Ch 16 Notes

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Contents

16 Multiple Integrals

16.1 Double Integrals over Rectangles

- The double integral of f over the rectangle R is

$$\iint_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

  if this limit exists.

- If $f(x,y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$V = \iint_R f(x,y)dA$$

- Midpoint rule for double integrals:

$$\iint_R f(x,y)dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\Delta A$$

  where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

- Fubini's Theorem: If f is continuous on the rectangle $R = \{(x,y) \mid a \leq x \leq b, c \leq y \leq d\}$, then

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

  More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

- $$\iint\limits_R g(x)h(y)dA = \int_a^b g(x)dx \int_c^d h(y)dy \qquad \text{where } R = [a,b] \times [c,d]$$

16.2 Double Integrals over General Regions

- If F is integrable over R, then we define the double integral of f over D by
$$\iint\limits_D f(x,y)dA = \iint\limits_R F(x,y)dA \qquad \text{where } F \text{ is given by Equation 1}$$
- If f is continuous on a type I region D such that
$$D = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$
then
$$\iint\limits_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$
- Type II plane regions:
$$D = \{(x,y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$$
- If D is a type II region,
$$\iint\limits_D f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$
- Properties of double integrals
  - $$\iint\limits_D [f(x,y) + g(x,y)]dA = \iint\limits_D f(x,y)dA + \iint\limits_D g(x,y)dA$$
  - $$\iint\limits_D cf(x,y)dA = c\iint\limits_D f(x,y)dA$$
    where c is a constant
  - If $f(x,y) \ge g(x,y)$ for all (x,y) in D, then
$$\iint\limits_D f(x,y)dA \ge \iint\limits_D g(x,y)dA$$
- If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries, then
$$\iint\limits_D f(x,y)dA = \iint\limits_{D_1} f(x,y)dA + \iint\limits_{D_2} f(x,y)dA$$
- $$\iint\limits_D 1dA = A(D)$$
- If $m \le f(x,y) \le M$ for all (x,y) in D, then
$$mA(D) \le \iint\limits_D f(x,y)dA \le MA(D)$$

16.3 Double Integrals in Polar Coordinates

- Recall:

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$$

- Change to polar coordinates in a double integral: If f is continuous on a polar rectangle R given by $0 \le a \le r \le b, \alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint\limits_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r dr d\theta$$

- If f is continuous on a polar region of the form

$$D = \{(r,\theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$

then

$$\iint\limits_D f(x,y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r dr d\theta$$

16.4 Applications of Double Integrals

- mass of a lamina:

$$m = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint\limits_D \rho(x,y)dA$$

- Total charge in a given area:

$$Q = \iint\limits_D \sigma(x,y)dA$$

- Moment of a lamina about the x axis:

$$M_x = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} y_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint\limits_D \rho(x,y)dA$$

- About the y axis:

$$M_y = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint\limits_D x\rho(x,y)dA$$

- The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x,y)$ are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint\limits_D x\rho(x,y)dA \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint\limits_D y\rho(x,y)dA$$

where the mass m is given by

$$m = \iint\limits_D \rho(x,y)dA$$

- Moment of intertia about x axis:

$$I_x = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint\limits_D y^2\rho(x,y)dA$$

- About the y axis:

$$I_y = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D x^2 \rho(x,y)dA$$

- Moment of inertia about the origin, or polar moment of intertia:

$$I_0 = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \left[(x_{ij}^*)^2 + (y_{ij}^*)^2\right] \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D (x^2+y^2)\rho(x,y)dA$$

  where $I_0 = I_x + I_y$

- Radius of gyration of a lamina about an axis is the number R such that

$$mR^2 = I$$

- Radius of gyration $\overline{\overline{y}}$ with respect to x axis and radius of gyration $\overline{\overline{x}}$ with respect to the y axis are given by

$$m\overline{\overline{y}}^2 = I_x \qquad m\overline{\overline{x}}^2 = I_y$$

- Expected values: if X and Y are random variables with joint density function f, we defined the X-mean and Y-mean, or expected values of X and Y as

$$\mu_1 = \iint_{\mathbb{R}^2} xf(x,y)dA \qquad \mu_2 = \iint_{\mathbb{R}^2} yf(x,y)dA$$

- A single random variable is normally distributed if its probability density function is of the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

  where μ is the mean and σ is the standard deviation.

16.5 Surface Area

- The surface area of a surface S is

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta T_{ij}$$

- The area of the surface with equation $z = f(x,y), (x,y) \in D$, where f_x and f_y are continuous, is

$$A(S) = \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}dA$$

  which is also

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA$$

16.6 Triple Integrals

- The triple integral of f over the box B is

$$\iiint_B f(x,y,z)dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)dV$$

  if this limit exists.

- If we choose the sample point to be (x_i, y_j, z_k), we get

$$\iiint_B f(x,y,z)dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_i, y_j, z_k)\Delta V$$

- Fubini's theorem for triple integrals: If f is continuous on the rectangular box $B = [a,b]\times[c,d]\times[r,s]$, then

$$\iiint_B f(x,y,z)dV = \int_r^s \int_c^d \int_a^b f(x,y,z)dxdydz$$

- A solid region E is said to be of type 1 if it lies between the graphs of two continuous functions of x and y, that is,

$$E = \{(x,y,z) \mid (x,y)\in D, u_1(x,y) \le z \le u_2(x,y)\}$$

- If E is a type 1 region:

$$\iiint_E f(x,y,z)dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz\right] dA$$

- If the projection of D of E onto the xy plane is a type I plane region, then

$$E = \{(x,y,z) \mid a \le x \le b, g_1(x) \le y \le g_2(x), u_1(x,y) \le z \le u_2(x,y)\}$$

  , and

$$\iiint_E f(x,y,z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdydx$$

- If D is a type II plane region, then

$$E = \{(x,y,z) \mid c \le y \le d, h_1(y) \le x \le h_2(y), u_1(x,y) \le z \le u_2(x,y)\}$$

  , and

$$\iiint_E f(x,y,z)dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdxdy$$

- A solid region E is of type 2 if:

$$E = \{(x,y,z) \mid (y,z)\in D, u_1(y,z) \le x \le u_2(y,z)\}$$

  where D is the projection of E onto the yz plane. The back surface is $x = u_1(y,z)$ and the front surface is $x = u_2(y,z)$, and

$$\iiint_E f(x,y,z)dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dx\right] dA$$

- A type 3 region is of the form:

$$E = \{(x, y, z) \mid (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$$

where D is the projection of E onto the xz plane, $y = u_1(x, z)$ is the left surface, and $y = u_2(x, z)$ is the right surface. Thus,

$$\iiint_E f(x, y, z)dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)dy\right] dA$$

- If $f(x, y, z) = 1$ for all points in E, then:

$$V(E) = \iiint_E dV$$

16.7 Triple Integrals in Cylindrical Coordinates

- Recall:

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta \qquad z = z \qquad \tan\theta = \frac{y}{x}$$

- Triple integration in cylindrical coordinates:

$$\iiint_E f(x, y, z)dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r dz dr d\theta$$

16.8 Triple Integrals in Spherical Coordinates

- Recall:

$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi \qquad \rho^2 = x^2 + y^2 + z^2$$

- Triple integral in spherical coordinates:

$$\iiint_E f(x, y, z)dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi d\rho d\theta d\phi$$

where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$$

16.9 Change of Variables in Multiple Integrals

- We can write the substitution rule as:

$$\int_a^b f(x)dx = \int_c^d f(g(u))g'(u)du$$

where $x = g(u)$ and $a = g(c), b = g(d)$ which is also

$$\int_a^b f(x)dx = \int_c^d f(x(u))\frac{dx}{du}du$$

- The Jacobian of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

- Approximation to the area ΔA of R:

$$\Delta A \approx \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v$$

where the Jacobian is evaluated at (u_0, v_0)

- Change of variables in a double integral: Suppose that T is a C^1 transformation whole Jacobian is nonzero and that T maps a region S in the uv plane onto a region R in the xy plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then:

$$\iint\limits_R f(x,y)dA = \iint\limits_S f(x(u,v)y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv$$

- If:

$$x = g(u,v,w) \qquad y = h(u,v,w) \qquad z = k(u,v,w)$$

then the Jacobian of T is given by:

$$\frac{\partial(x,y,z)}{\partial u,v,w} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

- Change of variables for triple integrals:

$$\iiint\limits_R f(x,y,z)dV = \iiint\limits_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dudvdw$$