MPS PRACTICE PROBLEM SET 4

SOLUTIONS

1. The harmonic oscillators are arranged in a line ⇒ they are distinguishable by their position in the line ⇒ Maxwell-Boltzmann statistics applies.

- Label the oscillators A, B, C, D.

We can enumerate the states by specifying the energy eigenvalue for each oscillator.

States with total energy $3\hbar\omega$ can be formed as $\frac{1}{2}\hbar\omega + \frac{1}{2}\hbar\omega + \frac{1}{2}\hbar\omega + \frac{3}{2}\hbar\omega$.

- So the possible states are:

A	B	C	D
$\frac{1}{2}\hbar\omega$	$\frac{1}{2}\hbar\omega$	$\frac{1}{2}\hbar\omega$	$\frac{3}{2}\hbar\omega$
$\frac{1}{2}\hbar\omega$	$\frac{1}{2}\hbar\omega$	$\frac{3}{2}\hbar\omega$	$\frac{1}{2}\hbar\omega$
$\frac{1}{2}\hbar\omega$	$\frac{3}{2}\hbar\omega$	$\frac{1}{2}\hbar\omega$	$\frac{1}{2}\hbar\omega$
$\frac{3}{2}\hbar\omega$	$\frac{1}{2}\hbar\omega$	$\frac{1}{2}\hbar\omega$	$\frac{1}{2}\hbar\omega$

- Since the system is isolated, all four states are equally likely to be occupied.

$\Rightarrow$ probability that the first oscillator has energy $\frac{1}{2}\hbar$

$$= 3 \times \frac{1}{4} = \frac{3}{4}$$

[We could also do the counting of states as :

$\frac{1}{2}\hbar\omega$	$\frac{3}{2}\hbar\omega$
A B C	D
A B D	C
A C D	B
B C D	A

] .

2(a) For a system in contact with a heat reservoir, the probability that a state with energy E_n is occupied is

$$P_n = \frac{e^{-E_n/k_B T}}{Z}$$

and the mean energy is

$$\overline{E} = \sum_n E_n P_n .$$

- Stating with the right hand side of the expression given,

$$k_B T^2 \frac{\partial}{\partial T} \ln Z$$

$$= k_B T^2 \frac{1}{Z} \frac{\partial Z}{\partial T}$$

$$= k_B T^2 \frac{1}{Z} \frac{\partial}{\partial T} \sum_n e^{-E_n/k_B T}$$

$$= \cancel{k_B T^2} \frac{1}{Z} \sum_n \left(-\frac{E_n}{\cancel{k_B}}\right)\left(-\frac{1}{\cancel{T^2}}\right) e^{-E_n/k_B T}$$

$$= \frac{1}{Z} \sum_n E_n e^{-E_n/k_B T}$$

$$= \sum_n E_n \frac{e^{-E_n/k_B T}}{Z}$$

$$= \sum_n E_n P_n$$

$$= \overline{E} .$$

(b) Since the particles are distinguishable, Maxwell - Boltzmann statistics apply, and the canonical partition function for N identical particles is

$$Z = (Z_1)^N,$$

where Z_1 is the canonical partition function for <u>one</u> of the particles.

. Since each particle has energy eigenstates with energy eigenvalues 0 and ε,

$$Z_1 = e^{-0/k_BT} + e^{-\varepsilon/k_BT}$$
$$= 1 + e^{-\varepsilon/k_BT}$$

$$\Rightarrow Z = \left(1 + e^{-\varepsilon/k_BT}\right)^N$$

(c) Using (a)

$$\bar{E} = k_B T^2 \frac{\partial}{\partial T} \ln Z$$

$$= k_B T^2 \frac{\partial}{\partial T} \ln\left[\left(1 + e^{-\varepsilon/k_B T}\right)^N \right]$$

$$= k_B T^2 \frac{\partial}{\partial T} N \ln\left(1 + e^{-\varepsilon/k_B T}\right)$$

$$= N k_B T^2 \frac{1}{\left(1 + e^{-\varepsilon/k_B T}\right)} \frac{\partial}{\partial T}\left(1 + e^{-\varepsilon/k_B T}\right)$$

$$= N \cancel{k_B} \cancel{T^2} \frac{1}{\left(1 + e^{-\varepsilon/k_B T}\right)} \left(-\frac{\varepsilon}{\cancel{k_B}}\right)\left(-\frac{1}{\cancel{T^2}}\right) \cdot e^{-\varepsilon/k_B T}$$

$$= N \frac{\varepsilon e^{-\varepsilon/k_B T}}{\left(1 + e^{-\varepsilon/k_B T}\right)}$$

[Note: if we multiply the top and bottom by $e^{\varepsilon/k_B T}$, we can write this as

$$\bar{E} = \frac{N\varepsilon}{1 + e^{\varepsilon/k_B T}}.$$

(d). $$\bar{E} = \frac{\varepsilon e^{-\varepsilon/k_B T}}{(1 + e^{-\varepsilon/k_B T})}.$$

In the limit $T \to 0$, $\frac{1}{T} \to \infty$

$$\Rightarrow \quad e^{-\varepsilon/k_B T} \to e^{-\infty} = 0$$

$$\Rightarrow \quad \bar{E} \Rightarrow \frac{\varepsilon \cdot 0}{1 + 0} = 0.$$

• Explanation: in the limit $T \to 0$, the system relaxes into the lowest possible energy state (it is no longer being "kicked" by the heat reservoir into higher energy state). This lowest energy state is that in which all N particles are in a state with energy 0

$$\Rightarrow \quad \bar{E} = N \cdot 0 = 0.$$