# Chapter 17 Notes

John Yang

December 18, 2021

## Contents



# 17 Vector Calculus

## 17.1 Vector Fields

- Def: Let $D$ be a set in $\mathbb{R}^2$ (a plane region). A vector field on $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point $(x, y)$ in $D$ a two-dimensional vector $\mathbf{F}(x, y)$
- Def: Let $E$ be a subset of $\mathbb{R}^3$. A vector field on $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point $(x, y, z)$ in $E$ a three-dimensional vector $\mathbf{F}(x, y, z)$
- Recall that the gradient of a scalar function $f$ of two variables $\nabla f$ is defined by

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Therefore $\nabla f$ is really a vector field on $\mathbb{R}^2$ and is called a gradient vector field. Likewise, if $f$ is a scalar function of 3 variables, its gradient is a vector field on $\mathbb{R}^3$ given by

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

## 17.2 Line Integrals

- Def: if $f$ is defined on a smooth curve $C$ given by

$$x = x(t) \qquad y = y(t) \qquad a \leq t \leq b$$

then the line integral of $f$ along $C$ is

$$\int_C f(x, y)ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

if this limit exists.

- If $f$ is a continuous function, then the limit always exists and the line integral is given by:

$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$$

  The value of the line integral does not depend on the parameterization of the curve, provided that the curve is traversed exactly once as $t$ increases from $a$ to $b$.

- Line integrals of $f$ along $C$ with respect to $x$ and $y$:

$$\int_C f(x,y)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*,y_i^*)\Delta x_i = \int_a^b f(x(t),y(t))x'(t)dt$$

$$\int_C f(x,y)dy = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*,y_i^*)\Delta y_i = \int_a^b f(x(t),y(t))y'(t)dt$$

- Recall that the vector representation of the line segment that starts at $\mathbf{r}_0$ and ends at $\mathbf{r}_1$ is given by

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad 0 \le t \le 1$$

- For line integrals in space, where $C$ is a curve given by

$$x = x(t) \qquad y = y(t) \qquad z = z(t) \qquad a \le t \le b$$

  or

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

  then the line integral of $f$ along $C$ is given by

$$\int_C f(x,y,z)ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*,y_i^*,z_i^*)\Delta s_i = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}dt$$

- We evaluate integrals of the form

$$\int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$$

  by expressing everything $(x, y, z, dx, dy, dz)$ in terms of the parameter $t$.

- Def: Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve $C$ given by a vector function $\mathbf{r}(t), a \le t \le b$. Then the line integral of $\mathbf{F}$ along $C$ is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt = \int_C \mathbf{F}\cdot\mathbf{T}ds$$

- We have:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C Pdx + Qdy + Rdz \qquad \text{where} \quad \mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

## 17.3 The Fundamental Theorem for Line Integrals

- Theorem: Let $C$ be a smooth curve given by the vector function $\mathbf{r}(t), a \le t \le b$. Let $f$ be a differentiable function of two or three variables whos gradient vector $\nabla f$ is continuous on $C$. Then

$$\int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

- Theorem: $\int_C \mathbf{F} \cdot d\mathbf{r}$ is an independent path in $D$ IFF $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path $C$ in $D$.
- Theorem: Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region $D$. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in $D$, then $\mathbf{F}$ is a conservative vector field on $D$; that is, there exists a function $f$ such that $\nabla f = \mathbf{F}$
- Theorem: If $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a conservative vector field, where $P$ and $Q$ have continuous first-order partial derivatives on a domain $D$, then throughout $D$ we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

- Theorem: Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region $D$. Suppose that $P$ and $Q$ have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

## 17.4 Green's Theorem

- Green's Theorem gives the relationship b/w a line integral around a simple closed curve $C$ and a double integral over the plane region $D$ bounded by $C$.
- We use the convention that the positive orientation of $C$ means traversing $C$ once counterclockwise.
- Green's Theorem: Let $C$ be a positively oriented, piecewise-smooth, simple closed curve in the plane and let $D$ be the region bounded by $C$. If $P$ and $Q$ have continuous partial derivatives on an open region that contains $D$, then

$$\int_C P dx + Q dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

  - Note: the notation $\oint_C P dx + Q dy$ is sometimes used to show that it is a closed path integral.
- To find the area of $D$:

$$A = \oint_C x dy = -\oint_C y dx = \frac{1}{2} \oint_C x dy - y dx$$

## 17.5 Curl and Divergence

- If $\mathbf{F} - P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of $P$, $Q$, and $R$ all exist, then the curl of $F$ is the vector field on $\mathbb{R}^3$ defined by

$$\text{curl } \mathbf{F} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

- $$\text{curl } \mathbf{F} = \nabla \times \mathbf{F}$$
- Theorem: If $f$ is a function of three variables that has continuous second-order partial derivatives, then

$$\text{curl}(\nabla f) = \mathbf{0}$$

- Theorem: If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

- Divergence: if $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $\frac{\partial P}{\partial x}$, $\frac{\partial Q}{\partial y}$, and $\frac{\partial R}{\partial z}$ exist, then the divergence of $\mathbf{F}$ is the function of three variables defined by

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

  It can also be written as

$$\text{div } \mathbf{F} = \boldsymbol{\nabla} \cdot \mathbf{F}$$

- Theorem: If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $P$,$Q$, and $R$ have continuous second-order partial derivatives, then

$$\text{div curl } \mathbf{F} = 0$$

- Vector form of Green's Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint D(\text{curl } \mathbf{F}) \cdot \mathbf{k} dA$$

- Which is also

$$\oint_C \mathbf{F} \cdot \mathbf{n} ds = \iint D\text{div } \mathbf{F}(x, y) dA$$

## 17.6 Parametric Surfaces and Their Areas

- Given the vector function

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

  The set of all points $(x, y, z)$ in $\mathbb{R}^3$ such that

$$x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v)$$

  and $(u, v)$ varies throughout $D$ is called a parametric surface $S$

- A surface of revolution can be represented parametrically with

$$x = x \qquad y = f(x)\cos\theta \qquad z = f(x)\sin\theta$$

- Given a parametric surface $S$, if $u$ is kept constant by $u = u_0$, then $\mathbf{r}(u_0, v)$ defines the grid curve $C_1$ on $S$. The tangent vector to $C_1$ at a point $P_0$ is given by

$$\mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

  If $v$ is kept constant by $v = v_0$, the grid curve $C_2$ given by $\mathbf{r}(u, v_0)$ lies on $S$ and its tangent vector at $P_0$ is given by

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}$$

  If $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$ then the surface $S$ is called smooth. For a smooth surface, $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to the tangent plane.

- Def: If a smooth parametric surface $S$ is given by the equation

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \qquad (u, v) \in D$$

  and $S$ is covered just once as $(u, v)$ ranges throughout the parameter domain $D$, then the surface area of $S$ is

$$A(S) = \iint D|\mathbf{r}_u \times \mathbf{r}_v| dA$$

  where

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \qquad r_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

- Special case, where $z = f(x,y)$ where $(x,y) \in D$ and $f$ has continuous partial derivatives, we have the parametric equations

$$x = x \qquad y = y \qquad z = f(x,y)$$

so

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial f}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial f}{\partial y}\right)\mathbf{k}$$

and

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$$

which gives

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

and the surface area is

$$A(S) = \iint D\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

## 17.7 Surface Integrals

- The surface integral of $f$ over the surface $S$ is given by

$$\iint Sf(x,y,z)dS = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij} = \iint Df(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v| dA$$

- for a surface $S$ with $z = g(x,y)$ the surface integral becomes

$$\iint Sf(x,y,z)dS = \iint Df(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 dA}$$

- Def: If $F$ is a continuous vector field defined on an oriented surface $S$ with unit normal vector $\mathbf{n}$, then the surface integral of $\mathbf{F}$ over $S$ is given by

$$\iint S\mathbf{F} \cdot d\mathbf{S} = \iint S\mathbf{F} \cdot \mathbf{n} dS = \iint D\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA = \iint D\left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

which is also called the flux of $\mathbf{F}$ across $S$.

## 17.8 Stokes' Theorem

- Stokes' Theorem: Let $S$ be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve $C$ with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains $S$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint S\text{curl } \mathbf{F} \cdot d\mathbf{S}$$

## 17.9 The Divergence Theorem

- Divergence Theorem: Let $E$ be a simple solid region and let $S$ be the boundary surface of $E$, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains $E$. Then

$$\iint S\mathbf{F} \cdot d\mathbf{S} = \iiint E\text{div } \mathbf{F} dV$$

## 17.10 Summary of Chapter 17

- All main results of Chapter 17 are higher-order versions of the Fundamental Theorem of calculus.
- Fundamental Theorem of Calculus:

$$\int_a^b F'(x)dx = F(b) - F(a)$$

- Fundamental Theorem for Line Integrals:

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

- Green's Theorem:

$$\iint D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P dx + Q dy$$

- Stokes' Theorem:

$$\iint S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

- Divergence Theorem:

$$\iiint E \text{div } \mathbf{F} dV = \iint S \mathbf{F} \cdot d\mathbf{S}$$