NAME: J. Yang

12.

1. Determine if the sequences below converge. If they do find the limits $n \to \infty$

   A. $\frac{\sin n}{n}$ B. ne^{-n}

Bc 14
Bu 12
BB 26
R 15
BBR 41
T 41

2.

   A. Show that if $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$, $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$

   B. If $a_1 = 1$ and $a_{n+1} = 1 + \frac{1}{1+a_n}$, show that $\lim_{n\to\infty} a_n = \sqrt{2}$.

   Hint: Use part A.

3.

   A. Calculate $\sum_{n=0}^{\infty} ar^n$

   B. Under what conditions is this series convergent and what does it converge to?

4. For what values of p is the sum $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

5. Estimate $\sum_{n=1}^{\infty} \frac{1}{n^2}$ using the sum of the first 9 terms, then approximating the rest.

6. Determine if the series below are convergent

   A. $\sum_{n=0}^{\infty} \frac{1}{2^n + 1}$

   B. $\sum_{n=1}^{\infty} \frac{1}{n^2 + n}$

7. For what value of x is the series $\sum_{n=1}^{\infty} n!x^n$ convergent?

8. Obtain the radius of convergenceand interval of convergence for $\sum_{n=1}^{\infty} \frac{(-a)^n x^n}{\sqrt{n+1}}$.

9. Obtain a power series representation for ln(1-x).

10. Write down the Taylor and Maclaurin series of the function *f(x)*.

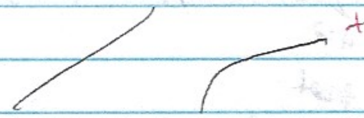
11. Obtain the Taylor series of e^x at a and 0.

Ch. 11 Test

1) a) $\frac{\sin n}{n}$ n^{th} term test ~

$\lim_{n\to\infty} \frac{\sin n}{n} \overset{L'Hopital}{=} \frac{\cos n}{1} \neq 0 \to$ diverges ~

b) $ne^{-n} = \frac{n}{e^n}$



+1

$\lim_{n\to\infty} \frac{n}{e^n} = \frac{1}{e^n} = 0$ +1 ★

2) a) $\lim_{n\to\infty} a_{2n} = L$, $\lim_{n\to\infty} a_{2n+1} = L$ ~~

$\{a_n\}$ converges & $\lim_{n\to\infty} a_n = L$ ~~ ~~

is a_{2n} & a_{2n+1} ? ★

b) ? ★ ~ ~ ~ ~ ~ ~

3) a) $\sum_{n=0} ar^n = \frac{a}{1-r}$ ★ ~ ~ ~ ~

b) converges is $|r|<1$ to $\frac{a}{1-r}$ +1

4) $\sum_{n=1}^{\infty} \frac{1}{n^p} \to$ p-series, $p>1$ +1 ~

5) $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \frac{1}{81} + \dots$ integral ~

$= 1.5397\ldots \to 1.6?\ 1.75?$ ~

6) a) dir. comp. $\sum \frac{1}{2^n} = (\frac{1}{2})^n \to$ geo, conv. +1

$\sum \frac{1}{2^n+1}$ conv $\quad \frac{1}{2^n+1} < \frac{1}{2^n}$ +

b) $\frac{1}{n^2} \to$ p-series $\to$ converges +

$\frac{1}{n^2+n} < \frac{1}{n^2} \quad \sum \frac{1}{n^2+n}$ +1

converges

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7) $\sum_{n=1}^{\infty} n! x^n \quad \lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right| < 1$

$= \lim_{n\to\infty} |(n+1)x| < 1$ +1

Can only converge if $x = 0$ +1

8) $\lim_{n\to\infty} \left| \frac{(-a)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-a)^n x^n} \right|$ +1

$= \lim_{n\to\infty} \left| \frac{-a \cdot x \cdot \sqrt{n+1}}{\sqrt{n+2}} \right| < 1$

$|-ax| < 1$ +1

$-ax < 1$

$ax \leq 1$

$ax > -1 \quad -1 < ax < 1$

$-\frac{1}{a} < x < \frac{1}{a}$

$\boxed{R = \frac{2}{a}}$ ~

at $x = -\frac{1}{a}$, $\frac{(-a)^n \cdot (-\frac{1}{a})^n}{\sqrt{n+1}}$

$= \frac{1}{\sqrt{n+1}} \quad \frac{1}{n^{1/2}} \to$ diverges

at $x = \frac{1}{a}$ $\quad (-a)^n \cdot (\frac{1}{a})^n$

$\frac{(-1)^n}{\sqrt{n+1}}$ alt. series converges ★

IOC $\boxed{-\frac{1}{a} < x \leq \frac{1}{a}}$

9) $\ln(1-x)$ ~ ~ ★

10) Taylor $\frac{(x-a)^n \cdot f^{(n)}(a)}{n!}$

$\sum_{n=0}^{\infty} \frac{f^{(n)}(a) \cdot (x-a)^n}{n!}$ +1

Maclaurin: $\sum_{n=0} \frac{f^{(n)}(0) \cdot (x)^n}{n!}$ +1

✓

11) e^x $f(a) = e^a, e^a, e^a \ldots$

$f(0) = e^0 = 1, 1, 1, 1 \ldots$

at $c=a$:

$$e^x = \frac{(1)\cdot e^a}{0!=1} + \frac{(x-a)\cdot e^a}{1} + \frac{(x-a)^2 e^a}{2!}$$

$$= e^a + (x-a)e^a + \frac{(x-a)^2 e^a}{2} + \frac{(x-a)^3 e^a}{6} + \ldots$$

at $c=0$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$$

Sigma notation

0

1) b) 2) (a)(b) 3) a) 9)

1) ne^{-n}, if $\lim_{n\to\infty} ne^{-n}$ exists, the seq converges

$\lim_{n\to\infty} \frac{n}{e^n} \to$ l'Hopital's rule $\lim_{n\to\infty} \frac{1}{e^n} = 0$ converges. (0)

2) ?

3) $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \ldots = S_n$ $+ ar^n$

$rS_n = ar + ar^2 + \ldots ar^{n+1}$ +1

$= ar + ar^n + ar^n$

$S_n - rS_n = a - ar^n$ +1

$S_n(1-r) = a - ar^n$

$S_n = \frac{a(1-r^n)}{1-r}$ +1

~ divergence/convergence condition

9) $\frac{d}{dx}\ln(1-x) = -\frac{1}{1-x} = \frac{1}{x-1}$

$= -\sum_{n=0}^{\infty} x^n$ +1 $\ln(1-x) = \int\left(-\sum_{n=0}^{\infty} x^n\right)dx$

$$= -C - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + 1$$

explicitly state starting condition 5/

1) a) 2) (a)(b)

3) $\sum_{n=0}^{\infty} ar^n$ diverges if $|r| \geq 1$ and converges if $|r| < 1$ +1

4) proof

5) 8) $R = \frac{1}{a}$ IOC $\left(-\frac{1}{a}, \frac{1}{a}\right]$ +1

9) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ +1

11) $f^n(x) = e^a, e^a, e^a, \ldots$

$f^n(0) = 1, 1, 1, 1, \ldots$

e^x at $a = \sum \frac{f^{(n)}(a)\cdot(x-a)^n}{n!}$ +1

$= \sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n$

e^x at 0: $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)\,x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ +1

5) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \underbrace{1 + \frac{1}{4} + \frac{1}{9} + \ldots \frac{1}{81}}_{=1.54} + \int_{10}^{\infty} \frac{1}{x^2}dx$ +1

$$\int_{10}^{\infty} \frac{dx}{x^2} = \lim_{b\to\infty}\int_{10}^{b} \frac{dx}{x^2} = \lim_{b\to\infty}\left(-\frac{1}{x}\right)_{10}^{b}$$

$$= \lim_{b\to\infty}\left(-\frac{1}{b} + \frac{1}{10}\right) = \frac{1}{10}$$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2} = 1.54 + 0.1 = \boxed{1.64}$ +1 7/

1) a) converges to 0 by squeeze theorem

$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$

$\lim_{n\to\infty} \frac{1}{n} = 0$ $0 \leq \frac{\sin n}{n} \leq 0 \Rightarrow 0$ +2

4) $\int_1^{\infty} \frac{dn}{n^p} = \left.\frac{n^{1-p}}{1-p}\right|_1^{\infty}$ +1

3/

2) a) $\lim_{n\to\infty} a_{2n} = L$, $\exists N_1 \to |a_{2n} - L| < \varepsilon$ for $n > N_1$, and since $\lim_{n\to\infty} a_{2n+1} = L$, $\exists$ $N_2 \to |a_{2n+1} - L| < \varepsilon$ for $n > N_2$

Let $N = \max\{2N_1, 2N_2 + 1\}$ and let $n > N$.

If n is even, $n = 2m$, $m > N_1$, $|a_n - L| = |a_{2m} - L| < \varepsilon$

If n is odd, $n = 2m+1$, $m > N_2$, $|a_n - L| = |a_{2m+1} - L| < \varepsilon$

$\therefore \{a_n\}$ converges and $\lim_{n\to\infty} a_n = L$

+6

b) If $a_1 = 1$ and $a_{n+1} = 1 + \frac{1}{1+a_n}$, show that $\lim_{n\to\infty} a_n = \sqrt{2}$

- odd terms increase
- even terms decrease.
  - two seqs are bounded monotonic b/w 1 and 2.

$$a_{n+2} = 1 + \frac{1}{1+1+1/(1+a_n)} = \frac{4+3a_n}{3+2a_n}$$

$$a_{2n+2} = 1 + \frac{1}{1+1+1/(1+a_{2n})} = \frac{4+3a_{2n}}{3+2a_{2n}}$$

$$L = \frac{4+3L}{3+2L} \to L^2 = 2 \quad L = \sqrt{2}$$

$$\lim_{n\to\infty} a_n = \sqrt{2}$$

+6

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J. Yang ~~Bu 14~~ Bu 12 R 15 Tp ~~…~~ 41

BB 26 BBR 41

1.

A. $\frac{\sin n}{n}$ converges to zero (*1pt*) by the squeeze theorem

$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$, $\lim_{n\to\infty} \frac{1}{n} = 0$, $0 \le \frac{\sin n}{n} \le 0$ (*1pt*)

B. $ne^{-n} = \frac{n}{e^n}$. e^n approaches ∞ faster than n does.(*1pt*)

E.g., use L'Hospital's rule. Therefore, $\lim_{n\to\infty} ne^{-n} = 0$ (*1pt*)

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2.

A. Since $\lim_{n\to\infty} a_{2n} = L$, $\exists\ N_1 \Rightarrow |a_{2n} - L| < \varepsilon$ for $n > N_1$ (*1pt*) and since $\lim_{n\to\infty} a_{2n+1} = L$, $\exists\ N_2 \Rightarrow$ $|a_{2n+1} - L| < \varepsilon$ for $n > N_2$ (*1pt*).

Let $N = \max\{2N_1, 2N_2+1\}$ and let $n > N$.(*1pt*) I(*1pt*)f n is even $n = 2m$, $m > N_1$, $|a_n - L| = |a_{2m} - L| < \varepsilon$ (*1pt*).

If n is odd $n = 2m+1$, $m > N_2$

, $|a_n - L| = |a_{2m+1} - L| < \varepsilon$ (*1pt*).

Therefore, $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$ (*1pt*)

B. If $a_1 = 1$ and $a_{n+1} = 1 + \frac{1}{1+a_n}$, show that $\lim_{n\to\infty} a_n = \sqrt{2}$.

When calculated, you will notice that the odd terms are increasing but the even ones are decreasing. Proof by induction confirms that(*1pt*). However, all terms lie between 1 and 2. The two series are bounded monotonic sequences(*1pt*).

Algebraic manimulations give

$$a_{n+2} = 1 + \frac{1}{1+1+1/(1+a_n)} = \frac{4+3a_n}{3+2a_n} \text{ (1pt)}$$

$$a_{2n+2} = 1 + \frac{1}{1+1+1/(1+a_n)} = \frac{4+3a_{2n}}{3+2a_{2n}} \text{ (1pt)}$$

Taking the limist of both sides give $L = \frac{4+3L}{3+2L}$ (*1pt*)

Solving this for L gives $L^2 = 2$, $L = \sqrt{2}$ (*1pt*)

Thus $\lim_{n\to\infty} a_n = \sqrt{2}$.

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3.

A. $s_n = a + ar + ar^2 + \ldots + ar^{n-1}$

$rs_n = ar + ar^2 + \ldots + ar^n$ (*1pt*)

$s_n - rs_n = a - ar^n$ (*1pt*)

$s_n = \frac{a(1-r^n)}{1-r}$ (*1pt*)

$\sum_{n=0}^{\infty} ar^n$ diverges if $r \ge 1$ but converges if $r < 1$ (*1pt*)

B. $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ if $r < 1$ (*1pt*)

4. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ (*1pt*). E.g., $\int_1^{\infty} \frac{dn}{n^p} = \frac{n^{1-p}}{1-p}\Big]_1^{\infty}$ (*1pt*)

Clearly, this result converges only for $p > 1$.

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5.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \frac{1}{81} + \int_{10}^{\infty} \frac{dx}{x^2}$$

$= 1.54 - \frac{1}{x}\Big]_{10}^{\infty} = 1.64$ (**2 pts**) (*1pt* for each line)

6.

A. $\sum_{n=0}^{\infty} \frac{1}{2^n+1}$ convergent (*1pt*) Comparison: $\frac{1}{2^n+1} < \frac{1}{2^n}$ (*1pt*)

B. $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ convergent (*1pt*) Comparison: $\frac{1}{n^2+n} < \frac{1}{n^2}$ (*1pt*)

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7. $\sum_{n=1}^{\infty} n! x^n$

$\frac{(n+1)!x^{n+1}}{n!\, x^n} = (n+1)x < 1$ (*1pt*)

We need $x < \frac{1}{n+1}$ i.e. $x=0$ (*1pt*)

8. $\sum_{n=1}^{\infty} \frac{(-a)^n x^n}{\sqrt{n+1}}$. As $n \to \infty$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-a)^{n+1}x^{n+1}}{\sqrt{n+2}}}{\frac{(-a)^n x^n}{\sqrt{n+1}}}\right| = \left|\frac{\sqrt{n+1}}{\sqrt{n+2}}ax\right| \to |ax| \text{ (1pt+1pt)}$$

The radius convergence is $1/a$ and the interval of convergence is $\left(-\frac{1}{a}, \frac{1}{a}\right]$ (*1pt*)

5

9. We know from question 3 that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ (*1pt*)

$$\ln(1-x) = -\int \frac{dx}{1-x} = -\sum_{n=0}^{\infty} \int x^n dx = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

(*1pt+1pt*)

10.

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ Taylor series (*1pt*)

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$ MacLaurin series (*1pt*)

11. Taylor series of e^x at a and 0

$f(x) = \sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n$ (*1pt*) and

$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ since $\frac{de^x}{dx} = e^x$ (*1pt*)

7

Mesut B. Çakır

∃ ⇒ "there exists"