



MPS TEST S2 2022 - practice problems

The Physics of Particles (University of Western Australia)



THE UNIVERSITY OF
**WESTERN
AUSTRALIA**

Physics

TEST 1, SEMESTER 2, 2022

**PHYS2002
Many Particle Systems**

FAMILY NAME: _____ GIVEN NAMES: _____

STUDENT ID:

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 SIGNATURE: _____

This Paper Contains: **4 pages (including title page) plus an equation sheet**
Time allowed: **45 minutes**

INSTRUCTIONS:

Attempt all questions in the answer booklet provided. To receive full marks you should show all reasoning and working and define any symbols you introduce.

Marks for parts of questions are displayed in square brackets in the right-hand margin. There are a total of 45 marks available.

You may NOT bring notes or other written material, in paper or electronic form, into the test.

Approved scientific calculators may be used.

This test is worth 20% of the unit.

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Questions

1. Consider a system with a single chemical species with chemical potential μ , and suppose we introduce a thermodynamic potential J defined by $J = E - \mu N$.

(a) Prove that J is “naturally” a function of the variables S , V , and μ (i.e. show that in an infinitesimal quasistatic process, the change in J depends only on the changes in these variables for an arbitrary system). [3 marks]

(b) Hence determine the three equations of state involving the partial derivatives of J . [4 marks]

2. Consider a system of fixed volume V consisting of a fixed number of N particles of a single chemical species in contact with a heat reservoir at temperature T . The system is initially partitioned into two subsystems 1 and 2 by a rigid and impenetrable partition. The volumes and particle numbers of the two subsystems are V_1 , N_1 and V_2 , N_2 respectively. The free energy of the total composite system, $F_{\text{total}}(T, V, N)$, is the sum of the free energies $F_1(T, V_1, N_1)$ and $F_2(T, V_2, N_2)$, the free energies of the subsystems 1 and 2 respectively.

Suppose that the partition is made free to move. The direction of movement will be determined by the condition that the free energy of the system must not increase.

- (a) If the volume of subsystem 1 is changed by an infinitesimal amount dV_1 , show that the change in the free energy of the combined system is

$$dF_{\text{total}} = \left(\frac{\partial F_1(T, V_1, N_1)}{\partial V_1} - \frac{\partial F_2(T, V_2, N_2)}{\partial V_2} \right) dV_1.$$

[4 marks]

- (b) Show that the requirement that $dF_{\text{total}} < 0$ means that the partition will move toward the side of lower pressure. [4 marks]

see over

3. Consider a system of three identical particles, each of which can occupy three energy eigenstates with energies $0, 2\epsilon$ and 4ϵ .

(a) Consider the case where the particles are distinguishable, and labelled A, B and C. If the system is isolated and has energy 6ϵ , determine the probability that particle A has zero energy. [5 marks]

(b) If the particles are indistinguishable bosons and in contact with a heat reservoir at temperature T , give an expression for the canonical partition function in terms of $y = e^{-\epsilon/k_B T}$. [5 marks]

(c) If the particles are indistinguishable fermions (i.e. they obey the Pauli exclusion principle) in contact with a heat reservoir at temperature T , give an expression for the canonical partition function in terms of $y = e^{-\epsilon/k_B T}$. [5 marks]

4. For a system in contact with a heat reservoir, the mean energy can be computed as

$$\bar{E} = k_B T^2 \frac{\partial}{\partial T} \ln Z(T, V, N),$$

where $Z(T, V, N)$ is the canonical partition function.

Consider a particle in a harmonic oscillator potential. The energy eigenstates have energies $\epsilon_n = (n + \frac{1}{2})\hbar\omega$, $n = 0, 1, 2, \dots$, where ω is the angular frequency of oscillation.

(a) Compute the canonical partition function for the particle.

Note: $1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$. [5 marks]

(b) Determine the average energy of the particle when it is in contact with a heat reservoir of temperature T . [5 marks]

(c) Using (b), determine the high and low temperature limits of the average energy, and give a physical explanation for these.

Note: $e^x \approx 1 + x$ for $|x| \ll 1$. [Hint: the harmonic oscillator has two ‘degrees of freedom’.] [5 marks]

END OF TEST

MPS EQUATION SHEET FOR TESTS AND EXAMS

- Notation: ν denotes number of moles, N denotes number of particles, system states are generically labeled n , and single particle states are generically labeled i .

$\nu = N/N_A$, where N_A is Avagadro's number.

- Boltzmann constant $k_B = 1.38 \times 10^{-23} \text{ J K}^{-1} = 8.62 \times 10^{-5} \text{ eV K}^{-1}$
- $\hbar = 6.6 \times 10^{-16} \text{ eV s} = 1.05 \times 10^{-34} \text{ J s}$.
- Avagadro's number $N_A = 6.02 \times 10^{23} \text{ mole}^{-1}$.
- If $p = |\vec{p}|$, $\int d^3\vec{p} f(p) = 4\pi \int_0^\infty dp p^2 f(p)$, $\int d^2\vec{p} f(p) = 2\pi \int_0^\infty dp p f(p)$.
- Define $I(n) \equiv \int_0^\infty dx x^n e^{-\alpha x^2}$. Then $I(0) = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$, $I(1) = \frac{1}{2\alpha}$, $I(2) = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$, $I(3) = \frac{1}{2\alpha^2}$.
- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$
- $1 + x + x^2 + \dots + x^n = \frac{1-x^{n+1}}{1-x}$
- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$, $e^x \approx 1 + x$ for $|x| \ll 1$.
- $\ln(1+x) \approx x$ for $|x| \ll 1$.
- Time independent Schrödinger equation:

$$-\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \psi(x, y, z) = E \psi(x, y, z).$$

- 1st law of thermodynamics: $dE = \delta Q + \delta W + \sum_i \mu_i dN_i$.

For quasistatic processes: $\delta Q = T dS$, $\delta W = -p dV$.

- Entropy: $S \equiv S(E, V, N)$

$$\frac{\partial S(E, V, N)}{\partial E} = \frac{1}{T}, \quad \frac{\partial S(E, V, N)}{\partial V} = \frac{p}{T}, \quad \frac{\partial S(E, V, N)}{\partial N_i} = -\frac{\mu_i}{T}.$$

- Free energy: $F(T, V, N) = E - T S$,

$$\frac{\partial F(T, V, N)}{\partial T} = -S, \quad \frac{\partial F(T, V, N)}{\partial V} = -p, \quad \frac{\partial F(T, V, N)}{\partial N_i} = \mu_i.$$

- For a system in contact with a heat and particle reservoir: $\Phi(T, V, \mu) = E - TS - \mu N$,

$$\frac{\partial \Phi(T, V, \mu)}{\partial T} = -S, \quad \frac{\partial \Phi(T, V, \mu)}{\partial V} = -p, \quad \frac{\partial \Phi(T, V, \mu)}{\partial \mu} = -\bar{N}.$$

- For an ideal gas, $pV = \nu RT = Nk_B T$, and $E = \nu c_V T$.
- For an isolated system (E, V, N fixed), all states are equally likely, and $S(E, V, N) = k_B \ln \Omega(E, V, N)$, where $\Omega(E, V, N)$ is the number of states accessible to the system.

- For a system in equilibrium with a heat reservoir at temperature T ,

$$P_n^{(eq)} = \frac{e^{-E_n/k_B T}}{Z(T, V, N)},$$

where $Z(T, V, N) = \sum_n e^{-E_n/k_B T}$ is the canonical partition function for the system, involving a sum over all system states n with energies E_n .

$$F(T, V, N) = -k_B T \ln Z(T, V, N).$$

- For a system in equilibrium with a heat and particle reservoir,

$$P_n^{(eq)} = \frac{e^{-(E_n - \mu N_n)/k_B T}}{\mathcal{Z}(T, V, \mu)},$$

where $\mathcal{Z}(T, V, \mu) = \sum_n e^{-(E_n - \mu N_n)/k_B T}$ is the grand canonical partition function for the system, and μ is the chemical potential.

$\Phi(T, V, \mu) = -k_B T \ln \mathcal{Z}(T, V, \mu)$ is the thermodynamic potential appropriate to a system in contact with a heat and particle reservoir.

- Maxwell-Boltzmann statistics: identical distinguishable particles.
- Fermi-Dirac statistics: identical indistinguishable particles obeying the Pauli exclusion principle.
- Bose-Einstein statistics: identical indistinguishable particles, any number of which can be placed in a given single particle state.
- For N identical particles obeying Maxwell-Boltzmann statistics, $Z(T, V, N) = (Z_1)^N$, where Z_1 is the canonical partition function for a single particle.
For modified Maxwell-Boltzmann statistics (approximation to allow for indistinguishability), $Z(T, V, N) \approx \frac{1}{N!} (Z_1)^N$.
- Grand canonical partition functions for Fermi-Dirac and Bose-Einstein statistics:

$$\mathcal{Z}_{FD}(T, V, \mu) = \prod_i (1 + e^{-(\epsilon_i - \mu)/k_B T}), \quad \mathcal{Z}_{BE}(T, V, \mu) = \prod_i (1 - e^{-(\epsilon_i - \mu)/k_B T})^{-1},$$

where i labels single particle states, with energies ϵ_i .

$$\ln \mathcal{Z}_{FD}(T, V, \mu) = \sum_i \ln(1 + e^{-(\epsilon_i - \mu)/k_B T}), \quad \ln \mathcal{Z}_{BE}(T, V, \mu) = - \sum_i \ln(1 - e^{-(\epsilon_i - \mu)/k_B T}).$$

- Distribution functions (mean occupation numbers for single particle states) for Fermi-Dirac and Bose-Einstein statistics:

$$(\bar{N}_i)_{FD} = \frac{1}{e^{(\epsilon_i - \mu)/k_B T} + 1}, \quad (\bar{N}_i)_{BE} = \frac{1}{e^{(\epsilon_i - \mu)/k_B T} - 1}.$$