# Chapter 14 Notes - MC

## John Yang

### August 18, 2021

## Contents

## 14 Vector Functions

### 14.1 Vector Functions and Space Curves

- vector-valued functions/vector functions - a function whose domain is a set of real numbers and whose range is a set of vectors. Written in terms of its components as a parametric equation:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

- Limits of a vector function: If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

provided the limits of the component functions exist.

- Space curves: suppose that $f$, $g$, and $h$ are continuous real-valued functions on an interval $I$. Then the space curve is the set $C$ of all points $(x, y, z)$ in space, where

$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

and $t$ varies throughout the interval $I$.

### 14.2 Derivatives and Integrals of Vector Functions

- derivative of a vector-valued function:

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if the limit exists.

- Theorem 2: If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where $f$, $g$, and $h$ are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

- Theorem 3: Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, $c$ is a scalar, and $f$ is a real valued function. Then:

$$\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$$

$$\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$$

$$\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$$

$$\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$$

$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$$

$$\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$$

(chain rule)

- Integral of a vector function:

$$\int_a^b \mathbf{r}(t)dt = \left( \int_a^b f(t)dt \right)\mathbf{i} + \left( \int_a^b g(t)dt \right)\mathbf{j} + \left( \int_a^b h(t)dt \right)\mathbf{k}$$

## 14.3 Arc Length and Curvature

- Length of a curve in 3D space:

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}dt = \int_a^b |\mathbf{r}'(t)|dt$$

- curvature of a curve is defined as:

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$$

where $\mathbf{T}$ is the unit tangent vector.

- 

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

- Theorem 10: The curvature of the curve given by the vector function $\mathbf{r}$ is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

- Equations for unit tangent, unit normal and binormal vectors, and curvature:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \qquad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times r''(t)|}{|\mathbf{r}'(t)|^3}$$

## 14.4   Motion in Space: Velocity and Acceleration

- Velocity:
$$\mathbf{v}(t) = \mathbf{r}'(t)$$

- speed is the magnitude of velocity.

- Parametric equations of trajectory:
$$x = (v_0 \cos \alpha)t \qquad y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2$$

- Tangential and normal components of acceleration:
$$\mathbf{a} = v'(\mathbf{T}) + \kappa v^2 \mathbf{N}$$

- Kepler's laws:

  - A planet revolves around the sun in an elliptical orbit with the sun at one focus.
  - The line joining the sun to a planet sweeps out equal areas in equal times.
  - The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of orbit.