Multivariable Calculus Concise Review

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11 Parametric Equations and Polar Coordinates

11.1 Curves Defined by Parametric Equations

- Parameter - 3rd variable that x and y are both a function of:

$$x = f(t) \text{ and } y = g(t)$$

- Points along the curve $(x, y) = (f(t), g(t))$
- Graphing calculators can be used to produce parametric curves that you wouldn't be able to make by hand.
- Equation 1: parametric equations for a cycloid:

$$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta) \qquad \theta \in \mathbb{R}$$

11.2 Calculus with Parametric Curves

- Equation 1: first derivative of a parametric equation:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{if} \quad \frac{dx}{dt} \neq 0$$

- Second derivative of a parametric equation:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \neq \frac{\frac{d^2}{dt^2}}{\frac{d^2x}{dt^2}}$$

- Equation 2: arc length of a curve:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

- Equation 3/Theorem 5: arc length of a parametric curve:

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

- Equation 6: surface area of a rotated parametric curve about the x axis:

$$S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

11.3 Polar Coordinates

- polar coordinates - (r, θ)
- Theta is always ccw
- Equations 1 and 2: polar coordinates:

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

- Derivative of a polar curve:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

11.4 Areas and Lengths in Polar Coordinates

- Equation 1: area of a sector of a circle: $A = \frac{1}{2}r^2\theta$
- Equations 3 and 4: polar area:
$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta = \int_a^b \frac{1}{2}r^2 d\theta$$
- Equation 5: polar arc length:
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

11.5 Conic Sections

- Equation 1: vertical parabola with focus $(0, p)$ and directrix $y = -p$:
$$x^2 = 4py$$
- Equation 2: horizontal parabola with focus $(p, 0)$ and directrix $x = -p$:
$$y^2 = 4px$$
- Equation 3: general form of an ellipse:
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
- Equation 4: horizontal ellipse with foci $(\pm c, 0)$, verticies $(\pm a, 0)$, where $c^2 = a^2 - b^2$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a \geq b > 0$$
- Equation 5: vertical ellipse with foci $(0, \pm c)$, verticies $(0, \pm a)$, where $c^2 = a^2 - b^2$
$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad a \geq b > 0$$
- Equation 6: general form of a hyperbola:
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
- Equation 7: hyperbola with horizontal transverse axis, with foci $(\pm c, 0)$, verticies $(\pm a, 0)$, asymptotes $y = \pm\frac{b}{a}x$, where $c^2 = a^2 + b^2$:
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
- Equation 8: hyperbola with vertical transverse axis, foci $(0, \pm c)$, verticies $(0, \pm a)$, asymptotes $y = \pm\frac{a}{b}x$, where $c^2 = a^2 + b^2$:
$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

11.6 Conic Sections in Polar Coordinates

- Theorem 1: Let F be a fixed point (called the focus) and l be a fixed line (called the directrix) in a plane. Let e be a fixed positive number (called the eccentricity). The set of all points P in the plane such that
$$\frac{|PF|}{|Pl|} = e$$
is a conic section. (That is, the ratio of the distance from F to the distance from l is the constant e). The conic is:

- (a) an ellipse if $e < 1$
- a parabola if $e = 1$
- a hyperbola if $e > 1$

- Theorem 6: A polar equation of the form

$$r = \frac{ed}{1 \pm e\cos\theta} \qquad \text{or} \qquad r = \frac{ed}{1 \pm e\sin\theta}$$

represents a conic section with eccentricity e. The conic is an ellipse if $e < 1$, parabola if $e = 1$, or a hyperbola if $e > 1$

  - d is the distance from focus to directrix

- $e = \dfrac{c}{a}$ where $c^2 = a^2 + b^2$
- Kepler's laws:
  - 1 - A planet revolves around the sun in an elliptical orbit with the sun at one focus.
  - 2 - The line joining the sun to a planet sweeps out equal areas in equal times.
  - 3 - The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.
- Equation 7: The polar equation of an ellpise with focus at the origin, semimajor axis a, eccentricity e, and directirx $x = d$ can be written in the form:

$$r = \frac{a(1-e^2)}{1 + e\cos\theta}$$

- Equation 8: The perihelion distance from a planet to the sun is $a(1-e)$ and the aphelion distance is $a(1+e)$

12 Infinite Sequences and Series

12.1 Sequences

- sequence - a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \cdots, a_n, \cdots$$

- a_1 - first term; a_2 - second term; a_n - nth term
- For infinite series, every term a_n has a successor a_{n+1}
- Notation - the sequence $\{a_1, a_2, a_3, \cdots\}$ can also be written as

$$\{a_n\} \text{ or } \{a_n\}_{n=1}^{\infty}$$

- Definition 1: limits of sequences:

$$\lim_{n\to\infty} a_n = L$$

  - This means: as n becomes very large, the terms of the sequence $\{a_n\}$ approach L.
- can also be written as

$$a_n \to L \text{ as } n \to \infty$$

- If the limit at infinity exists, the sequence is convergent/converges. Otherwise, it is divergent/diverges.

- Definition 2: A more precise definition of the limit of a sequence:

$$\lim_{n\to\infty} a_n = L$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$\text{if } n > N \text{ then } |a_n - L| < \varepsilon$$

- Theorem 3: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.
- Equation 4:

$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \text{ if } r > 0$$

- Definition 5: $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$\text{if } n > N \text{ then } a_n > M$$

- Limit laws for sequences: If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then:

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \quad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \text{ if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \text{ if } p > 0 \text{ and } a_n > 0$$

- Squeeze Theorem can be adapted for sequences:

$$\text{If } a_n \leq b_n \leq c_n \text{ for } n \geq n_0 \text{ and } \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L, \text{ then } \lim_{n\to\infty} b_n = L$$

- Theorem 6: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$
- Theorem 7: If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

- Equation 8 (example 10):

$$a_n = \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot n \cdot \cdots \cdot n}$$

(ex. 10)

- Equation 9 (example 11): The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

- Definition 10: A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is **monotonic** if it is either increasing or decreasing.

- Definition 11: A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leq M \text{ for all } n \geq 1$$

  It is bounded below if there is a number m such that

$$m \leq a_n \text{ for all } n \geq 1$$

  If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**

- Theorem 12: Monotonic Sequence Theorem: Every bounded, monotonic sequence is convergent.
- Proof of theorem 12: Suppose $\{a_n\}$ is an increasing sequence. Since $\{a_n\}$ is bounded, the set $S = \{a_n | n \geq 1\}$ has an upper bound. By the Completeness Axiom it has a least upper bound L. Given $\varepsilon > 0$, $L - \varepsilon$ is *not* an upper bound for S (since L is the *least* upper bound). Therefore

$$a_N > L - \varepsilon$$

  For some integer N. But the sequence is increasing so $a_n \geq a_N$ for every $n > N$. Thus if $n > N$, we have

$$a_n > L - \varepsilon$$

  so

$$0 \leq L - a_n < \varepsilon$$

  since $a_n \leq L$. Thus,

$$|L - a_n| < \varepsilon \text{ whenever } n > N$$

  so $\lim_{n\to\infty} a_n = L$. A similar proof can be applied if $\{a_n\}$ is decreasing.

12.2 Series

- Equation 1: infinite series/series can be written as:

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

- Partial sums:

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

  e.g.

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$s_4 = a_1 + a_2 + a_3 + a_4$$

- Def 2: given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, its nth partial sum is denoted as above.
  - If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called convergent and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \text{ or } \sum_{n=1}^{\infty} a_n = s$$

  - The number s is called the sum of the series. If the sequence $\{s_n\}$ is divergent, then the series is divergent.
- Geometric series:

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \text{ where } a \neq 0$$

- Equation 3: sum of a geometric series
$$s_n = \frac{a\left(1 - r^n\right)}{1 - r}$$
- Equation 4 (example 2): The geometric series
$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$
is convergent if $|r| < 1$ and its sum is
$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r} \text{ where } |r| < 1$$
If $|r| \geq 1$, the geometric series is divergent.
- Equation 5 (example 7):
$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}$$
- Theorem 6: If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$
  - Note: The converse of this theorem is not always true!
- Equation 7: Nth term test: If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- Theorem 8: If Σa_n and Σb_n are convergent series, then so are the series $\Sigma c a_n$ (where c is a constant), $\Sigma\left(a_n + b_n\right)$, and $\Sigma\left(a_n - b_n\right)$, and
$$\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$$
$$\sum_{n=1}^{\infty} \left(a_n + b_n\right) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$
$$\sum_{n=1}^{\infty} \left(a_n - b_n\right) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

12.3 The Integral Test and Estimates of Sums

- The Integral Test: Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent IFF the proper integral $\int_1^{\infty} f(x)dx$ is convergent.
  - CONDITIONS: continuous, positive, decreasing function
  - The integral from 1 to ∞ of the function must be convergent for the series to be convergent.
- Equation 1: P-series test: The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$
- Equation 2: Remainder Estimate for the Integral Test: Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and Σa_n is convergent. If $R_n = s - s_n$, then
$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx$$
- Equation 3 (example 5):
$$s_n + \int_{n+1}^{\infty} f(x)dx \leq s \leq s_n + \int_n^{\infty} f(x)dx$$

- Equation 4:

$$a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x)dx$$

- Equation 5:

$$\int_1^n f(x)dx \leq a_1 + a_2 + \cdots + a_{n-1}$$

  - Both eqns 4 and 5 depend on the fact that f is decreasing and positive.

12.4 The Comparison Tests

- The comparison test: Suppose that Σa_n and Σb_n are series with positive terms.
  - If Σb_n is convergent and $a_n \leq b_n$ for all n, then Σa_n is also convergent.
  - If Σb_n is divergent and $a_n \geq b_n$ for all n, then Σa_n is also divergent.
- The Limit comparison test: Suppose that Σa_n and Σb_n are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

  where c is a finite number and $c > 0$, then either both series converge or both series diverge.

12.5 Alternating Series

- The alternating series test: If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \text{ where } b_n > 0$$

  satisfies

$$\text{(i) } b_{n+1} \leq b_n \text{ for all } n$$
$$\text{(ii) } \lim_{n\to\infty} = 0$$

  then the series is convergent.

- Alternating series Estimation Theorem: If $s = \Sigma(-1)^{n-1} b_n$, where $b_n > 0$, is the sum of an alternating series that satisfies

$$b_{n+1} \leq b_n \text{ and } \lim_{n\to\infty} = 0$$

  then

$$|R_n| = |s - s_n| \leq b_{n+1}$$

12.6 Absolute Convergence and the Ratio and Root Tests

- Definition 1: A series Σa_n is called absolutely convergent if the series of absolute values $\Sigma |a_n|$ is convergent.
- Definition 2: A series Σa_n is called conditionally convergent if it is convergent but not absolutely convergent.
- Theorem 3: If a series Σa_n is absolutely convergent, then it is convergent.
- The ratio test:
  - If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
  - If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
  - If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of Σa_n

- The Root Test:
  - If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
  - If $\lim_{n\to\infty} \sqrt[n]{|a_n} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
  - if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the root test is inconclusive.

12.7 Strategy for Testing Series

- Classify series according to form in order to determine convergence or divergence.
- If the series is of the form $\Sigma 1/n^p$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leq 1$.
- Geometric series: Σar^n; converges if $|r| < 1$ and diverges if $|r| \geq 1$
- Series similar to geo or p-series: use a comparison test to determine.
- If the limit at infinity is immediately obvious not to be 0, use the nth term test.
- If the series contains $(-1)^n$, use the alternating series test.
- Series with factorials or other products: use the ratio test.
- If the series is in the form of $(b_n)^n$, use the root test.
- If $a_n = f(n)$ and $\int_1^{\infty} f(x)dx$ is easily evaluated, use the integral test as long as the function is continuous, positive, and decreasing.

12.8 Power Series

- (Equation 1) Power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

  where x is the variable and the c_ns are the coefficients of the series.

- (Equation 2): Power series with all coefficients as 1.

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

- Equation 3: power series centered about a

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

- Theorem 4: For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, there are only three possibilities:
  - The series converges only when $x = a$
  - The series converges for all x
  - There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$
- R is the radius of convergence of the power series. Interval of convergence is the interval that contains all x for which the series converges.
- Check endpoint convergence!

12.9 Representations of Functions an Power Series

- Equation 1: geometric series with $a = 1$ and $r = x$:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \text{ for } |x| < 1$$

- Theorem 2: If the power series $\Sigma c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + C_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

  is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

$$\text{(i) } f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$\text{(ii) } \int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} = c_n\frac{(x-a)^{n+1}}{n+1}$$

  The radii of convergence of the power series in Equations (i) and (ii) are both R.

12.10 Taylor and Maclaurin Series

- Theorem 5: If f has a power series representation/expansion at a, that is if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \text{ for } |x-a| < R$$

  then its coefficients are given by:

$$c_n = \frac{f^{(n)}(a)}{n!}$$

- Equation 6: Taylor series about a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

- Equation 7: Maclaurin series, which is a taylor series about $a = 0$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

- Theorem 8: If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0$$

  for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

- Equation 9: Taylor's Inequality: If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x) \leq \frac{M}{(n+1)!}|x-a|^{n+1} \text{ for } |x-a| \leq d$$

- Equation 10:

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \text{ for every real number } x$$

- Equation 11:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ for all } x$$

- Equation 12: the number e is a sum of the infinite series:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

- Equation 15: power series of $\sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \text{ for all } x$$

- Equation 16: power series of $\cos x$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^2 n}{(2n)!} \text{ for all } x$$

- Equation 17: The binomial series: If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

- Table 1: Important Maclaurin series and their radii of convergence

Series	Radius
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$R = 1$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$R = \infty$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$R = \infty$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$R = \infty$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$R = 1$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$R = 1$
$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$	$R = 1$

12.11 Applications of Taylor Polynomials

- Two main ways taylor polynomials are applied:
  - 1: Approximation - computers often use taylor polynomials to approximate values of functions because it's a simpler algorithm and the error can be brought very small.
  - 2: Physics: Taylor polynomials can be used to simply visualize/predict how a complicated function will behave. Also helpful in optics and other applications of small angle approximation.

13 Vectors and the Geometry of Space

13.1 Three-Dimensional Coordinate Systems

- Coordinates - (x, y, z)

- 3D space is split into octants
- Projections - easiest way to visualize is that the object/shape/line/point is in a glass box. If you look at the box from the chosen plane or angle and trace a 2D outline of it, it is the projection.
- The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) | x, y, z \in \mathbb{R}\}$ is the set of all ordered triples of real numbers, denoted by $\mathbb{R}^3$
- Distance formula in three dimensions:

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - Z_1)^2}$$

- Sphere - set of all points in 3D space a certain distance from the center.
  - Sphere with center $C(h, k, l)$ and radius r is given by:

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

  - Sphere at the origin is given by:

$$x^2 + y^2 + z^2 = r^2$$

13.2 Vectors

- Vectors - values with magnitudes and directions
- vectors are expressed in boldface and/or with an arrow over the letter: $\mathbf{a}$, $\vec{a}$, $\vec{\mathbf{a}}$
- the magnitude of a vector is expressed: $|\mathbf{a}|$
- Vector addition - head to tail, take the magnitude of the resultant vector from the beginning of the first vector to the end of the second vector
- Definition of scalar multiplication: If c is a scalar and $\mathbf{v}$ is a vector, then the scalar multiple $c\mathbf{v}$ is the vector whose lengthe is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite to $\mathbf{v}$ if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.
- Components of a vector: Equation 1: Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\mathbf{a}$ with representation $\vec{AB}$ is:

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

- The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

- The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

- If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \qquad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

- For three dimensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

- V_2 is the set of all 2-D vectors. V_3 is the set of all 3-D vectors. V_n is the set of all n-dimensional vectors.
- Properties of vectors. If $\mathbf{a}$,$\mathbf{b}$, and$\mathbf{c}$ are vectors in V_n and c and d are scalars, then:
  - $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$
  - $\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}$
  - $\mathbf{a}+\mathbf{0}=\mathbf{a}$
  - $\mathbf{a}+(-\mathbf{a})=0$
  - $c(\mathbf{a}+\mathbf{b})=c\mathbf{a}+c\mathbf{b}$
  - $(c+d)\mathbf{a}=c\mathbf{a}+d\mathbf{a}$
  - $(cd)\mathbf{a}=c(d\mathbf{a})$
  - $1\mathbf{a}=\mathbf{a}$
- Unit vectors: $\mathbf{i}=\langle 1,0,0\rangle$, $\mathbf{j}=\langle 0,1,0\rangle$, $\mathbf{k}=\langle 0,0,1\rangle$
- Use unit vectors to express components: $\mathbf{a}=\langle a_1,a_2,a_3\rangle=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$
- General unit vector expresses direction
$$\mathbf{u}=\frac{\mathbf{a}}{|\mathbf{a}|}$$

13.3 The Dot Product

- Definition 1: If $\mathbf{a}=\langle a_1,a_2,a_3\rangle$, and $\mathbf{b}=\langle b_1,b_2,b_3\rangle$, then the dot product of $\mathbf{a}$ and $\mathbf{b}$ is the scalar $\mathbf{a}\cdot\mathbf{b}$ given by
$$\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2+a_3b_3$$
- Other names for dot product: scalar product, inner product
- Properties of the dot product: If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are vectors in V_3 and c is a scalar, then:
  - $\mathbf{a}\cdot\mathbf{a}=|\mathbf{a}|^2$
  - $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$
  - $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$
  - $(c\mathbf{a})\cdot\mathbf{b}=c(\mathbf{a}\cdot\mathbf{b})=\mathbf{a}\cdot(c\mathbf{b})$
  - $\mathbf{0}\cdot\mathbf{a}=0$
- Theorem 3: If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then
$$\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$$
- Corollary 6: If θ is the angle between the nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then
$$\cos\theta=\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$
- Equation 7: Two vectors $\mathbf{a}$ and $\mathbf{b}$,are orthogonal IFF $\mathbf{a}\cdot\mathbf{b}=0$
- Direction angles of a nonzero vector $\mathbf{a}$ are the angles α, β, and γ that $\mathbf{a}$ makes with the positive x-, y-, and z-axes respectively. The cosines of the direction angles are called direction cosines.
- Direction angles are given by:
$$\cos\alpha=\frac{a_1}{|\mathbf{a}|}\qquad\cos\beta=\frac{a_2}{|\mathbf{a}|}\qquad\cos\gamma=\frac{a_3}{|\mathbf{a}|}$$
  and,
$$\frac{1}{|\mathbf{a}|}\mathbf{a}=\langle\cos\alpha,\cos\beta,\cos\gamma\rangle$$

- Projections: think of it like a shadow.
- Scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ (aka component of $\mathbf{b}$ along $\mathbf{a}$):

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$$

- Vector projection of $\mathbf{b}$ onto $\mathbf{a}$:

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$$

13.4 The Cross Product

- Definition of the cross product: If $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, then the cross product of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$$

- Cross product is also called vector product or external product.
- Cross product is only defined when both $\mathbf{a}$ and $\mathbf{b}$ are 3-D vectors.
- Determinant form of the cross product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- Theorem 8: The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.
- Direction of the external product: use curling rhr - fingers curl from $\mathbf{a}$ to $\mathbf{b}$, thumb is the direction of the cross product.
- Theorem 9: magnitude of the cross product: If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leq \theta \leq \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

- Corollary 10: Two nonzero vector $\mathbf{a}$ and $\mathbf{b}$ are parallel IFF

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

- The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.
- Cross products of unit vectors:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$
$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

- Cross product is neither commutative nor associative.
- Properties of the cross product: If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and c is a scalar, then:
  - $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
  - $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
  - $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
  - $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
  - $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
  - $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$
- Scalar triple product of vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- The scalar triple product is the volume of the parallelepiped determined by vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ and is given by:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

13.5 Equations of Lines and Planes

- Let the line L be any line in 3D space, which is determined when there is a point on L, $P_0(x_0, y_0, z_0)$, and we know the direction of L. Let $P(x, y, z)$ be any point on L and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P. Let $\mathbf{v}$ be a vector parallel to L. The vector equation of L is given by

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

  where t is a parameter.

- Parametric equations for a line L through the point (x_0, y_0, z_0) and parallel to the direction vector $\langle a, b, c\rangle$ are:

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

- Symmetric equations of L:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

- The line segment from $\mathbf{r_0}$ to $\mathbf{r_1}$ is given by the vector equation

$$\mathbf{r}(t) = (1 - t)\mathbf{r_0} + t\mathbf{r_1} \quad 0 \le t \le 1$$

- A plane is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and an orthogonal normal vector $\mathbf{n}$. The vector equation of the plane is given by

$$\mathbf{n}\cdot(\mathbf{r} - \mathbf{r_0}) = 0$$

  or

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}$$

- Scalar equation of the plane through point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c\rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

- The distance D from any point $P_1(x_1, y_1, z_1)$ to a plane $ax + by + cz + d = 0$ is given by

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

13.6 Cylinders and Quadric Surfaces

- A cylinder is a surface that consists of all lines (called rulings) that are parallel to a given line and pass through a given plane curve.

- A quadric surface is the graph of a second-degree equation in three variables x, y, and z. The most general form of a quadric surface is:

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

- It can also take one of two standard forms:

$$Ax^2 + By^2 + Cz^2 + J = 0 \qquad \text{or} \qquad Ax^2 + By^2 + Iz = 0$$

Common quadric surfaces (see page 877 for images):

- Ellipsoid - all traces are ellipses. Equation is given by:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

  If $a = b = c$, then the ellipsoid is a sphere.

- Elliptic paraboloid: horizontal traces are ellipses and vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

- Hyperbolic paraboloid: Horizontal traces are hyperbolas and vertical traces are parabolas.

$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

- Cone: Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

- Hyperboloid of one sheet: Horizontal traces are ellipses and vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

- Hyperboloid of two sheets: Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate the two sheets.

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

14 Vector Functions

14.1 Vector Functions and Space Curves

- vector-valued functions/vector functions - a function whose domain is a set of real numbers and whose range is a set of vectors. Written in terms of its components as a parametric equation:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

- Limits of a vector function: If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

provided the limits of the component functions exist.

- Space curves: suppose that f, g, and h are continuous real-valued functions on an interval I. Then the space curve is the set C of all points (x, y, z) in space, where

$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

and t varies throughout the interval I.

14.2 Derivatives and Integrals of Vector Functions

- derivative of a vector-valued function:

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if the limit exists.

- Theorem 2: If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

- Theorem 3: Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real valued function. Then:
  - $$\frac{d}{dt}[\mathbf{u}(t)+\mathbf{v}(t)] = \mathbf{u}'(t)+\mathbf{v}'(t)$$
  - $$\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$$
  - $$\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t)+f(t)\mathbf{u}'(t)$$
  - $$\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t)+\mathbf{u}(t)\cdot\mathbf{v}'(t)$$
  - $$\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$$
  - $$\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$$

    (chain rule)
- Integral of a vector function:
  $$\int_a^b \mathbf{r}(t)dt = \left(\int_a^b f(t)dt\right)\mathbf{i} + \left(\int_a^b g(t)dt\right)\mathbf{j}\left(\int_a^b h(t)dt\right)\mathbf{k}$$

14.3 Arc Length and Curvature

- Length of a curve in 3D space:
  $$L = \int_a^b \sqrt{[f'(t)]^2+[g'(t)]^2+[h'(t)]^2}dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dx}{dt}\right)^2}dt = \int_a^b |\mathbf{r}'(t)|dt$$
- curvature of a curve is defined as:
  $$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$$
  where $\mathbf{T}$ is the unit tangent vector.
- $$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$
- Theorem 10: The curvature of the curve given by the vector function $\mathbf{r}$ is
  $$\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$
- Equations for unit tangent, unit normal and binormal vectors, and curvature:
  $$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \qquad \mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t)$$
  $$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t)\times r''(t)|}{|\mathbf{r}'(t)|^3}$$

14.4 Motion in Space: Velocity and Acceleration

- Velocity:
$$\mathbf{v}(t) = \mathbf{r}'(t)$$
- speed is the magnitude of velocity.
- Parametric equations of trajectory:
$$x = (v_0 \cos\alpha)t \qquad y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$$
- Tangential and normal components of acceleration:
$$\mathbf{a} = v'(\mathbf{T}) + \kappa v^2 \mathbf{N}$$
- Kepler's laws:
  - A planet revolves around the sun in an elliptical orbit with the sun at one focus.
  - The line joining the sun to a planet sweeps out equal areas in equal times.
  - The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of orbit.

15 Partial Derivatives

15.1 Functions of Several Variables

- A function f of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the domain of f and its range is the set of values that f takes on, that is $f(x, y)|(x, y) \in D$
- If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.
- The level curves of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).
  - A level curve is the set ofa ll points in the domain of f at which f takes on a given value k. (Think of countour maps, equipotential lines)

15.2 Limits and Continuity

- Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the limit of $f(x, y)$ as x, y approaches (a, b) is L and we write
$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$
if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that
$$\text{if} \qquad (x, y) \in D \qquad \text{and} \qquad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \qquad \text{then} \qquad |f(x, y) - L| < \varepsilon$$
- If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.
- A function f of two variables is called continuous at (a, b) if
$$\lim_{(x,y)\to(a,b) f(x,y) = f(a,b)}$$
We say f is continuous on D if f is continuous at every point (a, b) in D.
- If f is defined on a subset D of $\mathbb{R}^n$, then $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ means that for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that
$$\text{if} \qquad x \in D \qquad \text{and} \qquad 0 < |\mathbf{x} - \mathbf{a}| < \delta \qquad \text{then} \qquad |f(\mathbf{x}) - L| < \varepsilon$$

15.3 Partial Derivatives

- If f is a function of two variables, its partial derivatives are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

- If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

- Finding partial derivatives of $z = f(x, y)$:
  - To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
  - To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.
- Clairaut's Theorem: suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

- 3D Laplace Equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

15.4 Tangent Planes and Linear Approximations

- Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

- If $z = f(x, y)$, then f is differentiable at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

  where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

- Theorem: If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).
- The total differential dz is defined by:

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

15.5 The Chain Rule

- Case 1: suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

  or

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

- Case 2: Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

- General Version: Suppose that u is a differentiable function of the n variables $x_1, x_2, \cdots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \cdots, t_m$. Then u is a function of $t_1, t_2, \cdots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \cdots, m$

- Implicit diffentiation:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

where $y = f(x)$ and $F(x, f(x)) = 0$

- $$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

15.6 Directional Derivatives and the Gradient Vector

- The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h\to 0}\frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

- Theorem: If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b$$

  - If the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis, then we can write $\mathbf{u} = \langle\cos\theta, \sin\theta\rangle$ and the previous eqn becomes:

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

- If f is a function of two variables x and y, then the gradient of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

- The equation of the directional derivative of a differentiable function can thus be written as:

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y)\cdot\mathbf{u}$$

- The directional derivative of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c\rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = \lim_{h\to 0}\frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

- Using vector notation:

$$D_{\mathbf{u}}f(\mathbf{x}_0) = \lim_{h\to 0}\frac{f(\mathbf{x_0} + h\mathbf{u}) - f(\mathbf{x}_0)}{h}$$

- For a function of three variables, the gradient vector:

$$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

- The directional derivative of a function of three variables:

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \nabla f(x, y, z) \cdot \mathbf{u}$$

- Theorem: Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

- The tangent plane to a level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ is the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$. The equation of the tangent plane is thus:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

- The normal line to the surface S at P is the line passing through P and perpendicular to the tangent plane. The direction of the normal line is therefore given by the gradient vector $\nabla F(x_0, y_0, z_0)$; its symmetric equations are given by:

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

15.7 Maximum and Minimum Values

- A function of two variables has a local maximum at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b). [This means that $F(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a local maximum value. If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b), then f has a local minimum at (a, b) and $f(a, b)$ is a local minimum value.

- Theorem: If f has a local maximum or minimum at (a, b) and the first order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

- Second derivatives test: Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

  - If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
  - If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
  - If $D < 0$, the $f(a, b)$ is not a local maximum or minimum.

- Extreme value theorem for functions of two variables: If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

- To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps one and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

15.8 Lagrange Multipliers

- Method of Lagrange Multipliers: To find the maximum and minimum values of $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x,y,z)=k$]:
  1. Find all values of x,y,z, and λ such that
  $$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$
  and
  $$g(x,y,z) = k$$
  2. Evaluate f at all the points (x,y,z) that result from step 1. The largest of these values is the maximum value of f; the smallest is the minimum value of f.
- With two constraints, $g(x,y,z)=k$ and $h(x,y,z)=c$, there exist Lagrange Multipliers, constants λ and μ such that
$$\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0) + \mu \nabla h(x_0,y_0,z_0)$$

16 Multiple Integrals

16.1 Double Integrals over Rectangles

- The double integral of f over the rectangle R is
$$\iint_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$
if this limit exists.
- If $f(x,y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z=f(x,y)$ is
$$V = \iint_R f(x,y)dA$$
- Midpoint rule for double integrals:
$$\iint_R f(x,y)dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\Delta A$$
where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.
- Fubini's Theorem: If f is continuous on the rectangle $R = \{(x,y) \mid a \leq x \leq b, c \leq y \leq d\}$, then
$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$
More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.
- $$\iint_R g(x)h(y)dA = \int_a^b g(x)dx \int_c^d h(y)dy \qquad \text{where } R = [a,b]\times[c,d]$$

16.2 Double Integrals over General Regions

- If F is integrable over R, then we define the double integral of f over D by

$$\iint_D f(x,y)dA = \iint_R F(x,y)dA \qquad \text{where } F \text{ is given by Equation 1}$$

- If f is continuous on a type I region D such that

$$D = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

then

$$\iint_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

- Type II plane regions:

$$D = \{(x,y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$$

- If D is a type II region,

$$\iint_D f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

- Properties of double integrals
  -
  $$\iint_D [f(x,y) + g(x,y)]dA = \iint_D f(x,y)dA + \iint_D g(x,y)dA$$
  -
  $$\iint_D cf(x,y)dA = c\iint_D f(x,y)dA$$

  where c is a constant
  - If $f(x,y) \ge g(x,y)$ for all (x,y) in D, then

  $$\iint_D f(x,y)dA \ge \iint_D g(x,y)dA$$

- If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries, then

$$\iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA$$

-
$$\iint_D 1dA = A(D)$$

- If $m \le f(x,y) \le M$ for all (x,y) in D, then

$$mA(D) \le \iint_D f(x,y)dA \le MA(D)$$

16.3 Double Integrals in Polar Coordinates

- Recall:

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$$

- Change to polar coordinates in a double integral: If f is continuous on a polar rectangle R given by $0 \le a \le r \le b, \alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)r dr d\theta$$

- If f is continuous on a polar region of the form

$$D = \{(r,\theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$

then

$$\iint_D f(x,y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)r dr d\theta$$

16.4 Applications of Double Integrals

- mass of a lamina:

$$m = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D \rho(x,y)dA$$

- Total charge in a given area:

$$Q = \iint_D \sigma(x,y)dA$$

- Moment of a lamina about the x axis:

$$M_x = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D \rho(x,y)dA$$

- About the y axis:

$$M_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D x\rho(x,y)dA$$

- The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x,y)$ are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\rho(x,y)dA \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\rho(x,y)dA$$

where the mass m is given by

$$m = \iint_D \rho(x,y)dA$$

- Moment of intertia about x axis:

$$I_x = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D y^2\rho(x,y)dA$$

- About the y axis:

$$I_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D x^2\rho(x,y)dA$$

- Moment of inertia about the origin, or polar moment of intertia:

$$I_0 = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \left[(x_{ij}^*)^2 + (y_{ij}^*)^2\right] \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D (x^2+y^2)\rho(x,y)dA$$

  where $I_0 = I_x + I_y$

- Radius of gyration of a lamina about an axis is the number R such that

$$mR^2 = I$$

- Radius of gyration $\overline{\overline{y}}$ with respect to x axis and radius of gyration $\overline{\overline{x}}$ with respect to the y axis are given by

$$m\overline{\overline{y}}^2 = I_x \qquad m\overline{\overline{x}}^2 = I_y$$

- Expected values: if X and Y are random variables with joint density function f, we defined the X-mean and Y-mean, or expected values of X and Y as

$$\mu_1 = \iint_{\mathbb{R}^2} xf(x,y)dA \qquad \mu_2 = \iint_{\mathbb{R}^2} yf(x,y)dA$$

- A single random variable is normally distributed if its probability density function is of the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

  where μ is the mean and σ is the standard deviation.

16.5 Surface Area

- The surface area of a surface S is

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta T_{ij}$$

- The area of the surface with equation $z = f(x,y), (x,y) \in D$, where f_x and f_y are continuous, is

$$A(S) = \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}dA$$

  which is also

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}dA$$

16.6 Triple Integrals

- The triple integral of f over the box B is

$$\iiint_B f(x,y,z)dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)dV$$

  if this limit exists.

- If we choose the sample point to be (x_i, y_j, z_k), we get

$$\iiint_B f(x,y,z)dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_i, y_j, z_k)\Delta V$$

- Fubini's theorem for triple integrals: If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then
$$\iiint\limits_B f(x, y, z)dV = \int_r^s \int_c^d \int_a^b f(x, y, z)dxdydz$$
- A solid region E is said to be of type 1 if it lies between the graphs of two continuous functions of x and y, that is,
$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$
- If E is a type 1 region:
$$\iiint\limits_E f(x, y, z)dV = \iint\limits_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dz\right] dA$$
- If the projection of D of E onto the xy plane is a type I plane region, then
$$E = \{(x, y, z) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x), u_1(x, y) \leq z \leq u_2(x, y)\}$$
, and
$$\iiint\limits_E f(x, y, z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dzdydx$$
- If D is a type II plane region, then
$$E = \{(x, y, z) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y), u_1(x, y) \leq z \leq u_2(x, y)\}$$
, and
$$\iiint\limits_E f(x, y, z)dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dzdxdy$$
- A solid region E is of type 2 if:
$$E = \{(x, y, z) \mid (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$$
where D is the projection of E onto the yz plane. The back surface is $x = u_1(y, z)$ and the front surface is $x = u_2(y, z)$, and
$$\iiint\limits_E f(x, y, z)dV = \iint\limits_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)dx\right] dA$$
- A type 3 region is of the form:
$$E = \{(x, y, z) \mid (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$$
where D is the projection of E onto the xz plane, $y = u_1(x, z)$ is the left surface, and $y = u_2(x, z)$ is the right surface. Thus,
$$\iiint\limits_E f(x, y, z)dV = \iint\limits_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)dy\right] dA$$
- If $f(x, y, z) = 1$ for all points in E, then:
$$V(E) = \iiint\limits_E dV$$

16.7 Triple Integrals in Cylindrical Coordinates

- Recall:
$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta \qquad z = z \qquad \tan\theta = \frac{y}{x}$$
- Triple integration in cylindrical coordinates:
$$\iiint\limits_E f(x, y, z)dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)rdzdrd\theta$$

16.8 Triple Integrals in Spherical Coordinates

- Recall:

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi \qquad \rho^2 = x^2 + y^2 + z^2$$

- Triple integral in spherical coordinates:

$$\iiint_E f(x, y, z)dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi d\rho d\theta d\phi$$

where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}$$

16.9 Change of Variables in Multiple Integrals

- We can write the substitution rule as:

$$\int_a^b f(x)dx = \int_c^d f(g(u))g'(u)du$$

where $x = g(u)$ and $a = g(c), b = g(d)$ which is also

$$\int_a^b f(x)dx = \int_c^d f(x(u))\frac{dx}{du}du$$

- The Jacobian of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

- Approximation to the area ΔA of R:

$$\Delta A \approx \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \Delta u \Delta v$$

where the Jacobian is evaluated at (u_0, v_0)

- Change of variables in a double integral: Suppose that T is a C^1 transformation whole Jacobian is nonzero and that T maps a region S in the uv plane onto a region R in the xy plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then:

$$\iint_R f(x, y)dA = \iint_S f(x(u, v)y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv$$

- If:

$$x = g(u, v, w) \qquad y = h(u, v, w) \qquad z = k(u, v, w)$$

then the Jacobian of T is given by:

$$\frac{\partial(x, y, z)}{\partial u, v, w} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

- Change of variables for triple integrals:

$$\iiint_R f(x, y, z)dV = \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| dudvdw$$

17 Vector Calculus

17.1 Vector Fields

- Def: Let D be a set in $\mathbb{R}^2$ (a plane region). A vector field on $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$
- Def: Let E be a subset of $\mathbb{R}^3$. A vector field on $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$
- Recall that the gradient of a scalar function f of two variables ∇f is defined by

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

  Therefore ∇f is really a vector field on $\mathbb{R}^2$ and is called a gradient vector field. Likewise, if f is a scalar function of 3 variables, its gradient is a vector field on $\mathbb{R}^3$ given by

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

17.2 Line Integrals

- Def: if f is defined on a smooth curve C given by

$$x = x(t) \qquad y = y(t) \qquad a \le t \le b$$

  then the line integral of f along C is

$$\int_C f(x, y)ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

  if this limit exists.

- If f is a continuous function, then the limit always exists and the line integral is given by:

$$\int_C f(x, y)ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$$

  The value of the line integral does not depend on the parameterization of the curve, provided that the curve is traversed exactly once as t increases from a to b.

- Line integrals of f along C with respect to x and y:

$$\int_C f(x, y)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta x_i = \int_a^b f(x(t), y(t))x'(t)dt$$

$$\int_C f(x, y)dy = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta y_i = \int_a^b f(x(t), y(t))y'(t)dt$$

- Recall that the vector representation of the line segment that starts at $\mathbf{r}_0$ and ends at $\mathbf{r}_1$ is given by

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad 0 \le t \le 1$$

- For line integrals in space, where C is a curve given by

$$x = x(t) \qquad y = y(t) \qquad z = z(t) \qquad a \le t \le b$$

  or

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

  then the line integral of f along C is given by

$$\int_C f(x, y, z)ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta s_i = \int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}dt$$

- We evaluate integrals of the form

$$\int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$$

by expressing everything (x, y, z, dx, dy, dz) in terms of the parameter t.

- Def: Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t), a \leq t \leq b$. Then the line integral of $\mathbf{F}$ along C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_C \mathbf{F} \cdot \mathbf{T}ds$$

- We have:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C Pdx + Qdy + Rdz \qquad \text{where} \quad \mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

17.3 The Fundamental Theorem for Line Integrals

- Theorem: Let C be a smooth curve given by the vector function $\mathbf{r}(t), a \leq t \leq b$. Let f be a differentiable function of two or three variables whos gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

- Theorem: $\int_C \mathbf{F} \cdot d\mathbf{r}$ is an independent path in D IFF $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.
- Theorem: Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$
- Theorem: If $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

- Theorem: Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

17.4 Green's Theorem

- Green's Theorem gives the relationship b/w a line integral around a simple closed curve C and a double integral over the plane region D bounded by C.
- We use the convention that the positive orientation of C means traversing C once counterclockwise.
- Green's Theorem: Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

  - Note: the notation $\oint_C Pdx + Qdy$ is sometimes used to show that it is a closed path integral.
- To find the area of D:

$$A = \oint_C xdy = -\oint_C ydx = \frac{1}{2}\oint_C xdy - ydx$$

17.5 Curl and Divergence

- If $\mathbf{F} - P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the curl of F is the vector field on $\mathbb{R}^3$ defined by
$$\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$
-
$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F}$$
- Theorem: If f is a function of three variables that has continuous second-order partial derivatives, then
$$\text{curl}(\nabla f) = \mathbf{0}$$
- Theorem: If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.
- Divergence: if $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $\frac{\partial P}{\partial x}$, $\frac{\partial Q}{\partial y}$, and $\frac{\partial R}{\partial z}$ exist, then the divergence of $\mathbf{F}$ is the function of three variables defined by
$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$
It can also be written as
$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$$
- Theorem: If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P,Q, and R have continuous second-order partial derivatives, then
$$\text{div curl } \mathbf{F} = 0$$
- Vector form of Green's Theorem:
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k} dA$$
- Which is also
$$\oint_C \mathbf{F} \cdot \mathbf{n} ds = \iint_D \text{div } \mathbf{F}(x,y) dA$$

17.6 Parametric Surfaces and Their Areas

- Given the vector function
$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$
The set of all points (x,y,z) in $\mathbb{R}^3$ such that
$$x = x(u,v) \qquad y = y(u,v) \qquad z = z(u,v)$$
and (u,v) varies throughout D is called a parametric surface S
- A surface of revolution can be represented parametrically with
$$x = x \qquad y = f(x)\cos\theta \qquad z = f(x)\sin\theta$$
- Given a parametric surface S, if u is kept constant by $u = u_0$, then $\mathbf{r}(u_0, v)$ defines the grid curve C_1 on S. The tangent vector to C_1 at a point P_0 is given by
$$\mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$
If v is kept constant by $v = v_0$, the grid curve C_2 given by $\mathbf{r}(u, v_0)$ lies on S and its tangent vector at P_0 is given by
$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}$$
If $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$ then the surface S is called smooth. For a smooth surface, $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to the tangent plane.

- Def: If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k} \qquad (u,v) \in D$$

and S is covered just once as (u,v) ranges throughout the parameter domain D, then the surface area of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA$$

where

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \qquad r_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

- Special case, where $z = f(x,y)$ where $(x,y) \in D$ and f has continuous partial derivatives, we have the parametric equations

$$x = x \qquad y = y \qquad z = f(x,y)$$

so

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial f}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial f}{\partial y}\right)\mathbf{k}$$

and

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$$

which gives

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

and the surface area is

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA$$

17.7 Surface Integrals

- The surface integral of f over the surface S is given by

$$\iint_S f(x,y,z)dS = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij} = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v| dA$$

- for a surface S with $z = g(x,y)$ the surface integral becomes

$$\iint_S f(x,y,z)dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}dA$$

- Def: If F is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the surface integral of $\mathbf{F}$ over S is given by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

which is also called the flux of $\mathbf{F}$ across S.

17.8 Stokes' Theorem

- Stokes' Theorem: Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

- Proof of Stokes' Theorem:

$$\iint (\boldsymbol{\nabla} \times \mathbf{F}) d\mathbf{S} = \oint \mathbf{F} \cdot d\mathbf{r}$$

$$\begin{aligned}\iint (\boldsymbol{\nabla} \times \mathbf{F}) d\mathbf{S} &= \iint \left[\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \mathbf{k} \cdot d\mathbf{S} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \mathbf{j} \cdot d\mathbf{S} + \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \mathbf{i} \cdot d\mathbf{S} \right] \\ &= \iint \left[\left(\frac{\partial F_y}{\partial x} dxdy - \frac{\partial F_x}{\partial y} dxdy \right) + \left(\frac{\partial F_x}{\partial z} dzdx - \frac{\partial F_z}{\partial x} dzdx \right) + \left(\frac{\partial F_z}{\partial y} dydz - \frac{\partial F_y}{\partial z} dydz \right) \right] \\ &= \iint \left[\left(\frac{\partial F_x}{\partial dz} dz + \frac{\partial F_x}{\partial y} dy \right) dx + \left(\frac{\partial F_y}{\partial x} dx + \frac{\partial F_y}{\partial z} dz \right) dy + \left(\frac{\partial F_z}{\partial x} dx + \frac{\partial F_y}{\partial y} dy \right) dz \right] \\ &= \oint (F_x dx + F_y dy + F_z dz) = \oint \mathbf{F} \cdot d\mathbf{r}\end{aligned}$$

17.9 The Divergence Theorem

- Divergence Theorem: Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F} dV$$

- Proof of the Divergence theorem:

$$\iiint \boldsymbol{\nabla} \cdot \mathbf{F} dV = \oiint \mathbf{F} \cdot d\mathbf{S}$$

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot (\mathbf{i} F_x + \mathbf{j} F_y + \mathbf{k} F_z) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Since $i \cdot i = 1, i \cdot j = 0$ etc. Also, $dV = dxdydz$

$$\iiint \boldsymbol{\nabla} \cdot \mathbf{F} dV = \iiiint \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dxdydz = \iiint (\partial F_x dydz - \partial F_y dxdz + \partial F_z dxdy)$$

$$= \oiint (F_x dS_x + F_y dS_y + F_z dS_z) = \oiint \mathbf{F} \cdot d\mathbf{S}$$

Note the negative sign in front of ∂F_y since $dxdz = -dzdx$ because they are cross products.

17.10 Summary of Chapter 17

- All main results of Chapter 17 are higher-order versions of the Fundamental Theorem of calculus.
- Fundamental Theorem of Calculus:

$$\int_a^b F'(x) dx = F(b) - F(a)$$

- Fundamental Theorem for Line Integrals:

$$\int_C \boldsymbol{\nabla} f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

- Green's Theorem:
$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P dx + Q dy$$
- Stokes' Theorem:
$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$
- Divergence Theorem:
$$\iiint_E \text{div } \mathbf{F} dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$

18 Second-Order Differential Equations

18.1 Second-Order Linear Equations

- A second-order linear differential equation has the form
$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x)$$
where P, Q, R, and G are continuouos function.
- Homogeneous linear equations are where $G(x) = 0$:
$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$
The equation is nonhomogeneous if $G(x) \neq 0$ for some x.
- Theorem: If $y_1(x)$ and $y_2(x)$ are both solutions of the linear homogeneous equation $P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$ and c_1 and c_2 are constants, then the function
$$y(x) = c_1y_1(x) + c_2y_2(x)$$
is also a solution of the equation.
- Theorem: if y_1 and y_2 are linearly independent solutions of a second-order linear homogeneous equation, and $P(x)$ is never 0, then the general solution is given by
$$y(x) = c_1y_1(x) + c_2y_2(x)$$
where c_1 and c_2 are arbitrary constants.
- Two equations are linearly independent if neither is a constant multiple of the other.
- It is difficult to find solutions to most second-order diff eqs, but it is always possible to do so when
$$ay'' + by' + cy = 0$$
- Consider the equation
$$ar^2 + br + c = 0$$
which is called the auxiliary equation or characteristic equation of the diff eq $ay'' + by' + cy = 0$. The roots can be found using the quadratic formula:
$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$
- Based on the discriminant $b^2 - 4ac$, there are three cases:

– Case 1: $b^2 - 4ac > 0$. If the roots r_1 and r_2 of the auxiliary equation $ar^2 + br + c = 0$ are real and unequal, then the general solution of $ay'' + by' + cy = 0$ is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

– Case 2: $b^2 - 4ac = 0$. If the auxiliary equation $ar^2 + br + c = 0$ only has one real root r, then the general solution of $ay'' + by' + cy = 0$ is

$$y = c_1 e^{rx} + c_2 x e^{rx}$$

– Case 3: $b^2 - 4ac < 0$. If the roots of the auxiliary equation $ar^2 + br + c = 0$ are the complex numbers $r_1 = \alpha + i\beta, r_2 = \alpha - i\beta$, then the general solution of $ay'' + by' + cy = 0$ is

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

18.2 Nonhomogeneous Linear Equations

- Nonhomogeneous equations take the form

$$ay'' + by' + cy = G(x)$$

where a, b, and c are constants and G is a continuous function. The equation

$$ay'' + by' + cy = 0$$

is called the complimentary equation.

- Theorem: The general solution of the nonhomogeneous diff eq $ay'' + by' + cy = G(x)$ can be written as

$$y(x) = y_p(x) + y_c(x)$$

where y_p is a particular solution of the nonhomogeneous equation and y_c is the general solution of the complimentary equation.

- The method of undetermined coefficients:
  – If $G(x) = e^{kx}P(x)$ where P is a polynomial of degree n, then try $y_p(x) = e^{kx}Q(x)$, where $Q(x)$ is an nth degree polynomial (whose coefficients are determined by substituting in the differential equation).
  – If $G(x) = e^{kx}P(x)\cos mx$ or $G(x) = e^{kx}P(x)\sin mx$, where P is an nth degree ploynomial, then try

$$y_p(x) = e^{kx}Q(x)\cos mx + e^{kx}R(x)\sin mx$$

where Q and R are nth degree polynomials.
  – Modification: If any term of y_p is a solution of the complimentary equation, multiply y_p by x (or by x^2 if necessary).

18.3 Applications of Second-Order Differential Equations

- Vibrating springs and Hooke's law:

$$m\frac{d^2x}{dt^2} = -kx$$

The general solution is $x(t) = c_1 \cos \omega t + c_2 \cos \omega t = A\cos(\omega t + \delta)$ where

$$\omega = \sqrt{\frac{k}{m}} \qquad \text{(frequency)}$$

$$A = \sqrt{c_1^2 + c_2^2} \qquad \text{(amplitude)}$$

$$\cos \delta = \frac{c_1}{A} \qquad \sin \delta = -\frac{c_2}{A} \qquad \text{(phase angle)}$$

- Damped vibrations:
$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$
- Forced vibrations:
$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$
where $F(t)$ is an external force.
- LRC circuits:
$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = V(t)$$

18.4 Series Solutions

- Many diff eqs can't be solved explicitly, but we can use the power series
$$y = f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$
- Substitute this expression into the diff eq and determine the value of the coefficients.

Other information

Coordinate Systems

- Cartesian Coordinates: x, y, z
$$dA_x = dydz \qquad dA_y = dzdx \qquad dA_z = dxdy$$
$$dV = dxdydz$$
- Cylindrical Coordinates: r, θ, z
$$dA_r = rd\theta dz \qquad dA_\theta = dzdr \qquad dA_z = rdrd\theta$$
$$dV = rdrd\theta dz$$
- Spherical coordinates: ρ, θ, ϕ
$$dA_\rho = \rho^2 \sin\phi d\phi d\theta \qquad dA_\theta = \rho d\rho d\phi \qquad dA_\phi = \rho \sin\phi d\rho d\theta$$
$$dV = \rho^2 \sin\phi d\phi d\theta d\rho$$