CDS 131 Homework 0

Mathematical Preliminaries via the Structured Singular Value

Winter 2025

In this course, we'll use techniques from linear algebra, analysis, and optimization. In order to get warmed up on these topics and learn a little bit about control along the way, we'll perform a cursory analysis of an important tool in robust control theory: the *structured singular value*. We'll provide only a little bit of background on the structured singular value, and the rest will be up to you! Don't worry if the questions seem difficult - this assignment is meant to be challenging and to give us a reference for what people's backgrounds are. Feel free to refer to your favorite linear algebra, analysis, or optimization book to complete the questions - the rest is self-contained.

Background

The (complex) structured singular value is a function from the set of $n \times n$ complex matrices to the reals that helps us understand the "gain" of matrices with structured uncertainty. The first step towards defining the structured singular value is to define a set of matrices $\underline{\Delta} \subseteq \mathbb{C}^{n\times n}$. Let $r_1, ..., r_S$ and $m_1, ..., m_F$ be positive integers for which $\sum_{i=1}^{S} r_i + \sum_{j=1}^{F} m_j = n$. Then, define a set $\underline{\Delta} \subseteq \mathbb{C}^{n\times n}$ as

$$\underline{\Delta} := \{\text{blkdiag}(\delta_1 I_{r_1}, ..., \delta_S I_{r_S}, \Delta_{S+1}, ..., \Delta_{S+F}) : \delta_i \in \mathbb{C},\ \Delta_{s+j} \in \mathbb{C}^{m_j \times m_j}\}, \tag{1}$$

where I_k represents the $k \times k$ identity matrix. In short, $\underline{\Delta}$ is the set of block diagonal matrices with *repeated scalar blocks* of dimensions $r_i \times r_i$ (these are the blocks $\delta_i I_{r_i}$) and and *full blocks* of dimensions $m_j \times m_j$ (these are the blocks Δ_{S+j}). Given a matrix $M \in \mathbb{C}^{n\times n}$ and a set $\underline{\Delta} \subseteq \mathbb{C}^{n\times n}$ of the form above, one defines the structured singular value of M, $\mu_{\underline{\Delta}}(M)$, as follows.

Definition 1 (Structured Singular Value)**.** *For $M \in \mathbb{C}^{n\times n}$, $\mu_{\underline{\Delta}}(M)$ is defined,*

$$\mu_{\underline{\Delta}}(M) := \frac{1}{\inf\{\overline{\sigma}(\Delta) : \Delta \in \underline{\Delta} \text{ and } \det(I - M\Delta) = 0\}}, \tag{2}$$

unless no $\Delta \in \underline{\Delta}$ makes $I - M\Delta$ singular, in which case $\mu_{\underline{\Delta}}(M) := 0$.

Note that here, we use $\overline{\sigma}(M)$ to denote the maximum singular value of M. Based on this definition, $\mu_{\underline{\Delta}}(M)$ depends both on M and on the set $\underline{\Delta}$. Now, let's get started on our analysis of $\mu_{\underline{\Delta}}$!

Problems

Note: In the following problems, you can assume for simplicity that one does not encounter the case where no Δ makes $I - M\Delta$ singular.

1. Compute $\mu_{\underline{\Delta}}(M)$ in the case where $\underline{\Delta}$ is *unstructured*, i.e. $\underline{\Delta} = \mathbb{C}^{n\times n}$.

2. Recall that the spectral radius of a matrix $M \in \mathbb{C}^{n\times n}$ is defined,

$$\rho(M) := \max_i |\lambda_i(M)|. \tag{3}$$

Define the set $B_{\underline{\Delta}} = \{\Delta \in \underline{\Delta} : \overline{\sigma}(\Delta) \leq 1\}$. Prove that the structured singular value can be calculated as the following function of spectral radius:

$$\mu_{\underline{\Delta}}(M) = \sup_{\Delta \in B_{\underline{\Delta}}} \rho(\Delta M). \tag{4}$$

Now, consider the special case where $\underline{\Delta} = \{\delta I_n : \delta \in \mathbb{C}\}$. In this case, show that $\mu_{\underline{\Delta}}(M) = \rho(M)$.

3. Let's consider some additional methods of computing μ. Define the following subset of $\mathbb{C}^{n\times n}$:

$$\underline{D} = \{\text{blkdiag}(D_1, ..., D_S, d_{S+1}I_{m_1}, ..., d_{S+F}I_{m_F} : D_i \in \mathbb{C}^{r_i\times r_i}, D_i \succ 0, d_{S+j} \in \mathbb{R}_{>0}\}. \tag{5}$$

   Prove that, for all $D \in \underline{D}$,

$$\mu_{\underline{\Delta}}(M) = \mu_{\underline{\Delta}}(D^{\frac{1}{2}}MD^{-\frac{1}{2}}). \tag{6}$$

   Then, show that,

$$\mu_{\underline{\Delta}}(M) \leq \inf_{D\in\underline{D}} \overline{\sigma}(D^{\frac{1}{2}}MD^{-\frac{1}{2}}). \tag{7}$$

4. Fix a matrix $M \in \mathbb{C}^{n\times n}$. For the set $\underline{D}$ introduced in part (3), show that the following set is convex for each fixed $\beta \in \mathbb{R}$:

$$\{D \in \underline{D} : \overline{\sigma}(D^{\frac{1}{2}}MD^{-\frac{1}{2}}) \leq \beta\}. \tag{8}$$

   Hint: Rewrite as a linear matrix inequality. Such inequalities are amenable to implementation in convex optimization solvers!