# Bertskas exercise 1.1

1. We assume $x_k$ integer.

$$
\begin{aligned}
J_4(x_4) &= 0 \\
J_3(x_3) &= \min_{u_3}(x_3^2 + u_3^2 + J_4(x_3+u_3)) \\
&= \min_{u_3}(x_3^2 + u_3^2) \\
&= x_3^2 \qquad u_3(x_3) = 0 \\
J_2(x_2) &= \min_{u_2}(x_2^2 + u_2^2 + J_3(x_2+u_2)) \\
&= \min_{u_2}(x_2^2 + u_2^2 + (x_2+u_2)^2) \\
&= \min_{u_2}(2x_2^2 + 2u_2(u_2+x_2)) \\
&= 2x_2^2 + 2\min_{u_2}((u_2+x_2-x_2)(u_2+x_2)) \\
&= 2x_2^2 + 2\min(0, 1-x_2, 2(2-x_2), 3(3-x_2), 4(4-x_2), 5(5-x_2))) \\
&= 2x_2^2 \qquad x_2 \le 1 \\
&= 2x_2^2 + 1 - x_2 \qquad 1 \le x_2 \le 3 \\
&= 2x_2^2 + 2(2-x_2) \qquad 3 \le x_2 \le 5 \\
&= 2x_2^2 + 3(3-x_2) \qquad 5 \le x_2 \le 7 \\
&= 2x_2^2 + 4(4-x_2) \qquad 7 \le x_2 \le 9 \\
&= 2x_2^2 + 5(5-x_2) \qquad 9 \le x_2
\end{aligned}
$$

   Etc

2. We assume $u_k, x_k$ real.

$$
\begin{aligned}
J_4(x_4) &= 0 \\
J_3(x_3) &= \min_{u_3}(x_3^2 + u_3^2 + J_4(x_3+u_3)) \\
&= \min_{u_3}(x_3^2 + u_3^2) \\
&= x_3^2 \qquad u_3(x_3) = 0 \\
J_2(x_2) &= \min_{u_2}(x_2^2 + u_2^2 + J_3(x_2+u_2)) \\
&= \min_{u_2}(x_2^2 + u_2^2 + (x_2+u_2)^2) \\
&= \min_{u_2}(2x_2^2 + 2u_2(u_2+x_2)) \\
&= 2x_2^2, u_2 = -x_2 \qquad x_2 \le 0 \\
&= 3/2x_2^2, u_2 = -1/2x_2 \qquad 0 \le x_2 \le 10 \\
&= 2x_2^2 + 50 - 10x_2, u_2 = 5 - x_2 \qquad 10 \le x_2
\end{aligned}
$$

# Bertsekas exercise 3.2

We follow the PMP approach of Kappen's tutorial pg. 7.

1. Construct $H$:

$$H(t,u,x,\lambda) = \exp(-\beta t)\sqrt{u} + \lambda(\alpha x - u)$$

2. Construct $H^*$:

$$\begin{aligned} u^* &= armin_u H = \frac{1}{4\lambda^2}\exp(-2\beta t) \\ H^*(t,x,\lambda) &= H(t,u^*,x,\lambda) = \frac{1}{4\lambda}\exp(-2\beta t) + \lambda\alpha x \end{aligned}$$

3. Hamilton equations

$$\begin{aligned} \dot{x} &= \frac{\partial H^*}{\partial \lambda} = -\frac{1}{4\lambda^2}\exp(-2\beta t) + \alpha x \\ \dot{\lambda} &= -\frac{\partial H^*}{\partial x} = -\lambda\alpha \end{aligned}$$

with boundary condition $x(0) = S$ and $x(T) = 0$. Note, that this is slightly different from the PMP derivation in the text, where we have an end condition for $\lambda$ instead of for $x$.

4. Solve the Hamilton equations. The solution for $\lambda$ is

$$\lambda(t) = A\exp(-\alpha t)$$

Substituting in the equation for $x$ we get

$$\dot{x} = \frac{-1}{4A^2}\exp(2\alpha t - 2\beta t) + \alpha x$$

We try the solution $x(t) = C(t)\exp(\alpha t)$, because we know that $x = C\exp(\alpha t)$ is the solution of the homogeneous equation. Substitution yields an equation for $C$ that we can solve:

$$\begin{aligned} \dot{C} &= \frac{-1}{4A^2}\exp(\alpha t - 2\beta t) \\ C(t) &= \frac{-1}{4A^2}\frac{1}{\alpha - 2\beta}\exp(\alpha t - 2\beta t) + C_0 \end{aligned}$$

Substituting this solution in the solution for $x(t)$ we can find the two constants $A$ and $C_0$ from the boundary conditions $X(0) = S$ and $X(T) = 0$. The final result is

$$\begin{aligned} x(t) &= \frac{S\exp(\alpha t)}{\exp((\alpha - 2\beta)T) - 1}\left(\exp((\alpha - 2\beta)T) - \exp(\alpha - 2\beta)t)\right) \\ u(t) &= \frac{S(\alpha - 2\beta)}{\exp((\alpha - 2\beta)T) - 1}\exp(2(\alpha - \beta)t) \end{aligned}$$

Note, that when there is no discounting ($\beta = 0$) the optimal spending increases with age which is due to the interest on the capital.

## Extra exercises

1. The mass on the spring problem with maximal end velocity is identical to maximal end position, except that the end cost $\phi(x) = -x_2$. The differential equation for $\psi$ (ICML sheets pg. 20) are solved as

$$\psi_1(t) = \sin(t - T)\psi_2(t) = -\cos(t - T)$$

with boundary condition $(\psi_(T), \psi_2(T)) = (0, -1)$. The optimal control is $u(t) = -\text{sign}(\psi_2(t)) = \text{sign}(\cos(t - T))$

$$
\begin{array}{cc}
u = 1 & 0 < t < \pi/2 \\
u = -1 & \pi/2 < t < 3\pi/2 \\
u = 1 & 3\pi/2 < t < 2\pi
\end{array}
$$

2. (a) The PMP formalism is given in Kappen's tutorial text on page. 7. The recipe is
   i. Construct the Hamiltonian
   $$H(t, x, u, \lambda) = -R(t, x, u) + \lambda f(t, u, x) = -\frac{1}{2}u^2 + \lambda u$$
   ii. Construct the optimized Hamiltonian
   $$H^*(t, x, \lambda) = H(t, x, u^*, \lambda) = \frac{1}{2}\lambda^2 \qquad u^* = \lambda$$
   iii. Solve the Hamilton equations of motion
   $$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H^*}{\partial \lambda} = \lambda \\ \frac{d\lambda}{dt} &= -\frac{\partial H^*}{\partial x} = 0 \end{aligned}$$
   with boundary conditions $x(t_0)$ and $\lambda(t = T) = -x(T)$[1]. The solution for $\lambda$ is constant $\lambda(t) = \lambda$ with $\lambda = -x(T)$. The solution for $x(t)$ is
   $$x(t) = x(t_0) + \lambda(t - t_0)$$
   Combining these two results, we get $\lambda = -x(T) = -x(t_0) - \lambda(T - t_0)$, or
   $$\lambda = \frac{-x(t_0)}{1 + T - t_0}$$
   which is the optimal control law.

   (b) We use the Bellman equation 1.18 on page 10 of Kappens tutorial.
   $$\begin{aligned} -\partial_t J &= \min_u \left( \frac{1}{2}u^2 + u\partial_x J + \frac{1}{2}\nu\partial_x^2 J \right) \\ &= -\frac{1}{2}(\partial_x J)^2 + \frac{1}{2}\nu\partial_x^2 J \end{aligned}$$
   where we have computed the optimal control as a function of $J$: $u = -\partial_x J$. We have to solve with boundary condition $J(x, T) = \frac{1}{2}x^2$. This is a partial differential equation, because $J$ is a function of $x$ and $t$. This is not easy to solve, but since the problem is linear quadratic, we know that
   $$J(x, t) = \frac{1}{2}P(t)x^2 + \alpha(t)x + \beta(t)$$

[1] Note, that $\phi(x) = \frac{1}{2}x^2$ so that $\phi_x = x$.

with boundary conditions $P(T) = 1, \alpha(T) = 0, \beta(T) = 0$.
Substitution and setting terms proportional to $x^2, x$ and constant equal to zer yields directly:

$$\begin{aligned}
\dot{P} &= P^2 \\
\dot{\alpha} &= \alpha P \\
\dot{\beta} &= \frac{1}{2}\alpha^2 - \frac{1}{2}\nu P
\end{aligned}$$

The solution is:

$$\begin{aligned}
P(t) &= \frac{-1}{t+c} \\
\alpha(t) &= 0
\end{aligned}$$

The solution for $\beta$ is irrelevant for the control. $c$ is an integration constant. Since $P(T) = 1$ we find $c = -1 - T$ and $P(t) = \frac{1}{1+T-t}$.
Finally, the control solution is

$$u = -\frac{\partial J}{\partial x} = \frac{-x}{1+T-t}$$

(c) The path integral solution is given in Kappen's tutorial on pg. 19.

3. (a)

$$\begin{aligned}
\rho(y,t|x,0) &= \frac{1}{\sqrt{2\pi\nu t}} \exp\left(-\frac{(y-x)^2}{2\nu t}\right) \\
\frac{\partial}{\partial t}\rho &= -\frac{1}{2t}\rho + \frac{(y-x)^2}{2\nu t^2}\rho \\
\frac{\partial}{\partial x}\rho &= -\frac{(y-x)}{\nu t}\rho \\
\frac{\partial^2}{\partial x^2}\rho &= \frac{1}{\nu t}\rho + \left(\frac{(y-x)}{\nu t}\right)^2 \rho = -\frac{1}{\nu}\frac{\partial}{\partial t}\rho
\end{aligned}$$

(b)

$$\begin{aligned}
J(x,t) &= -\lambda \log \psi(x,t) \\
\psi(x,t) &= \int dy \rho(y,T|x,t) \exp(-\phi(y)/\lambda) \\
\exp(-\phi(y)/\lambda) &= \delta(y-1) + \delta(y+1) \\
\psi(x,T) &= \rho(1,T|x,0) + \rho(-1,T|x,0) \\
&= \frac{1}{\sqrt{2\pi\nu T}}\left(\exp\left(-\frac{(1-x)^2}{2\nu T}\right) + \exp\left(-\frac{(1+x)^2}{2\nu T}\right)\right) \\
&= \frac{1}{\sqrt{2\pi\nu T}} \exp\left(-\frac{1+x^2}{2\nu T}\right)\left(\exp\left(\frac{x}{\nu T}\right) + \exp\left(-\frac{x}{\nu T}\right)\right) \\
&= \frac{2}{\sqrt{2\pi\nu T}} \exp\left(-\frac{1+x^2}{2\nu T}\right) \cosh\left(\frac{x}{\nu T}\right) \\
J(x,0) &= \lambda \log \sqrt{\frac{\pi\nu T}{2}} + \frac{R}{2T}(1+x^2) - \nu R \log\cosh\left(\frac{x}{\nu T}\right)
\end{aligned}$$

(c)

$$u = -\frac{1}{R}\frac{dJ}{dx} = -\frac{x}{T} + \frac{1}{T}\tanh\left(\frac{x}{\nu T}\right)$$