

# Chapter 11 Notes - MC

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## 11 Parametric Equations and Polar Coordinates

### 11.1 Curves Defined by Parametric Equations

- Parameter - 3rd variable that  $x$  and  $y$  are both a function of:

$$x = f(t) \text{ and } y = g(t)$$

- Points along the curve  $(x, y) = (f(t), g(t))$
- Graphing calculators can be used to produce parametric curves that you wouldn't be able to make by hand.
- Equation 1: parametric equations for a cycloid:

$$x = r(\theta - \sin \theta) \quad y = r(1 - \cos \theta) \quad \theta \in \mathbb{R}$$

### 11.2 Calculus with Parametric Curves

- Equation 1: first derivative of a parametric equation:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if } \frac{dx}{dt} \neq 0$$

- Second derivative of a parametric equation:

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{\frac{d}{dt} \left( \frac{dy}{dx} \right)}{\frac{dx}{dt}} \neq \frac{d^2}{dt^2}$$

- Equation 2: arc length of a curve:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

- Equation 3/Theorem 5: arc length of a parametric curve:

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

- Equation 6: surface area of a rotated parametric curve about the  $x$  axis:

$$S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

### 11.3 Polar Coordinates

- polar coordinates -  $(r, \theta)$
- Theta is always ccw
- Equations 1 and 2: polar coordinates:

$$x = r \cos \theta \qquad y = r \sin \theta$$

$$r^2 = x^2 + y^2 \qquad \tan \theta = \frac{y}{x}$$

- Derivative of a polar curve:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta}$$

### 11.4 Areas and Lengths in Polar Coordinates

- Equation 1: area of a sector of a circle:  $A = \frac{1}{2}r^2\theta$
- Equations 3 and 4: polar area:

$$A = \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta = \int_a^b \frac{1}{2} r^2 d\theta$$

- Equation 5: polar arc length:

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

### 11.5 Conic Sections

- Equation 1: vertical parabola with focus  $(0, p)$  and directrix  $y = -p$ :

$$x^2 = 4py$$

- Equation 2: horizontal parabola with focus  $(p, 0)$  and directrix  $x = -p$ :

$$y^2 = 4px$$

- Equation 3: general form of an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

- Equation 4: horizontal ellipse with foci  $(\pm c, 0)$ , vertices  $(\pm a, 0)$ , where  $c^2 = a^2 - b^2$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad a \geq b > 0$$

- Equation 5: vertical ellipse with foci  $(0, \pm c)$ , vertices  $(0, \pm a)$ , where  $c^2 = a^2 - b^2$

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad a \geq b > 0$$

- Equation 6: general form of a hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

- Equation 7: hyperbola with horizontal transverse axis, with foci  $(\pm c, 0)$ , vertices  $(\pm a, 0)$ , asymptotes  $y = \pm \frac{b}{a}x$ , where  $c^2 = a^2 + b^2$ :

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

- Equation 8: hyperbola with vertical transverse axis, foci  $(0, \pm c)$ , vertices  $(0, \pm a)$ , asymptotes  $y = \pm \frac{a}{b}x$ , where  $c^2 = a^2 + b^2$ :

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

## 11.6 Conic Sections in Polar Coordinates

- Theorem 1: Let  $F$  be a fixed point (called the focus) and  $l$  be a fixed line (called the directrix) in a plane. Let  $e$  be a fixed positive number (called the eccentricity). The set of all points  $P$  in the plane such that

$$\frac{|PF|}{|Pl|} = e$$

is a conic section. (That is, the ratio of the distance from  $F$  to the distance from  $l$  is the constant  $e$ ). The conic is:

- (a) an ellipse if  $e < 1$
- a parabola if  $e = 1$
- a hyperbola if  $e > 1$

- Theorem 6: A polar equation of the form

$$r = \frac{ed}{1 \pm e \cos \theta} \quad \text{or} \quad r = \frac{ed}{1 \pm e \sin \theta}$$

represents a conic section with eccentricity  $e$ . The conic is an ellipse if  $e < 1$ , parabola if  $e = 1$ , or a hyperbola if  $e > 1$

- Kepler's laws:

- 1 - A planet revolves around the sun in an elliptical orbit with the sun at one focus.
- 2 - The line joining the sun to a planet sweeps out equal areas in equal times.

- 3 - The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.
- Equation 7: The polar equation of an ellipse with focus at the origin, semimajor axis  $a$ , eccentricity  $e$ , and directrix  $x = d$  can be written in the form:

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta}$$

- Equation 8: The perihelion distance from a planet to the sun is  $a(1 - e)$  and the aphelion distance is  $a(1 + e)$