大家好，这篇是有关台大机器学习课程作业六的详解。

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作业地址:
https://www.csie.ntu.edu.tw/~htlin/course/ml15fall/

参考资料:
https://blog.csdn.net/a1015553840/article/details/51085129
http://www.vynguyen.net/category/study/machine-learning/page/6/
http://book.caltech.edu/bookforum/index.php
http://beader.me/mlnotebook/
https://blog.csdn.net/qian1122221/article/details/50130093
https://acecoooool.github.io/blog/

Problem 1

首先计算p_n

$$\begin{aligned}
p_n &= \theta(-y_n(Az_n+B)) \\
&= \frac{1}{1+\exp(y_n(Az_n+B))}
\end{aligned}$$

现在对式子进行化简

$$\begin{aligned}
F(A,B) &= \frac{1}{N}\sum_{n=1}^{N}\ln\Big(1+\exp\Big(-y_n\Big(Az_n+B\Big)\Big)\Big) \\
&= \frac{1}{N}\sum_{n=1}^{N}\ln\Big(\frac{1+\exp\Big(y_n\Big(Az_n+B\Big)\Big)}{\exp\Big(y_n\Big(Az_n+B\Big)\Big)}\Big) \\
&= -\frac{1}{N}\sum_{n=1}^{N}\ln\Big(\frac{\exp\Big(y_n\Big(Az_n+B\Big)\Big)}{1+\exp\Big(y_n\Big(Az_n+B\Big)\Big)}\Big) \\
&= -\frac{1}{N}\sum_{n=1}^{N}\ln\Big(1-p_n\Big)
\end{aligned}$$

对原式进行处理

$$y_n(Az_n+B) = \begin{pmatrix} y_n z_n \\ y_n \end{pmatrix}^T \begin{pmatrix} A \\ B \end{pmatrix} \triangleq \begin{pmatrix} y_n z_n \\ y_n \end{pmatrix}^T C$$

那么

$$
\begin{aligned}
p_n &= \theta(-y_n(Az_n+B)) \\
&= \theta\Big(-\begin{pmatrix} y_n z_n \\ y_n \end{pmatrix}^T C\Big) \\
\nabla_C p_n &= p_n(1-p_n)\nabla_C\Big(-\begin{pmatrix} y_n z_n \\ y_n \end{pmatrix}^T C\Big) \\
&= -p_n(1-p_n)\begin{pmatrix} y_n z_n \\ y_n \end{pmatrix} \qquad (1)
\end{aligned}
$$

所以

$$
\begin{aligned}
\nabla_C F(A,B) &= \nabla_C\Big(-\frac{1}{N}\sum_{n=1}^N \ln\Big(1-p_n\Big)\Big) \\
&= -\frac{1}{N}\sum_{n=1}^N \nabla_C \ln\Big(1-p_n\Big) \\
&= -\frac{1}{N}\sum_{n=1}^N (-1)\frac{1}{1-p_n}\nabla_C p_n \\
&= -\frac{1}{N}\sum_{n=1}^N (-1)\frac{1}{1-p_n}(-p_n)(1-p_n)\begin{pmatrix} y_n z_n \\ y_n \end{pmatrix} \\
&= -\frac{1}{N}\sum_{n=1}^N \begin{pmatrix} p_n y_n z_n \\ p_n y_n \end{pmatrix}
\end{aligned}
$$

Problem 2

现在要计算Hessian矩阵，由上一题可知

$$
\begin{aligned}
\frac{\partial F(A,B)}{\partial A} &= -\frac{1}{N}\sum_{n=1}^N y_n z_n p_n \\
\frac{\partial F(A,B)}{\partial B} &= -\frac{1}{N}\sum_{n=1}^N y_n p_n
\end{aligned}
$$

在计算$\frac{\partial^2 F(A,B)}{\partial A^2}$, $\frac{\partial^2 F(A,B)}{\partial B^2}$, $\frac{\partial^2 F(A,B)}{\partial A \partial B}$之前，由(1)可得$\frac{\partial p_n}{\partial A}$, $\frac{\partial p_n}{\partial B}$为

$$
\begin{aligned}
\frac{\partial p_n}{\partial A} &= p_n(1-p_n)(-y_n)z_n \\
\frac{\partial p_n}{\partial B} &= p_n(1-p_n)(-y_n)
\end{aligned}
$$

接下来分别计算上述三个式子，注意$y_n^2=1$

$$
\begin{aligned}
\frac{\partial^2 F(A,B)}{\partial A^2} &= \frac{\partial}{\partial A}\Big(-\frac{1}{N}\sum_{n=1}^{N} y_n z_n p_n\Big) \\
&= -\frac{1}{N}\sum_{n=1}^{N} y_n z_n \frac{\partial p_n}{\partial A} \\
&= -\frac{1}{N}\sum_{n=1}^{N} y_n z_n p_n(1-p_n)(-y_n) z_n \\
&= \frac{1}{N}\sum_{n=1}^{N} y_n^2 z_n^2 p_n(1-p_n) \\
&= \frac{1}{N}\sum_{n=1}^{N} z_n^2 p_n(1-p_n) \\
\frac{\partial^2 F(A,B)}{\partial B^2} &= \frac{\partial}{\partial B}\Big(-\frac{1}{N}\sum_{n=1}^{N} y_n p_n\Big) \\
&= -\frac{1}{N}\sum_{n=1}^{N} y_n \frac{\partial p_n}{\partial A} \\
&= -\frac{1}{N}\sum_{n=1}^{N} y_n p_n(1-p_n)(-y_n) \\
&= \frac{1}{N}\sum_{n=1}^{N} y_n^2 p_n(1-p_n) \\
&= \frac{1}{N}\sum_{n=1}^{N} p_n(1-p_n) \\
\frac{\partial^2 F(A,B)}{\partial A \partial B} &= \frac{\partial}{\partial B}\Big(-\frac{1}{N}\sum_{n=1}^{N} y_n z_n p_n\Big) \\
&= -\frac{1}{N}\sum_{n=1}^{N} y_n z_n \frac{\partial p_n}{\partial B} \\
&= -\frac{1}{N}\sum_{n=1}^{N} y_n z_n p_n(1-p_n)(-y_n) \\
&= \frac{1}{N}\sum_{n=1}^{N} y_n^2 z_n p_n(1-p_n) \\
&= \frac{1}{N}\sum_{n=1}^{N} z_n p_n(1-p_n)
\end{aligned}
$$

结合这几个式子，我们可知

$$
H(F) = \begin{pmatrix} \frac{1}{N}\sum_{n=1}^{N} z_n^2 p_n(1-p_n) & \frac{1}{N}\sum_{n=1}^{N} z_n p_n(1-p_n) \\ \frac{1}{N}\sum_{n=1}^{N} z_n p_n(1-p_n) & \frac{1}{N}\sum_{n=1}^{N} p_n(1-p_n) \end{pmatrix}
$$

Problem 3

首先回顾下Gaussian kernel的形式

$$K(x,x^{'})=\exp(-\gamma||x-x^{'}||^2)$$

所以如果$\gamma\to\infty$，那么$K(x,x^{'})\to 0$，从而kernel matrix $K\to 0$ ，注意最后的0是零矩阵的意思。现在回顾讲义上β的式子

$$\beta=(\lambda I+K)^{-1}y$$

现在$K\to\infty$，那么

$$\beta\to\lambda^{-1}y$$

Problem 4

本题的目的是将条件极值改写为无条件极值，先看下本题的条件。

$$-\epsilon-\xi_n^{\vee}\le y_n-w^T\phi(x_n)-b\le\epsilon+\xi_n^{\wedge}$$

由几何意义可知，

$$\text{当}\,y_n-w^T\phi(x_n)-b\ge 0\text{时，}\ \xi_n^{\vee}=0,\ \xi_n^{\wedge}=\max\Big(0,|w^Tz_n+b-y_n|-\epsilon\Big)$$
$$\text{当}\,y_n-w^T\phi(x_n)-b< 0\text{时，}\ \xi_n^{\wedge}=0,\ \xi_n^{\vee}=\max\Big(0,|w^Tz_n+b-y_n|-\epsilon\Big)$$

所以

$$(\xi_n^{\vee})^2+(\xi_n^{\wedge})^2=\Big(\max\Big(0,|w^Tz_n+b-y_n|-\epsilon\Big)\Big)^2$$

所以原问题可以转化为以下问题

$$\min_{b,w}\frac{1}{2}w^Tw+C\sum_{n=1}^N\Big(\max\Big(0,|w^Tz_n+b-y_n|-\epsilon\Big)\Big)^2$$

Problem 5

对Problem 4最后的结果进行改写

$$\min_b\left(\min_w\frac{1}{2}w^Tw+C\sum_{n=1}^N\Big(\max\Big(0,|w^Tz_n+b-y_n|-\epsilon\Big)\Big)^2\right)$$

对于第一个最小化问题$\min_w\frac{1}{2}w^Tw+C\sum_{n=1}^N\Big(\max\Big(0,|w^Tz_n+b-y_n|-\epsilon\Big)\Big)^2$，由Representer Theorem可知，该问题的最优解为

$$w_* = \sum_{m=1}^{N} \beta_m z_m$$

带入上式可得，现在问题转化为

$$\min_{b} \frac{1}{2} w_*^T w_* + C \sum_{n=1}^{N} \Big(\max\Big(0, |w_*^T z_n + b - y_n| - \epsilon\Big)\Big)^2$$

将$\beta_1, \ldots, \beta_N$视为参数，结合$K(x_n, x_m) = (\varphi(x_n))^T(\varphi(x_m))$，该问题转化为

$$\min_{b,\beta} F(b, \beta) = \frac{1}{2} \sum_{m=1}^{N} \sum_{n=1}^{N} \beta_n \beta_m K(x_n, x_m) + C \sum_{n=1}^{N} \Big(\max\Big(0, |\sum_{m=1}^{N} \beta_m K(x_n, x_m) + b - y_n| - \epsilon\Big)\Big)^2$$

题目中记$s_n = \sum_{m=1}^{N} \beta_m K(x_n, x_m) + b$，所以上式可以变形为

$$\min_{b,\beta} F(b, \beta) = \frac{1}{2} \sum_{m=1}^{N} \sum_{n=1}^{N} \beta_n \beta_m K(x_n, x_m) + C \sum_{n=1}^{N} \Big(\max\Big(0, |s_n - y_n| - \epsilon\Big)\Big)^2$$

现在计算$\frac{\partial F(b,\beta)}{\partial \beta_m}$，分两种情形讨论：

当$|s_n - y_n| - \epsilon \le 0$时，$C \sum_{n=1}^{N} \Big(\max\Big(0, |s_n - y_n| - \epsilon\Big)\Big)^2 = 0$

$$\begin{aligned}\frac{\partial F(b, \beta)}{\partial \beta_i} &= \sum_{n=1}^{N} \beta_n K(x_n, x_i) \\ &= \sum_{n=1}^{N} \beta_n K(x_n, x_i)\end{aligned}$$

当$|s_n - y_n| - \epsilon > 0$时，$C \sum_{n=1}^{N} \Big(\max\Big(0, |s_n - y_n| - \epsilon\Big)\Big)^2 = C \sum_{n=1}^{N} \Big(|s_n - y_n| - \epsilon\Big)^2$

$$\begin{aligned}\frac{\partial F(b, \beta)}{\partial \beta_i} &= \sum_{n=1}^{N} \beta_n K(x_n, x_i) + 2C \sum_{n=1}^{N} (|s_n - y_n| - \epsilon) \mathrm{sign}(s_n - y_n) \frac{\partial s_n}{\partial \beta_i} \\ &= \sum_{n=1}^{N} \beta_n K(x_n, x_i) + 2C \sum_{n=1}^{N} (|s_n - y_n| - \epsilon) \mathrm{sign}(s_n - y_n) K(x_n, x_i) \\ &= \sum_{n=1}^{N} (\beta_n + 2C(|s_n - y_n| - \epsilon) \mathrm{sign}(s_n - y_n) K(x_n, x_i)\end{aligned}$$

如果统一起来，可以写成

$$\frac{\partial F(b, \beta)}{\partial \beta_i} = \sum_{n=1}^{N} \Big(\beta_n + 2C [\![|s_n - y_n| - \epsilon > 0]\!] \mathrm{sign}(s_n - y_n)\Big) K(x_n, x_i)$$

其中

$$[\![|s_n-y_n|-\epsilon]\!]=\begin{cases}1 & |s_n-y_n|-\epsilon>0\\ 0 & |s_n-y_n|-\epsilon\le 0\end{cases}$$

Problem 6

我们把$E_{\text{test}}(g_t)=\frac{1}{M}\sum_{m=1}^{M}(g_t(\tilde{x}_m)-\tilde{y}_m)^2=e_t(t=0,1,2,\ldots,T)$这个式子展开，记

$$z_t=\frac{2}{M}\sum_{m=1}^{M}g_t(\tilde{x}_m)\tilde{y}_m$$

注意

$$\frac{1}{M}\sum_{m=1}^{M}(g_t(\tilde{x}_m))^2=s_t$$

那么对$t=0,1,2,\ldots,T$，我们有

$$\begin{aligned}\frac{1}{M}\sum_{m=1}^{M}(g_t(\tilde{x}_m)-\tilde{y}_m)^2&=e_t\\ \frac{1}{M}\sum_{m=1}^{M}(g_t(\tilde{x}_m))^2-\frac{2}{M}\sum_{m=1}^{M}g_t(\tilde{x}_m)\tilde{y}_m+\sum_{m=1}^{M}y_m^2&=e_t\\ s_t-z_t+\sum_{m=1}^{M}y_m^2&=e_t\end{aligned}$$

我们要求的量是z_t，已知的量是s_t,e_t，还有两个条件为$g_0(x)=0,s_0=\frac{1}{M}\sum_{m=1}^{M}(g_0(\tilde{x}_m))^2=0$，所以

$$\begin{aligned}z_0&=0\\ s_0-z_0+\sum_{m=1}^{M}y_m^2&=e_0\\ \sum_{m=1}^{M}y_m^2&=e_0-s_0=e_0\end{aligned}$$

所以

$$\begin{aligned}z_t&=s_t+\sum_{m=1}^{M}y_m^2-e_t=s_t+e_0-e_t\\ \sum_{m=1}^{M}g_t(\tilde{x}_m)\tilde{y}_m&=\frac{M}{2}z_t=\frac{M}{2}(s_t+e_0-e_t)\end{aligned}$$

Problem 7

设两个点的坐标为$(x_1,y_1),(x_2,y_2),y_1=x_1^2,y_2=x_2^2$，由公式可知，最小二乘解为

$$w=\frac{x_1y_1+x_2y_2-2\frac{x_1+x_2}{2}\frac{y_1+y_2}{2}}{(x_1-\frac{x_1+x_2}{2})^2+(x_2-\frac{x_1+x_2}{2})^2}=\frac{(x_1-x_2)(y_1-y_2)}{(x_1-x_2)^2}=\frac{y_1-y_2}{x_1-x_2}=\frac{x_1^2-x_2^2}{x_1-x_2}=x_1+x_2,$$
$$b=\frac{y_1+y_2}{2}-w\frac{x_1+x_2}{2}=\frac{x_1^2+x_2^2}{2}-(x_1+x_2)\frac{x_1+x_2}{2}=-x_1x_2$$

因为x_1,x_2服从$[0,1]$上的均匀分布，所以

$$\begin{aligned}\mathbb{E}w&=\mathbb{E}(x_1+x_2)=\mathbb{E}(x_1)+\mathbb{E}(x_2)=1\\ \mathbb{E}b&=\mathbb{E}(-x_1x_2)=-\mathbb{E}(x_1)\mathbb{E}(x_2)=-\frac{1}{2}\times\frac{1}{2}=-\frac{1}{4}\\ \bar{g}(x)&=x-\frac{1}{4}\end{aligned}$$

Problem 8

$$\min_w E_{\text{in}}^u(w)=\frac{1}{N}\sum_{n=1}^N u_n(y_n-w^Tx_n)^2$$

由于$u_n\ge 0$，所以可以对$E_{\text{in}}^u(w)$进行如下处理

$$E_{\text{in}}^u(w)=\frac{1}{N}\sum_{n=1}^N u_n(y_n-w^Tx_n)^2=\frac{1}{N}\sum_{n=1}^N(\sqrt{u_n}y_n-w^T\sqrt{u_n}x_n)^2$$

现在记$(\tilde{x}_n,\tilde{y}_n)=\sqrt{u_n}(x_n,y_n)$，那么$E_{\text{in}}^u(w)$可以转化为

$$E_{in}^u(w)=\frac{1}{N}\sum_{n=1}^N(\tilde{y}_n-w^T\tilde{x}_n)^2$$

这样就转化为常规形式。

Problem 9

我们知道$g_1(x)$的正确率为99%，只在negative example上预测错误，根据讲义8第11到13页可知

$$\frac{u_+^{(2)}}{u_-^{(2)}}=\frac{1}{99}$$

Problem 10

首先回顾假设的形式

$$g_{s,i,\theta}(x)=s\cdot\text{sign}(x_i-\theta)(i\in\{1,2,\ldots,d\})$$

先考虑两种最极端的情况，$\theta < L, \theta \geq R$，在这两种情形下，$\text{sign}(x_i - \theta)$或者都为$1$，或者全为$-1$，所以在这两种条件下一共有两个$g(x)$，注意这种情形是和$i$无关，最后计算的时候要注意这点。

现在考虑$L \leq \theta < R$，根据题目中的定义，决定$\text{sign}(x_i - \theta)$只是$\theta$相对于$x_i$的位置，所以对于

$$\theta \in [k, k+1), k \in \{L, L+1, \ldots, R-1\}$$

$\text{sign}(x_i - \theta)$表示的都是同一个函数，因此一共有$R - L$种$\text{sign}(x_i - \theta)$，

由于$s \in \{+1, -1\}$，所以$g_{s,i,\theta}(x) = s \cdot \text{sign}(x_i - \theta)$一共有$2(R-L)$种。我们现在考虑的是一个维度上的，因为一共有$d$个维度，每个维度代表一种分类器，最后加上最开始讨论的全1或者全-1的情况，所以一共有

$$2d(R-L)+2$$

此题将$d = 2, L = 1, R = 6$带入可得

$$2 \times 2 \times 5 + 2 = 22$$

Problem 11

先计算$g_t(x)g_t(x^{'})$

$$\begin{aligned} g_t(x)g_t(x^{'}) &= (s_t.\text{sign}(x_i - \theta_t))(s_t.\text{sign}(x_i^{'} - \theta_t)) \\ &= \text{sign}(x_{t_i} - \theta_t)\text{sign}(x_{t_i}^{'} - \theta_t) \\ & t_i\text{的含义为}g_t(x)\text{对应的}i \end{aligned}$$

所以

$$\begin{aligned} K_{ds}(x, x^{'}) &= (\phi_{ds}(x))^T \phi_{ds}(x^{'}) \\ &= \sum_{t=1}^{|\mathcal{G}|} g_t(x)g_t(x^{'}) \\ &= \sum_{t=1}^{|\mathcal{G}|} \text{sign}(x_{t_i} - \theta_t)\text{sign}(x_{t_i}^{'} - \theta_t) \\ & t_i\text{的含义为}g_t(x)\text{对应的}i \end{aligned}$$

现在考虑$\text{sign}(x_{t_i} - \theta_t)\text{sign}(x_{t_i}^{'} - \theta_t)$，分两种情况考虑，如果$\theta_t \in [\min(x_i, x_i^{'}), \max(x_i, x_i^{'}))$，那么$\text{sign}(x_{t_i} - \theta_t)\text{sign}(x_{t_i}^{'} - \theta_t)$异号，其余情况$\text{sign}(x_{t_i} - \theta_t)\text{sign}(x_{t_i}^{'} - \theta_t)$同号，总结如下

$$\text{sign}(x_{t_i} - \theta_t)\text{sign}(x_{t_i}^{'} - \theta_t) = \begin{cases} -1, & \theta_t \in [\min(x_{t_i}, x_{t_i}^{'}), \max(x_{t_i}, x_{t_i}^{'})) \\ 1, & \text{其他} \end{cases}$$

所以上述求和式中$\sum_{t=1}^{|\mathcal{G}|} \text{sign}(x_{t_i} - \theta_t)\text{sign}(x_{t_i}^{'} - \theta_t)$中$+1, -1$的数量取决于$x_{t_i}, x_{t_i}^{'}$，在$[\min(x_{t_i}, x_{t_i}^{'}), \max(x_{t_i}, x_{t_i}^{'}))$中，一共有$|x_{t_i} - x_{t_i}^{'}|$个整数（注意输入为整数），所以使得$\text{sign}(x_{t_i} - \theta_t)\text{sign}(x_{t_i}^{'} - \theta_t) = -1$的$t$的数量为

$$2\sum_{j=1}^{d} |x_j - x_j^{'}| = 2||x - x^{'}||_1$$

这里乘以2是因为还要考虑s有两种可能。注意到

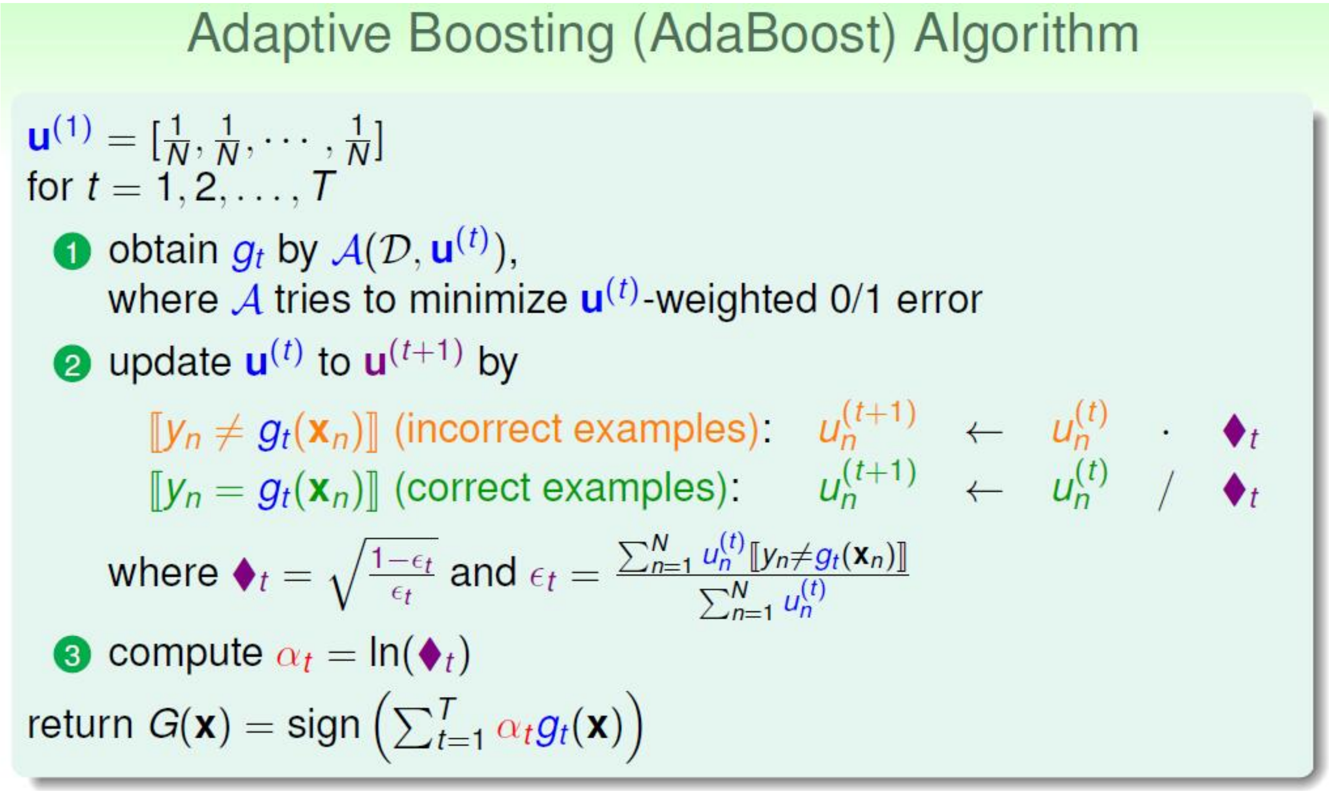
$$\sum_{t=1}^{|\mathcal{G}|}|\text{sign}(x_{t_i}-\theta_t)\text{sign}(x^{'}_{t_i}-\theta_t)|=|\mathcal{G}|$$

所以满足$\text{sign}(x_{t_i}-\theta_t)\text{sign}(x^{'}_{t_i}-\theta_t)=1$的$t$一共有$|\mathcal{G}|-2||x-x^{'}||_1$个，因此

$$\begin{aligned}K_{ds}(x,x^{'})&=\sum_{t=1}^{|\mathcal{G}|}\text{sign}(x_{t_i}-\theta_t)\text{sign}(x^{'}_{t_i}-\theta_t)\\&=|\mathcal{G}|-2||x-x^{'}||_1-2||x-x^{'}||_1\\&=|\mathcal{G}|-4||x-x^{'}||_1\\&=2d(R-L)-4||x-x^{'}||_1+2\end{aligned}$$

Problem 12

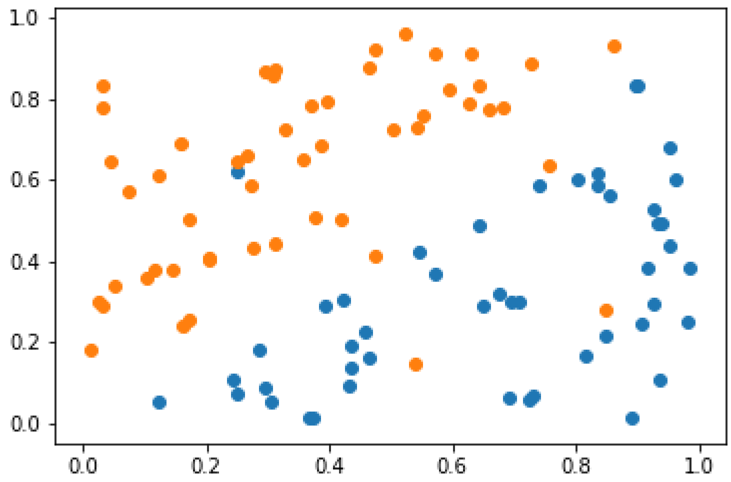
首先回顾逐步增强法：



题目的思路是这样的，利用decision stump来产生原始模型，然后用Adaptive Boosting算法得到最终结果，先作图看下。

```
# -*- coding: utf-8 -*-
"""
Created on Mon Apr  8 10:39:08 2019

@author: qinzhen
```

```
"""

import numpy as np
import matplotlib.pyplot as plt

#读取数据
train = np.genfromtxt('hw2_adaboost_train.dat')
test = np.genfromtxt('hw2_adaboost_test.dat')
#作图
plt.scatter(train[:, 0][train[:, 2] == 1], train[:, 1][train[:, 2] == 1])
plt.scatter(train[:, 0][train[:, 2] == -1], train[:, 1][train[:, 2] == -1])
plt.show()
```



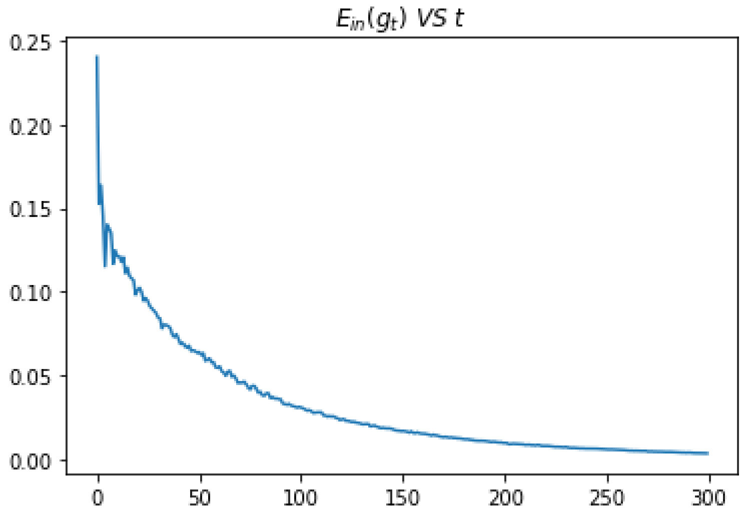

```
#按第一个下标排序
train1 = np.array(sorted(train, key=lambda x:x[0]))

#按第二个下标排序
train2 = np.array(sorted(train, key=lambda x:x[1]))

#获得theta
x1 = train1[:, 0]
theta1 = np.append(np.array(x1[0] - 0.1), (x1[:-1] + x1[1:])/2)
theta1 = np.append(theta1, x1[-1] + 0.1)

x2 = train1[:, 1]
theta2 = np.append(np.array(x2[0]-0.1), (x2[:-1] + x2[1:])/2)
theta2 = np.append(theta2, x2[-1]+0.1)

#合并theta
theta = np.c_[theta1, theta2]

y = train[:, 2]
X = train[:, :2]

def decision_stump(X, y, U, theta):
    """
    X为训练数据，y为标签，U为权重，theta为stump
    """
```

```python
    #向量化执行计算
    n = theta.shape[0]
    m = X.shape[0]
    #将X复制按横轴n份
    X = np.tile(X, (n, 1))
    #s=1
    y1 = np.sign(X - theta)
    #s=-1
    y2 = np.sign(X - theta) * (-1)
    #计算加权错误
    error1 = np.sum((y1!=y) * U, axis=1)
    error2 = np.sum((y2!=y) * U, axis=1)
    #计算最小错误对应的下标
    i1 = np.argmin(error1)
    i2 = np.argmin(error2)
    #判断哪个误差更小
    if error1[i1] < error2[i2]:
        s = 1
        index = i1
        error = error1[i1] / m
    else:
        s = -1
        index = i2
        error = error2[i2] / m
    return s, index, error

def decision_stump_all(X, y, U, theta):
    """
    对两个维度分别使用decision_stump，取误差较小的维度
    """
    #维度1
    X1 = X[:, 0]
    theta1 = theta[:, 0].reshape(-1, 1)
    s1, i1, e1 = decision_stump(X1, y, U, theta1)

    #维度2
    X2 = X[:, 1]
    theta2 = theta[:, 1].reshape(-1, 1)
    s2, i2, e2 = decision_stump(X2, y, U, theta2)

    if(e1 < e2):
        return e1, s1, 0, i1
    else:
        return e2, s2, 1, i2

def Adaptive_Boosting(X, y, theta, T=300):
    n = X.shape[0]
    u = np.ones(n) / n

    #记录需要的数据
    Alpha = np.array([])
    U = np.array([])
    Epsilon = np.array([])
```

```
    Ein = np.array([])
    G = np.array([])

    for t in range(T):
        #计算当且最优的参数
        ein, s, d, index = decision_stump_all(X, y, u, theta)
        #计算误差
        epsilon = u.dot((s * np.sign(X[:, d] - theta[:, d][index])) != y) / np.sum(u)
        #计算系数
        k = np.sqrt((1 - epsilon) / epsilon)
        #找到错误的点
        i1 = s * np.sign(X[:, d] - theta[:, d][index]) != y
        #更新权重
        u[i1] = u[i1] * k
        #找到正确的点
        i2 = s * np.sign(X[:, d] - theta[:, d][index]) == y
        #更新权重
        u[i2] = u[i2] / k
        #更新alpha
        alpha = np.log(k)

        #存储数据
        Ein = np.r_[Ein, ein]
        if(t == 0):
            U = np.array([u])
        else:
            U = np.r_[U, np.array([u])]
        Epsilon = np.r_[Epsilon, epsilon]
        Alpha = np.r_[Alpha, alpha]
        g = [[s, d, index]]

        if(t == 0):
            G = np.array(g)
        else:
            G = np.r_[G, np.array(g)]
    return Ein, U, Epsilon, Alpha, G
```

训练数据

```
#训练数据
T = 300
Ein, U, Epsilon, Alpha, G = Adaptive_Boosting(X, y, theta, T=T)

#problem 12
t = np.arange(T)

plt.plot(t, Ein)
plt.title("$E_{in}(g_t)\ vs\ t$")
plt.show()
print("Ein(g1) =", Ein[0], ",alpha1 =", Alpha[0])
```




```
Ein(g1) = 0.0024000000000000002 ,alpha1 = 0.5763397549691927
```

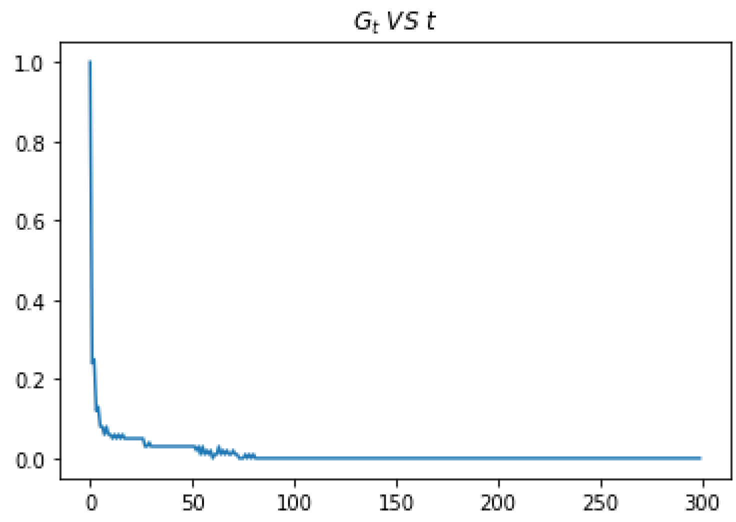
Problem 13

$E_{\text{in}}(g_t)$在逐渐变小，因为Adaptive Boosting算法每次对错误的点增加权重，正确的点减小权重，所以每一次比前一次的分类效果都会逐渐变好。

problem 14

```
#problem 14
def predict(X, y, G, Alpha, t, theta):
    """
    利用前t个alpha, g计算Ein(Gt)
    """

    s = G[:t, 0]
    d = G[:t, 1]
    theta_ = G[:t, 2]
    alpha = Alpha[:t]

    result = []
    for i in range(t):
        s1 = s[i]
        d1 = d[i]
        t1 = theta_[i]
        result.append(s1*np.sign(X[:, d1] - theta[:, d1][t1]))
    r = alpha.dot(np.array(result))

    return np.mean(np.sign(r) != y)

T = 300
t = np.arange(T)
G1 = [predict(X, y, G, Alpha, i, theta) for i in t]
```

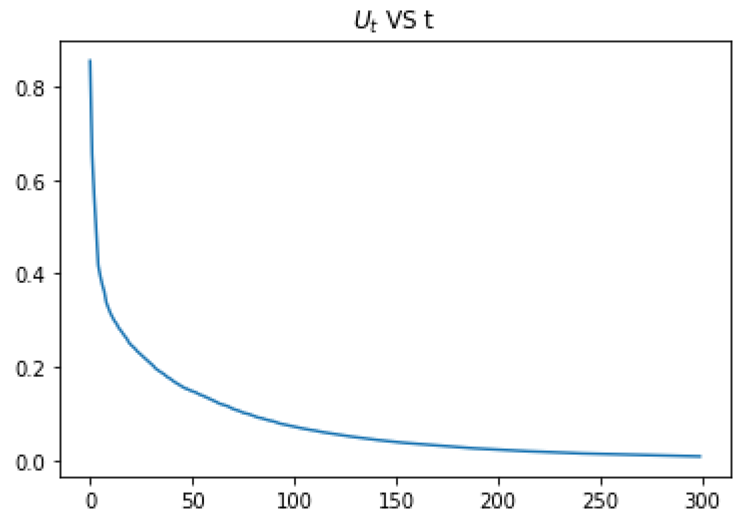
```
plt.plot(t, G1)
plt.title("$G_t\ VS\ t$")
plt.show()

print("Ein(G) =", G1[-1])
```




```
Ein(G) = 0.0
```

problem 15

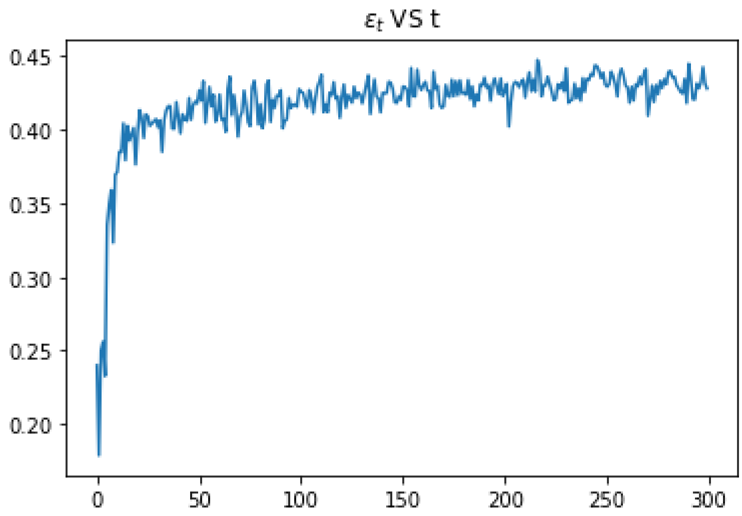
```
#problem 15
U1 = U.sum(axis=1)

plt.plot(t, U1)
plt.title('$U_t$ VS t')
plt.show()

print("U2 =", U1[1], "UT =", U1[-1])
```



```
U2 = 0.6545039637744691 UT = 0.008596775074963087
```

problem 16

```
#problem 16
plt.plot(t, Epsilon)
plt.title('$\epsilon_t$ VS t')
plt.show()

print("minimun epsilon =", np.min(Epsilon))
```

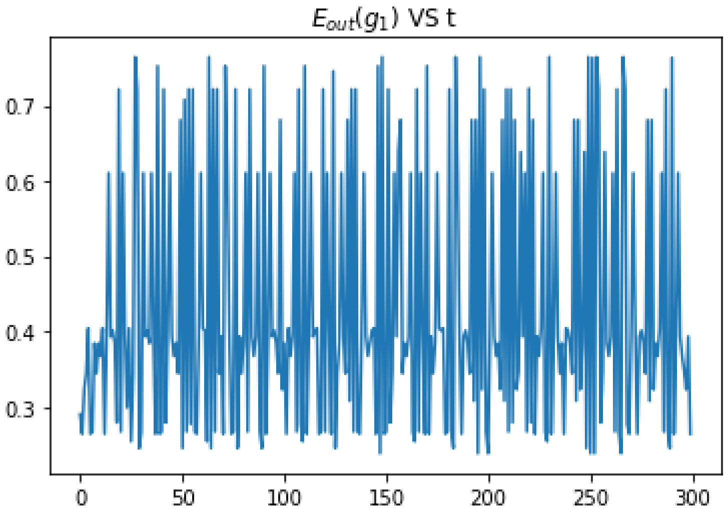



```
minimun epsilon = 0.1787280701754386
```

problem 17

```
#problem 17
Xtest = test[:, :2]
ytest = test[:, 2]
#获得参数
s = G[:, 0]
d = G[:, 1]
theta_ = G[:, 2]

g = []
for i in range(300):
    s1 = s[i]
    d1 = d[i]
    t1 = theta_[i]

    g.append(np.mean(s1*np.sign(Xtest[:, d1] - theta[:, d1][t1]) != ytest))
```

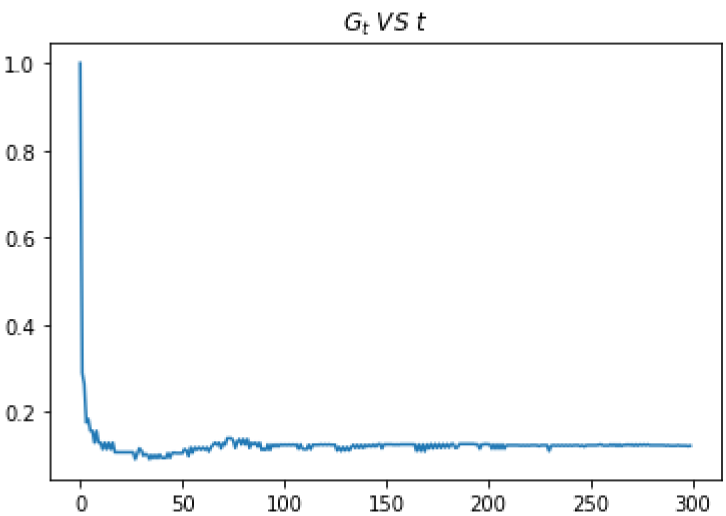
```
plt.plot(t, g)
plt.title('$E_{out}(g_1)$ VS t')
plt.show()

print("Eout(g1) =", g[0])
```




```
Eout(g1) = 0.29
```

problem 18

```
#problem 18
t = np.arange(T)
G2 = [predict(Xtest, ytest, G, Alpha, i, theta) for i in t]

plt.plot(t, G2)
plt.title("$G_t\ VS\ t$")
plt.show()

print("Ein(G) =", G2[-1])
```



```
Ein(G) = 0.123
```

Problem 19

首先这题需要计算高斯核矩阵，所以我们需要计算$[||x^{(i)}-x^{(j)}||^2]_{i,j}$，下面介绍向量化计算的方法：

假设

$$X=\begin{bmatrix} —(x^{(1)})^T— \\ —(x^{(2)})^T— \\ \vdots \\ —(x^{(m)})^T— \end{bmatrix} \in \mathbb{R}^{m\times d}, Y=\begin{bmatrix} —(y^{(1)})^T— \\ —(y^{(2)})^T— \\ \vdots \\ —(y^{(n)})^T— \end{bmatrix} \in \mathbb{R}^{n\times d}$$

其中$x^{(i)},y^{(i)}\in \mathbb{R}^d$，现在的问题是如何高效计算矩阵$D\in\mathbb{R}^{m\times n}$，其中

$$D_{i,j}=||x^{(i)}-y^{(j)}||^2$$

首先对$D_{i,j}$进行处理

$$\begin{aligned} D_{i,j}&=||x^{(i)}-y^{(j)}||^2 \\ &=(x^{(i)}-y^{(j)})^T(x^{(i)}-y^{(j)}) \\ &=(x^{(i)})^Tx^{(i)}-2(x^{(i)})^Ty^{(j)}+(y^{(j)})^Ty^{(j)} \end{aligned}$$

那么

$$\begin{aligned} D&=\begin{bmatrix} D_{1,1} & \ldots & D_{1,n} \\ \ldots & \ldots & \ldots \\ D_{m,1} & \ldots & D_{m,n} \end{bmatrix} \\ &=\begin{bmatrix} (x^{(1)})^Tx^{(1)}-2(x^{(1)})^Ty^{(1)}+(y^{(1)})^Ty^{(1)} & \ldots & (x^{(1)})^Tx^{(1)}-2(x^{(1)})^Ty^{(n)}+(y^{(n)})^Ty^{(n)} \\ \ldots & \ldots & \ldots \\ (x^{(m)})^Tx^{(m)}-2(x^{(m)})^Ty^{(1)}+(y^{(1)})^Ty^{(1)} & \ldots & (x^{(m)})^Tx^{(m)}-2(x^{(m)})^Ty^{(n)}+(y^{(n)})^Ty^{(n)} \end{bmatrix} \\ &=\begin{bmatrix} (x^{(1)})^Tx^{(1)} & \ldots & (x^{(1)})^Tx^{(1)} \\ \ldots & \ldots & \ldots \\ (x^{(m)})^Tx^{(m)} & \ldots & (x^{(m)})^Tx^{(m)} \end{bmatrix}+\begin{bmatrix} (y^{(1)})^Ty^{(1)} & \ldots & (y^{(n)})^Ty^{(n)} \\ \ldots & \ldots & \ldots \\ (y^{(1)})^Ty^{(1)} & \ldots & (y^{(n)})^Ty^{(n)} \end{bmatrix}-2\begin{bmatrix} (x^{(1)})^Ty^{(1)} & \ldots & (x^{(1)})^Ty^{(n)} \\ \ldots & \ldots & \ldots \\ (x^{(m)})^Ty^{(1)} & \ldots & (x^{(m)})^Ty^{(n)} \end{bmatrix} \\ &=\begin{bmatrix} (x^{(1)})^Tx^{(1)} \\ \ldots \\ (x^{(m)})^Tx^{(m)} \end{bmatrix}\underbrace{\begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}}_{1\times n\text{矩 阵}}+\underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}_{m\times 1\text{矩 阵}}\begin{bmatrix} (y^{(1)})^Ty^{(1)} & \ldots & (y^{(n)})^Ty^{(n)} \end{bmatrix}-2XY^T \end{aligned}$$

所以上述代码如下：

```
d1 = np.sum(X1 ** 2, axis=1).reshape(-1, 1)
d2 = np.sum(X2 ** 2, axis=1).reshape(1, -1)
dist = d1 + d2 - 2 * X1.dot(X2.T)
```

带入高斯核的计算公式可得：

```
K = np.exp(- gamma * dist)
```

回顾22次课件第4页，我们可以利用核矩阵计算β：

$$\beta = (\lambda I + K)^{-1} y \in \mathbb{R}^n$$

记

$$Z = \begin{bmatrix} z_1^T \\ \vdots \\ z_n^T \end{bmatrix} \in \mathbb{R}^{n \times d}, Z' = \begin{bmatrix} (z'_1)^T \\ \vdots \\ (z'_m)^T \end{bmatrix} \in \mathbb{R}^{m \times d}$$

对应核矩阵为$K' = (Z')Z^T \in \mathbb{R}^{m \times n}$

那么

$$w = \sum_{i=1}^{n} \beta_i z_i = Z^T \beta \in \mathbb{R}^d$$

预测结果为

$$\begin{aligned} Z' w &= Z' Z^T \beta \\ &= K' \beta \in \mathbb{R}^m \end{aligned}$$

对应代码如下：

```
# -*- coding: utf-8 -*-
"""
Created on Mon Apr  8 10:27:00 2019

@author: qinzhen
"""

import numpy as np
from scipy.linalg import inv

data = np.genfromtxt('hw2_lssvm_all.dat')

#获得K
def generateK(X1, X2, gamma):
    """
    返回X1，X2
    """
    d1 = np.sum(X1 ** 2, axis=1).reshape(-1, 1)
    d2 = np.sum(X2 ** 2, axis=1).reshape(1, -1)
    dist = d1 + d2 - 2 * X1.dot(X2.T)
    K = np.exp(- gamma * dist)

    return K
```

```python
n = int(data.shape[0] * 0.8)
m = data.shape[0] - n

#划分测试集训练集
trainx = data[:n,:][:, :-1]
trainy = data[:n,:][:, -1]
testx = data[n:,:][:, :-1]
testy = data[n:,:][:, -1]

#初始化参数
Gamma = [32, 2, 0.125]
Lambda = [0.001, 1, 1000]
#记录最优解
gammatrain = Gamma[0]
lambdatrain = Lambda[0]
gammatest = Gamma[0]
lambdatest = Lambda[0]
Ein = 1
Eout = 1

for i in Gamma:
    #计算核矩阵
    K = generateK(trainx, trainx, i)
    K1 = generateK(testx, trainx, i)
    for j in Lambda:
        #计算beta
        beta = inv(np.eye(n)*j + K).dot(trainy)
        #计算预测结果
        y1 = K.dot(beta)
        y2 = K1.dot(beta)
        ein = np.mean(np.sign(y1) != trainy)
        eout = np.mean(np.sign(y2) != testy)
        #更新最优解
        if(ein < Ein):
            Ein = ein
            gammatrain = i
            lambdatrain = j
        if(eout < Eout):
            Eout = eout
            gammatest = i
            lambdatest = j

#### Problem 19
print("minimum Ein =", Ein)
print("gamma =", gammatrain)
print("lambda =", lambdatrain)
```

```
minimum Ein = 0.0
gamma = 32
lambda = 0.001
```

Problem 20

```
#### Problem 20
print("minimum Eout =", Eout)
print("gamma =", gammatest)
print("lambda =", lambdatest)
```

```
minimum Eout = 0.39
gamma = 0.125
lambda = 1000
```

以下两题是证明Adaptive Boosting最终会导致$E_{\text{in}} \to 0$

Problem 21

首先看下题目中的条件，我们知道u_n^t的更新规则为

$$u_n^{t+1} = \begin{cases} u_n^t\sqrt{\frac{1-\epsilon_t}{\epsilon_t}}, & y_n g_t(x_n) = -1 \\ u_n^t/\sqrt{\frac{1-\epsilon_t}{\epsilon_t}}, & y_n g_t(x_n) = 1 \end{cases}$$

这个分段的式子可以合起来写为

$$u_n^{t+1} = u_n^t\left(\sqrt{\frac{1-\epsilon_t}{\epsilon_t}}\right)^{-y_n g_t(x_n)}$$

回顾课件我们知道

$$\begin{aligned} \alpha_t &= \ln\sqrt{\frac{1-\epsilon_t}{\epsilon_t}} \\ e^{\alpha_t} &= \sqrt{\frac{1-\epsilon_t}{\epsilon_t}} \end{aligned}$$

这样可以把上式改写为

$$u_n^{t+1} = u_n^t\left(\sqrt{\frac{1-\epsilon_t}{\epsilon_t}}\right)^{-y_n g_t(x_n)} = u_n^t e^{-y_n \alpha_t g_t(x_n)}$$

把这个式子递推下去可得

$$
\begin{aligned}
u_n^{t+1} &= u_n^t e^{-y_n\alpha_t g_t(x_n)} \\
&= u_n^{t-1} e^{-y_n(\sum_{i=t-1}^{t}\alpha_i g_i(x_n))} \\
&= \ldots \\
&= u_n^1 e^{-y_n(\sum_{i=1}^{t}\alpha_i g_i(x_n))} \\
&= \frac{1}{N} e^{-y_n(\sum_{i=1}^{t}\alpha_i g_i(x_n))}
\end{aligned}
$$

比较题目的的式子

$$
U_{t+1} = \frac{1}{N}\sum_{n=1}^{N}\exp\Big(-y_n\sum_{\tau=1}^{t}\alpha_\tau g_\tau(x_n)\Big)
$$

可得

$$
U_{t+1} = \sum_{n=1}^{N} u_n^{t+1}
$$

现在来证明题目中的结论，利用

$$
\begin{aligned}
u_n^{t+1} &= u_n^t e^{-y_n\alpha_t g_t(x_n)} \\
\epsilon_t &= \frac{\sum_{y_n\neq g_t(x_n)} u_n^t}{\sum_{n=1}^{N} u_n^t} \\
e^{\alpha_t} &= \sqrt{\frac{1-\epsilon_t}{\epsilon_t}}
\end{aligned}
$$

我们可得

$$
\begin{aligned}
U_{t+1} &= \sum_{n=1}^{N} u_n^{t+1} \\
&= \sum_{n=1}^{N} u_n^t e^{-y_n\alpha_t g_t(x_n)} \\
&= \sum_{y_n = g_t(x_n)} u_n^t e^{-\alpha_t} + \sum_{y_n\neq g_t(x_n)} u_n^t e^{\alpha_t} \\
&= \Big(\sum_{n=1}^{N} u_n^t\Big)\Big(e^{-\alpha_t}\frac{\sum_{y_n=g_t(x_n)} u_n^t}{\sum_{n=1}^{N} u_n^t} + e^{\alpha_t}\frac{\sum_{y_n\neq g_t(x_n)} u_n^t}{\sum_{n=1}^{N} u_n^t}\Big) \\
&= \Big(\sum_{n=1}^{N} u_n^t\Big)\Big(e^{-\alpha_t}(1-\epsilon_t) + e^{\alpha_t}\epsilon_t\Big) \\
&= U_t\Big(\sqrt{\frac{\epsilon_t}{1-\epsilon_t}}(1-\epsilon_t) + \sqrt{\frac{1-\epsilon_t}{\epsilon_t}}\epsilon_t\Big) \\
&= 2U_t\sqrt{\epsilon_t(1-\epsilon_t)}
\end{aligned}
$$

因为$\epsilon_t \le \epsilon < \frac{1}{2}$，所以由二次函数的性质可得

$$U_{t+1} = 2U_t\sqrt{\epsilon_t(1-\epsilon_t)} \le 2U_t\sqrt{\epsilon(1-\epsilon)}$$

最后补充证明下$E_{\text{in}}(G_T) \le U_{T+1}$，这里需要利用$G_T(x_n) = \text{sign}\Big(\sum_{\tau=1}^T \alpha_\tau g_\tau(x_n)\Big)$以及$[\![\text{sign}(x) \ne 1]\!] \le e^{-x}$

$$\begin{aligned}
E_{\text{in}}(G_T) &= \frac{1}{N}\sum_{n=1}^N [\![y_n \ne G_T(x_n)]\!] \\
&= \frac{1}{N}\sum_{n=1}^N [\![y_n G_T(x_n) \ne 1]\!] \\
&= \frac{1}{N}\sum_{n=1}^N [\![y_n \text{sign}\Big(\sum_{\tau=1}^T \alpha_\tau g_\tau(x_n)\Big) \ne 1]\!] \\
&= \frac{1}{N}\sum_{n=1}^N [\![\text{sign}\Big(\sum_{\tau=1}^T y_n\alpha_\tau g_\tau(x_n)\Big) \ne 1]\!] \\
&\le \frac{1}{N}\sum_{n=1}^N e^{-y_n\Big(\sum_{\tau=1}^T \alpha_\tau g_\tau(x_n)\Big)}
\end{aligned}$$

注意

$$U_{T+1} = \frac{1}{N}\sum_{n=1}^N e^{-y_n\sum_{\tau=1}^T \alpha_\tau g_\tau(x_n)}$$

所以

$$E_{\text{in}}(G_T) \le U_{T+1}$$

Problem 22

首先把题目给出的条件简单证明下，利用的结论是$1-x \le e^{-x}$

$$\sqrt{\epsilon(1-\epsilon)} = \sqrt{\frac{1}{4} - (\epsilon - \frac{1}{2})^2} = \frac{1}{2}\sqrt{1 - 4(\epsilon-\frac{1}{2})^2} \le \frac{1}{2}\sqrt{e^{-4(\epsilon-\frac{1}{2})^2}} = \frac{1}{2}e^{-2(\epsilon-\frac{1}{2})^2}$$

所以该结论成立。

利用上题$U_{t+1} \le U_t.2\sqrt{\epsilon(1-\epsilon)}, U_1 = 1$可得

$$U_{t+1} \le U_t.2\sqrt{\epsilon(1-\epsilon)} \le U_t e^{-2(\epsilon-\frac{1}{2})^2}$$

$$U_{t+1} \le U_t e^{-2(\epsilon-\frac{1}{2})^2} \le U_{t-1}e^{-2\times 2(\epsilon-\frac{1}{2})^2} \le \ldots \le U_1 e^{-2\times t(\epsilon-\frac{1}{2})^2} = e^{-2t(\epsilon-\frac{1}{2})^2}$$
$$U_{T+1} \le e^{-2T(\epsilon-\frac{1}{2})^2}$$

如果$e^{-2T(\epsilon-\frac{1}{2})^2} < \frac{1}{N}$，那么$E_{\text{in}}(G_T) \le U_{T+1} < \frac{1}{N}$，注意

$$E_{\text{in}}(G_T) = \frac{1}{N}\sum_{n=1}^{N}[\![y_n \neq G_T(x_n)]\!] = \frac{i}{N}$$
$$i = 1, \ldots, N$$

所以此时$E_{\text{in}}(G_T) = 0$，现在解$e^{-2T(\epsilon-\frac{1}{2})^2} < \frac{1}{N}$这个不等式可得

$$e^{-2T(\epsilon-\frac{1}{2})^2} < \frac{1}{N}$$
$$N < e^{2T(\epsilon-\frac{1}{2})^2}$$
$$\ln N < 2T(\epsilon - \frac{1}{2})^2$$
$$T = O(\log N)$$

所以结论成立。