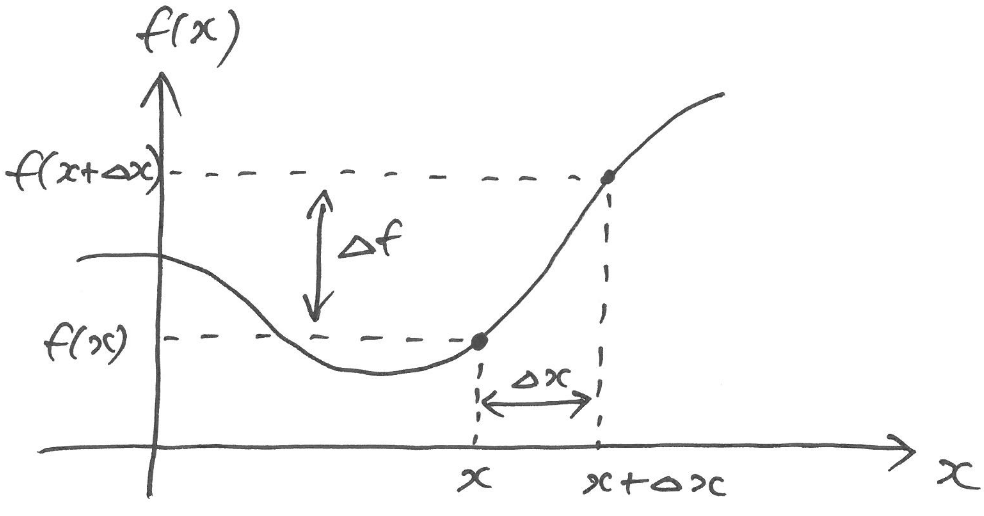
DIFFERENTIALS – EXACT & INEXACT

SINGLE VARIABLE

DEFINITION: THE DIFFERENTIAL df OF A FUNCTION f OF A SINGLE VARIABLE $f = f(x)$ IS DEFINED AS FOLLOWS

$$\boxed{df = \frac{\partial f}{\partial x}\, dx}$$

MEANING:



CONSIDER SMALL Δx, THEN

$$\frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} \approx \frac{\partial f}{\partial x}$$

OR $$f(x+\Delta x) - f(x) \simeq \frac{\partial f}{\partial x}\Delta x$$

OR $$\Delta f \simeq \frac{\partial f}{\partial x}\Delta x$$

IN THE LIMIT $\Delta x \to 0$ (THE INFINITESIMAL LIMIT)

$$\boxed{df = \frac{\partial f}{\partial x}dx}$$

INFINITESIMAL CHANGE IN f

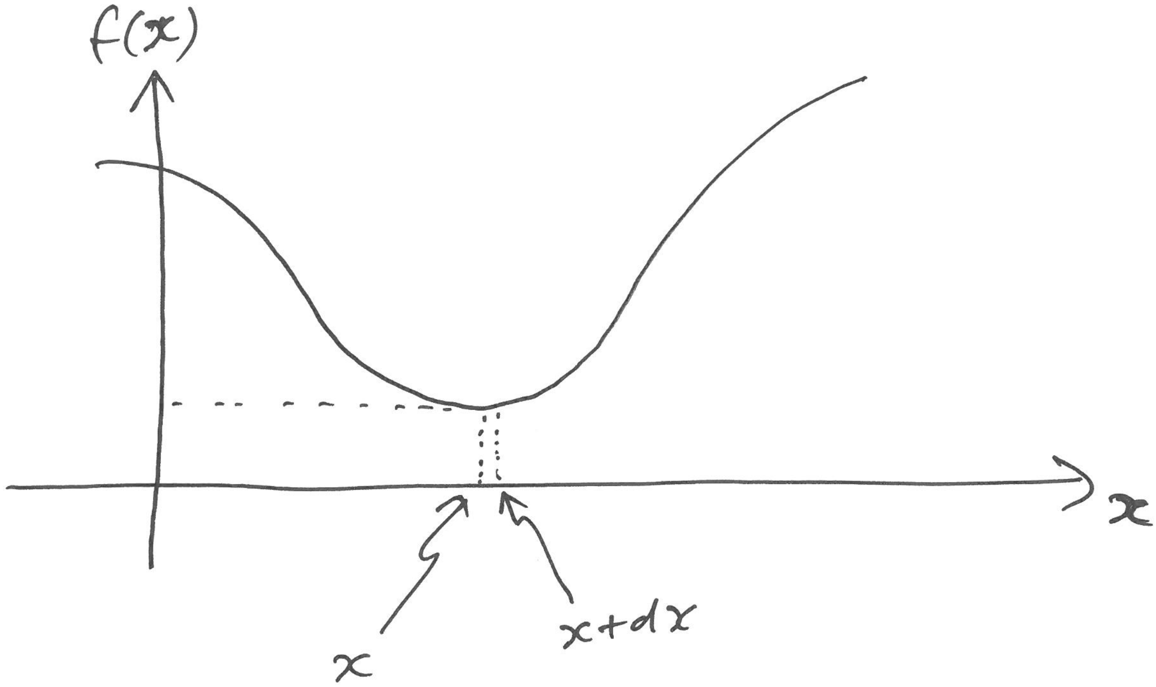
INFINITESIMAL CHANGE IN x

OR $$f(x+dx) - f(x) = \frac{\partial f}{\partial x}dx$$

OR $$\boxed{f(x+dx) = f(x) + \frac{\partial f}{\partial x}dx}$$

A POWER SERIES IN INFINITESIMAL dx
(YOU CAN IGNORE $(dx)^2$ & $(dx)^3$ etc)

SUPPOSE YOU WANT TO KNOW WHERE $f(x)$ IS STATIONARY (ie A MAX OR MIN)



THEN WE LOOK FOR THOSE PLACES WHERE f DOES NOT CHANGE WHEN WE CHANGE x BY A SMALL AMOUNT.

THAT IS, WE LOOK FOR $df = 0$ FOR <u>ARBITRARY</u> dx

SINCE $$df = \frac{\partial f}{\partial x}\, dx$$

THEN A STATIONARY POINT IS

$$0 = \frac{\partial f}{\partial x} dx \qquad dx \neq 0$$

$$\Rightarrow \quad \boxed{\frac{\partial f}{\partial x} = 0}$$

STATIONARY POINTS OCCUR WHERE THE DERIVATIVE IS ZERO.

<u>NOTICE</u>: IF $f(x) = g(x)\, h(x)$

THEN

$$df = \frac{\partial f}{\partial x} dx$$

$$= \frac{\partial}{\partial x}\left(g(x)\, h(x) \right) dx$$

$$= \left(\frac{\partial g}{\partial x} h + g \frac{\partial h}{\partial x} \right) dx$$

$$= \left(\frac{\partial g}{\partial x} dx \right) h + g \left(\frac{\partial h}{\partial x} dx \right)$$

$$= dg\, h + g\, dh$$

ie $$\boxed{df = dg\,h + g\,dh}$$

PRODUCT RULE

INTEGRATION : $x_1 \leq x \leq x_2$

$$\int_{x_1}^{x_2} df = \int_{x_1}^{x_2} \frac{\partial f}{\partial x} dx = f(x)\Big|_{x_1}^{x_2} = f(x_2) - f(x_1)$$

MULTIPLE VARIABLES

LET $f = f(u_1, u_2, u_3, \ldots, u_n)$, A FUNCTION

OF n INDEPENDENT VARIABLES u_i

WITH $i = 1, 2, \ldots, n$.

DEFINITION : THE DIFFERENTIAL OF f IS

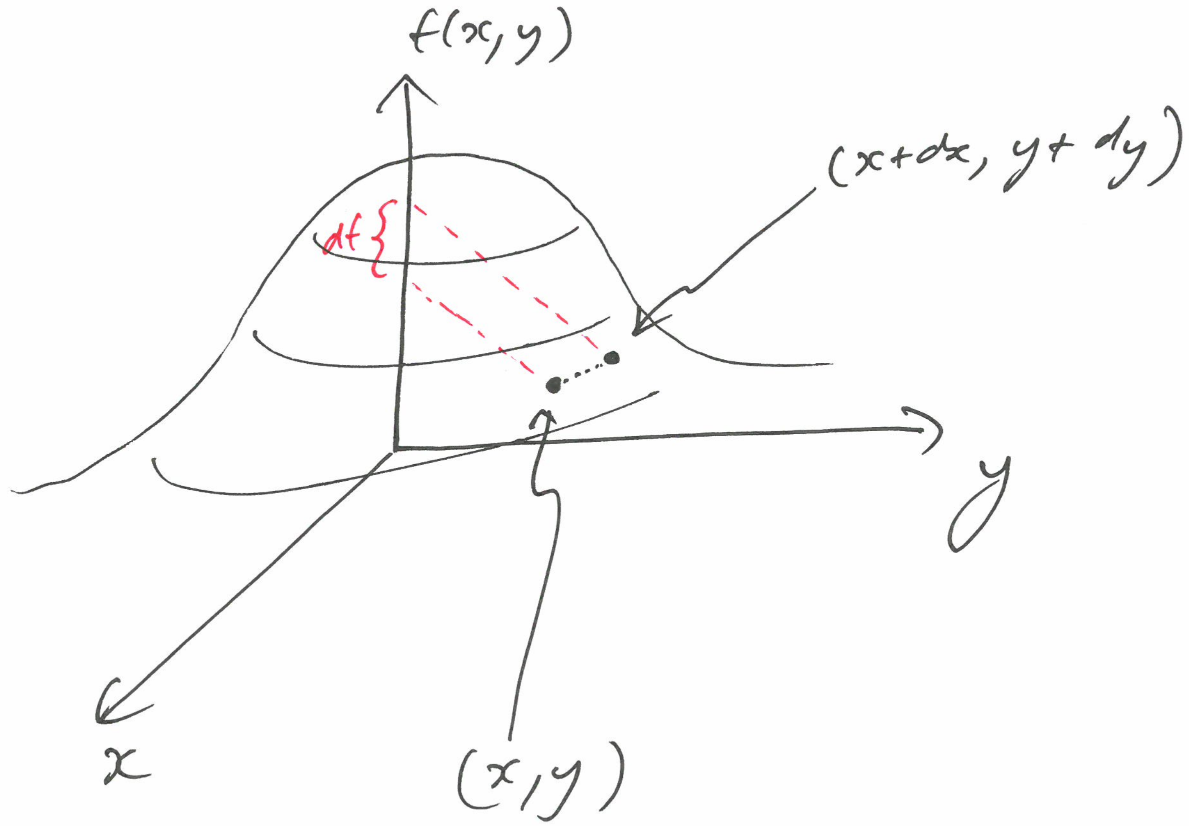
$$\boxed{df = \sum_{i=1}^{n} \frac{\partial f}{\partial u_i} du_i}$$

CONSIDER JUST 2 VARIABLES $f = f(x, y)$

THEN

$$\boxed{df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy}$$

MEANING:



AS YOU TAKE SMALL "STEPS" dx & dy

AWAY FROM THE POINT (x, y), THE

CHANGE IN THE FUNCTION IS df

$$\boxed{df = f(x+dx, y+dy) - f(x,y) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy}$$

INFINITESIMAL CHANGE IN f DUE TO INFINITESIMAL "STEPS" dx & dy IN x & y DIRECTIONS

INFINITESIMAL CHANGES IN x & y.

SUPPOSE YOU WISH TO FIND THE STATIONARY POINTS (ie LOCAL MAX OR MIN etc) WE SOLVE FOR $df = 0$ FOR ARBITRARY STEPS dx AND dy

ie $$0 = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$\Rightarrow \boxed{\frac{\partial f}{\partial x} = 0} \quad \& \quad \boxed{\frac{\partial f}{\partial y} = 0}$$

<u>AGAIN NOTICE</u> IF $f(x,y) = g(x,y)\,h(x,y)$

THEN

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$= \frac{\partial}{\partial x}(gh)\,dx + \frac{\partial}{\partial y}(gh)\,dy$$

$$= \left(\frac{\partial g}{\partial x}h + g\frac{\partial h}{\partial x}\right)dx + \left(\frac{\partial g}{\partial y}h + g\frac{\partial h}{\partial y}\right)dy$$

$$= h\left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy\right)$$

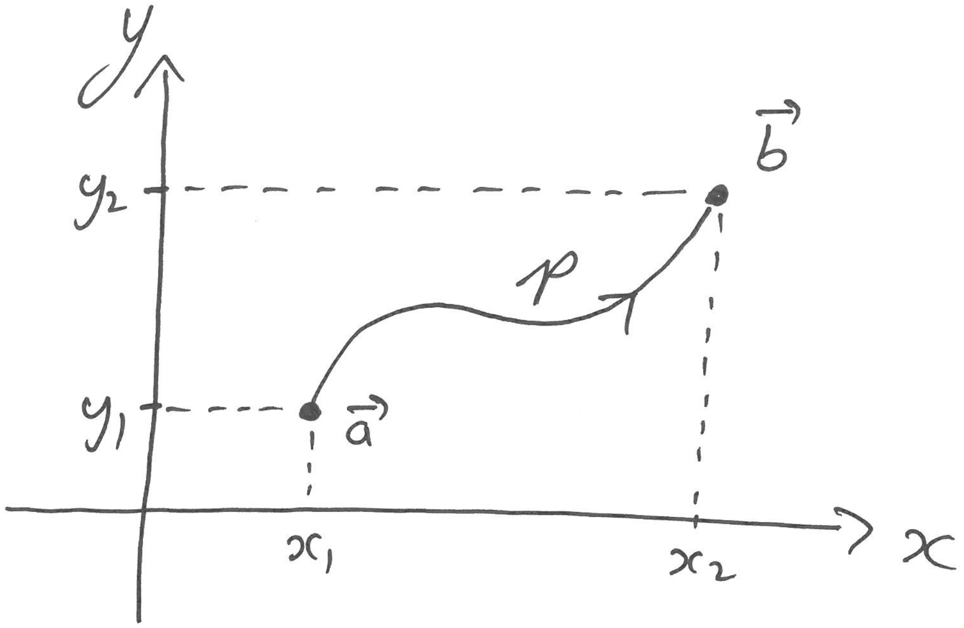
$$+ g\left(\frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy\right)$$

$$= h\,dg + g\,dh$$

$$\boxed{df = h\,dg + g\,dh}$$

PRODUCT RULE

INTEGRATION



PATH p FROM $\vec{a} = (x_1, y_1)$ TO $\vec{b} = (x_2, y_2)$

$$\int_p df = \int_p \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right)$$

$$= f(\vec{b}) - f(\vec{a})$$

$$= f(x_2, y_2) - f(x_1, y_1) \longleftarrow \text{TOTAL CHANGE}$$

PATH INDEPENDENT. df IS THE CHANGE IN A FUNCTION (IT IS THE "d" OF SOMETHING) $\propto$ ADDING UP (INTEGRATING) ALL THE CHANGES OVER ANY PATH JUST GIVES THE

TOTAL CHANGE $f(\vec{b}) - f(\vec{a})$.

df IS AN <u>EXACT DIFFERENTIAL</u>.

AN <u>EXACT DIFFERENTIAL</u> IS THE "d" OF SOME FUNCTION

<u>EXAMPLE</u> :

$$\boxed{\omega = y\,dx + x\,dy}$$

IS AN EXACT DIFFERENTIAL SINCE IT IS

$$\omega = d(xy) = dx\,y + y\,dx$$

AN EXAMPLE OF AN INEXACT DIFFERENTIAL IS

$$\boxed{\beta = x\,dy - y\,dx}$$

β CANNOT BE EXPRESSED AS THE "d" OF ANY FUNCTION

PROOF: SUPPOSE IT WAS POSSIBLE TO EXPRESS $\beta = df$ FOR SOME $f = f(x,y)$.

IF TRUE THEN

$$\beta = df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$= x\,dy - y\,dx$$

$$\Longrightarrow \boxed{\frac{\partial f}{\partial x} = -y} \quad \& \quad \boxed{\frac{\partial f}{\partial y} = x}$$

$$\frac{\partial f}{\partial x} = -y \Longrightarrow \boxed{f(x,y) = -xy + g(y)} \quad (1)$$

FOR SOME FUNCTION g OF y ONLY

$$\frac{\partial f}{\partial y} = x \Longrightarrow \boxed{f(x,y) = xy + h(x)} \quad (2)$$

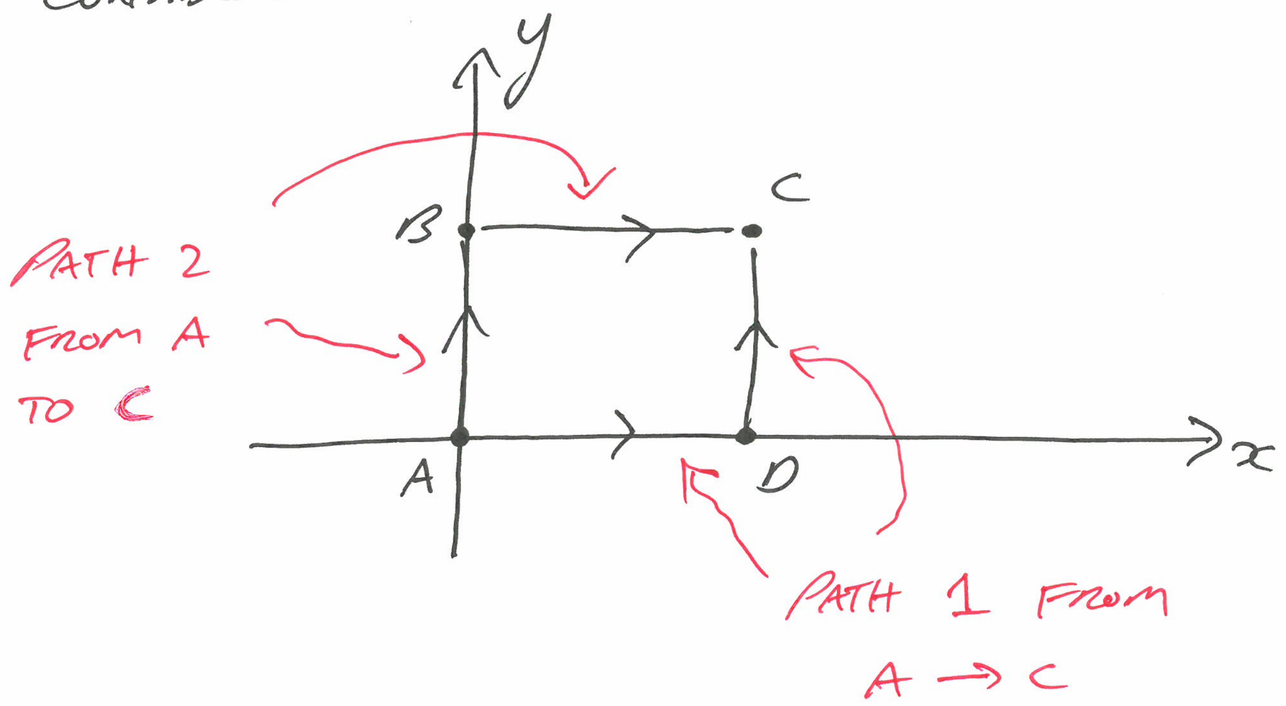
FOR SOME FUNCTION OF h OF x ONLY

EXPRESSIONS (1) & (2) ARE INCONSISTENT &
SO IT IS NOT POSSIBLE TO EXPRESS
β AS THE d OF ANY FUNCTION $f(x,y)$.

PATH DEPENDENCE OF AN INEXACT DIFFERENTIAL

LET $\qquad \beta = x\,dy - y\,dx$

& CONSIDER



PATH 1 : $A \to D \to C$

PATH 2 : $A \to B \to C$

$$\left.\begin{aligned} A &: (0,0) \\ B &: (0,1) \\ C &: (1,1) \\ D &: (1,0) \end{aligned}\right\} \text{COORDS OF POINTS}$$

INTEGRATE β FROM $A \to C$ VIA

PATH 1.

$$\int_{A\to D} \beta = \int_{A\to D} (x\,\cancelto{0}{dy} - y\,dx)$$

$y = 0, \; dy = 0$

$0 \le x \le 1$

$$= \int_0^1 (-y)\,dx = \int_0^1 0\,dx = 0$$

$$\int_{D\to C} \beta = \int_{D\to C} (x\,dy - y\,\cancelto{0}{dx})$$

$x = 1, \; dx = 0$

$0 \le y \le 1$

$$= \int_0^1 x\,dy = \int_0^1 1\,dy = 1$$

$$\int_{\text{PATH } 1} \beta = 0 + 1 = 1$$

INTEGRATE β FROM $A \to C$ VIA PATH 2

$$\int_{A\to B} \beta = \int_{A\to B} (x\,dy - y\,\cancel{dx}^{\,0})$$

$x = 0,\ dx = 0$

$0 \le y \le 1$

$$= \int_0^1 x\,dy = \int_0^1 0\,dy = 0$$

$$\int_{B\to C} \beta = \int_{B\to C} (x\,\cancel{dy}^{\,0} - y\,dx)$$

$y = 1,\ dy = 0$

$0 \le x \le 1$

$$= \int_0^1 (-y)\,dx = \int_0^1 (-1)\,dx = -1$$

$$\int_{\text{PATH 2}} \beta = 0 + (-1) = -1$$

$$\boxed{\int_{\text{PATH 1}} \beta \neq \int_{\text{PATH 2}} \beta}$$