Chapter 11 Notes - MC

John Yang

June 22, 2021

Contents

11 Infinite Sequences and Series

11.1 Sequences

- sequence - a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \cdots, a_n, \cdots$$

- a_1 - first term; a_2 - second term; a_n - nth term
- For infinite series, every term a_n has a successor a_{n+1}
- Notation - the sequence $\{a_1, a_2, a_3, \cdots\}$ can also be written as

$$\{a_n\} \text{ or } \{a_n\}_{n=1}^{\infty}$$

- Definition 1: limits of sequences:

$$\lim_{n\to\infty} a_n = L$$

  - This means: as n becomes very large, the terms of the sequence $\{a_n\}$ approach L.
- can also be written as

$$a_n \to L \text{ as } n \to \infty$$

- If the limit at infinity exists, the sequence is convergent/converges. Otherwise, it is divergent/diverges.

- Definition 2: A more precise definition of the limit of a sequence:

$$\lim_{n\to\infty} a_n = L$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$\text{if } n > N \text{ then } |a_n - L| < \varepsilon$$

- Theorem 3: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.
- Equation 4:

$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \text{ if } r > 0$$

- Definition 5: $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$\text{if } n > N \text{ then } a_n > M$$

- Limit laws for sequences: If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then:

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \quad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \text{ if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \text{ if } p > 0 \text{ and } a_n > 0$$

- Squeeze Theorem can be adapted for sequences:

$$\text{If } a_n \leq b_n \leq c_n \text{ for } n \geq n_0 \text{ and } \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L, \text{ then } \lim_{n\to\infty} b_n = L$$

- Theorem 6: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$
- Theorem 7: If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

- Equation 8 (example 10):

$$a_n = \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdot \dots \cdot n}{n \cdot n \cdot n \cdot \dots \cdot n}$$

(ex. 10)

- Equation 9 (example 11): The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

- Definition 10: A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is **monotonic** if it is either increasing or decreasing.

- Definition 11: A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leq M \text{ for all } n \geq 1$$

It is bounded below if there is a number m such that

$$m \leq a_n \text{ for all } n \geq 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**

- Theorem 12: Monotonic Sequence Theorem: Every bounded, monotonic sequence is convergent.

- Proof of theorem 12: Suppose $\{a_n\}$ is an increasing sequence. Since $\{a_n\}$ is bounded, the set $S = \{a_n | n \geq 1\}$ has an upper bound. By the Completeness Axiom it has a least upper bound L. Given $\varepsilon > 0$, $L - \varepsilon$ is *not* an upper bound for S (since L is the *least* upper bound). Therefore

$$a_N > L - \varepsilon$$

For some integer N. But the sequence is increasing so $a_n \geq a_N$ for every $n > N$. Thus if $n > N$, we have

$$a_n > L - \varepsilon$$

so

$$0 \leq L - a_n < \varepsilon$$

since $a_n \leq L$. Thus,

$$|L - a_n| < \varepsilon \text{ whenever } n > N$$

so $\lim_{n\to\infty} a_n = L$. A similar proof can be applied if $\{a_n\}$ is decreasing.

11.2 Series

11.3 The Integral Test and Estimates of Sums

11.4 The Comparison Tests

11.5 Alternating Series

11.6 Absolute Convergence and the Ratio and Root Tests

11.7 Strategy for Testing Series

11.8 Power Series

11.9 Representations of Functions an Power Series

11.10 Taylor and Maclaurin Series

11.11 Applications of Taylor Polynomials