Chapter 11 Notes - MC

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Contents

11 Infinite Sequences and Series

11.1 Sequences

- sequence - a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \cdots, a_n, \cdots$$

- a_1 - first term; a_2 - second term; a_n - nth term
- For infinite series, every term a_n has a successor a_{n+1}
- Notation - the sequence $\{a_1, a_2, a_3, \cdots\}$ can also be written as

$$\{a_n\} \text{ or } \{a_n\}_{n=1}^{\infty}$$

- Definition 1: limits of sequences:

$$\lim_{n\to\infty} a_n = L$$

  - This means: as n becomes very large, the terms of the sequence $\{a_n\}$ approach L.

- can also be written as

$$a_n \to L \text{ as } n \to \infty$$

- If the limit at infinity exists, the sequence is convergent/converges. Otherwise, it is divergent/diverges.

- Definition 2: A more precise definition of the limit of a sequence:

$$\lim_{n\to\infty} a_n = L$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$\text{if } n > N \text{ then } |a_n - L| < \varepsilon$$

- Theorem 3: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.
- Equation 4:

$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \text{ if } r > 0$$

- Definition 5: $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$\text{if } n > N \text{ then } a_n > M$$

- Limit laws for sequences: If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then:

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \quad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \text{ if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \text{ if } p > 0 \text{ and } a_n > 0$$

- Squeeze Theorem can be adapted for sequences:

$$\text{If } a_n \le b_n \le c_n \text{ for } n \ge n_0 \text{ and } \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L, \text{ then } \lim_{n\to\infty} b_n = L$$

- Theorem 6: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$
- Theorem 7: If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

- Equation 8 (example 10):

$$a_n = \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot n \cdot \cdots \cdot n}$$

(ex. 10)

- Equation 9 (example 11): The sequence $\{r^n\}$ is convergent if $-1 < r \le 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

- Definition 10: A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \ge 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \ge 1$. A sequence is **monotonic** if it is either increasing or decreasing.

- Definition 11: A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leq M \text{ for all } n \geq 1$$

It is bounded below if there is a number m such that

$$m \leq a_n \text{ for all } n \geq 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**

- Theorem 12: Monotonic Sequence Theorem: Every bounded, monotonic sequence is convergent.

- Proof of theorem 12: Suppose $\{a_n\}$ is an increasing sequence. Since $\{a_n\}$ is bounded, the set $S = \{a_n | n \geq 1\}$ has an upper bound. By the Completeness Axiom it has a least upper bound L. Given $\varepsilon > 0$, $L - \varepsilon$ is *not* an upper bound for S (since L is the *least* upper bound). Therefore

$$a_N > L - \varepsilon$$

For some integer N. But the sequence is increasing so $a_n \geq a_N$ for every $n > N$. Thus if $n > N$, we have

$$a_n > L - \varepsilon$$

so

$$0 \leq L - a_n < \varepsilon$$

since $a_n \leq L$. Thus,

$$|L - a_n| < \varepsilon \text{ whenever } n > N$$

so $\lim_{n\to\infty} a_n = L$. A similar proof can be applied if $\{a_n\}$ is decreasing.

11.2 Series

- Equation 1: infinite series/series can be written as:

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

- Partial sums:

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

e.g.

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$s_4 = a_1 + a_2 + a_3 + a_4$$

- Def 2: given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, its nth partial sum is denoted as above.
  - If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called convergent and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \text{ or } \sum_{n=1}^{\infty} a_n = s$$

  - The number s is called the sum of the series. If the sequence $\{s_n\}$ is divergent, then the series is divergent.

- Geometric series:

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \text{ where } a \neq 0$$

- Equation 3: sum of a geometric series

$$s_n = \frac{a(1-r^n)}{1-r}$$

- Equation 4 (example 2): The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

  is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ where } |r| < 1$$

  If $|r| \geq 1$, the geometric series is divergent.

- Equation 5 (example 7):

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

- Theorem 6: If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$
  - Note: The converse of this theorem is not always true!

- Equation 7: Nth term test: If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

- Theorem 8: If Σa_n and Σb_n are convergent series, then so are the series $\Sigma c a_n$ (where c is a constant), $\Sigma(a_n + b_n)$, and $\Sigma(a_n - b_n)$, and

$$\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$$

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

11.3 The Integral Test and Estimates of Sums

- The Integral Test: Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent IFF the proper integral $\int_1^{\infty} f(x)dx$ is convergent.
  - CONDITIONS: continuous, positive, decreasing function
  - The integral from 1 to ∞ of the function must be convergent for the series to be convergent.

- Equation 1: P-series test: The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$

- Equation 2: Remainder Estimate for the Integral Test: Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and Σa_n is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx$$

- Equation 3 (example 5):
$$s_n + \int_{n+1}^{\infty} f(x)dx \le s \le s_n + \int_{n}^{\infty} f(x)dx$$
- Equation 4:
$$a_2 + a_3 + \cdots + a_n \le \int_1^n f(x)dx$$
- Equation 5:
$$\int_1^n f(x)dx \le a_1 + a_2 + \cdots + a_{n-1}$$
  - Both eqns 4 and 5 depend on the fact that f is decreasing and positive.

11.4 The Comparison Tests

- The comparison test: Suppose that Σa_n and Σb_n are series with positive terms.
  - If Σb_n is convergent and $a_n \le b_n$ for all n, then Σa_n is also convergent.
  - If Σb_n is divergent and $a_n \ge b_n$ for all n, then Σa_n is also divergent.
- The Limit comparison test: Suppose that Σa_n and Σb_n are series with positive terms. If
$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$
where c is a finite number and $c > 0$, then either both series converge or both series diverge.

11.5 Alternating Series

- The alternating series test: If the alternating series
$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \quad \text{where } b_n > 0$$
satisfies
$$\text{(i) } b_{n+1} \le b_n \text{ for all } n$$
$$\text{(ii) } \lim_{n\to\infty} = 0$$
then the series is convergent.
- Alternating series Estimation Theorem: If $s = \Sigma(-1)^{n-1} b_n$, where $b_n > 0$, is the sum of an alternating series that satisfies
$$b_{n+1} \le b_n \text{ and } \lim_{n\to\infty} = 0$$
then
$$|R_n| = |s - s_n| \le b_{n+1}$$

11.6 Absolute Convergence and the Ratio and Root Tests

- Definition 1: A series Σa_n is called absolutely convergent if the series of absolute values $\Sigma |a_n|$ is convergent.
- Definition 2: A series Σa_n is called conditionally convergent if it is convergent but not absolutely convergent.
- Theorem 3: If a series Σa_n is absolutely convergent, then it is convergent.
- The ratio test:

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of Σa_n

- The Root Test:
  - If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
  - If $\lim_{n\to\infty} \sqrt[n]{|a_n} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
  - if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the root test is inconclusive.

11.7 Strategy for Testing Series

- Classify series according to form in order to determine convergence or divergence.
- If the series is of the form $\Sigma 1/n^p$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leq 1$.
- Geometric series: Σar^n; converges if $|r| < 1$ and diverges if $|r| \geq 1$
- Series similar to geo or p-series: use a comparison test to determine.
- If the limit at infinity is immediately obvious not to be 0, use the nth term test.
- If the series contains $(-1)^n$, use the alternating series test.
- Series with factorials or other products: use the ratio test.
- If the series is in the form of $(b_n)^n$, use the root test.
- If $a_n = f(n)$ and $\int_1^{\infty} f(x)dx$ is easily evaluated, use the integral test as long as the function is continuous, positive, and decreasing.

11.8 Power Series

- (Equation 1) Power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is the variable and the c_ns are the coefficients of the series.

- (Equation 2): Power series with all coefficients as 1.

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

- Equation 3: power series centered about a

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

- Theorem 4: For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:
  - The series converges only when $x = a$
  - The series converges for all x
  - There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$
- R is the radius of convergence of the power series. Interval of convergence is the interval that contains all x for which the series converges.
- Check endpoint convergence!

11.9 Representations of Functions an Power Series

- Equation 1: geometric series with $a = 1$ and $r = x$:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \text{ for } |x| < 1$$

- Theorem 2: If the power series $\Sigma c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + C_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

  is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

$$\text{(i) } f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$\text{(ii) } \int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} = c_n\frac{(x-a)^{n+1}}{n+1}$$

  The radii of convergence of the power series in Equations (i) and (ii) are both R.

11.10 Taylor and Maclaurin Series

- Equations 1-4: derivation of the taylor series
- Theorem 5: If f has a power series representation/expansion at a, that is if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \text{ for } |x-a| < R$$

  then its coefficients are given by:

$$c_n = \frac{f^{(n)}(a)}{n!}$$

- Equation 6: Taylor series about a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

- Equation 7: Maclaurin series, which is a taylor series about $a = 0$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

- Theorem 8: If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

- Equation 9: Taylor's Inequality: If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x) \leq \frac{M}{(n+1)!}|x-a|^{n+1} \text{ for } |x-a| \leq d$$

- Equation 10:

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \text{ for every real number } x$$

- Equation 11:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ for all } x$$

- Equation 12: the number e is a sum of the infinite series:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

- Equation 15: power series of $\sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \text{ for all } x$$

- Equation 16: power series of $\cos x$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^2 n}{(2n)!} \text{ for all } x$$

- Equation 17: The binomial series: If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

- Table 1: Important Maclaurin series and their radii of convergence

Series	Radius
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$R = 1$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$R = \infty$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$R = \infty$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$R = \infty$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$R = 1$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$R = 1$
$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$	$R = 1$

11.11 Applications of Taylor Polynomials

- Two main ways taylor polynomials are applied:
  - 1: Approximation - computers often use taylor polynomials to approximate values of functions because it's a simpler algorithm and the error can be brought very small.
  - 2: Physics: Taylor polynomials can be used to simply visualize/predict how a complicated function will behave. Also helpful in optics and other applications of small angle approximation.