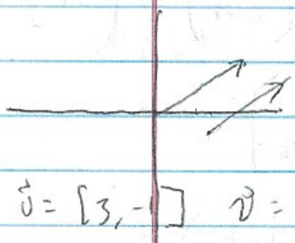
Ch. 1 - Vectors

§1.1 Geometry & algebra of vectors

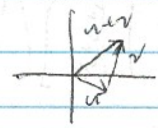
Ex.1 $A(-1,2)$ $B(3,4)$

$\vec{AB} = [4,2]$



Ex.2 $\vec{u} = [3,-1]$ $v = [1,4]$

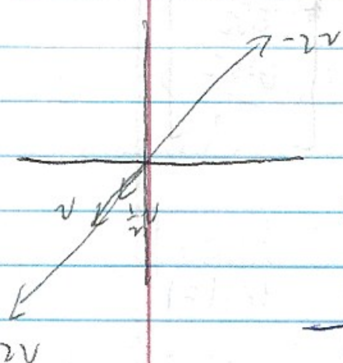
$u+v = [4,3]$

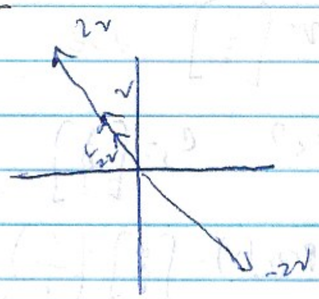


Ex.3 $v = [-2,4]$ $2v = [-4,8]$

$\frac{1}{2}v = [-1,2]$

$-2v = [4,-8]$





Ex.4 N/A

Ex.5 a) $3a + (5b - 2a) + 2(b-a)$

$= 3a + 5b - 2a + 2b - 2a$

$= -a + 7b$

b) $5x - a = 2(a + 2x)$

$5x - a = 2a + 4x$

$x = 3a$

Ex.6 N/A

Ex.7 N/A

Ex.8 N/A

Ex.9 $\mathbb{Z}_2^2 \rightarrow [0,0], [0,1], [1,0],$
$[1,1]$

$\mathbb{Z}_2^n$? $[0,0,0]$

$[0,0,0]$

...

nP_2 permutations?

Ex.10 $u = [1,1,0,1,0]$

$v = [0,1,1,1,0]$

$u+v = [1,0,1,0,0]$

Ex.11 N/A

Ex.12 3548 in $\mathbb{Z}_3$

keep dividing by 3 until you have the remainder 0, 1, 2

$\frac{3548}{3} = 1182$ R 2

$\therefore$ 3548 % 3 = 2

Ex.13 $\mathbb{Z}_3$ $2+2+1+2$

$= 1+1+2$

$= 1$

Ex.14 N/A

§1.2 dot product

Ex.15 $u \cdot v$, $u = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ $v = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$

$u \cdot v = -3 + 10 - 6 = 1$

Ex.16 prove that $(u+v)\cdot(u+v) = u \cdot u + 2(u \cdot v) + v \cdot v$

for all vectors u and v in $\mathbb{R}^n$

Since $u \cdot (v+w) = u \cdot v + u \cdot w$, we can foil out the expression

$(u+v)\cdot(u+v) = u \cdot u + u \cdot v + v \cdot u + v \cdot v$

Since $u \cdot v = v \cdot u$, $= \boxed{u \cdot u + 2u \cdot v + v \cdot v}$

QED

Ex.17 N/A

Ex.18 N/A

Ex.19 $v = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$ $\quad |v| = \sqrt{4+1+9} = \sqrt{14}$

$\hat{v} = \begin{bmatrix} \frac{2}{\sqrt{14}} \\ \frac{-1}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{bmatrix}$

Ex.20 $d(u,v)$ $\quad u = \begin{bmatrix} \sqrt{2} \\ 1 \\ -1 \end{bmatrix}$ $\quad v = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$

$d(u,v) = |u - v|$

$= \left| \begin{bmatrix} \sqrt{2} \\ -1 \\ 1 \end{bmatrix} \right| = \sqrt{2+1+1} = 2$

Ex.21 $u = [2, 1, -2]$ $\quad v = [1, 1, 1]$

$\theta = \cos^{-1}\left(\frac{u \cdot v}{|u||v|}\right) = \cos^{-1}\left(\frac{2+1-2}{\sqrt{4+1+4} \cdot \sqrt{3}}\right)$

$= \cos^{-1}\left(\frac{1}{3\sqrt{3}}\right) = 54.7°$

$\cos^{-1}\left(\frac{1}{3\sqrt{3}}\right) = 78.9$ (calc err)

Ex.22

0,0 $u = [1, 1, 0]$

$v = [0, 1, 1]$

$\theta = \cos^{-1}\left(\frac{1}{\sqrt{2} \cdot \sqrt{2}}\right)$

$= 60°$

Ex.23 N/A

Ex.24

a) $proj_u(v)$ $\quad v = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ $\quad u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$= u\left(\frac{u \cdot v}{u \cdot u}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\left(\frac{-2+3}{4+1}\right) = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

c) $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\quad u = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{bmatrix}$

$proj_u(v) = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix}\left(\frac{\frac{1}{2} + 1 + \frac{3}{\sqrt{2}}}{\frac{1}{4} + \frac{1}{4} + \frac{1}{2}}\right)$

$= \left(\frac{3}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{2}} + \frac{\sqrt{2}}{2\sqrt{2}}\right)[u]$

$= \frac{6+3\sqrt{2}}{2\sqrt{2}} u = \begin{bmatrix} \frac{6+3\sqrt{2}}{4\sqrt{2}} \\ \frac{6+3\sqrt{2}}{4\sqrt{2}} \\ \frac{6+3\sqrt{2}}{4} \end{bmatrix}$

b) $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\quad u = e_3$

e_3? $\quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad |e_3| = 1$

$proj_u(v) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \left(\frac{3}{1}\right) = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}\left(\frac{-2+3}{4+1}\right) = \frac{1}{5}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ \frac{1}{5} \end{bmatrix}$

c) $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix}\left(\frac{\frac{1}{2} + 1 + \frac{3}{\sqrt{2}}}{\frac{1}{4} + \frac{1}{4} + \frac{1}{2}}\right) = \left(\frac{1}{2} + 1 + \frac{3\sqrt{2}}{2}\right)\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$= \frac{3+3\sqrt{2}}{2}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{3+3\sqrt{2}}{2}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$

$= \frac{3+3\sqrt{2}}{4}\begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \end{bmatrix}$

EX.25 give a vector description of the mid point of $\overline{AB}$

$a=\overrightarrow{OA}$ $b=\overrightarrow{OB}$ $m=\overrightarrow{OM}$ if O is the origin

$AB=b-a$

$m-a=\overrightarrow{AM}=\frac{1}{2}\overrightarrow{AB}=\frac{1}{2}(b-a)$

$\therefore\ m=a+\frac{1}{2}(b-a)=\frac{1}{2}(a+b)$

§1.3 lines + planes

EX.26 N/A EX.27 N/A

EX.28 $P_0=(1,2,-1)$ $\vec{d}=\begin{bmatrix}5\\-1\\3\end{bmatrix}$

$x=p+td=\langle 1+5t,\ 2-t,\ -1+3t\rangle$

parametric: $x=1+5t$

$y=2-t$

$z=-1+3t$

EX.29 $\overrightarrow{PQ}=\langle 3,-4,1\rangle=\vec{d}$

$P_0=P(-1,5,0)$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-1\\5\\0\end{bmatrix}+t\begin{bmatrix}3\\-4\\1\end{bmatrix}$$

EX.30 $P(6,0,1)$ $\vec{n}=\begin{bmatrix}1\\2\\3\end{bmatrix}$

normal: $n\cdot(x-p)=0$

$n\cdot x=n\cdot p$

$$\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}6\\0\\1\end{bmatrix}$$

general: $x+2y+3z=6+3=9$

EX.31 $n=\begin{bmatrix}1\\2\\3\end{bmatrix}$ $P=\begin{bmatrix}6\\0\\1\end{bmatrix}$

vector & parametric equations

find 2 orthogonal vectors to n

"trial & error" to find $Q(9,0,0)$

$R(3,3,0)$

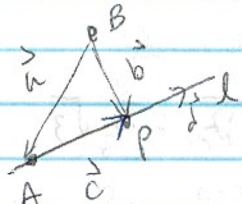
$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\0\\1\end{bmatrix}+s\begin{bmatrix}3\\0\\-1\end{bmatrix}+t\begin{bmatrix}-3\\3\\-1\end{bmatrix}$$

$x=6+3s-3t$

$y=3t$

$z=1-s-t$

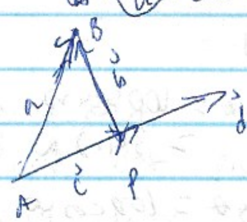
EX.32 dist. $B(1,0,2)$ to l, $A(3,1,1)$ $\vec{d}=\begin{bmatrix}-1\\1\\0\end{bmatrix}$



dist $=\vec{b}$

$\vec{c}=\text{proj}_{\vec{d}}\,a$

~~$\vec{b}=\vec{a}+\vec{c}=\vec{a}+\text{proj}_{d}\,a$~~



$c=\text{proj}_{d}\,a$

$\vec{b}=\vec{a}-\vec{c}$

$=\vec{a}-\text{proj}_{d}\,a$

$\vec{a}=AB=\begin{bmatrix}-2\\-1\\1\end{bmatrix}$

$c=\begin{bmatrix}-1\\1\\0\end{bmatrix}\left(\frac{2+1}{2}\right)=\begin{bmatrix}-3/2\\3/2\\0\end{bmatrix}$

$\vec{b}=\vec{a}-\vec{c}=\begin{bmatrix}-\frac{1}{2}\\-5/2\\1\end{bmatrix}$

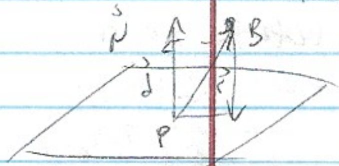
dist $=|\vec{b}|=\sqrt{\frac{1}{4}+\frac{25}{4}+1}=\sqrt{\frac{30}{4}}=\frac{1}{2}\sqrt{30}$

$c=\begin{bmatrix}-1\\1\\0\end{bmatrix}\left(\frac{2-1}{2}\right)=\begin{bmatrix}-\frac{1}{2}\\\frac{1}{2}\\0\end{bmatrix}$

$|\vec{b}|=\frac{1}{2}\sqrt{22}$

EX. 33 plane $x+y-z=1$

$B(1,0,2)$



choose a pt on the plane

$P(1,0,0)$

$\vec{d} = \text{proj}_{\vec{N}} \vec{r}$ $\quad \vec{r} = PB = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

$\vec{N} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

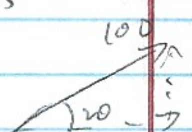
$\vec{d} = \text{proj}_{n} \vec{r} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \left(\frac{-2}{3} \right)$

$= \begin{bmatrix} -2/3 \\ -2/3 \\ 2/3 \end{bmatrix}$

$|\vec{d}| = \sqrt{\frac{4}{9}+\frac{4}{9}+\frac{4}{9}} = \frac{1}{3}\sqrt{12} = \frac{2}{3}\sqrt{3}$

EX. 34 N/A

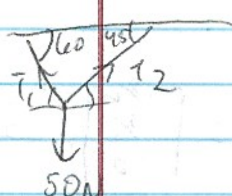
EX. 35



$F_y = F\sin\theta = 100\sin 20$
$= 34.2\text{ N}$

$F_x = F\cos\theta = 100\cos 20$
$= 94.0\text{ N}$

EX. 36



$T_1 \cos\theta_1 = T_2 \cos\theta_2$

$T_1 \sin\theta_1 + T_2 \sin\theta_2 = 50$

$\frac{1}{2} T_1 = \frac{\sqrt{2}}{2} T_2 \qquad T_1 = \sqrt{2}\, T_2$

$\frac{\sqrt{3}}{2} T_1 + \frac{\sqrt{2}}{2} T_2 = 50$

$\frac{\sqrt{6}}{2} T_2 + \frac{\sqrt{2}}{2} T_2 = 50$

$T_2 (\sqrt{6} + \sqrt{2}) = 100$

$T_2 = 25.9\text{ N}$

$T_1 = \sqrt{2}\, T_2 = 36.6\text{ N}$