Full Jayne's Cumming's Hamiltonian $\longrightarrow$ Interaction picture

$$\hat{H} = \underbrace{\overbrace{\hbar\omega_p \hat{a}^\dagger \hat{a}}^{\hat{H}_{field}} + \overbrace{\frac{\hbar\omega_a \hat{\sigma}_z}{2}}^{\hat{H}_{atom}}}_{\hat{H}_0} + \underbrace{\lambda\hbar(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-)}_{\hat{H}_1}$$

Switch to interaction picture Hamiltonian:

$$\hat{H}_{int}(t) = e^{\frac{i\hat{H}_0 t}{\hbar}} \hat{H}_1 e^{-\frac{i\hat{H}_0 t}{\hbar}}$$

using basis $|e, n-1\rangle$, $|g, n\rangle$

$$\hat{H}_0|e, n-1\rangle = \left(\hbar\omega_p(n-1) + \frac{\hbar\omega_a}{2}\right)|e, n-1\rangle$$

$$\hat{H}_0|g, n\rangle = \left(\hbar\omega_p n - \frac{\hbar\omega_a}{2}\right)|g, n\rangle$$

$$\hat{H}_1|e, n-1\rangle = \lambda\hbar\sqrt{n}\,|g, n\rangle$$

$$\hat{H}_1|g, n\rangle = \lambda\hbar\sqrt{n}\,|e, n-1\rangle$$

$$e^{\frac{i\hat{H}_0 t}{\hbar}}|e, n-1\rangle = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{i\hat{H}_0 t}{\hbar}\right)^k |e, n-1\rangle = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{it}{\hbar}\left(\hbar\omega_p(n-1) + \frac{\hbar\omega_a}{2}\right)\right)^k |e, n-1\rangle = e^{i(\omega_p(n-1) + \frac{\omega_a}{2})t}|e, n-1\rangle$$

$$e^{\frac{i\hat{H}_0 t}{\hbar}}|g, n\rangle = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{i\hat{H}_0 t}{\hbar}\right)^k |g, n\rangle = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{it}{\hbar}\left(\hbar\omega_p n - \frac{\hbar\omega_a}{2}\right)\right)^k |g, n\rangle = e^{i(\omega_p n - \frac{\omega_a}{2})t}|g, n\rangle$$

$$e^{-\frac{i\hat{H}_0 t}{\hbar}}|e, n-1\rangle = e^{-i(\omega_p(n-1) + \frac{\omega_a}{2})t}|e, n-1\rangle$$

$$e^{-\frac{i\hat{H}_0 t}{\hbar}}|g, n\rangle = e^{-i(\omega_p n - \frac{\omega_a}{2})t}|g, n\rangle$$

$$\Rightarrow \hat{H}_1 = \lambda\hbar\sqrt{n}\,|e, n-1\rangle\langle g, n| + \lambda\hbar\sqrt{n}\,|g, n\rangle\langle e, n-1|$$

$$e^{\frac{i\hat{H}_0 t}{\hbar}} = e^{i(\omega_p(n-1) + \frac{\omega_a}{2})t}|e, n-1\rangle\langle e, n-1| + e^{i(\omega_p n - \frac{\omega_a}{2})t}|g, n\rangle\langle g, n|$$

$$e^{-\frac{i\hat{H}_0 t}{\hbar}} = e^{-i(\omega_p(n-1) + \frac{\omega_a}{2})t}|e, n-1\rangle\langle e, n-1| + e^{-i(\omega_p n - \frac{\omega_a}{2})t}|g, n\rangle\langle g, n|$$

$$\Rightarrow e^{\frac{i\hat{H}_0 t}{\hbar}} \hat{H}_1 e^{-\frac{i\hat{H}_0 t}{\hbar}} = e^{i(\omega_p(n-1) + \frac{\omega_a}{2})t} \lambda\hbar\sqrt{n}\,|e, n-1\rangle\langle g, n| e^{-i(\omega_p n - \frac{\omega_a}{2})t} + e^{i(\omega_p n - \frac{\omega_a}{2})t} \lambda\hbar\sqrt{n}\,|g, n\rangle\langle e, n-1| e^{-i(\omega_p(n-1) + \frac{\omega_a}{2})t}$$

$$= e^{i(\omega_a - \omega_p)t} \underbrace{\lambda\hbar\sqrt{n}\,|e, n-1\rangle\langle g, n|}_{\hat{a}\hat{\sigma}_+} + e^{-i(\omega_a - \omega_p)t} \underbrace{\lambda\hbar\sqrt{n}\,|g, n\rangle\langle e, n-1|}_{\hat{a}^\dagger\hat{\sigma}_-} \qquad \text{let } \Delta = \omega_a - \omega_p$$

$$= \lambda\hbar\left(e^{i\Delta t}\hat{a}\hat{\sigma}_+ + e^{-i\Delta t}\hat{a}^\dagger\hat{\sigma}_-\right)$$