Eigenvalues of full JC Hamiltonian:

$$\hat{H} = \hbar\omega_p \hat{a}^\dagger\hat{a} + \frac{\hbar\omega_a}{2}\hat{\sigma}_z + \lambda\hbar(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-)$$

Assume definite number of quanta, n

Use basis $|e, n-1\rangle$ (excited atom, $n-1$ photons) and $|g, n\rangle$ (ground atom, n photons)

$$\hat{H}|e, n-1\rangle = \hbar\omega_p(n-1)|e, n-1\rangle + \frac{\hbar\omega_a}{2}|e, n-1\rangle + \lambda\hbar\sqrt{n}\,|g, n\rangle$$

$$\hat{H}|g, n\rangle = \hbar\omega_p n|g, n\rangle - \frac{\hbar\omega_a}{2}|g, n\rangle + \lambda\hbar\sqrt{n}\,|e, n-1\rangle$$

$\Rightarrow$ in this basis: $\hat{H} = \begin{pmatrix} \hbar\omega_p(n-1) + \frac{\hbar\omega_a}{2} & \lambda\hbar\sqrt{n} \\ \lambda\hbar\sqrt{n} & \hbar\omega_p n - \frac{\hbar\omega_a}{2} \end{pmatrix}$

Finding eigenvalues:

$$\det\begin{pmatrix} \hbar\omega_p(n-1) + \frac{\hbar\omega_a}{2} - E_n & \lambda\hbar\sqrt{n} \\ \lambda\hbar\sqrt{n} & \hbar\omega_p n - \frac{\hbar\omega_a}{2} - E_n \end{pmatrix} = 0$$

$$\left(\hbar\omega_p(n-1) + \frac{\hbar\omega_a}{2} - E_n\right)\left(\hbar\omega_p n - \frac{\hbar\omega_a}{2} - E_n\right) - \lambda^2\hbar^2 n = 0$$

$$\hbar^2\omega_p^2 n(n-1) - \frac{1}{2}\hbar^2\omega_p\omega_a(n-1) - \hbar\omega_p(n-1)E_n + \frac{1}{2}\hbar^2\omega_p\omega_a n - \frac{1}{4}\hbar^2\omega_a^2 - \frac{1}{2}\hbar\omega_a E_n - \hbar\omega_p n E_n + \frac{1}{2}\hbar\omega_a E_n + E_n^2 - \lambda^2\hbar^2 n = 0$$

$$E_n^2 - \hbar\omega_p(2n-1)E_n + \hbar^2\left(\omega_p^2 n(n-1) + \frac{1}{2}\omega_p\omega_a - \frac{1}{4}\omega_a^2 - \lambda^2 n\right) = 0$$

$$E_n = \frac{\hbar\omega_p(2n-1)}{2} \pm \frac{1}{2}\sqrt{\hbar^2\omega_p^2(2n-1)^2 - 4\hbar^2\left(\omega_p^2 n(n-1) + \frac{1}{2}\omega_p\omega_a - \frac{1}{4}\omega_a^2 - \lambda^2 n\right)}$$

$$= \frac{\hbar\omega_p(2n-1)}{2} \pm \frac{\hbar}{2}\sqrt{4n^2\omega_p^2 - 4n\omega_p^2 + \omega_p^2 - 4n^2\omega_p^2 + 4n\omega_p^2 - 2\omega_p\omega_a + \omega_a^2 + 4\lambda^2 n}$$

$$= \hbar\omega_p\left(n - \frac{1}{2}\right) \pm \frac{\hbar}{2}\sqrt{\omega_p^2 - 2\omega_p\omega_a + \omega_a^2 + 4\lambda^2 n}$$

$$= \hbar\omega_p\left(n - \frac{1}{2}\right) \pm \hbar\sqrt{\frac{(\omega_p - \omega_a)^2}{4} + \lambda^2 n}$$

If $\omega = \omega_p = \omega_a$ as we have assumed in the report, we get:

$E_n = \hbar\omega\left(n - \frac{1}{2}\right) \pm \lambda\hbar\sqrt{n}$ which is what we saw in the energy level diagram on page 2 of the report

Now if the coupling strength λ goes to 0

$E_n = \hbar\omega\left(n - \frac{1}{2}\right)$

so $|e, n-1\rangle$ and $|g, n\rangle$ become degenerate eigenstates of $\hat{H}$