Ch 18 Notes

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18 Second-Order Differential Equations

18.1 Second-Order Linear Equations

- A second-order linear differential equation has the form

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x)$$

  where P, Q, R, and G are continuouos function.

- Homogeneous linear equations are where $G(x) = 0$:

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

  The equation is nonhomogeneous if $G(x) \neq 0$ for some x.

- Theorem: If $y_1(x)$ and $y_2(x)$ are both solutions of the linear homogeneous equation $P(x)\frac{d^2y}{dx^2}+Q(x)\frac{dy}{dx}+R(x)y = 0$ and c_1 and c_2 are constants, then the function

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

  is also a solution of the equation.

- Theorem: if y_1 and y_2 are linearly independent solutions of a second-order linear homogeneous equation, and $P(x)$ is never 0, then the general solution is given by

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

  where c_1 and c_2 are arbitrary constants.

- Two equations are linearly independent if neither is a constant multiple of the other.

- It is difficult to find solutions to most second-order diff eqs, but it is always possible to do so when

$$ay'' + by' + cy = 0$$

- Consider the equation
$$ar^2 + br + c = 0$$
which is called the auxiliary equation or characteristic equation of the diff eq $ay'' + by' + cy = 0$. The roots can be found using the quadratic formula:
$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

- Based on the discriminant $b^2 - 4ac$, there are three cases:
  - Case 1: $b^2 - 4ac > 0$. If the roots r_1 and r_2 of the auxiliary equation $ar^2 + br + c = 0$ are real and unequal, then the general solution of $ay'' + by' + cy = 0$ is
  $$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$
  - Case 2: $b^2 - 4ac = 0$. If the auxiliary equation $ar^2 + br + c = 0$ only has one real root r, then the general solution of $ay'' + by' + cy = 0$ is
  $$y = c_1 e^{rx} + c_2 x e^{rx}$$
  - Case 3: $b^2 - 4ac < 0$. If the roots of the auxiliary equation $ar^2 + br + c = 0$ are the complex numbers $r_1 = \alpha + i\beta, r_2 = \alpha - i\beta$, then the general solution of $ay'' + by' + cy = 0$ is
  $$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

18.2 Nonhomogeneous Linear Equations

- Nonhomogeneous equations take the form
$$ay'' + by' + cy = G(x)$$
where a, b, and c are constants and G is a continuous function. The equation
$$ay'' + by' + cy = 0$$
is called the complimentary equation.

- Theorem: The general solution of the nonhomogeneous diff eq $ay'' + by' + cy = G(x)$ can be written as
$$y(x) = y_p(x) + y_c(x)$$
where y_p is a particular solution of the nonhomogeneous equation and y_c is the general solution of the complimentary equation.

- The method of undetermined coefficients:
  - If $G(x) = e^{kx}P(x)$ where P is a polynomial of degree n, then try $y_p(x) = e^{kx}Q(x)$, where $Q(x)$ is an nth degree polynomial (whose coefficients are determined by substituting in the differential equation).
  - If $G(x) = e^{kx}P(x)\cos mx$ or $G(x) = e^{kx}P(x)\sin mx$, where P is an nth degree ploynomial, then try
  $$y_p(x) = e^{kx}Q(x)\cos mx + e^{kx}R(x)\sin mx$$
  where Q and R are nth degree polynomials.
  - Modification: If any term of y_p is a solution of the complimentary equation, multiply y_p by x (or by x^2 if necessary).

18.3 Applications of Second-Order Differential Equations

- Vibrating springs and Hooke's law:

$$m\frac{d^2x}{dt^2} = -kx$$

The general solution is $x(t) = c_1 \cos \omega t + c_2 \cos \omega t = A \cos(\omega t + \delta)$ where

$$\omega = \sqrt{\frac{k}{m}} \qquad \text{(frequency)}$$

$$A = \sqrt{c_1^2 + c_2^2} \qquad \text{(amplitude)}$$

$$\cos \delta = \frac{c_1}{A} \qquad \sin \delta = -\frac{c_2}{A} \qquad \text{(phase angle)}$$

- Damped vibrations:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

- Forced vibrations:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

where $F(t)$ is an external force.

- LRC circuits:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = V(t)$$

18.4 Series Solutions

- Many diff eqs can't be solved explicitly, but we can use the power series

$$y = f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

- Substitute this expression into the diff eq and determine the value of the coefficients.