Many Particle Systems – Practice Problem Set 6

These problems are not for assessment. However, it is recommended that you attempt them as practice for the test and exam.

1. In lectures, we derived the following approximate expression for the canonical partition function for an ideal gas made up of N particles in a cubic container of side L :

$$Z(T,V,N) \approx \frac{1}{N!}\left(\frac{m\,k_B T L^2}{2\pi\hbar^2}\right)^{3N/2}.$$

The factor $\frac{1}{N!}$ arises from the attempt to account for the fact that the particles are indistinguishable (modified Maxwell-Boltzmann statistics), whilst our calculation of the factor in brackets was done assuming the particles are distinguishable (Maxwell-Boltzmann statistics).

The free energy of the ideal gas is computed as

$$F(T,V,N) = -k_B T\,\ln Z(T,V,N),$$

and the entropy is computed as

$$S(T,V,N) = -\frac{\partial F(T,V,N)}{\partial T},$$

yielding the result

$$S(T,V,N) = -k_B \ln N! + \frac{3N}{2}k_B \ln T + Nk_B \ln V + \frac{3N}{2}k_B + \frac{3N}{2}k_B \ln\left(\frac{m\,k_B}{2\pi\hbar^2}\right).$$

Using Stirling's approximation $\ln N! \approx N\ln N$ for large N, show that the entropy is an extensive quantity i.e. doubling N and V doubles the entropy.

Note: without the factor $\ln N!$, you would not be able to show entropy is an extensive quantity. Since we observe entropy to be extensive in thermodynamic systems, this shows that it is essential to account for the quantum indistinguishability of the particles in the ideal gas if we are to derive the correct thermodynamics.

2. In lectures, we defined the generating function

$$\tilde{\mathcal{Z}}(T,V,\alpha) = \sum_{N=1}^{\infty} Z(T,V,N)\alpha^N,$$

where $Z(T,V,N)$ is the canonical partition function for N particles in a volume V in contact with a heat reservoir at temperature T.

Consider a system of identical indistinguishable particles in which each particle has only two possible energy levels, ϵ_1 and ϵ_2.

(a) In the case where the identical indistinguishable particles are bosons (i.e do not obey the Pauli exclusion principle), compute the canonical partition function $Z(T, V, N)$ for the cases $N = 1$, $N = 2$ and $N = 3$.

(b) Using results found in lectures, the generating function in this case takes the form

$$\tilde{\mathcal{Z}}_{BE}(T, V, \alpha) = \frac{1}{(1 - \alpha\, e^{-\epsilon_1/k_B T})}\, \frac{1}{(1 - \alpha\, e^{-\epsilon_2/k_B T})}.$$

Compute the coefficients of α, α^2 and α^3 and show that they are equivalent to the canonical partition functions computed in part (a).
Note: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ for $|x| < 1$.

(c) In the case where the identical indistinguishable particles are fermions (which obey the Pauli exclusion principle), compute the canonical partition function $Z(T, V, N)$ for the cases $N = 1$, $N = 2$ and $N = 3$.

(d) Using results found in lectures, the generating function in this case takes the form

$$\tilde{\mathcal{Z}}_{FD}(T, V, \alpha) = (1 + \alpha\, e^{-\epsilon_1/k_B T})\,(1 + \alpha\, e^{-\epsilon_2/k_B T}).$$

Compute the coefficients of α, α^2 and α^3 and show that they are equivalent to the canonical partition functions computed in part (c).