General Physics II Mid-term Exam Answer

1. (25 points) Short answer questions. (Write down the process briefly)

   (a) (5 points) Give the full statement of "uniqueness theorem", including both the conductor and dielectric boundary conditions.

   （1）语言描述：给定区域 V 内导体之外的电荷分布 ρ, 给定导体上的总电荷 Q_i 或电势 Φ_i, 以及电介质边界上的 φ 或 $\frac{\partial\varphi}{\partial n}$，则区域 V 内的电场分布是唯一的。

   （2）公式描述：

   满足下述方程及边界条件的 φ 唯一

   在导体之外满足泊松方程：

$$\nabla^2\varphi_i = -\frac{\rho}{\epsilon_i}$$

   在电介质分界面上满足边界条件：

$$\varphi_i = \varphi_j \quad \& \quad \frac{\partial\varphi_i}{\partial n} = \frac{\partial\varphi_j}{\partial n}$$

   在电介质边界给定

$$\varphi|_S \quad xor \quad \frac{\partial\varphi}{\partial n}|_S$$

   在第 k 个导体上满足边界条件：

$$-\oint_{S_k} \frac{\partial\varphi}{\partial n} dS = Q_k \& \varphi_{S_k} = Const(unknown) \quad xor \quad \varphi_{S_k} = \Phi_k$$

   (b) (5 points) A dipole $\vec{p}$ is located at a distance d from the center of a grounded conducting sphere with radius a ($d > a$). The outside of the sphere is fulled with a dielectric material with dielectric constant ϵ. The direction of the dipole points toward the center of the conducting sphere (see the illustration below). Calculate the electric energy of such a configuration.

   电偶极子可视为由相距 l 带电量为 $\pm q$ 的两个电荷组成, 其中 $ql = p, l \to 0$。

   在介质中总带电量的电偶极子为 $\vec{p_t} = \frac{\epsilon_0}{\epsilon}\vec{p}$

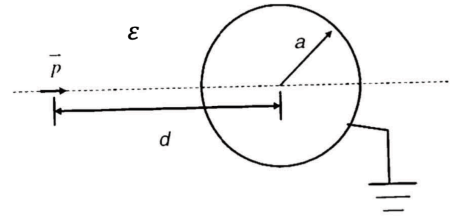


Figure 1.b

两个电荷及其在接地球形导体的像电荷的电荷量及位置分别为：

$$(+q_t, d-\frac{l}{2}) \to (-\frac{a}{d-\frac{l}{2}}q_t, \frac{a^2}{d-\frac{l}{2}}) \approx (-\frac{a}{d}q_t - \frac{a}{2d^2}q_t l, \frac{a^2}{d} + \frac{a^2}{d^2}\frac{l}{2})$$
$$(-q_t, d+\frac{l}{2}) \to (\frac{a}{d+\frac{l}{2}}q_t, \frac{a^2}{d+\frac{l}{2}}) \approx (\frac{a}{d}q_t - \frac{a}{2d^2}q_t l, \frac{a^2}{d} - \frac{a^2}{d^2}\frac{l}{2})$$

将像电荷系统保留至最低阶（电荷），但因此处电荷及电偶极子同阶，需要将像电荷视为在 $\frac{a^2}{d}$ 处的一个电偶极子与点电荷：

$$\vec{p}_{image} = \frac{a^3}{d^3}\vec{p_t}$$
$$q_{image} = -\frac{a}{d^2}|\vec{p_t}|$$

像电荷在 d 处产生的电场为 (以 $\vec{p}_{image}$ 的方向为 z 轴)：

$$\vec{E} = \frac{1}{4\pi\epsilon_0}[\frac{3|\vec{p}_{image}|cos\theta\hat{r} - \vec{p}_{image}}{r^3} + \frac{q_{image}\hat{r}}{r^2}]$$
$$= \frac{1}{4\pi\epsilon_0}[\frac{2\frac{a^3}{d^3}\vec{p_t}}{(d-\frac{a^2}{d})^3} + \frac{\frac{a}{d^2}\vec{p_t}}{(d-\frac{a^2}{d})^2}] = \frac{1}{4\pi\epsilon_0}\frac{(a^2+d^2)a\vec{p_t}}{(d^2-a^2)^3}$$

因此系统的静电能为：

$$W = -\frac{1}{2}\vec{p}\cdot\vec{E} = -\frac{1}{8\pi\epsilon}\frac{(a^2+d^2)a|\vec{p}|^2}{(d^2-a^2)^3} \tag{1}$$

[注：得到像电荷系统后有两种方法：一种是将其视为电荷系统展开为点电荷与电偶极子，因为其产生的电势等阶, 如上所示；一种是直接小量展开至 l 的最低阶 (电场以 $\vec{p}$ 为正)

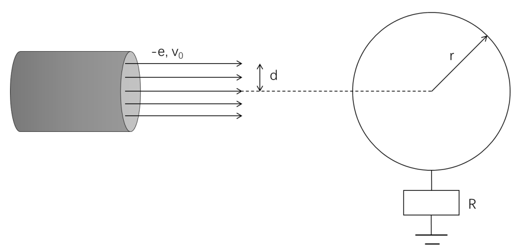
$$E = -\frac{1}{4\pi\epsilon_0}[\frac{(-\frac{a}{d}q_t - \frac{a}{2d^2}q_t l)}{(\frac{a^2}{d} + \frac{a^2}{d^2}\frac{l}{2} - d)^2} + \frac{(\frac{a}{d}q_t - \frac{a}{2d^2}q_t l)}{(\frac{a^2}{d} - \frac{a^2}{d^2}\frac{l}{2} - d)^2}]$$
$$\approx -\frac{1}{4\pi\epsilon_0}[\frac{(-adq_t)}{(a^2-d^2)^2}(1+\frac{1}{2d}l)(1-\frac{a^2}{d(a^2-d^2)}l) + \frac{(adq_t)}{(a^2-d^2)^2}(1-\frac{1}{2d}l)(1+\frac{a^2}{d(a^2-d^2)}l)]$$
$$\approx \frac{1}{4\pi\epsilon_0}\frac{(adq_t)}{(a^2-d^2)^2}[(1+\frac{1}{2d}l)(1-\frac{a^2}{d(a^2-d^2)}l) - (1-\frac{1}{2d}l)(1+\frac{a^2}{d(a^2-d^2)}l)]$$
$$= \frac{1}{4\pi\epsilon_0}\frac{(adq_t)}{(a^2-d^2)^2}[\frac{l}{d} - 2\frac{a^2}{d(a^2-d^2)}l] = \frac{1}{4\pi\epsilon_0}\frac{(a^2+d^2)ap_t}{(d^2-a^2)^3}]$$

(c) (7 points) Distant cathode rays continuously emit electrons with charge $-e$ and initial velocity v_0 towards a conductor sphere with a radius of r. When these electrons are first emitted, they are uniformly distributed and the particle number density is n, the maximum aiming distance is d. The conductor sphere leakages current through a resistance R. Find the voltage of the conductor sphere when it reaches a stable state. $d < r$. The mass of the electron is m, relativistic effects are not considered. All electrons that hit the conductor ball are absorbed.

Hint: Taking the center of the sphere as the origin, for one electron we have

$$\vec{l} \times \vec{F} = \frac{d\vec{L}}{dt}, \text{ with } \vec{L} = m\vec{l} \times \vec{v}$$

$\vec{l}$ is the distance from origin to the electron.



系统达到稳定状态时，单位时间内被球壳吸收的电量等于电流，其中 a 为在稳态时能被球体吸收（能打到球体上）的电子束输入半径。

$$U = -e\pi a^2 v_0 n R$$

由能量守恒与角动量守恒可得临界情况时的电子束半径（刚好不能打到球壳上，即电子距离球心距离为其半径时速度沿着切线方向）：

$$mv_0 a = mvr$$
$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 - eU$$

因此可得：

$$a^2 = \frac{mv_0^2}{mv_0^2/r^2 + 2e^2\pi v_0 n R}$$

考虑 a 与电子束最大半径的关系：

$$U = -\frac{1}{\frac{2e}{mv_0^2} + \frac{1}{e\pi v_0 n R r^2}}, (a < d)$$
$$U = -e\pi d^2 v_0 n R, (a \geq d)$$

(d) (8 points) A spherical capacitor with an inner diameter R_1 and an outer diameter R_2 is fulled with dielectric material. The dielectric constant of the filling in between changes under $\epsilon = \epsilon_0 + \epsilon_1 \cos^2\theta$, where θ is the polar angle. The outer sphere is grounded and the inner sphere is connected with voltage V.

(1) Calculate the capacitance C and energy.

(2) Calculate $\vec{E}, \vec{P}, \vec{D}$ and polarized charge density in the outer and inner spheres.

(1) 电介质分布与极角有关，由切向电场 E_t 连续可得在两球壳内 $\vec{E} = \frac{A\hat{r}}{r^2}$

由两球壳之间电压为 V 可得：

$$\int_{R_1/2}^{R_2/2} \frac{A}{r^2} dr = V, A = \frac{VR_1R_2}{2(R_2 - R_1)}$$

由 $\vec{D} = \epsilon \vec{E}$ 可得：

$$\vec{D} = (\epsilon_0 + \epsilon_1 \cos^2\theta) \frac{A\hat{r}}{r^2}$$

因此内球壳总带电量

$$\begin{aligned} Q &= \oint_S \vec{D} \cdot d\vec{S} = \int_0^\pi (\epsilon_0 + \epsilon_1 \cos^2\theta) \frac{VR_1R_2}{2(R_2 - R_1)} \frac{1}{(R_1/2)^2} 2\pi \frac{R_1}{2} \sin\theta \frac{R_1}{2} d\theta \\ &= \int_{-1}^{1} (\epsilon_0 + \epsilon_1 x^2) \frac{V\pi R_1 R_2}{(R_2 - R_1)} dx = \frac{2\pi V R_1 R_2}{(R_2 - R_1)} (\epsilon_0 + \frac{\epsilon_1}{3}) \end{aligned}$$

电容器电容：

$$C = \frac{Q}{V} = \frac{2\pi R_1 R_2}{(R_2 - R_1)} (\epsilon_0 + \frac{\epsilon_1}{3})$$

能量：

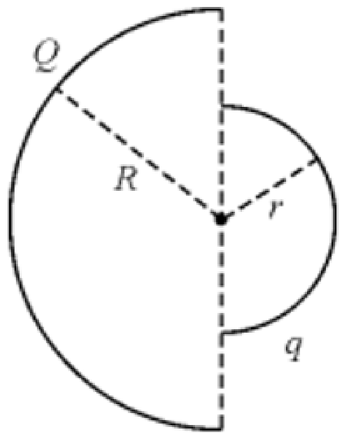
$$W = \frac{1}{2} C V^2 = \frac{\pi V^2 R_1 R_2}{(R_2 - R_1)} (\epsilon_0 + \frac{\epsilon_1}{3})$$

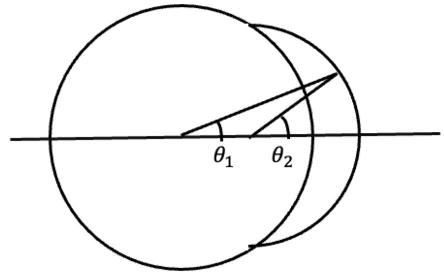
(2)$\vec{E}, \vec{D}, \vec{P}, \sigma_P$ 分别为：

$$\begin{aligned} \vec{E} &= \frac{VR_1R_2}{2(R_2 - R_1)} \frac{\hat{r}}{r^2} \\ \vec{D} &= (\epsilon_0 + \epsilon_1 \cos^2\theta) \frac{VR_1R_2}{2(R_2 - R_1)} \frac{\hat{r}}{r^2} \\ \vec{P} &= (\epsilon_1 \cos^2\theta) \frac{VR_1R_2}{2(R_2 - R_1)} \frac{\hat{r}}{r^2} \\ \sigma(R_1) &= -(\epsilon_1 \cos^2\theta) \frac{VR_1R_2}{2(R_2 - R_1)} \frac{1}{(R_1/2)^2} \\ \sigma(R_2) &= (\epsilon_1 \cos^2\theta) \frac{VR_1R_2}{2(R_2 - R_1)} \frac{1}{(R_2/2)^2} \end{aligned}$$

除两个球壳其他区域无极化电荷。

2. (15 points) Two evenly charged half-sphere shell (with radius R, r and charge Q, q, respectively) are placed as the following figure (left) shows.





(a) (3 points) If $R = r, Q = q$, find the force F acted on the right half sphere by the left one.

左半球壳对右半球壳的力可通过计算右半球壳每个小面元受到其余部分的力计算（每个小面元受到的力包括左半球壳的力以及右半球壳其余部分的力，求和时由于相互作用力大小相等方向相反相互抵消，因此只剩下左半球壳对右半球壳的力）:

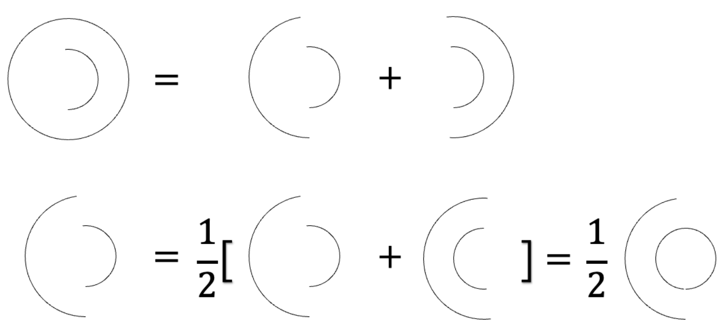
小面元产生的电场在面附近可视为无穷大平面板，因此 $E = \frac{\sigma}{2\epsilon_0}$

由于内部电场为 0，因此其余部分在小面元处产生的电场也为 $E = \frac{\sigma}{2\epsilon_0}$，方向沿着小面元。

因此右半球壳受到的静电力如下，方向垂直于分界面朝右。

$$F = \int_0^{\frac{\pi}{2}} 2\pi R \sin\theta R d\theta \sigma \frac{\sigma}{2\epsilon_0} cos\theta = \frac{Q^2}{8\pi\epsilon_0 R^2}$$

(b) (7 points) Redo (1) with arbitrary Q, q, and $R \neq r$.



2.(b)answer

不失一般性可假设 $r < R$($r > R$ 时计算小球壳受大球壳的力，由于力是相互的即可得大球壳受小球壳的力)。

补全大球壳，其内部电场为 0，小半球壳（右）受到的力为 0；补全后的大球壳对小球壳的力等于大半球壳（左）对小半球壳（右）的力加上大半球壳（右）对小半球壳（右）的力。（如图 2.(b)answer 第一行所示）因此大半球壳（左）对小半球壳（右）的力等于大半球壳（左）对小半球壳（左）的力。

因此因此大半球壳（左）对小半球壳（右）的力等于 $\frac{1}{2}$[大半球壳（左）对小半球壳（左）的力 + 大半球壳（左）对小半球壳（右）的力]=$\frac{1}{2}$ 大半球壳（左）对完整小球壳的力（如图 2.(b)answer 第二行所示）。

均匀电荷分布的完整球壳可视为点电荷，因此只需要计算大半球壳其球心处点电荷力的一半。

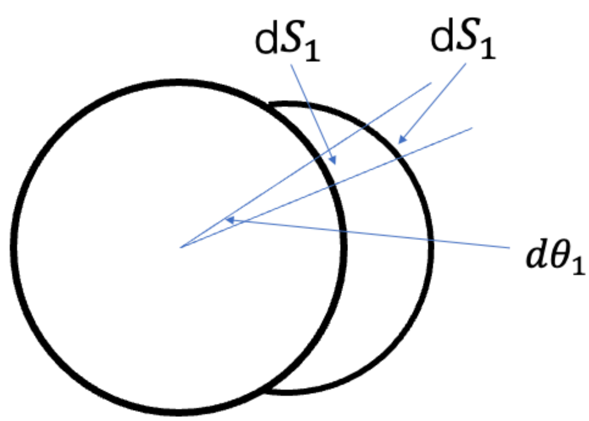
$$F = \frac{2q}{8\pi\epsilon_0}\int_0^{\frac{\pi}{2}} \frac{2\pi R sin\theta cos\theta R d\theta Q/(2\pi R^2)}{R^2} = \frac{qQ}{8\pi\epsilon_0 R^2}$$

当 $r > R$ 时同理得 $F = \frac{qQ}{8\pi\epsilon_0 r^2}$

(c) (5 points) Consider another situation, as shown in figure (right). A sphere shell with uniform charge distribution, total charge Q, and radius R_1 is connected with a spherical arc with radius $R_2, (R_2 < R_1)$. The distance between their centers is d. The area charge density of the arc is $\sigma = \sigma_0 cos(\theta_1 - \theta_2)$. Calculate the force acts on the arc by the shell.

方法一（利用高斯定理）:

考虑某一 θ_1 处的 $d\theta_1, d\phi_1$ 的小面元，封闭曲面由 dS_1, dS_2 以及由球心发射出的射线面组成 (如图 2(c)answer) 所示。其中 dS_1 无限接近完整球面但仍在球外。



2.(c)answer

由高斯定理 $\oint \vec{E} \cdot d\vec{S}$ 可得 (电场方向沿着径向，因此射线面为 0，电场与 dS_1 方向相同，电场与 dS_2 方向夹角为 $cos(\theta_1 - \theta_2)$):

$$|\vec{E}_{S_1}|dS_1 = |\vec{E}_{S_2}|cos(\theta_1 - \theta_2)dS_2$$

又因为两个小面元上的电场方向相同，两边再同乘 σ_0 可得 (等式右侧为小面元 dS_2 受到的力，由此等式可看出其等于电荷面密度为 σ_0 的 dS_1 受到的力。)

$$\vec{E}_{S_1}\sigma_0 dS_1 = \vec{E}_{S_2}\sigma_0 cos(\theta_1 - \theta_2)dS_2$$

因此电荷分布为 $\sigma_0 cos(\theta_1 - \theta_2)$ 的球弧所受的力等于电荷均匀分布为 σ_0 的球弧所受的力(此球弧为完成球壳的一部分)。

计算完整球的球弧受的力为:

$$F = \int_0^{\theta_{1max}} \frac{Q}{4\pi\epsilon_0} 2\pi R_1 sin\theta_1 R_1 d\theta_1 cos\theta_1 \sigma_0 / R_1^2 = \frac{Q\sigma_0}{4\epsilon_0} sin^2\theta_{1max}$$

由余弦定理得 $cos\theta_{1max} = \frac{R_1^2 + d^2 - R_2^2}{2R_1 d}$ 因此:

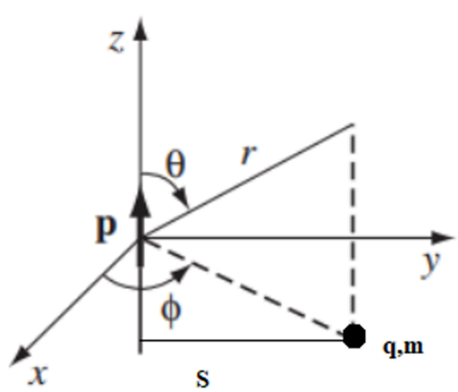
$$F = \frac{Q\sigma_0}{4\epsilon_0}[1 - (\frac{R_1^2 + d^2 - R_2^2}{2R_1 d})^2]$$

方法二（直接积分）：对球弧的一个个细圆环积分 (积分 θ_1, 设圆环上一点到完整球壳球心的距离为 x):

$$F = \int_0^{\theta_{1max}} \frac{Q}{4\pi\epsilon_0 x^2} 2\pi x sin\theta_1 \frac{x d\theta_1}{cos(\theta_1 - \theta_2)} cos\theta_1 \sigma_0 cos(\theta_1 - \theta_2) = \frac{Q\sigma_0}{4\epsilon_0} sin^2\theta_{1max}$$

需要说明的是圆环的宽度为 $\frac{x d\theta_1}{cos(\theta_1 - \theta_2)}$, 因 dS_2 与此处沿径向的面元夹角为 $cos(\theta_1 - \theta_2)$。

3. (10 points) A stationary electric dipole $\vec{p} = p\vec{e_z}$ at the origin. A positive point charge q (mass m) executes circular motion (radius s) at a constant speed in the field of the dipole. (Ignore the gravity. The electric potential of dipole is $\varphi = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$)



(a) (4 points) Characterize the plane of the orbit.

电偶极子的电场

$$\vec{E}=-\nabla\varphi=-\frac{1}{4\pi\epsilon_0}(\hat{r}\frac{\partial pcos\theta/r^2}{\partial r}+\hat{\theta}\frac{\partial pcos\theta/r^2}{\partial\theta})=\frac{2pcos\theta\hat{r}+p\sin\theta\hat{\theta}}{4\pi\epsilon_0 r^3}$$

利用单位向量间的关系 $\vec{p}=p\vec{e_z}=pcos\theta\hat{r}-psin\theta\hat{\theta}$,$\hat{r}=cos\theta\hat{z}+sin\theta\hat{\rho}$，其中 $\hat{\rho}$ 是垂直与 z 轴指向电荷的方向。

$$\vec{E}=\frac{3pcos\theta\hat{r}-\vec{p}}{4\pi\epsilon_0 r^3}=\frac{(3cos^2\theta-1)\vec{p}+3pcos\theta sin\theta\hat{\rho}}{4\pi\epsilon_0 r^3}$$

忽略重力的情况下，电荷绕着 z 轴转动时受的力只沿垂直轴向，因此:

$$3cos^2\theta-1=0, cos\theta=\pm\frac{\sqrt{3}}{3}$$

电荷带正电，垂直轴向力在 $cos\theta=-\frac{\sqrt{3}}{3}$ 才为吸引力。

因此电荷的运动轨道平面为:

$$z=-\frac{\sqrt{2}}{2}s$$

(b) (4 points) Find the speed and total energy of the charge.

带入 $cos\theta=-\frac{1}{\sqrt{3}}, r=\frac{\sqrt{3}s}{\sqrt{2}}$ 得到:

$$F=-\frac{qp\sqrt{2}}{4\pi\epsilon_0(\frac{\sqrt{3}s}{\sqrt{2}})^3}=-\frac{mv^2}{s}$$

因此运动速度为

$$v=\sqrt{\frac{pq}{3\sqrt{3}m\pi\epsilon_0 s^2}}$$

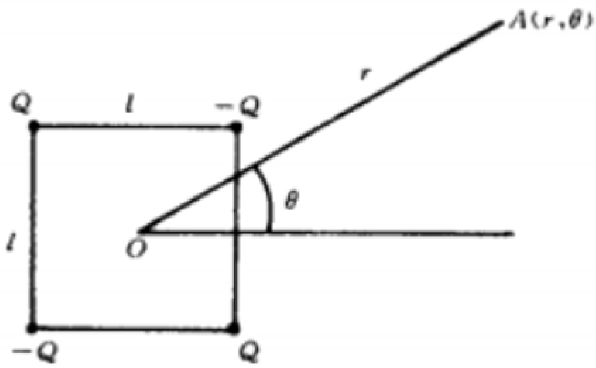
电荷的总能量为:

$$W=\frac{1}{2}mv^2+q\varphi=\frac{pq}{6\sqrt{3}\pi\epsilon_0 s^2}-\frac{q}{4\pi\epsilon_0}\frac{2p}{3\sqrt{3}s^2}=0$$

(c) (2 points) If we consider the gravity. How can we produce the proper extra electric field to maintain the motion of the point charge? (The motion when considering gravity and additional electric field is the same as in (a) and (b))

施加 $\vec{E}=\frac{mg}{q}\hat{z}$ 的匀强电场，可通过带电量为 $\sigma_0=\frac{2\epsilon_0 mg}{q}$ 的无穷大均匀带电板实现（面积远大于 πs^2 即可）

4. (10 points) Four point charges located at the vertexes of a square are called an electric quadrupole, as the following figure shows. Note that the formula of electric quadrupole cannot be used directly. A is a point in the plane of electric quadrupole with an arbitrary θ.



(a) (2 points) calculate the electric potential at A.

在 (r,θ) 处的电势为（其中 $x=rcos\theta, y=rsin\theta$）:

$$\varphi=\frac{Q}{4\pi\epsilon_0}[-\frac{1}{\sqrt{(x-l/2)^2+(y-l/2)^2}}+\frac{1}{\sqrt{(x+l/2)^2+(y-l/2)^2}}$$
$$-\frac{1}{\sqrt{(x+l/2)^2+(y+l/2)^2}}+\frac{1}{\sqrt{(x-l/2)^2+(y+l/2)^2}}]$$

(b) (2 points) Assume that A is far away from the quadrupole (i.e. $r>>l$), find an approximate solution of (a).

将 $x=rcos\theta, y=rsin\theta$ 带入，r 提出可得:

$$\varphi=\frac{Q}{4\pi\epsilon_0 r}[-(1+\frac{l}{r}(-cos\theta-sin\theta)+\frac{l^2}{4r^2})^{-1/2}+(1+\frac{l}{r}(cos\theta-sin\theta)+\frac{l^2}{4r^2})^{-1/2}$$
$$-(1+\frac{l}{r}(cos\theta+sin\theta)+\frac{l^2}{4r^2})^{-1/2}+(1+\frac{l}{r}(-cos\theta+sin\theta)+\frac{l^2}{4r^2})^{-1/2}]$$

将其展开至二阶项 $((1+x)^{-1/2}\approx 1-\frac{x}{2}+\frac{3x^2}{8})$:

$$\varphi=\frac{Q}{4\pi\epsilon_0 r}[-(1-\frac{l}{2r}(-cos\theta-sin\theta)-\frac{l^2}{8r^2}+\frac{3l^2}{8r^2}(1+sin2\theta))$$
$$+(1-\frac{l}{2r}(cos\theta-sin\theta)-\frac{l^2}{8r^2}+\frac{3l^2}{8r^2}(1-sin2\theta))$$
$$-(1-\frac{l}{2r}(cos\theta+sin\theta)-\frac{l^2}{8r^2}+\frac{3l^2}{8r^2}(1+sin2\theta))$$
$$+(1-\frac{l}{2r}(-cos\theta+sin\theta)-\frac{l^2}{8r^2}+\frac{3l^2}{8r^2}(1-sin2\theta))]=-\frac{3Ql^2sin2\theta}{8\pi\epsilon_0 r^3}$$

(c) (2 points) If a uniform electric field along the polar axis is applied, what is the energy of the electric quadrupole?

左上角的电荷电势为 φ_0, 则其余三个电荷的电势分别为 (顺时针):$\varphi_0-El,\varphi_0-El,\varphi_0$

因此此系统在外场下能量为

$$W_e=Q\varphi_0-Q(\varphi_0-El)+Q(\varphi_0-El)-Q\varphi_0=0$$

(d) (4 points) The applied electric field is in the plane of the electric quadrupole while is non-uniform. Calculate the energy of the electric quadrupole in the limit $l \to 0$.

左上角的电荷电势为 φ_0, 则其余三个电荷的电势分别为 (顺时针):

$$\varphi_1 = \varphi_0 + \frac{\partial \varphi_0}{\partial x} l + \frac{1}{2} \frac{\partial^2 \varphi_0}{\partial x^2} l^2$$
$$\varphi_2 = \varphi_0 + \frac{\partial \varphi_0}{\partial x} l + \frac{\partial \varphi_0}{\partial y} l + \frac{1}{2} \frac{\partial^2 \varphi_0}{\partial x^2} l^2 - \frac{\partial^2 \varphi_0}{\partial x \partial y} l^2 + \frac{1}{2} \frac{\partial^2 \varphi_0}{\partial y^2} l^2$$
$$\varphi_3 = \varphi_0 + \frac{\partial \varphi_0}{\partial y} l + \frac{1}{2} \frac{\partial^2 \varphi_0}{\partial y^2} l^2$$

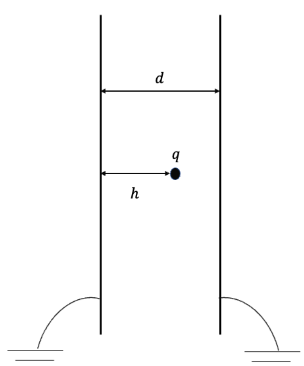
因此能量为

$$W_e = Q\varphi_0 - Q\varphi_1 + Q\varphi_2 - Q\varphi_3 = -Q \frac{\partial^2 \varphi_0}{\partial x \partial y} l^2$$

5. (10 points) Green's reciprocation theorem: If Φ is the potential due to a volume-charge density ρ within a volume V and a surface-charge density σ on the conducting surface S bounding the volume V, while Φ' is the potential due to another charge distribution ρ' and σ', then:

$$\int_V \rho \Phi' d^3\vec{x} + \int_S \sigma \Phi' dS = \int_V \rho' \Phi d^3\vec{x} + \int_S \sigma' \Phi dS.$$

Two infinite grounded parallel conducting planes are separated by a distance d. A point charge q is placed between the planes (The distance between the left plane and point charge is h, $h < d$). Use the Green's reciprocation theorem to calculate the total charge in each parallel conducting plane.



解法一（应用上述定理要求）:

上述定理即要求给出另一种简单的电荷分布，可以看到等式右侧在新的电荷分布 (ρ', σ') 除边界外无电荷分布时恒为 0（$\rho' = 0, \Phi|_S = 0$）；等式左侧可化简为

$$q\Phi' + \Phi'_{left} \int \sigma dS_{left} + \Phi'_{right} \int \sigma dS_{right} = q\Phi' + \Phi'_{left} Q_{left} + \Phi'_{right} Q_{right} = 0$$

因此只需确定两个极板上的电荷分布使左极板，电荷，右极板的电势确定。

可选取左极板电荷均匀分布（不接地），右极板接地。则三者的电势分别为 $Ed, E(d-h), 0$, 带入得 $Q_{left}Ed = -qE(d-h)$, 因此左极板电荷总量为 $Q_{left} = -q(1-h/d)$.

选取右极板电荷均匀分布（不接地），左极板接地。则三者的电势分别为 $0, Eh, Ed$, 带入得 $Q_{right}Ed = -qEh$, 因此右极板电荷总量为 $Q_{right} = -q(h/d)$

解法二 (等效为三个平行板):

将电荷 q 视为无穷多个 δq 聚集在一点，每一个 δq 会在两个极板上产生感应电荷，若在平行于两个极板的平面内移动此 δq，感应电荷也会随之移动，但总量不变，因此对于求两个极板的总电荷量来说在此平面内改变电荷分布是一个对称操作。

因此将电荷分布集中在一点变为在此平面均匀分布，满足 $\lim_{S\to\infty}\sigma_0 S = q$,

左右极板的电荷面密度分别为 σ_L, σ_R, 两侧极板接地，因此电荷只分布在左极板的右侧面和右极板的左侧面。三者满足 (选取高斯面包括左右两个极板内部的面，两个极板的电压都是 0):

$$\sigma_L + \sigma_R + \sigma_0 = 0$$
$$(-\sigma_L + \sigma_R + \sigma_0)h = (\sigma_L - \sigma_R + \sigma_0)(d-h)$$

由此可得 $\sigma_L = -\sigma_0(1-h/d), \sigma_R = -\sigma_0(h/d)$

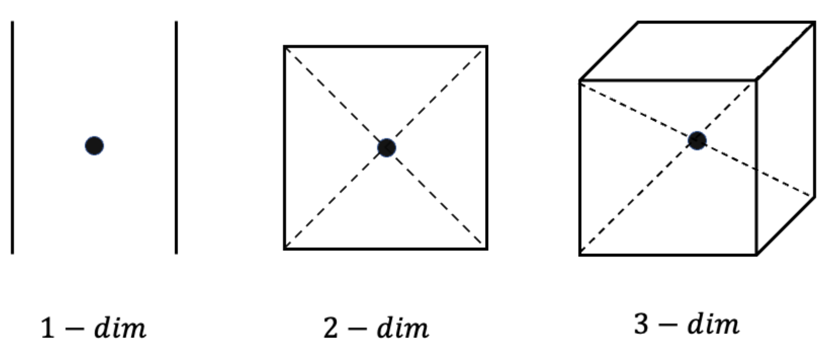
左右两个极板的电荷总量即为：$Q_{left} = -q(1-h/d), Q_{right} = -q(h/d)$

6. (10 points) Calculate the electric energy in each situation. Write the answer in the form of series (you don't need to calculate the value of series).

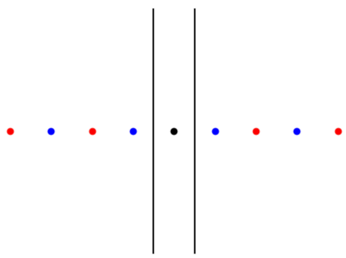
   (a) (2 points) A point charge is placed in the middle of two grounded infinite conductor planes (the distance between the two planes is l)

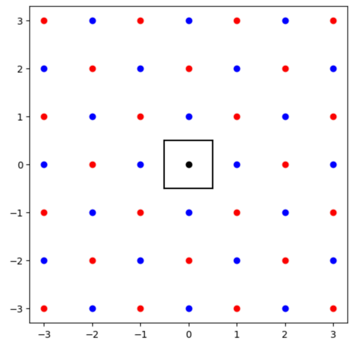
   (b) (3 points) A point charge is placed in the middle of a grounded infinitely long square conductor tube with side length l.

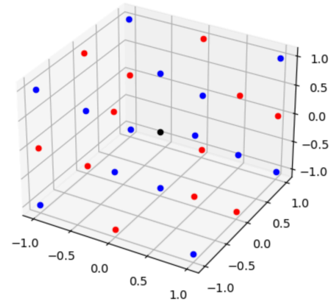
   (c) (5 points) A point charge is placed in the middle of a grounded cube with side length l.



不失一般性可假设电荷带正电，则三种情形的像电荷分布如三个图所示 (红色为正电荷，蓝色为负电荷)，因此其电势能为





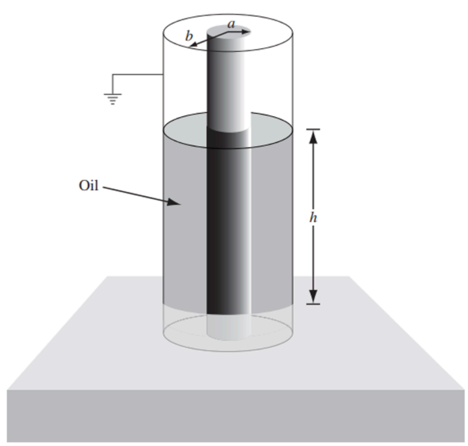


$$W_{1d} = \frac{q^2}{8\pi\epsilon_0 l}\left(\sum_{i=-\infty, i\neq 0}^{i=\infty} \frac{(-1)^i}{i}\right)$$

$$W_{2d} = \frac{q^2}{8\pi\epsilon_0 l}\left(\sum_{i,j=-\infty,(i,j)\neq(0,0)}^{i,j=\infty} \frac{(-1)^{(i+j)}}{\sqrt{(i^2+j^2)}}\right)$$

$$W_{3d} = \frac{q^2}{8\pi\epsilon_0 l}\left(\sum_{i,j,k=-\infty,(i,j,k)\neq(0,0,0)}^{i,j,k=\infty} \frac{(-1)^{(i+j+k)}}{\sqrt{(i^2+j^2+k^2)}}\right)$$

7. (10 points) Two long coaxial cylindrical metal tubes (inner radius a, outer radius b) stand vertically in a tank of dielectric oil (susceptibility χ_e, mass density ρ). The inner one is maintained at potential V, and the outer one is grounded.



(a) (7 points) Calculate the whole energy W_{tot} of this system when the oil rises to height h in the space between the tubes. (You need to consider the gravity)

此情形可看作两个电容器的并联。

在空气介质中电场强度为:$E = \frac{\lambda}{2\pi\epsilon_0 r}$，电压即为:$V = \int Edr = \frac{\lambda}{2\pi\epsilon_0}In\frac{b}{a}$, 带电量为 $Q = (H-h)\lambda$, 因此空气介质中的电容为 $C_1 = \frac{Q}{V} = \frac{2\pi\epsilon_0(H-h)}{In\frac{b}{a}}$

在油介质中电场强度为:$E = \frac{\lambda}{2\pi(\chi_e+1)\epsilon_0 r}$，电压即为:$V = \int Edr = \frac{\lambda}{2\pi(\chi_e+1)\epsilon_0}In\frac{b}{a}$, 带电量为 $Q = (h)\lambda$, 因此油介质中的电容为 $C_1 = \frac{Q}{V} = \frac{2\pi(\chi_e+1)\epsilon_0 h}{In\frac{b}{a}}$

因此总电容为

$$C = C_1 + C_2 = \frac{2\pi\epsilon_0(h\chi_e + H)}{In\frac{b}{a}}$$

电容器内存储的能量即为

$$W_e = \frac{1}{2}CV^2 = \frac{V^2\pi\epsilon_0(h\chi_e + H)}{In\frac{b}{a}}$$

重力势能为:

$$W_g = \int_0^h \rho g\pi(b^2 - a^2)xdx = \frac{1}{2}\rho gh^2\pi(b^2 - a^2)$$

总能量即为:

$$W_{tot} = W_e + W_g = \frac{V^2\pi\epsilon_0(h\chi_e + H)}{In\frac{b}{a}} + \frac{1}{2}\rho gh^2\pi(b^2 - a^2)$$

(b) (3 points) To what height the oil rises to reach equilibrium in the system.

若其高度改变 Δh, 则系统的能量改变量为 (其中 $-\Delta CV^2$ 为放电放出的热量):

$$\Delta W = \frac{dW_{tot}}{dh}\Delta h - \Delta CV^2$$

稳态时 $\Delta W = 0$, 因此:

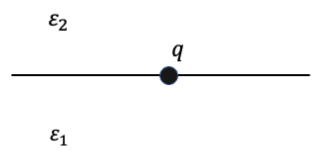
$$-\Delta h\frac{V^2\pi\epsilon_0\chi_e}{In\frac{b}{a}} + \Delta h\rho gh\pi(b^2 - a^2) = 0$$

化简得稳态时的高度为:

$$h = \frac{V^2\epsilon_0\chi_e}{\rho gIn\frac{b}{a}(b^2 - a^2)}$$

8. (10 points) Two kinds of dielectric materials are separated by an infinite plane. Consider the following types of charge distributions.

(a) (5 points) A point charge q is located on the interface between the two uniform infinite dielectrics. The interface is an infinite plane. The dielectric constants are ϵ_1 and ϵ_2 respectively. Calculate φ, $\vec{E}$, $\vec{D}$ in the medium and polarized surface charge density σ_P in the interface (except for the point charge position).



切线电场连续，全空间电场分布为 $\vec{E} = \frac{A\hat{r}}{r^2}$
由电位移矢量的高斯定理可得:

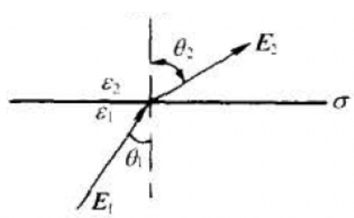
$$\epsilon_1 A/r^2 2\pi r^2 + \epsilon_2 A/r^2 2\pi r^2 = q, A = \frac{q}{2\pi(\epsilon_1 + \epsilon_2)}$$

因此可得 $\vec{E}, \varphi, \vec{D}$:

$$\varphi = \frac{q}{2\pi(\epsilon_1 + \epsilon_2)r}$$
$$\vec{E} = \frac{q\hat{r}}{2\pi(\epsilon_1 + \epsilon_2)r^2}$$
$$\vec{D} = \frac{\epsilon_2 q\hat{r}}{2\pi(\epsilon_1 + \epsilon_2)r^2}, (above \quad the \quad interface)$$
$$\vec{D} = \frac{\epsilon_1 q\hat{r}}{2\pi(\epsilon_1 + \epsilon_2)r^2}, (below \quad the \quad interface)$$

由于 $\vec{P}$ 平行于平面，因此 $\sigma_P = 0$

(b) (5 points) The dielectric constants of the two media are ϵ_1 and ϵ_2 respectively. There is a layer of free charges with a surface charge density of σ on their interface. The electric field strengths on both sides of this surface are $\vec{E_1}$ and $\vec{E_2}$ respectively. $|\vec{E_1}| = E_0$ is a known constant, while $|\vec{E_2}|$ is unknown. Their angles with the normal line of the interface are θ_1 and θ_2 respectively, as shown in the figure. Calculate the relationship between θ_1 and θ_2.



由边界条件：$E_{t1} = E_{t2}, D_{n2} - Dn1 = \sigma$ 可得：

$$E_1 sin\theta_1 = E_2 sin\theta_2, \epsilon_1 E_1 cos\theta_1 = \epsilon_2 E_2 cos\theta_2 - \sigma$$

化简得：

$$\epsilon_1 cot\theta_1 + \frac{\sigma}{E_0 sin\theta_1} = \epsilon_2 cot\theta_2$$