Poisson Summation Formula, Revisited

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1. THE FORMULA

For *sufficiently well-behaved* function $f : \mathbb{R} \to \mathbb{C}$, the Poisson summation formula reveals a somewhat surprising connection between the time domain representation f and its Fourier transform $\hat{f}$—given a fixed $T > 0$,

e.g. $f \in \mathcal{S}(\mathbb{R})$ can be a Schwartz function, though the result holds pointwise for a larger function space (see Section 3).

$$\sum_{n=-\infty}^{\infty} f(x + nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \hat{f}\left(\frac{n}{T}\right) \mathrm{e}^{\mathrm{i}2\pi n x/T}, \tag{PSF1}$$

for any $x \in \mathbb{R}$. In particular, if we set $x = 0$ and $T = 1$, then

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n),$$

which, to engineers, is more familiarly known as the fact that the Fourier transform of an impulse train is itself, i.e.

$$Ш^\wedge = Ш,$$

where

$$Ш(x) = \sum_{n=-\infty}^{\infty} \delta(x - n)$$

is the DIRAC COMB (impulse train) distribution and $Ш^\wedge$ is the Fourier transform of Ш as a tempered distribution.

ASIDE

The definition of Ш above should be understood as a *weakly* convergent series, in the sense that given any test function $\phi \in \mathcal{S}(\mathbb{R})$,

For the definition of **weak** convergence, see page 428 of [4] or section 3.11 of [5].

$$\begin{aligned} Ш\{\phi\} &= \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} \delta(x - n) \right) \phi(x)\,\mathrm{d}x = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x - n)\phi(x)\,\mathrm{d}x \\ &= \sum_{n=-\infty}^{\infty} \phi(n), \end{aligned} \tag{1.1}$$

where the switch of summation and integration is *symbolic* and can be justified from the decay of $\phi \in \mathcal{S}(\mathbb{R})$. More specifically, by definition of the Schwartz space,

$$|\phi(x)||x|^2 \leq C_1 \quad \text{and} \quad |\phi(x)| \leq C_2$$

for some constant $C_1, C_2 \in \mathbb{R}$, so

$$\sum_{n=-\infty}^{\infty} |\phi(n)| \leq \sum_{n=-\infty}^{\infty} \frac{C_1 + C_2}{1 + |n|^2},$$

Add 1 to avoid the pole at $n = 0$.

which is finite, so the interchange can be justified by the Fubini's Theorem. Similarly, its Fourier transform $\text{III}^\wedge$ should be understood as

$$\text{III}^\wedge\{\phi\} = \text{III}\{\hat{\phi}\} = \sum_{n=-\infty}^{\infty} \hat{\phi}(n).$$

However, we will intentionally *avoid* the use of distributions in the main discussion of this essay, because I think it is an *overkill* to use the theory of distributions, which requires a fair bit of effort to develop rigorously, to derive the basic form of the result. As we shall see, a clever use of the Poisson summation formula can lead to a much simpler derivation of the sampling theory.

2. PERIODIZATION & SAMPLING

Abstractly, the Poisson summation formula (PSF1) reveals a connection between the PERIODIZATION of f and SAMPLING of $\hat{f}$, though as we will see later, it actually tells us ~~a little~~ much more than this. But before moving on, we need to give a precise definition to them.

2.1 Periodization

On the LHS of (PSF1), $\sum_{n=-\infty}^{\infty} f(x+nT)$ is a PERIODIZATION, or PERIODIC SUMMATION, of the original function f with period T. If we define

or CONVOLUTION with an impulse train, if this is more familiar.

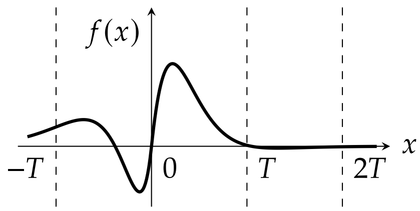
$$g(x) = \sum_{n=-\infty}^{\infty} f(x+nT),$$

and given the sum converges pointwise, we can see that g is T-periodic

$$\begin{aligned} g(x+T) &= \sum_{n=-\infty}^{\infty} f(x+T+nT) \\ &= \sum_{n=-\infty}^{\infty} f(x+(n+1)T) \\ &= g(x), \end{aligned}$$

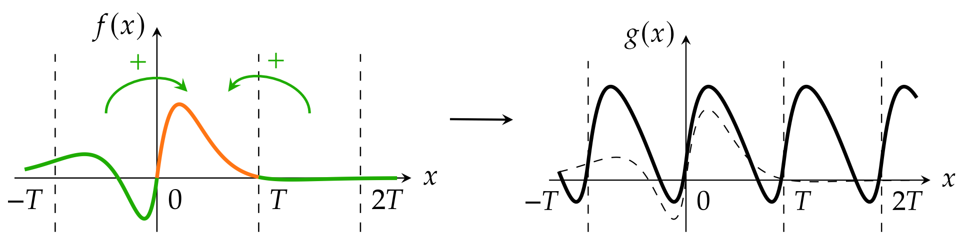
since $n \in \mathbb{Z}$ and the coefficient $(n+1)$ gets absorbed when the infinite sum runs across all the integers.

Visually, to periodize a function f defined on $\mathbb{R}$,



we can divide the domain $\mathbb{R}$ into segments of length T, then add the function value of f at each segment to the interval $[0, T)$, and finally take its periodic extension to $\mathbb{R}$. This gives us a T-periodic function g.

Of course, g might not converge or converge to a nice enough function so that its Fourier series converges pointwise, which is why we need f to be **well-behaved**.



2.2 Sampling

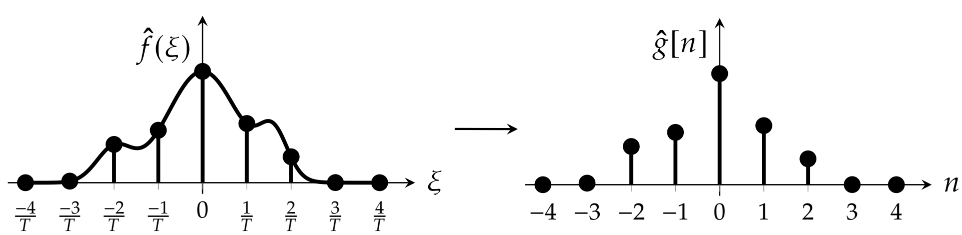
On the RHS of (PSF1), $\hat{f}\left(\frac{n}{T}\right) = \hat{f}\left(n\frac{1}{T}\right)$ is a SAMPLING or DISCRETIZATION of the spectrum $\hat{f}$ by $\frac{1}{T}$. If we define

or DECIMATION if $\hat{f}$ is discrete, as we will see later.

$$\hat{g}[n] = \hat{f}\left(\frac{n}{T}\right),$$

then $\hat{g}[n]$ is simply a discrete representation of the spectrum $\hat{f}(\xi)$.

We will only draw the real part of the spectrum because it's hard to visualize complex signals.



Using the sampled spectrum $\hat{g}[n]$ as Fourier coefficients, we can construct another T-periodic function $\tilde{g}$

$$\tilde{g}(x) = \sum_{n=-\infty}^{\infty} \hat{g}[n] e^{i2\pi nx/T},$$

which is in some sense also a periodization of the original function $f(x)$. The

Poisson summation formula (PSF1) essentially tell us the two T-periodic functions $g(x)$, defined in previous section, and $\frac{1}{T}\tilde{g}(x)$ coincide pointwise.

3. THE PROOF & CONDITIONS

With these definitions in mind, we will try to give a sufficient condition for the Poisson summation formula (PSF1) to hold true pointwise.

for more general results, see pages 15, 105, and 250 of [3].

For the result to hold pointwise, we want the Fourier transform $\hat{f}$ of f to be defined pointwise, so f needs to be absolutely integrable, i.e.

with either Riemann or Lebesgue construction

$$f \in L^1(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} \;\middle|\; \int_{-\infty}^{\infty} |f(x)| \, \mathrm{d}x < \infty \right\}.$$

Because we also need the T-periodization of f

$$g(x) = \sum_{n=-\infty}^{\infty} f(x + nT)$$

to be defined pointwise, f needs to have some further decay conditions. For its Fourier series expansion to converge pointwise, we either need to put constraints on the differentiability of f or constraints on $\hat{f}$ directly. The following condition would be one of the possibilities.

The condition here is actually adapted from the definition of the Wiener algebra $W(\mathbb{R})$ [2].

THEOREM (Poisson summation formula)

Let $T > 0$. Suppose f is continuous with sufficient decay such that

$$\sum_{n=-\infty}^{\infty} \sup_{x \in [0,T)} |f(x + nT)| < \infty \tag{PC1}$$

In fact, the condition doesn't depend on the value of T, see Section 11.5 of [1].

and $\hat{f}$ has sufficient decay such that

$$\sum_{n=-\infty}^{\infty} \sup_{\xi \in [0,1/T)} \left| \hat{f}\left(\xi + \frac{n}{T}\right) \right| < \infty \tag{PC2}$$

Notice that these two conditions implies $f, \hat{f} \in L^1(\mathbb{R})$, which will be explained later.

Then, the Poisson summation formula

$$\sum_{n=-\infty}^{\infty} f(x + nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \hat{f}\left(\frac{n}{T}\right) \mathrm{e}^{\mathrm{i}2\pi nx/T}$$

holds pointwise for any $x \in \mathbb{R}$ and both sums converge absolutely.

Proof. The first decay assumption (PC1) implies that the periodization

$$g(x) = \sum_{k=-\infty}^{\infty} f(x + kT)$$

converges absolutely. Furthermore, the continuity of f implies g is continuous. Since $g(x)$ is T-periodic, we can expand $g(x)$ in Fourier series.

$$g(x) = \sum_{n=-\infty}^{\infty} c[n]\mathrm{e}^{\mathrm{i}2\pi nx/T}.$$

If you are willing to consider f being defined by $\hat{f}$ (as an equivalent class $f \sim g$ iff the set $\{f(x) \neq g(x)\}$ has measure zero), the continuity of f in the condition can be omitted because $\hat{f} \in L^1(\mathbb{R})$ implies the (uniform) continuity of f.

It suffices to show that

$$c[n] = \frac{1}{T}\hat{f}\left(\frac{n}{T}\right) \tag{3.1}$$

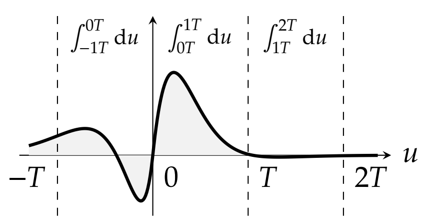
and the Fourier series converges pointwise. We can compute $c[n]$ directly.

$$\begin{aligned} c[n] &= \frac{1}{T}\int_0^T g(x)\mathrm{e}^{-\mathrm{i}2\pi nx/T}\,\mathrm{d}x \\ &= \frac{1}{T}\int_0^T \sum_{k=-\infty}^{\infty} f(x+kT)\mathrm{e}^{-\mathrm{i}2\pi nx/T}\,\mathrm{d}x. \end{aligned}$$

Assuming that we can exchange the integral and sum (which will be justified later),

$$\begin{aligned} c[n] &= \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_0^T f(x+kT)\mathrm{e}^{-\mathrm{i}2\pi nx/T}\,\mathrm{d}x & (3.2) \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{kT}^{(k+1)T} f(u)\mathrm{e}^{-\mathrm{i}2\pi n(u-kT)/T}\,\mathrm{d}u \qquad (u = x+kT). \end{aligned}$$

Notice that the sum of integrals $\sum_{k=-\infty}^{\infty}\int_{kT}^{(k+1)T}\mathrm{d}u$ is simply adding up disjoint integrals over the entire $\mathbb{R}$.



Therefore,

$$\begin{aligned} c[n] &= \frac{1}{T}\int_{-\infty}^{\infty} f(u)\mathrm{e}^{-\mathrm{i}2\pi n(u-kT)/T}\,\mathrm{d}u & (3.3) \\ &= \frac{1}{T}\int_{-\infty}^{\infty} f(u)\mathrm{e}^{-\mathrm{i}2\pi nu/T}\underbrace{\mathrm{e}^{\mathrm{i}2\pi nk}}_{=1}\,\mathrm{d}u \\ &= \frac{1}{T}\int_{-\infty}^{\infty} f(u)\mathrm{e}^{-\mathrm{i}2\pi nu/T}\,\mathrm{d}u \\ &= \frac{1}{T}\hat{f}\left(\frac{n}{T}\right), \end{aligned}$$

which is exactly (3.1). The only work remains is to justify some convergence issues. Note that

$$\sum_{k=-\infty}^{\infty} \int_0^T |f(x+kT)e^{-i2\pi nx/T}| \, dx \le \sum_{k=-\infty}^{\infty} \int_0^T \sup_{x\in[0,T)} |f(x+kT)| \, dx$$
$$= T \cdot \sum_{k=-\infty}^{\infty} \sup_{x\in[0,T)} |f(x+kT)| < \infty$$

This also justifies $f \in L^1(\mathbb{R})$. Similarly, $\hat{f} \in L^1(\mathbb{R})$.

by (PC1), so the switch of sum and integrals in (3.2) can be justified by Fubini's Theorem. Therefore, $\hat{f}\left(\frac{n}{T}\right)$ converges for any $n \in \mathbb{Z}$.

Furthermore, by assumption (PC2)

$$\sum_{n=-\infty}^{\infty} |c[n]| = \sum_{n=-\infty}^{\infty} \left| \frac{1}{T} \hat{f}\left(\frac{n}{T}\right) \right| < \infty \tag{3.4}$$

and continuity of $g(x)$, the Fourier series

$$g(x) = \sum_{n=-\infty}^{\infty} c[n] e^{i2\pi nx/T}.$$

converges pointwise, so the entire equality (PSF1) holds pointwise. □

4. FOURIER TRANSFORM & FOURIER SERIES

If we set $T = 1$ in (PSF1), we can obtain a simplified form of the formula

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{i2\pi nx}.$$

In this special case, the Poisson summation formula essentially tells us that if we periodize the function f with period $T = 1$, the Fourier coefficients $c[n]$ of the 1-periodic function

$$g(x) = \sum_{n=-\infty}^{\infty} f(x+n)$$

is exactly the same as the values of $\hat{f}(\xi)$ at integer values of ξ. That is, if we compute the Fourier series expansion of $g(x)$,

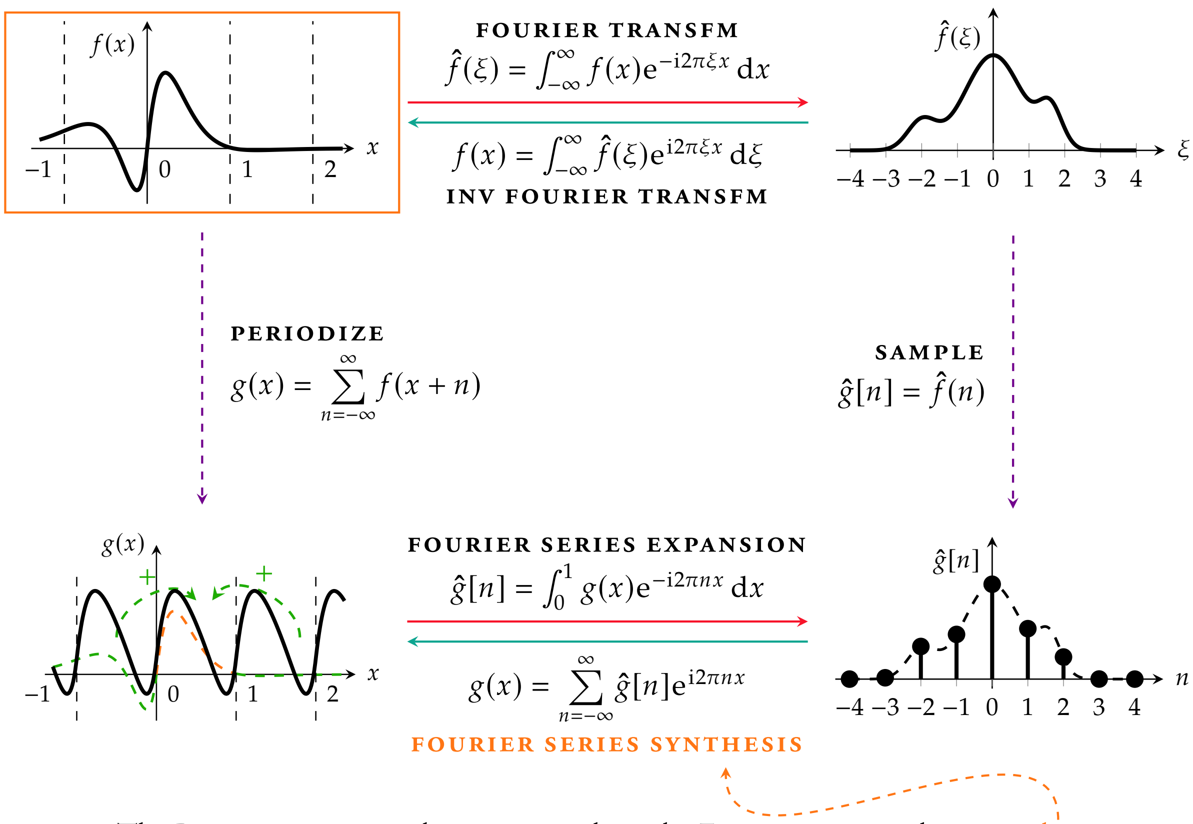
$$g(x) = \sum_{n=-\infty}^{\infty} c[n] e^{i2\pi nx},$$

we would notice that

$$c[n] = \hat{f}(n).$$

This result can be nicely summarized in a diagram.

Fig. 1: Poisson summation in a nutshell ($T = 1$)



The Poisson summation happens exactly at the Fourier series synthesis step here, though there is nothing prevent us taking the Fourier series expansion of the 1-periodic function $g(x)$ so that we can move back and forth between the time and frequency domain.

As we can see, periodization and sampling is the *link* between the Fourier transform and Fourier series. They allow us to move from the top to the bottom of the diagram. As long as the Poisson summation formula holds for the function f, the entire diagram commutes.

Drawing out the Poisson summation formula as a diagram has a significant advantage. Instead of periodizing f first as we did in the proof, we can take an alternate route: let us first take the Fourier transform of f

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i2\pi\xi x}\,dx.$$

Then, construct a sequence $\hat{g}[n]$ by sampling $\hat{f}(\xi)$ at integer values of ξ

$$\hat{g}[n] = \hat{f}(n).$$

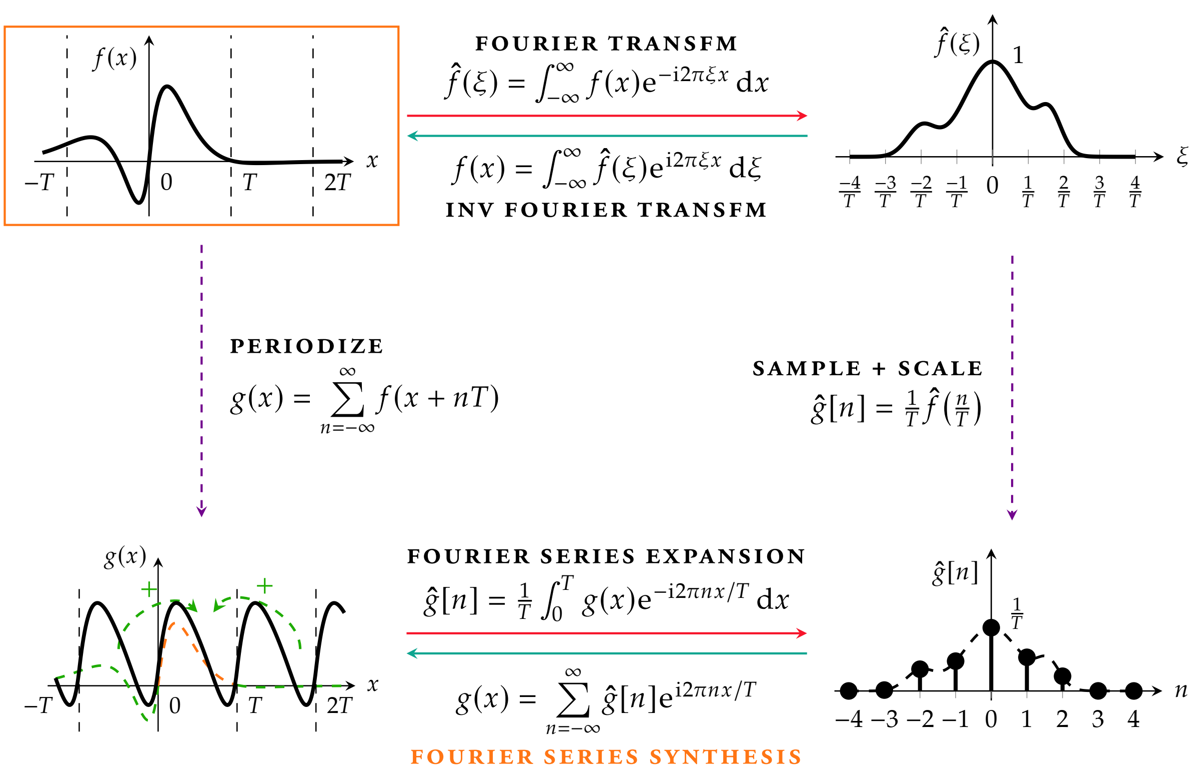
Finally, use $\hat{g}[n]$ as Fourier coefficients to construct a function $g(x)$,

$$g(x) = \sum_{n=-\infty}^{\infty} \hat{g}[n]\mathrm{e}^{\mathrm{i}2\pi nx}.$$

The diagram tells us that $g(x)$ is exactly the periodization of $f(x)$ with period $T = 1$!

If we remove the constraint that $T = 1$, there would be an extra scaling by $\frac{1}{T}$ after the sampling. The diagram will become slightly messier because of the extra T floating around.

Fig. 2: Poisson summation in a nutshell ($T > 0$)



The equality of the Poisson summation still occurs at the Fourier series synthesis step.

5. FOURIER TRANSFORM & DTFT

Have you noticed anything interesting in the two diagrams before?

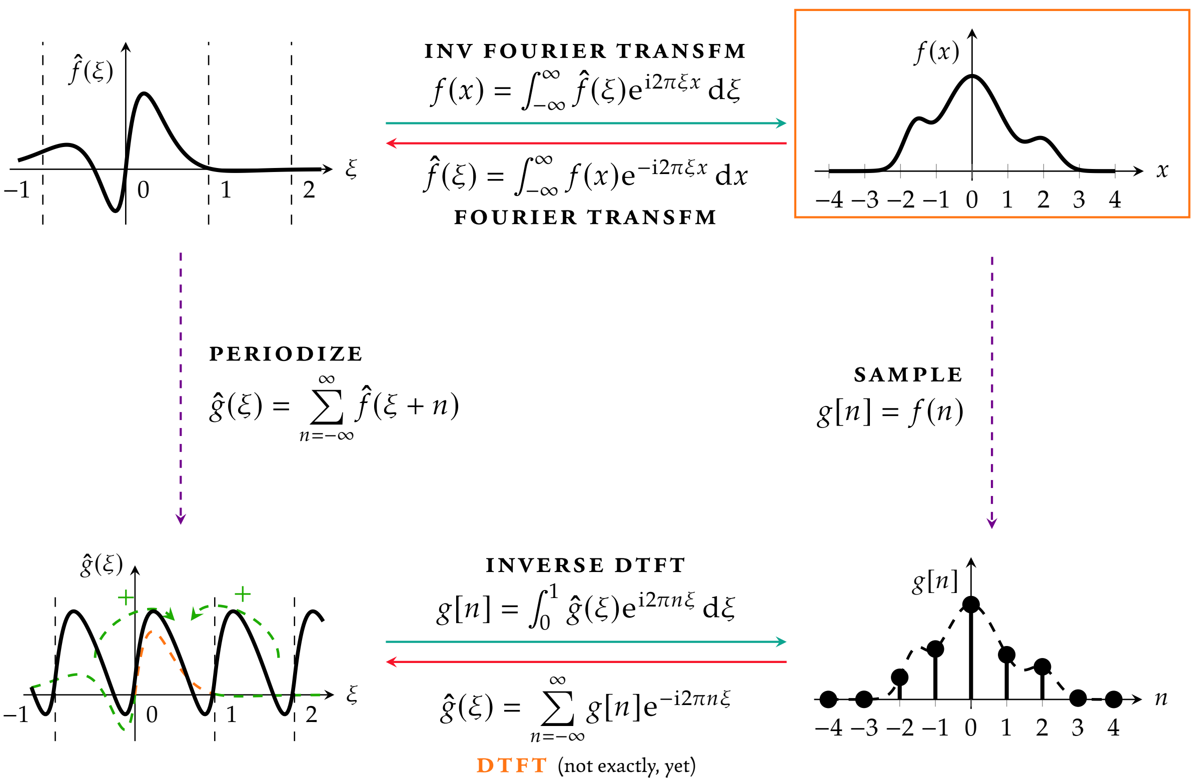
Because of the double arrows connecting the time and frequency domain, there is actually no problem switching the interpretation of f and $\hat{f}$, i.e. we can treat $\hat{f}$ as our function in time domain and f as its frequency domain representation.

given $\hat{f}$ satisfies the condition for f in (PSF1), though note that $f \in L^1(\mathbb{R})$ implies (uniform) continuity of $\hat{f}$, so our condition is symmetric for $f, \hat{f}$.

If not, we will discuss it in detail below.

Let us focus on the simpler case where $T = 1$ first. *If you are convinced that* the Poisson summation formula also works if we replace every i by $-\mathrm{i}$, then we can do some relabeling, switching ξ and x, i and $-\mathrm{i}$. This gives us a new diagram, which is in some sense the DUAL of the Poisson summation formula.

FIG. 3: Poisson summation, relabeled ($T = 1$)



After the relabeling, the LHS of the diagram is now the frequency domain and the RHS is the time domain. Because Fourier transform is unitary, the relabeling didn't change anything for the upper half of the diagram except a switch of order.

However, for the lower half of the diagram, we somehow obtained a frequency representation of a DISCRETE-TIME FUNCTION (or simply an infinite sequence) $g[n]$, which is directly sampled from the initial function $f(x)$ at the upper-right corner of the diagram.

5.1 Discrete-time Fourier transform

Therefore, based on this intuition, we can define the discrete-time Fourier transform (DTFT) of a discrete (discrete-time) function $f[n]$ to be

$$\hat{f}_1(\xi) = \sum_{n=-\infty}^{\infty} f[n]\mathrm{e}^{-\mathrm{i}2\pi n\xi}, \tag{DTFT1}$$

The 1 in the subscript denotes the periodicity of the spectrum.

if $f[n]$ is sampled from a continuous-time function $f(x)$ with sampling period $T = 1$. Ideally, the definition of DTFT should be independent of the sampling period T we use, but we will continue refining this definition in a moment.

We can immediately see that the current definition of the DTFT is always 1-periodic. The fact can be easily seen on the diagram, but we can also derive this result from the formula

$$\begin{aligned}
\hat{f}_1(\xi + 1) &= \sum_{n=-\infty}^{\infty} f[n]\mathrm{e}^{-\mathrm{i}2\pi n(\xi+1)} \\
&= \sum_{n=-\infty}^{\infty} f[n]\mathrm{e}^{-\mathrm{i}2\pi n\xi} \cdot \underbrace{\mathrm{e}^{-\mathrm{i}2\pi n}}_{=1} \\
&= \sum_{n=-\infty}^{\infty} f[n]\mathrm{e}^{-\mathrm{i}2\pi n\xi} = \hat{f}_1(\xi).
\end{aligned}$$

Under our current definition, the diagram tells us that if we sample a continuous-time function $f(x)$ with sampling period $T = 1$ to obtain a discrete function $g[n]$, and then take the DTFT, or more precisely (DTFT1), of the sampled function $g[n]$, its frequency domain representation $\hat{g}(\xi)$ will be exactly the periodization of the spectrum $\hat{f}(\xi)$ with period 1.

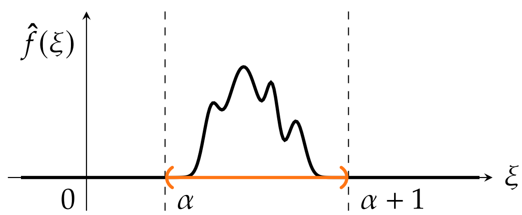
Furthermore, the diagram even tells us how to get back the discrete function $g[n]$ from its frequency domain representation, i.e. the inverse DTFT

$$g[n] = \int_0^1 \hat{g}(\xi)\mathrm{e}^{\mathrm{i}2\pi n\xi}\,\mathrm{d}\xi. \tag{IDTFT1}$$

5.2 The sampling theorem

If we know the support $\operatorname{supp}(\hat{f}) = \overline{\{\xi \in \mathbb{R} \mid \hat{f}(\xi) \neq 0\}}$ is contained in some open interval $I = (\alpha, \alpha+1)$ of length 1, i.e. $\operatorname{supp}(\hat{f}) \subset I$,

If $\hat{f}(\xi) \neq 0$ at the boundaries, e.g. $\xi = \alpha$, then the continuity of $\hat{f}$ will force $\hat{f}(\xi - \varepsilon) \neq 0$ for some $\varepsilon > 0$, so we use open interval here.



and we know exactly where the interval I is, then we can actually move from the bottom-left corner to the upper-left corner of the diagram by multiplying $\hat{g}(\xi)$ with the indicator function $\mathbb{1}_{(\alpha,\alpha+1)}$

$$\hat{f}(\xi) = \hat{g}(\xi) \cdot \mathbb{1}_{(\alpha,\alpha+1)}(\xi),$$

where

$$\mathbb{1}_{(\alpha,\alpha+1)}(\xi) = \begin{cases} 1 & \text{if } \xi \in (\alpha, \alpha+1) \\ 0 & \text{if } \xi \notin (\alpha, \alpha+1) \end{cases},$$

hence giving us a way to recover the original continuous function $f(x)$.

An alternate way to qualify this condition is to say that if $\hat{f}(\xi) \neq 0$ implies

$$\hat{f}(\xi + n) = 0$$

for any integer $n \in \mathbb{Z}$, i.e. there is no ALIASING in the frequency domain, then we can recover the original function $f(x)$. This is essentially a restricted case of the Nyquist–Shannon sampling theorem.

5.3 The justification of relabeling

Let us justify the statement we made before: the Poisson summation formula also works if we replace i with $-\mathrm{i}$ and do some relabeling. Because $T = 1$ is a special case of $T > 0$, we will prove the general result to save some work.

Since now the main operation is sampling, we will also replace T with $\frac{1}{T}$. Therefore, our goal is to show that

$$\sum_{n=-\infty}^{\infty} \hat{f}\left(\xi + \frac{n}{T}\right) = T \sum_{n=-\infty}^{\infty} f(nT)\mathrm{e}^{-\mathrm{i}2\pi nT\xi},$$

or equivalently,

$$\frac{1}{T}\sum_{n=-\infty}^{\infty}\hat{f}\left(\xi+\frac{n}{T}\right)=\sum_{n=-\infty}^{\infty}f(nT)\mathrm{e}^{-\mathrm{i}2\pi nT\xi}. \tag{PSF2}$$

The result is actually not hard to see. We will start with (PSF1),

$$\sum_{n=-\infty}^{\infty}f(x+nT)=\frac{1}{T}\sum_{n=-\infty}^{\infty}\hat{f}\left(\frac{n}{T}\right)\mathrm{e}^{\mathrm{i}2\pi nx/T}.$$

By setting $x=0$, we obtain a special case of the Poisson summation formula

$$\sum_{n=-\infty}^{\infty}f(nT)=\frac{1}{T}\sum_{n=-\infty}^{\infty}\hat{f}\left(\frac{n}{T}\right). \tag{5.1}$$

We will proceed from here with some basic properties of the Fourier transform.

Consider a function $f(x)$ that satisfy the condition for the Poisson summation formula with period T. Let us denote its Fourier transform by $\hat{f}(s)$.

You will see why we don't use ξ here in a moment.

Here, we define a helper function

$$h(x)=f(x)\mathrm{e}^{-\mathrm{i}2\pi x\xi}.$$

where $\xi\in\mathbb{R}$ is a constant. Since $|\mathrm{e}^{-\mathrm{i}2\pi x\xi}|=1$, the modulation does not change the decay property (PC1) of f. Because a modulation in the time domain is a translation (or convolution) in the frequency domain,

$$\hat{h}(s)=\hat{f}(s+\xi),$$

where the translation also does not change the decay property (PC2), we can apply the special case of the Poisson summation formula (5.1) on h and obtain the equality

$$\sum_{n=-\infty}^{\infty}h(nT)=\frac{1}{T}\sum_{n=-\infty}^{\infty}\hat{h}\left(\frac{n}{T}\right),$$

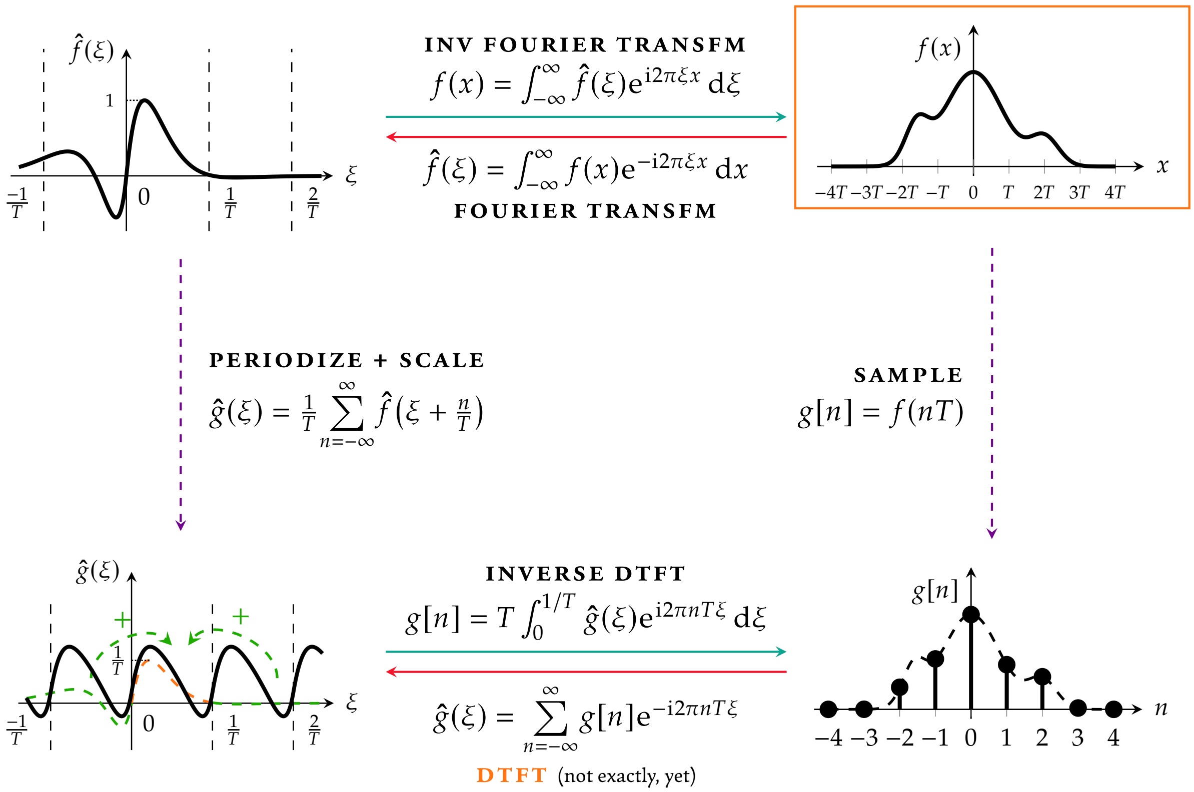
i.e.

$$\sum_{n=-\infty}^{\infty}f(nT)\mathrm{e}^{-\mathrm{i}2\pi nT\xi}=\frac{1}{T}\sum_{n=-\infty}^{\infty}\hat{f}\left(\xi+\frac{n}{T}\right),$$

which is exactly the result we want.

We can similarly make a diagram for the general case, which is simply relabeled from (FIG. 2) above.

FIG. 4: Poisson summation, relabeled ($T > 1$)



Similar to what we did before, from the diagram, we can define the general version of the discrete-time Fourier transform (DTFT) of a discrete function $f[n]$ to be

$$\hat{f}_{1/T}(\xi) = \sum_{n=-\infty}^{\infty} f[n]\mathrm{e}^{-\mathrm{i}2\pi nT\xi}, \tag{DTFT2}$$

if $f[n]$ is sampled from a continuous-time function $f(x)$ with sampling period $T > 0$. Similarly from the diagram, the corresponding inversion formula is

$$f[n] = T\int_0^{\frac{1}{T}} \hat{f}_{1/T}(\xi)\mathrm{e}^{\mathrm{i}2\pi nT\xi}\,\mathrm{d}\xi. \tag{IDTFT2}$$

5.4 The actual definition

There is still a problem, though. Our current definitions of the DTFT of a discrete function $f[n]$

$$\hat{f}_{1/T}(\xi) = \sum_{n=-\infty}^{\infty} f[n]\mathrm{e}^{-\mathrm{i}2\pi nT\xi}$$

and the corresponding inverse DTFT

$$f[n] = T \int_0^{\frac{1}{T}} \hat{f}_{1/T}(\xi) \mathrm{e}^{\mathrm{i}2\pi nT\xi} \, \mathrm{d}\xi.$$

still rely on the fact that the discrete function $f[n]$ is sampled from a continuous-time function $f(x)$ with some fixed sampling period $T > 0$ and sampling frequency $\xi_s = \frac{1}{T}$. To take the DTFT of a discrete function $f[n]$, we have to know the sampling period T or sampling frequency ξ_s beforehand.

To solve this problem, we will introduce the concept of a NORMALIZED FREQUENCY, denoted by ω. Given a sampling period T or sampling frequency $\xi_s = \frac{1}{T}$, the normalized frequency of an (unnormalized) frequency ξ in Hertz (Hz) is defined to be

$$\omega = 2\pi T\xi = 2\pi \frac{\xi}{\xi_s},$$

where the unit of ω is radians/sample.

If we know the sampling period T or sampling frequency ξ_s, we can recover the actual frequency ξ from normalized frequency by

$$\xi = \frac{\omega}{2\pi T} = \frac{\omega}{2\pi} \xi_s$$

However, even if we know nothing about the sampling period T, the definition of the normalized frequency still allows us to compute the DTFT of a discrete function $f[n]$ independent of the sampling period T. We just need to substitute

$$\omega = 2\pi T\xi$$

in previous formulas.

For DTFT, we have

$$\hat{f}_{2\pi}(\omega) = \sum_{n=-\infty}^{\infty} f[n] \mathrm{e}^{-\mathrm{i}\omega n}, \qquad \text{(DTFT3)}$$

which converges absolutely as long as $f[n]$ is absolutely summable, i.e.

$$f \in \ell^1(\mathbb{Z}) = \left\{ f : \mathbb{Z} \to \mathbb{C} \;\middle|\; \sum_{n \in \mathbb{Z}} |f[n]| < \infty \right\}$$

Because we are indexing using a new variable ω in (DTFT3), the frequency representation $\hat{f}_{2\pi}(\omega)$ is 2π-periodic instead of $\frac{1}{T}$-periodic, because

$$\begin{aligned}\hat{f}_{2\pi}(\omega+2\pi) &= \sum_{n=-\infty}^{\infty} f[n]\mathrm{e}^{-\mathrm{i}(\omega+2\pi)n} \\ &= \sum_{n=-\infty}^{\infty} f[n]\mathrm{e}^{-\mathrm{i}\omega n}\cdot \underbrace{\mathrm{e}^{-\mathrm{i}2\pi n}}_{=1} \\ &= \sum_{n=-\infty}^{\infty} f[n]\mathrm{e}^{-\mathrm{i}\omega n} = \hat{f}_{2\pi}(\omega)\end{aligned}$$

Similarly, to find the inverse DTFT of $\hat{f}_{2\pi}(\omega)$, we will perform a change of variable $\omega = 2\pi T\xi$, i.e.

$$\begin{aligned}f[n] &= T\int_0^{\frac{1}{T}} \hat{f}_{1/T}(\xi)\mathrm{e}^{\mathrm{i}2\pi nT\xi}\,\mathrm{d}\xi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \hat{f}_{1/T}\left(\frac{\omega}{2\pi T}\right)\mathrm{e}^{\mathrm{i}\omega n}\,\mathrm{d}\omega.\end{aligned}$$

Since

$$\begin{aligned}\hat{f}_{1/T}\left(\frac{\omega}{2\pi T}\right) &= \sum_{n=-\infty}^{\infty} f[n]\mathrm{e}^{-\mathrm{i}2\pi nT\frac{\omega}{2\pi T}} \\ &= \sum_{n=-\infty}^{\infty} f[n]\mathrm{e}^{-\mathrm{i}\omega n} = \hat{f}_{2\pi}(\omega),\end{aligned}$$

we have

$$f[n] = \frac{1}{2\pi}\int_0^{2\pi} \hat{f}_{2\pi}(\omega)\mathrm{e}^{\mathrm{i}\omega n}\,\mathrm{d}\omega. \tag{IDTFT3}$$

To understand the connection between the Fourier transform $\hat{f}(\xi)$ of $f(x)$ and the DTFT $\hat{f}_{2\pi}(\omega)$ of the sampled function $f[n] = f(nT)$, we need to go back to the Poisson summation formula (PSF2)

$$\frac{1}{T}\sum_{n=-\infty}^{\infty} \hat{f}\left(\xi+\frac{n}{T}\right) = \sum_{n=-\infty}^{\infty} f(nT)\mathrm{e}^{-\mathrm{i}2\pi nT\xi}$$

Substitute $\omega = 2\pi T\xi$ into the right-hand side, we get

$$\begin{aligned}\frac{1}{T}\sum_{n=-\infty}^{\infty} \hat{f}\left(\xi+\frac{n}{T}\right) &= \sum_{n=-\infty}^{\infty} f(nT)\mathrm{e}^{-\mathrm{i}\omega n} \\ &= \sum_{n=-\infty}^{\infty} f[n]\mathrm{e}^{-\mathrm{i}\omega n} \\ &= \hat{f}_{2\pi}(\omega).\end{aligned}$$

That is, the value of $\hat{f}_{2\pi}$ at normalized frequency ω is a periodized version of $\hat{f}$ evaluated at (unnormalized) frequency $\xi = \frac{\omega}{2\pi T}$ and scaled by $\frac{1}{T}$, i.e.

$$\hat{f}_{2\pi}(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \hat{f}\left(\frac{\omega}{2\pi T} + \frac{n}{T}\right).$$

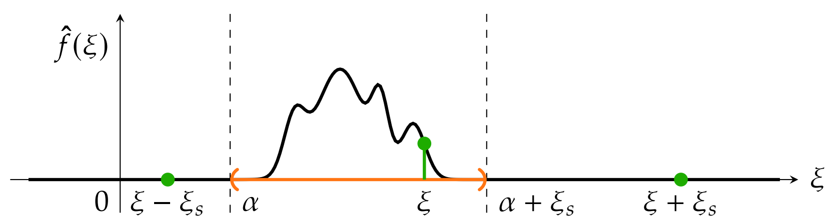
5.5 The sampling theorem (take 2)

Alternatively, we could say $\hat{f}_{2\pi}$ is an ALIASED version of $\hat{f}$, scaled by $\frac{1}{T}$. Whenever we query the frequency content of $\hat{f}$ at $\xi = \frac{\omega}{2\pi T}$ via the DTFT $\hat{f}_{2\pi}$, we would pick up some extra "aliased" components, i.e.

$$\hat{f}_{2\pi}(\omega) = \frac{1}{T}\left[\hat{f}(\xi) + \hat{f}\left(\xi + \frac{1}{T}\right) + \hat{f}\left(\xi - \frac{1}{T}\right) + \cdots\right],$$

where only one of them is needed, usually being the first term.

Similar to Section 5.2, if we know the support of $\hat{f}(\xi)$ is completely contained in some open interval $I = (\alpha, \alpha + \xi_s)$ of length $\xi_s = \frac{1}{T}$, or more precisely $\operatorname{supp}(\hat{f}) \subset I$,



then no aliasing can happen in this case, and we can obtain the frequency content because only one term is summed

$$\hat{f}_{2\pi}(\omega) = \frac{1}{T}\hat{f}(\xi),$$

though we still need to pay extra attention to the periodicity of $\hat{x}_{2\pi}$, because it is also possible that

$$\hat{f}_{2\pi}(\omega) = \frac{1}{T}\hat{f}\left(\xi + \frac{n}{T}\right)$$

for some $n \neq 0$.

In order to know exactly which frequency ξ we have queried from $\hat{f}$, it is necessary to know exact location of the interval $I = (\alpha, \alpha + \xi_s)$. From the figure above, it is not hard to see that the actual frequency ξ_α we queried via $\hat{f}_{2\pi}$ is

$$\xi_\alpha = \xi + \frac{n_\alpha}{T} = \frac{\omega}{2\pi T} + \frac{n_\alpha}{T},$$

where n_α satisfies the condition that $n_\alpha \in \mathbb{Z}$ and

$$f_\alpha = f + \frac{n_\alpha}{T} \in [\alpha, \alpha + \xi_s).$$

The values at the boundaries are necessarily 0 so they do not matter here.

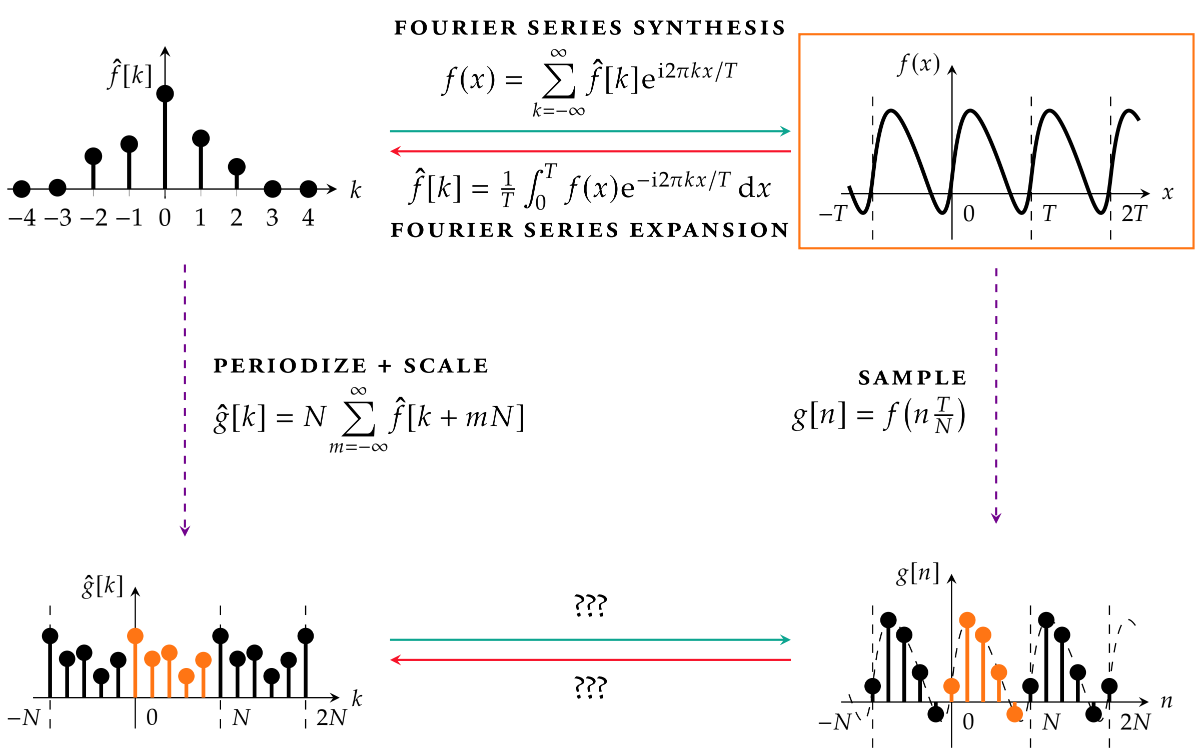
6. FOURIER SERIES & DFT

It turns out that we can go even further. What if we replace f with a T-periodic function, and replace the Fourier transform at the top with Fourier series?

6.1 Yet another relabeling

Let us consider the following diagram.

FIG. 5: Discrete Poisson summation ($T > 0, N > 1$)



The diagram is obtained by directly mimicking **FIG.** 4. We first created a discrete function $g[n]$ by sampling each T-cycle of f with N samples

$$g[n] = f\left(n\frac{T}{N}\right).$$

Why don't we use FIG. 2? A **duality** almost always implies going one way is much easier than following the other path.

Since we divide each T-cycle into evenly spaced intervals, g is N-periodic,

$$g[n+N] = f\left(n\frac{T}{N} + T\right) = f\left(n\frac{T}{N}\right) = g[n].$$

Similarly, we also applied a (discrete) periodization and scaling to the Fourier coefficients $\hat{f}[k]$ and obtained a new discrete function

$$\hat{g}[k] = N \sum_{m=-\infty}^{\infty} \hat{f}[k+mN],$$

which is also N-periodic.

The diagram naturally leads to a question for us to ponder upon:

> *Is there any connection between g and $\hat{g}$?*

The answer, of course, is YES, and the key to answer this question is hidden in the proof of the Poisson summation formula.

6.2 The inverse discrete Fourier transform

We will start with the Fourier series synthesis step, where

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}[k] \mathrm{e}^{\mathrm{i}2\pi kx/T}. \tag{6.1}$$

By substituting $x = nT/N$, we obtain

$$\begin{aligned} g[n] &= f\left(n\frac{T}{N}\right) \\ &= \sum_{k=-\infty}^{\infty} \hat{f}[k] \mathrm{e}^{\mathrm{i}2\pi nk/N}. \end{aligned}$$

Recall that in the proof of (PSF1) in Section 3, we used a trick to combine a sum of integrals of length T into an integral of the entire $\mathbb{R}$ (3.3). Here, we will do something similar but in reverse—we will break the sum across $\mathbb{Z}$ into a sum of sums of length N

$$g[n] = \sum_{m=-\infty}^{\infty} \sum_{k=mN}^{(m+1)N-1} \hat{f}[k] \mathrm{e}^{\mathrm{i}2\pi nk/N}.$$

This is valid as long as $\hat{f} \in \ell^1(\mathbb{Z})$, using Fubini's Theorem. Next, we perform a change of variable by setting $r = k - mN$ or $k = mN + r$, which yields

$$g[n] = \sum_{m=-\infty}^{\infty} \sum_{r=0}^{N-1} \hat{f}[mN+r] \mathrm{e}^{\mathrm{i}2\pi n(mN+r)/N}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{r=0}^{N-1} \hat{f}[mN+r] \mathrm{e}^{\mathrm{i}2\pi nr/N} \underbrace{\mathrm{e}^{-\mathrm{i}2\pi nm}}_{=1}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{r=0}^{N-1} \hat{f}[mN+r] \mathrm{e}^{\mathrm{i}2\pi nr/N}$$

Similar to (3.2), we will switch the two sums, which is again valid if $\hat{f} \in \ell^1(\mathbb{Z})$,

$$g[n] = \sum_{r=0}^{N-1} \underbrace{\sum_{m=-\infty}^{\infty} \hat{f}[mN+r]}_{=\frac{1}{N}\hat{g}[r]} \mathrm{e}^{\mathrm{i}2\pi nr/N} \tag{6.2}$$

$$= \frac{1}{N} \sum_{r=0}^{N-1} \hat{g}[r] \mathrm{e}^{\mathrm{i}2\pi nr/N}$$

Relabel r with k, we obtain

$$g[n] = \frac{1}{N} \sum_{k=0}^{N-1} \hat{g}[k] \mathrm{e}^{\mathrm{i}2\pi nk/N}, \tag{IDFT}$$

which, you may have recognized, is exactly the inverse DISCRETE FOURIER TRANSFORM of $g[n]$, but we have to define the forward transform first.

6.3 The discrete Fourier transform

Because g and $\hat{g}$ are both N-periodic, we only need N consecutive samples of g and $\hat{g}$ to define the entire sequence on $\mathbb{Z}$. Hence, we can represent g and $\hat{g}$ by two vectors $\mathbf{g}, \hat{\mathbf{g}} \in \mathbb{C}^N$

$$\mathbf{g} = \begin{bmatrix} g[0] \\ g[1] \\ \vdots \\ g[N-1] \end{bmatrix} \qquad \hat{\mathbf{g}} = \begin{bmatrix} \hat{g}[0] \\ \hat{g}[1] \\ \vdots \\ \hat{g}[N-1] \end{bmatrix}.$$

The advantage of representing g and $\hat{g}$ with vectors is that we can write (IDFT) compactly in matrix notations by defining an $N \times N$ matrix

$$\mathcal{E}_N = \frac{1}{\sqrt{N}} \begin{bmatrix} \mathrm{e}^{\mathrm{i}2\pi\cdot 0\cdot 0/N} & \mathrm{e}^{\mathrm{i}2\pi\cdot 0\cdot 1/N} & \cdots & \mathrm{e}^{\mathrm{i}2\pi\cdot 0\cdot (N-1)/N} \\ \mathrm{e}^{\mathrm{i}2\pi\cdot 1\cdot 0/N} & \mathrm{e}^{\mathrm{i}2\pi\cdot 1\cdot 1/N} & \cdots & \mathrm{e}^{\mathrm{i}2\pi\cdot 1\cdot (N-1)/N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{e}^{\mathrm{i}2\pi\cdot (N-1)\cdot 0/N} & \mathrm{e}^{\mathrm{i}2\pi\cdot (N-1)\cdot 1/N} & \cdots & \mathrm{e}^{\mathrm{i}2\pi\cdot (N-1)\cdot (N-1)/N} \end{bmatrix},$$

where

$$\mathcal{E}_N[n,k] = \frac{1}{\sqrt{N}} \mathrm{e}^{\mathrm{i}2\pi nk/N}.$$

With this definition, we can write (IDFT) as

$$\mathbf{g} = \frac{1}{\sqrt{N}} \mathcal{E}_N \hat{\mathbf{g}}. \tag{IDFT-M}$$

Note that the columns of $\mathcal{E}_N$ are orthonormal and $\mathcal{E}_N$ is unitary because

$$\mathcal{E}_N \mathcal{E}_N^* = \mathcal{E}_N^* \mathcal{E}_N = \mathrm{I}_N,$$

Since this is finite dimensional linear algebra, it can be checked by brute force.

where $\mathcal{E}_N^* = \overline{A}^\mathsf{T}$ is the conjugate transpose or the adjoint of $\mathcal{E}_N$, and I_N is the $N \times N$ identity matrix. Since $\mathcal{E}_N$ is already symmetric,

$$\begin{aligned}
\mathcal{E}_N^{-1} &= \mathcal{E}_N^* = \overline{\mathcal{E}_N} \\
&= \frac{1}{\sqrt{N}} \begin{bmatrix} \mathrm{e}^{-\mathrm{i}2\pi\cdot 0\cdot 0/N} & \mathrm{e}^{-\mathrm{i}2\pi\cdot 0\cdot 1/N} & \cdots & \mathrm{e}^{-\mathrm{i}2\pi\cdot 0\cdot (N-1)/N} \\ \mathrm{e}^{-\mathrm{i}2\pi\cdot 1\cdot 0/N} & \mathrm{e}^{-\mathrm{i}2\pi\cdot 1\cdot 1/N} & \cdots & \mathrm{e}^{-\mathrm{i}2\pi\cdot 1\cdot (N-1)/N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{e}^{-\mathrm{i}2\pi\cdot (N-1)\cdot 0/N} & \mathrm{e}^{-\mathrm{i}2\pi\cdot (N-1)\cdot 1/N} & \cdots & \mathrm{e}^{-\mathrm{i}2\pi\cdot (N-1)\cdot (N-1)/N} \end{bmatrix},
\end{aligned}$$

where

$$\mathcal{E}_N^{-1}[k,n] = \frac{1}{\sqrt{N}} \mathrm{e}^{-\mathrm{i}2\pi nk/N}.$$

By now, we have prepared everything to define the DISCRETE FOURIER TRANSFORM of an N-periodic discrete signal $g[n]$ or vector $\mathbf{g} \in \mathbb{C}^N$. First, we will define

$$\mathcal{F}_N = \mathcal{E}_N^{-1}$$

to be the $N \times N$ Fourier transform matrix.

By rewriting (IDFT-M), we have

$$\hat{\mathbf{g}} = \sqrt{N} \mathcal{F}_N \mathbf{g}, \tag{DFT-M}$$

or equivalently,

$$\hat{g}[k] = \sum_{n=0}^{N-1} g[n] \mathrm{e}^{-\mathrm{i}2\pi nk/N}. \tag{DFT}$$

Notice that in (IDFT-M) and (DFT-M) there is an extra factor $\sqrt{N}$ floating around. To solve this problem, we can define a unitary version of the discrete Fourier transform by removing this extra factor

$$\hat{\mathbf{g}} = \mathcal{F}_N \mathbf{g} \tag{UDFT-M}$$

$$\mathbf{g} = \mathcal{F}_N^{-1} \hat{\mathbf{g}} = \overline{\mathcal{F}_N} \hat{\mathbf{g}}. \tag{UIDFT-M}$$

By expanding the matrix notation, we have

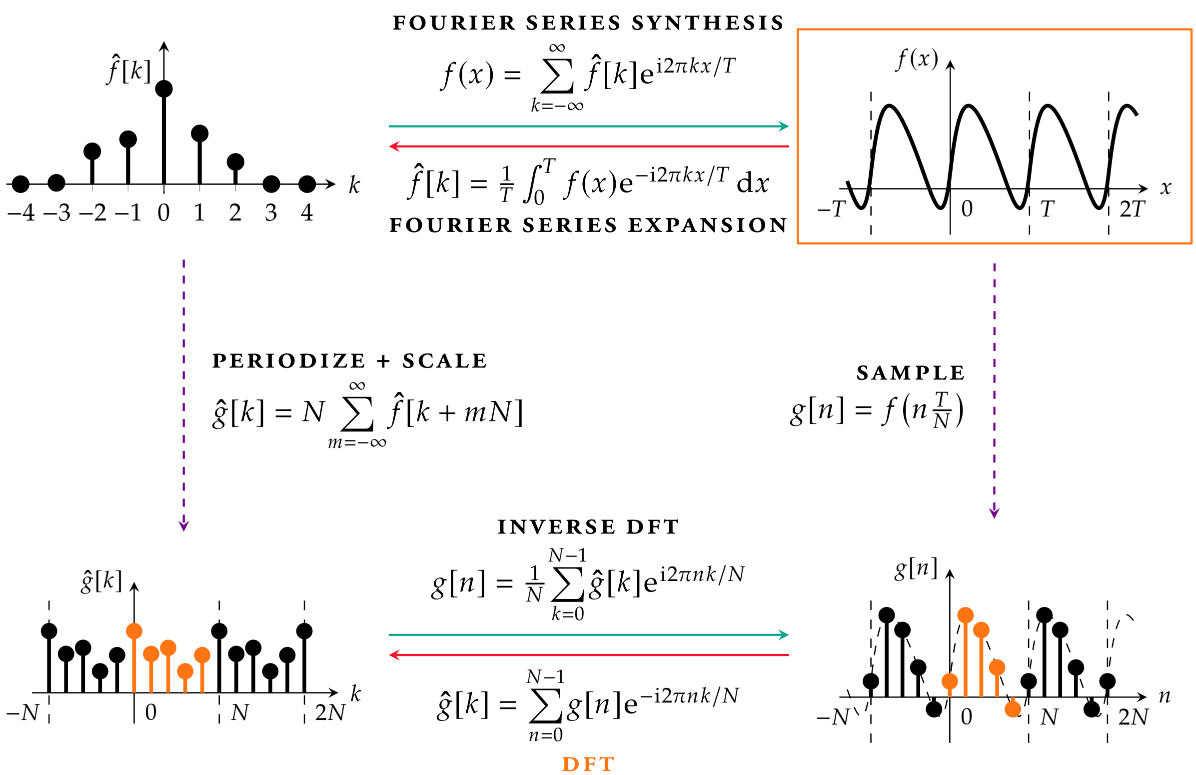
$$\hat{g}[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} g[n] \mathrm{e}^{-\mathrm{i}2\pi nk/N} \tag{UDFT}$$

$$g[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{g}[k] \mathrm{e}^{\mathrm{i}2\pi nk/N}. \tag{UIDFT}$$

6.4 The complete picture

As we can see, the missing pieces of FIG. 5 is exactly the DFT. By filling in the missing labels, we now have a complete picture of the relationship between Fourier series and DFT.

FIG. 6: Discrete Poisson summation ($T > 0, N > 1$)



6.5 The sampling theorem (take 3)

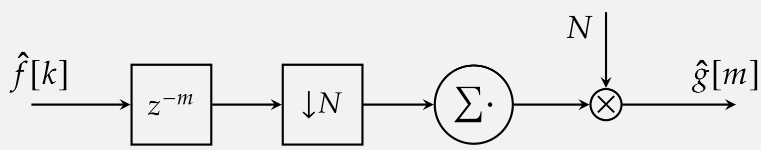
Analog to Section 5.5, we can try to figure out a condition for when we can recover the original function $f(x)$ from the sampled function $g[n]$.

When we query a frequency k from the DFT spectrum $\hat{g}[k]$, we obtain an aliased and scaled version of the Fourier coefficients $\hat{f}[k]$

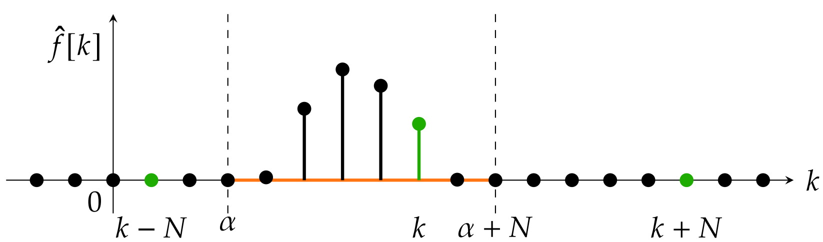
$$\hat{g}[k] = N \cdot \left(\hat{f}[k] + \hat{f}[k+N] + \hat{f}[k-N] + \cdots\right).$$

ASIDE

In block diagrams, if we want to obtain the DFT spectrum $\hat{g}[k]$ at $k = m$, then we can first delay $\hat{f}$ by m samples, then DECIMATE it by N, take the infinite sum, and scale by N.



If $\hat{f}[k] = 0$ except N consecutive terms $k = \alpha, \alpha+1, \ldots, \alpha+N-1$, i.e. f is synthesized using no more than N complex exponentials of consecutive frequencies,



then the samples of $\hat{f}[k]$ between α and $\alpha+N-1$ will be invariant to periodization, so we can recover the original function f by

$$f(x) = \frac{1}{N}\sum_{k=\alpha}^{\alpha+N-1} \hat{g}[k]\mathrm{e}^{\mathrm{i}2\pi kx/T}$$

if α is known beforehand. Note that here we treat $\hat{g}[k]$ as a periodic function rather than a vector.

6.6 Conditions for pointwise convergence

By substituting $\hat{g}[k]$ and $g[n]$ into (DFT), we obtain the discrete form of the Poisson summation formula

$$N \sum_{m=-\infty}^{\infty} \hat{f}[k+mN] = \sum_{n=0}^{N-1} f\left(n\frac{T}{N}\right) \mathrm{e}^{-\mathrm{i}2\pi nk/N}, \tag{DPSF1}$$

which is a direct analog of (PSF2).

Since we obtained (DFT) using finite dimensional linear algebra from (IDFT), there was no convergence issues in Section 6.3. If we want (DPSF1) to hold pointwise for any $k \in \mathbb{Z}$, we need to evaluate each step in our derivation of the (IDFT) in Section 6.2. First, for the LHS of (DPSF1) to be defined pointwise, it suffices to have

$$\sum_{m=-\infty}^{\infty} \hat{f}[k+mN] < \infty.$$

Unlike the continuous case, as long as we have $\hat{f} \in \ell^1(\mathbb{Z})$,

A similar condition to (PC1) is more restrictive since it implies $\hat{f} \in \ell^1(\mathbb{Z})$.

$$\sum_{m=-\infty}^{\infty} \hat{f}[k+mN] \le \sum_{m=-\infty}^{\infty} \left|\hat{f}[k+mN]\right| \le \sum_{k=-\infty}^{\infty} \left|\hat{f}[k]\right| < \infty.$$

This also allows us to exchange the double sum in (6.2). Since the RHS is a finite sum, there is no convergence issue.

For the Fourier coefficients $\hat{f}[k]$ in (6.1) to be defined pointwise, we need $f \in L^1[0,T)$, and for the Fourier series to converge to $f(x)$ for every $x \in \mathbb{R}$, we need f to be continuous and $\hat{f} \in \ell^1(\mathbb{Z})$. To summarize these observations,

THEOREM (Poisson summation formula for discrete $\hat{f}$)

Let $T > 0$ and $N > 1$. Suppose $f \in L^1[0,T)$ is T-periodic and continuous, with its Fourier coefficients $\hat{f} \in \ell^1(\mathbb{Z})$. Then, the discrete version of the Poisson summation formula

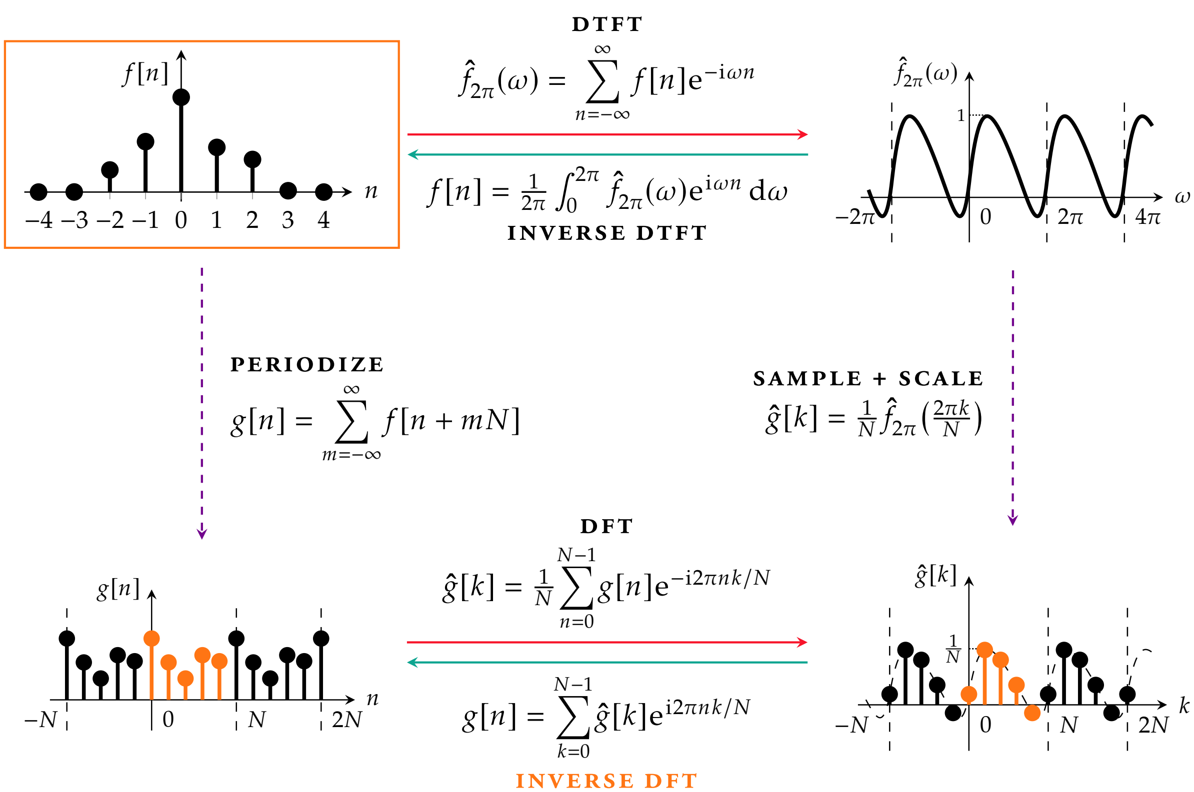
$$N \sum_{m=-\infty}^{\infty} \hat{f}[k+mN] = \sum_{n=0}^{N-1} f\left(n\frac{T}{N}\right) \mathrm{e}^{-\mathrm{i}2\pi nk/N}$$

holds pointwise for any $k \in \mathbb{Z}$ and the infinite sum on the LHS converges absolutely.

7. DTFT & DFT

Our next step should be very clear—we will relabel FIG. 6 by switching f and $\hat{f}$, $-\mathrm{i}$ and i, n and k, etc., though we will focus on a special case where $T = 2\pi$, because this corresponds directly to the DTFT indexed by normalized frequency ω.

FIG. 7: Discrete Poisson summation, relabeled ($N > 1$)



Notice that the scaling factor $\frac{1}{N}$ corresponds to $\frac{1}{T}$ in FIG. 2. Here we start with a discrete function $f[n]$. The relabeled diagram tells us that a (discrete) N-periodization of f

$$g[n] = \sum_{m=-\infty}^{\infty} f[n+mN]$$

followed a DFT (with different scaling factors)

$$\hat{g}[k] = \frac{1}{N}\sum_{n=0}^{N-1} g[n]\mathrm{e}^{-\mathrm{i}2\pi nk/N},$$

is equivalent to sampling the DTFT spectrum $\hat{f}_{2\pi}$ followed by a scaling of $1/N$

$$\hat{g}[k] = \frac{1}{N}\hat{f}_{2\pi}\left(\frac{2\pi k}{N}\right).$$

It shouldn't be a surprise that this relabeling would work. The justification is essentially the same as the proof in Section 3. By substituting $g[n]$ and $\hat{g}[k]$ in the inverse DFT step

$$g[n] = \sum_{k=0}^{N-1} \hat{g}[k]\mathrm{e}^{\mathrm{i}2\pi nk/N},$$

we obtain the second discrete form of the Poisson summation formula

$$\sum_{m=-\infty}^{\infty} f[n+mN] = \frac{1}{N}\sum_{k=0}^{N-1} \hat{f}_{2\pi}\left(\frac{2\pi k}{N}\right)\mathrm{e}^{\mathrm{i}2\pi nk/N}, \tag{DPSF2}$$

which is the analog of (PSF1).

To see why this is true, it suffices to show that the DFT $\hat{g}[k]$ of $g[n]$ is exactly

$$\hat{g}[k] = \frac{1}{N}\hat{f}_{2\pi}\left(\frac{2\pi k}{N}\right).$$

Proof. Let us take the DFT of $g[n]$,

$$\begin{aligned}\hat{g}[k] &= \frac{1}{N}\sum_{n=0}^{N-1} g[n]\mathrm{e}^{-\mathrm{i}2\pi nk/N} \\ &= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=-\infty}^{\infty} f[n+mN]\mathrm{e}^{-\mathrm{i}2\pi nk/N}\end{aligned}$$

Assume that $f \in \ell^1(\mathbb{Z})$ so that we can exchange the double summation,

$$\hat{g}[k] = \frac{1}{N}\sum_{m=-\infty}^{\infty}\sum_{n=0}^{N-1} f[n+mN]\mathrm{e}^{-\mathrm{i}2\pi nk/N}$$

With a change of variable $r = n + mN$, we obtain

$$\begin{aligned}\hat{g}[k] &= \frac{1}{N}\sum_{m=-\infty}^{\infty}\sum_{r=mN}^{(m+1)N-1} f[r]\mathrm{e}^{-\mathrm{i}2\pi(r-mN)k/N} \\ &= \frac{1}{N}\sum_{m=-\infty}^{\infty}\sum_{r=mN}^{(m+1)N-1} f[r]\mathrm{e}^{-\mathrm{i}2\pi rk/N}\underbrace{\mathrm{e}^{\mathrm{i}2\pi mk}}_{=1} \\ &= \frac{1}{N}\sum_{m=-\infty}^{\infty}\sum_{r=mN}^{(m+1)N-1} f[r]\mathrm{e}^{-\mathrm{i}2\pi rk/N}.\end{aligned}$$

Similar to (3.3) and the previous section, we combine the double summation into one, then relabel r with n

$$\begin{aligned}\hat{g}[k] &= \frac{1}{N}\sum_{r=-\infty}^{\infty} f[r]\mathrm{e}^{-\mathrm{i}2\pi rk/N} \\ &= \frac{1}{N}\sum_{n=-\infty}^{\infty} f[n]\mathrm{e}^{-\mathrm{i}2\pi nk/N}.\end{aligned}$$

Note that

$$\frac{1}{N}\hat{f}_{2\pi}\left(\frac{2\pi k}{N}\right) = \frac{1}{N}\sum_{n=-\infty}^{\infty} f[n]\mathrm{e}^{-\mathrm{i}2\pi nk/N} = \hat{g}[k],$$

which is exactly the derised result. □

Interestingly, unlike (DPSF1), we only used the assumption that $f \in \ell^1(\mathbb{Z})$ to prove the pointwise result of (DPSF2). This is because we don't need the inversion from $\hat{f}_{2\pi}(\omega)$ to $f[n]$, similar to what we did in (6.1), in the proof.

To summarize, we have the following theorem.

THEOREM (Poisson summation formula for discrete f)

Let $N > 1$. Suppose $f \in \ell^1(\mathbb{Z})$. Then, the second discrete version of the Poisson summation formula

$$\sum_{m=-\infty}^{\infty} f[n+mN] = \frac{1}{N}\sum_{k=0}^{N-1} \hat{f}_{2\pi}\left(\frac{2\pi k}{N}\right)\mathrm{e}^{\mathrm{i}2\pi nk/N}$$

holds pointwise for any $n \in \mathbb{Z}$ and the infinite sum on the LHS converges absolutely.

8. UNIFYING THE DFT

To prepare for the last section, we need to fix some minor details in the previous two sections. As you might have noticed, the definitions of DFT in FIG. 6 and FIG. 7 are not the same. The one in FIG. 6 has the normalizing factor $\frac{1}{N}$ in inverse DFT, but in FIG. 7, the factor is in the forward DFT.

We somehow got different transforms by following different paths. This is a really a minor annoyance, but it will become crucial later when we want to connect the dots.

Fortunately, this is easy to fix, notice that in (DPSF1)

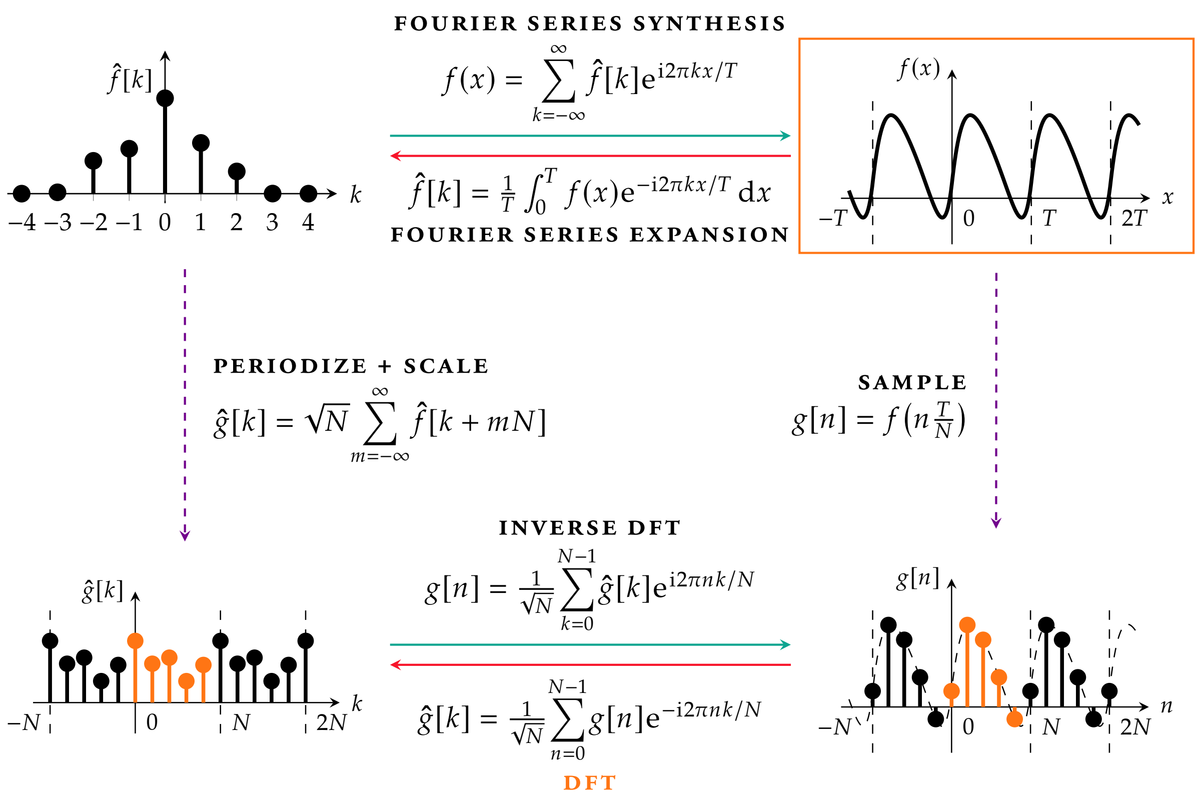
$$N \sum_{m=-\infty}^{\infty} \hat{f}[k+mN] = \sum_{n=0}^{N-1} f\left(n\frac{T}{N}\right) \mathrm{e}^{-\mathrm{i}2\pi nk/N},$$

the DFT is essentially the RHS of the equation, if we divide both side by $\sqrt{N}$, we can obtain a new version of the discrete Poisson summation using the unitary DFT defined in Section 6.3

$$\sqrt{N} \sum_{m=-\infty}^{\infty} \hat{f}[k+mN] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f\left(n\frac{T}{N}\right) \mathrm{e}^{-\mathrm{i}2\pi nk/N}. \tag{UDPSF1}$$

Therefore, we have

FIG. 8: Discrete Poisson summation, unitary DFT ($T > 0, N > 1$)



This version of the discrete Poisson summation formula breaks the symmetry with FIG. 4, but it allows us to unify the DFT used in the lower half of the diagram.

For (DPSF2)

$$\sum_{m=-\infty}^{\infty} f[n+mN] = \frac{1}{N}\sum_{k=0}^{N-1} \hat{f}_{2\pi}\left(\frac{2\pi k}{N}\right)\mathrm{e}^{\mathrm{i}2\pi nk/N},$$

we will do something different. We could multiply both sides by $\sqrt{N}$ and obtain

$$\sqrt{N}\sum_{m=-\infty}^{\infty} f[n+mN] = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} \hat{f}_{2\pi}\left(\frac{2\pi k}{N}\right)\mathrm{e}^{\mathrm{i}2\pi nk/N},$$

but this breaks the symmetry with FIG. 8. Instead, we simply break up

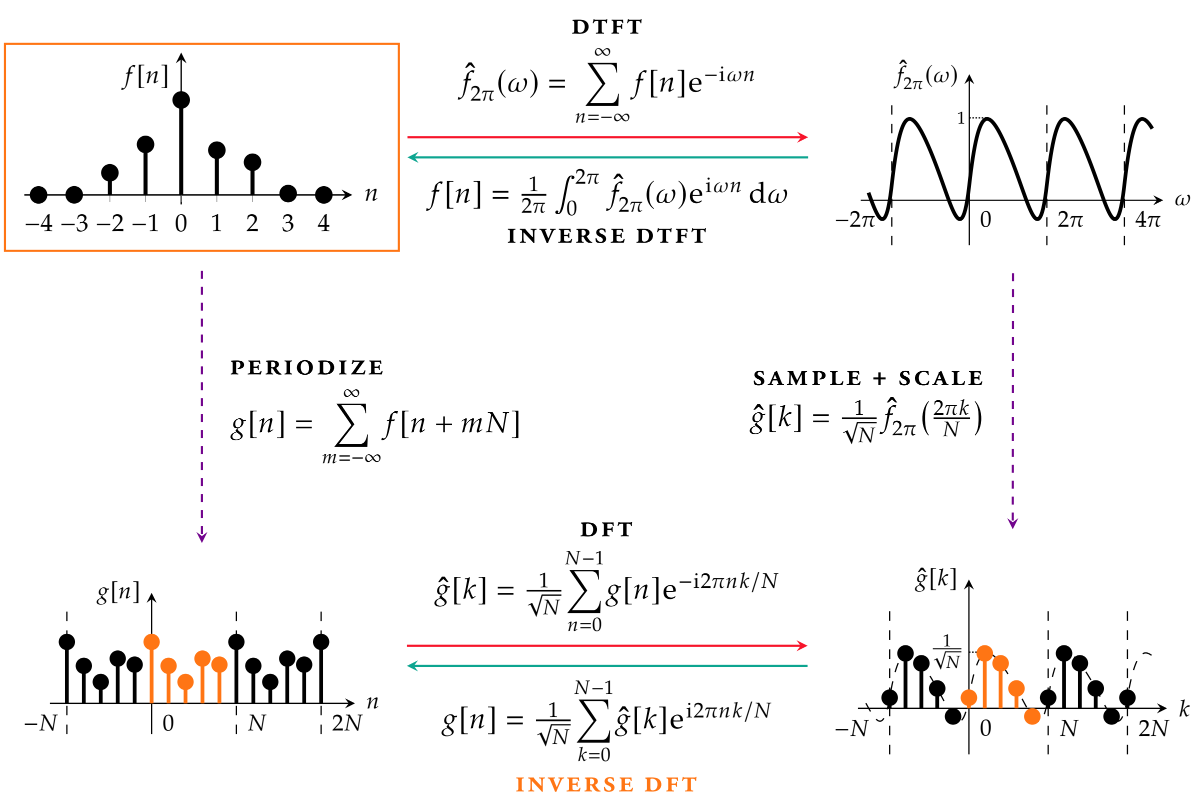
$$\frac{1}{N} = \frac{1}{\sqrt{N}}\cdot\frac{1}{\sqrt{N}}$$

and get

$$\sum_{m=-\infty}^{\infty} f[n+mN] = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} \frac{1}{\sqrt{N}}\hat{f}_{2\pi}\left(\frac{2\pi k}{N}\right)\mathrm{e}^{\mathrm{i}2\pi nk/N}, \qquad \text{(UDPSF2)}$$

which translates into

FIG. 9: Discrete Poisson summation, relabeled + unitary DFT ($N > 1$)



Φ. A SUMMARY: THE MISSING FACES

We are *almost* done, though there are still some missing pieces.

In the previous five sections, we have explored the relationships between

F1. Fourier transform and Fourier series (FIG. 2)

F2. Fourier transform and DTFT (FIG. 4)

F3. Fourier series and DFT (FIG. 8)

F4. DTFT and DFT (FIG. 9)

Each pair of transforms in the list is connected via periodization and sampling (with some extra scaling factors) in the time domain and frequency domain.

If we place the four Fourier transforms on the four corners of a SQUARE, the dashed lines in each of the figures naturally give us two ways to get from the Fourier transform to the DFT.

$$\begin{tikzcd} \mathbf{FT} \arrow[r, dashed, "\text{Fig. 2}"] \arrow[d, dashed, "\text{Fig. 4}"'] & \mathbf{FS} \arrow[d, dashed, "\text{Fig. 8}"] \\ \mathbf{DTFT} \arrow[r, dashed, "\text{Fig. 9}"'] & \mathbf{DFT} \end{tikzcd}$$

It is, however, quite unsatisfying to call it an end here. The SQUARE is a poor representation of what we have done so far. We have to compress an entire figure worth of information into a single arrow. The fascinating details of how we get from one transform to another are lost.

What we have explored is much more than a square—it is actually a CUBE [4, p.36], where each of the figures should be a surface, or one of the cube's FACES, rather than a line.

Notice something interesting: a CUBE should have SIX FACES, but we have only covered FOUR of them. There are still TWO of them missing. And the MISSING FACES turn out to be a very nice summary for the previous five sections.

In order to see the MISSING FACES, we need to draw out the entire cube first. Let us define

$$f_{\mathrm{FT}}(x) \qquad f_{\mathrm{FS}}(x) \qquad f_{\mathrm{DTFT}}[n] \qquad f_{\mathrm{DFT}}[n]$$

to be the time domain input of each Fourier transform variants, where f_{FS} is T-periodic and obtained from f_{FT} using (PSF1), f_{DTFT} is obtained from f_{FT} using (PSF2) by setting $T = \frac{1}{2\pi}$, and f_{DFT} is N-periodic, obtained from either f_{FS} using (UDPSF1) or f_{DTFT} using (UDPSF2).

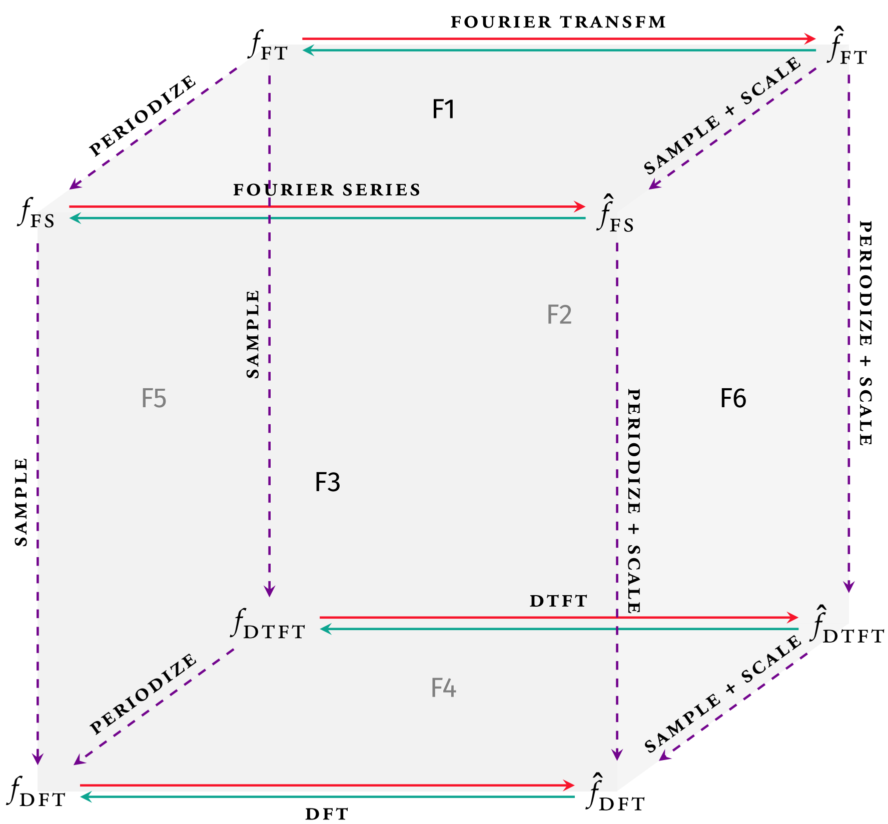
Furthermore, we define

$$\hat{f}_{\mathrm{FT}}(\xi) \qquad \hat{f}_{\mathrm{FS}}[k] \qquad \hat{f}_{\mathrm{DTFT}}(\omega) \qquad \hat{f}_{\mathrm{DFT}}[k]$$

to be their corresponding frequency domain representations.

Then by placing the eight functions on the eight vertices of the CUBE and connecting them with the arrows in our four figures, we have

FIG. 10: Poisson summation formula—the big picture



As we can see, the top, back, from, bottom faces of the cube correspond to F1, F2, F3, and F4, respectively.

Note that for DTFT (F2), we are using the version with normalized frequency ω, i.e. (DTFT3), so we need to set $T = \frac{1}{2\pi}$ in (PSF2), which gives us

$$2\pi \sum_{n=-\infty}^{\infty} \hat{f}_{\mathrm{FT}}(\omega + 2\pi n) = \sum_{n=-\infty}^{\infty} f_{\mathrm{FT}}\left(\frac{n}{2\pi}\right) \mathrm{e}^{-\mathrm{i}\omega n}. \tag{PSF2-N}$$

Similarly, we need to replace T with $\frac{1}{2\pi}$ in F2 as well.

After we fill in the four faces, we get the left (F5) and right (F6) faces of the cube for free. They are exactly what we have missed—the connections between

F5. Functions in the time domain

F6. Functions in the frequency domain

As a final summary, let us inspect the two MISSING FACES in detail. We will start with F5, the connections in the time domain.

FIG. 11: Fourier transforms in the time domain

$$\begin{tikzcd}[column sep=huge, row sep=huge]
f_{\mathrm{FT}} \arrow[r, dashed, "\substack{\textbf{PERIODIZE} \\ f_{\mathrm{FS}}(x) = \sum_{n=-\infty}^{\infty} f_{\mathrm{FT}}(x+nT)}"] \arrow[d, dashed, "\substack{\textbf{SAMPLE} \\ f_{\mathrm{DTFT}}[n] = f_{\mathrm{FT}}\left(\frac{n}{2\pi}\right)}"'] & f_{\mathrm{FS}} \arrow[d, dashed, "\substack{\textbf{SAMPLE} \\ f_{\mathrm{DFT}}[n] = f_{\mathrm{FS}}\left(n\frac{T}{N}\right)}"] \\
f_{\mathrm{DTFT}} \arrow[r, dashed, "\substack{f_{\mathrm{DFT}}[n] = \sum_{m=-\infty}^{\infty} f_{\mathrm{DTFT}}[n+mN] \\ \textbf{PERIODIZE}}"'] & f_{\mathrm{DFT}}
\end{tikzcd}$$

The second missing face is F6, the connections in the frequency domain.

FIG. 12: Fourier transforms in the frequency domain

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\hat{f}_{\mathrm{FT}} \arrow[r, dashed, "{\substack{\textsc{sample + scale} \\ \hat{f}_{\mathrm{FS}}[k] = \frac{1}{T}\hat{f}_{\mathrm{FT}}\left(\frac{k}{T}\right)}}"] \arrow[d, dashed, "{\substack{\textsc{periodize + scale} \\ \hat{f}_{\mathrm{DTFT}}(\omega) = 2\pi \sum_{m=-\infty}^{\infty} \hat{f}_{\mathrm{FT}}(\omega + 2\pi m)}}"'] & \hat{f}_{\mathrm{FS}} \arrow[d, dashed, "{\substack{\textsc{periodize + scale} \\ \hat{f}_{\mathrm{DFT}}[k] = \sqrt{N} \sum_{m=-\infty}^{\infty} \hat{f}_{\mathrm{FS}}[k + mN]}}"] \\
\hat{f}_{\mathrm{DTFT}} \arrow[r, dashed, "{\substack{\hat{f}_{\mathrm{DFT}}[k] = \frac{1}{\sqrt{N}}\hat{f}_{\mathrm{DTFT}}\left(\frac{2\pi k}{N}\right) \\ \textsc{sample + scale}}}"'] & \hat{f}_{\mathrm{DFT}}
\end{tikzcd}$$

And, sadly, this concludes our explorations of the Poisson summation formula.

From the surface, we might only see the connections between the Fourier transform and Fourier series using the formula itself. However, as we have discovered, the RELABELING trick is able to induce the connections between all four Fourier transforms—Fourier transform, Fourier series, DTFT, and DFT—almost for free.

also known as Fourier transform on $\mathbb{R}$, $\mathbb{R}/T\mathbb{Z}$, $\mathbb{Z}$, and $\mathbb{Z}/N\mathbb{Z}$, respectively.

Although this approach might be somewhat restrictive compared to the usual approach using the theory of distributions, I find it much more satisfying to derive from intuitions instead of introducing the delta distribution δ directly. Plus, it even gives us a way to *derive* the formula for DTFT and DFT easily. I hope you appreciate the elegance of this approach and find it enlightening.

❊

A. PERIODIC FOURIER TRANSFORM

At the very beginning, we mentioned that the special case of the Poisson summation formula is equivalent to

$$\text{Ш}^\wedge = \text{Ш}.$$

In fact, if we define the T-periodic Dirac comb as the weakly convergent series

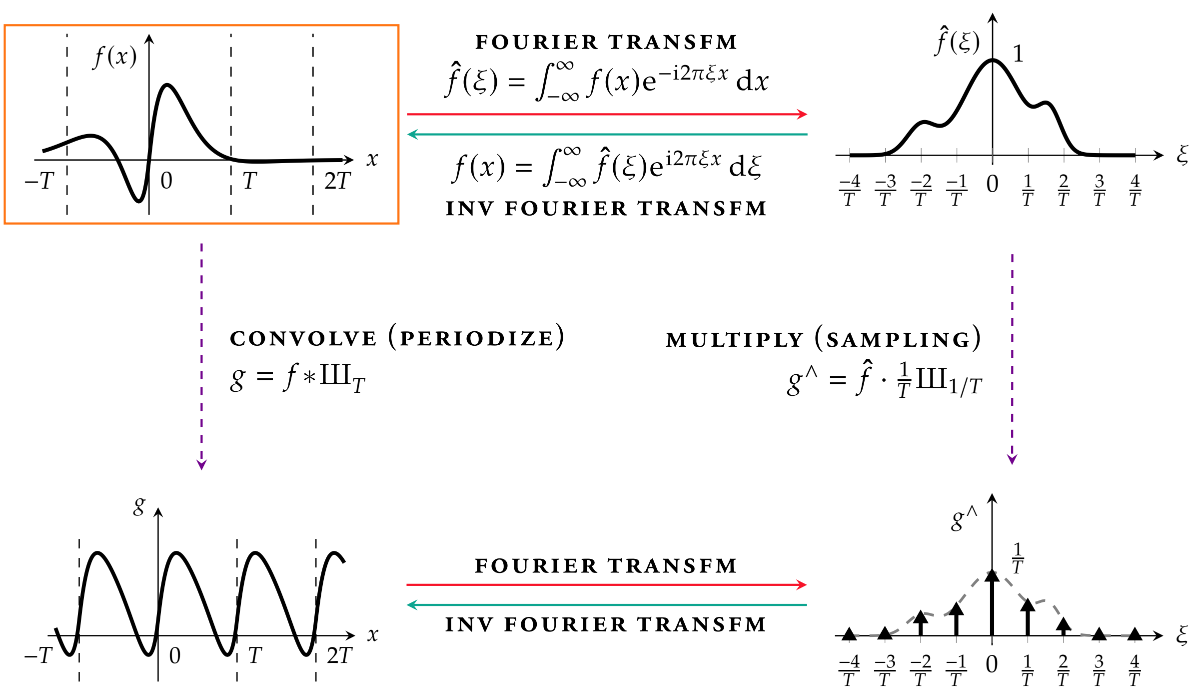
$$\text{Ш}_T(x) = \sum_{n=-\infty}^{\infty} \delta(x - nT),$$

then the general case of the Poisson summation formula translates to

$$\text{Ш}_T^\wedge = \frac{1}{T}\text{Ш}_{1/T},$$

which means the following diagram is equivalent to FIG. 2.

FIG. 13: Poisson summation as Dirac comb ($T > 0$)

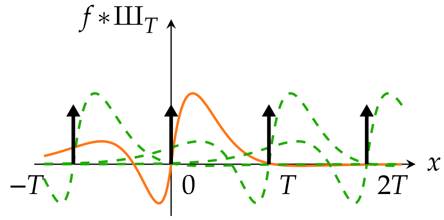


Note that here we denote the function f and its corresponding tempered distribution (a functional)

$$f\{\phi\} = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx$$

by the same symbol.

It shouldn't be hard to see intuitively why convolving f with a T-periodic Dirac comb



is the same as periodizing f with period T, though the diagram (FIG. 13) tells us something more—if the Fourier coefficients of g (as a periodic function) are

$$\hat{g}[n] = \frac{1}{T} \int_0^T g(x) e^{-i2\pi nx/T} \, dx,$$

then if we treat g as a tempered distribution, the distributional Fourier transform of g should be

Note that the ordinary Fourier transform of a periodic function usually diverges since it does not have any decay.

$$g^\wedge = \sum_{n=-\infty}^{\infty} \hat{g}[n] \delta\left(x - \frac{n}{T}\right).$$

This is why multiplying a $\frac{1}{T}$-periodic Dirac comb can be considered a SAMPLING of the spectrum, even though we can't use $g^\wedge$ directly as Fourier coefficients to recover $g(x)$ as a function.

It is actually not hard to see why the distributional Fourier transform $g^\wedge$ is consistent with the Fourier coefficients $\hat{g}[n]$ in this case. Consider a function f that satisfies the condition of (PSF1), then the T-periodic function

For more general results, see Section 7.7 of [4] and Chapter 11 of [6].

$$g(x) = \sum_{n=-\infty}^{\infty} f(x + nT)$$

is continuous and bounded, thus growing slower than any non-constant polynomials, so $g \in \mathcal{S}'(\mathbb{R})$ can be treated as a tempered distribution. For a test function $\phi \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned} g^\wedge\{\phi\} &= g\{\hat{\phi}\} \\ &= \int_{-\infty}^{\infty} g(x)\hat{\phi}(x) \, dx \\ &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \hat{g}[n] e^{i2\pi nx/T} \hat{\phi}(x) \, dx \end{aligned}$$

Since $\mathcal{S}(\mathbb{R})$ is invariant to Fourier transform, i.e. $\hat{\phi} \in \mathcal{S}(\mathbb{R})$, we can switch the summation and integral since

$$\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \hat{g}[n] \mathrm{e}^{\mathrm{i}2\pi n x/T} \hat{\phi}(x) \right| \mathrm{d}x = \sum_{n=-\infty}^{\infty} |\hat{g}[n]| \int_{-\infty}^{\infty} |\hat{\phi}(x)| \, \mathrm{d}x < \infty$$

by the decay of $\mathcal{S}(\mathbb{R})$ and (3.4). Therefore,

$$\begin{aligned} g^{\wedge}\{\phi\} &= \sum_{n=-\infty}^{\infty} \hat{g}[n] \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}2\pi n x/T} \hat{\phi}(x) \, \mathrm{d}x \\ &= \sum_{n=-\infty}^{\infty} \hat{g}[n] \phi\left(\frac{n}{T}\right). \end{aligned}$$

which is exactly the same as the distribution $h(x) = \sum_{n=-\infty}^{\infty} \hat{g}[n] \delta\left(x - \frac{n}{T}\right)$ since for any $\phi \in \mathcal{S}(\mathbb{R})$

$$\begin{aligned} h\{\phi\} &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \hat{g}[n] \delta\left(x - \frac{n}{T}\right) \phi(x) \, \mathrm{d}x \\ &= \sum_{n=-\infty}^{\infty} \hat{g}[n] \int_{-\infty}^{\infty} \delta\left(x - \frac{n}{T}\right) \phi(x) \, \mathrm{d}x \\ &= \sum_{n=-\infty}^{\infty} \hat{g}[n] \phi\left(\frac{n}{T}\right) = g^{\wedge}\{\phi\} \end{aligned}$$

by a similar argument as in (1.1).

B. LAPLACE TRANSFORM AND Z-TRANSFORM

Now that we have discovered the connections between the four Fourier transforms, it is natural to question whether the (bilateral) Laplace transform of a function $f : \mathbb{R} \to \mathbb{C}$

We will take a look at the unilateral version later.

$$F_{\mathrm{L}}(s) = \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-sx} \, \mathrm{d}x, \tag{LAP-B}$$

and (bilateral) z-transform of the T-sampled function $g[n] = f(nT)$

$$G_{\mathrm{Z}}(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n}, \tag{Z-B}$$

where $s, z \in \mathbb{C}$, $n \in \mathbb{Z}$, $T > 0$, have any kind of connections.

The answer, again, is YES, and it is exactly hidden in FIG. 4, though we can't perform our favorite relabeling trick on the diagram anymore, since the types of some variables do not match up and, more importantly, the inverse of Laplace transform and z-transform becomes a bit tricker than the inverse of Fourier transforms.

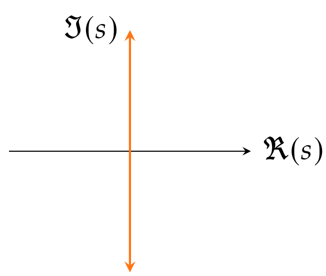
However, we can take an alternate route—by considering the (bilateral) Laplace transform as an EXTENSION of the Fourier transform and considering the (bilateral) z-transform as an EXTENSION of the DTFT.

B.1 Embedded Fourier transform

If we take $s = \mathrm{i}2\pi\xi$, where $\xi \in \mathbb{R}$, in (LAP-B), we can obtain

$$F_{\mathrm{L}}(\mathrm{i}2\pi\xi) = \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-\mathrm{i}2\pi\xi x} \, \mathrm{d}x,$$

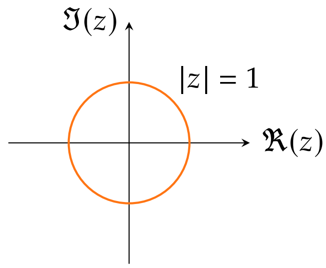
which is exactly the Fourier transform $\hat{f}_{\mathrm{FT}}(\xi)$ of f. This corresponds to the (dilated) imaginary axis in the s-domain



Similarly, the DTFT of g is also embedded in the z-domain. If we take $z = \mathrm{e}^{\mathrm{i}\omega}$, where $\omega \in \mathbb{R}$, in (Z-B)

$$G_{\mathrm{Z}}(\mathrm{e}^{\mathrm{i}\omega}) = \sum_{n=-\infty}^{\infty} g[n]\mathrm{e}^{-\mathrm{i}\omega n},$$

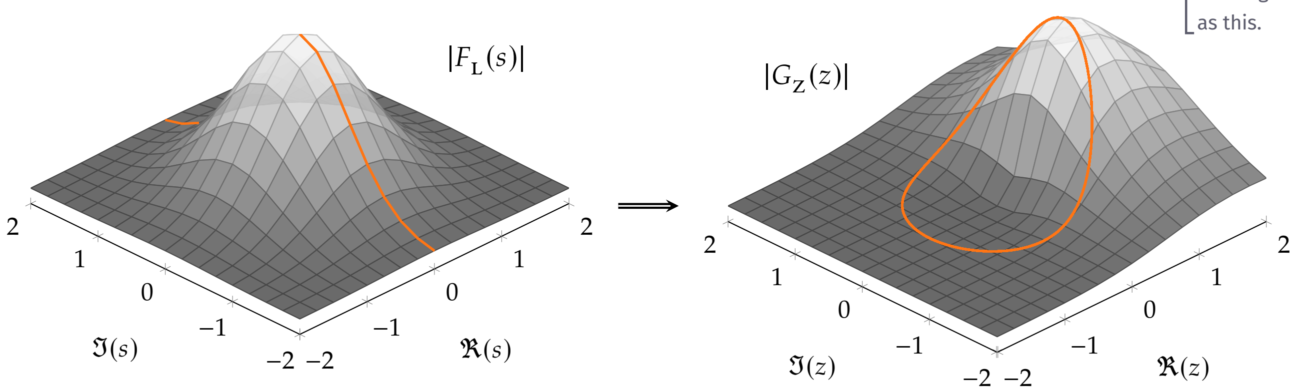
we obtain exactly (DTFT3). This corresponds to the unit circle centered at the origin in the z-domain.



If we know that $f(x)$, $F_{\mathrm{L}}(\mathrm{i}2\pi\xi)$ satisfies the condition for (PSF2), then we can conclude that the unit circle in the z-domain is exactly a periodized and scaled version of the (dilated) imaginary axis in the s-domain.

I'm using a Gaussian here because it's easier to visualize. The Laplace transform usually won't converge as nicely as this.

FIG. 14: s-domain vs. z-domain



To make the relationship more precise, let us take a look at (PSF2) again

$$\frac{1}{T}\sum_{k=-\infty}^{\infty} \hat{f}_{\mathrm{FT}}\left(\xi + \frac{k}{T}\right) = \sum_{n=-\infty}^{\infty} f(nT)\mathrm{e}^{-\mathrm{i}2\pi nT\xi}.$$

Note that the RHS of the equation is actually (DTFT2) instead of (DTFT3), which is equivalent to

$$G_{\mathrm{Z}}(\mathrm{e}^{\mathrm{i}2\pi T\xi}) = \sum_{n=-\infty}^{\infty} g[n]\mathrm{e}^{-\mathrm{i}2\pi nT\xi}.$$

Since we also know that

$$\hat{f}_{\mathrm{FT}}(\xi) = F_{\mathrm{L}}(\mathrm{i}2\pi\xi),$$

by substituting them into (PSF2), we obtain

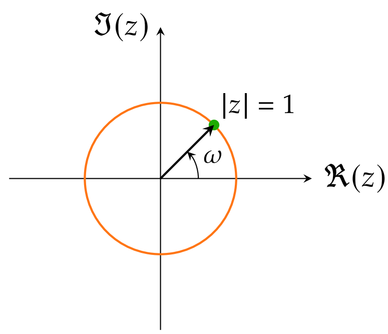
$$G_{\mathrm{Z}}(\mathrm{e}^{\mathrm{i}2\pi T\xi}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F_{\mathrm{L}}\left(\mathrm{i}2\pi\left(\xi + \frac{k}{T}\right)\right), \tag{EPSF1}$$

which is, in some sense, the basic form of an EXTENDED Poisson summation formula for the Laplace transform and z-transform.

This formula still has an issue. Because neither G_{Z} nor F_{L} takes a point on the complex plane as input, but rather parametrized by $\xi \in \mathbb{R}$, it is not clear which point on the circle corresponds to which point(s) on the line. This is easy to fix, though.

Since we are looking at points on a circle in the z-domain, we can index $z = \mathrm{e}^{\mathrm{i}\omega}$ by the angle $\omega = \arg(z) \in \mathbb{R}$ the point z makes with the real axis.

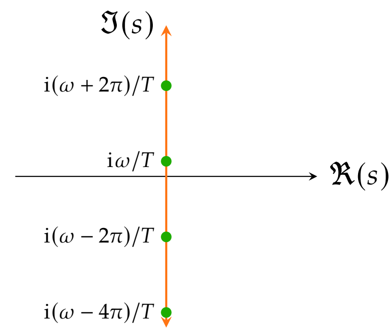
If you are worried about branch cuts, simply use the principal branch.



By substituting $\xi = \frac{\omega}{2\pi T}$ into (EPSF1), we obtain

$$\begin{aligned} G_{\mathrm{Z}}(\mathrm{e}^{\mathrm{i}\omega}) &= G_{\mathrm{Z}}\left(\mathrm{e}^{\mathrm{i}2\pi T(\omega/2\pi T)}\right) \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} F_{\mathrm{L}}\left(\mathrm{i}2\pi\left(\frac{\omega}{2\pi T} + \frac{k}{T}\right)\right) \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} F_{\mathrm{L}}\left(\mathrm{i}\left(\frac{\omega}{T} + \frac{2\pi k}{T}\right)\right). \end{aligned} \tag{EPSF2}$$

Therefore, to obtain the value of the z-transform of the sampled function g at $z = \mathrm{e}^{\mathrm{i}\omega}$, the imaginary axis of the Laplace transform of f is first periodized with period $\frac{2\pi}{T}$ then evaluated at $\frac{\omega}{T}$, and finally the result is scaled by $\frac{1}{T}$.



Alternatively, if we define

$$\tilde{F}_{\mathrm{L}}(s) = \frac{1}{T} F_{\mathrm{L}}\left(\frac{s}{T}\right) \tag{B.1}$$

for any $s \in \mathbb{C}$ to be the scaled and dilated s-domain, then we can also say that the z-transform evaluated at $z = \mathrm{e}^{\mathrm{i}\omega}$ is a 2π periodization of the imaginary axis of $\tilde{F}_{\mathrm{L}}(s)$, i.e.

Note that dilation and scaling in s-domain is equivalent to dilating the function by T in the time domain.

$$G_{\mathrm{Z}}(\mathrm{e}^{\mathrm{i}\omega}) = \sum_{k=-\infty}^{\infty} \tilde{F}_{\mathrm{L}}\big(\mathrm{i}(\omega + 2\pi k)\big).$$

B.2 The extension to $\mathbb{C}$

Even if $s = \sigma + \mathrm{i}2\pi\xi$, where $\sigma, \xi \in \mathbb{R}$, is not a pure imaginary number, we can still substitute it into (LAP-B) and obtain

$$F_{\mathrm{L}}(\sigma + \mathrm{i}2\pi\xi) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\sigma x}\mathrm{e}^{-\mathrm{i}2\pi\xi x}\,\mathrm{d}x.$$

For a fixed σ, this is exactly the Fourier transform of the function

$$f'(x) = f(x)\mathrm{e}^{-\sigma x}.$$

The $'$ here has nothing to do with derivatives.

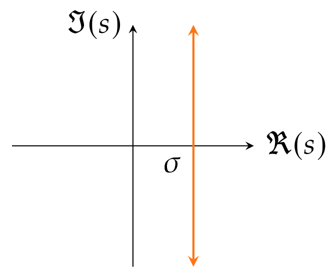
Since we are using the bilateral Laplace transform, if $\sigma \neq 0$, the convergence of the integral of Laplace transform might become an issue. We need to assume that the Fourier transform of f' converges absolutely, or equivalently $f' \in L^1(\mathbb{R})$, i.e. $\sigma + \mathrm{i}2\pi\xi$ is within the REGION OF CONVERGENCE (ROC) of F_{L}. This is also necessary for (PSF2) to hold true.

If we evaluate the Laplace transform F'_{L} of f' at $\mathrm{i}2\pi\xi$, we can see that

$$F'_{\mathrm{L}}(\mathrm{i}2\pi\xi) = \int_{-\infty}^{\infty} \big(f(x)\mathrm{e}^{-\sigma x}\big)\mathrm{e}^{-\mathrm{i}2\pi\xi x}\,\mathrm{d}x = F_{\mathrm{L}}(\sigma + \mathrm{i}2\pi\xi).$$

Therefore, the Fourier transform of f' corresponds to a vertical line in the s-

domain of f.



Is it possible that this line also corresponds to some circle in the z-domain of g somehow? To find out the answer, let us construct a new discrete function $g'[n]$ by sampling $f'(x)$ with sampling period T

$$g'[n] = f'(nT) = f(nT)\mathrm{e}^{-\sigma nT}.$$

Evaluating its z-transform at $z = \mathrm{e}^{i\omega}$, where $\omega \in \mathbb{R}$, gives us

$$\begin{aligned} G'_{\mathrm{Z}}(\mathrm{e}^{\mathrm{i}\omega}) &= \sum_{n=-\infty}^{\infty} g'[n]\mathrm{e}^{-\mathrm{i}\omega n} \\ &= \sum_{n=-\infty}^{\infty} f(nT)\mathrm{e}^{-\sigma nT}\mathrm{e}^{-\mathrm{i}\omega n} \\ &= G_{\mathrm{Z}}(\mathrm{e}^{\sigma T}\mathrm{e}^{\mathrm{i}\omega}) \end{aligned}$$

If we additionally know that f' satisfy the condition for (PSF2), we can use the result from (EPSF2) and conclude that

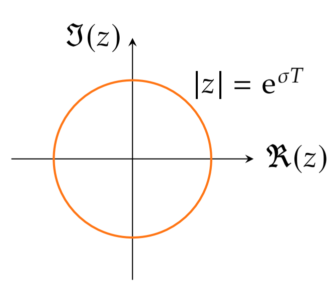
which makes the result less useful since Laplace transform is about **poles** and zeros. However, it is possible to extend the result using approximation procedures, see [3].

$$G'_{\mathrm{Z}}(\mathrm{e}^{\mathrm{i}\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F'_{\mathrm{L}}\left(\mathrm{i}\left(\frac{\omega}{T} + \frac{2\pi k}{T}\right)\right),$$

which is the equivalent to

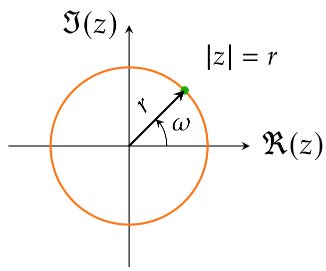
$$G_{\mathrm{Z}}(\mathrm{e}^{\sigma T}\mathrm{e}^{\mathrm{i}\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F_{\mathrm{L}}\left(\sigma + \mathrm{i}\left(\frac{\omega}{T} + \frac{2\pi k}{T}\right)\right). \tag{EPSF3}$$

As we can see, by periodizing and scaling the vertical line $\Re(s) = \sigma$ in the s-domain of f, we obtain the circle $|z| = \mathrm{e}^{\sigma T}$ in the z-domain of g.



Note that a similar argument to (3.4) implies that the DTFT of $g'[n]$ is absolutely convergent for any ω, i.e. the circle is contained in the ROC of $g[n]$.

To find out what a point $z = r\mathrm{e}^{\mathrm{i}\omega}$, where $r > 0, \omega \in \mathbb{R}$, on the z-domain of g corresponds to



we set

$$r = \mathrm{e}^{\sigma T}.$$

Since $r > 0$, we can take the natural log on both sides and give us

$$\log(r) = \sigma T \implies \sigma = \frac{1}{T}\log(r).$$

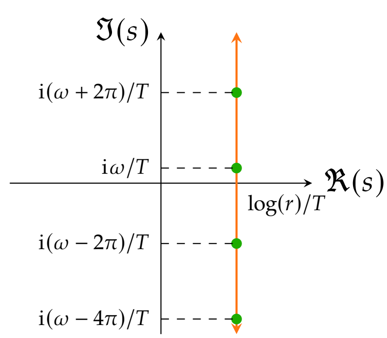
Substitute back to (EPSF3), we obtain

$$G_{\mathrm{Z}}(r\mathrm{e}^{\mathrm{i}\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F_{\mathrm{L}}\left(\frac{1}{T}\log(r) + \mathrm{i}\left(\frac{\omega}{T} + \frac{2\pi k}{T}\right)\right), \tag{EPSF4}$$

given that

$$f'(x) = f(x)r^{-x/T}$$

satisfies the condition for (PSF2). This is very similar to (EPSF2) except the periodization is applied on the vertical line $\Re(s) = \frac{1}{T}\log(r)$ in the s-domain F_{L}.



If we use the dilated and scaled s-domain $\tilde{F}_{\mathrm{L}}$ defined in (B.1), the result can also be written as

$$G_Z(re^{i\omega}) = \sum_{k=-\infty}^{\infty} \tilde{F}_L\big(\log(r) + i(\omega + 2\pi k)\big).$$

B.3 The unilateral transforms

Because of convergence issues, the unilateral Laplace transform

$$F_{UL}(s) = \int_0^{\infty} f(x)e^{-sx}\,dx, \tag{LAP-U}$$

and the unilateral z-transform

$$G_{UZ}(z) = \sum_{n=0}^{\infty} g[n]z^{-n}, \tag{Z-U}$$

is often used. However, notice that if we multiply $f(x)$ by the unit step function

$$u(x) = \begin{cases} 1 & , x \geq 0 \\ 0 & , x < 0 \end{cases},$$

then the bilateral Laplace transform of $f'(x) = f(x)u(x)$ will be the same as the unilateral transform of f

$$F_{UL}(s) = \int_{-\infty}^{\infty} f(x)u(x)e^{-sx}\,dx = F'_L(s).$$

If we sample f' using sampling period $T > 0$,

$$g'[n] = f'(nT) = f(nT)u(nT) = \begin{cases} g[n] & , n \geq 0 \\ 0 & , n < 0 \end{cases},$$

so we also have

$$G_{UZ}(z) = G'_Z(z).$$

This means that equation (EPSF4)

$$G_{UZ}(re^{i\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F_{UL}\left(\frac{1}{T}\log(r) + i\left(\frac{\omega}{T} + \frac{2\pi k}{T}\right)\right)$$

holds for the unilateral Laplace transform and z-transform as well, though the condition becomes that

Again, the $''$ here does not refer to derivatives.

$$f''(x) = f(x)u(x)r^{-x/T}$$

should satisfy the condition for (PSF2).

The condition for unilateral transforms is weaker in terms of the decay of f, but since we also need the continuity of f'' for pointwise equality, this condition also restricts that $f(x) = 0$ for some ε-neighborhood around $x = 0$ or f'' would have a discontinuity at $x = 0$.

C. FURTHER READING

[1] Ole Christensen, *An introduction to frames and Riesz bases*, Second edtion, ser. Applied and numerical harmonic analysis. Birkhäuser, 2016, ISBN: 978-3-319-25611-5.

[2] Karlheinz Gröchenig, "An uncertainty principle related to the Poisson summation formula," *Studia Mathematica*, vol. 121, no. 1, pp. 87–104, 1996. DOI: 10.4064/sm-121-1-87-104.

[3] Karlheinz Gröchenig, *Foundations of time-frequency analysis*. Birkhäuser Boston, 2001, ISBN: 0-8176-4022-3.

[4] David W. Kammler, *A First Course in Fourier Analysis*. Cambridge University Press, 2008, ISBN: 978-0-521-70979-8.

[5] Walter Rudin, *Functional analysis*, 2nd ed. McGraw-Hill, 1991, ISBN: 978-0-07-054236-5.

[6] A. H. Zemanian, *Distribution theory and transform analysis*. Dover Publications, 1987, ISBN: 978-0-486-65479-9.

D. REVISION HISTORY

Be cautious that this write-up is by no means perfect or free of errors. The following is a list of significant revisions made to this document. If you find any errors or have any suggestions, please write to me. Comments are welcome as well.

❦

DEC 31, 2019: The initial draft is (finally) complete, right before the year ends.

JAN 08, 2020: A link to my email is added at the very beginning.

JAN 24, 2020: Rewrite the ending paragraph.

JAN 26, 2020: A new section (Appendix B) about Laplace transform and z-transform is added.

FEB 18, 2020: Some more remarks about the Wiener algebra $W(\mathbb{R})$.

MAR 07, 2020: Some tweaks to matrix spacing.

APR 30, 2020: Further improvements to the ending, and added some more margin notes.