Ch. 13 - Vectors & the geometry of Space

§13.1 - 3D coordinate systems

Ex. 1

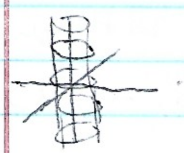
a) $z=5 \to$ plane, horizontal, at $z=5$ containing all x and all y

b) $y=5 \Rightarrow$ vertical plane at $y=5$ containing all x and all z

Ex. 2 (x,y,z) $x^2+y^2=1$ $z=3$

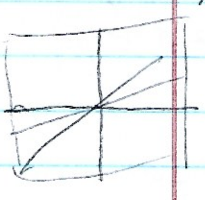
a) all points in the unit circle centered at $(0,0,3)$ in the plane $z=3$

b) it is a cylindrical surface



Ex. 3 $\Rightarrow$ $y=x$ in $\mathbb{R}^3$

plane, diagonal



in $\mathbb{R}^2$, $y=x$ is a line

in $\mathbb{R}^3$, $y=x$ is the plane that includes all z are the line.

Ex. 4 N/A

Ex. 5 ~~Eq.~~ Eq. distance from center $C(h,k,l)$

$x^2+y^2+z^2=r^2$

$(x-h)^2+(y-k)^2+(z-l)^2=r^2$

Ex. 6

$x^2+4x+4-4+y^2-6y+9-9$

$+z^2+2z+1-1+6=0$

$(x+2)^2+(y-3)^2+(z+1)^2=-6+4+9+1$

$=8$

Center $(-2,3,-1)$

Sphere b/c it takes the correct form.

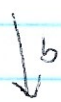
$r=\sqrt{8}=2\sqrt{2}$

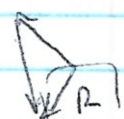
Ex. 7 $1 \le x^2+y^2+z^2 \le 4$

Volume b/w the sphere at center $(0,0,0)$ of radius 2 & sphere at origin of radius 1

§13.2 Vectors

Ex. 1 $\vec{a}+\vec{b}$





Ex. 2 $\vec{a}-2\vec{b}=\vec{a}+2(-\vec{b})$

Ex. 3 $\vec{a}=\langle -2-2,\ 1+3,\ 1-4\rangle$

$=\langle -4,4,-3\rangle$

Ex. 4 $\vec{a}=\langle 4,0,3\rangle$ $\vec{b}=\langle -2,1,5\rangle$

$|a|=\sqrt{4^2+0^2+3^2}=\sqrt{16+9}=5$

$\vec{a}+\vec{b}=\langle 2,1,8\rangle$

$\vec{a}-\vec{b}=\langle 6,-1,-2\rangle$

$3\vec{b}=\langle -6,3,15\rangle$

$2\vec{a}+5\vec{b}=\langle -2,5,31\rangle$

EX-5 $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ $\quad \vec{b} = 4\hat{i} + 7\hat{k}$

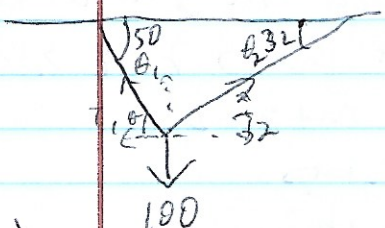
$2\vec{a} + 3\vec{b} = 2\hat{i} + 4\hat{j} - 6\hat{k} + 12\hat{i} + 21\hat{k}$

$= 14\hat{i} + 4\hat{j} + 15\hat{k}$

EX-6 $u = \frac{\vec{a}}{|\vec{a}|}$

$= \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{4+1+4}} = \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$

EX-7



$\vec{T}_1 + \vec{T}_2 = 100\hat{j}$

$\vec{T}_{1x} + \vec{T}_{2x} = 0$

$T_1 \cos\theta_1 = T_2 \cos\theta_2$ $\quad \frac{T_1}{T_2} = \frac{\cos\theta_2}{\cos\theta_1} = \frac{\cos 32}{\cos 50}$

$= 1.32$

$T_1 \sin\theta_1 + T_2 \sin\theta_2 = 100$ $\quad T_1 = 1.32\, T_2$

$1.32\, T_2 \sin\theta_1 + T_2 \sin\theta_2 = 100$

$T_2 = \frac{100}{\sin 32 + 1.32 \sin 50} = 64.9\ lb$

$T_1 = 1.32\, T_2 = 85.7\ lb$

$T_1 = 85.7$ @ $130°$

$T_2 = 64.9$ @ $32°$

§13.3 Dot Products

EX-1 N/A

EX-2 $a \cdot b = ab\cos\theta = 4 \cdot 6 \cdot \cos\frac{\pi}{3}$

$= 12$

EX-3 $\vec{a} = \langle 2, 2, -1 \rangle$ $\quad \vec{b} = (5, -3, 2)$

$a \cdot b = (2)(5) + (2)(-3) + (-1)(2)$

$= 10 - 6 - 2 = 2 = ab\cos\theta$

$\cos\theta = \frac{2}{|a||b|} = \frac{2}{\sqrt{4+4+1} \cdot \sqrt{25+9+4}}$

$= \frac{2}{18.5}$

$\theta = \cos^{-1}\left(\frac{2}{18.5}\right) = 83.8°$

EX-4 $a \cdot b = 0 \rightarrow$ orthogonal

$\langle 2, 2, -1 \rangle \cdot \langle 5, -4, 2 \rangle$

$= (2)(5) + (2)(-4) + (-1)(2)$

$= 10 - 8 - 2 = 0$ $\therefore$ they are orthogonal

EX-5 $\vec{a} = (1, 2, 3)$

$|a| = \sqrt{1+4+9} = \sqrt{14}$

$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) = 74.5°$

$\beta = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) = 57.7°$

$\gamma = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) = 36.7°$

Ex-6

$\text{comp}_{\vec{a}} \vec{b} = \frac{a \cdot b}{|a|} = \frac{(1)(-2) + (1)(3) + (2)(1)}{\sqrt{4+9+1}}$

$= 0.802$

$\text{proj}_a b = \frac{a \cdot b}{|a|^2} \cdot \vec{a} = \frac{3}{14}\langle -2, 3, 1 \rangle$

$= \langle \frac{-3}{7}, \frac{9}{14}, \frac{3}{14} \rangle$

EX.7 $W = F D \cos\theta = (70N)(100m)\cos 35$

$\approx 5.7 \text{ kJ}$

EX.8 $W = \langle 3, 4, 5 \rangle \cdot \langle 4-2, 6-1, 2-0 \rangle$

$= \langle 3, 4, 5 \rangle \cdot \langle 2, 5, 2 \rangle$

$= (3)(2) + (4)(5) + (5)(2)$

$= 6 + 20 + 10 = \boxed{36 \text{ J}}$

§13.4 cross product

EX.1 N/A

EX.2 $\vec{a} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$

$= \langle a_2a_3 - a_3a_2, a_3a_1 - a_1a_3, a_1a_2 - a_2a_1 \rangle$

$= \langle 0, 0, 0 \rangle$

EX.3

$\vec{a} = \vec{PQ} = \langle -3, 1, -7 \rangle$

$\vec{b} = \vec{PR} = \langle 0, -5, -5 \rangle$

$\vec{a} \times \vec{b} = \langle (1)(-5) - (-7)(-5), (-7)(0) - (-3)(-5),$

$(-3)(-5) - (1)(0) \rangle$

$= \langle -5 - 35, -15, 15 \rangle$

$= \langle -40, -15, 15 \rangle$

EX.4 $A = \frac{1}{2} |\vec{a} \times \vec{b}|$

$= \frac{1}{2} \sqrt{40^2 + 15^2 + 15^2}$

$= \frac{1}{2} \sqrt{2050} = \frac{5}{2} \sqrt{82}$

EX.5

$\vec{a} \cdot (\vec{b} \times \vec{c}) = \langle 1, 4, -7 \rangle \cdot ($

vol determinant form

$a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

& $a = \langle 1, 4, -7 \rangle$ $b = \langle 2, -1, 4 \rangle$

$c = \langle 0, -9, 18 \rangle$

$a \cdot (b \times c) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$

$= 1(-18 + 36) - 4(36 - 0) - 7(-18)$

$= 18 - 144 + 126 = 0$

0 volume means coplanar

EX.6

$\tau = rF\sin\theta = (0.25m)(40N)\sin 75$

$= 9.66 \text{ N}\cdot\text{m}$

§13.5 Equations of lines & planes

EX.1

$\vec{r} = \vec{r}_0 + t\vec{v} = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$

$= \langle 5 + t, 1 + 4t, 3 - 2t \rangle$

$x = 5 + t, \quad y = 1 + 4t, \quad z = 3 - 2t$

$t = -1 \to (4, -3, 7)$

$t = 1 \to (6, 5, 1)$

EX.2

direction of L $\to \vec{v} = \langle 1, -5, 4 \rangle$

$\frac{x-2}{1} = \frac{y-4}{-5} = \frac{z+3}{4} \quad (= t)$

symmetric

Parametric: $x = 2 + t$
$y = 4 - 5t$
$z = -3 + 4t$

b) when $z = 0$, $4t = 3$
$t = \frac{3}{4}$

$\left(\frac{11}{4}, \frac{1}{4}, 0\right)$

EX. 3

1st - find dir. vectors

$L_1 \rightarrow \frac{x-1}{1} = \frac{y+2}{3} = \frac{z-4}{-1}$

dir. vector $\langle 1, 3, -1 \rangle$

$L_2 \rightarrow \frac{x}{2} = \frac{y-3}{1} = \frac{z+3}{4} \rightarrow \langle 2, 1, 4 \rangle$

dir. vectors are $\neq$ ∴ not parallel

Solve sys. of eqn. to find intersection planes:

$1 + t = 2s$
$-2 + 3t = 3 + s$
$4 - t = -3 + 4s$

$$\left.\begin{array}{l} 2s - t = 1 \\ -s + 3t = 5 \\ 4s + t = 7 \end{array}\right\} \rightarrow \text{no solution}$$

EX. 4

Eqn of a plane:

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$2(x - 2) + 3(y - 4) + 4(z + 1) = 0$

EX. 5 $\quad PQ = \langle 2, -4, 4 \rangle = \vec{a}$

$PR = \langle 4, -1, -2 \rangle = \vec{b}$

$a \times b \rightarrow$ normal vector

$$= \begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = \langle 12, 20, 14 \rangle$$

plane $= 12(x-1) + 20(y-3) + 14(z-2) = 0$

EX. 6 $\quad x = 2 + 3t \qquad z = 5 + t$
$y = -4t$

$4x + 5y - 2z = 18$

$4(2 + 3t) + 5(-4t) - 2(5 + t) = 18$

$8 + 12t - 20t - 10 - 2t = 18$

$-10t = 20$
$t = 2$

$x = 2 + 3(2) = 8$
$y = -4(2) = -8$
$z = 5 + t = 7$

$(8, -8, 7)$

$t = -2 \quad x = 2 - 6 = -4$
$y = -4(-2) = 8$
$z = 5 - 2 = 3$

$(-4, 8, 3)$

EX. 7 $\quad x + y + z = 1 \rightarrow (1)$
$x - 2y + 3z = 1 \rightarrow (2)$

Find normal vectors, then use dot product!

$|A \cdot B| = |A||B|\cos\theta$

normal vectors - coefficients of x, y, z

(1) $\vec{n}_1 = \langle 1, 1, 1 \rangle$
$\vec{n}_2 = \langle 1, -2, 3 \rangle$

$\vec{n}_1 \cdot \vec{n}_2 = \langle 1, -2, 3 \rangle$

$\theta = \cos^{-1}\left(\frac{|\langle 1, -2, 3 \rangle|}{|\langle 1, 1, 1 \rangle||\langle 1, -2, 3 \rangle|}\right)$

$= \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = \boxed{54.7}$

$$\theta = \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|}\right)$$

$$= \cos^{-1}\left(\frac{1-2+3}{\sqrt{3}\,\sqrt{1+4+9}}\right) = 72^\circ$$

b) find a point on L & then find dir vector, which is the cross product of 2 normal vectors since the line is orthogonal to both.

Let $x=0$

$y+z=1$ $\quad z=1-y$

$-2y+3z=1$

$-2y+3-3y=1$

$-5y=4$ $\quad y=-\frac{4}{5}$

$z=\frac{9}{5}$

Point on L: $(0, -\frac{4}{5}, \frac{9}{5})$

dir vector $\vec{v} = \vec{n}_1 \times \vec{n}_2$

$$= \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= \langle 5, -2, -3 \rangle$$

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

$$\frac{x}{5} = \frac{y+\frac{4}{5}}{-2} = \frac{z-\frac{9}{5}}{-3}$$

Ex. 8 dist. $P_1(x_1, y_1, z_1)$

plane $ax+by+cz+d=0$

closest distance is normal

Find normal vector at plane.

$n = \langle a, b, c \rangle$

take any point $P_0(x_0, y_0, z_0)$

and let $\vec{b}$ be the vector $\overrightarrow{P_0P_1}$

$= \langle x_1-x_0, y_1-y_0, z_1-z_0 \rangle$

$$D = |\text{comp}_n b| = \frac{|n \cdot b|}{|n|}$$

$$= \frac{|a(x_1-x_0)+b(y_1-y_0)+c(z_1-z_0)|}{\sqrt{a^2+b^2+c^2}}$$

$ax_0+by_0+cz_0+d=0$

$$\therefore D = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

Ex. 9

$10x+2y-2z=5$ (1)

$5x+y-z=1$ (2)

find a point on any plane & find distance to other plane

let's declare $y=0$

plane 2 $\rightarrow (0, 0, -1)$

$$D = \frac{|0(10)+0(2)-1(-2)-5|}{\sqrt{10^2+2^2+2^2}}$$

$$= \frac{3}{2\sqrt{27}} = \frac{3}{6\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

Ex. 10

L_1 $\quad x=1+t \quad y=-2+3t \quad z=4-t$

L_2 $\quad x=2s \quad y=3+s \quad z=-3+4s$

Find 2 parallel planes that hold the two lines; find distance b/w the planes.

2 parallel planes have an orthogonal normal vector to both:

dir vectors:

$L_1 \to \vec{v}_1 = \langle 1, 3, -1 \rangle$

$\vec{v}_2 = \langle 2, 1, 4 \rangle$

$$n = v_1 \times v_2 = \begin{vmatrix} i & j & k \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = \langle 13, -6, -5 \rangle$$

$L_1 \to t = 0 : (1, -2, 4)$

$13(x-1) - 6(y+2) - 5(z-4) = 0$

$13x - 13 - 6y - 12 - 5z + 20 = 0$

$13x - 6y - 5z - 5 = 0 \to$ plane 1

pick a point on plane 2 → line 2

$s = 0 \to (0, 3, -3)$

dist $(0, 3, -3) \to$ plane 1

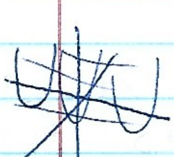
$$= \frac{|13(0) - 6(3) - 5(-3) - 5|}{\sqrt{13^2 + 6^2 + 5^2}}$$

$$= \frac{8}{\sqrt{230}} = 0.53$$

§13.6 Cylinders & Quadric Surfaces

$z = x^2$ EX-1

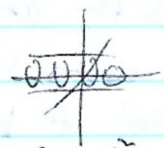
no y involvement → all y



EX-2

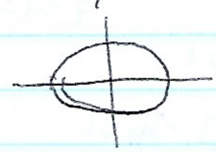
a) $x^2 + y^2 = 1 \to$ all z; circle r=1

b) $y^2 + z^2 = 1 \to$ all x; circle r=1

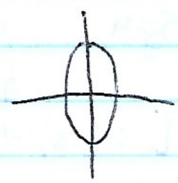


Ex-3 $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$

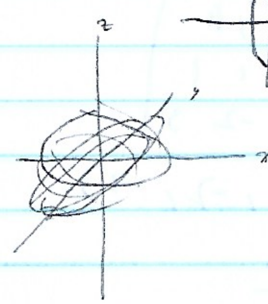
$x = 0 \to$



$y = 0$



$z = 0$

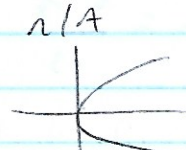


football shape

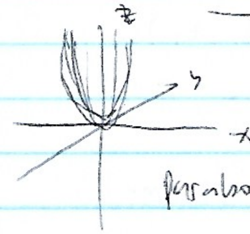
Ex-4 $z = 4x^2 + y^2$

$z = 0$ n/A

$x = 0$



$y = 0$

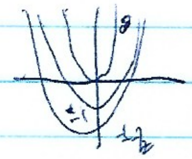


paraboloid cup thing

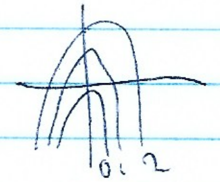
Ex. 5 $z = y^2 - x^2$

find traces in all planes with a parameter k

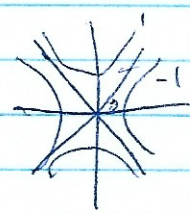
$x = k$, $y = k$, $z = k$



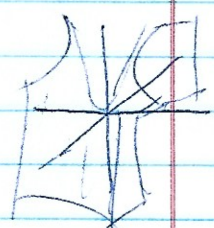
$x = k$



$y = k$

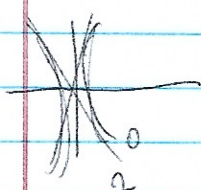


$z = k$

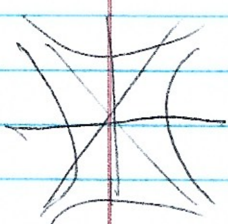


Ex. 6 $\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1$

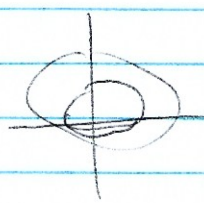
$x = k$

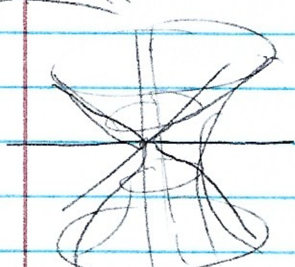


$y = k$



$z = k$





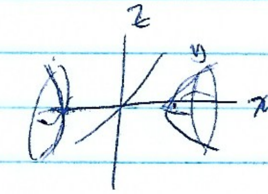
Ex. 7 $4x^2 - y^2 + 2z^2 + 4 = 0$

$4x^2 - y^2 + 2z^2 = -4$

$-x^2 + \frac{y^2}{4} - \frac{z^2}{2} = 0$

hyperboloid of 2 sheets

negative x and z → orientation



Ex. 8 $x^2 + 2z^2 - 6x - y + 10 = 0$

$(x-3)^2 - 9 + 2z^2 + 10 = y$

$\frac{y}{2} = \frac{(x-3)^2}{2} + z^2 + 1$

$(x-3)^2 + 2z^2 = y - 1$

+1 shift in y