Assignment - 10 Saketh Patibandla

Problem 4.2 Itô's Lemma:

Assume that W_t is Brownian motion. Use Itô's Lemma to find $df(t, W_t)$.

1) $f(t, W_t) = e^{t/2} \cos(W_t)$

Proof ÷ $df(t,x) = f_t\, dt + f_x\, dW_t + \frac{1}{2} f_{xx} \underbrace{(dW_t\, dW_t)}_{dt}$ (as W_t is a Brownian motion)

Similarly, let $f(t,x) = e^{t/2} \cos(x)$

$\therefore \frac{\partial f(t,x)}{\partial t} = f_t(t,x) = \frac{1}{2} e^{t/2} \cos(x) = \frac{1}{2} f(t,x)$

$\frac{\partial f}{\partial x}(t,x) = f_x(t,x) = -e^{t/2} \sin(x)$

$\frac{\partial^2 f(t,x)}{\partial x^2} = f_{xx}(t,x) = e^{-t/2} \cos(x) = -f(t,x)$

substituting these values in the Itô's Doeblin formula for $f(t, W_t)$ we get,

$df(t, W_t) = f_t(t, W_t)\, dt + f_x(t, W_t)\, dW_t + \frac{1}{2} f_{xx}(t, W_t)$

$df(t, W_t) = \cancel{\frac{1}{2} f(t, W_t)\, dt} + (-e^{t/2} \sin W_t)\, dW_t + \cancel{\frac{1}{2}(-f(t, W_t))dt}$

$= -e^{t/2} \sin W_t\, dW_t$

$\therefore$ it is the same as the question value.

2) $f(t, W_t) = [W_t + t]\, e^{-W_t - t/2}$

Solution: By using Ito-Doeblin equation

$$df(t,x) = f_t(t,x)\,dt + f_x(t,x) + \frac{1}{2} f_{xx}(t,x)\,\frac{dW_t\,dW_t}{dt}$$

Now, let $f(t,x) = [x+t)\, e^{-x-t/2}$

$$f_t(t,x) = x^{-e^{x-t/2}} + t\dot{e}^{-x-t/2}$$

$$\therefore f_t(t,x) = -\frac{1}{2}[xe^{-x-t/2}] + [-\frac{t}{2} e^{-x-t/2} + e^{-x-t/2}]$$

$$= \frac{-1}{2} e^{-x-t/2}(x+t) + e^{-x-t/2}$$

$$= \frac{-1}{2} f(t,x) + e^{-x-t/2}$$

$$fx(t,x) = [-xe^{-x-t/2} + e^{-x-t/2}] + (-te^{-x-t/2})$$

$$= -e^{-x-t/2}(x+t) + e^{-x-t/2}$$

$$\therefore f_x(t,x) = -f(t,x) + e^{-x-t/2}$$

$$f_{xx}(t,x) = -f_x(t,x) + (-e^{-x-t/2})$$

Putting the value of $f_x(t,x)$

$$f_{xx}(t,x) = -(-f(t,x) + e^{-x-t/2}) - e^{-x-t/2}$$

$$= f(t,x) - e^{-x-t/2} - e^{-x-t/2}$$

$$f_{xx}(t,x) = f(t,x) - 2e^{-x-t/2}$$

Substituting the values of $f_t(t,x)$, $f_x(t,x)$ & $f_{xx}(t,x)$, to $f(t,W_t)$

$$df(t,W_t) = \left(\frac{-1}{2} f(t,W_t) + e^{-W_t - t/2}\right) dt + \left(-f(t,W_t) + e^{-W_t - t/2}\right) dW_t + \frac{1}{2}\left(f(t,W_t) - 2e^{-W_t - t/2}\right) dt$$

$$= \cancel{\frac{-1}{2} f(t,W_t) dt} + \cancel{e^{-W_t - t/2} dt} + \left(-f(t,W_t) + e^{-W_t - t/2}\right) dW_t + \cancel{\frac{1}{2} f(t,W_t) dt} - \cancel{e^{-W_t - t/2} dt}.$$

$$\therefore \; df(t,W_t) = \left(-f(t,W_t) + e^{-W_t - t/2}\right) dW_t.$$