Time evolution in non-resonant case

$$\hat{H}_{int}(t) = \lambda\hbar\left(\hat{a}\hat{\sigma}_+ e^{i\Delta t} + \hat{a}^\dagger\hat{\sigma}_- e^{-i\Delta t}\right) \qquad \Delta = \omega_a - \omega_p$$

Assume definite number of quanta, n
Use basis $|e, n-1\rangle$ (excited atom, $n-1$ photons) and $|g, n\rangle$ (ground atom, n photons)

=> General state : $|\Psi(t)\rangle = c_{e,n-1}(t)|e, n-1\rangle + c_{g,n}(t)|g, n\rangle$

$$-i\hbar \frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{H}_{int}|\Psi(t)\rangle$$

$$-i\hbar\left(\dot{c}_{e,n-1}(t)|e, n-1\rangle + \dot{c}_{g,n}(t)|g, n\rangle\right) = \lambda\hbar e^{-i\Delta t}\sqrt{n}\, c_{e,n-1}(t)|g, n\rangle + \lambda\hbar e^{i\Delta t}\sqrt{n}\, c_{g,n}(t)|e, n-1\rangle$$

$$\Rightarrow -i\hbar\, \dot{c}_{e,n-1}(t) = \lambda\hbar e^{i\Delta t}\sqrt{n}\, c_{g,n}(t)$$

$$-i\hbar\, \dot{c}_{g,n}(t) = \lambda\hbar e^{-i\Delta t}\sqrt{n}\, c_{e,n-1}(t)$$

Solution to these coupled differential equations is derived in W.P. Schleich, "Quantum optics in phase space" Ch 15.3

$$c_{e,n-1}(t) = e^{\frac{-i\Delta t}{2}}\left\{\left[\cos(\lambda_n t) + \frac{i\Delta}{2\lambda_n}\sin(\lambda_n t)\right]c_{e,n-1}(0) - \frac{i\lambda\sqrt{n}}{\lambda_n}\sin(\lambda_n t)\, c_{g,n}(0)\right\}$$

$$c_{g,n}(t) = e^{\frac{i\Delta t}{2}}\left\{-\frac{i\lambda\sqrt{n}}{\lambda_n}\sin(\lambda_n t)\, c_{e,n-1}(0) + \left[\cos(\lambda_n t) - \frac{i\Delta}{2\lambda_n}\sin(\lambda_n t)\right]c_{g,n}(0)\right.$$

Where $\lambda_n = \sqrt{\left(\frac{\Delta}{2}\right)^2 + \lambda^2 n}$

From these expressions for the probability amplitudes we can find the probabilities of measuring the atom in the excited or ground state at any time, $t \geq 0$, if we know the initial state $|\Psi(0)\rangle$

$P_e(t) = |c_{e,n-1}(t)|^2$
$P_g(t) = |c_{g,n}(t)|^2$