CDS 131 Homework 4: Lyapunov & I/O Stability

Winter 2025

Due 2/3 at 11:59 PM

Instructions

This homework is divided into three parts:

1. *Optional Exercises: the exercises are entirely optional but are recommended to be completed before looking at the problems. They consist of easier, more computational questions to help you get a feel for the material.*
2. *Required Problems: the problems are the required component of the homework, and might require more work than the exercises to complete.*
3. *Optional Problems: the optional problems are some additional, recommended problems - some of these might go a little beyond the standard course material.*

All you need to turn in is the solutions to the required problems - the others are recommended but not required.

1 Optional Exercises

1.1 Some System Norm Computations

1. Consider a SISO, LTI system with input $u(\cdot)$, output $y(\cdot)$, and transfer function,

$$\hat{G}(s) = \frac{s+2}{4s+1}, \tag{1}$$

   Compute the norm $\sup_{\|u\|_\infty=1} \|y\|_\infty$ and find an input signal $u(\cdot)$ achieving this norm.

2. Compute the 1-norm of the impulse response map corresponding to the transfer function,

$$\hat{G}(s) = \frac{1}{\tau s+1},\ \tau > 0. \tag{2}$$

1.2 Delay-Invariant Norms

Recall that the transfer function for a time delay of time τ is $\hat{D}(s) = e^{-s\tau}$. A norm $\|\cdot\|$ on the space of transfer functions is delay-invariant if, for every transfer function $\hat{G}$ with $||\hat{G}|| < \infty$ and every $\tau > 0$, $||\hat{D}\hat{G}|| = ||\hat{G}||$. Is the 2-norm delay invariant? What about the ∞-norm? You may provide answers in the scalar (SISO) case.

1.3 Evaluating Potential Norms

Consider the set $C^1(\mathbb{R},\mathbb{R})$ of continuously differentiable, scalar signals. Which of the following qualifies as a norm on $C^1(\mathbb{R},\mathbb{R})$?

$$\sup_{t\in\mathbb{R}} |\dot{u}(t)|, \quad |u(0)| + \sup_{t\in\mathbb{R}} |\dot{u}(t)|, \quad \max\{\sup_{t\in\mathbb{R}} |u(t)|,\ \sup_{t\in\mathbb{R}} |\dot{u}(t)|\}, \quad \sup_{t\in\mathbb{R}} |u(t)| + \sup_{t\in\mathbb{R}} |\dot{u}(t)|. \tag{3}$$

2 Required Problems

2.1 Lyapunov Certificates of Robust Stability

When analyzing the stability of linear, time-invariant systems, we assumed that we had *perfect* knowledge of the matrix A determining the system's dynamics. In this problem, we'll introduce *uncertainty* into the system, and show that we can use the Lyapunov equation to get a certificate of *robust stability*. Consider an uncertainty set $\vec{\Delta}$, defined as the convex hull of a finite collection $\{\Delta_1, ..., \Delta_m\}$, where $\Delta_i \in \mathbb{R}^{n\times n}$,

$$\vec{\Delta} = \text{conv}\{\Delta_1, ..., \Delta_m\} = \left\{ \sum_{i=1}^{m} \lambda_i \Delta_i : \sum_{i=1}^{m} \lambda_i = 1, 0 \le \lambda_i \le 1,\ \lambda_i \in \mathbb{R} \right\} \subseteq \mathbb{R}^{n\times n}. \tag{4}$$

Consider an unforced, uncertain system $\dot{x}(t) = (A+\Delta)x(t)$, where $\Delta \in \vec{\Delta}$ is some unknown parameter. If $x_e = 0$ is exponentially stable for $\dot{x}(t) = (A+\Delta)x(t)$ for all $\Delta \in \vec{\Delta}$, we say that $x_e = 0$ is *robustly exponentially stable*. In this problem, we'll find a certificate for robust exponential stability using *linear matrix inequalities* (LMIs). Let $\mathbb{S}^n$ represent the set of $n \times n$ real, symmetric matrices and V represent a finite-dimensional vector space. A linear matrix inequality in $X \in V$ is a matrix inequality of the form $F(X) \preceq Q$, where $F : V \to \mathbb{S}^n$ is a linear map, $Q \in \mathbb{S}^n$, and $F(X) \preceq Q$ denotes the positive semidefinite constraint $0 \preceq Q - F(X)$. The linear matrix inequality $F(X) \preceq Q$ has a solution if there exists an $X \in V$ for which $F(X) \preceq Q$.

1. Let's begin by synthesizing an LMI to identify a Lyapunov function for the unforced system $\dot{x}(t) = Ax(t)$ without uncertainty. Show that A is Hurwitz if and only if the pair of linear matrix inequalities,
$$I \preceq P,\ A^\top P + PA \preceq -I, \tag{5}$$
in a symmetric matrix $P \in \mathbb{S}^n$, has a solution.

2. Now, let's introduce the uncertainty set $\vec{\Delta} = \text{conv}\{\Delta_1, ..., \Delta_m\}$. Prove there exists a finite collection of linear matrix inequalities for which the existence of a solution to the inequalities implies the origin of $\dot{x} = (A+\Delta)x,\ \Delta \in \vec{\Delta}$ is robustly exponentially stable.

3. Now, let's introduce control into the system. Consider the uncertain state equation,
$$\dot{x} = (A+\Delta)x + Bu, \tag{6}$$
where $u \in \mathbb{R}^m$ is a control input and $\Delta \in \vec{\Delta} = \text{conv}\{\Delta_1, ..., \Delta_m\}$. Devise a linear matrix inequality method for computing a matrix $K \in \mathbb{R}^{m\times n}$ for which the control law $u = Kx$ renders the equilibrium $x_e = 0$ of $\dot{x} = (A+\Delta)x + Bu$ robustly exponentially stable. Produce an example of $(A, B, \vec{\Delta})$ for which your method will be successful, and an example for which it will be unsuccessful. *Hints: Your LMIs do not have to directly give you K. How are the eigenvalues of A and $A^\top$ related?*

2.2 Some Special Lyapunov Equations

In this problem, we'll analyze some special cases of the continuous-time Lyapunov equation. Consider the continuous-time, SISO LTI system, $\dot{x} = Ax + Bu$, $y = Cx$, for which $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, and $y \in \mathbb{R}$.

1. Suppose there exists a positive definite matrix $P \in \mathbb{S}^n$, $P \succ 0$, and a constant $\alpha \in \mathbb{R}$, for which
$$A^\top P + PA \prec \alpha P. \tag{7}$$
Which region in the complex plane do the eigenvalues of A lie in? Write your answer in terms of α.

2. Let's design a basic feedback controller using the result of part (1). Suppose Equation (7) holds with $\alpha = 0$, and in addition, that $PB = C^\top$. Show that, for a feedback control law $u = -ky, k \ge 0$, the equilibrium $x_e = 0$ of the closed-loop system $\dot{x} = Ax - Bky$ is globally exponentially stable.

3. Now, suppose $A \in \mathbb{R}^{n\times n}$ and $P \in \mathbb{S}^n,\ P \succ 0$ satisfy the equality $A^\top P + PA = 0$. Which region of the complex plane do the eigenvalues of A lie in?

4. Assuming again that $PB = C^\top$, does the equality $A^\top P + PA = 0$ still guarantee exponential stability for a control law $u = -ky$, $k \ge 0$? If not, what additional conditions would you need?

2.3 LMI Ellipsoid Identification

Consider an unforced, LTI system $\dot{x}(t) = Ax(t),\ x \in \mathbb{R}^n$. A forward invariant set for this system is a set $\mathcal{S} \subseteq \mathbb{R}^n$ for which $x_0 \in \mathcal{S}$ implies $\varphi(t, t_0, x_0) \in \mathcal{S}$ for all $t \geq t_0$, where φ is the (unforced) state transition map. In this problem, we'll develop a technique for certifying the invariance of certain sets.

1. An ellipsoid in $\mathbb{R}^n$, centered at zero, can be defined in two ways:

$$\mathcal{E}_1 = \{x \in \mathbb{R}^n : x^\top S x \leq 1\}, \qquad \text{where } S \succ 0, \tag{8}$$

$$\mathcal{E}_2 = \{x \in \mathbb{R}^n : x = My, \|y\|_2 \leq 1\}, \qquad \text{where } \det M \neq 0. \tag{9}$$

   Given S for $\mathcal{E}_1$, explain how to find M for $\mathcal{E}_2$ such that $\mathcal{E}_1 = \mathcal{E}_2$. Given M for $\mathcal{E}_2$, explain how to find S for $\mathcal{E}_1$ such that $\mathcal{E}_1 = \mathcal{E}_2$. Support your claims with proofs.

2. Using a collection of linear matrix inequalities, explain how to find an ellipsoid $\mathcal{E} \subseteq \mathbb{R}^n$, centered at 0, that is forward invariant for $\dot{x} = Ax$ and satisfies $z^{(i)} \in \mathcal{E},\ i = 1, ..., m$ and $w^{(j)} \notin \mathcal{E},\ j = 1, ..., p$, for $z^{(i)}, w^{(j)}$ some fixed points in $\mathbb{R}^n$. *Note: you may leave your solution in terms of strict inequalities (e.g. $\prec, <$) for the purposes of this problem. For computer implementation, strict inequalities are not realizable. Think about how you can re-pose your problem with non-strict inequalities!*

3. Produce a simple example for which your method from part (2) has a solution, and an example for which it does not. You may present your answer in the form of a simple sketch or in the form of problem data.

2.4 Finite Impulse Response Systems

In this problem, we'll consider a special type of discrete-time system, called a *finite impulse response (FIR)* system. A SISO discrete-time, LTI system with impulse response map $H[\cdot] : \mathbb{Z} \to \mathbb{R}$ is said to be finite impulse response of horizon N if for all $k > N$, $H[k] = 0$. That is,

$$H[0] = h_0,\ H[1] = h_1,\ ...,\ H[N] = h_N,\ H[N + k] = 0,\ \forall k \geq 1, \tag{10}$$

where $h_0, ..., h_N \in \mathbb{R}$ are real constants.

1. Prove that the transfer function $\hat{H}$ of a SISO, FIR system, must have all of its poles at the origin. Calculate an upper bound on the degree of the denominator of $\hat{H}$ in terms of the FIR horizon.

2. Does a (SISO) transfer function $\hat{G} \in \mathcal{RH}_\infty$ with all poles at the origin correspond to an FIR system? Provide a proof or counterexample to support your claim.

3. Let's consider input signals of length $M \in \mathbb{Z}_{\geq 0}$—input signals which are zero for all integers $k \notin [0, M]$. Calculate—as completely as you can—the 2-norm to 2-norm gain of a SISO FIR system with horizon N, over the set of length M inputs,

$$\sup_{\|u\|_{\ell_2} = 1,\ \text{len}(u) = M} \|H * u\|_{\ell_2}. \tag{11}$$

   Prove there exists a length M input u achieving the supremum. *You shouldn't have to use the table of system norms to answer this question!*

4. For the same system as above, calculate the infimum,

$$\inf_{\|u\|_{\ell_2} = 1,\ \text{len}(u) = M} \|H * u\|_{\ell_2}. \tag{12}$$

   Prove there exists a length M input u achieving the infimum. When (if ever) is this infimum zero?

2.5 Column-Wise Separability

A *functional* is a map from a set into the reals, $\mathbb{R}$. Consider a space of maps, $\mathcal{F} = \{\hat{G} : \Omega \to \mathbb{C}^{p\times m},\ \Omega \subseteq \mathbb{C}\}$. Let $\hat{G}_i$ denote the i'th column of an element $\hat{G} \in \mathcal{F}$. A functional $f : \mathcal{F} \to \mathbb{R}$ is *column-wise separable* if there exist functionals $f_1, ..., f_m$ for which

$$f(\hat{G}) = \sum_{i=1}^{m} f_i(\hat{G}_i),\ \forall\, \hat{G} \in \mathcal{F}. \tag{13}$$

That is, a functional is column-wise separable if it can be equivalently written via a sum of a collection of functionals, each acting on a column of $\hat{G} \in \mathcal{F}$.

1. Consider $\mathcal{F} = \{\hat{G} : \Omega \to \mathbb{C}^{p\times m},\ \Omega \subseteq \mathbb{C}\}$. Show that the functional $f : \mathcal{F} \to \mathbb{R}$ mapping $\hat{G} \in \mathcal{F}$ to the square of its Frobenius norm at a point $s_0 \in \Omega$, $f(\hat{G}) = ||\hat{G}(s_0)||_F^2$, is column-wise separable.
2. Now, suppose $\mathcal{F} = \mathcal{RH}_2$, with elements of $\mathcal{F}$ being rational functions of finite $\mathcal{H}_2$-norm taking values in $\mathbb{C}^{p\times m}$. Consider the functional $f : \mathcal{F} \to \mathbb{R}$ taking $\hat{G} \in \mathcal{F}$ to the square of its $\mathcal{H}_2$-norm, $f(\hat{G}) = ||\hat{G}||_2^2$. Is this functional column-wise separable? What about the functional taking $\hat{G}$ to its $\mathcal{H}_\infty$-norm?
3. Suppose you want to compute a functional of a *large* matrix-valued function on a computer. Why might it be advantageous for the functional to be column-wise separable? Comment on the implications of your answer for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$-norms.

3 Optional Problems

3.1 The Sylvester Equation

In the previous chapter, we examined the continuous-time *Lyapunov equation.* This equation is an instance of a more general type of matrix equation called a *Sylvester equation.* A matrix equation of the form,

$$MX + XN = Q, \tag{14}$$

in an unknown matrix X, is said to be a Sylvester equation in X. In this problem, we'll analyze the set of Sylvester equations using an algebraic approach different to that which we've considered so far.

1. For matrices $M \in \mathbb{R}^{n\times n}$ and $N \in \mathbb{R}^{n\times n}$, the Kronecker product of M and N, denoted $M \otimes N$, is the matrix

   $$M \otimes N := \begin{bmatrix} m_{11}N & \dots & m_{1n}N \\ \vdots & \ddots & \vdots \\ m_{n1}N & \dots & m_{nn}N \end{bmatrix}. \tag{15}$$

   Let $X = [x_1, ..., x_n] \in \mathbb{R}^{n\times n}$ and $Q = [q_1, .., q_n] \in \mathbb{R}^{n\times n}$, where each x_i and q_i denotes a column of X and Q, respectively. Define,

   $$\mathrm{vec}(X) = \begin{bmatrix} x_1^\top & \dots & x_n^\top \end{bmatrix}^\top \in \mathbb{R}^{n^2} \tag{16}$$
   $$\mathrm{vec}(Q) = \begin{bmatrix} q_1^\top & \dots & q_n^\top \end{bmatrix}^\top \in \mathbb{R}^{n^2}. \tag{17}$$

   Using the Kronecker product, prove there exists a matrix $A \in \mathbb{R}^{n^2\times n^2}$ for which

   $$A\,\mathrm{vec}(X) = \mathrm{vec}(Q) \Longleftrightarrow MX + XN = Q. \tag{18}$$

2. Suppose the eigenvalues of M are $\lambda_1, ..., \lambda_n$ and the eigenvalues of N are $\mu_1, ..., \mu_n$. Show that the eigenvalues of $M \otimes N$ are the n^2 numbers $\lambda_i\mu_j,\ (i, j) \in \{1, ..., n\}^2$.
3. Using your answers to parts (1) and (2), show there exists a unique solution X to the Sylvester equation $MX + XN = Q$ if and only if $\lambda_i(M) + \lambda_j(N) \neq 0$ for all $(i, j) \in \{1, ..., n\}^2$. Specialize this result to conclude the CTLE test for a Hurwitz matrix.