MPS 2020 Test Solutions

The Physics of Particles (University of Western Australia)

Dear Class!

The solutions that follow are written in an a v. detailed form with the intention of improving your understanding.

They contain, in many cases, more material than is necessary. In a test/exam setting you must pick and choose the important parts to demonstrate your understanding. I hope this helps your revision.

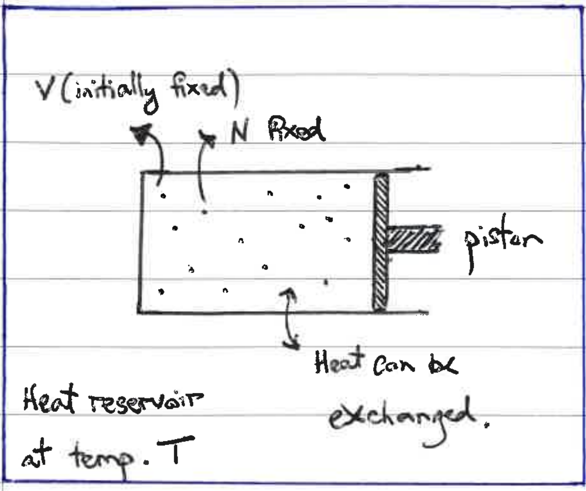
Cheers Daniel

Total marks 50

I

2020 Many Particle Systems Test – SOLUTIONS

Q1 Consider the following system, fixed in volume (initially) & with a fixed number of particles in equilibrium with a heat reservoir:



A sample of an ideal, monatomic gas is initially confined to this volume with the piston locked in position.

(a) Upon release of the piston, the gas in general will enter a non-equilibrium state. However, once equilibrium is re-established the direction of spontaneous change is determined by the requirement that $dF_{sys} < 0$, or $\Delta F_{sys} < 0$ (macroscopically).

The given expression for $F(T,V,N) = \frac{3}{2} N k_B T \ln\left(\frac{2\pi\hbar^2 N^{2/3}}{m k_B T}\right) - N k_B T \ln V$.

So that: $\Delta F = F(T, V_f, N) - F(T, V_i, N)$

$$= -N k_B T \left(\ln V_f - \ln V_i\right) = N k_B T \ln\left(V_i / V_f\right).$$

Now $\Delta F < 0 \Rightarrow \ln\left(\frac{V_i}{V_f}\right) < 0 \Rightarrow V_f > V_i$

So upon release, spontaneous change will occur so that the piston moves to increase the volume of the monatomic gas.

4 marks

(b). For the above process determine ΔS_{system}.

We have $F_{sys} = \bar{E}_{sys} - T S_{sys}$. with $\bar{E}_{sys} = \frac{3}{2} N k_B T$

and $\Delta F_{sys} = \Delta \bar{E}_{sys} - T \Delta S_{sys}$ as the process is isothermal.

Furthermore: $\Delta \bar{E}_{sys} = \frac{3}{2} N k_B \Delta T = 0$.

II

4 marks

So $\Delta F_{sys} = -T\Delta S_{sys} < 0 \Rightarrow \Delta S_{sys} > 0$.

So upon release of the piston the entropy of the monatomic gas must increase. ie microscopically, the larger volume leads to more quantum states acceseable to the gas atoms.

Q2. The system is N, indistinguishable (eg, as in a gas), identical, non-interacting, spin 0 bosons. It occupies a volume V and is in contact with a heat reservoir at temperature T. As they are BOSONS, there is no restriction as to how many particles can occupy a single particle eigenstate. The single particle spectrum consists of only two eigenstates $|0\rangle$ & $|\epsilon\rangle$.

(a) The list of the possible N-particle eigenstates for the system is as follows:

micro-state	$\lvert 0\rangle$	$\lvert \epsilon\rangle$	$E_{state} = N_{\lvert 0\rangle} 0 + N_{\lvert\epsilon\rangle}\epsilon$	$\lvert\psi\rangle$
1	$\underbrace{AAA\ldots A}_{N}$	–	0	$\lvert 0\rangle_1\lvert 0\rangle_2\lvert 0\rangle_3\cdots\lvert 0\rangle_N$
2	$\underbrace{AAA\ldots A}_{N-1}$	A	ϵ	$\lvert 0\rangle_1\lvert 0\rangle_2\cdots\lvert 0\rangle_{N-1}\lvert\epsilon\rangle_N + \lvert 0\rangle_1\lvert 0\rangle_2\cdots\lvert\epsilon\rangle_{N-1}\lvert 0\rangle_N + \cdots$
3	$\underbrace{AAA\ldots A}_{N-2}$	AA	2ϵ	$\lvert 0\rangle_1\lvert 0\rangle_2\cdots\lvert\epsilon\rangle_{N-1}\lvert\epsilon\rangle_N$ + appropriate terms
⋮	⋮	⋮	⋮	
N	A	$\overbrace{A\ldots AAA}^{N-1}$	$(N-1)\epsilon$	$\lvert 0\rangle_1\lvert\epsilon\rangle_2\cdots\lvert\epsilon\rangle_N + \lvert\epsilon\rangle_1\lvert 0\rangle_2\cdots\lvert\epsilon\rangle_N + \cdots$
N+1	–	$\underbrace{A\ldots AAA}_{N}$	$N\epsilon$.	$\lvert\epsilon\rangle_1\lvert\epsilon\rangle_2\cdots\lvert\epsilon\rangle_N$.

THIS COLUMN FOR INTEREST ONLY!!

4 marks.

So there are N+1, N particle energy eigenstates as above with energies $n\epsilon$ with n an integer $0 \leq n \leq N$.

III

(b). $$Z(T,V,N) = \sum_{\substack{N \text{ particle} \\ \text{states}}} e^{-E_{\text{state}}/k_BT} = 1 + e^{-\epsilon/k_BT} + e^{-2\epsilon/k_BT} + \ldots + e^{-N\epsilon/k_BT}$$

But $$e^{-\epsilon/k_BT} Z(T,V,N) = e^{-\epsilon/k_BT} + e^{-2\epsilon/k_BT} + \ldots e^{-(N+1)\epsilon/k_BT}.$$

$$\Rightarrow \left(1 - e^{-\epsilon/k_BT}\right) Z(T,V,N) = 1 - e^{-(N+1)\epsilon/k_BT}$$

3 marks

$$\Rightarrow Z(T,V,N) = \frac{1 - e^{-(N+1)\epsilon/k_BT}}{1 - e^{-\epsilon/k_BT}}$$

(closed form for full marks)

(c). $$F(T,V,N) = -k_BT \ln Z(T,V,N)$$

2 marks

$$= -k_BT \ln \left[\frac{1 - e^{-(N+1)\epsilon/k_BT}}{1 - e^{-\epsilon/k_BT}} \right]$$

(d) $$S = -\frac{\partial}{\partial T} F(T,V,N) = \frac{\partial}{\partial T} k_BT \ln\left(1 - e^{-(N+1)\epsilon/k_BT}\right) - \frac{\partial}{\partial T} k_BT \ln\left(1 - e^{-\epsilon/k_BT}\right)$$

$$= k_B \ln\left(1 - e^{-(N+1)\epsilon/k_BT}\right) + k_BT \cdot \frac{-1}{1 - e^{-(N+1)\epsilon/k_BT}} e^{-(N+1)\epsilon/k_BT} \cdot \frac{(N+1)\epsilon}{k_BT^2}$$

$$- k_B \ln\left(1 - e^{-\epsilon/k_BT}\right) - k_BT \cdot \frac{-1}{1 - e^{-\epsilon/k_BT}} e^{-\epsilon/k_BT} \cdot \frac{\epsilon}{k_BT^2}$$

2 marks

$$= k_B \ln\left(\frac{1 - e^{-(N+1)\epsilon/k_BT}}{1 - e^{-\epsilon/k_BT}}\right) + \frac{\epsilon}{T}\left[\frac{1}{e^{\epsilon/k_BT} - 1} - \frac{N+1}{e^{(N+1)\epsilon/k_BT} - 1}\right]$$

(e) Consider the $\lim_{T \to 0} S$, ie of the above function

As $T \to 0$, $1/T \to \infty$ so both $e^{-(N+1)\epsilon/k_BT}$ and $e^{-\epsilon/k_BT} \to 0$.

This means the first term in the above expression $\to k_B \ln 1 = 0$.

Next, consider $T\left(e^{\epsilon/k_BT} - 1\right) = T\left(1 + \frac{\epsilon}{k_BT} + \frac{1}{2!}\left(\frac{\epsilon}{k_BT}\right)^2 + \ldots - 1\right)$

then it is clear that the denominators in the 2nd term in the above function $\to \infty$.

IV

2 marks

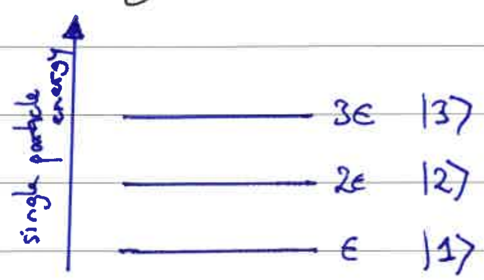
So ! $\lim_{T \to 0} S = 0$. On physical grounds this makes sense as the system "freezes out" into the N particle ground state. or microstate 1 in the table in (a). There is no thermal energy available $(k_B T \to 0)$ from the heat reservoir to excite the system into N particle microstates > 1.
As we know with complete certainty the state of the system, the entropy is 0 or $S = k_B \ln \Omega = k_B \ln 1 = 0 \ JK^{-1}$

(f). For N distinguishable particles

4 marks

$$Z(T,V,N) = Z(T,V,1)^N = \underline{\left(1 + e^{-\epsilon/k_B T}\right)^N}$$

Q3. Our system consists of 3 particles whose occupancy of a single particle energy eigenstate is restricted to being 0, 1 or 2.
The single particle energy spectrum is simply illustrated to the right :



if the particles are indentical but indistinguishable find $Z(T,V,3)$ for our 3 particle system.

Possible system configurations are as listed.

$\vert 1\rangle$	$\vert 2\rangle$	$\vert 3\rangle$	E_{state}
AA	A	–	4ϵ
AA	–	A	5ϵ
A	AA	–	5ϵ
–	AA	A	7ϵ
A	–	AA	7ϵ
–	A	AA	8ϵ
A	A	A	6ϵ.

$$Z(T,V,3) = \sum_{\substack{\text{three particle} \\ \text{energy eigenstate}}} e^{-E_{state}/k_B T}$$

V

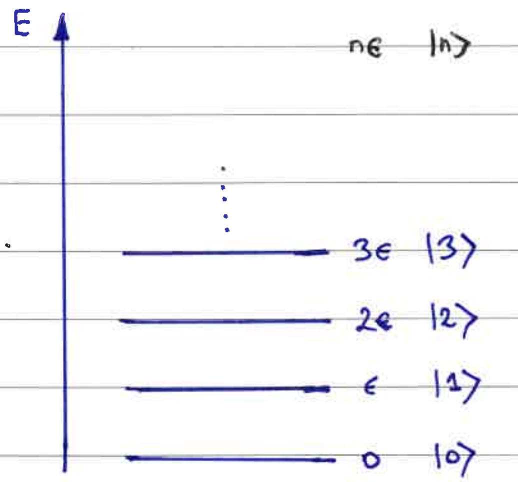
$$Z(T,V,3) = e^{-4\epsilon/k_BT} + 2e^{-5\epsilon/k_BT} + e^{-6\epsilon/k_BT} + 2e^{-7\epsilon/k_BT} + e^{-8\epsilon/k_BT}$$

8 marks

$$= z^4 + 2z^5 + z^6 + 2z^7 + z^8$$

where $z = e^{-\epsilon/k_BT}$

Q3. Here the system is 3 non-interacting particles, each with a single particle energy spectrum $n\epsilon$ with $n = 0, 1, 2, 3, \ldots$ as shown to the right. (like the simple Harmonic oscillator but without the zero point energy).



(a). The system is isolated from it's environment with total energy $E = 3\epsilon$. Under these circumstances, the micro canonical distribution applies and we need to list all the accessible micro-states such that the total energy $E = \epsilon_{\text{particle 1}} + \epsilon_{\text{particle 2}} + \epsilon_{\text{particle 3}}$ is 3ϵ. The entropy will then be $S = k_B \ln \Omega_s$.

(i) The 3 particles are distinguishable (as in sites in a lattice)

0	ϵ	2ϵ	3ϵ	microstate
AB	–	–	C	1
BC	–	–	A	2
CA	–	–	B	3
A	B	C	–	4, 5, 6, 7, 8, 9
–	ABC	–	–	10.

3! permutations of ABC leads to 6 microstates

3 marks

So there are 10 microstates s.t. the total system energy is 3ϵ.

$$\Rightarrow S = k_B \ln 10 \quad JK^{-1}$$

(ii) The particles are indistinguishable Bosons

3 marks

0	ϵ	2ϵ	3ϵ	microstate
AA	-	-	A	1
A	A	A	-	2
-	AAA	-	-	3

Only 3 microstates are accessible $\Rightarrow$ $S = k_B \ln 3 \ JK^{-1}$

(iii) The particles are indistinguishable Fermions. Here, the only state consistent with Pauli Exclusion Principle is

3 marks

0	ϵ	2ϵ
A	A	A

$\Rightarrow$ $S = k_B \ln 1 = 0 \ JK^{-1}$

(b). In the case that the particles are distinguishable & the system follows the microcanonical distribution s.t. $E = 3\epsilon$. Determine $\bar{N}_{|0\rangle}$

4 marks

$$\bar{N}_{|0\rangle} = \sum_{\substack{\text{microstates} \\ i}} N_i P_i = 2 \times \underbrace{\frac{3}{10}}_{\text{microstates } 1,2,3} + 1 \times \underbrace{\frac{6}{10}}_{\text{microstates } 4 \to 9} + 0 \times \underbrace{\frac{1}{10}}_{\text{microstate } 10}.$$

$$= 1.2$$

(c) Finally consider our 3 particle system when it is placed in contact with a heat reservoir at temperature T.
Determine $Z(T, V, 3)$ in the case where the particles are indistinguishable BOSONS.

(P.T.O).

Since the single particle energy spectrum is unbounded (countably infinite) it should be intuitively obvious that there will be an infinite number of 3-particle states contributing to $Z(T,V,3)$.

Some of the low energy 3-particle states are tabulated below:

$\lvert 0\rangle$	$\lvert 1\rangle$	$\lvert 2\rangle$	$\lvert 3\rangle$	....	E_{state}
AAA	—	—	—		0
AA	A	—	—		ϵ
AA	—	A	—		2ϵ
A	AA	—	—		2ϵ
AA	—	—	A		3ϵ
A	A	A			3ϵ
	AAA				3ϵ
⋮					⋮

Although it is tempting to speculate that $Z(T,V,3) \overset{?}{=} \sum_{n=0}^{\infty} n e^{-n\epsilon}$, this pattern is broken at higher n (try $n=8$).

The best we can do is appeal to the Generating Functional from lectures.

So let the occupation numbers of the single particle states be $(N_0, N_1, N_2, \dots)$
$\lvert 0\rangle \ \lvert 1\rangle \ \lvert 2\rangle \ \dots$

Then $$\mathcal{Z}(T,V,\alpha) = \sum_{N=0}^{\infty} \alpha^N Z(T,V,N)$$

$$= \sum_{(N_0,N_1,\dots)} \alpha^{N_0+N_1+\dots} e^{-(N_0\epsilon_0 + N_1\epsilon + N_2 \cdot 2\epsilon + \dots)/k_BT}$$

($\epsilon_0 = 0$)

$$= \left(\sum_{N_0=0}^{\infty} \alpha^{N_0} e^{-N_0\epsilon_0/k_BT}\right)\left(\sum_{N_1=0}^{\infty} \alpha^{N_1} e^{-N_1\epsilon/k_BT}\right)\dots$$

Let $\xi = e^{-\epsilon/k_BT}$

$$= \left(\frac{1}{1-\alpha}\right)\left(\frac{1}{1-\alpha\xi}\right)\left(\frac{1}{1-\alpha\xi^2}\right)\dots$$

$$= \prod_{n=0}^{\infty} \frac{1}{(1-\alpha\xi^n)}$$

So! (P.T.O)

To determine $Z(T,V,3)$ we could expand this series and collect the coefficients of α^3.

A more economical way of saying this is to evaluate the following for the case $N=3$.

4 marks

$$Z(T,V,N) = \frac{1}{N!}\frac{\partial^N}{\partial \alpha^N} \mathcal{Z}(T,V,\alpha)\Big|_{\alpha=0},$$

This is obviously more easier said than done. Any arguments hinting at this attracts some marks.