MPS Practice Problem Set 6 Solutions

The Physics of Particles (University of Western Australia)

MPS PRACTICE PROBLEM SET 6

SOLUTIONS

1.

$$\boxed{S(T,V,N) = -k_B \ln N! + \frac{3N}{2} k_B \ln T + N k_B \ln V + \frac{3N}{2} k_B + \frac{3N}{2} k_B \ln\left(\frac{m k_B}{2\pi\hbar^2}\right)}$$

USING STIRLING'S APPROXIMATION FOR LARGE N,

$-k_B \ln N! \approx -k_B N \ln N$, AND

IF WE ADD THIS TERM TO THE TERM

$N k_B \ln V$ WE HAVE

$$N k_B(\ln V - \ln N) = N k_B \ln\left(\frac{V}{N}\right)$$

AND SO THE EXPRESSION FOR S IS

$$\boxed{S(T,V,N) \approx N k_B \ln\left(\frac{V}{N}\right) + \frac{3N}{2} k_B \ln T + \frac{3N}{2} k_B + \frac{3N}{2} k_B \ln\left(\frac{m k_B}{2\pi\hbar^2}\right)}$$

IF WE DOUBLE THE SIZE OF THE SYSTEM, $V \rightarrow 2V$ & $N \rightarrow 2N$

WE FIND

$$S(T, 2V, 2N) = (2N)k_B \ln\left(\frac{2V}{2N}\right) + \frac{3(2N)}{2}k_B \ln T$$

$$+ \frac{3(2N)}{2}k_B + \frac{3(2N)}{2}k_B \ln\left(\frac{mk_B}{2\pi\hbar^2}\right)$$

$$= 2\,S(T, V, N)$$

AND THE EXPRESSION FOR THE ENTROPY CORRECTLY DOUBLES IF THE SIZE OF THE SYSTEM DOUBLES.

<u>NOTE</u>: IF THE $-k_B \ln N!$ FACTOR <u>WAS NOT PRESENT</u>, THEN

$$N k_B \ln V \longrightarrow (2N) k_B \ln(2V)$$

$$= 2Nk_B \ln V + 2Nk_B \ln 2$$

AND THEN

$$S(T, 2V, 2N) \rightarrow 2S(T, V, N) + 2Nk_B \ln 2$$

SO ENTROPY WOULD <u>NOT</u> BE AN EXTENSIVE QUANTITY. THE $-k_B \ln N!$ TERM COMING FROM THE RECOGNITION THE THE PARTICLES ARE INDISTINGUISHABLE IS AN ESSENTIAL FACTOR.

2. (a) DENOTE THE INDISTINGUISHABLE PARTICLES BY A. THE STATES ARE ENUMERATED IN TERMS OF OCCUPATION NUMBERS OF THE SINGLE PARTICLE STATES.

ALSO DEFINE

$$a := e^{-\varepsilon_1/k_B T}$$

$$b := e^{-\varepsilon_2/k_B T}$$

FOR $N=1$

SYSTEM STATE	LEVEL 1	LEVEL 2	ENERGY OF SYSTEM
STATE 1	A	—	ε_1
STATE 2	—	A	ε_2

$$Z(T,V,1) = \sum_{\text{STATES}} e^{-\varepsilon_{\text{STATE}}/k_B T}$$

$$= e^{-\varepsilon_1/k_B T} + e^{-\varepsilon_2/k_B T}$$

So

$$\boxed{Z(T,V,1) = a + b}$$

FOR $N = 2$

SYSTEM STATE	LEVEL 1	LEVEL 2	ENERGY OF SYSTEM
STATE 1	AA	–	$2\varepsilon_1$
STATE 2	A	A	$\varepsilon_1 + \varepsilon_2$
STATE 3	–	AA	$2\varepsilon_2$

$$Z(T,V,2) = e^{-2\varepsilon_1/k_BT} + e^{-(\varepsilon_1+\varepsilon_2)/k_BT} + e^{-2\varepsilon_2/k_BT}$$

$$= a^2 + ab + b^2$$

$$\boxed{Z(T,V,2) = a^2 + ab + b^2}$$

FOR $N=3$

SYSTEM STATE	LEVEL 1	LEVEL 2	ENERGY OF SYSTEM
STATE 1	AAA	–	$3\varepsilon_1$
STATE 2	AA	A	$2\varepsilon_1 + \varepsilon_2$
STATE 3	A	AA	$\varepsilon_1 + 2\varepsilon_2$
STATE 4	–	AAA	$3\varepsilon_2$

$$Z(T,V,3) = e^{-3\varepsilon_1/k_BT} + e^{-(2\varepsilon_1+\varepsilon_2)/k_BT} + e^{-(\varepsilon_1+2\varepsilon_2)/k_BT} + e^{-3\varepsilon_2/k_BT}$$

$$= a^3 + a^2 b + a b^2 + b^3$$

$$\boxed{Z(T,V,3) = a^3 + a^2 b + a b^2 + b^3}$$

(b)

$$\tilde{Z}_{BE}(T,V,\alpha) = \left(1-\alpha e^{-\varepsilon_1/k_BT}\right)^{-1}\left(1-\alpha e^{-\varepsilon_2/k_BT}\right)^{-1}$$

$$=(1-\alpha a)^{-1}(1-\alpha b)^{-1}$$

WE WISH TO EXPAND THE ABOVE EXPRESSION IN A POWER SERIES OF α AND READ OFF THE COEFFICIENTS OF α, α^2, AND α^3.

SO WE ONLY NEED TO EXPAND $\tilde{Z}_{BE}$ TO ORDER α^3

USING

$$(1-\alpha a)^{-1} = 1 + \alpha a + (\alpha a)^2 + (\alpha a)^3 + O(\alpha^4)$$

AND A SIMILAR EXPRESSION FOR $(1-\alpha b)^{-1}$

TO $O(\alpha^4)$ WE HAVE

$$\tilde{Z}_{BE} = (1-\alpha a)^{-1}(1-\alpha b)^{-1}$$

$$= (1+\alpha a+\alpha^2 a^2+\alpha^3 a^3)(1+\alpha b+\alpha^2 b^2+\alpha^3 b^3) + O(\alpha^4)$$

$$= 1 + \alpha b + \alpha^2 b^2 + \alpha^3 b^3$$
$$+ \alpha a + \alpha^2 ab + \alpha^3 ab^2$$
$$+ \alpha^2 a^2 + \alpha^3 a^2 b$$
$$+ \alpha^3 a^3 + O(\alpha^4)$$

$$\tilde{Z}_{BE} = 1 + \alpha(a+b) + \alpha^2(a^2+ab+b^2)$$
$$+ \alpha^3(a^3+a^2b+ab^2+b^3)$$
$$+ O(\alpha^4)$$

THE COEFFICIENT OF α IS $Z(T,V,1)$

THE COEFFICIENT OF α^2 IS $Z(T,V,2)$, &

THE COEFFICIENT OF α^3 IS $Z(T,V,3)$

IN AGREEMENT WITH PART (a)

2. (c) DENOTE THE INDISTINGUISHABLE PARTICLES BY A. THE STATES ARE ENUMERATED IN TERMS OF OCCUPATION NUMBERS OF THE SINGLE PARTICLE STATES.

ALSO DEFINE

$$a := e^{-\varepsilon_1/k_B T}$$

$$b := e^{-\varepsilon_2/k_B T}$$

FOR $N=1$

SYSTEM STATE	LEVEL 1	LEVEL 2	ENERGY OF SYSTEM
STATE 1	A	—	ε_1
STATE 2	—	A	ε_2

$$Z(T,V,1) = \sum_{\text{STATES}} e^{-\varepsilon_{\text{STATE}}/k_B T}$$

$$= e^{-\varepsilon_1/k_B T} + e^{-\varepsilon_2/k_B T}$$

So

$$\boxed{Z(T,V,1) = a + b}$$

FOR $N=2$

SYSTEM STATE	LEVEL 1	LEVEL 2	ENERGY OF SYSTEM
STATE 1	A	A	$\varepsilon_1 + \varepsilon_2$

$$Z(T, V, 2) = e^{-(\varepsilon_1 + \varepsilon_2)/k_B T}$$

$$= ab$$

So

$$\boxed{Z(T, V, 2) = ab}$$

<u>FOR $N = 3$</u> : IT IS NOT POSSIBLE TO PLACE 3 IDENTICAL INDISTINGUISHABLE FERMIONS INTO JUST 2 SINGLE PARTICLE STATES.

(d)

$$\tilde{Z}_{FD}(T,V,\alpha) = \left(1+\alpha e^{-\varepsilon_1/k_BT}\right)\left(1+\alpha e^{-\varepsilon_2/k_BT}\right)$$

$$= (1+\alpha a)(1+\alpha b)$$

$$= 1 + \alpha b + \alpha a + \alpha^2 ab$$

$$= 1 + \alpha(a+b) + \alpha^2(ab)$$

THE COEFFICIENT OF α IS $Z(T,V,1)$ AND THE COEFFICIENT OF α^2 IS $Z(T,V,2)$ AS FOUND IN PART (C). THERE ARE NO $\alpha^3, \alpha^4, \ldots$ TERMS, INDICATING THAT IT IS NOT POSSIBLE TO HAVE $3, 4, \ldots$ PARTICLES IN SUCH A SYSTEM.