Chapter 11 Notes - MC

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Contents

11 Infinite Sequences and Series

11.1 Sequences

- sequence - a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \cdots, a_n, \cdots$$

- a_1 - first term; a_2 - second term; a_n - nth term
- For infinite series, every term a_n has a successor a_{n+1}
- Notation - the sequence $\{a_1, a_2, a_3, \cdots\}$ can also be written as

$$\{a_n\} \text{ or } \{a_n\}_{n=1}^{\infty}$$

- Definition 1: limits of sequences:

$$\lim_{n\to\infty} a_n = L$$

  - This means: as n becomes very large, the terms of the sequence $\{a_n\}$ approach L.
- can also be written as

$$a_n \to L \text{ as } n \to \infty$$

- If the limit at infinity exists, the sequence is convergent/converges. Otherwise, it is divergent/diverges.

- Definition 2: A more precise definition of the limit of a sequence:

$$\lim_{n\to\infty} a_n = L$$

  if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$\text{if } n > N \text{ then } |a_n - L| < \varepsilon$$

- Theorem 3: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

- Equation 4:

$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \text{ if } r > 0$$

- Definition 5: $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$\text{if } n > N \text{ then } a_n > M$$

- Limit laws for sequences: If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then:

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \quad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \text{ if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \text{ if } p > 0 \text{ and } a_n > 0$$

- Squeeze Theorem can be adapted for sequences:

$$\text{If } a_n \leq b_n \leq c_n \text{ for } n \geq n_0 \text{ and } \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L, \text{ then } \lim_{n\to\infty} b_n = L$$

- Theorem 6: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

- Theorem 7: If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

- Equation 8 (example 10):

$$a_n = \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot n \cdot \cdots \cdot n}$$

  (ex. 10)

- Equation 9 (example 11): The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

- Definition 10: A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is **monotonic** if it is either increasing or decreasing.

- Definition 11: A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leq M \text{ for all } n \geq 1$$

It is bounded below if there is a number m such that

$$m \leq a_n \text{ for all } n \geq 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**

- Theorem 12: Monotonic Sequence Theorem: Every bounded, monotonic sequence is convergent.
- Proof of theorem 12: Suppose $\{a_n\}$ is an increasing sequence. Since $\{a_n\}$ is bounded, the set $S = \{a_n | n \geq 1\}$ has an upper bound. By the Completeness Axiom it has a least upper bound L. Given $\varepsilon > 0$, $L - \varepsilon$ is *not* an upper bound for S (since L is the *least* upper bound). Therefore

$$a_N > L - \varepsilon$$

For some integer N. But the sequence is increasing so $a_n \geq a_N$ for every $n > N$. Thus if $n > N$, we have

$$a_n > L - \varepsilon$$

so

$$0 \leq L - a_n < \varepsilon$$

since $a_n \leq L$. Thus,

$$|L - a_n| < \varepsilon \text{ whenever } n > N$$

so $\lim_{n\to\infty} a_n = L$. A similar proof can be applied if $\{a_n\}$ is decreasing.

11.2 Series

- Equation 1: infinite series/series can be written as:

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

- Partial sums:

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

e.g.

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$s_4 = a_1 + a_2 + a_3 + a_4$$

- Def 2: given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, its nth partial sum is denoted as above.
  - If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called convergent and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \text{ or } \sum_{n=1}^{\infty} a_n = s$$

  - The number s is called the sum of the series. If the sequence $\{s_n\}$ is divergent, then the series is divergent.

- Geometric series:

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \text{ where } a \neq 0$$

- Equation 3: sum of a geometric series

$$s_n = \frac{a\left(1 - r^n\right)}{1 - r}$$

- Equation 4 (example 2): The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

  is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r} \text{ where } |r| < 1$$

  If $|r| \geq 1$, the geometric series is divergent.

- Equation 5 (example 7):

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}$$

- Theorem 6: If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$
  - Note: The converse of this theorem is not always true!
- Equation 7: Nth term test: If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- Theorem 8: If Σa_n and Σb_n are convergent series, then so are the series $\Sigma c a_n$ (where c is a constant), $\Sigma\left(a_n + b_n\right)$, and $\Sigma\left(a_n - b_n\right)$, and

$$\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$$

$$\sum_{n=1}^{\infty} \left(a_n + b_n\right) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty} \left(a_n - b_n\right) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

11.3 The Integral Test and Estimates of Sums

- The Integral Test: Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent IFF the proper integral $\int_1^{\infty} f(x)dx$ is convergent.
  - CONDITIONS: continuous, positive, decreasing function
  - The integral from 1 to ∞ of the function must be convergent for the series to be convergent.
- Equation 1: P-series test: The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$
- Equation 2: Remainder Estimate for the Integral Test: Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and Σa_n is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx$$

- Equation 3 (example 5):
$$s_n + \int_{n+1}^{\infty} f(x)dx \leq s \leq s_n + \int_{n}^{\infty} f(x)dx$$
- Equation 4:
$$a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x)dx$$
- Equation 5:
$$\int_1^n f(x)dx \leq a_1 + a_2 + \cdots + a_{n-1}$$
  - Both eqns 4 and 5 depend on the fact that f is decreasing and positive.

11.4 The Comparison Tests

- The comparison test: Suppose that Σa_n and Σb_n are series with positive terms.
  - If Σb_n is convergent and $a_n \leq b_n$ for all n, then Σa_n is also convergent.
  - If Σb_n is divergent and $a_n \geq b_n$ for all n, then Σa_n is also divergent.
- The Limit comparison test: Suppose that Σa_n and Σb_n are series with positive terms. If
$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$
where c is a finite number and $c > 0$, then either both series converge or both series diverge.

11.5 Alternating Series

- The alternating series test: If the alternating series
$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \text{ where } b_n > 0$$
satisfies
$$\text{(i) } b_{n+1} \leq b_n \text{ for all } n$$
$$\text{(ii) } \lim_{n\to\infty} = 0$$
then the series is convergent.
- Alternating series Estimation Theorem: If $s = \Sigma(-1)^{n-1} b_n$, where $b_n > 0$, is the sum of an alternating series that satisfies
$$b_{n+1} \leq b_n \text{ and } \lim_{n\to\infty} = 0$$
then
$$|R_n| = |s - s_n| \leq b_{n+1}$$

11.6 Absolute Convergence and the Ratio and Root Tests

11.7 Strategy for Testing Series

11.8 Power Series

11.9 Representations of Functions an Power Series

11.10 Taylor and Maclaurin Series

11.11 Applications of Taylor Polynomials