参考文献

[1] A. W. van der Vaart. Asymptotic Statistics. Cambridge University Press, 2000.

問題

標本歪度 l_n は以下のように定義される。

$$l_n \equiv \frac{n^{-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^3}{\left(n^{-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2\right)^{3/2}}$$

$Z \equiv (X_1, X_1^2, X_1^3)$ の分散共分散行列 V は、

$$\begin{aligned}
V &= \mathbb{E}\left[\begin{pmatrix} (X_1-\alpha_1)(X_1-\alpha_1) & (X_1-\alpha_1)(X_1^2-\alpha_2) & (X_1-\alpha_1)(X_1^3-\alpha_3) \\ (X_1^2-\alpha_2)(X_1-\alpha_1) & (X_1^2-\alpha_2)(X_1^2-\alpha_2) & (X_1^2-\alpha_2)(X_1^3-\alpha_3) \\ (X_1^3-\alpha_3)(X_1-\alpha_1) & (X_1^3-\alpha_3)(X_1^2-\alpha_2) & (X_1^3-\alpha_3)(X_1^3-\alpha_3) \end{pmatrix}\right] \\
&= \mathbb{E}\left[\begin{pmatrix} X_1^2 - 2\alpha_1 X_1 + \alpha_1^2 & X_1^3 - \alpha_1 X_1^2 - \alpha_2 X_1 + \alpha_1\alpha_2 & X_1^4 - \alpha_1 X_1^3 - \alpha_3 X_1 + \alpha_1\alpha_3 \\ X_1^3 - \alpha_1 X_1^2 - \alpha_2 X_1 + \alpha_1\alpha_2 & X_1^4 - 2\alpha_2 X_1^2 + \alpha_2^2 & X_1^5 - \alpha_2 X_1^3 - \alpha_3 X_1^2 + \alpha_2\alpha_3 \\ X_1^3 - \alpha_1 X_1^3 - \alpha_3 X_1 + \alpha_1\alpha_3 & X_1^5 - \alpha_2 X_1^3 - \alpha_3 X_1^2 + \alpha_2\alpha_3 & X_1^6 - 2\alpha_3 X_1^3 + \alpha_3^2 \end{pmatrix}\right] \\
&= \begin{pmatrix} \alpha_2 - 2\alpha_1\alpha_1 + \alpha_1^2 & \alpha_3 - \alpha_1\alpha_2 - \alpha_2\alpha_1 + \alpha_1\alpha_2 & \alpha_4 - \alpha_1\alpha_3 - \alpha_3\alpha_1 + \alpha_1\alpha_3 \\ \alpha_3 - \alpha_1\alpha_2 - \alpha_2\alpha_1 + \alpha_1\alpha_2 & \alpha_4 - 2\alpha_2\alpha_2 + \alpha_2^2 & \alpha_5 - \alpha_2\alpha_3 - \alpha_3\alpha_2 + \alpha_2\alpha_3 \\ \alpha_4 - \alpha_1\alpha_3 - \alpha_3\alpha_1 + \alpha_1\alpha_3 & \alpha_5 - \alpha_2\alpha_3 - \alpha_3\alpha_2 + \alpha_2\alpha_3 & \alpha_6 - 2\alpha_3\alpha_3 + \alpha_3^2 \end{pmatrix} \\
&= \begin{pmatrix} \alpha_2 - \alpha_1^2 & \alpha_3 - \alpha_1\alpha_2 & \alpha_4 - \alpha_1\alpha_3 \\ \alpha_3 - \alpha_1\alpha_2 & \alpha_4 - \alpha_2^2 & \alpha_5 - \alpha_2\alpha_3 \\ \alpha_4 - \alpha_1\alpha_3 & \alpha_5 - \alpha_2\alpha_3 & \alpha_6 - \alpha_3^2 \end{pmatrix}
\end{aligned}$$

よって中心極限定理より以下が成り立つ。

$$\sqrt{n}\begin{pmatrix} X_1 - \alpha_1 \\ X_1^2 - \alpha_2 \\ X_1^3 - \alpha_3 \end{pmatrix} \rightsquigarrow \mathcal{N}\left(\begin{pmatrix} X_1 - \alpha_1 \\ X_1^2 - \alpha_2 \\ X_1^3 - \alpha_3 \end{pmatrix}\begin{pmatrix} \alpha_2 - \alpha_1^2 & \alpha_3 - \alpha_1\alpha_2 & \alpha_4 - \alpha_1\alpha_3 \\ \alpha_3 - \alpha_1\alpha_2 & \alpha_4 - \alpha_2^2 & \alpha_5 - \alpha_2\alpha_3 \\ \alpha_4 - \alpha_1\alpha_3 & \alpha_5 - \alpha_2\alpha_3 & \alpha_6 - \alpha_3^2 \end{pmatrix}\right)$$