Finding time evolution operator for interaction picture Hamiltonian in resonant case

$\hat{H}_{int} = \lambda\hbar(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-)$

Use basis $|e, n-1\rangle$ (excited atom, $n-1$ photons) and $|g, n\rangle$ (ground atom, n photons)

$\hat{H}_{int}|e, n-1\rangle = \lambda\hbar\sqrt{n}\,|g, n\rangle$
$\hat{H}_{int}|g, n\rangle = \lambda\hbar\sqrt{n}\,|e, n-1\rangle$

Time evolution operator: $\hat{U}(t) = e^{\frac{-i\hat{H}_{int}t}{\hbar}}$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{-i\hat{H}_{int}t}{\hbar}\right)^k$$

$$\hat{U}(t)|e, n-1\rangle = \sum_{k=0,2,\dots}^{\infty} \frac{(-1)^{\frac{k}{2}}}{k!}(\lambda\sqrt{n}\,t)^k|e, n-1\rangle - i\sum_{k=1,3,\dots}^{\infty} \frac{(-1)^{\frac{k-1}{2}}}{k!}(\lambda\sqrt{n}\,t)^k|g, n\rangle$$
$$= \cos(\lambda\sqrt{n}\,t)|e, n-1\rangle - i\sin(\lambda\sqrt{n}\,t)|g, n\rangle$$

$$\hat{U}(t)|g, n\rangle = -i\sum_{k=1,3,\dots}^{\infty} \frac{(-1)^{\frac{k-1}{2}}}{k!}(\lambda\sqrt{n}\,t)^k|e, n-1\rangle + \sum_{k=0,2,\dots}^{\infty} \frac{(-1)^{\frac{k}{2}}}{k!}(\lambda\sqrt{n}\,t)^k|g, n\rangle$$
$$= -i\sin(\lambda\sqrt{n}\,t)|e, n-1\rangle + \cos(\lambda\sqrt{n}\,t)|g, n\rangle$$

$\Rightarrow$ $\hat{U}(t) = \cos(\lambda\sqrt{n}\,t)|e, n-1\rangle\langle e, n-1| - i\sin(\lambda\sqrt{n}\,t)|e, n-1\rangle\langle g, n| - \cdots$
$\cdots - i\sin(\lambda\sqrt{n}\,t)|g, n\rangle\langle e, n-1| + \cos(\lambda\sqrt{n}\,t)|g, n\rangle\langle g, n|$

If the system starts in the state $|\Psi(0)\rangle = |g, n\rangle$

Then $|\Psi(t)\rangle = \hat{U}(t)|g, n\rangle = -i\sin(\lambda\sqrt{n}\,t)|e, n-1\rangle + \cos(\lambda\sqrt{n}\,t)|g, n\rangle$

$\Rightarrow$ Probability of measuring atom in excited state: $P_e(t) = |\langle e, n-1|\Psi(0)\rangle|^2 = \sin^2(\lambda\sqrt{n}\,t)$

Similar for ground state: $P_g(t) = \cos^2(\lambda\sqrt{n}\,t)$