Chapter 6 Notes - LA

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April 19, 2022

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6 Vector Spaces

6.1 Vector Spaces and Subspaces

- In the past, we studied vectors in a concrete situation, $\mathbb{R}^n$. Now, we generalize "vectors" by abstracting them into a general setting.
- Let V be a set on which two operations, called addition and scalar multiplication, have been defined. If $\mathbf{u}$ and $\mathbf{v}$ are in V, the sum of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u}+\mathbf{v}$, and if c is a scalar, the scalar multiple of $\mathbf{u}$ by c is denoted by $c\mathbf{u}$. If the following axioms hold for all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d, then V is called a vector space and its elements are vectors.

1. $\mathbf{u}+\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
4. There exists an element $\mathbf{0}$ in V, called a zero vector, such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$
5. For each $\mathbf{u}$ in V, there is an element $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$
6. $c\mathbf{u}$ is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$
9. $c(d\mathbf{u})=(cd)\mathbf{u}$
10. $1\mathbf{u}=\mathbf{u}$

- Let V be a vector space $\mathbf{u}$ a vector in V, and c a scalar.
  - $0\mathbf{u}=\mathbf{0}$

  - $c\mathbf{0} = \mathbf{0}$
  - $(-1)\mathbf{u} = -\mathbf{u}$
  - If $c\mathbf{u} = \mathbf{0}$, then $c = 0$ or $\mathbf{u} = \mathbf{0}$

- A subset W of a vector space V is called a subspace of V if W is itself a vector space with the same scalars, addition, and scalar multiplication as V.

- Let V be a vector space and let W be a nonempty subset of V. Then W is a subspace of V IFF the following conditions hold:

  - If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u} + \mathbf{v}$ is in W
  - If $\mathbf{u}$ is in W and c is a scalar, then $c\mathbf{u}$ is in W.

- If W is a subspace of a vector space V, then W contains the zero vector $\mathbf{0}$ of V.

- If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is called the span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and is denoted by $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ or $\text{span}(S)$. If $V = \text{span}(S)$, then S is called a spanning set of V and V is said to be spanned by S.

- Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in a vector space V.

  - $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is a subspace of V.
  - $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is the smallest subspace of V that contains $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$

6.2 Linear Independence, Basis, and Dimension

- A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space V is linearly dependent if there are scalars $c_1, c_2, \ldots, c_k$, at least one of which is not zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

  A set of vectors that is not linearly dependent is said to be linearly independent.

- A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space V is linearly dependent IFF at least one of the vectors can be expressed as a linear combination of the others.

- A set S of vectors in a vector space V is linearly dependent if it contains finitely many linearly dependent vectors. A set of vectors that is not linearly dependent is said to be linearly independent.

- A subset $\mathcal{B}$ of a vector space V is a basis for V if

  - $\mathcal{B}$ spans V and
  - $\mathcal{B}$ is linearly indepnedent.

- Let V be a vector space and let $\mathcal{B}$ be a basis for V. For every vector $\mathbf{v}$ in V, there is exactly one way to write $\mathbf{v}$ as a linear combination of the basis vectors in $\mathcal{B}$

- Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a basis for the vector space V. Let $\mathbf{v}$ be a vector in V, and write $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$. Then $c_1, c_2, \ldots, c_n$ are called the coordinates of $\mathbf{v}$ with respect to $\mathcal{B}$, and the column vector

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

  is called the coordinate vector of $\mathbf{v}$ with respect to $\mathcal{B}$.

- Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a basis for a vector space V. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in V and let c be a scalar. Then

  - $[\mathbf{u}+\mathbf{v}]_{\mathcal{B}} = [\mathbf{u}]_{\mathcal{B}} + [\mathbf{v}]_{\mathcal{B}}$
  - $[c\mathbf{u}]_{\mathcal{B}} = c[\mathbf{u}]_{\mathcal{B}}$

- $$[c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k]_{\mathcal{B}} = c_1[\mathbf{u}_1]_{\mathcal{B}} + \cdots + c_k[\mathbf{u}_k]_{\mathcal{B}}$$

- Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a basis for a vector space V and let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ be vectors in V. Then $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is linearly independent in V IFF $\{[\mathbf{u}_1]_{\mathcal{B}}, \ldots, [\mathbf{u}_k]_{\mathcal{B}}\}$ is linearly independent in $\mathbb{R}^n$.

- Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a basis for a vector space V.
  - Any set of more than n vectors in V must be linearly dependent.
  - Any set of fewer than n vectors in V cannot span V.

- The Basis Theorem: If a vector space V has a basis with n vectors, then every basis for V has exactly n vectors.

- A vector space V is called finite-dimensional if it has a basis consisting of finitely many vectors. The dimension of V, denoted by dim V, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. A vector space that has no finite basis is called infinite-dimensional.

- Let V be a vector space with dim $V = n$. Then:
  - Any linearly independent set in V contains at most n vectors.
  - Any spanning set for V contains at least n vectors.
  - Any linearly indpendent set of exactly n vectors in V is a basis for V.
  - Any spanning set for V consisting of exactly n vectors is a basis for V.
  - Any linearly independent set in V can be extended to a basis for V.
  - Any spanning set for V can be reduced to a basis for V.

- Let W be a subspace of a finite-dimensional vector space V. Then:
  - W is finite-dimensional and dim $W \leq$ dim V.
  - dim $W =$ dim V IFF $W = V$

6.3 Change of Basis

- Let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be bases for a vector space V. The $n \times n$ matrix whose columns are the coordinate vectors $[\mathbf{u}_1]_{\mathcal{C}}, \ldots, [\mathbf{u}_n]_{\mathcal{C}}$ of the vectors in $\mathcal{B}$ with respect to $\mathcal{C}$ is denoted by $P_{\mathcal{C}\leftarrow\mathcal{B}}$ and is called the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$. That is,
$$P_{\mathcal{C}\leftarrow\mathcal{B}} = [[\mathbf{u}_1]_{\mathcal{C}}[\mathbf{u}_2]_{\mathcal{C}} \cdots [\mathbf{u}_n]_{\mathcal{C}}]$$

- Let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be bases for a vector space V and let $P_{\mathcal{C}\leftarrow\mathcal{B}}$ be the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$. Then
  - $P_{\mathcal{C}\leftarrow\mathcal{B}[\mathbf{x}]_{\mathcal{B}}=[\mathbf{x}]_{\mathcal{C}}}$ for all $\mathbf{x}$ in V.
  - $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is the unique matrix P with the property that $P[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$ for all $\mathbf{x}$ in V.
  - $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is invertible and $(P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1} = P_{\mathcal{B}\leftarrow\mathcal{C}}$.

- Let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be bases for a vector space V. Let $B = [[\mathbf{u}_1]_{\mathcal{E}} \cdots [\mathbf{u}_n]_{\mathcal{E}}]$ and $C = [[\mathbf{v}_1]_{\mathcal{E}} \cdots [\mathbf{v}_n]_{\mathcal{E}}]$, where $\mathcal{E}$ is any basis for V. Then the row reduction applied to the $n \times 2n$ augmented matrix $[C|B]$ produces
$$[C|B] \rightarrow [I|P_{\mathcal{C}\leftarrow\mathcal{B}}]$$

6.4 Linear Transformations

- A linear transformation from a vector space V to a vector space W is a mapping $T : V \to W$ such that, for all $\mathbf{u}$ and $\mathbf{v}$ in V and for all scalars c,
  - $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
  - $T(c\mathbf{u}) = cT(\mathbf{u})$
- $T : V \to W$ is a linear transformation IFF

  $$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_kT(\mathbf{v}_k)$$

  for all $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in V and scalars $c_1, \ldots, c_k$.
- Let $T : V \to W$ be a linear transformation. Then:
  - $T(\mathbf{0}) = \mathbf{0}$
  - $T(-\mathbf{v}) = -T(\mathbf{v})$ for all $\mathbf{v}$ in V.
  - $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in V.
- Let $T : V \to W$ be a linear transformation and let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a spanning set for V. Then $T(\mathcal{B}) = \{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ spans the range of T.
- If $T : U \to V$ and $S : V \to W$ are linear transformations, then the composition of S with T is the mapping $S \circ T$, defined by

  $$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$$

  where $\mathbf{u}$ is in U.
- If $T : U \to V$ and $S : V \to W$ are linear transformations, then $S \circ T : U \to W$ is a linear transformation.
- $R \circ (S \circ T) = (R \circ S) \circ T$
- A linear transformation $T : V \to W$ is invertible if there is a linear transformation $T' : W \to V$ such that

  $$T' \circ T = I_V \quad \text{and} \quad T \circ T' = I_W$$

  In this case, T' is called an inverse for T.
- If T is an invertible linear transformation, then its inverse is unique.

6.5 The Kernel and Range of a Linear Transformation

- Let $T : V \to W$ be a linear transformation. The kernel of T, denoted $\ker(T)$, is the set of all vectors in V that are mapped by T to $\mathbf{0}$ in W. That is,

  $$\ker(T) = \{\mathbf{v} \text{ in } V : T(\mathbf{v}) = \mathbf{0}\}$$

  The range of T, denoted $\text{range}(T)$, is the set of all vectors in W that are images of vectors in V under T. That is,

  $$\text{range}(T) = \{T(\mathbf{v}) : \mathbf{v} \text{ in } V\} = \{\mathbf{w} \text{ in } W : \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \text{ in } V\}$$

- Let $T : V \to W$ be a linear transformation. Then:
  - The kernel of T is a subspace of V.
  - The range of T is a subspace of W.
- Let $T : V \to W$ be a linear transformation. The rank of T is the dimension of the range of T and is denoted by $\text{rank}(T)$. The nullity of T is the dimension of the kernel of T and is denoted by $\text{nullity}(T)$.

- The rank theorem: Let $T : V \to W$ be a linear transformation from a finite-dimensional vector space V into a vector space W. Then

$$\text{rank}(T) + \text{nullity}(T) = \dim V$$

- A linear transformation $T : V \to W$ is called one-to-one if T maps distinct vectors in V to distinct vectors in W. If range$(T) = W$, then T is called onto.

- $T : V \to W$ is one-tocloftone if, for all $\mathbf{u}$ and $\mathbf{v}$ in V,

$$\mathbf{u} \neq \mathbf{v} \text{ implies that } T(\mathbf{u}) \neq T(\mathbf{v})$$

- Which is to say, if $T : V \to W$ is one-to-one if, for all $\mathbf{u}$ and $\mathbf{v}$ in V,

$$T(\mathbf{u}) = T(\mathbf{v}) \text{ implies that } \mathbf{u} = \mathbf{v}$$

- $T : V \to W$ is onto if, for all $\mathbf{w}$ in W, there is at leeast one $\mathbf{v}$ in V such that

$$\mathbf{w} = T(\mathbf{v})$$

- A linear transformation $T : V \to W$ is one-to-one IFF $\ker(T) = \{\mathbf{0}\}$.

- Let $\dim V = \dim W = n$. Then a linear transformation $T : V \to W$ is one-to-one IFF it is onto

- Let $T : V \to W$ be a one-to-one linear transformation. If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a linearly indepnedent set in V, then $T(S) = \{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\}$ is a linearly independent set in W.

- Let $\dim V = \dim W = n$. Then a one-to-one linear transformation $T : V \to W$ maps a basis for V to a basis for W.

- A linear transformation $T : V \to W$ is invertible IFF it is one-to-one and onto.

- A lineart transformation $T : V \to W$ is called an isomorphism if it is one-to-one and onto. If V and W are two vector spaces such that there is an isomorphism from V to W, then we say that V is isomorphic to W and write $V \cong W$.

- Let V and W be two finite-dimensional vector spaces (over the same field of scalars). Then V is isomorphic to W IFF $\dim V = \dim W$.

6.6 The Matrix of a Linear Transformation

- Let V and W be two finite-dimensional vector spaces with bases $\mathcal{B}$ and $\mathcal{C}$, respectively, where $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. If $T : V \to W$ is a linear transformation, then the $m \times n$ matrix A defined by

$$A = [[T(\mathbf{v}_1)]_{\mathcal{C}} | [T(\mathbf{v}_2)]_{\mathcal{C}} | \cdots | [T(\mathbf{v}_n)]_{\mathcal{C}}]$$

  satisfies

$$A[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}}$$

  for every vector $\mathbf{v}$ in V.

- $[T]_{\mathcal{C} \leftarrow \mathcal{B}} [\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}}$

- $[T]_{\mathcal{B}} [\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{B}}$

- Let U, V, and W be finite-dimensional vector spaces with bases $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$, respectively. Let $T : U \to V$ and $S : V \to W$ be linear transformations. Then

$$[S \circ T]_{\mathcal{D} \leftarrow \mathcal{B}} = [S]_{\mathcal{D} \leftarrow \mathcal{C}} [T]_{\mathcal{C} \leftarrow \mathcal{B}}$$

- Let $T : V \to W$ be a linear transformation between n-dimensional vector spaces V and W and let $\mathcal{B}$ and $\mathcal{C}$ be bases for V and W, respectively. Then T is invertible IFF the matrix $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$ is invertible. In this case,
$$([T]_{\mathcal{C}\leftarrow\mathcal{B}})^{-1} = [T^{-1}]_{\mathcal{B}\leftarrow\mathcal{C}}$$
- Let V be a finite dimensional vector space with bases $\mathcal{B}$ and $\mathcal{C}$ and let $T : V \to V$ be a linear transformation. Then
$$[T]_{\mathcal{C}} = P^{-1}[T]_{\mathcal{B}}P$$
where P is the change-of-basis matrix from $\mathcal{C}$ to $\mathcal{B}$.
- Let V be a finite-dimensional vector space and let $T : V \to V$ be a linear transformation. Then T is called diagonalizable if there is a basis $\mathcal{C}$ for V such that the matrix $[T]_{\mathcal{C}}$ is a diagonal matrix.
- The Fundamental Theorem of invertible matrices: version 4.
  - A is invertible
  - $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$
  - $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
  - The reduced row echelon form of A is I_n.
  - A is the product of elementary matrices.
  - $\text{rank}(A) = n$
  - $\text{nullity}(A) = 0$
  - The column vectors of A are linearly independent
  - The column vectors of A span $\mathbb{R}^n$
  - The column vectors of A form a basis for $\mathbb{R}^n$
  - The row vectors of A are linearly independent
  - The row vectors of A span $\mathbb{R}^n$
  - The row vectors of A form a basis for $\mathbb{R}^n$
  - $\det A \neq 0$
  - 0 is not an eigenvalue of A
  - T is invertible.
  - T is one-to-one.
  - T is onto.
  - $\ker(T) = \{\mathbf{0}\}$
  - $\text{range}(T) = W$

6.7 Applications

- The set S of all solutions to $y' + ay = 0$ is a subspace of $\mathscr{F}$
- If S is the solution space of $y' + ay = 0$, then $\dim S = 1$ and $\{e^{-at}\}$ is a basis for S.
- Let S be the solution space of
$$y'' + ay' + by = 0$$
and let λ_1 and λ_2 be the roots of the characteristic equation $\lambda^2 + a\lambda + b = 0$.
  - If $\lambda_1 \neq \lambda_2$, then $\{e^{\lambda_1 t}, e^{\lambda_2 t}\}$ is a basis for S.
  - If $\lambda_1 = \lambda_2$, then $\{e^{\lambda_1 t}, te^{\lambda_1 t}\}$ is a basis for S.