

Chapter 17 Notes

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17 Vector Calculus

17.1 Vector Fields

- Def: Let D be a set in \mathbb{R}^2 (a plane region). A vector field on \mathbb{R}^2 is a function \mathbf{F} that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$
- Def: Let E be a subset of \mathbb{R}^3 . A vector field on \mathbb{R}^3 is a function \mathbf{F} that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$
- Recall that the gradient of a scalar function f of two variables ∇f is defined by

$$\nabla f(x, y) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

Therefore ∇f is really a vector field on \mathbb{R}^2 and is called a gradient vector field. Likewise, if f is a scalar function of 3 variables, its gradient is a vector field on \mathbb{R}^3 given by

$$\nabla f(x, y, z) = f_x(x, y, z) \mathbf{i} + f_y(x, y, z) \mathbf{j} + f_z(x, y, z) \mathbf{k}$$

17.2 Line Integrals

- Def: if f is defined on a smooth curve C given by

$$x = x(t) \quad y = y(t) \quad a \leq t \leq b$$

then the line integral of f along C is

$$\int_C f(x, y) ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta s_i$$

if this limit exists.

- If f is a continuous function, then the limit always exists and the line integral is given by:

$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The value of the line integral does not depend on the parameterization of the curve, provided that the curve is traversed exactly once as t increases from a to b .

- Line integrals of f along C with respect to x and y :

$$\int_C f(x, y) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta x_i = \int_a^b f(x(t), y(t)) x'(t) dt$$

$$\int_C f(x, y) dy = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta y_i = \int_a^b f(x(t), y(t)) y'(t) dt$$

- Recall that the vector representation of the line segment that starts at \mathbf{r}_0 and ends at \mathbf{r}_1 is given by

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \quad 0 \leq t \leq 1$$

- For line integrals in space, where C is a curve given by

$$x = x(t) \quad y = y(t) \quad z = z(t) \quad a \leq t \leq b$$

or

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

then the line integral of f along C is given by

$$\int_C f(x, y, z) ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*, z_i^*) \Delta s_i = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

- We evaluate integrals of the form

$$\int_C P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz$$

by expressing everything (x, y, z, dx, dy, dz) in terms of the parameter t .

- Def: Let \mathbf{F} be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Then the line integral of \mathbf{F} along C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_C \mathbf{F} \cdot \mathbf{T} ds$$

- We have:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy + R dz \quad \text{where} \quad \mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

17.3 The Fundamental Theorem for Line Integrals

- Theorem: Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C . Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

- Theorem: $\int_C \mathbf{F} \cdot d\mathbf{r}$ is an independent path in D IFF $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D .
- Theorem: Suppose \mathbf{F} is a vector field that is continuous on an open connected region D . If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D , then \mathbf{F} is a conservative vector field on D ; that is, there exists a function f such that $\nabla f = \mathbf{F}$
- Theorem: If $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D , then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

- Theorem: Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D . Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D$$

17.4 Green's Theorem

- Green's Theorem gives the relationship b/w a line integral around a simple closed curve C and a double integral over the plane region D bounded by C .
- We use the convention that the positive orientation of C means traversing C once counterclockwise.
- Green's Theorem: Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C . If P and Q have continuous partial derivatives on an open region that contains D , then

$$\int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

– Note: the notation $\oint_C Pdx + Qdy$ is sometimes used to show that it is a closed path integral.

- To find the area of D :

$$A = \oint_C xdy = - \oint_C ydx = \frac{1}{2} \oint_C xdy - ydx$$

17.5 Curl and Divergence

- If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on \mathbb{R}^3 and the partial derivatives of P , Q , and R all exist, then the curl of F is the vector field on \mathbb{R}^3 defined by

$$\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

•

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F}$$

- Theorem: If f is a function of three variables that has continuous second-order partial derivatives, then

$$\text{curl}(\nabla f) = \mathbf{0}$$

- Theorem: If \mathbf{F} is a vector field defined on all of \mathbb{R}^3 whose component functions have continuous partial derivatives and $\text{curl } \mathbf{F} = \mathbf{0}$, then \mathbf{F} is a conservative vector field.

- Divergence: if $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on \mathbb{R}^3 and $\frac{\partial P}{\partial x}$, $\frac{\partial Q}{\partial y}$, and $\frac{\partial R}{\partial z}$ exist, then the divergence of \mathbf{F} is the function of three variables defined by

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

It can also be written as

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$$

- Theorem: If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on \mathbb{R}^3 and P, Q , and R have continuous second-order partial derivatives, then

$$\operatorname{div} \operatorname{curl} \mathbf{F} = 0$$

- Vector form of Green's Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} dA$$

- Which is also

$$\oint_C \mathbf{F} \cdot \mathbf{n} ds = \iint_D \operatorname{div} \mathbf{F}(x, y) dA$$

17.6 Parametric Surfaces and Their Areas

- Given the vector function

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

The set of all points (x, y, z) in \mathbb{R}^3 such that

$$x = x(u, v) \quad y = y(u, v) \quad z = z(u, v)$$

and (u, v) varies throughout D is called a parametric surface S

- A surface of revolution can be represented parametrically with

$$x = x \quad y = f(x) \cos \theta \quad z = f(x) \sin \theta$$

- Given a parametric surface S , if u is kept constant by $u = u_0$, then $\mathbf{r}(u_0, v)$ defines the grid curve C_1 on S . The tangent vector to C_1 at a point P_0 is given by

$$\mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

If v is kept constant by $v = v_0$, the grid curve C_2 given by $\mathbf{r}(u, v_0)$ lies on S and its tangent vector at P_0 is given by

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}$$

If $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$ then the surface S is called smooth. For a smooth surface, $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to the tangent plane.

- Def: If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \quad (u, v) \in D$$

and S is covered just once as (u, v) ranges throughout the parameter domain D , then the surface area of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA$$

where

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \quad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

- Special case, where $z = f(x, y)$ where $(x, y) \in D$ and f has continuous partial derivatives, we have the parametric equations

$$x = x \quad y = y \quad z = f(x, y)$$

so

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial f}{\partial x}\right) \mathbf{k} \quad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial f}{\partial y}\right) \mathbf{k}$$

and

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x} \mathbf{i} - \frac{\partial f}{\partial y} \mathbf{j} + \mathbf{k}$$

which gives

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

and the surface area is

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA$$

17.7 Surface Integrals

- The surface integral of f over the surface S is given by

$$\iint_S f(x, y, z) dS = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(P_{ij}^*) \Delta S_{ij} = \iint_D f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| dA$$

- for a surface S with $z = g(x, y)$ the surface integral becomes

$$\iint_S f(x, y, z) dS = \iint_D f(x, y, g(x, y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} dA$$

- Def: If \mathbf{F} is a continuous vector field defined on an oriented surface S with unit normal vector \mathbf{n} , then the surface integral of \mathbf{F} over S is given by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA = \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA$$

which is also called the flux of \mathbf{F} across S .

17.8 Stokes' Theorem

- Stokes' Theorem: Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let \mathbf{F} be a vector field whose components have continuous partial derivatives on an open region in \mathbb{R}^3 that contains S . Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

17.9 The Divergence Theorem

- Divergence Theorem: Let E be a simple solid region and let S be the boundary surface of E , given with positive (outward) orientation. Let \mathbf{F} be a vector field whose component functions have continuous partial derivatives on an open region that contains E . Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F} dV$$

17.10 Summary of Chapter 17

- All main results of Chapter 17 are higher-order versions of the Fundamental Theorem of calculus.
- Fundamental Theorem of Calculus:

$$\int_a^b F'(x)dx = F(b) - F(a)$$

- Fundamental Theorem for Line Integrals:

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

- Green's Theorem:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C Pdx + Qdy$$

- Stokes' Theorem:

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

- Divergence Theorem:

$$\iiint_E \text{div } \mathbf{F} dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$