大家好，这篇是有关台大机器学习课程作业五的详解。

我的github地址：
https://github.com/Doraemonzzz

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http://doraemonzzz.com/

作业地址:
https://www.csie.ntu.edu.tw/~htlin/course/ml15fall/

参考资料:
https://blog.csdn.net/a1015553840/article/details/51085129
http://www.vynguyen.net/category/study/machine-learning/page/6/
http://book.caltech.edu/bookforum/index.php
http://beader.me/mlnotebook/
https://blog.csdn.net/qian1122221/article/details/50130093

Problem 1

回顾下问题的形式

$$\min_{w,b,\xi} : \frac{1}{2} w^T w + C \sum_{n=1}^{N} \xi_n$$
$$\text{subject to: } y_n (w^T x_n + b) \ge 1 - \xi_n$$
$$\xi_n \ge 0 (n = 1, \ldots, N)$$

x_n, y_n为变量，C为人为设定的超参数，其余量均为参数，注意$w \in \mathbb{R}^d$，所以我们的参数有

$$w = (w_1, \ldots, w_d), \xi_1, \ldots, \xi_N, d$$

一共$d + N + 1$个。限制条件为

$$y_n (w^T x_n + b) \ge 1 - \xi_n$$
$$\xi_n \ge 0 (n = 1, \ldots, N)$$

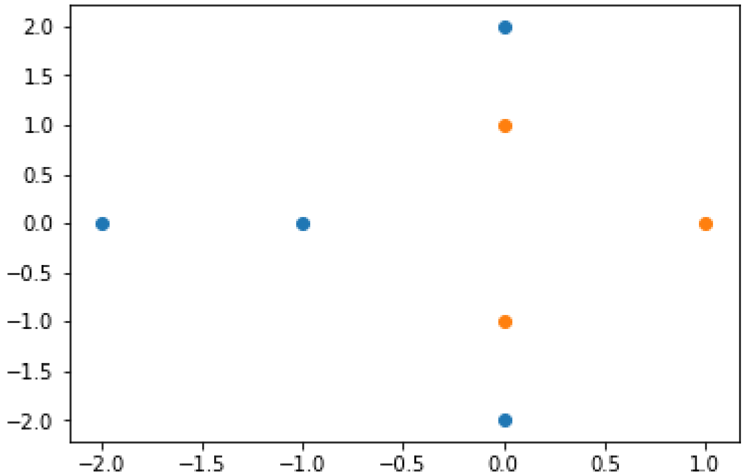
一共有$2N$个。

Problem 2

首先作图看下

```
# -*- coding: utf-8 -*-
"""
Created on Sat Mar 23 13:11:34 2019

@author: qinzhen
```

```
"""

import numpy as np
import matplotlib.pyplot as plt
from sklearn import svm

####Problem 2

#原始图
x = np.array([[1, 0],[0, 1],[0, -1],[-1, 0],[0, 2],[0, -2],[-2, 0]])
z = np.array([-1, -1, -1, +1, +1, +1, +1])

x1 = x[z>0][:, 0]
y1 = x[z>0][:, 1]
x2 = x[z<0][:, 0]
y2 = x[z<0][:, 1]

plt.scatter(x1,y1)
plt.scatter(x2,y2)
plt.show()
```



可以看到，如果用二次曲线的话，应该可以分类，现在做特征转换之后的图像。

转换之后的标记为+1的点为

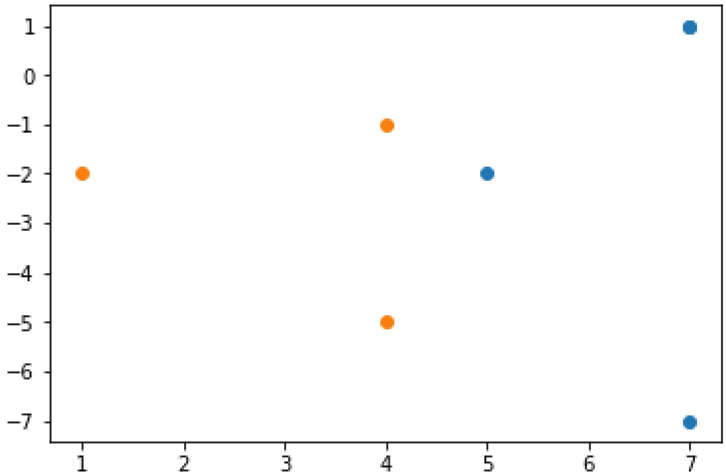
$$z_4=(5,-2),z_5=(7,-7),z_6=(7,1),z_7=(7,1)$$

转换之后的标记为+1的点为

$$z_1=(1,-2),z_2=(4,-5),z_3=(4,-1)$$

```
#特征转换之后的图
def phi_1(x):
    return x[1] ** 2 - 2 * x[0] + 3

def phi_2(x):
    return x[0] ** 2 - 2 * x[1] - 3

X = []
```

```python
for i in x:
    X.append([phi_1(i), phi_2(i)])
X = np.array(X)

X1 = X[z>0][:, 0]
Y1 = X[z>0][:, 1]
X2 = X[z<0][:, 0]
Y2 = X[z<0][:, 1]

plt.scatter(X1,Y1)
plt.scatter(X2,Y2)
plt.show()
```

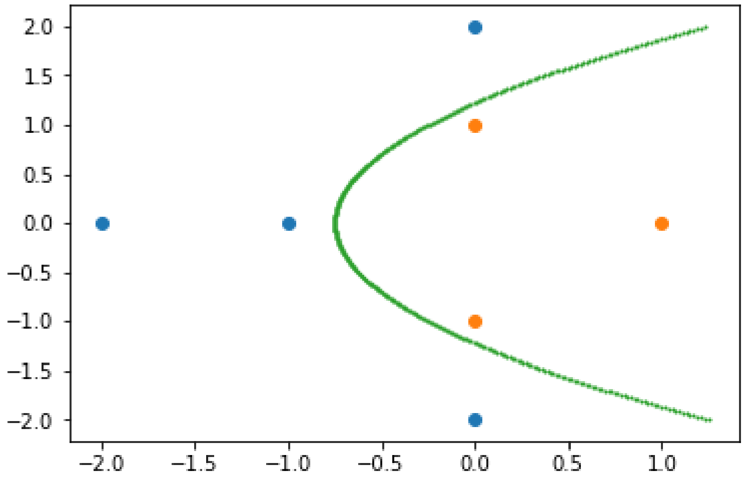


从这个图像中可以看出，最大间隔分类器为$\phi_1(x) = 4.5$，将$\phi_1(x) = x_2^2 - 2x_1 + 3$带入可得最大间隔分类器为

$$x_2^2 - 2x_1 + 3 = 4.5$$
$$x_1 = \frac{x_2^2 - 1.5}{2}$$

最后看下曲线的图。

```python
#曲线图
y3 = np.arange(-2, 2, 0.01)
x3 = np.array([(i * i - 1.5) / 2 for i in y3])

plt.scatter(x1, y1)
plt.scatter(x2, y2)
plt.scatter(x3, y3, s=1)
plt.show()
```



Problem 3

利用sklearn处理即可，这里设置参数shrinking=False，参数含义如下：

https://stats.stackexchange.com/questions/24414/svm-options-in-scikit-learn

```
clf = svm.SVC(kernel='poly', degree=2, coef0=1, gamma=1, C=1e10, shrinking=False)
clf.fit(x, z)
```

```
SVC(C=10000000000.0, cache_size=200, class_weight=None, coef0=1,
  decision_function_shape='ovr', degree=2, gamma=1, kernel='poly',
  max_iter=-1, probability=False, random_state=None, shrinking=False,
  tol=0.001, verbose=False)
```

看下哪几个向量为支持向量。

```
clf.support_
```

```
array([1, 2, 3, 4, 5])
```

这说明第2到6个向量均为支持向量，再来看下对偶问题的系数。

```
clf.dual_coef_
```

```
array([[-0.64491963, -0.76220325,  0.88870349,  0.22988879,  0.2885306 ]])
```

这些系数分别支持向量对应的系数，非支持向量对应的系数为0，这里还要注意一点，对偶问题的系数为$y_n\alpha_n$，所以如果我们要得到原系数，就要乘以y_n

```
z[clf.support_] * clf.dual_coef_[0]
```

```
array([0.64491963, 0.76220325, 0.88870349, 0.22988879, 0.2885306 ])
```

所以

$$(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7) = (0, 0.64491963, 0.76220325, 0.88870349, 0.22988879, 0.2885306, 0)$$

支持向量为

$$x_2, x_3, x_4, x_5, x_6$$

Problem 4

为了求得曲线方程，我们需要利用以下几个式子

$$\begin{aligned} w &= \sum_{\alpha_n^*>0} y_n \alpha_n^* z_n^T \\ b &= y_s - w^T z_s \\ g(x) &= \text{sign}\Big(\sum_{\alpha_n^*>0} y_n \alpha_n^* K(x_n, x) + b\Big) \end{aligned}$$

注意

$$K(x_n, x) = (1 + x_n^T x)^2$$

首先计算b，可以直接获得：

```
b = clf.intercept_[0]
b
```

```
-1.6663314053609206
```

接着计算$K(x_n,x) = (1+x_n^T x)^2$，首先找出支持向量：

```
x[clf.support_]
```

```
array([[ 0,  1],
       [ 0, -1],
       [-1,  0],
       [ 0,  2],
       [ 0, -2]])
```

然后计算上式，记

$$x = (x^{(1)}, x^{(2)})$$

那么每一项分别为：

$$
\begin{gathered}
(1+x^{(2)})^2 \\
(1-x^{(2)})^2 \\
(1-x^{(1)})^2 \\
(1+2x^{(2)})^2 \\
(1-2x^{(2)})^2
\end{gathered}
$$

最后计算$\alpha_n y_n$，直接调用即可：

```
clf.dual_coef_[0]
```

```
array([-0.64491963, -0.76220325,  0.88870349,  0.22988879,  0.2885306 ])
```

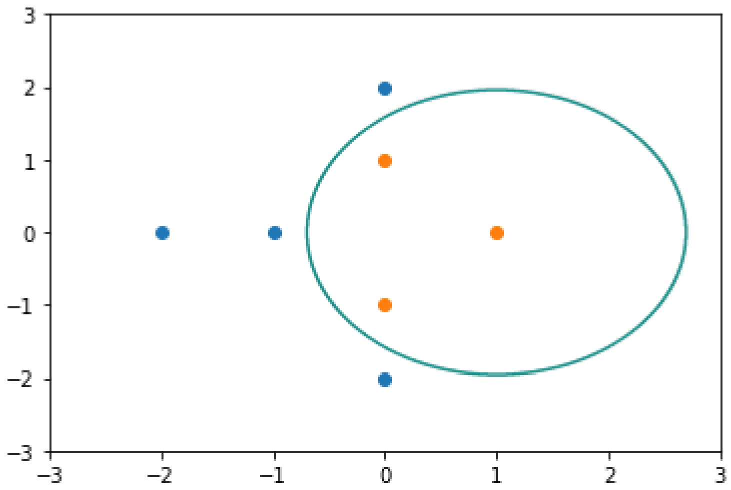
所以曲线方程为：

$$
-0.64491963(1+x^{(2)})^2-0.76220325(1-x^{(2)})^2+0.88870349(1-x^{(1)})^2+0.22988879(1+2x^{(2)})^2+0.2885306(1-2x^{(2)})^2-1.6663314053609206=0
$$

Problem 5

这里作图来看：

```
#点的数量
n = 1000
r = 3

#作点
a = np.linspace(-r, r, n)
b = np.linspace(-r, r, n)

#构造网格
A, B = np.meshgrid(a, b)
X = np.c_[A.reshape(-1, 1), B.reshape(-1, 1)]
label = np.reshape(clf.predict(X), A.shape)

#绘制等高线
plt.contour(A, B, label, 0)

plt.scatter(x1, y1)
plt.scatter(x2, y2)
plt.show()
```



可以看到，图像和之前的抛物线是不一样的。

Problem 6

结合课件的推导，我们可知

$$L(R, c, \lambda) = R^2 + \sum_{n=1}^{N} \lambda_n (||x_n - c||^2 - R^2)$$
$$\lambda_n \geq 0$$

所以在$||x_n - c||^2 \leq R^2$条件下

$$\sum_{n=1}^{N} \lambda_n (||x_n - c||^2 - R^2) \leq 0$$
$$\max\{\sum_{n=1}^{N} \lambda_n (||x_n - c||^2 - R^2)\} = 0$$

从而

$$\min_{R \in \mathbb{R}, c \in \mathbb{R}^d} \max_{\lambda_n \geq 0} L(R, c, \lambda) = \min_{R \in \mathbb{R}, c \in \mathbb{R}^d} R^2$$

Problem 7

将上述问题转化为对偶问题

$$\min_{R \in \mathbb{R}, c \in \mathbb{R}^d} \max_{\lambda_n \geq 0} L(R, c, \lambda) = \max_{\lambda_n \geq 0} \min_{R \in \mathbb{R}, c \in \mathbb{R}^d} L(R, c, \lambda)$$

所以现在可以对$L(R, c, \lambda)$求无条件极值，分别求偏导可得

$$
\begin{aligned}
\frac{\partial L(R,c,\lambda)}{\partial c} &= \frac{\partial[R^2+\sum_{n=1}^N \lambda_n(||x_n-c||^2-R^2)]}{\partial c} \\
&= \frac{\partial[R^2+\sum_{n=1}^N \lambda_n(x_n^T x_n-2x_n^T c+c^T c-R^2)]}{\partial c} \\
&= \frac{\partial[R^2+\sum_{n=1}^N \lambda_n(x_n^T x_n-2x_n^T c+c^T c-R^2)]}{\partial c} \\
&= \sum_{n=1}^N \lambda_n \frac{\partial(x_n^T x_n-2x_n^T c+c^T c-R^2)}{\partial c} \\
&= \sum_{n=1}^N \lambda_n(2c-2x_n) \\
&= 2(c\sum_{n=1}^N \lambda_n-\sum_{n=1}^N \lambda_n x_n)
\end{aligned}
$$

令$\frac{\partial L(R,c,\lambda)}{\partial c}=0$可得

$$
c\sum_{n=1}^N \lambda_n-\sum_{n=1}^N \lambda_n x_n=0
$$

如果

$$
\sum_{n=1}^N \lambda_n \neq 0
$$

那么

$$
c=\frac{\sum_{n=1}^N \lambda_n x_n}{\sum_{n=1}^N \lambda_n}
$$

接着关于R求偏导

$$
\begin{aligned}
\frac{\partial L(R,c,\lambda)}{\partial R} &= \frac{\partial[R^2+\sum_{n=1}^N \lambda_n(||x_n-c||^2-R^2)]}{\partial R} \\
&= 2R-2\sum_{n=1}^N \lambda_n R
\end{aligned}
$$

令$\frac{\partial L(R,c,\lambda)}{\partial R}=0$可得

$$
\sum_{n=1}^N \lambda_n=1
$$

结合这个条件，关于c的条件可以简化

$$c=\frac{\sum_{n=1}^N \lambda_n x_n}{\sum_{n=1}^N \lambda_n}=\sum_{n=1}^N \lambda_n x_n$$

Problem 8

将$\sum_{n=1}^N \lambda_n=1, c=\sum_{n=1}^N \lambda_n x_n$这两个条件带入$L(R,c,\lambda)$，先带入$\sum_{n=1}^N \lambda_n=1$

$$\begin{aligned}L(R,c,\lambda)&=R^2+\sum_{n=1}^N \lambda_n(||x_n-c||^2-R^2)\\&=\sum_{n=1}^N \lambda_n||x_n-c||^2+R^2-R^2\sum_{n=1}^N \lambda_n\\&=\sum_{n=1}^N \lambda_n||x_n-c||^2\end{aligned}$$

再对$\sum_{n=1}^N \lambda_n||x_n-c||^2$进行处理可得

$$\begin{aligned}L(R,c,\lambda)&=\sum_{n=1}^N \lambda_n||x_n-c||^2\\&=\sum_{n=1}^N \lambda_n(x_n^Tx_n-2x_n^Tc+c^Tc)\\&=\sum_{n=1}^N \lambda_n x_n^Tx_n-2(\sum_{n=1}^N \lambda_n x_n^T)c+c^Tc\sum_{n=1}^N \lambda_n\\&=\sum_{n=1}^N \lambda_n x_n^Tx_n-2(\sum_{n=1}^N \lambda_n x_n^T)c+c^Tc\end{aligned}$$

再带入$c=\sum_{n=1}^N \lambda_n x_n$

$$\begin{aligned}L(R,c,\lambda)&=\sum_{n=1}^N \lambda_n x_n^Tx_n-2c\sum_{n=1}^N \lambda_n x_n^T+c^Tc\\&=\sum_{n=1}^N \lambda_n x_n^Tx_n-2(\sum_{n=1}^N \lambda_n x_n^T)(\sum_{n=1}^N \lambda_n x_n)+(\sum_{n=1}^N \lambda_n x_n^T)(\sum_{n=1}^N \lambda_n x_n)\\&=\sum_{n=1}^N \lambda_n x_n^Tx_n-(\sum_{n=1}^N \lambda_n x_n^T)(\sum_{n=1}^N \lambda_n x_n)\end{aligned}$$

所以问题转换为

$$\text{在条件} \sum_{n=1}^{N} \lambda_n = 1, \lambda_n \geq 0 \text{下，}$$

$$\text{最小化} f(\lambda) = \sum_{n=1}^{N} \lambda_n x_n^T x_n - (\sum_{n=1}^{N} \lambda_n x_n^T)(\sum_{n=1}^{N} \lambda_n x_n)$$

Problem 9

这题是要利用$z_n = \phi(x_n)$以及$K(x_n, x_m)$来简化问题，先对$f(\lambda)$进行处理

$$\begin{aligned} f(\lambda) &= \sum_{n=1}^{N} \lambda_n x_n^T x_n - (\sum_{n=1}^{N} \lambda_n x_n^T)(\sum_{n=1}^{N} \lambda_n x_n) \\ &= \sum_{n=1}^{N} \lambda_n x_n^T x_n - \sum_{n=1}^{N} \sum_{m=1}^{N} \lambda_n \lambda_m x_n^T x_m \end{aligned}$$

将x_n替换为$z_n = \phi(x_n)$，然后代入$K(x_n, x_m) = z_n^T z_m$可得

$$\begin{aligned} f(\lambda) &= \sum_{n=1}^{N} \lambda_n z_n^T z_n - \sum_{n=1}^{N} \sum_{m=1}^{N} \lambda_n \lambda_m z_n^T z_m \\ &= \sum_{n=1}^{N} \lambda_n K(x_n, x_n) - \sum_{n=1}^{N} \sum_{m=1}^{N} \lambda_n \lambda_m K(x_n, x_m) \end{aligned}$$

Problem 10

由Problem 6的推导过程我们可知，

$$\lambda_n(||x_n - c||^2 - R^2) = 0$$
$$n = 1, \ldots, N$$

这里讨论的是特征转换之后的问题，所以上述问题可以修改为

$$\lambda_n(||z_n - c||^2 - R^2) = 0$$
$$n = 1, \ldots, N$$

所以如果存在$\lambda_i \neq 0$，那么

$$||z_i - c||^2 - R^2 = 0$$

结合$c = \sum_{n=1}^{N} \lambda_n z_n$，可得

$$
\begin{aligned}
R^2 &= ||z_i - c||^2 \\
&= (z_i - \sum_{n=1}^{N} \lambda_n z_n)^T (z_i - \sum_{n=1}^{N} \lambda_n z_n) \\
&= z_i^T z_i - 2\sum_{n=1}^{N} \lambda_n z_n^T z_i + \sum_{n=1}^{N} \sum_{m=1}^{N} \lambda_n \lambda_m z_n^T z_m \\
&= K(x_i, x_i) - 2\sum_{n=1}^{N} \lambda_n K(x_n, x_i) + \sum_{n=1}^{N} \sum_{m=1}^{N} \lambda_n \lambda_m K(x_n, x_m)
\end{aligned}
$$

Problem 11

首先看下现在的问题，假设假设$x_n \in \mathbb{R}^k$

$$
\min_{w,b,\xi} \ \frac{1}{2} w^T w + C \sum_{n=1}^{N} \xi_n^2 \\
\text{subject to } y_n(w^T x_n + b) \ge 1 - \xi_n
$$

转换成hard-margin的关键问题在于把$\frac{1}{2} w^T w + C \sum_{n=1}^{N} \xi_n^2$写成$\frac{1}{2} \tilde{w}^T \tilde{w}$

$$
\frac{1}{2} w^T w + C \sum_{n=1}^{N} \xi_n^2 = \frac{1}{2}(w^T w + 2C \sum_{n=1}^{N} \xi_n^2)
$$

从这点可以想到

$$
\tilde{w} = \begin{bmatrix} w \\ \sqrt{2C}\xi_1 \\ \ldots \\ \sqrt{2C}\xi_N \end{bmatrix}
$$

从而

$$
\tilde{w}^T \tilde{w} = w^T w + 2C \sum_{n=1}^{N} \xi_n^2
$$

这样变换之后也要对条件进行变换，将条件化为$y_n(w^T \tilde{x}_n + b) \ge 1$的形式

$$
y_n(w^T x_n + b) \ge 1 - \xi_n \Leftrightarrow \\
y_n(w^T x_n + b + y_n \xi_n) \ge 1 \Leftrightarrow \\
y_n(w^T x_n + \sqrt{2C}\xi_n(\frac{1}{\sqrt{2C}} y_n) + b) \ge 1
$$

结合$\tilde{w}$的式子，我们可以定义$\tilde{x}_n$，注意$x_n \in \mathbb{R}^k$

$$
\tilde{x}_n = \begin{bmatrix} x_n \\ 0 \\ \ldots \\ \frac{1}{\sqrt{2C}} y_n \\ \ldots \\ 0 \end{bmatrix} \in \mathbb{R}^{k+N}
$$

其中$\tilde{x}_n$的第$k+1$到$k+N$的分量中，除了第$k+n$个为$\frac{1}{\sqrt{2C}}y_n$，其余均为0，这样就把问题转化为

$$
\min_{w,b,\xi} \frac{1}{2}\tilde{w}^T\tilde{w}
$$
$$
\text{subject to } y_n(\tilde{w}^T x_n + b) \ge 1
$$

如果已经计算出了$\tilde{w}$，那么由于

$$
\tilde{w} = \begin{bmatrix} w \\ \sqrt{2C}\xi_1 \\ \ldots \\ \sqrt{2C}\xi_N \end{bmatrix}
$$

所以取$\tilde{w}$的前k个分量即可得到w。

Problem 12

只要看这些映射对应的Gram矩阵是否半正定即可，设K, K_1, K_2对应的Gram矩阵为M, M_1, M_2

(a)$K(x, x^{'}) = K_1(x, x^{'}) + K_2(x, x^{'})$可得$M = M_1 + M_2$

$$
y^T M y = y^T(M_1 + M_2)y = y^T M_1 y + y^T M_2 y \ge 0
$$

所以$K(x, x^{'}) = K_1(x, x^{'}) + K_2(x, x^{'})$为kernerl

(b)$K(x, x^{'}) = K_1(x, x^{'}) - K_2(x, x^{'})$可得$M = M_1 - M_2$

这个一看就不是kernel，反例也很好构造

$$
M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, M = M_1 - M_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}
$$

显然M不是半正定的，所以$K(x, x^{'}) = K_1(x, x^{'}) - K_2(x, x^{'})$不是kernel

(c)首先计算下式子的形式。

$$
\begin{aligned}
K(x,x^{'}) = K_1(x,x^{'})K_2(x,x^{'}) &= \phi_1(x)^T\phi_1(x^{'})\phi_2(x)^T\phi_2(x^{'}) \\
&= \sum_{i=1}^{n}\phi_1^i(x)\phi_1^i(x^{'})\sum_{j=1}^{n}\phi_2^j(x)\phi_2^j(x^{'}) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\phi_1^i(x)\phi_1^i(x^{'})\phi_2^j(x)\phi_2^j(x^{'}) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}(\phi_1^i(x)\phi_2^j(x))(\phi_1^i(x^{'})\phi_2^j(x^{'}))
\end{aligned}
$$

根据这个形式，可以做以下定义

$$
\Phi^i(x) = \phi_1^i(x)(\phi_2^1(x),\ldots,\phi_2^n(x)) \\
\Phi(x) = (\Phi^1(x),\ldots,\Phi^n(x))^T
$$

我们来计算下这个式子

$$
(\Phi^i(x))^T(\Phi^i(x^{'})) = \sum_{j=1}^{n}(\phi_1^i(x)\phi_2^j(x))(\phi_1^i(x^{'})\phi_2^j(x^{'})) \\
(\Phi(x))^T\Phi(x^{'}) = \sum_{i=1}^{n}(\Phi^i(x))^T(\Phi^i(x^{'})) = \sum_{i=1}^{n}\sum_{j=1}^{n}(\phi_1^i(x)\phi_2^j(x))(\phi_1^i(x^{'})\phi_2^j(x^{'})) = K(x,x^{'})
$$

根据核函数的定义可知，$K(x,x^{'}) = K_1(x,x^{'})K_2(x,x^{'})$为kernel

(d)$K(x,x^{'}) = K_1(x,x^{'})/K_2(x,x^{'})$

这个也不是kernel，反例如下

$$
M_1 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, M_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, M = M_1/M_2 = \begin{bmatrix} \frac{1}{2} & 2 \\ 2 & 3 \end{bmatrix}
$$

M行列式小于0，所以必然不是半正定的，所以$K(x,x^{'}) = K_1(x,x^{'})/K_2(x,x^{'})$不是kernel

所以这题答案为(a)(c)。

Problem 13

设K, K_1对应的Gram矩阵为M, M_1

(a)

$$
M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, M = \begin{bmatrix} (1-1)^2 & (1-0)^2 \\ (1-0)^2 & (1-1)^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
$$

M的行列式小于0，所以$K(x,x^{'}) = (1-K_1(x,x^{'}))^2$不是kernel

(b)

$$
M = 1126M_1
$$

因为M_1半正定，所以M半正定，从而$K(x,x^{'})=1126K_1(x,x^{'})$是kernel

(c)

$$M_1=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},M=\begin{bmatrix}e^{-1} & 1\\1 & e^{-1}\end{bmatrix}$$

M的行列式小于0，所以$K(x,x^{'})=\exp(-K_1(x,x^{'}))$不是kernel

(d)利用$f(x)=\frac{1}{1-x}$在$(0,1)$区间的泰勒展开可得

$$\begin{aligned}K(x,x^{'})&=(1-K_1(x,x^{'}))^{-1}\\&=\sum_{i=0}^{+\infty}(K_1(x,x^{'}))^i\end{aligned}$$

由上题的(b)我们知道$K_1(x,x^{'})^i$为kernel，再有上题的(a)我们知道kernel的和也为kernel，所以

$$K(x,x^{'})=\sum_{i=0}^{+\infty}(K_1(x,x^{'}))^i$$

也为kernel

Problem 14

首先回顾下对偶问题对应的QP问题

$$\begin{gathered}\underset{\alpha\in R^N}{\text{minimize}}:\frac{1}{2}\alpha^TQ_D\alpha-1_N^T\alpha\\\text{subject to: }A_D\alpha\ge u\end{gathered}$$

$$Q_D=\begin{bmatrix}y_1y_1K_{11} & \ldots & y_1y_NK_{1N}\\y_2y_1K_{12} & \ldots & y_2y_NK_{2N}\\\ldots & \ldots & \ldots\\y_Ny_1K_{N1} & \ldots & y_Ny_NK_{NN}\end{bmatrix},A_D=\begin{bmatrix}y^T\\-y^T\\I_{N\times N}\\-I_{N\times N}\end{bmatrix},u=\begin{bmatrix}0\\0\\0_{N\times N}\\C\times I_{N\times N}\end{bmatrix}$$

我们来看如果用新的kernel，$\tilde{K}(x,x^{'})=pK(x,x^{'})+q$，并且$\tilde{C}=\frac{C}{p}$会产生什么情况：

$$
\begin{aligned}
\tilde{Q}_D &= \begin{bmatrix} y_1y_1\tilde{K}_{11} & \dots & y_1y_N\tilde{K}_{1N} \\ y_2y_1\tilde{K}_{12} & \dots & y_2y_N\tilde{K}_{2N} \\ \dots & \dots & \dots \\ y_Ny_1\tilde{K}_{N1} & \dots & y_Ny_N\tilde{K}_{NN} \end{bmatrix} \\
&= \begin{bmatrix} y_1y_1(pK_{11}+q) & \dots & y_1y_N(pK_{1N}+q) \\ y_2y_1(pK_{12}+q) & \dots & y_2y_N(pK_{2N}+q) \\ \dots & \dots & \dots \\ y_Ny_1(pK_{N1}+q) & \dots & y_Ny_N(pK_{NN}+q) \end{bmatrix} \\
&= pQ_D + q(y_1,\dots,y_N)^T(y_1,\dots,y_N)
\end{aligned}
$$

$$
\tilde{A}_D = A_D
$$

$$
\tilde{u} = \begin{bmatrix} 0 \\ 0 \\ 0_{N\times N} \\ \tilde{C}\times I_{N\times N} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0_{N\times N} \\ \frac{C}{p}\times I_{N\times N} \end{bmatrix} = \frac{1}{p}u
$$

接着我们来计算$\alpha^T\tilde{Q}_D\alpha$，注意

$$
(y_1,\dots,y_N).\alpha = 0
$$

所以

$$
\alpha^T\tilde{Q}_D\alpha = \alpha^T pQ_D\alpha + \alpha^T q(y_1,\dots,y_N)^T(y_1,\dots,y_N)\alpha = p\alpha^T Q_D\alpha
$$

所以我们的目标函数为

$$
\frac{1}{2}\alpha^T\tilde{Q}_D\alpha - 1_N^T\alpha = \frac{1}{2}p\alpha^T\tilde{Q}_D\alpha - 1_N^T\alpha = \frac{1}{p}[\frac{1}{2}(p\alpha)^T Q_D(p\alpha) - 1_N^T(p\alpha)]
$$

由于p为常数，所以目标函数可以简化为

$$
\frac{1}{2}(p\alpha)^T Q_D(p\alpha) - 1_N^T(p\alpha)
$$

再看下限制条件

$$
\tilde{A}_D\alpha \ge \tilde{u} \Leftrightarrow A_D\alpha \ge \frac{1}{p}u \Leftrightarrow A_D(p\alpha) \ge u
$$

我们令$\overline{\alpha} = p\alpha$，那么问题可以转换为

$$
\begin{gathered}
\underset{\alpha \in R^N}{\text{minimize}}: \frac{1}{2}\overline{\alpha}^T Q_D \overline{\alpha} - 1_N^T \overline{\alpha} \\
\text{subject to: } A_D \overline{\alpha} \ge u \\
Q_D = \begin{bmatrix} y_1 y_1 K_{11} & \ldots & y_1 y_N K_{1N} \\ y_2 y_1 K_{12} & \ldots & y_2 y_N K_{2N} \\ \ldots & \ldots & \ldots \\ y_N y_1 K_{N1} & \ldots & y_N y_N K_{NN} \end{bmatrix}, A_D = \begin{bmatrix} y^T \\ -y^T \\ I_{N\times N} \\ -I_{N\times N} \end{bmatrix}, u = \begin{bmatrix} 0 \\ 0 \\ 0_{N\times N} \\ C \times I_{N\times N} \end{bmatrix}
\end{gathered}
$$

可以看出，这个问题和原问题是一致的，记原问题的最优解为α^*，那么该问题的最优解$\tilde{\alpha}^*$满足以下条件

$$
\alpha^* = p\tilde{\alpha}^*
$$

我们根据这个条件开始讨论问题。

回顾下soft-margin得到的计算公式

$$
\begin{gathered}
w^* = \sum_{n=1}^{N} \alpha_n^* y_n z_n \\
\beta_n^* = C - \alpha_n^* (n = 1, \ldots, N) \\
b^* = y_m - w^{*T} z_m = y_m - (\sum_{n=1}^{N} \alpha_n^* y_n z_n)^T z_m = y_m - \sum_{n=1}^{N} \alpha_n^* y_n K(x_n, x_m) \\
\text{其中} z_m \text{对应的} \alpha_m^* \ne 0 \\
g(x) = \text{sign}(w^{*T} z + b) = \text{sign}((\sum_{n=1}^{N} \alpha_n^* y_n z_n)^T z + b) = \text{sign}(\sum_{n=1}^{N} \alpha_n^* y_n K(x_n, x) + b^*)
\end{gathered}
$$

如果我们将kernel换成

$$
\tilde{K}(x, x^{'}) = pK(x, x^{'}) + q
$$

利用之前的条件$\alpha^* = p\tilde{\alpha}^*$以及$\tilde{C} = \frac{C}{p}, \sum_{n=1}^{N} \alpha_n^* y_n = 0$，可以对变换kernel之后的问题求解

$$
\begin{gathered}
\tilde{w}^* = \sum_{n=1}^{N} \tilde{\alpha}_n^* y_n z_n = \frac{1}{p}(\sum_{n=1}^{N} \alpha_n^* y_n z_n) = \frac{w^*}{p} \\
\tilde{\beta}_n^* = \tilde{C} - \tilde{\alpha}_n^* = \frac{C}{p} - \frac{\alpha_n^*}{p} = \frac{1}{p}(C - \alpha_n^*) = \frac{\beta_n}{p} (n = 1, \ldots, N)
\end{gathered}
$$

$$
\begin{aligned}
\tilde{b}^* &= y_m - \sum_{n=1}^{N} \tilde{\alpha}_n^* y_n \tilde{K}(x_n, x_m) \\
&= y_m - \sum_{n=1}^{N} \frac{\alpha_n^*}{p} y_n (pK(x_n, x_m) + q) \\
&= y_m - \sum_{n=1}^{N} \alpha_n^* y_n K(x_n, x_m) - \frac{q}{p} \sum_{n=1}^{N} \alpha_n^* y_n \\
&= y_m - \sum_{n=1}^{N} \alpha_n^* y_n K(x_n, x_m) \\
&= b^*
\end{aligned}
$$

$$
\begin{aligned}
\tilde{g}(x) &= \text{sign}(\sum_{n=1}^{N} \tilde{\alpha}_n^* y_n \tilde{K}(x_n, x) + \tilde{b}^*) \\
&= \text{sign}[\sum_{n=1}^{N} \frac{\alpha_n^*}{p} y_n (pK(x_n, x) + q) + b^*] \\
&= \text{sign}[\sum_{n=1}^{N} \alpha_n^* y_n K(x_n, x) + \frac{q}{p} \sum_{n=1}^{N} \alpha_n^* y_n + b^*] \\
&= \text{sign}(\sum_{n=1}^{N} \alpha_n^* y_n K(x_n, x) + b^*) \\
&= g(x)
\end{aligned}
$$

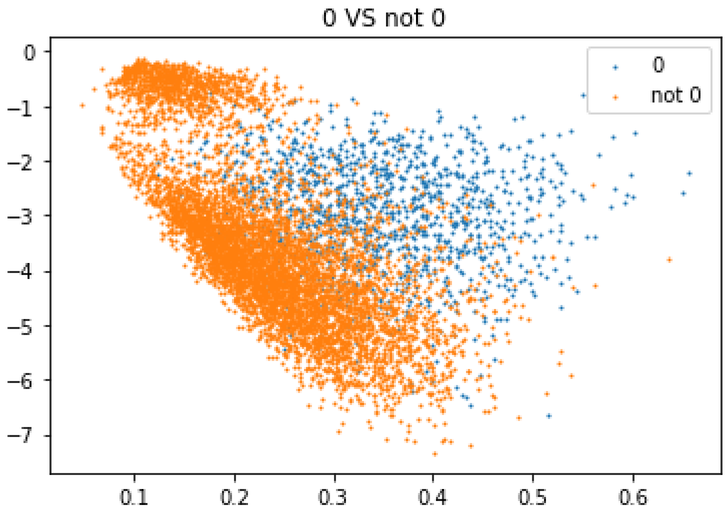
这说明这样变换之后我们的分类器没有变。

Problem 15

首先读取数据并作图

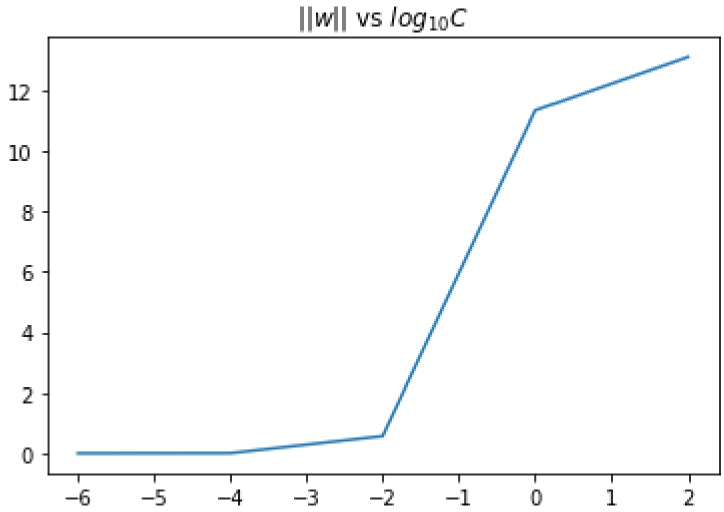
```python
# -*- coding: utf-8 -*-
"""
Created on Wed Mar 27 13:42:18 2019

@author: qinzhen
"""

####Problem 15
import matplotlib.pyplot as plt
import numpy as np
from sklearn import svm

def transformdata(file):
    data = np.genfromtxt(file)
    y, X = data[:, 0], data[:, 1:]
    return X, y
```

```
train = "featurestrain.txt"
X_train, y_train = transformdata(train)
test = "featurestest.txt"
X_test,y_test = transformdata(test)

#作图
plt.scatter(X_train[y_train == 0][:, 0], X_train[y_train == 0][:, 1], s=1, label='0')
plt.scatter(X_train[y_train != 0][:, 0], X_train[y_train != 0][:, 1], s=1, label='not 0')
plt.title('0 VS not 0')
plt.legend()
plt.show()
```



接着训练模型并作图：

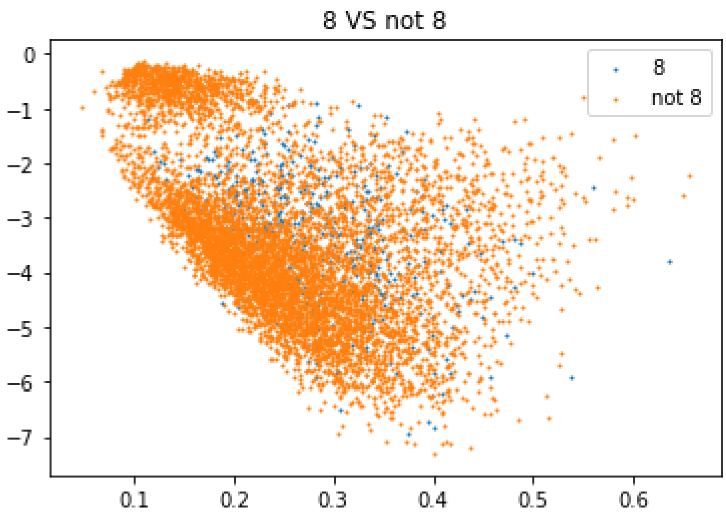
```
#训练模型并作图
y_train_1 = (y_train==0).astype("int")
C = [-6, -4, -2, 0, 2]
W = []

for i in C:
    c = 10 ** i
    clf = svm.SVC(kernel="linear", C=c)
    clf.fit(X_train, y_train_1)
    w = clf.coef_
    W.append(np.linalg.norm(w))

plt.plot(C, W)
plt.title("$||w||$ vs $log_{10}C$")
plt.show()
```



Problem 16

注意这题是8和not 8，首先作图

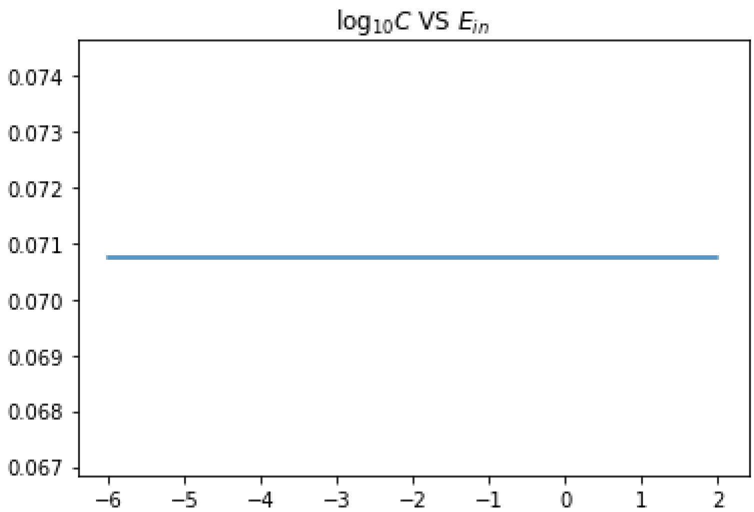
```
####Problem 16
plt.scatter(X_train[y_train == 8][:, 0], X_train[y_train == 8][:, 1], s=1, label='8')
plt.scatter(X_train[y_train != 8][:, 0], X_train[y_train != 8][:, 1], s=1, label='not 8')
plt.title('8 VS not 8')
plt.legend()
plt.show()
```



注意17题要计算$\sum_{n=1}^{N}\alpha_n$，所以我们这题把系数也算出来，由于sklearn只能计算对偶系数$y_n\alpha_n$，所以这里要把标签转换为$+1,-1$，方便计算。

```
#训练
y_train_8 = 2 * (y_train==8).astype("int") - 1
C = [-6, -4, -2, 0, 2]
Ein = []
alpha = []
for i in C:
```

```
    c = 10 ** i
    clf = svm.SVC(kernel='poly', degree=2, coef0=1, gamma=1, C=c)
    clf.fit(X_train, y_train_8)
    e = np.mean(clf.predict(X_train) != y_train_8)
    #支持向量的索引
    support = clf.support_
    #计算系数
    coef = np.sum(clf.dual_coef_[0] * y_train_8[support])
    alpha.append(coef)
    Ein.append(e)

#作图
plt.plot(C,Ein)
plt.show()
```

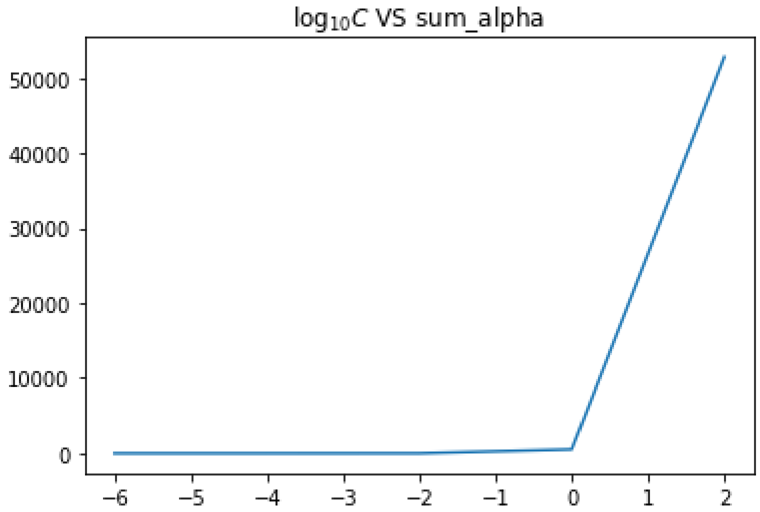


比较神奇的是这题的E_{in}都一样。

Problem 17

利用上题的数据作图即可。

```
####Problem 17
plt.plot(C, alpha)
plt.title("$\log_{10}C$ VS sum_alpha")
plt.show()
```



可以看到尽管上题的E_{in}一致，但是这里$\sum_{n=1}^{N}\alpha_n$会增加。

Problem 18

这题需要计算权重w，因为涉及到高斯核，无法直接计算。

现在假设有n个样本，那么：

$$w=\sum_{i=1}^{n}\alpha_i y_i z_i, z_i=\Phi\left(x_i\right)$$
$$\|w\|^2=\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i y_i \alpha_j y_j z_i^T z_j=\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i y_i \alpha_j y_j z_i^T z_j K\left(x_i, x_j\right)$$

其中

$$K(x_i, x_j)=\exp\left(-\gamma||x_i-x_j||^2\right)$$

记

$$\bar{y}=\begin{bmatrix}\alpha_1 y_1\\ \vdots\\ \alpha_n y_n\end{bmatrix}\in\mathbb{R}^n, D\in\mathbb{R}^{n\times n}, D_{ij}=||x_i-x_j||^2$$

那么

$$K=\exp(-\gamma D)$$

并且

$$\|w\|^2=\bar{y}^T K\bar{y}$$

注意只有支持向量部分的系数非零，所以上述计算步骤只要对支持向量进行即可（clf.support_为支持向量的索引），所以只要计算出D即可，这里介绍向量化的计算方法，考虑更一般的问题：

假设

$$X=\begin{bmatrix} —(x^{(1)})^T— \\ —(x^{(2)})^T— \\ \vdots \\ —(x^{(m)})^T— \end{bmatrix} \in \mathbb{R}^{m\times d}, Y=\begin{bmatrix} —(y^{(1)})^T— \\ —(y^{(2)})^T— \\ \vdots \\ —(y^{(n)})^T— \end{bmatrix} \in \mathbb{R}^{n\times d}$$

其中$x^{(i)},y^{(i)}\in\mathbb{R}^d$，现在的问题是如何高效计算矩阵$D\in\mathbb{R}^{m\times n}$，其中

$$D_{i,j}=||x^{(i)}-y^{(j)}||^2$$

首先对$D_{i,j}$进行处理

$$\begin{aligned} D_{i,j} &= ||x^{(i)}-y^{(j)}||^2 \\ &= (x^{(i)}-y^{(j)})^T(x^{(i)}-y^{(j)}) \\ &= (x^{(i)})^T x^{(i)} - 2(x^{(i)})^T y^{(j)} + (y^{(j)})^T y^{(j)} \end{aligned}$$

那么

$$\begin{aligned} D &= \begin{bmatrix} D_{1,1} & \ldots & D_{1,n} \\ \ldots & \ldots & \ldots \\ D_{m,1} & \ldots & D_{m,n} \end{bmatrix} \\ &= \begin{bmatrix} (x^{(1)})^T x^{(1)} - 2(x^{(1)})^T y^{(1)} + (y^{(1)})^T y^{(1)} & \ldots & (x^{(1)})^T x^{(1)} - 2(x^{(1)})^T y^{(n)} + (y^{(n)})^T y^{(n)} \\ \ldots & \ldots & \ldots \\ (x^{(m)})^T x^{(m)} - 2(x^{(m)})^T y^{(1)} + (y^{(1)})^T y^{(1)} & \ldots & (x^{(m)})^T x^{(m)} - 2(x^{(m)})^T y^{(n)} + (y^{(n)})^T y^{(n)} \end{bmatrix} \\ &= \begin{bmatrix} (x^{(1)})^T x^{(1)} & \ldots & (x^{(1)})^T x^{(1)} \\ \ldots & \ldots & \ldots \\ (x^{(m)})^T x^{(m)} & \ldots & (x^{(m)})^T x^{(m)} \end{bmatrix} + \begin{bmatrix} (y^{(1)})^T y^{(1)} & \ldots & (y^{(n)})^T y^{(n)} \\ \ldots & \ldots & \ldots \\ (y^{(1)})^T y^{(1)} & \ldots & (y^{(n)})^T y^{(n)} \end{bmatrix} - 2\begin{bmatrix} (x^{(1)})^T y^{(1)} & \ldots & (x^{(1)})^T y^{(n)} \\ \ldots & \ldots & \ldots \\ (x^{(m)})^T y^{(1)} & \ldots & (x^{(m)})^T y^{(n)} \end{bmatrix} \\ &= \begin{bmatrix} (x^{(1)})^T x^{(1)} \\ \ldots \\ (x^{(m)})^T x^{(m)} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}}_{1\times n\text{矩阵}} + \underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}_{m\times 1\text{矩阵}} \begin{bmatrix} (y^{(1)})^T y^{(1)} & \ldots & (y^{(n)})^T y^{(n)} \end{bmatrix} - 2XY^T \end{aligned}$$

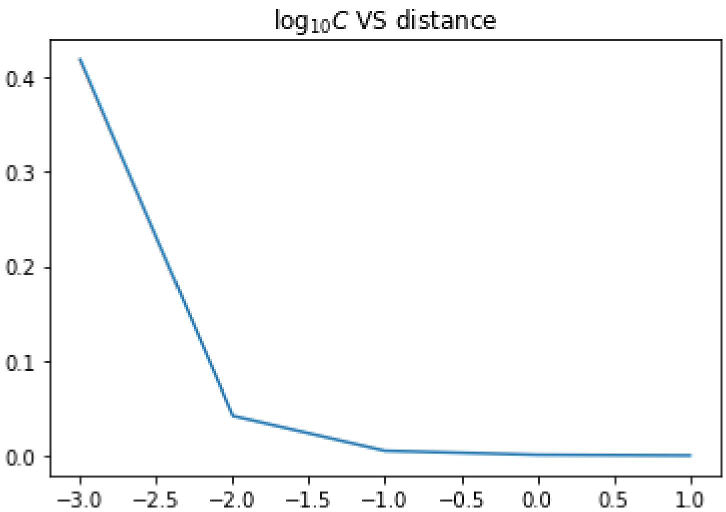
利用numpy的广播机制上式可以简写如下：

```
#计算距离矩阵
d1 = np.sum(X ** 2, axis=1).reshape(-1, 1)
d2 = np.sum(Y ** 2, axis=1).reshape(1, -1)
dist = d1 + d2 - 2 * X.dot(Y.T)
```

利用上述方法计算$||w||^2$即可，最后注意距离为$\frac{1}{||w||}$：

```
####Problem 18
C = [-3, -2, -1, 0, 1]
Distance = []
```

```
#将标签修改为-1, 1
y = 2 * y_train_1 - 1

for i in C:
    c = 10**i
    clf = svm.SVC(kernel='rbf', gamma=1, C=c)
    clf.fit(X_train, y)
    X = X_train[clf.support_]
    #距离矩阵
    d1 = np.sum(X ** 2, axis=1).reshape(-1, 1)
    d2 = np.sum(X ** 2, axis=1).reshape(1, -1)
    dist = d1 + d2 - 2 * X.dot(X.T)
    #Kernel矩阵
    K = np.exp(- c * dist)
    #计算anyn
    y1 = clf.dual_coef_[0] * y[clf.support_]
    w2 = y1.dot(K).dot(y1.T)
    #计算距离
    distance = 1 / np.sqrt(w2)
    Distance.append(distance)

plt.plot(C, Distance)
plt.title("$\log_{10}C$ VS distance")
plt.show()
```

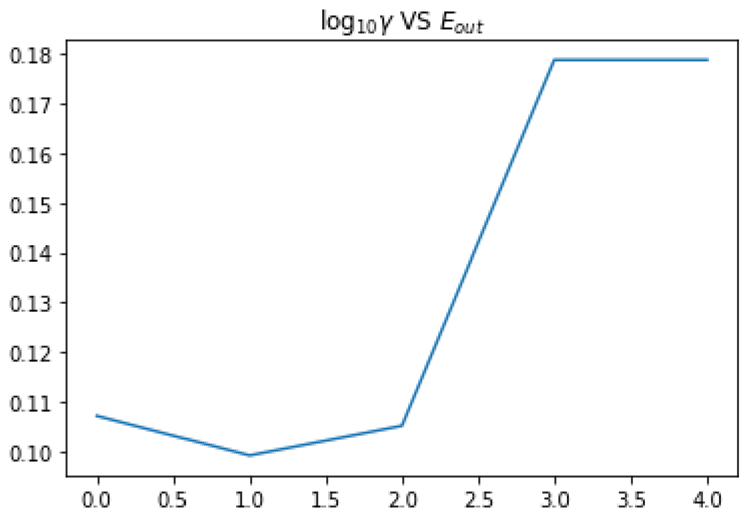


随着C增加，距离在减少。

Problem 19

```
####Problem 19
y_test_1 = (y_test == 0)

Gamma = range(5)
Eout = []

for i in Gamma:
```

```
    gamma = 10**i
    clf = svm.SVC(kernel='rbf', gamma=gamma, C=0.1)
    clf.fit(X_train, y_train_1)
    e = np.mean(clf.predict(X_test) != y_test_1)
    Eout.append(e)

plt.plot(Gamma, Eout)
plt.title("$\log_{10}\gamma$ VS $E_{out}$")
plt.show()
```



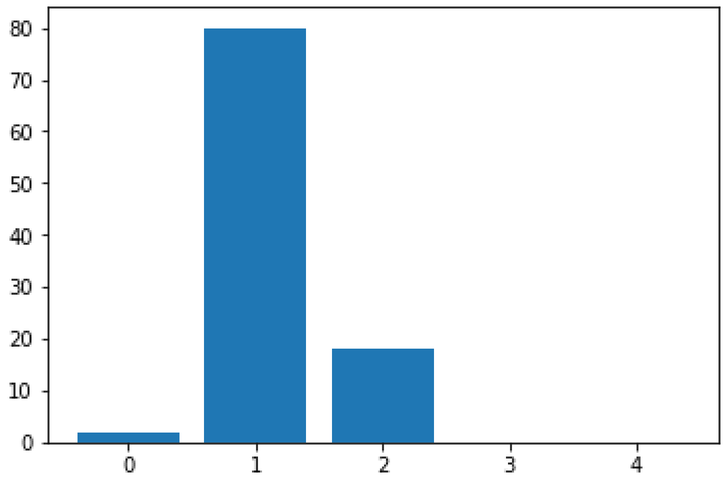
可以看到，$\log_{10}\gamma=1$时，E_{out}最小，说明C不能太大，也不能太小。

Problem 20

这题要构造交叉验证集，多次实验，做直方图，先做一些预处理。

```
####Problem 20
from sklearn.model_selection import train_test_split

#对数据合并，方便调用train_test_split函数
Data = np.c_[X_train, y_train]
N = 100
#记录最小Eval对应的gamma的索引的次数
Cnt = np.zeros(5)
Gamma = range(5)

for _ in range(N):
    #划分数据
    train_set, val_set = train_test_split(Data, test_size=0.2)
    #取特征
    X_train = train_set[:, :2]
    #取标签
    y_train = train_set[:, 2]
    X_val = val_set[:, :2]
    y_val = val_set[:, 2]
    Eval = []
```

```python
    for i in Gamma:
        gamma = 10 ** i
        clf = svm.SVC(kernel='rbf', gamma=gamma, C=0.1)
        clf.fit(X_train, y_train)
        e = np.mean(clf.predict(X_val) != y_val)
        Eval.append(e)
    #找到最小Eval对应的索引
    index = np.argmin(Eval)
    #对应索引次数加1
    Cnt[index] += 1

plt.bar(Gamma, Cnt)
plt.show()
```



可以看到大部分最优解对应的$\log_{10}\gamma$都为1，和上一题说明同一个道理，说明γ不能太大，也不能太小。

附加题

Problem 21

这题的问题是求hard-margin SVM对偶问题的对偶问题，我们来看一下。

首先回顾下hard-margin SVM对偶问题。

$$
\begin{aligned}
\underset{\alpha\in\mathbb{R}^N}{\text{minimize}}:&\ \frac{1}{2}\sum_{m=1}^{N}\sum_{n=1}^{N} y_n y_m \alpha_n \alpha_m x_n^T x_m - \sum_{n=1}^{N}\alpha_n \\
\text{subject to:}&\ \sum_{n=1}^{N} y_n\alpha_n = 0, \alpha_n \ge 0 (n=1,\ldots,N)
\end{aligned}
$$

根据拉格朗日乘子法，我们将上述问题转化为

$$
\min_{\alpha\in\mathbb{R}^N}\max_{\lambda_i\ge 0}: \frac{1}{2}\sum_{m=1}^{N}\sum_{n=1}^{N} y_n y_m \alpha_n \alpha_m x_n^T x_m - \sum_{n=1}^{N}\alpha_n + \lambda_0\left(\sum_{n=1}^{N} y_n\alpha_n\right) - \sum_{n=1}^{N}\lambda_n\alpha_n
$$

注意这里如下事实成立：

$$\min\max = \max\min$$

所以上述问题可以转化为

$$\max_{\lambda_i\ge 0}\min_{\alpha\in\mathbb{R}^N}: \frac{1}{2}\sum_{m=1}^N\sum_{n=1}^N y_n y_m\alpha_n\alpha_m x_n^T x_m - \sum_{n=1}^N\alpha_n + \lambda_0(\sum_{n=1}^N y_n\alpha_n) - \sum_{n=1}^N\lambda_n\alpha_n$$

这个最小值问题转化为为一个无条件极值，现在做以下记号

$$f(\lambda,\alpha) = \frac{1}{2}\sum_{m=1}^N\sum_{n=1}^N y_n y_m\alpha_n\alpha_m x_n^T x_m - \sum_{n=1}^N\alpha_n + \lambda_0(\sum_{n=1}^N y_n\alpha_n) - \sum_{n=1}^N\lambda_n\alpha_n$$

关于α_i求偏导可得

$$\begin{aligned}\frac{\partial f}{\partial\alpha_i} &= \frac{1}{2}\sum_{n=1}^N y_n y_i\alpha_n x_n^T x_i + \frac{1}{2}\sum_{m=1}^N y_i y_m\alpha_m x_i^T x_m - 1 + \lambda_0 y_i - \lambda_i\\ &= \sum_{n=1}^N y_n y_i\alpha_n x_i^T x_n - 1 + \lambda_0 y_i - \lambda_i\end{aligned}$$

令偏导为0可得

$$\sum_{n=1}^N y_n y_i\alpha_n x_i^T x_n = 1 - \lambda_0 y_i + \lambda_i$$

对这个式子作以下变形可得

$$\begin{bmatrix} y_i y_1 x_i^T x_1 & \ldots & y_i y_N x_i^T x_N\end{bmatrix}\begin{bmatrix}\alpha_1\\ \ldots\\ \alpha_N\end{bmatrix} = 1 - \lambda_0 y_i + \lambda_i \tag{1}$$

这个形式是见过的，我们回顾下Learning from data第八章的28页，我们知道hard-margin SVM对偶问题也可以写成如下的形式

$$\begin{gathered}\underset{\alpha\in\mathbb{R}^N}{\operatorname{minimize}}: \frac{1}{2}\alpha^T Q_D\alpha - 1_N^T\alpha\\ \text{subject to: } A_D\alpha\ge 0_{N+2}\\ Q_D = \begin{bmatrix} y_1y_1x_1^Tx_1 & \ldots & y_1y_Nx_1^Tx_N\\ y_2y_1x_2^Tx_1 & \ldots & y_2y_Nx_2^Tx_N\\ \ldots & \ldots & \ldots\\ y_Ny_1x_N^Tx_1 & \ldots & y_Ny_Nx_N^Tx_N\end{bmatrix}\text{ and } A_D = \begin{bmatrix} y^T\\ -y^T\\ I_{N\times N}\end{bmatrix}\end{gathered}$$

(1)的等式左边的第一个向量对应着Q_D的每一行，所以如果我们对(1)式中i从1取到N，可以把(1)的条件转化为

$$
Q_D\alpha = 1_N - \lambda_0 y + \lambda
$$
$$
\alpha = \begin{bmatrix} \alpha_1 \\ \ldots \\ \alpha_N \end{bmatrix}, y = \begin{bmatrix} y_1 \\ \ldots \\ y_N \end{bmatrix}, \lambda = \begin{bmatrix} \lambda_1 \\ \ldots \\ \lambda_N \end{bmatrix}
$$

在这个记号下，对原问题进行处理

$$
\begin{aligned}
\sum_{m=1}^N \sum_{n=1}^N y_n y_m \alpha_n \alpha_m x_n^T x_m &= \alpha^T Q_D \alpha \\
\sum_{n=1}^N \alpha_n &= 1_N^T \alpha \\
\lambda_0 (\sum_{n=1}^N y_n \alpha_n) &= \lambda_0 \alpha^T y \\
\sum_{n=1}^N \lambda_n \alpha_n &= \alpha^T \lambda
\end{aligned}
$$

所以

$$
f(\lambda, \alpha) = \frac{1}{2}\alpha^T Q_D \alpha - 1_N^T \alpha + \lambda_0 \alpha^T y - \alpha^T \lambda
$$

将$Q_D\alpha = 1_N - \lambda_0 y + \lambda$带入可得

$$
\begin{aligned}
f(\lambda, \alpha) &= \frac{1}{2}\alpha^T Q_D \alpha - 1_N^T \alpha + \lambda_0 \alpha^T y - \alpha^T \lambda \\
&= \frac{1}{2}\alpha^T (1_N - \lambda_0 y + \lambda) - 1_N^T \alpha + \lambda_0 \alpha^T y - \alpha^T \lambda \\
&= -\frac{1}{2}\alpha^T 1_N + \frac{1}{2}\lambda_0 \alpha^T y - \frac{1}{2}\alpha^T \lambda \\
&= -\frac{1}{2}\alpha^T (1_N - \lambda_0 y + \lambda)
\end{aligned}
$$

如果Q_D可逆，那么

$$
\alpha = Q_D^{-1}(1_N - \lambda_0 y + \lambda)
$$

代入可得

$$
\begin{aligned}
-\frac{1}{2}\alpha^T (1_N - \lambda_0 y + \lambda) &= -\frac{1}{2}[Q_D^{-1}(1_N - \lambda_0 y + \lambda)]^T (1_N - \lambda_0 y + \lambda) \\
&= -\frac{1}{2}(1_N - \lambda_0 y + \lambda)^T Q_D^{-1} (1_N - \lambda_0 y + \lambda)
\end{aligned}
$$

所以问题转化为

$$
\begin{aligned}
&\underset{\lambda_i \geq 0}{\operatorname{maxmize}}: -\frac{1}{2}(1_N-\lambda_0 y+\lambda)^T Q_D^{-1}(1_N-\lambda_0 y+\lambda)\\
&\text{subject to: } \lambda_i \geq 0 (n=0,1,\ldots,N)
\end{aligned}
$$

把负号提出来，可以转化为标准的QP问题

$$
\begin{aligned}
&\underset{\lambda_i \geq 0}{\operatorname{minmize}}: \frac{1}{2}(1_N-\lambda_0 y+\lambda)^T Q_D^{-1}(1_N-\lambda_0 y+\lambda)\\
&\text{subject to: } \lambda_i \geq 0 (n=0,1,\ldots,N)
\end{aligned}
$$

可以看到，这和最开始的问题是非常类似的。