx\ (\text{max.} -\text{ve})=W\left\{-\frac{x(L-x)}{L}+\frac{x}{L}(L-x)\frac{2}{L}x\right\}$$

$$=\frac{Wx}{L^2}\left\{-L^2+Lx+2Lx-2x^2\right\}$$

$$=-\frac{Wx}{L^2}(L-x)(L-2x) \qquad (15·25)$$

In order to plot the maximum bending moment diagram, vary the value of x. Eq. 15·25 is a third degree polynomial. To find the position of the section where absolute maximum negative B.M. occurs, differentiate Eq. 15·25 with respect to x and equate it to zero.

Thus

$$(L-x)(L-2x)-x(L-2x)-2x(L-x)=0$$

or

$$6x^2-6Lx+L^2=0$$

which gives $\quad x=0·211L \quad$ or $\quad =0·789L \qquad (15·26)$

Substituting the value of x in Eq. 15·25, we get

$$(-)\ M_{max\cdot max}=-0·096\ WL$$

(b) Maximum positive bending moment diagram

From Fig. 15·13(e), it is clear that the maximum positive bending moment at section X occurs when the point load is over the central hinge, its value being

$$Mx(\text{max.}+\text{ve})=W\left[\frac{x}{L}(L-x)-\frac{x(L-x)}{L}\cdot\frac{1}{L-x}\cdot\frac{L}{2}\right]$$

$$=\frac{Wx}{2L}(L-2x) \qquad (15{\cdot}27)$$

which is the equation of parabola.

$$\therefore \quad \frac{dMx}{dx}=0=(L-2x)-2x$$

or $$x=L/4$$

Substituting in Eq. 15·27

$$(+)M_{max\cdot max}=+\frac{WL}{16}=+0{\cdot}0625\,WL$$

Fig. 15·13 (f) shows the maximum positive and maximum negative bending moment diagrams due to point load W.

(8) MAXIMUM BENDING MOMENT DIAGRAM DUE TO UNIFORMLY DISTRIBUTED LOAD

From Fig. 15·12(e) it is evident that the area of the negative portion of the I.L. diagram is equal to the area of the positive portion of the I.L. diagram for Mx. Thus, the net area of I.L. diagram for Mx is zero. *Hence it is concluded that there will be no bending moment anywhere in the girder due to its self weight or due to uniformly distributed load occuping the whole span.* The I.L. diagram for Mx has a zero ordinate at the point O, distant m from the left support. Let us first find the value of m.

From triangle $dx_1 f$, $$oo_1=\frac{x(L-x)}{L}\cdot\frac{1}{L-x}(L-m)$$

From triangle $de_1 f$, $$oo_1=\frac{x(L-x)}{L}\cdot\frac{2}{L}m$$

Equating the two we get

$$\frac{L-m}{L-x}=\frac{2m}{L}$$

or $$m=\frac{L^2}{3L-2x} \qquad (15{\cdot}28)$$

$$\therefore \text{ Ordinate } oo_1=\frac{x(L-x)}{L}\cdot\frac{2}{L}\cdot\frac{L^2}{3L-2x}=\frac{2x(L-x)}{(3L-2x)}$$

For the maximum negative bending moment at the section, the portion do should be loaded with the U.D.L. while portion of should be empty. Similarly, for maximum positive B.M. at X, portion of should be loaded while do should be empty.

Area of $-$ve portion of I.L. diagram$=\Delta dx_1 f-\Delta do_1 f$

$$=\frac{1}{2}\cdot L\cdot\frac{x(L-x)}{L}-\frac{1}{2}\cdot L\cdot\frac{2x(L-x)}{3L-2x}$$

$$=\frac{x(L-x)(L-2x)}{2(3L-2x)}$$

If w is the intensity of U.D.L., we get

$$(\pm)\,Mx_{(max)}=\frac{wx(L-x)(L-2x)}{2(3L-2x)} \qquad (15{\cdot}29)$$

To find the value of x at which absolute maximum positive or negative B.M. occurs, differentiate Eq. 15·29 with respect to x and equate to zero. Putting $x=nL$ in Eq. 15·29 we get

$$Mx_{(max)}=wL^2\frac{n(1-n)(1-2n)}{2(3-2n)}$$

$$\therefore \quad \frac{dMx_{(max.)}}{dn}=0=(3-2n)[(1-n)(1-2n)-n(1-2n)-2n(1-n)]$$
$$+2n(1-n)(1-2n)$$

or $$8n^3-24n^2+18n-3=0$$

$$\therefore \quad n=0{\cdot}234 \quad \text{or} \quad x=0{\cdot}234\,L$$

Substituting the value of x in Eq. 15·29, we get

$$(\pm)M_{max\cdot max}=\pm0{\cdot}01883\,wL^2 \qquad (15{\cdot}30)$$

The loaded length $m=\dfrac{L^2}{3L-2(0{\cdot}234L)}=0{\cdot}395\,L$

Fig. 15·13 (g) shows the maximum positive and maximum negative bending moment diagrams due to U.D.L.

(9) INFLUENCE LINE FOR SHEAR FORCE

Consider the section X distant x from the left support [Fig. 15·14(a)]. When the unit load is so placed that $\alpha L<x$, we have

$$Fx=+\alpha+p(L-x)-\frac{pL}{2}$$

or $$Fx=\alpha+p\left(\frac{L}{2}-x\right) \qquad [15{\cdot}31(a)]$$

or $$Fx=fx+p\left(\frac{L}{2}-x\right) \qquad (15{\cdot}31)$$

where $\alpha=fx=$S.F. at X by considering the girder to be simply supported.

Now the equation of the parabola is

$$y=\frac{4d}{L^2}x(L-x)$$

$$\therefore \quad \tan\theta = \frac{dy}{dx} = \frac{8d}{L^2}\left(\frac{L}{2} - x\right) \tag{15·32}$$

or $\quad\theta$=inclination of the tangent at X', to the horizontal.

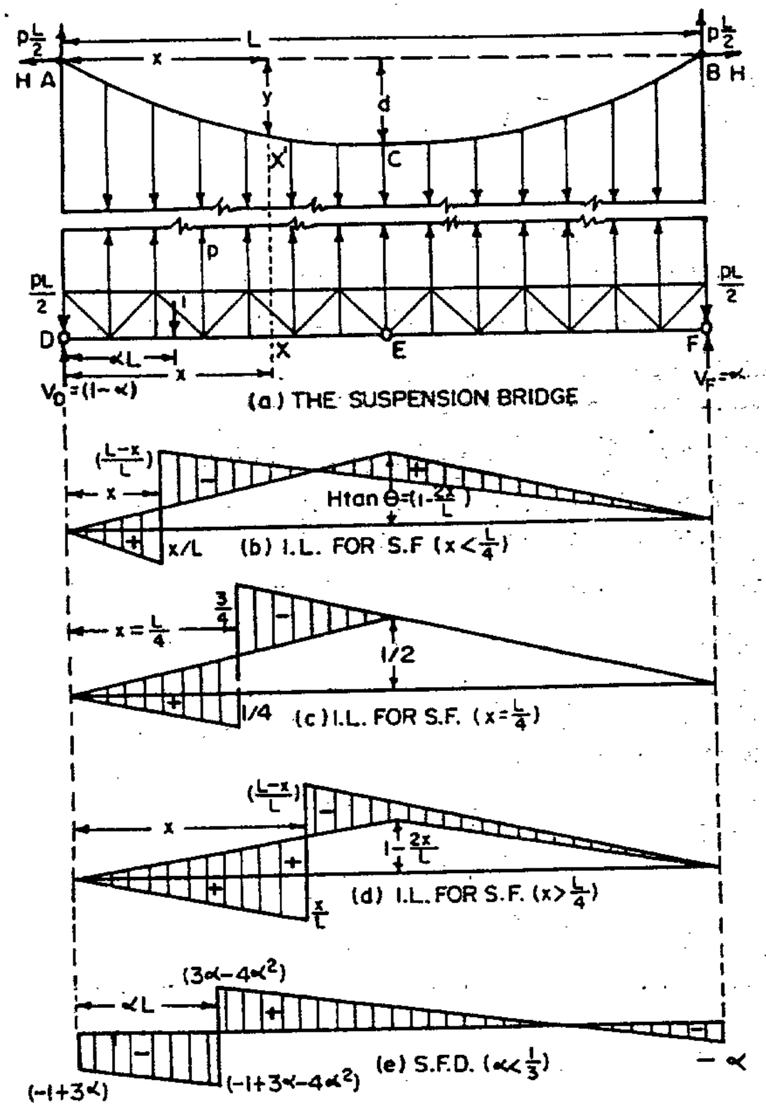


Fig. 15·14.
Three hinged stiffening girder.

But $\quad H = \dfrac{pL^2}{8d} \quad$ or $\quad \dfrac{8d}{L^2} = \dfrac{p}{H}$

$$\therefore \quad \tan\theta = \frac{p}{H}\left(\frac{L}{2} - x\right)$$

or $\quad p\left(\dfrac{L}{2} - x\right) = H\tan\theta$

Substituting the value of $p\left(\dfrac{L}{2} - x\right)$ in Eq. 15·31, we get

$$Fx = fx + H\tan\theta \tag{15·33}$$

Thus, the shear force at X also consists of two parts : the S.F. due to the simply-supported action of the girder and the S.F. due to the effect of horizontal reaction. To plot the I.L. for S.F. at X, superimpose I.L. for fx over I.L. for $H\tan\theta$, θ being constant for the section X.

When αL is lesser than x, $fx=\alpha$, and therefore, fx and $H\tan\theta$ ordinates are additive. When αL is greater than x, $fx=-(1-\alpha)$ and hence fx and $H\tan\theta$ ordinates are substractive. The I.L. for S.F. at X is shown in Fig. 15·14(*b*), 15·14(*c*) and 15·14(*d*), illustrating three possibilities, depending upon the position X.

When the unit load is at the central hinge,

$$H = \frac{\alpha L}{2d} = \frac{L}{4d}$$

$$H\tan\theta = \frac{L}{4d}\cdot\frac{8d}{L^2}\left(\frac{L}{2} - x\right)$$

$$= \frac{2}{L}\left(\frac{L-2x}{2}\right) = \left(1 - \frac{2x}{L}\right) \tag{15·34}$$

(10) SHEAR FORCE DIAGRAM

$$Fx = -(1-\alpha) + \frac{pL}{2} - px \;\vdots\; +1$$

where $\quad p = \dfrac{4\alpha}{L}$, from Eq. 15·24

$$\therefore \quad Fx = -(1-\alpha) + \frac{4\alpha}{L}\left(\frac{L}{2} - x\right) \vdots\; +1$$

When $\quad x=0$, $F = -(1-\alpha) + 2\alpha = (-1+3\alpha)$

When $\quad x = \alpha L$

$F = -1 + 3\alpha - 4\alpha^2$

or $\quad = 3\alpha - 4\alpha^2$

When $\quad x = L$, $F = -(1-\alpha) - 2\alpha + 1 = -\alpha$

Fig. 15·14(*e*) shows the S.F.D. for a typical case when α is less than $\frac{1}{3}$.

Example 15·8. *Derive from first principles the bending moment diagram for symmetrical suspension bridge with three pinned stiffening girder of length L subjected to a point load W at a distance a from the central pin. Draw to scale the bending moment diagram for the value of a that gives the largest bending moment under the load.*

Solution.

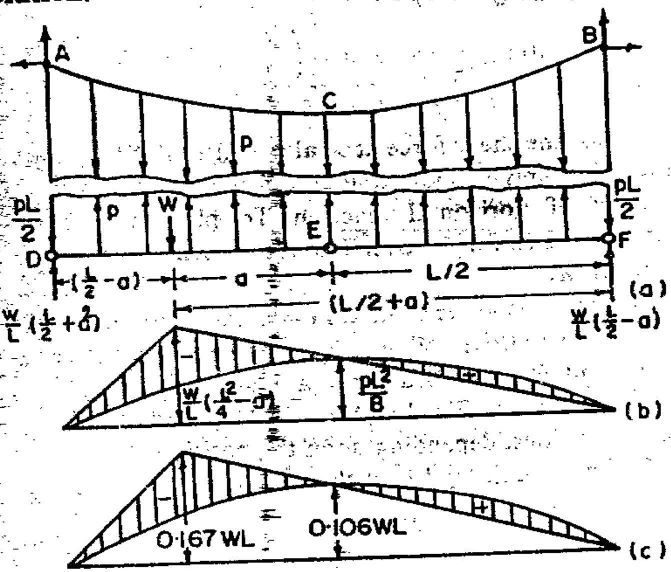


Fig. 15·15.

Fig. 15·15 (*a*) shows the free body diagrams of the cable and the three hinged stiffening girder. For the girder, the simply supported reactions at the ends can be obtained by taking moments about *D* and *F*. Thus,

$$V_D=\frac{W}{L}\left(\frac{L}{2}+a\right)$$

and
$$V_F=\frac{W}{L}\left(\frac{L}{2}-a\right) \qquad (1)$$

Taking moments about the central hinge, we get

$$M_E=0=-\frac{W}{L}\left(\frac{L}{2}-a\right)\frac{L}{2}+\frac{pL}{2}\cdot\frac{L}{2}-p\left(\frac{L}{2}\right)\left(\frac{L}{4}\right)$$

or
$$p=\frac{4W}{L^2}\left(\frac{L}{2}-a\right) \qquad (2)$$

The bonding moment at any point is given by

$$M_x=\mu_x+Hy$$

Since y varies parabolically with x, the $H.y$ diagram will be a parabola having a maximum ordinate of $\frac{pL^2}{8d}\cdot d=\frac{pL^2}{8}$ under the central hinge.

The μx diagram will be a triangle having a maximum ordinate of $\frac{W}{L}\left(\frac{L}{2}-a\right)\left(\frac{L}{2}+a\right)=\frac{W}{L}\left(\frac{L^2}{4}-a^2\right)$ under the load.

The bending moment diagram is shown in Fig. 15·15 (*b*).

The net B.M. under the load is given by

$$M=-\frac{W}{L}\left(\frac{L}{2}+a\right)\left(\frac{L}{2}-a\right)+\frac{pL}{2}\left(\frac{L}{2}-a\right)-\frac{p}{2}\left(\frac{L}{2}-a\right)^2$$

Substituting the value of p from (2), we get

$$M=-\frac{W}{L}\left(\frac{L^2}{4}-a^2\right)+\frac{2W}{L}\left(\frac{L}{2}-a\right)^2-\frac{2W}{L^2}\left(\frac{L}{2}-a\right)^3$$

For maxima, differentiate M with respect to a.

$$\therefore \quad \frac{dM}{da}=0=-\frac{2Wa}{L}+\frac{4W}{L}\left(\frac{L}{2}-a\right)(-1)-\frac{6W}{L^2}\left(\frac{L}{2}-a\right)^2(-1)$$

or
$$a-2\left(\frac{L}{2}-a\right)+\frac{3}{L}\left(\frac{L^2}{4}+a^2-aL\right)=0$$

or
$$3a^2=\frac{L^2}{4}$$

$$\therefore \quad a=\pm 0{\cdot}288\ \mathbf{L}.$$

The bending moment diagram for this value of a is shown in Fig. 15·15 (*c*), in which the ordinate under load

$$=\frac{W}{L}\left\{\frac{L^2}{4}-(0{\cdot}288\ L)^2\right\}=0{\cdot}167\ \mathbf{WL},$$

and the ordinate under the central hinge

$$=\frac{4W}{L^2}\left(\frac{L}{2}-0{\cdot}288\ L\right)\frac{L^2}{8}=0{\cdot}106\ \mathbf{WL}.$$

Example 15·9. *The three hinged stiffening girder of a suspension bridge of 100 m span subjected to two point loads of 10 kN each placed at 20 m and 40 m respectively from the left hand hinge. Determine the B.M. and S.F. in the girder at section 30 m from each end. Also, determine the maximum tension in the cable which has a central dip of 10 m.*

Solution

To find the simply supported reactions V_D and V_F, take moments about hinges *D* and *F*. Consider the equilibrium of the girder alone.

Thus,

$$V_F=\frac{1}{100}\left\{(10\times40)+(10\times20)\right\}=6 \text{ kN}$$

$$V_D=\frac{1}{100}\left\{(10\times60)+(10\times80)\right\}=14 \text{ kN}$$

In order to find H, take moments about the central hinge E.

Thus

$$M_E=0=\mu_E+Hd$$

where $\mu_E=(-6\times50)$, and $d=10$ m

$\therefore$ $M_E=(-6\times50)+10H=0$

or $H=30$ kN.

The equation of the parabola is

$$y=\frac{4d}{L^2}x(L-x)=\frac{4\times10}{100\times100}x(100-x)$$

$$y=\frac{4x}{1000}(100-x)$$

At $x=30$ $\quad y=\frac{4\times30}{1000}(100-30)=8\cdot4$ m

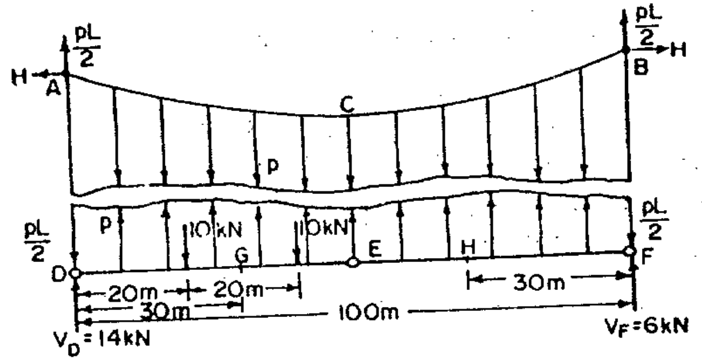


Fig. 15·16

Thus at points G and H, distant 30 m from D and F respectively, $y=8\cdot4$ m.

The B.M. at any point is given by

$$M=\mu_X+Hy$$

$$\therefore\quad M_G=\{(-14\times30)+(10\times10)\}+(30\times8\cdot4)$$

$$=-68 \text{ kN-m}$$

and $\quad M_H=(-6\times30)+(30\times8\cdot4)=+72$ kN-m

The S.F. at any point is given by

$$F_X=f_X+H\tan\theta$$

Now $\quad y=\frac{4x}{1000}(100-x)=\frac{4}{10}x-\frac{4x^2}{1000}$

$$\therefore\quad \frac{dy}{dx}=\tan\theta=\frac{4}{10}-\frac{8x}{1000}$$

At $x=30$ m, $\tan\theta=\frac{4}{10}-\frac{8\times30}{1000}=0\cdot16$

Similarly, at $x=70$ m (*i.e.* 30 m from R.H.),

$\tan\theta=-0\cdot16$ (*i.e.* anticlockwise)

Hence $\quad F_G=(-14+10)+(30\times0\cdot16)=+0\cdot8$ kN

$F_H=(+6)+(30)(-0\cdot16)=+1\cdot2$ kN

Now $\quad H=\frac{pL^2}{8d}$

$$\therefore\quad p=\frac{8d}{L^2}\cdot H=\frac{8\times10}{100\times100}\times30=0\cdot24 \text{ kN/m}$$

Vertical reactions at the ends of the cables

$$=\frac{pL}{2}=\frac{0\cdot24\times100}{2}=12 \text{ kN}$$

$\therefore$ Maximum tension in the cable

$$=\sqrt{(12)^2+(30)^2}=32\cdot4 \text{ kN.}$$

Example 15·10. *A suspension bridge of 250 m span has two three hinged stiffening girders supported by two cables having a central dip of 25 m. The width of the roadway is 8 m. The roadway carries a dead load of ½ kN per sq. metre extending over the whole span, and a live load of 1 kN per sq. metre extending over the left hand half of the bridge. Find the B.M. and S.F. at point 60 m and 200 m from the left hinge. Also, calculate the maximum tension in the cable.*

Solution

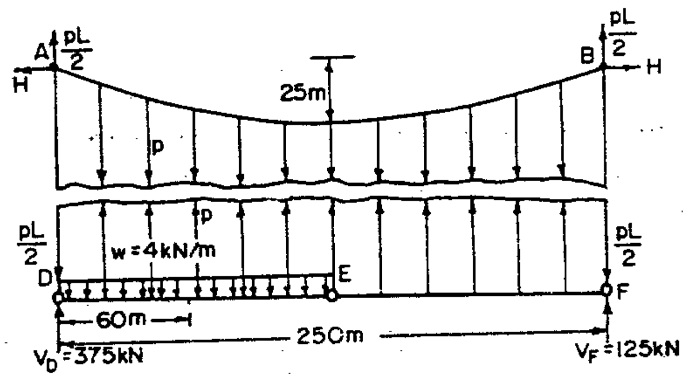


Fig. 15·17

The free body diagrams for the cable and the stiffening girder are shown in Fig. 15·17. Since there will be no B.M. and S.F. any where in the girder due to uniformly distributed dead load, only live load has been shown.

Live load per metre run by girder

$$=w=\tfrac{1}{2}[8\times 1\times 1]=4 \text{ kN/m}$$

$$V_F=\frac{1}{250}(4\times 125\times 62{\cdot}5)=125 \text{ kN}$$

$$\therefore \quad V_D=(4\times 125)-125=375 \text{ kN}$$

$$M_E=3=\mu_E+Hy$$

$$(-125\times 125)+25H=0$$

or $$H=\frac{125\times 125}{25}=625 \text{ kN}$$

The equation of the cable is

$$y=\frac{4d}{L^2}\,x(L-x)=\frac{4\times 25}{250\times 250}\,x(250-x)$$

or $$x=\frac{x}{625}(250-x)$$

$$\therefore \quad \frac{dy}{dx}=\tan\theta=\frac{250-2x}{625}$$

At $x=60$ m, $y=\dfrac{60}{625}(250-60)=18{\cdot}25$

and $$\tan\theta=\frac{250-120}{625}=0{\cdot}208$$

At $x=200$, $y=\dfrac{200}{625}(250-200)=16$

and $$\tan\theta=\frac{250-400}{625}=-0{\cdot}24$$

$$\therefore \; M_{60}=\mu+Hy=\left\{(-375\times 60)+\left(\frac{4\times 60\times 60}{2}\right)\right\}+(625\times 18{\cdot}25)$$

$$=-3894 \text{ kN-m}$$

and $$M_{200}=(-125\times 50)+(625\times 16)=+3750 \text{ kN-m}$$

$$F_{60}=f_{60}+H\tan\theta$$

$$=(-375+60\times 4)+(625\times 0{\cdot}208)=-5 \text{ kN}$$

$$F_{200}=(+125)+(625)(-0{\cdot}24)=-25 \text{ kN.}$$

Again, the equivalent U.D.L. transferred to the cable due to the live load

$$=\frac{8d}{L^2}H=\frac{8\times 25}{240\times 250}\times 625=2 \text{ kN/m}$$

Equivalent U.D.L. transferred to the cable due to dead load, per metre run

$$=\tfrac{1}{2}[\tfrac{1}{2}\times 8\times 1]=2 \text{ kN/m}$$

$\therefore$ Total $p=2+2=4$ kN/m

Maximum tension in the cable is given by

$$T=\frac{pL}{2}\sqrt{1+\frac{L^2}{16d^2}}$$

$$=\frac{4\times 250}{2}\sqrt{1+\frac{250\times 200}{16\times 25\times 25}}=1345 \text{ kN}$$

Example 15·11. *A suspension bridge cable of span 80 m and central dip 8 m is suspended from the same level at two towers. The bridge cable is stiffened by a three hinged stiffening girder which carries a single concentrated load of 10 kN at a point 20 m from one end. Sketch the S.F. diagram for the girder.*

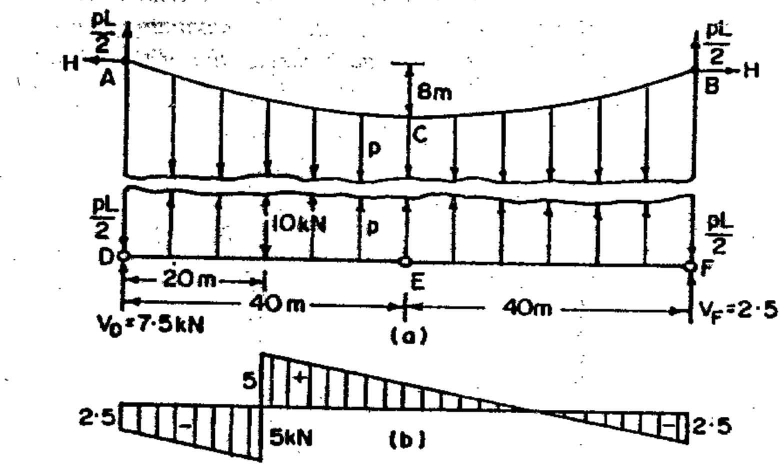


Fig. 15·18

Solution

The simply supported reactions are given by

$$V_D=\frac{1}{80}\left[10\times 60\right]=7{\cdot}5 \text{ kN}$$

$$V_F=\frac{1}{80}\left[10\times 20\right]=2{\cdot}5 \text{ kN}$$

To find the value of p consider the equilibrium of the stiffening girder and find the bending moment about E.

$$M_E=0=\left(-2{\cdot}5\times 40\right)-\left(p\times 40\times 20\right)+\left(\frac{p\times 80}{2}\times 40\right)$$

$$\therefore \quad p=\frac{100}{800}=0{\cdot}125 \text{ kN/m}$$

$$\therefore \quad \frac{pL}{2}=0{\cdot}125\times\frac{80}{2}=5 \text{ kN}$$

At any section distant x from D, the S.F. is given by

$$F_x=-7{\cdot}5\times5-0{\cdot}125\,x\,\vdots\,+10$$

At $x=0$, $\quad F_x=-7{\cdot}5+5=-2{\cdot}5$

At $x=20$, $\quad F_x=-7{\cdot}5+5-(0{\cdot}125\times20)\,\vdots\,+10$

$$=-5 \text{ or } +5$$

At $x=80$, $\quad F_x=-7{\cdot}5+5-0{\cdot}125\times80+10=-2{\cdot}5$

The complete S.F. diagram is shown in Fig. 15·18(*b*).

Example 15·12. *A suspension bridge cable hangs between two points A and B separated horizontally by 120 m and with B 20 m above A. The lowest point in the cable is 4 below A. The cable supports a stiffening girder weighing ⅓ kN/m run which is hinged vertically below the lowest point of the cable. Calculate the maximum tension A, B and the lowest point of the cable. Calculate the maximum tension which occurs in the cable when a 10 kN load crosses the girder from A to B.*

Solution

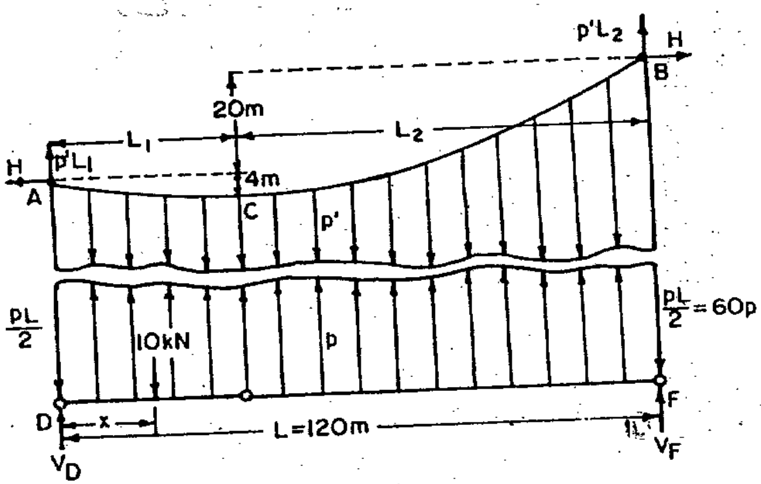


Fig. 15·19

Given : $\quad d_1=4$ m ; $d_2=20+4=24$ m ; $L=120$ m

$$\frac{L_1}{L_2}=\sqrt{\frac{d_1}{d_2}}=\sqrt{\frac{4}{24}}=0{\cdot}408 \qquad \ldots(i)$$

$$L_1+L_2=120 \text{ m}$$

From (*i*) and (*ii*)

$$L_1=35 \text{ m} \,;\; L_2=85 \text{ m}$$

Let the 10 kN load be at a distance of x from D.

Then the simply supported reactions due to the live load of 10 kN are :

$$V_D=\frac{1}{120}\left(120-x\right)10=\left(10-\frac{x}{12}\right)$$

$$V_F=\frac{1}{120}\left(10\,x\right)=\frac{x}{12}$$

In order to find the value of p, take moments about E, of all forces to the right of it. Thus

$$M_E=0=\left(-\frac{x}{12}\times85\right)-\frac{p}{2}\left(85\right)^2+\left(60p\right)\left(75\right)$$

or $$17{\cdot}5p-\frac{x}{12}=0 \qquad (iii)$$

The above equation gives the relationship between the load position and the value of p. The maximum tension in the cable depends upon the maximum value of p. The maximum value of p will evidently occur when the load is on the central hinge, *i.e.*, when $x=35$ m. Thus, from (*iii*), we get

$$p=\frac{x}{12\times17{\cdot}5}=\frac{35}{12\times17{\cdot}5}=0{\cdot}167 \text{ kN/m}$$

Also, the U.D.L. transferred to the hangers due to dead load $=\frac{1}{3}$ kN/m $=0{\cdot}333$ kN/m

$\therefore$ Total U.D.L. $=p'=0{\cdot}167+0{\cdot}333=0{\cdot}5$ kN/m.

$\therefore$ The maximum tension will occur at B.

To find V_B, take moments about A

$$V_B=\frac{1}{L}\left[\frac{p'L^2}{2}+H(d_2-d_1)\right]$$

or $$V_B=\frac{1}{L}\left[\frac{p'L^2}{2}+\frac{p'L_2^2}{2d_2}(d_2-d_1)\right]$$

From which $V_B=p'L_2$

Similarly, $\quad V_A=p'L_1$

$$\therefore \quad T_B=\sqrt{H^2+(p'L_2)^2}$$

But $$H=\frac{p'L_2^2}{2d_2} \quad \text{[Eq. 15·12](a)}$$

$$\therefore \quad T_B=p'L_2\sqrt{\frac{L_2^2}{4d_2^2}+1}$$

$$=(0{\cdot}5\times85)\sqrt{\frac{85\times85}{4\times24\times24}+1}=\mathbf{86{\cdot}5 \text{ kN}}$$

Example 15·13. *A suspension cable, stiffened with a three hinged girder, has 100 m span and 10 m dip. The girder carries a load of 0·4 kN/m. A live load of 10 kN rolls from left to right. Determine (i) the maximum B.M. anywhere in the girder, (ii) the maximum tension in the cable.*

Solution

(*a*) *Maximum B.M.*

From Fig. 15·13, the absolute maximum negative B.M., due to point load, in the girder occurs at $x=0{\cdot}211\ L=0{\cdot}211\times100=21{\cdot}1$ m from either end, is value being :

$(-)\ M_{max\cdot max}=0{\cdot}096\ WL=0{\cdot}096\times10\times100=$ **96 kN-m**

The absolute maximum positive B.M. occurs at $x=0{\cdot}25\ L$ $=25$ m from either end, its value being

$$(+)M_{max\cdot max}=0{\cdot}0625\ WL=0{\cdot}0625\times10\times100$$
$$=\mathbf{62{\cdot}5\ kN\text{-}m}$$

$\therefore$ Greatest B.M. $=-$**96 kN-M** at 21·1 m from either ends. It should be noted that there will be no B.M. anywhere in the girder due to U.D.L. covering the whole span.

(*b*) *Maximum cable tension*

From Fig. 15·13 (*c*), the maximum value of *H*, due to point load, occurs when the load is on the central hinge, its value being

$$H=W\ \frac{L}{4d}=\frac{10\times100}{4\times10}=\mathbf{25\ kN}$$

p (due to U.D.D.L.)$=0{\cdot}4$ **kN-m.**

$$H \text{ due to U.D.D.L.}=\frac{pL^2}{8d}=\frac{0{\cdot}4\times100\times100}{8\times10}=\mathbf{50\ kN}$$

$\therefore$ Total $H=25+50=$ **75 kN**

$$T_{max}=\sqrt{V^2+H^2}$$
$$=H\sqrt{1+\frac{16d^2}{L^2}}$$
$$=75\sqrt{1+\frac{16\times100}{100\times100}}=\mathbf{76\ kN.}$$

Example 15·14. *A suspension bridge cable hangs between two points A and B separated horizontally by 120 m and with A 20 m above B. The lowest point in the cables is 4 m below B. The cable supports a stiffening girder which is hinged vertically below A, B and the lowest point in the cable. Find the position and magnitude of the*

largest bending moment which a point load of 10 kN can induce in the girder together with the position of the load.

Solution

Given $d_1=20+4=24$ m ; $d_2=4$ m ; $L=120$ m ;

$$\therefore\quad \frac{L_1}{L_2}=\sqrt{\frac{d_1}{d_2}}=\sqrt{\frac{24}{4}}=2{\cdot}45 \qquad (1)$$

$$L_1+L_2=120\text{ m} \qquad (2)$$

$$\therefore\quad L_1=85\text{ m};\ L_2=35\text{ m}$$

Let the point load *W* be at a distance αL from *D*. The simply supported reactions are

$$V_D=W\ (1-\alpha);\ V_E=W\alpha$$

The bending moment at any point is given by

$$M=\mu+Hy.$$

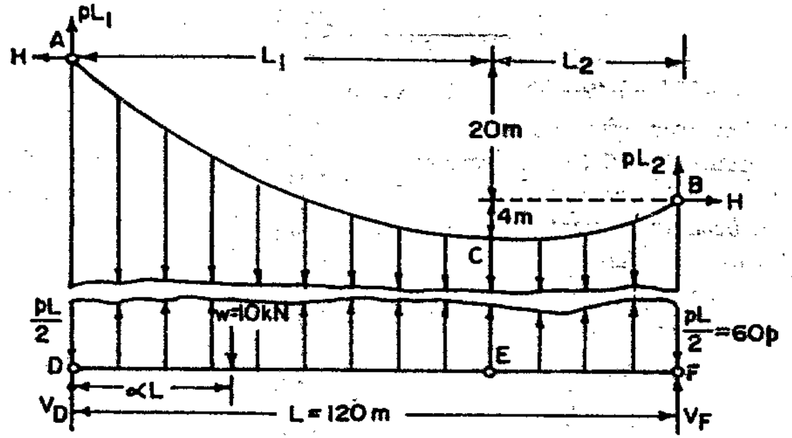


Fig. 15·20.

By inspection, the maximum B.M. occurs under the point when the load is between *D* and *E*. Let us first find *p*.

$$M_B=0=-W\alpha\ (35)+\frac{p\times120}{2}\times35-\frac{p}{2}\ (35)^2$$

or
$$p=\frac{W\alpha}{42{\cdot}5} \qquad (1)$$

The maximum B.M. under the load is given by

$$M=-W(1-\alpha)\ (\alpha L)+\frac{pL}{2}\ (\alpha L)-\frac{p}{2}\ (\alpha L)^2$$

Susbtituting the value of *p*, we get

$$M=-W\alpha L\ \ (1-\alpha)+\frac{W\alpha^2L^2}{85}-\frac{W\alpha^3L^2}{85} \qquad (2)$$

For maxima, $\frac{dM}{d\alpha}=0$

$$-WL+2W\alpha L+\frac{2W\alpha L^2}{85}-\frac{3W\alpha^2L^2}{85}=0$$

Cancelling W and substituting the value of L, we get

$$-1+2\alpha+\frac{2\times120\alpha}{85}-\frac{3\times120\alpha^2}{85}=0$$

or $\alpha^2-1.14\alpha+0.236=0$

∴ $\alpha=0.18$; $\alpha L=0.18\times120=21.6$ m

Substituting the value of W, α and L in (2), we get

$$M_{max}=-10\times0.18\times120(1-0.18)+10\frac{(0.18\times120)^2}{85}$$

$$-\frac{10\times0.18(0.18\times120)^2}{85}$$

$$=-177+54.9-9.9=\mathbf{-132\ kN\text{-}m}$$

at 21·6 m from left support.

Example 15·15. *A symmetrical three pinned stiffening girder of a suspension bridge is 600 ft. long. Find the magnitude of the largest bending moment that can be exerted by a moving load 20 tons uniformly distributed over a length of 30 ft. Indicate the position of the load for this condition.*

Solution

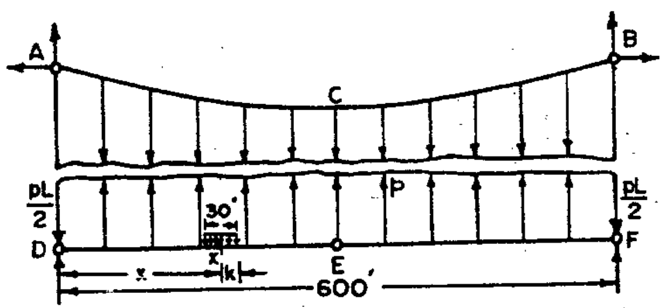


Fig. 15·21

Let the absolute maximum B.M. occur at a point X distant x from D, when the head of the load is ahead of it by a distance k.

Keeping the position of the section fixed, let us first find out the value of k to get maximum B.M. at X.

The simply supported reactions are :

$$V_F=\frac{20}{600}(x+k-15)\text{ and }V_D=20-V_F$$

To find p, take moments about E. Thus

$$M_E=0=\left\{-\frac{20}{600}\left(x+k-15\right)\times300\right\}+\left\{p\times300\times300\right\}-\frac{p}{2}\left(300\right)^2$$

$$\therefore\quad p=\frac{x+k-15}{4500}\qquad(1)$$

The bending moment at the fixed section is given by

$$Mx=-V_D.x+\frac{pL}{2}x-\frac{px^2}{2}+\frac{20}{30}\frac{(30-k)^2}{2}$$

$$=-\left\{20x-\frac{x}{30}\left(x+k-15\right)\right\}+\left\{\frac{x+k-15}{4500}\times300x\right\}$$

$$-\left\{\frac{x+k-15}{4500}\cdot\frac{x^2}{2}\right\}+\left\{\frac{(30-k)^2}{3}\right\}\qquad(2)$$

For Maxima, $\frac{dMx}{dk}=0$

$$\therefore\quad 0=+\frac{x}{30}+\frac{x}{15}-\frac{x^2}{9000}+\frac{2(30-k)(-1)}{3}$$

or $$\frac{x}{10}-\frac{x^2}{9000}-20+\frac{2}{3}k=0$$

$$k=-\frac{3}{20}x+\frac{x^2}{6000}+30\qquad(3)$$

Substituting this value of k in (2), we get

$$Mx=-20x+\frac{x}{30}\left\{x-15+\left(-\frac{3}{20}x+\frac{x^2}{6000}+30\right)\right\}$$

$$+\frac{x}{15}\left\{x-15+\left(-\frac{3}{20}x+\frac{x^2}{6000}+30\right)\right\}$$

$$-\frac{x^2}{9000}\left\{x-15+\left(-\frac{3}{20}x+\frac{x^2}{6000}+30\right)\right\}$$

$$+\frac{1}{3}\left\{30-\left(-\frac{3}{20}x+\frac{x^2}{6000}+30\right)\right\}^2$$

Simplifying and rearranging, we get

$$Mx=-18.5x+\frac{17x^2}{200}+\frac{7x^2}{1200}-\frac{17x^3}{180000}-\frac{x^4}{108,000,00}\qquad(4)$$

For getting the absolute maximum B.M., treat x as variable,

Thus $\frac{Mx}{dx}=0$, for maxima

$$\therefore\quad 0=-18.5+\frac{17x}{100}+\frac{7x}{600}-\frac{17x^2}{60000}-\frac{x^3}{27,000,000}$$

$$0=18.5-\frac{109}{600}x+\frac{17x^2}{60,000}+\frac{x^3}{27,000,000}$$

Using Newton's method, try $x=150$

$$\therefore \quad f(x)=18.5-27.25+6.37+0.12=-2.26$$

$$\therefore \quad f'(x)=-\frac{109}{600}+\frac{17x}{30,000}+\frac{x^2}{9,000,000}$$

$$=-0.18+0.085+\ldots\ldots=-0.1$$

$$\therefore \quad \text{Better value of } x=150-\frac{-2.26}{-0.1}=150-22.6=127.4$$

Try $x=127.4$

$$\therefore \quad f(x)=18.5-\frac{109\times127.4}{600}+\frac{17(127.4)^2}{60,000}+\frac{(127.4)^3}{27,000,000}$$

$$=0.0309$$

$$f'(x)=-0.18+0.072+\ldots\ldots=0.108$$

$$\therefore \quad \text{Better value of } x=127.4-\frac{0.0309}{-0.108}=127.7$$

Use approximate value of $x=128$ **ft.**

Thus the absolute maximum B.M. occurs at a section 128 ft. from the left support. To get the value of absolute maximum B.M. substitute $x=128$ ft in Eq. 4. Thus.

$$M_{max-max}=(-18.5\times128)+\frac{17}{200}(128)^2+\frac{7}{1200}(128)^2$$

$$=\frac{17}{180,000}(128)^3-\frac{(128)^4}{108,000,000}$$

$$=-1081 \text{ t-ft.}$$

This occurs when $k=\frac{3}{20}\times128+\frac{(128)^2}{6000}+30$

$$=13.5 \text{ ft.}$$

Hence absolute maximum bending moment occurs at a section 128 ft. from left support, when the head of the load is at a distance (128+13.5)=141.5 ft. from the support.

15.7. TWO HINGED STIFFENING GIRDER

A two hinged stiffening girder is a statically indeterminate structure, since there are three unknowns to be determined (*i*) the reaction V_D, at the hinged support *D*, (*ii*) the reaction V_F at the hinged support *F*, and (*iii*) the pull *p* exerted by the cables. Only two equations of statical equilibrium are available. In the case of a three hinged girder, an additional equation $M_E=0$ was available at the central hinge. The problem of two hinged girder can be solved approximately by the *strain energy method.* The solution by strain

energy is not within the scope of the book. The problem will therefore, be solved approximately by assuming the girder to be infinitely rigid, so that the pull $p=\frac{W}{L}$, where W is the point load placed anywhere on the girder. On the assumption, the pull in the cable is *constant*, irrespective of the load position, while in the case of a three hinged girder, $p=\frac{4\alpha}{L}$ and depends upon the load position.

(1) INFLUENCE LINE FOR H

For the cable, $H=\frac{pL^2}{8d}$

Consider a unit load placed at distance αL from the left support. Then

$$p=\frac{W}{L}=\frac{1}{L} \qquad (15.35)$$

$$\therefore \quad H=\frac{L^2}{L.8d}=\frac{L}{8d} \qquad (15.36)$$

Thus, the horizontal pull at the abutments of the cable is constant, and does not vary with the load position. The influence lines for *p* and *H* will thus be rectangles, as shown in Fig. 15.22 (*b*) and (*c*) respectively.

(2) BENDING MOMENT DIAGRAM

The bending moment at any section *X*, distant α from *D* is given by

$$M_x=\mu_x+H.y$$

The μ_x-diagram is a triangle having a maximum ordinate of $\frac{\alpha L(L-\alpha L)}{L}=\alpha L(1-\alpha)$ under the unit load.

The *H.y* diagram is a parabola since *y* varies parabolically with *x*, having a maximum value of the ordinate equal to $\frac{L}{8d}\,d=\frac{L}{8}$ at the middle of the span.

At the point *X*, the ordinate of the *H.y* diagram will be

$$\frac{L}{8d}\cdot y=\frac{L}{8D}\cdot\frac{4d}{L^2}\,x(L-x)=\frac{x}{2L}(L-x)$$

The B.M.D. is shown in Fig 15.22(*d*).

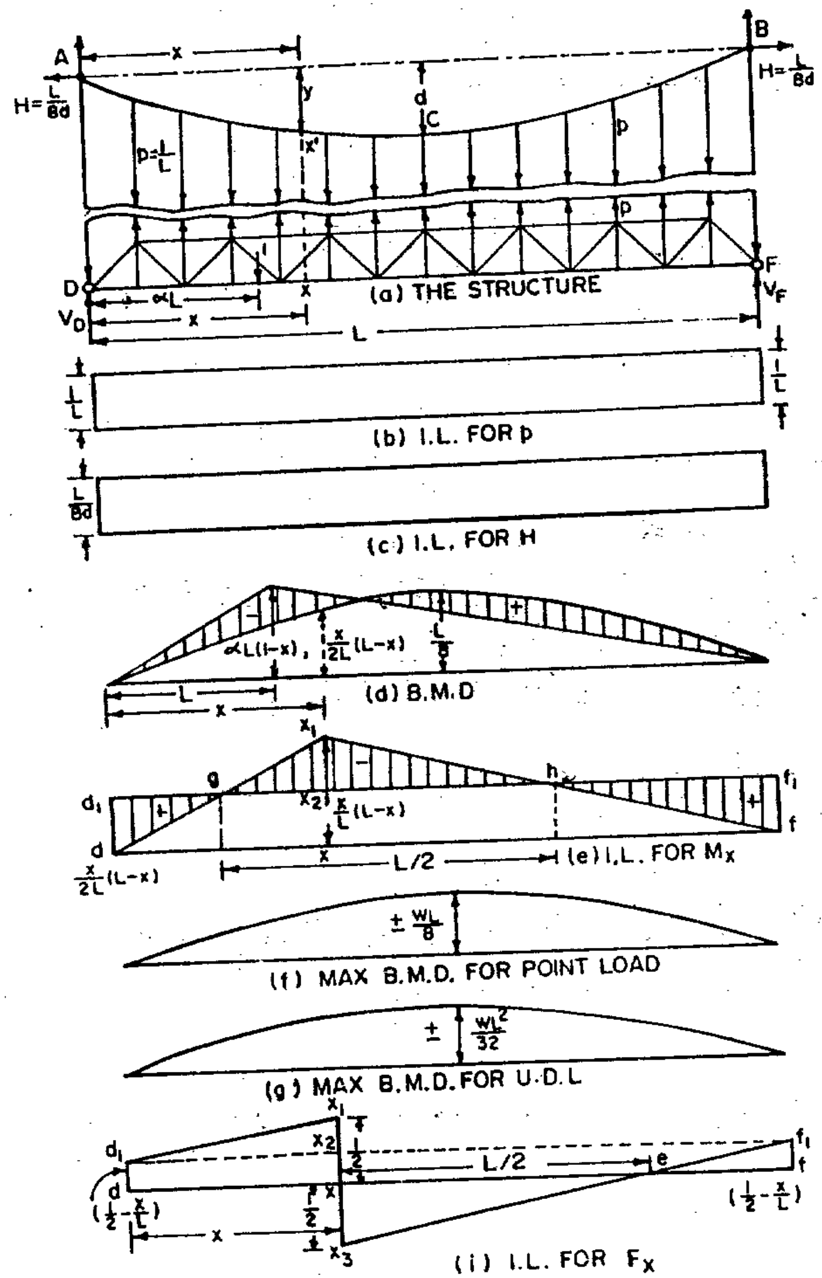


Fig. 15·22
Two hinged stiffening girder.

(3) INFLUENCE LINE FOR BENDING MOMENT

The B.M. at section X distant x from D is given by

$$M_x=\mu_x+H.y$$

Thus the I.L. for B.M. is obtained by superimposing I.L. for μ_x on the I.L. for $H.y$.

The I.L. for μ_x is a triangle having a maximum ordinate of $\frac{x(L-x)}{L}$ under the section.

The value of y is fixed for a given section X. Also the value of H is fixed, and does not vary with the load position. Hence the quantity $H.y$ is fixed, its value being

$$H.y=\frac{L}{8d}\cdot\frac{4d}{L^2}x(L-x)=\frac{x}{2L}(L-x) \qquad (15·37)$$

Thus the I.L. for $H.y$. is a *rectangle* having an ordinate of $\frac{x}{2L}.(L-x)$. Fig. 15·22(*e*) shows the I.L. for bending moment at X.

By the inspection of the Fig. 15·22 (*e*), we have

$$d_1d=f_1f=\tfrac{1}{2}x_1x=x_1x_2$$

$$\therefore \quad d_1g=gx_2 \text{ and } x_2h=hf_1$$

$$\therefore \quad gx_2+x_2h=d_1g+hf_1$$

But $\quad (gx_2+x_2h)+(d_1g+hf_1)=L$

$$\therefore \quad gx_2+x_2h=\frac{L}{2} \qquad (15·38)$$

And area of Δgx_1h=area of $\Delta dd_1g+\Delta ff_1h$

or Total positive area=total negative area.

Thus, in the case of two hinged girder also, there will be no bending moment any where in the girder due to uniformly distributed dead load or live load covering the whole span.

(4) MAXIMUM BENDING MOMENT DIAGRAM DUE TO SINGLE POINT LOAD W

When a point load W rolls over the girder, the maximum +ve B.M. at X will occur [Fig. 15·22(*e*)] when the load is either on the left or right hinge, and its value is equal to $+\frac{Wx}{2L}(L-x)$.

The maximum negative B.M. at X will occur when the load is on the section itself, its value being equal to $-\frac{Wx}{2L}(L-x)$.

$$\therefore \quad (\pm)M_{max}=\frac{Wx}{2L}(L-x) \qquad (15·39)$$

The variation is thus parabolic. Fig. 15·22 (*f*) shows the max B.M.D.

For absolute max. B.M.,

$$\frac{dM_{max}}{dx}=\frac{W}{2}-\frac{Wx}{L}=0$$

$$\therefore \quad x=\frac{L}{2}$$

$$\therefore \quad \pm M_{max.max}=\frac{W}{2L}\cdot\frac{L}{2}\cdot\frac{L}{2}=\frac{WL}{8}. \qquad (15·40)$$

(5) MAXIMUM BENDING MOMENT DIAGRAM DUE TO UNIFORMLY DISTRIBUTED LOAD

By inspection of Fig. 15·22(*e*), the maximum negative B.M. at *X* will occur when only portion *gh* is loaded, while the maximum positive B.M. at *X* will occur when the portion d_1g and hf_1 are loaded keeping portion *gh* empty. In either case, the maximum B.M. is given by

$$(\pm)M_{max}=\frac{w}{2}\cdot\frac{L}{2}\cdot\frac{x}{2L}(L-x)=\frac{wx}{8}(L-x) \qquad (15·41)$$

The variation is parabolic. Fig. 15·22(*g*) shows the maximum B.M.D. The absolute maximum B.M. will evidently occur at $x=\frac{L}{2}$.

$$\therefore \quad \pm M_{max.max}=\frac{w}{8}\cdot\frac{L}{2}\cdot\frac{L}{2}=\frac{wL^2}{32} \qquad (15·42)$$

(6) INFLUENCE LINE FOR SHEAR FORCE

The S.F. at any section *X*, distant *x* from the left hinge is given by

$$F_x=f_x+H\tan\theta$$

where f_x=Shear force at *X* due to simply supported actions.

The I.L. for f_x will have zero ordinates at the ends, and ordinates of $\frac{x}{L}$ and $\frac{L-x}{L}$ and the section.

The I.L. for $H\tan\theta$, in which both *H* and $\tan\theta$ are constant, will be rectangle having the ordinate :

$$H\tan\theta=\frac{L}{8d}\cdot\frac{8d}{L^2}\left(\frac{L}{2}-x\right)=\left(\frac{1}{2}-\frac{x}{L}\right) \qquad (15·43)$$

Thus, the I.L. for F_x will be given by superimposing the I.D. for F_x over the I.L. for $H\tan\theta$, as shown in Fig. 15·22 (*i*). From Fig. 15·22(*i*), we have :

$$dd_1=ff_1=\left(\frac{1}{2}-\frac{x}{L}\right);\ x_1x_2=\frac{x}{L};$$

$$x_2x_3=\left(1-\frac{x}{L}\right);\ xx_2=\left(\frac{1}{2}-\frac{x}{L}\right)$$

$$\therefore \quad xx_1=xx_2-x_2x_1=\left(\frac{1}{2}-\frac{x}{L}\right)+\frac{x}{L}=\frac{1}{2}$$

$$xx_3=x_2x_3-xx_2=\left(1-\frac{x}{L}\right)-\left(\frac{1}{2}-\frac{x}{L}\right)=\frac{1}{2}.$$

Now $\frac{xe}{ef}=\frac{xx_3}{ff_1}=\frac{\frac{1}{2}}{\frac{1}{2}-\frac{x}{L}}=\frac{1}{2\left(\frac{1}{2}-\frac{x}{L}\right)}$

$$xe+ef=(L-x)$$

$$\therefore \quad \frac{ef}{2\left(\frac{1}{2}-\frac{x}{L}\right)}+ef=(L-x)$$

From which $ef=\left(\frac{L}{x}-x\right)$

$$\therefore \quad xe=(L-x)-\left(\frac{L}{2}-x\right)=\frac{L}{2}$$

$$\therefore \quad \text{Area } xx_3e=\frac{1}{2}\cdot\frac{L}{2}\cdot\frac{1}{2}=\frac{L}{8}$$

Also, Area (dd_1x_1x)+area $(eff_1)=\frac{1}{2}x\left[\frac{1}{2}-\frac{x}{L}+\frac{1}{2}\right]$

$$+\left[\frac{1}{2}\left(\frac{L}{2}-x\right)\left(\frac{1}{2}-\frac{x}{L}\right)\right]$$

$$=\frac{L}{8}.$$

Hence the area of positive S.F. is equal to the area of negative shear force. It can, therefore, be concluded that there will be no S.F. anywhere in the girder due to uniformly distributed dead and/or live load covering the whole span.

15·8. TEMPERATURE STRESS IN TWO HINGED GIRDER

In the case of a cable subjected to a change of temperature *t*, the change in the dip *d* is given by Eq. 15·18.

$$\delta d=\frac{3}{16}\cdot\frac{L^2}{d}\,\alpha t \qquad (1)$$

The horizontal pull *H* and the U.D.L. *p* carried by the suspenders depend directly or indirectly on the value of the dip *d*.

Let $\quad \delta p$=change in pull *p*, due to change in δd.

Due to this change in δp, the change in the maximum deflection at the centre of the girder is given by

$$\delta(\Delta)=\frac{5}{384}\frac{\delta p.L^4}{EI}$$

where Δ=deflection at the centre of the girder.

EI=flexural rigidity of the girder.

Now, when dip d increases (*i.e.* when the cable sags), the suspenders become loose. The suspenders can remain taut only if the girder sags by an equal amount at every point. Thus, we have the compatibility equation.

$$\delta(\Delta)=\delta d$$

$$\therefore \quad \frac{5}{384}\frac{\delta p.L^4}{EI}=\frac{3}{16}\cdot\frac{L^2}{d}\alpha.t$$

or

$$\delta p=\frac{72}{5}\frac{EI\alpha t}{L^2 d} \qquad (15{\cdot}43)$$

$\therefore$ Increase in B.M. at the girder$=\dfrac{\delta p.L^2}{8}$

Increase in the stress in the girder$=\dfrac{\delta p.L^2}{8}\cdot\dfrac{D}{2I}$

$$=\frac{72}{5}\frac{EI\alpha t}{L^2 d}\cdot\frac{L^2 D}{16I}$$

$$=\frac{9}{10}\cdot\frac{D}{d}\cdot E\alpha t \qquad (15{\cdot}45)$$

where D=height of the girder.

Thus, the change in the stress is independent of the span and the moment of inertia, and depends on $\dfrac{D}{d}$ ratio.

Example 15·16. *A suspension bridge with two hinged stiffening girder has a span of 100 m and the cables have a central dip of 10 m. The stiffening girders are 4 m deep, and have moment of inertia equal to $1{\cdot}64\times10^{10}$ mm^4. If the temperature falls through 22 Kelvin, calculate (i) flange stress. (ii) increase in the tension in the cable. Take $E=2\times10^5$ N/mm^2 and $\alpha=11\times10^{-6}$ per 1 K.*

Solution

From Eq. 15·45. stress in the flanges of the stiffening girder due to fall of temperature

$$=\frac{9}{10}\cdot\frac{D}{d}\,E\alpha t$$

$$=\frac{9}{10}\times\frac{4}{10}\times2\times10^5(11\times10^{-6})\times22$$

$$=17{\cdot}42\text{ N-mm}^2=17{\cdot}42\times10^3\text{ kN/mm}^2$$

The increase in load δp on the girder is given by

$$\delta p=\frac{72}{5}\;EI\;\frac{\alpha t}{L^2 d}$$

where $EI=(2\times10^5)(1{\cdot}64\times10^{10})=3{\cdot}28\times10^{15}$ N-mm^2

$=3{\cdot}28\times10^6$ kN-m^2

$$\therefore \quad \delta p=\frac{72}{5}(3{\cdot}28\times10^6)\frac{11\times10^{-6}\times22}{100\times100\times10}=0{\cdot}1143\text{ kN-m}$$

$$\delta H=\frac{\delta p\ L^2}{8d}=\frac{0{\cdot}1143\ (100)^2}{8\times10}=14{\cdot}29\text{ kN}$$

$\therefore$ Change in the cable tension is

$$\delta T=\delta H\sqrt{1+\left(\frac{4d}{L}\right)^2}$$

$$=14{\cdot}29\sqrt{1+\left(\frac{4\times10}{100}\right)^2}=\mathbf{15{\cdot}39\ kN}$$

Example 15·17: *A suspension cable has a span of 160 m and a central dip of 16 m, and is suspended from the same level at both towers. The bridge is stiffened by a stiffening girder hinged at the end supports. The girder carries a single concentrated load of 8 kN at a point 40 m from left end. Assuming equal tensions in the suspension hangers, calculate (i) the horizontal tension in the cable and (ii) the maximum positive and negative bending moments.*

If the 8 kN load rolls from left to right, what will be the value of absolute maximum B.M. and S.F. and where do they occur ?

Solution. (Fig. 15·22)

(*i*) Load per metre run in the hangers$=p=\dfrac{W}{L}$

$$=\frac{8}{160}=0{\cdot}05\text{ kN/m}$$

Horizontal tension in the cable$=\dfrac{pL^2}{8d}$

$$=\frac{0{\cdot}05\times160\times160}{8\times16}=\mathbf{10\ kN}$$

(*ii*) The simply supported reactions are

$$V_F=\frac{8\times40}{160}=2\text{ kN}\ ;\ V_D=8-2=6\text{ kN}$$

The maximum negative B.M. will occur under the load,

$$\therefore \quad (-)M_{max.}=(-V_D\times 40)+\left(\frac{pL}{2}\times 40\right)-\frac{p}{2}(40)^2$$

$$=-240+\frac{0{\cdot}05\times 160\times 40}{2}-\frac{0{\cdot}05}{2}\times 1600$$

$$=\mathbf{-120\ kN\text{-}m}$$

The maximum positive B.M. will occur in the portion to the right of the load. Measuring x from F, we have

$$M=-V_F.x+\frac{pL}{2}\,x-\frac{p}{2}x^2$$

$$=-2x+\frac{0{\cdot}05\times 160}{2}\,x-\frac{0{\cdot}05}{2}x^2$$

$$=2x-\frac{0{\cdot}05}{2}x^2$$

For maxima, $\frac{dM}{dx}=0=2-0{\cdot}05\,x$

$$\therefore \quad x=\frac{2}{0{\cdot}05}=40 \text{ m}$$

$$\therefore \quad (+)M_{max.}=2x-\frac{0{\cdot}05}{2}x^2=(2\times 40)+\frac{0{\cdot}05}{2}(40)^2$$

$$=\mathbf{+40\ kN\text{-}m}$$

(*iii*) Absolute maximum ($\pm$) B.M. occurs at the mid span.

$$(\pm)M_{max\text{-}max}=\frac{WL}{8}=\frac{8(160)}{8}=\mathbf{160\ kN\text{-}m}$$

(*iv*) From the I.L. for S.F. (Fig. 15·22), it is clear that the absolute maximum S.F. occurs under the load, its value being equal half the load $=\frac{1}{2}\times 8=$ **4 kN**, irrespective of the position of the load.

PROBLEMS

1. A cable is suspended between two points which are at the same level 120 m apart horizontally. The cable carries uniform load of 15 N per horizontal metre, and two concentrated loads, one of 900 N at 40 m horizontally from one end and the other of 300 N at 40 m horizontally from the other end. Determine the horizontal distance from one end to the lowest point and maximum tension in the cable.

2. The cables of a suspension bridge of 100 m span are suspended from piers which are 12 metres and 6 metres respectively above the lowest point of the cable. The load carried by each cable is 1 kN/m of span. Find (*i*) the length of the cable between the piers, (*ii*) the horizontal pull in the cable, (*iii*) tension in the cable at the piers, (*iv*) the pressure on the piers assuming that the cable pass over smooth pulleys fixed to the top of the piers and that the back stay at the lower pier makes an angle of 60° with the vertical and that the higher pier makes an angle of 45° with the vertical.

3. A suspension cable has a span of 400 ft measured horizontally and the level of the left-hand end A is 8 ft below the level of the right-hand end B. A load of 12 tons is carried by the cable at a point C which is 150 ft horizontally from B and 21 ft below the level of A. The weight of cable may be taken as 3 lb/ft of horizontal distance. Determine the horizontal and vertical reactions at the ends and the maximum tension in the cable.

4. A suspension cable of 160 m span and 16 m central dip carries a load of $\frac{1}{2}$ kN per lineal horizontal metre, calculate the maximum and minimum tension in the cable. Find the horizontal and vertical forces in each pier under the following alternative conditions :

(*i*) if the cable passes over frictionless rollers on the top of the piers.

(*ii*) if the cable is firmly clamped to saddles carried on frictionless rollers on the tops of the piers. In each case the backstay is inclined at 30° with the horizontal.

5. An unstiffened suspension cable carries a total load of 40 kN uniformly distributed over a span of 160 m. The suspension and anchor cables are attached to saddles free to move horizontally on the piers, one saddle being 12 m and the other 20 m above the lowest point on the cable. The anchor cables are inclined at 45° to the vertical and their weight may be neglected. Determine the greatest and the least tension in the suspension cables, the greatest thrust on a pier and the tension in an anchor cable.

6. A suspension cable of span L, having in the shape of a symmetrical parabola is strengthened by a stiffening girder pinned at the abutments and at the centre. Show that for the passage across the bridge of a uniformly distributed load longer than the span the maximum $\pm$ shearing force occurs at the abutments. Find the magnitude of these forces and of the maximum $\pm$shearing force at the centre of the span and state the corresponding loading conditions in all cases. (*U.L.*)

7. The roadway for a suspension bridge of span $2L$ is stiffened by longitudinal girders of length L, pin-joined together at the centre of the span and hinged at their outer ends the abutments.

Their girders are supported by a large number of vertical tie rods attached to suspending chains, the lengths of tie rods being such that each chain is the form of a parabola with the axis vertical.

Show that the greatest hogging and sagging bending moments set up in the bridge by concentrated load W advancing across the bridge are $\frac{WL}{8}$ and $\frac{WL}{3\sqrt{3}}$. (*Cambridge*)

8. The cable of a suspension bridge have a span of 160 m and a central dip of 20 m. Each cable is stiffened by girder hinged at the ends at midspan to constrain the cable to retain its parabolic shape. There is a uniform dead load of 1/4 kN per horizontal metre of span over the whole of the girder and in addition a load of 3/4 kN per horizontal metre and 40 m long.

Determine the maximum cable tension when the live load is situated on the left hand of the stiffening girder with its right hand end over the central hinge. Sketch the S.F. and B.M. diagrams for the girder showing on them the maximum positive and negative values.

9. The towers of a 500 ft span suspension bridge are of unequal height, one is 60 ft and the other 20 ft above the lowest point of the cable, which is immediately above the inner pin of a three-pinned stiffening girder hinged at the towers. Find the maximum tension in the cable due to a point load of W crossing the bridge. *(U.L.)*

10. A suspension bride cable hangs between two points A and B separated horizontally by 300 ft and with B 50 ft above A. The lowest point in the cable is 10 ft below A. The cable supports a stiffening girder weighing 1/4 t/ft run which is hinged vertically below A, B and the lowest point of the cable. Calculate the maximum tension which occurs in the cable when a 20 t load crossess the girder from A to B. *(St. Andrews)*

11. The towers of a suspension bridge with a three pinned stiffening girder are 15 m and 10 m high respectively and are 50 m apart. The cable dips 4 m below the top of the 10 m tower and its lowest point is immediately above the pin in the stiffening girder. Find the position and magnitude of the largest bending moment which a point load of 4 kN can induce in the girder together with position of the load.

12. A steel cable 2 cm diameter is stretched across two poles 100 metres apart. If the central dip is 2 m at a temperature of 58°F, calculate the stress intensity in the cable. Calculate the fall of temperature necessary to raise the stress to 550 kg/cm². Weight of steel$=7{\cdot}8$ g/cm³ and $\alpha=6{\cdot}2\times10^{-6}$ per 1°F.

13. A suspension bridge with two hinged stiffening girder has a span of 150 metres, the cables having a central dip of 15 metres at 65°F. If the stiffening girder is 5 m deep, calculate the flange stress due to a fall of 25°F in temperature at the central section, given I for the section$=4{\cdot}5\times10^{6}$ cm⁴. Find also the increase in the horizontal tension in the cable. Take $E=2{\cdot}10\times10^{6}$ kg/cm² and $\alpha=6{\cdot}2\times10^{-6}$ per °F.

14. The two parts of the three pinned stiffening girder of a suspension bridge are 400 ft and 300 ft long respectively. Find the position and magnitude of the maximum bending moment due to a uniformly distributed load w per foot run for 100 ft on both sides of the inner pin. *(U.L.)*

Answers

1. 46·67 m ; 6750 kN.
2. (*i*) 102·22 m, (*ii*) 143·1 kN, (*iii*) 155·5 kN, 146 kN. (*iv*) 167·63 kN, 115·77 kN.

3. $V_A=6{\cdot}81$ tons. $V_B=5{\cdot}725$ tons, $H=47{\cdot}9$ tons, $T_{max}=48{\cdot}4$ tons.
4. Max. tension : 107·7 kN, Minimum tension$=H=100$ kN
   (*i*) 6·7 kN horizontally, 93·8 kN vertically.
   (*ii*) 97·7 kN vertically, No force horizontally.
5. (*i*) Greatest tension : 56·4 kN,
   (*ii*) Least tension$=H=51{\cdot}6$ kN,
   (*iii*) Greatest thrust on the pier$=74{\cdot}18$ kN,
   (*iv*) Tension in anchor cables$=73$ kN.
8. $T_{max}=95$ kN
   $M_{max}=-135$ kN-m at 56 m from left support.
   $=+225$ kN-m at 120 m from left support.
9. $H=3{\cdot}35$ W, $T_{max}=3{\cdot}584$ W.
10. 166 tons.
11. $M_{max}=-21{\cdot}91$ kN-m at 12·15 m from left support.
12. 489 kg/cm² ; 48°F.
13. 97·6 kg/cm², 828 kg.
14. 7410 w at 174 ft from the remote end of the 400 ft length.

16

Arches

14·1. INTRODUCTION

An arch may be looked upon as a curved girder, either a solid rib or braced, supported at its ends and carrying transverse loads which are frequently vertical. Since the transverse loading at any section normal to the axis of the girder is at an angle to the normal face, an arch is subjected to three restraining forces : (*i*) thrust, (*ii*) shear force and (*iii*) bending moment. Depending upon the number of hinges, arches may be divided into for classes (Fig. 16·1) :

1. Three hinged arch
2. Two hinged arch
3. Single hinged arch
4. Fixed arch (hingeless arch).

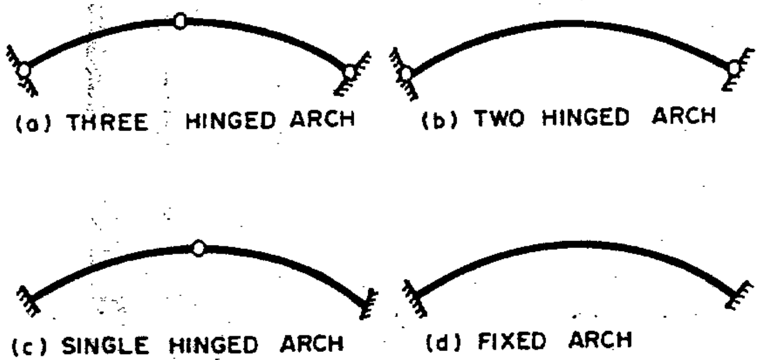


Fig. 16·1.
Type of arches

A three hinged arch is a statically determinate structure while the rest three arches are statically indeterminate. In bridge construction, especially in railroad bridges, the more frequently used arches are the two-hinged and the fixed end ones.

16·2. LINEAR ARCH (THEORETICAL ARCH)

Consider a system of jointed linkwork inverted about AB, with loads as shown in Fig. 16·12 (*a*). Under a given system of loading, every link will be in a state of compression. The magnitudes of pushes (or thrusts) T_1, T_2, T_3...... etc. can be known by the rays *Od*, *Oe*, etc., in the force polygon. For any arch under a given system of loading, the final lines of actions of thrust T_1, T_2, T_3...... etc., in the respective segments can be plotted by the usual graphical methods if the horizontal reactions at A and B are *known*. The *line of thrust* (*i.e.*, the actual lines of action of thrusts T_1, T_2, T_3 etc.) is known as the *theoretical arch* or *linear arch*.

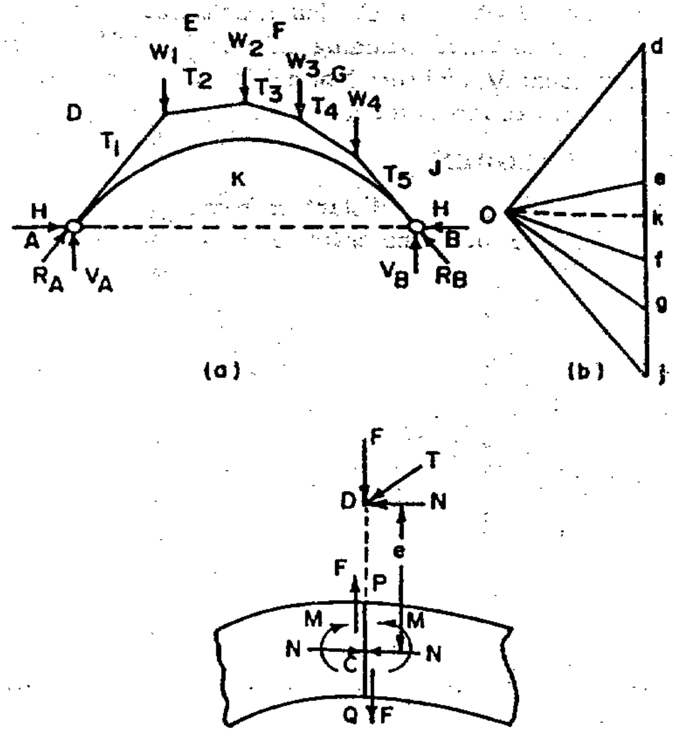


Fig. 16·2.
Theoretical arch.

It is, however, not possible to construct the actual arch of the shape of theoretical arch. The moving loads will change the shape of the theoretical arch, and it cannot be made to change its shape to suit the varying load positions. In actual practice, therefore, an arch is made parabolic, circular or elliptic in shape.

Consider a cross-section PQ of the arch [Fig. 16·2 (c)]. Let T be the resultant thrust acting through D along the linear arch. The thrust T is neither normal to the cross-section nor does it act through centre C of the cross-section.

The resultant thrust T can be resolved normal and tangential to the section PQ. Let N be the normal component and F be the tangential component. Evidently, the tangential component F will cause shear force at the section PQ. The normal component N acts eccentrically, the eccentrically e being equal to CD. Thus, the action of N acting at D is twofold : (i) a normal thrust N at C, and (ii) a bending moment $M=N.\ e$ at C. Hence, unlike beams, a section of arch is subjected to three straining actions : (i) Shear force F, (ii) Bending moment M, and (iii) Normal thrust N. The shear force F is also sometimes known as the *radial shear*.

16·3. EDDY'S THEOREM

Consider a section at P distant x from A, of an arch, shown in Fig. 16·3. Let the other co-ordinate of P be y. For the given

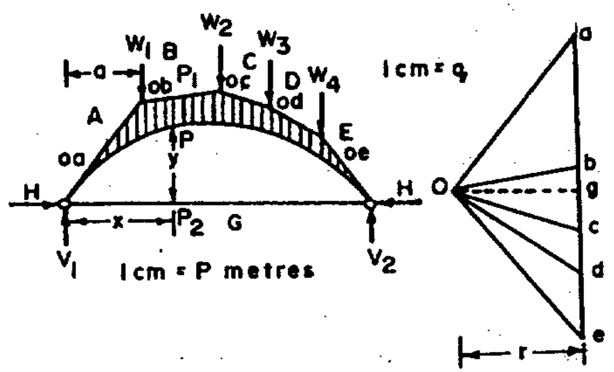


Fig. 16·3.

system of loads, the linear arch can be constructed (if H is known). Since funicular polygon represents the bending moment diagram to ...ne scale, the vertical intercept P_1P_2 at the section P will give the bending moment due to external load system. If the arch is drawn to a scale of 1 cm$=p$ m, load diagram is plotted to a scale 1 cm$=q$

N and if the distance of pole O from the load line is r, the scale of bending moment diagram will be 1 cm$=p.q.r.$ N-m.

Now, theoretically, the B.M. at P is given by

$$M_P=-V_1x+W_1(x-a)+Hy$$
$$=\mu_x+Hy$$

where $\mu x=-V_1x+W_1(x-a)$

=Usual bending moment at a section due to load system on a simply supported beam.

From Fig. 16·3, we have,

$$\mu x=-(P_1P_2)\times\text{scale of B.M. diagram}$$
$$=-P_1P_2\,(p.q.r)$$

and $$Hy=(PP_2)\times\text{scale of B.M. diagram}$$
$$=PP_2(p.q.r.)$$

Hence $$M_P=\mu x+Hy=-P_1P_2(p.q.r.)+PP_2(p.q.r.)$$
$$=-(PP_1)(p.q.r.)$$

Hence the ordinate between the linear arch and the actual arch gives the bending moment. This is known as Eddy's theorem and may be stated as below :

"The bending moment at any section of an arch is equal to the vertical intercept between the linear arch and the centre line of the actual arch".

16·4. THREE HINGED ARCH

A three hinged arch is a statically determinate structure, having a hinge at each abutment or springing, and also at the crown. There are in all four reaction components (two at each hinge, *i.e.*, H and V). Three equations are available from the static equilibrium and one additional equation is available from the fact that the B.M. at the hinge at the crown is zero. Thus, the value of H can be easily calculated for any given load system.

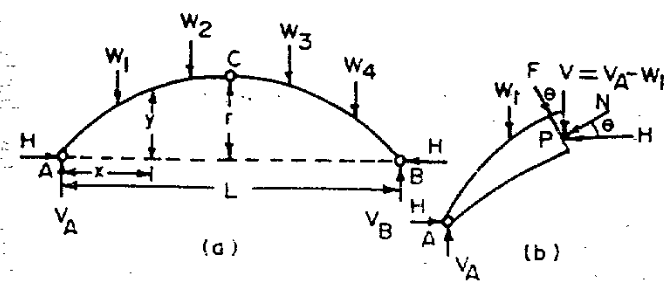


Fig. 16·4.

The arch shown in Fig. 16·4 (*a*), is subjected to a number of loads W_1, W_2, etc. Let the reactions at A and B be (H, V_A) and (H, V_B) respectively. Since the B.M. at C is zero, we have,

$$M_C=\mu_C+Hy=0$$

$$\therefore \quad H=-\frac{\mu_C}{y} \qquad ...(16·1)$$

The value of H is thus known. The value of V_A can be known by taking moments, of all forces, about B. Similarly, V_B can be known. After having known the reaction components, the value of radial shear (F) and normal thrust (N) at any section P can be easily calculated, with reference to Fig. 16·4 (*b*) where equilibrium of left portion PA has been shown. The vertical and horizontal sections on the section P are : $V=V_A-W_1$ and $H=H$.

Now, resolving *along* the section at P, we get

$$F=H\sin\theta-V\cos\theta\uparrow \qquad ...(16·2)$$

Similarly, resolving normal to the section, we get

$$N=H\cos\theta+V\sin\theta \qquad [(16·2\ (a)]$$

THREE HINGED PARABOLIC ARCH

The equation of a parabola, with origin at the left hand hinge A [Fig. 16·4 (*a*)] can be written as

$$y=kx(L-x) \qquad (i) \text{ where } k \text{ is a constant}$$

At $\quad x=\frac{L}{2}$, let $y=r=$central rise.

Substituting in (*i*), we get

$$r=k\,\frac{L}{2}\left(L-\frac{L}{2}\right)=k\,\frac{L^2}{4}$$

$$\therefore \quad k=\frac{4r}{L^2}$$

$$\therefore \quad y=\frac{4r}{L^2}\,x\,(L-x) \qquad ...(16·3)$$

This is the equation of a parabolic arch.

According to Eddy's theorem, the vertical intercept between the linear arch and the centre line of the actual arch gives the B.M. at a section. Due to uniformly distributed load, the linear arch will be a parabola. It will pass through the hinge at the crown. The centre line of the actual arch is also parabolic, passing through the central hinge. These two parabolas pass through three common

points and hence they overlap each other. *Therefore a parabolic arch will not have B.M. due to U.D.L. It will be subjected to pure compression.*

THREE HINGED CIRCULAR ARCH

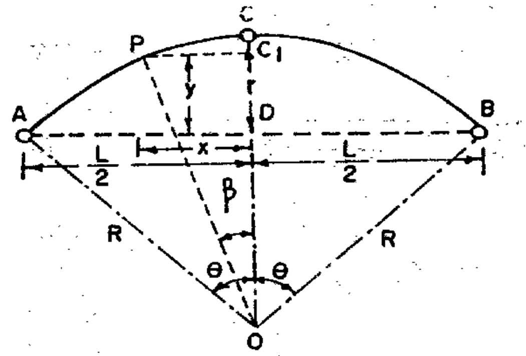


Fig. 16·5.

Let us now consider the centre line of the arch to be segment of a circle of radius R, subtending an angle of 2θ at the centre. It is always convenient to have the origin at D, the middle of the span. Let (x, y) be the co-ordinates of the point P. Draw line PC_1 parallel to AB.

Then $\quad OP^2=OC_1^2+PC_1^2$

or $\quad R^2=\{y+(R-r)\}^2+x^2 \qquad ...(16·4)$

Equation (16·4) connects y with x.

Also, $\quad r(2R-r)=\frac{L}{2}\cdot\frac{L}{2}=\frac{L^2}{4} \qquad ...(16·5)$

From equation (16·5), the value of the radius can be calculated for the known values of the span and the rise.

The co-ordinates of P (*i.e.* x and y) can also be expressed as trigonometric functions. Thus, if OP makes an angle β with OC.

$$x=OP\sin\beta=R\sin\beta$$

and $\quad y=C_1D=OC_1-OD=R\cos\beta-R\cos\theta=R(\cos\beta-\cos\theta)$

Example 16·1. *A parabolic arch hinged at the springings and crown has a span of 20 m. The central rise of the arch is 4 m. It is loaded with a uniformly distributed load of intensity 2 kN/m on the left 8 m length. Calculate (a) the direction and magnitude of reaction at the hinges, (b) the bending moment, normal thrust and shear at 4 m* *and 15 m from the left end, and (c) maximum positive and negative bending moments.*

Solution

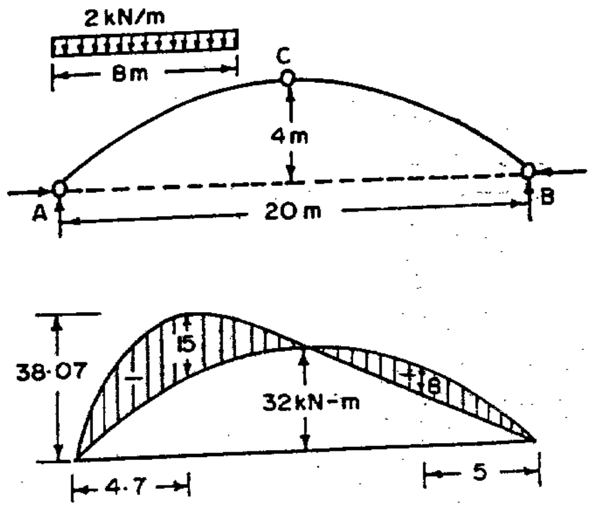


Fig. 16·6.

(*a*) For vertical reaction at *A*, take moments about *B*. Thus

$$V_A\times 20=2\times 8(20-4)$$

$$V_A=12{\cdot}8\text{ kN}$$

Hence $\quad V_B=8\times 2-12{\cdot}8=3{\cdot}2\text{ kN}$

Since the bending moment at the hinge *C* is zero, we have

$$M_C=(-3{\cdot}2\times 10)+H\times 4=0$$

$$\therefore \quad H=\frac{32{\cdot}0}{4}=8\text{ kN}$$

$\therefore$ Reaction at $A=R_A=\sqrt{V_A^2+H^2}=\sqrt{(12{\cdot}8)^2+8^2}=15{\cdot}09$ kN

It inclination with the horizontal is given by

$$\tan\theta_A=\frac{V_A}{H}=\frac{12{\cdot}8}{8}=1{\cdot}6\,;\quad\therefore\quad \theta_A=58$$

$\therefore$ Reaction at $B=R_B=\sqrt{V_B^2+H^2}=\sqrt{(3{\cdot}2)^2+8^2}=8{\cdot}62$ kN

Its inclination with the horizontal is given by,

$$\tan\theta_B=\frac{V_B}{H}=\frac{3{\cdot}2}{8}=0{\cdot}4\,;\quad\therefore\quad\theta_B=21^\circ\,48'$$

The magnitude and direction of the reaction at the crown will be the same as that of the reaction and the hinge *B*, *i.e.*, 8·62 kN at 21° 48′ with the horizontal. This is due to the fact that there is no

loading between *B* and *C*. *The reaction at B passes through C since* $M_C=0$.

(*b*) The bending moment diagram is shown in Fig. 16·6(*b*). The equation of the parabola is

$$y=\frac{4r}{L^2}\,x(L-x)=\frac{4\times 4}{400}x(20-x)=\frac{x}{25}(20-x)$$

and $$\frac{dy}{dx}=\frac{20-2x}{25}$$

At $x=4$ m, $\quad y=\frac{4}{25}(20-4)=2{\cdot}56$ m

$$\tan\theta=\frac{dy}{dx}=\frac{20-2\times 4}{25}=0{\cdot}48$$

$\therefore \quad \theta=25^\circ\,38'$

$\therefore \quad \sin\theta=0{\cdot}433$ and $\cos\theta=0{\cdot}901$

$$M_A=-(12{\cdot}8\times 4)+(8\times 2{\cdot}56)+(4\times 2\times 2)$$

$$=\mathbf{-14{\cdot}72\text{ kN-m}}$$

Vertical shear at the section,

$$V=12{\cdot}8-2\times 4=4{\cdot}8\text{ kN}$$

and $\quad H=8$ kN

From Fig. 16·7(*a*),

$$F=H\sin\theta-V\cos\theta(\uparrow)$$

$$=8\times 0{\cdot}433-4{\cdot}8\times 0{\cdot}901=-0{\cdot}861\text{ kN}$$

Hence $\quad$ **$F=0{\cdot}861$ kN** $\uparrow\downarrow$

and $\quad N=H\cos\theta+V\sin\theta$

$$=8\times 0{\cdot}901+4{\cdot}8\times 0{\cdot}433=\mathbf{9{\cdot}286\text{ kN}}$$

At $x=15$, $\quad y=\frac{15}{25}(20-15)=3{\cdot}0$ m

$$\frac{dy}{dx}=\tan\theta=\frac{20-2\times 15}{15}=-0{\cdot}4$$

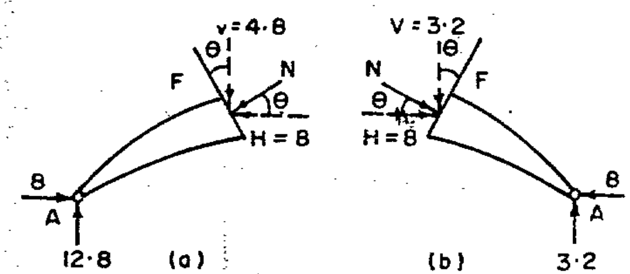


Fig. 16·7.

$\therefore$ $\theta=21° 48'$ (inclination with BA)

$\sin\theta=0\cdot3714$ and $\cos\theta=0\cdot9285$

$M_{15}=(-3\cdot2\times5)+8\times3\cdot0=+8\cdot0$ kN-m

From Fig. 16·7(*b*),

$$F=H\sin\theta-V\cos\theta\uparrow$$
$$=8\times0\cdot3714-3\cdot2\times0\cdot9285$$
$$=2\cdot97-2\cdot97=0$$

and
$$N=H\cos\theta+V\sin\theta$$
$$=8\times0\cdot9285+3\cdot2\times0\cdot3714$$
$$=\mathbf{8\cdot616\ kN.}$$

(*c*) *Maximum positive and negative B.M.*

Maximum negative B.M. will occur somewhere under the U.D.L. Let it occur at x from the left hinge.

$$Mx=(-12\cdot8\times x)+\frac{2x^2}{2}+8y$$
$$=-12\cdot8x+x^2+\frac{8x}{25}(20-x)$$
$$\frac{dMx}{dx}=0=-12\cdot8+2x+\frac{32}{5}-\frac{16}{25}x=0$$

from which $x=4\cdot7$ m

$$\therefore\quad M_{max}(-ve)=-12\cdot8\times4\cdot7+4\cdot7^2+\frac{8}{25}(4\cdot7)(20-4\cdot7)$$
$$=\mathbf{15\ kN\text{-}m.}$$

The maximum positive B.M. will evidently occur somewhere in the portion BC for which the equation of B.M. is given by,

$$Mx=-3\cdot2x+8y,\ x \text{ being measured from } B$$
$$=-3\cdot2x+\frac{8x}{25}(20-x)$$
$$\therefore\quad\frac{dMx}{dx}=0=-3\cdot2+\frac{32}{5}-\frac{16x}{25}$$

from which $x=5$.

Hence maximum positive B.M. occurs where the radial shear is zero.

$$M_{max}(+ve)=-3\cdot2\times5+\frac{8\times5}{25}(20-5)$$
$$=\mathbf{+8\ kN\text{-}m.}$$

Example 16·2. *A symmetrical parabolic arch with a central hinge, of rise r and span L, is supported at its ends on pins at the same level. What is the value of the horizontal thrust when a load W which is uniformly distributed horizontally covers the whole span?*

Show also that with this loading there is no bending moment any point in the arch rib.

Solution.

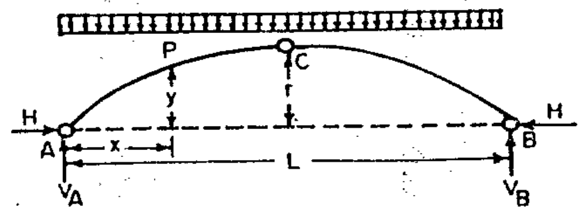


Fig. 16·8.

The vertical reaction at A and B will be each equal to $\frac{W}{2}$

For horizontal, thrust, taking moment about C.

$$H\times r=\frac{W}{2}\times\frac{L}{2}-\frac{W}{2}\times\frac{L}{4}=\frac{WL}{8}$$

or
$$H=\frac{WL}{8r}$$

Let us now consider any section at distance x from A.

Equation of the parabola is given by.

$$y=\frac{4r}{L^2}x(L-x).$$

$$\therefore\quad Mx=Hy-V_A.x+\frac{W}{L}\cdot\frac{x^2}{2}$$
$$=\frac{WL}{8r}\cdot\frac{4r}{L^2}x(L-x)-\frac{W}{2}x+\frac{Wx^2}{2L}$$
$$=\frac{Wx}{2}-\frac{Wx^2}{2L}-\frac{Wx}{2}+\frac{Wx^2}{2L}=0.$$

Hence bending moment at any point in the arch rib is zero.

Example 16·3. *An arch in the form of a parabola with axis vertical has hinges at the abutments and the vertex. The abutments are at different levels, the horizontal span being L and the heights of vertex above the abutments being h_1 and h_2.*

Show that the horizontal thrust due to a load w/unit length uniformly distributed across the span is

$$\frac{wL^2}{2(\sqrt{h_1}+\sqrt{h_2})^2}\qquad\text{(Cambridge)}$$

Solution.

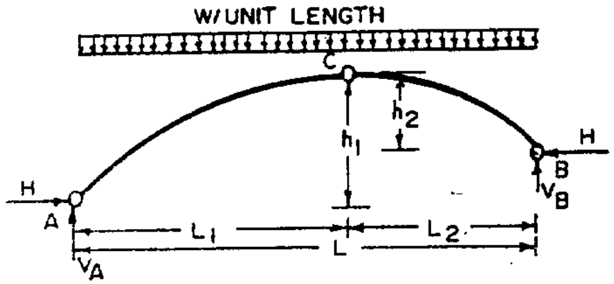


Fig. 16·9.

Let L_1 be the distance of the vertex from the left hand abutment. With C as origin, the equation of parabola is $y=kx^2$.

For CA, therefore, $h_1=kL_1^2$

or $$k=\frac{h_1}{L_1^2}$$

For CB, we have $h_2=k\,(L-L_1)^2$

or $$k=\frac{h_2}{(L-L_1)^2}$$

Equating the two, we get

$$\frac{h_1}{L_1^2}=\frac{h_2}{(L-L_1)^2}$$

∴ $$L_1^2h_2=(L-L_1)^2h_1$$

or $$L_1\sqrt{h_2}=(L-L_1)\sqrt{h_1}$$

or $$L_1=\frac{L\sqrt{h_1}}{\sqrt{h_1}+\sqrt{h_2}} \qquad (16·6)$$

Taking moments about B,

$$Hh_1=V_A\frac{L\sqrt{h_1}}{\sqrt{h_1}+\sqrt{h_2}}-\frac{w}{2}\,\frac{h_1L^2}{(\sqrt{h_1}+\sqrt{h_2})^2} \qquad (i)$$

By taking moments about B,

$$H(h_1-h_2)+\frac{wL^2}{2}=V_AL$$

∴ $$V_A=\frac{H(h_1-h_2)}{L}+\frac{wL}{2} \qquad (ii)$$

Substituting the value of V_A in (*i*) we get

$$Hh_1=\left[\frac{H(h_1-h_2)}{L}+\frac{wL}{2}\right]\frac{L\sqrt{h_1}}{\sqrt{h_1}+\sqrt{h_2}}-\frac{w}{2}\,\frac{h_1L^2}{(\sqrt{h_1}+\sqrt{h_2})^2}$$

or $$H\left[h_1-\frac{(h_1-h_2)\sqrt{h_1}}{\sqrt{h_1}+\sqrt{h_2}}\right]=\frac{wL^2\sqrt{h_1}}{2(\sqrt{h_1}+\sqrt{h_2})}-\frac{wh_1L^2}{2(\sqrt{h_1}+\sqrt{h_2})^2}$$

Simplifying and rearranging, we get

$$H\sqrt{h_1h_2}=\frac{wL^2}{2(\sqrt{h_1}+\sqrt{h_2})^2}\sqrt{h_1h_2}$$

or $$H=\frac{wL^2}{2(\sqrt{h_1}+\sqrt{h_2})^2}.$$

Example 16·4. *A three hinged parabolic arch of 20 metre span and 4 m central rise carries a point load of 4 kN at 4 m horizontally from the left hand hinge. Calculate the normal thrust and shear force at the section under the load. Also, calculate the maximum B.M. positive and negative.*

Solution.

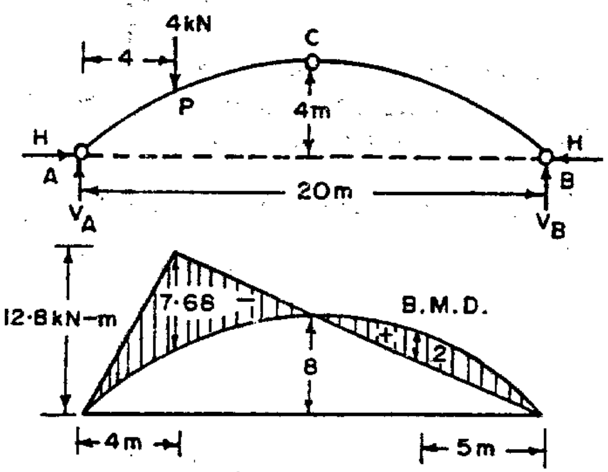


Fig. 16·10.

Taking A as origin, we have the equation of the parabola

$$y=\frac{4r}{L^2}x(L-x)=\frac{4\times4}{20\times20}x(20-x)$$

$$=\frac{x}{25}(20-x)$$

Taking moments about B, we get

$$20V_A+4\times16=0$$

or $$V_A=\frac{64}{20}=3·2\text{ kN}\uparrow$$

∴ $$V_B=4-3·2=0·8\uparrow$$

Taking moments about C, we get

$$M_C=4H-(0·8\times10)=0$$

or $$H=\frac{8}{4}=2\ \text{kN}$$

Now, bending moment at any section is

$$M_x=\mu_x+Hy$$

The μ_x-diagram is a triangle having maximum ordinate $=3{\cdot}2\times 4=12{\cdot}8$ kN-m under the point load. The Hy diagram is a parabola having a maximum ordinate $=2\times 4=8$ kN-m under the central hinge.

At $$x=4,\ y=\frac{4}{25}(20-4)=2{\cdot}56$$

Under the point load,

$$M_P=(-3{\cdot}2\times 4)+(2\times 2{\cdot}56)$$
$$=-7{\cdot}68\ \text{kN-m}$$

This is evidently the maximum negative B.M.

The maximum positive B.M. will occur somewhere in the portion BC. Measuring x from B, the equation of B.M. for the portion BC is,

$$M_x=-0{\cdot}8x+2y$$
$$=-0{\cdot}8x+2\frac{x}{25}(20-x)$$

$$\therefore\quad \frac{dM_x}{dx}=-0{\cdot}8+\frac{40}{25}-\frac{4}{25}x=0$$

which gives $$x=5\ \text{m}$$

$$\therefore\quad M_{max.}(+\text{ve})=(-0{\cdot}8\times 5)+\frac{2}{25}\times 5(20-5)=+2\ \text{kN-m}$$

The equation of the parabola is $y=\frac{x}{25}(20-x)$

$$\therefore\quad \tan\theta=\frac{dy}{dx}=\frac{20}{25}-\frac{2x}{25}$$

$$\therefore\quad \tan\theta\ (\text{at } x=4\ \text{m})=0{\cdot}8-0{\cdot}32=0{\cdot}48$$

$$\therefore\quad \theta=25°38'\ ;\ \sin\theta=0{\cdot}433\ ;\ \cos\theta=0{\cdot}901$$

Considering the point load slightly to the right of P, we get

$$F=H\sin\theta-V_A\cos\theta$$
$$=(2\times 0{\cdot}433)-(3{\cdot}2\times 0{\cdot}901)=-2{\cdot}017=\mathbf{2{\cdot}017}\uparrow\downarrow$$

and $$N=H\cos\theta+V_A\sin\theta$$
$$=(2\times 0{\cdot}901)+(3{\cdot}2\times 0{\cdot}433)=\mathbf{3{\cdot}188\ kN}.$$

Example 16·5. *A symmetrical three hinged circular arch has a span of 16 m and a rise to the central hinge of 4 m. It carries a verti-*

cal load of 16 kN at 4 m from the left hand end. Find tude of the thrust at the springings, (b) the reactions at th.. bending moment of 6 m from the left hand hinge, and (d) the positive and negative bending moment.

Solution.

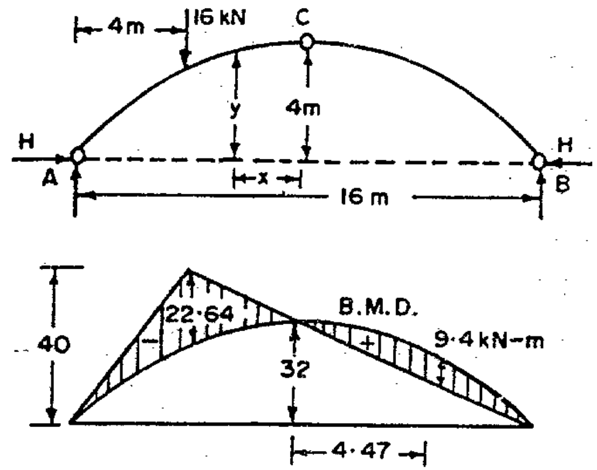


Fig. 16·11.

By property of a circle,

$$4(2R-4)=8\times 8$$

From which $$R=10\ \text{m}.$$

Let y be the rise at any point at a distance x from the centre. Then

$$R^2=x^2+\{(R-r)+y\}^2$$

or $$10^2=x^2+\{(10-4)+y\}^2$$

or $$(6+y)^2=100-x^2$$

or $$y=(100-x^2)^{1/2}-6 \qquad (i)$$

For vertical reaction at A, take moments about B. Thus,

$$V_A=\frac{16\times 12}{16}=12\ \text{kN}$$
$$V_B=4\ \text{kN}$$

(*a*) Taking moments about C,

$$H\times 4=(12\times 8)-16\times 4=32$$
$$H=\mathbf{8\ kN}$$

(*b*) Reaction at $A=\sqrt{V_A^2+H^2}=\sqrt{144+64}=\mathbf{14{\cdot}42\ kN}$

Its inclination with the horizontal is given by

$$\tan\theta=\frac{12}{8}=1{\cdot}5, \quad \therefore\ \theta=56^\circ\ 18'.$$

Reaction at $B=\sqrt{V_B^2+H^2}=\sqrt{16+64}=\mathbf{8{\cdot}94\ kN}$

Its inclination with the horizontal is given by

$$\tan\theta=\frac{4}{8}=0{\cdot}5, \quad \therefore\ \theta=26^\circ\ 34'.$$

(c) At 6 m from left hand hinge, $x=(8-6)=2$ m

$$\therefore\quad y=(100-2^2)^{1/2}-6=9{\cdot}8-6=3{\cdot}8 \text{ m}.$$

$$\therefore\quad M=(-12\times 6)+(8\times 3{\cdot}8)+(16\times 2)$$

$$=\mathbf{-9{\cdot}6\ kN\text{-}m}$$

(d) The maximum negative bending moment will occur under the load, where

$$x=(8-4)=4 \text{ m}$$

$$\therefore\quad y=(100-4^2)^{1/2}-6=3{\cdot}17 \text{ m}.$$

$$\therefore\quad M_{max.}(-ve)=(-12\times 4)+(8\times 3{\cdot}17)=\mathbf{-22{\cdot}64\ kN\text{-}m.}$$

Maximum positive B.M. will occur somewhere in CB. Let it occur at a distance x from C, on the right hand side,

$$y=(100-x^2)^{1/2}-6$$

$$\therefore\quad M_X=-4(8-x)+\{(100-x^2)^{1/2}-6\}$$

$$\therefore\quad \frac{dMx}{dx}=+4+\frac{8(-2x)}{2(100-x^2)^{1/2}}=0$$

or $\quad (100-x^2)^{1/2}=2x$

or $\quad 5x^2=100$

$$\therefore\quad x=\sqrt{20}=4{\cdot}47 \text{ m}$$

$$\therefore\quad y=(106-4{\cdot}47^2)^{1/2}-6=2{\cdot}94 \text{ m}$$

$$\therefore\quad M_{max}(+ve)=8\times 2{\cdot}94-4(8-4{\cdot}47)$$

$$=23{\cdot}82-14{\cdot}12=\mathbf{9{\cdot}4\ kN\text{-}m.}$$

Example 16·6. *A three hinged circular arch consists of a portion AC of radius 3m and rise of the hinge C with respect to the left abutment is 3 m. The right hand portion CB is of radius 8 m and the horizontal distance BC is 7 m. If a concentrated load of 10 kN acts at 6 m from the left hand end, determine the reactions at the hinges and maximum bending moment on the arch.*

Solution.

(Fig. 16·12)

The rise of the crown above the hinge B

$$=8-\sqrt{8^2-7^2}=4{\cdot}13 \text{ m}$$

Taking moments about B,

$$H(4{\cdot}13-3)+V_A\times 10=10\times 4$$

Taking moments about C,

$$H\times 3=V_A\times 3$$

or $\quad H=V_A$

Substituting in equation (i)

$$11{\cdot}13V_A=40$$

or $\quad V_A=3{\cdot}59 \text{ kN}=H$

and $\quad V_B=10-3{\cdot}59=6{\cdot}41 \text{ kN}$

Reaction at $A=\sqrt{(3{\cdot}59)^2+(3{\cdot}59)^2}=\mathbf{5{\cdot}08\ kN}$

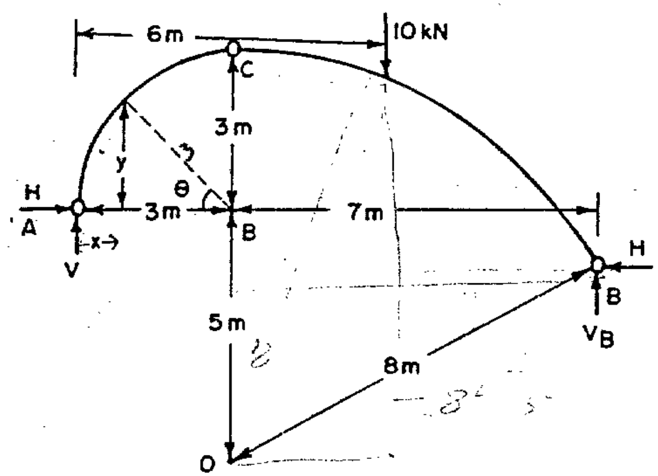


Fig. 16·12

Its inclination with the horizontal, $\tan\theta=\dfrac{3{\cdot}59}{3{\cdot}59}=1$

$$\therefore\quad \theta=45^\circ$$

Reaction at $B=\sqrt{(6{\cdot}41)^2+(3{\cdot}59)^2}=\mathbf{7{\cdot}35\ kN}$

Its inclination with the horizontal, $\tan\theta=\dfrac{6{\cdot}41}{3{\cdot}59}=1{\cdot}723$

$$\therefore\quad \theta=60^\circ\ 43'$$

Let the maximum positive B.M. in portion AC occur at any point at horizontal distance x from A,

$$x=3(1-\cos\theta) \text{ and } y=3\sin\theta$$

$$\therefore\quad M_X=3{\cdot}59\times 3\sin\theta-3{\cdot}59\times 3(8-\cos\theta)$$

$$=10{\cdot}77(\sin\theta-1+\cos\theta)$$

$$\frac{dM}{d\theta}=10{\cdot}77\,(\cos\theta-\sin\theta)=0.$$

$\therefore\quad \tan\theta=1 \qquad \text{or} \qquad \theta=45°$

$\therefore\quad M_{max.}\,(+ve)=10{\cdot}77\,(\sin 45°-1+\cos 45°)$

$\qquad =\mathbf{4{\cdot}45\ kN\text{-}m}$

The maximum negative B.M. in portion BC will evidently occur just below the load.

Height of point of application of load above $O=\sqrt{8^2-3^2}$
$=7{\cdot}42$ m.

Height of the hinge B above $O=\sqrt{8^2-7^2}=3{\cdot}87$ **m**

$\therefore$ Rise of the point of application of the load above the hinge B.

$$=7{\cdot}42-3{\cdot}87=3{\cdot}55 \text{ m}$$

$\therefore\quad M_{max.}\,(-ve)=-6{\cdot}41\times4+3{\cdot}59\times3{\cdot}55=-12{\cdot}90$ kN-m

Hence, the maximum B.M. over the span $=-\mathbf{12{\cdot}90}$ **kN-m**

Example 16·7. *A frame shown in Fig. 16·13 is hinged to the ground at A and E hinged also at C and has rigid corners at B and D. Find the reactions at A and E when a uniformly distributed load of 40 kN per metre run covers BD and draw the bending moment diagram, figuring significant values on the diagram.*

Solution

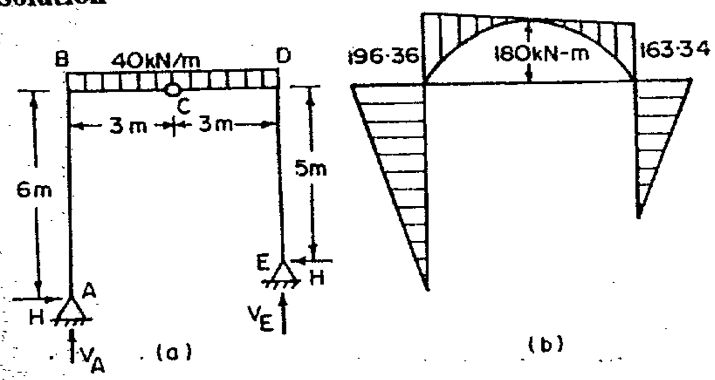


Fig. 16·13

Taking moments about E,

$$(H\times1)+(40\times6\times3)=6V_A$$

or $\qquad H+720=6V_A \qquad (1)$

Taking moments about C,

$$(H\times6)+(40\times3\times1{\cdot}5)=V_A\times3$$

or $\qquad 12H+360=6V_A \qquad (2)$

Equating (1) and (2), we get

$$H=32{\cdot}727 \text{ kN}$$

Substituting in (1), we get

$$32{\cdot}727+720=6V_A$$

or $\qquad V_A=125{\cdot}45$ kN

$\therefore\qquad V_E=(6\times40)-125{\cdot}45=114{\cdot}55$ kN

Bending moment at the head of the column BA

$$=H\times6=+32{\cdot}727\times6=196{\cdot}36 \text{ kN-m}$$

Bending moment at the head of column DE

$$=+32{\cdot}727\times5\times+163{\cdot}64 \text{ kN-m}$$

On BD the B.M. at any distance x from B is given by

$$M_x=+196{\cdot}36-125{\cdot}45\,x+\frac{40x^2}{2}$$

which is evidently zero at the central hinge C where $x=3$ m. The bending moment diagram is shown in Fig. 15·13 (b).

16·4. MOVING LOADS ON THREE HINGED ARCHES

(1) INFLUENCE LINE FOR H

Let us consider a unit load rolling from A to B. At any instant let the load be at a distance αL from A. The vertical reactions at A and B will be $(1-\alpha)$ and α respectively.

For H, equate M_C to zero.

$$\text{Thus } M_C=0=(H\times r)-\alpha.\frac{L}{2}$$

or $$H=\frac{\alpha L}{2r}$$

Thus, H varies linearly with α.

At A, $\alpha L=0$ and hence $H=0$

At, $\mathbb{C}$ $\alpha L=\dfrac{L}{2}$, and hence $H=\dfrac{L}{4r}$

The I.L. diagram for H will, therefore, be a triangle having maximum ordinate of $\dfrac{L}{4r}$ under the central hinge, as shown in Fig. 16·14 (b).

(2) INFLUENCE LINE FOR B.M. AT P

Let us now draw the I.L. from B.M. at P distance x from A

$$M_P=\mu+H.y$$

Thus, the influence line for M_P consists of (i) I.L. for μ, and (ii) I.L. for Hy. The I.L. for μ will be a triangle having a maximum ordinate of $\frac{x(L-x)}{L}$ under the section. The I.L. for H_y will also be a triangle having a maximum ordinate

$$=\frac{L}{4r}\,y=\frac{L}{4r}\times\frac{4r}{L^2}\,x(L-x)=\frac{x}{L}\,(L-x)$$

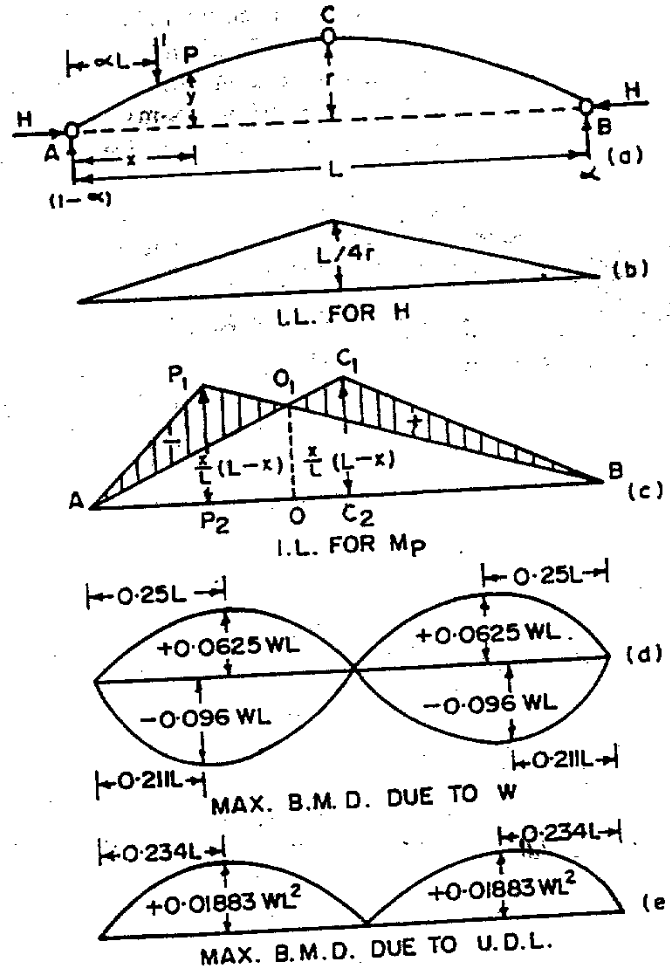


Fig. 16·14

It must be noted here that both x and y are fixed quantities for P. The I.L. for M_P will be obtained by superimposing the two ... in Fig. 16·14 (c).

Using the influence lines, let us now plot the maximum positive and negative bending moment diagram due to (*i*) single point load W and (*ii*) U.D.L.

(3) MAXIMUM BENDING MOMENT DIAGRAM DUE TO SINGLE POINT LOAD W

By the inspection of the I.L. for M_P, it is clear that *maximum negative B.M.* will occur when the point load is at the section P.

Thus, $M_{max.}(-ve)=-\frac{Wx(L-x)}{L}+\frac{Wx(L-x)}{L}\cdot\frac{2}{L}\cdot x$

$$=-\frac{Wx(L-x)}{L^2}\,(L-2x)$$

$$=-\frac{Wx(L-x)(L-2x)}{L^2} \qquad (i)$$

To get the absolute maximum negative B.M., differentiate (*i*) with respect to x and equate it to zero.

Thus $\frac{dM_{max.}}{dx}=0=(L-x)(L-2x)-x(L-2x)-2x(L-x)=0$

or $\qquad 6x^2-6Lx+L^2=0$

which gives $\qquad x=(0.5\pm0.289)L$

$\qquad =0.211\ L$ or $0.789\ L$.

Substituting the value of x in (*i*), we get

$$\therefore\quad M_{max.max.}(-ve)=-\frac{WL}{6\sqrt{3}}=-0.096\ WL$$

The maximum negative bending moment diagram will be a third degree polynomial, as shown in Fig 16.14(*d*) below the base.

The maximum positive bending moment at P will occur when the load W is at the central hinge, as is clear from the I.L. diagram for M_P.

Thus $M_{max.}(+ve)=\frac{Wx(L-x)}{L}-\frac{Wx(L-x)}{L}\cdot\frac{L}{(L-x)}\cdot\frac{L}{2}$

$$=\frac{Wx}{2L}\,(2L-2x-L)=\frac{Wx(L-2x)}{2L} \qquad (ii)$$

To get the absolute maximum positive B.M. differentiate (*ii*) with respect to x and equate it to zero.

Thus, $\frac{dM_{max.}}{dx}=0=L-4x$

$$\therefore\quad x=\frac{L}{4}=0.25\ L$$

Substituting the value of x in (*ii*), we get

$$M_{max\cdot max}\ (+\text{ve}) = +\frac{WL}{16} = +0{\cdot}0625\ WL.$$

The maximum positive bending moment diagram will be a second degree polynomial (*i.e.* parapola), as shown in Fig 16·14(*d*) above the base.

(4) MAXIMUM BENDING MOMENT DIAGRAM DUE TO U.D.L.

Maximum negative B.M. at P will occur when the AO is loaded, while the maximum positive B.M. at P will occur when the span BO is loaded. Since the ordinate P_1P_2 and C_1C_2 are equal the area of Δs AP_1B and AC_1B are equal and hence area of Δs AP_1O_1 and BC_1O_1 will be equal. *Therefore, the maximum negative B.M. at P will be equal to the maximum positive B.M. at P.*

Thus $M_{max} = \pm w \times$ (area of triangle AP_1O_1) (1)

Let us, therefore, locate the point O first. Let the distance $AO = a$, and $OB = (L-a)$.

Now $OO_1 = C_1C_2 \times \frac{2}{L} \times a = \frac{2a}{L} C_1C_2$

Also, $OO_1 = P_1P_2 \cdot \frac{1}{(L-x)} \times (L-a) = \frac{L-a}{L-x} P_1P_2$

Equating the two, we get

$$\frac{2a}{L} C_1C_2 = \frac{L-a}{L-x} P_1P_2$$

or $$\frac{2a}{L} = \frac{L-a}{L-x} \text{ since } C_1C_2 = P_1P_2$$

$$\therefore \quad a = AO = \frac{L^2}{3L-2x} \quad (2)$$

Hence, the ordinate $OO_1 = \frac{2}{L} \cdot \frac{L^2}{(3L-2x)} \cdot \frac{x(L-x)}{L}$

$$= \frac{2x(L-x)}{(3L-2x)}$$

Now area AP_1O_1 = area AP_1B − area AO_1B

$$= \frac{1}{2} L \cdot \frac{x(L-x)}{L} - \frac{1}{2} L \cdot \frac{2x(L-x)}{(3L-2x)}$$

$$= \frac{x(L-x)(L-2x)}{2(3L-2x)} \quad (3)$$

Substituting the value in (1), we get

$$M_{max} = \pm \frac{wx(L-x)(L-2x)}{2(3L-2x)} \quad (4)$$

To obtain absolute maximum (±) B.M., differentiate (4) with respect to x and equate it to zero. Putting $x = nL$ in (4) we get

$$M_{max} = \pm wL^2 \frac{n(1-n)(1-2n)}{(6-4n)}$$

$$\therefore \quad \frac{dM_{max}}{dn} = 0$$

$$= (6-4n)[(1-n)(1-2n) - n(1-2n) - 2n(1-2n)] + 4n(1-n)(1-2n)$$

or $$8n^3 - 24n^2 + 18n - 3 = 0$$

From which, $n = 0{\cdot}234$

or $x = 0{\cdot}234L$

Substituting the value of x in (4), we get

$M_{max\cdot max}(\pm) = \pm 0{\cdot}01883\ wL^2$

The loaded length $AO = \frac{L^2}{3L - 2(0{\cdot}234L)} = 0{\cdot}395L.$ (5)

The maximum negative and positive diagrams are shown in Fig. 16·14 (*c*).

(5) INFLUENCE LINES FOR RADIAL SHEAR (F) AND NORMAL THRUST (N)

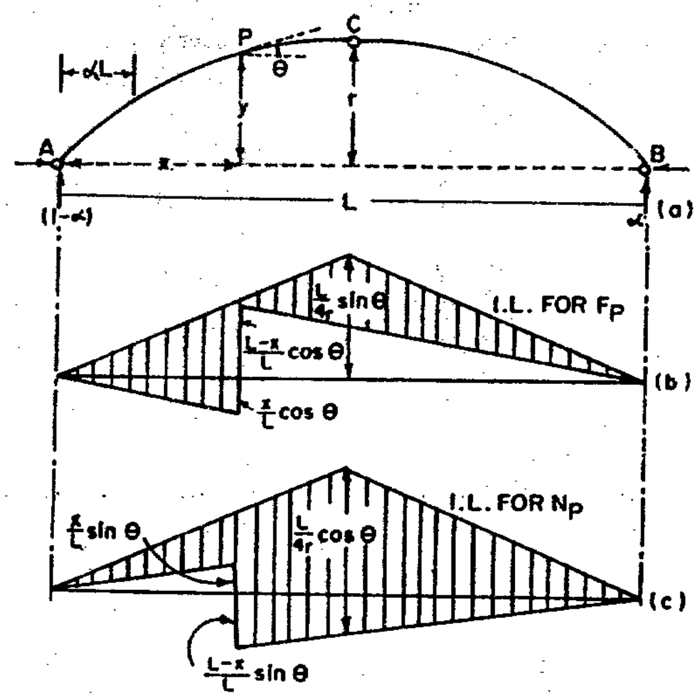


Fig. 16·15

Let us now draw the influence line diagrams for radial shear (F) and normal thrust (N) at a section P distant x from A. When the load is in AP, consider the equilibrium of the portion PB [Fig. 16·16 (b)] and when the load is in PB, consider the equilibrium of the portion AP [Fig. 16·16(a)]. When the unit load is at a distance αL from A, the reactions at A and B will be $(1-\alpha)$ and α respectively while the horizontal thrust H will be equal $\frac{\alpha L}{2r}$ as proved earlier.

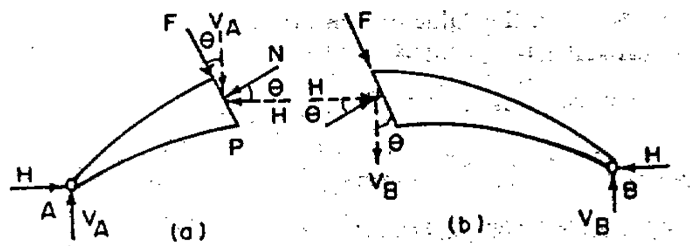


Fig. 16·16

When the load is between A and P, consider Fig. 16·16 (b) from which

$$F_P = V_B \cos\theta + H \sin\theta\ (\downarrow) \quad \text{(I)}$$

and $$N_P = H\cos\theta - V_B \sin\theta(\rightarrow) \quad \text{(II)}$$

Again, when the load is between P and B, we get from Fig. 13·16(a).

$$F_P = H\sin\theta - V_A\cos\theta(\uparrow) \quad \text{(III)}$$

and $$N_P = H\cos\theta + V_A \sin\theta(\leftarrow) \quad \text{(IV)}$$

By the inspection of equation (I) and (III) it is clear that the influence line for F_P can be obtained by superimposing $V\cos\theta$ diagram ($i.e.$ I.L. for S.F. for simple beam, every ordinate of which is multiplied by cos θ) on $H\sin\theta$ diagram, keeping in view the fact that both the quantities are additive when the load is in AP and are subtractive when the load is in BP. Fig. 16·15 (b) shows the I.L. diagram for F_P.

Similarly, I.L. for N_P can be obtained by superimposing $V\sin\theta$ diagram on $H\cos\theta$ diagram keeping in view the fact that both the quantities are subtractive when the load is in AP and are additive when the load is in BP. Fig. 16·15(c) shows the I.L. diagram for N_P.

Example 16·8. *A three hinged parabolic arch has a span of 40 m and a central rise of 8 m. Five wheel loads of 4, 4, 6, 6 and 5*

tonnes spaced 2, 3, 2 and 3 m in order, cross the arch from left to right with the 4 kN load leading. When the leading load is 25 m from the left hand hinge, calculate the horizontal thrust in the arch. Also, calculate the bending moment, normal thrust and shear force at the section under the tail load.

Solution

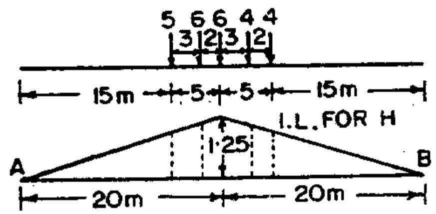


Fig. 16·17.

Fig. 16·17 shows the I.L. diagram for H having a maximum ordinate $=\frac{L}{4r}=\frac{40}{4\times 8}=1{\cdot}25$ at the central hinge. The loads have been shown in the required position. The influence line ordinates under the various loads are :

Under first 4 kN load, ordinate$=\frac{1{\cdot}25}{20}\times 15=0{\cdot}938$

Under next 4 kN load, ordinate$=\frac{1{\cdot}25}{20}\times 17=1{\cdot}062$

Under 6 kN load, ordinate$=1{\cdot}25$

Under next 6 kN load, ordinate$=\frac{1{\cdot}25}{20}\times 18=1{\cdot}125$

Under last 5 kN load, ordinate$=\frac{1{\cdot}25}{20}\times 15=0{\cdot}938$

$\therefore\ H=(4\times 0{\cdot}938)+4(1{\cdot}062)+(6\times 1{\cdot}25)+(6\times 1{\cdot}125)+(5\times 0{\cdot}938)$
$=3{\cdot}75+4{\cdot}25+7{\cdot}50+6{\cdot}75+4{\cdot}69=$ **26·94 kN**

At the section under the tail load :

Considering the tail load slightly to the right side of the section, $V_A=\frac{1}{40}[(4\times 15)+(4\times 17)+(6\times 20)+(6\times 22)+(5\times 25)]$
$=12{\cdot}625.$

The equation of the parabola is

$$=\frac{4r}{L^2}x\,(L-x)=\frac{4\times 8}{1600}x\,(40-x)=\frac{x}{50}(40-x)$$

$$\therefore \quad \frac{dy}{dx}=\tan\theta=\frac{4}{5}-\frac{x}{25}$$

At $\quad x=15,\; y=\frac{15}{50}(40-15)=7{\cdot}5$

$$\tan\theta=\frac{4}{5}-\frac{15}{25}=0{\cdot}2$$

$\therefore \quad \theta=11^\circ\,18'$; $\sin\theta=0{\cdot}196$; $\cos\theta=0{\cdot}981$

$$M=-(12{\cdot}625\times 15)+(26{\cdot}94\times 7{\cdot}5)=+\mathbf{12{\cdot}68\ kN\text{-}m}$$

$$F=H\sin\theta-V_A\cos\theta=26{\cdot}94\times 0{\cdot}196-12{\cdot}625\times 0{\cdot}981$$

$$=-7{\cdot}12\text{ kN}$$

or $\quad F=\mathbf{7{\cdot}12\ kN}\downarrow\uparrow$

$$N=H\cos\theta+V_A\sin\theta=26{\cdot}94\times 0{\cdot}981+12{\cdot}625\times 0{\cdot}196$$

$$=\mathbf{28{\cdot}89\ kN}$$

Example 16·9. *A three hinged parabolic arch has a horizontal span of 30 m with central rise of 5 m. A point load of 10 kN moves cross from left to right. Calculate the maximum positive and negative moment at the section 8 m from the left hinge.*

Also, calculate the position and amount of the absolute maximum B.M. that may occur in the arch.

Solution. (Fig. 16·14)

The equation to the parabola is

$$y=\frac{4r}{L^2}\,x(L-x)=\frac{4\times 5}{30\times 30}x(30-x)=\frac{x}{45}(30-x)$$

At $\quad x=8,\; y=\frac{8}{45}\,(30=8)=3{\cdot}91$ m.

By inspection of I.L. for B.M. at any section, it is clear that maximum negative B.M. occurs when the load is on the section while maximum positive B.M. at the section occurs when the load is on the central hinge.

When the load is on the section :

$$V_B=\frac{10\times 8}{30}=\frac{8}{3}\text{ kN}$$

For H, $\quad M_C=0=-\frac{8}{3}\times 15+H\times 5$

or $\quad H=\frac{40}{5}=8$ kN

$$\therefore \quad M_{max}\,(-\text{ve})=-\frac{8}{3}\times 22+(8\times 3{\cdot}91)=-\mathbf{27{\cdot}39\ kN\text{-}m}$$

When the load is on the central hinge :

$$V_A=\frac{10}{2}=5\text{ kN}$$

For H, $\quad M_C=0=-5\times 15+H\times 5$

$\therefore \quad H=15$ kN

$\therefore \quad M_{max}\,(+\text{ve})=-5\times 8+15\times 3{\cdot}91=+18{\cdot}65$

The absolute maximum bending moments are $+0{\cdot}0625\,WL$ and $-0{\cdot}096\,WL$. Out of these two, the negative bending moment is greater.

Hence $M_{max\cdot max\cdot}=-0{\cdot}096\,WL$

$$=-0{\cdot}096\times 10\times 30$$

$$=\mathbf{-28{\cdot}8\ kN\text{-}m.}$$

This occurs at $0{\cdot}211\,L=6{\cdot}33$ m from either end-hinge.

16·6. TWO HINGED ARCH

A two hinged arch is statically indeterminate to single degree, since there are four reaction components to be determined while the number of equations available from statical equilibrium is only three. Considering H to be the redundant reaction, it can be found out by the use of Castigliano's theorem of least work.

Thus, assuming the horizontal span remaining unchanged, we have,

$$\frac{\partial U}{\partial H}=0$$

where U is the total strain energy stored in the arch. Here also, the strain energy stored due to thrust and shear will be considered negligible in comparison to that due to bending.

$$U=\int\frac{M^2ds}{2EI}$$

$$\therefore \quad \frac{\partial U}{\partial H}=\int\frac{2M}{2EI}\cdot\frac{\partial M}{\partial H}\cdot ds$$

$$=\int\frac{M}{EI}\,\frac{\partial M}{\partial H}\cdot ds$$

Now $\quad M=\mu+Hy\;;\;\frac{\partial M}{\partial H}=y$

$$\therefore \quad \frac{\partial U}{\partial H}=0=\int\frac{(\mu+Hy)y}{EI}\,ds. \qquad \text{(I)}$$

$$\therefore \quad H\int\frac{y^2ds}{EI}=-\int\frac{\mu y ds}{EI}$$

or
$$H=-\frac{\int\frac{\mu yds}{EI}}{\int\frac{y^2ds}{EI}}$$

Taking $dx=ds\cos\theta$, and $I=I_0\sec\theta$ where I_0 is the moment of inertia at the crown, we get

$$H=-\frac{\int\mu.ydx}{\int y^2dx} \quad (16\cdot7)$$

If the two hinges are forced a distance λ apart by the thrust, λ must be added to the right hand side of equation (1). Thus,

$$\int\frac{(\mu+Hy)y}{EI}ds=\lambda$$

or
$$H=\frac{\lambda EI-\int\mu yds}{\int y^2.ds} \quad (16\cdot8)$$

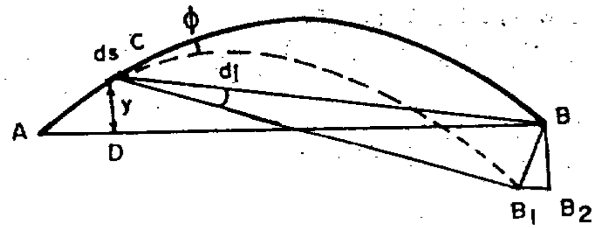


Fig. 16·18.

Alternative method :

Equation 16·7 can also be obtained by the consideration of the flexural deformation of a curved rib.

Let ACB represent the centre line of a curved rib subjected to variable B.M. Let us find the horizontal and vertical displacements of end B with reference to A. Consider the effect of B.M. on an element of length ds. Let this element turn through an angle di, the part AC of the rib being unchanged. The chord CB will therefore turn to a position CB_1 through an angle di. B_1B_2, thus, gives the horizontal displacement while BB_2 gives the vertical displacement of B.

Now
$$\begin{aligned}B_1B_2&=BB_1\cos BB_1B_2\\&=(CB.di)\cos BCD\\&=di.(CB)\cos BCD=di(CD)\\&=ydi\end{aligned} \quad (1)$$

But
$$\frac{M}{I}=\frac{E}{R}=E.\frac{di}{ds}$$

$$\therefore \quad di=\frac{Mds}{EI} \quad (2)$$

Substituting in (1), we get

$$B_1B_2=y.\frac{Mds}{EI}=\frac{Myds}{EI}$$

The total horizontal displacement of $B=\int\frac{Myds}{EI}$ (3)

It can be proved, similarly, that the total vertical displacement of $B=\int\frac{Mxds}{EI}$.

Now, for two hinged arch having no yielding of the supports, we have

$$\int\frac{Myds}{EI}=0$$

or
$$\int\frac{(\mu+Hy)yds}{EI}=0$$

From which, $$H=-\frac{\int\mu yds}{\int y^2ds} \quad (16\cdot7)$$

It is to be noted that in the above equation, $\int y^2ds$ is the property of an arch while $\int\mu yds$ depends both on the property of the arch as well as on the loading.

16·7. TWO HINGED PARABOLIC ARCH : EXPRESSION FOR H.

Consider a two hinged parabolic arch of horizontal span L and central rise r, subjected to a point load W at a distance αL from the left support.

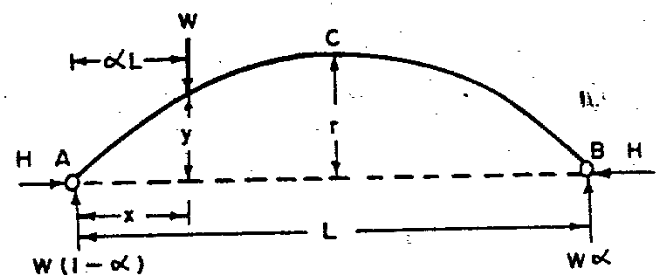


Fig. 16·19.

The equation of arch is $y=\frac{4r}{L^2}\alpha(L-x)$

Now $$H=-\frac{\int_0^L \mu y dx}{\int_0^L y^2 dx} \quad (1)$$

The numerator $=\int_0^L \mu y dx=\int_L^{\alpha L} \mu y dx+\int_{\alpha L}^L \mu y dx=a+b \quad (2)$

The quantity $a=\int_0^{\alpha L} \mu y dx=-\int_0^{\alpha L} W(1-\alpha)x.\ \frac{4r}{L^2}x(L-x)dx$

$$=-\frac{(1-\alpha)4rW}{L^2}\left(\frac{Lx^3}{3}-\frac{x^4}{4}\right)_0^{\alpha L}$$

$$=-\frac{(1-\alpha)4rW}{L^2}\left(\frac{L^4x^3}{3}-\frac{L^4\alpha^4}{4}\right) \quad (3)$$

The quantity $b=-\int_{\alpha L}^L W\alpha(L-x).\frac{4r}{L^2}x(L-x)dx$

$$=-\frac{4r\alpha W}{L^2}\int_{\alpha L}^L (L-x)^2 x dx$$

$$=-\frac{4r\alpha W}{L^2}\int_{\alpha L}^L (L^2x+x^3-2Lx^2)dx$$

$$=-\frac{4r\alpha W}{L^2}\left[\left(\frac{L^2.L^2}{2}+\frac{L^4}{4}-\frac{2LL^3}{3}\right)-\left(\frac{L^2.\alpha^2.L^2}{2}+\frac{\alpha^4L^4}{4}-\frac{2L\alpha^3L^3}{3}\right)\right]$$

$$=-\frac{4r\alpha WL^4}{12L^2}(1-6\alpha^2-3\alpha^4+8\alpha^3) \quad (4)$$

Substituting the values of a and b in (2), we get

$$\text{The numerator}=-\left[\frac{(1-\alpha)4rW}{L^2}\left(\frac{L^4\alpha^3}{3}-\frac{\alpha^4L^4}{4}\right)+\frac{4r\alpha WL^2}{12}\left(1-6\alpha^2-3\alpha^4+8\alpha^3\right)\right] \quad (5)$$

Again, the denominator $=\int_0^L y^2dx=\frac{16r^2}{L^4}\int_0^L x^2(L-x)^2dx$

$$=\frac{16r^2}{L^4}\int_0^L (x^2L^2+x^4-2Lx^3)dx$$

$$=\frac{16r^2}{L^4}\left(\frac{L^5}{3}+\frac{L^5}{5}-\frac{L^5}{2}\right)$$

$$=\frac{16r^2L}{30}(10+6-15)$$

$$=\frac{8}{15}r^2L \quad (16{\cdot}9)$$

Substituting in (1), we get

$$H=-\frac{\left[\frac{(1-\alpha)4rW}{L^2}\left(\frac{L^4\alpha^3}{3}-\frac{\alpha^4L^4}{4}\right)+\frac{4r\alpha}{12}WL^2(1-6\alpha^2-3\alpha^4-8\alpha^3)\right]}{\frac{8}{15}r^2L}$$

which reduces to

$$H=\frac{5}{8}W\frac{L}{r}\alpha(1-\alpha)(1+\alpha-\alpha^2) \quad (16{\cdot}10)$$

16·8. TWO HINGED CIRCULAR ARCH : EXPRESSION FOR H

Let θ be the half angle subtended by the arch at the centre. Let the load W be acting at a section which makes an angle ϕ with the centre line.

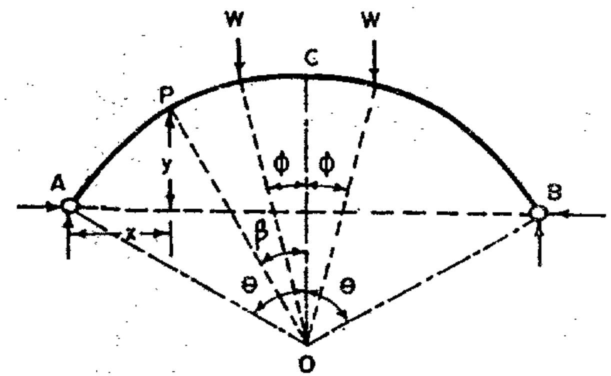


Fig. 16·20

Consider any point P subtending an angle β with the centre line.

The co-ordinates of P are given by

$$x=R(\sin\theta-\sin\beta) \quad (i)$$

and $$y=R(\cos\beta-\cos\theta) \quad (ii)$$

Also, $ds=Rd\beta$

Now, $$\int_A^B y^2ds=2\int_0^\theta R^2(\cos\beta-\cos\theta)^2\,Rd\beta$$

$$=2R^3\int_0^\theta (\cos^2\beta-2\cos\beta\cos\theta+\cos^2\theta)d\beta.$$

$$=2R^3\left\{\int_0^\theta \cos^2\beta d\beta-2\cos\theta\int_0^\theta \cos\beta d\beta+\cos^2\theta\int_0^\theta d\beta\right\}$$

which on simplification gives

$$\int_A^B y^2\, ds=\frac{R^3}{2}(4\theta\cos^2\theta+2\theta-3\sin 2\theta) \qquad (16.11)$$

$$=R^3(2\theta+\theta\cos 2\theta-1.5\sin 2\theta) \qquad [16.11\ (a)]$$

To find μyds, assume an equal load W placed symmetrically on the other side so that the integrations may be simplified. In that case $V_A=V_B=W$.

$$\therefore \quad \int_A^B \mu yds=2\left\{\int_0^\phi \mu yds+\int_\phi^\theta \mu yds\right\}=2\{a+b\} \qquad (1)$$

The integral $a=\int_0^\phi \mu yds$

$$=-\int_0^\phi R^3W(\sin\theta-\sin\phi)(\cos\beta-\cos\theta)d\beta$$

$$-=WR^3(\sin\theta-\sin\phi)\int_0^\phi(\cos\beta-\cos\theta)\,d\beta$$

$$=-WR^3(\sin\theta-\sin\phi)(\sin\beta-\beta\cos\theta)_0^\phi$$

$$=-WR^3\left(\sin\theta\sin\phi-\sin^2\phi-\frac{\phi\sin 2\theta}{2}+\phi\sin\phi\cos\theta\right) \qquad (2)$$

The second integral $=b=-WR^3\int_\phi^\theta\{\sin\theta(\cos\phi-\cos\theta)-\sin\beta\cos\beta+\cos\theta\sin\beta\}d\beta$

which on simplification gives

$$(b)=-WR^3\left[\sin^2\theta-\frac{\theta\sin 2\theta}{2}+\frac{\cos 2\theta}{4}-\cos^2\theta-\sin\theta\sin\phi+\frac{\phi\sin 2\theta}{2}-\frac{\cos 2\phi}{4}+\cos\theta\cos\phi\right] \qquad (3)$$

Adding (2) and (3) and simplifying, we get

$$\int_A^B \mu yds=-WR^3[\sin^2\theta-\sin^2\phi-2\cos\theta(\cos\theta-\cos\phi+\sin\theta-\phi\sin\phi)] \qquad (4)$$

Hence,
$$H=\frac{\int_A^B \mu y\, ds}{\int_A^B y^2\, ds}$$

$$=\frac{2WR^3[\sin^2\theta-\sin^2\phi-2\cos\theta(\cos\theta-\cos\phi+\theta\sin\theta-\phi\sin\phi)]}{R^3(4\theta\cos^2\theta+2\theta-3\sin 2\theta)}$$

Hence H for a single isolated W is given by

$$H=\frac{W[\sin^2\theta-\sin^2\phi-2\cos\theta(\cos\theta-\cos\phi+\theta\sin\theta-\phi\sin\phi)]}{(4\theta\cos^2\theta+2\theta-3\sin 2\theta)} \qquad (16.12)$$

Semi-circular Arch

If $\theta=90°=\frac{\pi}{2}$, equation (16.12) reduces to

$$H=\frac{W\cos^2\phi}{\pi} \qquad (16.13)$$

If the load is applied at the centre, we get

$$H=\frac{W}{\pi}=0.318\,W \qquad (16.14)$$

Example 16.10. *A parabolic arch, hinged at the ends has a span 30 m and rise 5 m. A concentrated load of 12 kN acts at 10 m from the left hinge. The second moment of area varies as the secant of the slope of the rib axis. Calculate the horizontal thrust and the reactions at the hinges. Also, calculate the maximum bending moment anywhere on the arch.*

Solution. Fig. (16.19)

$$V_A=\frac{12\times 20}{30}=8 \text{ kN}$$

$$V_B=12-8=4 \text{ kN}$$

$$H=\frac{\int\mu y\,dx}{\int y^2\,dx}$$

The equation of the parabola with A as origin, is

$$y=\frac{4r}{L^2}x(L-x)=\frac{4\times 5}{900}x(30-x)=\frac{x}{45}(30-x)$$

For AC, $\mu=-8x$

For CB, $\mu=-4(30-x)$

$$\therefore \quad =-\int_0^{30}\mu y\,dx=\int_0^{10}8xydx+\int_{14}^{30}4(30-x)y\,dx$$

$$=\int_0^{10}\frac{8x^2}{45}(30-x)\,dx+\int_{10}^{30}\frac{4x}{45}(30-x)^2\,dx$$

$$=\frac{8}{45}\left[\frac{30x^3}{3}-\frac{x^4}{4}\right]_0^{10}+\frac{4}{45}\left[450x^2+\frac{x^4}{4}-20x^3\right]_{10}^{30}$$

$$=\frac{44{,}000}{9}$$

$$\int_0^{30}y^2dx=\int_0^{30}\frac{x^2(30-x)^2}{45^2}dx$$

$$=\frac{1}{45^2}\int_0^{30}(900x^2+x^4-60x^3)dx$$

$$=\frac{1}{45^2}\left[300x^3+\frac{x^5}{5}-15x^4\right]_0^{30}$$

$$=400$$

$$\therefore \quad H=\frac{44000}{9\times4000}=12{\cdot}22 \text{ kN}$$

Reaction at $A=R_A=\sqrt{8^2+12{\cdot}22^2}=14{\cdot}61$ kN

Its inclination with the horizontal is given by

$$\tan\theta_A=\frac{8}{12{\cdot}22}=0{\cdot}655$$

or $\theta_A=33°\ 14'$

Reaction at B, $R_B=\sqrt{4^2+12{\cdot}22^2}=12{\cdot}85$ kN

Its reaction with the horizontal is given by

$$\tan\theta_B=\frac{4}{12{\cdot}22}=0{\cdot}327$$

or $\theta_B=18°\ 6'$

Maximum negative B.M. will occur in AC, just below the load.

Rise of the arch at the point of application of the load is given by

$$y=\frac{x}{45}(30-x)=\frac{10}{45}(30-10)=\frac{40}{9}\text{ m}$$

$$\therefore M_{max}(-\text{ve})=12{\cdot}22\times\frac{40}{9}-8\times10=-25{\cdot}69 \text{ kN-m}$$

Let the maximum positive moment occur at distance x from B

$$M_x=-4x+12{\cdot}22\times\frac{x(30-x)}{45}$$

$$\therefore \frac{dM_x}{dx}=0=-4+\frac{12{\cdot}22\times30}{45}-\frac{12{\cdot}22}{45}\times2x$$

From which $x=7{\cdot}65$ m

$$\therefore \quad M_{max}(+\text{ve})=-4\times7{\cdot}65+12{\cdot}22\times\frac{7{\cdot}65(30-7{\cdot}65)}{45}$$

$$=-30{\cdot}60+46{\cdot}40=15{\cdot}80 \text{ kN-m}$$

$\therefore$ Maximum bending moment is $-25{\cdot}69$ **kN-m** which occurs below the load.

Example 16·11. *A parabolic two hinged arch has a span of 32 metres and a rise of 8 m. A uniformly distributed load of 1 kN/m covers 8 m horizontal length of the left side of the arch. If $I=I_0 \sec\theta$ where θ is the inclination of the arch of the section to the horizontal, and I_0 is the moment of inertia of the section at the crown, find out the horizontal thrust at hinges and bending moment at 8 m from the left hinge. Also find out normal thrust and radial shear at this section.*

Solution. (Fig. 16·19)

Taking moments about B for vertical reaction at A,

$$V_A\times32=8\times1\times28$$

or $V_A=7$ kN ; and $V_B=8-7=1$ kN

The rise of the arch at any section distant x from A is given by

$$y=\frac{4\times8}{32^2}x(32-x)=\frac{x}{32}(32-x)$$

$$-\int_0^L \mu y\,dx=\int_0^8\left(7x-\frac{x^2}{2}\right)\frac{x(32-x)}{32}\,dx+\int_8^{32}1(32-x)\,\frac{x(32-x)}{32}dx$$

$$=\int_0^8\left(7x^2-\frac{x^3}{2}-\frac{7}{32}x^3+\frac{x^4}{64}\right)dx+\int_8^{32}\left(32x+\frac{x^3}{32}-2x^2\right)dx$$

$$=\left[\frac{7x^3}{3}-\frac{x^4}{8}-\frac{7x^4}{128}+\frac{x^5}{320}\right]_0^8+\left[16x^2+\frac{x^4}{128}-\frac{2x^3}{3}\right]_8^{32}$$

$$=8^3\left[\frac{7}{3}-1-\frac{7\times8}{128}+\frac{8^2}{320}\right]+\left[\left(16\times32^2+\frac{32^4}{128}-\frac{2\times32^3}{3}\right)-\left(16\times8^2+\frac{8^4}{128}-\frac{2\times8^2}{3}\right)\right]$$

$$=2477{\cdot}07$$

$$\int_0^L y^2dx=\int_0^{32}\frac{x^2(32-x)^2}{32^2}\,dx$$

$$=\frac{1}{32^2}\int_0^{32}(1024x^2+x^4-64x^3)\,dx$$

$$=\frac{1}{32^2}\left[\frac{1024x^3}{3}+\frac{x^5}{5}-\frac{64x^4}{4}\right]_0^{32}$$

$$=\frac{1024\times32}{3}+\frac{32^3}{5}-\frac{64\times32^2}{3}=1092{\cdot}16$$

$$\therefore \quad H=-\frac{\int_0^L \mu y\,dx}{\int_0^L y^2dx}=\frac{2477{\cdot}07}{1092{\cdot}16}=2{\cdot}27 \text{ kN}$$

Now, $\quad y=\dfrac{x}{32}(32-x)$

or $\quad \dfrac{dy}{dx}=\tan\theta=\dfrac{32-2x}{32}=1-\dfrac{x}{16}$

At $\quad x=8,\ y=\dfrac{8}{32}(32-8)=6$ m

and $\quad \tan\theta=1-\dfrac{8}{16}=0{\cdot}5$

$\therefore \quad \theta=26°\ 34'$; $\sin\theta=0{\cdot}447$; $\cos\theta=0{\cdot}894$

B.M. (at $x=8$)$=(-1\times24)+2{\cdot}27\times6=-24+13{\cdot}62$
$=-10{\cdot}38$ kN-m

Vertical shear$=V=1$ kN
Normal thrust$=N=H\cos\theta-V\sin\theta=2{\cdot}27\times0{\cdot}894-1\times0{\cdot}447$
$=1{\cdot}583$ kN-m.
Radial shear$=F=H\sin\theta+V\cos\theta=2{\cdot}27\times0{\cdot}447+1\times0{\cdot}894$
$=1{\cdot}909$ kN ↓ ↑

16·9. MOVING LOADS ON TWO HINGED ARCHES

(1) INFLUENCE LINE FOR H

Let us consider a unit point load at a distance αL from A. The vertical reaction at A will be $(1-\alpha)$ while the vertical reaction at B will be equal to α. For this load position the horizontal thrust, as proved earlier, is given by

$$H=\frac{5}{8}\cdot\frac{L}{r}\alpha(1-\alpha)(1+\alpha-\alpha^2) \qquad (16{\cdot}10)$$

Since α is the variable, it is clear that equation 14·10 is a fourth degree polynomial. The I.L. for H can very easily be plotted by giving α different values.

At A, $\alpha=0$; $\therefore$ $H=0$

At B, $\alpha=1$; $\therefore$ $H=0$

At C, $\alpha=\frac{1}{2}$ $\therefore$ $H=\frac{5}{8}\frac{L}{r}\frac{1}{2}\cdot\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{4}\right)=\frac{25}{128}\frac{L}{r}$ (16·15)

The I.L. diagram is shown in Fig. 16·21 (b).

(2) INFLUENCE LINE FOR B.M.

The B.M. at any point P distance x from A is given by

$$H_P=\mu+Hy$$

Thus, the I.L. for M_P can be obtained by superimposing the I.L. for μ on the I.L. for Hy. The I.L. for μ will have maximum ordinate of $\frac{x(L-x)}{L}$ under P. The I.L. for Hy will be similar or I.L. for H, with maximum ordinate of $\frac{25}{128}\frac{L}{r}y$ under the crown. The I.L. for M_P is shown in Fig. 16·21 (c).

(3) INFLUENCE LINE FOR N_P AND F_P

Influence lines of normal thrust and radial shear at R can be obtained exactly in the same manner as that for a three hinged arch except for the difference that the $H\sin\theta$ and $H\cos\theta$ diagram will

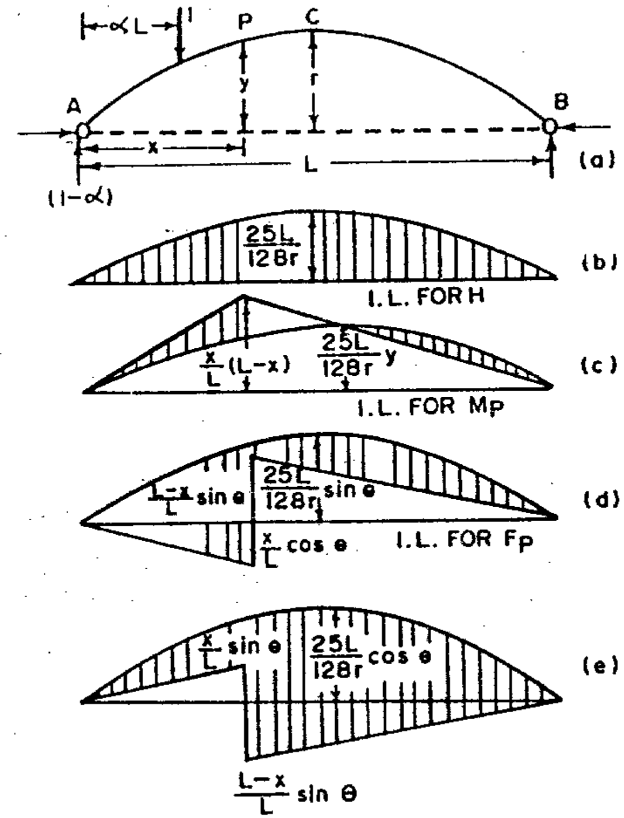


Fig. 16·21

be curved. The influence lines for F_P and N_P are shown in Fig. 16·21 (d) and 16·21 (e) respectively.

Example 16·12. *A semi-circular arch of constant section and span 2R is pinned at both supports. Find what part of the span must be covered by a uniformly distributed load w/unit length so as to produce maximum sagging bending moment at the mid-span.*

Solution

Let a unit point load be placed at a point D subtending an angle ϕ with the central line. The co-ordinates of the point D are given by

$$x=R(1-\sin\phi)$$
$$y=R\cos\phi$$

It has been proved in § 16·7 that the horizontal thrust for a two hinged semicircular arch is given by

$$H=\frac{W\cos^2\phi}{\pi}=\frac{\cos^2\phi}{\pi}, \text{ when } W=1.$$

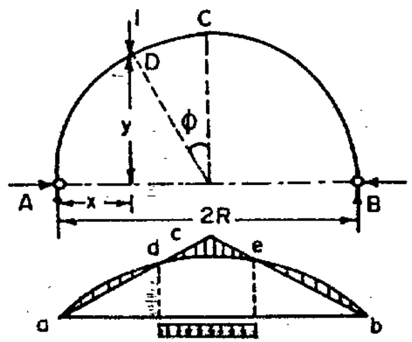


Fig. 16·22.

Let us now draw the influence line for B.M. at B, the crown. The bending moment is given by

$$M_C=\mu+Hy=\mu+HR$$

The influence line for μ will be at triangle having a maximum ordinate$=\frac{R\times R}{2R}=\frac{R}{2}$ under C. The influence line for HR will be a cosine curve having a maximum ordinate$=\frac{\cos^2 0°}{\pi}R=\frac{R}{\pi}$ under C. The influence line for M_C is thus the result of superimposition of the two, as shown in Fig. 16·22.

The influence line for M_C will have zero ordinates at points d and e. To find the position of those points, write the equation of B.M. at C and equate it to zero.

The vertical reaction at B is given by

$$V_B=\frac{1\times R(1-\sin\phi)}{2R}=\frac{1-\sin\phi}{2}$$

$$M_C=-V_BR+HR$$

$$=-\left(\frac{1-\sin\phi}{2}\right)R+\frac{\cos^2\phi}{\pi}R$$

Equating this to zero, we get

$$2\cos^2\phi+\pi\sin\phi-\pi=0.$$

The solution of which is $\sin\phi=0·571$

or $$\phi=34°\ 50'$$

Hence the load between $\phi=+34°\ 50'$ and $-34°\ 50'$ gives the position for the maximum bending moment at the mid-span.

Example 16·13. *A two hinged parabolic arch has a span of 30 m and a central rise of 5 m. Calculate the maximum positive and negative bending moment at a section distant 10 m from the left support, due to a single point load of 10 kN rolling from left to right.*

Solution

The equation of the parabola is

$$y=\frac{4\times5x}{30\times30}(30-x)=\frac{x}{45}(30-x)$$

At $x=10$, $$y=\frac{10}{45}(30-10)=4·44 \text{ m}$$

The I.L. diagram for M_P is shown in Fig. 16·23.

The ordinate (of I.L. of μ) under the load$=\frac{10\times20}{30}$

$$=6·67 \text{ kN-m}$$

The ordinate (of Hy diagram) under the crown$=\frac{25L}{128r}y.$

$$=\frac{25}{128}\times\frac{30}{5}\times4·44$$

$$=5·2 \text{ kN-m.}$$

For any load position at distance αL from A, the value of H is given by

$$H=\frac{5W}{8}\frac{L}{r}\alpha(1-\alpha)(1+\alpha-\alpha^2) \qquad (16·9)$$

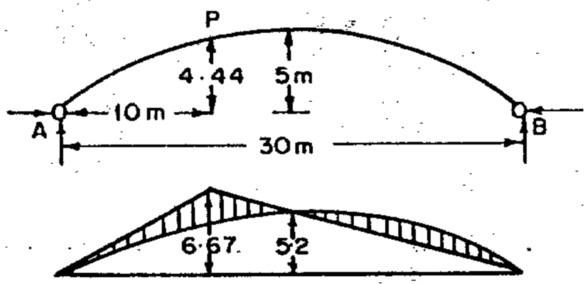


Fig. 16·23.

Maximum negative B.M. at P will evidently occur when the load is on the section P. In that case $\alpha L=10$ m.

or $$\alpha=\frac{10}{30}=\frac{1}{3}.$$

$$\therefore \quad H=\frac{5\times10}{8}\times\frac{30}{5}\times\frac{1}{3}\left(\frac{2}{3}\right)\left(1+\frac{1}{3}-\frac{1}{9}\right)=10{\cdot}19$$

Also, $V_A=W(1-\alpha)=10\times\left(1-\frac{1}{3}\right)=\frac{20}{3}$

$$\therefore \quad M_{max}(-\text{ve})=-\frac{20}{3}\times10+10{\cdot}19\times4{\cdot}44=\mathbf{21{\cdot}43\ kN\text{-}m.}$$

Maximum positive B.M. will occur when the load is some where in CB. Let the load be at a distance αL from A. Then,

$$H=\frac{5}{8}\times10\times\frac{30}{3}\,\alpha(1-\alpha)(1+\alpha-\alpha^2)$$

$$=37{\cdot}5\,(\alpha-2\alpha^3+\alpha^4)$$

$$V_A=W(1-\alpha)=10(1-\alpha)$$

Now $\quad M_P=-10(1-\alpha)\,10+37{\cdot}5(\alpha-2\alpha^3+\alpha^4)4{\cdot}44$

$$\frac{dM_P}{dx}=0=+100+37{\cdot}5\times4{\cdot}44(1-6\alpha^2+4\alpha^3)$$

or $\quad (1-6\alpha^2+4\alpha^3)+0{\cdot}6=0$

or $\quad 4\alpha^3-6\alpha^2+1{\cdot}6=0$

Solving this by trial and error, we get

$$\alpha=0{\cdot}71$$

$\therefore$ Distance of the load from $A=0{\cdot}71\times30=21{\cdot}3$ m.

Distance of the load from B $\quad =8{\cdot}7$ m.

H for this load position $\quad =37{\cdot}5(0{\cdot}71-2\times0{\cdot}358+0{\cdot}225)$

$\quad =9{\cdot}35$ kN

$V=10(1-0{\cdot}71)=2{\cdot}9$ kN

$\therefore$ $M_{max}(+\text{ve})$ at $P=-2{\cdot}9\times10+9{\cdot}35\times4{\cdot}44=+\mathbf{12{\cdot}5\ kN\text{-}m.}$

16·10. TEMPERATURE EFFECTS

(1) Three Hinged Arch

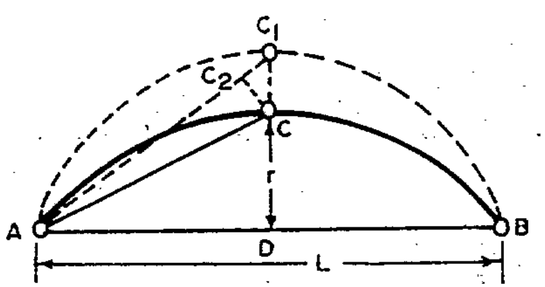


Fig. 16·24.

Due to increase in temperature, the length of the arch ACB will increase. Since the two end hinges are rigidly fixed, the crown C will rise from C to C_1. Thus AC_1B will be new centre line of the arch. *No temperature stresses will, therefore, be induced.* Let us find out the value of CC_1 for a given increase (or decrease) in the temperature.

AC_1 is the position of the chord. Make $AC_2=AC$. Then increase in the length of the chord $=AC_1-AC_2=C_1C_2=AC.\ \alpha t.$

$$\therefore \quad CC_1=C_1C_2\sec CC_1C_2$$

$$=C_1C_2\sec DCA,\ \text{since } CAC_1 \text{ is small}$$

$$=(AC.\ \alpha\ t)\frac{AC}{AD}=\frac{AC^2}{CD}.\ \alpha\ t$$

$$=\frac{(AD^2+DC^2)}{CD}\ \alpha\ t=\frac{\left(\frac{L^2}{4}+r^2\right)}{r}\ \alpha\ .\ t$$

$$=\frac{L^2+4r^2}{4r}\alpha t \qquad (16{\cdot}16)$$

(2) Two Hinged Arch

Since there is no central hinge in the case of two hinged arch the end hinges will exert a horizontal thrust on the arch to prevent the ends from moving out when the temperature of the arch increases. Due to this horizontal thrust, there will be bending moment at all the sections.

Let H_t be the horizontal thrust induced due to a rise in temperature by $t°$. The increase in horizontal span of arch $=L\alpha t$ where α = co-efficient of thermal expansion. The bending moment on any element at a height y is $M=H_ty$. We have already seen that total increase in span due to bending of curved bar $=\int_0^L \frac{My}{EI}\,ds.$

Evidently, $\quad \int_0^L \frac{M\cdot y^2}{EI}\,ds=L\ \alpha\ t$

or $\quad \int_0^L \frac{H_ty^2}{EI}\,ds=L\ \alpha\ t$

or $$H_t=\frac{L\ \alpha\ t}{\int_0^L \frac{y^2ds}{EI}}=\frac{EI\ .\ L\ \alpha\ t}{\int_0^L y^2ds} \qquad (16{\cdot}17)$$

Example 16·14. *A two hinged parabolic arch of span 40 m and rise 8 m is subjected to a temperature rise of 22 K. Calculate the maximum bending stress at the crown due to the temperature rise if*

$\alpha=11\times10^{-6}$ per 1 °K and $E=2{\cdot}1\times10^5$ N/mm². The rib section is symmetrical and 1 m deep.

Solution

The equation of the parabola is

$$y=\frac{4r}{L^2}\,x(L-x)=\frac{4\times8}{40\times40}\,x(40-x)$$

$$=\frac{x}{50}\,(40-x)$$

$$E=2{\cdot}1\times10^5\ \text{N/mm}^2=2{\cdot}1\times10^8\ \text{KN/m}^2$$

$$EI\,.\,L\,\alpha\,t=(2{\cdot}1\times10^8)\,I\,(40\times11\times10^{-6}\times22)$$

$$=2{\cdot}0328\times10^6\,I\ \text{kN-m}^3$$

(where I is in m⁴ units)

$$\int_0^L y^2ds=\int_0^{40}\frac{x^2}{2500}\,(40-x)^2\,dx$$

$$=\int_0^{40}\frac{x^2}{2500}\,(1600+x^2-80x)\,dx$$

$$=\frac{1}{2500}\left(1600\,\frac{x^3}{3}+\frac{x^5}{5}-\frac{80x^4}{4}\right)_0^{40}$$

$$=\frac{(40)^3}{2500}\left(\frac{1600}{3}+\frac{1600}{5}-\frac{80\times40}{4}\right)\text{m}^5$$

$$=1360\ \text{m}^5$$

$$H_t=\frac{EI\,L\,\alpha t}{\int_0^L y^2ds}=\frac{2{\cdot}0328\times10^6 I}{1360}$$

$$=1495\,I\ \text{kN}\qquad\text{(where } I \text{ is in m}^4\text{ units)}$$

$$=1495\,I\times8$$

Maximum B.M. at crown $=1495I\times8$

$$=11960\,I\ \text{kN-m}$$

Maximum bending stress at crown

$$f=\frac{M}{Z}=\frac{11960\,I}{\frac{I}{0{\cdot}5}}\qquad\left(\text{where } Z=\frac{I}{y}=\frac{I}{0{\cdot}5}\right)$$

$$=11960\times0{\cdot}5=5980\ \text{kN/m}^2$$

$$=5{\cdot}98\ \text{N/mm}^2.$$

Example 16·15. *A steel two hinged circular arch rib has a span of 30 m and a rise of 3 m. The rib section is uniform throughout with an overall depth of 0·7 m. Neglecting all effect except those due to bending, find, from first principles, the bending stress at the crown due*

to a temperature change of 30 K. Take $E=2\times10^5$ N/mm² and $\alpha=11\times10^{-6}$ per 1 K.

Solution

The radius R of the arch is given by

$$r(2R-r)=\frac{L^2}{4},\qquad\text{(where } r=\text{central rise)}$$

$$2r-3=\frac{30\times30}{4\times3}=75$$

$$\therefore\qquad R=\frac{1}{2}\,(75+3)=39\ \text{m.}$$

$$\sin\theta=\frac{L}{2R}=\frac{30}{2\times39}=0{\cdot}3846$$

$$\theta=22{\cdot}62^\circ=0{\cdot}3948\ \text{radian}$$

$$\cos\theta=0{\cdot}9231\ ;\ \sin2\theta=0{\cdot}710\ ;\ \cos2\theta=0{\cdot}7041$$

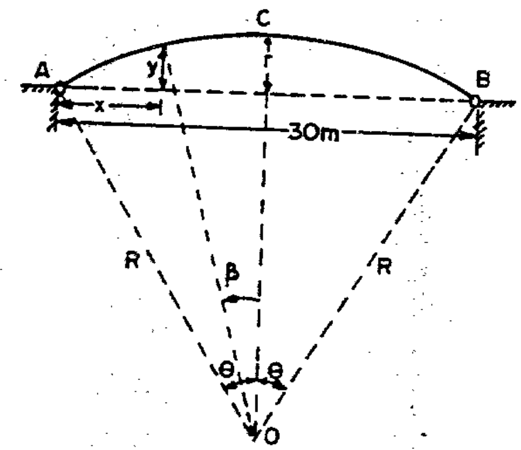


Fig. 16·25.

The y co-ordinate of any point P is given by

$$y=R\,(\cos\beta-\cos\theta)$$

Also, $$ds=R\,d\beta$$

$$\int_0^L y^2ds=2\int_0^\theta R^2\,(\cos\beta-\cos\theta)^2R\,d\beta$$

$$=2R^3\int_0^\theta(\cos^2\beta-2\cos\beta\cos\theta+\cos^2\theta)\,d\beta$$

$$=R^3\int_0^\theta\left\{(1+\cos2\beta)-4\cos\theta\cos\beta+2\cos^2\theta\right\}d\theta$$

$$=R^3\left[\beta+\frac{\sin 2\beta}{2}-4\cos\theta\sin\beta+2\beta\cos^2\theta\right]_0^\theta$$

$$=R^3\left[\theta+\frac{\sin 2\theta}{2}-4\cos\theta\sin\theta+2\theta\cos^2\theta\right]$$

$$=R^3\left[\theta-\frac{3}{2}\sin 2\theta+\theta(1+\cos 2\theta)\right]$$

$$=R^3\left[2\theta-1{\cdot}5\sin 2\theta+\theta\cos 2\theta\right]$$

Which is the same as Eq. 16·11(*a*) derived earlier. Substituting the numerical values, we get

$$\int_0^{L'} y^2 ds=R^3\left[0{\cdot}7896-1{\cdot}5\times0{\cdot}710+0{\cdot}3948\times0{\cdot}7041\right]$$

$$=0{\cdot}002579\ R^3=0{\cdot}002579(39)^3=152{\cdot}98\ \text{m}^3$$

$$E=2\times10^5\ \text{N/mm}^2=2\times10^8\ \text{kN/m}^2$$

$$EIL.\alpha t=(2\times10^8)(I)\ (30\times11\times10^{-6}\times30)$$

$$=1{\cdot}98\times10^6\ I\ \text{kN-m}^3$$

(where I is in m^4 units)

$$\therefore\qquad H_t=\frac{EIL.\alpha t}{\int y^2 ds}=\frac{1{\cdot}98\times10^6\ I}{152{\cdot}98}=12943\ I$$

The max. B.M. at the crown

$$=H_t r=12943\ I\times3$$

$$=38829\ I\ \text{kN-m}$$

Maximum bending stress

$$F=\frac{M}{Z}=\frac{38829\ I}{I/0{\cdot}35}$$

$$=13590\ \text{kN/m}^2$$

$$=\mathbf{13{\cdot}59\ N/mm^2}$$

Example 16·16. *A semicircular ring of radius R and uniform flexural stiffness, loaded by equal and opposite forces R at the points A and B, is shown in Fig. 14·96. Show that the separation of points A and B is* $\frac{\pi WR^3}{2EI}$, *if only bending deformations are taken into consideration. If the effect of axial pull is also taken into account, what will be the separation of the two points ?*

Solution

Consider any element of length ds, subtending an angle $d\theta$ at the centre. At any angular position θ, we have,

$$M=Wy=W\,R\sin\theta$$

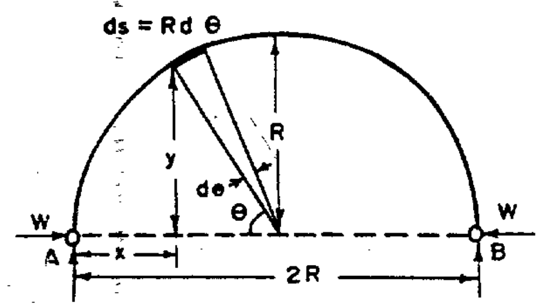


Fig. 16·26

Normal thrust $=N=W\sin\theta$.

Now, the *total* strain energy is given by

$$U=\int\frac{M^2 ds}{2EI}+\int\frac{N^2 ds}{2AE}$$

$$=\int_0^\pi\frac{W^2R^2\sin^2\theta}{2EI}\ .\ Rd\theta+\int_0^\pi\frac{W^2\sin^2\theta}{2AE}\ .\ Rd\theta$$

$$=W^2\left[\frac{R^3}{2EI}+\frac{R}{2AE}\right]\int_0^\pi\sin^2\theta\,d\theta=\frac{W^2\pi}{4}\left[\frac{R^3}{EI}+\frac{R}{AE}\right]$$

Now, according to Castigliano's first theorem, the separation of points A and B is given by

$$\delta=\frac{\partial U}{\partial W}=\frac{W\pi}{2}\left(\frac{R^3}{EI}+\frac{R}{AE}\right)=\frac{\pi WR^3}{2EI}\left(1+\frac{I}{AR^2}\right)\qquad(1)$$

The above expression gives the separation of the two points when both bending and axial pull are taken into consideration. If however, the axial pull is neglected, we get

$$\delta=\frac{\pi WR^3}{2EI}.$$

It is to be noted that the term $\frac{I}{AR^2}$ in equation (1) is extremely small in comparison to unity. This is why, the strain energy due to normal thrust is generally neglected in many cases.

Example 16·17. *The segmental rib ACB shown in Fig. 16·27 is of constant section throughout and carries a load W at its mid-point C. Ends A and B are tied together with a tie rod whose extensibility*

$$AE=\frac{50EI}{R^3}$$

where EI is the flexural rigidity of the rib AB. Calculate the load in the tie. *(St. Andrews)*

Solution

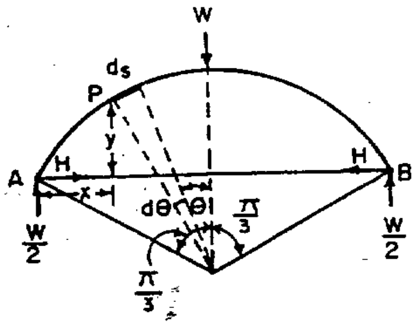


Fig. 16·27.

Let the tension in the tie rod be H.

If U is the total strain energy due to bending, we have

$$\frac{\partial U}{\partial H}=\text{shortening of tie rod}=-\frac{HL}{AE}=-\frac{H\sqrt{3}R.R^2}{50EI}=-\frac{\sqrt{3}HR^3}{50EI} \quad (1)$$

Consider an element of length $ds=Rd\theta$. Let x and y be the co-ordinates of the element the origin being taken at A. We have thus,

$$x=R\left(\sin\frac{\pi}{3}-\sin\theta\right)=R\left(\frac{\sqrt{3}}{2}-\sin\theta\right)$$

$$y=R\left(\cos\theta-\cos\frac{\pi}{3}\right)=R\left(\cos\theta-\frac{1}{2}\right)$$

$$M_P=-\frac{W}{2}R\left(\frac{\sqrt{3}}{2}-\sin\theta\right)+HR\left(\cos\theta-\frac{1}{2}\right)$$

$$\frac{\partial M_P}{\partial H}=R\left(\cos\theta-\frac{1}{2}\right)$$

$$\therefore \quad \frac{\partial U}{\partial H}=2\int_0^{\pi/3}M_P\cdot\frac{\partial M_P}{\partial H}\cdot\frac{1}{EI}\cdot ds$$

$$=\frac{2}{EI}\int_0^{\pi/3}\left\{-\frac{W}{2}R\left(\frac{\sqrt{3}}{2}-\sin\theta\right)+HR\left(\cos\theta-\frac{1}{2}\right)\right\}\times\left\{R\left(\cos\theta-\frac{1}{2}\right)\right\}Rd\theta$$

$$=\frac{2R^3}{EI}\int_0^{\pi/3}\left[-\frac{W\sqrt{3}}{4}\cos\theta+\frac{W\sin\theta\cos\theta}{2}+H\cos^2\theta-\frac{H\cos\theta}{2}+\frac{W\sqrt{3}}{8}-\frac{W\sin\theta}{4}-\frac{H\cos\theta}{2}+\frac{H}{4}\right]d\theta$$

$$=\frac{2R^3}{EI}\left[-\frac{W\sqrt{3}}{4}\sin\theta-\frac{W}{8}\cos 2\theta+\frac{H\theta}{2}+\frac{H\sin 2\theta}{4}-\frac{H\sin\theta}{2}+\frac{W\sqrt{3}}{8}\theta+\frac{W\cos\theta}{4}-\frac{H\sin\theta}{2}+\frac{H\theta}{4}\right]_0^{\pi/3}$$

$$=\frac{2R^3}{EI}\left[-0{\cdot}085\,W+0{\cdot}135\,H\right]$$

Equating (1) and (2), we have

$$\frac{2R^3}{EI}\left[-0{\cdot}085\,W+0{\cdot}135\,H\right]=-\frac{\sqrt{3}HR^3}{50EI}$$

From which, $H=$**0·56 W.**

16·11. REACTION LOCUS FOR TWO HINGED ARCH

The reaction locus is a line which gives the point of intersection of the two reactions for any position of an isolated load.

Parabolic Arch

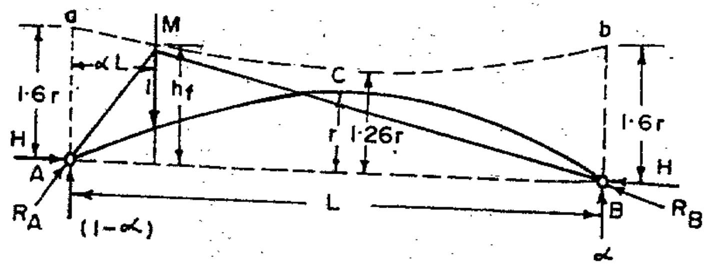


Fig. 16·28.

In Fig. 16·28, aMb shows the reaction locus due to a single point load rolling from A to B. The vertical line through the point P (where the unit load is acting at any instant) intersects the reactions locus at the point M. Thus, MA and MB gives the directions of reaction at A and B. Thus, we get the line of pressure AMB (or the linear arch) at once. The B.M. at any point is then equal to the horizontal thrust multiplied by the vertical distance between the line of pressure and the centre line of the arch at the point.

Let $h_f=MN=$ordinate of reaction locus at any point P.

Then, by triangle of force AMN, we have

$$\frac{AN}{H}=\frac{MN}{V_A}$$

or

$$\frac{\alpha L}{H}=\frac{h_f}{1-\alpha}$$

or

$$h_f=\frac{\alpha L(1-\alpha)}{H}$$

But $$H=\frac{5}{8}\,\frac{L}{r}\,\alpha(1-\alpha)(1+\alpha-\alpha^2)$$

Hence $$h_f=\frac{\alpha L(1-\alpha)}{\frac{5}{8}\,\frac{L}{r}\,\alpha(1-\alpha)(1+\alpha-\alpha^2)}=\frac{8}{5}\,\frac{r}{1+\alpha-\alpha^2}$$

or $$h_f=\frac{1{\cdot}6r}{1+\alpha-\alpha^2} \qquad (16{\cdot}18)$$

Equation 16·18 gives the equation of the reaction-locus, which can be plotted by giving α different values. Thus,

When $\alpha=0$, $h_f=1{\cdot}6r$

$\alpha=1$, $h_f=1{\cdot}6r$

$\alpha=\frac{1}{2}$ $h_f=1{\cdot}28r$

Semi-circular Arch

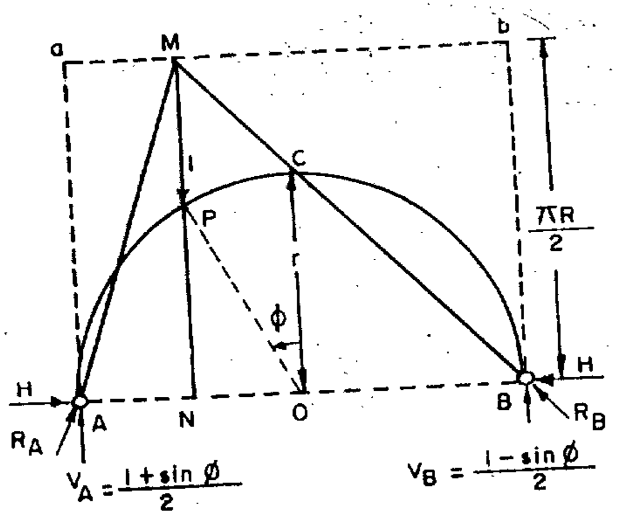


Fig. 16·29.

For semi-circular arch, we have

$$H=\frac{\cos^2\phi}{\pi}=\left(\frac{1-\sin^2\phi}{\pi}\right) \qquad (16{\cdot}13)$$

Now $$\frac{AN}{H}=\frac{MN}{V_A} \qquad (1)$$

But $$AN=AO-ON=R(1-\sin\phi)$$

$$V_A=\frac{BN}{AB}=\frac{R(1+\sin\phi)}{2R}\quad\left(\frac{1+\sin\phi}{2}\right)$$

Substituting in (1), we get

$$h_f=MN=\frac{AN}{H}V_A=\left\{\frac{R(1-\sin\phi)}{\frac{(1-\sin^2\phi)}{\pi}}\right\}\frac{(1+\sin\phi)}{2}$$

or $$h_f=\frac{\pi R}{2} \qquad (16{\cdot}19)$$

Thus, the ordinate h_f is constant and does not depend on the load position. The reaction locus is thus a straight line parallel to AK, and having ordinate to $\frac{\pi R}{2}$.

16·12 FIXED ARCH

In the case of a fixed arch, there are six reaction components (*i.e.* V, H and M_F at either end) to be determined. Since only three equations are available from static equilibrium, fixed arch is statically indeterminate to *third degree*. We must have, therefore, three equations from the consideration of elastic deformations. Since the ends A and B are position-fixed and direction-fixed we have the following conditions to satisfy :

(*a*) Horizontal movement of B with respect to $A=0$.

e.g. $$\int_A^B \frac{My}{EI}ds=0 \qquad (1)$$

or for finite elements, $\sum_A^B \frac{My}{EI}ds=0$

(*b*) Vertical movement of B with respect to $A=0$.

i.e. $$\int_A^B \frac{Mx}{EI}ds=0 \qquad (2)$$

or $$\sum_A^B \frac{Mx}{EI}ds=0.$$

(*c*) Change of slope of the end B with respect to $A=0$.

Now $$\frac{M}{I}=\frac{E}{R}=E\,\frac{di}{ds}$$

$$\therefore \quad di=\frac{Mds}{EI}$$

$$\therefore \quad \int_A^B \frac{Mds}{EI}=0 \qquad (3)$$

or $$\sum_A^B \frac{Mds}{EI}=0$$

Thus we have got three additional equations, the simultaneous solution of which gives the required results.

Fig. 16·30 shows a fixed parabolic arch subjected to a single point load W at a distance αL from A. The equation for B.M. at any point p distant x from A may be written as

$$M=\mu+M_A+(M_B-M_A)\frac{x}{L}+Hy$$

where M_A and M_B are the fixing moments at ends A and B.

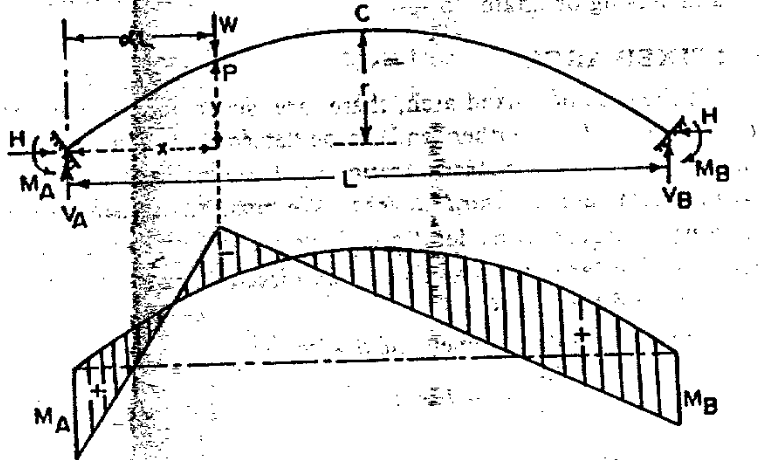


Fig. 16·30

Substituting the value of M in equations (1), (2) and (3) we get

$$\int\frac{My}{EI}\,ds=\int\frac{\mu y}{EI}\,ds+M_A\int\frac{yds}{EI}+\frac{M_B-M_A}{L}\int\frac{xyds}{EI}+H\int\frac{y^2}{EI}ds=0 \quad (16·20)$$

$$\int\frac{Mx}{EI}\,ds=\int\frac{\mu x}{EI}\,ds+M_A\int\frac{xds}{EI}+\frac{M_B-M_A}{L}\int\frac{x^2ds}{EI}+H\int\frac{xy}{EI}\,ds=0 \quad (16·21)$$

and $$\int\frac{M}{EI}\,ds=\int\frac{\mu}{EI}\,ds+M_A\int\frac{ds}{EI}+\frac{M_B-M_A}{L}\int\frac{xds}{EI}+H\int\frac{yds}{EI}=0 \quad (16·22)$$

From the above three equations, M_A, M_B and H can be determinad.

Example 16·18. *A circular arched rib of uniform cross-section is fixed at A and B and is subjected to vertical loads as shown in Fig. 16·31 (a). Find the magnitudes of the vertical reactions at A and B.*

Solution

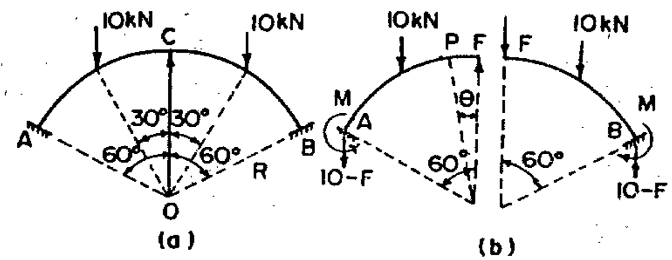


Fig. 16·31.

Due to the skew-symmetrical loading, there will be no deflection and bending moment at the midspan. There is no resultant load on the span, and hence there will be no horizontal reaction at the supports. The arch can, therefore, be splitted into two halves, as shown in Fig. 16·31 (*b*). Let the S.F. at the midspan be F. Take F as the redundant.

Consider any point P subtending an angle θ with the central line. We have

$$M_P=-FR\sin\theta;\ +10\,R\,(\sin\theta-\sin 30°)$$

$$\frac{\partial M_P}{\partial F}=-R\sin\theta.$$

Also, $$ds=Rd\theta$$

Now $$\frac{\partial U}{\partial F}=\int_0^{60°}M_P.\ \frac{\partial M_P}{\partial F}\frac{I}{EI}ds$$

or $$EI\frac{\partial U}{\partial F}=FR^3\int_0^{60°}\sin^2\theta.d\theta-10R^3\int_{30}^{60°}(\sin^2\theta-\sin 30°\sin\theta)d\theta$$

$$=FR^3\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]_0^{60°}-10R^3\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}+\sin 30°\ c\ s\quad\right]_{30°}^{60°}$$

$$=FR^3\left[30°-\frac{\sqrt{3}}{8}\right]-10R^3\left[\left\{30°-\frac{\sqrt{3}}{8}+\frac{1}{2}\ \frac{1}{2}\right\}-\left\{15°-\frac{\sqrt{3}}{8}+\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right\}\right]$$

$$=FR^3(0·524-0·216)-10R^3[0·524-0·216+0·25-0·262+0·216.-0·432]$$

$$=0·308\ FR^3-0·8\ R^3$$

Equating this to zero, we get

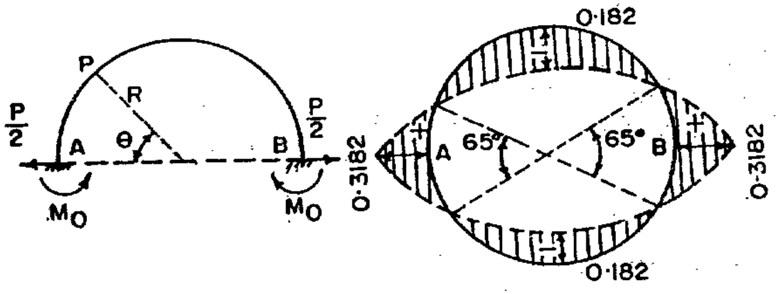
$$F=\frac{0·8}{0·308}=2·6\text{ kN}$$

$\therefore$ Reaction at $A = 10 - 2{\cdot}6 = 7{\cdot}4 \uparrow$ kN

$\therefore$ Reaction at $B = 7{\cdot}4 \downarrow$ kN.

Example 16·19. *A thin circular proving ring of radius 10 cm and uniform flexural stiffuess EI carries concentrated load 10 kN applied at the ends of a diameters. Find the maximum bending moment in the ring, and estimate the separation of the loaded points. The thickness of the ring in its own plane is 20 mm and the breadth is 40 mm. Take $E = 2 \times 10^5$ N/mm²*

Solution



(a) (b)

Fig. 16·32.

Let the load be P and radius be R.

Due to the symmetrical loading, it is evident that there will be no axial load in the ring at the loaded points A and B. Cut the ring in two parts at the level of AB and fix the point A and B. The upper half ring [Fig. 16·32(*a*)] carries load of $\frac{1}{2}P$ and moments M_0 at A and B. The only unknown is M_0.

At any point P, the B.M. is given by

$$M_P = M_0 - \frac{1}{2} PR \sin \theta \qquad (1)$$

$$\therefore \quad \frac{\partial M_P}{\partial M_0} = 1$$

Also $\quad ds = R d\theta$

$$\therefore \quad \frac{\partial U}{\partial M_0} = \int_0^{\pi} M_P \frac{\partial M_P}{\partial M_0} ds = 0$$

or $$\int_0^{\alpha} \left(M_0 - \frac{1}{2} PR \sin \theta \right) R d\theta = 0$$

$$\left(M_0 \theta + \frac{1}{2} PR \cos \theta \right)_0^{\pi} = 0$$

$$\therefore \quad M_0 = \frac{PR}{\pi} \qquad (2)$$

$$\therefore \quad M_P = \frac{PR}{\pi} - \frac{1}{2} PR \sin \theta$$

which is the expression for B.M. at any point.

At $\quad \theta = 0, \quad M = +\dfrac{PR}{\pi}$

$$= +\frac{10000 \times 100}{\pi} = 3{\cdot}182 \times 10^5 \text{ N-mm}$$

$$= 0{\cdot}3182 \text{ kN-m}$$

At $\quad \theta = 90°, \quad M = \dfrac{PR}{\pi} - \dfrac{PR}{2}$

$$= (10000 \times 100)(0{\cdot}318 - 0{\cdot}5)$$

$$= -1{\cdot}82 \times 10^5 \text{ N-mm} = -0{\cdot}182 \text{ kN-m}$$

For points of contraflexure, equate M_P to zero.

Thus $$\frac{PR}{\pi} - \frac{PR}{2} \sin \theta = 0$$

which gives $\quad \theta = 32{\cdot}5°$

The B.M. diagram is shown in Fig. 16·32 (*b*).

The deflection of loaded points is given by

$$\delta = \frac{\partial U}{\partial P}$$

$$= \int_0^{\pi} \left(M_0 - \frac{1}{2} PR \sin \theta \right) \left(-\frac{1}{2} R \sin \theta \right) \frac{R d\theta}{EI}$$

$$= \left[\frac{\pi^2 - 8}{4\pi} \right] \frac{PR^3}{EI}$$

where $\quad I = \dfrac{1}{12} \times 40 \times 20^3 = 2{\cdot}67 \times 10^4 \text{ mm}^4$

$$\therefore \quad \delta = \frac{1{\cdot}870}{4\pi} \times \frac{1000(100)^3}{2{\cdot}1 \times 10^5 \times 2{\cdot}67 \times 10^4}$$

$$= 0{\cdot}265 \text{ mm.}$$

16·13. THREE HINGED SPANDRAL BRACED ARCH

Fig. 16·33(*a*) shows a three hinged spandril braced arch. The structure is statically determinate, since the horizontal reaction at the hinges can be determined by taking moments about the central hinge. The stresses in various members, due to static loading, can be easily found out by the method of sections. We shall draw the influence lines for the forces in the members of the panel DE.

(*a*) **I.L. for H**

The I.L. for H can be obtained exactly in the same manner as that for a three hinged arch. Thus, the I.L. for H will be a triangle having a central ordinate of $\frac{L}{4r}=\frac{60}{4\times12\cdot5}=1\cdot2$, as shown in Fig. 16·33 (*b*).

(*b*) **I.L. for P_{DE}**

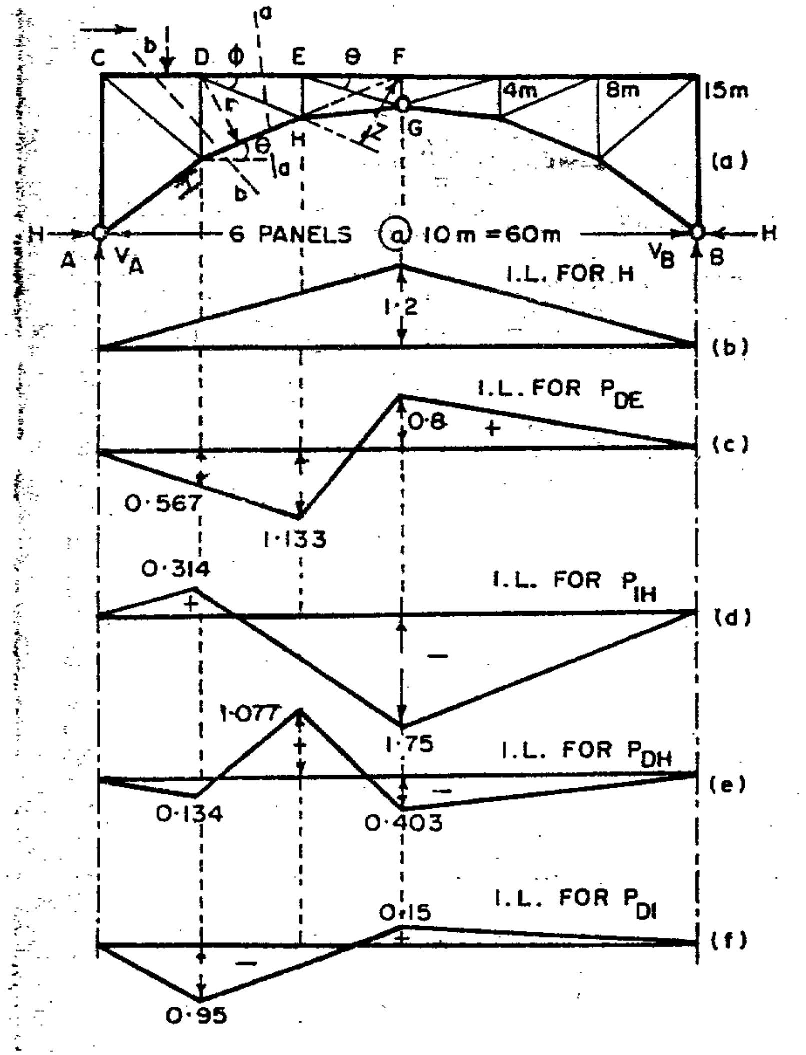


Fig. 16·33.

Pass a section aa and consider the equilibrium of all the forces to the left of it. Taking moments about point H, we get

$$P_{DE}=\frac{M_H}{EH}=\frac{M_H}{4}$$

When the unit load is at D, $V_A=\frac{5}{6}$, and $H=\frac{1\cdot2}{30}\times10=0\cdot4$

$$\therefore\quad P_{DE}=\frac{1}{4}\left[\left(\frac{5}{6}\times20\right)-\left(0\cdot4\times11\right)-\left(1\times10\right)\right]=0\cdot567$$

(comp.)

When the unit load is at E, $V_A=\frac{4}{6}=\frac{2}{3}$.

and $$H=\frac{1\cdot2}{30}\times20=0\cdot8$$

$$\therefore\quad P_{DE}=\frac{1}{2}\left[\left(\frac{2}{3}\times20\right)-(0\cdot8\times11)\right]=1\cdot133 \text{ (comp.)}$$

When the load is at F, $V_A=0\cdot2$ and $H=1\cdot2$

$$\therefore\quad P_{DE}=\frac{1}{4}\left[\left(1\cdot2\times11\cdot0\right)-\left(0\cdot5\times20\right)\right]=0\cdot8 \text{ (tension)}$$

Thus, the stress in DE changes sign as the unit load moves from E to F, as shown in Fig. 16·33 (*c*).

(*c*) **I.L. for P_{IH}**

Consider the equilibrium of all the forces to the left of the section aa and take moments about D.

$$\therefore\quad P_{IH}=\frac{M_D}{r}$$

where r=perpendicular distance of IH from D.

Prolong IH so as to meet the top boom. Let θ be the inclination of IH with horizontal.

$$\tan\theta=\frac{8-4}{10}=0\cdot4\ ;\ \therefore\ \theta=21°48'\ ;\ \sin\theta=0\cdot371$$

Since $EH=\frac{1}{2}DI$, the member IH will meet the top boom in IF.

Now $r=DF\sin\theta=20\times0\cdot371=7\cdot42$ m

$$\therefore\quad P_{IH}=\frac{M_D}{7\cdot42}$$

When the unit load is at D, $V_A=\frac{5}{6}$ and $H=\frac{1\cdot2}{30}\times10=0\cdot4$

$$\therefore\quad P_{IH}=\frac{1}{7\cdot42}\left[\left(\frac{5}{6}\times10\right)-\left(0\cdot4\times15\right)\right]=0\cdot314 \text{ (tension)}$$

When the unit load is at F, $V_A=0{\cdot}5$ and $H=1{\cdot}2$.

$$\therefore \quad P_{IH}=\frac{1}{7{\cdot}42}\left[\left(1{\cdot}2\times15\right)-\left(0{\cdot}5\times10\right)\right]=1{\cdot}75 \text{ (comp.)}$$

The I.L. for P_{IH} is shown in Fig. 16·30(d).

(d) I.L. for P_{DH}

Consider the equilibrium of the forces of the left of the section aa and take moments about the point E where the members DE and IH meet. Thus

$$P_{DH}=\frac{M_F}{z}$$

where z is perpendicular distance of DH from F.

The inclination ϕ of DH is given by

$$\tan\phi=\frac{4}{10}=0{\cdot}4\;;\;\therefore\;\phi=21°48' \text{ and } \sin\phi=0{\cdot}371$$

$$z=DF\sin\phi=20\times0{\cdot}371=7{\cdot}42 \text{ m}$$

$$\therefore \quad P_{DH}=\frac{M_F}{7{\cdot}42}$$

When the unit load is at D, $V_A=\frac{5}{6}$ and $H=0{\cdot}4$

$$\therefore \quad P_{DH}=\frac{1}{7{\cdot}42}\left[\left(0{\cdot}4\times15\right)-\left(\frac{5}{6}\times30\right)+\left(1\times20\right)\right]=0{\cdot}134 \text{ (comp.)}$$

When the unit load is at E, $V_A=\frac{2}{3}$ and $H=0{\cdot}8$.

$$\therefore \quad P_{DH}=\frac{1}{7{\cdot}42}\left[\left(\frac{2}{3}\times30\right)-\left(0{\cdot}8\times15\right)\right]=1{\cdot}077 \text{ (tension)}$$

When the unit load is at F, $V_A=0{\cdot}5$ and $H=1{\cdot}2$

$$\therefore \quad P_{DH}=\frac{1}{7{\cdot}42}\left[\left(1{\cdot}2\times15\right)-\left(0{\cdot}5\times30\right)\right]=0{\cdot}403 \text{ (comp.)}$$

The I.L. for P_{DH} is shown in Fig. 16·33 (e).

(e) I.L. for P_{DI}

Pass a section bb and consider the equilibrium of all the forces to the left of it.

$$\text{Thus,} \quad P_{DI}=\frac{M_F}{DF}=\frac{M_F}{20}$$

When the unit load is at D, $V_A=\frac{5}{6}$ and $H=0{\cdot}4$

$$\therefore \quad P_{DI}=\frac{1}{20}\left[\left(\frac{5}{6}\times30\right)-\left(0{\cdot}4\times15\right)\right]=0{\cdot}95 \text{ (comp.)}$$

When the unit load is at F, $V_A=0{\cdot}5$ and $H=1{\cdot}2$

$$\therefore \quad P_{DI}=\frac{1}{20}\left[\left(1{\cdot}2\times15\right)-\left[0{\cdot}5\times30\right)\right]=0{\cdot}15 \text{ (tension)}$$

The I.L. for P_{DI} is shown in Fig. 11·33 (f).

PROBLEMS

1. State briefly what do you understand by an arch.

A parabolic arch rib, 20 m span and 3 m rise is hinged at the abutments and the crown and carries a point load of 10 kN at 7·5 m from the left hand hinge. Calculate the horizontal thrust and the bending moment at a section 7·5 m form right-hand hinge. What is the value of the greatest bending moment in the arch, and where does it occur ?

2. A three-hinged parabolic arch has a span of 2·4 m and a rise to the central hinge of 4 m. The arch is loaded with two vertical 20 kN loads symmetrically situated on either side of the central hinge at 3 m horizontally from the hinge. Calculate the values of the maximum positive and negative bending moments in the arch stating where these occur.

3. A three hinged parabolic arch of 30 meter span and 6 m central rise carries a point load of 6 kN at 8 m horizontally from the left hand hinge. Calculate the normal thrust and shear force at the section under the load. Also calculate the maximum B.M. positive and negative.

4. A three hinged parabolic arch rib has a span of 84 m and a rise of 18 m to the central pin at the crown. The rib carries load of intensity 2 kN per m uniformly distributed horizontally over a length of 1/3 of the span from the left hand springing. Calculate the bending moments in the rib at the quarter span points. *(Based on U.L.)*

5. A parabolic three hinged arch of span L and rise c is hinged at the springings and the centre of the span. It carries a uniformly distributed load of intensity w per unit of horizontal length over a length x (where x between 1/4 L and 1/2 L) extending from the left-hand end.

Derive expressions for :

(a) the vertical components of reactions at the ends.
(b) the horizontal thrust.
(c) the bending moment at the two 1/4 points (1/4 L from each end).
(d) the maximum bending moment at the left hand 1/4 point. *(U.L.)*

6. A three-hinged arch rib has the form of a quadrant of a circle with the chord horizontal and joining springing 20 m apart. The rib carries concentrated load of 20 kN acting through 1/4 span.

Calculate :

(a) the end reactions,
(b) the reaction at the crown.
(c) the bending moment at the load point.
(d) the bending moment at the other 1/4 span point.

7. Fig. 16·34 shows a three pinned parabolic arch, the hinge B being at the highest point and being the vertex of the parabola.

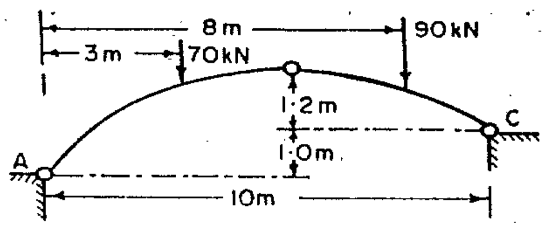


Fig. 16·34

Find (*a*) the horizontal distance of the hinge *B* from one end, (b) horizontal and vertical reaction at the abutments *A* and *C*, and (*c*) the bending moment at the point of application of the 70 kN load.

8. An arch in the form of a parabola with axis vertical has hinges at the abutments and the vertex. The abutments are at different levels, the horizontal span being 80 m and the heights of the vertex above the abutments being 16 m. Calculate :

(*a*) The horizontal thrust, and (b) maximum negative B.M. due to a U.D.L. of 2·5 kN/m run over the whole span.

9. A symmetrical 3-pinned parabolic arch has a span of 50 m and a rise of [illegible] m. Find the maximum bending moment at a quarter point of the arch caused by a uniformly distributed load of 10 kN per m run which can occupy any portion of the span. Indicate the position of the load for this condition.

(*U.L.*)

10. A three-hinged parabolic arch has horizontal span of 240 ft. and a rise of 24 ft. Derive from first principles the influence line for the horizontal thrust at the abutments and bending moment at the quarter span. Plot these influence line on squared paper.

Find the maximum value of the horizontal thrust and the bending moment at quarter-span produced by a moving uniformly distributed load of 1 ton per horizontal foot, 60 ft long. In each case, state the position of load for which maximum value occurs.

11. A three hinged parabolic arch has a horizontal span of 40 m with a central rise of 5 m. A point load of 8 kN moves across from left to right. Calculate the maximum positive and negative B.M. at the section 10 m from the left hand hinge. Also, calculate the position and amount of the absolute maximum B.M. that may occur in the arch.

12. A three hinged parabolic arch has a span of 60 metres and a central rise of 10 metres. Five wheel loads of 5, 6, 4, 7 and 3 kN spaced, 4, 3, 3 and 4 m in order cross the arch from left to right with the 4 kN load leading. When the leading load is at the central hinge, calculate the horizontal thrust in the arch. Also, calculate the bending moment, normal thrust, and shear force at the section under the 7 kN load.

13. Fig 16·35 shows the dimensions of a three-hinged arch with hinges at *A*, *B* and *C*. The distributed load of 1 ton per ft. of span acts on the half span *AB*, and there is also a concentrated load of 10 tons at *E*. Determine.

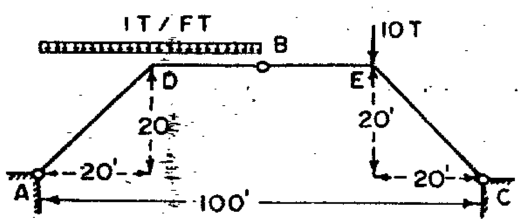


Fig. 16·35

(*a*) the horizontal thrust in the arch,
(*b*) the bending moments at *D* and *E*,

(*c*) the maximum bending moment in the arch stating where it occurs.

(*A.M.I. Struct. E*)

14. A parabolic arch, hinged at the ends has a span of 60 m and a rise of 12 m. A concentrated load of 8 kN acts at 15 m from the left hinge. The second moment of area varies as the secant of the slope of the rib axis. Calculate the horizontal thrust and the reactions at the hinge. Also, calculate the maximum bending moments anywhere on the arch.

15. A parabolic two hinged arch has a span of 80 metres and a rise of 10 m. A uniformly distributed load of 2·5 kN/m covers half of the span. If $I=I_0 \sec\theta$, find out the horizontal thrust at the hinges and radial shear at this section.

16. A parabolic two hinged arch has a span *L* and central rise *r*. Calculate the horizontal thrust at the hinges due to (*a*) U.D.L. *w* over the whole span, and (*b*) U.D.L. *w* over half the span.

17. A two hinged semicircular arch of radius 10 m is subjected to a load of 10 kN acting on the section subtending an angle of 45° with the central line of the arch at its centre. Working from first principles, calculate (*a*) the horizontal thrust at the hinges, (*b*) the vertical reactions at the hinges, (*c*) maximum positive and negative bending moments.

18. A two hinged parabolic arch has a span of 40 m and a central rise of 8 m. Calculate the maximum positive and negative B.M. at a section distant 12 m from the left hinge, due to a single point load of 6 kN rolling from left to right.

19. A circular arched rib, of uniform section, hinged at the springings, has a span of 40 m and a central rise of 5 m. If the rib section is symmetrical and has a depth of 80 cm, calculate the maximum bending stress due to a rise of temperature of 60°F ; $\alpha=0{\cdot}0000062$ per °F and $E=2{\cdot}1\times10^6$ kg/cm².

20. A uniform arch rib covers a span of 40 m, the centre line of the rib being the segment of a circle subtending an angle of 120° at the centre. The arch is pinned at the two supports and carries a vertical load of 8 kN at the crown of the arch.

Calculate the reactions at the supports and construct the bending moment diagram for the arch.

21. A two hinged semicircular arch, of uniform flexural stiffness *EI* carries central vertical load *W*. Show that the horizontal thrust at support is $\dfrac{W}{\pi}$

22. A steel bar of constant *EI* is bent into the form of a semicircle of large radius *r*, and is attached by end hinges to two rigid anchorages. Find the bending moment developed at its centre by a rise in temperature *t* if the coefficient of linear expansion is α. Find also the value of the centrally applied point load which would cause the same central bending moment.

Answers

1. $H=12{\cdot}5$ kN ; $M=+9{\cdot}38$ kN-m ; $M_{max}=-28{\cdot}13$ kN-m
2. $M_{max.}(-)=-11{\cdot}25$ kN-m at 9 m from either hinge
   $M_{max.}(+)=+20$ kN-m at 4 m from either hinge
3. 2·72 kN ↑ ↓ ; 5·28 kN ; 6 kN-m ; 16·44 kN-m

4. −245 kN-m at first quarter ; +98 kN-m at third quarter

5. (*a*) $wx-\frac{wx^2}{2L}$; $\frac{wx^2}{2L}$ (*b*) $\frac{wx^2}{4c}$

(*c*) $-\frac{w}{32}(8Lx-10x^2-L^2)$; $+\frac{wx^2}{16}$

(*d*) $-\frac{3}{160}wL^2$ when $x=\frac{2}{5}L$

5. (*a*) 19·25 ; 13·05 kN
   (*b*) 13·05 kN
   (*c*) −36·27 kN-m
   (*d*) +13·73 kN-m
7. (*a*) 5·76 m from A
   (*b*) H=120 kN ; V_A=78·5 kN ; V_C=81·5 kN
   (*c*) −35·9 kN-m
8. (*a*) H=163·3 kN ; M_{max} (−ve)=1364 kN-m
9. M_{max}=∓1875 kN-m ; load length : Left 37·5 m or right 40 m
10. H=131·25 t for the load placed centrally on the arch.
 M_{max}=−928 ton-ft for the extending from 22·5 ft. to 82·5 ft. from nearer springing.
11. 20 kN-m ; 30 kN-m ; −30·72 kN-m ; 8·44 m from either ends
12. 29·30 kN ; −32·32 kN-m ; 31·26 kN ; 5·60 kN ↑ ↓
13. (*a*) 36·25 tons ;
   (*b*) M_D=+135 t-ft ; M_E=+315 t-ft.
   (*c*) M_{max}=M_E=+314 t-ft.
14. H=5·56 kN ; V_A=6 kN ; V_B=2 kN ; M_{max}=−39·96 kN-m.
15. 100 kN ; −250 kN-m ; 103·1 kN ; 0
16. (*a*) $\frac{wL^2}{8r}$ (b) $\frac{wL^2}{16r}$
17. (*a*) 1·592 kN (b) V_A=8·54 kN ; V_B=1·46 kN
   (*c*) M_{max} (−ve)=13·47 kN-m; M_{max} (+ve)=2·03 kN-m
18. 10·4 kN-m.; 18·4 kN-m
19. 81·4 kg/cm²
20. H=5·11 kN ; M_{max}=21·07 kN-m (−ve)
22. (*a*) $M=\frac{4EI\alpha t}{\pi r}$ (hogging) ;
   (b) $W=\frac{8EI\alpha t}{1\cdot1416r^2}$.

SECTION 3

ADVANCED STRENGTH OF MATERIALS

17

Bending of Curved Bars

17·1. INTRODUCTION : BARS WITH SMALL INITIAL CURVATURE

In *'simple bending'* the relations between the straining actions and the stresses and strains were established for a 'straight beam'. The well known formula :

$$\frac{M}{I}=\frac{f}{y}=\frac{E}{R}$$

is sometimes called the *'straight-beam formula'*. The results of simple bending can be applied, with sufficient accuracy, to the beams or bars having small initial curvature.

It is common practice to distinguish rods or bars of small and large initial curvature. The chief characteristic of such a division is the ratio of the depth of the section h in the plane of curvature to the radius of curvature R_0 of the rod axis. If this ratio $\frac{h}{R_0}$ is 0·2 and less, it is taken that the rod has a small curvature. For a rod of large curvature, the ratio $\frac{h}{R_0}$ is comparable with unity. This division is purely conventional.

Let us now take the case of a beam or bar with small initial curvature R_0 about the neutral surface EF (Fig. 17·1). Let the beam $ABCD$, under the action of pure bending, be bent to $A'B'C'D'$. The radius of curvature decreases from R_0 to R, and the central angle increases from θ to $(\theta+\delta\theta)$.

Consider any surface GH, distant y from the neutral surface EF.

The treatment that follows is based on the same assumptions as those of simple bending of straight bars.

$$\text{Strain in } GH=\frac{G'H'-GH}{GH}$$

$$=\frac{(R+y)(\theta+\delta\theta)-(R_0+y)\theta}{(R_0+y)\theta}$$

$$=\frac{R(\theta+\delta\theta)+y\theta+y\delta\theta-R_0\,\theta-y\theta}{(R_0+y)\theta}$$

$$=\frac{R(\theta+\delta\theta)+y\delta\theta-R_0\,\theta}{(R_0+y)\theta} \quad ...(i)$$

But $$EF=E'F'=R_0\theta=R(\theta+\delta\theta) \quad ...(ii)$$

$$\therefore \quad \frac{R_0-R}{R}=\frac{\delta\theta}{\theta} \quad ...(iii)$$

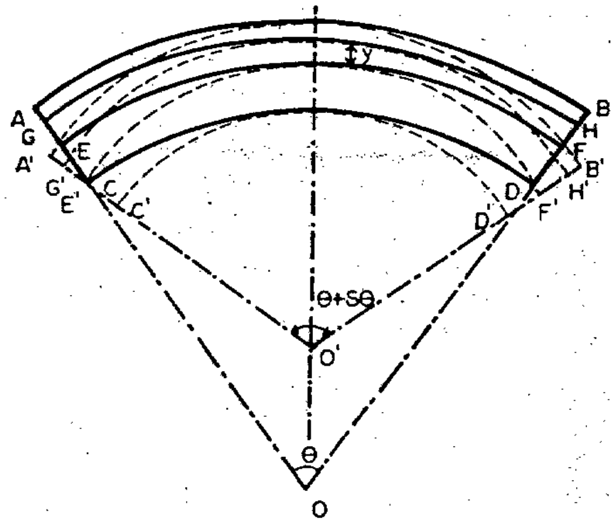


Fig. 17·1
Beam with small initial curvature

Substituting (*ii*) in (*i*), we get

$$\text{Strain in } GH=\varepsilon=\frac{y\delta\theta}{(R_0+y)\theta}$$

Substituting the value of $\frac{\delta\theta}{\theta}$, from (*iii*), we get

$$\varepsilon=\frac{y}{R_0+y}\cdot\frac{R_0-R}{R} \quad ...(17·1)$$

Assuming y to be small as compared to R_0, we have $R_0+y\simeq R_0$.

Hence $$\varepsilon\simeq\frac{y}{R_0}\;\frac{R_0-R}{R}$$

$$\varepsilon=y\left[\frac{1}{R}-\frac{1}{R_0}\right] \quad ...(17·2)$$

$$\therefore \quad \text{Stress } f=E\varepsilon=Ey\left[\frac{1}{R}-\frac{1}{R_0}\right]$$

or $$\frac{f}{y}=E\left[\frac{1}{R}-\frac{1}{R_0}\right] \quad ...(17·3)$$

But $\frac{f}{y}=\frac{M}{I}$ (from simple theory of bending).

$$\therefore \quad \frac{f}{y}=\frac{M}{I}=E\left[\frac{1}{R}-\frac{1}{R_0}\right]. \quad ...(17·4)$$

17·2. BARS WITH LARGE INITIAL CURVATURE

In the above analysis, it is assumed that y is negligible in comparison to the initial radius of curvature R_0. However, there are practical cases of bars, such as hooks, links and rings, etc., which have large initial curvature (or small radius of curvature). In such a case, the dimensions of the cross-section are not very small in comparison with either the radius of curvature or with the length of the bar. The treatment that follows is based on the following assumptions by Winkler.

1. Transverse sections which are plane before bending remain plane after bending.
2. Longitudinal fibres of the bar, parallel to the central axis exert no pressure on each other.
3. The working stresses are below the limit of proportionality.
4. The line joining the centroids of the cross-sections of the bar, called the centre line, is the plane curve and that the cross-sections have an axis of symmetry in this plane. The bar is subjected to the action of forces lying in the plane of symmetry so that bending takes place in this plane.

Consider a curved beam of constant cross-section, subjected to *pure bending* produced by couples M applied at the ends. Let AB and CD be two adjacent cross-sections of the beam subtending a small angle $d\theta$ at the centre of curvature, before bending. Let the bending moment M cause the plane CD to rotate through $\Delta d\theta$ (Fig. 17·2), changing the centre of curvature from O to O'. Let the distance of the *centroidal axis* from the centre of curvature be changed from initial value of R to ρ. Consider any fibre PQ distant y from the centroidal axis. The section CD rotates about the point H ; hence the layer GH is the neutral layer. It should be noted that the quantities R, ρ and y are measured from the centroidal axis ; and *not* from the neutral axis.

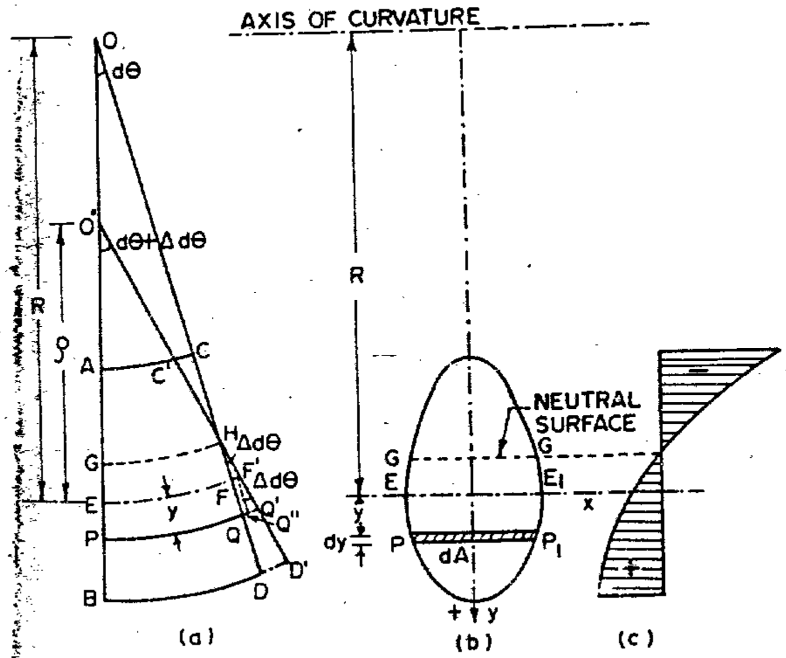


Fig. 17·2. Beam with large initial curvature.

Let ε_0=strain of the centroidal fibre

ε=strain of any other fibre PQ distant y from the centroidal layer.

$\therefore$
$$\varepsilon_0=\frac{FF'}{EE} \text{ or } FF'=\varepsilon_0(EF)=\varepsilon_0\, R\, d\theta \qquad \ldots(1)$$

$$\varepsilon=\frac{QQ'}{PQ}=\frac{QQ''+Q'Q''}{PQ}=\frac{FF'+Q'Q''}{PQ}$$

or
$$\varepsilon=\frac{\varepsilon_0\, R\, d\theta+y.\Delta d\theta}{(R+y)d\theta}=\frac{\varepsilon_0\, R+y\left(\dfrac{\Delta d\theta}{d\theta}\right)}{R+y} \qquad \ldots(2)$$

Let ω=angular strain=$\dfrac{\Delta d\theta}{d\theta}$

Substituting this and adding and subtracting ε_0 to the numerator of (2), we get

$$\varepsilon=\frac{\varepsilon_0\, R+\varepsilon_0\, y+y(\omega)-\varepsilon_0 y}{R+y}$$

or
$$\varepsilon=\varepsilon_0+(\omega-\varepsilon_0)\,\frac{y}{R+y} \qquad \ldots(17{\cdot}5)$$

Within the elastic property of the material,

$$f=E\,\varepsilon=E\left[\varepsilon_0+(\omega-\varepsilon_0)\,\frac{y}{R+y}\right] \qquad \ldots(17{\cdot}6)$$

where f=bending stress or normal stress (also known as circumferential stress)

It will be assumed, for simplicity, that the section of the beam is symmetrical about the plane of curvature. The y-axis is then the axis of symmetry of the section [Fig. 17·2 (*b*)] and the moment of elementary forces $f\,.\,dA$ with respect to this axis is zero.

There are two unknowns in Eq. 17·6 : ε_0 and ω. To determine these quantities we use the two equations of statics which state that the sum of the normal forces distributed over a cross-section is equal to zero and the moment of these forces is equal to the external moment M. These equations are :

$$\int_A f\,.\,dA=\int_A E\left[\varepsilon_0+(\omega-\varepsilon_0)\frac{y}{R+y}\right]dA=0$$

and
$$\int_A f\,.\,y dA=\int_A Ey\left[\varepsilon_0+(\omega-\varepsilon_0)\,\frac{y}{R+y}\right]dA=M \qquad \ldots(\text{II})$$

The above equations may be simplified as follows :

From (I),
$$\int_A f.dA=\int_A E\varepsilon_0 dA+\int_A E(\omega-\varepsilon_0)\frac{y}{R+y}\,dA$$

or
$$\varepsilon_0=\int_A dA=-(\omega-\varepsilon_0)\int_A \frac{y}{R+y}\,dA$$

Also, from (II),
$$M=E\varepsilon_0\int_A ydA+(\omega-\varepsilon_0)\int_A \frac{y^2}{R+y}\,dA \qquad \ldots(\text{IV})$$

But $\displaystyle\int_A dA=A\ ;\ \int_A y.dA=0$

and let $\displaystyle\int_A \frac{y}{R+y}\,dA=-mA$ or $\displaystyle m=-\frac{1}{A}\int\frac{y}{R+y}\,dA \qquad \ldots(17{\cdot}7)$

$$\therefore \int_A \frac{y^2}{R+y}\,dA=\int\left(y-R\,\frac{y}{R+y}\right)dA=-R\int\frac{y}{R+y}\,dA=m.A.R \qquad \ldots(17\cdot8)$$

Substituting these in (III) and (IV), we get

$$\varepsilon_0=(\omega-\varepsilon_0)m \qquad \ldots(a)$$

and

$$M=E(\omega-\varepsilon_0)m\;AR \qquad \ldots(b)$$

Solving these two, we get

$$(\omega-\varepsilon_0)=\frac{M}{EmAR} \qquad \ldots(17\cdot9)$$

$$\varepsilon_0=\frac{M}{E.A.R} \qquad \ldots(17\cdot10)$$

and

$$\omega=\frac{1}{EA}\left(\frac{M}{R}+\frac{M}{mR}\right) \qquad \ldots(17\cdot11)$$

Substituting these values in Eq. 17·6, we get

$$f=\frac{M}{A.R}\left(1+\frac{1}{m}\,\frac{y}{R+y}\right) \qquad \ldots(17\cdot12)$$

The above expression for f is generally known as the *Winkler-Bach formula*. The distribution of f, given by Eq. 17·12 is *hyperbolic* (and not linear as in the case of straight beams) and is shown in Fig 17·2 (*c*).

In the above expression the quantity m is a pure number, and is the property of each particular shape of the cross-section, defined by Eq. 17·7. Its value can be determined by performing the integration or by graphical method. The quantity mA is called the *modified area* of the cross-section.

Reduction of the above formula for the case of the straight beam

Eq. 17·12 can be reduced for the case of a straight beam. Rewriting it,

$$f=\frac{M}{AR}+\frac{M}{mAR}\cdot\frac{y}{R+y}$$

$$=\frac{M}{AR}+\frac{M}{\int\frac{y^2}{R+y}\,.\,dA}\cdot\frac{y}{R+y}$$

$$=\frac{M}{AR}+\frac{My}{\left(1+\frac{y}{R}\right)\int\frac{y^2dA}{1+\frac{y}{R}}} \qquad \ldots(17\cdot13)$$

For a straight beam, R is infinitely large. Hence Eq. 17·13 reduces to

$$f=0+\frac{My}{\int y^2dA}=\frac{My}{I} \qquad \ldots(17\cdot14)$$

where $\int y^2dA=I$

Eq. 17·14 is the same as our usual bending stress formula.

Location of neutral axis

The neutral axis can be located by equating Eq. 17·12 to zero, since the bending stress at the neutral axis is zero.

$$\therefore \qquad 1+\frac{1}{m}\cdot\frac{y_0}{R+y_0}=0$$

or

$$y_0=-\frac{m.R}{1+m} \qquad \ldots(17\cdot15)$$

where y_0=distance of neutral axis from the centroidal layer.

The negative sign suggests that the neutral axis is towards the centre of curvature (see sign conventions below).

Sign Convention

The following sign convention will be followed :

1. A bending moment M will be taken as *positive* if it *decreases* the radius of curvature, and negative if it increases the radius of curvature.

2. y is positive when measured towards the convex side of beam, and negative when measured towards the concave side (or towards the centre of curvature).

3. With the above sign convention, if f comes out to be positive it will denote tensile stress while negative sign will mean compressive stress.

17·3. ALTERNATIVE EXPRESSION FOR f

In the previous treatment, the integrals

$$\int_A\frac{y}{R+y}\,dA \text{ and } \int_A\frac{y^2}{R+y}\,dA$$

were expressed as $-mA$ and mAR respectively, and substituted in Eqs. III and IV to get the final expression for f. We shall now denote the above integrals in terms of a new parameter A' defined below :

$$\int\frac{y}{R+y}\,dA=\int\left(1-\frac{R}{y+R}\right)dA=\int dA-\int\frac{R}{y+R}\,dA$$

But $$\int dA = A$$

and let $$\int_A \frac{R}{y+R}\,dA = A' \qquad \text{...(17·16)}$$

Hence $$\int \frac{y}{R+y}\,dA = A - A' \qquad \text{...(17·17)}$$

Also $$\int \frac{y^2}{R+y}\,dA = \int (y-R)\,dA + \int \frac{R^2}{y+R}\,dA$$
$$= \int y.dA - R\int dA + R\int \frac{R}{y+R}\,dA$$
$$= 0 - RA + RA' = R(A'-A) \qquad \text{...(17·18)}$$

Substituting these values in Eqs. III and IV of the previous article, and simplifying as before, we get the following alternative expression for f:

$$f = \frac{M}{R(A'-A)}\left(\frac{A'}{A} - \frac{R}{y+R}\right) \qquad \text{...(17·19)}$$

Comparing Eqs. 17·7 and 17·17, we have

$$\int \frac{y}{R+y}\,dA = -mA = A - A'$$

$$\therefore \quad m = \frac{A'-A}{A} = \left(\frac{A'}{A} - 1\right) \qquad \text{...(17·20)}$$

or $$A' - A = mA \qquad \text{...[17·20 (a)]}$$

Substituting these values of $(A'-A)$ and $\frac{A'}{A}$ in Eq. 17·19, we get

$$f = \frac{M}{RAm}\left(m + 1 - \frac{R}{y+R}\right) = \frac{M}{RAm}\left(m + \frac{y}{y+R}\right)$$

or $$f = \frac{M}{AR}\left(1 + \frac{1}{m}\,\frac{R}{R+y}\right)$$

which is the same as Eq. 17·12.

17·4. DETERMINATION OF FACTOR m FOR VARIOUS SECTIONS

We shall now determine the values of factor m for various shapes of cross-section by evaluating the integral defined by Eq. 17·7.

(1) RECTANGULAR CROSS-SECTION

Consider a rectangular normal section of a curved beam. The width parallel to the axis of curvature is b while the dimension measured along the direction of radius of curvature is d. Let R be the distance of the centroidal axis from the axis of curvature.

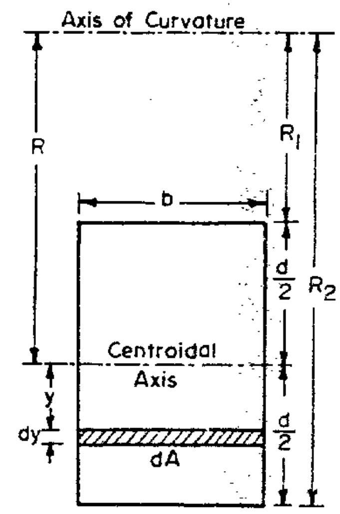


Fig. 17·3

From Eq. 17·7,

$$m = -\frac{1}{A}\int_A \frac{y}{R+y}\,dA = -\frac{1}{bd}\int_{-d/2}^{+d/2} \frac{y}{R+y}\,.\,bdy$$
$$= -\frac{1}{d}\int_{-d/2}^{+d/2} \frac{y}{R+y}\,dy \qquad \text{...(1)}$$
$$= -\frac{1}{d}\int_{-d/2}^{+d/2}\left(1 - \frac{R}{R+y}\right)dy$$
$$= -\frac{1}{d}\Big[y - R\log_e(R+y)\Big]_{-d/2}^{+d/2}$$

or $$m = \frac{R}{d}\log_e\left(\frac{R+d/2}{R-d/2}\right) - 1 = \frac{R}{d}\log_e\left(\frac{R_2}{R_1}\right) - 1 \qquad \text{...(17·21)}$$

Also $$\log_e\left(\frac{R+d/2}{R-d/2}\right) = \frac{d}{R}\left[1 + \frac{1}{3}\left(\frac{d}{2R}\right)^2 + \frac{1}{5}\left(\frac{d}{2R}\right)^4 + \ldots\right]$$

$$\therefore \quad m = \frac{1}{3}\left(\frac{d}{2R}\right)^2 + \frac{1}{5}\left(\frac{d}{2R}\right)^4 \qquad \text{...(17·22)}$$

(2) TRAPEZOIDAL CROSS-SECTION

Consider an elementary strip of width b, distant y from the centroidal axis. Let r be its distance from the axis of rotation.

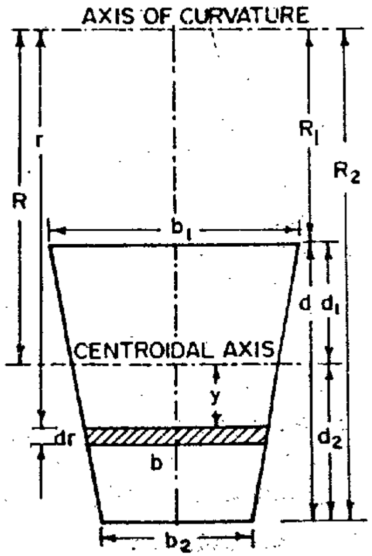


Fig. 17·4. Trapezoidal section.

$\therefore$ $$r=(R+y) \qquad ...(i)$$

$$b=b_2+\frac{b_1-b_2}{d}\left(R_2-r\right) \qquad ...(ii)$$

$$dA=b.dr=\left\{b_2+\frac{b_1-b_2}{d}\left(R_2-r\right)\right\}dr \qquad ...(iii)$$

Now $$m=-\frac{1}{A}\int_A \frac{y}{R+y}\,dA=-\frac{1}{A}\int_A\left(1-\frac{R}{R+y}\right)dA$$

$$=-1+\frac{R}{A}\int_A\frac{dA}{r}$$

$$=-1+\frac{R}{A}\int_{R_1}^{R_2}\left\{b_2+\frac{b_1-b_2}{d}\left(R_2-r\right)\right\}\frac{dr}{r}$$

$$=-1+\frac{R}{A}\left[\left\{b_2+\left(b_1-b_2\right)\frac{R_2}{d}\right\}\log_e\frac{R_2}{R_1}-\left(b_1-b_2\right)\right] \qquad ...(17\text{·}23)$$

For rectangular section, $b_1=b_2=b$. Hence the above expression reduces to

$$m=-1+\frac{Rb}{A}\log_e\frac{R_2}{R_1}=\frac{R}{d}\log_e\left(\frac{R_2}{R_1}\right)-1$$

which is the same as Eq. 17·21.

(3) TRIANGULAR SECTION

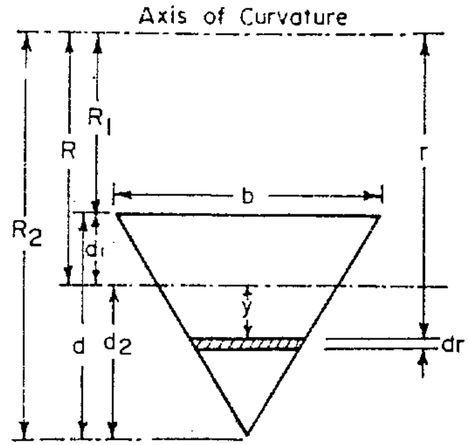


Fig. 17·5. Triangular section.

$$r=R+y\,;$$

Width of elementary strip $=\frac{b}{d}\,(R_2-r)$

$\therefore$ $$dA=\frac{b}{d}\,(R_2-r)dr$$

$\therefore$ $$m=-\frac{1}{A}\int\frac{y}{R+y}\,dA=-1+\frac{R}{A}\int\frac{dA}{r}$$

$$=-1+\frac{R}{A}\int_{R_1}^{R_2}\frac{b}{d}(R_2-r)\,\frac{dr}{r}$$

$$=-1+\frac{R}{A}\left[\left\{\frac{bR_2}{d}\log_e\frac{R_2}{R_1}\right\}-b\right] \qquad ...(17\text{·}24)$$

Alternatively, the above expression can also be obtained by putting $b_1=b$ and $b_2=0$ in Eq. 17·23.

(4) CIRCULAR SECTION

Let r be the radius of the circle

$$y=r\sin\theta\,;\; x=r\cos\theta$$
$$dy=r\cos\theta\,d\theta$$

$\therefore$ $$dA=2x.dy=2r\cos\theta.r\cos\theta.d\theta$$
$$=2r^2\cos^2\theta.d\theta$$

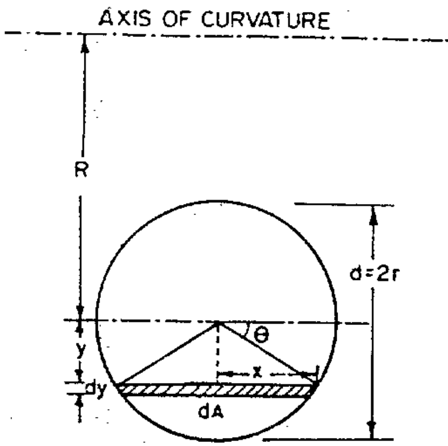


Fig. 17·6. Circular section.

Now
$$m=-\frac{1}{A}\int_{-\pi/2}^{+\pi/2}\frac{y}{R+y}\,dA$$

$$=-\frac{1}{\pi r^2}\int_{-\pi/2}^{+\pi/2}\frac{r\sin\theta}{R+r\sin\theta}\,.2r^2\cos^2\theta.d\theta$$

$$=-\frac{2r}{\pi}\int_{-\pi/2}^{+\pi/2}\frac{\sin\theta\cos^2\theta\,d\theta}{R+r\sin\theta}\qquad ...(1)$$

Putting $\frac{R}{r}=k$, we get

$$m=-\frac{2}{\pi}\int_{-\pi/2}^{+\pi/2}\frac{\sin\theta-\sin^3\theta}{\sin\theta+k}\,d\theta$$

$$=\frac{2}{\pi}\int_{-\pi/2}^{+\pi/2}\left[\sin^2\theta-k\sin\theta+(k^2-1)-\frac{k(k^2-1)}{\sin\theta+k}\right]d\theta$$

$$=\frac{2}{\pi}\left[\frac{\pi}{2}+(k^2-1)\,\pi\right]-2k\sqrt{k^2-1}$$

$$=1+2(k^2-1)-2k\sqrt{k^2-1}$$

$$=-1+2k^2-2k\sqrt{k^2-1}$$

$$=1+2\left(\frac{R}{r}\right)^2-2\left(\frac{R}{r}\right)\sqrt{\left(\frac{R}{r}\right)^2-1}\qquad ...[17\cdot25\ (a)]$$

Alternatively, if $\frac{1}{R+r\sin\theta}$ is expanded in the converging series, and each term then under the integral (1) is integrated separately the final result is

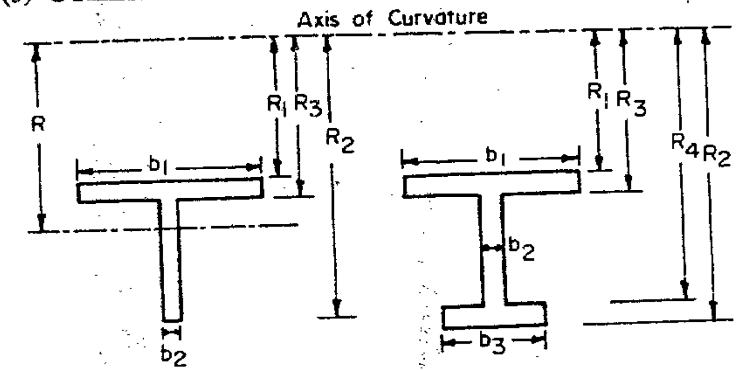
$$m=\frac{1}{4}\left(\frac{r}{R}\right)^2+\frac{1}{8}\left(\frac{r}{R}\right)^4+\frac{5}{64}\left(\frac{r}{R}\right)^6+\ldots\ldots\qquad ...[17\cdot25\ (b)]$$

Hollow circular section

For a hollow circular section of inner radius r_1 and outer radius r_2, m is given by

$$m=-1+\frac{2R}{r_2^2-r_1^2}\left[\sqrt{R^2-r_1^2}-\sqrt{R^2-r_2^2}\right]\qquad ...[17\cdot25\ (c)]$$

(5) OTHER SECTIONS



(a) T-Section (b) I-Section

Fig. 17·7

For the *T*-section [Fig. 17·7 (*a*)]

$$m=\frac{R}{A}\left(b_1\log_e\frac{R_3}{R_1}+b_2\log_e\frac{R_2}{R_3}\right)-1\qquad ...(17\cdot26)$$

For the *I*-section [17·7 (*b*)]

$$m=\frac{R}{A}\left(b_1\log_e\frac{R_3}{R_1}+b_2\log_e\frac{R_4}{R_3}+b_3\log_e\frac{R_2}{R_4}\right)-1\qquad ...(17\cdot27)$$

Example 17·1. *A curved beam, rectangular in cross-section is subjected to pure bending with couple of +400 N-m. The beam has width of 20 mm, and depth of 40 mm and is curved in a plane parallel* *to the depth. The mean radius of curvature is 5 mm. Find the position of the neutral axis, and the ratio of the maximum to the minimum stress. Also, plot the variation of the bending stress across the section.*

Solution.

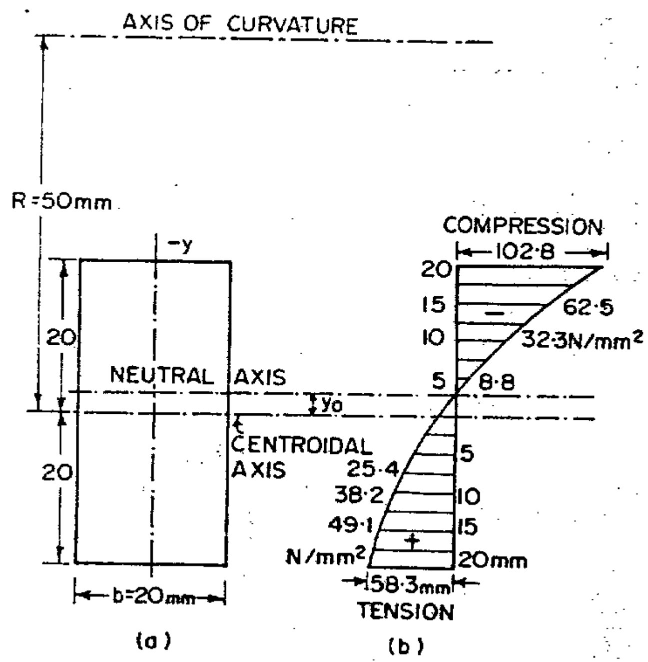


Fig. 17·8

From Eq. 17·21,

$$m=\frac{R}{d}\log_e\left(\frac{R_2}{R_1}\right)-1$$

Here, $R=50$, $R_1=50-20=30$ and $R_2=50+20=70$ mm.
$d=40$ mm.

$$m=\frac{50}{40}\log_e\left(\frac{70}{30}\right)-1=\frac{50}{40}\left(0{\cdot}847\right)-1=0{\cdot}0591$$

Alternatively, from Eq. 17·22, considering first two terms,

$$m=\frac{1}{3}\left(\frac{40}{100}\right)^2+\frac{1}{5}\left(\frac{40}{100}\right)^4=0{\cdot}0585$$

The stress is given by Eq. 17·12

$$f=\frac{M}{AR}\left(1+\frac{1}{m}\;\frac{y}{R+y}\right)$$

$$=\frac{400000}{40\times20\times50}\left(1+\frac{1}{0{\cdot}0591}\;\frac{y}{50+y}\right)$$

$$\therefore \quad f=10\left(1+16{\cdot}92\frac{y}{50+y}\right) \qquad \ldots(1)$$

The maximum stress evidently occurs at the inner-most fibre where $y=-20$ mm.

$$\therefore \quad f_{max}=10\left[1+16{\cdot}92\frac{(-20)}{50-20}\right]=-102{\cdot}8 \text{ N/mm}^2$$

$$=102{\cdot}8 \text{ N/mm}^2 \text{ (compressive)}$$

Similarly, minimum stress occurs at $y=+20$ mm

$$f_{min}=10\left[1+16{\cdot}92\frac{(20)}{50+20}\right]=58{\cdot}3 \text{ N/mm}^2 \text{ (tensile)}$$

$$\text{Ratio } \frac{f_{max}}{f_{min}}=\frac{102{\cdot}8}{58{\cdot}3}=1{\cdot}76$$

To find the position of a N.A., we have, from (1),

$$f=0=10\left[1+16{\cdot}92\frac{y_0}{50+y_0}\right]$$

$$\therefore\ 16{\cdot}92\,y_0=-50-y_0$$

or $$y_0=-\frac{50}{17{\cdot}92}=-2{\cdot}79 \text{ mm}$$

(*i.e.*, towards the centre of curvature).

The stress distribution can be plotted by substituting of various values of y in (1). Fig. 17·8 shows the stress distribution.

Example 17·2. *Solve example 17·1 assuming the section to be circular, of diameter 40 mm. The mean radius of curvature of the beam is 50 mm.*

Solution.

m is given by Eq. 17·25 (b),

$$m=\frac{1}{4}\left(\frac{d}{2R}\right)^2+\frac{1}{8}\left(\frac{d}{2R}\right)^4+\ldots$$

$$=\frac{1}{4}\left(\frac{40}{100}\right)^2+\frac{1}{8}\left(\frac{40}{100}\right)^4+\ldots=0{\cdot}0432$$

Stress f is given by Eq. 17·12

$$f=\frac{M}{AR}\left(1+\frac{1}{m}\,\frac{y}{R+y}\right)$$

where $A=\frac{\pi}{4}\,(40)^2=1257\text{ mm}^2$

$$f=\frac{400\times10^3}{1257\times50}\left[1+\frac{1}{0{\cdot}0432}\,\frac{y}{50+y}\right]$$

$$=6{\cdot}364\left[1+23{\cdot}15\,\frac{y}{50+y}\right] \quad \ldots(2)$$

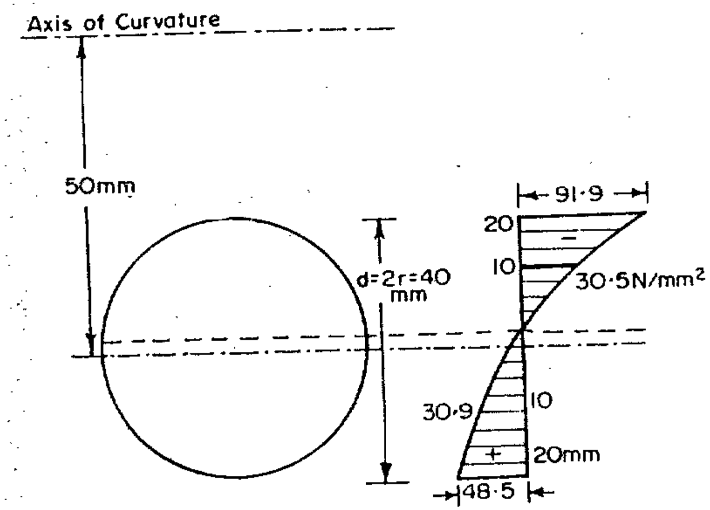


Fig. 17·9

$$\therefore \quad f_{max}=6{\cdot}364\left[1+23{\cdot}15\,\frac{(-20)}{50-20}\right]=-91{\cdot}9\text{ N/mm}^2$$

$$=\mathbf{91{\cdot}9\ N/mm^2}\text{ (compressive)}$$

$$f_{min}=6{\cdot}364\left[1+23{\cdot}15\,\frac{20}{50+20}\right]=+\mathbf{48{\cdot}5\ N/mm^2}$$

Ratio $\frac{f_{max}}{f_{min}}=\frac{91{\cdot}9}{48{\cdot}5}\simeq\mathbf{1{\cdot}9}$

The position of neutral axis is given by

$$f=0=6{\cdot}364\left[1+23{\cdot}15\,\frac{y_0}{50+y_0}\right]$$

or $\quad 23{\cdot}15\,y_0=50-y_0$

$$\therefore \quad y_0=-\frac{50}{24{\cdot}15}=-\mathbf{2{\cdot}07\ mm.}$$

The stress distribution across the section is shown in Fig. 17·9.

Example 17·3. *Solve example 17·1 if the beam section is trapezoidal with dimensions shown in Fig. 17·10.*

Solution.

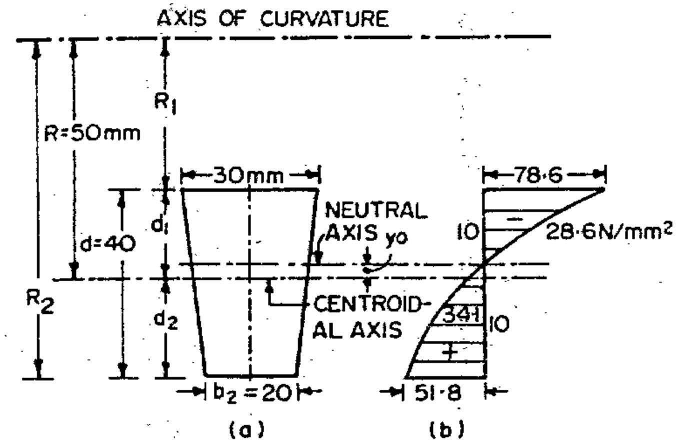


Fig 17·10

The distance d_1 of the centroidal axis from the side b_1 of a trapezium is given by

$$d_1=\frac{b_1+2b_2}{b_1+b_2}\times\frac{d}{3} \quad \ldots(17{\cdot}28)$$

$$=\frac{30+40}{50}\times\frac{40}{3}=18{\cdot}6\text{ mm}$$

$$\therefore \quad d_2=40-18{\cdot}6=21{\cdot}4\text{ mm}$$

$$R_2=R+d_2=50+21{\cdot}4=71{\cdot}4\text{ mm}$$

$$R_1=R-d_1=50-18{\cdot}6=31{\cdot}4\text{ mm}$$

$$A=(b_1+b_2)\,\frac{d}{2}=(30+20)\,\frac{40}{2}=1000\text{ mm}^2$$

From Eq. 17·23,

$$m=-1+\frac{R}{A}\left[\left\{b_2+(b_1-b_2)\,\frac{R_2}{d}\right\}\log_e\frac{R_2}{R_1}-(b_1-b_2)\right]$$

$$=-1+\frac{50}{1000}\left[\left\{20(30-20)\frac{71{\cdot}4}{40}\right\}\log_e\frac{71{\cdot}4}{31{\cdot}4}\,(-30-20)\right]$$

$$=0{\cdot}0547$$

The stress is given by Eq. 17·12,

$$f=\frac{M}{AR}\left(1+\frac{1}{m}\ \frac{y}{R+y}\right)$$

$$=\frac{400\times10^3}{1000\times50}\left[1+\frac{1}{0\cdot0547}\ \frac{y}{50+y}\right]$$

$$=8\left[1+18\cdot28\ \frac{y}{50+y}\right] \qquad ...(1)$$

$$f_{max}=8\left[1+18\cdot28\ \frac{-18\cdot6}{50-18\cdot6}\right]=-78\cdot6 \text{ N/mm}^2$$

$$=78\cdot6 \text{ N/mm}^2 \text{ (compressive)}$$

$$f_{min}=8\left[1+18\cdot28\ \frac{21\cdot4}{50+21\cdot4}\right]=51\cdot8 \text{ N/mm}^2 \text{ (tensile)}$$

$$\text{Ratio } \frac{f_{max}}{f_{min}}=\frac{78\cdot6}{51\cdot8}=\mathbf{1\cdot52}$$

The position of $N.A.$ is given by

$$f=0=8\left[1+18\cdot28\ \frac{y_0}{50+y_0}\right]$$

or $\quad 18\cdot28\ y_0=-50-y_0$

$$\therefore \quad y_0=-\frac{50}{19\cdot28}=-2\cdot59 \text{ mm.}$$

The pressure distribution across the section is shown in Fig. 17·10 (*b*).

17·5. BENDING OF CURVED BAR BY FORCES ACTING IN THE PLANE OF SYMMETRY

Upto this stage, we have discussed the case of 'pure bending', *i.e.* bending of a curved bar by the action of pure couples. However, a curved beam may be subjected to a system of forces of P_1, P_2, ...P_n keeping it in equilibrium. It is assumed that these forces act in the plane of the centre line, which is the plane of symmetry of the bar. If we consider any normal section of the beam, the resultant force to the right of it may not be along the section, but may be inclined to the section. This inclined force can be resolved along the section and normal to the section. Therefore, as in the case of an arch, the section of the curved beam may be subjected to three straining actions :

(*i*) A normal force N, acting normal to the section ;

(*ii*) A shearing force F, acting along or tangential to the section, and

(*iii*) A bending moment M acting in the plane of symmetry (*i.e.*, symmetrical bending of a curved bar).

The stress at a point in a section will therefore be equal to the algebraic sum of the direct stress f_0 due to the normal force N and the bending stress f_b. It is assumed that the normal force N acts through the centroid of the section and that the stress caused due to this is equal to $\frac{N}{A}$ at each point on the area. Hence the final stress at any point is given by

$$f=f_0+f_b=\frac{N}{A}+\frac{M}{AR}\left(1+\frac{1}{m}\ \frac{y}{R+y}\right) \qquad ...(17\cdot29)$$

N is assumed to be plus if it produces tensile stress, and minus if it produces compressive stress.

The shearing force F produces shearing stresses, the distribution of which is assumed to be the same as for a straight bar.

17·9. STRESSES IN HOOKS

The results of the previous article can now be applied to find the stresses in the horizontal section through the centre of curvature of a hook carrying a vertical load P.

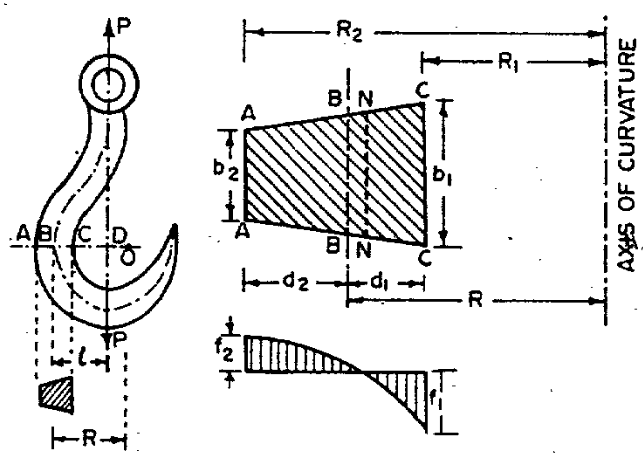


Fig. 17·11. Stresses in a hook.

The horizontal section AC, passing through the centre of curvature is the most highly stressed section. BB is the centroidal axis of the horizontal section which may be of trapezoidal or any other shape. The load P which the hook supports acts eccentrically with respect to the centroid of the section. Hence this force causes (*i*) bending moment $M=P\times l$ (being negative since it increases the radius of curvature) and (*ii*) a tensile force P acting through the centroid of the section. Hence the stress at point on the horizontal section is given by Eq. 17·29.

$$f=\frac{P}{A}+\frac{M}{AR}\left(1+\frac{1}{m}\;\frac{y}{R+y}\right)$$

or

$$f=\frac{P}{A}-\frac{Pl}{AR}\left(1+\frac{1}{m}\;\frac{y}{R+y}\right) \qquad \ldots(17·30)$$

(Since $M=-Pl$).

Designating y as positive when measured towards the convex side and negative when measured towards concave side, stress f_1 at CC where $y=-d_1$ is given by

$$f_1=\frac{P}{A}-\frac{Pl}{AR}\left(1-\frac{1}{m}\;\frac{d_1}{R-d_1}\right) \qquad \ldots(17·31)$$

Similarly, the stress f_2 at AA, where $y=+d_2$ is given by

$$f_2=\frac{P}{A}-\frac{Pl}{AR}\left(1+\frac{1}{m}\;\frac{d_2}{R+d_2}\right) \qquad \ldots(17·32)$$

The bending action alone causes compressive stress at AA and tensile stress at CC, while the normal force P causes uniform tensile stress over the whole section. In a well-designed hook, both the stresses f_1 and f_2 are not very different.

If $l=R$ (*i.e.* centre line of the load passing through the centre of the curvature of the hook) the above two equations reduce to the following simplified form :

$$f_1=\frac{P}{A}-\frac{PR}{AR}\left(1-\frac{1}{m}\;\frac{d_1}{R-d_1}\right)=\frac{P}{Am}\;\frac{d_1}{R-d_1} \qquad \ldots(17·33)$$

and

$$f_2=\frac{P}{A}-\frac{PR}{AR}\left(1+\frac{1}{m}\;\frac{d_2}{R+d_2}\right)=-\frac{P}{Am}\;\frac{d_2}{R+d_2} \qquad \ldots(17·34)$$

Eqs. 17·33 and 17·34 clearly suggest that the final stress of CC (intrados) is tensile (positive) while the stress at AA (extrados) is compressive (negative). If, however, l is more than R, f_1 is slightly reduced and f_2 slightly increased from the corresponding values given

by Eqs. 17·33 and 17·34. In a well-designed hook, the centre of load passes through the centre of curvature.

Example 17·4. *A central horizontal section of a hook is a symmetrical trapezium 50 mm deep, the inner width being 60 mm and the outer width being 30 mm. Estimate the extreme intensities of stress when the hook carries a load of 27 kN, the load line passing 40 mm from the inside edge of the section and the centre of curvature being in the load line. Also, plot the stress distribution across the section.*

Solution.

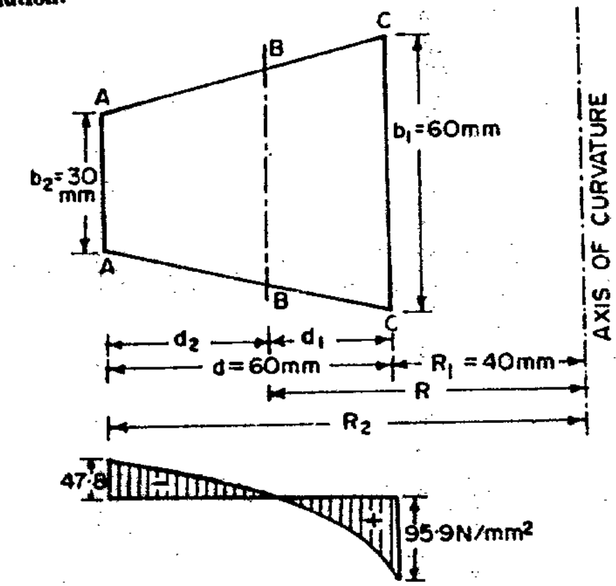


Fig. 17·12

The distance d_1 of the centroidal axis from the side b_1 or the trapezium is given by

$$d_1=\frac{b_1+2b_2}{b_1+b_2}\times\frac{d}{3}$$

$$=\frac{60+60}{60+30}\times\frac{60}{3}=26·7 \text{ mm}$$

$$d_2=d-d_1=60-26·7=33·3 \text{ mm}$$

$$R=R_1+d_1=40+26·7=66·7 \text{ mm}$$

$$R_2=R_1+d=40+60=100 \text{ mm}$$

$$A=(b_1+b_2)\;\frac{d}{2}=(60+30)\;\frac{60}{2}=2700 \text{ mm}^2.$$

From Eq. 17·13,

$$m=-1+\frac{R}{A}\left[\left\{b_2+(b_1-b_2)\frac{R_2}{d}\right\}\log_e\frac{R_2}{R_1}-(b_1-b_2)\right]$$

$$=-1+\frac{66{\cdot}7}{2700}\left[\left\{30+(60-30)\frac{100}{70}\right\}\log_e\frac{100}{40}-(60-30)\right]$$

$$=0{\cdot}06975.$$

The bending stress at any point is given by Eq. 17·12

$$f_b=\frac{M}{AR}\left(1+\frac{1}{m}\frac{y}{R+y}\right)$$

But $M=-P\times l$ (here $l=R=66{\cdot}7$ mm

$=-27000\times 66{\cdot}7$ N-mm)

$$\therefore \quad f_b=-\frac{27000\times 66{\cdot}7}{2700\times 66{\cdot}7}\left[1+\frac{1}{0{\cdot}06975}\times\frac{y}{66{\cdot}7+y}\right]$$

or

$$f_b=-10\left(1+14{\cdot}34\frac{y}{66{\cdot}7+y}\right)$$

Also, direct stress

$$f_o=\frac{P}{A}=\frac{27000}{2700}=10 \text{ N/mm}^2.$$

(*i*) At the section *CC*,

$$f_b=-10\left(1-14{\cdot}34\frac{26{\cdot}7}{66{\cdot}7-26{\cdot}7}\right)$$

$=85{\cdot}9$ N/mm² (tensile)

$f_o=10$ N/mm² (tensile)

$\therefore$ Total stress $f=f_b+f_s=85{\cdot}9+10=$ **95·9 N/mm** (tensile).

(*ii*) At the section *AA*,

$$f_b=-10\left(1+14{\cdot}34\frac{33{\cdot}3}{66{\cdot}7+33{\cdot}3}\right)=-57{\cdot}8 \text{ N/mm}^2$$

$f_o=+10$ N/mm²

$$\therefore \quad f=f_b+f_o=-57{\cdot}8+10=-47{\cdot}8 \text{ N/mm}^2$$

=47·8 N/mm² (compressive)

At any point distant y from the centroidal axis, the stress is given by

$$f=f_b+f_o=-10\left(1+\frac{14{\cdot}34y}{66{\cdot}7+y}\right)+10$$

At $y=-10$, $f=-10\left(1-\frac{14{\cdot}34\times 10}{66{\cdot}7-10}\right)+10$

$=15{\cdot}3+10=25{\cdot}3$ N/mm²

At $y=-20$, $f=-10\left(1-\frac{14{\cdot}34\times 20}{66{\cdot}7-20}\right)+10$

$=51{\cdot}4+10=61{\cdot}4$ N/mm²

At $y=0$, $f=-10(1+0)+10=0$

At $y=+10$, $f=-10\left(1+\frac{14{\cdot}34\times 10}{66{\cdot}7+10}\right)+10$

$=-28{\cdot}7+10=-18{\cdot}7$ N/mm²

At $y=+20$, $f=-100\left(1+\frac{14{\cdot}34\times 20}{66{\cdot}7\times 20}\right)+10$

$=-43{\cdot}1+10=-33{\cdot}1$ N/mm²

At $y=+30$, $f=-10\left(1+\frac{14{\cdot}34\times 30}{66{\cdot}7+30}\right)+10$

$=-54{\cdot}5+10=-44{\cdot}5$ N/mm²

The stress distribution is shown in Fig. 17·12.

Example 17·5. *A central horizontal section of a hook is an I-section with diamension shown in Fig. 17·13. The hook carries a load P, the load line passing 40 mm from the inside edge of the section, and the centre of curvature being in the load line. Determine the magnitude of the load P if the maximum stress in the hook is not to exceed the permissible stress of 120 N/mm². What will be the maximum compressive stress in hook for that value of the load?*

Solution.

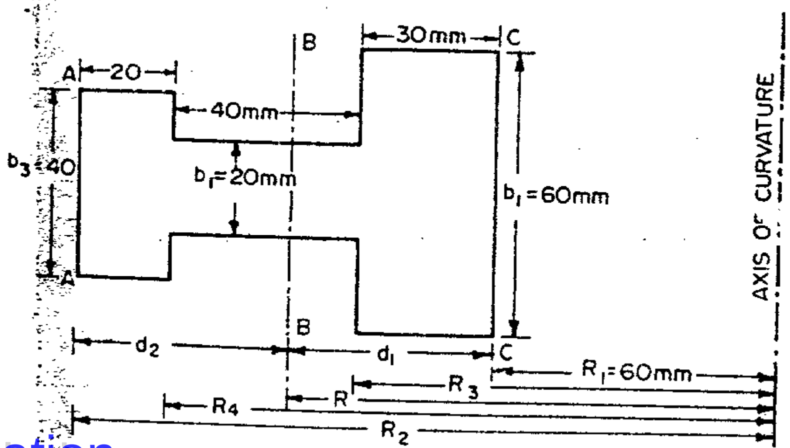


Fig. 17·13

$A=(40\times20)+(20\times40)+(60\times30)=3400$ mm²

To find the position of the centroidal axis take moments of individual areas, about CC. Thus,

$$3400\,d_1=(60\times30\times15)+(40\times20\times50)+(40\times20\times80)$$

$\therefore\quad d_1=38{\cdot}5$ mm; $d=20+40+30=90$ mm

$\therefore\quad d_1=90-38{\cdot}5=51{\cdot}5$ mm

$R_1=60$ mm ; $R=R_1+d_1=60+38{\cdot}5=98{\cdot}5$ mm

$R_2=R_1+d=60+90=150$; $R_3=R_1+30=90$ mm

$R_4=R_1+30+40=60+30+40=130$ mm

From Eq. 17·27,

$$m=\frac{R}{A}\left(b_1\log_e\frac{R_3}{R_1}+b_2\log_e\frac{R_4}{R_3}+b_3\log_e\frac{R_2}{R_4}\right)-1$$

$$=\frac{98{\cdot}5}{3400}\left\{60\log_e\frac{90}{60}+20\log_e\frac{130}{90}+40\log_e\frac{150}{130}\right\}-1$$

$$=0{\cdot}0837.$$

The bending stress at any point is given by Eq. 17·27,

$$f_b=\frac{M}{AR}\left[1+\frac{1}{m}\,\frac{y}{R+y}\right]$$

But $M=-P\times l$ (here $l=R=98{\cdot}5$ mm)

$$\therefore\quad f_b=-\frac{P\times98{\cdot}5}{3400\times98{\cdot}5}\left[1+\frac{1}{0{\cdot}0857}\times\frac{y}{98{\cdot}5+y}\right]$$

or

$$f_b=-\frac{P}{3400}\left[1+\frac{11{\cdot}95y}{98{\cdot}5+y}\right] \qquad ...(1)$$

Maximum bending stress occurs at

$$y=-38{\cdot}5 \text{ mm}$$

$$\therefore\quad (f_b)_{max}=-\frac{P}{3400}\left[1-\frac{11{\cdot}95\times38{\cdot}5}{95{\cdot}5-38{\cdot}5}\right]=+\frac{P}{510} \qquad ...(2)$$

Also, $f_0=\dfrac{P}{A}=\dfrac{P}{3400}$

$$\therefore\quad f_{max}=(f_b)_{max}+f_0=\frac{P}{510}+\frac{P}{3400}=\frac{P}{443{\cdot}5}$$

But this is not to exceed the permissible stress of 120 N/mm²

$$\therefore\quad \frac{P}{443{\cdot}5}=120$$

or $P=120\times443{\cdot}5=$ **53217 N.**

Max. compressive stress occurs at $y=d_2=51{\cdot}5$ mm

$$f=-\frac{P}{3400}\left[1+\frac{11{\cdot}95\times51{\cdot}5}{98{\cdot}5+51{\cdot}5}\right]+\frac{P}{3400}$$

$$=-\frac{53217}{3400}\times\frac{11{\cdot}95\times51{\cdot}5}{98{\cdot}5+51{\cdot}5}$$

$$=\mathbf{64{\cdot}2\ N/mm^2\ (comp.).}$$

17·7. STRESSES IN RING SUBJECTED TO CONCENTRATED LOAD

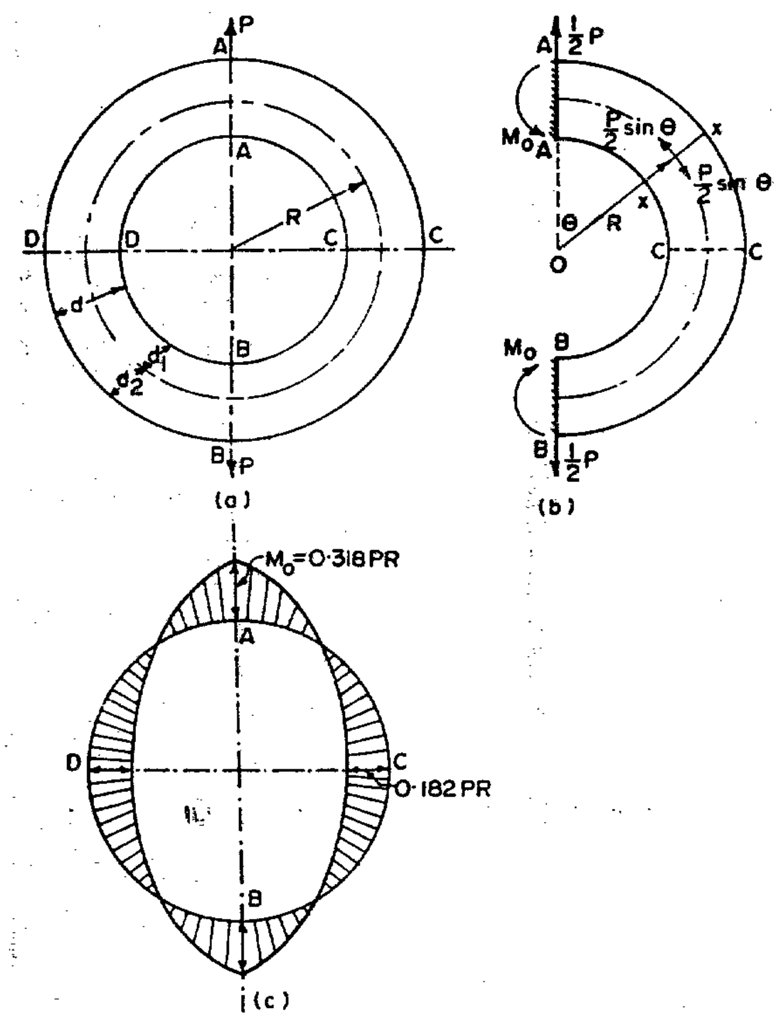


Fig. 17·14. Stresses in a closed ring.

Consider a ring subjected to a pull (or push) through its centre, as shown in Fig. 17·14 (*a*). At any radial section, such as XX, it is subjected to a bending moment M, a radial shear F and a normal pull (or thrust) P. From the condition of symmetry, the distribution of stress in two halves of ring will be the same. Due to symmetrical loading, it is evident that there will be no axial load in the ring at the loaded points A and B. Cut the ring in two parts, through A and B, and fix the ends A and B. Each half of the ring carries loads of $\frac{1}{2}P$ and moments M_0 at ends A and B [Fig. 17·14 (*b*)].

The problem of determining moment M at any section is statically indeterminate. The problem can approximately be solved by neglecting the effect of initial curvature in determining an expression for the elastic rotation of any section in terms of the moment at the section. The error introduced due to this simplification is relatively small. However, while calculating the bending stress, after the bending moment has been found, the curvature has to be taken into account since it has significant effect on its value. If the method of strain energy is used in determining the value of M_0 (and hence M), the impact of the simplification is to neglect the strain energy due to thrust. A straight beam, subjected to transverse loading, is subjected to bending moment and shearing force at any section and the strain energy at any point is predominantly due to bending. A section of a curved beam, however, has strain energy due to both bending as well as thrust. By considering, for the purposes of determining M and M_0, the beam to be straight, we indirectly neglect the effect of thrust in the expression the strain energy.

Consider a section XX, such that the radius OX makes an angle θ with OA. The section XX is subjected to a bending moment M given by

$$M=M_0-\tfrac{1}{2}P\,(R\sin\theta) \qquad \ldots(17{\cdot}35)$$

where R=radius of the centre line of the ring.

In addition to this, the section is subjected to a shearing force of magnitude $\frac{P}{2}\cos\theta$, and a direct pull $\frac{P}{2}\sin\theta$. The most important stresses are those arising from bending and direct stress at the inner and outer edges of the ring at the sections where the bending moments and the direct stress reach their extreme value.

In order to find the value of M_0, use the principle of minimum strain energy :

$$\frac{\partial U}{\partial M_0}=0$$

Consider a small element of length ds of the ring, subtending at angle $d\theta$ at the centre.

$$dU=\frac{1}{2}\,\frac{M^2}{EI}\,ds$$

$$\therefore \quad U=\int_0^\pi \frac{1}{2}\,\frac{M^2}{EI}\,ds$$

$$\therefore \quad \frac{\partial U}{\partial M_0}=\int_0^\pi \frac{M}{EI}\,\frac{\partial M}{\partial M_0}\,ds=0$$

But $\quad M=M_0-\tfrac{1}{2}PR\sin\theta\;;\; ds=Rd\theta$

$$\therefore \quad \frac{\partial M}{\partial M_0}=1$$

$$\therefore \quad \frac{\partial U}{\partial M_0}=\int_0^\pi \left(M_0-\frac{1}{2}PR\sin\theta\right)Rd\theta=0$$

$$\therefore \quad \left(M_0\theta-\tfrac{1}{2}PR\cos\theta\right)_0^\pi=1$$

or $$M_0=\frac{PR}{\pi}=0{\cdot}318\,PR \qquad \ldots(17{\cdot}36)$$

Substituting the value of M_0 in Eq. 17·35, we get

$$M=\frac{PR}{\pi}-\frac{1}{2}PR\sin\theta=PR\left(\frac{1}{\pi}-\frac{1}{2}\sin\theta\right) \qquad \ldots(17{\cdot}37)$$

It should be noted that the moment M_0 is positive, since it reduces the radius of curvature. At the section CC,

when $\quad \theta=\frac{\pi}{2}$, we have

$$M=PR\left(\frac{1}{\pi}-\frac{1}{2}\right)=-0{\cdot}182PR \qquad \ldots(17{\cdot}38)$$

The bending moment at CC is, therefore, negative. The bending moment is zero at a section given by

$$M=0=PR\left(\frac{1}{\pi}-\frac{1}{2}\sin\theta\right)$$

or $$\sin\theta=\frac{2}{\pi} \qquad \ldots[17{\cdot}39\,(a)]$$

or $$\theta=32{\cdot}5^\circ$$

The complete B.M. diagram is shown in Fig. 17·14. Knowing M, the bending stress at any point can be determined by Eq. 17·12.

The stress at any point will be equal to the algebraic sum of the stress due to direct pull $\frac{1}{2}P\sin\theta$ and the bending stress f_b.

Thus $$f=f_o+f_b=\frac{P\sin\theta}{2A}+\frac{M}{AR}\left(1+\frac{1}{m}\frac{y}{R+y}\right) \quad ...(17·40)$$

Let d_1= distance of the extreme inside edge of the cross-section from the centre line.

d_2= distance of the extreme outside edge [Fig. 17·14 (*a*)].

(*a*) *At the intrados of any section*, $y=-d_1$

$$\therefore \quad f_i=\frac{P\sin\theta}{2A}+\frac{M}{AR}\left(1-\frac{1}{m}\frac{d_1}{R-d_1}\right) \quad ...(17·41)$$

where $M=PR\left(\frac{1}{\pi}-\frac{1}{2}\sin\theta\right)$

At $\theta=0$, Eq. 17·41 reduces to

$$f_i=\frac{PR}{\pi AR}\left(1-\frac{1}{m}\frac{d_1}{R-d_1}\right)=\frac{P}{\pi A}\left(1-\frac{1}{m}\frac{d_1}{R-d_1}\right) \quad ...[17·42\ (a)]$$

Since $\frac{1}{m}\frac{d_1}{R-d_1}$ is always greater than 1, the above expression is negative. Hence the maximum compressive stress, given by the above equation at $\theta=0$, is

$$(f_c)_{max}=\frac{P}{\pi A}\left(\frac{1}{m}\frac{d_1}{R-d_1}-1\right) \quad ...(17·42)$$

Similarly, at $\theta=\frac{\pi}{2}$, Eq. 17·41 reduces to

$$f_i=\frac{P}{2A}+\frac{PR\left(\frac{1}{\pi}-\frac{1}{2}\right)}{AR}\left(1-\frac{1}{m}\frac{d_1}{R-d_1}\right)$$

or $$f_i=\frac{P}{2A}-\frac{0·182\,PR}{AR}\left(1-\frac{1}{m}\frac{d_1}{R-d_1}\right)$$

or $$f_i=\frac{P}{2A}-\frac{0·182P}{A}\left(1-\frac{1}{m}\frac{d_1}{R-d_1}\right) \quad ...(17·43)$$

It can be seen that f_i given by Eq. 17·43 has both the terms positive $\left(\frac{1}{m}\frac{d_1}{R-d_1}>1\right)$, and hence f_i at $\theta=\frac{\pi}{2}$ is tensile.

Thus, the stress at the intrados changes from a compressive value at $\theta=0$ to a tensile value at $\theta=\pi/2$.

(*b*) *At the extrados of any section*, $y=+d_2$

$$\therefore \quad f_e=\frac{P\sin\theta}{2A}+\frac{M}{AR}\left(1+\frac{1}{m}\frac{d_2}{R+d_2}\right) \quad ...(17·44)$$

where $M=PR\left(\frac{1}{\pi}-\frac{1}{2}\sin\theta\right)$

At $\theta=0$, the above expression reduces to

$$f_e=\frac{P}{\pi A}\left(1+\frac{1}{m}\frac{d_2}{R+d_2}\right) \quad ...(17·45)$$

This is wholly tensile.

At $\theta=\pi/2$, Eq. 17·44 reduces to

$$f_e=\frac{P}{2A}+\frac{PR\left(\frac{1}{\pi}-\frac{1}{2}\right)}{AR}\left[1+\frac{1}{m}\frac{d_2}{R+d_2}\right]$$
$$=\frac{P}{2A}-\frac{0·182\,P}{A}\left(1+\frac{1}{m}\frac{d_2}{R+d_2}\right) \quad ...(17·46)$$

This will be evidently compressive, since the second term of R.H.S. (*i.e.* bending stress) is always more than the first term.

The absolute maximum tensile stress anywhere in the ring may be given either by Eq. 17·43 or Eq. 17·44 or Eq. 17·45. For a ring whose mean radius R is large compared to the dimensions of cross-section, Eq. 17·43 gives the greatest tensile stress, where if R is small, owing to greater curvature, greatest tension may be given by Eq. 17·45. There is, however, a critical value of R at which the tensile stress given by Eqs. 17·43 and 17·45 are equal.

Example 17·6. *A closed ring of mean radius 120 mm is subjected to a pull of 20 kN the line of action of which passes through its centre. The ring is circular in cross-section with a radius equal to 40 mm. Find the maximum value of tensile and compressive stresses in the ring.*

Solution. (Fig. 17·14)

The factor m for a circular section is given by Eq. 17·25 (*a*)

$$m=-1+2\left(\frac{R}{r}\right)^2-2\left(\frac{R}{r}\right)\sqrt{\left(\frac{R}{r}\right)^2-1}$$

where $\frac{R}{r}=\frac{120}{40}=3$

$$\therefore \quad m=-1+2(3)^2-2(3)\sqrt{3^2-1}=0·02943$$

Alternatively, from Eq. 17·25 (*b*),

$$m=\frac{1}{4}\left(\frac{r}{R}\right)^2+\frac{1}{8}\left(\frac{r}{R}\right)^4+\frac{5}{64}\left(\frac{r}{R}\right)^6+\ldots=0.02943$$

$$\therefore \quad \frac{1}{m}=\frac{1}{0.02943}=33.98\simeq 34$$

$$A=\pi r^2=\pi(40)^2=5027 \text{ mm}^2.$$

(*a*) At $\theta=0$, the stress at intrados is given by Eq. 17·25 (*a*),

$$f_i=\frac{P}{\pi A}\left(1-\frac{1}{m}\,\frac{d_1}{R-d_1}\right)$$

$$=\frac{20000}{\pi(5027)}\left[1-34\,\frac{40}{120-40}\right]=-20.3 \text{ N/mm}^2$$

$$=20.3 \text{ N/mm}^2 \text{ (compressive)} \qquad \ldots(i)$$

The stress at extrados is given by Eq. 17·45,

$$f_i=\frac{P}{\pi A}\left[1+\frac{1}{m}\,\frac{d_2}{R+d_2}\right]$$

$$=\frac{20000}{\pi(5027)}\left[1+\frac{34\times 40}{120+40}\right]$$

$$=12.03 \text{ N/mm}^2 \text{ (tensile)} \qquad \ldots(ii)$$

(*b*) At $\theta=\frac{\pi}{2}$, the stress at intrados is given by Eq. 17·43

$$f_i=\frac{P}{2A}-\frac{0.182\,P}{A}\left(1-\frac{1}{m}\,\frac{d_1}{R-d_1}\right)$$

$$=\frac{20000}{2\times 5027}-\frac{0.182\times 20000}{5027}\left[1-\frac{34\times 40}{120-40}\right]$$

$$=1.99+11.59=13.58 \text{ N/mm}^2 \text{ (tensile)} \qquad \ldots(iii)$$

The stress at the extrados is given by Eq. 17·46,

$$f_e=\frac{P}{2A}-\frac{0.182\,P}{A}\left[1+\frac{1}{m}\,\frac{d_2}{R+d_2}\right]$$

$$=\frac{20000}{2\times 5027}-\frac{0.182\times 20000}{5027}\left[1+\frac{34\times 40}{120+40}\right]$$

$$=1.99-6.88=-4.89$$

$$=4.89 \text{ N/mm}^2 \text{ (compressive)} \qquad \ldots(iv)$$

From (*i*) to (*iv*), we get

Max. compressive stress $=20.3$ N/mm²

Max. tensile stress $=13.58$ N/mm².

17·8. STRESSES IN SIMPLE CHAIN LINKS

Fig. 17·15 (*a*) shows a simple chain link, consisting of semi-circular ends and straight sides connecting them. The approximate theory of stress distribution for the case of rings can be extended to make an estimate of stress in a link.

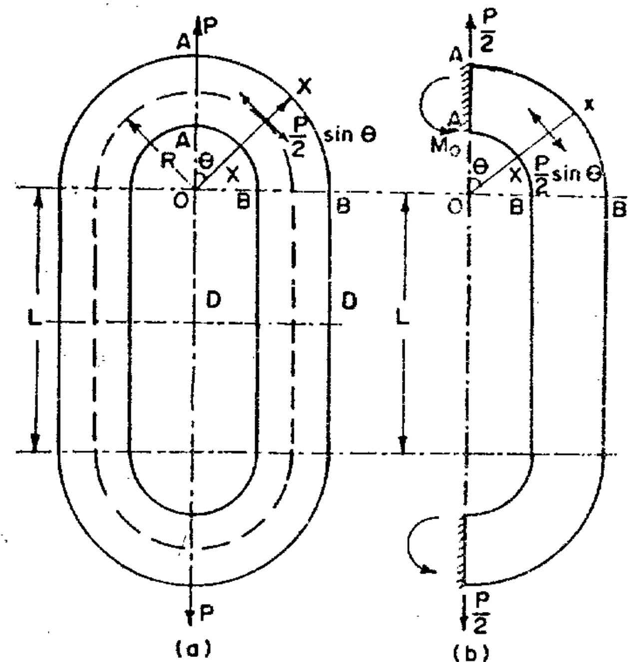


Fig. 17·15

Let R=mean radius of curvature of the circular portion

L=length of the straight portion.

Consider half portion the link, subjected to pull $P/2$ as shown in Fig. 17·15 (*b*).

At any section XX, the bending moment M is given by

$$M=M_0-\frac{PR}{2}\sin\theta \qquad \ldots(1)$$

At BB, moment $=M_1=M_0-\frac{PR}{2}$ ...(2)

Now $\quad U=U_{circular}+U_{straight}$

$$U=2\int_0^{\pi/2}\frac{M^2 ds}{2EI}+2\int_0^{L/2}\frac{M^2 dx}{2EI}$$

$$\therefore \quad \frac{\partial U}{\partial M_0}=2\int_0^{\pi/2}\frac{M}{EI}\frac{\partial M}{\partial M_0}ds+2\int_0^{L/2}\frac{M}{EI}\frac{\partial M}{\partial M_0}dx=0$$

or

$$\int_0^{\pi/2} M\frac{\partial M}{\partial M_0}\,ds+\int_0^{L/2} M\frac{\partial M}{\partial M_0}\,dx=0 \quad ...(3)$$

For circular portion, $M=M_0-\frac{PR}{2}\sin\theta$

$$\therefore \quad \frac{\partial M}{\partial M_0}=1\ ;\ ds=Rd\theta$$

$$\therefore \quad \int_0^{\pi/2} M\frac{\partial M}{\partial M_0}\,dx=\int_0^{\pi/2}\left(M_0-\frac{PR}{2}\sin\theta\right)Rd\theta$$

$$=R\left(M_0\theta+\frac{PR}{2}\cos\theta\right)_0^{\pi/2}$$

$$=M_0R\,\frac{\pi}{2}-\frac{PR^2}{2} \quad ...(4)$$

For the straight portion, $M=M_1=M_0-\frac{PR}{2}$

$$\therefore \quad \frac{\partial M}{\partial M_0}=1$$

$$\therefore \quad \int_0^{L/2} M\frac{\partial M}{\partial M_0}\,dx=\int_0^{L/2}\left(M_0-\frac{PR}{2}\right)dx$$

$$=\left(M_0-\frac{PR}{2}\right)\frac{L}{2} \quad ...(5)$$

Substituting in (3), we get

$$M_0R\,\frac{\pi}{2}-\frac{PR^2}{2}+\left(M_0-\frac{PR}{2}\right)\frac{L}{2}=0$$

$$\therefore \quad M_0(\pi R+L)-\frac{PR}{2}\,(L+2R)=0$$

or

$$M_0=\frac{PR}{2}\,\frac{L+2R}{L+\pi R} \quad ...(17\cdot47)$$

Substituting in (1), the moment at any section in the curved portion is given by

$$M=M_0-\frac{PR}{2}\sin\theta=\frac{PR}{2}\cdot\frac{L+2R}{L+\pi R}-\frac{PR}{2}\sin\theta$$

$$=\frac{PR}{2}\left[\frac{L+2R}{L+\pi R}-\sin\theta\right] \quad ...(17\cdot48)$$

At $\theta=\frac{\pi}{2}$,

$$M=M_1=\frac{PR}{2}\left[\frac{L+2R}{L+\pi R}-1\right]$$

$$=\frac{PR}{2}\,\frac{2R-\pi R}{L+\pi R} \quad ...(17\cdot49)$$

(M_1 is evidently negative since $2R$ is lesser than πR)

In addition to the moment M, any section in the curved portion is subjected to a pull of $\frac{P}{2}\sin\theta$. This pull increases to a value of $\frac{P}{2}$ in the straight portion.

Stresses in the curved portion

The stress at any section is the algebraic sum of bending stress and the direct stress :

$$f=f_0+f_b=\frac{P}{2A}\sin\theta+\frac{M}{AR}\left(1+\frac{1}{m}\,\frac{y}{R+y}\right)$$

or

$$f=\frac{P}{2A}\sin\theta+\frac{P}{2A}\left[\frac{L+2R}{L+\pi R}-\sin\theta\right]\left[1+\frac{1}{m}\,\frac{y}{R+y}\right] \quad ...(17\cdot50)$$

(*a*) At the section AA, $\theta=0$

$$\therefore \quad f_i=\frac{P}{2A}\left[\frac{L+2R}{L+\pi R}\right]\left[1+\frac{1}{m}\,\frac{y}{R+y}\right] \quad ...(17\cdot51)$$

At the intrados, $y=-d_1$.

$$\therefore \quad f=\frac{P}{2A}\left[\frac{L+2R}{L+\pi R}\right]\left[1-\frac{1}{m}\,\frac{d_1}{R-d_1}\right] \quad ...(17\cdot52)$$

This is negative (*i.e.* compressive) since $\frac{1}{m}\,\frac{d_1}{R+d_1}$ is always greater than 1.

At the extrados, $y=+d_2$

$$\therefore \quad f_e=\frac{P}{2A}\left[\frac{L+2R}{L+\pi R}\right]\left[1+\frac{1}{m}\,\frac{d_2}{R+d_2}\right] \quad ...(17\cdot53)$$

This is tensile.

(*b*) At section BB, $\theta=\pi/2$ (just at the junction of curved and straight portion).

$$\therefore \quad f=\frac{P}{2A}+\frac{P}{2A}\left[\frac{L+2R}{L+\pi R}-1\right]\left[1+\frac{1}{m}\,\frac{y}{R+y}\right]$$

$$=\frac{P}{2A}+\frac{P}{2A}\left[\frac{2R-\pi R}{L+\pi R}\right]\left[1+\frac{1}{m}\,\frac{y}{R+y}\right]$$

$$=\frac{P}{2R}-\frac{PR}{2A}\left[\frac{\pi-2}{L+\pi R}\right]\left[1+\frac{1}{m}\frac{y}{R+y}\right] \quad ...(17.54)$$

At the intrados, $y=-d_1$

$$\therefore \quad f_i=\frac{P}{2A}-\frac{PR}{2A}\left[\frac{\pi-2}{L+\pi R}\right]\left[1-\frac{1}{m}\frac{d_1}{R-d_1}\right] \quad ...(17.55)$$

This is tensile, since both the terms of the R.H.S. will finally come out to be positive.

At the extrados, $y=+d_2$

$$\therefore \quad f_e=\frac{P}{2R}-\frac{PR}{2A}\left[\frac{\pi-2}{L+\pi R}\right]\left[1+\frac{1}{m}\frac{d_2}{R+d_2}\right] \quad ...(17.56)$$

Stresses in the straight portion

$$f=\frac{P}{2A}+\frac{M_1}{I}y$$

$$=\frac{P}{2A}+\frac{PR}{2I}\left[\frac{2R-\pi R}{L+\pi R}\right]y$$

$$=\frac{P}{2A}-\frac{PR^2}{2I}\left[\frac{\pi-2}{L+PR}\right]y \quad ...(17.57)$$

At the intrados $y=-d_1$

$$\therefore \quad f_i=\frac{P}{2A}+\frac{PR^2}{2I}\left[\frac{\pi-2}{L+\pi R}\right]d_1 \quad ...(17.58)$$

This is tensile.

At the extrados, $y=+d_2$

$$\therefore \quad f_e=\frac{P}{2A}-\frac{PR^2}{2I}\left[\frac{\pi-2}{L+\pi R}\right]d_2 \quad ...(17.59)$$

Example 17.7. *A simple link shown in Fig. 17.15 consists of a semi-circular ends of mean radius of curvature 80 mm and the length of straight sides also equal to 80 mm. The radius of circular cross-section of the link is 40 mm. Estimate the stresses in the link when a pull of 100 kN is applied at the ends.*

Solution.

$$\pi=Ar^2=\pi(40)^2=5027 \text{ mm}^2$$

$$\frac{r}{R}=\frac{40}{80}=0.5$$

$$m=\frac{1}{4}\left(\frac{r}{R}\right)^2+\frac{1}{8}\left(\frac{r}{R}\right)^4+\frac{5}{64}\left(\frac{r}{R}\right)^6+......=0.0715$$

$$\therefore \quad \frac{1}{m}=13.97 \simeq 14\ ;\ L=80 \text{ mm}\ ;\ R=80 \text{ mm.}$$

(a) Section AA

Stress at intrados is given by Eq. 17.52,

$$f_i=\frac{P}{2A}\left[\frac{L+2R}{L+\pi R}\right]\left[1-\frac{1}{m}\frac{d_1}{R+d_1}\right]$$

$$=\frac{100000}{2\times5027}\left[\frac{80+160}{80+80\pi}\right]\left[1-\frac{14\times40}{80-40}\right]=-93.7$$

$=93.7$ N/mm² (compressive)

The stress at extrados is given by Eq. 17.52

$$f_e=\frac{P}{2A}\left[\frac{L+2R}{L+\pi R}\right]\left[1+\frac{1}{m}\frac{d_2}{R-d_2}\right]$$

$$=\frac{100000}{2\times5027}\left[\frac{80+160}{80+80\pi}\right]\left[1+14\frac{40}{80+40}\right]$$

$=40.8$ N/mm².

(b) Section BB (curved portion)

Stress at the intrados is given by Eq. 17.55,

$$f_i=\frac{P}{2A}-\frac{PR}{2A}\left[\frac{\pi-2}{L+\pi R}\right]\left[1-\frac{1}{m}\frac{d_1}{R+d_1}\right]$$

$$=\frac{100000}{2\times5027}-\frac{100000\times80}{2\times5027}\left[\frac{\pi-2}{80+80\pi}\right]\left[1-\frac{14\times40}{80-40}\right]$$

$=9.96+35.64$

$=45.60$ N/mm² (tensile)

Stress at the extrados is given by Fig. 17.56,

$$f_e=\frac{P}{2\pi}-\frac{PR}{2A}\left[\frac{\pi-2}{L+\pi R}\right]\left[1+\frac{1}{m}\frac{d_2}{R+d_2}\right]$$

$$=\frac{100000}{2\times5027}-\frac{100000\times80}{2\times5027}\left[\frac{\pi-2}{80+80\pi}\right]\left[1+\frac{14\times40}{80+40}\right]$$

$=9.96-15.54=-5.58$

$=5.58$ N/mm² (compressive).

(c) Straight Portion

The stress at the intrados is given by Eq. 17.58.

$$f_i=\frac{P}{2A}+\frac{PR^2}{2I}\left[\frac{\pi-2}{L+\pi R}\right]d_1$$

Here $I=\frac{\pi}{4}r^4=\frac{\pi}{4}(40)^4=2.01\times10^6$ mm²

$$\therefore \quad f_i=\frac{100000}{2\times5027}+\frac{100000(80)^2}{2\times2.01\times10^6}\left[\frac{\pi-2}{80+80\pi}\right]\times40$$

$=9.96+21.94=31.9$ N/mm² (tensile)

The stress at the extrados is given by Eq. 17·59,

$$f_e = \frac{P}{2A} - \frac{PR^2}{2I}\left[\frac{\pi-2}{L+\pi R}\right] d_2$$

Since $d_1 = d_2 = 40$ mm

$$f_e = 9{\cdot}96 - 21{\cdot}94 = -11{\cdot}98$$
$$= \mathbf{11{\cdot}98\ N/mm^2}\ \text{(compressive)}$$

Hence the maximum tensile stress in the link is 45·6 N/mm² just at the junction of the curved and straight portion, while the maximum compressive stress is 93·7 N/mm².

PROBLEMS

1. A curved beam, whose centre line is a circular arc of radius 60 mm, is formed of a tube of radius 20 mm outside and thickness 2·5 mm. Determine the greatest tensile and compressive stresses set up by a bending moment of 200 kN/mm tending to increase the curvature.

2. Fig. 17·16 shows the normal section of a curved beam. Find the dimension b_1 and b_3 so that the maximum and minimum stresses developed in the section due to pure bending are numerically equal. Given : $b_1 + b_3 = 50$ mm.

3. Calculate the greatest tensile and compressive stresses in the hook shown in Fig. 17·17. if it carries a load $P = 10$ kN.

4. A ring with a mean radius of curvature of 25 mm is subjected to a load of 2000 N as shown in Fig. 17·18. The ring is made of circular section of 10 mm radius. Calculate the circumferential stress on the inside of the fibre of the ring at A and at B.

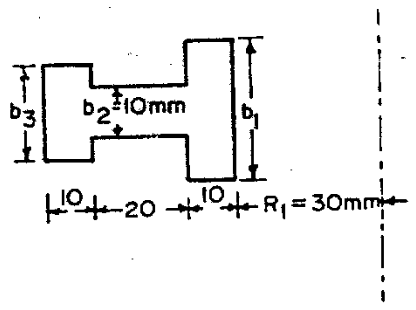


Fig 17·16

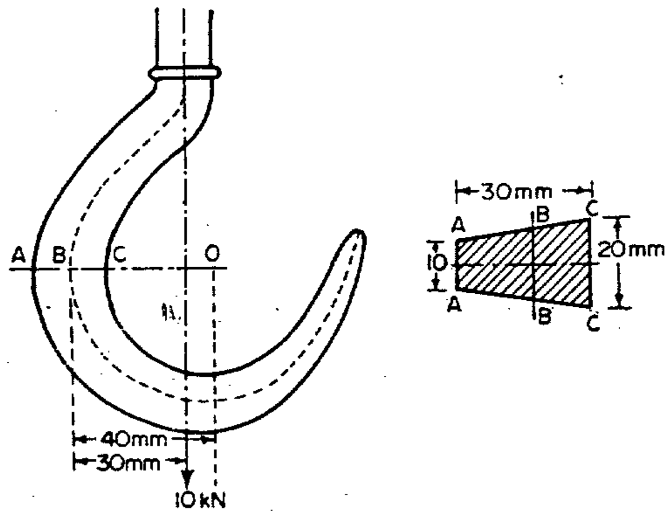


Fig. 17·17

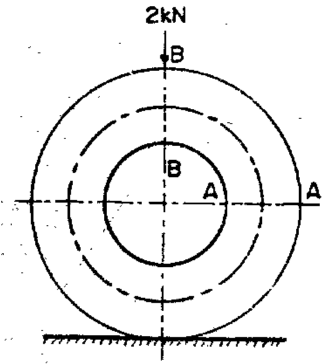


Fig. 17·18

ANSWERS

1. 65·6 N/mm² ; −91·6 N/mm².
2. $b_1 = 36{\cdot}7$ mm ; $b_3 = 13{\cdot}3$ mm.
3. 184·2 N/mm² ; −99 N/mm².
4. $f_A = -19{\cdot}9$ N/mm², $f_B = 29{\cdot}1$ N/mm².

18

Stresses Due to Rotation

18·1. ROTATING RING OR WHEEL RIM

Stresses are set up in circular rings, wheel rims, circular discs and cylinders, etc. on account of rotation about their axis of symmetry. The analysis of the stresses set up in a rotating member such a pulleys, fly wheels etc. can be made on the basis of certain simplified assumptions. We shall take first the case of a thin ring rotating about an axis through its centre of gravity and perpendicular to its central plane.

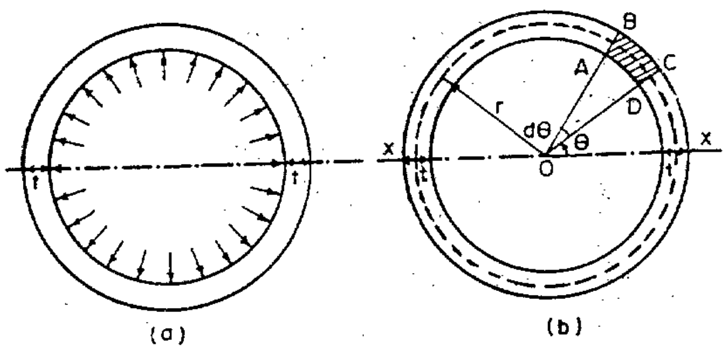


Fig. 18·1. Rotating Ring.

Let r be the mean radius of the ring. The rotation will cause hoop stress (hoop tension) in it, due to its inertia. Assuming the cross-sectional dimensions to be small compared to the radius, the hoop tension will be nearly uniform.

Let ω=Angular velocity in radians.

v=linear velocity.

$$\omega=\frac{v}{r} \qquad \ldots(18\cdot1)$$

Every point on the rim will have a radial inward acceleration equal to $\omega^2 r$ or equal to $\frac{v^2}{r}$. This inward radial acceleration will give rise so inward radial force, known as *centripetal force*. The centripetal force is resisted by an equal and oppsite force P, known as *centrifugal force*, produced due to inertia of the ring. This centrifugal force acts like an internal pressure in a thin cylinder trying to burst it out into two halves.

Consider an element $ABCD$, subtending an angle $\delta\theta$ at the centre. Then the centrifugal force δP on the elementary volume $ABCD$ is

$$\delta P=\text{mass}\times\text{acceleration}$$

$$=\left(\frac{\rho r\,.\,d\theta\,.\,l\,.\,t}{g}\right)\frac{v^2}{r}=\frac{\rho}{g}\,l\,t\,v^2\,d\theta$$

where, ρ=unit weight of material

g=acceleration due to gravity

l=length of ring

t=thickness of ring.

Let the ring burst about XX.

∴ Component of force perpendicular to XX, trying to burst it

$$=\frac{\rho}{g}\,l\,t\,v^2\,d\theta\,.\,\sin\theta.$$

∴ Total force $=\int_0^\pi \frac{\rho}{g}\,l\,t\,v^2\sin\theta\,d\theta$

$$=-\frac{\rho}{g}\,l\,t\,v^2\Big[\cos\theta\Big]_0^\pi=\frac{2\rho}{g}\,l\,t\,v^2 \qquad \ldots(1)$$

Let f=hoop stress (tensile) produced in the ring to resist this bursting action.

Then total resisting force=stress×total resisting area

$$=f(2\,t\,l) \qquad \ldots(2)$$

Equating (1) and (2), we get

$$f(2\,t\,l)=\frac{2\rho}{g}\,l\,t\,v^2$$

or

$$f=\frac{\rho}{g}\,v^2=\frac{\rho}{g}\,\omega^2 r^2 \qquad \ldots(18\cdot2)$$

Thus, the stress induced is independent of the thickness of the ring, and wholly depends upon the velocity. Hence increasing the section of puleys, flywheels, etc. does not decrease the stress.

For a rim of a given material (*i.e.*, given permissible value of (f_A), the limiting velocity is given by

$$v=\sqrt{\frac{f_A\, g}{\rho}} \qquad \ldots(18{\cdot}3)$$

where f_A=allowable stress.

For pulleys and wheel rims made of cast iron, the limiting speed is in the vicinity of 27 m/sec. Strong wheel rims, made channel shaped and wound round with high tensile steel wire, may have limiting speed as high as 80 m/sec.

Example 18·1. *The rim of a steel flywheel 1 m diameter and 200 mm wide is rotating at 2400 revolutions per minute. Calculate (i) hoop stress developed, (ii) shrinkage in the width of the rim, and (iii) extension of the circumference due to rotation.*

What will be the speed in R.P.M. at which the rim will burst?

Take $E=2\times10^5$ N/mm².

Poisson's ratio $=\frac{1}{3}$

Ultimate stress for steel=400 N/mm²

Unit weight of steel $=78{\cdot}5\times10^{-6}$ N/mm³ (78·5 kN/m³).

Solution.

(*i*) $$\omega=\frac{2\pi n}{60}=\frac{2\times2400}{60}=80\,\pi \text{ radians/sec.}$$

$$f=\frac{\rho}{g}\,\omega^2 r^2=\frac{78{\cdot}5\times10^{-6}}{9810}\,(80\pi)^2\,(500)^2$$

$$=\mathbf{126{\cdot}4\ N/mm^2}.$$

(*ii*) In the above treatment, the effect of radial stress is neglected. Hence

Lateral strain $$=-\frac{f}{mE}=-\frac{1}{3}\times\frac{126{\cdot}4}{2\times105}$$

$$=-2{\cdot}1\times10^{-4} \text{ (i.e. compressive)}$$

But lateral strain $$=\frac{\text{change in width}}{\text{original width}}$$

$\therefore$ Change in width$=2{\cdot}1\times10^{-4}\times200$ mm

$$=\mathbf{0{\cdot}042\ mm}.$$

(*iii*) Circumferential strains$=\frac{f}{E}=\frac{126{\cdot}4}{2\times10^5}=6{\cdot}32\times10^{-4}$

$$\therefore \quad \frac{\pi\delta d}{\pi d}=\frac{\delta d}{d}=6{\cdot}32\times10^{-4}$$

or $$\pi\delta d=(6{\cdot}32\times10^{-4})(\pi\times1000)\text{ mm}$$

$$=\mathbf{1{\cdot}98\ mm}.$$

Let N be the speed in r.p.m.

Then $$\frac{400}{126{\cdot}4}=\left(\frac{N}{2400}\right)^2$$

$$\therefore \quad N=2400\sqrt{\frac{400}{126{\cdot}4}}$$

$$=\mathbf{4270\ r.p.m.}$$

18·2. ROTATING DISC

Let us take the case of a circular disc rotating about its axis. It is assumed that the disc is of uniform thickness and that the thickness is so small compared with its diameter that there is no variation of stress along the thickness. At the free flat surfaces there can be no stress normal to these faces and there can be no shear stress on or perpendicular to these faces. Thus the direction of axis is the direction of zero principal stress. The displacement of any point due to strain must be radial. The radial and circumferential stresses, therefore, represent the principal stresses.

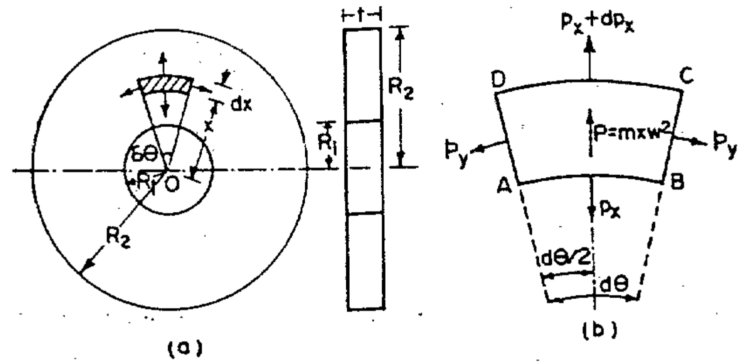


Fig. 18·2. Rotating disc.

Consider an element $ABCD$ of the disc, at a radius x, subtending an angle $d\theta$ at the centre, and of radial width dx. Volume of element$=x\,.\,d\theta\,.\,dx\,.\,t$ (approx.).

$$\therefore \quad \text{Mass of element}=\frac{\rho}{g}\,.\,x\,.\,dx\,.\,d\theta\,.\,t$$

$\therefore$ Centrifugal force on it, due to rotation is

$$P = \text{mass} \times \text{acceleration}$$

$$= \left(\frac{\rho}{g}\, x\, dx\, .\, d\theta\, .\, t\right)\left(\omega^2\, x\right)$$

$$= \frac{\rho}{g}\, x^2\, \omega^2 t\, dx\, d\theta \qquad \ldots(1)$$

Let p_x = radial stress at radius x

$p_x + dp_x$ = radial stress at radius $(x+dx)$

p_y = circumferential stress at radius x.

Resolving the forces along the centre line of the element, we get for the equilibrium,

$$p_x . x\, d\theta . t - (p_x + dp_x)(x+dx) d\theta . t + 2p_y . dx . t \sin \frac{d\theta}{2} - P = 0$$

Taking $\sin \frac{d\theta}{2} \simeq \frac{d\theta}{2}$, substituting the value of P from (1) and neglecting infinitesimal quantities of higher order, we get

$$-p_x . dx . d\theta . - x . d\theta . dp_x + p_y dx d\theta - \frac{\rho}{g}\, x^2 \omega^2 dx d\theta = 0$$

or
$$p_y = \frac{\rho}{g}\, \omega^2 x^2 + p_x + \frac{x\, dp_x}{dx} \qquad \ldots(2)$$

or
$$p_y = \frac{\rho}{g}\, \omega^2 x^2 + \frac{d}{dx}\,(x p_x) \qquad \ldots(2a)$$

and
$$p_y - p_x = x\, \frac{dp_x}{dx} + \frac{\rho}{g}\, \omega^2 x^2 \qquad \ldots(3)$$

The other relation between p_y and p_x can be obtained by the considerations of strains.

Let u = radial displacement of radius OA

(*i.e.* x increased to $x+u$)

$u + du$ = radial displacement of radius OD

(*i.e.* width dx increased to $dx + du$)

$\therefore$ Circumferential strain at x

$$= e_y = \frac{2\pi(x+u) - 2\pi x}{2\pi x} = \frac{u}{x}$$

But this is equal to $\frac{p_y}{E} - \frac{p_x}{mE}$

$$\therefore \qquad \frac{u}{x} = \frac{p_y}{E} - \frac{p_x}{mE} \qquad \ldots(4)$$

Also, Radial strain at x

$$= e_x = \frac{(dx+du) - dx}{dx} = \frac{du}{dx}$$

But this is equal to $\frac{p_x}{E} - \frac{1}{m}\, \frac{p_y}{E}$

$$\therefore \qquad \frac{du}{dx} = \frac{p_x}{E} - \frac{1}{m}\, \frac{p_y}{E} \qquad \ldots(5)$$

From (4),
$$u = \frac{x}{E}\left(p_y - \frac{p_x}{m}\right)$$

Differentiating,
$$\frac{du}{dx} = \frac{1}{E}\left(p_y - \frac{p_x}{m}\right) + \frac{x}{E}\left[\frac{dp_y}{dx} - \frac{1}{m}\, \frac{dp_x}{dx}\right]$$

Substituting this value of $\frac{du}{dx}$ in (5), we get

$$\frac{1}{E}\left(p_y - \frac{p_x}{m}\right) + \frac{x}{E}\left[\frac{dp_y}{dx} - \frac{1}{m}\, \frac{dp_x}{dx}\right] = \frac{p_x}{E} - \frac{1}{m}\, \frac{p_y}{E}$$

or
$$p_y\left(1 + \frac{1}{m}\right) - p_x\left(1 + \frac{1}{m}\right) + x\left[\frac{dp_y}{dx} - \frac{1}{m}\, \frac{dp_x}{dx}\right] = 0$$

or
$$(p_y - p_x)\left(1 + \frac{1}{m}\right) + x\left[\frac{dp_y}{dx} - \frac{1}{m}\, \frac{dp_x}{dx}\right] = 0$$

Substituting the value of $p_y - p_x$ from (3), we get

$$\left(x\, \frac{dp_x}{dx} + \frac{\rho}{g}\, \omega^2 x^2\right)\left(\frac{m+1}{m}\right) + x\, \frac{dp_y}{dx} - \frac{x}{m}\, \frac{dp_x}{dx} = 0$$

or
$$x\, \frac{dp_x}{dx} + \frac{\rho}{g}\, \omega^2 x^2 + \frac{m}{m+1}\, x\, \frac{dp_y}{dx} - \frac{x}{x+1}\, \frac{dp_x}{dx} = 0$$

or
$$x\, \frac{dp_x}{dx}\left\{1 - \frac{1}{m+1}\right\} + \frac{mx}{m+1}\, \frac{dp_y}{dx} + \frac{\rho}{g}\, \omega^2 x^2 = 0$$

$$\therefore \qquad \frac{dp_x}{dx} + \frac{dp_y}{dx} + \frac{m+1}{m}\, \frac{\rho\omega^2 x}{g} = 0$$

$$\frac{d(p_x + p_y)}{dx} + \frac{m+1}{m}\, \frac{\rho\omega^2 x}{g} = 0$$

Integrating it, we get

$$(p_x + p_y) + \frac{m+1}{m}\, \frac{\rho}{g}\, \omega^2\, \frac{x^2}{2} = \text{constant} = 2A \text{ (say)} \qquad \ldots(6)$$

$$\therefore \qquad p_y = 2A - p_x - \frac{m+1}{m}\, \frac{\rho\omega^2}{g}\, \frac{x^2}{2} \qquad \ldots(7)$$

Substituting this value of p_Y in Eq. (3),

$$2A-p_X-\frac{m+1}{m}\,\frac{\rho\omega^2}{g}\,\frac{x^2}{2}-p_X=x\,\frac{dp_X}{dx}+\frac{\rho}{g}\omega^2x^2$$

or $$2p_X+x\,\frac{dp_X}{dx}=2A-\frac{m+1}{m}\,\frac{\rho\omega^2}{g}\,\frac{x^2}{2}-\frac{\rho\omega^2}{g}x^2$$

or $$2x.\,p_X+x^2\,\frac{dp_X}{dx}=2Ax-\frac{\rho\omega^2x^3}{g}\left(\frac{3m+1}{2m}\right)$$

or $$\frac{d(x^2p_X)}{dx}=2Ax-\frac{3m+1}{2m}\,\frac{\rho\omega^2}{g}x^3$$

Integrating,

$$x^2p_X=Ax^2-\frac{3m+1}{2m}\,\frac{\rho\omega^2}{g}\,\frac{x^4}{4}+B$$

$$p_X=A-\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}\,x^2+\frac{B}{x^2} \qquad ...(8)$$

where B is another constant of integration.

Substituting the value of p_X in (7), we get

$$p_Y=\left[2A-\frac{m+1}{m}\,\frac{\rho\omega^2}{g}\,\frac{x^2}{2}\right]-\left[A-\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}x^2+\frac{B}{x^2}\right]$$

$$=A-\frac{m+3}{8m}\,\frac{\rho\omega^2}{g}\,x^2-\frac{B}{x^2} \qquad ...(9)$$

To summarise we get the following expression for p_X and p_Y (from 8 and 9) :

$$p_X=A+\frac{B}{x^2}-\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}x^2 \qquad ...(18\cdot4)$$

$$p_Y=A-\frac{B}{x^2}-\frac{m+3}{8m}\,\frac{\rho\omega^2}{g}\,x^2 \qquad ...(18\cdot5)$$

The constants A and B are to be evaluated from the various cases of boundary conditions. We will take the following common cases :

(*i*) Disc with a central hole

(*ii*) Solid disc.

18·3. DISC WITH A CENTRAL HOLE

Let us take a hollow disc, *i.e.*, disc with a central hole with internal radius R_1 and external radius R_2 as shown in Fig. 18·2.

The boundary conditions are as as follows :

At $\quad x=R_1,\ p_X=0$

At $\quad x=R_2,\ p_X=0$

Substituting these in Eq. 18·4, we get

$$0=A+\frac{B}{R_1^2}-\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}\,R_1^2 \qquad ...(1)$$

and $$0=A+\frac{B}{R_2^2}-\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}\,R_2^2 \qquad ...(2)$$

Subtracting (1) from (2), we get

$$B\left(\frac{1}{R_2^2}-\frac{1}{R_1^2}\right)-\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}\,(R_2^2-R_1^2)=0$$

or $$B\left(\frac{R_1^2-R_2^2}{R_1^2R_2^2}\right)=\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}\,(R_2^2-R_1^2)$$

$$\therefore\quad B=-\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}\,R_1^2R_2^2 \qquad ...(18\cdot6)$$

Substituting this in (1), we get

$$A=\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}\,R_2^2+\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}\,R_1^2$$

$$=\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}\,(R_1^2+R_2^2) \qquad ...(18\cdot7)$$

Substituting the values of A and B in Eq. 18·4, we get

$$p_X=\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}\,(R_1^2+R_2^2)-\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}\,\frac{R_1^2R_2^2}{x^2}-\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}\,x^2 \qquad ...(18\cdot8)$$

or $$p_X=\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}\left(R_1^2+R_2^2-\frac{R_1^2R_2^2}{x^2}-x^2\right) \qquad ...(18\cdot9)$$

Similarly, substituting the values of A and B in Eq. 18·5, we get

$$p_Y=\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}\,(R_1^2+R_2^2)+\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}\,\frac{R_1^2R_2^2}{x^2}-\frac{m+3}{8m}\,\frac{\rho\omega^2}{g}\,x^2$$

or $$p_Y=\frac{\rho\omega^2}{8mg}\left[(3m+1)(R_1^2+R_2^2)+(3m+1)\,\frac{R_1^2R_2^2}{x^2}-(m+3)x^2\right] \qquad ...(18\cdot10)$$

For getting maximum value of p_X,

$$\frac{dp_X}{dx}=0=\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}\left\{-2x+\frac{2R_1^2R_2^2}{x^3}\right\}$$

$$\therefore\quad R_1^2R_2^2=x^4$$

or $$R=\sqrt{R_1R_2} \qquad ...(18\cdot11)$$

Substituting this value of x in Eq. 18·9,

$$(p_X)_{max}=\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}\left(R_1^2+R_2^2-\frac{R_1^2R_2^2}{R_1R_2}-R_1R_2\right)$$

$$=\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}\,(R_2-R_1)^2 \qquad \text{...(18·12)}$$

Inspection of Eq. 18·10 shows that p_Y goes on increasing as x decreases and hence p_Y is maximum at $x=R_1$.

$$\therefore\ (p_Y)_{max}=\frac{\rho\omega^2}{8mg}\left[3(m+1)(R_1^2+R_2^2)+(3m+1)R_2^2-(m+3)R_1^2\right]$$

$$=\frac{\rho\omega^2}{4mg}\left[(3m+1)R_2^2+(m-1)R_1^2\right] \qquad \text{...(18·13)}$$

The variations p_X and p_Y with x are shown in Fig. 18·3.

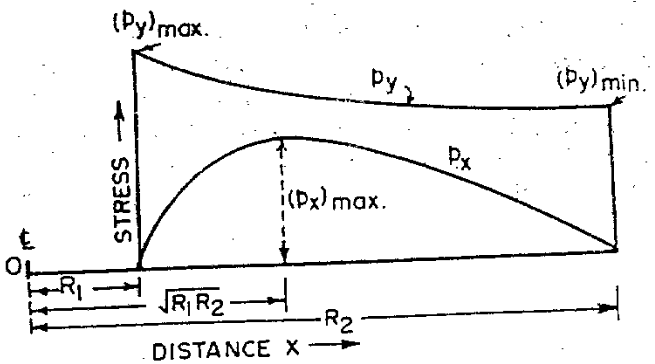


Fig. 18·3. Variations of p_X and p_Y in a hollow disc.

If R_1 is very small so that R_1^2 is negligible as compared to R_2^2, we get

$$(p_Y)_{max}=\frac{(3m+1)\rho\omega^2}{4mg}R_2^2 \qquad \text{...(18·14)}$$

This is double the value of $(p_Y)_{max}$ for a solid disc (see Eq. 18·19). *Hence even a small hole in the disc doubles the value of maximum hoop tension.*

Also, when R_1 approaches R_2, such that

$R_1 \simeq R_2 \simeq R$, we get

$$(p_Y)_{max}=\frac{\rho\omega^2}{4mg}\times 4mR^2=\frac{\rho}{g}\omega^2R^2 \qquad \text{...(18·15)}$$

which is the same as Eq. 26·2 obtained for the case of a ring.

18·4. SOLID DISC

Let the solid disc have radius equal to R.

Then $R_2=R\ ;\ R_1=0.$

At $x=0$, p_X is having some finite value. But if $x=0$ is substituted in Eq. 18·4, it gives infinite value of p_X. This suggests that the constant $B=0$, in Eq. 18·4.

Also, at $x=R$, $p_X=0$

$$\therefore\quad 0=A-\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}R^2$$

$$\therefore\quad A=\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}R^2 \qquad \text{...(18·16)}$$

Substituting these values of A and B, Eqs. 18·4 and 18·5, we get

$$p_X=\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}R^2-\frac{3m+1}{3m}\,\frac{\rho\omega^2}{g}x^2$$

$$=\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}(R^2-x^2) \qquad \text{...(18·17)}$$

and

$$p_Y=\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}R^2-\frac{m+3}{8m}\,\frac{\rho\omega^2}{g}x^2$$

$$=\frac{\rho\omega^2}{8mg}\left[(3m+1)\,R^2-(m+3)x^2\right] \qquad \text{...(18·18)}$$

By inspection, p_X and p_Y are maximum at $x=0$

$$(p_X)_{max.}=(p_Y)_{max.}=\frac{(3m+1)\rho\omega^2}{8mg}R^2 \qquad \text{...(18·19)}$$

Also at $x=R$

$$p_Y=\frac{\rho\omega^2}{8mg}\left[(3m+1-m-3)R^2\right]=\frac{(m-1)\rho\omega^2}{4mg}R^2 \qquad \text{...(18·20)}$$

The variations of p_X and p_Y are shown in Fig. 18·4.

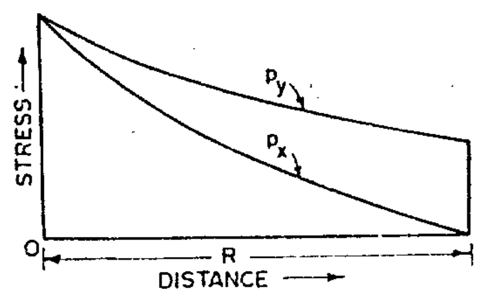


Fig. 18·4. Variations of p_X and p_Y in a solid disc.

18·5. PERMISSIBLE SPEED OF A SOLID DISC

From Eq. 18·19 the maximum value of p_X and p_Y is given by

$$(p_X)_{max}=(p_Y)_{max}=\frac{(3m+1)\rho\omega^2}{8mg}R^2$$

We shall now apply various theories of failure to find the permissible speed ω. For the treatment that follows, $(p_Y)_{max}$ and $(p_X)_{max}$ will be designated by p_X and p_Y respectively, for simplicity.

(1) Maximum principal stress theory

Let f = simple direct stress at elastic failure

$$p_Y \leqslant f$$

or

$$\frac{(3m+1)\rho\omega^2}{8mg}R^2 \leqslant f$$

From which the maximum speed is given by

$$\omega=\frac{1}{R}\sqrt{\frac{8mgf}{(3m+1)\rho}} \quad \ldots(18·21)$$

(2) Maximum principal strain theory

$$p_Y-\frac{p_X}{m}\leqslant f$$

But $p_Y=p_X$

$$\therefore \quad p_Y\left(1-\frac{1}{m}\right)\leqslant f$$

or

$$\frac{(3m+1)\rho\omega^2R^2}{8mg}\cdot\frac{m-1}{m}\leqslant f$$

From which $$\omega=\frac{m}{R}\sqrt{\frac{8gf}{(3m+1)(m-1)\rho}} \quad \ldots(18·22)$$

(3) Maximum shear stress theory

$$f \geqslant \frac{p_X+p_Y}{2}>p_X$$

$$\therefore \quad \frac{(3m+1)\rho\omega^2R^2}{8mg}\leqslant f$$

From which $$\omega=\frac{1}{R}\sqrt{\frac{8mgf}{(3m+1)\rho}} \quad \ldots(18·23)$$

(4) Maximum strain energy theory

$$f^2 \geqslant p_X^2+p_Y^2-\frac{2}{m}p_Xp_Y$$

But $p_X=p_Y$

$$\therefore \quad f^2 \geqslant 2p_X^2\left(1-\frac{1}{m}\right)=\frac{2}{m}p_X^2(m-1)$$

$$\therefore \quad \left[\frac{(3m+1)\rho\omega^2R^2}{8mg}\right]^2\frac{2(m-1)}{1}\leqslant f^2$$

From which $$\omega=\left[\frac{1}{R}\sqrt{\frac{8mgf}{(3m+1)\rho}}\right]\left[\frac{m}{2(m-1)}\right]^{1/4} \quad \ldots(18·24)$$

(5) Maximum shear strain energy theory

$$p_X^2+p_Y^2-p_Xp_Y\leqslant f^2$$

But $p_X=p_Y$

$$\therefore \quad p_X\leqslant f$$

$$\therefore \quad \frac{(3m+1)\rho\omega^2R^2}{8mg}\leqslant f$$

From which $$\omega=\frac{1}{R}\sqrt{\frac{mgf}{(3m+1)\rho}} \quad \ldots(18·25)$$

It is to be noted that Eqs. 18·21, 18·23 and 18·25 give the same value of ω.

Example 18·2. *A flat steel disc of uniform thickness and of 1 m diameter rotates at 2400 r.p.m. Determine the intensities of principal stresses.*

Take $\rho=7·85\times10^{-5}$ N/mm³ and $m=3$.

Solution.

The intensities of radial and hoop stresses are given by Eqs. 18·4 and 18·5.

$$p_X=A+\frac{B}{x^2}-\frac{3m+1}{8m}\frac{\rho\omega^2}{g}x^2 \quad \ldots(1)$$

$$p_Y=A-\frac{B}{x^2}-\frac{m+3}{8}\frac{\rho\omega^2}{g}x^2 \quad \ldots(2)$$

Since p_X cannot have infinite value at $x=0$, we get $B=0$.

Also, at $x=R$, $p_X=0$

$$\therefore \quad 0=A-\frac{3m+1}{8m}\frac{\rho\omega^2}{g}R^2$$

$$\therefore \quad A=\frac{3m+1}{8m}\frac{\rho w^2}{g}R^2$$

Here $m=3$; $g=9810$ mm/sec² ; $\rho=7·85\times10^{-5}$ N/mm³

$$\omega=\frac{2\pi N}{60}=\frac{2\pi\ 2400}{60}=80\pi=251 \text{ radians/sec}$$

$$\therefore \quad \frac{3m+1}{8m}\cdot\frac{\rho\omega^2}{g}=\frac{3\times3+1}{8\times3}\cdot\frac{7{\cdot}85\times10^{-5}}{9810}(251)^2=0{\cdot}21\times10^{-3} \quad \ldots(3)$$

$$A=0{\cdot}21\times10^{-3}(500)^2=52{\cdot}5 \quad \ldots(4)$$

Also,
$$\frac{m+3}{8m}\cdot\frac{\rho\omega^2}{g}=\frac{3+3}{8\times3}\cdot\frac{7{\cdot}85\times10^{-5}}{9810}(251)^2=0{\cdot}126\times10^{-3} \quad \ldots(5)$$

Substituting the values in (1) and (2), we get

$$p_X=52{\cdot}5-0{\cdot}21\times10^{-3}\,x^2 \quad \ldots(6)$$

$$p_Y=52{\cdot}5-0{\cdot}126\times10^{-3}\,x^2 \quad \ldots(7)$$

At $x=0$, $p_X=p_Y=$ **52·5 N/mm²**

At $x=R=500$,

$$p_X=52{\cdot}5-0{\cdot}21\times10^{-3}(500)^2=0$$

and
$$p_Y=52{\cdot}5-0{\cdot}126\times10^{-3}(500)^2$$
$$=\mathbf{21\ N/mm^2}$$

It should be noted that the above values of p_X and p_Y are also the principal stresses.

Example 18·3. *Solve Example 18·2 if the disc has a central hole 200 mm diameter.*

Solution.

$$R_1=100\text{ mm},\ R_2=500\text{ mm}$$

The stress intensities are given by

$$p_X=A+\frac{B}{x^2}-\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}\,x^2 \quad \ldots(1)$$

and
$$p_Y=A-\frac{B}{x^2}-\frac{m+3}{8m}\,\frac{\rho\omega^2}{g}\,x^2 \quad \ldots(2)$$

where $\frac{3m+1}{8m}\,\frac{\rho\omega^2}{g}=0{\cdot}21\times10^{-3}$ and $\frac{m+3}{8m}\,\frac{\rho\omega^2}{g}=0{\cdot}126\times10^{-3}$ from the previous example.

At $x=100$ mm, $p_X=0$

$$\therefore \quad 0=A+\frac{B}{10000}-0{\cdot}21\times10^{-3}\,(10000)$$

or
$$10000\,A+B=0{\cdot}21\times10^5 \quad \ldots(3)$$

Also, at $x=500$, $p_X=0$

$$\therefore \quad 0=A+\frac{B}{250000}-0{\cdot}21\times10^{-3}\,(250000) \quad \ldots(4)$$

$$250000+B=131{\cdot}25\times10^5$$

From (3) and (4), we get

$$A=54{\cdot}6 \text{ and } B=52{\cdot}5\times10^4$$

Hence the stresses are given by

$$p_X=54{\cdot}6-\frac{52{\cdot}5\times10^4}{x^2}-0{\cdot}21\times10^{-3}\,x^2 \quad \ldots(5)$$

and
$$p_Y=54{\cdot}6+\frac{52{\cdot}5\times10^4}{x^2}-0{\cdot}21\times10^{-3}\,x^2 \quad \ldots(6)$$

At $x=100$, $p_Y=\left(p_Y\right)_{max}=54{\cdot}6+52{\cdot}5-2{\cdot}1$
$=\mathbf{105\ N/mm^2}$

At $x=500$, $p_Y=54{\cdot}6+\dfrac{52{\cdot}5\times10^4}{250000}-0{\cdot}21\times10^{-3}\,(250000)$
$=54{\cdot}6+2{\cdot}1-52{\cdot}5=\mathbf{4{\cdot}2\ N/mm^2}$

p_X is maximum at $x=\sqrt{R_1R_2}=\sqrt{100\times500}=223{\cdot}6$ mm.

$$(p_X)_{max}=54{\cdot}6-\frac{52{\cdot}5\times10^4}{(223{\cdot}6)^2}-0{\cdot}21\times10^{-3}\,(223{\cdot}6)^2$$
$$=5{\cdot}46-10{\cdot}5-10{\cdot}5=33{\cdot}6\ \text{N/mm}^2.$$

18·6. DISC OF UNIFORM STRENGTH

A disc of uniform strength is the one in which the values of radial and circumferential stresses are equal in magnitude for all values x. This suggests that the disc of uniform strength must have a varying thickness, such as shown in Fig. 18·5.

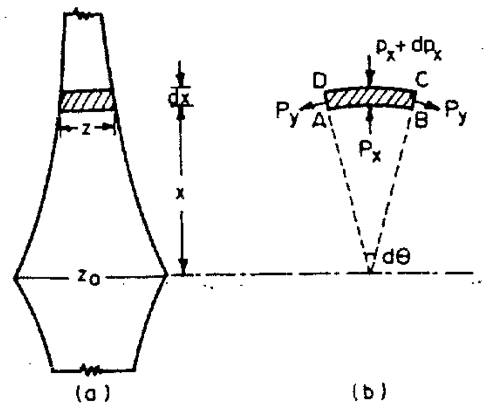


Fig. 18·5. Disc of uniform strength.

Let $\quad z=$ thickness at a radial distance x.

Consider an element $ABCD$ as before.

Centrifugal force $=P=\left(\frac{\rho}{g}\ x\ dx\ d\theta\ z\right)\omega^2 x$

$$=\frac{\rho}{g}\ x^2\omega^2 z.dx.d\theta \quad ...(1)$$

Resolving the forces radially, we get

$$p_X\ x\ d\theta.z-(p_X+dp_X)(x+dx)\ d\theta\ (z+dz)+2p_Y\ dx.z.\ \sin\frac{d\theta}{2}-P=0$$

Taking $\sin\frac{d\theta}{2}\simeq\frac{d\theta}{2}$, substituting the value of P from (1) and neglecting infinitesimal quantities of higher order, we get

$$z\ p_Y=\frac{\rho}{g}\ x^2\omega^2 z+\frac{d}{dx}\ (z.x.p_X) \quad ...(18{\cdot}26)$$

Let $p_X=p_Y=p$ (constant)

$$\therefore \quad zp=\frac{\rho}{g}\ x^2\omega^2\ z+p\ \frac{d}{dx}\ (zx)$$

$$\therefore \quad zp=\frac{\rho}{g}\ x^2\omega^2\ z+pz+px\ \frac{dz}{dx}$$

or
$$\frac{dz}{dx}+\frac{\rho}{g}\ \frac{\omega^2}{p}\ z\ x=0$$

or
$$\frac{dz}{z}+\frac{\rho}{g}\ \frac{\omega^2}{p}\ x\ dx=0$$

Integrating, $\log_e z=-\frac{\rho\omega^2}{gp}\ \frac{x^2}{2}+\log_e A$

where $\log_e A$ is a constant of integration.

At $x=0$, $z=z_0$

$$\therefore \quad \log_e A=0+\log_e z_0$$

Substituting in (2), we get

$$\log_e z=-\frac{\rho\omega^2}{gp}\ \frac{x^2}{2}+\log_e z_0$$

or
$$\log_e \frac{z}{z_0}=-\frac{\rho\omega^2}{gp}\ \frac{x^2}{2}$$

$$\therefore \quad \frac{z}{z_0}=e^{-\frac{\rho\omega^2}{2gp}x^2}$$

$$\therefore \quad z=z_0\ e^{-\frac{\rho\omega^2}{2gp}x^2} \quad ...(18{\cdot}27)$$

Eq. 18·27 gives the variation of thickness z.

18·7. ROTATING CYLINDER

A cylinder may be defined as a disc of large thickness. Thus the thickness t corresponds to the length of the cylinder. Let us assume that the length of the axis is great compared to the radius. We shall confine ourselves to the stresses about the region of the central circular section perpendicular to the axis of the cylinder. It is further assumed that the plane sections of the cylinder, when stationary, remain plane even during rotation, *i.e.* the axial strain e_X is constant and is independent of x. Let the direction of x be along a radius, direction of z be along the axis and the direction y be perpendicular to the two. Let p_X, p_Y and p_Z be the normal stresses in x, y and z directions respectively. If we take an element $ABCD$ (Fig. 18·2) of thickness dz, at the centre of the cylinder axis, there will be no shear stress due to symmetry. Hence the stresses p_X, p_Y and p_Z are the principle stresses.

As in Eq. 2 of § 18·2 the equation for the forces acting radially will be

$$p_Y=\frac{\rho}{g}\ \omega^2\ x^2=p_X+x\ \frac{dp_X}{dx} \quad ...(1)$$

Also, as before, $e_X=\frac{du}{dx}$ and $e_Y=\frac{u}{x}$

Hence we have

Radial strain, $$e_X=\frac{du}{dx}=\frac{1}{E}\left(p_X-\frac{p_Y+p_Z}{m}\right) \quad ...(2)$$

Circumferential strain,

$$e_Y=\frac{u}{x}=\frac{1}{E}\left(p_Y-\frac{p_X+p_Z}{m}\right) \quad ...(3)$$

Axial strain, $$e_Z=\frac{1}{E}\left(p_Z-\frac{p_X+p_Y}{m}\right) \quad ...(4)$$

If it is assumed that plane sections remain plane, e_Z is constant with respect to x. From (4),

$$p_Z=\frac{p_X+p_Y}{m}+E\ e_Z \quad ...(5)$$

$$\therefore \quad \frac{dp_Z}{dx}=\frac{1}{m}\left[\frac{pdx}{dx}+\frac{dp_Y}{dx}\right] \quad ...(6)$$

Now, from (3),

$$uE = x\left(p_y - \frac{p_x+p_z}{m}\right) = xp_y - \frac{xp_x}{m} - \frac{xp_z}{m}$$

$$\therefore\ E\frac{du}{dx} = p_y + x\frac{dp_y}{dx} - \frac{1}{m}\left[p_x+p_z\right] - \frac{x}{m}\left[\frac{dp_x}{dx}+\frac{dp_z}{dx}\right]$$

Substituting the value of $E\frac{du}{dx}$ from (2), we get

$$p_x - \frac{p_y}{m} - \frac{p_z}{m} = p_y + x\frac{dp_y}{dx} - \frac{p_x}{m} - \frac{p_z}{m} - \frac{x}{m}\left[\frac{dp_x}{dx}+\frac{dp_z}{dx}\right]$$

$$\therefore\ p_x + \frac{p_x}{m} - p_y - \frac{p_y}{m} = x\left[\frac{dp_y}{dx} - \frac{1}{m}\left(\frac{dp_x}{dx} - \frac{dp_z}{dx}\right)\right]$$

or $$(p_x - p_y)\left(1+\frac{1}{m}\right) = x\left[\frac{dp_y}{dx} - \frac{1}{m}\left(\frac{dp_x}{dx}+\frac{dp_z}{dx}\right)\right]$$

Substituting the value of $\frac{dp_z}{dx}$ from (6), we get

$$(p_x - p_y)\left(1+\frac{1}{m}\right) = x\left[\frac{dp_y}{dx} - \frac{1}{m}\left\{\frac{dp_x}{dx} + \frac{1}{m}\left(\frac{dp_x}{dx}+\frac{dp_y}{dx}\right)\right\}\right]$$

or $$(p_x - p_y)\left(1+\frac{1}{m}\right) = x\left[\frac{dp_y}{dx}\left(1-\frac{1}{m^2}\right) - \frac{1}{m}\frac{dp_x}{dx}\left(1+\frac{1}{m}\right)\right]$$

$$\therefore\ p_x - p_y = x\left[\frac{dp_y}{dx}\left(1-\frac{1}{m}\right) - \frac{1}{m}\frac{dp_x}{dx}\right] \quad \ldots(7)$$

Substituting the value of $p_x - p_y$ from (1), we get

$$-x\frac{dp_x}{dx} - \frac{\rho}{g}\omega^2 x^2 = x\left[\frac{dp_y}{dx}\left(1-\frac{1}{m}\right) - \frac{1}{m}\frac{dp_x}{dx}\right]$$

$$\therefore\ \frac{dp_x}{dx}\left(1-\frac{1}{m}\right) + \frac{dp_y}{dx}\left(1-\frac{1}{m}\right) = -\frac{\rho}{g}\omega^2 x$$

Hence $$\frac{dp_x}{dx} + \frac{dp_y}{dx} = -\frac{\rho\omega^2 x}{g}\cdot\frac{m}{m-1}.$$

Integrating,

$$p_x + p_y = -\frac{\rho\omega^2}{g}\cdot\frac{m}{m-1}\cdot\frac{x^2}{2} + 2A$$

where $2A$ = constant of integration.

Also, from (1), $p_x - p_y = -\frac{\rho}{g}\omega^2 x^2 - x\frac{dp_x}{dx}$

Adding this to (8), we get

$$2p_x = -\frac{\rho\omega^2 x^2}{g}\left[\frac{m}{2(m-1)}+1\right] - x\frac{dp_x}{dx} + 2A$$

$$\therefore\ 2xp_x + x^2\frac{dp_x}{dx} = 2Ax - \frac{\rho\omega^2 x^3}{g}\left[\frac{3m-2}{(m-1)}\right]$$

Integrating,

$$x^2 p_x = Ax^2 - \frac{\rho\omega^2 x^4}{8g}\left(\frac{3m-2}{m-1}\right) + B$$

$$p_x = A + \frac{B}{x^2} - \frac{\rho\omega^2 x^2}{8g}\left(\frac{3m-2}{m-1}\right) \quad \ldots(9)$$

Substituting this in (8), we get

$$p_y = \left[-\frac{\rho\omega^2}{g}\cdot\frac{m}{m-1}\cdot\frac{x^2}{2} + 2A\right] - \left[A + \frac{B}{x^2} - \frac{\rho\omega^2 x^2}{8g}\left(\frac{3m-2}{m-1}\right)\right]$$

$$= A - \frac{B}{x^2} - \frac{\rho\omega^2 x^2}{8g}\cdot\frac{m+2}{m-1} \quad \ldots(10)$$

To summarize, the values of p_x, p_y and p_z are given by the following equations :

$$p_x = A + \frac{B}{x^2} - \frac{\rho\omega^2 x^2}{8g}\left(\frac{3m-2}{m-1}\right) \quad \ldots(18{\cdot}28)$$

$$p_y = A - \frac{B}{x^2} - \frac{\rho\omega^2 x^2}{8g}\left(\frac{m+2}{m-1}\right) \quad \ldots(18{\cdot}29)$$

and $$p_z = \frac{p_x+p_y}{m} + Ee_z \quad \ldots(18{\cdot}30)$$

Thus, to find the values of p_x, p_y and p_z we have to first determine the constants A, B and $E\,e_z$, depending upon the boundary conditions. Two cases will be considered : (1) hollow cylinder, (2) solid cylinder.

18·8. HOLLOW CYLINDER

Let R_1 = internal radius

R_2 = external radius of the hollow cylinder.

Boundary conditions are :

$p_x = 0$ at $x = R_1$ and also at $x = R_2$

Substituting these in Eq. 18·28, we get

$$A + \frac{B}{R_1^2} - \frac{\rho\omega R_1^2}{8g}\left(\frac{3m-2}{m-1}\right) = 0 \quad \ldots(1)$$

and $$A + \frac{B}{R_2^2} - \frac{\rho\omega R_2^2}{8g}\left(\frac{3m-2}{m-1}\right) = 0 \quad \ldots(2)$$

Substituting (2) from (1),

$$B\left(\frac{1}{R_1^2} - \frac{1}{R_2^2}\right) = \frac{\rho\omega^2}{8g}\cdot\frac{3m-2}{m-1}\left(R_1^2 - R_2^2\right)$$

$$\therefore \quad B=-\frac{\rho\omega^2}{8g}\cdot\frac{3m-2}{m-1}R_1^2R_2^2 \quad ...[18\cdot31\ (a)]$$

Substituting this value of B in (1), we get

$$A=\frac{\rho\omega^2}{8g}\cdot\frac{3m-2}{m-1}R_2^2+\frac{\rho\omega^2R_1^2}{8g}\cdot\frac{3m-2}{m-1}$$

or
$$A=\frac{\rho\omega^2}{8g}\cdot\frac{3m-2}{m-1}\left(R_2^2+R_1^2\right) \quad ...[18\cdot31\ (b)]$$

Substituting the values of A and B in Eq. 18·28, we get

$$p_X=\frac{\rho\omega^2}{8g}\left(\frac{3m-2}{m-1}\right)\left(R_2^2+R_1^2\right)-\frac{\rho\omega^2}{8g}\left(\frac{3m-2}{m-1}\right)\frac{R_1^2R_2^2}{x^2}-\frac{\rho\omega^2x^2}{8g}\left(\frac{3m-2}{m-1}\right)$$

or
$$p_X=\frac{\rho\omega^2}{8g}\left[\frac{3m-2}{m-1}\right]\left[R_1^2+R_2^2-\frac{R_1^2R_2^2}{x^2}-x^2\right] \quad ...(18\cdot32)$$

Similarly, substituting the values of A and B in Eq. 18·32, we get

$$p_Y=\frac{\rho\omega^2}{8g}\cdot\frac{3m-2}{m-1}(R_2^2+R_1^2)+\frac{\rho\omega^2}{8g}\cdot\frac{3m-2}{m-1}\cdot\frac{R_1^2R_2^2}{x^2}-\frac{\rho\omega^2x^2}{8g}\left(\frac{m+2}{m-1}\right)$$

$$=\frac{\rho\omega^2}{8g}\left[\frac{3m-2}{m-1}\left\{R_1^2+R_2^2+\frac{R_1^2R_2^2}{x^2}\right\}-x^2\left(\frac{m+2}{m+1}\right)\right] \quad ...(18\cdot33)$$

For maximum value of p_X, $\frac{dp_X}{dx}=0$. This gives, from Eq. 18·32,

$$\frac{2R_1^2R_2^2}{x^2}-2x=0$$

or
$$x=\sqrt{R_1R_2} \quad ...(18\cdot34)$$

$$\therefore \quad \left(p_X\right)_{max}=\frac{\rho\omega^2}{g}\left(\frac{3m-2}{m-1}\right)\left[R_1^2+R_2^2-R_1R_2-R_1R_2\right]$$

$$=\frac{\rho\omega^2}{8g}\left(\frac{3m-2}{m-1}\right)\left(R_2-R_1\right)^2 \quad ...(18\cdot35)$$

By inspection of Eq. 18·23, p_Y will be maximum at minimum value of x, *i.e.* at $x=R_1$.

$$\therefore \quad \left(p_Z\right)_{max}=\frac{\rho\omega^2}{8g}\left[\frac{3m-2}{m-1}\left\{R_1^2+R_2^2+R_2^2\right\}-R_1^2\left(\frac{m+2}{m-1}\right)\right]$$

$$=\frac{\rho\omega^2}{4g(m-1)}\left[(3m-2)R_2^2+(m-2)R_1^2\right] \quad ...(18\cdot36)$$

After having known the values of p_Y and p_X, let us now compute the axial stress p_Z from Eq. 18·30 in which the constant Ee_Z is still to be computed.

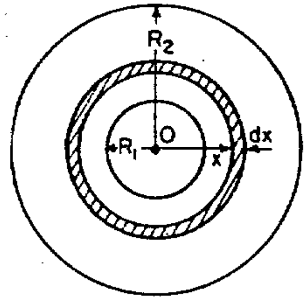


Fig. 18·6

Consider a hollow cylinder of internal radius R_1 and external radius R_2. Take an elementary ring of radius x and thickness dx as shown in Fig. 18·6.

Axial force on elementary ring $=2\pi x\,dx.p_Z$

$\therefore$ Total axial force $=F_Z=\int_{R_1}^{R_2}2\pi x dx.p_Z$

Substituting the value of p_Z from Eq. 18·30,

$$F_Z=\int_{R_1}^{R_2}2\pi x dx\left\{\frac{p_X+p_Y}{m}+E\,e_Z\right\}$$

Substituting the value of p_X and p_Z from Eqs. 18·28 and 18·29, we get

$$F_Z=2\pi\int_{R_1}^{R_2}x\,dx\left[Ee_Z+\frac{1}{m}\left\{A+\frac{B}{x^2}-\frac{\rho\omega^2x^2}{8g}\left(\frac{3m-2}{m-1}\right)+A-\frac{B}{x^2}-\frac{\rho\omega^2x^2}{8g}\left(\frac{m+2}{m-1}\right)\right\}\right]$$

$$=2\pi\int_{R_1}^{R_2}x\,dx\left[Ee_Z+\frac{1}{m}\left\{2A-\frac{\rho\omega^2x^2}{8g\,(m-1)}\times(3m-2+m+2)\right\}\right]$$

or
$$F_Z=2\pi\int_{R_1}^{R_2}x\,dx\left[Ee_Z+\frac{1}{m}\left\{2A-\frac{\rho\omega^2\,x^2\,.\,m}{2g(m-1)}\right\}\right] \quad ...(18\cdot37)$$

In the case of free ends, total axial force is equal to zero.

$$\therefore \quad \int_{R_1}^{R_2}x\,dx\,E.e_Z+\int_{R_1}^{R_2}\frac{2Ax\,dx}{m}-\int_{R_1}^{R_2}\frac{\rho\omega^2\,x^3}{2g(m-1)}dx=0$$

$$\therefore \quad \left[Ee_Z\frac{x^2}{2}+\frac{Ax^2}{m}-\frac{\rho\omega^2\,x^4}{8g\,(m-1)}\right]_{R_1}^{R_2}=0$$

$$\therefore \frac{Eez}{2}\left(R_2^2-R_1^2\right)+\frac{A}{m}\left(R_2^2-R_1^2\right)-\frac{\rho\omega^2}{8g(m-1)}\left(R_2^4-R_1^4\right)=0$$

$$\therefore \quad Eez=\frac{\rho\omega^2}{4g(m-1)}\left(R_1^2+R_2^2\right)-\frac{2A}{m}$$

Substituting the value of A from Eq. 26·31 (*b*),

$$Eez=\frac{\rho\omega^2}{4g(m-1)}\left(R_1^2+R_2^2\right)-\frac{\rho\omega^2}{4gm}\cdot\frac{3m-2}{m-1}\left(R_1^2+R_2^2\right)$$

$$=\frac{\rho\omega^2}{4g(m-1)}\left(R_1^2+R_2^2\right)\left(1-\frac{3m-2}{m}\right)$$

$$=-\frac{\rho\omega^2}{2gm}\left(R_1^2+R_2^2\right) \qquad \ldots(18·38)$$

Substituting the values of Eez (Eq. 18·38), px (Eq. 18·32) and pY (Eq. 18·33) in Eq. 18·30, we get

$$pz=\frac{px+pY}{m}+Eez$$

$$\text{or } pz=\frac{1}{m}\left[\frac{\rho\omega^2}{4g}\left(\frac{3m-2}{m-1}\right)\left(R_1^2+R_2^2\right)-\frac{\rho\omega^2x^2}{8g(m-1)}\left(3m-2+m+2\right)\right]-\frac{\rho\omega^2}{2gm}\left(R_1^2+R_2^2\right)$$

$$=\frac{\rho\omega^2}{2gm}\left[\frac{3m-2}{2m-2}-1\right]\left[R_1^2+R_2^2\right]-\frac{\rho\omega^2x^2}{2g(m-1)}$$

$$=\frac{\rho\omega^2}{2g(m-1)}\left[\frac{R_1^2+R_2^2}{2}-x^2\right] \qquad \ldots(18·39)$$

By inspection, pz is maximum at $x=R_1$

$$\therefore \quad (pz)_{max}=\frac{\rho\omega^2}{2g(m-1)}\left[\frac{R_2^2-R_1^2}{2}\right] \qquad \ldots(18·40)$$

18·9. SOLID CYLINDER ($R_1=0$)

The values of px, pY and pz are given by Eqs. 18·28, 18·29 and 18·30 respectively. At $x=0$, these give infinite values of px, pY and pz. Since the stresses are finite, we get $B=0$.

To get the value of A, we apply the condition that $px=0$ at $x=R_2=R$ (say).

Hence from Eq. 18·28,

$$px=0=A-\frac{\rho\omega^2R^2}{8g}\left(\frac{3m-2}{m-1}\right)$$

$$\therefore \quad A=\frac{\rho\omega^2R^2}{8g}\left(\frac{3m-2}{m-1}\right) \qquad \ldots(18·41)$$

Substituting this in Eq. 18·28

$$px=\frac{\rho\omega^2R^2}{8g}\left(\frac{3m-2}{m-1}\right)-\frac{\rho\omega^2x^2}{8g}\left(\frac{3m-2}{m-1}\right)$$

$$=\frac{\rho\omega^2}{8g}\cdot\frac{3m-2}{m-1}\left(R^2-x^2\right) \qquad \ldots(18·42)$$

This will be evidently maximum at the centre of the cylinder.

$$\therefore \quad (px)_{max}=\frac{\rho\omega^2R^2}{8g}\cdot\frac{3m-2}{m-1} \qquad \ldots(18·43)$$

Also, substituting the value of A in Eq. 18·29, we get

$$pY=\frac{\rho\omega^2R^2}{8g}\left(\frac{3m-2}{m-1}\right)-\frac{\rho\omega^2x^2}{8g}\left(\frac{m+2}{m-1}\right)$$

$$=\frac{\rho\omega^2}{8g(m-1)}\left[(3m-2)R^2-(m+2)x^2\right] \qquad \ldots(18·44)$$

This is maximum at $x=0$

$$\therefore \quad (pY)_{max}=\frac{\rho\omega^2R^2}{8g}\cdot\frac{3m-2}{m-1} \qquad \ldots(18·45)$$

Thus at the centre of the cylinder, $pY=px$.

To get the value of pz, consider the axial force on an elementary ring of radius x and thickness dx (Fig. 18·6). We have, similar to Eq. 18·37.

$$2\pi\int_0^R x.dx\left[Eez+\frac{1}{m}\left(2A-\frac{\rho\omega^2x^2m}{2g(m-1)}\right)\right]=0$$

$$\therefore \int_0^R x\,dx\,Eez+\int_0^R \frac{2Axdx}{m}-\int_0^R \frac{\rho\omega^2x^3}{2g(m-1)}\,dx=0$$

$$\therefore \quad Eez\,\frac{R^2}{2}+\frac{A}{m}R^2-\frac{\rho\omega^2R^4}{8g(m-1)}=0$$

$$\text{or} \qquad Eez=\frac{\rho\omega^2R^2}{4g(m-1)}-\frac{2A}{m}$$

Substituting the value of A from Eq. 18·41,

$$ez=\frac{\rho\omega^2R^2}{4g(m-1)}-\frac{\rho\omega^2R^2}{4gm}\left(\frac{3m-2}{m-1}\right)$$

$$=\frac{\rho\omega^2R^2}{4g(m-1)}\left[1-\frac{3m-2}{m}\right]$$

$$=-\frac{\rho\omega^2R^2}{2gm} \qquad \ldots(18·46)$$

Substituting the value of Eez (Eq. 18·46), px (Eq. 18·42) and pY (Eq. 18·43) in Eq. 18·30, we get

$$\therefore \quad pz=\frac{1}{m}\left[\frac{\rho\omega^2}{8g}\cdot\frac{3m-2}{m-1}\left(R^2-x^2\right)+\frac{\rho\omega^2}{8g(m-1)}\left\{(3m-2)R^2-(m+2)x^2\right\}\right]-\frac{\rho\omega^2R^2}{2gm}$$

$$\therefore \quad p_Z=\frac{\rho\omega^2}{2mg}\left[\frac{1}{4(m-1)}\left\{2(3m-2)\,R^2-4mx^2\right\}-R^2\right] \quad \ldots(18{\cdot}47)$$

This is maximum at $x=0$

$$\therefore \quad (p_Z)_{max}=\frac{\rho\omega^2}{2gm}\left[\frac{2(3m-2)R^2}{4(m-1)}-R^2\right]$$

$$=\frac{\rho\omega^2R^2}{2gm}\cdot\frac{6m-4-4m+4}{4(m-1)}$$

$$=\frac{\rho\omega^2R^2}{4g(m-1)}. \quad \ldots(18{\cdot}48)$$

PROBLEMS

1. Determine the hoop stress developed in the rim of a fly-wheel, 600 mm diameter, rotating at 3000 revolutions per minute. Take the unit weight of the material as $7{\cdot}6\times10^{-5}$ N/mm³.

2. Determine the maximum radial and circumferential stresses in a flat steel disc, 600 mm. diameter, and uniform thickness, rotating at 3000 revolutions per minute. Take $\rho=7{\cdot}85\times10^{-5}$ N/mm³ and $m=3$.

3. Solve problem 2, if there is a pin hole at the centre of the disc.

4. Solve problem 2 if the disc is hollow having internal radius equal to 150 mm.

5. A circular hollow disc of external radius R and internal radius of half the external revolves at a constant speed ω radians/sec. Find the expressions for maximum values of radial and circumferential stresses.

ANSWERS

1. $68{\cdot}7$ N/mm²
2. $(p_X)_{max}=(p_Y)_{max}=29{\cdot}5$ N/mm²
3. $(p_X)_{max}=29{\cdot}5$ N/mm² ; $(p_Y)_{max}=59$ N/mm²
4. $(p_X)_{max}=7{\cdot}4$ N/mm² ; $(p_Y)_{max}=62{\cdot}2$ N/mm²
5. $(p_X)_{max}=\dfrac{(3m+1)\rho\omega^2R^2}{32mg}$;

   $(p_Y)_{max}=\dfrac{\rho\omega^2R^2}{16mg}\,(3+13m)$.

19

Vibrations and Critical Speeds

19·1. INTRODUCTION

If a body, held in equilibrium by elastic constraints, is momentarily disturbed from its equilibrium position, it begins to vibrate. The nature of the vibration depends upon several factors such as inertia of the system, stiffness, elastic forces and the amount of disturbance. When the external forces displaces the body from its equilibrium position, work is done against the external elastic forces resisting deformation. This work is stored momentarily as the strain energy in constraints. When the external force is removed, the body tries to regain its original equilibrium position and thus changes the strain energy into kinetic energy. The body thus vibrates. If the external force does not take part in vibrations except for the initial momentary displacement, the vibrations are called *free* or *natural vibrations*. The vibrations continue indefinitely. However, some frictional forces are always present which gradually *damp* the vibrations. If, however, the external disturbing force is *periodically* acting on the body, vibrations having the same frequency as that of the disturbing force will be set up. Such vibrations are called *forced vibrations*.

The vibrations may further be classified as (*i*) linear and (*ii*) angular or torsional. Linear vibrations are either longitudinal or transverse.

19·2. LINEAR VIBRATIONS : SIMPLE HARMONIC MOTION

Consider a point P moving along the circumference of a circle, with a constant angular velocity ω (Fig. 19·1). Consider a diameter AB of the circle. Let C be the projection of point P on the line AB. As the point P rotates, point C will oscillate or vibrate between A and B. The motion of C along AB is called *simple harmonic motion*.

Let r=radius of the circle

$x=OC$=distance of the point C from O=Displacement of C.

$\therefore \quad x=r\cos\phi$

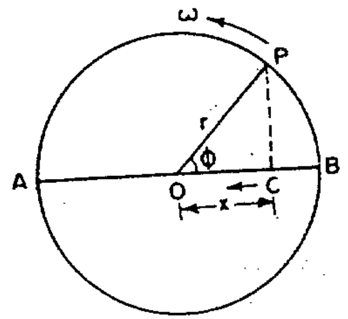


Fig. 19·1. Simple Harmonic Motion.

If the time is measured from the instant when P is at B, $\phi=\omega t$

$\therefore \quad x=r\cos\omega t$

$\therefore \quad v=-\dfrac{dx}{dt}=r\omega\sin\omega t$

(The minus sign indicates that v increases as x decreases)

$\therefore$ Acceleration $f=\dfrac{dv}{dt}=r\omega^2\cos\omega t$ ...(19·1)

or $f=\omega^2.x$

Thus acceleration of point C is proportional to its distance or displacement from the mean position (fixed point) O.

From Eq. 19·1,

$$\omega^2=\frac{\text{acceleration of } C}{\text{displacement of } C \text{ from the centre}}=\frac{f}{x}$$

or $$\omega=\sqrt{\frac{\text{acceleration}}{\text{displacement}}}=\sqrt{\frac{f}{x}} \quad \text{...[19·2 (a)]}$$

Let T=time of one complete 'to and fro' vibration in seconds

Then $$\omega=\frac{2\pi}{T}$$

or $$T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}=2\pi\sqrt{\frac{x}{f}} \quad \text{...[19·3 (a)]}$$

Number of complete vibrations, n, is given by

$$n=\frac{1}{T} \text{ per second} \quad \text{...[19·4 (a)]}$$

If W=weight of the vibrating body

Then force=mass × acceleration

$$=\frac{W}{g}\times\text{acceleration}$$

Let k=*stiffness* of the supports$=\dfrac{\text{force}}{\text{displacement}}$

$$\therefore \quad k=\frac{\text{force}}{\text{displacement}}=\frac{W}{g}\times\frac{\text{acceleration}}{\text{displacement}}=\frac{W}{g}\times\omega^2$$

or $$\omega=\sqrt{\frac{k.g}{W}} \quad \text{...(19·2)}$$

Hence $$T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{W}{k.g}} \quad \text{...(19·3)}$$

$$\therefore \quad n=\frac{1}{T}=\frac{1}{2\pi}\sqrt{\frac{k.g}{W}} \quad \text{...(19·4)}$$

If n=complete vibrations (or revolutions) per minute

$$n=60\,n=\frac{30}{\pi}\sqrt{\frac{k.g}{W}} \quad \text{...(19·5)}$$

19·3. LONGITUDINAL VIBRATIONS

(a) HELICAL SPRING [Fig. 19·2 (a)].

Consider a helical spring [Fig. 19·2 (a)] supporting a weight W. *If the weight W is pulled downwards momentarily by an amount x and then released, it will have simple harmonic motion.* Let δ be the extension of the spring due to the static of load W.

Then stiffness $k=\dfrac{W}{\delta}$ ...(i)

Substituting this in Eq. 19·4, we get

$$n=\frac{1}{2\pi}\sqrt{\frac{k.g}{W}}=\frac{1}{2\pi}\sqrt{\frac{g}{\delta}} \quad \text{...(19·6)}$$

Let k=stiffness in N/mm

δ=static extension in mm

g=9810 mm/sec/sec.

$$\therefore \quad n=\frac{1}{2\pi}\sqrt{\frac{981.0}{\delta}}=\frac{15.7636}{\sqrt{\delta}} \simeq \frac{15.8}{\sqrt{\delta}} \text{ (per second)} \quad \ldots[19.7\ (a)]$$

$$N=\frac{945.82}{\sqrt{\delta}} \simeq \frac{946}{\sqrt{\delta}} \text{ vibrations/minute} \quad \ldots[19.7\ (b)]$$

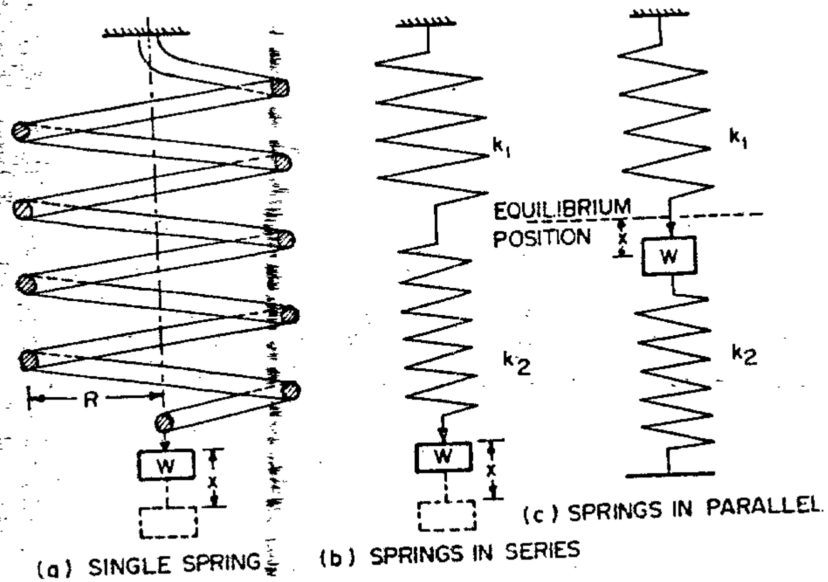


Fig. 19·2. **Longitudinal vibrations of a helical spring.**

If, however, the stiffness k is expressed in lb/inch, δ as the extension in inches and $g=32.2\times12$ inch/sec/sec, we get

$$n=\frac{1}{2\pi}\sqrt{\frac{32.2\times12}{\delta}}=\frac{3.13}{\sqrt{\delta}} \text{ vibrations per second} \quad \ldots[19.7\ (c)]$$

and
$$N=60n=\frac{187.8}{\sqrt{\delta}} \text{ vibrations/minute} \quad \ldots[19.7\ (d)]$$

(b) COMPOUND SPRINGS

(i) Springs in Series [Fig. 19·2 (b)]

Let two springs of stiffness k_1 and k_2 be arranged in series, and carry a weight W at its end, as shown in Fig. 19·2 (b). Let δ be the extension of the system, due to the static load W.

Then $\quad \delta=\delta_1+\delta_2=\dfrac{W}{k_1}+\dfrac{W}{k_2}$

$$\therefore \quad \delta=W\left(\frac{1}{k_1}+\frac{1}{k_2}\right) \quad \ldots(i)$$

If k=resultant stiffness of the whole system, we have

$$\delta=\frac{W}{k} \quad \ldots(ii)$$

Equating the two, we get

$$\frac{1}{k}=\frac{1}{k_1}+\frac{1}{k_2}=\frac{k_1+k_2}{k_1k_2}$$

Substituting this in Eq. 27·4, we get

$$n=\frac{1}{2\pi}\sqrt{\frac{kg}{W}}=\sqrt{\frac{g}{W}\left(\frac{k_1k_2}{k_1+k_2}\right)} \text{ (per second)}$$

(ii) Springs in Parallel [Fig. 19·2 (c)]

If the two springs are arranged in parallel, one end of each spring being fixed and the other end independently connected to a mass of weight W, each spring is deformed by an amount δ. Hence

$$W=k_1\delta+k_2\delta=\delta(k_1+k_2) \quad \ldots(i)$$

where $k_1\delta$ is the restoring force for the first spring and $k_2\delta$ is the restoring force for the other spring.

Also $\quad W=k\delta \quad \ldots(ii)$

where k is the stiffness of the composite system.

Equating the two, we get

$$k=k_1+k_2$$

Substituting this in Eq. 19·4, we get

$$n=\frac{1}{2\pi}\sqrt{\frac{kg}{W}}=\frac{1}{2\pi}\sqrt{\frac{g}{W}\cdot(k_1+k_2)}$$

Note. δ=static deflection of the system under load W.

x=initial displacement of the system to set it in simple harmonic vibrations. The value of n (or N) will be independent of the magnitude x. The value δ (or δ_1, δ_2) directly depends upon the stiffness k (or k_1, k_2).

(c) ELASTIC ROD OF NEGLIGIBLE WEIGHT, LOADED AT THE ENDS

Consider an elastic rod of length L and uniform cross-sectional area A subjected to the downward load W. Let the weight of the rod be negligible in comparison to the weight W. If the weight W is given a displacement of x from the equilibrium position and then released, vibrations will be set up in the rod.

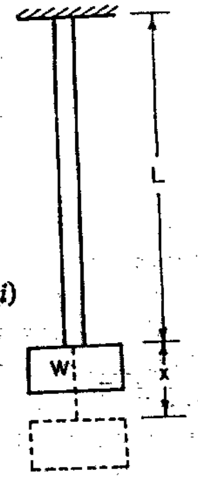


Fig. 19·3

Let P=restoring force

From Hooke's law,

$$x=\frac{PL}{AE}$$

or $$P=\frac{AEx}{L} \qquad ...(i)$$

But P=mass × acceleration

Also, from Eq. 19·3 (a),

$$\text{Acceleration}=\frac{4\pi^2 x}{T}$$

$$P=\frac{W}{g}\times\frac{4\pi^2 x}{T^2} \qquad ...(ii)$$

Equating (i) and (ii), we get

$$\frac{AEx}{L}=\frac{W}{g}\cdot\frac{4\pi^2 x}{T^2}$$

or $$\frac{1}{T^2}=\frac{EAg}{4LW\pi^2}$$

or $$n=\frac{1}{T}=\frac{1}{2\pi}\sqrt{\frac{EAg}{WL}} \qquad ...(19·8)$$

If E is in N/mm^2 ; W is in N ; A is in mm^2 ; g=9810 mm/sec^2 and L is in mm, we get

$$n=\frac{1}{2\pi}\sqrt{\frac{9810\,EA}{WL}}\simeq 15·8\sqrt{\frac{EA}{WL}} \qquad ...(19·9)$$

$$\therefore \quad N\simeq 946\sqrt{\frac{EA}{WL}} \text{ vibrations/min.} \qquad ...(19·10)$$

If, however, E is in lb/in^2 ; W is in lbs ; A is in^2 ; L is in inches and g=32·2×12 in inch/sec^2,

$$n=\frac{1}{2\pi}\sqrt{\frac{32·2\times12\,EA}{WL}}=3·13\sqrt{\frac{EA}{WL}} \qquad ...[19·9\ (a)]$$

and $$N=187·8\sqrt{\frac{EA}{WL}} \qquad ...[19·10\ (a)]$$

(b) EFFECT OF THE WEIGHT OF THE ROD

Consider a rod of length L, fixed at one end. The fixed end forms a node or stationary point. The remainder of the rod has a longitudinal vibratory movement.

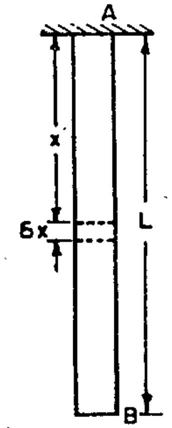


Fig. 19·4

Let v=velocity of the free end B.

∴ Velocity at a distance x from the fixed end

$$=\frac{v}{L}x$$

Consider a small length δx of the rod. The kinetic energy of the length δx will be

$$=\frac{1}{2}\frac{\rho A\delta x}{g}\left(\frac{vx}{L}\right)^2$$

where ρ=unit weight of the rod.

If W_1=total weight of the rod,

$$\rho=\frac{W_1}{AL}$$

∴ K.E. of δx length

$$=\frac{1}{2}\frac{W_1}{AL}\cdot\frac{A\delta x}{g}\left(\frac{vx}{L}\right)^2=\frac{W_1\delta x}{2gL^3}v^2x^2$$

∴ K.E. of the whole rod

$$=\int_0^L\frac{W_1v^2}{2gL^3}x^2\,dx=\frac{1}{3}\left(\frac{W_1v^2}{2g}\right) \qquad ...(19·11)$$

Thus, the K.E. due to the weight of the rod is equal to $\frac{1}{3}$ of the K.E. due to its weight considered to be concentrated at its end. Hence from Eq. 19·8,

$$n=\frac{1}{2\pi}\sqrt{\frac{EAG}{\frac{1}{3}W_1L}} \qquad ...(19·12)$$

If a weight W is also acting at the rod, we get

$$n=\frac{1}{2\pi}\sqrt{\frac{EAG}{(W+\frac{1}{3}W_1)L}} \qquad ...(19·13)$$

(c) NON-UNIFORM ROD

Let the rod consist of several parts of lengths L_1, L_2, L_3......and areas of cross-sections A_1, A_2, A_3... Let a weight W. large enough in comparison to the weight of the rod, be attached to its place end.

Let δ_1, δ_2, δ_3...... be the static deflection on displacement of each length, such that

$$\delta=\delta_1+\delta_2+\delta_3......$$

$$=W\left[\frac{L_1}{A_1E}+\frac{L_2}{A_2E}+\frac{L_3}{A_3E}+......\right]$$

or

$$\delta=\frac{W}{E}\Sigma\frac{L}{A}$$

$$\therefore \quad k=\text{stiffness}=\frac{W}{\delta}=\frac{E}{\Sigma\dfrac{L}{A}} \qquad ...(19\cdot14)$$

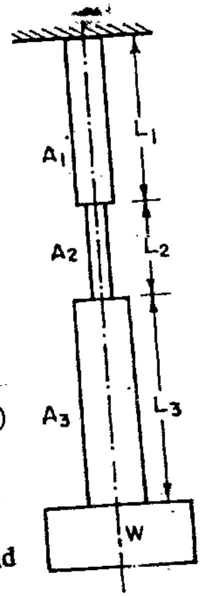


Fig. 19·5

Substituting this in Eq. 19·4, we get

$$n=\frac{1}{2\pi}\sqrt{\frac{kg}{W}}=\frac{1}{2\pi}\sqrt{\frac{Eg}{W\Sigma\dfrac{L}{A}}} \text{ per second} \qquad ...(19\cdot15)$$

19·4. TRANSVERSE VIBRATIONS

A bar or shaft is said to have transverse vibrations when all the particles of the bar move along a straight path perpendicular to the axis of the bar. We shall consider different cases.

(a) CONCENTRATED LOAD AT THE END OF A LIGHT CANTILEVER

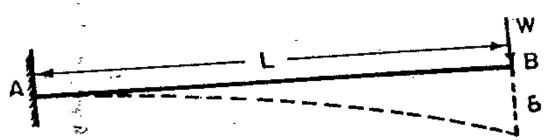


Fig. 19·6

Consider a cantilever AB of length L, subjected to a point load W. If the system is given an initial displacement to end B in the direction of W, the system will be subjected to transverse vibrations

Let $\quad \delta=\text{deflection at the free end}=\dfrac{WL^3}{3EI}$

or

$$W=\frac{3EI\delta}{L^3}$$

$$\therefore \text{ Stiffness } k=\frac{W}{\delta}=\frac{3EI}{L^3} \qquad ...(19\cdot16)$$

Hence from Eq. 19·4,

$$n=\frac{1}{2\pi}\sqrt{\frac{k.g}{W}}$$

$$=\frac{1}{2\pi}\sqrt{\frac{g}{\delta}} \quad \text{or} \quad =\frac{1}{2\pi}\sqrt{\frac{3EIg}{L^3W}} \qquad ...(19\cdot17)$$

If $\quad g=9810 \text{ mm/sec}^2$ and δ is in mm

$$n=\frac{1}{2\pi}\sqrt{\frac{9810}{\delta}}\simeq\frac{15\cdot8}{\sqrt{\delta}} \text{ per second} \qquad ...(19\cdot18)$$

$$N=60\ n\simeq\frac{946}{\sqrt{\delta}} \text{ per minute} \qquad ...(19\cdot19)$$

In F.P.S. units,

$$n=\frac{3\cdot13}{\sqrt{\delta}} \text{ per second} \qquad ...[19\cdot18\ (a)]$$

$$N=\frac{187\cdot8}{\sqrt{\delta}} \text{ per minute} \qquad ...[19\cdot19\ (a)]$$

(b) CONCENTRATED LOAD ON A SIMPLY SUPPORTED LIGHT BEAM

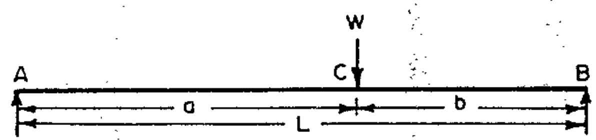


Fig. 19·7

Deflection under the concentrated load is given by

$$\delta=\frac{Wa^2b^2}{3EIL}$$

$$\therefore \quad \text{Stiffness}=k=\frac{W}{\delta}=\frac{3EIL}{a^2b^2} \qquad ...(19\cdot20)$$

Substituting this in Eq. 19·4, we get

$$n=\frac{1}{2\pi}\sqrt{\frac{k.g}{W}}$$

$$=\frac{1}{2\pi}\sqrt{\frac{g}{\delta}} \quad \text{or} \quad =\frac{1}{2\pi}\sqrt{\frac{3EILg}{a^2b^2W}} \qquad ...(19\cdot21)$$

Hence taking $g=9810$ mm/sec^2 and y in mm

$$n \simeq \frac{15\cdot 8}{\sqrt{\delta}} \text{ per second} \qquad \ldots[19\cdot 21\ (a)]$$

and

$$N = \frac{946}{\sqrt{\delta}} \text{ per minute} \qquad \ldots[19\cdot 21\ (b)]$$

(c) CONCENTRATED LOAD ON A LIGHT BEAM FIXED AT THE ENDS

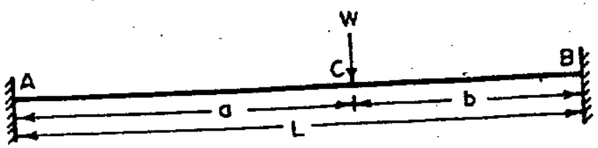


Fig. 19·8

Deflection under the load is given by

$$\delta = \frac{Wa^3b^3}{3EIL^3}$$

$$\therefore \text{ Stiffness} = k = \frac{W}{\delta} = \frac{3EIL^3}{a^3b^3}$$

$$\therefore \qquad n = \frac{1}{2\pi}\sqrt{\frac{k.g}{W}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} \quad \text{or} \quad = \frac{1}{2\pi}\sqrt{\frac{3EIL^3.g}{Wa^3b^3}} \qquad \ldots(19\cdot 22)$$

Hence taking $g=9810$ mm/sec^2 and y in mm

$$n \simeq \frac{15\cdot 8}{\sqrt{\delta}} \text{ per second} \qquad \ldots[19\cdot 22\ (a)]$$

and

$$N \simeq \frac{946}{\sqrt{\delta}} \text{ per minute} \qquad \ldots[19\cdot 22\ (b)]$$

19·5. TRANSVERSE VIBRATIONS OF A UNIFORMLY LOADED BEAM OR SHAFT

Consider a uniformly loaded beam or shaft of length L, subjected to transverse vibrations. The natural frequency of the vibrations can be approximately calculated by equating the strain energy which the beam would have in its static deflected position to the kinetic

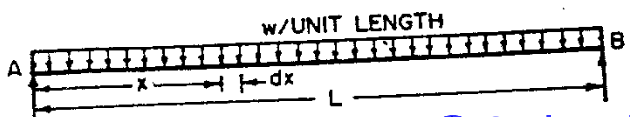


Fig. 19·9

energy which the system would have in passing through in undeflected position when vibrating with an amplitude equal at every point to the static deflection at that point.

Consider an elementary length dx at distance x from one end. Let y be the static deflection there, given by

$$y = \frac{w}{24EI}\left[x^4 - 2Lx^3 + L^3x\right]$$

Weight on the elementary length $= w\,dx$

$\therefore$ Strain energy of length of $dx = \frac{1}{2}(wdx)(y)$

$\therefore$ Strain energy of the whole beam

$$= \int_0^L \frac{1}{2}\, wdx\, y = \frac{1}{2}\, w \int_0^L \frac{w}{24EI}\left(x^4 - 2Lx^3 + L^3x\right) dx$$

$$= \frac{w}{48EI}\left[\frac{x^5}{5} - \frac{2Lx^4}{4} + \frac{L^3x^2}{2}\right]_0^L$$

$$= \frac{w^2L^5}{240EI} \qquad \ldots(1)$$

Again, the maximum velocity of the elementary length $= 2\pi\, yn$ where n = frequency of vibrations.

$$\therefore \text{ K.E. of elementary length} = \frac{1}{2} \cdot \frac{wdx}{g}(2\pi\, yn)^2$$

$$\therefore \text{ K.E. of the whole beam} = \frac{1}{2} \cdot \frac{w}{g}\int_0^L (2\pi\, yn)^2\, dx$$

$$= \frac{2\pi^2 n^2 w}{g}\int_0^L \left[\frac{w}{24EI}\left\{x^4 - 2Lx^3 + L^3x\right\}\right]^2 dx$$

$$= \frac{\pi^2 n^2 w^3}{288gE^2I^2}\int_0^L \left(x^8 + 4L^2x^6 - 4Lx^7 + L^6x^2 + 2L^3x^5 - 4L^4x^4\right) dx$$

$$= \frac{\pi^2 n^2 w^3}{288gE^2I^2}\left[\frac{L^9}{9} + \frac{4L^9}{7} - \frac{4L^9}{8} + \frac{L^9}{3} + \frac{2L^9}{6} - \frac{4L^9}{5}\right]$$

$$= \frac{31\pi^2 n^2 w^3 L^9}{288 \times 630E^2I^2g} \qquad \ldots(2)$$

Equating (1) and (2), we get

$$\frac{w^2L^5}{240EI} = \frac{31\pi^2 n^2 w^3 L^9}{288 \times 630E^2I^2g}$$

or

$$n^2 = 2\cdot 47\ \frac{EIg}{wL^4}$$

or

$$n = 1\cdot 572\sqrt{\frac{EIg}{wL^4}} \qquad \ldots(19\cdot 23)$$

Substituting δ=central deflection$=\dfrac{5}{384}\dfrac{wL^4}{EI}$

we get
$$n=1{\cdot}572\sqrt{\frac{5g}{384\delta}}=0{\cdot}179\sqrt{\frac{g}{\delta}} \text{ per sec} \qquad \ldots(19{\cdot}24)$$

Taking $g=9810$ mm/sec² and δ in mm, we get
$$n=\frac{17{\cdot}73}{\sqrt{\delta}} \text{ per sec} \qquad \ldots(19{\cdot}25)$$

and
$$N=\frac{1064}{\sqrt{\delta}} \text{ per min} \qquad \ldots(19{\cdot}26)$$

19·6. TRANSVERSE VIBRATIONS OF A BEAM OR SHAFT WITH SEVERAL POINT LOADS

(1) First Method

We have seen that for a point load, the value of $N \simeq \dfrac{946}{\sqrt{\delta}}$ per minute, while for uniformly distributed load, which is equivalent to infinite number of point loads, $K \simeq \dfrac{1064}{\sqrt{\delta}}$. Hence for any intermediate system of loading, consisting of a number of point loads, N will be between $\dfrac{946}{\sqrt{\delta}}$ to $\dfrac{1064}{\sqrt{\delta}}$ per minute. A practical value of N may be taken as
$$N=\frac{1000 \text{ to } 1050}{\sqrt{\delta_{max}}} \text{ per minute} \qquad \ldots(19{\cdot}27)$$

where δ_{max}=maximum deflection of the beam of shaft under a given system of loading.

(2) Second Method (Dunkerley's Method)

This method was suggested by Prof. Dunkerley. Let the shaft be subjected to a number of point loads $W_1, W_2, W_3, \ldots\ldots, W_n$, along with a uniformly distributed load (consisting of the self weight). Let $N_1, N_2, N_3, \ldots, N_n, N_w$, respectively, be the frequencies per minute) of the shaft when acted upon by *any one load*, other loads being absent. Then the frequency N of the shaft subjected to the combined loading is given by
$$\frac{1}{N^2}=\frac{1}{N_1^2}+\frac{1}{N_2^2}+\ldots\ldots\frac{1}{N_n^2}+\frac{1}{N_w^2} \qquad \ldots(19{\cdot}28)$$

where N_1 = frequency of the shaft when only load W_1 is acting

N_2=frequency of the shaft when only load W_2 is acting

⋮

N_n=frequency of the shaft when only W_n is acting

N_w=frequency of the shaft when only uniformly distributed load w/unit length is acting.

Now
$$N_1 \simeq \frac{946}{\sqrt{\delta_1}}\ ;\quad N_2 \simeq \frac{946}{\sqrt{\delta_2}}\ ;\quad N_n \simeq \frac{946}{\sqrt{\delta_n}}$$

and
$$N_w \simeq \frac{1064}{\sqrt{\delta_w}}$$

Hence
$$N \simeq \frac{946}{\sqrt{\delta_1+\delta_2+\ldots\delta_n+\dfrac{\delta_w}{1{\cdot}26}}} \qquad \ldots(19{\cdot}29)$$

where δ_w=maximum deflection due to uniformly distributed load acting alone on the shaft

δ_1=static deflection under load W_1 when W_1 is acting alone

δ_n=static deflection under load W_n, when W_n is acting alone.

(3) Third Method

In this method, it is assumed that the shape of the deflection curve of the vibrating shaft is similar to the shape of the static deflection curve.

Then K.E. of whole beam$=\Sigma\,\dfrac{1}{2}\,\dfrac{W}{g}\,(\omega y)^2$
$$=\frac{1}{2}\,\frac{\omega^2}{g}\,\Sigma W y^2 \qquad \ldots(i)$$

Strain energy of the whole beam$=\Sigma\frac{1}{2}W.y=\frac{1}{2}\Sigma Wy \qquad \ldots(ii)$

$$\therefore \quad \frac{1}{2}\,\frac{\omega^2}{g}\,\Sigma W\ y^2=\frac{1}{2}\,\Sigma Wy$$

or
$$\omega^2=\frac{g\Sigma Wy}{\Sigma Wy^2}$$
$$n=\frac{\omega}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{g\Sigma Wy}{\Sigma Wy^2}} \text{ per second} \qquad \ldots(19{\cdot}30)$$

or
$$N=\frac{30}{\pi}\sqrt{\frac{g\Sigma Wy}{\Sigma Wy^2}} \text{ per minute} \qquad \ldots(19{\cdot}31)$$

where y=*whole* deflection under each load resulting from the action of all the loads.

Example 19·1. *Obtain from first principles an expression for the fundamental natural frequency of transverse vibration of a cantilever of length L and weight w per unit length, it being assumed that the vibration deflection curve is of the same form as the static deflection curve.*

Hence find the frequency of transverse vibration of a steel turbine blade of uniform section, 150 mm long, having a weight of 0·02 N/mm length and least moment of inertia of 2540 mm⁴. Ignore the effect of centrifugal loading. Take $E=2\times10^5$ N/mm².

Solution.

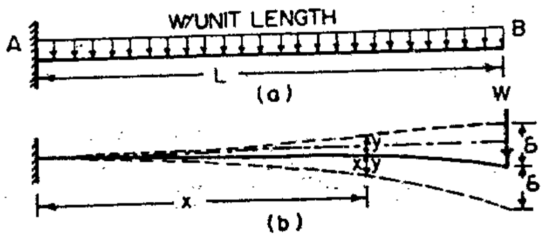


Fig. 19·10

For the sake of simplicity in the computation, we will consider he curve of elastic deflection as that caused by the dynamical equivalent of wL=a virtual load W concentrated at the free end of the cantilever, as shown in Fig. 19·10 (*b*).

Let the amplitude of vibration be δ at B and y at any section X, above and below the static deflection curve. The frequency of vibration will, however, be independent of the magnitude of y and δ.

Due to the equivalent point load W,

$$EI\,\frac{d^2y}{dx^2}=W(L-x)$$

$$\therefore\quad EI\,\frac{dy}{dx}=W\left(Lx-\frac{x^2}{2}\right)+0$$

$$EIy=W\left(\frac{Lx^2}{2}-\frac{x^3}{6}\right)+0$$

$$\therefore\quad \text{At } x=L,\quad EI\delta=W\left(\frac{L^3}{2}-\frac{x^3}{6}\right)=\frac{WL^3}{3}$$

$$\text{Also,}\quad EIy=\frac{W}{6}\left(3Lx^2-x^3\right)$$

$$\therefore\quad \frac{y}{\delta}=\frac{3Lx^2-x^3}{2L^3}\qquad\ldots(1)$$

Thus, with the help of a virtual load W, we could very easily find the ratio of the static deflections at two points.

Now consider the original U.D.L. on the cantilever.

Let v=velocity of vibration at X

V=velocity of vibration at end B

Since $v \propto y$, we have $\frac{v}{V}=\frac{y}{\delta}$

or
$$v=\frac{y}{\delta}\,V\qquad\ldots(2)$$

Consider an element of length dx at X

$$\text{K.E. of element}=\frac{1}{2}\left(\frac{wdx}{g}\right).\,v^2$$

$$=\frac{1}{2}\left(\frac{wdx}{g}\right)\left(\frac{yV}{\delta}\right)^2$$

$$=\frac{1}{2}\,\frac{wdx}{g}\,V^2\left\{\frac{3Lx^2-x^3}{2L^3}\right\}^2$$

$\therefore$ Total K.E. of the cantilever

$$=\frac{wV^2}{8gL^6}\int_0^L(9L^2x^4+x^6-6Lx^5)\,dx$$

$$=\frac{wV^2}{8gL^6}\left[\frac{9L^2x^5}{5}-Lx^6+\frac{x^7}{7}\right]_0^L$$

$$=\left(\frac{33}{140}\,wL\right)\frac{V^2}{2g}\qquad\ldots(3)$$

Hence dynamical equivalent of wL concentrated at B

$$=\frac{33}{140}\,wL=W$$

Now strain energy due to W

$$=\frac{1}{2}\,W\delta$$

$$\text{K.E. due to } W=\frac{WV^2}{2g}$$

$$\therefore\quad \frac{1}{2}\,W\delta=\frac{WV^2}{2g}$$

or
$$V^2=g\,.\,\delta=\omega^2\,.\,\delta^2$$

$$\therefore\quad \omega^2=\frac{g}{\delta}=(2\pi n)^2$$

$$\therefore \quad n=\frac{1}{2\pi}\sqrt{\frac{g}{\delta}}=\frac{1}{2\pi}\sqrt{\frac{k.g}{W}}$$

where k=stiffness $=\frac{W}{\delta}$.

Substituting, $W=\frac{33}{140}\,wL$, we get

$$n=\frac{1}{2\pi}\sqrt{\frac{k.g}{\frac{33}{140}\,wL}} \quad ...(4)$$

In the present case, $k=\frac{W}{\delta}=\frac{3EI}{L^3}$

$$\therefore \quad n=\frac{1}{2\pi}\sqrt{\frac{3EIg}{\frac{33}{140}\,wL^4}}=\frac{1}{2\pi}\sqrt{\frac{140\;EIg}{11\;wL^4}} \text{ per second}$$

$$...(19\cdot32)$$

Numerical part :

L=150 mm ; $E=2\times10^5$ N/mm²

I=2540 mm⁴ ; g=9810 mm/sec²

w=0·02 N/mm

$$\therefore \quad n=\frac{1}{2\pi}\sqrt{\frac{140\times2\times10^5\times2540\times9810}{11\times0{\cdot}02(150)^4}}$$

=398 vibrations per second.

Example 19·2. *A beam of 6 m length is simply supported at the ends and carries a central load of 20 kN. The moment of inertia of the beam is 8250×10^4 mm⁴ units. Calculate the natural frequency of vibrations. Neglect the effect of the weight of the beam. Take $E=2\times10^5$ N/mm².*

Solution.

From Eq. 19·4 (*a*),

$$n=\frac{1}{2\pi}\sqrt{\frac{k.g}{w}}$$

where k=stiffness of the beam

Central deflection $=\delta=\frac{WL^3}{48\,EI}$

$$\therefore \quad k=\frac{W}{\delta}=\frac{48EI}{L^3}$$

$$\therefore \quad n=\frac{1}{2\pi}\sqrt{\frac{48EIg}{WL^3}}$$

Substituting the values, we get

$$n=\frac{1}{2\pi}\sqrt{\frac{48\times2\times10^5\times8250\times10^4\times9810}{20000\times(6000)^3}}$$

=6·75 vibrations per second.

Example 19·3. *If the beam of Example 19·2 weighs 500 N/m, determine the frequency of vibrations, taking into account the effect the self weight of the beam.*

Solution.

Let w=weight of beam per unit length

Let W_1=dynamical equivalent of the uniformly distributed load wL, concentrated at the mid span.

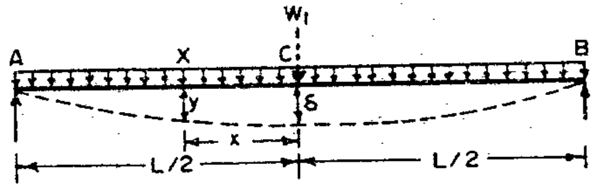


Fig. 19·11

Let the amplitude of vibration be δ at centre point C, and y at any section X, above and below the static deflection curve. Due to equivalent point load W_1.

$$EI\frac{d^2y}{dx^2}=-\frac{W_1}{2}\left(\frac{L}{2}-x\right)$$

$$\therefore \quad EI\frac{dy}{dx}=-\frac{W_1}{2}\left(\frac{L}{2}x-\frac{x^2}{2}\right)+0$$

$$\left(\text{Since } \frac{dy}{dx}=0 \text{ at } x=0\right)$$

and

$$EIy=-\frac{W_1}{2}\left(\frac{L}{2}\,\frac{x^2}{x}-\frac{x^2}{2}\right)+B$$

$$\text{At } x=\frac{L}{2},\ y=0 \qquad \therefore\ B=\frac{W_1L^3}{48}$$

$$\therefore \quad EIy=-\frac{W_1}{2}\left(\frac{L}{2}\,\frac{x^2}{6}-\frac{x^3}{2}\right)+\frac{W_1L^3}{48} \quad ...(1)$$

At $x=0,\ y=\delta$

$$\therefore \quad EI\delta=\frac{W_1L^3}{48} \quad ...(2)$$

Hence from (1) and (2),

$$\frac{y}{\delta}=1-6\left(\frac{x}{L}\right)^2+4\left(\frac{x}{L}\right)^3 \qquad ...(3)$$

Now consider the original uniformly distributed load.

Let v=velocity of vibration at X

V=velocity of vibration at centre C

Since $v \propto y$, we have $\frac{v}{V}=\frac{y}{\delta}$ ...(4)

$$\therefore \qquad v=\frac{y}{\delta}V$$

Consider an element of length dx at X

$$\text{K.E. of element}=\frac{1}{2}\left(\frac{wdx}{g}\right)v^2$$

$$=\frac{1}{2}\left(\frac{wdx}{g}\right)\left(\frac{yV}{\delta}\right)^2$$

$$=\frac{1}{2}\,\frac{wdx}{g}\,.\,V^2\left\{1-6\left(\frac{x}{L}\right)^2+4\left(\frac{x}{L}\right)^3\right\}^2$$

$\therefore$ K.E. of total beam,

$$=\frac{2wV^2}{2g}\int_0^{L/2}\left\{1-6\left(\frac{x}{L}\right)^2+4\left(\frac{x}{L}\right)^3\right\}^2$$

$$=\left(\frac{17wL}{35}\right)\frac{V^2}{2g} \qquad ...(5)$$

K.E. of the whole beam due to dynamically equivalent point load W_1 is equal to $W_1\,.\,\frac{V^2}{2g}$.

$$\therefore \qquad W_1=\frac{17wL}{35}.$$

Thus the uniformly distributed load can be replaced by an equivalent load $\frac{17}{35}wL$ *placed at the centre of the span.*

If W is the other point load acting at the centre in addition to the self weight of the beam, the total point load at the centre of span$=W+\frac{17}{35}wL$. Substituting this in Eq. 19·4 (*a*),

$$n=\frac{1}{2\pi}\sqrt{\frac{kg}{\left(W+\frac{17}{35}\,wL\right)}} \qquad ...(19·33)$$

where k=stiffness of beam.

Now due to point load W,

$$\delta=\frac{WL^3}{48\,EI}$$

$$\therefore \qquad k=\frac{W}{\delta}=\frac{48\,EI}{L^3}$$

$$=\frac{48\times2\times10^5\times8250\times10^4}{(6000)^3}$$

$$=3667 \text{ N/mm}$$

$$wL=500\times6=3000 \text{ N}$$

Substituting this in Eq. 19·33,

$$n=\frac{1}{2\pi}\sqrt{\frac{3667\times9810}{20000+\frac{17}{35}\times3000}}$$

$$=5·52 \text{ vibrations per second.}$$

Example 19·4. *A uniform steel beam, 2 m long, is simply supported at its ends and carries loads 1000 N at distance of 500 mm from such support. Determine the lowest natural frequency for the system if the mass of the beam itself may be neglected. The moment of inertia of the beam section*$=2\times10^5$ *mm*^{4}*. Take* $E=2\times10^5$ *N/mm*^{2}*.*

Solution.

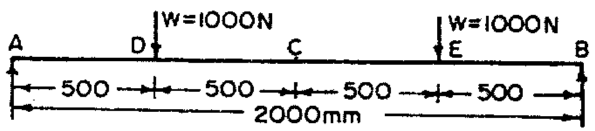


Fig. 19·12

Due to symmetrical loading, the deflections under each load will be equal. Also, slope at the centre will be zero.

$$R_A=R_B=1000 \text{ N}$$

$$EI\frac{d^2y}{dx^2}=-1000\,x \;\vdots\; +1000(x-500)$$

$$EI\frac{dy}{dx}=-1000\,\frac{x^2}{2}+A \;\vdots\; +1000\frac{(x-500)^2}{2}$$

At $\qquad x=\frac{L}{2}=1000$ mm, $\frac{dy}{dx}=0$

$$\therefore \qquad 0=-500\times(1000)^2+A+500(1000-500)^2$$

$$\therefore \qquad A=3·75\times10^8$$

$$\therefore \qquad EI\frac{dy}{dx}=-500x^2+3·75\times10^8 \;\vdots\; +500(x-500)^2$$

$$\therefore \quad EIy=-500\frac{x^3}{3}+3\cdot75\times10^8x+B \;\vdots\; +500\,\frac{(x-500)^3}{3}$$

At $\quad x=0, \quad y=0 \quad \therefore \quad B=0$

Hence $\quad x=500$ mm

$$EIy=-\frac{500}{3}(500)^3+3\cdot75\times10^8(500)=1667\times10^8$$

or
$$y=\frac{1667\times10^8}{2\times10^5\times2\times10^5}=4\cdot17 \text{ mm under each load.}$$

Now from Eq. 27·30,

$$n=\frac{1}{2\pi}\sqrt{\frac{g\Sigma W.y}{\Sigma Wy^2}}$$

Since loads are equal, this reduces to

$$n=\frac{1}{2\pi}\sqrt{\frac{g}{y}}=\frac{1}{2\pi}\sqrt{\frac{9810}{4\cdot17}}$$

$=7\cdot33$ vibrations per second.

Example 19·5. *Find the frequency of the beam of Example 19·4 using first and second methods of § 19·6.*

Solution.

(*i*) *First method*

From Fig. 19·12, y_{max} (or δ_{max}) occurs at the mid-span

($x=1000$ mm)

$$\therefore \quad EIy_{max}=-\frac{500}{3}(1000)^3+3\cdot75\times10^8(1000)+\frac{500}{3}(1000-500)^3$$

$$=2292\times10^8$$

$$\therefore \quad y_{max}=\delta_{max}=\frac{2922\times10^8}{2\times10^5\times2\times10^5}=5\cdot73 \text{ mm}$$

Hence from Eq. 19·27,

$$n=\frac{946}{\sqrt{\delta_{max}}} \text{ (when only point loads are acting)}$$

or
$$n=\frac{15\cdot8}{\sqrt{\delta_{max}}}=\frac{15\cdot8}{\sqrt{5\cdot73}}$$

$=6\cdot6$ vibrations per second.

(*ii*) *Second method*

From Eq. 19·29,

$$N=\frac{946}{\sqrt{\delta_1+\delta_2+\ldots}}$$

$$n=\frac{15\cdot8}{\sqrt{\delta_1+\delta_2}}=\frac{15\cdot8}{\sqrt{4\cdot17+4\cdot17}}$$

$=5\cdot47$ vibrations per second.

Example 19·6. *A helical spring has both ends securely fixed, one vertically above the other, and a mass is attached to the spring at some intermediate point. Show that the frequency of the vibrations is a minimum when the load point is midway between the fixed ends.*

A helical spring has a stiffness of 4N/mm when one end is fixed and the load is applied to the free end. Determine the minimum value of the frequency when both ends are fixed and a mass weighing 200 N is applied to the spring.

Solution.

Let the load W be applied at a distance L_1 from the fixed end A, or at distance L_2 from the other fixed end B, such that

$$L_1+L_2=L$$

Due to load W, let W_1 be the tension in the upper portion, and W_2 be the compression in the lower portion of the spring such that

$$W=W_1+W_2.$$

Now, in general,

$$n=\frac{1}{2\pi}\sqrt{\frac{kg}{W}}$$

where k=stiffness.

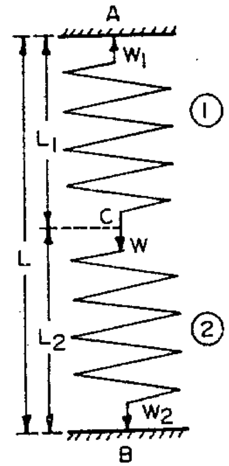


Fig. 19·13

Hence, for the first length,

$$n_1=\frac{1}{2\pi}\sqrt{\frac{k_1g}{W_1}}=\frac{1}{2\pi}\sqrt{\frac{W_1}{\delta_1}\cdot\frac{g}{W_1}}$$
$$=\frac{1}{2\pi}\sqrt{\frac{g}{\delta_1}}=\frac{1}{2\pi}\sqrt{\frac{g}{CL_1W_1}}$$

where δ_1=extension of the length

C=a constant

Similarly, $n_2=\dfrac{1}{2\pi}\sqrt{\dfrac{g}{CL_2W_2}}$

For equal minimum values, $n_1=n_2$.

$\therefore \quad (L_1W_1)_{max}=(L_2W_2)_{max}$

Now, if $W_1>\dfrac{W}{2}$, $W_2<\dfrac{W}{2}$

Also, if $L_1>\dfrac{L}{2}$, $L_2<\dfrac{L}{2}$

Hence for equal $(LW)_{max}$ and minimum n values,

$$L_1W_1=L_2W_2=\frac{L}{2}\times\frac{W}{2}$$

$$\therefore \quad L_1=L_2=\frac{L}{2} \text{ (proved)}$$

Now $\quad k=4$ N/mm

For the First Spring,

$$k=\frac{W_1}{\delta_1}=\frac{W/2}{\delta_1}=\frac{200/2}{\delta_1}=\frac{100}{\delta_1}$$

$$\therefore \quad \delta_1=\frac{100}{k}=\frac{100}{4}=25 \text{ mm.}$$

Similarly, if δ_2 is the compression of second spring, we get

$$k=\frac{W_2}{\delta_2}=\frac{W/2}{\delta_2}=\frac{200/2}{\delta_2}=\frac{100}{\delta_2}$$

or $$\delta_2=\frac{100}{k}=\frac{100}{4}=25 \text{ mm (as expected)}$$

$$\therefore \quad n_1=n_2=n=\frac{1}{2\pi}\sqrt{\frac{g}{\delta}}=\frac{1}{2\pi}\sqrt{\frac{9810}{25}}$$

$$=3{\cdot}15 \text{ vibrations per second.}$$

19·7. CRITICAL OR WHIRLING SPEED OF SHAFT

When a shaft is rotating in bearings, the initial crookedness, the dead weight of the shaft, and vibrations, etc. cause some deflection with the result that the centre line of the shaft do not coincide with the mathematically straight axis of rotation. Due to this, centrifugal forces will be developed producing a bending moment on the shaft tending to deflect it further, until they are balanced by the restoring forces arising from the stiffness of the shaft. As the speed of rotation increases, a limit will be reached when the centifugal force will exceed the limit of elastic forces. At this stage, instability will follow and the deflection and stress, unless prevented, will increase until fracture occurs. The speed which just gives balance between the two sets of forces, is called the *critical speed or whirling speed* of the shaft. The centrifugal forces may be regarded as having as neutralising effect upon the elastic forces which tend to return the shaft to its natural shape so that when whirling occurs the effective stiffness of the shaft is reduced to zero. If, however, the speed of the shaft is increased from a value below the critical to a value higher than the critical, in a short time so that the deflections do not get opportunity

to increase indefinitely, the shaft restores the stability. Many shafts are designed to run above the whirling speed.

Consider a shaft simply supported at the ends and carrying a central point load W. Let the weight of the shaft be negligible.

Let e=initial difference between the geometrical axis and the axis of rotation

y=increase in the displacement due to rotation.

The centrifugal force of the rotating mass

$$=\frac{W}{g}(e+y)\omega^2 \qquad \ldots(1)$$

where ω=angular speed of the shaft, in radians/sec

Restoring force$=k\,.\,y$ $\qquad \ldots(2)$

where k=stiffness of the shaft

Equating the two, we get

$$\frac{W}{g}(e+y)\omega^2=ky$$

or $$y\left(k-\frac{W}{g}\omega^2\right)=\frac{W}{g}e\omega^2$$

or $$y=\frac{\frac{We}{g}\omega^2}{k-\frac{W}{g}\omega^2}=\frac{e}{\frac{kg}{W\omega^2}-1} \qquad \ldots(19{\cdot}34)$$

The above equation gives the deflection due to rotation, at any angular speed ω. At the whirling speed $\omega=w_c$, the deflection y becomes infinitely great. This gives

$$\frac{kg}{W\omega_c^2}-1=0$$

or $$\omega_c^2=\frac{kg}{W} \qquad \ldots(19{\cdot}35)$$

But $$\frac{W}{k}=\text{static deflection}=\delta$$

$$\therefore \quad \omega_c^2=\frac{g}{\delta}$$

or $$\omega_c=\sqrt{\frac{g}{\delta}} \qquad \ldots(19{\cdot}36)$$

or $$n_c=\frac{1}{2\pi}\sqrt{\frac{g}{\delta}} \text{ revolutions per second} \qquad \ldots(19{\cdot}37)$$

or $$n_c=\frac{30}{\pi}\sqrt{\frac{g}{\delta}} \text{ per minute} \qquad \ldots[19{\cdot}37\,(a)]$$

Thus we find the critical speed n_c is equal to the natural frequency of vibration of the system. Similarly it can be shown that the critical speed of an unloaded shaft, taking into account its self weight, or critical speed of a shaft carrying uniformly distributed load is also equal to the natural frequency of vibration. Hence if a shaft carries a number of point loads, the method or expression for finding out the natural frequency N also apply for the present case of finding the critical speed (*i.e.* Eqs. 19·27, 19·28 and 19·29 apply for the crititical speed also).

Taking $g=9810$ mm/sec², and δ in mm, Eq. 19·27 reduces to

$$n_c=\frac{15\cdot8}{\sqrt{\delta}} \qquad \ldots(19\cdot38)$$

and

$$N_c=\frac{946}{\sqrt{\delta}} \text{ R.P.M.} \qquad \ldots[19\cdot38\,(a)]$$

Again, substituting $\frac{kg}{W}=\omega_c^2$ in Eq. 19·34, we get

$$y=\frac{e}{\dfrac{\omega_c^2}{\omega^2}-1}=-\frac{e\omega^2}{\omega^2-\omega_c^2} \qquad \ldots(19\cdot39)$$

It is evident from Eq. 19·37 that y becomes negative if the speed of rotation ω is greater than the critical speed ω_c. In other words, the shaft tries to straighten out. At very high speed, $y=-e$, and the geometrical axis and the axis of rotation will coincide. This is the principle of the flexible shaft of the De Laval steam tubine.

Example 19·7. *A shaft 20 mm diameter and 500 mm between the long bearing at its ends, carries a wheel weighing 100 N midway between the bearings. Neglecting the increase of the stiffness due to attachment of the wheel to the shaft, find the critical speed of rotation, and the maximum bending stress when the shaft is rotating at 4/5 of this speed, if the centre of gravity of the wheel is 0·4 mm from the centre of the shaft. Take $E=2\times10^5$ N/mm².*

Solution.

When shaft is supported on long bearings, it has an effect of fixidity at the end. Hence the shaft may be considered to a fixed beam with a central point load. The central deflection δ for such a case is given by

$$\delta=\frac{WL^3}{192EI}$$

$$\therefore \text{ Stiffness, } k=\frac{W}{\delta}=\frac{192EI}{L^3} \qquad \ldots(1)$$

$$I=\frac{\pi}{64}(20)^4=0\cdot785\times10^4\text{ mm}^4$$

Again,

$$N_c=\frac{60}{2\pi}\sqrt{\frac{k\,.\,g}{W}}$$

$$=\frac{30}{\pi}\sqrt{\frac{192EIg}{WL^3}}$$

$$=\frac{30}{\pi}\sqrt{\frac{192\times2\times10^5\times0\cdot785\times10^4\times9810}{100\,(500)^3}}$$

$$=\mathbf{4645 \text{ revolutions per minute.}}$$

Again, from Eq. 19·39,

$$y=\frac{e\omega^2}{\omega_c^2-\omega^2}$$

Putting $\omega=4/5\,\omega_c=0\cdot8\,\omega_c$ and $e=0\cdot4$ mm, we get

$$y=\frac{0\cdot4\,(0\cdot8\,\omega_c)^2}{\omega_c^2-(0\cdot8\omega_c)^2}=\frac{0\cdot4\times0\cdot64}{1-0\cdot64}=0\cdot71\text{ mm}$$

$\therefore$ Central centrifugal bending force

$$=k.y=0\cdot71\,k$$

$$\therefore \quad \text{B.M.}=M=\frac{1}{8}(0\cdot71\,k)\,L$$

$$=\frac{0\cdot71\,L}{8}\times\frac{192\,EI}{L^3}=\frac{0\cdot71\times24\,EI}{L^2}$$

$$\therefore \quad f=\frac{M}{Z}=\frac{M}{I}\times10=\frac{0\cdot71\times24\,E\times10}{L^2}$$

$$=\frac{7\cdot1\times24\times2\times10^5}{500\times500}$$

$$=\mathbf{136\cdot3\text{ N/mm}^2.}$$

Note: If the bearings are of short length, or if they have spherical seatings, it is taken as simply supported at the ends.

19·8. TORSIONAL VIBRATIONS

(a) SHAFT OR ROD CARRYING A LOAD W AT ITS END

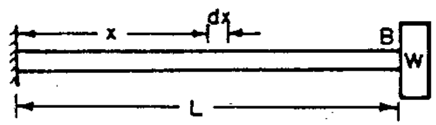


Fig. 19·14

Consider a shaft fixed at one end carrying a load W at the other end. The fixed end prevents any twisting strain at that end and hence form a fixed or stationary node. If the shaft is twisted by the rotation of the weight W, it will have vibratory movement in which every part at a given instant moves in a circle about the axis in the same sense as the applied twist. The torsional rigidity of the shaft will resist this twisting and the resisting couple will try to restore the body to its mean position. If the external torque is removed after giving an initial twist by amount θ, every point on the shaft will have simple harmonic torsional oscillations or vibrations.

Let c = torsional rigidity of the shaft

= torque per unit twist

or
$$c=\frac{T}{\theta} \qquad \ldots(1)$$

where T = torque

θ = twist in radius = amplitude of torsional vibrations.

$\therefore$ Restoring couple $=c.\theta$

Work done $=\frac{1}{2}\,(c\theta)\theta=\frac{1}{2}c.\theta^2 \qquad \ldots(2)$

Let ω = angular velocity in radians/sec $=2\pi n\theta$

where n = vibrations/second

$\therefore$ Kinetic energy $=\frac{1}{2}I\,\omega^2 \qquad \ldots(3)$

where I = mass moment of inertia of the weight W

$=\frac{W}{g}K^2$ (where K = radius of gyration of the weight W)

Equating (2) and (3),

$$\tfrac{1}{2}I\omega^2=\tfrac{1}{2}c\,\theta^2$$

$$\therefore \qquad \omega=\theta\sqrt{\frac{c}{I}} \qquad \ldots(19{\cdot}40)$$

But $\omega=2\pi n\theta$

$$\therefore \qquad n=\frac{1}{2\pi}\sqrt{\frac{c}{I}} \qquad \ldots(19{\cdot}41)$$

If L is the length of the shaft, we have

$$\frac{T}{J}=\frac{N\theta}{L}$$

But $T=c.\theta$

$$\therefore \qquad \frac{c.\theta}{J}=\frac{N\theta}{L}$$

or
$$c=\frac{NJ}{L} \qquad \ldots(4)$$

where J = polar moment of inertia of the shafts $=\frac{\pi d^4}{32}$

Substituting this in Eq. 19·41, we get

$$n=\frac{1}{2\pi}\sqrt{\frac{NJ}{IL}} \qquad \ldots(19{\cdot}42)$$

If the shaft consists of two or more parts of lengths L_1, L_2, etc. and polar moments of inertia J_1, J_2, etc., we have

$$\frac{1}{c}=\frac{1}{N}\left(\frac{L_1}{J_1}+\frac{L_2}{J_2}+\ldots\ldots\right)$$

or
$$\frac{1}{c}=\frac{1}{c_1}+\frac{1}{c_2}+\ldots\ldots \qquad \ldots[19{\cdot}42\ (a)]$$

$$\therefore \qquad n=\frac{1}{2\pi}\sqrt{\frac{N}{I\left(\frac{L_1}{J_1}+\frac{L_2}{J_2}+\ldots\ldots\right)}} \qquad \ldots[19{\cdot}42\ (b)]$$

In the above analysis, it has been assumed that the (mass) moment of inertia of the weight is so great that the (mass) moment of inertia of the shaft is negligible.

(b) EFFECT OF THE WEIGHT OF THE SHAFT

Let us now take the case of an unloaded shaft, having a self weight of w per unit length. Let the mass moment of inertia of the shaft be I_1. If ω is the angular frequency of vibration at the free end, the angular frequency of vibration at point distant x from the fixed end (Fig. 19·14) will be $\frac{\omega x}{L}$.

Mass moment of inertia of length $dx=\frac{dx}{L}\,J_1$

$\therefore$ K.E. of length $dx=\frac{1}{2}\left(\frac{dx}{L}\,I_1\right)\left(\frac{\omega x}{L}\right)^2$

$\therefore$ Total K.E. of shaft$=\int_0^L \frac{1}{2}\,\frac{dx}{L}\,I_1\left(\frac{\omega x}{L}\right)^2$

$$=\frac{1}{2}\,\frac{I_1\omega^2}{L^3}\left[\frac{L^3}{3}\right]$$

$$=\frac{1}{3}\left(\frac{1}{2}\,I_1\omega^2\right)$$

Thus the effect of the weight of the shaft is accounted for by adding $\frac{1}{3}I_1$ to the I of the weight securely fixed at the end.

Thus, if a shaft having mass moment of inertia I_1 carries a load W having mass moment of inertia I, the frequency of vibrations is given by

$$n=\frac{1}{2\pi}\sqrt{\frac{NJ}{(I+\frac{1}{3}I_1)L}} \qquad \ldots(19\cdot44)$$

(b) TWO LOADS ACTING ON THE SHAFT

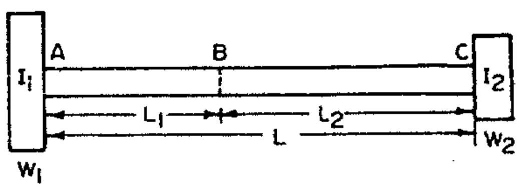


Fig. 19·15

If a freely supported shaft carries two weights W_1 and W_2 at its ends, the node will be somewhere between them. At the node point B, at a distance L_1 from one end and L_2 from the other, the torsional vibration due to either load will be equal. The shaft may thus be treated as fixed at the node point.

If I_1 and I_2 are the mass moments of inertia of the two weights, we have

$$n=\frac{1}{2\pi}\sqrt{\frac{NJ}{I_1L_1}}=\frac{1}{2\pi}\sqrt{\frac{NJ}{I_2L_2}}$$

$\therefore$ $L_1I_1=L_2I_2$

or $\frac{L_2}{L_1}=\frac{I_1}{I_2}$ $\ldots(19\cdot45)$

Also, $L_1+L_2=L$

Hence L_1 and L_2 can be found.

From Eq. 19·45, we observe that the node divides the length L inversely as the moments of inertia of the loads.

Again, from Eq. 19·45,

$$\frac{L_1+L_2}{L_2}=\frac{I_2+I_1}{I_1}$$

or $$L_2=\frac{LI_1}{I_1+I_2}$$

$\therefore$ $$n=\frac{1}{2\pi}\sqrt{\frac{NJ}{I_2L_2}}=\frac{1}{2\pi}\sqrt{\frac{NJ}{LI_1I_2}(I_1+I_2)} \qquad \ldots(19\cdot46)$$

or $$n=\frac{1}{2\pi}\sqrt{\frac{NJ}{L}\left(\frac{1}{I_1}+\frac{1}{I_2}\right)} \qquad \ldots[19\cdot46\,(a)]$$

Squaring, we get

$$n^2=\frac{1}{4\pi^2}\left[\frac{NJ}{I_1L}+\frac{NJ}{I_2L}\right]=n_1^2+n_2^2 \qquad \ldots(19\cdot47)$$

where $n_1=\frac{1}{2\pi}\sqrt{\frac{NJ_1}{LI_1}}$ and $n_2=\frac{1}{2\pi}\sqrt{\frac{NJ_2}{LI_2}}$.

Example 19·8. *A flywheel weighing 20 kN has a radius of gyration 1 m, and is fixed at one end of a shaft 100 mm in diameter and 1 metre long. A pulley weighing 12 kN and of radius of gyration of 600 mm is fixed at the other end of the shaft. Calculate the natural frequency of torsional vibrations. Take $N=0{\cdot}82\times10^5$ N/mm² for the material of the shaft. Neglect the mass moment of inertia of the shaft. Find also the position of node.*

Solution. (Fig. 19·15)

$W_1=20000\ N$; $K_1=1$ m$=1000$ mm

$$I_1=\frac{W_1}{g}\,K_1^2=\frac{20000}{9810}\,(1000)^2=2{\cdot}04\times10^6$$

$W_2=12000\ N$; $K_2=600$ mm

$$I_2=\frac{W_2}{g}\,K_2^2=\frac{12000}{9810}\,(600)^2=0{\cdot}44\times10^6$$

Now $I_1L_1=I_2L_2$

$\therefore$ $2{\cdot}04\times10^6\times L_1=0{\cdot}44\times10^6\,L_2$ $\ldots(i)$

$L_1+L_2=L=1000$ $\ldots(ii)$

From (i) and (ii), $L_1=177{\cdot}4$ mm and $L_2=822{\cdot}6$ mm.

Hence distance of node$=$**177·4 mm** from the flywheel.

$$\therefore \qquad n=\frac{1}{2\pi}\sqrt{\frac{NJ}{I_1L_1}}$$

where $J=\dfrac{\pi}{32}\quad d^4=\dfrac{\pi}{32}\quad(100)^4=9{\cdot}817\times10\ \text{mm}^3$

$$\therefore \qquad n=\frac{1}{2\pi}\sqrt{\frac{0{\cdot}82\times10^5\times9{\cdot}817\times10^6}{2{\cdot}04\times10^6\times177{\cdot}4}}$$

$=7{\cdot}51$ per second.

Alternatively, from Fig. 19·16,

$$n=\frac{1}{2\pi}\sqrt{\frac{NJ}{LI_1I_2}(I_1+I_2)}$$

$$=\frac{1}{2\pi}\sqrt{\frac{0{\cdot}82\times10^5\times9{\cdot}817\times10^6}{1000\times2{\cdot}04\times10^6\times44\times10^6}(2{\cdot}04+0{\cdot}44)10^6}$$

$=7{\cdot}51$ per second.

Example 19 9. *The flywheel of an engine driving a dynamo weighs 300 lb and has a radius of gyration of 10 in; the armature weighs 220 lb and has a radius of gyration of 8 in. The driving shaft has an effective length of 18 in. and is 2 in. diameter, and a spring coupling is incorporated at one end, having a stiffness of $0{\cdot}25\times10^5$ lb/in per radian Neglecting the inertia of the coupling and shaft, calculate the natural frequency of torsional vibration of the system. What would be the natural frequency if the spring coupling were omitted ? Take $N=11{\cdot}9\times10^6$ lb/in².* (U.L.)

Solution.

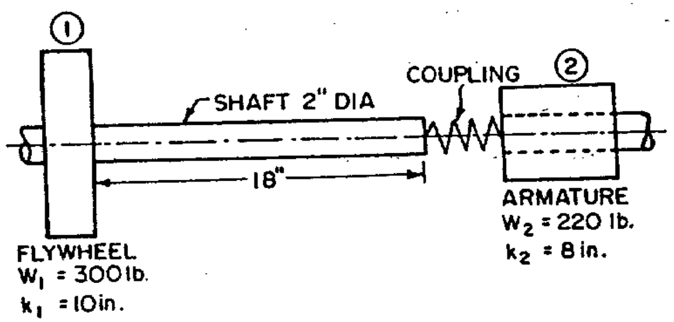


Fig. 19·16

(a) Coupling omitted

In general assuming that no nodal point occurs between 1 and 2, and have

$$n=\frac{1}{2\pi}\sqrt{\frac{c}{I_1+I_2}}$$

where c=Torsional stiffness of shaft

$$=\frac{NJ}{L}=\frac{11{\cdot}9\times10^6\times\pi(2)^4}{32\times18}$$

$$=1{\cdot}035\times10^6\ \text{lb. in./radian}$$

$$I_1=\frac{W_1}{g}K_1^2=\frac{300}{32{\cdot}2\times12}(10)^2$$

$$I_2=\frac{W_2}{g}K_2^2=\frac{200}{32{\cdot}2\times12}(8)^2$$

$$\therefore \qquad I_1+I_2=\frac{1}{32{\cdot}2\times12}\Big[300(10)^2+220(8)^2\Big]=114{\cdot}5$$

$$\therefore \qquad n=\frac{1}{2\pi}\sqrt{\frac{1{\cdot}035\times10^9}{114{\cdot}5}}=15{\cdot}15 \text{ per second.}$$

(b) Coupling included

Let c=combined stiffness of the shaft and coupling in series

c_1=stiffness of shaft

c_2=stiffness of coupling

$$\frac{1}{c}=\frac{1}{c_1}+\frac{1}{c_2}=\frac{1}{1{\cdot}035\times10^5}+\frac{4}{10^6}=\frac{4{\cdot}966}{10^6}$$

$$\therefore \qquad n=\frac{1}{2\pi}\sqrt{\frac{c}{I_1+I_2}}$$

$$=\frac{1}{2\pi}\sqrt{\frac{10^6}{4{\cdot}966\times114{\cdot}5}}=6{\cdot}69 \text{ per second.}$$

Example 19·10. *An engine shaft is directly coupled to the shaft of a dynamo. The engine shaft has a diameter of 60 mm and an effective length of 300 mm, while the dynamo shaft has a diameter of 50 mm and an effective length of 200 mm. The flywheel weighs 2500 N and has a radius of gyration of 350 mm and the armature weighs 1500 N and its radius of gyration is 250 mm. Neglecting the inertia of the coupling and of the shafts, determine the position of the node and the natural frequency of torsional oscillations. For both the shafts, take $N=0{\cdot}8\times10^5$ N/mm².*

Solution.

The shaft has a diameter $d_1=60$ mm for a length $L_1=300$ mm, and a diameter $d_2=50$ mm for a length $L_2=200$ mm. Let us first find an equivalent length for the length L_2 to have a uniform diameter d_1. Fig. 19·17(a) shows the actual shaft while Fig. 19·17(b) shows the equivalent shaft of uniform diameter $d_1=60$ mm.

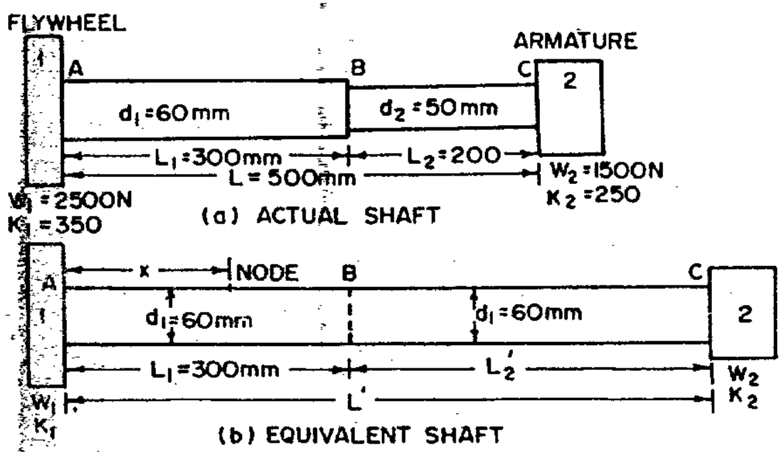


Fig. 19·17

If c is the stiffness, we have

$$c=\frac{NJ}{L}=\frac{N\pi d^4}{32\,L} \text{ or } c \propto \frac{d^4}{L}$$

Let the equivalent length of diameter $d_1=6$ cm be L'_2. Then

$$\frac{d_2^4}{L_2}=\frac{d_1^4}{L'_2}$$

$$\therefore \quad L'_2=\left(\frac{d_1}{d_2}\right)^4 L_2=\left(\frac{60}{50}\right)^4\times 200=415 \text{ mm}$$

$\therefore$ Total length $L'=L_1+L'_2=300+415=715$ mm.

Let the node be at a distance x from W_1.

Then $\quad \dfrac{W_1}{g}K_1^2\,x=\dfrac{W_2}{g}K_2^2\,(715-x)$

or $\quad 2500(350)^2\,x=1500(250)^2\,(715-x)$

or $\quad x=0{\cdot}306(715-x)$

From which $x=167{\cdot}5$ mm.

$$\therefore \quad n=\frac{1}{2\pi}\sqrt{\frac{NJ_1}{I_1 x}}=\frac{1}{2\pi}\sqrt{\frac{N\pi\, d_1^4 g}{32W_1K_1^2 x}}$$

$$=\frac{1}{2\pi}\sqrt{\frac{0{\cdot}8\times10^5\times\pi(60)^4\times9810}{32\times2500(350)^2\,(167{\cdot}5)}}$$

$=22{\cdot}1$ per second.

Example 19·11. *A light elastic shaft AB of uniform diameter, supported freely in bearings, carries wheel at each end and it is found that the natural frequency of torsional vibrations is 40 per second. A third wheel is mounted on the shaft at a point C, such that $AC=\frac{3}{4}\,AB$. If all the wheels have the same (mass) moment of inertia, determine the natural frequency of torsional vibrations.* (U.L.)

Solution.

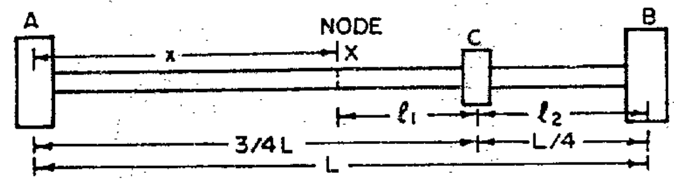


Fig. 19·18

If the shaft carries only one weight, having mass moments of inertia I, we have

$$n=\frac{1}{2\pi}\sqrt{\frac{c}{I}}=\frac{1}{2\pi}\sqrt{\frac{NJ}{l.I}}$$

$$\therefore \quad n^2=\frac{1}{4\pi^2}\cdot\frac{NJ}{l.I}=\frac{K}{l.I} \qquad \ldots(1)$$

where $K=\dfrac{NJ}{4\pi^2}$ and l=distance between the mass and the node.

Again, if a shaft carries two weights of equal mass moments of inertia, the node will be at the middle point of the shaft.

Then $\quad n=\dfrac{1}{2\pi}\sqrt{\dfrac{NJ}{I.l_1}}$

or $\quad n^2=\dfrac{NJ}{4\pi^2}\cdot\dfrac{1}{I(L/2)}$

where l_1=distance of node from the mass$=L/2$

$$\frac{NJ}{4\pi^2}=K.$$

$$\therefore \qquad n^2=\frac{2K}{IL}$$

But $\qquad n=40$ (given)

$$\therefore \qquad (40)^2=\frac{2K}{IL}$$

or $$K=800\,LI \qquad \ldots(2)$$

Again, when there are three weights acting, let the distance of the node point X be x from A. Then for the portion AX,

$$n^2=\frac{K}{xI} \qquad (3) \text{ from } (1)$$

Also, for the portion XCB

$$n^2=\frac{K}{(l_1+l_2)I}$$

where $\quad l_1$=distance of mass C from node $X=\frac{3}{4}L-x$

$\qquad l_2$=distance of mass B from node $X=L-x$

$$\therefore \qquad n^2=\frac{K}{\{(\frac{3}{4}L-x)+(L-x)\}I} \qquad \ldots(4)$$

Equating (3) and (1), we get

$$x=\tfrac{3}{4}L-x+L-x=\frac{7}{4}L-2x$$

or $$x=\frac{7}{12}L$$

$$\therefore \qquad n^2=\frac{K}{xI} \qquad \text{but} \quad \frac{K}{I}=800L \qquad (\text{from } 2)$$

$$\therefore \qquad n^2=\frac{800\,L}{\left(\frac{7}{12}L\right)}=\frac{800\times 12}{7}=\frac{9600}{7}$$

$$\therefore \qquad n=10\sqrt{\frac{96}{7}}=37{\cdot}4 \text{ per second.}$$

PROBLEMS

1. A horizontal cantilever of length L is clamped at one end and carries a load W at the other. Derive an expression for the time period of vibration of the cantilever when the load is given a small vertical displacement. Neglect the weight of the cantilever.

A horizontal flat steel strip 12 mm wide and 6 mm thick is clamped at one end with 12 mm side horizontal, and carries a weight

of 5 N at the free end. Find the distance of the weight from the fixed end if the frequency of natural vibrations of the strip is 50/sec. Take $E=2{\cdot}1\times10^5$ N/mm^2.

2. A 2 in. diameter steel AB, 8 ft. 3 in. long, is supported in two short bearings 6 ft. apart, one being at the end A of the shaft. The shaft carries three concentrated loads as under :

Load in lb.	180	360	60
Distance from A in ft.	2	4	8

Obtain a first approximation to the fundamental frequency of transverse vibration of the loaded shaft. Neglect the weight of the shaft. Take $E=30\times10^6$ lb/in^2. *(U.L.)*

3. A uniform vertical bar of steel of length L and cross-sectional area A, is fixed at the upper end and is extended a distance x by a load W at the lower end. If the rod is subjected to longitudinal vibrations, show that at any instant, when the additional extension is x, the change of potential energy, measured from the rest position of the load is $\frac{1}{2}\frac{AEx^2}{L}$ and, from the energy equation, deduce the natural period of vibration.

Find the length of the bar to give a frequency of 100 vibrations per second when A=100 sq. mm and W=100 N. Take $E=2\times10^5$ N/mm^2.

4. Solve problem 3 if the weight of the bar is 600 N.

5. A beam of length L is fixed at the ends and weighs w per unit length. Obtain an expression for the natural frequency of vibration if it carries a central point load W.

6. A close-coiled helical spring is fixed at its upper end and hangs vertically. A circular metal disc is fixed axially to the lower end of the spring. The times of vertical oscillations and for angular oscillations about the vertical axis are found to be equal.

Show that $\frac{E}{N}=\left(\frac{\text{Diameter of disc}}{\text{Mean diameter of coil}}\right)^2$, where E and N are elastic constants.

If the spring is made of wire 3 mm diameter and has 50 turns of 50 mm mean diameter, find the weight of the time of oscillations being 1 sec. Neglect the weight of the spring. Take $N=0{\cdot}8\times10^5$ N/mm^2.

ANSWERS

1. $L=139$ mm.
2. $n=8{\cdot}36$ vibrations per second.
3. $T=2\pi\sqrt{\dfrac{WL}{EAg}}$ seconds ; $L=498$ mm.
4. 415 mm.
5. $n=\dfrac{1}{2\pi}\sqrt{\dfrac{192\,EIg}{L^3\left(W+\dfrac{13}{35}\,wL\right)}}$.
6. $W=32{\cdot}2$ N.

20

Flat Circular Plates

20·1. INTRODUCTION

Flat plates are usually supported at its edges and are subjected to loads normal to their flat faces. The bending of such a plate differs from that of a beam in that the plate bend in all planes normal to the flat surface whereas the beam may be assumed to bend in one plane only. In addition to this, the bending of the plate in one plane is greatly influenced by the bending in all other planes ; hence the general theory of bending of plates is quite complicated. However, we shall consider here only an approximate theory analogous to the simple Bernoulli-Euler theory of flexure in beams. The cases which are covered by this theory include most of those that are of practical interest.

20·2. SYMMETRICALLY LOADED CIRCULAR PLATE

We shall start with the simplest case : a circular plate loaded symmetrically with respect to the central axis. The treatment that follows has been given by *Grashof* based on early investigations by *Poisson*. Grashof took the maximum strain as the measure of elastic strength.

Assumptions. The theory is based on the following assumptions :

1. The plate is of uniform thickness, and the thickness is small in comparison with the diameter.

2. The central deflection is small, and does not exceed say about one-fifth of the thickness of the plate.

3. Loading is symmetrical, so that stress and strain are symmetrical about an axis perpendicular to the plate and through its centre.

4. The plane midway between the faces of the plate is unstressed or unextended *i.e.*, the middle plane is the neutral plane.

5. The elements of the plate originally straight and perpendicular to the middle plane remain straight and become perpendicular to the middle surface when strained.

6. Only longitudinal and lateral stresses are considered. The normal stresses across planes parallel to the middle surface are neglected.

7. The material is homogeneous and isotropic, and follows Hooke's law.

Fig. 20·1 (*a*) shows the section of a thin plate after being strained. The concave side of the plate will be in compression while the convex side will be in tension. The middle plane, shown by dotted line, will be unstrained and will be a *neutral plane*. Let x and z directions be the radial and circumferential directions while y be the direction normal to the neutral plane.

Consider a point P, distant x radially from the vertical central axis, before straining. The line AB through P, originally vertical, is inclined at θ to the vertical axis OV. Let y be the distance of the point P from the middle plane of the plate. After straining, the radius at P will increase to $x+\theta y$.

Hence circumferential strain e_z at a depth y from the neutral plane is

$$e_z=\frac{2\pi(x+\theta y)-2\pi x}{2\pi x}=\frac{\theta y}{x} \qquad \ldots(1)$$

Let $\rho=\dfrac{x}{\theta}$ be the radius of curvature at P

Let $$e_z=\frac{y}{\rho} \qquad \ldots[1(a)]$$

Similarly, if we consider a section at $(x+\delta x)$ radially from O, originally vertical but become inclined at $\theta+\delta\theta$ after being strained, the distance δx at a depth y is increased to $(\delta x+y.\delta\theta)$. Hence the radial strain e_x is

$$e_x=\frac{y.\delta\theta}{\delta x}=y\,.\,\frac{d\theta}{dx}=\frac{y}{\rho'} \qquad \ldots(2)$$

where $\rho'=\dfrac{d\theta}{dx}$.

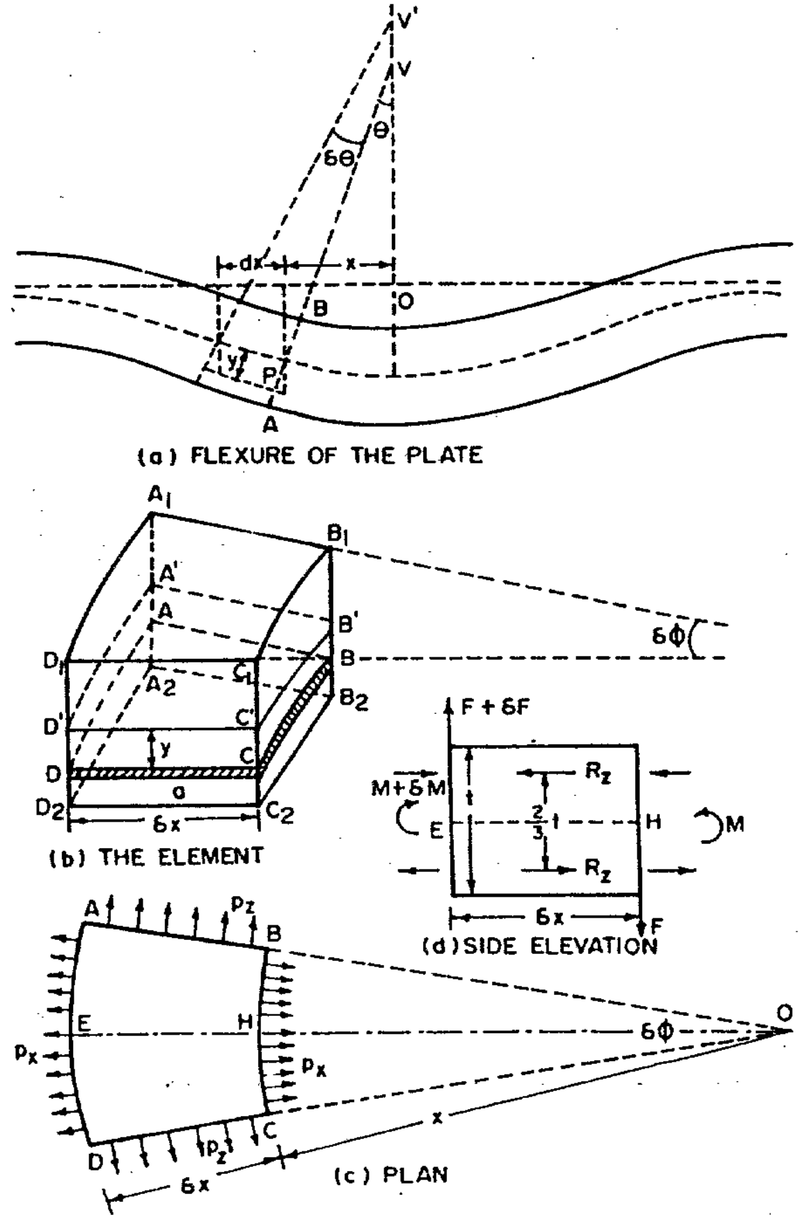


Fig. 20·1. Flat circular plate symmetrically loaded.

If p_x and p_z are radial and circumferential stresses, we have

$$e_z = \frac{\theta y}{x} = \frac{1}{E}\left(p_z - \frac{p_x}{m}\right) \quad \ldots(3)$$

and

$$e_x = y\frac{d\theta}{dx} = \frac{1}{E}\left(p_x - \frac{p_z}{m}\right) \quad \ldots(4)$$

where m=Poisson's ratio.

From Eqs. (3) and (4), we get

$$p_z = \frac{mE}{m^2-1}\left(m\frac{\theta}{x} + \frac{d\theta}{dx}\right)y \quad \ldots(20{\cdot}1)$$

$$p_x = \frac{mE}{m^2-1}\left(\frac{\theta}{x} + m\frac{d\theta}{dx}\right)y \quad \ldots(20{\cdot}2)$$

From Eqs. 20·1 and 20·2, it is clear that both circumferential stress as well as radial stress vary linearly with y. Thus the stress distribution across a plane is similar to the bending stress distribution in a beam.

Let us now compute resultant circumferential stress and resultant radial stress on an element included between radii x and $x+\delta x$, and between two vertical meridian planes inclined at a very small angle $\delta\theta$ to each other. Fig. 20·1 (*b*) shows a pictorial view of such an element while Fig. 20·1 (*c*) shows the plane of a horizontal plane $ABCD$ taken at a distance y below the neutral plane $A'B'C'D'$ [Fig. 20·1 (*b*)]. Since the plane $ABCD$ is below the neutral plane, both circumferential as well as radial stresses have been shown tensile in Fig. 20·1 (*c*). The distance y is taken positive when measured below the neutral surface (N.S.) and the tensile stress is taken as positive. Similarly, y measured above the N.S. is taken negative and the compressive stress is taken as negative.

In Fig. 20·1 (*c*), the stresses p_z on faces AB and CD are inclined at $\frac{\pi}{2} - \frac{\delta\phi}{2}$ to the middle radius EHO. Hence the resultant elementary force in the direction EHO is equal to

$$2p_z \,.\, \delta a \sin\frac{\delta\phi}{2} \simeq p_z \,.\, \delta a \,.\, \delta\phi \quad \ldots(\text{I})$$

where δa=elementary area of face AB or CD shown hatched in Fig. 20·1 (*b*).

Since p_z is of opposite sign on the opposite faces of the neutral surface, *the total force parallel to EHO resulting from p_z is zero.*

The elementary force given by (I) is tensile below the neutral surface and compressive above the neutral surface. Hence total moment M_1 of the couple formed by the above elementary forces, about an axis in the neutral plane and perpendicular to EHO is

$$M_1 = \delta\phi\, \Sigma y \,.\, p_z \,.\, \delta a \quad \ldots(\text{II})$$

Substituting the value of p_z from Eq. 28·1, we get

$$M_1 = \delta\phi.\, \frac{mE}{m^2-1}\left(m\frac{\theta}{x} + \frac{d\theta}{dx}\right)\Sigma y^2 \,.\, \delta a$$

Let t=thickness of the plate

Then $\Sigma y^2\, \delta a$=moment of inertia of rectangular face AB or CD

$$= \frac{1}{12}\,\delta x t^3$$

$$\therefore \quad M_1 = \delta\phi.\, \frac{mE}{m^2-1}\left(m\frac{\theta}{x} + \frac{d\theta}{dx}\right) \cdot \frac{1}{12}\,\delta x . t^3$$

or

$$M_1 = \frac{1}{12}\,\delta x.\delta\phi.t^3\, \frac{mE}{m^2-1}\left(m\frac{\theta}{x} + \frac{d\theta}{dx}\right) \quad \ldots(20{\cdot}3)$$

or

$$M_1 = R_z \times \tfrac{2}{3}t \quad \ldots[20{\cdot}3\,(a)]$$

where $R_z = \delta\phi.\Sigma p_z.\delta a$=total force is direction EHO on one side of the neutral plane, Fig. 20·1 (*d*).

$\frac{2}{3}t$=lever arm at which the equal and opposite forces R_z act, Fig. 20·1 (*d*).

Sign convention. If the vertex V is formed *above* the plate, θ is taken in positive. Hence $\frac{d\theta}{dx}$ is positive, making the plate convex downwards. Thus if the element [Fig. 20·1 (*b*)] is viewed from the $D_1C_1C_2D_2$, the moment M_1 due to the circumferential forces is counter-clockwise.

Let us now consider faces BC and AD on which radial stresses p_z act. If an elementary area δa is considered on face BC, the elementary force on this area, resolved in the direction EHO is approximately equal to $p_x\, \delta a$. This force is tensile if δa is considered below the neutral plane. Thus, the total force on the face BC due to radial stress is zero. However, the couple formed by equal and opposite elementary force on opposite sides of the neutral plane is given by,

$$M = \Sigma p_x.y.\delta a$$

Substituting the value of p_z from Eq. 20·2, we get

$$M = \frac{mE}{m^2-1}\left(\frac{\theta}{x} + m\frac{d\theta}{dx}\right)\Sigma y^2\, \delta a$$

For the face BC of width $x.\delta\phi$ and height t, we have

$$\Sigma\, y^2\, \delta a = \frac{1}{12}\,(x.\delta\phi)\, t^3$$

$$\therefore \quad M = \frac{mE}{m^2-1}\left(\frac{\theta}{x} + m\,\frac{d\theta}{dx}\right)\cdot\frac{1}{12}\; x\,\delta\phi\, t^3$$

or
$$M = \frac{x\,\delta\phi\, t^3}{12}\cdot\frac{mE}{m^2-1}\left(\frac{\theta}{x} + m\frac{d\theta}{dx}\right) \qquad \ldots(20{\cdot}4)$$

If θ and $\dfrac{d\theta}{dx}$ are positive, the moment M will be in the counter-clockwise direction as marked in Fig. 20·1 (d).

Now consider the face AD on which p_x act outwards. Let $(M+\delta M)$ be the moment due to the stress p_x acting on it. This moment $(M+\delta M)$ can be expressed in terms of $(x+\delta x)$ and $(\theta+\delta\theta)$. However,

$$\delta M = \frac{dM}{dx}\,.\,\delta x$$

Hence differentiating Eq. 20·4, we get

$$\delta M = \frac{dM}{dx}.\delta x = \frac{1}{12}.\delta x.\delta\phi.t^3\,\frac{mE}{m^2-1}\left(\frac{d\theta}{dx} + m\,\frac{d\theta}{dx} + mx\,\frac{d^2\theta}{dx^2}\right) \qquad \ldots(20{\cdot}5)$$

If θ and $\dfrac{d\theta}{dx}$ are positive, making the plate convex downwards, the moment $(M+\delta M)$ due to p_x for face AD will be *clockwise*.

Thus, face BC has a moment M in the counter-clockwise direction, while face AD has a moment $(M+\delta M)$ in the clockwise direction. Hence the *net* moment δM (given by Eq. 20·5) due to radial stresses acting on faces BC and AD will be clockwise and will be opposed to the moment M_1 due to circumferential stresses. Hence the internal moment of resistance of the element will be the algebraic sum of the moment M_1 and the moment δM $(=M_2$ say) given by the following expressions :

$$M_1 = \frac{1}{12}\;\delta x.\delta\phi.t^3\,\frac{mE}{m^2-1}\left(m\;\frac{\theta}{x} + \frac{d\theta}{dx}\right) \qquad \ldots(\text{i}) \ldots(20{\cdot}3)$$

(counter-clockwise)

$$M_2 = \delta M = \frac{1}{12}\;\delta x.\delta\phi.t^3\,\frac{mE}{m^2-1}\left(\frac{d\theta}{dx} + m\,\frac{d\theta}{dx} + mx\,\frac{d^2\theta}{dx^2}\right) \qquad \ldots(\text{ii}) \ldots(20{\cdot}3)$$

(clockwise)

The resultant of the two moments M_1 and M_2 is balanced by the external forces, included of reactions, acting on the plate. We shall now consider the following cases of circular plates with uniform pressure in its face :

(1) circular plate freely supported at its circumference,

(2) circular plate freely supported at its circumference, with a central circular hole,

(3) circular plate clamped at its circumference.

20·3. CIRCULAR PLATE FREELY SUPPORTED AT ITS CIRCUMFERENCE

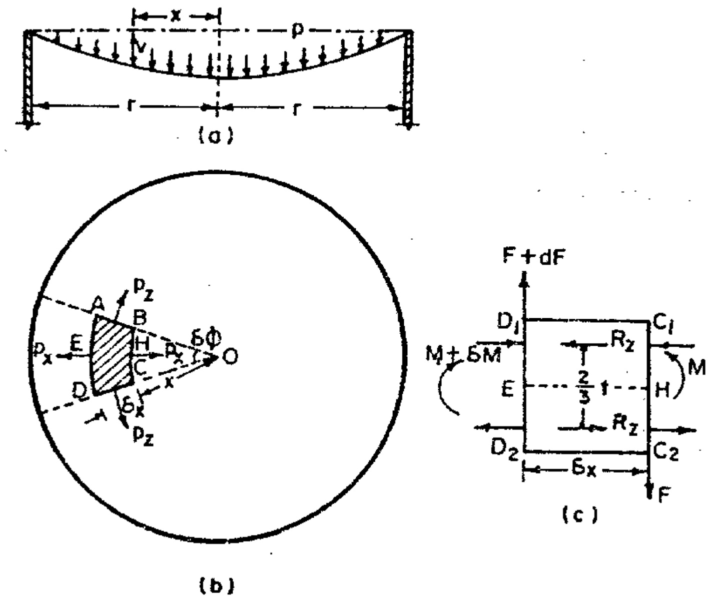


Fig. 20·2. Circular plate freely supported at its circumference.

Let p = uniformly distributed load or pressure normal to the face of the plate

P = total load on the plate $= p.\pi r^2$

or
$$p = \frac{P}{\pi r^2} \qquad \ldots(1)$$

Consider a concentric plate with a radius x. The external vertical force on it is equal to $p\pi x^2$. Hence the shear force F on the face BC is

$$F = p\pi x^2\,.\,\frac{\delta\phi}{2\pi} = \tfrac{1}{2}px^2\,\delta\phi \qquad \ldots(2)$$

Similarly, the shear force $F+\delta F$ on face AD is

$$F+\delta F=\tfrac{1}{2}p(x+\delta x)^2\,\delta\phi \qquad ...(3)$$

These forces F and $F+\delta F$ have been marked on Fig. 20·2 (c).

Neglecting small quantities of second order, the moment of external forces about an axis in the neutral plane and perpendicular to EH is given by

$$M=F.\delta x=\tfrac{1}{2}px^2.\delta\phi.\delta x \qquad ...(20·6)$$

This moment is in the clockwise direction.

Thus, there are three moments acting on the element : M_1, M_2 and M. For the equilibrium, the algebraic sum of the three must be equal to zero. Since M_1 is in the anticlockwise direction, while M_2 and M are in the clockwise direction, we have

$$M_1-M_2-M=0$$

Substituting the values from Eqs. 20·4, 20·5 and 20·6, we get

$$\frac{1}{12}\,\delta x.\delta\phi.t^3\,\frac{mE}{m^2-1}\left(m\,\frac{\theta}{x}+\frac{d\theta}{dx}\right)-\frac{1}{12}\,\delta x.\delta\phi.t^3\,\frac{mE}{m^2-1}\left(\frac{d\theta}{dx}+m\,\frac{d\theta}{dx}+mx\,\frac{d^2\theta}{dx^2}\right)-\tfrac{1}{2}px^2\,\delta\phi\,\delta x=0$$

Dividing by $\frac{1}{12}\,\delta x.\delta\phi.\frac{Em^2t^3}{m^2-1}$, we get

$$\frac{\theta}{x}+\frac{1}{m}\,\frac{d\theta}{dx}-\frac{1}{m}\,\frac{d\theta}{dx}-\frac{d\theta}{dx}-x\,\frac{d^2\theta}{dx^2}=\frac{6(m^2-1)}{Em^2t^3}\,x^2.p$$

or $$x\,\frac{d^2\theta}{dx^2}+\frac{d\theta}{dx}-\frac{\theta}{x}=-\,\frac{6(m^2-1)}{Em^2t^3}\,x^2.p$$

Putting $k=\frac{3(m^2-1)pr^2}{Em^2t^3}$ ...(20·7)

we get, $$x\,\frac{d^2\theta}{dx^2}+\frac{d\theta}{dx}-\frac{\theta}{x}=-2k\,\frac{x^2}{r^2}$$

The complete solution of the above differential equation is given by Eqs. 20·8 and 20·9 as under :

$$\frac{\theta}{x}=A+\frac{B}{x^2}-\frac{1}{4}\,k\,\frac{x^2}{r^2} \qquad ...(20·8)$$

and $$\frac{d\theta}{dx}=A-\frac{B}{x^2}-\frac{3}{4}\,k\,\frac{x^2}{r^2} \qquad ...(20·9)$$

where A and B are constants of integration, to be determined by the boundary conditions.

At $x=0$, $\theta=0$. Hence from Eq. 20·8, $B=0$.

From Eq. 20·2, we have

$$p_x=\frac{mE}{m^2-1}\left(\frac{\theta}{x}+m\,\frac{d\theta}{dx}\right)y$$

Substituting the values of $\frac{\theta}{x}$ and $\frac{d\theta}{dx}$ from Eqs. 20·8 and 20·9,

$$p_x=\frac{mEy}{m^2-1}\left(A-\frac{1}{4}\,k\,\frac{x^2}{r^2}+mA-\frac{3m}{4}\,k\,\frac{x^2}{r^2}\right)$$

At $x=r$, $p_x=0$

$$\therefore\quad p_x=0=A-\frac{1}{4}\,k\,\frac{r^2}{r^2}+mA-\frac{3m}{4}\,k\,\frac{r^2}{r^2}$$

or $$A=\frac{3m+1}{m+1}\cdot\frac{k}{4} \qquad ...(20·10)$$

Substituting the values of B and A in Eqs. 20·8 and 20·9,

$$\frac{\theta}{x}=\frac{k}{4}\left(\frac{3m+1}{m+1}-\frac{x^2}{r^2}\right) \qquad ...[20·8\ (a)]$$

and $$\frac{d\theta}{dx}=\frac{k}{4}\left(\frac{3m+1}{m+1}-\frac{3x^2}{r^2}\right) \qquad ...[20·9\ (a)]$$

Substituting these values in Eqs. 20·1 and 20·2, we get

$$p_z=\frac{3}{4}\,\frac{py}{mt^3}\left[(3m+1)r^2-(m+3)x^2\right] \qquad ...(20·11)$$

and $$p_x=\frac{3}{4}\,\frac{py}{mt^3}\left(3m+1\right)\left(r^2-x^2\right) \qquad ...(20·12)$$

By inspection, both p_x and p_z decrease as x increases. Hence maximum values of p_z and p_x occur at the centre of the plate, where $x=0$, on either side of the plate where $y=\pm\frac{1}{2}t$.

$$\therefore\quad \left[(p_x)_{max.}=(p_z)_{max.}=\pm\frac{3pr^2}{8t^2}\cdot\frac{3m+1}{m}\right] \qquad ...(20·13)$$

The radial and circumferential strains are found as under

$$e_x=x\,\frac{d\theta}{dx}=\frac{k}{4}\left(\frac{3m+1}{m+1}-\frac{3x^2}{r^2}\right)y \qquad ...(20·14)$$

and $$e_z=\frac{\theta}{x}\,y=\frac{k}{4}\left(\frac{3m+1}{m+1}-\frac{x^2}{r^2}\right)y \qquad ...(20·15)$$

These are maximum at $x=0$ and $y=\pm\frac{t}{2}$

$$\therefore\quad (e_x)_{max.}=(e_z)_{max.}=\pm\frac{3}{8}\,\frac{(m-1)(3m+1)}{Em^2t^2}.pr^2 \qquad ...(20·16)$$

Shear stress distribution

The shear stress distribution across the thickness of the slab can be roughly determined following the method followed in the case of straight beams. The expression for the shear stress at plane distant y from the neutral plane may be expressed as

$$q=\frac{FA\bar{y}}{bI}$$

where F=shear force on the element$=\frac{1}{2}\,px^2.\delta\phi$

$$I=\frac{1}{12}\,x.\delta\phi.t^3$$

$$b=x\,\delta\phi$$

$$A=(\delta\phi)x\left(\frac{t}{2}-y\right)$$

$$\bar{y}=\left(\frac{t}{4}+\frac{y}{2}\right)$$

$$\therefore\quad q=\frac{\left(\frac{1}{2}\,p\,x^2.\delta\phi\right)\left\{\left(x\,\delta\phi\right)\left(\frac{t}{2}-y\right)\right\}\left(\frac{t}{4}+\frac{y}{2}\right)}{\left(x\,\delta\phi\right)\left(\frac{1}{12}\,x\,\delta\phi.t^3\right)}$$

or

$$q=\frac{3px}{t^3}\left(\frac{t^2}{4}-y^2\right)\quad\ldots(20\cdot17)$$

Maximum shear occurs at the neutral plane, $y=0$

$$\therefore\quad q_{max.}=\frac{3}{4}\,\frac{p.x}{t}\quad\ldots(20\cdot18)$$

$$q_{av.}=\frac{\frac{1}{2}px^2\delta\phi}{x\,\delta\phi.t}=\frac{1}{2}\,\frac{px}{t}$$

$$\therefore\quad\frac{q_{max.}}{q_{av.}}=\frac{3}{2}$$

At $x=r$, $q_{max.}=\frac{3}{4}\,\frac{pr}{t}$.

It should be noted that we have neglected the shear stress (Eq. 20·18) while finding out expressions for p_x and p_z. In addition to this, we have also neglected the vertical direct compressive stress varying from p at the upper face to zero at the lower face of the plate.

Deflection of the plate

Let v=deflection of the neutral surface, at a radius x.

$$\therefore\quad\tan\theta=-\frac{dv}{dx}$$

Since θ is usually small, $\tan\theta\simeq\theta$

$$\therefore\quad-\frac{dv}{dx}=\theta\quad\ldots(1)$$

Substituting the value of θ from Eq. 20·8 (a), we get

$$-\frac{dv}{dx}=\frac{k}{4}\left(\frac{3m+1}{m+1}\,x-\frac{x^3}{r^2}\right)$$

Integrating,

$$v=-\frac{k}{4}\left(\frac{3m+1}{m+1}\,\frac{x^2}{2}-\frac{x^4}{4r^2}+c\right)\quad\ldots(2)$$

where c is a constant of integration.

At $x=r$, $v=0=-\frac{k}{4}\left(\frac{3m+1}{m+1}\,\frac{r^2}{2}-\frac{r^4}{4r^2}+c\right)$

$$\therefore\quad c=-\frac{r^2}{4}\,\frac{5m+1}{m+1}$$

Substituting this and the value of k in (2), we get

$$v=-\frac{3}{8}\,\frac{(m^2-1)p}{Em^2t^3}\left(\frac{3m+1}{m+1}\,r^2x^2-\frac{x^4}{2}-\frac{5m+1}{m+1}\,\frac{r^4}{2}\right)\quad\ldots(20\cdot19)$$

This is maximum at $x=0$

$$\therefore\quad v_{max.}=\frac{3}{16}\,\frac{(m-1)(5m+1)pr^4}{Em^2\,t^3}\quad\ldots(20\cdot20)$$

20·8. CIRCULAR PLATE WITH CENTRAL HOLE FREELY SUPPORTED AT ITS CIRCUMFERENCE

Let the plate of radius r have a central hole of radius r_0. Let p be the intensity of uniformly distributed load.

The radial stress p_x is given by Eq. 20·2,

$$p_x=\frac{mE}{m^2-1}\left(\frac{\theta}{x}+m\,\frac{d\theta}{dx}\right)y$$

Substituting the value of $\frac{\theta}{x}$ and $\frac{d\theta}{dx}$ from Eqs. 20·8 and 20·9,

$$p_x=\frac{mE}{m^2-1}\left[\left(A+\frac{B}{x^2}-\frac{1}{4}\,k\,\frac{x^2}{r^2}\right)+m\left(A-\frac{B}{x^2}-\frac{3}{4}\,k\,\frac{x^2}{r^2}\right)\right]y\quad\ldots(1)$$

where $k=\frac{3(m^2-1)pr^2}{Em^2\,t^3}$ (Eq. 20·7)

The boundary conditions for the present case are

$p_x=0$ at $x=r_0$

and $p_x=0$ at $x=r$.

Applying boundary conditions in Eq. (1), we get

$$A(m+1)+\frac{B}{r_0^2}(1-m)-\frac{k}{4}\cdot\frac{r_0^2}{r^2}(1+3m)=0 \quad \text{...(2)}$$

and
$$A(m+1)+\frac{B}{r^2}(1-m)-\frac{k}{4}(1+3m)=0 \quad \text{...(3)}$$

Solving (2) and (3), we get

$$A=\frac{3m+1}{4(m+1)}k\left(1+\frac{r_0^2}{r^2}\right) \quad \text{...(4)}$$

and
$$B=\frac{3m+1}{4(m-1)}k\,r_0^2 \quad \text{...(5)}$$

Substituting these values of A and B, and of k, we get the following expressions for px and pz.

$$px=\frac{3}{4}\cdot\frac{3m+1}{mt^3}pr^2y\left[1+\frac{r_0^2}{r^2}-\frac{r_0^2}{x^2}-\frac{x^2}{r^2}\right] \quad \text{...(20·21)}$$

and
$$pz=\frac{3}{4}\cdot\frac{py}{mt^3}\left[(3m+1)r^2\left(1+\frac{r_0^2}{x^2}+\frac{r_0^2}{r^2}\right)-(m+3)x^2\right] \quad \text{...(20·22)}$$

At $x=r_0$, pz is given by

$$pz=\frac{3}{4}\cdot\frac{py}{mt^3}\left[(3m+1)r^2\left(2+\frac{r_0^2}{r^2}\right)-(m+3)r_0^2\right]$$

If r_0 is extremely small, so as to have only a pin hole at the centre of the slab, the maximum circumferential stress at $y=\pm\frac{1}{2}t$ is given by

$$(pz)_{max.}=\pm\frac{3}{4}\cdot\frac{pr^2}{t^2}\cdot\frac{3m+1}{m} \quad \text{...(20·23)}$$

Thus the intensity of stress at the centre is twice that for the plate with no hole (Eq. 28·13).

20·5. CIRCULAR PLATE CLAMPED AT ITS CIRCUMFERENCE

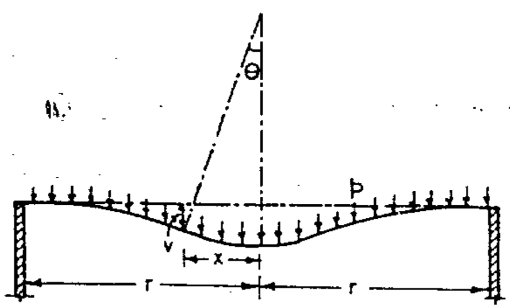


Fig. 20·3

The boundary conditions for this case are :

$$\theta=0 \text{ at } x=0$$

and
$$\theta=0 \text{ at } x=r$$

Substituting these in Eq. 20·8, we get

$$B=0$$

and
$$A=\frac{k}{4}$$

Substituting these values in Eqs. 20·8 and 20·9, we get

$$\frac{\theta}{x}=\frac{k}{4}\left(1-\frac{x^2}{r^2}\right) \quad \text{...(20·28)}$$

$$\frac{d\theta}{dx}=\frac{k}{4}\left(1-\frac{3x^2}{r^2}\right) \quad \text{...(20·25)}$$

where $k=\dfrac{3(m^2-1)pr^2}{Em^2t^3}$ (Eq. 20·7)

Substituting the values of $\frac{\theta}{x}$, $\frac{d\theta}{dx}$ and k in Eqs. 20·1 and 20·2, we get

$$px=\frac{3}{4}\cdot\frac{py}{mt^3}\left\{(m+1)r^2-(3m+1)x^2\right\} \quad \text{...(20·26)}$$

and
$$pz=\frac{3}{4}\cdot\frac{py}{mt^3}\left\{(m+1)r^2-(m+3)x^2\right\} \quad \text{...(20·27)}$$

At the centre of the plate $x=0$, the radial and circumferential stresses at $y=\pm\frac{1}{2}t$ are given by

$$px=pz=\pm\frac{3}{8}\cdot\frac{m+1}{m}\cdot\frac{pr^2}{t^2} \quad \text{...(20·28)}$$

At the circumference $x=r$, the stresses are

$$px=-\frac{3}{2}\cdot\frac{pr^2}{t^3}y \quad \text{...[20·29 (a)]}$$

$$pz=-\frac{3}{2}\cdot\frac{pr^2}{mt^3}y \quad \text{...[20·30 (a)]}$$

From Eqs. 20·29 (a) and 20·30 (a), it is evident that the circumferential stress pz reaches $\frac{1}{m}$ of the radial stress px at $x=r$. The maximum values are given at $y=\pm\frac{t}{2}$.

$$px=\mp\frac{3}{4}\cdot\frac{r^2}{t^2}p \quad \text{...[20·30 (a)]}$$

and
$$pz=\mp\frac{3}{4}\cdot\frac{r^2}{mt^2}p \quad \text{...[20·30 (b)]}$$

The greatest intensity of bending stress in the plate is thus the radial stress px at $x=r$ [Eq. 20·30 (*a*)].

Similarly, the maximum strain is the radial strain at $x=r$ and $y=\pm\frac{t}{2}$, given by

$$ex=\pm y\frac{d\theta}{dx}=\mp\frac{3}{4}\cdot\frac{m^2-1}{Em^2}\cdot\frac{r^2}{t^2}\,p \qquad \ldots(20·31)$$

or *alternatively*,

$$ex=\frac{px}{E}-\frac{1}{m}\frac{pz}{E}=\frac{px}{E}-\frac{1}{m}\cdot\frac{1}{m}\frac{px}{E}$$

$$=\frac{px}{E}\cdot\frac{m^2-1}{m^2} \qquad \ldots[20·31\ (a)]$$

$$=\mp\frac{3}{4}\frac{r^2}{t^2}\cdot\frac{p}{E}\cdot\frac{m^2-1}{m^2}=\mp\frac{3}{4}\frac{m^2-1}{Em^2}\cdot\frac{r^2}{t^2}\cdot p$$

Deflection of the plate

As in article 20·3,

$$-\frac{dv}{dx}=\theta=\frac{3}{4}\frac{(m^2-1)p}{Em^2t^3}(r^2x-x^3)$$

Integrating,

$$v=-\frac{3}{4}\frac{(m^2-1)p}{Em^2t^3}\left(\frac{r^2x^2}{2}-\frac{x^4}{4}+C\right) \qquad \ldots(1)$$

where C is the constant of integration.

At $x=r,\ v=0$

$$\therefore\quad 0=-\frac{3}{4}\frac{(m^2-1)p}{Em^2t^3}\left(\frac{r^4}{2}-\frac{r^4}{4}+C\right)$$

$$\therefore\quad C=-\frac{r^4}{4}$$

Substituting in (1), we get

$$v=-\frac{3}{4}\frac{(m^2-1)p}{Em^2t^3}\left(\frac{r^2x^2}{2}-\frac{x^4}{4}-\frac{r^4}{4}\right) \qquad \ldots(20·32)$$

The maximum deflection occurs at $x=0$.

$$\therefore\quad v_{max}=\frac{3}{4}\cdot\frac{(m^2-1)p}{Em^2t^3}\cdot\frac{r^4}{4}=\frac{3}{16}\frac{(m^2-1)}{Em^2t^3}\cdot pr^4 \qquad \ldots(20·33)$$

Example 20·1. *A cylinder 500 mm internal diameter has a flat end 30 mm thick. Find the greatest intensity of stress in the end if the pressure in the cylinder is 1 N/mm². The end may be taken as freely supported.*

Also, find what intensity of simple direct stress would produce (i) the same maximum strain, (ii) the same maximum strain energy and (iii) the same maximum shear strain energy. Take m=3.

Solution.

The greatest intensity of stress is given by Eq. 20·13,

$$(px)_{max}=(pz)_{max}=\frac{3pr^2}{8t^2}\left(\frac{3m+1}{m}\right)$$

$$=\frac{3\times1(250)^2}{8(30)^2}\left(\frac{3\times3+1}{3}\right)$$

$$=86·8\ \text{N/mm}^2.$$

(*i*) *Simple stress to produce the same maximum strain*

Let p=simple stress

e=maximum strain

$$\therefore\quad e=\frac{p}{E}=\frac{1}{E}\left(px-\frac{pz}{m}\right)$$

$$\therefore\quad p=px-\frac{pz}{m}=px\left(1-\frac{1}{m}\right)$$

Since $px=pz=86·8$

$$\therefore\quad p=86·8(1-\tfrac{1}{3})=57·9\ \text{N/mm}^2.$$

(*ii*) *Simple stress to produce same shear strain energy*

$$\frac{p^2}{2E}=\frac{1}{2E}\left(px^2+pz^2-\frac{2px\,pz}{m}\right)$$

or
$$p=\sqrt{px^2+pz^2-\frac{2px\,pz}{m}}=px\sqrt{2\left(1-\frac{1}{m}\right)}$$

$$=86·8\sqrt{2(1-\tfrac{1}{3})}=100·2\ \text{N/mm}^2.$$

(*iii*) *Simple stress to produce the same shear strain energy*

$$\frac{m+1}{3mE}p^2=\frac{(m+1)}{3mE}\left(px^2+pz^2-px.pz\right)$$

or
$$p=\sqrt{px^2+pz^2-px\,pz}=px\ (\text{since } px=pz)$$

$$\therefore\quad p=86·8\ \text{N/mm}^2.$$

Example 20·2. *Solve Example 20·1 if the flat end is assumed to be fixed at the edges.*

Solution.

The maximum radial stress is given by Eq. 20·29 (*b*),

$$px=\frac{3}{4}\frac{r^2}{t^2}p=\frac{3}{4}\left(\frac{250}{30}\right)^2\times1·0=52\ \text{N/mm}^2$$

$$\therefore\quad p=\frac{1}{m}p=\frac{52}{3}=17·4\ \text{N/mm}^2.$$

(*a*) *Simple stress to produce the same maximum strain*

$$p=px-\frac{pz}{m}=52-\frac{17·4}{3}=46·2\ \text{N/mm}^2.$$

(*b*) *Simple stress to produce same maximum strain energy*

$$p=\sqrt{px^2+pz^2-\frac{2px\,pz}{m}}$$

But $$pz=\frac{px}{m}$$

$$\therefore \quad p=px\sqrt{1+\frac{1}{m^2}-\frac{2}{m^2}}=52\sqrt{1+\frac{1}{9}-\frac{2}{9}}$$

$$=49 \text{ N/mm}^2.$$

(*c*) *Simple stress to produce same shear strain energy*

$$p=\sqrt{px^2+pz^2-px\,pz}=px\sqrt{1+\frac{1}{m^2}-\frac{1}{m}}$$

$$=52\sqrt{1+\frac{1}{9}-\frac{1}{3}}=45{\cdot}9 \text{ N/mm}^2.$$

21

Unsymmetrical Bending

21·1. INTRODUCTION

In the simple theory of bending using the well known flexure formula $\frac{M}{I}=\frac{f}{y}$; it is assumed that the neutral axis of the cross-

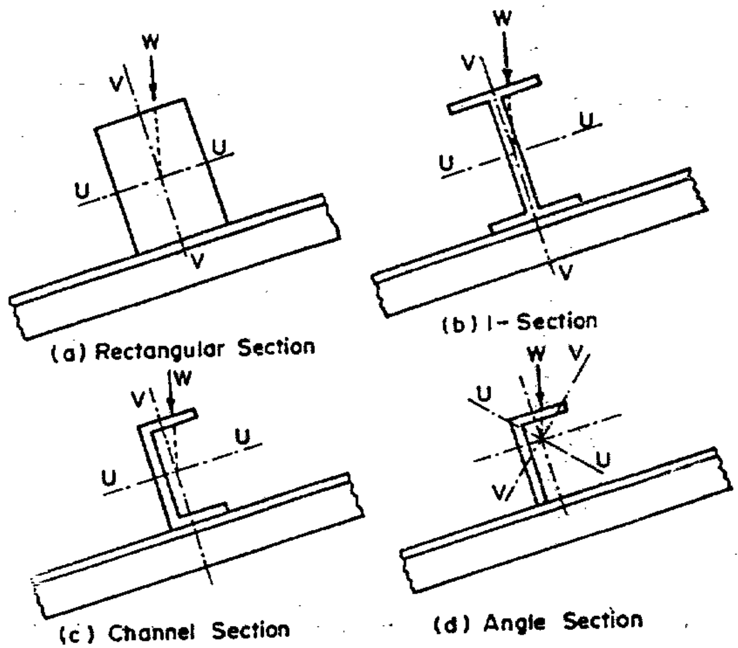


Fig. 21·1. Unsymmetrical bending.

section of the beam is perpendicular to the plane of loading. This condition implies that the plane of loading, or the plane of bending, is coincident with, or parallel to, plain containing a *principal centroidal axis* of inertia of the cross-section of the beam. If, however, the plane of loading or that of bending, does not lie in (or parallel to) a plane that contains the principal centroidal axes of the cross-section, the bending is called *unsymmetrical bending*. Fig. 21·1 shows some cases of unsymmetrical bending in which the plane of load W is vertical and do not coincide with the principal centroidal axes UU and VV. In the case of unsymmetrical bending, the direction of the neutral axis will not be perpendicular to the plane of bending.

21·2. CENTROIDAL PRINCIPAL AXES OF A SECTION

The centroidal principal axes of a section are defined as a pair of rectangular axes through the centre of gravity of a plane area such that the *product of inertia* is zero.

Let U-U, V-V=Principal centroidal axes

X-X, Y-Y=Any pair of centroidal rectangular axes

α=angle between U-U and X-X axes (Fig. 21·2).

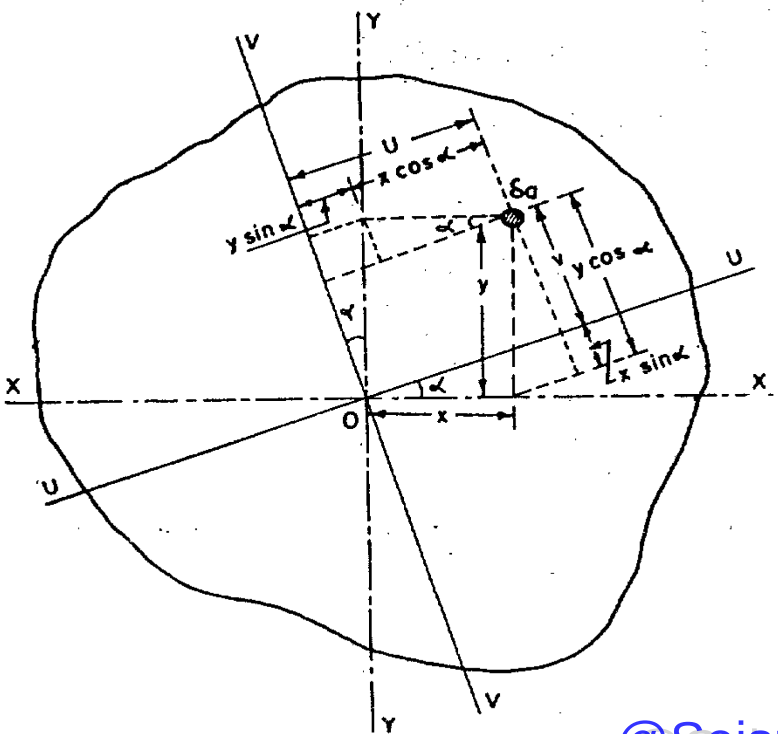


Fig. 21·2. Principal axes.

If U-U, V-V are the principal axes, the product of inertia $\Sigma u.v.\delta a=0$, where δa is an elementary area with co-ordinates u and v referred to the principal axes. If a plane area has an axis of symmetry, it is obviously a principal axis, since the axis of symmetry has to satisfy the condition $\Sigma uv\,\delta a=0$ about it. In general, however, a plane area may not have any axis of symmetry. In that case the principal axes may be located provided its properties about any pair of rectangular axes X-X, Y-Y are known.

Let x, y be the co-ordinates of an elementary area δa, with respect to the X-Y axes, and u, v be the corresponding co-ordinates with respect to the principal axes U-V.

By definition, $I_{XX}=\Sigma y^2\,\delta a$; $I_{YY}=\Sigma x^2 \delta a$; $I_{XY}=\Sigma\, xy\,\delta a$

Similarly, $I_{UU}=\Sigma\, v^2\,\delta a$; $I_{VV}=\Sigma\, u^2\,\delta a$; $I_{UV}=\Sigma\, uv\,\delta a$

The relationships between x, y and u, v co-ordinates are

$$u=x\cos\alpha+y\sin\alpha$$

and
$$v=y\cos\alpha-x\sin\alpha.$$

Hence
$$I_{UU}=\Sigma\, v^2.\delta a=\Sigma\,(y\cos\alpha-x\sin\alpha)^2\,\delta\alpha$$
$$=\cos^2\alpha\,\Sigma y^2\,\delta a+\sin^2\alpha\,\Sigma x^2\,\delta a$$
$$-2\sin\alpha\cos\alpha\,\Sigma xy\,\delta a$$
$$=I_{XX}\cos^2\alpha+I_{YY}\sin^2\alpha-I_{XY}\sin 2\alpha \qquad ...(21·1)$$

$$I_{VV}=\Sigma u^2\,\delta a=\Sigma(x\cos\alpha+y\sin\alpha)^2\,\delta a$$
$$=\sin^2\alpha\,\Sigma y^2\,\delta a+\cos^2\alpha\,\Sigma x^2\,\delta x$$
$$+2\sin\alpha\cos\alpha\,\Sigma xy\,\delta a$$
$$=I_{XX}\sin^2\alpha+I_{YY}\cos^2\alpha+I_{XY}\sin 2\alpha \qquad ...(21·2)$$

and
$$I_{UV}=\Sigma uv\,\delta a=\Sigma(x\cos\alpha+y\sin\alpha)(y\cos\alpha-x\sin\alpha)\,\delta a$$
$$=\cos^2\alpha\,\Sigma xy\,\delta a-\sin^2\alpha\,\Sigma xy\,\delta a$$
$$+\sin\alpha\cos\alpha\,\{\Sigma y^2\,\delta a-\Sigma x^2\,\delta a\}$$
$$=\cos^2\alpha.I_{XY}-\sin^2\alpha.I_{XY}+\sin\alpha\cos\alpha\,\{I_{XX}-I_{YY}\}$$
$$=\left(\frac{I_{XX}-I_{YY}}{2}\right)\sin 2\alpha+I_{XY}\cos 2\alpha \qquad ...(21·3)$$

Since U-U and V-V are the principal axes

$$I_{UV}=0=\left(\frac{I_{XX}-I_{YY}}{2}\right)\sin 2\alpha+I_{XY}\cos 2\alpha$$

or
$$\tan 2\alpha=-\frac{2I_{XY}}{I_{XX}-I_{YY}} \qquad ...(21·4)$$

Knowing I_{XX}, I_{YY} and I_{XY}, the angle α can be calculated from Eq. 21·4.

Substituting α in Eqs. 21·1 and 21·2, the moment of inertia about the principal axes can be determined.

Analytical Solution

Analytical expressions for I_{UU} and I_{VV} can be derived by rewriting Eqs. 21·1 and 21·2 in the following alternative forms :

$$I_{UU}=\frac{I_{XX}+I_{YY}}{2}+\frac{I_{XX}-I_{YY}}{2}\cos 2\alpha-I_{XY}\sin 2\alpha \qquad ...[21·1\ (a)]$$

and $$I_{VV}=\frac{I_{XX}+I_{YY}}{2}-\frac{I_{XX}-I_{YY}}{2}\cos 2\alpha+I_{XY}\sin 2\alpha \qquad ...[21·2\ (a)]$$

Also, from Eq. 21·4, we have

$$\sin 2\alpha=\frac{-I_{XY}}{\sqrt{\left(\frac{I_{XX}-I_{YY}}{2}\right)^2+I_{XY}^2}} \qquad ...[21·4\ (a)]$$

and $$\cos 2\alpha=\frac{I_{XX}-I_{YY}}{\sqrt{\left(\frac{I_{XX}-I_{YY}}{2}\right)^2+I_{XY}^2}} \qquad ...[21·4\ (b)]$$

Substituting the values of sin 2α and cos 2α in Eqs. 21·1 (*a*) and 21·2 (*b*), we get the following final expression for I_{UU} and I_{VV}

$$I_{UU}=\frac{I_{XX}+I_{YY}}{2}+\sqrt{\left(\frac{I_{XX}-I_{YY}}{2}\right)^2+I_{XY}^2} \qquad ...(21·5)$$

$$I_{VV}=\frac{I_{XX}+I_{YY}}{2}-\sqrt{\left(\frac{I_{XX}-I_{YY}}{2}\right)^2+I_{XY}^2} \qquad ...(21·6)$$

Thus knowing I_{XX}, I_{YY} and I_{XY}, the principal moments of inertia I_{UU} and I_{VV} can be calculated from the above analytical expressions. It should be noted that the moments of inertia of a section about its principal axes have maximum and minimum values respectively.

21·3. GRAPHICAL METHOD FOR LOCATING PRINCIPAL AXES

Eqs. 21·5 and 21·6 can also be solved by the following graphical methods :

(1) Mohr-circle

(2) Circle of inertia.

1. MOHR CIRCLE

A close inspection of Eqs. 21·5 and 21·6 would reveal that the expressions for I_{UU} and I_{VV} are similar (or analogous) to the following well known expressions for the principal stresses :

$$\sigma_1=\frac{\sigma_X+\sigma_Y}{2}+\sqrt{\left(\frac{\sigma_X-\sigma_Y}{2}\right)^2+\tau^2}$$

and $$\sigma_2=\frac{\sigma_X+\sigma_Y}{2}-\sqrt{\left(\frac{\sigma_X-\sigma_Y}{2}\right)^2+\tau^2}$$

$$\tan\theta=\frac{2\tau}{\sigma_X-\sigma_Y}.$$

Hence I_{UU} and I_{VV}, represented by Eqs. 21·5 and 21·6 can be determined by a Mohr-circle of construction—similar to the Mohr stress circle employed for the determination of the principal stresses.

Fig. 21·3 shows the Mohr-circle construction for principal axes and principal moment of inertia, wherein :

$OA=I_{XX}$; $OB=I_{YY}$

$AD=-I_{XY}$ or $BD'=+I_{XY}$.

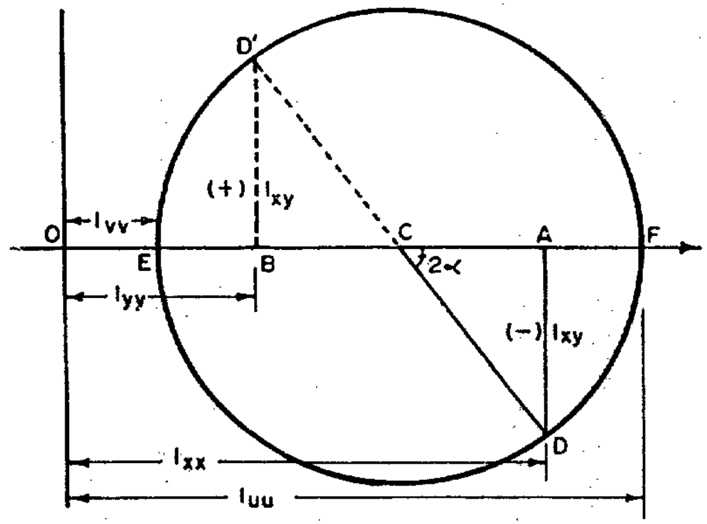


Fig. 21·3. Mohr-circle construction for principal axes.

Bisect *AB* at *C*. With *C* as centre and *CD* (or *CD'*) as radius, draw the circle cutting the horizontal axis *OC* at *E* and *F*. Then $\angle ACD=2\alpha$ and *OE* and *OF* represent the minimum and maximum principal moment of inertia.

Proof. From Fig. 21·3, we have

$$OC=\frac{1}{2}\left(I_{XX}+I_{YY}\right)$$

$$CA=\frac{1}{2}\left(I_{XX}-I_{YY}\right)$$

$$\therefore\quad OE=I_{VV}=OC-EC=OC-D'C$$

$$=\frac{1}{2}\left(I_{XX}+I_{YY}\right)-\sqrt{\left(\frac{I_{XX}-I_{YY}}{2}\right)^2+I_{XY}^2}$$

Similarly,

$$OF=I_{UU}=OC+CF=OC+CD$$

$$=\frac{1}{2}\left({}_{XX}+I_{YY}\right)+\sqrt{\left(\frac{I_{XX}-I_{YY}}{2}\right)^2+I_{XY}^2}.$$

These are the same as Eqs. 21·5 and 21·6 respectively. It should be noted that I_{XY} is plotted below the line OA if it is negative (such as line AD), and plotted above the line OA if it is positive. If I_{XY} is negative, the X-X axis makes an angle $2\alpha=\angle ACD$ with U-U axis in the *clockwise direction* (or U-U axis is inclined at 2α with X-X axis in anticlockwise direction). Similarly, if I_{XY} is positive, $\angle BCD'=2\alpha$ is the angle with the U-U axis makes with the X-X axis in the clockwise direction. In general, therefore, the direction of the principal axis is given by the angle *measured from* the inclined radial line (such as CD or CD') *towards* the horizontal line CF or CB, as the case may be.

2. CIRCLE OF INERTIA

An alternative graphical method to determine I_{UU} and I_{VV} is to construct what is commonly known as '*circle of inertia*', '*dyadic circle*' or '*Mohr-Land construction*' (Fig. 21·4).

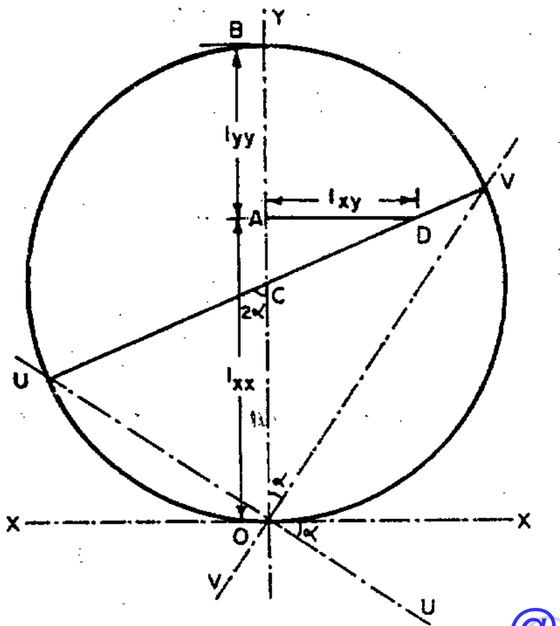


Fig. 21·4. Circle of Inertia (Dyadic Circle).

Let O be the centroid of the section and X-X, Y-Y be any set of rectangular axes passing through it. Make $OA=I_{XX}$ and $AB=I_{YY}$. Draw a circle with BO as the diameter. Hence $BC=CO=\frac{1}{2}(I_{XX}+I_{YY})$ where C is the centre of the circle. At A, erect perpendicular $AD=I_{XY}$ to the right if I_{XY} is positive, or to the left if I_{XY} is negative. Join C and D and prolong it to meet the circle in U and V. Join OU and OV. Then OU is the U-axis and OV is the V-axis. Also, $UD=I_{UU}$ and $DV=I_{VV}$.

Proof. $CA=CB-AB=\frac{1}{2}\left(I_{XX}+I_{YY}\right)-I_{XY}=\frac{1}{2}\left(I_{XX}-I_{YY}\right)$

$$AD=I_{XY}$$

$$\therefore \quad CD=\sqrt{\left(\frac{I_{XX}-I_{YY}}{2}\right)^2+I_{XY}^2}$$

$$\therefore \quad UD=UC+CD$$

or $$UD=\frac{I_{XX}+I_{YY}}{2}+\sqrt{\left(\frac{I_{XX}-I_{YY}}{2}\right)^2+I_{XY}^2}=I_{UU}$$

and $$DV=CV-CD$$

or $$DV=\frac{I_{XX}+I_{YY}}{2}-\sqrt{\left(\frac{I_{XX}-I_{YY}}{2}\right)^2+I_{XY}^2}=I_{VV}$$

Also, $\tan 2\alpha=\frac{AD}{AC}=\frac{I_{XY}}{\frac{1}{2}(I_{XX}-I_{YY})}=\frac{2I_{XY}}{I_{XX}-I_{YY}}$ (numerically).

21·4. MOMENTS OF INERTIA REFERRED TO ANY SET OF RECTANGULAR AXES

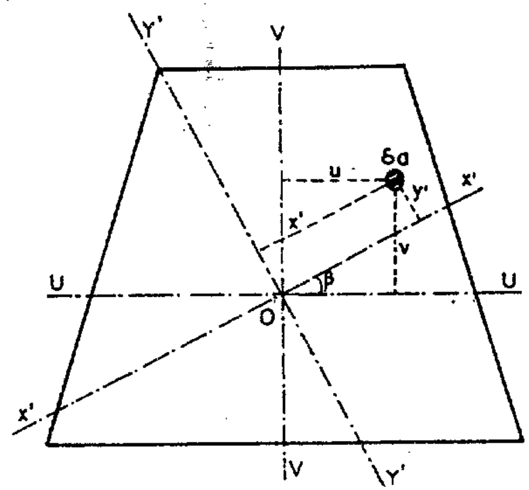


Fig. 21·5

The discussions of § 21·2 and § 21·3 can now be generalised to find the moments of inertia referred to any set of rectangular axes X'-Y' inclined at β to the principal centroidal axes. Refer Fig. 21·5. Consider an elementary area δa. Let its co-ordinates be u, v with respect to U-V axes and x'-y' with respect to X'-Y' axes.

Then $\quad x'=u\cos\beta+v\sin\beta$

$\qquad\quad y'=v\cos\beta-u\sin\beta$

Now $\quad I_{X'X'}=\Sigma y'^2\,\delta a=\Sigma(v\cos\beta-u\sin\beta)^2\delta a$

$$=\cos^2\beta\,\Sigma v^2\delta a+\sin^2\beta\,\Sigma u^2\delta a-2\sin\beta\cos\beta\,\Sigma uv\,\delta a$$

$$=I_{UU}\cos^2\beta+I_{VV}\sin^2\beta \qquad \ldots(21·7)$$

(Since $\quad I_{UV}=\Sigma uv\,\delta a=0$)

Similarly, $\quad I_{Y'Y'}=\Sigma x'^2\,\delta a=\Sigma(u\cos\beta+v\sin\beta)^2\,\delta a$

$$=\cos^2\beta\,\Sigma u^2\,\delta a+\sin^2\beta\,\Sigma v^2\,\delta a+2\sin\beta\cos\beta\,uv\,\delta a$$

$$=I_{UU}\sin^2\beta+I_{VV}\cos^2\beta \qquad \ldots(21·8)$$

Adding Eqs. 21·7 and 21·8, we get

$$I_{X'X'}+I_{Y'Y'}=I_{UU}+I_{VV}$$

Also, from adding Eqs. 21·1 and 21·2, we get

$$I_{UU}+I_{VV}=I_{XX}+I_{YY}$$

Hence $I_{XX}+I_{YY}=I_{X'X'}+I_{Y'Y'}=I_{UU}+I_{VV} \qquad \ldots(21·9)$

Thus the sum of moments of inertia about any set of rectangular axes is constant.

Example 21·1. *Determine the principal moments of inertia for an unequal angle section $60\times40\times6$ mm shown in Fig. 21·6.*

Solution.

Let O be the centroid of the section. Let the X-axis be at a distance C_X from face PQ, and Y-axis be at a distance C_Y from face PR.

Area $\quad A=A_1+A_2=(40\times6)+(54\times6)$

$$=240+324=564\text{ mm}^2$$

$$C_X=\frac{(40\times6\times3)+(54\times6\times33)}{564}=20·2\text{ mm}$$

$$C_Y=\frac{(40\times6\times20)+(54\times6\times3)}{564}=10·2\text{ mm}$$

$$I_{PQ}=(\tfrac13\times6\times60^3)+(\tfrac13\times34\times6^3)=43·44\times10^4\text{ mm}^4$$

$$I_{XX}=I_{PQ}-A.C_X^2=43·44\times10^4-564(20·2)^2$$

$$=20·34\times10^4\text{ mm}^4$$

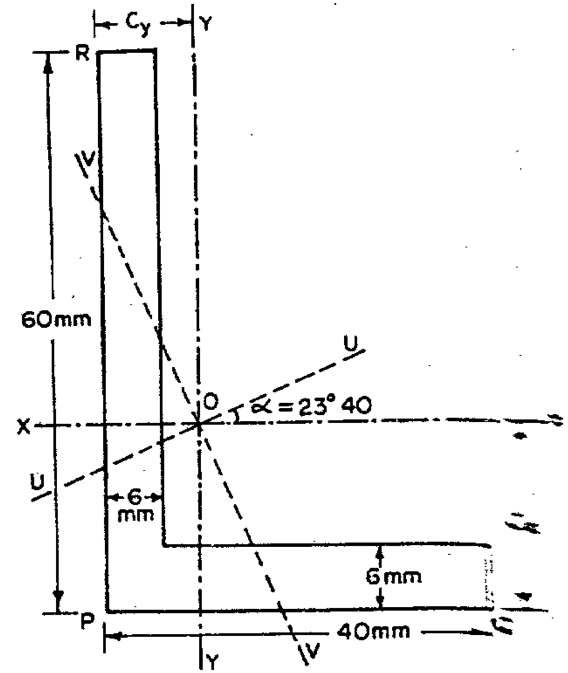


Fig. 21·6

$$I_{PR}=(\tfrac13\times54\times6^3)+(\tfrac13\times6\times40^3)=\text{[illegible]}$$

$$I_{YY}=I_{PR}-A.C_Y^2=13·19\times10^4-\text{[illegible]}$$

$$=7·33\times10^4\text{ mm}^4$$

and $\qquad I_{XY}=A_1.x_1.y_1+A_2.x_2.y_2$

where (x_1, y_1) are the co-ordinates of C.G. of area [illegible]

and $\quad (x_2, y_2)$ are the co-ordinates of C.G. of area [illegible]

$\therefore \qquad I_{XY}=\{240(20-10·2)(3-20·2)\}$

$$+\{324\text{[illegible]}\}$$

$$=-4·05\times10^4-2·99\times10^4=\text{[illegible]}$$

From Fig. 21·4, the positions of principal [illegible]

$$\tan 2\alpha=-\frac{2I_{XY}}{I_{XX}-I_{YY}}$$

$$=\frac{2\times7·04\times10^4}{(20·34-7·33)10^4}=1·085$$

$\therefore \qquad 2\alpha=47°\ 20'$

or $\qquad \alpha=23°\ 40'$ (anticlockwise)

$$\frac{I_{XX}+I_{YY}}{2}=\frac{(20{\cdot}34+7{\cdot}33)10^4}{2}=13{\cdot}84\times10^4$$

$$\frac{I_{XX}-I_{YY}}{2}=\frac{(20{\cdot}34-7{\cdot}33)10^4}{2}=6{\cdot}5\times10^4$$

$$\sqrt{\left(\frac{I_{XX}-I_{YY}}{2}\right)^2-I_{XY}^2}=\sqrt{(6{\cdot}5\times10^4)^2+(-7{\cdot}04\times10^4)^2}$$
$$=9{\cdot}58\times10^4$$

Hence from Eqs. 21·5 and 21·6,

$$I_{UU}=13{\cdot}84\times10^4+9{\cdot}58\times10^4=23{\cdot}42\times10^4 \text{ mm}^4$$
$$I_{VV}=13{\cdot}84\times10^4-9{\cdot}58\times10^4=4{\cdot}26\times10^4 \text{ mm}^4.$$

Check

$$I_{XX}+I_{YY}=I_{UU}+I_{VV}$$
$$(20{\cdot}34\times10^4+7{\cdot}33\times10^4)=(23{\cdot}42\times10^4+4{\cdot}26\times10^4)$$
$$27{\cdot}67\times10^4\simeq27{\cdot}68\times10^4.$$

Example 21·2. *Find I_{UU} and I_{VV} graphically by Mohr-circle method, for the data of previous problem.*

Solution.

$I_{XX}=20{\cdot}34\times10^4$ mm^4 ; $I_{YY}=7{\cdot}33\times10^4$ mm^4 ; $I_{XY}=-7{\cdot}04\times10^4$ mm^4 as calculated in the previous problem.

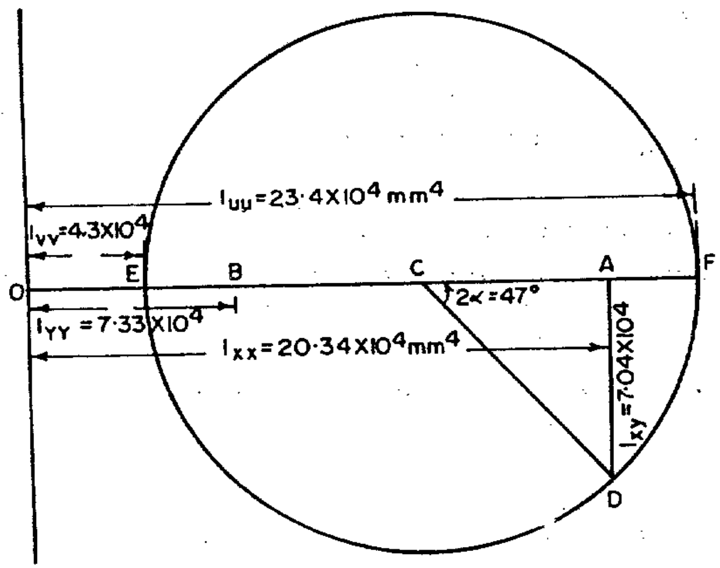


Fig. 21·7

Make $OB=I_{YY}=7{\cdot}33\times10^4$ mm^4 ; $OA=I_{XX}=20{\cdot}34\times10^4$ mm^4.

Plot $AD=-7{\cdot}04\times10^4$ downward. Find C midway between A and B. Join CD. With C as centre and CD at radius, draw a circle cutting the abscissa at E and F.

Then $OF=I_{UU}=23{\cdot}4\times10^4$ mm^4
$OE=I_{VV}=4{\cdot}3\times10^4$ mm^4
$2\alpha=47°$
$\alpha=23°\ 30'$ (anticlockwise).

Example 21·3. *Determine I_{UU} and I_{VV} graphically using the dyadic circle method.*

Solution. (Fig. 21·8)

Let O be the centroid of the section.

Make $OA=I_{XX}=20{\cdot}34\times10^4$ and $AB=I_{YY}=7{\cdot}33\times10^4$ mm^4. At A, erect perpendicular $AD=I_{XY}=7{\cdot}04\times10^4$ to the left side. From point C as centre and CB as radius, draw the dyadic circle. Join CD and prolong it to both the sides, cutting the circle in U and V. Then, by measurement,

$UD=I_{UU}=23{\cdot}4\times10^4$ mm^4 ; $DV=I_{VV}=4{\cdot}3\times10^4$ mm^4
and $\alpha=\angle XOU=23°\ 40'$.

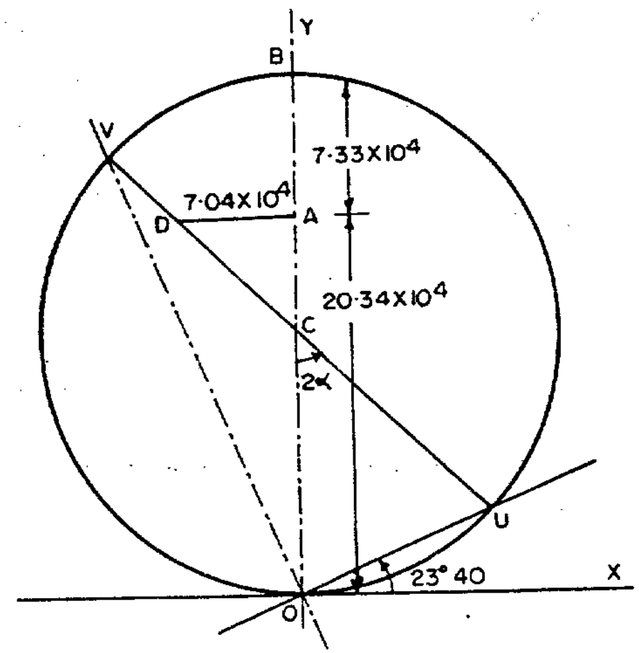


Fig. 21·8

21·5. BENDING STRESS IN BEAM SUBJECTED TO UNSYMMETRICAL BENDING

In the case of simple bending, where the plane of loading (or bending) coincides with one of the principal plane, the neutral axis is perpendicular to the principal plane and passes through the centroid of the section. In the case of unsymmetrical bending, neutral axis is not perpendicular to the plane of bending. The bending stress at any point in the beam subjected to unsymmetrical bending can be determined by following methods :

1. Resolution of bending moment into two components along principal axes.
2. Resolution of bending moment into two components along any rectangular axes through the centroid.
3. Locating neutral axis of the section.

21·6. RESOLUTION OF BENDING MOMENT INTO TWO COMPONENTS ALONG PRINCIPAL AXES

Let the plane of bending (M) be inclined at an angle θ with one of the pincipal planes. The bending moment M can be resolved

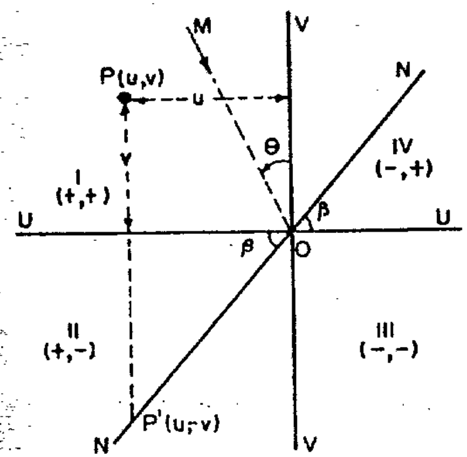


Fig. 21·9. Resolution of bending moment into two components along principal axes.

into components : $M \cos\theta$ along plane V-V and $M \sin\theta$ along the plane U-U. Having resolved the bending moments in the two components the simple theory of bending can then be applied to bending

in occurring in the principal planes. The intensity of bending stress at any point $P(u, v)$ will be the algebraic sum of the stress due to the component bending moments.

For the component $M \cos\theta$, bending takes place about V-V for which U-U becomes the neutral axis. For the component $M \sin\theta$, bending takes place about U-U for which V-V becomes the neutral axis. Hence the final bending stress at P is

$$f_b = \frac{M\cos\theta}{I_{UU}} \cdot v + \frac{M\sin\theta}{I_{VV}} \cdot u \qquad \ldots(21{\cdot}10)$$

The co-ordinates u and v will be positive in that quadrant of U-V planes in which bending moment is applied. Thus, u is positive in quadrants I and II, while v is positive in quadrants I and IV. Since the co-ordinates of any extreme point of section are known, the bending stress can be calculated. It should be noted that the component $M \cos\theta$ causes compression for all points above U-U axis and tension at points below U-U axis. Similarly, $M \sin\theta$ causes compression for points to the left of V-V axis and tension for the points to the right of V-V axis. Hence the points of quadrant I are subjected to a resultant bending stress which is wholly compressive, while those in quadrant III to wholly tensile stress. Angle θ is taken to be positive when measured in an anticlockwise direction with the +ve V-axis. The method is suitable to those sections which have at least one axis of symmetry which is also of the principal axes. In such a case, the points at extreme distances from the principal axis can be located by visual inspection.

21·7. RESOLUTION OF B.M. INTO ANY TWO RECTANGULAR AXES THROUGH THE CENTROID

The most general method of finding the bending stress at any point is to resolve it along any two rectangular axes passing through the centroid of the section. Let X-X and Y-Y be the centroidal axes (Fig. 21·10).

The resolved component of M *along* the Y-Y axis (also called as the moment *about* X-X axis) is designed as M_{XX} and is equal to $M \cos\theta$. Similarly, the resolved component of M *along* X-X axis (also called the moment *about* y-y axis) is designated as M_{YY} and is equal to $M \sin \theta$.

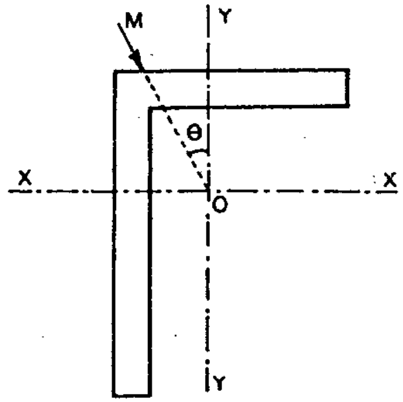


Fig. 21·10. Resolution of M into any two rectangular axes through the centroid.

The bending stress f_b at any point $P(x, y)$ can be expressed as

$$f_b = a_1 x + b_1 y \qquad \ldots(21{\cdot}11)$$

Since the variation of bending stress is linear.

a_1 and b_1 are constants, to be determined.

Now, $\quad M_{XX}$ = bending moment about X-axis

$$= \int f_b \delta a . y = \int (a_1 x + b_1 y) y \delta a$$

$$= a_1 \int xy \, \delta a + b_1 \int y^2 \, \delta a$$

$$= a_1 \, I_{XY} + b_1 \, I_{XX} \qquad \ldots(1)$$

M_{XY} = bending moment about y-axis

$$= \int f_b \, \delta a . x = \int (a_1 x + b_1 y) x \, \delta a$$

$$= a_1 \int x^2 \, \delta a + b_1 \int xy \, \delta a$$

$$= a_1 . I_{YY} + b_1 . I_{XY} \qquad \ldots(2)$$

Solving (1) and (2), we get

$$a_1 = \frac{M_{YY} . I_{XX} - M_{XX} . I_{XY}}{I_{XX} . I_{YY} - I_{XY}^2} \qquad \ldots(3)$$

and $$b_1 = \frac{M_{XX} . I_{YY} - M_{YY} . I_{XY}}{I_{XX} . I_{YY} - I_{XY}^2} \qquad \ldots(4)$$

Substituting the values of a_1 and b_1 in Eq. 21·11, we get

$$f_b = \frac{M_{YY} \, I_{XX} - M_{XX} \, I_{XY}}{I_{XX} I_{YY} - I_{XY}^2} \, . \, x + \frac{M_{XX} I_{YY} - M_{YY} . I_{XY}}{I_{XX} I_{YY} - I_{XY}^2} \, . \, y \qquad \ldots(21{\cdot}12)$$

Thus the bending stress f_b can be calculated at any point whose co-ordinates (x, y) are known. The method is specially suitable for sections in which the web and flanges are parallel to x-x and y-y axes.

21·8. LOCATION OF NEUTRAL AXIS

As stated earlier in the case of unsysmmetrical bending, the neutral axis is neither perpendicular to the plane of bending, nor perpendicular to any of the principal planes.

Let θ = Inclination of the plane of bending to the V-V axis.

β = Inclination of the neutral axis, with the U-U axis.

The neutral axis can be located by two methods :

1. Analytical method
2. Graphical method : Momental ellipse.

Analytical Method

Fig. 21·9 shows the neutral axis N-N, inclined at an angle β with the U-U axis. At any point (such as P') on it, the bending stress is equal to zero. Hence equating Eq. 21·10 to zero, we get

$$f_b = 0 = \frac{M \cos \theta}{I_{UU}} \, v + \frac{M \sin \theta}{I_{VV}} \, u$$

$$\therefore \qquad v = -u \, \frac{I_{UU}}{I_{VV}} \tan \theta \qquad \ldots(21{\cdot}13)$$

Eq. 21·13 is the equation of the neutral axis N-N which is a straight line. It is clear that when $v=0$, $u=0$; hence the neutral axis passes through the centroid of the section.

From Fig. 21·9, $\tan \beta = \dfrac{v}{u}$

But $\qquad -\dfrac{v}{u} = \dfrac{I_{UU}}{I_{VV}} \tan \theta$, from Eq. 21·13.

Hence $\qquad \tan \beta = \dfrac{I_{UU}}{I_{VV}} \tan \theta \qquad \ldots(21{\cdot}14)$

Thus the N.A. can be located from Eq. 21·14.

Let $\qquad I_{NN}$ = moment of inertia of the beam about the neutral axis.

Thus, from Eq. 21·7, treating x'-x' axis as the neutral axis, we have

$$I_{NN}=I_{UU}\cos^2\beta+I_{VV}\sin^2\beta \qquad \ldots(21\cdot15)$$

The neutral axis is inclined at β with the U-axis, while the plane of loading is inclined at θ with the V-axis. Hence, the plane of loading is inclined at angle $(90-\theta+\beta)$ with the neutral axis. If a line is drawn perpendicular to the neutral axis, the plane of bending will be inclined at $(\beta-\theta)$ to the line. Hence the component of bending moment M *along* the axis will be given by

$$M_{NN}=M\cos(\beta-\theta) \qquad \ldots(21\cdot16)$$

where M_{NN} = component of bending moment *along* a line perpendicular to the neutral axis

= bending moment *about* the neutral axis.

If y_N = perpendicular distance of any point from the neutral axis, we have

$$f_b=\frac{M\cos(\beta-\theta)}{I_{NN}}.y_N \qquad \ldots(21\cdot17)$$

The bending stress f_b will be positive or negative depending upon the position of the point relative to the neutral axis and the direction of bending.

21·9. GRAPHICAL METHOD : MOMENTAL ELLIPSE

From Eq. 21·9, we have

$$I_{X'X'}+I_{Y'Y'}=I_{UU}+I_{VV}$$

This may be written in terms of the radii of gyration as under :

$$k_{X'X'}^2+k_{Y'Y'}^2=k_{UU}^2+k_{VV}^2 \qquad \ldots(21\cdot18)$$

If k_{UU} and k_{VV} are known, $k_{X'X'}$ and $k_{Y'Y'}$ can be determined graphically by the construction of *momental ellipse or ellipse of inertia.* Refer Fig. 21·11.

Set off the principal axes UU and VV through the centroid O of the section. Draw the inner circle with radius $OA=k_{VV}$ (assuming $k_{VV}<k_{UU}$) and outer circle with radius $OB=k_{UU}$. Set off axes OX' and OY' at inclination θ with OU and OV respectively. OX' cuts the circles at C and D. Through C and D, draw lines CP and DP parallel to OV and OU respectively, meeting in a point P which is a point of an ellipse. Change the value of θ (by rotation of axis X'

and Y') and get a number of such points P. Join them to get an ellipse. From Fig. 21·11 (*a*),

$$OP^2=OE^2+EP^2.$$

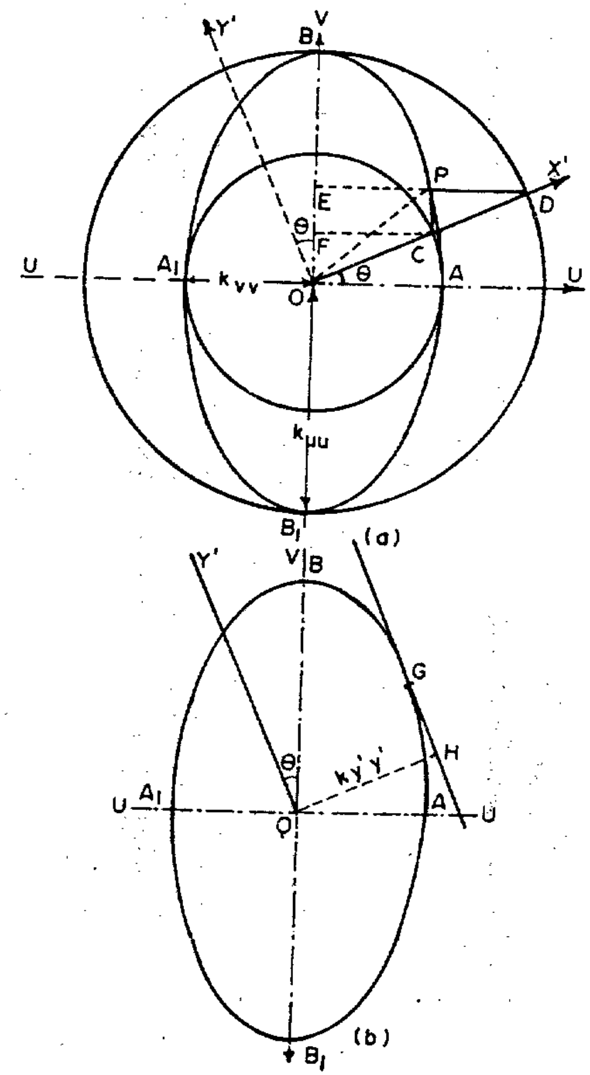


Fig. 21·11. Momental ellipse.

But $\qquad OE=OD\sin\theta=k_{UU}\sin\theta$

and $\qquad EP=CF=OC\cos\theta=k_{VV}\cos\theta$

$\therefore \qquad OP^2=k_{UU}^2.\sin^2\theta+k_{VV}^2\cos^2\theta \qquad \ldots(1)$

But, from Eq. 21·7,

$$I_{Y'Y'} = I_{UU}\sin^2\theta + I_{VV}\cos^2\theta$$

or $$k_{Y'Y'}^2 = k_{UU}^2\sin^2\theta + k_{VV}^2\cos^2\theta \quad \ldots(2) \quad \ldots(21\cdot19)$$

Comparing (1) and (2), we find

$$OP = k_{Y'Y'}$$

Again, the co-ordinates of points P are

$$u = EP = k_{VV}\cos\theta \quad \ldots(3)$$

$$v = OE = k_{UU}\sin\theta \quad \ldots(4)$$

From (3) and (4), we get

$$\frac{u^2}{k_{VV}^2} + \frac{v^2}{k_{UU}^2} = \cos^2\theta + \sin^2\theta = 1 \quad \ldots(21\cdot20)$$

This is the equation of an ellipse having k_{UU} and k_{VV} as its semi-major and semi-minor axes.

Fig. 21·11 (*b*) shows the complete ellipse. In order to find graphically the value of $k_{Y'Y'}$ corresponding to any value θ, the axis OY' is set off at θ with the OV axis. A tangent GH is then drawn to the ellipse, parallel to the OY' axis. A line OH is drawn *perpendicular* to the tangent. It can be shown that OH is equal to $k_{Y'Y'}$.

Proof. Let m = slope of line OY' with OU

$$= \tan(90+\theta) = -\cot\theta$$

If the equation of an ellipse is $\dfrac{u^2}{a^2} + \dfrac{v^2}{b^2} = 1$, then the equation to its tangent, with a slope m, is given by

$$v = mu \pm \sqrt{b^2 + a^2m^2}$$

For the present case, $m = -\cot\theta$; $b = k_{UU}$ and $a = k_{VV}$

$$\therefore \quad v = -u\cot\theta \pm \sqrt{k_{UU}^2 + k_{VV}^2\cot^2\theta} \quad \ldots(21\cdot21)$$

The perpendicular distance OH from O to this tangent is given by

$$OH = \frac{\pm\sqrt{b^2 + a^2m^2}}{\sqrt{1+m^2}}$$

$$= \frac{\pm\sqrt{k_{UU}^2 + k_{VV}^2\cot^2\theta}}{\sqrt{1+\cot^2\theta}}$$

$$\therefore \quad OH^2 = \frac{k_{UU}^2 + k_{VV}^2\cot^2\theta}{\operatorname{cosec}^2\theta} = k_{UU}^2\sin^2\theta + k_{VV}^2\cos^2\theta$$

But $$k_{UU}^2\sin^2\theta + k_{VV}^2\cos^2\theta = k_{Y'Y'}^2 \quad \ldots(21\cdot19)$$

$$OH = k_{Y'Y'}.$$

Hence, we draw a very important conclusion : *To find the radius of gyration about any axis, draw a tangent to the momental ellipse, in a direction parallel to that axis. Then the perpendicular distance between the tangent and the origin gives the required radius of gyration.*

The above conclusion will now be utilized to find the radius of gyration k_{NN} about the neutral axis N-N. To do this, we must first locate the position of the neutral axis.

Fig. 21·12 shows the momental ellipse. Let OM represent the plane of loading inclined at an angle θ to the OV axis. Let ON be

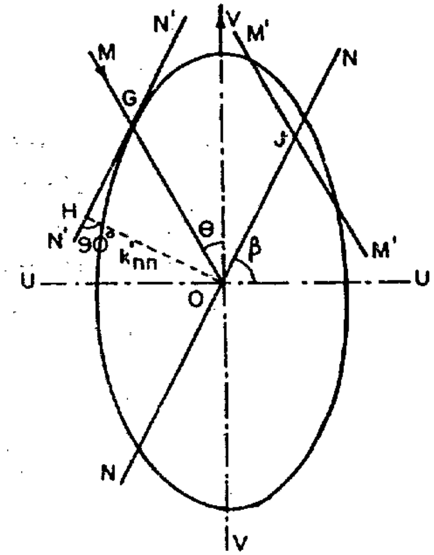


Fig. 21·12. Determination of k_{NN}.

the direction of the neutral axis, inclined at β to the OU axis. It is first required to determine the position of the neutral axis ON graphically.

From Eq. 21·14, we have

$$\tan\beta = \frac{I_{UU}}{I_{VV}}\tan\theta$$

$$\therefore \quad \tan\beta = \frac{k_{UU}^2}{k_{VV}^2}\tan\theta \quad \ldots(21\cdot22)$$

or $$\tan\beta\cot\theta = \frac{k_{UU}^2}{k_{VV}^2} \quad \ldots[21\cdot22\ (a)]$$

Now the slope m of the line $ON = \tan\beta$.

Slope m' of the line $OM = \tan(90+\theta) = -\cot\theta$

$\therefore \qquad mm' = -\tan\beta \cot\theta$

But $\qquad \tan\beta\cot\theta = \dfrac{k_{UU}^2}{k_{VV}^2}$; from Eq. 21·22 (*a*).

Hence
$$mm' = -\frac{k_{UU}^2}{k_{VV}^2} \qquad \ldots(21{\cdot}23)$$

Eq. 21·23 suggests that the lines OM and ON are the two conjugate diameters of the ellipse. If an ellipse has the equation $\dfrac{u^2}{a^2}+\dfrac{v^2}{b^2}=1$, then, from the property of the ellipse, the product mm of the slopes of any two *conjugate diameters* is given by

$$mm' = -\frac{b^2}{a^2} = -\frac{k_{UU}^2}{k_{VV}^2}$$

Thus it is concluded that the neutral axis is in a direction of a diameter which is conjugate to the diameter in the direction loading.

Again, if the direction of any diameter OM of the ellipse is known, the direction of any diameter conjugate to it can be drawn by drawing any diameter $M'M'$ parallel to OM, bisecting it at J, and joining O and J. Thus OJ prolonged gives the direction ON of the neutral axis.

Having located the neutral axis ON, a line $N'N'$ is drawn tangentially to the ellipse and parallel to the neutral axis ON. The perpendicular OA then gives the radius of gyration k_{NN} about the neutral axis.

Knowing the radius of gyration k_{NN}, the moment of inertia I_{NN} about the neutral axis is calculated from the relation

$$I_{NN} = A.k_{NN}^2 \qquad \ldots(21{\cdot}24)$$

where A = area of cross-section of the section.

The bending stress f_b at any point is then calculated from Eq. 21·17,

$$f_b = \frac{M\cos(\beta-\theta)}{I_{NN}} \cdot y_N.$$

Example 21·4. *A beam of rectangular section, 80 mm wide and 120 mm deep is subjected to a bending moment of 12 kN-m. The trace of the plane of loading is inclined at 45° to the Y-Y axis of the section. Locate the neutral axis of the section and calculate the maximum bending stress induced in the section.*

Solution.

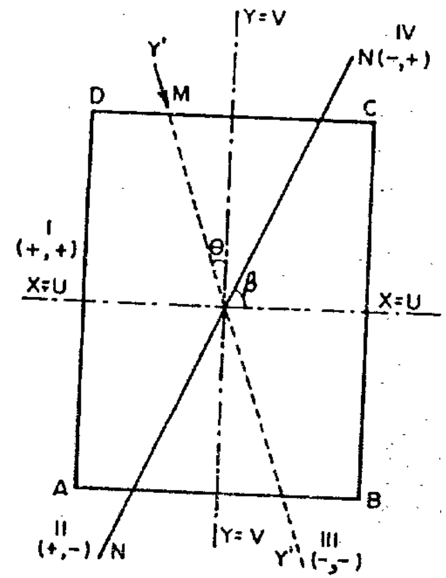


Fig. 21·13

Let the plane of loading (bending) be inclined at an angle θ with a Y-Y axis and the neutral axis NN be inclined at β with the X-X axis.

$\theta = 45°$ (Given)

$M = 12000 \times 1000 = 12 \times 10^6$ N/mm

$$I_{XX} = I_{UU} = \frac{1}{12}bd^3 = \frac{1}{12} \times 80 \times 120^3 = 11{\cdot}52 \times 10^6 \text{ mm}^4$$

$$I_{YY} = I_{VV} = \frac{1}{12}db^3 = \frac{1}{12} \times 120 \times 80^3 = 5{\cdot}12 \times 10^6 \text{ mm}^4$$

From Eq. 29·14, the inclination β of the N.A. is given by

$$\tan\beta = \frac{I_{UU}}{I_{VV}}\tan\theta = \frac{11{\cdot}52\times10^6}{5{\cdot}12\times10^6}\times\tan 45° = 2{\cdot}25.$$

$\therefore \qquad \beta = 66°$

This gives the location of the neutral axis.

By inspection, maximum stress will occur either at B or at D, whichever is more distant from the N.A.

The stress is given by Eq. 21·10

$$f_b=\frac{M\cos\theta}{I_{UU}}\cdot v+\frac{M\sin\theta}{I_{VV}}\ u$$

$$=\frac{M\cos\theta}{I_{XX}}\cdot y+\frac{M\sin\theta}{I_{YY}}\ x$$

where x and y are the coordinates of the point. The coordinates x and y will be positive in that quadrant of X-Y planes in which the bending moment is applied. From this point of view, both the coordinates of point D will be *positive*, while those of point B will be negative.

Thus, for point B, $x=-40$ and $y=-60$

For point D, $x=+40$ and $y=+60$

$$(f_b)_B=-\frac{12\times10^6\cos 45^\circ}{11{\cdot}52\times10^6}\times60-\frac{12\times10^6\sin 45^\circ}{5{\cdot}12\times10^6}\times40$$

$$=-110{\cdot}5\ \text{N/mm}^2$$

(*i.e.* tensile)

$$(f_b)_D=+\frac{12\times10^6\cos 45^\circ}{11{\cdot}52\times10^6}\times60+\frac{12\times10^6\sin 45^\circ}{5{\cdot}13\times10^6}\times40$$

$$=+110{\cdot}5\ \text{N/mm}^2$$

(*i.e.* compressive)

Alternative Solution

$I_{NN}=I_{UU}\cos^2\beta+I_{VV}\sin^2\beta$ (Eq. 21·15)

$$=11{\cdot}52\times10^6\cos^2 66^\circ+5{\cdot}12\times10^6\sin^2 66^\circ$$

$$=1{\cdot}91\times10^6+4{\cdot}28\times10^6=6{\cdot}19\times10^6\ \text{mm}^4$$

Also, $(y_N)_B=x\sin\beta+y\cos\beta$

$$=-40\sin 66^\circ-60\cos 66^\circ=-61\ \text{mm}.$$

Hence from Eq. 21·17,

$$(f_b)_B=\frac{M\cos(\beta-\theta)}{I_{NN}}\cdot y_N$$

$$=-\frac{12\times10^6\cos(66^\circ-45^\circ)}{6{\cdot}19\times10^6}\times61$$

$$=-110{\cdot}4\ \text{N/mm}^2\ (\textit{i.e.}\ \text{tensile}).$$

Example 21·5. *A 60 mm×40 mm×6 mm unequal angle is placed with the longer leg vertical, and is used as a beam. It is subjected to a bending moment of 12 kN-cm acting in the vertical plane through the centroid of the section. Determine the maximum bending stress induced in the section.*

Solution.

We have found the properties of this section in example 21·1. For the position of the angle as shown in Fig. 21·14 (*a*), the various parameters are as follows :

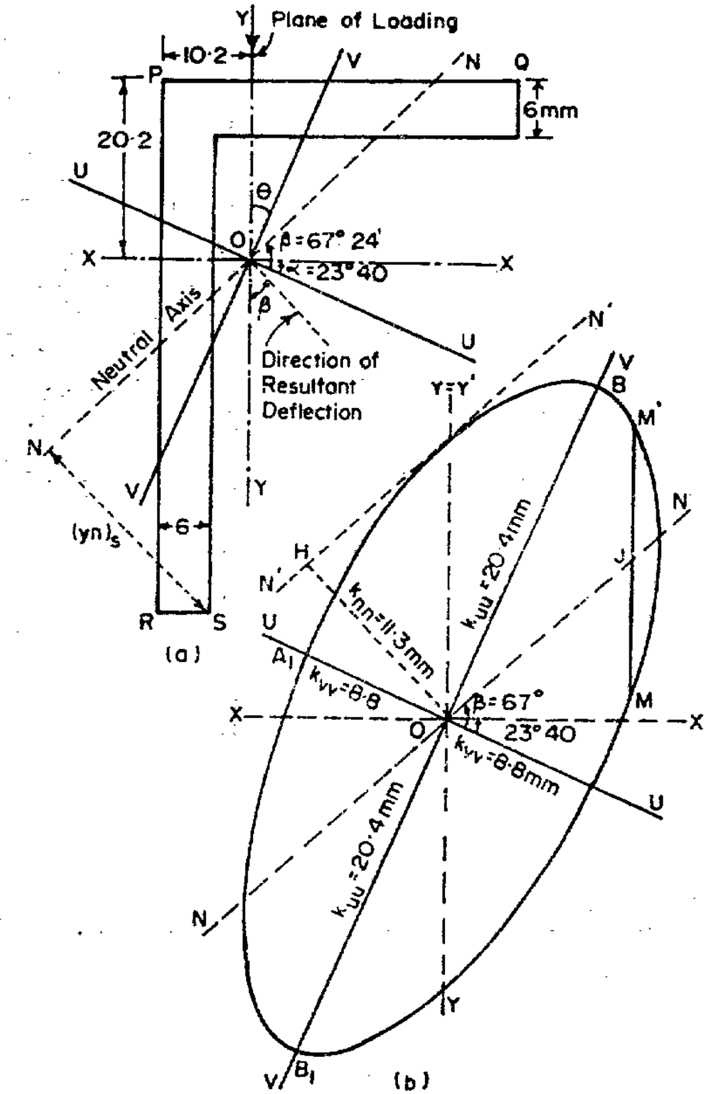


Fig. 21·14

$C_X=20{\cdot}2$ mm ; $C_Y=10{\cdot}2$ mm

$A=564$ mm^2

$I_{XX}=20{\cdot}34\times10^4$ mm^4 ; $I_{YY}=7{\cdot}33\times10^4$ mm^4 ;
$I_{XY}=+7{\cdot}04\times10^4$ mm^4

$I_{UU}=23{\cdot}42\times10^4$ mm^4 ; $I_{VV}=4{\cdot}26\times10^4$ mm^4 ; $\alpha=23°\ 40'$

The plane of loading is vertical. Hence Y' axis and Y axis coincide.

$\therefore\quad \theta=\alpha=23°\ 40'$.

(a) Analytical Solution

The inclination β of the neutral axis N-N with the U-U axis is given by

$$\tan\beta=\frac{I_{UU}}{I_{VV}}\tan\theta=\frac{23{\cdot}42\times10^6}{4{\cdot}26\times10^6}\tan 23°\ 40'=2{\cdot}4$$

$\therefore\quad \beta=67°\ 24'$

$$I_{NN}=I_{UU}\cos^2\beta+I_{VV}\sin^2\beta$$
$$=23{\cdot}42\times10^4\cos^2 67°\ 24'+4{\cdot}26\times10^4\sin^2 67°\ 24'$$
$$=3{\cdot}46\times10^4+3{\cdot}64\times10^4=7{\cdot}1\times10^4\text{ mm}^4$$

Since point S is farthest from the N.A., it will be stressed maximum. The distance S from N.A. is given by

$$(y_N)_S=u.\sin\beta+v\cos\beta$$

where u and v are the coordinates of point S referred to U-V axes. If (x, y) are the coordinates of S referred to x-y axes, we have

$$u=y\sin\alpha-x\cos\alpha$$

and $$v=y\cos\alpha+x\sin\alpha$$

where $$x=-10{\cdot}2-6=-4{\cdot}2\text{ mm}$$

and $$y=-60-20{\cdot}2=-39{\cdot}8\text{ mm}.$$

(Both x and y are negative since S is in the second quadrant with respect to the X-Y axes, the plane of loading being reckoned as situated in the first quadrant).

$$\therefore\quad u=\{-39{\cdot}8\sin 23°\ 40'+4{\cdot}2\cos 23°\ 40'\}=-12{\cdot}2\text{ mm}$$
$$v=\{-39{\cdot}8\cos 23°\ 40'-4{\cdot}2\sin 23°\ 40'\}=-38{\cdot}2\text{ mm}$$

$$\therefore\quad (y_N)_S=u\sin\beta+v\cos\beta=-12{\cdot}2\sin 67°\ 24'-38{\cdot}2\cos 67°\ 24$$
$$=-25{\cdot}9\text{ mm}.$$

Hence from Eq. 21·17,

$$(f_b)_S=\frac{M\cos(\beta-\theta)}{I_{NN}}\,y_N$$

$$=\frac{12\times10^4\cos(67°\ 24'-23°\ 40')}{7{\cdot}1\times10^4}\times25{\cdot}9$$

$$=-31{\cdot}7\text{ N/mm}^2$$

i.e., **$31{\cdot}7$ N/mm² (tensile).**

Alternatively,

From Eq. 21·10,

$$(f_b)_S=\frac{M\cos\theta}{I_{UU}}.v+\frac{M\sin\theta}{I_{VV}}.u$$

$$=-\frac{12\times10^4\cos 23°\ 24'}{23{\cdot}42\times10^4}\times38{\cdot}2-\frac{12\times10^4\sin 23°\ 24'}{4{\cdot}26\times10^4}\times12{\cdot}2$$

$$=-31{\cdot}7\text{ N/mm}^2.$$

Graphical Solution [Fig. 21·14 (*b*)]

$$k_{UU}=\sqrt{\frac{I_{UU}}{A}}=\sqrt{\frac{23{\cdot}42\times10^4}{564}}=20{\cdot}4\text{ mm}$$

$$k_{VV}=\sqrt{\frac{I_{VV}}{A}}=\sqrt{\frac{4{\cdot}26\times10^4}{564}}=8{\cdot}8\text{ mm}$$

Draw the U-U axis inclined at 23° 40′ with X-X axis in clockwise direction. Similarly, set off V-V axis. Draw the momental ellipse, making $OB=k_{UU}=20{\cdot}4$ mm and $AO=k_{VV}=8{\cdot}8$ mm.

To find the direction of neutral axis, draw any vertical line MM', and find its middle point J. Then OJ gives the direction of neutral axis NN. By measurement, $\beta=67°$.

Draw tangent $N'N'$ parallel to the neutral axis. Draw OH perpendicular to $N'N'$.

Then $\quad OH=k_{NN}=11{\cdot}3$ mm (by measurement)

$\therefore\quad I_{NN}=A.k_{NN}^2=564\ (11{\cdot}3)^2=7{\cdot}2\times10^4$ mm^4

From Fig. 21·14 (*a*), $(y_D)_S=26$ mm (by measurement)

$$\therefore\quad (f_b)_S=\frac{M\cos(\beta-\theta)}{I_{NN}}\,(y_N)_S$$

$$=\frac{12\times10^4\cos(67°-23°\ 40')}{7{\cdot}2\times10^4}\times2{\cdot}6$$

$$=3{\cdot}15\text{ N/mm}^2\text{ (tensile).}$$

21·10. THE Z-POLYGON

In the case of simple bending, the strength of a beam depends upon its section modulus Z. In the case of unsymmetrical bending,

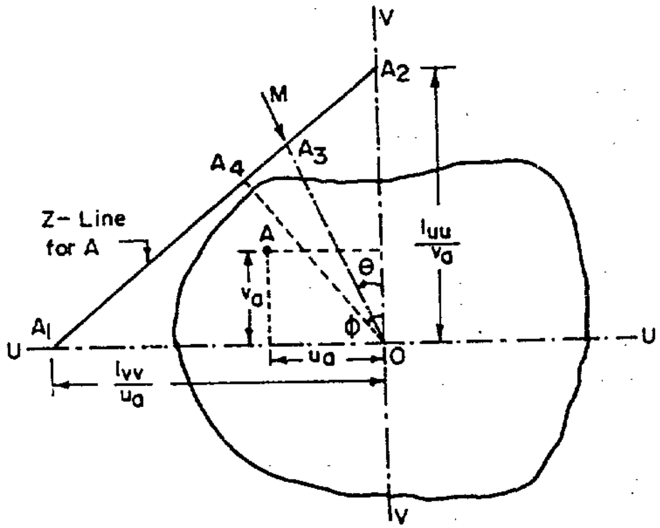


Fig. 21·15. The *Z*-line.

the section modulus Z *for any point* in the section depends also on the position of the plane of loading. It is interesting to study the variation of Z for any point as the direction of the plane of loading varies. It will be proved below that the variation of Z, for a point, is linear with the varying value of θ, and if such variations are plotted for some key points of the section, a polygon is obtained. Such a polygon is known as *Z-polygon*, and is very useful in finding out the minimum value of Z for the section and the corresponding position of the plane of loading.

From Eq. 21·10, the bending stress at any point A having co-ordinates u_A and v_A with reference to the principal axes, is given by

$$f_b=\frac{M\cos\theta}{I_{UU}}\,v_A+\frac{M\sin\theta}{I_{VV}}\,u_A \qquad \ldots(1)$$

(where θ is the angle of plane of loading OM with OV axis)

$$\therefore \qquad f_b=M\left[\frac{v_A\cos\theta}{I_{UU}}+\frac{u_A\sin\theta}{I_{VV}}\right]=\frac{M}{Z} \qquad \ldots(21{\cdot}25)$$

where Z=section modulus of the sections for the point A, given by

$$\frac{1}{Z}=\frac{v_A\cos\theta}{I_{UU}}+\frac{u_A\sin\theta}{I_{VV}} \qquad \ldots(21{\cdot}26)$$

or
$$v_A\frac{Z\cos\theta}{I_{UU}}+u_A\frac{Z\sin\theta}{I_{VV}}=1. \qquad \ldots[21{\cdot}27\ (a)]$$

Putting $X\cos\theta=v$ and $Z\sin\theta=u$, we get

$$v\,.\,\frac{v_A}{I_{UU}}+u\,.\,\frac{u_A}{I_{VV}}=1 \qquad \ldots(21{\cdot}27)$$

This is the equation of straight line which gives the variation of Z with θ. The straight line A_1A_2 (Fig. 21·15) is called the Z-line for the point A. The intercepts of this straight line UU and VV axes are $\frac{I_{VV}}{u_A}$ and $\frac{I_{UU}}{v_A}$ respectively. Hence in order to draw the Z-line for A, set off $OA_1=\frac{I_{VV}}{u_A}$ on UU axis, and $OA_2=\frac{I_{UU}}{v_A}$ on the VV axis. Join A_1, A_2, which is the Z-line for the point A. The minimum value of Z is given by the perpendicular OA_4, inclined at an angle ϕ with the OV axis.

$$\therefore \qquad Z_{min}=OA_4.$$

Maximum bending stress at A will be $\frac{M}{OA_4}$ when the plane of bending is inclined at θ to the principal axis OV.

It will be useful to plot such Z-lines for some key points of a given section, getting what is known as the Z-polygon. We shall take the case of a rectangular section $ABCD$ of width b and depth d, to plot the Z-polygon. (Fig. 21·16).

The principal axes U-U and V-V of a rectangular section coincide with the usual XX and YY-axes passing through its centroid. For the Z-line for A, the distance $OP=\frac{I_{VV}}{u_A}$ and $OQ=\frac{I_{UU}}{v_A}$.

But
$$I_{VV}=I_{YY}=\frac{1}{12}\,db^3\,;\quad u_A=\frac{b}{2}$$
$$I_{UU}=I_{XX}=\frac{1}{12}\,bd^3\,;\quad v_A=\frac{d}{2}$$

$$\therefore \qquad OP=\frac{1}{12}\,db^3\times\frac{2}{b}=\frac{1}{6}\,db^2=\text{usual } Z_{YY}$$
$$OQ=\frac{1}{12}\,bd^3\times\frac{2}{d}=\frac{1}{6}\,bd^2=\text{usual } Z_{XX}$$

Similarly, the Z-lines for B, C and D are respectively obtained as the line QR, RS and SP. Then $PQRS$ is the required Z-polygon, for the rectangular section. The Z-polygon provides, at a glance, the position of the plane of bending for the maximum and minimum strength.

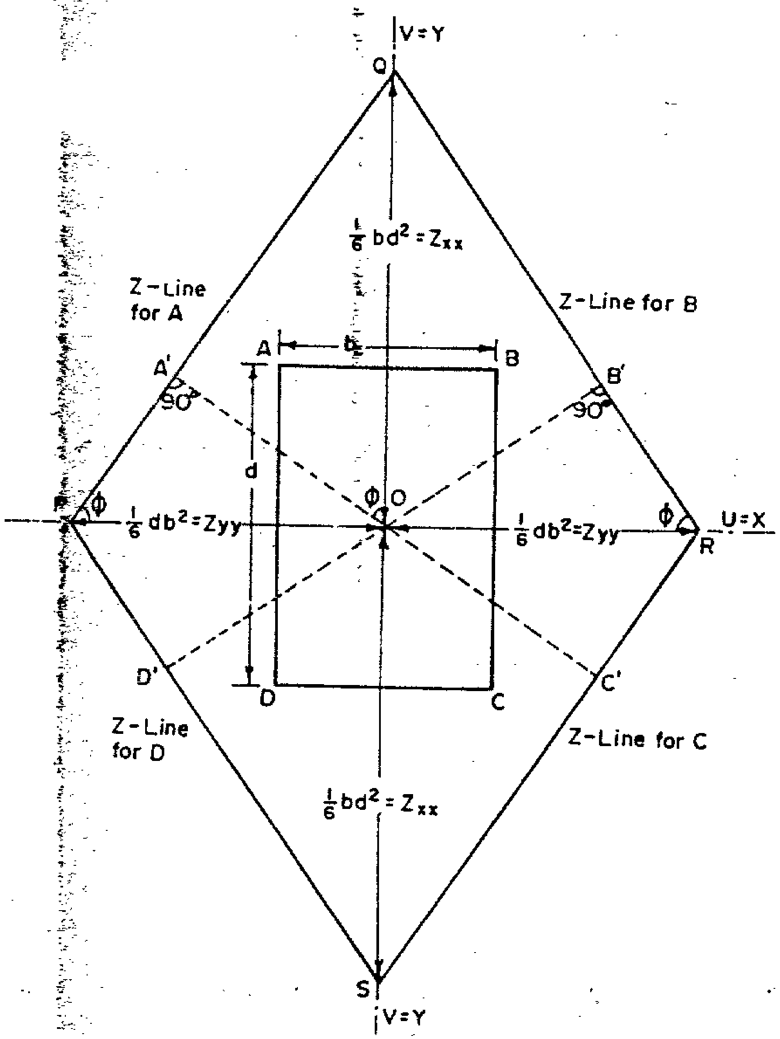


Fig. 21·16. *Z*-polygon for rectangular section.

For the rectangular section, the position of plane of loading for maximum strength is along YY axis since the value of Z along

this axis is $\frac{1}{6}bd^2$ and is the maximum. Similarly, the position of plane of loading for minimum strength is along $A'C'$ (or $B'D'$), inclined at ϕ with the YY axis, given by,

$$\tan\phi=\frac{A'Q}{A'O}=\frac{OQ}{OP}=\frac{Z_{XX}}{Z_{YY}}=\frac{1}{6}bd^2\div\frac{1}{6}db^2$$

or $$\tan\phi=\frac{d}{b} \quad ...(21·28)$$

Now $$Z_{min}=OA'=OQ\cos\phi$$
$$=Z_{XX}\cos\phi$$

$$=\frac{1}{6}bd^2\frac{b}{\sqrt{b^2+d^2}}=\frac{b^2d^2}{6\sqrt{b^2+d^2}} \quad ...(21·29)$$

21·11. DEFLECTION OF BEAM UNDER UNSYMMETRICAL BENDING

Fig. 21·17 shows the plane of loading OM inclined at θ to the OV axis. Let the neutral axis be inclined at β with the OU axis. The resolved component of bending moment in the VV direction is $M\cos\theta$, while in the U direction it is equal to $M\sin\theta$.

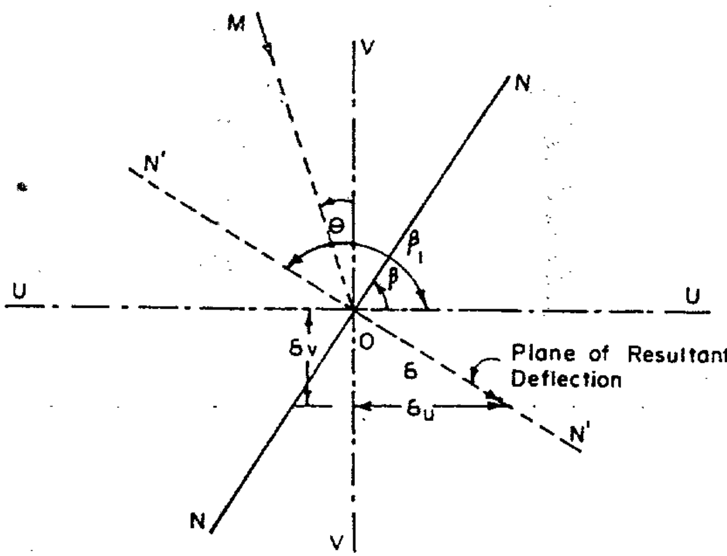


Fig. 21·17

The deflection of the beam in any direction, due to a bending moment M_1 is given by

$$\delta=\int_0^L \frac{M_1 m_1}{EI}\,dx$$

where m_1=moment due to unit load acting at the point in the direction of the desired deflection

dx=elementary length of beam, measured along the span of the beam.

Hence the deflection of the beam in the direction of axis VV is given by

$$\delta_V=\int_0^L \frac{M\cos\theta}{EI_{UU}}\,.\,m_V\,dx \qquad \ldots(1)\ \ldots[21\cdot30\,(a)]$$

The deflection in the direction of axis UU is given by

$$\delta_U=\int_0^L \frac{M\sin\theta}{EI_{VV}}\,.\,m_U\,dx \qquad \ldots(2)\ \ldots[21\cdot30\,(b)]$$

The resultant deflection δ is then given by

$$\delta=\left[\delta_U^2+\delta_V^2\right]^{1/2} \qquad \ldots(21\cdot31)$$

In Eqs. (1) and (2) above, $m_U=m_V=m$.

Let β_1=angle which the resultant deflection in the direction $N'N'$ makes with the UU axis.

Then, $\tan\beta_1=-\dfrac{\delta_U}{\delta_V}$

$$=-\frac{\int_0^L \frac{Mm\sin\theta}{EI_{VV}}\,dx}{\int_0^L \frac{Mm\cos\theta}{EI_{UU}}\,dx}$$

$$\therefore \quad \tan\beta_1=-\frac{I_{UU}}{I_{VV}}\tan\theta \qquad \ldots(3)$$

From Eq. 21·14,

$$\tan\beta=\frac{I_{UU}}{I_{VV}}\tan\theta \qquad \ldots(4)$$

Comparing (3) and (4), we get

$$\tan\beta_1=-\tan\beta=\tan(90+\beta)$$

$$\therefore \quad \beta_1=90+\beta.$$

Hence the resultant deflection occurs in a direction $N'N'$, which is perpendicular to the neutral axis NN for any given direction of loading.

Let us take the case of a simply supported beam subjected to uniformly distributed load.

Then,
$$\delta_U=\frac{5}{384}\,\frac{w\sin\theta.L^4}{EI_{VV}} \qquad \ldots(5)$$

$$\delta_V=\frac{5}{384}\,\frac{w\cos\theta.L^4}{EI_{UU}} \qquad \ldots(6)$$

$$\delta=\left[\delta_U^2+\delta_V^2\right]^{1/2}$$

$$=\frac{5}{384}\,\frac{wL^4}{E}\left[\frac{\sin^2\theta}{I_{VV}^2}+\frac{\cos^2\theta}{I_{UU}^2}\right]^{1/2}$$

$$=\frac{5}{384}\,\frac{wL^4.\cos\theta}{EI_{UU}}\left[1+\left(\frac{I_{UU}}{I_{VV}}\tan\theta\right)^2\right]^{1/2}$$

$$=\frac{5}{384}\,\frac{wL^4\cos\theta}{EI_{UU}}\left[1+\tan^2\beta\right]^{1/2}$$

$$\left(\text{Since } \frac{I_{UU}}{I_{VV}}\tan\theta=\tan\beta,\text{ from Eq. 21·14}\right)$$

$$\delta=\frac{5}{384}\,\frac{wL^4}{EI_{UU}}\cos\theta.\sec\beta$$

$$=\frac{5}{384}\,\frac{wL^4}{EI_{UU}}\cdot\frac{\cos\theta}{\cos\beta}\cdot\frac{\cos(\beta-\theta)}{\cos(\beta-\theta)}$$

$$=\frac{5}{384}\,\frac{wL^4}{EI_{UU}}\cdot\frac{\cos\beta\cos(\beta-\theta)}{\cos^2\beta\,[\cos\theta+\sin\beta\sin\theta]}$$

$$=\frac{5}{384}\,\frac{wL^4}{EI_{UU}}\cdot\frac{\cos(\beta-\theta)\cos\theta}{\cos^2\beta\cos\theta\,(1+\tan\beta\tan\theta)}$$

Substituting the value of $\tan\theta=\dfrac{I_{VV}}{I_{UU}}.\tan\beta$, we get

$$\theta=\frac{5}{384}\,\frac{wL^4}{EI_{UU}}\,\frac{\cos(\beta-\theta)}{\cos^2\beta\left[1+\frac{I_{VV}}{I_{UU}}\tan^2\beta\right]}$$

$$=\frac{5}{384}\,\frac{wL^4}{EI_{UU}}\,\frac{\cos(\beta-\theta)}{\cos^2\beta\left[\frac{I_{UU}\cos^2\beta+I_{VV}\sin^2\beta}{I_{UU}\cos^2\beta}\right]}$$

$$=\frac{5}{384}\,\frac{wL^4}{E}\,\frac{\cos(\beta-\theta)}{[I_{UU}\cos^2\beta+I_{VV}\sin^2\beta]}$$

But $[I_{UU}\cos^2\beta+I_{VV}\sin^2\beta]=I_{NN}$ (Eq. 21·15)

$$\therefore \quad \delta=\frac{5}{384}\,\frac{w.\cos(\beta-\theta).L^4}{EI_{NN}} \qquad \ldots(21\cdot32)$$

In the above expression, $w.\cos(\beta-\theta)$ is the component of the resultant uniformly distributed load along the direction $N'N'$ perpendicular to the neutral axis.

Example 21·6. *A 60 mm × 40 mm × 6 mm unequal angle is placed with the longer leg vertical and is used as a beam simply supported at the ends, over a span of 2 m. If it carries a uniformly distributed load of such magnitude as to produce the maximum bending moment of 0·12 kN-m determine the maximum deflection of the beam.*

Take $E = 2{\cdot}1 \times 10^5\ N/mm^2$.

Solution.

The properties of the section are known from example 21·6.

Thus, $I_{NN} = 7{\cdot}1 \times 10^4\ \text{mm}^2$; $\beta = 67°\ 24'$; $\theta = \alpha = 23°\ 40'$

Now, for a simply supported beam, maximum B.M. at the centre of the span, is gived by

$$M = \frac{wL^2}{8}$$

$$\therefore \quad w = \frac{8M}{L^2} = \frac{8 \times 0{\cdot}12 \times 10^6}{(2000)^2} = 0{\cdot}24 \text{ N/mm} \quad \ldots(i)$$

From Eq. 21·32, the maximum resultant deflection is given by

$$\delta = \frac{5}{384} \frac{w \cos(\beta - \theta)\, L^4}{EI_{NN}}$$

$$= \frac{5}{384} \frac{0{\cdot}24 \cos(67°\ 24' - 23°\ 40')\,(2000)^4}{2 \times 10^5\,(7{\cdot}1 \times 10^4)}$$

$$= \mathbf{2{\cdot}54\ mm.}$$

The maximum (resultant) deflection takes place in a direction perpendicular to the neutral axis. If β_1 is the inclination of the plane of maximum deflection, we have, [Fig. 21·14 (*a*)],

$$\beta_1 = 67°\ 24' - 23°\ 40' = 43°\ 44'.$$

Example 21·7. *Draw Z-polygon for a rolled steel joist (RSJ) having the following properties [Fig. 21·18 (a)] :*

Depth of section $(h) = 200$ *mm*

Width of flange $(b) = 100$ *mm*

Thickness of flange $= 7{\cdot}3$ *mm*

Thickness of web $= 5{\cdot}4$ *mm*

$$I_{UU} = 1696{\cdot}6 \times 10^4\ mm^4$$
$$I_{VV} = 115{\cdot}4 \times 10^4\ mm^4$$
$$Z_{UU} = 169{\cdot}7 \times 10^3\ mm^3$$
$$Z_{VV} = 23{\cdot}1 \times 10^3\ mm^3$$

Hence find the maximum bending stress due to a bending moment of 1800 N-m. What is the inclination of the plane of loading to give the maximum bending stress ?

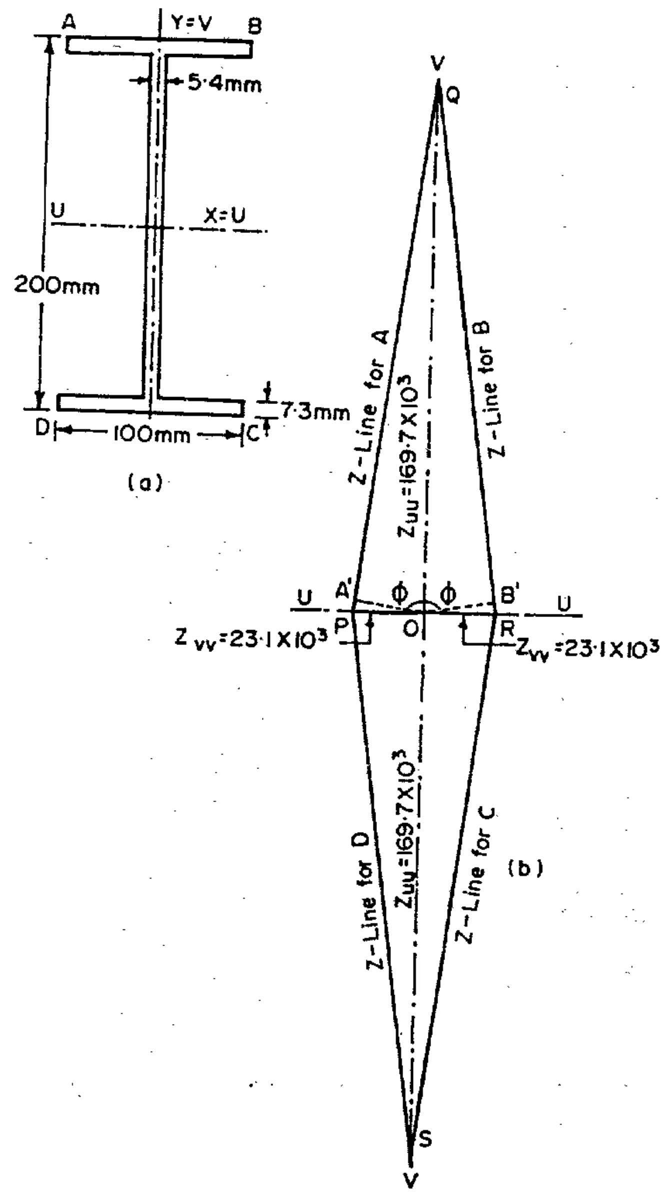


Fig. 21·8. *Z*-polygon for *I*-joist.
(Example 21·7)

Solution. Fig. 21·18 (*b*).

Since the key points *ABCD* form the corners of a rectangle of size 100 mm × 200 mm, the *Z*-polygon for a *I*-beam will be similar in shape to the *Z*-polygon for a rectangular beam (Fig. 21·16).

Hence to get the points *P*, *Q*, *R* and *S* of the *Z*-polygon [Fig. 21·18 (*b*)], make $OQ=OS=Z_{UU}=169·7\times10^3$ mm³ units, and

$$OR=OP=Z_{VV}=23·1\times10^3 \text{ mm}^3$$

The maximum strength is obtained along the direction *V-V*, having a section modulus $Z_{UU}=169·7\times10^3$. To find the minimum strength (or maximum bending stress), drop perpendiculars OA' or OB'. Let the inclination of OA' or OB' be ϕ with the *V-V* axis.

Then, $$Z_{min}=OA'=OP\sin\phi \quad \ldots(i)$$

Now $$\tan\phi=\frac{Z_{UU}}{Z_{VV}}=\frac{169·7\times10^3}{23·1\times10^3}=7·34$$

$\therefore$ $$\phi=82°\ 15' \text{ and } \sin\phi=0·991$$

$\therefore$ $$Z_{min}=OP\sin\phi=Z_{VV}\sin\phi=23·1\times10^3\times0·991$$
$$=22·9\times10^3 \text{ mm}^3$$

$\therefore$ $$(f_b)_{max}=\frac{M}{Z_{min}}=\frac{1800\times1000}{22·9\times10^3}=78·6 \text{ N/mm}^2.$$

The plane of bending is inclined at $\phi=82°\ 15'$ with the *V-V* axis, to give the above maximum bending stress.

Example 21·8. *For the angle section of example 21·5, draw the Z-polygon. Hence determine (i) maximum bending stress due to a bending moment of 0·12 kN-m acting in the vertical plane through the centroid of the section, (ii) the absolute maximum bending stress due to a bending moment of 0·12 kN-m, and the corresponding position of plane of loading.*

Solution. (Fig. 21·19).

From example 21·5, we have $\alpha=23°\ 40'$. Hence set of *U-U* axis at 23° 40′ with the *X*-axis, in clockwise direction. Make *V*-axis perpendicular to *U*-axis.

The *U* and *V* coordinates of key-points *A*, *B*, *C*, *D* and *E*, referred to *U-V* axes, can either be calculated, or determined by direct measurement. Since the *Z*-polygon method is essentially a graphical method, it is advisable to determine these coordinates by graphical measurements from the drawing (Fig. 21·19). Regarding the sign of *U* and *V*, it should be noted that *U* and *V* are positive in

that quadrant in which the plane of loading lies. The four quadrants I, II, III and IV) have been marked in Fig. 21·19, taking the quadrant of the plane of loading as the first quadrant. According to this, the coordinates of *A*, *B*, *C*, *D* and *E* are found to be as follows :

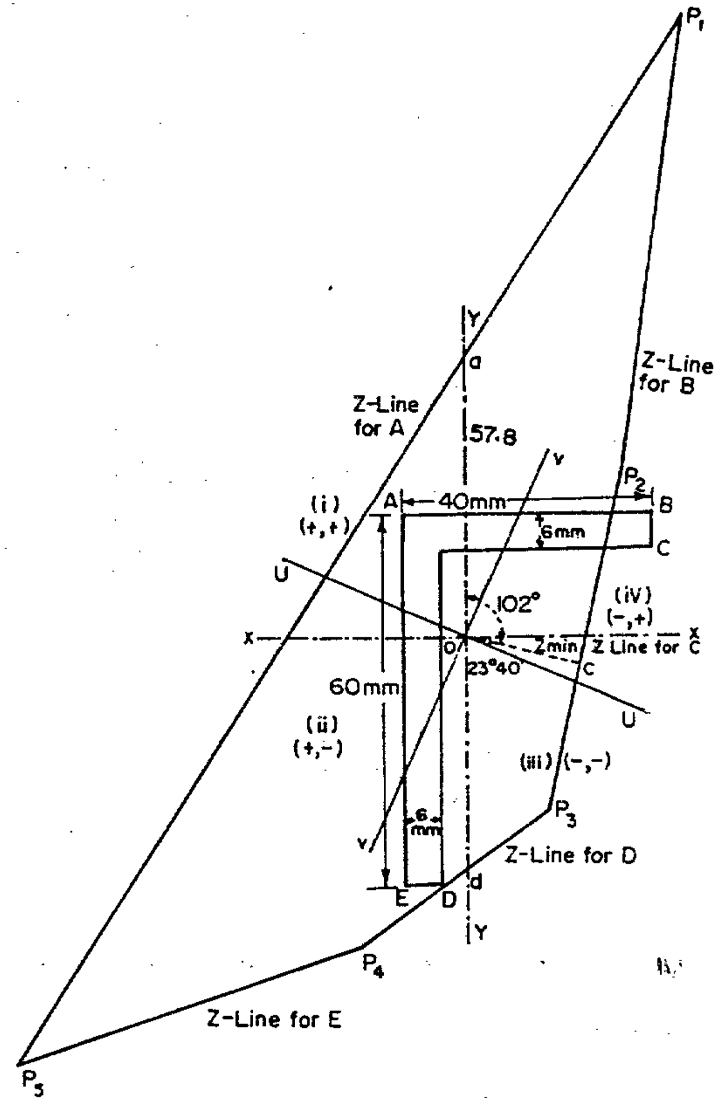


Fig. 21·19. (Example 21·8).

	(u) mm	(v) mm
A	$+17{\cdot}4$	$+14{\cdot}2$
B	$-19{\cdot}6$	$+31{\cdot}0$
C	$-22{\cdot}2$	$+25{\cdot}3$
D	$-12{\cdot}2$	$-38{\cdot}2$
E	$-\ 7{\cdot}0$	$-41{\cdot}0$

From example 21·5,

$$I_{UU}=23{\cdot}42\times10^4\ \text{mm}^4$$
$$I_{VV}=4{\cdot}26\times10^4\ \text{mm}^4$$

The equation of Z-line for A is given by

$$v\,.\,\frac{v_A}{I_{UU}}+u\,.\,\frac{u_A}{I_{VV}}=1$$

$$v\,.\,\frac{14{\cdot}2}{23{\cdot}42\times10^4}+u\,.\,\frac{17{\cdot}4}{4{\cdot}26\times10^4}=1$$

or $$v+6{\cdot}74\,u=16493 \qquad \ldots(1)$$

Equation of Z-line for B is given by

$$v\,.\,\frac{31{\cdot}0}{23{\cdot}42\times10^4}+u\,.\,\frac{(-19{\cdot}6)}{4{\cdot}26\times10^4}=1$$

or $$v-3{\cdot}48\,u=7555 \qquad \ldots(2)$$

Equation of Z-line for C is given by

$$v\,.\,\frac{(+25{\cdot}3)}{23{\cdot}42\times10^4}+u\,\frac{(-22{\cdot}2)}{4{\cdot}26\times10^4}=1$$

or $$v-4{\cdot}824=9257 \qquad \ldots(3)$$

Equation of Z-line for D is given by

$$v\,.\,\frac{(-38{\cdot}2)}{23{\cdot}42\times10^4}+u\,\frac{(-12{\cdot}2)}{4{\cdot}26\times10^4}=1$$

or $$v+1{\cdot}764=-6131 \qquad \ldots(4)$$

And equation of Z-line for E is given by

$$v\,.\,\frac{(-41{\cdot}0)}{23{\cdot}42\times10^4}+u\,\frac{(-7{\cdot}0)}{4{\cdot}26\times10^4}=1$$

or $$v+0{\cdot}94\,u=-5712. \qquad \ldots(5)$$

Let P_1, P_2, P_3, P_4 and P_5 be the corner points of the Z-polygon.

The point P_1, is the point of intersection of Z-lines for points A and B, and hence, its co-ordinates are found by simultaneously solving equations (1) and (2). Hence co-ordinates of point P_1 are :

$$u=+8{\cdot}73 \text{ and } v=+106{\cdot}1\ \text{mm}.$$

Solving equations (2) and (3), the co-ordinates of point P_2 are

$$u=-12{\cdot}5\ ;\ v=-32{\cdot}1$$

Solving equations (3) and (4), the co-ordinates of point P_3 are

$$u=-23{\cdot}4\ ;\ v=-20{\cdot}2$$

Solving equations (4) and (5), the co-ordinates of point P_4 are

$$u=-4{\cdot}8\ ;\ v=-52{\cdot}7$$

Solving equations (5) and (1), the co-ordinates of point P_5 are

$$u=+38{\cdot}2\ ;\ v=-93{\cdot}0$$

Knowing these co-ordinate, the Z-polygon $P_1P_2P_3P_4P_5$ can be plotted, as shown in Fig. 21·19.

(*i*) For the loading in the vertical plane, the section modulus Z for A is given by $Oa=5{\cdot}78\times10^3$ mm³ and that of D is given by $Od=3{\cdot}75\times10^3$ mm³. Hence the *minimum* section modulus$=3{\cdot}75\times10^3$ mm³. The maximum bending stress is, therefore, given by

$$(f_b)_{max}=\frac{M}{Z}=\frac{0{\cdot}12\times10^6}{3{\cdot}75\times10^3}=32\ \text{N/mm}^2.$$

(*ii*) For absolute maximum bending stress, we have to find the absolute minimum section modulus. If perpendiculars are drawn from O to the various Z-lines, we find that the minimum Z is equal to OC.

$$\therefore\quad Z_{min}=OC=1{\cdot}88\times10^3\ \text{mm}^3$$

$$\therefore\quad (f_b)_{max\cdot max}=\frac{0{\cdot}12\times10^6}{1{\cdot}88\times10^3}=63{\cdot}8\ \text{N/mm}^2.$$

The corresponding direction of loading is at 102° with the vertical line YY, as marked in Fig. 21·19.

PROBLEMS

1. Determine the principal moments of inertia for an unequal angle section 200 mm × 150 mm × 10 mm.

2. A 4 in. × 4 in. × ½ in. steel angle is used as a cantilever of length 3 ft. and carries an end load. One leg of the angle is horizontal and the load at the end is vertical with its line of action passing through the centroid of the section. Determine the maximum allowable load if the bending stress is not to exceed 7½ tons/in.² and find also the vertical deflection at the end due to this load. Assume all corners of the angle to be left square. $E=13500$ tons/in.². (*U.L.*)

3. A cantilever consists of 3 in. × 3 in. × ½ in. angle with the top face AB horizontal (Fig. 21·20). It carries a load of 400 lb. at a distance of 3 ft. from the fixed end, the line of action of the load

passing through the centroid of the section and inclined at 30° to the vertical. Determine the stress at the corners A, B and C at the fixed end and also the position of the neutral axis. Given the following : $A=2{\cdot}753$ in.^{2} ; $I_{XX}=I_{YY}=2{\cdot}18$ in.^{4} ; $I_{UU}=3{\cdot}44$ in.^{4} ; $I_{VV}=0{\cdot}92$ in.^{4}. (*U.L.*)

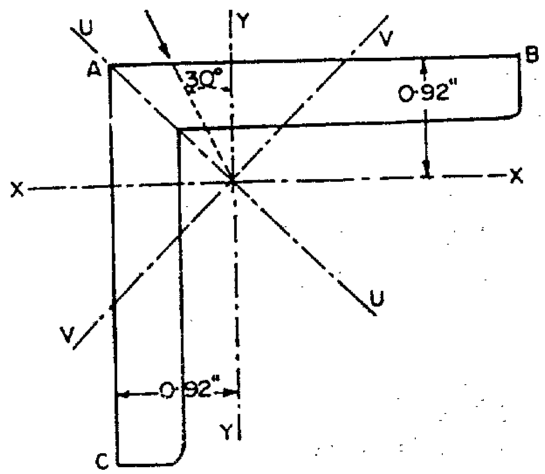


Fig. 21·20

ANSWERS

1. $I_{UU}=17\times10^6$ mm^4 ; $I_{VV}=3{\cdot}51\times10^6$ mm^4.
2. $W_{max.}=705$ lb ; $\delta_V=0{\cdot}0993$ in.
3. $(f_b)_A=19{,}650$ lb/in.^{2}
   $(f_b)_B=10{,}140$ lb/in.^{2}
   $(f_b)_C=14{,}740$ lb/in.^{2}.

Neutral axis inclined at 40° 34′ to the XX-axis in the anticlockwise direction.

22

Elementary Theory of Elasticity

22·1. STATE OF STRESS AT A POINT : STRESS TENSOR

Force systems acting on an elastic body in equilibrium are of two kinds : body forces and surface forces. Forces distributed over the surface of the body, such as the pressure of one body on another or hydrostatic pressure are called *surface forces*. Such forces are applied externally at the boundaries of the body, and dimensionally, a surface force is defined as force per unit area. Forces distributed over the volume of a body, such as gravitational forces, magnetic forces, seepage forces, or in the case of a body in motion, inertia forces, are called *body forces*. Dimensionally, a body force is taken as a force per unit volume.

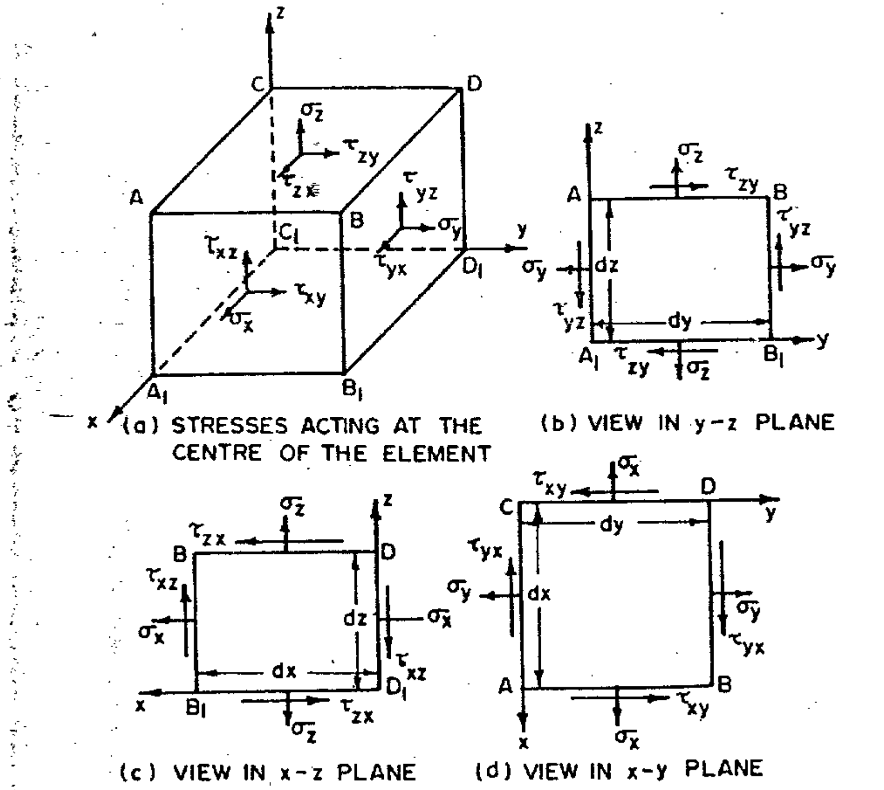
The total stress field on any three dimensional element is determined by the following stresses :

$$\begin{pmatrix} \sigma_{XX} & \tau_{XY} & \tau_{XZ} \\ \tau_{YX} & \sigma_{YY} & \tau_{YZ} \\ \tau_{ZX} & \tau_{ZY} & \sigma_{ZZ} \end{pmatrix} \text{ or } \begin{pmatrix} \sigma_X & \tau_{XY} & \tau_{XZ} \\ \tau_{XY} & \sigma_Y & \tau_{ZY} \\ \tau_{XZ} & \tau_{YZ} & \sigma_Z \end{pmatrix}$$

These nine stress components, as given by this group of square matrix of stresses, are the components of a mathematical entity called the stress tensor, with a symmetrical matrix relative to its main diagonal (upper left to lower right). The main diagonal elements of the stress tensor are the normal stress components and the off-diagonal elements are shear stresses.

Each stress component in it is represented by its magnitude, direction as well as the position of the plane on which it is acting. For example, σ_{XX} (or simply σ_X) signifies the normal stress acting on the face of the element that is perpendicular to x-axis, and the stress is acting in the x-direction. Similarly, the shearing stress τ_{XY} denotes a stress acting on the face of an element that is perpendicular to

x-axis, the stress acting in the direction of y-axis. The stress τ_{YZ} denotes a stress acting on the face of an element that is perpendicular to y-axis, the stress acting in the direction of z-axis. Thus, at a point, there are three normal stresses : σ_X (or σ_{XX}), σ_Y (or σ_{YY}) and σ_Z (or σ_{ZZ}) and six shearing stresses : τ_{XY}, τ_{YX} ; τ_{YZ}, τ_{ZY} ; τ_{ZX} and τ_{XZ} as shown in Fig. 22·1 (a).


(a) STRESSES ACTING AT THE CENTRE OF THE ELEMENT (b) VIEW IN y–z PLANE

(c) VIEW IN x–z PLANE (d) VIEW IN x–y PLANE

Fig. 22·1. Three dimensional stress field.

Fig. 22·1 (a) shows the stresses acting at the centre of an elemental volume of size dx, dy and dz. Figs. 22·1 (b), (c) and (d) show the views in z-y, x-z and x-y planes respectively. Considering the equilibrium of the elemental volume, and applying equation $\Sigma M_Z=0$ (where M_Z represents the moment about z-axis), we get from Fig. 22·1 (d),

$$(\tau_{YX}.dx\ dz)dy=(\tau_{XY}.dy.dz)dx$$

or $$\tau_{YX}=\tau_{XY} \quad ...(22·1)$$

Similarly, from Figs. 22·1 (b) and (c), we get

$$\tau_{YZ}=\tau_{ZY} \quad ...[22·2\ (a)]$$

and $$\tau_{ZX}=\tau_{XZ} \quad ...[22·2\ (b)]$$

Thus, out of six shearing stresses there are only three independent shearing stresses and total independent stresses are therefore *six* only (*i.e* three normal stresses and three shearing stresses).

22·2. EQUILIBRIUM EQUATIONS

Fig. 22·2 shows an elemental volume of size dx, dy and dz, with the nine stress components acting at the centre of the element. The stress on each face will be equal to the stress at the centre increased or reduced by the distance from the centre to the face times the spatial derivative of the stress. For example, normal stress component σ_X acting at the centre will be increased to $\left(\sigma_X+\frac{\partial\sigma_X}{\partial x}\cdot\frac{dx}{2}\right)$ at the face ABB_1A_1 and decreased to $\left(\sigma_X-\frac{\partial\sigma_X}{\partial x}\cdot\frac{dx}{2}\right)$ at the face CDD_1C_1.

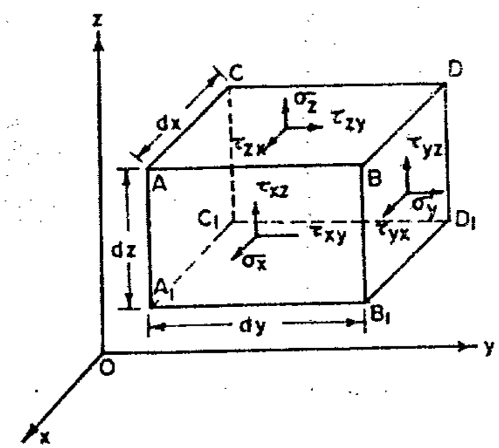


Fig. 22·2

If X, Y and Z denote the components of body forces per unit volume, in the three corresponding directions, then the equation of equilibrium obtained by summing all the forces acting on the element in the x-direction is :

$$\left\{\left(\sigma_X+\frac{\partial\sigma_X}{\partial x}\cdot\frac{dx}{2}\right)dy\,dz-\left(\sigma_Y-\frac{\partial\sigma_X}{\partial x}\cdot\frac{dx}{2}\right)dy\,dz\right\}$$
$$+\left\{\left(\tau_{YX}+\frac{\partial\tau_{YX}}{\partial y}\cdot\frac{dy}{2}\right)dx\,dz-\left(\tau_{YX}-\frac{\partial\tau_{YX}}{\partial y}\cdot\frac{dy}{2}\right)dx\,dz\right\}$$
$$+\left\{\left(\tau_{ZX}+\frac{\partial\tau_{ZX}}{\partial z}\cdot\frac{dz}{2}\right)dx\,dy-\left(\tau_{ZX}-\frac{\partial\tau_{ZX}}{\partial z}\cdot\frac{dz}{2}\right)dx\,dy\right\}$$
$$+X.dx.dy.dz=0$$

Dividing all the terms by $dx.dy.dz$ and simplifying, we get

$$\frac{\partial\sigma_X}{\partial x}+\frac{\partial\tau_{YX}}{\partial y}+\frac{\partial\tau_{ZX}}{\partial z}+X=0 \qquad \text{...(I)} \quad \text{...[22·3 (a)]}$$

Similarly, in the y-direction, the balance of forces requires that

$$\left\{\left(\sigma_Y+\frac{\partial\sigma_Y}{\partial y}\cdot\frac{dy}{2}\right)dx\,dz-\left(\sigma_Y-\frac{\partial\sigma_Y}{\partial y}\cdot\frac{dy}{2}\right)dx\,dz\right\}$$
$$+\left\{\left(\tau_{ZY}+\frac{\partial\tau_{ZY}}{\partial z}\cdot\frac{dz}{2}\right)dy\,dx-\left(\tau_{ZY}-\frac{\partial\tau_{ZY}}{\partial z}\cdot\frac{dz}{2}\right)dy\,dx\right\}$$
$$+\left\{\left(\tau_{XY}+\frac{\partial\tau_{XY}}{\partial x}\cdot\frac{dx}{2}\right)dy\,dz-\left(\tau_{XY}-\frac{\partial\tau_{XY}}{\partial x}\cdot\frac{dx}{2}\right)dy\,dz\right\}$$
$$+Y.dx.dy.dz=0$$

which on simplification reduce to :

$$\frac{\partial\sigma_Y}{\partial y}+\frac{\partial\tau_{ZY}}{\partial z}+\frac{\partial\tau_{XY}}{\partial x}+Y=0 \qquad \text{...(II)} \quad \text{...[22·3 (b)]}$$

Lastly, in the z-direction, we have

$$\left(\sigma_Z+\frac{\partial\sigma_Z}{\partial z}\cdot\frac{dz}{2}\right)dx\,dy-\left(\sigma_Z-\frac{\partial\sigma_Z}{\partial z}\cdot\frac{dz}{2}\right)dz\,dy\right\}$$
$$+\left\{\left(\tau_{XZ}+\frac{\partial\tau_{XZ}}{\partial x}\cdot\frac{dx}{2}\right)dz\,dy-\left(\tau_{XZ}+\frac{\partial\tau_{XZ}}{\partial x}\cdot\frac{dx}{2}\right)dz\,dy\right\}$$
$$+\left\{\left(\tau_{YZ}+\frac{\partial\tau_{YZ}}{\partial y}\cdot\frac{dy}{2}\right)dz\,dx-\left(\tau_{YZ}-\frac{\partial\tau_{YZ}}{\partial y}\cdot\frac{dy}{2}\right)dz\,dx\right\}$$
$$+Z.dx.dy.dz=0$$

which on simplification reduces to

$$\frac{\partial\sigma_Z}{\partial z}+\frac{\partial\tau_{XZ}}{\partial x}+\frac{\partial\tau_{YZ}}{\partial y}+Z=0 \qquad \text{...(III)} \quad \text{...[22·3 (c)]}$$

Eqs. 22·3 (*a*), (*b*) and (*c*) are the three equilibrium equations which are also sometimes written in the following order :

$$\frac{\partial\sigma_X}{\partial x}+\frac{\partial\tau_{YX}}{\partial y}+\frac{\partial\tau_{ZX}}{\partial z}+X=0 \qquad \text{...(22·4)}$$

$$\frac{\partial\tau_{XY}}{\partial x}+\frac{\partial\sigma_Y}{\partial y}+\frac{\partial\tau_{ZY}}{\partial z}+Y=0 \qquad \text{...(22·5)}$$

$$\frac{\partial\tau_{XZ}}{\partial x}+\frac{\partial\tau_{YZ}}{\partial y}+\frac{\partial\sigma_Z}{\partial z}+Z=0 \qquad \text{...(22·6)}$$

However, we have seen in § 22·1 that there are six independent stress components acting at a point and the complete solution of the problem requires the determination of these six stress components. Thus, there are six unknowns, and only three *equations of equilibrium* are available. Thus the problem of elasticity is strictly of indeterminate nature. These equations of static equilibrium must be supplemented with *equations of compatibility of deformations* (§ 22·4) to get the complete solution. In addition to this, the final solution should satisfy the boundary conditions (§ 22·5).

22·3. STRAIN COMPONENTS : STRAIN TENSOR

Let u, v and w be the displacements in x, y and z directions respectively. For a three dimensional case, there are six strain components :

ε_X, ε_Y, ε_Z, γ_{XY}, γ_{YZ} and γ_{ZX}.

The three linear strain components are defined by :

$$\varepsilon_X=\frac{\partial u}{\partial x} \qquad \text{...[22·7 (a)]}$$

$$\varepsilon_Y=\frac{\partial v}{\partial y} \qquad \text{...[22·7 (b)]}$$

$$\varepsilon_Z=\frac{\partial w}{\partial z} \qquad \text{...[22·7 (c)]}$$

In order to find the other strain components (called the *shearing strain* components), consider a plane lamina of size dx, dy in the x-y plane. The lines OA and OB, originally orthogonal to each other, are displaced to positions $O'A'$ and $O'B'$ respectively. The shearing strain is equal to the change in the angle at O.

Displacement of A in x-direction $=u+\dfrac{\partial u}{\partial x}\cdot dx$

Displacement of B in y-direction $=v+\dfrac{\partial v}{\partial y}\cdot dy$

Displacement of A in y-direction $= v + \dfrac{\partial v}{\partial x} \cdot dx$

Displacement of B in x-direction $= u + \dfrac{\partial u}{\partial y} \cdot dy$

$\therefore$ Total change in the angle at O

$$= \gamma_{XY} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad \text{...[22·8 (a)]}$$

Similarly,
$$\gamma_{YZ} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \quad \text{...[22·8 (b)]}$$

and
$$\gamma_{ZX} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \quad \text{...[22·8 (c)]}$$

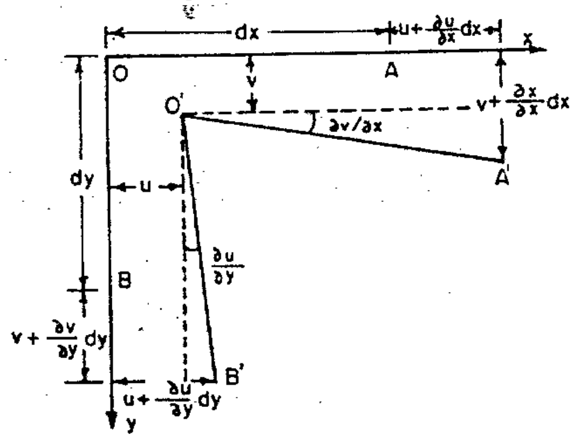


Fig. 22·3. Shear Strains.

It can be shown that linear strain of a diagonal is equal to half the shearing strain. Thus, if ε_{XY}, ε_{YZ} and ε_{ZX} represent the linear strains of the diagonals of a plane lamina, we have

$$\varepsilon_{XY} = \tfrac{1}{2}\gamma_{XY} \; ; \; \varepsilon_{YZ} = \tfrac{1}{2}\gamma_{YZ} \; ; \; \varepsilon_{ZX} = \tfrac{1}{2}\gamma_{ZX} \quad \text{...(22·9)}$$

Therefore, the *strain tensor*, consisting of nine strain components, can be represented as under :

$$\begin{pmatrix} \varepsilon_{XX} & \varepsilon_{XY} & \varepsilon_{XZ} \\ \varepsilon_{YX} & \varepsilon_{YY} & \varepsilon_{YZ} \\ \varepsilon_{ZX} & \varepsilon_{ZY} & \varepsilon_{ZZ} \end{pmatrix} \text{ or } \begin{pmatrix} \varepsilon_X & \tfrac{1}{2}\gamma_{XY} & \tfrac{1}{2}\gamma_{XZ} \\ \tfrac{1}{2}\gamma_{YX} & \varepsilon_Y & \tfrac{1}{2}\gamma_{YZ} \\ \tfrac{1}{2}\gamma_{ZX} & \tfrac{1}{2}\gamma_{ZY} & \varepsilon_Z \end{pmatrix}$$

22·4. COMPATIBILITY EQUATIONS

The equations resulting from the application of strain equations are known as the compatibility equations, or Saint-Venant's equations.

Differentiating Eq. 22·7 (*a*) twice with respect to y, Eq. 22·7 (*b*) twice with respect to x and Eq. 22·8 (*a*) once with respect to x and then with respect to y, we get

$$\frac{\partial^2 \varepsilon_X}{\partial y^2} = \frac{\partial^3 u}{\partial x . \partial y^2} \quad \text{...(i)}$$

$$\frac{\partial^2 \varepsilon_Y}{\partial x^2} = \frac{\partial^3 v}{\partial y . \partial x^2} \quad \text{...(ii)}$$

$$\frac{\partial^2 \gamma_{XY}}{\partial x \partial y} = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y} \quad \text{...(iii)}$$

By inspection from (*i*), (*ii*) and (*iii*), we get

$$\frac{\partial^2 \varepsilon_X}{\partial y^2} + \frac{\partial^2 \varepsilon_Z}{\partial x^2} = \frac{\partial^2 \gamma_{YZ}}{\partial x . \partial y} \quad \text{...[22·10 (a)]}$$

Similarly,

$$\frac{\partial^2 \varepsilon_Y}{\partial z^2} + \frac{\partial^2 \varepsilon_Z}{\partial y^2} = \frac{\partial^2 \gamma_{YZ}}{\partial y . \partial z} \quad \text{...[22·10 (b)]}$$

and
$$\frac{\partial^2 \varepsilon_Z}{\partial x^2} + \frac{\partial^2 \varepsilon_X}{\partial z^2} = \frac{\partial^2 \gamma_{ZX}}{\partial z . \partial x} \quad \text{...[22·10 (c)]}$$

Again, from Eq. 22·7 (*a*),

$$\frac{\partial^2 \varepsilon_X}{\partial y \partial z} = \frac{\partial^3 u}{\partial x \partial y \partial z} \quad \text{...(iv)}$$

From Eq. 22·8 (*a*), $\dfrac{\partial^2 \gamma_{XY}}{\partial z . \partial x} = \dfrac{\partial^3 v}{\partial x^2 \partial z} + \dfrac{\partial^3 u}{\partial y \partial z \partial x}$ ...(v)

From Eq. 22·8 (*b*), $\dfrac{\partial^2 \gamma_{YZ}}{\partial x^2} = \dfrac{\partial^3 w}{\partial x^2 \partial y} + \dfrac{\partial^3 v}{\partial x^2 \partial z}$ ...(vi)

and from Eq. 22·8 (*c*), $\dfrac{\partial^2 \gamma_{ZX}}{\partial y \partial x} = \dfrac{\partial^3 w}{\partial x^2 \partial y} + \dfrac{\partial^3 u}{\partial y \partial z \partial x}$ ...(vii)

From (*iv*), (*v*), (*vi*) and (*vii*), we have

$$2\frac{\partial^2 \varepsilon_X}{\partial y \partial z} = \frac{\partial^2 \gamma_{XY}}{\partial z \partial x} - \frac{\partial^2 \gamma_{YZ}}{\partial x^2} + \frac{\partial^2 \gamma_{ZX}}{\partial y \partial x}$$

or
$$2\frac{\partial^2 \varepsilon_X}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{YZ}}{\partial x} + \frac{\partial \gamma_{ZX}}{\partial y} + \frac{\partial \gamma_{XY}}{\partial z}\right) \quad \text{...[22·10 (d)]}$$

Similarly,

$$2\frac{\partial^2 \varepsilon_Y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{YZ}}{\partial x} - \frac{\partial \gamma_{ZX}}{\partial y} + \frac{\partial \gamma_{XY}}{\partial z}\right) \quad \text{...[22·10 (e)]}$$

and $$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y}=\frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x}+\frac{\partial \gamma_{zx}}{\partial y}-\frac{\partial \gamma_{xy}}{\partial z}\right) \qquad \ldots[22\cdot10\ (f)]$$

Eqs. 22·10 are the six compatibility equations.

22·5. BOUNDARY CONDITION EQUATIONS

The solution of an elasticity problem is obtained by the solution of equilibrium and compatibility equations, but the final solution must also satisfy the boundary condition equations. In order to derive the boundary condition equations, consider a boundary plane ABC [Fig. 22·4 (*a*)] with l, m and n as the direction cosines of the external normal to its surface at any point. $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ be the components of surface forces per unit area *on the elementary area* ABC. Fig. 22·4 (*b*) shows the nine stress components on the face OBC, OAC and OAB. If the elemental volume is considered to be shrunk to a point, these nine stress components are assumed to act at the point.

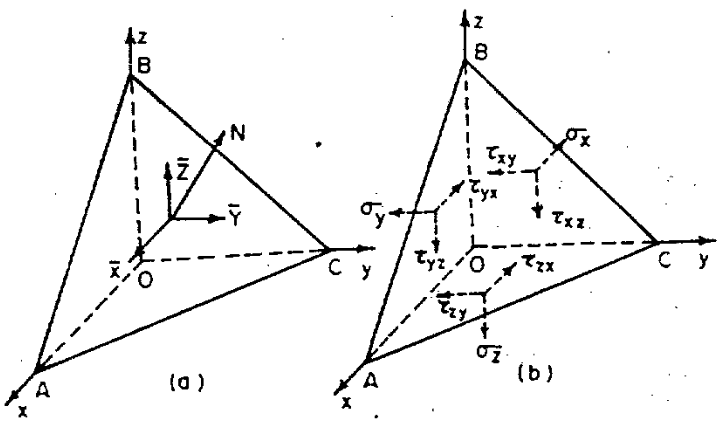


Fig. 22·4. Boundary conditions.

Let area $ABC=ds$

Area $OBC=ds.\cos(N, x)=ds.l$

$OAB=ds.\cos(N, y)=ds.m$

$OAC=ds.\cos(N, z)=ds.n$

Resolving all the forces in x-direction, and equating them to zero, we get

$$\Sigma X=0=\bar{X}\,ds-\sigma_x.ds.l-\tau_{yx}.ds\,m-\tau_{zx}.ds.n$$

or $$\bar{X}=\sigma_x.l+\tau_{yx}.m+\tau_{zx}.n \qquad \ldots[22\cdot11\ (a)]$$

Similarly, $$\bar{Y}=\sigma_y.m+\tau_{zy}.n+\tau_{xy}.l \qquad \ldots[22\cdot11\ (b)]$$

and $$\bar{Z}=\sigma_z.n+\tau_{xz}.l+\tau_{yz}.m \qquad \ldots[22\cdot11\ (c)]$$

These are the *boundary condition equations* which are also sometimes written in the matrix form :

$$\begin{bmatrix}\bar{X}\\ \bar{Y}\\ \bar{Z}\end{bmatrix}=\begin{bmatrix}\sigma_x & \tau_{yx} & \tau_{zx}\\ \tau_{xy} & \sigma_y & \tau_{zy}\\ \tau_{xz} & \tau_{yz} & \sigma_z\end{bmatrix}\begin{bmatrix}l\\ m\\ n\end{bmatrix} \qquad \ldots(22\cdot11)$$

22·6. GENERALISED HOOKE'S LAW : HOMOGENEITY AND ISOTROPY

In its simplest form, Hooke's law states that within the elastic limit, stress is proportional to strain. Following are the generalised Hooke's law equations (due to Navier-Stokes), connecting the strain components to stress components :

$$\varepsilon_x=C_{11}\sigma_x+C_{12}\sigma_y+C_{13}\sigma_z+C_{14}\tau_{xy}+C_{15}\tau_{yz}+C_{16}\tau_{zx} \ldots[22\cdot12\ (a)]$$

$$\varepsilon_y=C_{21}\sigma_x+C_{22}\sigma_y+C_{23}\sigma_z+C_{24}\tau_{xy}+C_{25}\tau_{yz}+C_{26}\tau_{zx} \ldots[22\cdot12\ (b)]$$

$$\varepsilon_z=C_{31}\sigma_x+C_{32}\sigma_y+C_{33}\sigma_z+C_{34}\tau_{xy}+C_{35}\tau_{yz}+C_{36}\tau_{zx} \ldots[22\cdot12\ (c)]$$

$$\gamma_{xy}=C_{41}\sigma_x+C_{42}\sigma_y+C_{43}\sigma_z+C_{44}\tau_{xy}+C_{45}\tau_{yz}+C_{46}\tau_{zx} \ldots[22\cdot12\ (d)]$$

$$\gamma_{yz}=C_{51}\sigma_x+C_{52}\sigma_y+C_{53}\sigma_z+C_{54}\tau_{xy}+C_{55}\tau_{yz}+C_{56}\tau_{zx} \ldots[22\cdot12\ (e)]$$

$$\gamma_{zx}=C_{61}\sigma_x+C_{62}\sigma_y+C_{63}\sigma_z+C_{64}\tau_{xy}+C_{65}\tau_{yz}+C_{66}\tau_{zx} \ldots[22\cdot12\ (f)]$$

These equations contain 36 ***elastic constants***. These elastic constants are independent of the stress components at a point. An elastic body is said to be *homogeneous* if there are only 36 elastic constants, and they are the same at all points within a region. A point in a material is called *isotropic* if its elastic constants are the *same in all directions* at the point. By a series of rotations of axes, it can be shown that the number of independent elastic constants for a homogeneous, isotropic body are only 2. The considerations of homogeneity restrict the total elastic constants within a region to a finite value of 36, while the considerations of isotropy further reduce these constants to only 2. The generalised Hooke's law equations (Eqs. 22·12) then reduce to

$$\varepsilon_X = C_{11}\sigma_X + C_{12}(\sigma_Y + \sigma_Z)$$
$$\varepsilon_Y = C_{11}\sigma_Y + C_{12}(\sigma_Z + \sigma_X)$$
$$\varepsilon_Z = C_{11}\sigma_Z + C_{12}(\sigma_X + \sigma_Y)$$
$$\gamma_{XY} = 2(C_{11} - C_{12})\tau_{XY} \qquad \ldots(22{\cdot}13)$$
$$\gamma_{YZ} = 2(C_{11} - C_{12})\tau_{YZ}$$

and $$\gamma_{ZX} = 2(C_{11} - C_{12})\tau_{ZX}$$

In order to evaluate the values of the two constants C_{11} and C_{12} let us take the case of uniaxial stress in the x-direction, with

$$\sigma_Y = \sigma_Z = 0$$

From which $C_{11} = \dfrac{\varepsilon_X}{\sigma_X} = \dfrac{1}{E}$

where E=Young's modulus of elasticity.

Also, $$\varepsilon_Y = \varepsilon_Z = C_{12}\sigma_X = C_{12}E.\varepsilon_X$$

$$\therefore \quad C_{12} = \frac{1}{E} \cdot \frac{\varepsilon_Y}{\varepsilon_X} = -\frac{\mu}{E}$$

where μ=Poisson's ratio$=\dfrac{1}{m}$.

Substituting these in Eq. 22·13, we get the final equations as under :

$$\varepsilon_X = \frac{1}{E}\Big[\sigma_X - \mu(\sigma_Y + \sigma_Z)\Big] \qquad \ldots[22{\cdot}14\ (a)]$$

$$\varepsilon_Y = \frac{1}{E}\Big[\sigma_Y - \mu(\sigma_Z + \sigma_X)\Big] \qquad \ldots[22{\cdot}14\ (b)]$$

$$\varepsilon_Z = \frac{1}{E}\Big[\sigma_Z - \mu(\sigma_X + \sigma_Y)\Big] \qquad \ldots[22{\cdot}14\ (c)]$$

$$\gamma_{XY} = \frac{2(1+\mu)}{E}\tau_{XY} \qquad \ldots[22{\cdot}14\ (d)]$$

$$\gamma_{YZ} = \frac{2(1+\mu)}{E}\tau_{YZ} \qquad \ldots[22{\cdot}14\ (e)]$$

$$\gamma_{ZX} = \frac{2(1+\mu)}{E}\tau_{ZX} \qquad \ldots[22{\cdot}14\ (f)]$$

22·7. TWO DIMENSIONAL PROBLEMS

(*a*) **Plane Stress.** If a *thin* plate is uniformly loaded by forces applied at the boundary, parallel to the plane (say xy plane) of the plate, the stress components $\sigma_Y = \tau_{XY} = \tau_{YZ} = 0$ on both the faces of the plate. Such a state of stress is called the *plane stress*. The components of stress are σ_X, σ_Z and τ_{XZ}. These components are independent of y (*i.e.* they do not vary with y). The Hooke's law equations for plane stress case are :

$$\varepsilon_X = \frac{1}{E}\Big(\sigma_X - \mu\sigma_Z\Big)$$

$$\varepsilon_Z = \frac{1}{E}\Big(\sigma_Z - \mu\sigma_X\Big)$$

$$\sigma_Y = -\frac{\mu}{E}\Big(\sigma_X + \sigma_Z\Big)$$

$$\gamma_{XZ} = \frac{2(1+\mu)}{E}\tau_{XZ}$$

and $$\gamma_{XY} = \gamma_{YZ} = 0 \quad (\because \quad \tau_{XY} = \tau_{YZ} = 0)$$

(*b*) **Plane Strain.** There are many problems in which one dimension (say, y-direction) is very large in comparison to the other two. If such bodies are loaded by forces which are perpendicular to the longitudinal elements (in the long direction y) and do not vary along the length, all cross-sections will be in the same condition. The state of affairs existing in the xz plane through a point holds for all planes parallel to it. Such a case is known as *plane strain* case. The strain components ε_Y, γ_{XY} and γ_{ZY} will each be zero. The other strain components ε_X, ε_Z and γ_{XZ} are given by the following Hooke's law equations :

$$\varepsilon_Y = 0 = \frac{1}{E}\Big[\sigma_Y - \mu(\sigma_Z + \sigma_X)\Big]$$

$$\therefore \quad \sigma_Y = \mu(\sigma_Z + \sigma_X) \qquad \ldots(22{\cdot}16)$$

$$\therefore \quad \varepsilon_X = \frac{1}{E}\Big[\sigma_X - \mu\sigma_Y - \mu\sigma_Z\Big]$$

$$= \frac{1}{E}\Big[\sigma_X - \mu\sigma_Z - \mu^2(\sigma_X + \sigma_Z)\Big]$$

or $$\varepsilon_X = \frac{1-\mu^2}{E}\Big[\sigma_X - \frac{\mu}{1-\mu}\sigma_Z\Big] \qquad \ldots[22{\cdot}17\ (a)]$$

Similarly, $\epsilon_z=\dfrac{1-\mu^2}{E}\left[\sigma_z-\dfrac{\mu}{1-\mu}\sigma_x\right]$ ...[22·17 (*b*)]

and $\gamma_{xz}=\dfrac{2(1+\mu)}{E}\tau_{xz}$ ...[22·17 (*c*)]

$\gamma_{xy}=\gamma_{yz}=0$ (Hence $\tau_{xy}=\tau_{yz})=0$

Equilibrium Equations. For both plane stress as well as plane strain case, the equilibrium equations are :

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xz}}{\partial z}+X=0$$

$$\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\sigma_z}{\partial z}+Z=0 \qquad ...(22·18)$$

22·8. COMPATIBILITY EQUATION IN TWO DIMENSIONAL CASE

For the two dimensional case, the six compatibility equations (Eq. 22·10) evidently reduce to one single equation :

$$\frac{\partial^2\epsilon_x}{\partial z^2}+\frac{\partial^2\epsilon_z}{\partial x^2}=\frac{\partial^2\gamma_{xz}}{\partial x\partial z} \qquad ...(22·19)$$

Compatibility equation in terms of Stress

The above compatibility equation in terms of strains can be converted into the compatibility equation in terms of stress. We shall consider both the cases : plane stress case and plane strain case.

(1) Plane Stress Case

Substituting the value of ϵ_x, ϵ_z and γ_{xz} from the Hooke's law quations (Eq. 22·15 to Eq. 22·19), we get

$$\frac{\partial^2}{\partial z^2}(\sigma_x-\mu\sigma_z)+\frac{\partial^2}{\partial x^2}(\sigma_z-\mu\sigma_z)=2(1+\mu)\frac{\partial^2\tau_{xz}}{\partial x\partial z} \qquad ...(i)$$

Again, differentiating first of the equilibrium equations [Eq. 22·18 (*a*)] with respect to x, and the second [Eq. 22·18 (*b*)] with respect to z and adding them together, we get

$$2\frac{\partial^2\tau_{xz}}{\partial x\partial z}=-\left(\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_z}{\partial z^2}\right)-\left(\frac{\partial X}{\partial x}+\frac{\partial Z}{\partial z}\right) \qquad ...(ii)$$

Substituting the value of $\dfrac{2\partial^2\tau_{xz}}{\partial x\partial z}$ in (*i*), we get

$$\frac{\partial^2}{\partial z^2}(\sigma_x-\mu\sigma_z)+\frac{\partial^2}{\partial x^2}(\sigma_z-\mu\sigma_x)+(1+\mu)\left(\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_z}{\partial z^2}\right)$$

$$+(1+\mu)\left(\frac{\partial X}{\partial x}+\frac{\partial Z}{\partial z}\right)=0$$

This, on simplification, gives

$$\frac{\partial^2\sigma_x}{\partial z^2}+\frac{\partial^2\sigma_z}{\partial x^2}+\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_z}{\partial z^2}=-(1+\mu)\left(\frac{\partial X}{\partial x}+\frac{\partial Z}{\partial z}\right)$$

or $$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)(\sigma_x+\sigma_z)=-(1+\mu)\left(\frac{\partial X}{\partial x}+\frac{\partial Z}{\partial z}\right) \qquad ...(22·20)$$

If body forces are absent, or constant, we have

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)(\sigma_x+\sigma_z)=0 \qquad ...(22·21)$$

which is the required compatibility equation in terms of stress, for the plane stress case.

(2) Plane Strain Case

Substituting the values of ϵ_x, ϵ_z and γ_{xz} (Eq. 22·17), we get from Eq. 22·19,

$$(1-\mu^2)\frac{\partial^2\sigma_x}{\partial z^2}-\mu(1+\mu)\frac{\partial^2\sigma_z}{\partial z^2}+(1-\mu^2)\frac{\partial^2\sigma_z}{\partial x^2}-\mu(1+\mu)\frac{\partial^2\sigma_x}{\partial x^2}$$

$$=2(1+\mu)\frac{\partial^2\tau_{xz}}{\partial x\partial z} \qquad ...(iii)$$

Substituting the value of $\dfrac{2\partial^2\tau_{xz}}{\partial x\partial z}$ as found in (*ii*), we get

$$(1-\mu^2)\frac{\partial^2\sigma_x}{\partial z^2}-\mu(1+\mu)\frac{\partial^2\sigma_z}{\partial z^2}+(1-\mu^2)\frac{\partial^2\sigma_z}{\partial x^2}-\mu(1+\mu)\frac{\partial^2\sigma_x}{\partial x^2}$$

$$+(1+\mu)\left[\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_z}{\partial z^2}+\frac{\partial X}{\partial x}+\frac{\partial Z}{\partial z}\right]=0$$

which, on simplification, reduces to,

$$\frac{\partial^2\sigma_x}{\partial z^2}+\frac{\partial^2\sigma_z}{\partial z^2}+\frac{\partial^2\sigma_z}{\partial x^2}+\frac{\partial^2\sigma_x}{\partial x^2}=-\frac{1}{1-\mu}\left(\frac{\partial X}{\partial x}+\frac{\partial Z}{\partial z}\right)$$

or $$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)(\sigma_x+\sigma_z)=-\frac{1}{1-\mu}\left(\frac{\partial X}{\partial x}+\frac{\partial Z}{\partial z}\right) \qquad ...(22·22)$$

If the body forces are absent, or constant, we have

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)(\sigma_x+\sigma_z)=0 \qquad ...(22·28)$$

which is the same as Eq. 22·21 found for the plane stress case. Thus in case of constant body forces (or no body forces), same compatibility equation holds both for the case of plane stress and for case of plane strain. Hence the stress distribution is the same in both the cases, provided the shape of the boundary and the external forces are the same. Also, the stress distribution is the same for all the isotropic materials, since Eq. 22·21 or 22·23 do not contain any elastic constant. The photo-elastic method of determination of stress-distribution is based on this conclusion.

22·9. STRESS FUNCTION

The solution of a two dimensional problem of elasticity reduces to the integration of the differential equations of equilibrium together with the compatibility equation and the boundary condition equations. The equations are :

$$\frac{\partial \sigma x}{\partial x}+\frac{\partial \tau xz}{\partial z}=0 \qquad \text{...[22·24 (a)]}$$

$$\frac{\partial \tau xz}{\partial x}+\frac{\partial \sigma z}{\partial z}+Z=0 \qquad \text{...[22·24 (b)]}$$

and
$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)(\sigma x+\sigma z)=0 \qquad \text{...[22·24 (c)]}$$

(where Z is the body force per unit volume in the Z-direction. It is assumed that the weight of the body is directed towards Z-axis, so that *X* is zero.)

It is usual to reduce the above three equations into one single equation in terms of the so-called '*stress function*' ϕ defined by

$$\sigma x=\frac{\partial^2\Phi}{\partial z^2} \qquad \text{...[22·25 (a)]}$$

$$\sigma z=\frac{\partial^2\Phi}{\partial x^2} \qquad \text{...[22·25 (b)]}$$

and
$$\tau xz=-\frac{\partial^2\Phi}{\partial x \partial z}-Z.x \qquad \text{...[22·25 (c)]}$$

The above function Φ is called the *Airy's Stress Function* and was introduced by G.B. Airy in 1862. It can be easily seen that the stress function Φ defined by Eq. 22·25 satisfies the equilibrium equations [Eq. 22·24 (*a*), (*b*)]. Performing differentiation on Eq. 22·25, we get

$$\frac{\partial \sigma x}{\partial x}=\frac{\partial^3\Phi}{\partial x \partial z^2}\ ;\ \frac{\partial \tau xz}{\partial z}=-\frac{\partial^3\Phi}{\partial x \partial z^2}$$

$$\frac{\partial \sigma z}{\partial z}=\frac{\partial^3\Phi}{\partial z \partial x^2}\ ;\ \frac{\partial \tau xz}{\partial x}=-\frac{\partial^3\Phi}{\partial x^2 \partial z}$$

$$\therefore\quad \frac{\partial \sigma x}{\partial x}+\frac{\partial \tau xz}{\partial z}=\frac{\partial^3\Phi}{\partial x \partial z^2}-\frac{\partial^3\Phi}{\partial x \partial z^2}=0 \text{ (satisfied)}$$

$$\frac{\partial \tau xz}{\partial x}+\frac{\partial \sigma z}{\partial z}+Z=-\frac{\partial^3\Phi}{\partial x^2 \partial z}-Z+\frac{\partial^3\Phi}{\partial z \partial x^2}+Z=0 \text{ (satisfied)}$$

Again, substituting the proper derivatives of stress components as function of Airy's stress function in the compatibility equation, [Eq. 22·24 (*c*)], we get

$$\frac{\partial^4\Phi}{\partial x^4}+2\frac{\partial^4\Phi}{\partial x^2 \partial z^2}+\frac{\partial\Phi^4}{\partial z^4}=0 \qquad \text{...(22·26)}$$

or
$$\nabla^2\left[\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial z^2}\right]=0$$

or
$$\nabla^4\ \Phi=0$$

This is a biharmonic differential equation of fourth order, and is known as the compatibility equation in terms of stress function. The solution of Eq. 22·26 must also satisfy the boundary conditions.

In most of the problems, it is usual to find the stresses due to body forces and those due to boundary forces separately. In that case, Eq. 22·25 (*c*) is modified to

$$\tau xz=-\frac{\partial^2\Phi}{\partial x \partial z}.$$

The solution of two dimensional problems is thus reduced to the integration of the differential equation 22·26 having regard to the boundary conditions. In case of long rectangular strips, the solution of the biharmonic equation is done in the form of polynomials of various degrees.

22·10. EQUILIBRIUM EQUATIONS IN POLAR COORDINATES

In most of the problems, it is more convenient to use polar coordinates. An infinitesimal element in the polar coordinate system is shown in Fig. 22·5.

Let σ_r=normal stress component in radial direction.
σ_θ=normal stress component in circumferential direction.
$\tau_{r\theta}$=shear stress component acting tangentially to the four surfaces.

Let the stresses on the four faces be further defined by suffixes 1, 2, 3 and 4 respectively.

Resolving the forces in θ direction and noting that

$$\sin\frac{d\theta}{2} \simeq \frac{d\theta}{2} \text{ and } \cos\frac{d\theta}{2} \simeq 1$$

and assuming body forces in θ direction to be zero, we get

$$\{(\sigma_\theta)_2-(\sigma_\theta)_4\}\, dr \cos\frac{d\theta}{2}+\{(\tau_{r\theta})_2+(\tau_{r\theta})_4\}\, dr \,.\, \sin\frac{d\theta}{2} +\{(\tau_{r\theta})_1-(\tau_{r\theta})_3\}\, r\, d\theta=0$$

or $$\{(\sigma_\theta)_2-(\sigma_\theta)_4\}\, dr+\{(\tau_{r\theta})_2+(\tau_{r\theta})_4\}\, dr \,.\, \frac{d\theta}{2}+\{(\tau_{r\theta})_1-(\tau_{r\theta})_3\}\, r d\theta=0.$$

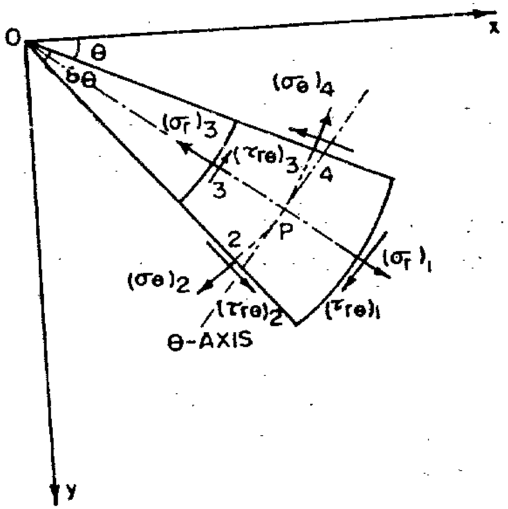


Fig. 22·5. Stresses in polar co-ordinates.

Dividing by $dr.d\theta$, we get

$$\frac{(\sigma_\theta)_2-(\sigma_\theta)_4}{d\theta}+\frac{(\tau_{r\theta})_2+(\tau_{r\theta})_4}{2}+\frac{\{(\tau_{r\theta})_1-(\tau_{r\theta})_3\}\, r}{dr}=0 \quad \ldots(i)$$

Assuming the element to be shrunk to a point, we have

$$\frac{(\sigma_\theta)_2-(\sigma_\theta)_4}{d\theta}=\frac{\partial\sigma_\theta}{\partial\theta}$$

$$\frac{(\tau_{r\theta})_2+(\tau_{r\theta})_4}{2}=\tau_{r\theta}$$

and $$\frac{\{(\tau_{r\theta})_1-(\tau_{r\theta})_3\}\, r}{dr}=\frac{\partial(\tau_{r\theta})}{\partial r}=\tau_{r\theta}+r\frac{\partial\tau_{r\theta}}{\partial r}$$

Substituting in (i), we get

$$\frac{\partial\sigma_\theta}{\partial\theta}+\tau_{r\theta}+\left(\tau_{r\theta}+r\frac{\partial\tau_{r\theta}}{\partial r}\right)=0$$

which gives

$$\frac{1}{r}\,\frac{\partial\sigma_\theta}{\partial\theta}+\frac{\partial\tau_{r\theta}}{\partial r}+\frac{2\tau_{r\theta}}{r}=0 \quad \ldots(\text{I}) \quad \ldots[22\cdot27\ (a)]$$

which is the equilibrium equation in θ direction.

Similarly, resolving in r-direction, we get

$$\{(\sigma_r)_1-(\sigma_r)_3\}\, r\, d\theta-\{(\sigma_\theta)_2+(\sigma_\theta)_4\}\, dr \,.\, \sin\frac{d\theta}{2} +\{(\tau_{r\theta})_2-(\tau_{r\theta})_4\}\, dr \,.\, \cos\frac{d\theta}{2}=0$$

or $$\{(\sigma_r)_1-(\sigma_r)_3\}\, r\, d\theta-\{(\sigma_\theta)_2+(\sigma_\theta)_4\}\, dr \,.\, \frac{d\theta}{2}+\{(\tau_{r\theta})_2-(\tau_{r\theta})_4\}\, dr=0$$

Dividing by $dr.d\theta$, we get

$$\frac{\{(\sigma_r)_1-(\sigma_r)_3\}\, r}{dr}-\frac{(\sigma_\theta)_2+(\sigma_\theta)_4}{2}+\frac{(\tau_{r\theta})_2-(\tau_{r\theta})_4}{d\theta}=0$$

or $$\frac{\partial(\sigma_r)r}{\partial r}-\sigma_\theta+\frac{\partial\tau_{r\theta}}{\partial\theta}=0$$

or $$r\frac{\partial\sigma_r}{\partial r}+\sigma_r-\sigma_\theta+\frac{\partial\tau_{r\theta}}{\partial\theta}=0$$

which may be rewritten as

$$\frac{\partial\sigma_r}{\partial r}+\frac{(\sigma_r-\sigma_\theta)}{r}+\frac{1}{r}\,\frac{\partial\tau_{r\theta}}{\partial\theta}=0 \quad \ldots(\text{II}) \quad \ldots[22\cdot27\ (b)]$$

which is the second equilibrium equation. If, however, R is the body force per unit volume in the r-direction, one more term $R.rd\theta.dr$ will be added to the right hand side, and Eq. (II) will be modified as

$$\frac{\partial\sigma_r}{\partial r}+\frac{\sigma_r-\sigma_\theta}{r}+\frac{1}{r}\,\frac{\partial\tau_{r\theta}}{\partial\theta}+R=0 \quad \ldots(\text{II}\ a) \quad \ldots[22\cdot27\ (c)]$$

22·11. COMPATIBILITY EQUATION AND STRESS FUNCTION IN POLAR CO-ORDINATES

It can be shown that the compatibility equation in terms of stress components, in polar co-ordinates, is given by

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\,\frac{\partial}{\partial r}+\frac{1}{r^2}\,\frac{\partial^2}{\partial\theta^2}\right)(\sigma_r+\sigma_\theta)=0 \quad \ldots(22\cdot28)$$

The stress function Φ is defined in terms of stress components in polar co-ordinates as follows :

$$\sigma_r=\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} \quad \ldots[22\cdot29\ (a)]$$

$$\sigma_\theta=\frac{\partial^2\Phi}{\partial r^2} \quad \ldots[22\cdot29\ (b)]$$

and
$$\tau_{r\theta}=\frac{1}{r^2}\frac{\partial\Phi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\Phi}{\partial r\partial\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right) \quad \ldots[22\cdot29\ (c)]$$

To yield a possible stress distribution, this stress function must ensure that the condition of compatibility (Eq. 22·28) is satisfied. In cartesian co-ordinates, this condition is $\nabla^4\Phi=0$ (Eq. 22·26). A corresponding condition in polar co-ordinates can be obtained by the substitutions : $r^2=x^2+z^2$ and $\theta=\tan^{-1}\frac{z}{x}$. Thus, the compatibility equation, in terms of Φ, in polar co-ordinates becomes :

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}\right)=0$$

or
$$\nabla_r^2(\nabla_r^2\Phi)=0 \quad \ldots(22\cdot30)$$

Example 22·1. *Given the following stress function*

$$\Phi=\frac{H}{\pi}\, z\tan^{-1}\frac{x}{z}$$

determine the stress components σx, σz *and* τxz.

Solution.

By successive differentiation of the stress function, we get

$$\frac{\partial\Phi}{\partial z}=\frac{H}{\pi}\left[-\frac{xz}{x^2+z^2}+\tan^{-1}\frac{x}{z}\right] \quad \ldots(i)$$

$$\frac{\partial^2\Phi}{\partial z^2}=\frac{H}{\pi}\frac{1}{(x^2+z^2)^2}\left[2xz^2-xz^2-x^3-xz^2-x^3\right]$$

$$=-\frac{2H}{\pi}\frac{x^3}{(x^2+z^2)^2} \quad \ldots(ii)$$

$$\frac{\partial^3\Phi}{\partial z^3}=\frac{H}{\pi}\frac{8x^3z}{(x^2+z^2)^3} \quad \ldots(iii)$$

$$\frac{\partial^4\Phi}{\partial z^4}=\frac{H}{\pi}\frac{8x^5-40x^3z^2}{(x^2+z^2)^4} \quad \ldots(iv)$$

$$\frac{\partial^3\Phi}{\partial z^2\partial x}=-\frac{2H}{\pi}\frac{3x^2z^2-x^4}{(x^2+z^2)^3} \quad \ldots(v)$$

$$\frac{\partial^4\Phi}{\partial z^2\partial x^2}=\frac{H}{\pi}\frac{64x^3z^2-24xz^4-8x^5}{(x^2+z^2)^4} \quad \ldots(vi)$$

Similarly,

$$\frac{\partial\Phi}{\partial x}=\frac{H}{\pi}\frac{z^2}{x^2+z^2} \quad \ldots(vii)$$

$$\frac{\partial^2\Phi}{\partial x^2}=-\frac{2H}{\pi}\frac{xz^2}{(x^2+z^2)^2} \quad \ldots(viii)$$

$$\frac{\partial^3\Phi}{\partial x^3}=\frac{2H}{\pi}z^2\cdot\frac{3x^2-z^2}{(x^2+z^2)^3} \quad \ldots(ix)$$

$$\frac{\partial^4\Phi}{\partial x^4}=\frac{H}{\pi}\cdot\frac{24xz^4-24x^3z^2}{(x^2+z^2)^4} \quad \ldots(x)$$

Now
$$\sigma x=\frac{\partial^2\Phi}{\partial z^2}$$

$$=-\frac{2H}{\pi}\frac{x^3}{(x^2+z^2)^2} \quad \text{(Answer)}$$

$$\sigma z=\frac{\partial^2\Phi}{\partial x^2}=-\frac{2H}{\pi}\frac{xz^2}{(x^2+z^2)^2} \quad \text{(Answer)}$$

and
$$\tau xz=\frac{\partial^2\Phi}{\partial x\partial z}=-\frac{2H}{\pi}\cdot\frac{x^2z}{(x^2+z^2)^2} \quad \text{(Answer)}$$

Check for equilibrium equations

The above solution will be acceptable only if the stress components satisfy the equilibrium equations. The equilibrium equation are

$$\frac{\partial\sigma x}{\partial x}+\frac{\partial\tau xz}{\partial z}=0 \quad \ldots(I)$$

$$\frac{\partial\sigma x}{\partial z}+\frac{\partial\tau zx}{\partial x}=0 \quad \ldots(II)$$

Now
$$\frac{\partial\sigma z}{\partial x}=\frac{H}{\pi}\frac{2x^4-6x^2z^2}{(x^2+z^2)^3}$$

$$\frac{\partial\tau xz}{\partial z}=-\frac{H}{\pi}\frac{2x^4-6x^2z^2}{(x^2+z^2)^3}$$

$$\frac{\partial\sigma z}{\partial z}=\frac{H}{\pi}\frac{4xz^3-4x^3z}{(x^2+z^2)^3}$$

$$\frac{\partial\tau xz}{\partial x}=-\frac{H}{\pi}\frac{4xz^3-4x^3z}{(x^2+z^2)^3}.$$

Substituting the above values in the equilibrium equations, we see that they are satisfied.

Check for compatibility equation

$$\frac{\partial^4\Phi}{\partial x^4}+\frac{2\partial^4\Phi}{\partial x^2\partial z^2}+\frac{\partial^4\Phi}{\partial z^4}=0$$

Substituting the values, we get

$$\frac{H}{\pi}\,\frac{1}{(x^2+z^2)^4}\Big[24xz^4-24x^3z^2+64x^3z^2-24xz^4$$
$$-8x^5+8x^5-40x^3z^2\Big]=0$$

Thus the compatibility equation is also satisfied.

Hence the values of σx, σz and τxz found above : acceptable.

Example 22·2. *Given the following stress function*

$$\Phi=\frac{P}{\pi}\,r\theta.\cos\theta$$

determine the stress components : σ_r, σ_θ *and* $\tau_{r\theta}$.

Solution.

The stress components, by definition of Φ, are given as follows :

$$\sigma_r=\frac{1}{r}\,\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\,\frac{\partial^2\Phi}{\partial\theta^2} \quad ...(i)$$

$$\sigma_\theta=\frac{\partial^2\Phi}{\partial r^2} \quad ...(ii)$$

$$\tau_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial\Phi}{\partial\theta}\right)=\frac{1}{r^2}\,\frac{\partial\Phi}{\partial\theta}-\frac{1}{r}\,\frac{\partial^2\Phi}{\partial r\partial\theta} \quad ...(iii)$$

The various derivatives are as follows :

$$\frac{\partial\Phi}{\partial r}=\frac{P}{\pi}\,\theta\cos\theta$$

$$\frac{\partial^2\Phi}{\partial r^2}=0$$

$$\frac{\partial\Phi}{\partial\theta}=\frac{P}{\pi}\,r(-\theta\sin\theta+\cos\theta)$$

$$\frac{\partial^2\Phi}{\partial\theta^2}=-\frac{Pr}{\pi}\,(\theta\cos\theta+2\sin\theta)$$

$$\frac{\partial^2\Phi}{\partial r\partial\theta}=\frac{P}{\pi}\,(-\theta\sin\theta+\cos\theta).$$

Substituting the values in (*i*), (*ii*) and (*iii*), we get

$$\sigma_r=\frac{1}{r}\,\frac{P}{\pi}\,\theta\cos\theta-\frac{1}{r^2}\,\frac{P}{\pi}\,r(\theta\cos\theta+2\sin\theta)$$

$$=\frac{1}{r}\,\frac{P}{\pi}\,\theta\cos\theta-\frac{1}{r}\,\frac{P}{\pi}\,\theta\cos\theta-\frac{1}{r}\,\frac{P}{\pi}\,2\sin\theta$$

$$=-\frac{2}{r}\,\frac{P}{\pi}\sin\theta \text{ (Answer)}$$

$$\sigma_\theta=\frac{\partial^2\Phi}{\partial r^2}=0 \text{ (Answer)}$$

$$\tau_{r\theta}=\frac{1}{r^2}\,\frac{P}{\pi}\,r(-\theta\sin\theta+\cos\theta)-\frac{1}{r}\,\frac{P}{\pi}(-\theta\sin\theta+\cos\theta)$$

$$=0 \quad \text{(Answer)}$$

It can be shown that the above values satisfy both equilibrium equations as well as compatibility equation.

22·12. SOLUTION OF TWO DIMENSIONAL PROBLEMS BY POLYNOMIALS

In the case of two dimensional problems, the solution of problem (when the body forces are absent or are constant) consists of the integration of the differential equation

$$\frac{\partial^4\Phi}{\partial x^4}+2\,\frac{\partial^4\Phi}{\partial x^2\partial z^2}+\frac{\partial^4\Phi}{\partial z^4}=0 \quad ...(22{\cdot}26)$$

The solution so obtained must satisfy the boundary conditions. Eq. 22·26 can be solved by means of polynomials of various degrees by suitably adjusting their coefficients. We shall consider here second and third degree polynomials.

(*i*) Second degree polynomial

Let the solution of Eq. 22·26 be represented by the following polynomial of second degree

$$\Phi_2=\frac{a_2}{2}\,x^2+b_2xz+\frac{c_2}{2}\,z^2 \quad ...(22{\cdot}31)$$

where suffix 2 denotes the second degree of the polynomial, and *a*, *b* and *c* are constants. It can be seen that Φ, given above satisfies Eq. 22·26. The corresponding stress components (Eq. 22·25) are

$$\sigma x = \frac{\partial^2 \Phi_2}{\partial z^2} = c_2$$

$$\sigma z = \frac{\partial^2 \Phi_2}{\partial x^2} = a_2$$

and
$$\tau xz = -\frac{\partial^2 \Phi_2}{\partial x \partial z} = -b_2.$$

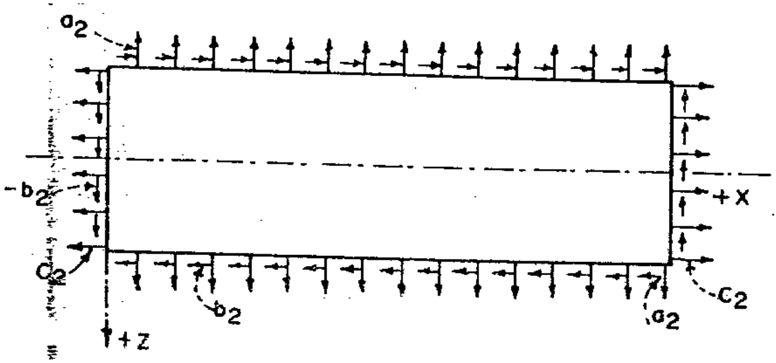


Fig. 22·6

This shows that the above stress components do not depend upon the co-ordinates x and z, *i.e.* they are constant throughout the body. Thus, the stress function represented by Eq. 22·31 represents a state of uniform tensions (or compressions) in two perpendicular directions accompained with uniform shear, as shown in Fig. 22·6.

(*ii*) **Third degree polynomial**

Let the solution of Eq. 22·26 be represented by the following polynomial of third degree :

$$\Phi_3 = \frac{a_3}{3\,.\,2}x^3 + \frac{b_3}{2}x^2 z + \frac{c_3}{2}xz^2 + \frac{d_3}{3\,.\,2}z^3 \quad \ldots(22\text{·}32)$$

It can be shown that Φ_3 given above satisfies Eq. 22·26. The corresponding stress components are :

$$\sigma x = \frac{\partial^2 \Phi_3}{\partial z^2} = c_3 x + d_3 z$$

$$\sigma z = \frac{\partial^2 \Phi_3}{\partial x^2} = a_3 x + b_3 z \quad \ldots(a)$$

and
$$\tau xz = -\frac{\partial^2 \Phi_3}{\partial x \partial z} = -b_3 x - c_3 z.$$

It should be noted that we are completely free in choosing the magnitudes of the coefficients a_3, b_3, c_3 and d_3 since Eq. 22·26 is satisfied whatever values these coefficients have. Choosing all coefficients except d_3 equal to zero, we get from (*a*),

$$\sigma x = d_3 z \quad \ldots(22\text{·}33)$$

$$\sigma z = 0 \text{ and } \tau xz = 0.$$

Eq. 22·33 evidently represents a case of pure bending, as shown in Fig. 22·7. At $z=-h$, $\sigma x = -d_3 h$ and at $z=+h$, $\sigma x = +d_3 h$. The variation of σx with z is linear.

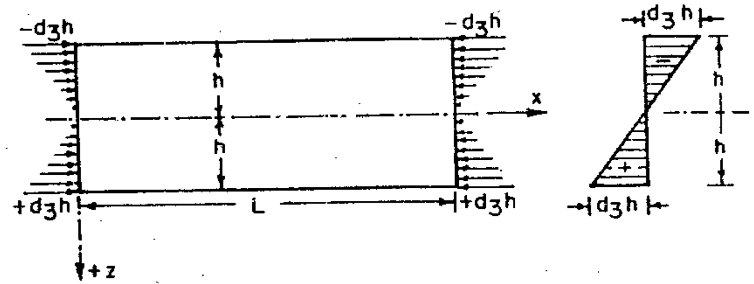


Fig. 22·7

Similarly, if all the coefficients except b_3 are zero, we get from (*a*),

$$\sigma x = 0$$

$$\sigma z = b_3 z \quad \ldots(22\text{·}34)$$

$$\tau xz = -b_3 x.$$

The stresses represented by Eq. 22·34 vary as shown in Fig. 22·8.

The σz stress is constant with x (*i.e.* constant along the span L of the beam), but varies with z at a particular section. At $z=+h$, $\sigma x = b_3 h$ (*i.e.* tensile), while at $z=-h$, $\sigma z = -b_3 h$ (*i.e.* compressive). σx is zero everywhere. Shear stress τxz is zero at $x=0$ and is equal to $-b_3 L$ at $x=L$. At any other section, the shear stress is proportional to x.

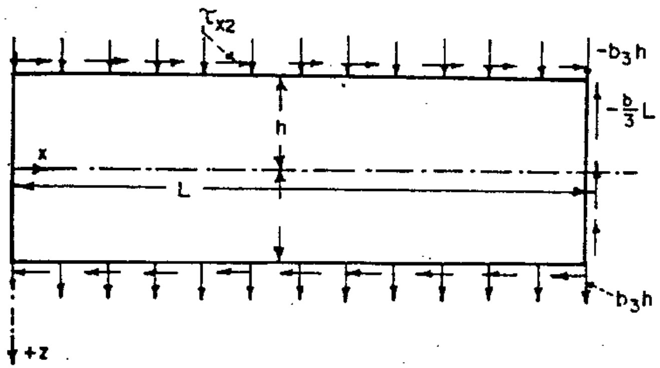


Fig. 22·8

Example 22·3. *Given the following stress function :*

$$\Phi = -\frac{F}{d^3}\, xz^2\,(3d-2z).$$

Determine the stress components and sketch their variations in a region included in $z=0$, $z=d$, $x=0$, on the side x positive.

Solution

The given stress function may be rewritten as

$$\Phi = -\frac{3F}{d^2}\, xz^2 + \frac{2F}{d^3}\, xz^3 \qquad ...(i)$$

$$\therefore \qquad \frac{\partial^2\Phi}{\partial z^2} = -\frac{6Fx}{d^2} + \frac{12F}{d^3}\, xz$$

$$\frac{\partial^2\Phi}{\partial x^2} = 0$$

and
$$\frac{\partial^2\Phi}{\partial x\partial z} = -\frac{6Fz}{d^2} + \frac{6F}{d^3}\, z^2$$

Hence
$$\sigma_x = \frac{\partial^2\Phi}{\partial z^2} = -\frac{\partial Fx}{d^2} + \frac{12F}{d^3}\, x.z \qquad ...(ii)$$

$$\sigma_z = \frac{\partial^2\Phi}{\partial x^2} = 0 \qquad ...(iii)$$

and
$$\tau_{xz} = -\frac{\partial^2\Phi}{\partial x\partial z} = \frac{6Fz}{d^2} - \frac{6F}{d^3}\, z^2 \qquad ...(iv)$$

$\therefore$ Eqs. (*ii*), (*iii*) and (*iv*) give the values for the three stress components. Let us find their values at certain boundary points.

(*i*) *Variation of* σ_x

From (*ii*), it is clear that σ_x varies linearly with x, and at a given section it varies linearly with z.

At $\quad x=0$ and $z=\pm d$, $\sigma_x=0$

At $\quad x=L$ and $z=0$, $\quad \sigma_x = -\dfrac{6FL}{d^2}$

At $\quad x=L$ and $z=+d$, $\sigma_x = -\dfrac{6FL}{d^2} + \dfrac{12F}{d^3}\, Ld = \dfrac{6FL}{d^2}$

At $\quad x=L$ and $z=-d$, $\sigma_x = -\dfrac{6FL}{d^2} - \dfrac{12F}{d^3}\, Ld = -\dfrac{18FL}{d^2}$

The variation of σ_x is shown in Fig. 22·9.

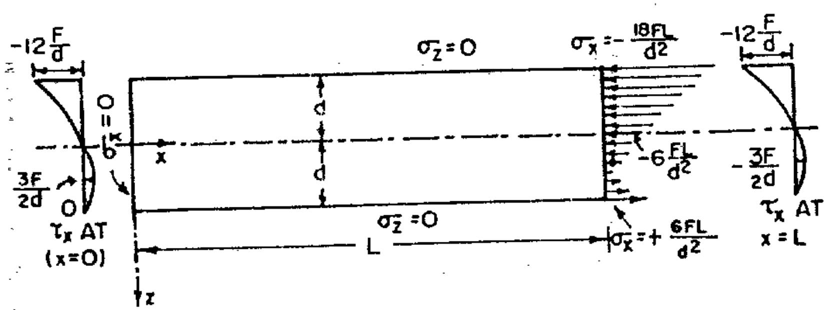


Fig. 22·9

(*ii*) *Variation of* σ_z

σ_z is zero for all values of x.

(*iii*) *Variation of* τ_{xz}

From (*iv*), it is clear that variation of τ_{xz} is parabolic with z. However, τ_{xz} is independent of x, and is thus constant along the length, corresponding to a given value of z.

At $z=0$, $\qquad \tau_{xz}=0$

At $z=+d$, $\quad \tau_{xz} = \dfrac{6F}{d^2}\,.\,d - \dfrac{6F}{d^3}\, d^2 = 0$

At $z=-d$, $\quad \tau_{xz} = -\dfrac{6F}{d^2}\,.\,d - \dfrac{6F}{d^3}(-d)^2 = -\dfrac{12F}{d}$

The variation of τ_{xz} is shown in Fig. 22·9.

Example 22·4. *Investigate what problem of plane stress is satisfied by the stress function*

$$\Phi=\frac{3F}{4d}\left[xz-\frac{xz^3}{3d^2}\right]+\frac{P}{2}z^2$$

applied to the region included in $z=0$, $z=d$, $x=0$ on the side x positive.

Solution.

The given stress function may be written as

$$\Phi=\frac{3F}{4d}xz-\frac{1}{4}\frac{Fxz^3}{d^3}+\frac{P}{2}z^2 \quad \ldots(i)$$

$$\frac{\partial^2\Phi}{\partial x^2}=0$$

$$\frac{\partial^2\Phi}{\partial z^2}=-\frac{3\times2}{4d}\frac{Fxz}{d^3}+\frac{2P}{2}=P-1{\cdot}5\frac{F}{d^3}xz$$

and $$\frac{\partial^2\Phi}{\partial x\partial z}=\frac{3F}{4}-\frac{3}{4}\frac{Fz^2}{d^3}$$

Hence the stress components are :

$$\sigma x=\frac{\partial^2\Phi}{\partial z^2}=P-1{\cdot}5\frac{F}{d^3}xz \quad \ldots(ii)$$

$$\sigma z=\frac{\partial^2\Phi}{\partial x^2}=0 \quad \ldots(iii)$$

$$\tau xz=-\frac{\partial^2\Phi}{\partial x\partial z}=\frac{3}{4}\frac{Fz^2}{d^3}-\frac{3}{4}\frac{F}{d} \quad \ldots(iv)$$

(*i*) *Variation of* σx

$$\sigma x=P-1{\cdot}5\frac{F}{d^3}xz$$

When $x=0$ and $z=0$ or $\pm d$, $\sigma x=P$ (*i.e.* constant across the section)

When $x=L$ and $z=0$, $\sigma x=P$

When $x=L$ and $z=+d$, $\sigma x=P-1{\cdot}5\frac{FL}{d^2}$

When $x=L$ and $z=-d$, $\sigma x=P+1{\cdot}5\frac{FL}{d^2}$

Thus, at $x=L$, the variation of σx is linear with z.

The variation of σx is shown in Fig. 22·10.

Variation of σz

$$\sigma z=\frac{\partial^2\Phi}{\partial x^2}=0$$

Thus σz is zero for all values of x and z.

(*ii*) *Variation of* τxz

$$\tau xz=\frac{3Fz^2}{4d^3}-\frac{3}{4}\frac{F}{d}.$$

Thus τxz varies parabolically with z. However, it is independent of x, *i.e.* its value is the same for all values of x.

At $z=0$, $\tau xz=-\frac{3}{4}\frac{F}{d}$

At $z=\pm d$, $\tau xz=\frac{3}{4}\frac{F}{d^3}(d)^2-\frac{3}{4}\frac{F}{d}=0.$

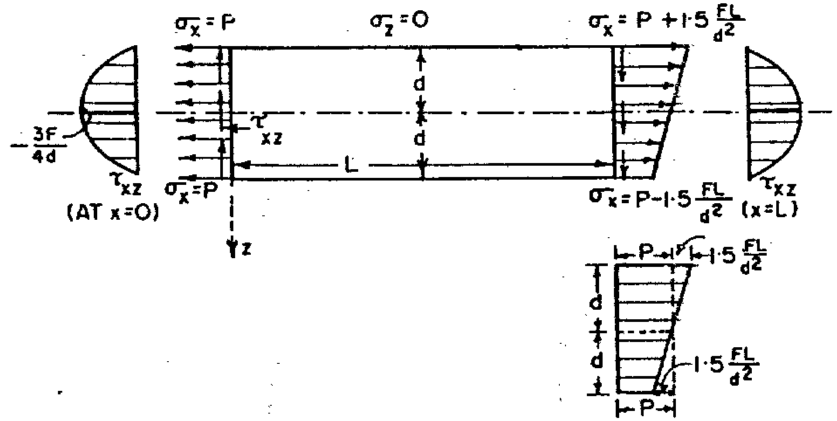


Fig. 22·10

22·13. BENDING OF A CANTILEVER LOADED AT THE END

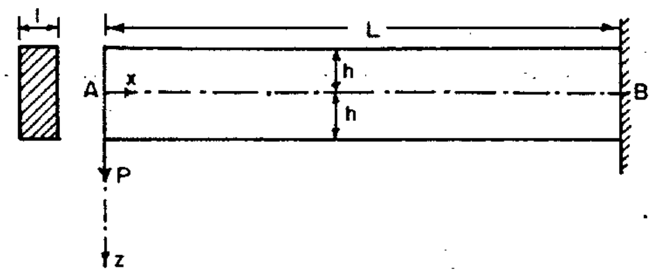


Fig. 22·11 Cantilever loaded at the end.

Consider a cantilever of span L, subjected to a point load P, which may be considered to be the resultant of the shearing forces across the section at $x=0$. Let the depth of the section of the cantilever be $2h$ and width unity.

The normal stress σx at any point in the cantilever may be taken to be proportional to x and z.

Hence $$\sigma x=\frac{\partial^2\Phi}{\partial z^2}=c_1xz \quad ...(1)$$

where c_1 is a constant, to be determined.

Integrating the above equation twice with respect to z, we get

$$\frac{d\Phi}{dz}=c_1\frac{xz^2}{2}+f_1x$$

and $$\Phi=c_1\frac{xz^3}{6}+z.f_1x+f_2x \quad ...(2)$$

where f_1x and f_2x are the functions of x.

Eq. (2) gives the stress function Φ. It should satisfy the compatibility equation $\nabla^4\Phi=0$.

$$\frac{\partial^4\Phi}{\partial x^4}+2\frac{\partial^4\Phi}{\partial x^2\partial z^2}+\frac{\partial^4\Phi}{\partial z^4}=0 \quad ...(3)$$

From (2), $$\frac{\partial^4\Phi}{\partial x^4}=z\frac{d^4f_1x}{dx^4}+\frac{d^4f_2x}{dx^4}$$

$$\frac{\partial^4\Phi}{\partial x^2\partial z^2}=0 \text{ and } \frac{\partial^4\Phi}{\partial z^4}=0$$

Substituting these in (3), we get

$$z\frac{d^4f_1x}{dx^4}+\frac{d^4f_2x}{dx^4}=0 \quad ...(4)$$

This must be satisfied for all values of x and z. If z is not to be zero, we get from (4),

$$\frac{d^4f_1x}{dx^4}=0 \text{ or } f_1x=c_2x^3+c_3x^2+c_4x+c_5 \quad ...(5)$$

and $$\frac{d^4f_2x}{dx^4}=0 \text{ or } f_2x=c_6x^3+c_7x^2+c_8x+c_9 \quad ...(6)$$

Thus, we have to determine nine constants $c_1, c_2, \ldots c_9$.

Substituting the values of f_1x and f_2x in (2), we get

$$\Phi=\frac{c_1xz^3}{6}+z(c_2x^3+c_3x^2+c_4x+c_5)+(c_6x^3+c_7x^2+c_8x+c_9) \quad ...(7)$$

Hence $$\sigma z=\frac{\partial^2\Phi}{\partial x^2}=6(c_2z+c_6)x+2(c_3z+c_7) \quad ...(I)$$

and $$\tau xz=-\frac{\partial^2\Phi}{\partial x\partial z}=-\frac{c_1z^2}{2}-3c_2x^2-2c_3x-c_4 \quad ...(II)$$

Boundary conditions

Let us now apply boundary conditions :

(*a*) *First condition*

$\sigma z=0$ at $z=\pm h$, for all values of x.

Hence from (I), we get

$$0=6(c_2h+c_6)x+2(c_3h+c_7) \quad ...(i)$$

and $$0=6(-c_2h+c_6)x+2(-c_3h+c_7) \quad ...(ii)$$

From (*i*) and (*ii*), it is evident that the constants c_2, c_3, c_6 and c_7 are each zero for non-zero value of x.

Substituting these values in Eq. 7 and (II), we get

$$\Phi=\frac{c_1}{6}xz^3+z(c_4x+c_5)+(c_8x+c_9) \quad ...(7\,a)$$

and $$\tau xz=-\frac{c_1}{2}z^2-c_4 \quad ...(\text{II } a)$$

(*b*) *Second condition*

τxz is zero at the free boundary of top and bottom of the cantilever, *i.e.* at $z=\pm h$

$$\therefore \quad \tau xz=0=-\frac{c_1}{2}(\pm h)^2-c_4$$

$$\therefore \quad c_4=-\frac{c_1h^2}{2} \quad ...(\text{III})$$

Substituting this in II (*a*), we get

$$\tau xz=-\frac{c_1}{2}z^2+\frac{c_1h^2}{2}=\frac{c_1}{2}(h^2-z^2) \quad ...(\text{II } b)$$

(*c*) *Third condition*

The end load P is the resultant of the shearing forces acros the section at $x=0$.

$$P=\int_{-h}^{+h}\tau xz.dz=\int_{-h}^{+h}\frac{c_1}{2}(h^2-z^2).dz$$

or $$P=\tfrac{2}{3}c_1h^3$$

From which $\qquad c_1=\frac{P}{\frac{2}{3}h^3}=\frac{P}{I_{YY}}$ ...(8)

where $\quad I_{YY}=\frac{1}{12}\times1\times(2h)^3=\frac{2}{3}h^3$

Final stresses

Thus, the final stresses are as follows :

(*i*) $\qquad \sigma_X=c_1xz=\frac{Pxz}{I_{YY}}=\frac{M.z}{I_{YY}}$ ...(IV)

This is the same as used in the simple theory of bending.

(*ii*) $\qquad \sigma_Y=0$

(*iii*) $\qquad \tau_{XZ}=\frac{c_1}{2}\left(h^2-z^2\right)=\frac{P}{2I_{YY}}\left(h^2-z^2\right)$ ...(V)

This shows that the external force must be distributed parabolically on the external face.

SECTION 4
MISCELLANEOUS TOPICS

23

Welded Joints

23·1. GENERAL

Welding is a process of jointing two similar pieces of metal by fusion or pressure. A metallic bond is established between the two pieces. This bond has the same mechanical and physical properties as the parent metal. A number of methods are used for the process of fusion. The oxyacetylene or gas welding and electric arc welding are the most important of these methods. The metal at the joint is melted by the heat generated from either an electric are or an oxy-acetylene flame and fuses with metal from a welding rod. After cooling, the parent metal (base metal) and the weld metal form a continuous and homogeneous joint. The welded connections have become so reliable that they are replacing riveted joints, both in structural as well as machine design. Before proceeding further, we shall discuss here, in short, the advantages and disadvantages of welded connections.

Advantages

1. Welded joints are economical, from the points of view of cost of labour and materials, both. The filler plates, gusset plates, connecting angles, etc. are eliminated in welded joints. The smaller sizes of members, compared to those which may be used in riveted connections from the practical point of view, may be used here.

2. The efficiency of the welded joint is 100% as compared to an efficiency of 75 to 100% in case of riveted joints.

3. The fabrication of a complicated structure is easier by connection as in case of a circular steel pipe. The alterations or additions in existing structure are facilitated by it.

4. The welding provides very rigid joints. This is in keeping with the modern trend of providing rigid frames.

5. The noise associated with the riveting work is a source of a great nuisance. This is avoided in welding operation.

6. When riveting is done in populated localities, safety precautions to protect the public from the flying rivets has to be taken. No such precautions are necessary in case of welding operation.

7. The welded structures look more pleasing in comparison to the riveted ones.

Disadvantages

Not withstanding the advantages narrated above, the welded connections have a number of disadvantages in comparison to the riveted connections. The same have been narrated below :

1. No provision for expansion and contraction is kept in welded connection and, therefore, there is possibility of cracks developing in such structures.

2. Due to uneven heating and cooling of the members during welding the members may distort resulting in additional stresses.

3. The inspection of welding work is more difficult and costlier than the riveting work. The welding work requires a skilled person, while, a semi-skilled person can do the riveting work.

4. On account of extreme heat, fatigue may take place.

23·2. TYPES OF WELDS

In structural practice, the following types of welds are used :

1. Butt weld or groove weld.
2. Fillet weld.
3. Plug or slot weld.

1. Butt Weld or Groove Weld

When the joining members are to be jointed in such a way that they form a T or butt against each other, butt weld is used.

The butt weld is usually made convex on either sides. This extra area is called *reinforcement*. The reinforcement varies from 1 to 3 mm.

The common types of butt welds have been shown in Fig. 23·1. The square butt joints shown in Fig. 23·1 (*a*) and (*b*) are used for thickness less than 8 mm. The effective thickness of the weld, called *throat thickness*, is less than the thickness T of the plates jointed. It is taken as $\frac{5}{8}\,T$. In single V-butt joint [Fig. 23·1 (*c*)], the throat thickness is taken at $\frac{3}{4}\,T$. The butt welds of Fig. 23·1 (*d*) and (*e*) are fully effective. In Fig. 23·1 (*e*), a backing strip has been used. As a rule, in single-*V*, single-*U*, single-*J* butt welds, with backing strips, and in which the welding has been done from one side, full penetration is not possible, and the effective throat thickness is taken equal to $\frac{5}{8}$ *T*.

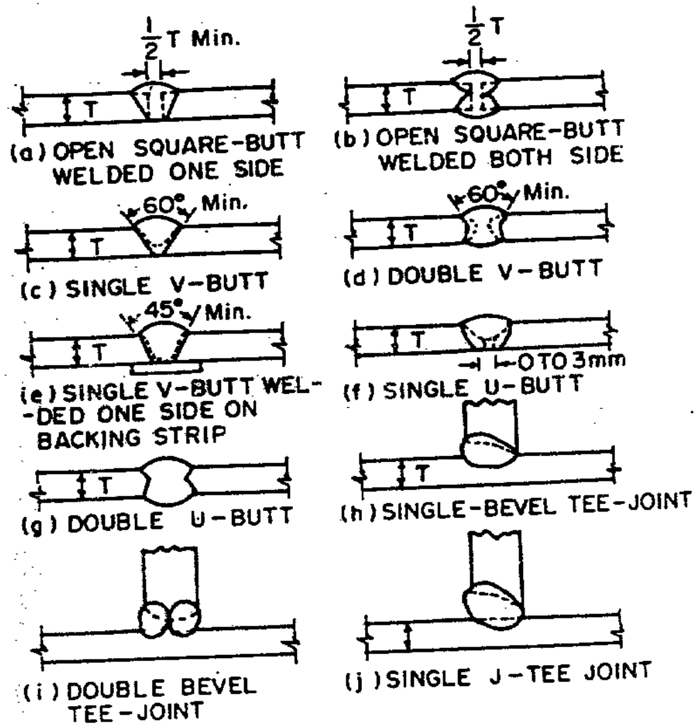


Fig. 23·1. Butt welds.

In all the above mentioned welds with backing strips and in case of double-*V*, double-*U* and double-*J* butt welds, full penetration is possible and the effective thickness of the throat is taken equal to the thickness of the plates joined. Whenever two plates of different thickness are joined, the thickness of the thinner plates must be taken into account

2. Fillet Welds

When the lapped plates are to be joined, fillet welds are used. These are generally of right-angled triangle shape. The outer surface is generally made convex.

The sides containing the right angles are called *legs*. The size of a fillet weld is specified by the minimum leg length. The *throat* or *throat distance* of the fillet weld is equal to the perpendicular distance of the corner from the hypotenuse, the reinforcement being neglected [Fig 23·2 (*a*)]. The throat thickness *t* is, therefore, equal to $\frac{\text{min. leg length}}{\sqrt{2}} = 0{\cdot}707 \times \text{minimum leg length}$.

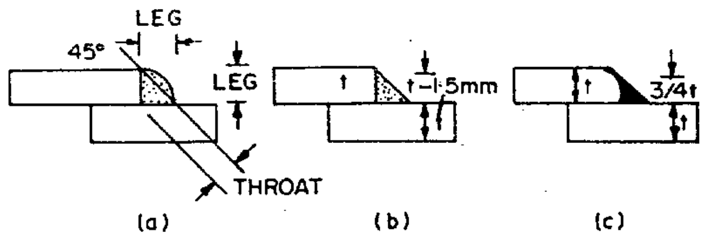


Fig. 23·2. Fillet welds.

For angles other than 90°, throat thickness is given by :

Maximum throat thickness $= k \times$ minimum leg length.

The value of *k* is given in the table below :

Angle	60° to 90°	91° to 100°	101° to 106°	107° to 113°	114° to 120°
k	0·7	0·65	0·6	0·55	0·50

The maximum size of a fillet weld at the square edge of a plate [Fig. 23·2 (*b*)] is 1·5 mm less than the plate thickness and in case of a weld at the rounded edges of flanges or the toe of an angle is kept three-fourths the thickness of the edge [Fig. 23·2 (*c*)].

When the fillet weld is placed parallel to the direction of the force, on both sides of the member, it is called side fillet weld. When the weld is placed at the end of the member, such that it is perpendicular to the direction of the force, it is called end fillet weld (Fig. 23·3).

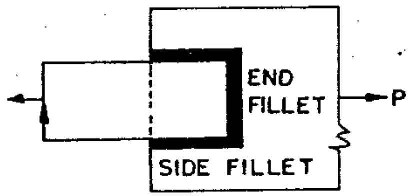


Fig. 23·3

The *effective length* of the fillet weld is taken as the overall length minus twice the weld size. The effective length should not be less than four times the size of the weld, otherwise the weld size must be taken as one-fourth of its effective length. If only the side welds are used, the length of each side fillet weld must not be less than the perpendicular distance between the two.

3. Plug or Slot Weld

Whenever sufficient space is not available for providing the necessary length of the fillet weld, the plug or slot weld is provided to strengthen the connection. The slot or plug weld is also used for equalising the stress in plates and to prevent buckling in case of wide plates.

A circular or a slotted hole is made into one of the members to be jointed. The weld metal is then filled in the hole. Otherwise a fillet weld is provided along the edge of the hole [Fig. 23·4 (*a*)].

Fig. 23·4

The minimum diameter and the width of the hole should not be less than the thickness of the part containing the hole plus 8 mm, the maximum diameter and width being limited to 2·25 times the thickness of the plate punched.

23·3. STRENGTH OF WELDS

1. BUTT WELDS

For full penetration butt welds [Fig. 23·1 (*c*), (*d*), (*e*), (*g*)] the strength of the welds is equal to that of the parent metal. In such cases, no calculations are needed. Even then if the strength is to be found, it can be found by the following formula :

$$P=f_w \times t \times L$$

where P=strength of weld.

t=effective thickness, including reinforcement.

L=length of the weld.

f_w=working stress in tension or compression.

For full penetration welds, the effective thickness is equal to the thickness of the thinner plate. If the full penetration weld has not been used the effective thickness must be determined.

It must be noted here that a butt weld is generally subjected to either tensile or compressive stress.

The allowable stress in butt weld is taken as that for the parent metal. Thus for mild steel conforming to IS : 226-1962 as parent metal and with electrodes conforming to IS : 814-1963, the allowable stresses in the welds, as recommended in IS : 816-1956 are as under :

	Kind of stress	*Max. permissible value*
1.	Tension on section through throat of butt weld	142 N/mm² (1420 kg/cm²)
2.	Compression on section through throat of butt weld	142 N/mm² (1420 kg/cm²)
3.	Fibre stress in bending	
	(*a*) Tension	157·5 N/mm² (1575 kg/cm²)
	(*b*) Compression	157·5 N/mm² (1575 kg/cm²)
4.	Shear on section through throat of butt and fillet welds	102·5 N/mm² (1025 kg/cm²)

Maximum permissible value of stresses for shear and tension are reduced to 80% of those given above if the welding is done at site (field). When effects of wind and/or earth-quake forces are considered, maximum permissible values of stress are increased by 20%.

2. FILLET WELDS

(*a*) Side fillets

When a side fillet weld is subjected to a load P, it is subjected to shear stresses only. The maximum stress will develop at the throat and failure will occur by shear along the throat. If we assume a uniform distribution of shear stress,

$$P=\text{Stress}\times\text{Area}$$

$$\text{Area}=\text{Throat thickness}\times\text{Length}$$

$$=0{\cdot}707\times h\times L$$

$$\therefore \quad P=0{\cdot}707\times h\times L\times f$$

or $$\text{Stress } f=\frac{P}{0{\cdot}707\, hL}.$$

(*b*) End fillets

If a load P is applied to an end fillet weld as shown in Fig. 23·5 it will be subjected to different types of stresses on different planes. The plane DB will be subjected to tensile stress and the plan AC to shearing stress. In both the cases the stress will be numerically equal to $\frac{P}{hL}$. The section AD is subjected to tensile and shear stresses both, each equal to $\frac{P/\sqrt{2}}{Lh/\sqrt{2}}=\frac{P}{Lh}$.

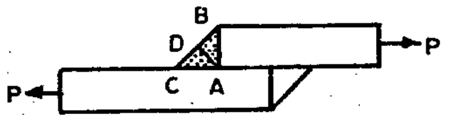


Fig. 23·5

It can be proved that the maximum tension on this section is $\frac{1{\cdot}06P}{Lh}$ and the maximum shear stress is $\frac{1{\cdot}12\,P}{Lh}$. As the material is weaker in shear we shall consider the shear stress only. It may be noted that the shear stress developed in this case is less than that developed in case of sides fillets. So for convenience in design, no distinction is made between the end fillets.

The allowable stress in shear is taken to $10{\cdot}25$ N/mm^2.

3. PLUG OR SLOT WELDS

The strength of a plug or slot weld is equal to the cross-sectional area of the plug or slot, multiplied by the allowable stress.

If fillet weld is provided in the plug or slot, its effective length is taken equal to the average length of the throat of the fillet. Generally, it is taken equal to the length of a line running parallel to the vertical lag of the weld at one-fourth the leg dimension h from it. For a hole of diameter d the length of the fillet is $\pi\left(d-\frac{h}{2}\right)$. The maximum length is limited to 10 times the thickness of the material.

Example 23·1. *A 100 mm × 10 mm plate is to be welded to another plate 150 mm × 10 mm by the fillet welding on three sides as shown in Fig. 23·6. The size of the weld is 6 mm. Find out the necessary overlap of the plate if the smaller plate is to develop full strength. The allowable stress in plates 142 N/mm².*

Solution.

The total load taken by the smaller plate

$=100\times10\times142=142000$ N $=142$ kN

Throat thickness $=\frac{h}{\sqrt{2}}=\frac{6}{\sqrt{2}}=4{\cdot}24$ mm

Allowable load per lineal mm $=4{\cdot}24\times102{\cdot}5=434{\cdot}6$ N

$\therefore$ Total length of weld required $=\frac{142000}{434{\cdot}6}=327$ mm

The length of end fillet $=100$ mm

$\therefore$ Length to be provided inside fillet $=327-100=227$ mm

$\therefore$ Overlap $=x=\frac{227}{2}=113{\cdot}5$ m

Provide an overlap of 120 mm.

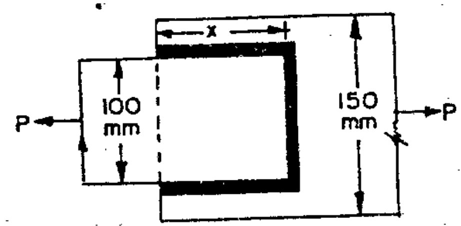


Fig. 23·6

This is more than the distance between two side fillets.

Example 23·2. *A tie bar 120 mm × 10 mm is to be connected to the other of size 120 mm × 15 mm. If the tie bars are to be loaded by 160 kN, find out the size of the end fillets such that the stresses in both the end fillets are same.*

Solution.

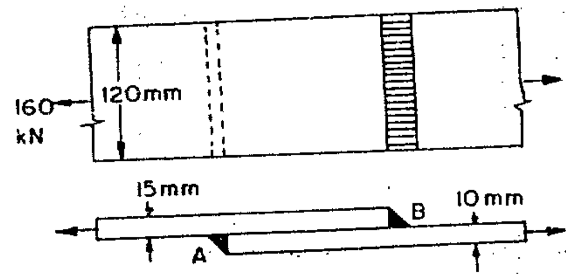


Fig. 23·7

The portion of plates between the welds stretch by the same amount. Therefore, the strain and hence the stress in both the plates are same. The force carried by each plate will be proportional to its thickness. Thus if the 10 mm thick plate carries P_1 force, the 15 mm thick plate will carry $1.5P_1$. Therefore, to keep the stresses same in both the end welds, we must keep the size of the welds in proportion to the thickness of the respective plates.

Let the size of the lower weld $= h$

$\therefore$ Size of the upper weld $= 1.5h$

The length of the welds in each case $= 120$ mm

Strength of lower weld $= 0.707 \times h \times 120 \times 102.5 = 8696h$ N

Strength of upper weld $= 0.707 \times 1.5\,h \times 120 \times 102.5 = 13044h$ N

Total load to be carried by the tie bar $= 160000$ N

$\therefore$ $8696\,h + 13044\,h = 160000$

$$\therefore \quad h = \frac{160000}{21740} = 7.36 \text{ mm}$$

We shall keep $h = 8$ mm. Therefore, the size of the lower weld $= 8$ mm and the size of the upper weld $= 12$ mm.

The maximum allowable size of lower weld $= t - 1.5 = 10 - 1.5 = 8.5 < 8$ mm.

Similarly, the size of upper weld B is lesser than the maximum allowable.

Example 23.3. *A tension member consisting of two channel section 200 mm × 75 mm @ 22.1 kg/m back to back is to be connected to gusset plate. Design the welded joint for the condition that the section is loaded to its full strength. A = 2821 sq. mm, thickness of the flange = 11.4 mm and the thickness of the web = 6.1 mm.*

Solution.

In case of rolled section, the size of the weld is taken of three-fourth of the thickness.

$$\therefore \quad \text{Size of the weld} = \frac{3}{4} \times 6.1 = 4.6 \text{ mm}$$

We shall provide 4 mm weld.

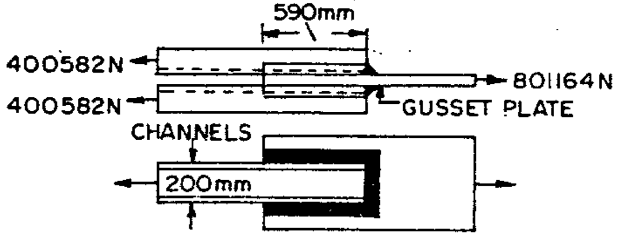


Fig. 23.8

Strength of weld per lineal cm $= 0.707 \times 4 \times 102.5 = 290$ N

The load to be carried by each channel $= 2821 \times 142$
$= 400582$ N

Total length of weld required for one channel

$$= \frac{400582}{290} \simeq 1380 \text{ mm}$$

Length of the end weld $= 200$ cm

Therefore, length of side welds $= 1380 - 200 = 1180$ mm

$$\therefore \quad \text{Overlap} = \frac{1180}{2} = 590 \text{ mm.}$$

Example 23.4. *An I-section is built-up by welding a 250 mm × 15 mm web plate to two 150 mm × 15 mm flange plates by 8 mm fillet welds. Find out maximum shearing force which may be permitted if the mean shearing stress in web and maximum shear stress in weld are not to exceed 100 N/mm².*

Solution.

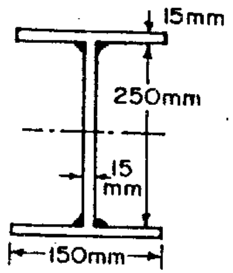


Fig. 23.9

The moment of inertia of the section about xx is,

$$I_{xx} = \frac{1}{12}\left[150 \times 280^3 - 135 \times 250^3\right] = 98.62 \times 10^6 \text{ mm}^4$$

Shear stress q at the section passing through welds is given by

$$q = \frac{F}{I_{xx}\,b}\,(A\bar{y})$$

where b = Effective thickness $= 2 \times$ throat thickness
$= 2 \times 8 \times 0.707 = 11.3$ mm

$A\bar{y}$ = Moment of the flange area about xx
$= 150 \times 15(125 + 7.5) = 298125$ mm³

q = Allowable shear stress $= 100$ N/mm²

$$100 = \frac{F}{98.62 \times 10^6 \times 11.3}\left[298125\right]$$

From which $F=373805$ N$=373{\cdot}8$ kN

The maximum shear force limited on the web

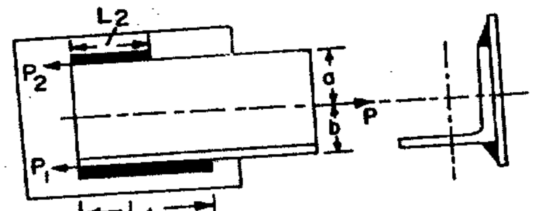
$=250\times15\times100=375000$ N$=375$ kN

$\therefore$ The max. allowable shearing force$=373{\cdot}8$ kN.

23·4. FILLET WELDING OF UNSYMMETRICAL SECTIONS, AXIALLY LOADED

Uptill now we have considered the fillet welding of symmetrical sections. In case of unsymmetrical sections like angles and Tee which are loaded along the axis passing through their centroid, the weld lengths are so arranged that the gravity axis of the weld lines coincides with the neutral axis. This will avoid eccentricity of loading and hence the bending moment.

Let us consider an angle section subjected to load P, welded to a gusset plate as shown in Fig. 23·10.



(a) (b)

Fig. 23·10

Let L_1 and L_2 be the required lengths of welds on the two faces, and P_1 and P_2 be the resisting forces exerted by the respective welds. These are assumed to act along the edges of the angle.

Taking moments about the line of action of P_2 we obtain,

$(a+b)\times P_1=Pa.$ $\qquad\therefore\quad P_1=\dfrac{Pa}{a+b}$ ...(1)

Similarly, taking moments about the line of action of P_1,

$(a+b)\times P_2=Pb.$ $\qquad\therefore\quad P_2=\dfrac{Pb}{a+b}$ ...(2)

If s is the strength of the weld per unit length of the weld,

$$L_1=\frac{P_1}{s}=\frac{Pa}{s(a+b)}\text{ and }L_2=\frac{Pb}{s(a+b)}.$$

Sometimes it is not possible to accommodate the required length of the weld on the sides of the section. In such cases, the end fillets are also provided. The procedure for analysing such a ...

Example 23·5. *An equal angle 65 mm×65 mm @ 9·4 kg/m of thickness 10 mm carries a load of 160 kN, applied along its centroidal axis. The angle is to be welded to a gusset plate. Find out the lengths of the side fillet welds required at the heel and the toe of angle. Its C.G. is at 19·7 mm from its heel.*

Solution.

Taking moments about the line of action of P_2

$65P_1=160\times45{\cdot}3$ $\qquad\therefore\quad P_1=111{\cdot}5$ kN

$\therefore\quad P_2=P-P_1=160-111{\cdot}5=48{\cdot}49$ kN.

The maximum size of the weld for a rounded edge at the toe of the angle is three-fourths of the thickness, *i.e.* $\frac{3}{4}\times10=7{\cdot}5$ mm.

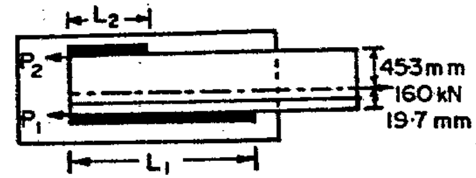


Fig. 23·11

Strength of weld per lineal mm$=0{\cdot}707\times7{\cdot}5\times102{\cdot}5=543{\cdot}5$ N

$$\therefore\quad L_1=\frac{P_1}{543{\cdot}5}=\frac{111{\cdot}5\times1000}{543{\cdot}5}=205\text{ mm}$$

and $$L_2=\frac{P_2}{543{\cdot}5}=\frac{48{\cdot}49\times1000}{543{\cdot}5}=89{\cdot}2\text{ mm}$$

These values are effective and must be increased by twice the weld size, *i.e.*, $2\times7{\cdot}5=15$ mm to get the actual lengths of the welds.

Example 23·6. *A tie bar consisting of a single angle 60 mm× 60 mm×10 mm is to be welded to a gusset plate. The tie bar carries a load of 150 kN along its centroidal axis. Design the joint if both the side fillets and end fillets are to be provided. The centroidal axis lies at 18·5 mm from the heel of the angle.*

Solution.

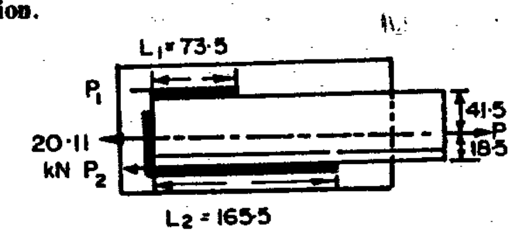


Fig. 23·12

The maximum size of the fillet weld at the end, along the square edge of the angle will be 1·5 mm less than the thickness of the angle. Therefore, the maximum size of the end fillet$=10-1{\cdot}5$ $=8{\cdot}5$ mm.

The maximum size of the side fillets, along the rounded edge

$$=\frac{3}{4}\times 10=7{\cdot}5 \text{ mm}$$

We shall provide 7·5 mm weld throughout.

Strength of the weld per cm length

$$=0{\cdot}707\times 7{\cdot}5\times 102{\cdot}5=543{\cdot}5 \text{ N}$$

The end weld will be placed symmetrical about the line of action of the load in order to avoid eccentricity. The maximum length of the end weld is, therefore, equal to $2\times 18{\cdot}5=37$ mm.

The strength of the end weld

$$=543{\cdot}5\times 37=20110 \text{ N}$$

$$=20{\cdot}11 \text{ kN}$$

Taking moments about the line of action of force P_1, we get

$$60P_2=150\times 41{\cdot}5-20{\cdot}11\times 41{\cdot}5$$

From which $P_2=89{\cdot}94$ kN

$$\therefore \quad P_1=150-89{\cdot}94-20{\cdot}11=39{\cdot}95 \text{ kN}$$

$$\therefore \quad \text{Length } L_1=\frac{P_1}{543{\cdot}5}=\frac{39{\cdot}95\times 1000}{543{\cdot}5}=73{\cdot}5 \text{ mm}$$

and $\quad$ $$\text{Length } L_2=\frac{P_2}{543{\cdot}5}=\frac{89{\cdot}94\times 1000}{543{\cdot}5}=165{\cdot}5 \text{ mm.}$$

Twice the size of the weld, *i.e.*, $2\times 7{\cdot}5=15$ mm must be added to above lengths to get the actual lengths of the side fillets.

23·5. WELDED JOINT SUBJECTED TO BENDING MOMENT

If the load P acting on a welded joint does not pass through the centroid of weld lines, it will subject the welded joint to bending moment in addition to the direct load. Consider I-beam welded to the flange of a column and subjected to a load P at eccentricity e shown in Fig. 23·13.

Let L be the length of the weld and t be the throat thickness. The joint is subjected to a load P and a momet $P\times e$.

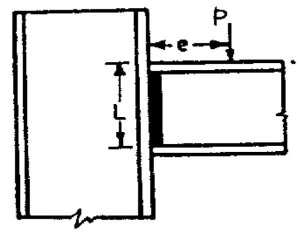


Fig. 23·13

The area resisting direct stress$=2\times L\times t$.

$\therefore$ Direct throat stress $f_d=\dfrac{P}{2Lt}$. It will act in the direction of the load, *i.e.* vertically in this case.

Section modulus of both the weld lines$=\dfrac{2\times t\times L^2}{6}=\dfrac{L^2t}{3}$

Bending stress, $f_b=\dfrac{M}{Z}=\dfrac{P.e\times 3}{L^2t}=\dfrac{3P.e}{L^2t}$. The stress acts in horizontal direction.

Therefore, the direct and bending stresses act at right angles to each other. The resultant is given by

$$f=\sqrt{f_d^2+f_b^2}.$$

Example 23·7. *A bracket consisting of an I-section is connected to the flaage of a vertical column by means of two side fillets 250 mm deep and 8 mm thick, as shown in Fig. 23·13. The bracket carries a load of 160 kN at an eccentricity of 60 mm. Calculate the throat stress in the weld.*

Solution.

Throat thickness$=\dfrac{8}{\sqrt{2}}=5{\cdot}67$ mm

Direct stress $f_d=\dfrac{160000}{2\times 250\times 5{\cdot}67}=56{\cdot}4$ N/mm²

Bending moment$=166000\times 60=96\times 10^5$ N-mm

Z_{xx} of the weld lines$=2\times\dfrac{1}{6}\times 5{\cdot}67(250)^2$

$$=118125 \text{ mm}^3$$

Bending stress $f_b = \frac{M}{Z_{xx}} = \frac{96 \times 10^5}{118125} = 81.3$ N/mm²

Resultant stress $f = \sqrt{f_d^2 + f_b^2} = \sqrt{(56.4)^2 + (81.3)^2}$
$= 98.9$ N/mm².

Example 23.8. *A bracket carrying a load of 120 kN is connected to a column by means of two horizontal fillet welds, each 150 mm long and 10 mm thick. The load acts at 80 mm from the face of the column. Find the throat stress.*

Solution.

Fig. 23.14 shows the arrangement of the bracket.

Throat thickness $= \frac{10}{\sqrt{2}} = 7.07$ mm

Direct force $= 120$ kN

Direct stress $f_d = \frac{120 \times 1000}{2 \times 150 \times 7.07} = 56.6$ N/mm², acting in vertical direction.

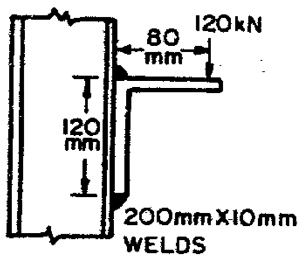


Fig. 23.14

Moment $= 120000 \times 80 = 96 \times 10^5$ N-mm

The forces due to bending in the two welds will form a resistin couple $= 150 \times 7.07 \times f_b \times 120 = 1.27 \times 10^5 f_b$

$\therefore \quad 1.27 \times 10^5 f_b = 96 \times 10^5$

From which $f_b = 75.4$ N/mm² acting in vertical direction.

$\therefore$ Resultant stress $= \sqrt{(75.4)^2 + (56.5)^2} = 94.3$ N/mm².

23.6. WELDED JOINT SUBJECTED TO TORSION

If a bracket carrying a eccentrically applied load 'P' is connected to the column as shown in Fig. 23.15 the welds will be subjected to torsion. The difference between this sort of bracket connection and that shown in Fig. 23.13 must be noted. In this case the moment is acting in a plane containing the welds, while in previous case, the

moment was acting in a plane perpendicular to the welds. The method of analysis for welded joint subjected to torsion is similar to that for riveted joints.

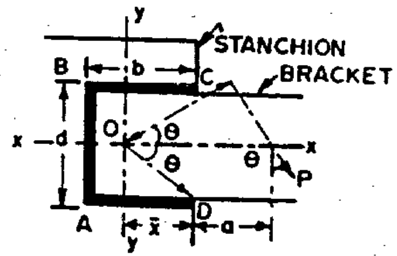


Fig. 23.15

Let the load be transmitted to the stanchion by means of weld of throat thickness t on top, bottom and one side of the bracket.

Total throat area $= t(d+2b)$

Direct throat stress $= \frac{P}{t(d+2b)}$.

To find out the torsional stress in the welds, the position of centroid of the welds must be determined. Let $\bar{x}$ be the distance of centroid from the face of the stanchion CD.

$$\bar{x} = \frac{2b \times t \times \frac{b}{2} + d \times t \times b}{2b \times t + d \times t} = \frac{b^2 + bd}{2b + d}$$

$\therefore$ Eccentricity $e = (a + \bar{x}) \cos \theta$

Twisting moment $T = P \times e$

It will be assumed that the stresses developed due to torsion at any point of the weld will be proportional to its distance from the centroid and that it will act at right angles to the radius vector.

If T is the twisting moment and f_t is the stress in weld caused due to twisting,

$$f_t = \frac{T \cdot r}{J}$$

where J is the polar moment of inertia of the weld sections out the centroid O.

$$J = I_{zz} = I_{xx} + I_{yy}$$

Here $I_{xx} = 2 \times bt \times \frac{d^2}{4} + \frac{td^3}{12}$

and $I_{yy} = \frac{2tb^3}{12} + 2bt\left(\bar{x} - \frac{b}{2}\right)^2 + td\left(b - \bar{x}\right)^2$

The maximum stress will occur at C and D. The direct stress will act in the direction of the load and the stress due to torsion will act at right angles to the direction of the radius vector. The resultant of the two will give the total stress at the point.

Example 23·9. *A circular shaft of diameter 120 mm is welded to a rigid plate by a fillet weld of size 6 mm. If a torque of 8 kN·m applied to the shaft find the maximum stress in the weld.*

Solution.

We shall first derive relation between the stresses developed and the torque applied for such a shaft. Let d be its diameter and h be the size of the weld.

Considering a small area δa of the weld as shown in Fig. 23·16, $J=\Sigma\, \delta a=\left(\frac{d}{2}\right)^2$ (assuming the size of the weld negligible in comparison to the diameter of the shaft).

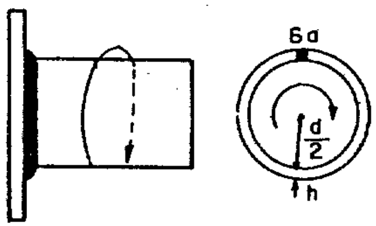


Fig. 23·16

$$\therefore \qquad J=\left(\frac{d^2}{4}\right)\Sigma\, \delta a$$

Throat thickness $=\frac{h}{\sqrt{2}}$

$$\Sigma\, \delta a=\pi d\times\left(\frac{h}{\sqrt{2}}\right)$$

$$\therefore \qquad J=\frac{d^2}{4}\times\frac{\pi . dh}{\sqrt{2}}=\frac{\pi d^3 h}{4\sqrt{2}}$$

Stress $=\frac{T.r}{J}=T\,.\,\frac{d}{2}\times\frac{4\sqrt{2}}{\pi d^3 h}$

$$=\frac{2\sqrt{2}\,T}{\pi d^2 h}$$

Substituting the values, we get

Maximum stress $=\frac{2\sqrt{2}(8\times 1000\times 1000)}{\pi(120)^2(6)}=83{\cdot}4$ N/mm².

Example 23·10. *A bracket is subjected to a load of 100 kN as shown in Fig. 23·17. A bracket is welded to a stanchion by means of three lines of weld on three sides as indicated in the figure. Find out size of the welds so that the load is carried safely.*

Solution.

If $\bar{x}$ is the distance of centroid of weld area from AB,

$$\bar{x}=\frac{2\times 120\, t\times 60}{2\times 120\, t+240\, t}=30 \text{ mm}$$

$\therefore$ Eccentricity of the load $=100+120-30=190$ mm

$$I_{xx}=\frac{t\,(240)^3}{12}+2\times 120\, t\,(120)^2$$

$$=460{\cdot}8\times 10^4\, t \text{ mm}^4$$

where t = thickness of the weld.

$$I_{yy}=240\times t(30)^2+2\times\frac{1}{12}\,t(120)^3+2\times 120\, t\,(30)^2$$

$$=72\times 10^4\, t \text{ mm}^4$$

$$\therefore \qquad J=I_{zz}=I_{xx}+I_{yy}=460{\cdot}8\times 10^4\, t+72\times 10^4\, t$$

$$=532{\cdot}8\times 10^4\, t \text{ mm}^4$$

Area $=2\times 120\times t+240\, t=480\, t$ mm²

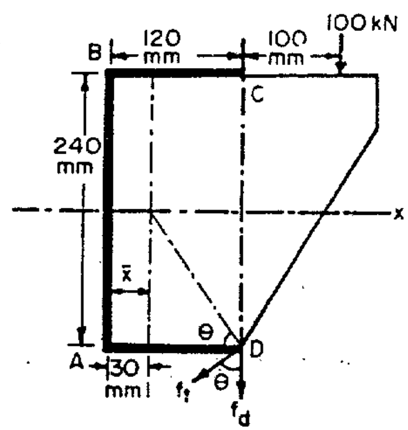


Fig. 23·17

The maximum stress due to torsion will occur either at C or D.

Length of radius vector for C or D

$$=\sqrt{(120)^2+(90)^2}=150 \text{ mm}$$

Maximum stress due to torsion

$$f_t=\frac{Tr}{J}=\frac{100000\times 190\times 150}{532{\cdot}8\times 10^4\, t}$$

$$=\frac{534{\cdot}9}{t}\ \text{N/mm}^2$$

Direct stress, $f_d=\dfrac{10000}{480\,t}=\dfrac{208{\cdot}3}{t}$ N/mm²

The angle between the stresses is θ, as shown in Fig. 23·17.

$$\cos\theta=-\frac{90}{150}=-0{\cdot}6$$

∴ Resultant stress

$$f_r=\sqrt{f_t^2+f_d^2-2f_t\,.\,f_d\cos\theta}$$

$$=\sqrt{\left(\frac{534{\cdot}9}{t}\right)^2+\left(\frac{208{\cdot}3}{t}\right)^2+2\left(\frac{534{\cdot}9}{t}\right)\left(\frac{208{\cdot}3}{t}\right)\times 0{\cdot}6}$$

$$=\frac{680{\cdot}6}{6}\ \text{N/mm}^2$$

Allowable shear stress $=102{\cdot}5$ N/mm²

$$\therefore\quad 112{\cdot}5=\frac{680{\cdot}6}{t}$$

From which $t=6{\cdot}64$ mm

Size of weld $=\sqrt{2}\,.\,t=9{\cdot}39$ mm.

Hence provided 10 mm size weld.

PROBLEMS

1. A 150 mm × 15 plate is welded to other plate by two side welds 12 cm each and end fillet of 100 mm length. Find the safe axial load to which this joint may be subjected if the size of the weld is 7 mm.

2. A 100 mm × 10 plate is welded to other by means of two end fillets and two side fillets of 8 mm as shown in Fig. 23·18. If the plate is loaded to its full strength, find out the required overlap length.

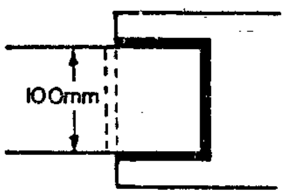


Fig. 23·18

3. An equal angle 75 mm × 75 mm @ 11·0 kg/m is subjected to a load for 180 kN, whose line of action passes through the centroid of the section, which is 22·2 mm from the heel. This angle is to be welded to a gusset plate. If the size of the weld is to be 8 mm, find the length of the size fillet welds.

4. An I-section is made up of a 200 mm × 10 mm thick web plate welded to two flange plates 120 mm × 10 mm thick by means of fillet welds to size 6 mm. Calculate the maximum shear force which this section can resist.

5. Fig 23·19 shows a 10 mm angle bracket 100 mm wide welded to the flange of a steel stanchion. It carries a vertical load

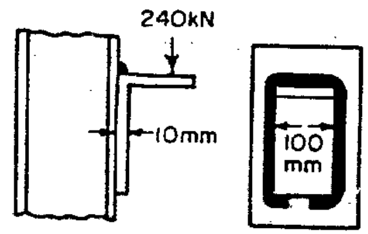


Fig. 23·19

of 240 kN. The connection consists of continuous 10 mm weld extending along the top and both sides and returned at the bottom of the bracket. Treating the 240 kN load as a vertical shear load (*i.e.*, neglecting bending moment), calculate the depth of bracket, taking 110 N/mm² as the working stress in the transverse weld and 79 N/mm² in the longitudinal weld. (*U.L.*)

6. An I-section bracket carrying 120 kN load, is connected to by a column as shown in Fig. 23·20 means of two side fillet welds 200 mm deep. The load is eccentric by 70 mm. Calculate the size of the fillet weld.

7. A bracket consisting of a Tee-section 150 mm × 150 mm and 10 mm is connected to a column as shown in Fig. 23·20. The bracket carries 150 kN load at 80 mm eccentricity. If the size of the weld is 6 mm, find out the maximum throat stress.

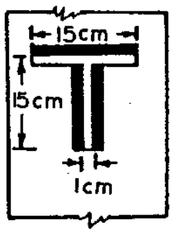


Fig. 23·20

8. The bracket shown in Fig. 23·21 is welded to a stanchion by side fillet welds on three sides indicated by heavy lines. Calculate the maximum forces per inch of weld metal when the bracket carries the load of 200 kN acting as shown.

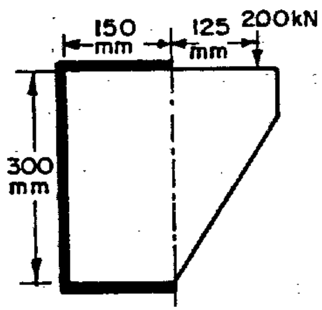


Fig. 23·21

9. A bracket is welded to a stanchion by fillet welds, having a throat thickness of 9 mm and a load of 180 kN is applied in the plane of the bracket as shown in Fig. 23·22. The weld extends round three sides and has the given dimensions. Determine the maximum tress on the throat of the weld.

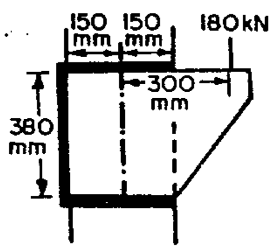


Fig. 23·22

10. The dimensions of a plate bracket welded to the face of a stanchion are given in Fig. 23·23. Assuming a maximum weld stress of 6 tons/in^2 on the throat of the fillet, determine the maximum permissible value of W. *(I.S.E.)*

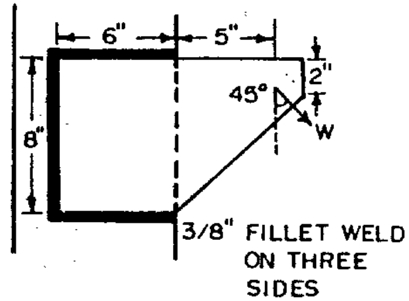


Fig. 23·23

ANSWERS

1. 172·5 kN
2. 72·5 mm
3. 218 mm and 91·5 mm
5. 160 mm
6. 9·5 mm
8. 1100 N/mm^2
9. 1800 N/mm^2
10. 8·03 ton.

24

Method of Tension Coefficients

24·1. INTRODUCTION

In Volume 1, we have studied two methods of analysis of plane frames : (*i*) method of joints, and (*ii*) method of sections. We can also find out the forces in the members of plane frame, by the *graphical method*. We now introduce a more general method, called the '*method of tension coefficients*' which is equally applicable to both plane frames as well as space frames. The method of *tension coefficient*, first introduced by Prof. R.V. Southwell, is in effect a neat and systematic presentation of the 'method of joints'. The method is particularly useful to space frames in which other methods prove to be cumbersome and tedious. In this chapter, we shall introduce the method, and illustrate its applications for plane frames. Its application for the *space frames* has been illustrated in chapter 25.

24·2. TENSION COEFFICIENTS

The *tension coefficient* for a member of a frame is defined as the *pull* or *tension* in that member divided by its length. Thus,

$$t=\frac{T}{L} \qquad \ldots(24{\cdot}1)$$

where t is the tension coefficient for the member, T is the pull in the member and L is its length.

Consider a member AB of a pin-jointed perfect frame in equilibrium, under a given system of external forces (or reactions) acting at the joints. Let T_{AB} be the resulting pull in the member.

Let (x_A, y_A) be the coordinates of A and (x_B, y_B) be the coordinates of joint B, referred to suitable axes of reference.

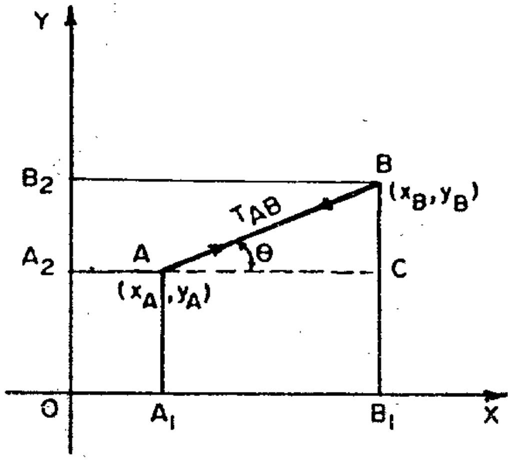


Fig. 24·1

Component of pull T_{AB} in x-direction is

$$T_{AB}\,.\cos\theta=T_{AB}\frac{AC}{AB}=T_{AB}\frac{(x_B-x_A)}{L_{AB}}$$

$$=t_{AB}(x_B-x_A) \qquad \ldots(1)$$

where t_{AB}=tension coefficient for $AB=\dfrac{T_{AB}}{L_{AB}}$.

Similarly, component of T_{AB} in y-direction is

$$T_{AB}\,.\sin\theta=T_{AB}\,.\frac{CB}{AB}=T_{AB}\frac{(y_B-y_A)}{L_{AB}}$$

$$=t_{AB}(y_B-y_A) \qquad \ldots(2)$$

Thus, we observe that the component of forces in the members along the x and y directions can very easily we expressed in terms of *tension coefficients* and the coordinates of the ends of the members with reference to the chosen reference axes.

The length of the member AB is obviously given by

$$L_{AB}=\sqrt{(x_B-x_A)^2+(y_B-y_A)^2} \qquad \ldots(24{\cdot}2)$$

A member which is in *compression* will have a *negative tension coefficient*.

24·3. ANALYSIS OF PLANE FRAMES

Let us now apply the method of tension coefficient for the analysis of plane frame. Let us consider a joint A, where a number of members AB, AC, AD...... etc. are meeting. Let P_A be the external force acting at the joint A, and let X_A and Y_A be the components of this force P, in x and y directions. Since the joint is in equilibrium under the external force P_A and the *pulls* in the members meeting at joint A, the algebraic sum of *resolved* parts of these forces in any direction must be equal to zero.

Resolving the forces in x and y directions, and equating the algebraic sum of these forces in each direction to zero, we get

$$t_{AB}(x_B-x_A)+t_{AC}(x_C-x_A)+t_{AD}(x_D-x_A)......+X_A=0 \quad ...[24\cdot3\ (a)]$$

and $$t_{AB}(y_B-y_A)+t_{AC}(y_C-y_A)+t_{AD}(y_D-y_A)......+Y_A=0 \quad ...[24\cdot3\ (b)]$$

The above equations may also be written in *compact* form as under :

$$\Sigma t\,(x_F-x_N)+X_A=0 \quad ...[24\cdot4\ (a)]$$

and $$\Sigma t\,(y_F-y_N)+Y_A=0 \quad ...[24\cdot4\ (b)]$$

where (x_F, y_F) are the coordinates of the *far end* of each member and (x_N, y_N) are the coordinates of the *near end* of each member meeting at the joint. The end of the member at the joint under consideration is known as the *near end.*

Thus, we obtain two equations at each joint, in which *tension coefficients* are the only unknowns. If the frame has j joints, we will have $2j$ such equations, the solution of which will yield tension coefficients for each member. If there are n members in a perfect frame, we have

$$n=2j-3$$

There will be n tension coefficients, for which we require only $(2j-3)$ equations, while available equations are $2j$. The three surplus equations will be useful either for determining the external reactions acting on the frame or for applying the check.

In order to apply the method, it is assumed that all the members are in tension. In the final solution, a member in compression will then automatically get *negative tension coefficient.* However,

in setting the equations, utmost care must be taken in assigning correct sign to relevant terms. The term will be taken as positive or negative according as they tend to move the joint in the positive or negative directions of the axes of reference. The origin should be so selected that the coordinates of various joints can be written easily.

The method will now be illustrated with the help of few examples.

Example 24·1. *A plane frame consists of two members AB and CB, hinged at A and C to the wall, as shown in Fig. 24·2. Determine the forces in the two members due a vertical force P applied at joint P.*

Solution.

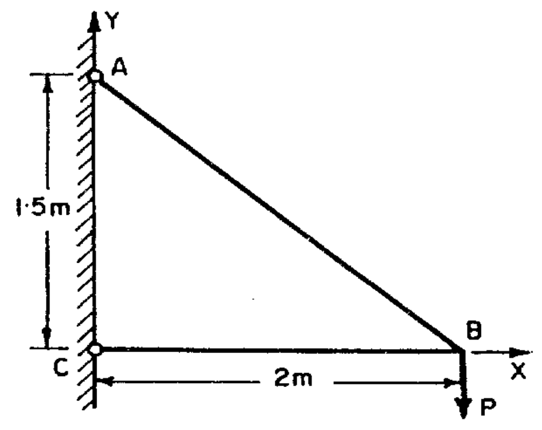


Fig. 24·2

Let us take the origin at joint C, and CX and CY be the axes of reference. The coordinates of the three joints are

$$C(0, 0)\ ;\ B(2, 0)\ ;\ A(0, 1\cdot5).$$

There are only two members and therefore, there will be only two tension coefficients. Let us therefore take joint B and set two equations at that joint, assuming that every member is in a state of tension, exerting a pull on the joint, though in the present case member BA will be in tension while member BC will be in compression. The tension coefficient for BC will automatically workout to be negative.

Length $L_{BA}=\sqrt{(0-2)^2+(1\cdot5-0)^2}=2\cdot5$ m

$L_{BC}=2$ m (given)

At the joint B, we have the following two equations in x and y directions :

$$t_{BA}\ (x_A-x_B)+t_{BC}\ (x_C-x_B)+0=0$$

and $$t_{BA}\ (y_A-y_B)+t_{BC}\ (y_C-y_B)-P=0$$

(Negative sign has been placed before P since force P acts in the negative y-direction)

Substituting the values, we get

$$t_{BA}\ (0-2)+t_{BC}\ (0-2)=0$$

or $$t_{BA}+t_{BC}=0 \qquad ...(1)$$

and $$t_{BA}(1{\cdot}5-0)+t_{BC}(0-0)-P=0$$

or $$1{\cdot}5t_{BA}=P \qquad ...(2)$$

Solving (1) and (2), we get

$$t_{BA}=\frac{P}{1{\cdot}5}\ \text{kN/m}$$

and $$t_{BC}=-\frac{P}{1{\cdot}5}\ \text{kN/m}$$

Minus sign suggests that member BC will be in compression.

∴ Force in member BA

$$=T_{BA}=t_{BA}\ .\ L_{BA}$$

$$=\frac{P}{1{\cdot}5}\times 2{\cdot}5=1{\cdot}6667P \text{ (tension).}$$ **Ans.**

and Force in member BC

$$=T_{BC}=t_{BC}\times L_{BC}=-\frac{P}{1{\cdot}5}\times 2$$

$$=-1{\cdot}3333P=1{\cdot}3333\ \text{P (comp.).}$$

Example 24·2. *A truss, shown in Fig. 24·3 is loaded with two point loads of 2P and P kN at joints B and C. Determine the forces in all the members.*

Solution.

Given : $L_{AB}=8$ m ; $L_{BC}=4$ m ; $L_{CD}=8$ m and $L_{AD}=12$ m.

Let us keep the origin at A, with positive reference directions as shown in Fig. 24·3

Now $AE=AB\cos 60°=8\cos 60°=4$ m $=FD$

$BE=AB\sin 60°=8\sin 60°=6{\cdot}9282$ m

$ED=12-4=8$ m

∴ $BD=(BE^2+ED^2)^{1/2}=10{\cdot}583$ m

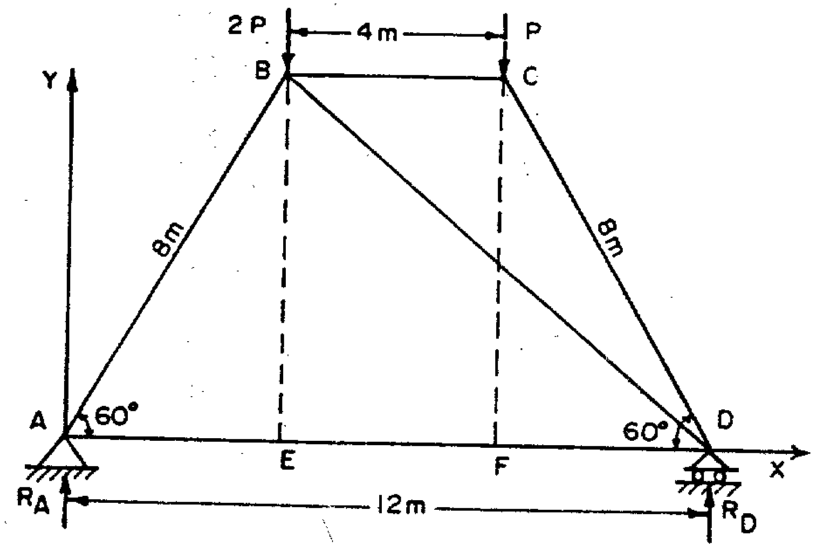


Fig. 24·3

The coordinates of various points are as under :

A, (0, 0) ; B, (4, 6·9282) ; C, (8 m, 6·9282) ; D, (12, 0).

To find the reaction R_D at D, take moments about A.

$$\therefore\quad R_D=\frac{1}{12}\ [2P\times 4+P\times 8]=1{\cdot}3333P$$

$$\therefore\quad R_A=2P+P-1{\cdot}3333P=1{\cdot}6667P$$

There are four joints, and hence eight equations will be available, while there are only five unknown tension coefficients. Let us set down the equations for each joint assuming that every member is in state of tension, exerting a pull at the joint, though actually all members except AD are in compression.

Joint A :

$$t_{AB}(x_B-x_A)+t_{AD}(x_D-x_A)=0$$

and $$t_{AB}(y_B-y_A)+t_{AD}(y_D-y_A)+R_A=0$$

Substituting the values, we get

$$4t_{AB}+12t_{AD}=0 \qquad ...(1)$$

and $$6{\cdot}928t_{AB}+1{\cdot}6667P=0 \qquad ...(2)$$

Solving, we get $t_{AB}=-0{\cdot}2406P$ kN/m

and $t_{AD}=+0{\cdot}0802P$ kN/m.

$\therefore \quad T_{AB}=t_{AB} \cdot L_{AB}=-0{\cdot}2406\times 8P=-1{\cdot}9246P$ kN (*i.e.* compression)

$T_{AD}=t_{AD} \cdot L_{AD}=+0{\cdot}0802P\times 12=+0{\cdot}9624P$ kN (tension)

Joint B :

$$t_{BA}(x_A-x_B)+t_{BD}(x_D-x_B)+t_{BC}(x_C-x_B)=0$$

and $$t_{BA}(y_A-y_B)+t_{BD}(y_D-y_B)+t_{BC}(y_C-y_B)-2P=0$$

Substituting the values, we get

$$-4t_{BA}+8t_{BD}+4t_{BC}=0 \qquad ...(3)$$

and $$-6{\cdot}9282t_{BA}-6{\cdot}9282t_{BD}-2P=0 \qquad ...(4)$$

From (4), $t_{BA}+t_{BD}=-0{\cdot}2887P$

But $t_{BA}=t_{AB}=-0{\cdot}2406P$

$\therefore \quad t_{BD}=-0{\cdot}2887P+0{\cdot}2406P=-0{\cdot}0481P$

Hence from (3), $-t_{BA}+2t_{BD}+t_{BC}=0$

or $$0{\cdot}2406P-2\times 0{\cdot}0481P+t_{BC}=0$$

From which $t_{BC}=-0{\cdot}1444P$

Hence $T_{BC}=-0{\cdot}1444P\times 4=-0{\cdot}5776P$ (*i.e.* compression)

and $T_{BD}=-0{\cdot}0481P\times 10{\cdot}583=-0{\cdot}509P$ (*i.e.* compression).

Joint C :

$$t_{CB}(x_B-x_C)+t_{CD}(x_D-x_C)=0$$

$$t_{CB}(y_B-y_C)+t_{CD}(y_D-y_C)-P=0$$

Substituting the values, we get

$$-4t_{CB}+4t_{CD}=0 \qquad ...(5)$$

and $$-6{\cdot}9282\ t_{CD}-P=0 \qquad ...(6)$$

Solving, we get $t_{CD}=-0{\cdot}1444P$

and $t_{CB}=t_{CD}=-0{\cdot}1444P$ (which checks the earlier result).

$\therefore \quad T_{CD}=-0{\cdot}1444P\times 8=-1{\cdot}1552P$ (*i.e.* compression)

Thus the values of forces in all the members have been obtained. Additional two equations available at joint D can be used to check the values of T_{DB} and T_{DA} obtained earlier.

Sometimes, it is preferable to arrange all the computations in a tabular form shown below.

(+for tension ; −for compression)

Joint		*Equation*	*Member*	*t*	*L (m)*	*T*
A (0, 0)	*x*	$4t_{AB}+12t_{AD}=0$	*AB*	$-0{\cdot}2406P$	8	$-1{\cdot}9246P$
	y	$6{\cdot}928t_{AB}+1{\cdot}6667P=0$	*AD*	$+0{\cdot}0802P$	12	$+0{\cdot}9624P$
B (4, 6·9282)	*x*	$-4t_{BA}+8t_{BD}+4t_{BC}=0$	*BC*	$-0{\cdot}1444P$	4	$-0{\cdot}5776P$
	y	$-6{\cdot}9282t_{BA}-6{\cdot}9282t_{BD}-2P=0$	*BD*	$-0{\cdot}0481P$	10·583	$-0{\cdot}509P$
C (8, 6·9282)	*x*	$-4t_{CB}+4t_{CD}=0$	*CD*	$-0{\cdot}1444P$	8	$-0{\cdot}1552P$
	y	$-6{\cdot}9282t_{CD}-P=0$				

Example 24·3. *Fig. 23·4 shows a Warren type cantilever truss alongwith the imposed loads. Determine the forces in all the members, using the method of tension coefficients.*

Solution.

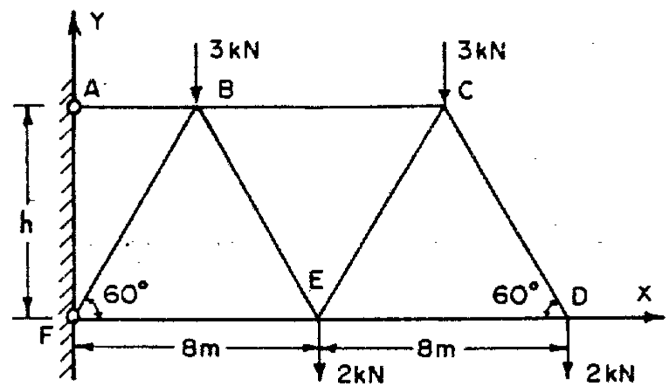


Fig. 24·4

$$AF=h=\sqrt{8^2-4^2}=6{\cdot}928 \text{ m}$$

Let us take origin at F, and select x-axis and y-axis along FD and FA respectively. The coordinates of various points are as under :

Joint	x	y
F	0	0
A	0	6·928
B	4	6·928
C	12	6·928
D	16	0
E	8	0

Joint D : The two equations are

$$t_{DE}(x_E-x_D)+t_{DC}(x_C-x_D)+X_D=0 \quad \ldots(i)$$

$$t_{DE}(y_E-y_D)+t_{DC}(y_C-y_D)+Y_D=0 \quad \ldots(ii)$$

Substituting the values, we obtain

$$-8t_{DE}-4t_{DC}=0 \quad \ldots(1)$$

and $$6·928\, t_{DC}-2=0 \quad \ldots(2)$$

From (2), $t_{DC}=+0·289 \quad \ldots(a)$

From (1), $t_{DE}=-\frac{1}{2}t_{DC}=-0·144 \quad \ldots(b)$

Joint C : The two equations are

$$t_{CD}(x_D-x_C)+t_{CE}(x_E-x_C)+t_{CB}(x_B-x_C)+X_C=0 \quad \ldots(iii)$$

$$t_{CD}(y_D-y_C)+t_{CE}(y_E-y_C)+t_{CB}(y_B-y_C)+Y_C=0 \quad \ldots(iv)$$

Substituting the values, we get

$$4t_{CD}-4t_{CE}-8t_{CB}=0 \quad \ldots(3)$$

and $$-6·928t_{CD}-6·928t_{CE}-3=0 \quad \ldots(4)$$

From (4), $t_{CE}=-t_{CD}-0·433=-0·289-0·433=-0·772 \quad \ldots(c)$

From (3), $t_{CB}=0·5t_{CD}-0·5t_{CE}=(0·5\times0·289)+(0·5\times0·722)$

$$=+0·5055 \quad \ldots(d)$$

Joint E : The two equations are

$$t_{EC}(x_C-x_E)+t_{ED}(x_D-x_E)+t_{EB}(x_B-x_E)+t_{EF}(x_F-x_E)+X_E=0 \quad \ldots(v)$$

$$t_{EC}(y_C-y_E)+t_{ED}(y_D-y_E)+t_{EB}(y_B-y_E)+t_{EF}(y_F-y_E)+Y_E=0 \quad \ldots(vi)$$

Substituting the values, we get

$$4t_{EC}+8t_{ED}-4t_{EB}-8t_{EF}=0 \quad \ldots(5)$$

$$6·928t_{EC}+6·928t_{EB}-2=0 \quad \ldots(6)$$

From (2), $t_{EB}=0·2887-t_{EC}=0·2887+0·722=+1·011 \quad \ldots(e)$

From (5), $t_{EF}=0·5t_{EC}+t_{ED}-0·5t_{EB}$

$$=(-0·5\times0·722)+(-0·144)-(0·5\times1·011)$$

$$=-0·4323$$

Joint B : The two equations are

$$t_{BC}(x_C-x_B)+t_{BE}(x_E-x_B)+t_{BF}(x_F-x_B)+t_{BA}(x_A-x_B)+X_B=0$$

$$t_{BC}(y_C-y_B)+t_{BE}(y_E-y_B)+t_{BF}(y_F-y_B)+t_{BA}(y_A-y_B)+Y_B=0$$

Substituting the values, we get

$$8t_{BC}+4t_{BE}-4t_{BF}-4t_{BA}=0 \quad \ldots(7)$$

$$-6·928t_{BE}-6·928t_{BF}-3=0 \quad \ldots(8)$$

From (8), $t_{BF}=-t_{BE}-0·433=-1·011-0·433=-1·444 \quad \ldots(f)$

From (7), $t_{BA}=2t_{BC}+t_{BE}-t_{BF}=(2\times0·5055)+1·011+1·444$

$$=+1·732$$

The values of tension coefficients and forces in various members are tabulated below.

(+for tension ; −for compression)

Member	Length (m)	t	T (kN)
AB	4	+3·466	+13·864
BC	8	+0·5055	+4·044
CD	8	+0·289	+2·312
DE	8	−0·144	−1·152
EF	8	−1·011	−8·084
FB	8	−1·444	−11·548
EC	8	−0·722	−5·776
EB	8	+1·011	+8·086

PROBLEMS

1. Using the method of tension coefficients, find forces in the members of the triangular type cantilever truss shown in Fig. 20·10 (Vol. 1).

2. Using the method of tension coefficients, determine the forces in the members of the crane structure shown in Fig. 20·17 (Vol. 1).

3. Using the method of tension coefficients, determine the forces in the members of the frame shown in Fig. 20·18 (Vol. 1).

4. Using the method of tension coefficients, determine the forces in the members of the frame shown in Fig. 20·19 (Vol. 1).

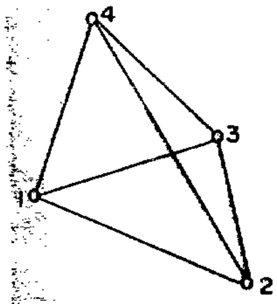
5. Find the forces in all the members of the frame shown in Fig. 20·23 (Vol. 1). The frame is supported on a pinned support at *P* and roller support at *R*.

25

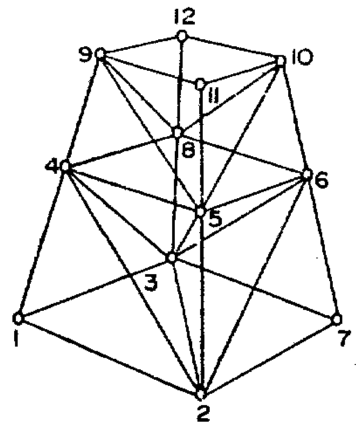
Space Frames

25·1. INTRODUCTION

A *space frame* or *space truss* is a three dimensional assemblage of line members, each member being joined at its ends to the foundation or to other members by frictionless ball-and-socket joints. The simplest space frame consists of six members joined to form a tetrahedron [Fig. 25·1 (*a*)]. By beginning with six members forming a tetrahedron, a stable space frame can be constructed by successive addition of three new members and a joint. One such frame is shown in [Fig. 25·1 (*b*)].



(*a*) Basic tetrahedron.



(*b*) Typical space frame.

Fig. 25·1. Simple space frames.

In order to form a stable (or rigid) space frame, a sufficient number of members have to be arranged in a suitable manner explained above, starting with a basic tetrahedron. The original tetrahedron consists of *six members* and *four joints*. Since for each additional joint there are three additional members, the relationship between the number of members (n) and number of joints is given by

$$n-6=3(j-4)$$

or
$$\boxed{n=3j-6} \qquad \ldots(25.1)$$

The above equation give the *minimum* number of members required to build a *stable* truss or frame. If a truss or frame has less than this number of members, it cannot be stable. If it has more members, then the truss or frame will be termed *internally statically indeterminate*.

25.2. METHOD OF TENSION COEFFICIENTS APPLIED TO SPACE FRAMES

The method of tension coefficients, explained in the previous chapter, can be extended to space frames by writing the equation of equilibrium of forces at each joint, in each direction. However, in space frame, there are three axes of reference : x-axis, y-axis and z-axis, the first two axes being considered in a horizontal plane and z-axis in a vertical plane. Thus, there will be *three equations* in place of two equations used for plane frames.

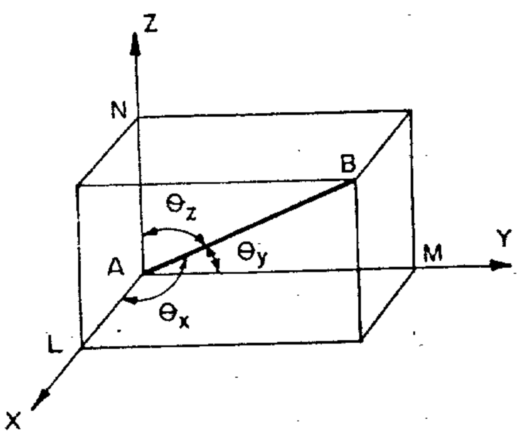


Fig. 25·2

Let us consider a member AB of a space frame, carrying a tensile force T_{AB}. Let (x_A, y_A, z_A) be coordinates of point A, and (x_B, y_B, z_B) be the coordinates of point B, referred to the three axes

X, Y, Z of reference. Let the line AB be inclined at angles $\theta_x, \theta_y,$ … repectively to the three reference axes. Evidently,

$$\boxed{L_{AB}=[(x_B-x_A)^2+(y_B-y_A)^2+(z_B-z_A)^2]^{1/2}} \qquad \ldots(25.2$$

The direction cosines are :

$$l_{AB}=\cos\theta_x\,;\ m_{AB}=\cos\theta_y \text{ and } n_{AB}=\cos\theta_z \qquad \ldots(25.$$

As shown in Fig. 25.2, the components of AB along X, Y an… Z axes are :

$$AL=(x_B-x_A)\,;\ AM=(y_B-y_A)\,;\ AN=(z_B-z_A)$$

Hence,
$$AL=(x_B-x_A)=AB\cos\theta_x=l_{AB}\,.\,L_{AB}$$
$$AM=(y_B-y_A)=AB\cos\theta_y=m_{AB}\,.\,L_{AB}$$
$$AN=(z_B-y_A)=AB\cos\theta_z=n_{AB}\,.\,L_{AB}$$

The resolved component of force T_{AB} along x-direction is

$$T_{AB}\cos\theta_x=T_{AB}\frac{(x_B-x_A)}{L_{AB}}=t_{AB}(x_B-x_A) \qquad \ldots($$

where t_{AB}=tension coefficient for AB.

Similarly, the resolved component of force T_{AB} along y and … directions are

$$T_{AB}\cos\theta_y=T_{AB}\frac{(y_B-y_A)}{L_{AB}}=t_{AB}(y_B-y_A) \qquad \ldots($$

and
$$T_{AB}\cos\theta_z=T_{AB}\frac{(z_B-z_A)}{L_{AB}}=t_{AB}(z_B-z_A) \qquad \ldots($$

Let P_A be the external force at joint A, and let X_A, Y_A and … be the resolved component of this force in X, Y and Z directio… respectively. Let $AB, AC, AD, \ldots\ldots$ be the members meeting at t… joint A. Since the joint is in equilibrium under the external force … and the pulls in the members meeting at the joint A, the algebraic su… of *resolved* parts of these forces in any direction must be equal … zero.

Hence resolving the forces in X, Y and Z directions and equati… the algebraic sum of these forces in each direction to zero, we get t… following three equations for joint A :

$$t_{AB}(x_B-x_A)+t_{AC}(x_C-x_A)+t_{AD}(x_D-x_A)+\ldots\ldots+X_A=0$$
$$t_{AB}(y_B-y_A)+t_{AC}(y_C-y_A)+t_{AD}(y_D-y_A)\ldots\ldots+Y_A=0 \qquad \ldots(25$$
$$t_{AB}(z_B-y_A)+t_{AC}(z_C-z_A)+t_{AD}(z_D-z_A)\ldots\ldots+Z_A=0$$

The above equations may also be written in compact formas under :

$$\Sigma t(x_F-x_N)+X_A=0$$
$$\Sigma t(y_F-y_N)+Y_A=0 \quad ...(25\cdot5)$$
$$\Sigma t(z_F-z_N)+Z_A=0$$

where (x_F, y_F, z_F) are the coordinates of the *far end* of each member and (x_N, y_N, z_N) are the coordinates of the *near end* of each member meeting at the joint. The end of the member at the joint under consideration is known as the *near end*.

If there are j joints, there will be $3j$ such equations, while the number of unknowns (tension coefficients) will be equal to number of members n, where $n=3j-6$. Thus, we will have six *surplus equations* which can be utilised either for determining the external reactions or for verifying the results.

The method will now be illustrated with the help of few examples.

25·3. ILLUSTRATIVE EXAMPLES

Example 25·1. *A pair of shear leg has length of each leg as 5 m, and the distance between their feet is 4 m. The line joining the feet of the legs is 7 m from the foot of the guy rope. If the length of the guy rope is 10 m, find the thrust in each leg and the pull in the guy rope when a load of 100 kN is suspended from the head.*

Solution.

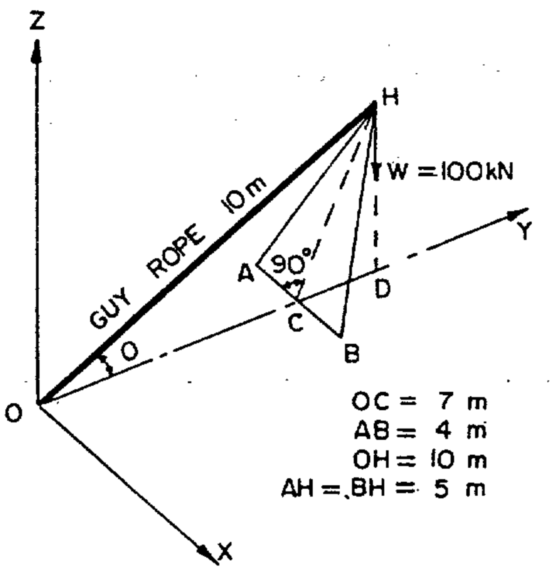


Fig. 25·3

Let the origin pass through the foot of the guy rope, and let the Y-axis pass through the mid-point C of the line AB joining the feet of the two legs, as shown in Fig. 25·3, along with the direction of X and Z axes. H is the head of the shear leg. Let the guy rope make an angle θ with the Y-axis.

Now $CH=\sqrt{AH^2-AC^2}=\sqrt{5^2-2^2}=\sqrt{21}=4\cdot5826$ m

From ΔOCH, $\cos\theta=\dfrac{OH^2+OC^2-CH^2}{2OC.OH}$

$$=\frac{10^2+7^2-21}{2\times7\times10}=0\cdot9143$$

$\theta=23\cdot9°$; $\sin\theta=0\cdot4051$

From H, drop a perpendicular HD on Y-axis.

$\therefore\quad OD=OH\cos\theta=10\times0\cdot9143=9\cdot143$ m

$HD=OH\sin\theta=10\times0\cdot4051=4\cdot051$ m

$\therefore\quad CD=OD-OC=9\cdot143-7=2\cdot143$

Hence the co-ordinates of various points are as under :

Point	*x*	*y*	*z*
O	0	0	0
H	0	9·143	4·051
A	−2	7	0
B	+2	7	0

At the head H, the three equations are as under :

$$t_{HA}(x_A-x_H)+t_{HB}(x_B-x_H)+t_{HO}(x_O-x_H)=0 \quad ...(i)$$
$$t_{HA}(y_A-y_H)+t_{HB}(y_B-y_H)+t_{HO}(y_O-y_H)=0 \quad ...(ii)$$
$$t_{HA}(z_A-z_H)+t_{HB}(z_B-z_H)+t_{HO}(z_O-z_H)+Z_H=0 \quad ...(iii)$$

Substituting the values, we get

$$-2t_{HA}+2t_{HB}=0 \quad ...(1)$$
$$-2\cdot143t_{HA}-2\cdot143t_{HB}-9\cdot143t_{HO}=0 \quad ...(2)$$
and
$$-4\cdot051(t_{HA}+t_{HB}+t_{HO})-100=0 \quad ...(3)$$

From (1), $t_{HA}=t_{HB}$ (as expected)

From (2), $-(2\cdot143+2\cdot143)t_{HA}=9\cdot143t_{HO}$

From which $t_{HA}=-2\cdot1332t_{HO}$

From (3), $-4\cdot051(-2\cdot1332-2\cdot1332+1)\,t_{HO}=100$

From which $t_{HO}=7\cdot5573$

$\therefore\quad t_{HA}=t_{HB}=-2\cdot1332\times7\cdot5573=-16\cdot1212$

Hence *pull* in guy rope$=t_{HO}\times L_{HO}=7\cdot5573\times10\simeq$ **75·57 kN**

Force in each shear leg$=-16.1212\times5=-80.61$ kN

$\therefore$ *Thrust* in each leg$=$**80.61 kN.**

Example 25.2. *A space frame consists of six members: AF, BE, BF, FE, EC and FD. The frame is pinned to a vertical wall at ABCD in such a way that ABCD form a square as shown in Fig. 24.4. Also, ABEF is a rectangle in a horizontal plane. Using method of tension coefficients, find forces in each member due to a load of 100 kN applied at E acting towards the joint D.* (Based on U.L.)

Solution.

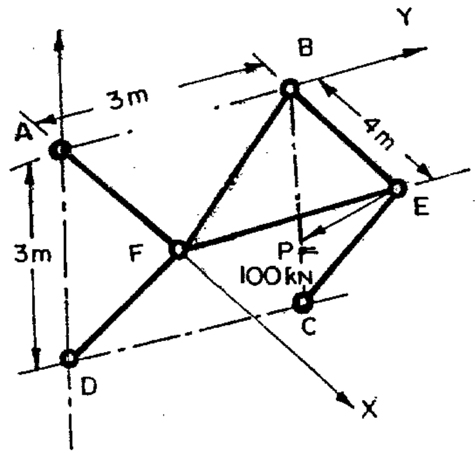


Fig. 25.4

Select the origin at A. Let X-axis be directed along AF, Y-axis be directed along AB and Z-axis be directed vertically through A. Thus, $ABEF$ is in the X-Y plane which is a horizontal plane.

Here, the load of 100 kN is in an inclined direction which does not pass through any of the three axes. Hence it is essential to find its components along X, Y and Z directions.

Length $FD=\sqrt{3^2+4^2}=5$ m.

Fig. 25.5 (a) shows plane $DFEC$, in which the load of 100 kN acts along ED. Evidently, angle $\theta=\tan^{-1}\frac{3}{5}$, from which $\theta=30.964°$, sin $\theta=0.5145$ and cos $\theta=0.8575$.

$\therefore$ Resolved component of the force along EF (*i.e.* alo Y-direction)

$=100\sin\theta=100\times0.5145=51.45$ kN

Resolved component of the force along EC

$=100\cos\theta=100\times0.8575=85.75$ kN

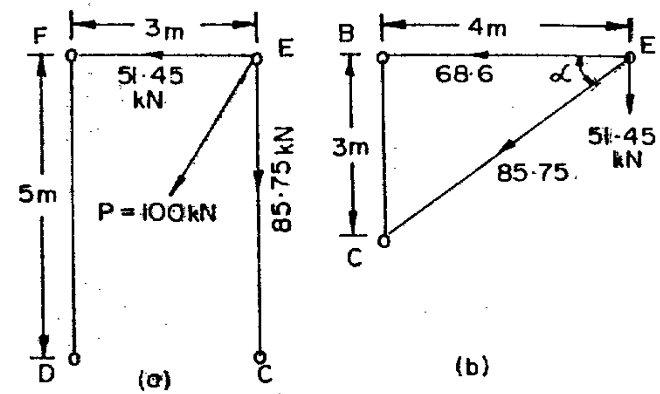


Fig. 25.5

Fig. 25.5 (b) shows plane BEC, in which BC is vertical, BE horizontal, and force in EC is 85.75 kN.

$$\cos\alpha=\frac{BE}{EC}=\frac{4}{5}\ ;\ \sin\alpha=\frac{BC}{EC}=\frac{3}{5}$$

$\therefore$ Resolved component of force in EC, along EB (*i.e.* alo X-direction)

$$=85.75\times\frac{4}{5}=68.6 \text{ kN}$$

Resolved component of force in EC, along vertical direction

$$=85.75\times\frac{3}{5}=51.45 \text{ kN}$$

Thus, resolved components of force P along the th directions are

$P_X=-68.6$ kN

$P_Y=-51.45$ kN

$P_Z=-51.45$ kN

The coordinates of various points are as under :

Point	x	y	z
E	4	3	0
F	4	0	0
A	0	0	0
B	0	3	0
C	0	3	-3
D	0	0	-3

Joint E :

The three equations at the joint E are :

$$t_{EF}\,(x_F-x_E)+t_{EC}\,(x_C-x_E)+t_{EB}\,(x_B-x_E)+P_x=0 \qquad ...(i)$$

$$t_{EF}\,(y_F-y_E)+t_{EC}\,(y_C-y_E)+t_{EB}\,(y_B-y_E)+P_y=0 \qquad ...(ii)$$

$$t_{EF}\,(z_F-z_E)+t_{EC}\,(z_C-z_E)+t_{EB}\,(z_B-z_E)+P_z=0 \qquad ...(iii)$$

Substituting the values, we get

$$-4t_{EC}-4t_{EB}-68{\cdot}6=0 \qquad ...(1)$$

$$-3t_{EF}-51{\cdot}45=0 \qquad ...(2)$$

$$-3t_{EC}-51{\cdot}45=0 \qquad ...(3)$$

From (2), $t_{EF}=-17{\cdot}15$

From (3), $t_{EC}=-17{\cdot}15$

From (1), $t_{EB}=0$

$\therefore$ $T_{EF}=-17{\cdot}15\times3=\mathbf{-51{\cdot}45\ kN}$

$T_{EC}=-17{\cdot}15\times5=\mathbf{-85{\cdot}75\ kN}$

$T_{EB}=0.$

Joint F : The three equations are

$$t_{FA}\,(x_A-x_F)+t_{FB}\,(x_B-x_F)+t_{FD}\,(x_D-x_F)+t_{FE}\,(x_E-x_F)=0$$

$$t_{FA}\,(y_A-y_F)+t_{FB}\,(y_B-y_F)+t_{FD}\,(y_D-y_F)+t_{FE}\,(y_E-y_F)=0$$

$$t_{FA}\,(z_A-z_F)+t_{FB}\,(z_B-z_F)+t_{FD}\,(z_D-z_F)+t_{FE}\,(z_E-z_F)=0$$

Substituting the values, we get

$$-4t_{FA}-4t_{FB}-4t_{FD}=0 \qquad ...(4)$$

$$3t_{FB}+3t_{FE}=0 \qquad ...(5)$$

$$-3t_{FD}=0 \qquad ...(6)$$

From (6), $t_{FD}=0$

From (5), $t_{FB}=-t_{FE}=-t_{EF}=+17{\cdot}15$

From (4), $-4t_{FA}-4\times17{\cdot}15=0$

or $t_{FA}=-17{\cdot}15$

Hence $T_{FD}=0$

$T_{FB}=+17{\cdot}15\times5=\mathbf{+85{\cdot}75\ kN}$

$T_{FA}=-17{\cdot}15\times4=\mathbf{-68{\cdot}60\ kN}.$

Example 25·3. *The feet of a tripod resting on a smooth ground are tied by horizontal bars forming a triangle BCD, as shown in Fig. 25·6 (a), where E is the mid-point of CD and F is the mid-point of BE. The apex A [Fig. 25·6 (b)] of the tripod is 3 m vertical above point F. Determine the forces in all the members due to a load of 100 kN suspended from apex A.*

Solution.

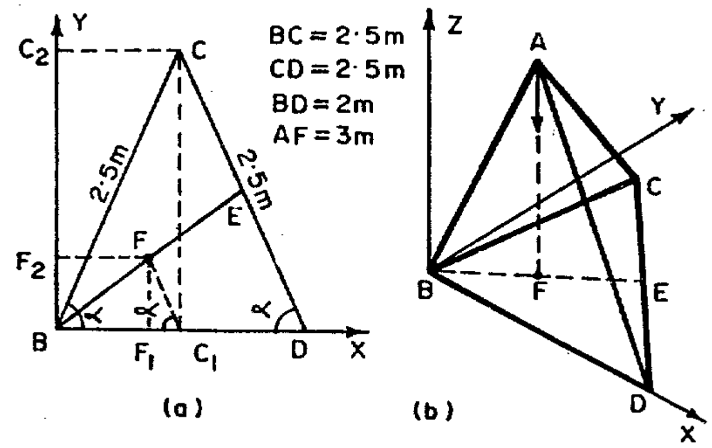


Fig. 25·6

Fig. 25·6 (a) shows the plan of the base triangle BCD in the horizontal plane (x-y plane). Let B be the origin, and let x-axis be directed along BD ; y-axis will be perpendicular to x-axis, while z-axis will be directed vertically upwards.

Drop CC_1 perpendicular to BD. Obviously, $BC_1=C_1D=1$ m.

Also, $CC_1=\sqrt{(2{\cdot}5)^2-(1)^2}=2{\cdot}29$ m.

$$\cos\alpha=\frac{1}{2{\cdot}5}=0{\cdot}4\ ;\ \sin\alpha=\frac{2{\cdot}29}{2{\cdot}5}=0{\cdot}916$$

Since F is the mid-point of BE and C_1 is the mid-point of BD, FC_1 will be parallel to ED and will be equal to half of ED.

$\therefore$ $FC_1=\frac{1}{2}\,ED=\frac{1}{4}\,CD=\frac{1}{4}\times2{\cdot}5=0{\cdot}625$ m

From F, drop perpendiculars FF_1 and FF_2 on x and y-axes.

From $\triangle FF_1C_1$, $FF_1=FC_1\sin\alpha=0{\cdot}625\times0{\cdot}916=0{\cdot}5725$ m

$F_1C_1=FC_1\cos\alpha=0{\cdot}625\times0{\cdot}4=0{\cdot}25$ m

$\therefore$ $BF_1=1-0{\cdot}25=0{\cdot}75$

Since point A is vertically above F, the x and y coordinates of point A will be the same as that of point F.

Alternatively, the coordinates of point F may be found as under :

(*i*) *For E* : $x_E=\dfrac{x_C+x_D}{2}=\dfrac{1+2}{2}=1{\cdot}5$

$$y_E=\frac{y_C+y_D}{2}=\frac{2{\cdot}29+0}{2}=1{\cdot}145$$

$$z_E=\frac{z_C+z_D}{2}=\frac{0+0}{2}=0$$

(*ii*) *For F* : $x_F=\dfrac{x_B+x_E}{2}=\dfrac{0+1\cdot5}{2}=0\cdot75$

$$y_F=\frac{y_B+y_E}{2}=\frac{0+1\cdot145}{2}=0\cdot5725$$

$$z_F=\frac{z_B+z_E}{2}=\frac{0+0}{2}=0$$

Hence the co-ordinates of various points are as under :

Point	*x*	*y*	*z*
B	0	0	0
C	1	2·29	0
D	2	0	0
A	0·75	0·5725	3

Length $AB=\sqrt{(x_B-x_A)^2+(y_B-y_A)^2+(z_B-z_A)^2}$

$$=\sqrt{(0-0\cdot75)^2+(0-0\cdot5725)^2+(0-3)^2}=3\cdot145 \text{ m}$$

Length $AC=\sqrt{(1-0\cdot75)^2+(2\cdot29-0\cdot5725)^2+(0-3)^2}$

$$=3\cdot466 \text{ m}$$

Length $AD=\sqrt{(2-0\cdot75)^2+(0-0\cdot5775)^2+(0-3)^2}=3\cdot3 \text{ m}$

Joint A : The three equations are

$$t_{AB}(x_B-x_A)+t_{AC}(x_C-x_A)+t_{AD}(x_D-x_A)+X_A=0 \quad \ldots(i)$$
$$t_{AB}(y_B-y_A)+t_{AC}(y_C-y_A)+t_{AD}(y_D-y_A)+Y_A=0 \quad \ldots(ii)$$
$$t_{AB}(z_B-z_A)+t_{AC}(z_C-z_A)+t_{AD}(z_D-z_A)+Z_A=0 \quad \ldots(iii)$$

Substituting the values, we get

$$-0\cdot75t_{AB}+0\cdot25t_{AC}+1\cdot25t_{AD}=0 \quad \ldots(1)$$
$$-0\cdot5725t_{AB}+1\cdot7175t_{AC}-0\cdot5725t_{AD}=0 \quad \ldots(2)$$
$$-3(t_{AB}+t_{AC}+t_{AD})-100=0 \quad \ldots(3)$$

Solving (1), (2) and (3), we get

$t_{AB}=-16\cdot667$; $t_{AC}=-8\cdot333$, $t_{AD}=-8\cdot333$.

Joint B :

Because of *smooth* ground, the reactions at *B*, *C* and *D* will be wholly *vertical*. The threee equations are

$$t_{BA}(x_A-x_B)+t_{BC}(x_C-x_B)+t_{BD}(x_D-x_B)+X_B=0 \quad \ldots(iv)$$
$$t_{BA}(y_A-y_B)+t_{BC}(y_C-y_B)+t_{BD}(y_D-y_B)+Y_B=0 \quad \ldots(v)$$
$$t_{BA}(z_A-z_B)+t_{BC}(z_C-z_B)+t_{BD}(z_D-z_B)+Z_B=0 \quad \ldots(vi)$$

Substituting the values, we get

$$0\cdot75t_{BA}+1t_{BC}+2t_{BD}=0 \quad \ldots(4)$$
$$0\cdot5725t_{BA}+2\cdot29t_{BC}=0 \quad \ldots(5)$$
$$3t_{BA}+R_B=0 \quad \ldots(6)$$

From (6), $R_B=-3t_{BA}=3\times16\cdot667=$ **50 kN**

From (5), $t_{BC}=-0\cdot25t_{BA}=+4\cdot167$

From (4), $t_{BD}=\frac{1}{2}[0\cdot75\times16\cdot667-4\cdot167]=+4\cdot167$

Joint D :

$$t_{DA}(x_A-x_D)+t_{DB}(x_B-x_D)+t_{DC}(x_C-x_D)+X_D=0 \quad \ldots(vii)$$
$$t_{DA}(y_A-y_D)+t_{DB}(y_B-y_D)+t_{DC}(y_C-y_D)+Y_D=0 \quad \ldots(viii)$$
$$t_{DA}(z_A-z_D)+t_{DB}(z_B-z_D)+t_{DC}(z_C-z_D)+Z_D=0 \quad \ldots(ix)$$

Substituting the values

$$-1\cdot25t_{DA}-2t_{DB}-t_{DC}=0 \quad \ldots(7)$$
$$0\cdot5725t_{DA}+2\cdot29t_{DC}=0 \quad \ldots(8)$$
$$3t_{DA}+R_D=0 \quad \ldots(9)$$

From (9), $R_D=-3t_{DA}=3\times8\cdot333=$ **25 kN**

From (8), $t_{DC}=-\dfrac{0\cdot5725}{2\cdot29}\,t_{DA}=+2\cdot083$

From (7), $t_{DB}=-\frac{1}{2}(t_{DC}+1\cdot25t_{DA})=-\frac{1}{2}(2\cdot083-1\cdot25\times8\cdot333)$

$$=4\cdot167 \text{ (check)}$$

Hence the values of tension coefficients and forces in all the six members are as under :

Member	*t*	*L*	*T (kN)*
AB	−16·667	3·145	−52·42
AC	−8·333	3·466	−28·88
AD	−8·333	3·30	−27·50
BC	+4·167	2·5	+10·42
CD	+2·083	2·5	+5·21
DB	+4·167	2	+8·33

Example 25·4. *A space frame shown in Fig. 25·7 is supported at A, B, C and D in a horizontal plane, through ball joints. The member EF is horizontal, and is at a height of 3 m above the base. The loads at the joints E and F, shown in the figure act in a horizontal plane. Find the forces in all the members of the frame.*

Solution.

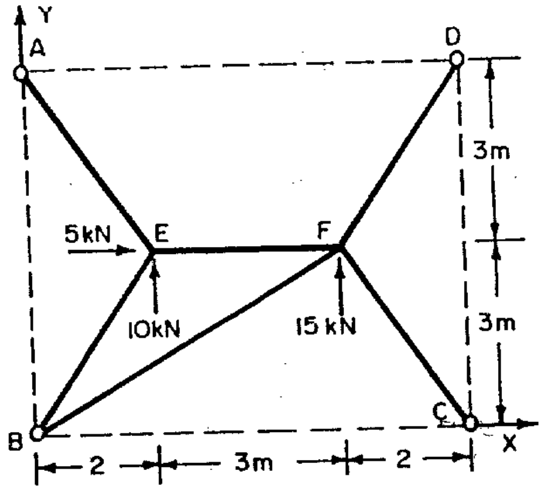


Fig. 25·7

Let the origin be at B, with X and Y axes along BC and BA respectively, and let Z-axis be directed vertically. The coordinates of various points are as under :

Point	X	Y	Z
B	0	0	0
C	7	0	0
D	7	6	0
A	0	6	0
E	2	3	3
F	5	3	3

$$L_{AE}=\sqrt{(2-0)^2+(3-6)^2+(3-0)^2}=4{\cdot}6904 \text{ m}$$
$$=L_{BE}=L_{CF}=L_{DF}$$
$$L_{BF}=\sqrt{(5-0)^2+(3-0)^2+(3-0)^2}=6{\cdot}5574 \text{ m}$$
$$L_{EF}=3 \text{ m (given).}$$

Joint E : The three equations at joint E as follows :

$$t_{EA}(x_A-x_E)+t_{EB}(x_B-x_E)+t_{EF}(x_F-x_E)+X_E=0 \quad \ldots(i)$$
$$t_{EA}(y_A-y_E)+t_{EB}(y_B-y_E)+t_{EF}(y_F-y_E)+Y_E=0 \quad \ldots(ii)$$
$$t_{EA}(z_A-z_E)+t_{EB}(z_B-z_E)+t_{EF}(z_F-z_E)+Z_E=0 \quad \ldots(iii)$$

Substituting the values, we obtain

$$-2t_{EA}-2t_{EB}+3t_{EF}+5=0 \quad \ldots(1)$$
$$3t_{EA}-3t_{EB}+10=0 \quad \ldots(2)$$
$$-3t_{EA}-3t_{EB}=0 \quad \ldots(3)$$

From (3), $t_{EA}=-t_{EB}$

From (2), $t_{EA}+t_{EA}=-\frac{10}{3}$

or $t_{EA}=-\frac{5}{3}$ ...(*a*)

$\therefore$ $t_{EB}=+\frac{5}{3}$ ...(*b*)

From (1), $\left(2\times\frac{5}{3}\right)-\left(2\times\frac{5}{3}\right)+3t_{EF}+5=0$

From which $t_{EF}=-\frac{5}{3}$ ...(*c*)

Joint F : The three equations are

$$t_{FD}(x_D-x_F)+t_{FC}(x_C-x_F)+t_{FB}(x_B-x_F)+t_{FE}(x_E-x_F)+X_F=0 \quad \ldots(iv)$$
$$t_{FD}(y_D-y_F)+t_{FC}(y_C-y_F)+t_{FB}(y_B-y_F)+t_{FE}(y_E-y_F)+Y_F=0 \quad \ldots(v)$$
$$t_{FD}(z_D-z_F)+t_{FC}(z_C-z_F)+t_{FB}(z_B-z_F)+t_{FE}(z_E-z_F)+Z_F=0 \quad \ldots(vi)$$

Substituting the values, we get

$$2t_{FD}+2t_{FC}-5t_{FB}-3t_{FE}=0 \quad \ldots(4)$$
$$3t_{FD}-3t_{FC}-3t_{FB}+15=0 \quad \ldots(5)$$
$$-3t_{FD}-3t_{FC}-3t_{FB}=0 \quad \ldots(6)$$

From (4), $t_{FD}+t_{FC}-2{\cdot}5\ t_{FB}=-2{\cdot}5$ ...(*d*)

From (5), $t_{FD}-t_{FC}-t_{FB}=-5$ ...(*e*)

From (6), $t_{FD}+t_{FC}+t_{FB}=0$ ...(*f*)

From (*e*) and (*f*), $t_{FD}=-2{\cdot}5$

From (*d*) and (*e*), $t_{FB}=+0{\cdot}7143$

From (*f*), $t_{FC}=+1{\cdot}7857$

Hence the tension coefficients and forces in the various members wil be as shown in the table below.

Member	*Length* (*m*)	t	T (kN)
EA	4·6904	−1·6667	−7·817
EB	4·6904	+1·6667	+7·817
EF	3·0	−1·6667	+5·0
FC	4·6904	+1·7857	+8·376
FD	4·6904	−2·50	−11·726
FB	6·5574	+0·7143	+4·684

PROBLEMS

1. Find forces in all the members of the space frame shown in Fig. 25·8. Take $AB=4$ m, $AD=5$ m and $AF=6$ m.

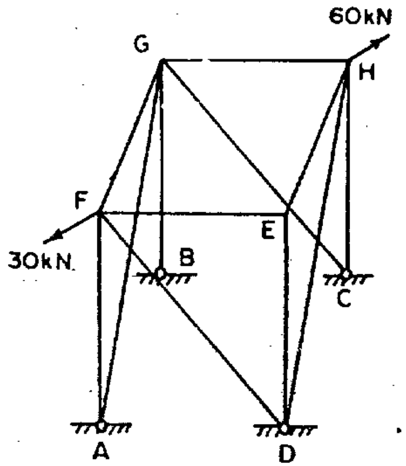


Fig. 25·8

2. Fig 25·9 shows the plan of a tripod, the feet A, B and C being in the same horizontal plane and the apex D being 3·75 m

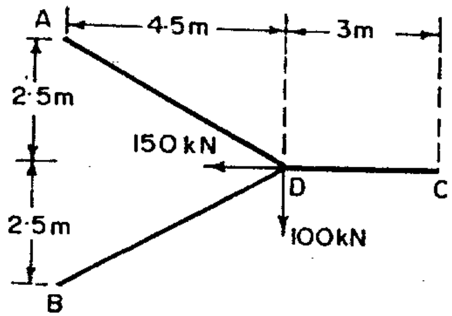


Fig. 25·9

above the plane. Horizontal loads of 100 kN and 150 kN are applied at D in the directions shown. Find the forces in the members assuming that all joints are pin-joints. (*Based on U.L.*)

3. A frame padestal shown in Fig. 25·10 is simply supported at B and has two reaction supports at A and C. Determine the reac-

tion and forces in all the members due to horizontal force 100 kN acting at F. Take $AB=BC=4$ m.

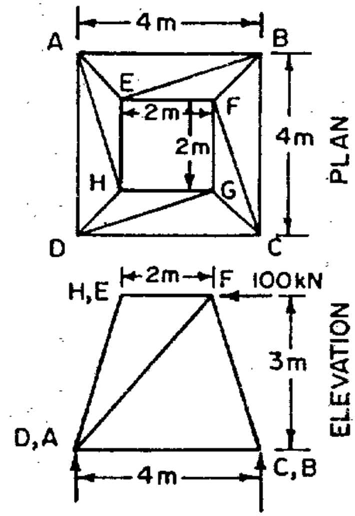


Fig. 25·10

4. Fig. 25·11 shows two views of a tripod bracket. All connections are pinned. Find the forces in magnitude and nature in the three members due to a vertical load of 100 kN acting at O.

(*Based on U.L.*)

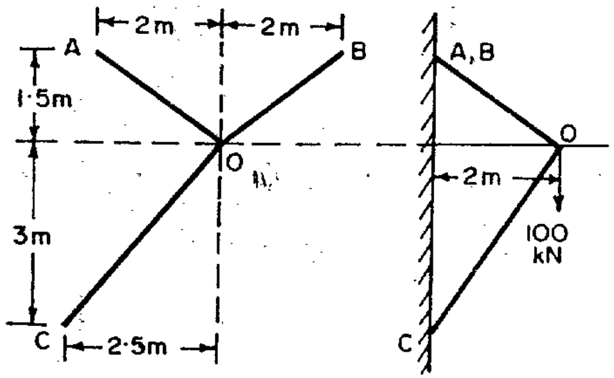


Fig. 25·11

ANSWERS

1. $T_{EA}=0$; $T_{FD}=+37{\cdot}5$; $T_{FE}=0$; $T_{FG}=+18{\cdot}74$; $T_{GA}=-30$
$T_{GB}=+28{\cdot}11$; $T_{GC}=-60$; $T_{GH}=+46{\cdot}85$; $T_{HE}=0$
$T_{HD}=+40$; $T_{HC}=-37{\cdot}48$; $T_{ED}=0$
2. $T_{DA}=+63{\cdot}7$; $T_{DC}=+96$; $T_{DB}=-191{\cdot}1$
3. $T_{AB}=-75$; $T_{BC}=-25$; $T_{CD}=0$; $T_{DA}=+25$;
$T_{EA}=+35{\cdot}36$; $T_{EB}=-79{\cdot}06$; $T_{EF}=-100$; $T_{EH}=0$;
$T_{EB}=0$; $T_{FC}=0$; $T_{FG}=0$; $T_{GC}=0$; $T_{GD}=0$; $T_{HD}=0$;
$T_{HA}=0$
4. $T_{OA}=+80{\cdot}04$ kN ; $T_{OB}=-8{\cdot}89$ kN ; $T_{OC}=-97{\cdot}5$ kN.

26

Plastic Theory

26·1. INTRODUCTION

A structure may reach its limit of usefulness through instability, fatigue or excessive deflection. Alternatively, if none of these failure modes occur, then the structure will continue to carry load beyond the elastic limit until it reaches its ultimate load through plastic deformation, and then collapse. Plastic analysis is based on this mode of failure. The concept of ductility of structural steel forms the basis for the plastic theory of bending.

The rigorous analysis of a structure according to the theory of elasticity demands that the stress satisfy two sets of conditions : (1) the equilibrium conditions and (2) compatibility conditions. The first set of conditions must be invariably satisfied in any material. However, the second condition ceases to be valid as soon as plastic yielding occurs. The elastic method of design assumes that a frame will become useless as soon as yield stress is reached. The working stress is, therefore, kept much below the yield stress. The design so produced gives a structure of unknown ultimate strength. The elastic methods of analysis are also very cumbersome, specially for redundant frames. In plastic method of design, the limit load of a system is a statically determinate quantity. The limit load is independent of all imperfections of the structure, such as faulty length of bars, settlement of supports and residual stresses caused by rolling or welding. The plastic method of design gives an economical design. The margin of safety provided in this method is not less than that provided according to the past practice.

The need for the study of plastic behaviour was appreciated by A.E.H. Love in 1892. The posibility of the development of plastic hinge was first suggested by G.V. Kazinczy in 1914. Prof. H. Maier-Leibnitz of Germany carried out load tests on encastre and continuous beams, carrying them out of the elastic into the plastic range, and showed that the ultimate capacity was not effected by settlement of supports of continuous beams. Further work was done by Vander Brock in United States and J.F. Baker and his associates in Great Britain.

26·2. THE DUCTILITY OF STEEL

The plastic theory is based on the ductility of steel. Through ductility, structural steel has capacity of absorbing large deformation beyond elastic limit without the danger of fracture. However, in the plastic range, the behaviour of steel depends strongly not only on its chemical composition but also on the mechanical and thermal treatments to which it has been subjected.

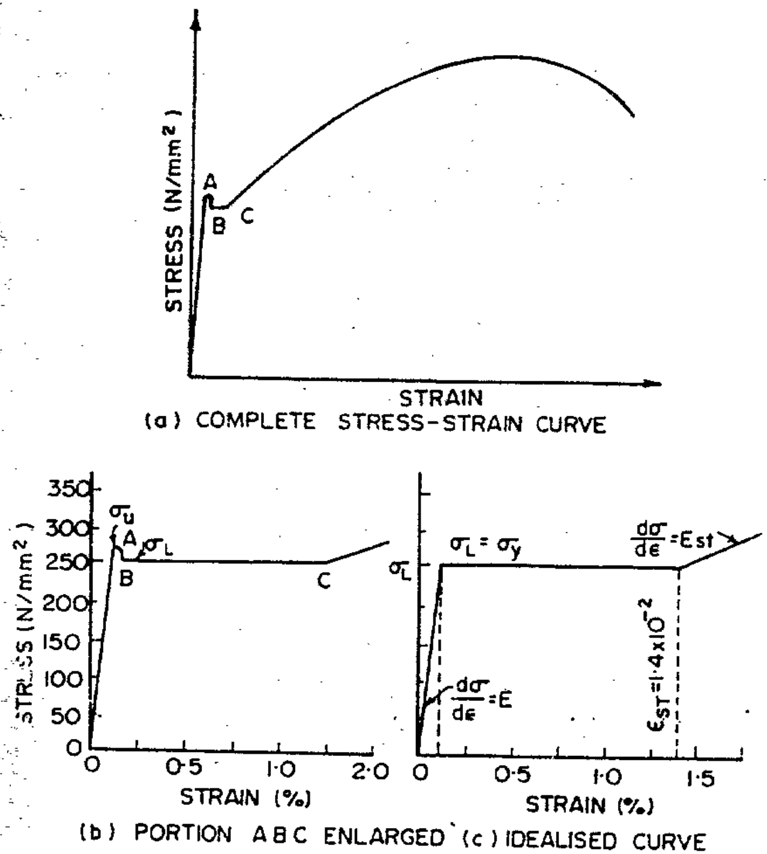


Fig. 26·1.
Stress-Strain curves.

Fig. 26·1 (*a*) shows the complete stress-strain curve of mild steel. Fig. 26·1 (*b*) shows the portion *ABC* enlarged. It will be seen that the stress-strain relation is linear in the elastic range. The upper yield is reached at point *A*, and then the stress suddenly drops to lower yield stress at *B*. The strain then increases upto *C* at constant stress. This represents the plastic range. Beyond *C* the strain increases with further increase of stress and the material is said to be in strain hardening range. For ordinary steel the elastic strain is about 1/12 to 1/15 of strain at the beginning of the strain hardening and about 1/200 of maximum strain.

Experience shows that the metal of rolled beams does not usually exhibit any upper yield point and that even when an upper yield point exists, it can be removed by cold working such as straightening. Hence the theory of plastic bending is based on the assumption of a steel without upper yield point. The strain upto point *C* is about 1·5%. In plastic design, at ultimate load the critical strains will not have exceeded about 1·5% elongation. Hence the strain hardening range is neglected in simple theory of plastic bending. This reduces complications in the calculations, and still leaves available a major portion of reserve ductility of steel which can be used as an added margin of safety. Fig. 26·1 (*c*) shows the idealised stress-strain curve which forms the basis of plastic design.

26·3. ULTIMATE LOAD CARRYING CAPACITY OF MEMBERS CARRYING AXIAL FORCES

Consider three bars *OA*, *OB* and *OC*, meeting at a common point *O* and hinged at the other ends *A*, *B* and *C* respectively (Fig. 26·2). Let a vertical load *P* be applied at the point *O*. We shall first solve the problem by elastic method.

Let P_1 be the tensile force in *OB* and P_2 be the force in *OA* and *OC*. Point *O* moves vertically to *O′* after the application of the load. Let Δ_1 and Δ_2 be the axial deformations of rods *BO* and *AO* (or *CO*) respectively. The dotted lines show the deformation portion of the structure.

From Statics,

$$P_1+2P_2 \cos \theta=P \quad \ldots(1)$$

The structure is statically indeterminate to single degree. The second equation is obtained from the compatibility of deformations :

$$\Delta_2=\Delta_1 \cos \theta$$

or

$$\frac{P_2L_2}{AE}=\frac{P_1L_1}{AE} \cos \theta \quad \ldots[2\ (a)]$$

where L_1=Length of member OB and L_2=Length of OA.
A=Area cross-section of each bar (assumed equal).

But
$$\frac{L_1}{L_2}=\cos\theta$$

$\therefore$
$$\frac{P_2}{P_1}=\frac{L_1}{L_2}\cos\theta=\cos^2\theta \qquad ...(2)$$

From (1) and (2), we get
$$P_1=\frac{P}{1+2\cos^3\theta} \text{ and } P_2=\frac{P\cos^2\theta}{1+2\cos^3\theta}$$

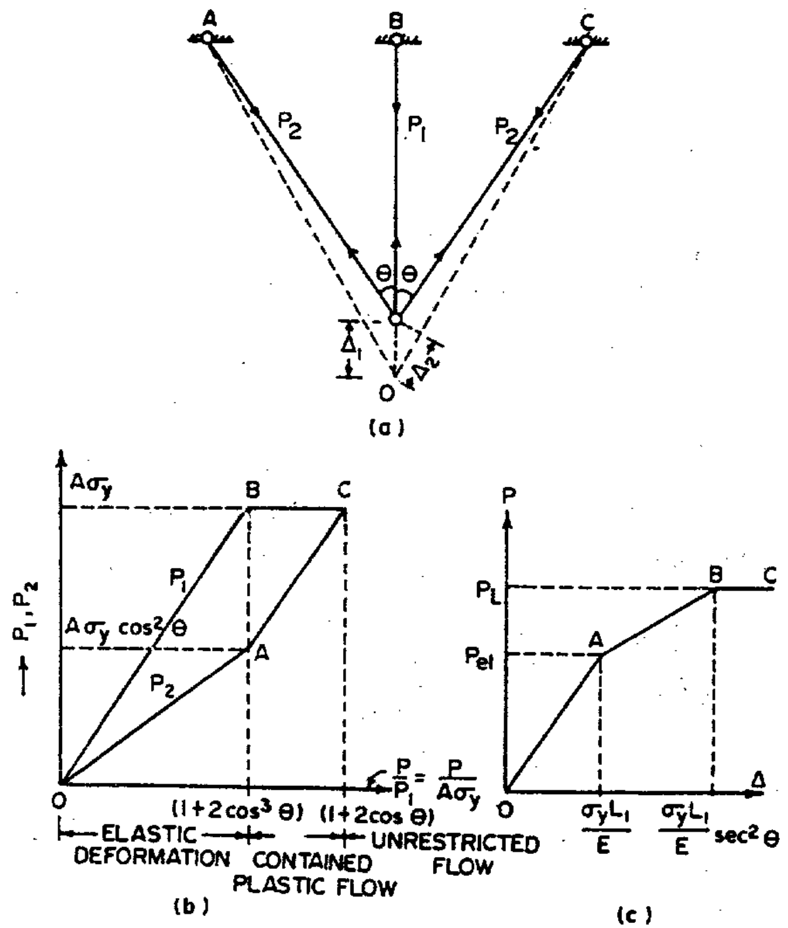


Fig. 26·2.

Since $\cos\theta$ is always less than unity, it is clear that P_1 is greater than P_2. From the point of view of elastic theory, the system

fails (or becomes useless) when the central bar reaches yield stress σ_Y. At the yied condition, the force P_1 in the central bar will be equal to $A.\sigma_Y$ while the force P_2 (from Eq. 2) will be equal to $A.\sigma_Y.\cos^2\theta$. Hence the total load, called the ***elastic limit load*** is

$$P_{el}=P_1+2P_2\cos\theta=A\sigma_Y+2A\sigma_Y\cos^3\theta$$

or
$$P_{el}=A\sigma_Y(1+2\cos^3\theta) \qquad ...(26·1)$$

The corresponding elongation of OB is given by
$$\Delta_{el}=\frac{\sigma_Y L_1}{E} \qquad ...(26·2)$$

In Fig. 26·2 (*b*), OB and OA represent the increase in P_1 and P_2 respectively, as the external force is increased from zero to P.

Let us now analyse the structure on the basis of Plastic theory. Eq. (1) above holds good in this case also. However, when the load P_1 in the central rod reaches a value $A\sigma_Y$, the structure does not collapse, but continues to take further load. When the external load is increased above P_{el} the force in the central bar retains the constant value of $A\sigma_Y$, while the force P_2 increases further from its value of $A\sigma_Y\cos^2\theta$ to a value $A\sigma_Y$ till the bars OA and OC reach the yield point. When all the three bars have yielded, we have
$$P_1=P_2=A\sigma_Y \qquad ...(3)$$

From (1) and (3), the load at the yield, called the ***plastic collapse*** or ***limit load***, is given by
$$P_L=P_1+2P_2\cos\theta=A\sigma_Y+2A\sigma_Y.\cos\theta$$

or
$$P_L=A\sigma_Y(1+2\cos\theta) \qquad ...(26·3)$$

When all the three bars have reached the plastic stage,
$$\Delta_2=\frac{\sigma_Y L_2}{E}.$$

$\therefore$
$$\Delta_1=\frac{\Delta_2}{\cos\theta}=\frac{\sigma_Y L_2}{E\cos\theta}=\frac{\sigma_Y L_1}{E\cos^2\theta}$$

Hence the vertical deflection of the joint O is given by
$$\Delta_L=\frac{\sigma_Y L_1}{E\cos^2\theta}=\frac{\sigma_Y L_1}{E}\sec^2\theta \qquad ...(26·4)$$

It is to be observed that this deflection is $\sec^2\theta$ times the elastic deflection Δ_{el} given by Eq. 26·2. If $\theta=45°$, we get
$$\Delta_L=2\Delta_{el}=\frac{2\sigma_Y L_1}{L}.$$

Fig. 26·2 (*e*) shows a plot between the load P *versus* the vertical deflection $\Delta(=\Delta_1)$.

When the external load P_{el} is increased, the central rod is fully in plastic stage while the end rods are still in elastic stage. This condition is generally called the *contained plasic flow condition* represented by lines BC and AC of Fig. 26·2 (*b*). When all the three rods have become plastic, the condition is known as the *unrestricted flow condition*.

The load P_L given by Eq. 26·3 may be considered as the *failure load*. The *service load* may be taken as a certain portion $\frac{1}{Q}$ of load P_L, where Q is a safety factor, usually called a *load factor* in plastic analysis. Thus,

$$Q=\frac{P_L}{P} \quad ...(26·5)$$

where P=service load.

The saving achieved by designing according to the plastic theory instead of elastic theory is equal to

$$\left(1-\frac{\text{Area by plastic theory}}{\text{Area by elastic theory}}\right)\times 100=\left(1-\frac{1+2\cos^3\theta}{1+2\cos\theta}\right)100$$

When $\theta=45°$, this amount to about 29%.

Example 26·1. *A rigid beam ABC is kept in horizontal position by three rods as shown. All the three rods are made of the same material and have equal area of cross-section of 200 mm². The length of the outer rod is 1 m while the length of the central rod is 2 m. Calculate the collapse load for the structure, applied at the centre of the beam. Take the stress at yield equal to 250 N/mm² and $E=2\times 10^5$ N/mm².*

Solution.

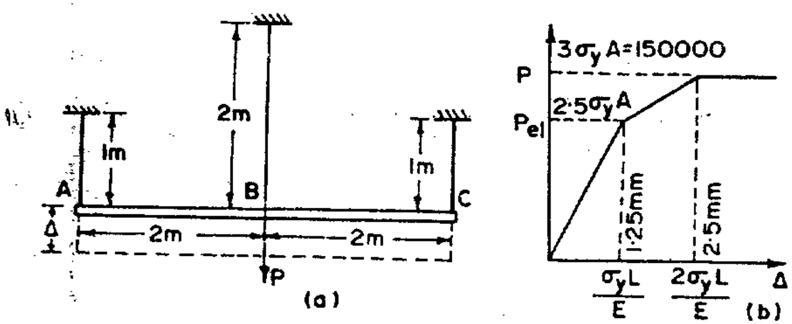


Fig. 26·3

Let P_1=force in each of the outer rods

P_2=force in the central rod.

From Statics $2P_1+P_2=P$. ...(1)

Since the beam remains horizontal, the vertical extensions Δ of all the three rods will be equal. This gives

$$\frac{P_1\times 1}{AE}=\frac{P_2\times 2}{AE}$$

or $$P_1=2P_2 \quad ...(2)$$

Eq. 2 shows that the load carried by each of the outer rods is twice the load carried by the central load. Thus the outer rods will yield first. The yield load carried by each of the outer rods will be equal to $\sigma_Y.A=250\times 200=50000$ N.

The corresponding deflection Δ_{el} is equal to $\frac{\sigma_Y L}{E}$

$$=\frac{250\times 1000}{2\times 10^5}=1{\cdot}25 \text{ mm}$$

and the load carried by the central rod is equal to

$$\tfrac{1}{2}P_1=\tfrac{1}{2}\times 50000=25000 \text{ N}.$$

Thus, according to the elastic analysis, the total load carrying capacity is given by

$$P_{el}=2P_1+P_2=(50000\times 2)+25000=125000 \text{ N} \quad ...(3)$$

This is evidently equal to $2{\cdot}5\,\sigma_Y.A$.

However, the structure does not collapse when the force in each of the outer rods is 50000 N. When the external load is increased above 125000 N, the outer rods continue to carry a constant force of 50000 N each, while the force in the central rod increases from a value of 25000 N to a value of $\sigma_Y.A=25000\times 2=50000$ N, when plastic stage is reached in it. Thus, according to the plastic theory, the collapse load is given by

$$P_L=2P_1+P_2=(50000\times 2)+50000=150000 \text{ N}.$$

This is evidently equal to $3\sigma_Y.A$.

The deflection at this stage is given by

$$\Delta_L=\frac{P_2L_2}{AE}=\frac{50000\times 2000}{200\times 2\times 10^5}=2{\cdot}5 \text{ mm}.$$

Fig. 26·3 (*b*) shows the complete load-deflection diagram.

Example 26·2. *A rigid beam ABCD is hinged at A and being supported by two vertical rods attached at B and C, as shown in Fig. 26·4. Determine* (*i*) *the load P when first yield occurs in any of* *the bars and (ii) when the whole arrangement collapse. Each rod has an area of section of 200 mm². Take $\sigma_Y = 252\ N/mm^2$.*

Solution.

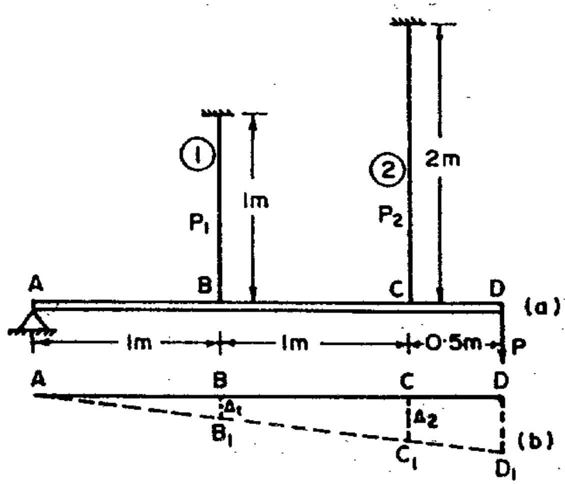


Fig. 26·24

(a) Load at fist yield

From statics, $P_1+P_2+R=P$ ...(1)

where P_1 and P_2 are the forces in rods at B and C, and R is the reaction at A, assumed vertically upwards.

Taking moments about A, we get

$$P+2P_2=2{\cdot}5\ P \qquad ...(2)$$

Fig. 26·3 (*b*) shows the deformed shape of the arrangement. If Δ_1 and Δ_2 are the vertical extensions of the two bars,

$$\frac{\Delta_1}{\Delta_2}=\frac{1}{2}$$

or $$2\Delta_1=\Delta_2$$

or $$2\,\frac{P_1\times 1}{AE}=\frac{P_2(1{\cdot}5)}{AE}$$

or $$P_2=\frac{2}{1{\cdot}5}\,P_1 \qquad ...(3)$$

Eq. (3) shows that P_2 is greater than P_1. Since the area of sections of both the bars are equal, it is evident that yield first occurs in bar 2. As the load P is increased, the force P_2 will go on increasing till yield occurs in it. Hence, at the first yield,

$$P_2=\sigma_Y.A=252\times 200=50400\ \text{N}.$$

Corresponding value of P_1 is

$$P_1=\frac{1{\cdot}5}{2}\,P_2=0{\cdot}75\ P_2$$

Substituting these in Eq. (2) the load at the first yield is given by

$$P_{el}=\frac{1}{2{\cdot}5}[P_1+2P_2]$$

$$=\frac{2{\cdot}75}{2{\cdot}5}\ P_2=\frac{2{\cdot}75}{2{\cdot}5}\times 50400=55500\ \text{N}$$

Thus according to the elastic solution, the load carrying capacity of structure is 55500 N.

(b) Load at complete collapse

When the load P is further increased, the load in rod 1 increases while the load in rod 2 remains constant at a value of 50400 N. When yield is reached in bar 1, the whole structure collapses. The force P_1 corresponding to yield in the first rod is evidently equal to $\sigma_Y.A=252\times 200=50400$ N. Substituting these values of P_1 and P_2 (each equal to 50400 N) in Eq. (2), we get

$$P_L=\frac{1}{2{\cdot}5}\,[P_1+2{\cdot}5\ P_2]$$

$$=\frac{3{\cdot}5}{2{\cdot}5}\times 50400=70500\ \text{N}.$$

It should be noted that in plastic analysis, compatibility equation 3 is no longer useful. However, the equilibrium equations (Eqs. 1 and 2) still hold good. This shows the simplicity of plastic analysis.

Example 26·3. *Compute the ultimate load P at the collapse of the structure shown in Fig. 26·5. All the four rods have equal area of cross-section.*

Solution.

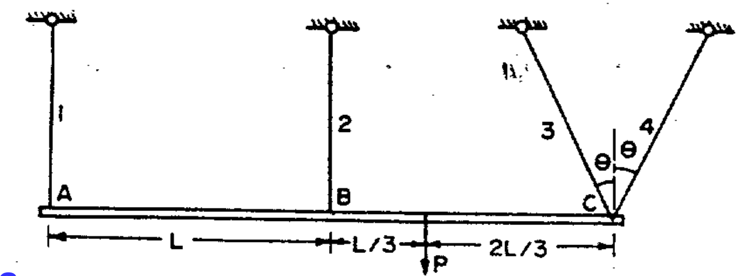


Fig. 26·5

The structure will collapse when it is turned into a mechanism. There are two possibilities, and both of these should be investigated. In the first possibility, rods 1 and 2 become plastic and the collapse may take place by rotation about point C. In the second possibility, rods 4, 3 and 2 may become plastic and failure may take place by rotation about A. The free body diagrams for both these possibilities are shown in Fig. 26·6 (a) and (b).

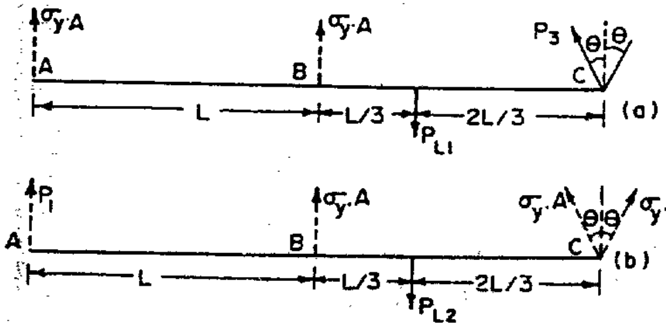


Fig. 26·6

The first possibility of collapse is shown in Fig. 26·6 (a). By inspection, rod 1 will yield first, when the force in it is $P_1=\sigma_Y.A$. With the further increase in the external load, P_1 will remain constant at $\sigma_Y.A$ while P_2 will increase till it also becomes equal to $\sigma_Y.A$. At this stage, the structure will turn into mechanism, and collapse will take place by rotation about C. Just before such collapse, we get, by taking moments about C,

$$\sigma_Y.A.2L+\sigma_Y.A.L=P_{L1}\left(\frac{2L}{3}\right)$$

or $$P_{L1}=\frac{9}{2}\sigma_Y.A \qquad \ldots(1)$$

Let us now consider the second possibility, when the rod 4, 3 and 2 yield, and collapse takes place by rotation about A. At the yield stage, force carried by each of these rods is equal to $\sigma_Y.A$. Hence we get by taking moments about A,

$$\sigma_Y.A.L+2\sigma_Y.A\cos\theta.2L=P_{L2}\cdot\frac{4}{3}L$$

or $$P_{L2}=\frac{3}{4}\sigma_Y.A\,(1+4\cos\theta) \qquad \ldots(2)$$

It will be seen that P_{L2} is less than P_{L2} for all values of θ. Hence the collpase load is given by Eq. (2).

26·4. PLASTIC BENDING OF BEAMS

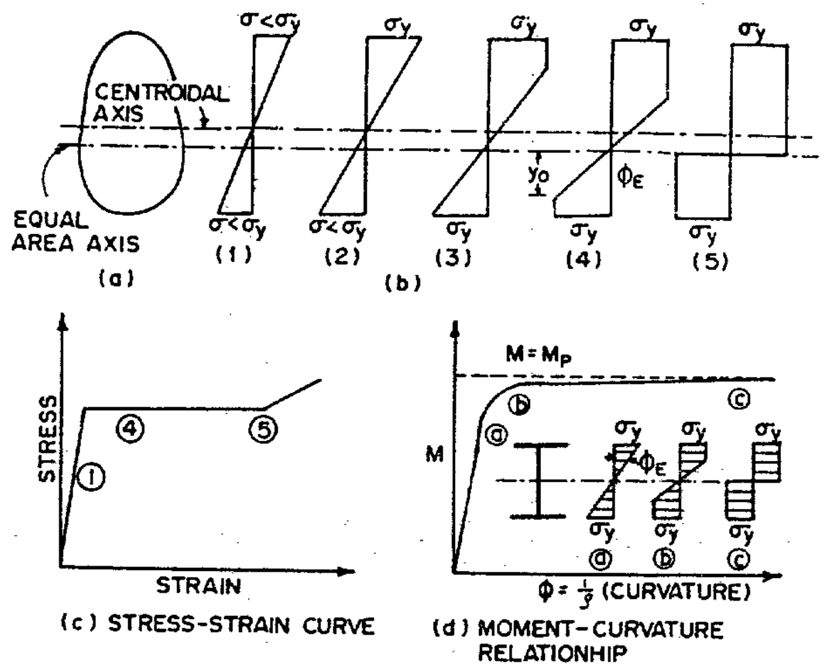


Fig. 26·7.
Plastic bending of beams.

Let a beam be subjected to an increasing bending moment M (pure bending). The beam has atleast one axis of symmetry so that bending is symmetrical about that axis. When the bending stresses are within the elastic range, the bending stress distribution will be as shown in Fig. 26·7 (b-1). The neutral axis will pass through the centroid of the section. As the moment is increased, yield stress will appear either in the topmost (or in the bottom most fibre, as the case may be) with the neutral axis still passing through the centroid of the section [Fig. 26·7 (b-2)]. The moment at which the first yield has occurred is called the ***yield moment*** (M_Y). With further increase of M, the yield will also occur in the bottom fibre and it will spread inwards in the top portion [Fig. 26·7 (b-3)]. The neutral axis no longer passes through the centroid, its location being determined by the fact that the total compressive force is equal to the total tensile force over the cross-section.

Further increase of bending moment will cause the yield to spread further inwards towards neutral axis. A stage is ultimately reached when the yield spreads right up to the neutral axis and the section becomes *fully plastic* [Fig. 26·7 (*b*-5)]. The corresponding bending moment is called the *fully plastic moment* and is denoted by M_P. Neglecting strain hardening in the outer fibres, no further increase in the bending moment can be attained. The plastic moment, therefore, represents the limiting strength of the beam in bending. The neutral axis in case of fully plastic section will pass through the equal area axis. In case of sections having two axes of symmetry, the location of neutral axis in elastic and fully plastic conditions remain unchanged. When the fully plastic moment is reached, the section will act as a hinge permitting rotation. With further increase of the load, the yield will spread in longitudinal direction.

Moment curvature relationship

The curvature ϕ is the relative rotation of two sections a unit distance apart. As in the elastic bending, we have, according to first approximation :

$$\phi=\frac{1}{\rho}=\frac{\varepsilon}{y_0}=\frac{\sigma_Y}{Ey_0}$$

or $$\phi E=\frac{\sigma_Y}{y_0} \qquad \text{...[26·7 (a)]}$$

where y_0=distance of farthest still-elastic fibre and ε=maximum elastic strain.

The curvature at any given stage can thus be obtained from the stess distribution. The curvature of a partially plastic section is controlled by the deformations of still-elastic interior fibres. Fig. 26·7 (*d*) shows the moment-curvature relationship. This moment-curvature relationship is of great importance in the plastic theory. When an unloaded beam is subjected to increasing bending moment, the curvature first increases linearly with bending moment as represented by *Oa*. This is elastic range. With the appearance of yield under the section of yield moment M_Y, the linear relation no longer holds good. As the bending moment is increased further the curvature will increase at a faster rate which corresponds to the spread of yield in inward direction. As the bending moment approaches its fully plastic value the curvature will tend to infinity. This corresponds to fully plastic section. When, at a particular section, the bending moment reaches the value M_P, the bending moment on either side of it will be lesser than M_P. With the

attainment of fully plastic moment at a section, the curvature at this section becomes infinitely large. Thus a finite change of slope can occur over an indefinitely small length of the member at this section. This section will therefore, act as if a hinge has been inserted in the member at this section.

26·5. EVALUATION OF FULLY PLASTIC MOMENT

The moment of resistance developed by a fully plastic section is called the *fully plastic moment* M_P. The following simplifying assumptions are made for evaluation of fully plastic moment (Baker, 1956) :

1. The material obeys Hooke's law until the stress reaches the upper yield value ; on further straining the stress drops to the lower yield value and thereafter remains constant.
2. The upper and lower yield stresses and the modulus of elasticity have the same values of compression as in tension.
3. The material is homogeneous and isotropic in both the elastic and plastic states.
4. Plane transverse sections remain plane and normal to the longitudinal axial after bending, the effect of shear being neglected.
5. There is no resultant axial force on the beam.
6. The cross-section of beam is symmetrical about an axis through its centroid parallel to the plane of bending.
7. Every layer of the material is free to expand and contract longitudinally and laterally under stress as if separated from the other layers.

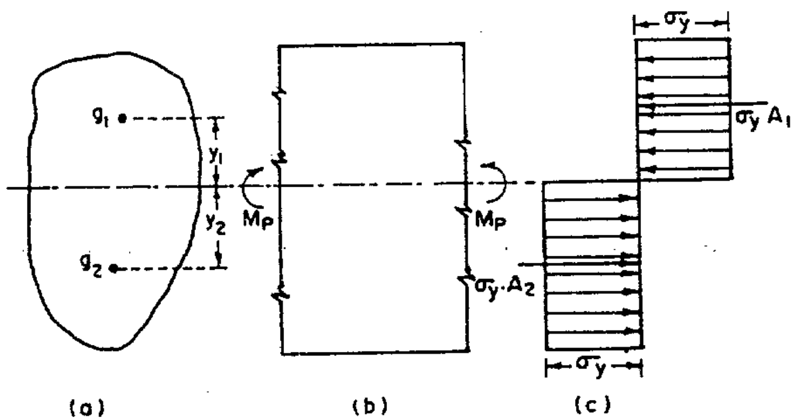


Fig. 26·8.
Evaluation of fully plastic moment.

Consider a cross-section of a beam subjected to a fully plastic moment M_P of sagging nature. Under the action of this fully plastic moment, every fibre of the cross-section will be stressed to the yield stress σ_Y and the stress distribution will be rectangular as shown in Fig. 26·8 (*c*). The stress in the fibres above the neutral axis will be of compressive nature while the stress of fibres located below the neutral axis will be of tensile nature. Let A_1 be the area of the portion of the section situated above the neutral axis, its C.G. (g_1) being at y_1 from the N.A. Similarly, let A_2 be the tensile area, with its C.G. (g_2) situated at y_2 below the neutral axis.

The compressive force acting over the cross-section $= A_1 \times \sigma_Y$

Total tensile force acting over the cross-section $= A_2 \times \sigma_Y$

$$A_1 \sigma_Y = A_2 \sigma_Y \qquad \ldots(2)$$

or $$A_1 = A_2$$

But $$A = A_1 + A_2 \quad \text{(total area)}$$

$\therefore$ $$A_1 = A_2 \frac{A}{2}.$$

Thus, the neutral axis divides the section into two equal parts, i.e. it passes through the *equal area axis.*

Again these two forces should form a couple such that its magnitude is equal to the externally applied moment M_P. Hence

$$A_1 \sigma_Y . y_1 + A_2 \sigma_Y . y_2 = M_P \qquad \ldots(2)$$

But $$A_1 = A_2 = \frac{A}{2}$$

$\therefore$ $$M_P = \sigma_Y \frac{A}{2} (y_1 + y_2) \qquad \ldots(26·6)$$

or $$M_P = \sigma_Y . Z_P \qquad \ldots(26·7)$$

where $$Z_P = \frac{A}{2} (y_1 + y_2) \qquad \ldots(26·8)$$

Z_P is the first moment of area about neutral axis is termed as *plastic section modulus*. It should be noted that the fully plastic moment M_P is *constant* for a particular cross-section of a given material.

Also, yield moment M_Y (*i.e.*, the moment at which the first yield occurs, section still being elastic) is given by

$$M_Y = \sigma_Y Z \qquad \ldots(26·9)$$

where Z = elastic section modulus.

The ratio of plastic moment to the yield moment is called the *shape factor S*.

$$S = \frac{M_P}{M_Y} = \frac{\sigma_Y Z_P}{\sigma_Y Z} = \frac{Z_P}{Z} \qquad \ldots(26·10)$$

EVALUATION OF SHAPE FACTOR

The shape factor is the property of a section and depends solely on the shape of the cross-section. We shall evaluate the shape factor of some of the standard sections.

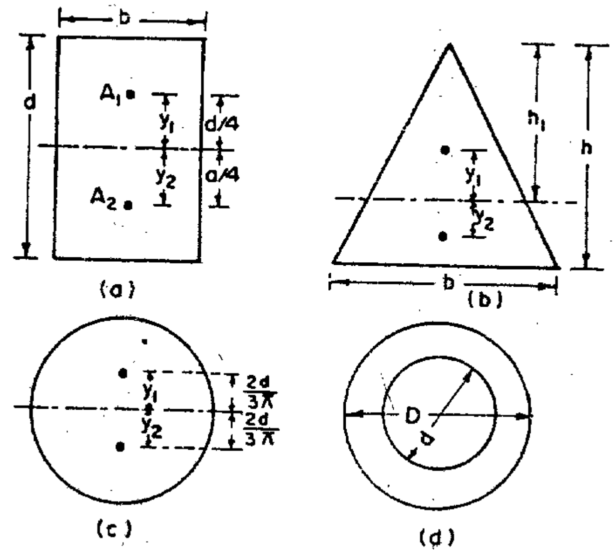


Fig. 26·9

(*a*) **Rectangular Section** [Fig. 26·9 (*a*)]

Elastic section modulus, $Z_P = \frac{bd^2}{6}$

Plastic section modulus, $Z_P = \frac{A}{2} (y_1 + y_2)$

$$= \frac{b \times d}{2} \left[\frac{d}{4} + \frac{d}{4} \right] = \frac{bd^2}{4} \qquad \ldots(26·11)$$

$\therefore$ $$S = \frac{Z_P}{Z} = \frac{bd^2}{4} \div \frac{bd^2}{6} = 1·5$$

(*b*) **Triangular Section** [Fig. 26·9 (*b*)]

$$I = \frac{bh^3}{36}$$

The distance of extreme fibre from the elastic neutral axis $= \frac{2}{3} h$

$\therefore$ $$Z = \frac{bh^3}{36} \times \frac{3}{2h} = \frac{bh^2}{24}$$

For locating the equal area axis, equate the area on either side. Let the equal area axis be at distance h_1 from the apex.

$$\therefore \quad \frac{b_1 h_1}{2} = \frac{1}{2} \cdot \frac{bh}{2}$$

But $\quad \dfrac{h_1}{h} = \dfrac{b_1}{b} \quad$ or $\quad b_1 = \dfrac{bh_1}{h}$

$$\frac{bh_1}{h} \cdot \frac{h_1}{2} = \frac{1}{2} \cdot \frac{bh}{2}$$

or $\quad h_1 = \dfrac{h}{\sqrt{2}}$. Similarly $b_1 = \dfrac{bh}{2} = \dfrac{b}{\sqrt{2}}$

Now $\quad y_1 = \dfrac{h_1}{3} = \dfrac{h}{3\sqrt{2}} = 0.235\, h$

and
$$y_2 = \frac{h-h_1}{3} \cdot \frac{b_1+2b}{b_1+b} = \frac{h-h/\sqrt{2}}{3} \cdot \frac{2b+b/\sqrt{2}}{b+b/\sqrt{2}}$$
$$= \frac{8-5\sqrt{2}}{6}\, h = 0.155\, h$$

$$\therefore \quad Z_P = \frac{A}{2}\left(y_1+y_2\right) = \frac{bh}{4}\left\{0.235h + 0.155h\right\}$$
$$= 0.098\, bh^2$$

$$\therefore \quad S = \frac{Z_P}{Z} = 0.098\, bh^2 \times \frac{24}{bh^2} = \mathbf{2.34}.$$

(*c*) **Circular Section** [Fig. 26.9 (*c*)]

$$Z = \frac{\pi}{32} d^3$$

$$Z_P = \frac{A}{2}\left(y_1+y_2\right), \quad \text{where } A = \frac{\pi}{4} d^2$$

$y_1 = y_2 =$ distance of C.G. of semi-circle from N.A.

$$= \frac{2d}{3\pi}$$

$$\therefore \quad Z = \frac{1}{2} \times \frac{\pi}{4} d^2 \left(\frac{2d}{3\pi} + \frac{2d}{3\pi}\right) = \frac{d^3}{6} \quad \text{...(26.13)}$$

$$\therefore \quad S = \frac{Z_P}{Z} = \frac{d^3}{6} \div \frac{\pi}{32} d^2 = \mathbf{1.7}.$$

(*b*) **Hollow Circular Section** [Fig. 26.9 (*d*)]

$d =$ internal diameter

$D =$ external diameter.

Let $\quad \dfrac{d}{D} = k$

$$Z = \frac{\frac{\pi}{64}\left[D^4 - d^4\right]}{D/2} = \frac{\frac{\pi}{64}\left[D^4 - (kD)^4\right]}{D/2}$$
$$= \frac{\pi}{64} D^3 \left(1-k^4\right)$$

For a circular section, $Z_P = \dfrac{d^3}{6}$ in general.

Hence for a hollow circular section,

$$Z = \frac{D^3}{6} - \frac{d^3}{6} = \frac{D^3}{6} - \frac{(kD)^3}{6}$$
$$= \frac{D^3}{6}\left[1-k^3\right] \quad \text{...(26.14)}$$

$$\therefore \quad S = \frac{Z_P}{Z} = \frac{D^3}{6}\left[1-k^3\right] \div \frac{\pi}{32} D^3 \left[1-k^4\right]$$
$$= 1.7\left(\frac{1-k^3}{1-k^4}\right) \quad \text{...(26.15)}$$

26.6. PLASTIC HINGE

A plastic hinge is a zone of yielding due to flexure in a structural member. At those sections where plastic hinges are located, the member acts as if it were hinged, except with a constant restraining moment M_P. Just like any ordinary hinge, the plastic hinge allows the rotation of members on its two sides without change in curvature of members. The plastic hinge is capable of resisting rotation until fully plastic moment is developed and then permitting rotation of any magnitude while the bending moment remains constant at M_P.

The hinge extends over a length of member that is dependent on loading and the geometry. The hinge length ΔL is the length of the beam over which the bending moment is greater than the yield moment M_Y. However, in all of its length ΔL the section are not plastic to its full depth. To illustrate this we shall consider the case of a simply supported beam loaded with a central point load W, the section of the beam being rectangular.

Let the yielded portion (*i.e.* the plastic hinge) extreme points distant x from either end. The moment at these extreme points is M_Y and the moment at other points beyond these is less than M_Y.

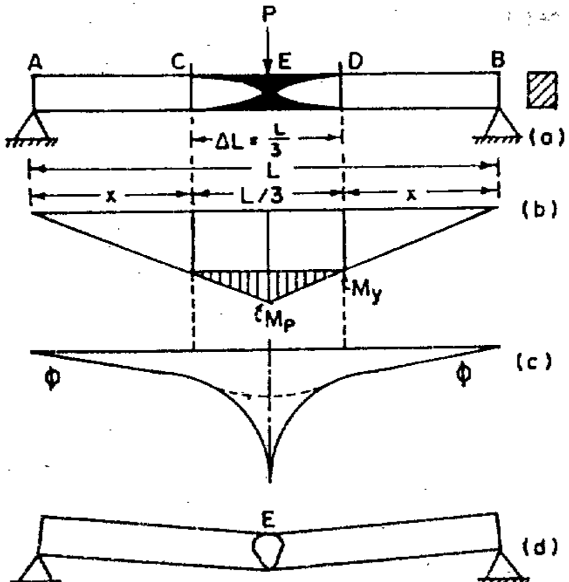


Fig. 26·10

From Fig. 26·10 (*b*), we get

$$\frac{M_P}{M_Y}=\frac{L/2}{x}$$

But $\quad \frac{M_P}{M_Y}=S=\frac{3}{2}$ for a rectangular section

$\therefore \quad \frac{3}{2}=\frac{L}{2x}$

or $\quad x=\frac{L}{3}$

Hence $\quad \triangle L=L-2x=L-\frac{2L}{3}=\frac{L}{3}.$

Similarly, it can be shown that if the beam is of I-section, the length $\triangle L$ is about $\frac{L}{8}$. The length $\triangle L$, in fact, represents the length of elasto-plastic zone.

Because of the shape of the moment curvature diagram [Fig. 26·7 (*d*)], the curvature remains very small near ends C and D of the plastic region, while in the the neighbourhood of point E, the curvature is extremely high as shown in Fig. 26·10 (*e*). The beam, therefore, deforms very nearly as if it consisted of two rigid portions connected by a hinge in E [Fig. 26·10 (*d*)]. *In most of the analytical work, it is assumed that all plastic rotations occur at a point, i.e. length of the hinge approaches zero.*

26·7. LOAD FACTOR

The load factor is the ratio of the collapse load to the working load :

$$Q=\frac{W_C}{W} \text{ or } \frac{W_L}{W}$$

where $\quad Q$=Load factor

W_C or W_L=Collapse load or limit load

W=Working load.

The value of load factor depends upon type of loading, the end conditions of the supports and the cross-section of the member.

Let $\quad M_{max}$=maximum bending moment corresponding to working load W

M_P=fully plastic moment corresponding to collapse load M_C.

Since bending moment *at a given section* is directly proportional to load, we have

$$M \propto W \text{ or } M=aW \qquad \ldots(1)$$

$\left(\text{For simply supported beam } M=\frac{WL}{4} \text{ and hence } a=\frac{L}{4}\right)$

Similarly, $\quad M_P \propto W_P$

or $\quad M_P=aW_C=a.QW \qquad \ldots(2)$

$$\frac{M_P}{M_{max}}=Q \qquad \ldots(I)$$

Now elastic section modulus required $=Z=\frac{M_{max}}{f_t} \qquad \ldots(3)$

where f_t=allowable stress in bending.

Plastic section modulus required $=Z_P=\frac{M_P}{\sigma_Y} \qquad \ldots(4)$

$$\therefore \quad \frac{Z_P}{Z}=\frac{M_P}{\sigma_Y} \div \frac{M_{max}}{f_t}=\frac{M_P}{M_{max}} \cdot \frac{f_t}{\sigma_Y} \qquad \ldots(II)$$

But $\quad \frac{M_P}{M_{max}}=Q$; $\frac{Z_P}{Z}=S$

and $\quad \frac{\sigma_Y}{f_t}=F$=factor of safety elastic method.

Substituting these in (II), we get

$$S=\frac{Q}{F}$$

or $\quad Q=S \times F \qquad \ldots(26\cdot16)$

The above relation shows that the load factor is equal to the shape factor multiplied by the factor of safety used in elastic design. If $f=160$ N/mm² ; $\sigma_Y=252$ N/mm² and $S=1.5$, we get

$$Q=1.15\times\frac{253}{160}=1.82.$$

26.8. METHODS OF LIMIT ANALYSIS : BASIC THEOREMS

In the elastic method of analysis, three conditions must be satisfied : (1) continuity condition, (2) equilibrium condition, and (3) limiting stress condition. Thus, an elastic analaysis requires that the deformations must be compatible, the structure should be in equilibrium and the bending moments anywhere in the structure should be less than M_Y (or the stress should be less than σ_Y).

Compared to this, an analysis according to the plastic method must satisfy the following fundamental conditions :

1. **Mechanism condition.** The ultimate load or collapse load is reached when a mechanism is formed. There must, however, be just enough plastic hinges that a mechanism is formed.

2. **Equilibrium condition.** The summation of the forces and moments acting on a structure must be equal to zero.

3. **Plastic moment condition.** The bending moment anywhere must not exceed the fully plastic moment.

It should, however, be noted that all the three conditions cannot be satisfied in one operation. Two theorems have been evolved which must be satisfied to ensure that all the conditions are fulfilled. The general method of limit analysis and design are based on the two fundamental theorems evolved by Greenberg and Prager. The first theorem, called the *static or lower bound theorem*, furnishes a lower boundary for the limit load, while second theorem called the *kinematic or upper bound theorem* gives an upper boundary for the limit load.

Basic Theorems

1. Static theorem or lower bound theorem

The static theorem states that *for a given frame and loading, if there exists any distribution of bending moment throughout the frame which is both safe and statically admissible, with a set of loads W, the value of W must be less than or equal to the collapse load* W_C.

The distribution of bending moment, such that it satisfies all the conditions of equilibrium is called *statically admissible distribution*. If the distribution of bending moment is such that the fully

plastic moment is not exceeded anywhere in frame, it is called *safe distribution*.

2. Kinematic or upper bound theorem

The upper bound theorem states that *for a given frame subjected to a set of loads W, the value of W which is found to correspond to any assumed mechanism will always be greater than or equal to the actual collapse load* W_C. This theorem satisfies the equilibrium condition as well as mechanism condition, and provides the upper bound or limit of collapse load. If the values of W corresponding to a number of mechanisms for a given frame under given set of loading are found, the collapse load W_C will be the smallest of all these found.

Methods of Analysis

Based on the above two theorems, there are two basic methods of limit analysis : (1) static method, and (2) kinematic method.

1. Static method. Static method is based on the static or lower bound theorem according to which a *load computed on the basis of an assumed equilibrium moment diagram in which the moments not greater than* M_P, *is less than or at best equal to the true ultimate load*. In this method, a moment diagram is sketched in such a way that the conditions of equilibrium are satisfied. The moments must either be less than or equal to M_P. If a mechanism is formed, then the solution of equilibrium equation will give true collapse load. If the mechanism is not formed, the moment at some of the sections will have to be increased so as to obtain a mechanism, *i.e.* the existing load will have to be increased. The load will become equal to the collapse load when a mechanism is formed. The procedure for application of static theorem is as follows :

1. Convert the structure into statically determine structure by removing the redundant forces.
2. Draw free bending moment diagram for the structure.
3. Draw the bending moment diagram for the redundant forces.
4. Draw the composite bending moment diagram in such a way that a mechanism is obtained.
5. Find out the value of collapse load by solving equilibrium equations.
6. Check the moments to ensure that $M \leqslant M_P$. If it is so, correct value of collapse load is obtained.

The method is suitable only for simple structures. For complicated frames, the method becomes very difficult and, therefore, kinematic method is preferred.

2. Kinematic or Mechanism method. Kinematic method is based on the kinematic or upper bound theorem according to which *a load computed on the basis of an assumed mechanism will always be greater than or at best equal to the true ultimate load.* For the application of this method, it is very essential to know the possible types and number of mechanisms. There are four types of independent mechanisms (Fig. 26·11) : (*i*) beam mechanism, (*ii*) panel mechanism, (*iii*) gable mechanism, and (*iv*) joint mechanism. Various combination of the independent mechanisms may be made to obtain certain number of composite mechanisms.

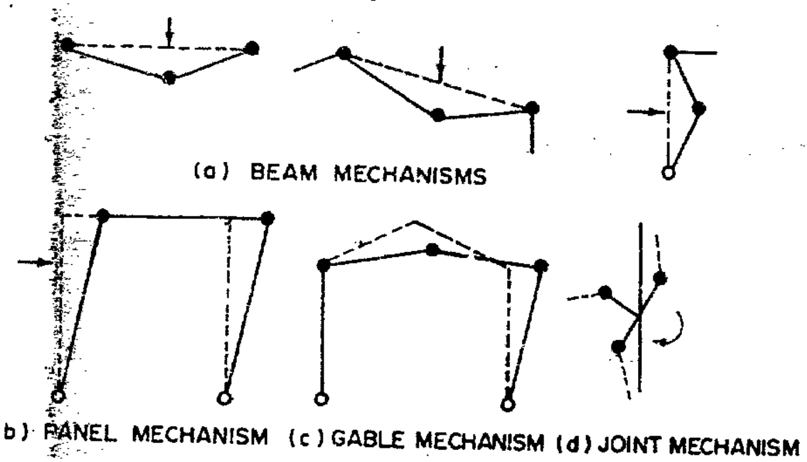


Fig. 26·11.
Types of independent mechanisms.

For a particular structure with a loading, the number of independent mechanisms is given by

$$N = n - T \qquad \ldots(26\cdot17)$$

where N = number of independent mechanisms
n = number of possible hinges
T = number of redundancies.

A number of possible collapse mechanisms may be obtained by the combination of independent mechanisms. The correct mechanism will be the one which results in the lowest possible load (upper

bound theorem) and for which the moment does not exceed the plastic moment at any section of the structure (lower bound theorem). The procedure of application of the kinematic theorem is as follows :

1. Determine the location of possible plastic hinges.
2. Select possible independent and composite mechanisms.
3. Solve equilibrium equation by virtual displacements method for the lowest load.
4. Check that $M \leq M_P$.

Principle of virtual work. It is stated as follows :

"*If a deformable structure in equilibrium under the action of a system of external forces is subjected to a virtual deformation compatible with its conditions of suppport, the work done by these forces on the displacements associated with the virtual deformation is equal to the work done by the internal stresses on the strains associated with this deformation.*" This principle has wide utility for the structure at collapse. During collapse there is no change in the elastic strain energy stored in the beam since the bending moment and, therefore, the curvature remains the same. So the work done during small motion of collapse mechanism is equal to the work absorbed by the plastic hinge. The work absorbed in the hinges is always positive irrespective of the sign of B.M.

26·9. DETERMINATION OF COLLAPSE LOAD FOR SOME STANDARD CASES OF BEAMS

1. Simply supported beam carrying a concentrated load W

Let the beam section have a plastic moment of resistance M_P. We shall solve the problem by both the methods.

(*a*) *Static Method* :

The maximum bending moment of $\dfrac{Wab}{L}$ evidently occurs under the load. When the load is increased to the collapse load W_C, the maximum bending moment will be equal to $\dfrac{W_C.ab}{L}$, as shown in Fig. 26·12 (*b*). This should evidently be equal to the plastic moment of resistance M_P.

$$\therefore \qquad \frac{W_C.ab}{L} = M_P$$

$$\text{or} \qquad W_C = \frac{M_P.L}{ab}$$

Since the bending moment nowhere in the beam exceeds M_P, the load given by the above expression is the true collapse load.

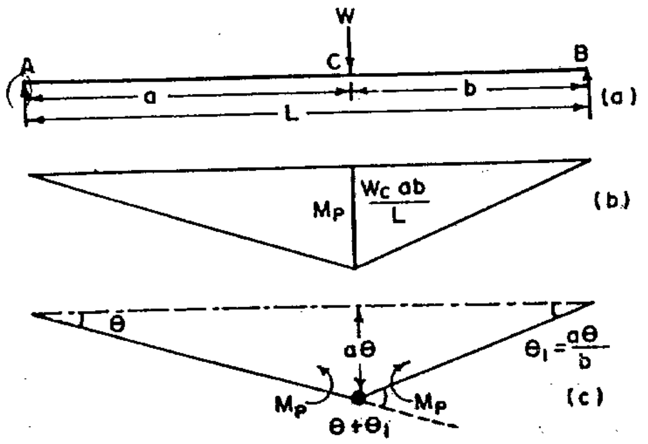


Fig. 26·12

(*b*) *Kinematic Method* :

The collapse mechanism beam mechanism) is shown in Fig. 26·12 (*c*). Collapse will occur when a hinge is formed under the load.

Let θ angle of rotation of the left portion of the beam.

Deflection below the load$=a\theta$.

Hence angle of rotation of the right portion of the beam

$$=\theta_1=\frac{a\theta}{b}$$

Rotation of the hinge under the action of plastic moment

$$=\theta+\theta_1=\theta+\frac{a\theta}{b}=\theta\cdot\frac{L}{b}$$

The work absorbed by the hinge$=M_P\cdot\frac{\theta L}{b}$

The work done by the load$=W_C.a\theta$

Equating the two, we get

$$W_C.a\theta=M_P\cdot\frac{\theta\cdot L}{b}$$

or $$W_C=\frac{M_P.L}{ab}, \text{ as before.}$$

The value of B.M. anywhere does not exceed M_P, and hence above value of collapse load is correct.

If the load is acting at the centre of the beam, $a=b=L/2$ and hence

$$W_C=\frac{M_P\cdot L}{\frac{L}{2}\cdot\frac{L}{2}}=\frac{4M_P}{L}$$

2. Simply supported beam carrying uniformly distributed load

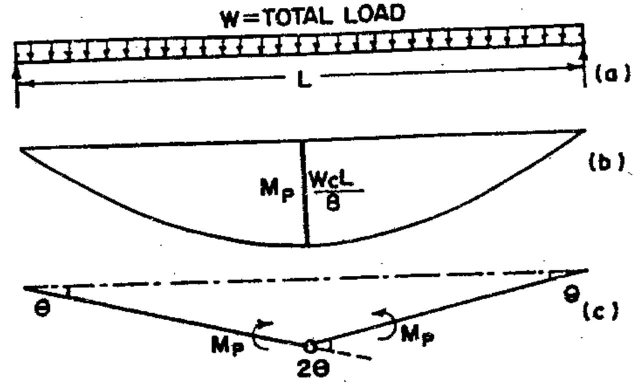


Fig. 26·13

Let W=total U.D.L.

M_P=full plastic moment of resistance of the beam section.

(*a*) *Static method* :

The maximum bending moment of $\frac{WL}{8}$ will occur at the centre of the beam. When the load W is increased to the collapse load W_C, the maximum bending moment at the centre will be equal to $\frac{W_C.L}{8}$, and a hinge will form there to get a collapse mechanism.

For equilibrium, the maximum, moment must be equal to the plastic moment of resistance M_P.

$$\therefore \quad \frac{W_C.L}{8}=M_P$$

or $$W_C=\frac{8M_P}{L}.$$

(*b*) *Kinematic method* :

The collapse mechanism is shown in Fig. 26·13 (*c*). The central hinge will rotate through an angle 2θ, while the beam will deflect by $\frac{L}{2}\theta$ vertically downwards at its centre.

Average vertical moment of the U.D.L. $=\frac{1}{2}\cdot\frac{L}{2}\theta=\frac{L\theta}{4}$

$\therefore$ Work done by the load $=W_C\cdot\frac{L\theta}{4}$

Work absorbed by the hinge $=M_P\cdot 2\theta$

$$\therefore \quad W_C\cdot\frac{L\theta}{4}=M_P\cdot 2\theta$$

or
$$W_C=\frac{8M_P}{L}, \text{ as before.}$$

The bending moment nowhere exceeds M_P and hence the assumed mechanism and the collapse load is correct.

3. Propped cantilever with eccentric concentrated load

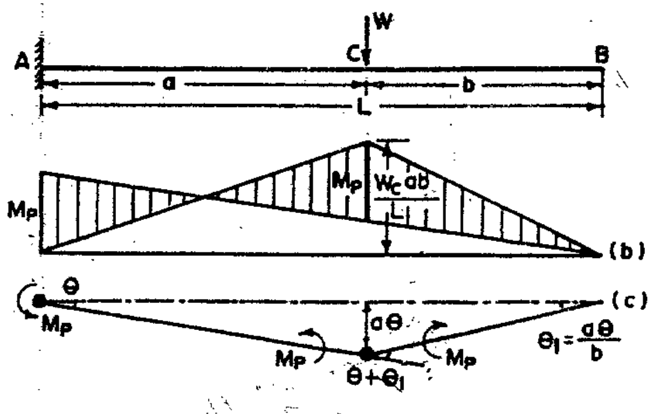


Fig. 26·14

The shape of the bending moment diagram during elastic stage will be the same as that shown in Fig. 26·13 (*b*). Since the static B.M. at *C* is greater than at *A*, the plastic hinge will first develop at *C* and then at *A*. The structure will then be converted into a mechanism and it will ultimately collapse. During collapse, the moments at both *A* and *C* will be M_P.

(*i*) *Static Method* :

From Fig. 26·14 (*b*), the equilibrium equation is

$$\frac{W_C.ab}{L}=M_P+M_P\cdot\frac{b}{L}=M_P\left(\frac{L+b}{L}\right)$$

or
$$M_C=M_P\,\frac{L+b}{ab}$$

The moment anywhere is not more than M_P. Hence the above expression gives correct value of the collapse load.

(*ii*) *Kinematic method* :

The collapse mechanism is shown in Fig. 26·14 (*c*). During collapse, one hinge will be formed at the fixed end *A* and the other hinge will be formed at *C*. Let the rotation of the left portion be θ. The rotation of the right portion will be $\theta_1=\theta\,\frac{a}{b}$. Thus, the hinge at *A* will rotate through angle θ while the hinge at *C* will rotate through $\theta+\theta_1$. The collapse load will move down by $a\theta$. The equilibrium equation is, therefore :

$$W_C.a\theta=M_P.\theta+M_P\left(\theta+\frac{a\theta}{b}\right)$$

$$W_C.a\theta=M_P\,\frac{\theta}{b}\,(b+b+a)=M_P\cdot\frac{\theta}{b}\,(L+b)$$

$$\therefore \quad W_C=M_P\,\frac{(L+b)}{ab}, \text{ as before.}$$

If the load acts at the centre of the cantilever, $a=b=\frac{L}{2}$

$$\therefore \quad W_C=\frac{M_P\left(L+\frac{L}{2}\right)}{\frac{L}{2}\cdot\frac{L}{2}}=\frac{6M_P}{L}.$$

4. Propped cantilever carrying U.D.L.

Fig. 26·15 (*b*) shows the bending moment diagram at collapse. One plastic hinge will form at the fixed end *A* and the other at *C*. The exact location of *C* is to be determined. Let M_C be the simply supported bending moment at *C*, distant *x* from *B*.

Then
$$M_C=-\left(\frac{W_C}{2}x-\frac{W_C}{L}\cdot\frac{x^2}{2}\right)=-\left(M_P+M_P\,\frac{x}{L}\right)$$

or
$$M_P\left(1+\frac{x}{L}\right)=\frac{W_C.x}{L}\left(\frac{L}{2}-\frac{x}{2}\right)$$

or
$$M_P(L+x)=W_C\,\frac{x}{2}\,(L-x)$$

or
$$M_P=\frac{W_C}{2}\cdot\frac{x(L-x)}{L+x}=\frac{W_C}{2}\cdot\frac{Lx-x^2}{L+x} \quad \ldots(1)$$

For maxima,

$$\frac{\partial M_P}{\partial x}=0=x^2+2Lx-L^2$$

This gives $\quad x=L(\sqrt{2}-1)=0{\cdot}414\,L$

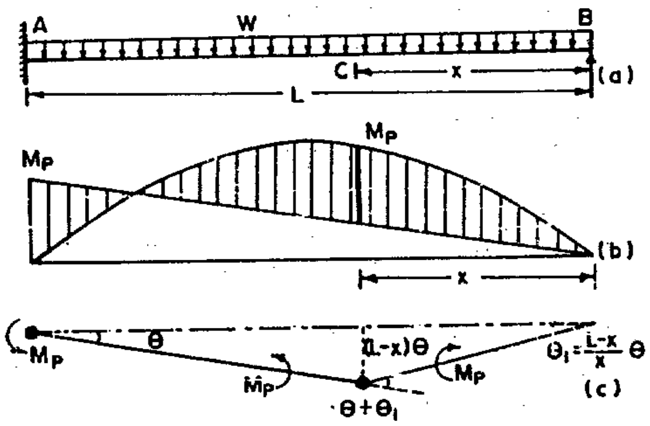


Fig. 26·15

(a) *Static method* :

The equilibrium equation is

$$M_P=\frac{W_C}{2}\cdot\frac{x(L-x)}{L+x},\text{ from (1)}$$

$$=\frac{W_C}{2}\cdot\frac{L(\sqrt{2}-1)}{\sqrt{2}}\cdot\frac{L(2-\sqrt{2})}{L}=\frac{W_C L(3-2\sqrt{2})}{2}$$

or
$$M_C=\frac{2M_P}{L(3-2\sqrt{2})}=\frac{2M_P}{L}(3+2\sqrt{2})=11{\cdot}656\ \frac{M_P}{L}$$

(b) *Kinematic method* :

The rotation of left portion is θ while that of right portion is $\theta_1=\frac{L-x}{x}.\theta$. The hinge at C will, therefore, rotate through $\theta+\frac{L-x}{x}\theta$. The downward movement of the load is equal to $(L-x)\ \theta$, where $x=4{\cdot}414\ L$.

$$\text{Work done by the load}=\frac{W_C(L-x)\ \theta}{2}=\frac{W_C L(2-\sqrt{2})\ \theta}{2}$$

Work absorbed by the hinges

$$=M_P\ \theta+M_P\left[\theta+\frac{L-x}{x}\theta\right]$$

$$=M_P\ \theta\left[1+\frac{L}{x}\right]=M_P\ .\ \theta\ \frac{\sqrt{2}L}{L(\sqrt{2}-1)}$$

Hence the equilibrium equation is

$$\frac{W_C\ L(2-\sqrt{2})\ \theta}{2}=M_P\ \theta\ \frac{\sqrt{2}L}{L(\sqrt{2}-1)}$$

$$\therefore\qquad W_C=\frac{M_P\sqrt{2}}{\sqrt{2}-1}\times\frac{2}{L(2-\sqrt{2})}=\frac{2M_P}{L}(3+2\sqrt{2})$$

$$=\frac{11{\cdot}656\ M_P}{L},\text{ as before.}$$

5. Fixed beam carrying an eccentric point load

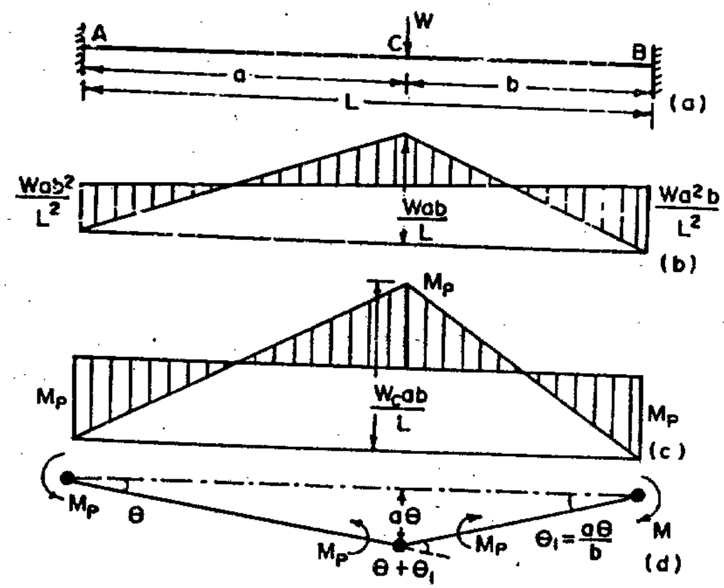


Fig. 26·16

Fig. 26·16 (*b*) shows the bending moment diagram for elastic stage. The bending moment at B is the greatest. As the load is increased the plastic hinge will first form at B, then at C and finally at A. At this stage, the beam will be converted into a mechanism and ultimately it will collapse.

(1) *Static method* :

From Fig. 26·16 (*c*), the equilibrium equation is

$$\frac{W_C.ab}{L}=M_P+M_P=2M_P$$

$$\therefore\qquad W_C=\frac{2M_P.L}{ab}.$$

(2) *Kinematic method* :

The rotation of the hinge at $A=\theta$. The load W_C will, therefore, move downward by $a\theta$. The rotation of hinge at B is $\theta_1=\frac{a\theta}{b}$. The rotation of hinge at $C=\theta+\theta_1=\theta+\frac{a\theta}{}=\frac{L\theta}{}$

Work done by the load $= W_C . a\theta$

Work absorbed by the hinges $= M_P\theta + M_P \cdot \frac{L\theta}{b} + M_P \frac{a\theta}{b}$

$$= M_P\,\theta\left(1+\frac{L}{b}+\frac{a}{b}\right) = \frac{2L}{b}\,M_P.\theta$$

$$\therefore \quad W_C.a\theta = \frac{2L}{b}.M_P.\theta$$

$$\therefore \quad W_C = \frac{2M_P.L}{ab}$$

If, however, the load is placed at the middle of the beam

$$a = b = \frac{L}{2}$$

$$\therefore \quad W_C = \frac{2M_P.L}{\frac{L}{2}\cdot\frac{L}{2}} = \frac{8M_P}{L}$$

If the beam carries uniformly distributed load, it can be shown that the collapse load,

$$W_C = \frac{16M_P}{L}.$$

6. Three span continuous beam with U.D.L.

Let the total uniformly distributed load on each span be W. A continuous beam will collapse in the same manner as fixed beam by the formation of three plastic hinges, two at the supports and one between the supports of any span. The failure of one span will result in the failure of the whole structure.

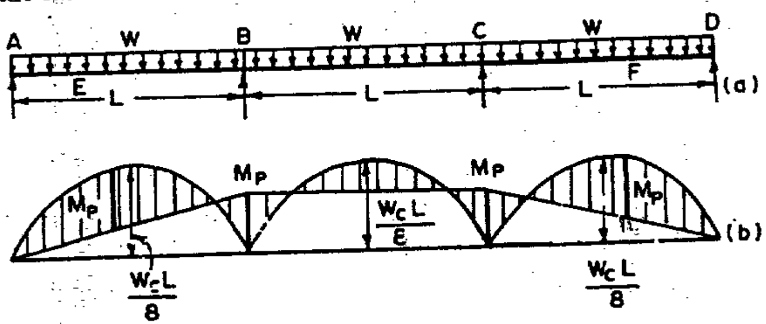


Fig. 26·17

Fig. 26·17 (*b*) shows the bending moment diagram at collapse. During the elastic stage the ordinates of bending moment diagram will be $\frac{WL}{10}$ at the inner supports and $\frac{WL}{8}$ at the mid span. When

the collapse load is applied, the plastic hinges will form at E, B, C and F, and the beams AB and CD will collapse. The beam BC can still take more load, but for all practical purposes the continuous beam has been rendered useless. The spans AB and CD may be looked upon as propped cantilevers with uniformly distributed loading. The collapse load is, therefore, equal to $11{\cdot}656\,\frac{M_P}{L}$ and the hinges in the end span form at 0·414 L from the outer supports. The plastic bending moment diagram can now be drawn as shown in Fig. 26·17 (*b*).

Example 26·4. *Calculate the plastic section modulus, shape factor and plastic moment of the following sections :*

(*a*) *ISMB 200* [*Fig. 26·18* (*a*)] *having the following properties :*

$I_{xx} = 2235{\cdot}4\ cm^4$; $Z_{xx} = 223{\cdot}5\ cm^3$; $A = 32{\cdot}33\ cm^2$;

Thickness of web $= 5{\cdot}7$ *mm* ; *Thickness of flange* $= 10{\cdot}8$ *mm.*

(*b*) *ISHT 150* [*Fig. 26·18* (*b*)] *having the following properties :*

$I_{xx} = 573{\cdot}7\ cm^4$; $A = 37{\cdot}42$ *sq. cm and distance of C.G. from the top is 26·6 mm.*

Take the yield stress for mid steel as 253 N/mm².

Solution.

(*a*) *I-section : Given* : $I_{xx} = 2235{\cdot}4 \times 10^4\ mm^4$; $Z_{xx} = 223{\cdot}5 \times 10^3\ mm^3$ *and* $A = 3233\ mm^2$

$$Z_P = \frac{A}{2}(y_1 + y_2)$$

Since the equal area axis coincides with the centroidal axis, y_1 and y_2 are equal. To find y_1 of the upper half area, we have

$$y_1 = \frac{100\times10{\cdot}8(100-5{\cdot}4) + 5{\cdot}7(100-10{\cdot}8)(100-10{\cdot}8)\frac{1}{2}}{(100\times10{\cdot}8)+(5{\cdot}7)(100-10{\cdot}8)}$$

$$= \frac{102168+22676}{1080+508{\cdot}4} = 78{\cdot}6 \text{ mm}$$

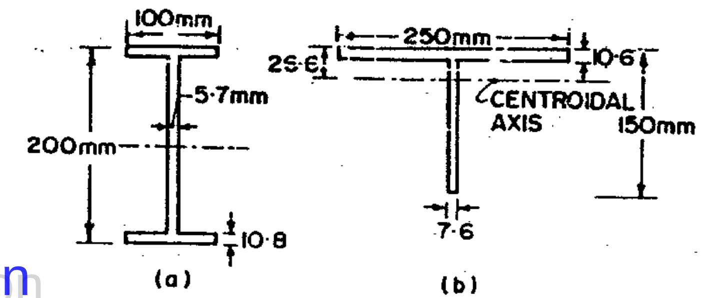


Fig. 26·18

$$\therefore \quad Z_P=\frac{A}{2}(y_1+y_2)=A.y_1=3233\times78{\cdot}6=254106 \text{ mm}^3$$

$$Z=223{\cdot}5\times10^3 \text{ mm}^3=223500 \text{ mm}^3$$

$$\therefore \quad S=\frac{Z_P}{Z}=\frac{254106}{223500}=\mathbf{1{\cdot}14}$$

$$M_P=Z_P.\sigma_Y=254106\times253=64{\cdot}29\times10^6 \text{ N-mm}$$

$$=\mathbf{64{\cdot}29 \text{ kN-m.}}$$

(*b*) **Tee Section**

Given $I_{xx}=573{\cdot}7\times10^4$ mm^4 ; $A=3742$ mm^2

Elastic section modulus $Z=\dfrac{573{\cdot}7\times10^4}{150-26{\cdot}6}=46491$ mm^3

Let the equal area axis pass through the flange at distance x below the top fibre.

$$250x=250(10{\cdot}6-x)+(150-10{\cdot}6)\times7{\cdot}6 \quad \text{or} \quad 500x=3709$$

From which $x=7{\cdot}42$ mm

y_1=Distance of C.G. of the top area from equal area axis

$$=\frac{7{\cdot}42}{2}=3{\cdot}71 \text{ mm}$$

Distance of bottom of flange from equal area axis

$$=10{\cdot}6-7{\cdot}42=3{\cdot}18 \text{ mm.}$$

y_2=distance of C.G. of bottom area from the equal area axis

$$=\frac{250\times3{\cdot}18\times(3{\cdot}18)\frac{1}{2}+(150-10{\cdot}6)7{\cdot}6\{(150-10{\cdot}6)\frac{1}{2}+3{\cdot}18\}}{\frac{1}{2}(3742)}$$

$$=41{\cdot}94 \text{ mm.}$$

$$\therefore \quad Z_P=\frac{A}{2}(y_1+y_2)=\frac{3742}{2}[3{\cdot}71+41{\cdot}94]=\mathbf{85417 \text{ mm}^3}$$

$$S=\frac{Z_P}{Z}=\frac{85417}{46491}=\mathbf{1{\cdot}84}$$

$$M_P=Z_P.\sigma_Y=85417\times253=21{\cdot}61\times10^6 \text{ N-mm}$$

$$=\mathbf{21{\cdot}61 \text{ kN-m.}}$$

Example 26·5. *A beam of rectangular cross-section $b\times d$ is subjected to a bending moment $0{\cdot}9$ M_P. Find out the depth of the elastic core.*

Solution.

Let the total depth of the elastic core=$2x$. Therefore, the depth of the plastic zone=$\dfrac{d}{2}-x$, on either side.

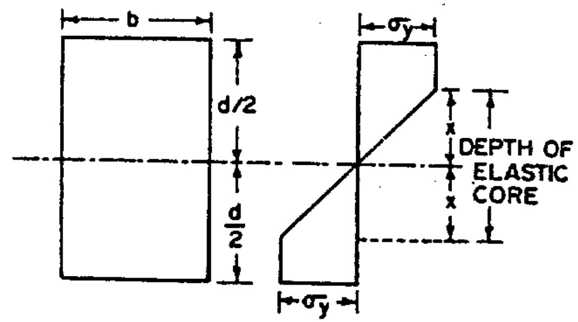


Fig. 26·19

Distance of C.G. of force on one side from N.A.

$$=\frac{b\left(\frac{d}{2}-x\right)\times\sigma_Y\times\left\{x+\frac{1}{2}\left(\frac{d}{2}-x\right)\right\}+bx.\frac{\sigma_Y}{2}\cdot\frac{2}{3}x}{b\left(\frac{d}{2}-x\right)\sigma_Y+bx\,\frac{\sigma_Y}{2}}$$

$$=\frac{\left(\frac{bd}{2}-bx\right)\left(\frac{d}{4}+\frac{x}{2}\right)+\frac{bx^2}{3}}{\frac{bd}{2}-bx+\frac{bx}{3}}$$

$$=\frac{3d^2-4x^2}{12(d-x)}$$

Total internal moment of the forces about N.A.

$$=2\times\left\{b\left(\frac{d}{2}-x\right)+\frac{bx}{2}\right\}.\sigma_Y\times\frac{3d^2-4x^2}{12(d-x)}$$

$$=\frac{3d^2-4x^2}{12}\times b\times\sigma_Y$$

Externally applied moment

$$=0{\cdot}9\,M_P=0{\cdot}9\,\frac{bd^2}{4}\,\sigma_Y$$

Equating the two, we get

$$\frac{3d^2-4x^2}{12}\,b.\sigma_Y=0{\cdot}9\times\frac{bd^2}{4}\,\sigma_Y$$

or $\quad 3d^2-4x^2=2{\cdot}7d^2$

$\therefore \quad 4x^2=0{\cdot}3\,d^2$

or $\quad x=0{\cdot}274\,d$

Hence depth of elastic core=$2x$=**0·548 d.**

Example 26·6. *A fixed beam of span L carries a uniformly distributed load W on the left half portion. Determine the value of W at collapse. The elastic moment of resistance of the beam is M_P.*

Solution.

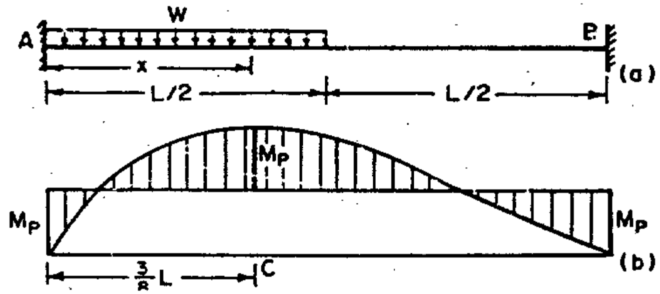


Fig. 26·20

Let the maximum free bending moment occur at C, distant x from A.

$$\therefore \quad M_P = -\frac{3Wx}{4} + \frac{2W}{L}\cdot\frac{x^2}{2} = -\frac{3Wx}{4} + \frac{Wx^2}{L}$$

For maxima,

$$\frac{dM_x}{dx} = 0 = -\frac{3W}{4} + \frac{2Wx}{L}$$

$$\therefore \quad x = \frac{3}{8}\ L$$

$$\therefore \quad M_{max} = -\frac{3W}{4}\times\frac{3}{8}\ L + \frac{W}{L}\left(\frac{3}{8}\ L\right)^2 = -\frac{9}{64}WL$$

At collapse, it becomes equal to $\frac{9}{64}W_C.L$ (Numerically)

Hence as per static method, the equilibrium equation is

$$\frac{9}{64}\ W_C L = M_P + M$$

$$\therefore \quad W_C = \frac{2M_P}{L}\times\frac{64}{9} = \frac{128}{9}\ \frac{M_P}{L} \text{ (Answer).}$$

Example 26·7. *A beam ABC of span L is fixed at the ends A and C, and carries a point load at a distance $\frac{L}{4}$ from the left end. Find the value of the load at collapse if the left half of the beam has a plastic moment of resistance $2M_P$ and the right half has a plastic moment M_P.*

Solution.

There are two possible collapse mechanisms. *In the first mechanism* [Fig. 26·21 (*b*)], hinges may form at A, D and C. The equilibrium equation is

$$W_C\ .\ \frac{3}{4}\ L\theta = 2M_P.3\theta + 2M_P.4\theta + M_P\theta$$

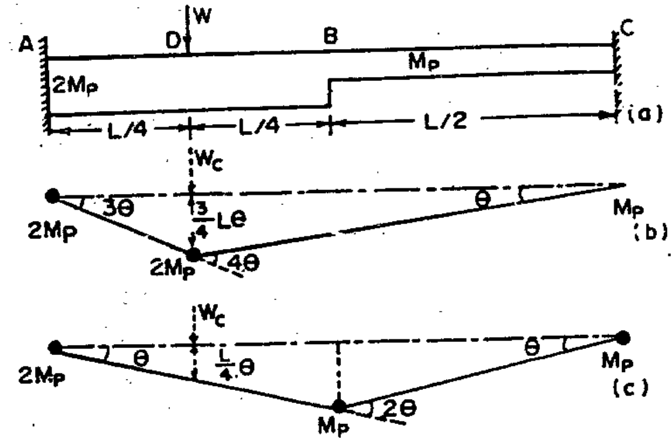


Fig. 26·21

$$\therefore \quad W_C = \frac{20M_P}{L}$$

In the second mechanism [Fig. 26·21 (*c*)], hinges may form at A, B and C. The hinge at B will work corresponding to the least moment of resistance at B, *i.e.* M_P. The equilibrium equation is

$$W_C\ .\ \frac{L}{4}\ \theta = 2M_P.\ \theta + M_P.2\theta + M_P\theta$$

$$\therefore \quad W_P = \frac{20M_P}{L}$$

The mechanism is the one which gives the minimum of the collapse load (upper bound theorem). However, in the present case, both mechanisms give equal collapse load of $\frac{20\ M_P}{L}$.

Example 26·8. *A simply supported beam consists of a steel rod of diameter 30 mm. The span of the beam is 2 m. The steel rod is bored for a length of 0·5 m at each end. Find the diameter of the bore so that plastic hinges may form simultaneously at A, B and C, as shown in Fig. 26·21. If $\sigma_y = 253$ N/mm², find the collapse load.*

Solution.

When the plastic hinge is formed, the bending moment at the centre $= \frac{W_C \times 2}{2} = \frac{W_C}{4}$ = plastic moment of unbored rod $= M_P$.

Bending moment at B and C

$$= \frac{W_C}{2}\times\frac{1}{2} - \frac{W_C}{2}\times\frac{1}{2}\times\frac{1}{4} = \frac{3}{16}\ W_C = M_{P_1}.$$

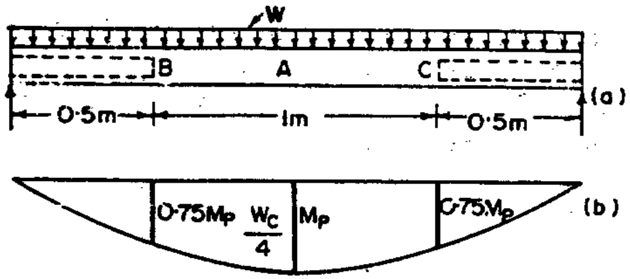


Fig. 26·22

For the plastic hinges to form at B and C, the fully plastic moment of the bored steel rod must be $\frac{3}{16}W_C$.

Now $M_P : M_{P_1} = \frac{W_C}{4} : \frac{3}{16}W_C = 1 : 0{\cdot}75$.

For unbored steel rod, $Z_P = \frac{d^3}{6} = \frac{(30)^3}{6} = 4500\ \text{mm}^3$.

Let d_1 be the inner diameter and d the outer diameter of the bored tube. Plastic section modulus of bored tube is

$$Z_{P_1} = \frac{d^3 - d_1^3}{6} = \frac{30^3 - d_1^3}{6} = \frac{27000 - d_1^3}{6}$$

Since the ratio of M_P and M_{P_1} is $1 : 0{\cdot}75$, the ratio of Z_P and Z_{P_1} must also be $1 : 0{\cdot}75$.

$$\therefore \quad Z_{P_1} = 0{\cdot}75\ Z_P$$

or $$\frac{27000 - d_1^3}{6} = 0{\cdot}75 \times 4500$$

From which $d_1 = 18{\cdot}9$ **mm**

Now $$M_P = Z_P \times \sigma_Y = 4500 \times 253 = 1{\cdot}1385 \times 10^6\ \text{N-mm}$$
$$= 1{\cdot}1385\ \text{kN-m}$$

At collapse, $$M_P = \frac{W_C}{4}\ \text{kN-m}$$

$$\therefore \quad \frac{W_C}{4} = 1{\cdot}1385\ \text{kN}$$

or $$W_C = 1{\cdot}1385 \times 4 = 4{\cdot}55\ \text{kN}.$$

Example 26·9. *Determine the value of W at collapse, for a three span continuous beam of constant M_P, loaded as shown in Fig. 26·23.*

Solution.

The three possible failure mechanisms are shown in Fig. 26·23.

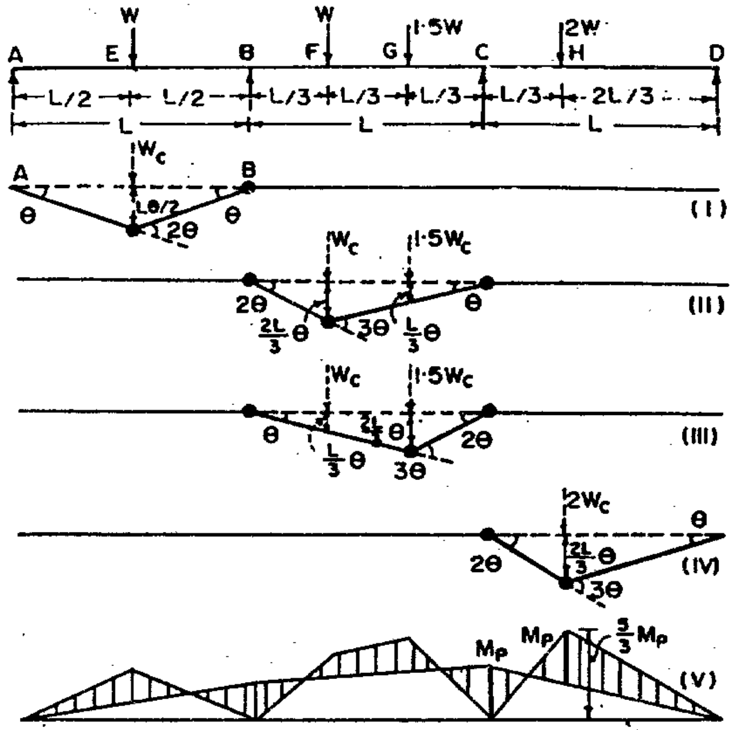


Fig. 26·23

(*I*) *Let the span AB collapse first*

The equilibrium equation is

$$W_C \cdot \frac{L\theta}{2} = M_P . 2\theta + M_P\theta$$

or $$W_C = \frac{6M_P}{L}. \qquad \ldots(1)$$

(*II*) *Let the span BC collapse first*

Hinges are formed at B, F and C [Fig. 26·23 (*II*)]

The equilibrium equation is

$$W_C \cdot \frac{2L}{3}\ \theta + 1{\cdot}5\ W_C \cdot \frac{L}{3}\ \theta = M_C\ 2\theta + M_P . 3\theta + M_P\theta$$

$$W_P = \frac{36}{7}\ \frac{M_P}{L} \qquad \ldots(2)$$

(*III*) *Let the span BC collapse first*

Hinges formed at B, G and C [Fig. 26·23 (*III*)]

The equilibrium equation is

$$W_C \cdot \frac{L}{3}\ \theta + 1{\cdot}5\ W\ \frac{2}{3}\ L\theta = M_P.\theta + M_P.3\theta + M_P.2\theta$$

or $$W_C = 4{\cdot}5\ \frac{M_P}{L}. \qquad \ldots(3)$$

(*IV*) *Let the span CD collapse first*

The equilibrium equation is

$$2W_C \cdot \tfrac{2}{3} L\theta = M_P \, . \, 2\theta + M_P \, . \, 3\theta$$

$$\therefore \qquad W_C = \frac{15}{4}\ \frac{M_P}{L} \qquad \ldots(4)$$

The actual collapse load will be the least of these four. Hence

$$W_C = \frac{15}{4}\ \frac{M_P}{L} \text{ (Answer)}$$

The bending moment diagram corresponding to this value of W_C is shown. Since the B.M. nowhere exceeds M_P, the above value of W_C is correct.

26·10. PORTAL FRAMES

In the case of a portal frame, at least four hings (natural+plastic) are necessary to convert it into a mechanism. In general, one hinge more than the number of redundancies will be required. In the case of a portal frame hinged at both the legs, where redundancy is one, two plastic hinges will be required for the mechanism to form. Similarly, for a portal frame fixed at the base, four plastic hinges are necessary. The degree of redundancy can be found by the expression :

$$T = 3a + R - 3,$$

where T=number of redundancies

a=number of areas completely enclosed by the members

R=total number of reaction components.

In the case of a simple single span portal frame, $a=0$. If the legs are fixed, $R=3+3=6$. Hence $T=6-3=3$. Thus a fixed portal frame has 3 redundancies. For finding out the number of redundancies of other cases, reader is advised to go through chapter 6. The number of independent mechanisms is given by Eq. 26·17. In addition to these, a number of combined mechanisms may be possible, and all of these should be tried to determine the minimum collapse load.

Example 26·10. *Determine the value of W at collapse for the portal frame loaded as shown in Fig. 26·24. All the members have the same plastic moment of resistance M_P.*

Solution.

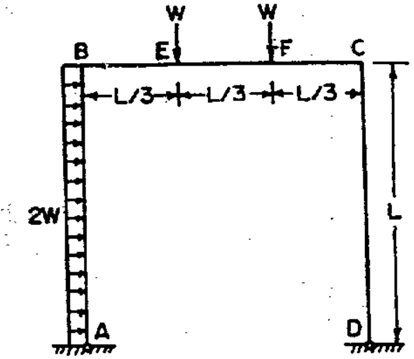


Fig. 26·24

The total number of independent machanisms are given by Eq. 26·17 :

$$N = n - T$$

where n=number of possible hinges

=1 (at B)+1 (at C)+1 (at E or F)+1 (anywhere on AB)=4

T=number of redundancies

=(2+2)−3=1

$\therefore \quad N = 4 - 1 = 3.$

Thus there are three independent mechanisms :

Beam mechanisms : 2 (one for AB and other for BC)

Panel mechanism : 1

Total 3

In addition to these, there may be one combined beam and panel mechanism. All the four mechanisms are shown in Fig. 26·25.

(*a*) *Beam mechanism BC* [Fig. 26·25 (*a*)]

$$W_C\ \frac{2L}{3}\ \theta + W_C\ \frac{L}{3}\ \theta = M_P 2\theta + M_P . 3\theta + M_P\theta$$

$$\therefore \qquad W_C = \frac{6M_P}{L}$$

(*b*) *Beam mechanism AB* [Fig. 26·25 (*b*)]

$$2\ W_C \, . \, \tfrac{1}{2}\left(\frac{L}{2}\ \theta\right) = M_P . 2\theta + M_P\theta$$

$$\therefore \qquad W_C = \frac{6M_P}{L}.$$

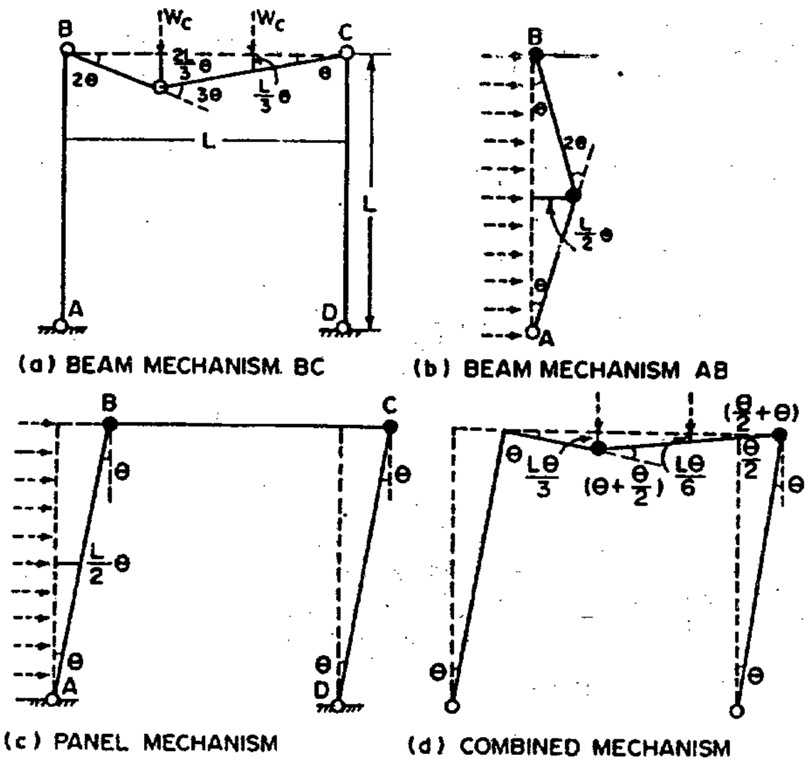


Fig. 26·25

(*c*) *Panel mechanism* [Fig. 26·25 (*c*)]

$$2W_C \cdot \frac{L\theta}{2} = M_P \cdot \theta + M_P \cdot \theta$$

$$\therefore \qquad W_C = \frac{2M_P}{L}.$$

(*d*) *Combined mechanism* [Fig. 26·25 (*d*)]

$$W_C \cdot \frac{L\theta}{3} + W_C \cdot \frac{L\theta}{6} + 2W_C \cdot \frac{L\theta}{2} = M_P \cdot \left(\theta + \frac{\theta}{2}\right) + M_P\left(\theta + \frac{\theta}{2}\right)$$

$$\therefore \qquad W_C = \frac{2M_P}{L}$$

The actual collapse load is the minimum of these.

$$W_C = \frac{2M_P}{L}.$$

Example 26·11. *Determine the value of W at collap for the portal frame shown in Fig. 26·26. All the members hav. .ne same plastic moment of resistance.*

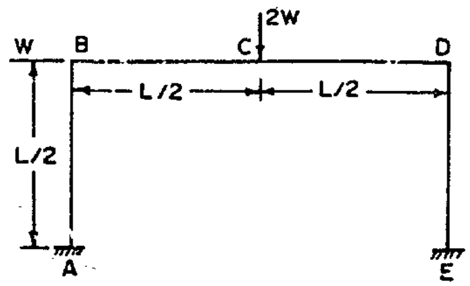


Fig. 26·26

Solution.

$$T = 3a + R - 3$$

$$a = 0\ ;\ R = 3 + 3 = 6\ ;$$

$$\therefore \quad T = 6 - 3 = 3.$$

Thus the frame is statically indeterminate to third degree. The total number of independent mechanisms are given by

$$N = n - T$$

where n = number of possible hinges

$= 5$ (one each at points A, B, C, D and E)

$$\therefore \qquad N = 5 - 3 = 2$$

Thus there are two independent mechanisms : (1) beam mechanism, and (2) panel mechanism. In addition to these, a combined mechanism, consisting of beam and panel mechanism is possible. All the three mechanisms are shown in Fig. 26·17 (*a*), (*b*) and (*c*) respectively.

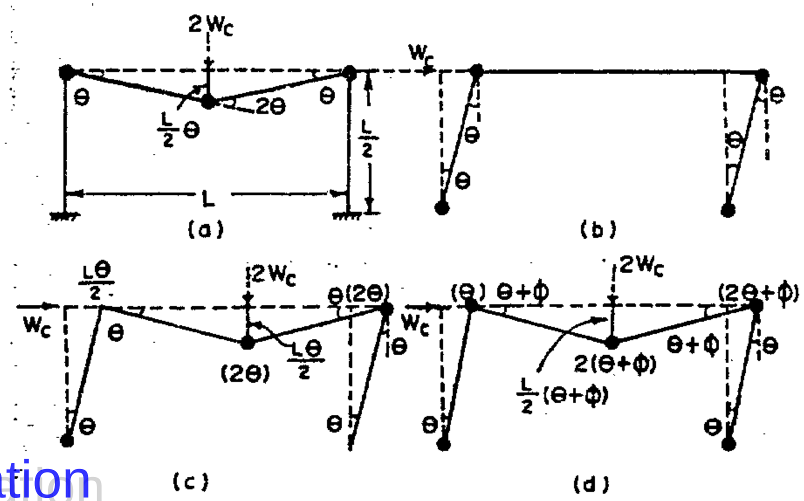


Fig. 26·27

(a) *Beam mechanism* [Fig. 26·27 (a)]

$$2W_C.\frac{L\theta}{2}=M_P\theta+M_P.2\theta+M_P\theta$$

$$W_C=\frac{4M_P}{L}.$$

(b) *Panel mechanism* [Fig. 26·27 (b)]

$$\therefore\ W_C.\frac{L\theta}{2}=M_P.\theta+M_P.\theta+M_P.\theta+M_P.\theta$$

$$\therefore\quad W_C=\frac{8M_P}{L}.$$

(c) *Combined mechanism* [Fig. 26·27 (c)]

$$2W_C.\frac{L\theta}{2}+W_C.\frac{L\theta}{2}=M_P.\theta+M_P.2\theta+M_P.2\theta+M_P.\theta.$$

$$W_C=\frac{4M_P}{L}$$

$\therefore$ Actual collapse load $=\frac{4W_P}{L}$. But since this load occurs in two mechanisms, the collapse mechanism will be combination of these two mechanisms, as shown in Fig. 26·27 (d).

In Fig. 26·27 (d), the equilibrium equation is

$$2W_P.\ \frac{L}{2}(\theta+\phi)+W_C.\frac{L\theta}{2}=M_P\theta+M_P\phi+M_P.2(\theta+\phi)+M_P(2\theta+\phi)+M_P.\theta$$

or $$W_C.\frac{L}{2}(3\theta+2\phi)=2M_P(3\theta+2\phi)$$

or $$W_C=\frac{4M_P}{L}\text{ which is the same as before.}$$

PROBLEMS

1. A beam of rectangular cross-section $b\times d$ is subjected to a bending moment 0·75 M_P. Find out the depth of the elastic core.

2. A beam *ABC*, 2·5 m long is suspended at *B* and *C* in horizontal position by means of two wires of 2 cm diameter. The bar is hinged at *A*. A load *P* is applied as shown in Fig. 26·28. Find out the value of *P* (a) when the yield appears in any of the bars, (b) when the whole arrangement collapses. Take $\sigma_Y=253$ N/mm².

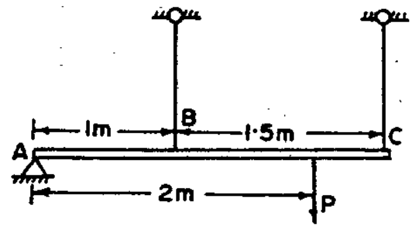


Fig. 26·28

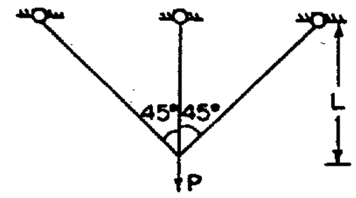


Fig. 26·29

3. A load *P* is supported by three rods as shown in Fig. 26·29. Find the value of *P* at collapse. All the bars are of the same area of cross-section, *A*. Show that this load is $\sqrt{2}$ times the elastic load.

4. For the structure shown in Fig. 26·30, compute the value of *P* (i) when the first yield occurs, (ii) when the whole arrangement collapses. The vertical bars have the same area of cross-section *A*. The horizontal beam is rigid.

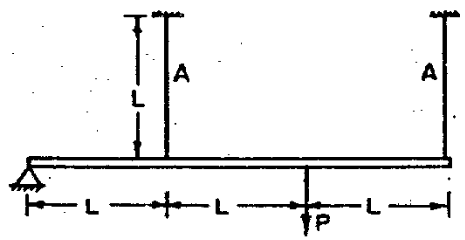


Fig. 26·30

5. (a) A simply supported beam carries uniformly distributed load. Prove that $\frac{W_C}{W_Y}=S$.

(b) Show that the fully-plastic moment of a beam of rectangular cross-section is 50% greater than the bending moment at which the beam reaches the limit of elasticity.

6. A beam fixed at both the ends carries uniformly distributed load. Prove that $\frac{W_C}{W_Y}=\frac{4}{3}S$

7. A simply supported beam of span *L* carries a central point load. Calculate the value of the load at collapse in terms of the plastic moment of resistance M_P.

8. If a propped cantilever, with a constant M_P, carries a central point load, determine its value at collapse.

9. A beam of span *L* and constant M_P, is fixed at its end. It carries a load *W*. Determine its value at collapse if (a) *W* is concen-

trated at the centre of the span, (*b*) W is uniformly distributed over the whole span.

10. A *ISWB* 600 @ 145·1 kg/m is supported over a length of 5 m. Its one end is fixed and the other is hinged. It is loaded with a uniformly distributed load W and a concentrated load 0·5 W at 2 m from the fixed end. Find the value of W for collapse to take place. If the load factor is 2, find out the value of safe working load. Given $Z_{xx}=3854{\cdot}2\ \text{cm}^3$; shape factor$=1{\cdot}14$ and $\sigma_Y=2530\ \text{kg/cm}^2$.

11. A fixed beam of span 6 m carries a uniformly distributed load W on the left half portion. If the fully plastic moment of the beam is 100 kN-m, find the value of the collapse load.

12. A uniform beam of rectangular cross-section is built in at each end and carries a vertical load at the mid-length. Determine the plastic zones and their extent at the collapse load of the beam.

13. A uniform beam of length L is built-in at one end and simply supported at the other. A load W is applied to the beam at a distance αL from the built-in end. If the fully plastic moment of the beam is M_P, find the value of W for collapse, and find the value of α for which the collapse load is a minimum.

14. A uniform beam of length $3L$ is built-up at each end and carries vertical loads W and $2W$ at the third points. If the plastic moment of the beam is M_P, estimate the value of W for complete collapse.

15. Find the value of W, for the beam shown in Fig. 26·31 so that collapse may take place. The plastic moment of the beam section is M_P.

Fig. 26·31

16. Find the collapse load W_C for the continuous beam shown in Fig. 26·32. The beam has constant plastic moment M_P.

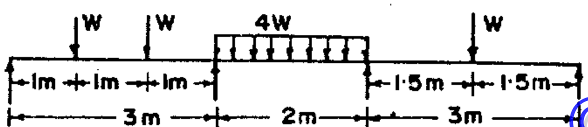


Fig. 26·32

17. A portal frame shown in Fig. 26·33 is hinged at the ends and has fully plastic moment M_P. It is loaded with a vertical load W and a horizontal load $\frac{W}{2}$ as shown in Fg. 26·33. Find the value of W at collapse.

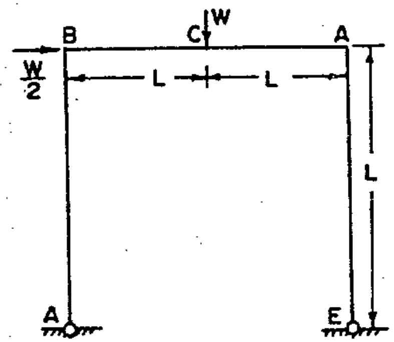


Fig. 26·33

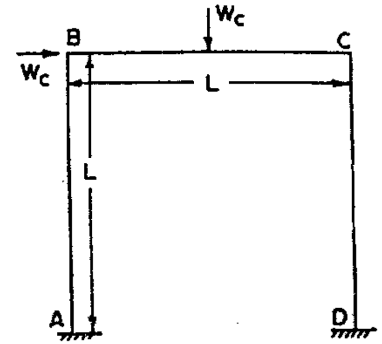


Fig. 26·34

18. A portal frame shown in Fig. 26·34 has uniform section throughout. Determine the value of the plastic moment of resistance in terms of the load parameter W_C at collapse.

19. A portal frame of height L and span L is hinged at the base and is of uniform plastic moment M_P. It carries a single central vertical load. Find the value of W at collapse.

20. A fixed rectangular portal frame of height L and span $2L$, is of uniform section with fully plastic moment M_P. A horizontal load W is applied at the top of the column and another load W is applied vertically at the centre of the beam. Find the value of W at collapse.

ANSWERS

1. 0·868 *d*.
2. (*a*) 115·2 kN (*b*) 139 kN.
3. $P_{el}=\left(1+\frac{1}{\sqrt{2}}\right)A.\sigma_Y$; $P_L=A\sigma_Y\,(1+\sqrt{2})$.
4. (*i*) $\frac{5}{3}\sigma_Y.A$ (*ii*) $2\sigma_Y.A$.
7. $\frac{6M_P}{L}$.
8. $\frac{4M_P}{L}$.

9. (a) $\frac{8M_P}{L}$; (b) $\frac{16M_P}{L}$.

10. $W_C=146$ t ; Safe, $W=73$ t.

11. $W_C=235{\cdot}6$ kN.

12. $\frac{L}{12}$ at either ends and $\frac{L}{6}$ at the middle.

13. $W_C=\frac{M_P}{L}\left[\frac{2-\alpha}{\alpha(1-\alpha)}\right]$; $\alpha=0{\cdot}586$.

14. $W_C=1{\cdot}2\frac{M_P}{L}$.

15. $W_C=3{\cdot}74$.

16. $W_C=\frac{4}{3}M_P$.

17. $\frac{8}{3}\cdot\frac{M_P}{L}$.

18. $M_P=W_C$.

19. $\frac{8M_P}{L}$.

20. $\frac{3M_P}{L}$.

27

Building Frames

27·1. INTRODUCTION

A building frame may contain a number of bays and may have several storeys. A multi-storeyed, multi-panelled frame is a complicated statically indeterminate structure. It consists of a number of beams and columns built monolithically, forming a network. The doors and walls are supported on beams which transmit the loads to the columns. A building frame is subjected to both the vertical as well as horizontal loads. The vertical loads consist of the dead weight of structural components such as beams, slabs, columns etc. and live load. The horizontal loads consist of the wind forces and earthquake forces. The ability of multi-storey building to resist the wind and other lateral forces depends upon tho rigidity of connections between the beams and columns. When the connections of beams and columns are fully rigid, the structure as a whole is capable of resisting the lateral forces acting on the structure.

In ordinary reinforced concrete skeleton buildings, a continuous beam is rigidly connected with columns. Due to this, the moments in the beam depend not only upon the number and length of spans composing the beam itself, but also upon the rigidity of columns with which it is connected. The bending moment at the end of any one span of the continuous beam cannot be transferred to the beam in the next span without subjecting the columns to bending. Instead of transmitting the bending moment in full to the beam in the next span, part of the moment is transferred to the columns above and below the beam, and the balance to the beam. Due to this, the effect of loading on one span upon the other spans is much lower than in ordinary continuous beams which are not connected to the columns.

27·2. SUBSTITUTE FRAME

The analysis of a multi-storeyed multi-panelled building frame is very cumbersome, since the frame contains a number of continuous beams and columns. As stated in the previous article, the effect of loading on the span upon other spans is much smaller. The moments

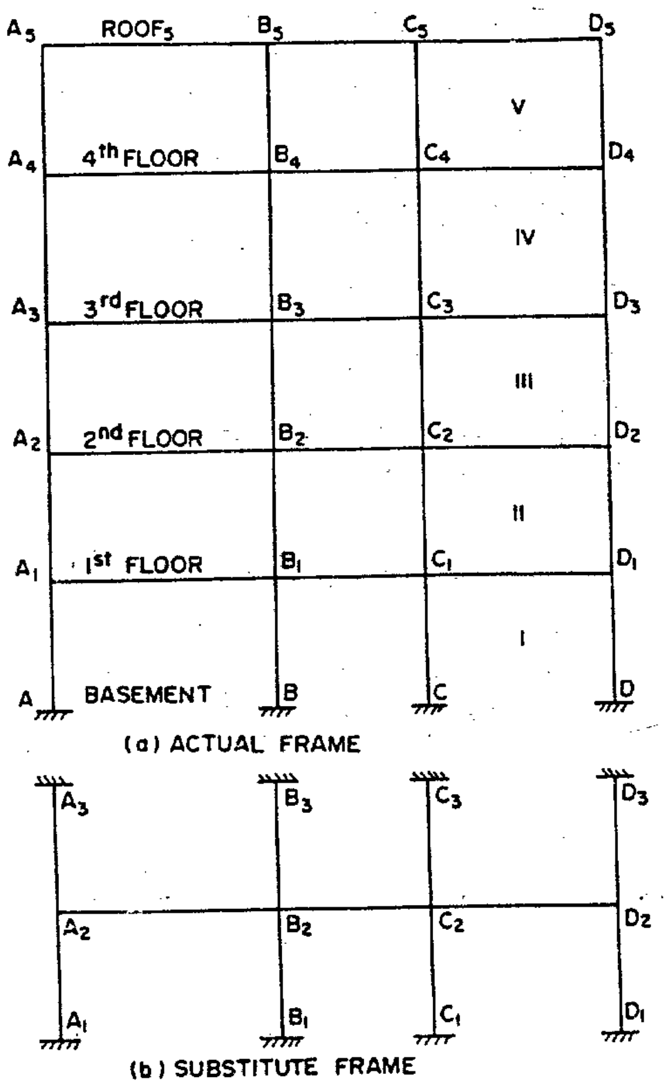


Fig. 27·1

in any beam and column are mainly due to the loads on spans very close to it. Loads on distant spans do not have appreciable effect. Due to this, a simple method of analysis, accurate enough for practical purpose, is used by analysing a small portion of the frame, called '*substitute frame*', rather than analysing the whole frame.

It has been found by exact analysis that the moments carried from floor to floor, through columns, are very small in comparison to beam moments. In other words, the moments in one floor have negligible effect on the moments of the floor above and below it. Therefore, a substitute frame consists of one floor, connected above and below with their four ends either hinged or fixed or restrained.

Fig. 27·1 (*a*) shows a building frame consisting of five storeys and three bays. Fig. 27·1 (*b*) shows the substitute frame of determining bending moment in the second floor. Generally, it is sufficient to consider two adjacent spans on each side of joint considered. The substitute frame gives the results which are safe for all practical purposes.

Types of substitute frames

Under ordinary conditions, the following four types of substitute frames are considered sufficient :

(*a*) Three span structure with two storey columns.

(*b*) Substitute frame for wall columns.

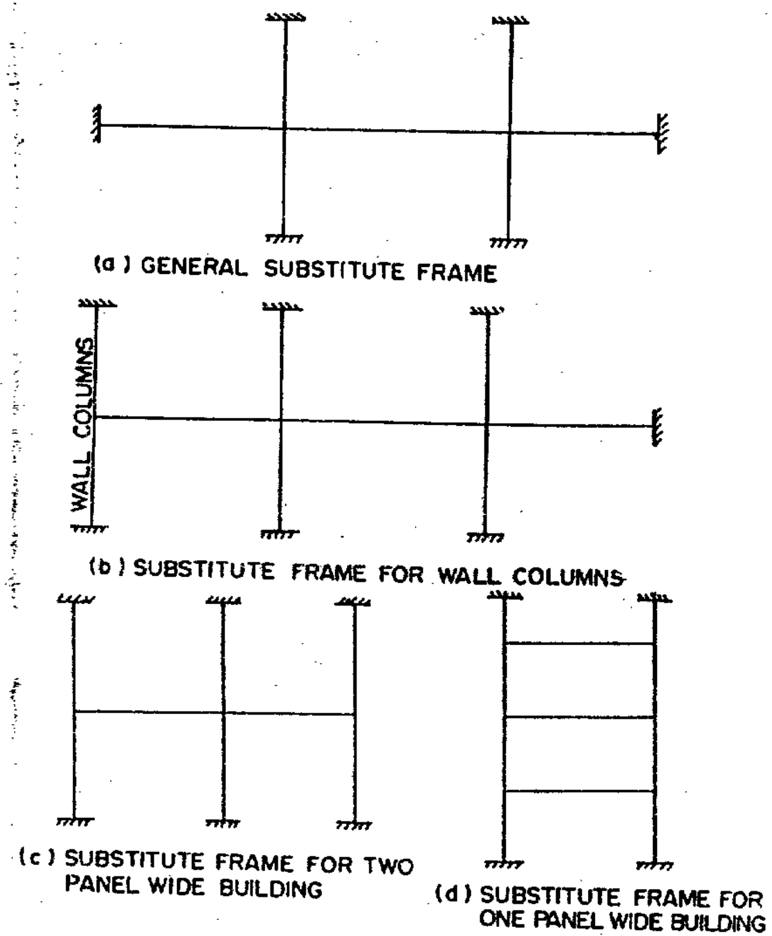
(*c*) Substitute frame for two panel wide building.

(*d*) Substitute frame for one panel wide building.

Fig. 27·2 (*a*) shows the most general substitute frame consisting of *three span, two-storey substitute structure* with irregular spacing of columns. Fig. 27·2 (*b*) shows the substitute frame for finding the bending moments in wall columns. This consists of three spans and three two-storeys columns, one of which is the wall column. Fig. 27·2 (*c*) shows the substitute frame for structures with two panels wide. Fig. 27·2 (*d*) shows the substitute frame for one span multi-storey frame.

End conditions for substitute frames

The restraining effect of any one member, upon other members forming a joint depends also upon the condition existing on the other end of the restraining member. The other end may have three conditions : (*i*) free to turn (*i.e.* hinged), (*ii*) partially restrained, or



(a) GENERAL SUBSTITUTE FRAME

(b) SUBSTITUTE FRAME FOR WALL COLUMNS

(c) SUBSTITUTE FRAME FOR TWO PANEL WIDE BUILDING

(d) SUBSTITUTE FRAME FOR ONE PANEL WIDE BUILDING

Fig. 27·2.
Various types of substitute frames.

(*iii*) partially fixed. The restraining effect is largest for the rigidly fixed conditions of the end and smallest for free end. It should be noted that the restraining effect of a fixed member is one-third larger than the restraining effect if it were free to turn. The rigidity of any member is expressed by the ratio $\frac{I}{L}$ where I is its moment of inertia and l is its length (for beam) or height (for column). If the loaded mumber has rigidity $\frac{I}{L}$ and the restraining member has rigidity $\frac{I_1}{l_1}$, then this restraining member is considered as fixed at

the other end if $\frac{I_1}{l_1} \div \frac{I}{l}$ is equal to or greater than 10. The end of a member is considered as a partly restrained when it runs into another joint composed of several members, a condition which is often found in concrete skeleton structure. No restraint exists if $\frac{I_1}{l_1} \div \frac{I}{l}=0$. The other ends of the member of the substitute frame are sometimes taken as hinged (except for columns fixed in the ground). This gives severest condition for a particular reaction under investigation. The moments obtained by assuming the ends hinged gives the moments nearest to the value obtained from full frame analysis and compensates to some extent for the error caused due to neglecting loads on members of distant span.

27·3. ANALYSIS FOR VERTICAL LOADS

A building frame is a three dimensional structure consisting of a number of bays in two directions at right angles to each other. A building structure may be assumed to be consisting of two sets of plane frames crossing each other at right angles. The vertical members (*i.e.* columns) are common to both these sets of frame. Each set of frames is analysed separately. Since moments in the vertical members occur in two planes, the stresses in columns should be found for moments acting in two planes simultaneously and the corresponding vertical loads.

(*a*) Maximum bending moments in beams

The magnitude of bending moments in beams and columns respectively depend upon their relative rigidity. *Generally*, the beams are made of the same dimensions in all the floors, while the dimensions of columns vary from storey to storey. Columns have smallest dimensions at the top and largest dimensions at the bottom. Due to this reason, the ratio of the rigidity of the beam to that of the column is larger in the upper floors than in the lower floors. The positive bending moments in the beams increase with decrease of the rigidity of the columns, while the negative B.M. in them increase with the increase in the rigidity of the columns. Due to this, the positive B.M. are the largest in the upper storeys where the columns are least rigid and the negative bending moments are maximum in the lower storeys where the columns are rigid.

In order to find the maximum moment in a given span of the beam, the substitute frame is so selected that span under investigation forms the centre span. This substitute frame may be moved from floor to floor. However, since the beams in all floors are made of the same dimensions and provided with same amount of steel, only one substitute frame may be sufficient when placed in a position in the structure for which the bending moments are the largest. The beams should be loaded with live loads as follows for maximum effects :

(*i*) *For maximum positive B.M.* At the mid-point C of a span AB, the loads should be placed on the span and on alternative spans, as shown in Fig. 27·3 (*a*).

(*ii*) *For maximum negative B.M.* At the mid-point C of a span AB, the span AB should be unloaded while load should be placed on spans adjacent to the span under consideration, as shown in Fig. 27·3 (*a*).

(*iii*) *For maximum negative B.M.* At the support A, loads should be placed on the two spans adjacent to the support as shown in Fig. 27·3 (*c*).

When the spans of the beams are not equal, substitute frames should be selected in which the largest span forms the centre span, and also frames in which the smallest span forms the centre span. Several trial computations may be necessary to get the frame for which the bending moments are maximum. To get the bending moment in the wall columns and wall beams, substitute frame shown in Fig. 27·2 (*b*) should be used.

The bending moments due to dead loads are found separately. The bending moments for dead and live loads are then added, and the beam is designed.

(*b*) **Maximum bending moment in columns**

The bending moments in columns increase with increase in their rigidity. Hence they are largest in the lower storeys, and smallest in the upper storey. The maximum compressive stresses in columns is found by combining maximum vertical loads with the maximum bending moments. The maximum tensile stresses in columns is found by combining the maximum bending moment with minimum vertical loads. Though the bending moment is smallest in the upper floors, its effect is much larger since the dimensions of the columns are the smallest there and also the vertical loads are much smaller than in lower storeys. Also the possibility of tensile stresses in columns is much larger in upper storeys than in lower storeys.

The maximum moments in columns occur when alternative spans are loaded as shown in Fig. 27·4 (*a*), (*b*). The corresponding axial loads are found. The column is designed to resist the stresses provided by every combination of axial load and the corresponding moment.

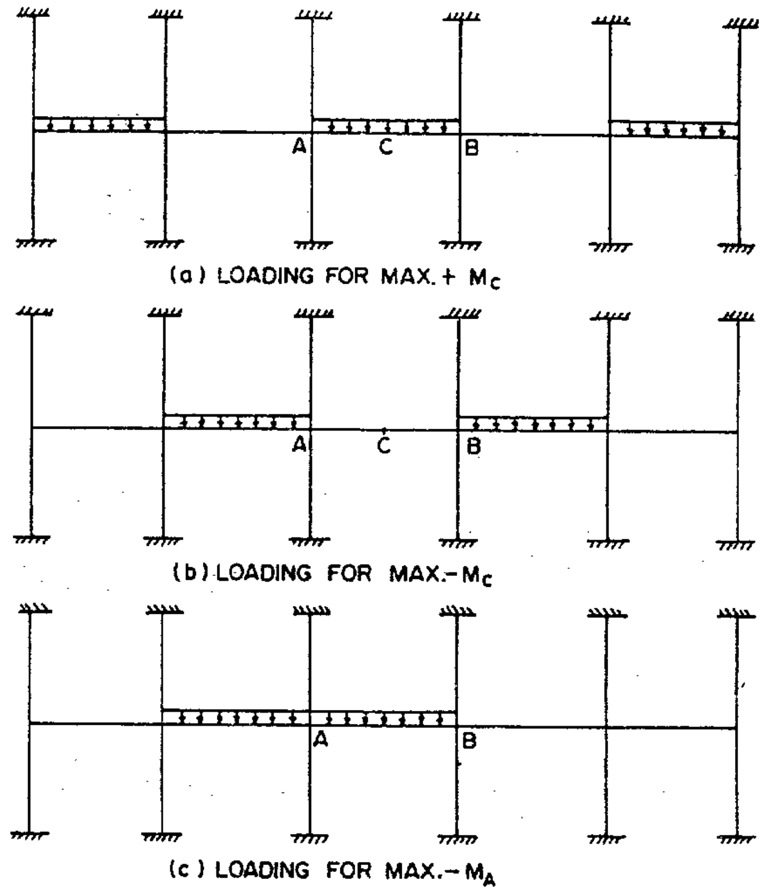


Fig. 27·3. Loading for bending moments in beams.

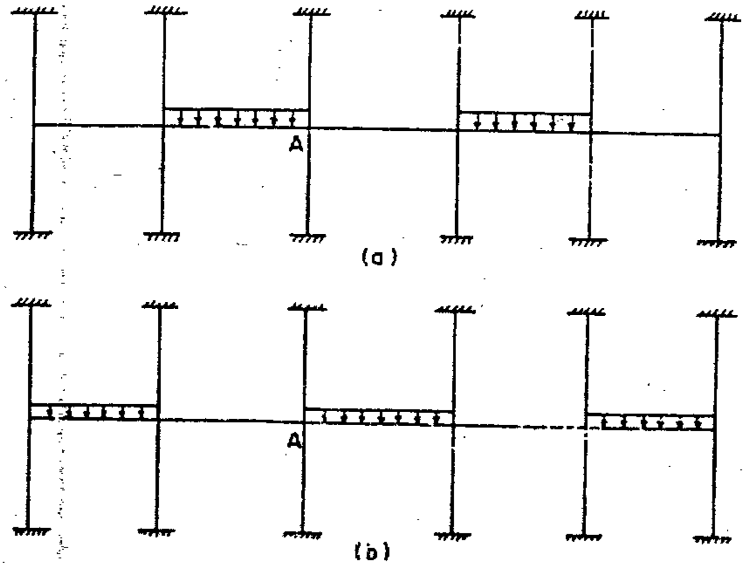


Fig. 27·4. Loading for max. B.M. at column A.

27·4. METHODS OF COMPUTING B M.

The bending moments in the beams and columns of a substitute frame may be computed by the following methods :

1. Slope deflection method.
2. Moment distribution method.
3. Building frame formulae.
4. Kani's method.

The slope-deflection method results in too many equations to be solved simultaneously. Hence moment distribution method, using two cycles is used. Taylor, Thomson and Smulski recommended the use of building frame formulae which they have developed using slope deflection equations.

Example 27·1. *Fig. 27·5 shows an intermediate frame of a multistoreyed building. The frames are spaced at 4 metres centre to centre. Analyse the frame taking live load of 4000 N/m² and dead oad as 3000 N/m², 3250 N/m² and 2750 N/m² for panels AB, BC and CD respectively. The self weight of the beams may be taken as under :*

Beams of 7 m span : 5000 N/m.

Beams of 5 m span : 3500 N/m.

Beams of 3·5 m span : 2500 N/m.

The relative stiffness of the members are marked on the figure itself.

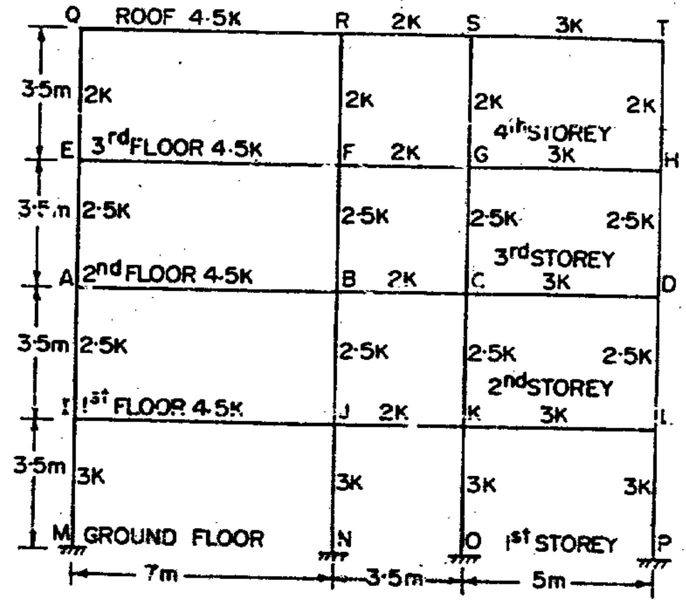


Fig. 27·5

Solution.

1. Substitute frame

Let us analyse the second floor *ABCD*. The substitute frame is shown in Fig. 27·6, assuming the far ends of the columns fixed. Other floors can be analysed in a similar manner.

2. Loading and fixed end moments

Since the frames are spaced @ 4 m c/c, the live loads transferred from the floors will be

$$4000 \times 4 = 16000 \text{ N/m.}$$

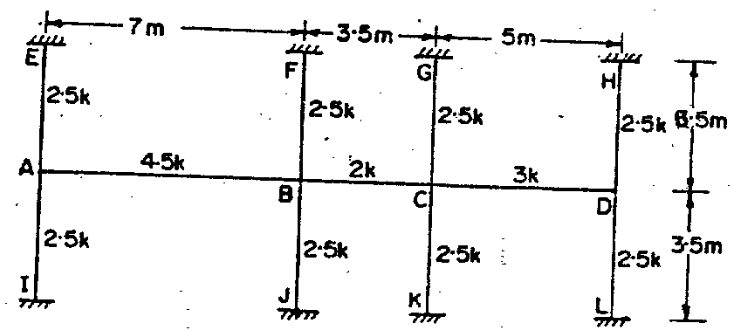


Fig. 27·6

The total dead load on a beam will be equal to dead load from the floors plus the dead load due to the self weight of the beam.

Thus, the total dead load on the beam *AB*, per metre run $=(3000 \times 4)+5000=17000$ N/m. Dead loads for other beams are tabulated in Table 27·1.

The fixed end moment is calculated from the following expressions

$$M_F = \pm \frac{wL^2}{12}$$

where *w* is the U.D.L. and *L* is the span of the beam. Clockwise moments are taken as positive and anticlockwise moments are taken as negative. The fixed end moments due to dead load, and due to dead and live load combined are tabulated in Table 27·1.

Table 27·1

Loading and Fixed End Moments

Member	*Dead load (N/m)*	*Live load (N/m)*	*F.E.M. due to dead load (N-m)*	*F.E.M. due to dead and live load combined (N-m)*
AB	17000	16000	69420	134750
BC	15500	16000	15820	32160
CD	14500	16000	30210	63540

3. Distribution factor

The distribution factors at a joint depends upon the relative stiffnesses of the members meeting at the joints. These are tabulated in Table 27·2 on next page.

Table 27·2

Distribution Factors

Joint	*Members*	*Relative stiffness*	*Sum*	*Distribution factors*
	AE	$2 \cdot 5K$		0·263
A	*AI*	$2 \cdot 5K$	$9 \cdot 5K$	0·263
	AB	$4 \cdot 5K$		0·474
	BA	$4 \cdot 5K$		0·392
B	*BF*	$2 \cdot 5K$	$11 \cdot 5K$	0·217
	BC	$2 \cdot 0K$		0·174
	BJ	$2 \cdot 5K$		0·217
	CB	$2K$		0·20
C	*CG*	$2 \cdot 5K$	$10K$	0·25
	CD	$3K$		0·30
	CK	$2 \cdot 5K$		0·25
	DC	$3K$		0·375
D	*DH*	$2 \cdot 5K$	$8K$	0·3125
	DL	$2 \cdot 5K$		0·3125

(A) MAXIMUM NEGATIVE B.M. AT SUPPORTS

4. Maximum Negative B.M. at Joint A

The condition of loading to obtain maximum negative B.M. at a joint *A* is as follows : Live load on *AB* only, while the dead load is on *AB* and *CD*. The effect of load on other spans is neglected. The moment distribution is carried out in Table 27·3. The distribution is done at Joint *B* and the carry over effect (*i.e.* half the moment) is transferred to joint *A*. After adding the total moment at *A*, distribution is done at *A*. The distribution at joint *B* has not been recorded in Table 27·3.

Table 27·3

Moment Distribution for —ve B.M. at A

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
D.F.	0·474	0·392	0·174	0·20	0·30	0·375
1. F.E.M. due to D.L.			−15820	+15820		
2. F.E.M. due to total load	−134750	+134750				
3. Distribution at B and carry over to A	−23310					
4. Add (2) and (3)	−158060					
5. Distribution	+74920					
6. Total [sum of (4) and (5)]	−83140					

5. Maximum Negative B.M. at Joint B

The loading conditions are : Live load on AB and BC, while dead load on whole of $ABCD$. The moment distribution is carried out in Table 27·4. In the first cycle, joints A and C are balanced, and half the moments are carried over to joint B for beams BA and BC respectively. In the second cycle, joint B is balanced and final moments are found. Thus, in the first cycle, unbalanced moment at C is $+1950$, the distributed moment to CB will be $-1950 \times 0{\cdot}2 = -200$, the carry over moment at $B = -200/2 = -100$. Similarly, the unbalanced moment at A is -134750, the distributed moment for $AB = +134750 \times 0{\cdot}474 = +63880$, the carried over moment to $B = +31940$. The total moments at BA and BC will be $+166690$ and -32260, leaving an unbalanced moment of $+134430$. The distributed moments to BA and BC will be $-134430 \times 0{\cdot}392 = -52700$, and $-134430 \times 0{\cdot}174 = -23390$ respectively.

Table 27·4

Moment Distribution for —ve B.M. at B

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
D.F.	0·474	0·392	0·174	0·20	0·30	0·375
1. F.E.M. due to D.L.					−30210	
2. F.E.M. due to total load	−134750	+134750	−32160	+32160		
3. Distribution at A and C and carry over to B.		−31940		−100		
4. Add (2) and (3)		+166690	−32260			
5. Distribution		−52700	−23390			
6. Total (sum of 4 and 5)		+113990	−55650			

6. Maximum Negative B.M. at C

The conditions of loadings are : Live load on BC and CD, and Dead load on $ABCD$. In the first cycle, joints B and D are balanced and effects are carried over to C. In the second cycle, joint C is balanced, as shown in Table 27·5.

7. Maximum Negative B.M. at D

The conditions for loadings are : Live load on CD and dead load on $ABCD$. In first cycle, joint C is balanced and its effect is carried over to D. In the second cycle, joint D is balanced as shown in Table 27·5.

Table 27·5

Moment Distribution for −ve B.M. at C

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
D.F.	0·474	0·392	0·174	0·20	0·30	0·375
1. F.E.M. due to D.L.	−69420	+69420				
2. F.E.M. due to total load			−32160	+32160	−63540	+63540
3. Distribution at B and D and carry over to C				−3240	−11910	
4. Add (2) and (3)				+28920	−75450	
5. Distribution				+9310	+13960	
6. Total (sum of 4 and 5)				+38230	−61490	

Table 27·6

Moment Distribution for −ve B.M. at D

Joint	A	B		C		D
Member	AB	BA	CB	CB	CD	DC
D.F.	0·474	0·392	0·174	0·20	0·30	0·375
1. F.E.M. due to D.L.			−15820	+15820		
2. F.E.M. due to total load					−63540	+63540
3. Distribution at C and carry over to D						+7160
4. Add (2) and (3)						+70700
5. Distribution						−26510
6. Total (sum of 4 and						+44190

(B) MAXIMUM POSITIVE B.M. AT MID-SPANS

8. Maximum +ve B.M. in mid-span of AB

The conditions of loadings are : Live load on AB and CD and Dead load on $ABCD$. In the first cycle, distribution is performed at joints A, B and C. Half of these distributed moments are carried over to the opposite ends, *i.e.* from A to B and B to A, and from C to B. In the second cycle, distribution is performed at A and B, as illustrated in Table 27·7. Thus the end moments at A and B for beam AB are −83140 and +105680 respectively. The free B.M. at mid-span of

$$AB=\frac{wL^2}{8}$$

$$=\frac{(17000+16000)(7)^2}{8}=202120 \text{ N-m}$$

$$\therefore \quad \text{Net B.M. at centre of } AB=202120-\frac{(83140+105680)}{2}$$

$$=107710 \text{ N-m}$$

Table 27·7

Moment Distribution for +ve B.M. at mid-span of AB

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
D.F.	0·474	0·392	0·174	0·20	0·30	0·375
1. F.E.M. due to D.L.			−15820	+15820		
2. F.E.M. due to total load	−134750	+134750			−63540	+63540
3. Distribution at A, B and C	+63870	−46620		+9540		
4. Carry over	−23310	+31940	+4770			
5. Distribution at A and B	+11050	−14390				
6. Total moments (sum of 1, 2, 3, 4, 5)	−83140	+105680				

9. Maximum +ve B.M. in mid-span of BC

The conditions for loadings are : Live load on *BC* and dead load on *ABCD*. In the first cycle, moments are distributed at *A*, *B*, *C* and *D*. These distributed moments are carried over from *A* to *B*, from *B* to *C*, from *C* to *B* and from *D* to *C*. Finally, the moment and distributed at joints *B* and *C*, as shown in Table 27·8.

Table 27·8

Moment Distribution for +ve B.M. at mid-span of BC

Joint	*A*	*B*		*C*		*D*
Member	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
D.F.	*0·474*	*0·392*	*0·174*	*0·20*	*0·30*	*0·375*
1. F.E.M. due to D.L.	−69420	+69420			−30210	+30210
2. F.E.M. due to total load			−32160	+32160		
3. Distribution at *A*, *B*, *C* and *D*	+32900		−6480	−390		−11330
4. Carry over		+16450	−200	−3240	−5660	
5. Distribution at *B* and *C*.			−2830	+1780		
6. Total moments			−41670	+30310		

Thus, the end moments at *B* and *C* are −41670 and +34190 respectively. The free B.M. at the centre of span *BC* is

$$=\frac{wL^2}{8}=\frac{(15500+16000)}{8}(3·5)^2=48230$$

∴ Net B.M. at centre of *BC*

$$=48230-\frac{41670+30310}{2}=12240 \text{ N-m.}$$

10. Maximum +ve B.M. in mid-span of CD

Conditions of loadings are : Live load on *CD* and *AB* and dead load on *ABCD*. In the first cycle, the moment distribution is done at joints *B*, *C* and *D*, and half the distributed moments are carried over to the opposite ends, *i.e.* from *D* to *C* and *C* to *D*, and from *B* to *C*. In the second cycle, distribution is performed at *C* and *D*, as illustrated in Table 27·9.

Table 27·9

Moment Distribution for the B.M. at Mid-span of CD

Joint	*A*	*B*		*C*		*D*
Member	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
D.F.	*0·474*	*0·392*	*0·174*	*0·20*	*0·30*	*0·375*
1. F.E.M. due to D.L.			−15820	+15820		
2. F.E.M. due to total load.	−134750	+134750			−63540	+63540
3. Distribution at *B*, *C* and *D*			−20690		+14320	−23820
4. Carry over.				−10350	−11910	+7160
5. Distribution at *C* and *D*.					−6680	−2680
					−54450	+44200

Thus, the end moments at *C* and *D* are −5445 and +4420 respectively. The free B.M. at the centre of span *CD*

$$=\frac{wL^2}{8}=\frac{(14500+16000)\times 5^2}{8}=95310 \text{ N-m}$$

∴ Net B.M. at the centre of *CD*

$$=95310-\frac{54450\times 44200}{2}=45980 \text{ N-m.}$$

(C) MAXIMUM NEGATIVE B.M. AT CENTRE OF SPANS

11. Maximum Negative B.M. at the Centre of Span BC

The condition for loadings are : Live loads on *AB* and *CD*, and dead load on *ABCD*. In the first cycle, moment distribution is carried out at all the four joints *A*, *B*, *C* and *D*. These moments are then carried over to joints *B* and *C* and from joints *A* and *B*, as well as between themselves. The second distribution is carried out at joints *B* and *C*, as shown in Table 27·16.

Table 27·10

Moment Distribution for Max. —ve B.M. at centre of Span BC

Joint	*A*	*B*		*C*		*D*
Member	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
D.F.	0·474	0·392	0·174	0·20	0·30	0·375
1. F.E.M. due to D.L.			−15820	+15820		
2. F.E.M. due to total load	−134750	+134750			−63540	+63540
3. Distribution at *A*, *B*, *C* and *D*	+63880	−46620	−20700	+9540	+14320	−23820
4. Carry over to *B* and *C*		+31940	+4770	−10350	−11910	
5. Distribution at *B* and *C*			−6380	+4450		
6. Final moments			−38130	+19460		

Thus the end moments at B and C are −38130 and +19460 respectively. Free B.M. at all the centre of span BC

$$=\text{Dead load intensity}\times\frac{L^2}{8}=\frac{15500(3\cdot5)^2}{8}=23730$$

∴ Net B.M. at centre of BC

$$=23730-\frac{38130+19460}{8}$$

$$=-5065.$$

12. Maximum Negative B.M. at Centre of Span AB and CD

Since spans AC and CD are large, free B.M. at their mid-span will be large. It will be seen that the net B.M. at the centre of these spans will either be positive, or will be negative but of negligible small magnitude. Due to this reason, these spans are not being investigated for maximum negative B.M. However, conditions for maximum negative B.M. at the centre of span AB will be when live load is on BC and dead load is on $ABCD$. Similarly, the loading condition for maximum negative B.M. at centre of CD will be when span BC is loaded with live load, and dead load is on $ABCD$.

(D) BENDING MOMENTS IN COLUMNS

For maximum B.M. in columns, alternate spans should be loaded with live load, while the whole floor is loaded with dead load. The two possible load conditions are shown in Fig. 27·4. For the present case, the loadings will be as shown in para (13) and (14) below. See Tables 27·11 and 27·12.

Table 27·12

Bending Moment in Columns

Joint	*A*	*B*		*C*		*D*
Column D.F.						
(a) Just above floor	0·263	0·217		0·25		0·3125
(b) Just below floor	0·263	0·217		0·25		0·3125
Horizontal Members	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
D.F.	0·474	0·392	0·174	0·20	0·30	0·375
1. F.E.M. due to D L.			−15820	+15820		
2. F.E.M. due to total load	−134750	+134750			−63740	+63740
3. Distribution	+63880	−46620	−20700	+9540	+14320	−23820
4. Carry over	−23310	+31940	+4770	−10350	−11910	+7160
5. New moments (total of 1, 2, 4)	−158060	+166690	−11050	+5470	−75650	+70900
6. Distribution to columns						
(*i*) Just above floor	+41570	−33770		+17550		−22160
(*ii*) Just below floor	+41570	−33770		+17550		−22160

13. Max. B.M. in Columns

The loading conditions are: Live load on AB and CD, and dead load on $ABCD$. The moment distribution is carried out as illustrated in Table 27·11. In the first cycle, distribution is done at all the four joints A, B, C and D. The carry over moments are then transferred to the appropriate points. These carried over moments are added to the original F.E.M. to get new moments at each joint. These new moments are distributed to the columns meeting at the joints.

14. Max. B.M. in Columns : Alternative Loading

The loading conditions are : Live load on BC and dead load on $ABCD$. The moment distribution is carried out as illustrated in Table 27.12. In the first cycle, distribution is done at all the four

Table 27.12

Bending Moments in Columns

Joint	*A*	*B*		*C*		*D*
Column D.F.						
(a) *Just above floor*	0.263	0.217		0.25		0.3125
(b) *Just below floor*	0.263	0.217		0.25		0.3125
Horizontal Members	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
D.F.	474	0.392	0.174	0.20	0.30	0.375
1. F.E.M. due to D.L.	−69420	+69420			−30210	+30210
2. F.E.M due to total load			−32160	+32160		
3. Distribution	+32900	−14600	−6480	−400	−580	−11320
4. Carry over	−7300	+16450	−200	−3240	−5660	−290
5. New moments (sum of 1, 2, 4)	−76720	+85870	−32360	+28920	−35870	+29920
6. Distribution to columns						
(a) Just above floor	+20180	−11610		+1740		−9350
(b) Just below floor	+20180	−11610		+1740		−9350

joints. The carry over moments then transferred to appropriate points. These carried over moments are added to the original

F.E.M. to get new moments at each joint. The new moments are distributed to the columns meeting at the joints.

27.5. ANALYSIS OF FRAMES SUBJECTED TO HORIZONTAL FORCES

A building frame is subjected to horizontal forces due to wind pressure and seismic effects. These horizontal forces cause axial forces in columns and bending moment in all the members of the frame. As stated earlier, a building frame is a highly indeterminate structure. The degree of indeterminacy of a building bent (Fig. 27.7) is found by providing a cut near mid-span of each beam. Each cut beam will thus have three unknown reaction components : moment (M), shear (F) and axial thrust (H). Each column with its cut beams will act as a cantilever, which is a statically determinate structure. Thus, if n is the number of beams in a bent, the degree of indeterminacy will be $3n$. For the building bent shown in Fig. 27.7, there are eight beams and hence the bent is statically indeterminate upto 24th degree. An ordinary 20 storey building with 20 storeys and 5 stacks of columns has 80 beams, thus having the degree of indeterminacy of 240.

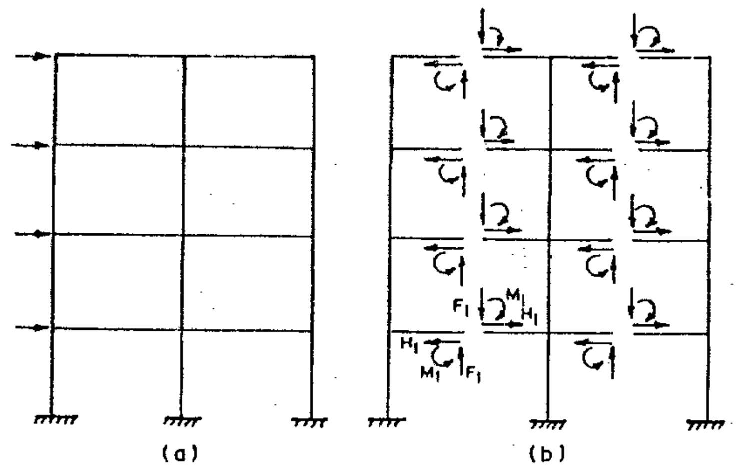


Fig. 27.7

Due to this reason, suitable assumptions are made so that the frame subjected to horizontal forces can be analysed by using simple principles of mechanics. Following approximate methods are commonly used for the analysis of building frames subjected to lateral forces :

1. Portal method
2. Cantilever method.

27·6. PORTAL METHOD

For the purposes of analysis, it is assumed that the horizontal forces are acting on the joints. The portal method is based on the following two important assumptions :

(*i*) the points of contraflexure in all the members lie at their mid-points, and

(*ii*) the horizontal shear taken by each interior column is double the horizontal shear taken by each of exterior column.

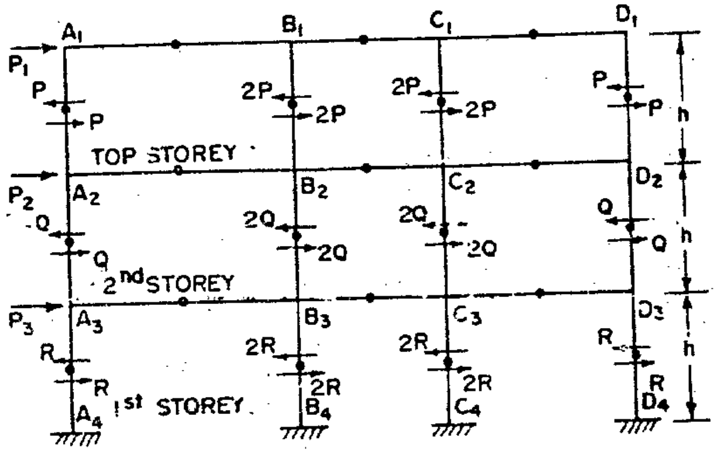


Fig. 27·8

Fig. 27·8 shows a three storey building frame with three spans. Let P_1, P_2, P_3 be the external horizontal forces acting at the joints of the wall columns. Under the action of horizontal forces, the frame will deflect. The point of contraflexure will lie at the middle of each member. Only horizontal shears will act at these points of contraflexure, since B.M. will be zero at these points.

Consider the top storey having vertical members A_1A_2, B_1B_2, C_1C_2 and D_1D_2. The horizontal shear for the outer columns A_1A_2 and D_1D_2 will be P each while that for the inner columns B_1B_2 and C_1C_2 will be $2P$ each, as marked.

The value of P is given by

$$P_1 = P + 2P + 2P + P$$

or
$$P = \frac{1}{6} P_1$$

Similarly, consider the second storey, where the exterior columns A_2A_3 and D_2D_3 have shear Q. The value of shear Q is found by

$$P_1 + P_2 = Q + 2Q + 2Q + Q$$

$$\therefore \qquad Q = \frac{1}{6}(P_1 + P_2)$$

Similarly, for the bottom storey, the shear R is given by

$$P_1 + P_2 + P_3 = R + 2R + 2R + R$$

$$\therefore \qquad R = \frac{1}{6}(P_1 + P_2 + P_3)$$

Knowing the horizontal shears at the point of contraflexure, the bending moment in the column can be easily found.

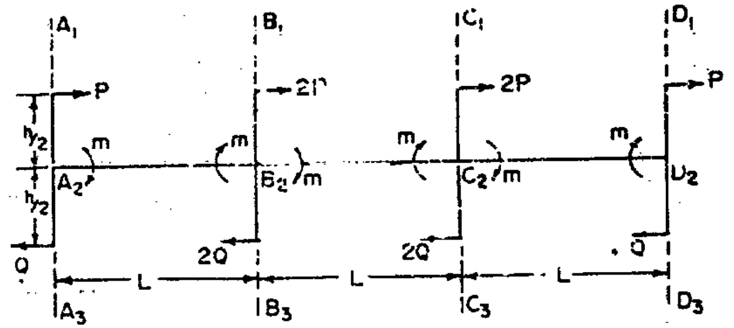


Fig. 27·9

Let us consider the floor $A_2B_2C_2D_2$ between third and second storey. The shear acting at the point of contraflexure are as shown in Fig. 27·9. The joint A_2 is subjected to clockwise moment of $Ph/2$ at A_2 in column A_2A_3, and to a clockwise moment equal to $Qh/2$ at A_2 in column A_2A_3. The beam A_2B_2 is thus required to resist a clockwise moment of $m = (P+Q)h/2$ at A_2. Similarly, at joint B, there will be a clockwise moment equal to $(2P+2Q)h/2$. But there are two beams to resist this. Hence clockwise moment in each beam will be $(P+Q)h/2$. Thus the ends of each beam receive the same clockwise moment of $(P+Q)h/2$, with the result that points of contraflexure will lie in the middle of the beams.

The moment m acting at each end of the beam A_2B_2, B_2C_2, C_2D_2 give rise to vertical reactions in columns. If L is the span of these beams, each beam will impose an upward pull of $\frac{2m}{L}$ on windward column and a push of $\frac{2m}{L}$ on Leeward column connected to the beam, for each span. The vertical reactions will neutralize for any intermediate column, provided span of beams on either side are equal. Only the end columns will experience vertical reactions. The windward column will have an upward pull of $\frac{2m}{L}$ and the Leeward column will have a downward push of $\frac{2m}{L}$

The method of analysis is illustrated in Example 27·2.

27·7. CANTILEVER METHOD

The cantilever method is based on the following assumptions :

(*i*) Points of contraflexure in each member lies at its mid-span or mid-height.

(*ii*) The direct stresses (axial stresses) in the columns, due to horizontal forces, are directly proportional to their distance from the centroidal vertical axis of the frame.

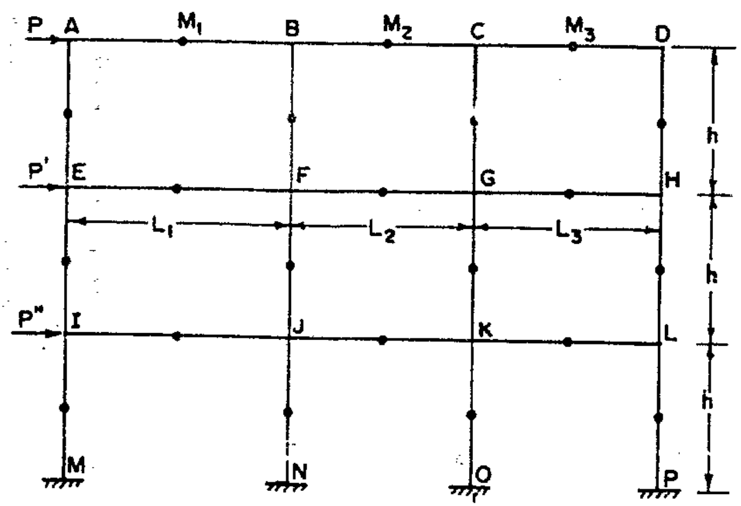


Fig. 27·10 (*a*)

Fig. 27·10 (*a*) shows a building frame subjected to horizontal forces. Fig. 27·10 (*b*) shows the top storey, upto the points of contraflexure of the columns. The reactions at the points of contraflexure will be direct and shear forces only. Let V_1, V_2, V_3 and V_4 be

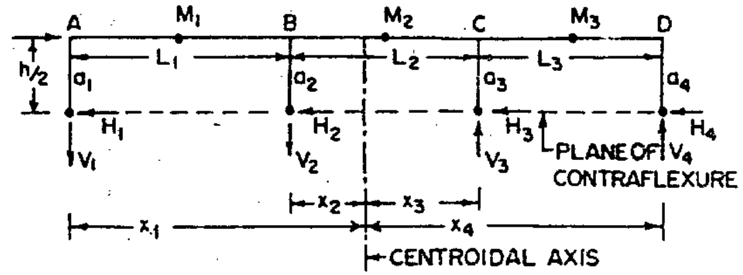


Fig. 27·10 (*b*)

the axial forces in the columns *AE*, *BF*, *CG* and *DH*, having areas of cross-sections a_1, a_2, a_3 and a_4 respectively.

From statics, we have

$$P=H_1+H_2+H_3+H_4 \quad \ldots(i)$$

From assumption 2, we have

$$\frac{V_1/a_1}{x_1}=\frac{V_2/a_2}{x_2}=\frac{V_3/a_3}{x_3}=\frac{V_4/a_4}{x_4} \quad \ldots(ii)$$

where x_1, x_2, x_3 and x_4 are the centroidal distances of the columns from vertical centroidal axis of the frame.

By taking moments about the point of intersection of the vertical centroidal axis and top beam, we get

$$(H_1+H_2+H_3+H_4)\,\frac{h}{2}=V_1.x_1+V_2.x_2+V_3.x_3+V_4.x_4$$

or
$$V_1.x_1+V_2.x_2+V_3.x_3+V_4.x_4=\frac{Ph}{2} \quad \ldots(iii)$$

From (*ii*) and (*iii*), axial forces V_1, V_2, V_3 and V_4 can be determined.

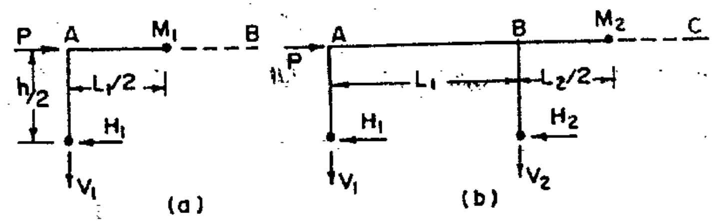


Fig. 27·11

In order to determine H_1, take moments about the point of contraflexure M_1 in beam AB [Fig. 27·11 (*b*)] :

$$H_1 \cdot \frac{h}{2} = V_1 \cdot \frac{L_1}{2}$$

$$\therefore \qquad H_1 = \frac{V_1 . L_1}{h} \qquad \ldots(a)$$

Similarly, taking moments about point of contraflexure M_2 in beam BC,

$$H_1 \cdot \frac{h}{2} + H_2 \cdot \frac{h}{2} = V_1 \left(L_1 + \frac{L_2}{2} \right) + V_2 \cdot \frac{L_2}{2}$$

$$\therefore \qquad (H_1 + H_2) = \frac{2\left[V_1 L_1 + (V_1 + V_2) \frac{L_2}{2} \right]}{h} \qquad \ldots(b)$$

Since h_1 is known from (*a*), H_2 can be determined. In a similar manner, H_3 and H_4 can be determined.

Example 27·2. *Analyse the building frame, subjected to horizontal forces, as shown in Fig. 27·12. Use portal method.*

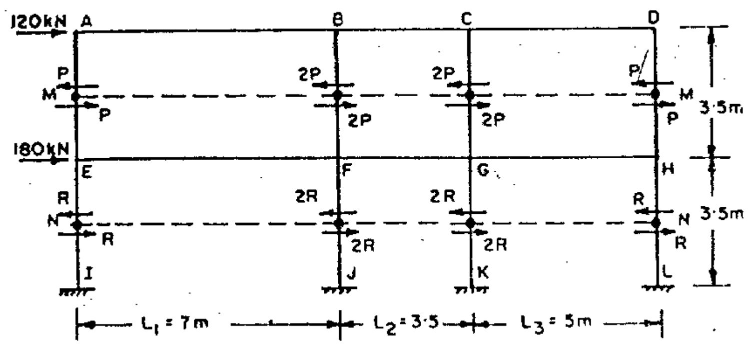


Fig. 27·12

Solution.

1. **Horizontal shear**

Let the horizontal shears in the exterior columns be P and in the interior columns be $2P$ for the top storey. Similarly, for the bottom storey, let the shears be R and $2R$ for the exterior and interior columns.

For the top storey, we have

$$P + 2P + 2P + P = 120$$

$$\therefore \qquad P = \frac{120}{6} = 20 \text{ kN.}$$

For the bottom storey, we have

$$R + 2R + 2R + R = 120 + 180$$

$$\therefore \qquad R = \frac{300}{6} = 50 \text{ kN.}$$

2. **Moments at the ends of columns**

For the top storey,

$$M_{EA} = M_{AE} = M_{HD} = M_{DH} = P \times \frac{h}{2} = 20 \times \frac{3·5}{2} = 35 \text{ kN-m}$$

$$M_{FB} = M_{BF} = M_{GC} = M_{CG} = 2P \times \frac{h}{2} = 20 \times 3·5 = 70 \text{ kN-m.}$$

For the bottom storey,

$$M_{IE} = M_{EI} = M_{LH} = M_{HL} = R \times \frac{h}{2} = 50 \times \frac{3·5}{2} = 87·5 \text{ kN-m}$$

$$M_{JF} = M_{FJ} = M_{KG} = M_{GK} = 2R \cdot \frac{h}{2} = 50 \times 3·5 = 175 \text{ kN-m.}$$

3. **Moments at the ends of the beams**

First floor beams

$$m_{EF} = M_{EA} + M_{EI} = 35 + 87·5 = 122·5 \text{ kN-m}$$

Similarly, $m_{FE} = M_{FG} = m_{GF} = m_{HG} = 122·5$,

since the point of contraflexure lies at the middle of each span.

In general, $m = (P + R) \dfrac{h}{2} = (20 + 50) \times \dfrac{3·5}{2} = 122·5$

Roof beams

$$m_{AB} = m_{BA} = m_{BC} = m_{CB} = m_{CD} = m_{DC} = P \cdot \frac{h}{2}$$

$$= 20 \times \frac{3·5}{2} = 35 \text{ kN-m.}$$

4. **Shear in beams**

Since no external vertical force is acting on the beam, shear F is given by

$$F = \frac{m_1 + m_2}{L}$$

where m_1 and m_2 are the moments at ends of the beam of span L.

Thus, $\qquad F_{EF} = \dfrac{122·5 + 122·5}{7} = 35 \text{ kN} \uparrow$

$$F_{FE} = 35 \text{ kN} \downarrow$$

$$F_{FG}=F_{GF}=\frac{122.5+122.5}{3.5}=70 \text{ kN}$$

$$F_{GH}=F_{HG}=\frac{122.5+122.5}{5}=49 \text{ kN}$$

$$F_{AB}=F_{BA}=\frac{35+35}{7}=10 \text{ kN}$$

$$F_{BC}=F_{CB}=\frac{35+35}{3.5}=20 \text{ kN}$$

$$F_{CD}=F_{DC}=\frac{35+35}{5}=14 \text{ kN.}$$

5. **Axial force in columns**

The axial forces in the columns will be as under :

Column AE=shear in beam AB=10 kN↑

Column EI=axial force in AE+shear in EF

$=10+35=45$ kN↑

Column DH=shear in beam DC=14 kN↓

Column HL=axial force in DH+shear in HG

$=14+49=63$ kN↓

Since the spans are not equal, interior columns will also have axial forces.

Column $BF=F_{BA}-F_{BC}=10-20=-10$ kN (*i.e.* ↑)

Column $FJ=(-10)+(F_{FE}-F_{FG})$

$=(-10)+(35-70)=-45$ kN (*i.e.* ↑)

Alternatively, axial force in BF

$$=\frac{2m'}{L_1}-\frac{2m'}{L_2}=\frac{2\times 35}{7}-\frac{2\times 35}{3.5}=-10 \text{ kN}$$

and axial force in column

$$FJ=(-1)+\left(\frac{2m}{L_1}-\frac{2m}{L_2}\right)$$

$$=(-10)+\left(\frac{2\times 122.5}{7}-\frac{2\times 122.5}{3.5}\right)$$

$=-45$ kN (*i.e.* ↑)

Axial force in $CG=\frac{3m'}{L_2}-\frac{2m'}{L_3}$

$$=\frac{2\times 35}{3.5}-\frac{2\times 35}{5}=6\ (\downarrow)$$

Axial force in column GK

$$=6+\left(\frac{2m}{L_2}-\frac{2m}{L_3}\right)$$

$$=6+\left(\frac{2\times 122.5}{3.6}-\frac{2\times 122.5}{5}\right)=27\downarrow$$

Check : Total axial force at the base

$=-45\ (\uparrow)-45\ (\uparrow)+27\ (\downarrow)+63\ (\downarrow)$

=zero.

Example 27.3. *Re-analyse the frame of example 27.2 by cantilever method, assuming that all the columns have the same area of cross-section.*

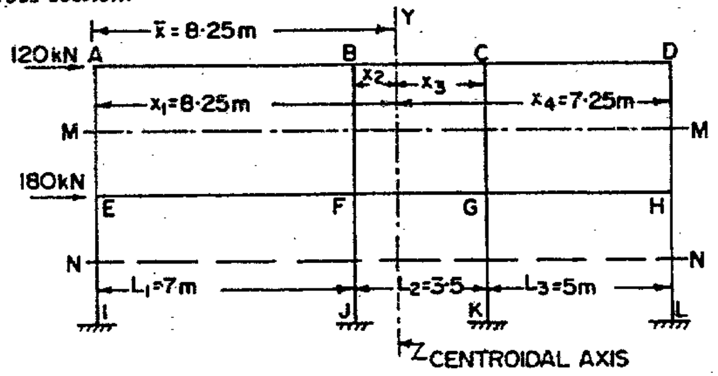


Fig. 27.13

Solution.

1. **Location of centroidal axis of the columns**

Let the centroidal axis be at a distance $\bar{x}$ from the windward column AEI. Taking moment of areas of the columns about AEI, we get

$$\bar{x}=\frac{(2\times 0)+(2\times 7)+(2\times 10.5)+(2\times 15.5)}{8}=8.25 \text{ m}$$

∴ $x_1=8.25(=\bar{x})$; $x_2=8.25-7=1.25$ m

$x_3=3.5-1.25=2.25$ m

$x_4=(7+3.5+5)-8.25=7.25$ m.

2. **Axial forces in columns of first storey**

Let the axial force in column $EI=V_1=V$.

Since the areas are equal, the axial forces in other columns will be in proportion to their distances from the centroidal axis.

Since there is point of contraflexure at the middle of column, *AE*,

$$M_{EA}=47{\cdot}6 \text{ kN-m}$$

$$\therefore \quad M_{BF}=M_{BA}+M_{BC}=47{\cdot}6+27{\cdot}4=75 \text{ kN-m}$$

$$M_{FB}=75 \text{ kN-m}$$

$$M_{CG}=M_{CD}=27{\cdot}4+29{\cdot}9=57{\cdot}3$$

$$M_{GC}=57{\cdot}5$$

$$M_{DH}=M_{DC}=29{\cdot}9 \text{ kN-m}$$

$$M_{HD}=29{\cdot}9 \text{ kN-m}.$$

(*b*) *Bottom storey*

$$M_{EI}+M_{EA}=M_{EF}$$

$$\therefore \quad M_{EI}=M_{EF}-M_{EA}=166{\cdot}8-47{\cdot}6=119{\cdot}2 \text{ kN-m}$$

$$M_{IE}=119{\cdot}2 \text{ kN-m}$$

$$M_{FJ}+M_{FB}=M_{FE}+M_{FG}$$

$$\therefore \quad M_{FJ}=166{\cdot}8+96-75=187{\cdot}8 \text{ kN-m}$$

Hence $M_{JF}=187{\cdot}8$ kN-m

$$M_{GK}=M_{GC}+M_{GF}+M_{GH}$$

$$\therefore \quad M_{GK}=96+10{\cdot}47-57{\cdot}5=143\ 2$$

$$\therefore \quad M_{KG}=143{\cdot}2 \text{ kN-m}$$

$$M_{HL}+M_{HD}=M_{GH}$$

$$\therefore \quad M_{HL}=104{\cdot}7-29{\cdot}9=74{\cdot}8$$

$$M_{LH}=74{\cdot}8.$$

Alternatively, the moment at the column ends can be found by first determining horizontal shears (H) at the point of contraflexure and multiplying there by half the height of the column.

Thus, $M_{AE}=H_1'\times\frac{h}{2}$; $M_{BF}=H_1'\times\frac{h}{2}$ etc.

Similarly, $M_{EI}=H_1\times\frac{h}{2}$; $M_{FJ}=H_2\times\frac{h}{2}$ etc.

The method of determining horizontal shears have been explained in § 27·7.

For example,

$$H_1'=\frac{V_1'L_1}{h}=13{\cdot}615\times\frac{7}{3{\cdot}5}=27{\cdot}23$$

$$H_2'=\frac{2\left[V_1'.L_1+(V_1'+V_2')\frac{L_2}{2}\right]}{h}-H_1'$$

$$=\frac{2\left[13{\cdot}615\times7+(13{\cdot}615+2{\cdot}063)+\frac{3{\cdot}5}{2}\right]}{3{\cdot}5}-27{\cdot}23$$

$$=42{\cdot}908$$

$$\therefore \quad M_{AE}=H_1'\times\frac{h}{2}=27{\cdot}23\times\frac{3{\cdot}5}{2}=47{\cdot}65$$

$$M_{BF}=H_2'\times\frac{h}{2}=42{\cdot}908\times\frac{3{\cdot}5}{2}=75$$

which is the same as found earlier.

27·8. FACTOR METHOD

This method is more accurate than either the portal method or cantilever method, and is more useful when the moments of inertia of various members (of columns and beams) are different. Both cantilever method as well as portal method assume uniform moments of inertia of members. These methods, therefore, depend on some stress assumptions, thus limiting the analysis to be based on equations of statics only. The factor method is based on assumptions regarding the elastic action of the structure. For analysis by factor method the relative stiffness 'K' ($=I/L$) for each member of the frame or structure should be known. The procedure consists of the following steps :

1. Calculate the girder factor 'g' for each joint from the following expression

$$g=\frac{\Sigma K_c}{\Sigma K} \qquad ...(27{\cdot}1)$$

where ΣK_c=Sum of relative stiffnesses of all column members at the joint considered

ΣK=Sum of relative stiffneses of all the members at the joint considered.

These values of girder factor 'g' are entered in a tabular form as shown in Table 27·13. The values are entered at the end of each girder meeting at that joint.

2. Calculate the column factor 'c' for all joints from the following expression :

$$c=1-g \qquad ...(27{\cdot}2)$$

where 'g'=girder factor of the joint.

The values of column factors are entered in Table 27·13 at the end of each column at the joint.

For columns which are fixed at the base, the column factor 'c' is taken as 1·00.

Fig. 27·16 shows a simple frame with two storeys and two bays, used for illustration purpose. The relative stiffnesses $k_1, k_2, k_3 \ldots\ldots k_{10}$ of all the ten members are entered on/near each member. Table 27·13 is used for computation of *column factor* (c), *girder factor* (g), *column moment factor* (C) and *girder moment factor* (G).

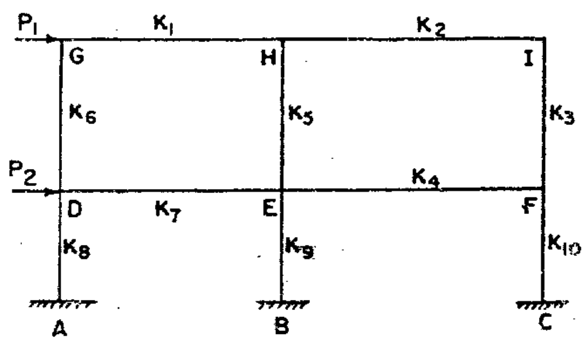


Fig. 27·16

3. As shown in Table 27·13, in the first column of the table, the name of all the joints are entered. The 2nd column contains all the members at each joint. In 3rd column, the corresponding girder or column factors are entered against each girder or column. In column 4 of the table, half the values of the column factor/girder factor of opposite end of the members are entered. For example, if c_1=column factor of member *DG*, it is entered in column 3 opposite *DG*, while column factor c_2 of member *GD* is entered in column 3 opposite member *GD*. Hence in column 4, half the column factor of opposite end, *i.e.* $c_2/2$ is entered opposite member *DG*. Similarly, for member *GD*, a value of $c_1/2$ is entered opposite it for the same reason. So in this way column no. 4 is entered. The values in column no. (3) and (4) of Table 27·13 for each member are added and entered in column no. 5. In column no. 6, the relative stiffness values $K=I/L$ for each member is entered.

4. The sum of columns (3) and (4), which is entered in column no. '5' is multiplied by relative stiffness of respective members (which are entered in column no. 6). This product is termed as *column moment factor* 'C' for columns and *girder moment factor* 'G' for girders. This is entered in column no. '7' of Table 27·13.

Table 27·13

Joint	*Member*	*Column/girder factors 'c' or 'g'*	*Half value of factor at opposite end of member*	(3)+(4)	$K=I/L$	*Column/girder moment factor C or G =(5)×(6)*
(1)	(2)	(3)	(4)	(5)	(6)	(7)
D	*DG*	c_1	$c_2/2$	$c_1+c_2/2$	K_6	
	DA	c_1	$c/2$ from *AD*	$c_1+(c/2)_{AD}$	K_8	
	DE	g_1	$g/2$ from *ED*	$g_1+(g/2)_{ED}$	K_7	
G	*GD*	c_2	$c_1/2$	$c_2+c_1/2$	K_6	
	GH	g_2	$g/2$ from *HG*	$g_2+(g/2)_{HG}$	K_1	

Note. Refer Table 27·14 for further illustration.

The column moment factor 'C', gives the relative values of moments at the ends of columns for each storey in which the column occurs. The sum of column end moments is equal to the horizontal shear on that storey multiplied by the storey height. Hence the column moment factors (C) are converted into end moments for columns by direct proportion for each storey. Similarly, the girder moment factor G, gives the relative values of moment at ends of each girder for the joint. The sum of girder end moment at each joint is equal to the sum of end moments in the columns at the joint. Hence the girder moment factors are converted into end moments for girders by direct proportion for each storey.

(5) **Calculation of column moments :**

(a) Total column moments (A) for each storey is found by the relation

$$A=\frac{H\times h}{\Sigma C} \qquad \ldots(27{\cdot}3)$$

where A=Total column moment for each storey

H=Total horizontal force above the storey considered

h=height of the storey considered

ΣC=Sum of ~~column end~~ moment factors ~~of that storey~~

Thus, for each storey, different column moments A_1, A_2......etc. are calculated.

(b) The column moment factor 'C' of each member is multiplied by the total column moment (A) of that storey in which the column occurs. For example, in Fig. 27·16 if we want to find the column moment M_{GD} of column GD, we have

$$M_{GD}=A_1\times C_{GD} \qquad \ldots(27{\cdot}4)$$

where A_1=Total column moments of first storey (Eq. 27·3) and

C_{GD}=column moment factor for column GD.

Similarly, moment M_{HE} in column HE of first storey is

$$M_{HE}=A_1\times C_{HE}$$

Also for column DA of ground storey,

$$M_{DA}=A_0\times C_{DA}$$

where A_0=total column moments of ground storey.

(6) **Calculations of Girder/beam moments**

(a) For calculation of girder/beam moments, a constant 'B' is found for each joint.

$$B=\frac{\text{Sum of column moments at the joint}}{\text{Sum of the girder moment factors at that joint}}$$

(b) This constant 'B' is multiplied by the girder moment factor (G) to obtain the girder moments.

For example (Fig. 27·16),

$$M_{DE}=B_D\times G_{DE}$$
$$M_{HI}=B_H\times G_{HI}$$
$$M_{IH}=B_I\times G_{IH}$$

and so on.

Here, B_D, B_H, B_I are the *constants* for joints D, H and I respectively.

The factor method of analysing the building frame has been illustrated in Example 27·4.

Example 27·4. *Analyse the frame as shown in 27·17 by factor method. Sketch the B.M.D. The relative 'K' value are written on the members.*

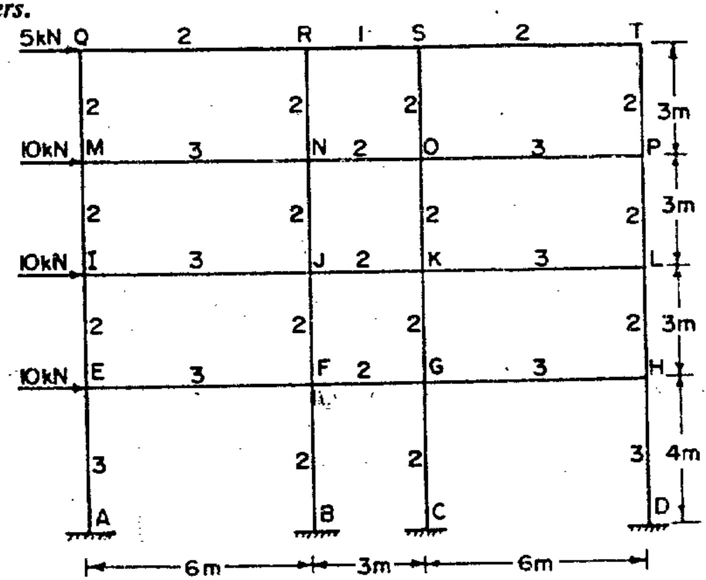


Fig. 27·17

Step 1. Calculate the girder factor 'g' at all joints by Eq. 27·1.

$$g=\frac{\Sigma K_C}{\Sigma K}$$

where ΣK_C=Sum of relative Stiffness of columns at that joint

ΣK=Sum of relative Stiffness of all the members at that joint

Joint Q $\quad g_Q=\dfrac{K_{QM}}{K_{QM}+K_{QR}}=\dfrac{2}{2+2}=0{\cdot}5$

Joint R $\quad g_R=\dfrac{K_{RN}}{K_{RV}+K_{RQ}+K_{RS}}=\dfrac{2}{2+2+1}=0{\cdot}4$

Joint S $\quad g_S=\dfrac{K_{SO}}{K_{SO}+K_{SR}+K_{ST}}=\dfrac{2}{2+1+2}=0{\cdot}4$

Joint T $\quad g_T=\dfrac{K_{TP}}{K_{TP}+K_{TS}}=\dfrac{2}{2+2}=0{\cdot}5$

Joint M $\quad g_M=\dfrac{K_{MI}+K_{MQ}}{K_{MI}+K_{MQ}+K_{MN}}=\dfrac{2+2}{2+2+3}=0{\cdot}57$

Joint N $\quad g_N=\dfrac{K_{NJ}+K_{NR}}{K_{NJ}+K_{NR}+K_{NM}+K_{NO}}=\dfrac{2+2}{2+2+3+2}=0{\cdot}44$

Joint O $\quad g_O=\dfrac{K_{OK}+K_{OS}}{K_{OK}+K_{OS}+K_{ON}+K_{OP}}=\dfrac{2+2}{2+2+2+3}=0{\cdot}44$

Joint P $\quad g_P=\dfrac{K_{PL}+K_{PT}}{K_{PL}+K_{PT}+K_{PO}}=\dfrac{2+2}{2+2+3}=0{\cdot}57$

Joint I $\quad g_I=\dfrac{K_{IE}+K_{IM}}{K_{IE}+K_{IM}+K_{IJ}}=\dfrac{2+2}{2+2+3}=0{\cdot}57$

Joint J $\quad g_J=\dfrac{K_{JF}+K_{JN}}{K_{JF}+K_{JN}+K_{JI}+K_{JK}}=\dfrac{2+2}{2+2+3+2}=0{\cdot}44$

Joint K $\quad g_K=\dfrac{K_{KO}+K_{KG}}{K_{KO}+K_{KG}+K_{KJ}+K_{KL}}=\dfrac{2+2}{2+2+3}=0{\cdot}44$

Joint L $\quad g_L=\dfrac{K_{LM}+K_{LP}}{K_{LH}+K_{LP}+K_{LK}}=\dfrac{2+2}{2+2+3}=0{\cdot}57$

Joint E $\quad g_E=\dfrac{K_{EA}+K_{EI}}{K_{EA}+K_{EI}+K_{EF}}=\dfrac{3+2}{3+2+3}=0{\cdot}63$

Joint F $\quad g_F=\dfrac{K_{FB}+K_{FJ}}{K_{FB}+K_{FJ}+K_{FE}+K_{FG}}=\dfrac{2+2}{2+2+3+2}=0{\cdot}44$

Joint G $\quad g_G=\dfrac{K_{GC}+K_{GK}}{K_{GC}+K_{GK}+K_{GF}+K_{GH}}=\dfrac{2+2}{2+2+2+3}=0{\cdot}44$

Joint H $\quad g_H=\dfrac{K_{HD}+K_{HL}}{K_{HD}+K_{HL}+K_{HG}}=\dfrac{3+2}{3+2+3}=0{\cdot}63$

These values of girder factors are written at the ends of girders beams meeting at each joint as shown in col. 3, Table 27·14.

Step 2. Calculate the column factor 'c' at the joints by the relation,

$$c=1-g \qquad \ldots(27{\cdot}2)$$

where g=girder factor at the joint

Joint Q $\quad c_Q=1-g_Q=1-0{\cdot}5=0{\cdot}5$

Joint R $\quad c_R=1-g_R=1-0{\cdot}4=0{\cdot}6$

Joint S $\quad c_S=1-g_S=1-0{\cdot}4=0{\cdot}6$

Joint T $\quad c_T=1-g_T=1-0{\cdot}5=0{\cdot}5$

Joint M $\quad c_M=1-g_M=1-0{\cdot}57=0{\cdot}43$

Joint N $\quad c_N=1-g_N=1-0{\cdot}44=0{\cdot}56$

Joint O $\quad c_O=1-g_O=1-0{\cdot}44=0{\cdot}56$

Joint P $\quad c_P=1-g_P=1-0{\cdot}57=0{\cdot}43$

Joint I $\quad c_I=1-g_I=1-0{\cdot}57=0{\cdot}43$

Joint J $\quad c_J=1-g_J=1-0{\cdot}44=0{\cdot}56$

Joint K $\quad c_K=1-g_K=1-0{\cdot}44=0{\cdot}56$

Joint L $\quad c_L=1-g_L=1-0{\cdot}57=0{\cdot}43$

Joint E $\quad c_E=1-g_E=1-0{\cdot}63=0{\cdot}37$

Joint F $\quad c_F=1-g_F=1-0{\cdot}44=0{\cdot}56$

Joint G $\quad c_G=1-g_G=1-0{\cdot}44=0{\cdot}56$

Joint H $\quad c_H=1-g_H=1-0{\cdot}63=0{\cdot}37$

$$\left.\begin{array}{ll} \textit{Joint A} & c_A=1{\cdot}00 \\ \textit{Joint B} & c_B=1{\cdot}00 \\ \textit{Joint C} & c_C=1{\cdot}00 \\ \textit{Joint D} & c_D=1{\cdot}00 \end{array}\right\}$$ For columns fixed at the base, column factor is taken as 1·00

These values of column factors are written at the end of columns meeting at the joint, and have been entered in column 3 of Table 27·14.

Step 3. Half the values of column/grider factors of the opposite ends are entered in column 4.

The values in col. 3 and col. 4 are added and entered in col. '5' of Table 27·14.

Step 4. Enter the values of relative stiffneses of all members in col. 6. These values of stiffnesses are multiplied by the values of column 5, to get the values of girder/column moment factors (G or C), and are entered in col. 7 of Table 27·14.

Table 27·14

Joints 1	Members 2	Girder/Column factor (c or g) 3	Half values of the factors from opposite end 4	(3)+(4) 5	Relative stiffness (K=I/L) 6	Girder Column Moment (C or G) factor =(5)×(6) 7
Q	QR	0·5	0·2	0·7	2	1·4
	QM	0·5	0·21	0·71	2	1·42
R	RQ	0·4	0·25	0·65	2	1·30
	RS	0·4	0·2	0·60	1	0·60
	RN	0·6	0·28	0·88	2	1·76
S	SR	0·4	0·2	0·6	1	0·6
	ST	0·4	0·25	0·65	2	1·3
	SO	0·6	0·28	0·88	2	1·76
T	TS	0·5	0·2	0·7	2	1·4
	TP	0·5	0·21	0·71	2	1·42
M	MN	0·57	0·22	0·79	3	2·37
	MI	0·43	0·21	0·64	2	1·28
	MQ	0·43	0·25	0·68	2	1·36
N	NM	0·44	0·28	0·72	3	2·16
	NO	0·44	0·22	0·66	2	1·32
	NR	0·56	0·3	0·86	2	1·72
	NJ	0·56	0·28	0·84	2	1·68

1	2	3	4	5	6	7
O	ON	0·44	0·22	0·66	2	1·32
	OP	0·44	0·28	0·72	3	2·16
	OS	0·56	0·30	0·86	2	1·72
	OK	0·56	0·28	0·84	2	1·68
P	PO	0·57	0·22	0·79	3	2·37
	PL	0·43	0·21	0·64	2	1·28
	PT	0·43	0·25	0·68	2	1·36
I	IJ	0·57	0·22	0·79	3	2·37
	IM	0·43	0·21	0·64	2	1·28
	IE	0·43	0·18	0·61	2	1·22
J	JI	0·44	0·28	0·72	3	2·16
	JK	0·44	0·22	0·66	2	1·32
	JN	0·56	0·28	0·84	2	1·68
	JF	0·56	0·28	0·84	2	1·68
K	KJ	0·44	0·22	0·66	2	1·32
	KL	0·44	0·28	0·72	3	2·16
	KO	0·56	0·28	0·84	2	1·68
	KG	0·56	0·28	0·84	2	1·68
L	LK	0·57	0·22	0·79	3	2·37
	LP	0·43	0·21	0·64	2	1·28
	LH	0·43	0·18	0·61	2	1·22

1	2	3	4	5	6	7
E	EF	0·63	0·22	0·85	3	2·55
	EI	0·37	0·21	0·58	2	1·16
	EA	0·37	0·5	0·87	3	2·61
F	FE	0·44	0·31	0·75	3	2·25
	FG	0·44	0·22	0·66	2	1·32
	FB	0·56	0·5	1·06	2	2·12
	FJ	0·56	0·28	0·84	2	1·68
G	GF	0·44	0·22	0·66	2	1·32
	GH	0·44	0·31	0·75	3	2·25
	GK	0·56	0·28	0·84	2	1·68
	GC	0·56	0·50	1·06	2	2·12
H	HG	0·63	0·22	0·85	3	2·55
	HL	0·37	0·21	0·58	2	1·16
	HD	0·37	0·50	0·87	3	2·61
A	AE	1·00	0·18	1·18	3	3·54
B	BF	1·00	0·28	1·28	2	2·56
C	CG	1·00	0·28	1·28	2	2·56
D	DH	1·00	0·18	1·18	3	3·54

Thus Table 27·14 is completed.

Step 5. Calculation of column moments

Total column moments for each storey

$$A=\frac{H\times h}{\Sigma C}$$

where H=Total horizontal force above the storey considered

h=height of the storey considered

ΣC=sum of *column end moment factors* of that storey.

Let A_3=Total column moments of third/top storey

A_2=Total column moments of second storey

A_1=Total column moments of first storey

A_0=Total column moments of ground storey.

$$\therefore \quad A_3=\frac{5\times 3}{C_{QM}+C_{MQ}+C_{RN}+C_{NR}+C_{SO}+C_{OS}+C_{TP}+C_{PT}}$$

$$=\frac{15}{1·42+1·36+1·76+1·72+1·76+1·72+1·42+13·6}$$

$$=1·2 \text{ kN-m}$$

$$A_2=\frac{(10+5)\times 3}{C_{MI}+C_{IM}+C_{NJ}+C_{JN}+C_{OK}+C_{KO}+C_{PL}+C_{LP}}$$

$$=\frac{45}{1·28+1·28+1·68+1·68+1·68+1·68+1·28+1·28}$$

$$=3·80 \text{ kN-m}$$

$$A_1=\frac{(10+10+5)\times 3}{C_{IL}+C_{LI}+C_{JF}+C_{FJ}+C_{KG}+C_{GK}+C_{LH}+C_{HL}}$$

$$=\frac{75}{1·22+1·16+1·68+1·68+1·68+1·68+1·42+1·16}$$

$$=6·422 \text{ kN-m}$$

$$A_0=\frac{(10+10+10+5)\times 4}{C_{EA}+C_{AE}+C_{FB}+C_{EF}+C_{GC}+C_{CG}+C_{HD}+C_{DH}}$$

$$=\frac{140}{2·61+3·54+2·12+2·56+2·12+2·56+2·61+3·54}$$

$$=6·464 \text{ kN-m.}$$

Column Moments :

TOP STOREY : $A_3=1·2$ kN-m

$M_{QM}=A_3\times C_{QM}=1·2\times 1·42=1·704$ kN-m

$M_{MQ}=A_3\times C_{MQ}=1·2\times 1·36=1·632$ kN-m

$M_{RN}=A_3\times C_{RN}=1·2\times 1·76=2·112$ kN-m

$M_{NR}=A_3\times C_{NR}=1·2\times 1·72=2·064$ kN-m

$M_{SO}=A_3\times C_{SO}=1.2\times 1.76=2.112$ kN-m

$M_{OS}=A_3\times C_{OS}=1.2\times 1.72=2.064$ kN-m

$M_{TP}=A_3\times C_{TP}=1.2\times 1.42=1.704$ kN-m

$M_{PT}=A_3\times C_{PT}=1.2\times 1.36=1.632$ kN-m

Second Storey : $A_2=3.80$ kN-m

$M_{MI}=3.80\times 1.28=4.864$ kN-m

$M_{IM}=3.80\times 1.28=4.864$ kN-m

$M_{NJ}=3.80\times 1.68=6.384$ kN-m

$M_{JN}=3.80\times 1.68=6.384$ kN-m

$M_{OK}=3.80\times 1.68=6.384$ kN-m

$M_{KO}=3.80\times 1.68=6.384$ kN-m

$M_{PL}=3.80\times 1.28=4.864$ kN-m

$M_{LP}=3.80\times 1.28=4.864$ kN-m

First Storey : $A_1=6.422$ kN-m

$M_{IE}=6.422\times 1.22=7.834$ kN-m

$M_{EI}=6.422\times 1.16=7.449$ kN-m

$M_{JF}=6.422\times 1.68=10.789$ kN-m

$M_{FJ}=6.422\times 1.68=10.789$ kN-m

$M_{KG}=6.422\times 1.68=10.789$ kN-m

$M_{GK}=6.422\times 1.68=10.789$ kN-m

$M_{LH}=6.422\times 1.22=7.834$ kN-m

$M_{HL}=6.422\times 1.16=7.449$ kN-m

Ground Storey : $A_0=6.464$ kN-m

$M_{EA}=6.464\times 2.61=16.871$ kN-m

$M_{AE}=6.464\times 3.54=22.882$ kN-m

$M_{FB}=6.464\times 2.12=13.703$ kN-m

$M_{BF}=6.464\times 2.56=16.547$ kN-m

$M_{GC}=6.464\times 2.12=13.703$ kN-m

$M_{CG}=6.464\times 2.56=16.547$ kN-m

$M_{HD}=6.464\times 2.61=16.871$ kN-m

$M_{DH}=6.464\times 3.54=22.882$ kN-m

***Step 6.* Calculation of beam moments**

(*a*) constant 'B' $=\dfrac{\text{Sum of column moments at the joint}}{\text{Sum of girder moment factors at that joint}}$

$\therefore$ *Joint* Q : $B_Q=\dfrac{M_{QM}}{G_{QR}}=\dfrac{1.704}{1.4}=1.217$

Joint R : $B_R=\dfrac{M_{RN}}{G_{RQ}+G_{RS}}=\dfrac{2.112}{1.30+0.60}=1.111$

Joint S : $B_S=\dfrac{M_{SO}}{G_{QR}+G_{ST}}=\dfrac{2.112}{0.6+1.3}=1.111$

Joint T : $B_T=\dfrac{M_{TP}}{G_{TS}}=\dfrac{1.704}{1.4}=1.217$

Joint M : $B_M=\dfrac{M_{MQ}+M_{MI}}{G_{MN}}=\dfrac{1.632+4.864}{2.37}=2.741$

Joint N : $B_N=\dfrac{M_{NR}+M_{NJ}}{G_{NM}+G_{NO}}=\dfrac{2.064+6.384}{2.16+1.32}=2.427$

$B_O=\dfrac{M_{OS}+M_{OK}}{G_{ON}+G_{OP}}=\dfrac{2.064+6.384}{1.32+2.16}=2.427$

$B_P=\dfrac{M_{PT}+M_{PL}}{G_{PO}}=\dfrac{1.632+4.864}{2.37}=2.741$

$B_I=\dfrac{M_{IE}+M_{IM}}{G_{IJ}}=\dfrac{7.834+4.864}{2.37}=5.357$

$B_J=\dfrac{M_{JN}+M_{JF}}{G_{JI}+G_{JK}}=\dfrac{10.789+6.384}{2.16+1.32}=4.934$

$B_K=\dfrac{M_{KO}+M_{KG}}{G_{KJ}+G_{KL}}=\dfrac{10.789+6.384}{1.32+2.16}=4.934$

$B_L=\dfrac{M_{LP}+M_{LH}}{G_{LK}}=\dfrac{7.834+4.864}{2.37}=5.357$

$B_E=\dfrac{M_{LI}+M_{EA}}{G_{EF}}=\dfrac{7.449+16.871}{2.55}=9.537$

$B_F=\dfrac{M_{FJ}+M_{FB}}{G_{FE}+G_{FG}}=\dfrac{10.789+13.703}{2.25+1.32}=6.86$

$B_G=\dfrac{M_{GK}+M_{GL}}{G_{GF}+G_{GH}}=\dfrac{10.789+13.703}{1.32+2.25}=6.86$

$B_H=\dfrac{M_{HB}+M_{HL}}{G_{HG}}=\dfrac{16.871+7.449}{2.55}=9.537$

(*b*) *BEAM MOMENTS*

Beam moments $=B\times$ Girder Moment Factor.

$M_{QR}=B_Q\times G_{QR}=1.217\times 1.4=1.704$ kN-m

$M_{RQ}=B_R\times G_{RQ}=1.111\times 1.30=1.444$ kN-m

$M_{RS}=B_R\times G_{RS}=1.111\times 0.60=0.667$ kN-m

$M_{SR}=B_S\times G_{SR}=1.111\times 0.60=0.667$ kN-m

$M_{ST}=B_S\times G_{ST}=1.111\times 1.3=1.444$ kN-m

$M_{TS}=B_T\times G_{TS}=1.217\times 1.4=1.704$ kN-m

$M_{MN}=B_M\times G_{MN}=2.741\times 2.37=6.496$ kN-m

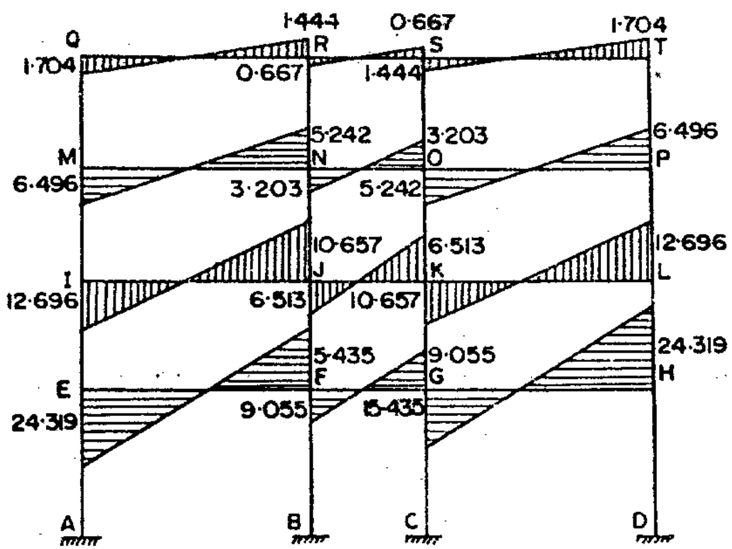
$M_{NM}=B_N\times G_{NM}=2.427\times 2.16=5.242$ kN-m

$M_{NO}=B_N\times G_{NO}=2.427\times 1.32=3.203$ kN-m

$M_{ON}=B_O\times G_{ON}=2.427\times 1.32=3.203$ kN-m

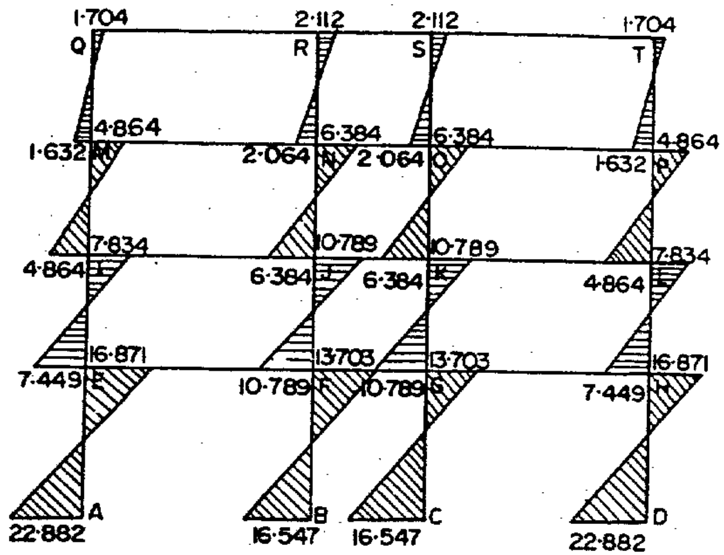
$$M_{OP}=B_O\times G_{OP}=2{\cdot}427\times2{\cdot}16=5{\cdot}242 \text{ kN-m}$$
$$M_{PO}=B_P\times G_{PO}=2{\cdot}741\times2{\cdot}37=6{\cdot}496 \text{ kN-m}$$
$$M_{IJ}=B_I\times G_{IJ}=5{\cdot}357\times2{\cdot}37=12{\cdot}696 \text{ kN-m}$$
$$M_{JI}=B_J\times G_{JI}=4{\cdot}934\times2{\cdot}16=10{\cdot}657 \text{ kN-m}$$
$$M_{JK}=B_J\times G_{JK}=4{\cdot}934\times1{\cdot}32=6{\cdot}513 \text{ kN-m}$$
$$M_{KJ}=B_K\times G_{KJ}=4{\cdot}934\times1{\cdot}32=6{\cdot}513 \text{ kN-m}$$
$$M_{KL}=B_K\times G_{KL}=4{\cdot}934\times2{\cdot}16=10{\cdot}657 \text{ kN-m}$$
$$M_{LK}=B_L\times G_{LK}=5{\cdot}357\times2{\cdot}37=12{\cdot}696 \text{ kN-m}$$
$$M_{EF}=B_L\times G_{EF}=9{\cdot}537\times2{\cdot}55=24{\cdot}319 \text{ kN-m}$$
$$M_{FE}=B_F\times G_{FE}=6{\cdot}86\times2{\cdot}25=15{\cdot}435 \text{ kN-m}$$
$$M_{FG}=B_F\times G_{FG}=6{\cdot}86\times1{\cdot}32=9{\cdot}055 \text{ kN-m}$$
$$M_{GF}=B_G\times G_{GF}=6{\cdot}86\times1{\cdot}32=9{\cdot}055 \text{ kN-m}$$
$$M_{GH}=B_G\times G_{GH}=6{\cdot}86\times2{\cdot}25=15{\cdot}435 \text{ kN-m}$$
$$M_{HG}=B_H\times G_{HG}=9{\cdot}537\times2{\cdot}55=24{\cdot}319 \text{ kN-m}$$

Thus the moments in all the columns and girder are found. Fig. 27·18 shows the B.M. diagram for girder/beams while Fig. 27·19 shows the B.M. diagram for columns.



B. M. D. FOR BEAMS

Fig. 27·18



B.M.D. FOR COLUMNS

Fig. 27·19

PROBLEMS

1. What do you understand by a substitute frame ? How do you select it ? Discuss in brief the method of analysis.

2. Explain the portal method for analysing a building frame subjected to horizontal forces.

3. Explain the cantilever method for analysing a building frame subjected to horizontal forces.

4. A two-span intermediate frame of a multi-storeyed building is shown in Fig. 27·20. The frames are spaced at 5 m intervals. The dead load and live load per metre run of the beam may be taken as 15 kN/m and 20 kN/m respectively. Analyse the frame using two cycle method of moment distribution.

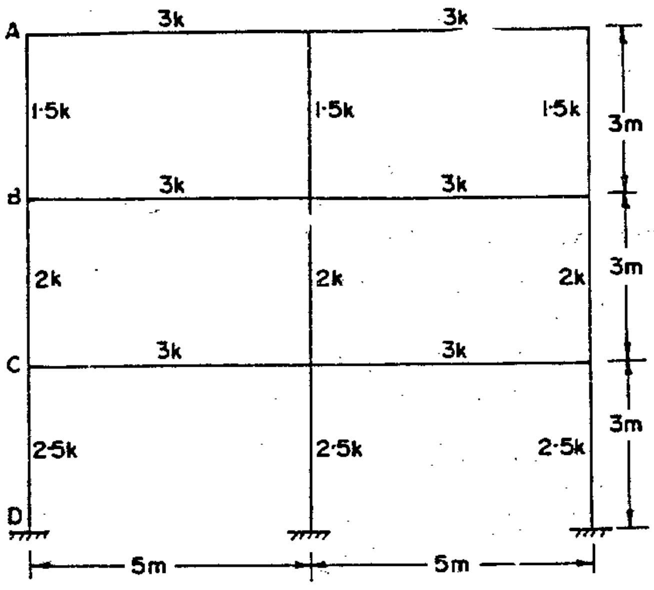


Fig. 27·20

5. If wind loads of 15 kN, 30 kN and 30 kN are acting at joint *A*, *B* and *C* respectively, analyse the frame. (Fig. 27·20) by (*a*) portal method, (*b*) cantilever method. Assume that all the columns have equal area of cross-section for the purpose of analysis.

28

Kani's Method

28.1. INTRODUCTION

In Chapter 9, we have discussed, the slope deflection method presented by G.A. Maney(1915), wherein the rotations and displacements of the joints are treated as *unknowns*. The simultaneous solution of the various slope deflection equations, alongwith the equilibrium equations gives the values of these *unknowns*. However, in many cases, specially in the cases of frames, the solutions become clumsy, though the solutions of these sets of linear simultaneous equations can be obtained by method such as relaxation technique, Gauss-Seidell iteration, Cramer's rule etc. In chapter 10, we have discussed the 'moment distribution method', given by Prof. Hardy Cross(1930). It should be noted that Hardy Cross method of moment distribution is a technique of solving the above mentioned equations *numerically*, without explicitly writing them, by using Gauss Seidell iteration. The quantities iterated in the 'moment distribution method' are the *increments* to the member end moments, instead of end-moment themselves.

We now introduce Kani's method, given by Dr. Gasper Kani(1947). The Kani's method is similar to the moment distribution method in that both these methods use Gauss-Seidell iteration procedure to solve the slope deflection equations, without explicitly writing them down. However the difference between the Kani's method and the moment distribution method is that Kani's method iterates the member end moments themselves rather than iterating their increments. Kani's method essentially consists of a single, simple numerical operation, *performed repeatedly* at the joints of a structure, in a chosen sequence.

Let us now develop the method for the following two cases: (*i*) Continuous beams and Frames without joint translation. (*ii*) Frames with sway.

28.2. CONTINUOUS BEAMS AND FRAMES WITHOUT JOINT TRANSLATION

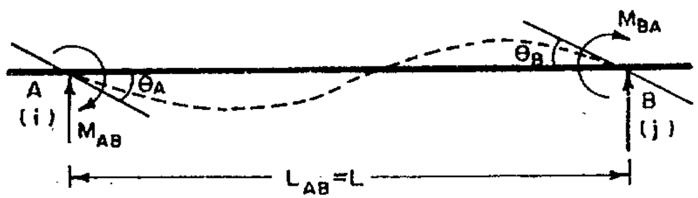


Fig. 28.1.

Fig. 28.1 shows the final deflected shape of a span AB of a continuous beam, under imposed loads. Let M_{AB} and M_{BA} be the moments developed, and θ_A and θ_B be the corresponding rotations at A and B, due to imposed loads. The slope deflection equation for span AB, at joint A is :

$$M_{AB} = M_{FAB} + 2EK_{AB}(2\theta_A + \theta_B) \quad ...(28.1)$$

where $$K_{AB} = \frac{I_{AB}}{L_{AB}}$$

$$\therefore \quad M_{AB} = M_{FAB} + 4\,EK_{AB}\,\theta_A + 2\,EK_{AB}\,.\,\theta_B$$

or $$M_{AB} = M_{FAB} + 2\,m_{AB} + m_{BA} \quad (28.2) \quad ..(I)$$

where, by definition, $m_{AB} = 2EK_{AB}\,.\,\theta_A$

$$\text{and } m_{BA} = 2EK_{AB}\,.\,\theta_B \quad ...(28.3)$$

Here, m_{AB} is called the *rotational contribution* of end A to M_{AB}, while m_{BA} is called the *rotational contribution* of end B to M_{AB}.

Now, in general, there may be many members meeting at end A, so that B is the common designation for *far ends* . For the equilibrium at the joint A, the algebraic sum of end moments of all the members meeting there must be zero.

Hence $$\sum_B M_{AB} = 0$$

Hence from Eq. 28.2, we obtain

$$\sum_B M_{FAB} + \sum_B (2\,m_{AB} + m_{BA}) = 0 \quad ...(28.4)$$

where the symbol $\sum_B$ represents the sum taken over all the adjacent joints B.

Now let M_{FA} be the resultant restraint moment at A, equal to the algebraic sum of all fixed end moments at A, given by

$$M_{FA} = \sum M_{FAB} \quad ...(28.5)$$

Hence we get from Eq. 28.4.

$$\sum m_{AB} = -\frac{1}{2}(M_{FA} + \sum m_{BA}) \quad ...(28.6)$$

In fact, Eq. 28.6 gives the algebraic sum of *rotational contribution* of all members meeting at A. Now, we know that m_{AB} for any member is proportional to its K value. The individual share of the members in the total rotational contribution can be found by distributing the total rotational contribution in proportion to their respective K-values.

Thus, for member AB,

$$m_{AB} = \frac{K_{AB}}{\sum_B K_{AB}} \sum_B m_{AB} \quad ...(28.7)$$

or $$m_{AB} = -\frac{1}{2}\left(\frac{K_{AB}}{\sum_B K_{AB}}\right)\left[M_{FA} + \sum_B m_{BA}\right]$$

or $$m_{AB} = R_{AB}\left(M_{FA} + \sum_B m_{BA}\right) \quad ...(28.8) \quad ...(II)$$

Where R_{AB} is known as *rotational factor* for AB given by

$$R_{AB} = -\frac{1}{2}\left(\frac{K_{AB}}{\sum_B K_{AB}}\right) \quad ...(28.9)$$

It is to be noted that the rotational factor R is equal to $-\frac{1}{2}$ times the distribution factors used in moment distribution.

Eqs (I) and (II) form the basis of Kani's method of solution wherein the support moment M_{AB} can be determined. Obviously, both m_{AB} and m_{BA} must be determined before M_{AB} can be found. This is accomplished by Gauss-Seidell iteration procedure outlined below. However, sometimes it is preferable to use symbol i for near end A and j for the far ends B. In that case, the various equations take the form listed below :

$$M_{ij} = M_{Fij} + 2\,E\,K_{ij}\,(2\,\theta_i + \theta_j) \quad ...(28.1)$$

where $K_{ij} = I_{ij} / L_{ij}$

$$M_{ij} = M_{Fij} + 2\,m_{ij} + m_{ji} \quad ..(I) \quad ...(28.2)$$

where $$m_{ij} = 2\,E\,K_{ij}\,\theta_i \text{ and } m_{ji} = 2\,E\,k_{ij}\,\theta_j \quad ...(28.3)$$

$$\sum_i M_{Fji} + \sum_i (2\,m_{ij} + m_{ji}) = 0 \quad ...(28.4)$$

$$M_{Fi} = \sum_j M_{Fij} \quad \text{...(28.5)}$$

$$\sum_j m_{ij} = -\frac{1}{2}\left(M_{Fi} + \sum_j m_{ji}\right) \quad \text{...(28.6)}$$

$$m_{ij} = R_{ij}\ (M_{Fi} + \sum_j m_{ji}) \quad \text{...(II) ..(28.8)}$$

where $$R_{ij} = -\frac{1}{2}\left(\frac{K_{ij}}{\sum_j K_{ij}}\right) \quad \text{...(28.9)}$$

Procedure for Kani's method

Step 1. Calculate fixed end moments (M_{Fij}) in all the members of the structure. Find the *resultant restraint moment* at each joint using Eq. 28.9. Enter these value of *resultant restraint moment* within the square or circle made at each joint (Fig. 28.2)

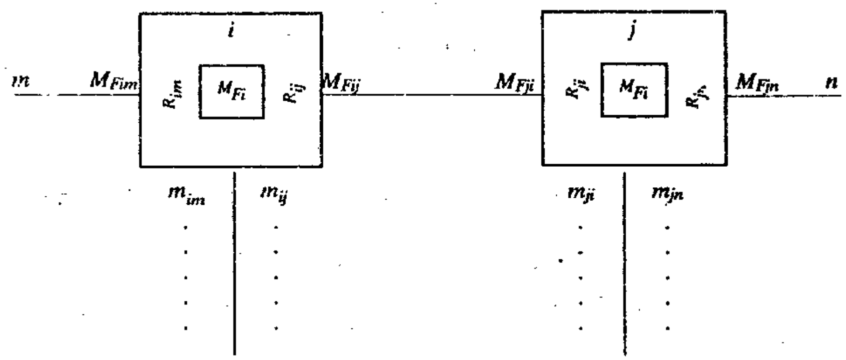


Fig. 28.2

Step 2 Calculate the K values and rotation factors R_{ij} for all members meeting at each joint. These values are entered outside the first square(or circle) but inside the second square(or circle), towards each member as shown in Fig. 28.2.

Step 3. Compute rotational contribution (m_{ij}) of the two ends of all the members by Gauss-Seidell iteration performed on Eq. 28.8(Eq.II), taking $m_{ij} = 0$ at all joints starting with the approximation that $\theta_i = 0$. Continue the iteration through several cycles till practically the same values of m_{ij} are obtained in two successive cycles. Each cycle gives improved approximation for the rotational contribution. All these values of m_{ij} are entered as shown in Fig. 28.2.

Step 4. Using Eq. 28.2(Eq.I), determine member end moments M_{ij}.

Example 28.1 illustrates the complete procedure.

Members with far ends hinged

If the end j of the member ij is hinged, then its *modified stiffness coefficient* is $3EK_{ij}$ as compared to the normal value of $4EK_{ij}$. Hence for calculating the factor R in Eq. 28.9, 0.75 times its actual K value should be used. Eq. 28.8 can then be used without any modification. The member end moment at the hinged end will be zero. However, half the initial F.E.M. at the joint j will have to be carried to joint i, with negative sign, and added to the F.E.M. of i, to get the modified value of F.E.M. at i. Thus, if M_{Fij} and M_{Fji} are the F.E.M's at i and j, computed by taking j to be fixed, then the modified fixed end moment at i will be as follows:

$$\text{Modified } M_{Fij} = M_{Fij} + \frac{1}{2}(-M_{Fji})$$

This modified fixed end moment at joint i becomes the starting F.E.M. at i before beginning the Kani's cycles.

Member with far end fixed : It should be noted that no iteration is performed at the fixed end. The rotational components of fixed joints at far ends are zero.

Example 28.1. *A continuous beam ABCD consists of three spans, and is loaded as shown in Fig. 28.3(a). Ends A and D are fixed. Determine the bending moments at the supports, using Kani's method. Also, plot the bending moment diagram and the deflected shape of the beam.*

Solution.

Step 1. Computation of fixed end moments **(kN-m *units*)**

$$M_{FAB} = -\frac{2\times 6^2}{12} = -6.0\,; \qquad M_{FBA} = +\frac{2\times 6^2}{12} = +6$$

$$M_{FBC} = -\frac{5\times 3\times 2^2}{5^2} = -2.4\,; \quad M_{FCB} = +\frac{5\times 2\times 3^2}{5^2} = +3.6$$

$$M_{FCD} = -\frac{8\times 5}{8} = -5.0\,; \qquad M_{FDC} = +\frac{8\times 5}{8} = +5.0$$

Step 2. Rotation factors

As pointed out earlier, rotation factor is equal to -0.5 times the distribution factor used in moment distribution. The relative stiffness, distribution factors and rotation factors are calculated in Table 28.1.

TABLE 28.1.

Joint	*Member*	*Relative Stiffness*	*Sum*	*Distribution factor* D_F	*Rotation factor* $R = -0.5 \times D_F$
B	*BA*	$I/6$	$17I/30$	5/17	− 0.147
	BC	$2I/5$		12/17	− 0.353
C	*CB*	$2I/5$	$3I/5$	2/3	− 0.333
	CD	$I/5$		1/3	− 0.167

It is to be noted that the sum of rotation factors at a joint is equal to − 0.5.

Thus $R_{BA} + R_{BC} = -(0.147 + 0.353) = -0.5$

Step 3. Resultant restraint moments

Compute resultant restraint moment at each joint by Eq. 28.5:

$$M_{Fi} = \sum_j M_{Fij}$$

Thus $M_{FB} = M_{FBA} + M_{FBC} = +6 - 2.4 = +3.6$ kN−m

$$M_{FC} = M_{FCB} + M_{FCD} = +3.6 - 5 = -1.4 \text{ kN−m}$$

Enter these values within the small square [(Fig. 28.3 (d)]

Step 4. Kani's Iteration cycles

Cycle 1 : Kani's iteration procedure can now be commenced, assuming all rotational components(m_{ij}) to be zero at all joints which will indirectly mean that $\theta_i = 0$. Note that m_{AB} and m_{DC} are permanently zero since ends *A* and *D* are fixed.

Applying Eq. 28.8 at joint *C* and assuming $m_{BC} = 0$, we get

$$m_{CB} = R_{CB}(M_{FC}) = -0.333(-1.4) = +0.466$$

and $m_{CD} = R_{CB}(M_{FC}) = -0.167(-1.4) = +0.234$

These values are now used in Eq. 28.8 for computing rotational components (m_{ij}) at joint *B*. Thus

$$m_{BC} = R_{BC}(M_{FB} + m_{CB}) = -0.353(+3.6+0.466) = -1.435$$

$$m_{BA} = R_{BA}(M_{FB} + m_{CB}) = -0.147(+3.6+0.466) = -0.598$$

Cycle 2 The values of the four rotational components found in cycle 1 will now be used to get better approximations for the rotational components at joint *C*. Applying Eq. 28.8 at joint *C*,

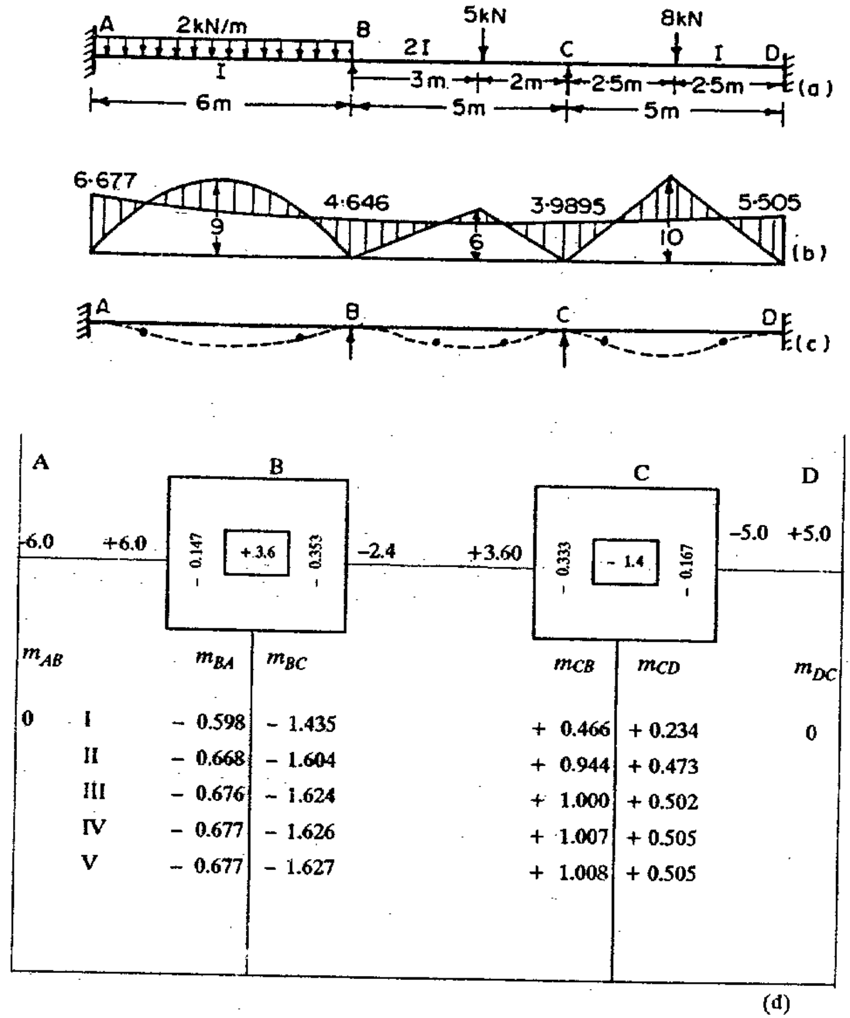


Fig. 28.3.

$$m_{CB} = R_{CB}(M_{FC} + m_{BC}) = -0.333(-1.4 - 1.435) = +0.944$$

$$m_{CD} = R_{CD}(M_{FC} + m_{BC}) = -0.167(-1.4 - 1.435) = +0.473$$

Similarly, applying Eq. 28.8 at joint *B*,

$$m_{BC} = R_{BC}(M_{FB} + m_{CB}) = -0.353(+3.6 + 0.944) = -1.604$$

$$m_{BA} = R_{BA}(M_{FB} + m_{CB}) = -0.147(+3.6 + 0.944) = -0.668$$

Cycle 3. Applying Eq. 28.8 at joint *C*,

$$m_{CB} = -0.333(-1.4 - 1.604) = +1.000$$

$$m_{CD} = -0.167(-1.4 - 1.604) = +0.502$$

At joint B, $m_{BC} = -0.353(+3.6 + 1.000) = -1.624$

$$m_{BA} = -0.147(+3.6 + 1.000) = -0.676$$

Cycle 4.

At C, $m_{CB} = -0.333(-1.4 - 1.624) = +1.007$

$$m_{CD} = -0.167(-1.4 - 1.624) = +0.505$$

At B, $m_{BC} = -0.353(+3.6 + 1.007) = -1.626$

$$m_{BA} = -0.147(+3.6 + 1.007) = -0.677$$

Cycle 5.

At, C, $m_{CB} = -0.333(-1.4 - 1.626) = +1.008$

$$m_{CD} = -0.167(-1.4 - 1.626) = +0.505$$

At B, $m_{BC} = -0.353(+3.6 + 1.008) = -1.627$

$$m_{BA} = -0.147(+3.6 + 1.008) = -0.677$$

The iteration is terminated at the end of 5th cycle as there is no change in the values of m_{ij} as compared to the corresponding values of 4th cycle.

Hence the final values of the rotational components are

$m_{BA} = -0.677$; $m_{BC} = -1.627$; $m_{CB} = +1.008$ and $m_{CD} = +0.505$

It should be clearly noted that we have actually solved the *displacement equations* in the above iteration, since from Eq. 28.3, we observe that

$$\theta_B = \frac{m_{BA}}{2EK_{BA}} = \frac{-0.677}{2E(I/6)} = -\frac{2.031}{EI}$$

and $$\theta_C = \frac{m_{CB}}{2EK_{CB}} = \frac{1.008}{2E(2I/5)} = +\frac{1.26}{EI}$$

Note that we obtained both these values of θ_B and θ_C by slope deflection method in Example 9.3 for the present beam.

Thus it is concluded that Kani's method indirectly solves the displacement equations. *This makes the solution self correcting.*

Step 5. Computation of final moments at joint

The final moments(M_{ij}) are computed from Eq. 28.2. The computations have been arranged in Table 28.2.

TABLE 28.2. Final moments

M_{ij}	M_{Fij}	$2m_{ij}$	m_{ji}	Sum (kn-m)
M_{AB}	− 6.0	0	− 0.677	− 6.677
M_{BA}	+ 6.0	− 1.354	0	+ 4.646
M_{BC}	− 2.4	− 3.254	+ 1.008	− 4.646
M_{CB}	+ 3.6	+ 2.016	− 1.627	+ 3.989
M_{CD}	− 5.0	+ 1.010	0	− 3.990
M_{DC}	+ 5.0	0	+ 0.505	+ 5.505

It will be noted that these values are practically the same as obtained by the slope deflection method. The B.M.D. and deflected shape of the beam are shown in Fig. 28.3(b) and (c) respectively.

Example 28.2 *Solve example 28.1 if the ends A and D are hinged (or simply supported).*

Step 1. Computation of fixed end moments

Considering ends of A and D as fixed, the fixed end moments at various joints will be as under(as calculated in the previous example).

$$M_{FAB} = -6.0\ ; \quad M_{FBA} = +6.0$$

$$M_{FBC} = -2.4\ ; \quad M_{FCB} = +3.6$$

$$M_{FCD} = -5.0\ ; \quad M_{FDC} = +5.0$$

Actually, ends A and D are free. Hence releasing ends A and D, the *modified moments* at A and D will be

$$m_{FBA} = +6.0 + \frac{1}{2}(+6.0) = +9.0 \quad \text{kN-m.}$$

$$m_{FCD} = -5.0 + \frac{1}{2}(-5.0) = -7.5 \quad \text{kN-m.}$$

Step 2. Rotation factors

Since end A is hinged, the stiffness of AB will be 3/4 times its actual K. Similarly, since end D is hinged, the stiffness of DC will be 3/4 times its actual K. Based on this, the relative stiffness, distribution factors and rotation factors are calculated in Table 28.3.

TABLE 28.3

Joint	Member	Relative Stiffness	Sum	Distribution factor D_F	Rotation factor $R = -0.5(D_F)$
B	BA	$\frac{3}{4}\cdot\frac{I}{6}$	$\frac{63\,I}{120}$	$\frac{15}{63}=\frac{5}{21}$	− 0.119
	BC	$\frac{2I}{5}$		$\frac{48}{63}=\frac{16}{21}$	− 0.381
C	CB	$\frac{2I}{5}$	$\frac{11\,I}{20}$	$\frac{8}{11}$	− 0.364
	CD	$\frac{3}{4}\cdot\frac{I}{5}$		$\frac{3}{11}$	− 0.136

Step 3. Resultant restraint moment

Compute resultant restraint moment at each joint by Eq. 28.5

$$M_{Fi} = \sum_j M_{Fij}$$

$\therefore$ $M_{FB} = M_{FBA} + M_{FBC} = 9.0 - 2.4 = +6.6$ KN-m.

$M_{FC} = M_{FCB} + M_{FCD} = 3.6 - 7.5 = -3.9$ kN-m.

Enter these values within the small square (Fig. 28.4)

Step 4. Kani's Iteration cycles

Cycle 1 : Kani's iteration procedure can now be commenced assuming all rotational components (m_{ij}) to be zero at all joints, which will indirectly mean that $\theta_i = 0$. Note that m_{AB} and m_{DC} are zero.

Applying 28.8 at joint C and assuming $m_{BC} = 0$ we get

$$m_{CB} = R_{CB}\,(M_{FC}) = -0.364\,(-3.9) = +1.420$$

$$m_{CD} = R_{CD}\,(M_{FC}) = -0.136\,(-3.9) = +0.530$$

These values are now used in Eq. 28.8 for computing rotational components (m_{ij}) at joint B. Thus

$$m_{BC} = R_{BC}\,(M_{FB} + m_{CB}) = -0.381\,(6.6 + 1.420) = -3.056$$

$$m_{BA} = R_{BA}\,(M_{FB} + m_{CB}) = -0.119(6.6 + 1.420) = -0.954$$

Cycle 2

At C, $m_{CB} = -0.364\,(-3.9 - 3.056) = +2.532$

$m_{CD} = -0.136\,(-3.9 - 3.056) = +0.946$

At B, $m_{BC} = -0.381\,(+6.6 + 2.532) = -3.479$

$m_{BA} = -0.119(+6.6 + 2.532) = -1.087$

Cycle 3

At C, $m_{CB} = -0.364\,(-3.9 - 3.479) = +2.686$

$m_{CD} = -0.136\,(-3.9 - 3.479) = +1.004$

At B, $m_{BC} = -0.381\,(+6.6 + 2.686) = -3.538$

$m_{BA} = -0.119\,(+6.6 + 2.686) = -1.105$

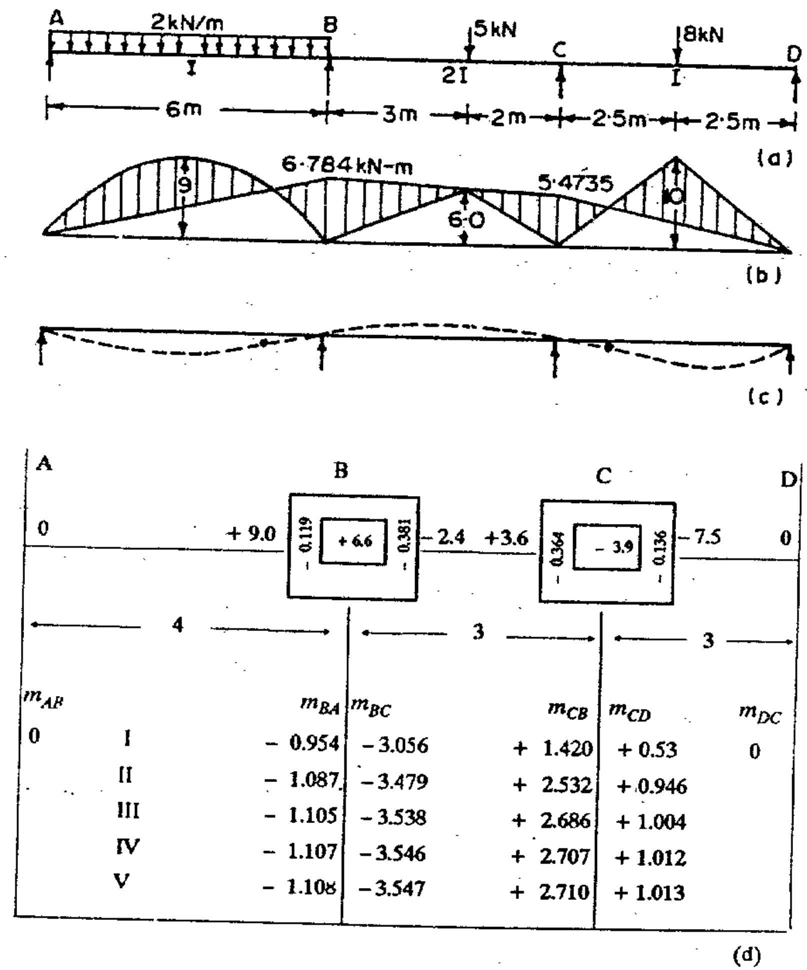


Fig. 28.4

Cycle 4

At C, $m_{CB} = -0.364\,(-3.9 - 3.538) = +2.707$

$m_{CD} = -0.136\,(-3.9 - 3.538) = +1.012$

At B, $m_{BC} = -0.381\,(+6.6 + 2.707) = -3.546$

$m_{BA} = -0.119\,(+6.6 + 2.707) = -1.107$

Cycle 5

At C, $m_{CB} = -0.364\,(-3.9 - 3.546) = +2.710$

$m_{CD} = -0.136\,(-3.9 - 3.546) = +1.013$

At B, $m_{BC} = -0.381\,(+6.6 + 2.710) = -3.547$

$m_{BA} = -0.119\,(+6.6 + 2.710) = -1.108$

The iteration may be terminated at the end of 5th cycle, as there is very little difference between the values of m_{ij} of 5th cycle and those of 4th cycle.

Step 5. Computation of final moments

Using Eq.28.2, the final moments at various joints can be computed, as shown in Table 28.4.

TABLE 28.4
Final moments

M_{ij}	M_{Fij}	$2m_{ij}$	m_{ji}	*Sum* (kN-m)
M_{BA}	+ 9.0	− 2.216	0	+ 6.784
M_{BC}	− 2.4	− 7.094	+ 2.710	− 6.784
M_{CB}	+ 3.6	+ 5.420	− 3.547	+ 5.473
M_{CD}	− 7.5	+ 2.026	0	− 5.474

The B.M.D. and the deflected shape of the beam are shown in Fig. 28.4(b) and (c) respectively.

Example 28.3 Solve example 28.1 if there is no support at end D.

Solution

Step 1 : Fixed end moments

Take Ends B and C as clamped(fixed) so that AB and BC are considered as fixed beam. The over hanging portion CD becomes a cantilever fixed at C. Hence :

$$M_{FAB} = -6.00 \text{ kN-m}\,;\quad M_{FBA} = +6.00 \text{ kN-m}$$

$$M_{FBC} = -2.40 \text{ kN-m}\,;\quad M_{FCB} = +3.6 \text{ kN-m}$$

$$M_{FCD} = -8 \times 2.5 = -20 \text{ kN-m}$$

Step 2. Rotation Factors

The stiffness of cantilever CD is zero since it has no resistance to rotation if an external moment is applied at the freely supported end C.

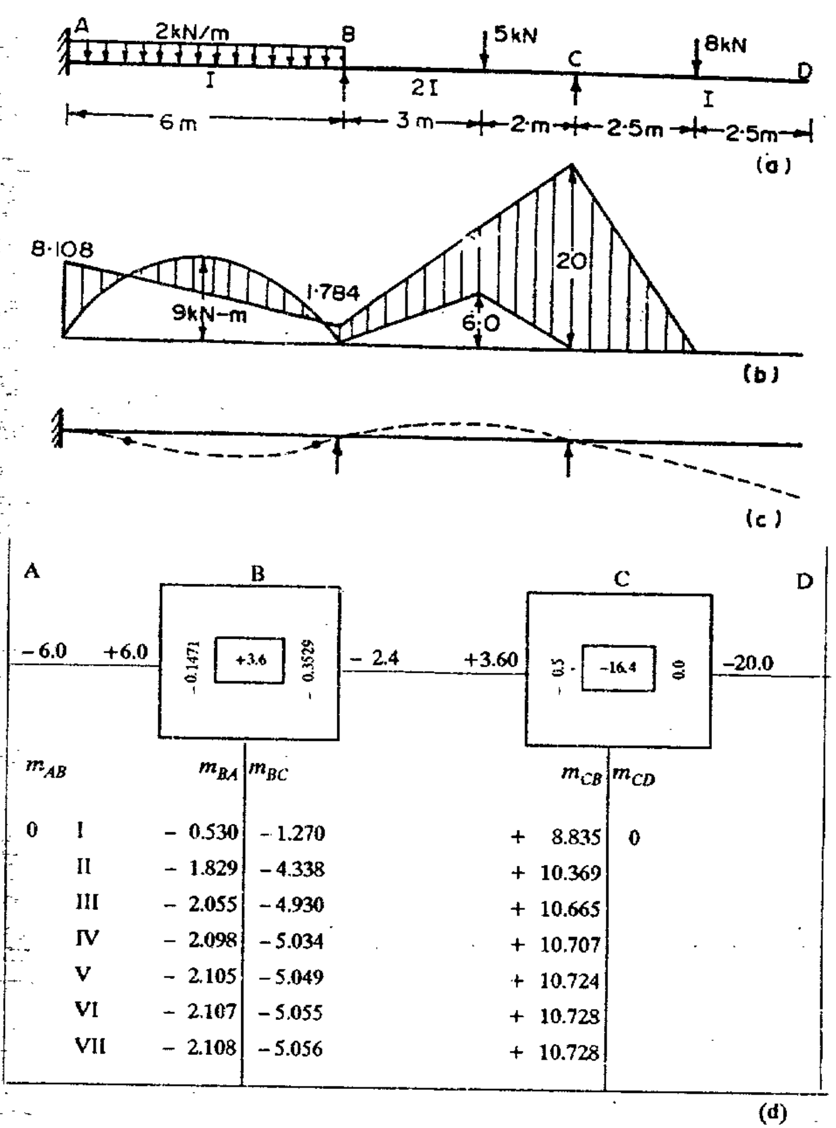


Fig. 28.5

The rotation factors are computed in Table 28.5 :

TABLE 28.5

Joint	Member	Relative stiffness	Sum	Distribution factor (D_F)	Rotation factor $R = -0.5\ D_F$
B	BA	$\frac{I}{6}$	$\frac{17I}{30}$	$\frac{5}{17}$	− 0.1471
	BC	$\frac{2I}{5}$		$\frac{12}{17}$	− 0.3529
C	CB	$\frac{2I}{5}$	$\frac{2I}{5}$	1	− 0.5
	CD	0		0	0.0

Step 3. Resultant Restraint moment

$$M_{FB} = +6.0 - 2.40 = +3.6 \text{ kN-m}$$

$$M_{FC} = +3.6 - 20 = -16.4 \text{ kN-m}$$

Step 4. Kani's Iteration Cycles

Cycle 1

The iteration cycles are to be performed at joints B and C only since $m_{AB} = 0$ (fixed end) and $m_{CD} = 0$ (fixed end of cantilever). To start with, let us assume $m_{CB} = 0$

$\therefore$ At joint B :

$$m_{BC} = R_{BC}(M_{FB} + m_{CB}) = -0.3529\,(+3.6 + 0) = -1.270$$

$$m_{BA} = R_{BA}(M_{FB} + m_{CB}) = -0.1471\,(+3.6 + 0) = -0.530$$

At joint C,

$$m_{CB} = R_{CB}(M_{FC} + m_{BC}) = -0.5\,(-16.4 - 1.270) = +8.835$$

Cycle 2

At joint B, $m_{BC} = -0.3529\,(+3.6 + 8.835) = -4.388$

$m_{BA} = -0.1471\,(+3.6 + 8.835) = -1.829$

At joint C, $m_{CB} = -0.5\,(-16.4 - 4.338) = +10.369$

Cycle 3.

At joint B, $m_{BC} = -0.3529\,(+3.6 + 10.369) = -4.930$

$m_{BA} = -0.1471\,(+3.6 + 10.369) = -2.055$

At joint C, $m_{CB} = -0.5\,(-16.4 - 4.930) = +10.665$

Cycle 4.

At joint B, $m_{BC} = -0.3529\,(+3.6 + 10.665) = -5.034$

$m_{BA} = -0.1471\,(+3.6 + 10.665) = -2.098$

At joint C, $m_{CB} = -0.5\,(-16.4 - 5.034) = +10.707$

Cycle 5.

At joint B, $m_{BC} = -0.3529\,(+3.6 + 10.707) = -5.049$

$m_{BA} = -0.1471\,(+3.6 + 10.707) = -2.105$

At joint C, $m_{CB} = -0.5\,(-16.4 - 5.049) = +10.724$

Cycle 6.

At joint B, $m_{BC} = -0.3529\,(+3.6 + 10.724) = -5.055$

$m_{BA} = -0.1471\,(+3.6 + 10.724) = -2.107$

At joint C, $m_{CB} = -0.5\,(-16.4 - 5.055) = +10.728$

Cycle 7.

At joint B, $m_{BC} = -0.3529\,(+3.6 + 10.728) = -5.056$

$m_{BA} = -0.1471\,(+3.6 + 10.728) = -2.108$

At joint C, $m_{CB} = -0.5\,(-16.4 - 5.056) = +10.728$

The iteration is terminated at the end of 7th cycle.

Step 5. Computation of Final moments (Table 28.6)

TABLE 28.6

M_{ij}	M_{Fij}	$2m_{ij}$	m_{ji}	Sum.
M_{AB}	− 6.0	0	− 2.108	− 8.108
M_{BA}	+ 6.0	− 4.216	0	+ 1.784
M_{BC}	− 2.4	− 10.112	+ 10.728	− 1.784
M_{CB}	+ 3.6	+ 21.456	− 5.056	+ 20.000
M_{CD}	− 20.0	0	0	− 20.000

The B.M.D. and the deflected shape of the beam are shown in Fig. 28.5(b) and (c) respectively.

Example 28.4 *Analyse the portal frame shown in Fig. 28.6 by Kani's method. Draw the B.M.D. and sketch the deflected shape of the frame. Take EI constant for all the members.*

Solution :

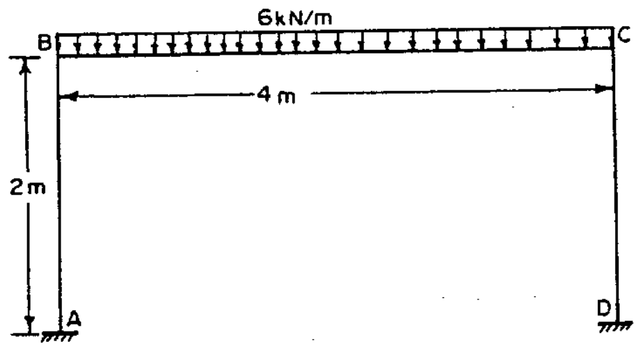


Fig. 28.6

The frame is symmetrical, and is symmetrically loaded. Hence it will not sway.

Step 1. Fixed End moments

$$M_{FBC} = -\frac{6\times 4^2}{12} = -8 \text{ kN-m}$$

$$M_{FCB} = +\frac{6\times 4^2}{12} = +8 \text{ kN-m}$$

$$M_{FAB} = 0\,; \quad M_{FDC} = 0.$$

Step 2. Rotation Factors

(TABLE 28.7)

Joint	Member	Relative Stiffness	Sum	D_F	$R = -0.5 \times D_F$
B (or C)	BA (or CD)	$\frac{I}{2} = \frac{2I}{4}$	$\frac{3I}{4}$	$\frac{2}{3}$	-0.3333
	BC (or CB)	$\frac{I}{4}$		$\frac{1}{3}$	-0.1667

Step 3. Resultant Restraint Moments

$$M_{FB} = -8 \quad ; \quad M_{FC} = +8.$$

Step 4. Kani's Iteration cycles.

In the case of frames there are horizontal members as well as vertical members. For horizontal members (*i.e.* beams) the rotational components (m_{ij}) are written below the beams, while for vertical members (*i.e.* columns) the rotational components are written along the columns, as illustrated in Fig. 28.7. Ends A and D are fixed. Hence $m_{AB} = 0$ and $m_{DC} = 0$. We will do the computations upto the accuracy of second decimal place.

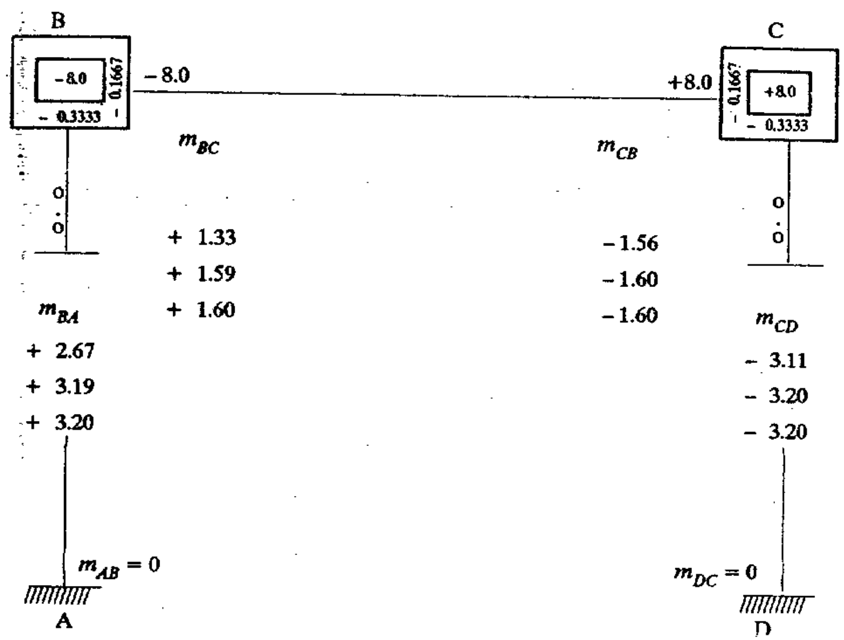


Fig. 28.7.

Cycle 1 : To start with, let $m_{CB} = 0$. Hence at joint B,

$$m_{BC} = R_{BC}(M_{FB} + m_{CB} + m_{AB}) = -0.1667(-8.0 + 0 + 0) = +1.33$$

$$m_{BA} = R_{BA}(M_{FB} + m_{CD} + m_{AB}) = -0.3333(-8.0 + 0 + 0) = +2.67$$

Hence at C,

$$m_{CB} = R_{CB}(M_{FC} + m_{BC} + m_{DC}) = -0.1667(-8.0 + 1.33 + 0) = -1.56$$

$$m_{CD} = R_{CD}(M_{FC} + m_{BC} + m_{DC}) = -0.3333(-8.0 + 1.33 + 0) = -3.11$$

Cycle 2

At joint B, $m_{BC} = -0.1667\,(-8.0 - 1.56) = +1.59$

$m_{BA} = -0.3333\,(-8.0 - 1.56) = +3.19$

At joint C, $m_{CB} = -0.1667\,(+8.0 + 1.59) = -1.60$

$m_{CD} = -0.3333\,(+8.0 + 1.59) = -3.20$

Cycle 3

At joint B, $m_{BC} = -0.1667\,(-8.0 - 1.60) = +1.60$

$m_{BA} = -0.3333\,(-8.0 - 1.60) = +3.20$

At joint C $m_{CB} = -0.1667\,(+8.0 + 1.60) = -1.60$

$m_{CD} = -0.3333\,(+8.0 + 1.60) = -3.20$

The iteration can now be terminated at the end of 3rd cycle.

Step 5. Computation of final moments (Table 28.8)

TABLE 28.8.

M_{ij}	M_{Fij}	$2m_{ij}$	m_{ji}	Sum.
AB	0	0	+ 3.20	+ 3.20
BA	0	+ 6.40	0	+ 6.40
BC	− 8.0	+ 3.20	− 1.60	− 6.40
CB	+ 8.0	− 3.20	+ 1.60	+ 6.40
CD	0	− 6.40	0	− 6.40
DC	0	0	− 3.20	− 3.20

The final B.M.D. and deflected shape are shown in Fig. 28.8(a) and (b) respectively.

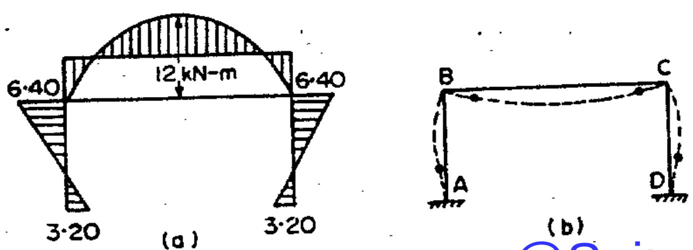


Fig. 28.8.

Example 28.5. *A continuous beam shown in Fig. 28.9 has rigidly fixed ends C and D, is pinned at E and has rigid joints at A and B. The members are of uniform section and material throughout. Sketch the bending moment diagram for the frame, showing all important values. Also, find the values of the horizontal and vertical reactions at D and E. Use Kani's method.*

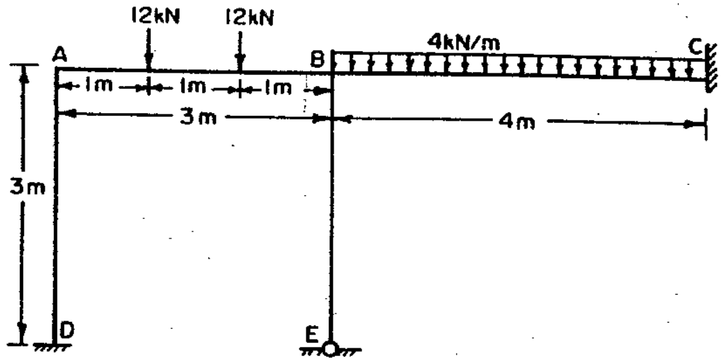


Fig. 28.9.

Solution

Step 1. Fixed End moments

$$M_{FAB} = -\frac{12 \times 1 \times 2^2}{3^2} - \frac{12 \times 2 \times 1^2}{3^2} = -8 \text{ kN-m}$$

$$M_{FBA} = +\frac{12 \times 2 \times 1^2}{3^2} + \frac{12 \times 1 \times 2^2}{3^2} = +8 \text{ kN-m}$$

$$M_{FBC} = -\frac{4 \times 4^2}{12} = -5.33 \text{ kN-m}$$

$$M_{FCB} = +\frac{4 \times 4^2}{12} = +5.33 \text{ kN-m}$$

Step 2. Rotation Factors.

TABLE 28.9

Joint	*Member*	*Relative stiffness*	*Sum*	D_F	*Rotation Factor* $R = -0.5\ D_F$
A	AD	$\frac{I}{3}$	$\frac{2I}{3}$	0.5	− 0.25
	AB	$\frac{I}{3}$		0.5	− 0.25
B	BA	$\frac{I}{3}$	$\frac{10I}{12}$	0.4	− 0.2
	BE	$\frac{3}{4}\cdot\frac{I}{3}=\frac{I}{4}$		0.3	− 0.15
	BC	$\frac{I}{4}$		0.3	− 0.15

Step 3. Resultant Restraint moments

$$M_{FA} = -8 \text{ kN-m}$$

$$M_{FB} = +8 - 5.33 = +2.67 \text{ kN-m.}$$

$$M_{FC} = +5.33$$

Step 4. Kani's Iteration cycles.

Ends D and E are fixed. Hence $m_{DA} = 0$ and $m_{CB} = 0$

Similarly, $m_{EB} = 0$

Cycle I : Let us assume that at all joints $m_{ij} = 0$ in the beginning.

At joint A, $m_{AB} = R_{AB}(M_{FA} + m_{DA} + m_{BA})$

$$= -0.25(-8 + 0 + 0) = +2.0$$

$$m_{AD} = R_{AB}(M_{FA} + m_{DA} + m_{BA})$$

$$= -0.25(-8 + 0 + 0) = +2.0$$

At Joint B, $m_{BA} = R_{BA}(M_{FB} + m_{AB} + m_{CB} + m_{EB})$

$$= -0.20(+2.67 + 2.0 + 0 + 0) = -0.93$$

$$m_{BE} = R_{BE}(M_{FB} + m_{AB} + m_{CB} + m_{EB})$$

$$= -0.15(+2.67 + 2.0 + 0 + 0) = -0.70$$

$$m_{BC} = R_{BC}(M_{FB} + m_{AB} + m_{CB} + m_{EB})$$

$$= -0.15(+2.67 + 2.0 + 0 + 0) = -0.70$$

Cycle II

At A, $m_{AB} = -0.25(-8 - 0.93) = +2.23$

$$m_{AD} = -0.25(-8 + 0.93) = +2.23$$

At B, $m_{BA} = -0.20(+2.67 + 2.23) = -0.98$

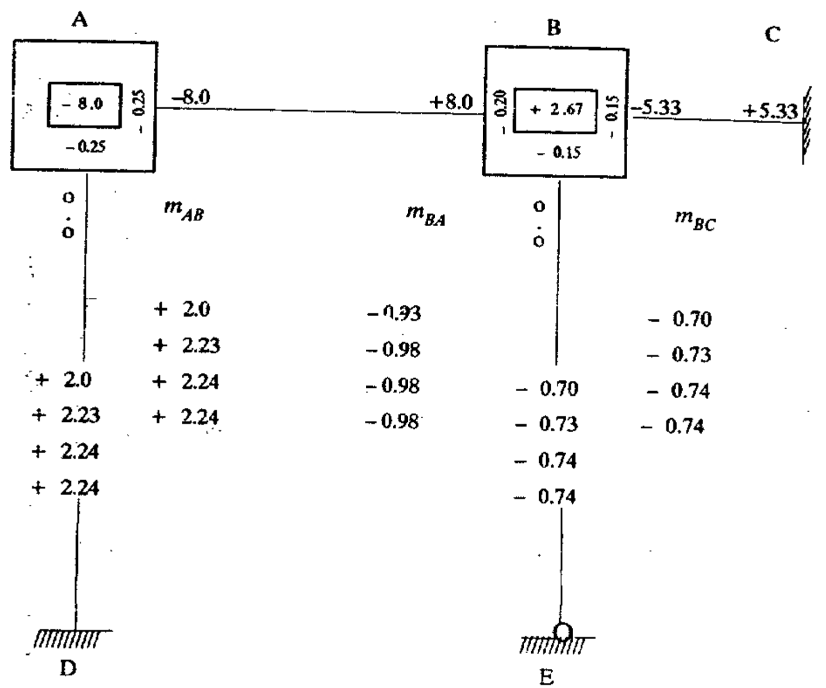


Fig. 28.10

$$m_{BE} = -0.15(+2.67 + 2.23) = -0.73$$

$$m_{BC} = -0.15(+2.67 + 2.23) = -0.73$$

Cycle III

At A, $m_{AB} = -0.25(-8 - 0.98) = +2.24$

$$m_{AD} = -0.25(-8 - 0.98) = +2.24$$

At B, $m_{BA} = -0.20(+2.67 + 2.24) = -0.98$

$$m_{BE} = -0.15(+2.67 + 2.24) = -0.74$$

$$m_{BC} = -0.15(+2.67 + 2.24) = -0.74$$

Cycle IV

At A, $m_{AB} = -0.25(-8 - 0.98) = +2.24$

$$m_{AD} = -0.25(-8 - 0.98) = +2.24$$

At B, $m_{BA} = -0.20\,(+2.67 + 2.24) = -0.98$

$m_{BE} = -0.15\,(+2.67 + 2.24) = -0.74$

$m_{BC} = -0.15\,(+2.67 + 2.24) = -0.74$

Step 5. Computation of final moments (Table 28.10)

TABLE 28.10

M_{ij}	M_{Fij}	$2\,m_{ij}$	m_{ji}	Sum
DA	0	0	+ 2.24	+ 2.24
AD	0	+ 4.48	0	+ 4.48
AB	− 8.0	+ 4.48	− 0.98	− 4.5
BA	+ 8.0	− 1.96	+ 2.24	+ 8.28
BC	− 5.33	− 1.48	0	− 6.81
BE	0	− 1.48	0	− 1.48
CB	+ 5.33	0	− 0.74	+ 4.59

The B.M.D. for the frame is shown in Fig. 28.11

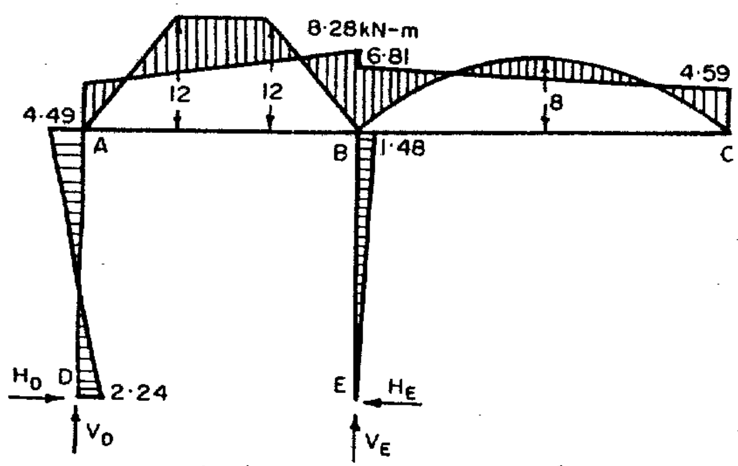


Fig. 28.11

Step 6. Computation of reactions

Considering the equilibrium of AD and taking moments about A,

$$H_D = \frac{M_{DA} + M_{AD}}{3} = \frac{2.24 + 4.49}{3} = 2.243 \text{ kN} \rightarrow$$

Similarly, taking moments about B, of all forces below B, we get

$$-1.48 + 3H_E = 0$$

$$\therefore \quad H_E = \frac{1.48}{3} = 0.493 \text{ kN} \leftarrow$$

Taking moments about B, of all forces to the right to B,

$$-6.81 + 4.59 - 4V_c + 4 \times 4 \times 2 = 0$$

$$\therefore \quad V_c = 7.42 \text{ kN} \uparrow$$

Taking moments about B, of all forces to the left to B,

$$8.28 + 2.24 + 3V_D - (3 \times 2.24) - (12 \times 1) - (12 \times 2) = 0$$

$$\therefore \quad V_D = 10.73 \text{ kN} \uparrow$$

Considering the vertical equilibrium of the whole frame :

$$V_E + 7.42 + 10.73 - 12 - 12 - (4 \times 4) = 0$$

$$V_E = 21.85 \text{ kN} \uparrow$$

28.3. SYMMETRICAL FRAMES

Symmetrical frames are those which are not only symmetrical about a vertical line through the mid-points of central beams, but have *symmetric load system*. Such a frame can be easily analysed by considering only half the frame and by taking modified K-values for half the symmetric beams. This is illustrated below.

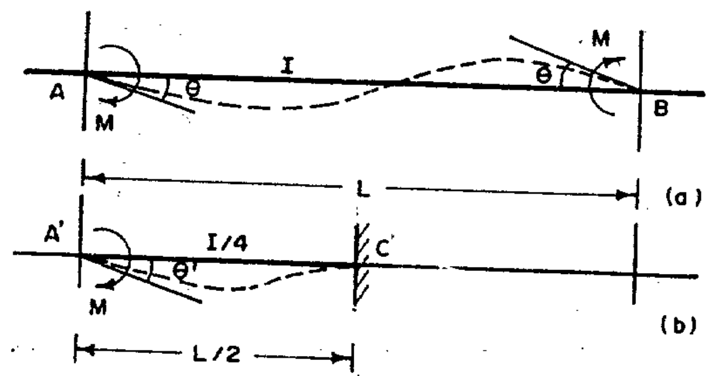


Fig. 28.12

Fig. 28.12(a) shows a prismatic beam of span L and sectional moment of inertia I, subjected to equal symmetric moments M at the ends A and B. For such a beam, we have

$$\theta = \frac{ML}{2EI}$$

Again, Fig. 28.12(*b*) shows another beam, $A'C'$ of span $L/2$ and sectional moment of inertia I/4, such that $K_{A'C'} = \frac{1}{2} K_{AB}$

The beam is fixed at C' and free to rotate at A'. *If a moment* M is applied at A', we have

$$\theta' = \frac{M(L/2)}{4E(I/4)} = \frac{ML}{EI}$$

$$\therefore \qquad \theta = \theta'$$

Hence for those frames which are symmetric about mid-points of the horizontal beams, the following procedure may be adopted.

(*i*) Find fixed end moments at various joints, due to imposed loading.

(*ii*) Replace the horizontal beams, about which the frame is symmetrical by *fictitious beams* which are of half the length and are fixed at the ends. For such fictitious beams, the relative stiffness is taken half the value of the actual beam.

(*iii*) Carry out the computations for half the frame only.

As an illustration, let us re-analyse the portal frame of example 28.4. The frame is symetrical about mid-point F of the beam BC. Hence replace BC by half the beam BF, fixed at F, but having relative stiffness equal to. $\frac{1}{2}\left(\frac{I}{4}\right) = \frac{I}{8}$. However, fixed end moments will remain the same as found earlier

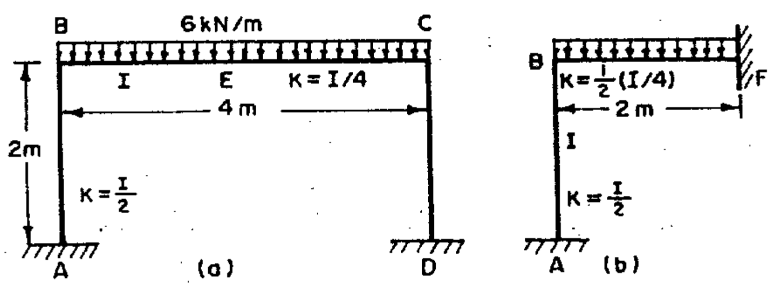


Fig. 28.13.

Hence the distribution factors are as follows :

$$R_{BC} = -0.5\left[\frac{I/8}{\frac{I}{8}+\frac{I}{2}}\right] = -0.1$$

$$R_{BA} = -0.5\left[\frac{I/2}{\frac{I}{8}+\frac{I}{2}}\right] = -0.4$$

$$M_{FBC} = -\frac{6\times 4^2}{12} = -8 \text{ kN-m as before.}$$

$$\therefore \qquad M_{FB} = -8 \text{ kN-m}$$

Also, $m_{EB} = 0$ and $m_{AB} = 0$ since A and F are fixed.

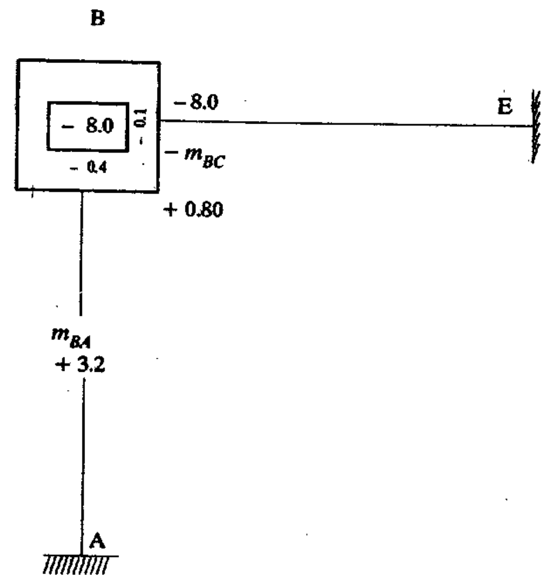


Fig. 28.14.

Now :, $m_{BC} = R_{BE}(M_{FB} + m_{EB} + m_{AB}) = -0.1(-8.0 + 0 + 0) = +0.8$

$m_{BA} = R_{BA}(M_{FB} + m_{EB} + m_{AB}) = -0.4(-8.0 + 0 + 0) = +3.2$

No further cycles are necessary since m_{EB} and m_{AB} are zero throughout.

$$\therefore \quad M_{BE} = M_{FBE} + 2m_{BE} + m_{EB} = -8 + 1.6 = -6.40 \text{ kN-m}$$

$$M_{BA} = M_{FBE} + 2m_{BA} + m_{AB} = 0 + (2\times 3.2) + 0 = +6.40 \text{ kN-m}$$

and $M_{AB} = M_{FAB} + 2m_{AB} + m_{BA} = 0 + 0 + 3.2 = +3.2$ kN-m.

We observe that though all the three values are the same as found earlier in Example 28.4, the labour has very much been reduced. By symmetry,

$M_{CB} = M_{BC} = -6.40$; $M_{CD} = M_{BA} = +6.4$; $M_{DC} = M_{AB} = +3.2$ kN-m.

28.4. FRAMES WITH SIDE SWAY

Frames may sway due to one of the following reasons :

(*i*) Eccentric or unsymmetrical loading on the portal frame.

(*ii*) Unsymmetrical out-line of portal frame.

(*iii*) Different end conditions of the columns of the portal frame.

(*iv*) Non-uniform section of the members of the frame.

(*v*) Horizontal loading on the columns of the frame.

(*vi*) Settlement of the supports of the frame.

(*vii*) A combination of the above.

Such frames undergo both *joint rotation* as well as *joint displacement* .

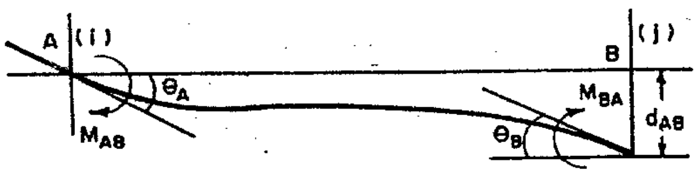


Fig. 28.15.

The slope deflection equations for member AB is

$$M_{AB} = M_{FAB} + \frac{2EI}{L_{AB}}\left(2\theta_A + \theta_B - \frac{3d_{AB}}{L_{AB}}\right)$$

or $$M_{AB} = M_{FAB} + 2EK_{AB}\left(2\theta_A + \theta_B - \frac{3d_{AB}}{L_{AB}}\right)$$

or $$M_{AB} = M_{FAB} + 2m_{AB} + m_{BA} + m'_{AB} \qquad ...(28.10)$$

where $m_{AB} = 2EK_{AB} . \theta_A$

$m_{BA} = 2EK_{BA} . \theta_B$

and $$m'_{AB} = -\frac{6EK_{AB}d_{AB}}{L_{AB}} \qquad ....(28.11)$$

Here m_{AB} and m_{BA}, are rotation contribution of ends A and B to the total moments M_{AB} while m'_{AB}, known as *displacement contribution*, constitute the contribution to M_{AB} by the lateral displacement d_{AB}.

For generalisation, replacing *A* by *i* and *B* by *j*, we have

$$M_{ij} = M_{Fij} + 2m_{ij} + m_{ji} + m'_{ij} \qquad ....(III) \quad ....(28.10)$$

$$m_{ij} = 2EK_{ij}\ \theta_i \ ; \ m_{ji} = 2EK_{ji}\ \theta_j$$

and $$m'_{ij} = -\frac{6EK_{ij}d_{ij}}{L_{ij}} \qquad ...(28.11)$$

Now, for the equilibrium of joint *i*,

$$\sum_j M_{ij} = 0$$

Hence from Eq. 28.10,

$$\sum_j M_{Fij} + \sum_j (2m_{ij} + m_{ji} + m'_{ij}) = 0 \qquad ...(28.12)$$

Introducing $M_{Fi} = \sum_j M_{Fij} =$ resultant restraint moment at *i*, we get

$$M_{Fi} + \sum_j (2m_{ij} + m_{ji} + m'_{ij}) = 0$$

or $$\sum_j m_{ij} = -\frac{1}{2}\left[M_{Fi} + \sum_j (m_{ji} + m'_{ij})\right] \qquad ....(28.13)$$

Eq. 28.13 is similar to Eq. 28.6, except that the term m'_{ij} (displacement contribution) has been included.

Eq. 28.13 gives the algebraic sum of *rotation contributions* of all members meeting at joint *i*. Now we know that m_{ij} for any member is proportional to its *K*-value. The individual share of the members in the total rotation contribution can be found by distributing the total rotation contribution in proportion to their respective *K*-values. Thus, for member *ij*,

$$m_{ij} = \frac{K_{ij}}{\sum_j K_{ij}} \sum_j m_{ij}$$

$$\therefore \quad m_{ij} = -\frac{1}{2}\left(\frac{K_{ij}}{\sum_j K_{ij}}\right)\left[M_{Fi} + \sum_j (m_{ji} + m'_{ij})\right]$$

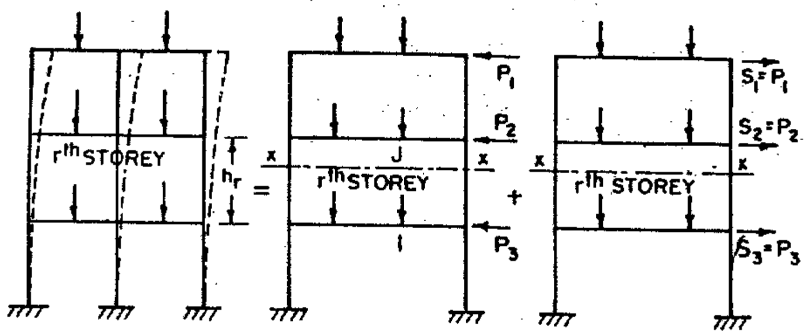
or $$m_{ij} = R_{ij}\left[M_{Fi} + \sum_j (m_{ji} + m'_{ij})\right] \qquad ...(IV) \quad ...(28.14)$$

where R_{ij} = rotation factor for $ij = -\frac{1}{2}\left(\frac{K_{ij}}{\sum_{ij} K_{ij}}\right)$

Thus, From Eq. 28.14, *rotation contribution* m_{ij} can be computed, provided the displacement contribution m'_{ij} is known. Let us, therefore, discuss the method for computing displacement contribution m'_{ij} by taking several cases.

CASE 1 : FRAME WITH COLUMNS OF EQUAL HEIGHT AND SUBJECTED TO VERTICAL LOADS ONLY

Fig. 28.16 shows the actual case of frame with vertical loading, which causes side sway. As done in the moment distribution method, the solution is accomplished in two steps.



(a) ACTUAL (b) NO SWAY (c) SWAY FORCES

Fig. 28.16.

Step 1. No Sway : Artificial restraints are applied so that the frame does not sway, and the frame is analysed as discussed previously.

Step 2. Sway forces : Sway forces are now applied, and the frame is analysed.

The, total solution may then be obtained as the albegraic sum of the solutions obtained in the above two steps.

Development of expression for displacement contribution m'_{ij}

Let us pass a horizontal section x-x. For the horizontal equilibrium of the part of the frame above the section x-x, the algebraic sum of column shears must vanish.

$$\sum_r Q_{ij} = 0$$

where $\sum_r$ denotes the sum of shears forall columns in r^{th} storey

and Q_{ij} = shear in column ij of r^{th} storey.

If h_r is the height of r^{th} storey,

$$Q_{ij} = \frac{M_{ij} + M_{ji}}{h_r} \quad ...(28.15)$$

But
$$M_{ij} = M_{Fij} + 2m_{ij} + m_{ji} + m'_{ij}$$
$$M_{ji} = M_{Fji} + 2m_{ji} + m_{ij} + m'_{ji}$$

Substituting these values in Eq. 28.15, and noting the fact that fixed end moment in the column ends are zero because of absence of intermediate horizontal loads, and also noting that

$m'_{ij} = m'_{ji} = -\frac{6EK_{ij}d_{ij}}{L_{ij}}$, we get

$$Q_{ij} = \frac{1}{h_r}\left(3m_{ij} + 3m_{ji} + 2m'_{ij}\right) \quad ...(28.16)$$

$$\therefore \quad \sum_r Q_{ij} = \sum_r \frac{1}{h_r}\left[3(m_{ij} + m_{ji}) + 2m'_{ij}\right] = 0$$

$$\therefore \quad \sum_r m'_{ij} = -\frac{3}{2}\sum_r (m_{ij} + m_{ji}) \quad ...(28.16)$$

This shows that the algebraic sum of displacement contribution of all the columns of r^{th} storey is equal to -1.5 times that of the rotation contributions of the two ends of the same columns.

Now since the m'_{ij} for any column is proportional to its K-value, the individual share of the members in the displacement contribution can be found by distributing the total displacement contribution in proportion to their respective K-values. Thus, for any column ij,

$$m'_{ij} = -1.5\left(\frac{K_{ij}}{\sum_r K_{ij}}\right)\sum_r (m_{ij} + m_{ji}) \quad ...(28.17)$$

Introducing a *displacement factor* D_{ij} for any column ij by the expression

$$D_{ij} = -1.5\left(\frac{K_{ij}}{\sum_r K_{ij}}\right) \quad ...(28.18)$$

We get from Eq. 28.17,

$$m'_{ij} = D_{ij}\sum_r (m_{ij} + m_{ji}) \quad ...(28.19) \quad ...(V)$$

From Eqs. 28.14 and 28.19, we note that the rotation contribution (m_{ij} and m_{ji}) and the displacement contribution m'_{ij} are interdependent. Equation 28.19 can also be solved by Gauss-Seidell iteration in conjunction with Eq. 28.14.

When once the rotation contributions and the displacement contribution are known, the final moments at various joints can be found by Eq. 28.10. Example 28.6 illustrates the complete procedure wherein solution has been obtained for a frame with sway in a single table.

CASE 2. FRAMES WITH COLUMNS OF EQUAL HEIGHT AND SUBJECTED TO HORIZONTAL LOADS

The procedure is the same as in the previous case, except that the *shear equation* is changed. For the horizontal equilibrium of the part of the frame above the section *x-x*, the *storey- shear* must be equal to applied horizontal forces above the section.

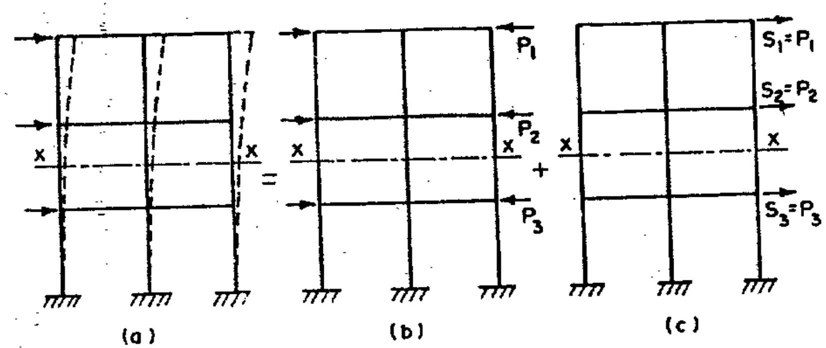


Fig. 28.17.

$$\therefore \quad Q_r = H_1 + H_2 + \ldots H_r$$

Hence the horizontal shear equation for r^{th} storey may be written as

$$Q_r + \sum_r \frac{1}{h_r}\,(M_{ij} + M_{ji}) = 0 \qquad \ldots(28.20)$$

Here ij denotes the columns of r^{th} storey.

But $\quad M_{ij} = M_{Fij} + 2\,m_{ij} + m_{ji} + m'_{ij}$

and $\quad M_{ji} = M_{Fij} + 2\,m_{ji} + m_{ij} + m'_{ji}$

Noting that the fixed end moments are zero in absence of intermediate horizontal loads, and also noting that $m'_{ij} = m'_{ji}$ we get

$$Q_r h_r + \sum_r \left[3\,(m_{ij} + m_{ji}) + 2\,m'_{ij}\right] = 0$$

or $$\sum_r m'_{ij} = -\frac{3}{2}\left[\frac{Q_r\,h_r}{3} + \sum_r (m_{ij} + m_{ji})\right] \qquad \ldots(28.21)$$

It should be noted that Eq. 28.21 differs from Eq. 28.16 of case 1, only by the extra term $\frac{Q_r h_r}{3} = M_r$ = *storey moment.*

Now since the m'_{ij} for any column is proportional to its K-value, the individual share of the members in the displacement contribution can be found by distributing the total displacement contribution in proportion to their respective K-values. Thus, for any column ij,

$$m'_{ij} = \frac{K_{ij}}{\sum_r K_{ij}} \sum_r m'_{ij}$$

$$m'_{ij} = -\frac{3}{2}\,\frac{K_{ij}}{\sum_r K_{ij}}\left[\frac{Q_r h_r}{3} + \sum_r (m_{ij} + m_{ji})\right]$$

or $$m'_{ij} = D_{ij}\left[M_r + \sum_r (m_{ij} + m_{ji})\right] \qquad \ldots(VI) \ \ldots 28.22$$

where $\quad M_r$ = storey moment = $\frac{Q_r h_r}{3}$.

Comparing this with Eq. 28.19, we observe that an extra term M_r has been introduced here, which needs to be calculated and added to the rotation contribution before m'_{ij} is calculated.

Column of equal height with hinged base.

If the columns of bottom storey, all of the same height, are *hinged* at the base instead of being fixed, these columns can be replaced by columns with fixed bases but with K-values 3/4 of those of corresponding actual members. Naturally, the factor $-\frac{3}{2}$ will be replaced by $\left(-\frac{3}{2} \times \frac{4}{3}\right) = -2$ in Eq. 28.18.

Hence $$D_{ij} = -2\,\frac{K_{ij}}{\sum_r K_{ij}} \qquad \ldots(28.18)(a)$$

If, however, some columns are hinged while others are fixed, in a given storey, it can be shown that

$$D_{ij} = -1.5 \frac{K_{ij}}{\sum_r m K_{ij}} \qquad ...(28.18)(b)$$

where $m = \frac{3}{4}$ for hinged columns and $m = 1$ for fixed columns.

CASE 3 : FRAMES WITH COLUMNS OF UNEQUAL HEIGHT

It may happen that the columns of any storey mostly the bottom storey are of unequal heights, subjected to horizontal loads. Let h_{ij} be the height of any column ij of r^{th} storey which is different from height h_r' of any arbitrarily chosen column of the same storey, where h_r' is a constant and is called the *storey height.*

Let C_{ij} = *reduction factor* for the *i-j* column $= \frac{h'_r}{h_{ij}}$ ...(28.23)

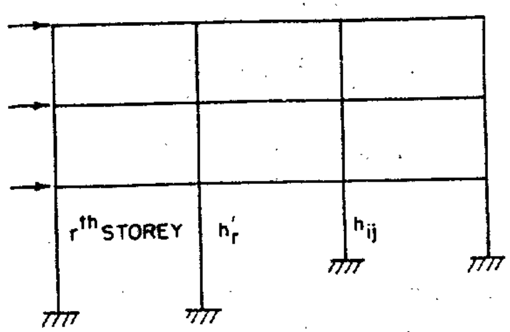


Fig. 28.18.

For the r^{th} storey, the horizontal shear equation is

$$Q_r + \sum_r \frac{1}{h_{ij}} (M_{ij} + M_{ji}) = 0 \qquad ...(28.24)$$

This is same as Eq. 28.20 except that the term h_r, representing constant storey height has been replaced by h_{ij} which is variable from column to column.

Eq. 28.24 can also be written as

$$Q_r . h'_r + \sum_r \frac{h'_r}{h_{ij}} (M_{ij} + M_{ji}) = 0$$

or $$Q_r . h'_r + \sum_r C_{ij} (M_{ij} + M_{ji}) = 0 \qquad ...(28.25)$$

But $$M_{ij} = M_{Fij} + 2 m_{ij} + m_{ji} + m'_{ij}$$

$$M_{ji} = m_{Fji} + 2 m_{ji} + m_{ij} + m'_{ji}$$

Noting that the fixed end moments are zero in absence of intermediate horizontal loads, and noting that $m_{ij}' = m_{ji}'$ *we get, as before*

$$Q_r h'_r + \sum_r C_{ij} (3 m_{ij} + 3 m_{ji} + 2 m'_{ij}) = 0$$

$$\therefore \quad \sum_r C_{ij} m_{ij}' = -\frac{3}{2} \left[\frac{Q_r h_r'}{3} + \sum_r (m_{ij} + m_{ji}) \right] \qquad ...28.26\ (a)$$

$$\sum_r C_{ij} m_{ij}' = -\frac{3}{2} \left[M_r + \sum_r C_{ij} (m_{ij} + m_{ji}) \right] \qquad ...(28.26)$$

where M_r = *storey moment* $= \frac{Q_r h_r'}{3}$

Now, by definition , $m_{ij}' = -\frac{6 E K_{ij} d_{ij}}{h_{ij}}$. Hence, m'_{ij} is proportional to $\frac{K_{ij}}{h_{ij}}$ and hence to $C_{ij} . K_{ij}$.

Hence $$\frac{m'_{ij}}{\sum_r C_{ij} m'_{ij}} = \frac{C_{ij} K_{ij}}{\sum_r C_{ij}^2 K_{ij}} \qquad ...(28.27)$$

Hence from Eqs 28.26 and 28.27

$$m_{ij}' = -1.5 \frac{C_{ij} K_{ij}}{\sum_r C_{ij}^2 K_{ij}} \left[M_r + \sum_r C_{ij} (m_{ij} + m_{ji}) \right] \qquad ...(28.28)(a)$$

or $$m_{ij}' = D_{ij} \left[M_r + \sum_r C_{ij} (m_{ij} + m_{ji}) \right] \qquad ...(28.28)$$

where $$D_{ij} = -1.5 \frac{C_{ij} K_{ij}}{\sum_r C_{ij}^2 K_{ij}} = \text{Displacement facter} \qquad ....(28.29)$$

If some columns are hinged and some are fixed, it can be sho..n that

$$D_{ij} = -1.5 \frac{C_{ij} K_{ij}}{\sum_r m\ C_{ij}^2 K_{ij}} \qquad ...(28.30)$$

where $m = \frac{3}{4}$ for hinged columns and $m = 1$ for columns with fixed ends. If, however, all the columns of the storey are hinged, $m = \frac{3}{4}$ for all the columns, and hence

$$D_{ij} = -2.0 \frac{C_{ij} K_{ij}}{\sum_r C_{ij}^2 K_{ij}} \quad ...(28.31)$$

Example 28.6.

Using Kani's method, analyse the portal frame shown in Fig. 28.19.

Solution

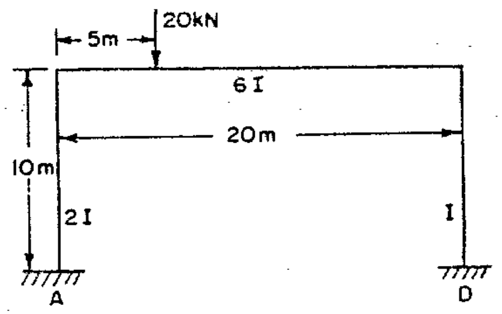


Fig. 28.19.

The frame is geometrically asymmetrical, and is asymmetrically loaded. Hence it will sway. However, since the legs are of equal length, and since there is no imposed horizontal load, it falls under case 1.

Step 1. Fixed End moments

$$M_{FBC} = -\frac{20 \times 5\,(15)^2}{(20)^2} = -56.25 \text{ kN-m}$$

$$M_{FCB} = +\frac{20 \times 5\,(15)^2}{(20)^2} = +18.75 \text{ kN-m}$$

Step 2 Rotation factors

Now $K_{ij} = \frac{I_{ij}}{L_{ij}}$ and $R_{ij} = -0.5 \frac{K_{ij}}{\sum K_{ij}}$

$$\therefore \quad K_{BA} = \frac{2I}{10}; \quad K_{BC} = \frac{6I}{20} = \frac{3I}{10} = K_{CB}; \quad K_{CD} = \frac{I}{10}$$

$$\therefore \quad R_{BA} = -0.5 \frac{2I/10}{\frac{2I}{10} + \frac{3I}{10}} = -0.5 \times \frac{2}{5} = -0.2$$

$$R_{BC} = -0.5 \frac{3I/10}{\frac{2I}{10} + \frac{3I}{10}} = -0.5 \times \frac{3}{5} = -0.3$$

$$R_{CB} = -0.5 \frac{3I/10}{\frac{3I}{10} + \frac{I}{10}} = -0.5 \times \frac{3}{4} = -0.375$$

$$R_{CD} = -0.5 \frac{I/10}{\frac{3I}{10} + \frac{I}{10}} = -0.5 \times \frac{1}{4} = -0.125$$

Step 3. Displacement factors

$$D_{ij} = -1.5 \frac{K_{ij}}{\sum_r K_{ij}}$$

$$\therefore \quad D_{BA} = -1.5 \left[\frac{K_{BA}}{K_{BA} + K_{CD}}\right] = -1.5 \left[\frac{2I/10}{\frac{2I}{10} + \frac{I}{10}}\right] = -1.5 \times \frac{2}{3} = -1.0$$

$$D_{CD} = -1.5 \left[\frac{K_{CD}}{K_{BA} + K_{CD}}\right] = -1.5 \left[\frac{I/10}{\frac{2I}{10} + \frac{I}{10}}\right] = -1.5 \times \frac{1}{3} = -0.5$$

It is to be noted that for a given storey, $\sum_r D_{ij} = -1.5$

Step 4. Resultant restraint moments

$$M_{FB} = -56.25 \text{ and } M_{FC} = +18.75$$

Steps. Kani's Iteration cycles

The fixed end moments, restraint moments and rotation factors are entered as usual. The displacement factors and displacement contributions are entered transverse to each column, as shown in Fig. 28.20.

Cycles. The *rotation contribution* m_{ij} at joint i is given by Eq. 28.14

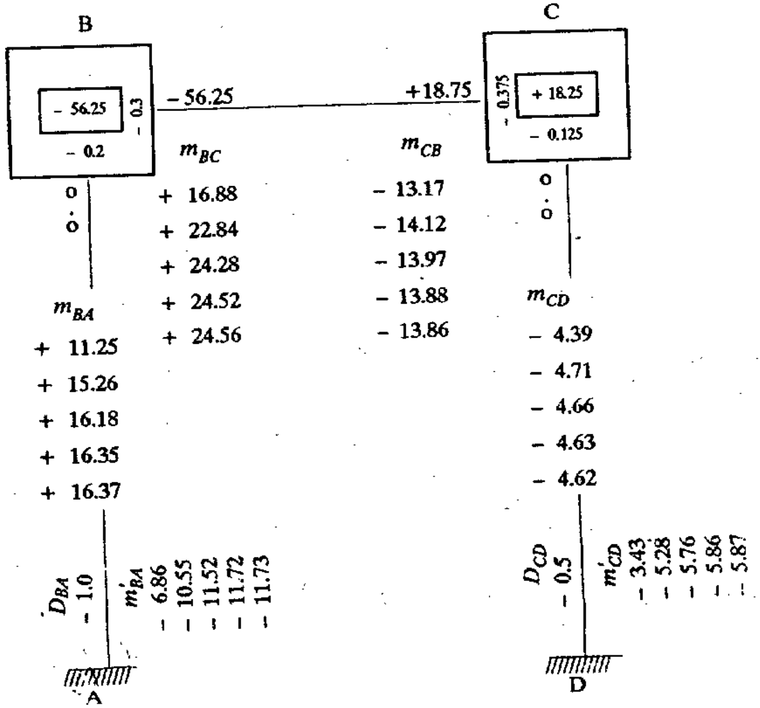


Fig. 28.20

$$m_{ij} = R_{ij}\left[M_{Fi} + \sum_j (m_{ji} + m'_{ij})\right]$$

To start with neither m_{ji} nor m'_{ij} are known, and hence these can be assumed to be zero.

At B, $m_{BC} = -0.3\,[-56.25 + 0 + 0] = +16.88$

$m_{BA} = -0.2\,[-56.25 + 0 + 0] = +11.25$

At C, $m_{CB} = R_{CB}\left[M_{FC} + m_{BC} + m_{DC} + m'_{CD}\right]$

$= -0.375\,[+18.25 + 16.88 + 0 + 0] = -13.17$

$m_{CD} = R_{CD}\left[M_{FC} + m_{BC} + m_{DC} + m'_{CD}\right]$

$= -0.125\,[18.25 + 16.88 + 0 + 0] = -4.39$

Displacement contribution The displacement contributions are found by Eq. 28.19, according to which the displacement contribution of any column is equal to displacement factor of that column multiplied by algebraic sum of rotation contributions of all the columns of that storey. Hence

$$m'_{BA} = D_{BA}\left[m_{BA} + m_{AB} + m_{CD} + m_{DC}\right]$$

$$= -1.0\,[+11.25 + 0 - 4.39 + 0] = -6.86$$

$$m'_{CD} = D_{CD}\left[m_{BA} + m_{AB} + m_{CD} + m_{DC}\right]$$

$$= -0.5\,[+11.25 + 0 - 4.39 + 0] = -3.43$$

This constitutes the first cycle.

Cycle 2

(*a*) *Rotation factors*

Joint B : $m_{BC} = R_{BC}\left[M_{FB} + m_{CB} + m_{AB} + m'_{BA}\right]$

$= -0.3\,[-56.25 - 13.17 + 0 - 6.86] = +22.84$

$m_{BC} = R_{BA}\left[M_{FB} + m_{CB} + m_{AB} + m'_{BA}\right]$

$= -0.2\,[-56.25 - 33.17 + 0 - 6.86] = +15.26$

Joint C : $m_{CB} = R_{CB}\left[M_{FC} + m_{BC} + m_{DC} + m'_{CD}\right]$

$= -0.375\,[+18.25 + 22.84 + 0 - 3.43] = -14.12$

$m_{CD} = R_{CD}\left[M_{FC} + m_{BC} + m_{DC} + m'_{CD}\right]$

$= -0.125[+18.25 + 22.84 + 0 - 3.43] = -4.71$

(*b*) *Displacement factors*

$m_{BA} = -1.0\,[+15.26 + 0 - 4.71 + 0] = -10.55$

$m_{CD} = -0.5\,[+15.26 + 0 - 4.71 + 0] = -5.28$

Cycle 3

(*a*) *Rotation factors*

Joint B : $m_{BC} = -0.3[-56.25 - 14.12 + 0 - 10.55] = +24.28$

$m_{BA} = -0.2[-56.25 - 14.12 + 0 - 10.55] = +16.18$

Joint C : $m_{CB} = -0.375[+18.25 + 24.28 + 0 - 5.28] = -13.97$

$m_{CD} = -0.125[+18.25 + 22.84 + 0 - 5.28] = -4.66$

(b) *Displacement factors*

$m'_{BA} = -1.0[+16.18 + 0 - 4.66 + 0] = -11.52$

$m'_{CD} = -1.5[+16.18 + 0 - 4.66 \div 0] = -5.76$

Cycle 4

(a) *Rotation Factors*

Joint B : $m_{BC} = -0.3[-56.25 - 13.97 + 0 - 11.52] = +24.52$

$m_{BA} = -0.2[-56.25 - 13.93 + 0 - 11.52] = +16.35$

Joint C : $m_{CB} = -0.375[+18.25 + 24.52 + 0 - 5.76] = -13.88$

$m_{CD} = -0.125[+18.25 + 24.52 + 0 - 5.76] = -4.63$

(b) *Diplacement Factors*

$m'_{BA} = -1.0[+16.35 + 0 - 4.63 + 0] = -11.72$

$m'_{CD} = -0.5[+16.35 + 0 - 4.63 + 0] = -5.86$

Cycle 5

(a) *Rotation factors*

Joint B : $m_{BC} = -0.3[-56.25 - 13.88 + 0 - 11.72] = +24.56$

$m_{BA} = -0.2[-56.25 - 13.88 + 0 - 11.72] = +16.37$

Joint C : $m_{CB} = -0.375[+18.25 + 24.56 + 0 - 5.86] = -13.86$

$m_{CD} = -0.125[+18.25 + 24.56 + 0 - 5.86] = -4.62$

(b) *Displacement Factors*

$m'_{BA} = -1.0[+16.35 + 0 - 4.62 + 0] = -11.73$

$m'_{CD} = -0.5[+16.35 + 0 - 4.62 + 0] = -5.87$

The iteration may be terminated now because the difference between the successive values is very small.

Step 6. Final moments

The final moments can be found by Eq. 28.10 :

$$M_{ij} = M_{Fij} + 2m_{ij} + m_{ji} + m'_{ij}$$

The computations are arranged in Tabular form below.

TABLE 28.11

M_{ij}	M_{Fij}	$2m_{ij}$	m_{ji}	m'_{ij}	Total (kN-m)
M_{BA}	0	+ 32.74	0	− 11.73	+ 21.01
M_{BC}	− 56.25	+ 49.12	− 13.86	−	− 20.99
M_{CB}	+ 18.25	− 27.72	+ 24.56	−	+ 15.09
M_{CD}	0	− 9.24	0	− 5.87	− 15.11
M_{AB}	0	0	+ 16.37	− 11.73	+ 4.64
M_{DC}	0	0	− 4.62	− 5.87	− 10.49

The B.M.D. for the frame is shown in Fig. 28.21

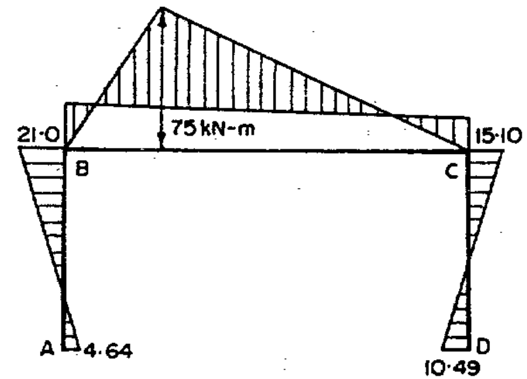


Fig. 28.21

Example 28.7.

Draw the bending moment diagram and sketch the deflected shape of the frame shown in Fig. 28.22.

Solution : The frame will evidently sway.

Step 1. Fixed end moments

$$M_{FBC} = -\frac{6(2)^2}{12} = -2.0 \text{ kN-m}$$

$$M_{FCB} = +2.0 \text{ kN-m}$$

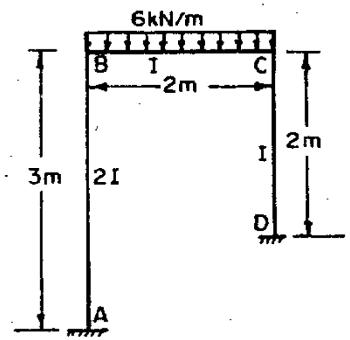


Fig. 28.22

Step 2. Rotation Factors

$$K_{ij} = \frac{I_{ij}}{L_{ij}} \quad \text{and} \quad R_{ij} = -0.5\,\frac{K_{ij}}{\sum\limits_j K_{ij}}$$

$$K_{BA} = \frac{2I}{3}\ ; \quad K_{BC} = \frac{I}{2} = K_{CB}\ ; \quad K_{CD} = \frac{I}{2}$$

$$\therefore \quad R_{BA} = -0.5\,\frac{2I/3}{\frac{2I}{3}+\frac{I}{2}} = -0.5 \times \frac{2}{3} \times \frac{6}{7} = -0.2857$$

$$R_{BC} = -0.5\,\frac{I/2}{\frac{2}{3}I+\frac{I}{2}} = -0.5 \times \frac{1}{2} \times \frac{6}{7} = -0.2143$$

$$R_{CB} = -0.5\,\frac{I/2}{\frac{I}{2}+\frac{I}{2}} = -0.5 \times \frac{1}{2} \times 1 = -0.25$$

$$R_{CD} = -0.5\,\frac{I/2}{\frac{I}{2}+\frac{I}{2}} = -0.5 \times \frac{1}{2} \times 1 = -0.25$$

Step 3. Displacement Factors

$$D_{ij} = -1.5\,\frac{C_{ij}\,K_{ij}}{\sum C_{ij}^2\,K_{ij}}, \quad \text{because of unequal legs.}$$

Let the reference height be taken as 3 m, so that

$$C_{BA} = \frac{3}{3} = 1 \quad \text{and} \quad C_{CD} = \frac{3}{2} = 1.5$$

$$\therefore \quad D_{BA} = -1.5\,\frac{C_{BA}\,.\,K_{BA}}{(C_{AB})^2\,.\,K_{AB} + (C_{CD})^2\,.\,K_{CD}} = -1.5\,\frac{1.5 \times \frac{2I}{3}}{(1)^2 \times \frac{2I}{3} + (1.5)^2\,\frac{I}{2}}$$

$$= -0.5581$$

$$D_{CD} = -1.5\,\frac{C_{CD}\,.\,K_{CD}}{(C_{BA})^2\,.\,K_{BA} + (C_{CD})^2\,.\,K_{CD}} = -1.5\,\frac{1.5 \times \frac{I}{2}}{(1)^2 \times \frac{2I}{3} + (1.5)^2\,\frac{I}{2}}$$

$$= -0.6279$$

Step 4. Resultant Restraint Moments

$M_{FB} = -2.0\ ; \quad M_{FC} = +2.0$ kN$-$m.

Step 5. Kani's Iteration cycles

Cycle 1.

The *rotation contribution* m_{ij} at a joint i is given by Eq 28.14

$$m_{ij} = R_{ij}\,[M_{Fi} + \sum_j (m_{ji} + m'_{ij})]$$

To start with, neither m_{ji} nor m'_{ji} are known, and hence these can be assumed to be zero.

At B : $m_{BC} = R_{BC}\,[M_{FB} + m_{CB} + m_{AB} + m'_{BA}]$

$= -0.2143\,[-2.0 + 0 + 0 + 0] = +0.429$

$m_{BA} = -0.2857\,[-2.0 + 0 + 0 + 0] = +0.571$

At C : $m_{CB} = R_{CB}\,[M_{FC} + m_{BC} + m_{DC} + m'_{CD}]$

$= -0.25[+2.0 + 0.429 + 0 + 0] = -0.607$

$m_{CD} = R_{CD}\,[M_{FC} + m_{BC} + m_{DC} + m'_{CD}]$

$= -0.25[+2.0 + 0.429 + 0 + 0] = -0.607$

Displacement Contribution

The displacement contributions given by Eq. 28.28, taking $M_r = 0$:

$$m'_{ij} = D_{ij}\,[\sum C_{ij}\,(m_{ij} + m_{ji})]$$

$$m'_{BA} = D_{BA}\left[C_{BA}\,(m_{BA} + m_{AB}) + C_{CD}\,(m_{CD} + m_{DC})\right]$$

$$= -0.5581[\,1\,(0.571 + 0) + 1.5\,(-0.607 + 0)] = +0.189$$

$$m'_{CD} = D_{CD}\left[1\,(m_{BA} + m_{AB}) + 1.5\,(m_{CD} + m_{DC})\right]$$

$$= -0.6279[\,1\,(0.571 + 0) + 1.5\,(-0.607 + 0)] = +0.213$$

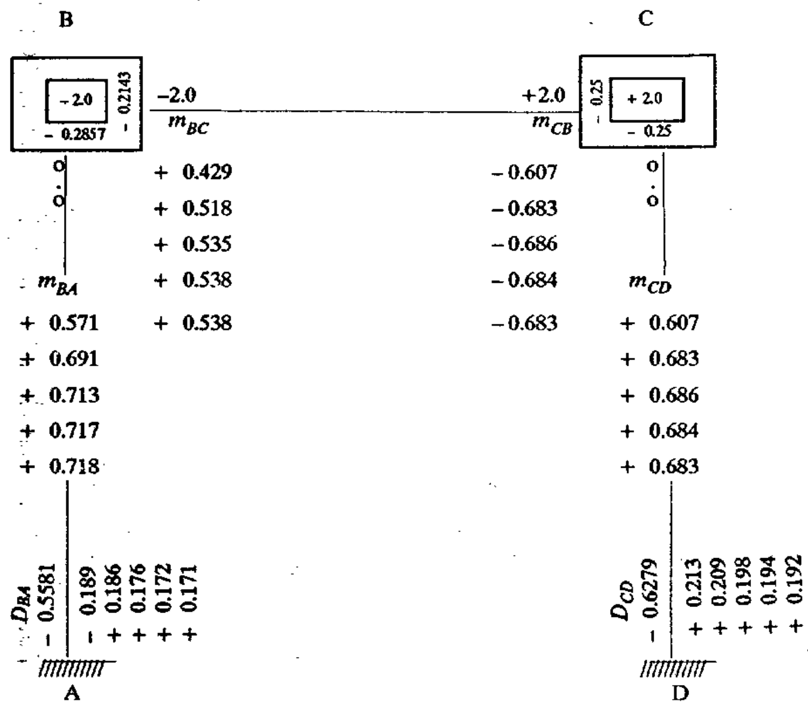


Fig. 28.23

Cycle 2. (*a*) *Rotation Factors*

Joint B : $m_{BC} = -0.2143\,[-2.0 - 0.607 + 0 + 0.189] = +0.518$

$m_{P4} = -0.2857\,[-2.0 - 0.607 + 0 + 0.189] = +0.691$

Joint C : $m_{CB} = -0.25\,[+2.0 + 0.518 + 0 + 0.213] = -0.683$

$m_{CD} = -0.25\,[+2.0 + 0.518 + 0 + 0.213] = -0.683$

(*a*) *Displacement Factors*

$m'_{BA} = -0.5581\,[1 \times 0.691 - 1.5 \times 0.683] = +0.186$

$m'_{CD} = -0.6279\,[1 \times 0.691 - 1.5 \times 0.683] = +0.209$

Cycle 3. (*a*) *Rotation Factors.*

Joint B : $m_{BC} = -0.2143\,[-2.0 - 0.683 + 0 + 0.186] = +0.535$

$m_{BA} = -0.2857\,[-2.0 - 0.683 + 0 + 0.186] = +0.713$

Joint C : $m_{CB} = -0.25\,[+2.0 + 0.535 + 0 + 0.209] = -0.686$

$m_{CD} = -0.25\,[+2.0 + 0.535 + 0 + 0.209] = -0.686$

(*b*) *Displacement Factors*

$m'_{BA} = -0.5581\,[+1 \times 0.713 + 0 - 1.5 \times 0.686 + 0] = +0.176$

$m'_{CD} = -0.6279\,[+1 \times 0.713 + 0 - 1.5 \times 0.686 + 0] = +0.198$

Cycle 4 (*a*) *Rotation Factors*

Joint B : $m_{BC} = -0.2143\,[-2.0 - 0.686 + 0 + 0.176] = +0.538$

$m_{BA} = -0.2857\,[-2.0 - 0.686 + 0 + 0.176] = +0.717$

Joint C : $m_{CB} = -0.25\,[+2.0 + 0.538 + 0 + 0.198] = -0.684$

$m_{CD} = -0.25\,[+2.0 + 0.538 + 0 + 0.198] = -0.684$

(*b*) *Displacement Factors*

$m'_{BA} = -0.5581\,[+1 \times 0.717 + 0 - 1.5 \times 0.684 + 0] = +0.172$

$m'_{CD} = -0.6279\,[+1 \times 0.717 + 0 - 1.5 \times 0.684 + 0] = +0.194$

Cycle 5 (*a*) *Rotation Factors*

Joint B : $m_{BC} = -0.2143\,[-2.0 - 0.684 + 0 + 0.172] = +0.538$

$m_{BA} = -0.2857\,[-2.0 - 0.684 + 0 + 0.172] = +0.718$

Joint C : $m_{CB} = -0.25\,[+2.0 + 0.538 + 0 + 0.194] = -0.683$

$m_{CD} = -0.25\,[+2.0 + 0.538 + 0 + 0.194] = -0.683$

(*b*) *Displacement Factors*

$m'_{BA} = -0.5581\,[+1 \times 0.718 + 0 - 1.5 \times 0.683 + 0] = +0.171$

$m'_{CD} = -0.6279\,[+1 \times 0.716 + 0 - 1.5 \times 0.684 + 0] = +0.192$

Step 6. Final moments

The final moment at any joint is found by Eq. 28.10 :

$M_{ij} = M_{Fij} + 2\,m_{ij} + m_{ji} + m'_{ij}$

The computations are arranged in Table 28.12

TABLE 28.12

M_{ij}	M_{Fij}	$2m_{ij}$	m_{ji}	m'_{ij}	*Total*
M_{BA}	0	+ 1.436	0	+ 0.172	+ 1.608
M_{BC}	– 2.0	+ 1.076	– 0.683	–	– 1.607
M_{CB}	+ 2.0	– 1.366	+ 0.538	–	+ 1.172
M_{CD}	0	– 1.366	0	+ 0.192	– 1.174
M_{AB}	0	0	+ 0.718	+ 0.172	+ 0.89
M_{DC}	0	0	– 0.683	+ 0.192	– 0.491

The B.M.D. and the deflected shape are shown in Fig. 28.24(*a*) and (*b*) respectively.

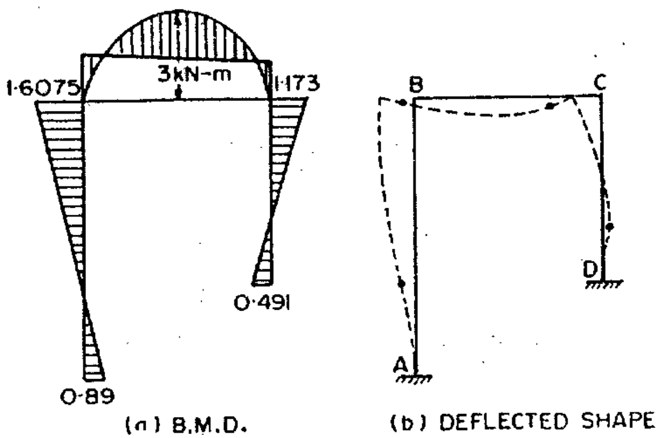


Fig. 28.24.

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