Chapter 15 Notes - MC

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15.1 Functions of Several Variables

- A function f of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the domain of f and its range is the set of values that f takes on, that is $f(x, y)|(x, y) \in D$
- If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.
- The level curves of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).
  - A level curve is the set ofa ll points in the domain of f at which f takes on a given value k. (Think of countour maps, equipotential lines)

15.2 Limits and Continuity

- Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the limit of $f(x, y)$ as x, y approaches (a, b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

  if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \qquad (x, y) \in D \qquad \text{and} \qquad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \qquad \text{then} \qquad |f(x, y) - L| < \varepsilon$$

- If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

- A function f of two variables is called continuous at (a, b) if

$$\lim_{(x,y)\to(a,b) f(x,y)=f(a,b)}$$

  We say f is continuous on D if f is continuous at every point (a, b) in D.

- If f is defined on a subset D of $\mathbb{R}^n$, then $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ means that for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \qquad x \in D \qquad \text{and} \qquad 0 < |\mathbf{x} - \mathbf{a}| < \delta \qquad \text{then} \qquad |f(\mathbf{x}) - L| < \varepsilon$$

15.3 Partial Derivatives

- If f is a function of two variables, its partial derivatives are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

- If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

- Finding partial derivatives of $z = f(x, y)$:
  - To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
  - To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

- Clairaut's Theorem: suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

- 3D Laplace Equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

15.4 Tangent Planes and Linear Approximations

- Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

- If $z = f(x, y)$, then f is differentiable at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

  where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

- Theorem: If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

- The total differential dz is defined by:

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

15.5 The Chain Rule

- Case 1: suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

  $$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

  or

  $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

- Case 2: Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

  $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

- General Version: Suppose that u is a differentiable function of the n variables $x_1, x_2, \cdots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \cdots, t_m$. Then u is a function of $t_1, t_2, \cdots, t_m$ and

  $$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

  for each $i = 1, 2, \cdots, m$

- Implicit diffentiation:

  $$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

  where $y = f(x)$ and $F(x, f(x)) = 0$

- $$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

15.6 Directional Derivatives and the Gradient Vector

- The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

  $$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

  if this limit exists.

- Theorem: If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and

  $$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$$

  - If the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis, then we can write $\mathbf{u} = \langle \cos\theta, \sin\theta\rangle$ and the previous eqn becomes:

    $$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

- If f is a function of two variables x and y, then the gradient of f is the vector function $\boldsymbol{\nabla} f$ defined by

  $$\boldsymbol{\nabla} f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

- The equation of the directional derivative of a differentiable function can thus be written as:

$$D_{\mathbf{u}}f(x,y) = \nabla f(x,y) \cdot \mathbf{u}$$

- The directional derivative of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c\rangle$ is

$$D_{\mathbf{u}}f(x_0,y_0,z_0) = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb, z_0+hc) - f(x_0,y_0,z_0)}{h}$$

  if this limit exists.

- Using vector notation:

$$D_{\mathbf{u}}f(\mathbf{x}_0) = \lim_{h\to 0} \frac{f(\mathbf{x_0}+h\mathbf{u}) - f(\mathbf{x}_0)}{h}$$

- For a function of three variables, the gradient vector:

$$\nabla f = \langle f_x, f_y, f_z\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

- The directional derivative of a function of three variables:

$$D_{\mathbf{u}}f(x_0,y_0,z_0) = \nabla f(x,y,z) \cdot \mathbf{u}$$

- Theorem: Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}}f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

- The tangent plane to a level surface $F(x,y,z) = k$ at $P(x_0,y_0,z_0)$ is the plane that passes through P and has normal vector $\nabla F(x_0,y_0,z_0)$. The equation of the tangent plane is thus:

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$$

- The normal line to the surface S at P is the line passing through P and perpendicular to the tangent plane. The direction of the normal line is therefore given by the gradient vector $\nabla F(x_0,y_0,z_0)$; its symmetric equations are given by:

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)} = \frac{y-y_0}{F_y(x_0,y_0,z_0)} = \frac{z-z_0}{F_z(x_0,y_0,z_0)}$$

15.7 Maximum and Minimum Values

- A function of two variables has a local maximum at (a,b) if $f(x,y) \leq f(a,b)$ when (x,y) is near (a,b). [This means that $F(x,y) \leq f(a,b)$ for all points (x,y) in some disk with center (a,b).] The number $f(a,b)$ is called a local maximum value. If $f(x,y) \geq f(a,b)$ when (x,y) is near (a,b), then f has a local minimum at (a,b) and $f(a,b)$ is a local minimum value.

- Theorem: If f has a local maximum or minimum at (a,b) and the first order partial derivatives of f exist there, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

- Second derivatives test: Suppose the second partial derivatives of f are continuous on a disk with center (a,b), and suppose that $f_x(a,b) = 0$ and $f_y(a,b) = 0$ [that is, (a,b) is a critical point of f]. Let

$$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

  - If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.
  - If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.
  - If $D < 0$, the $f(a,b)$ is not a local maximum or minimum.

- Extreme value theorem for functions of two variables: If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.
- To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps one and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

15.8 Lagrange Multipliers

- Method of Lagrange Multipliers: To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\boldsymbol{\nabla} g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:
  1. Find all values of x, y, z, and λ such that
  $$\boldsymbol{\nabla} f(x, y, z) = \lambda \boldsymbol{\nabla} g(x, y, z)$$
  and
  $$g(x, y, z) = k$$
  2. Evaluate f at all the points (x, y, z) that result from step 1. The largest of these values is the maximum value of f; the smallest is the minimum value of f.
- With two constraints, $g(x, y, z) = k$ and $h(x, y, z) = c$, there exist Lagrange Multipliers, constants λ and μ such that
$$\boldsymbol{\nabla} f(x_0, y_0, z_0) = \lambda \boldsymbol{\nabla} g(x_0, y_0, z_0) + \mu \boldsymbol{\nabla} h(x_0, y_0, z_0)$$