Chapter 13 Notes - MC

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13 Vectors and the Geometry of Space

13.1 Three-Dimensional Coordinate Systems

- Coordinates - (x, y, z)
- 3D space is split into octants
- Projections - easiest way to visualize is that the object/shape/line/point is in a glass box. If you look at the box from the chosen plane or angle and trace a 2D outline of it, it is the projection.
- The Cartesian product $\mathbb{R}\times\mathbb{R}\times\mathbb{R} = \{(x, y, z)|x, y, z \in \mathbb{R}\}$ is the set of all ordered triples of real numbers, denoted by $\mathbb{R}^3$
- Distance formula in three dimensions:

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - Z_1)^2}$$

- Sphere - set of all points in 3D space a certain distance from the center.
  - Sphere with center $C(h, k, l)$ and radius r is given by:

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

  - Sphere at the origin is given by:

$$x^2 + y^2 + z^2 = r^2$$

13.2 Vectors

- Vectors - values with magnitudes and directions
- vectors are expressed in boldface and/or with an arrow over the letter: $\mathbf{a}$, $\vec{a}$, $\vec{\mathbf{a}}$
- the magnitude of a vector is expressed: $|\mathbf{a}|$

- Vector addition - head to tail, take the magnitude of the resultant vector from the beginning of the first vector to the end of the second vector
- Definition of scalar multiplication: If c is a scalar and $\mathbf{v}$ is a vector, then the scalar multiple $c\mathbf{v}$ is the vector whose lengthe is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite to $\mathbf{v}$ if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.
- Components of a vector: Equation 1: Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\mathbf{a}$ with representation $\vec{AB}$ is:
$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$
- The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is
$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$
- The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is
$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$
- If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then
$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \qquad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$
$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$
- For three dimensional vectors,
$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$
$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$
$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$
- V_2 is the set of all 2-D vectors. V_3 is the set of all 3-D vectors. V_n is the set of all n-dimensional vectors.
- Properties of vectors. If $\mathbf{a}$,$\mathbf{b}$, and$\mathbf{c}$ are vectors in V_n and c and d are scalars, then:
  - $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
  - $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
  - $\mathbf{a} + \mathbf{0} = \mathbf{a}$
  - $\mathbf{a} + (-\mathbf{a}) = 0$
  - $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
  - $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
  - $(cd)\mathbf{a} = c(d\mathbf{a})$
  - $1\mathbf{a} = \mathbf{a}$
- Unit vectors: $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$
- Use unit vectors to express components: $\mathbf{a} = \langle a_1, a_2, a_3 \rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$
- General unit vector expresses direction
$$\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|}$$
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13.3 The Dot Product

- Definition 1: If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the dot product of $\mathbf{a}$ and $\mathbf{b}$ is the scalar $\mathbf{a} \cdot \mathbf{b}$ given by
$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$
- Other names for dot product: scalar product, inner product
- Properties of the dot product: If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are vectors in V_3 and c is a scalar, then:
  - $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
  - $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
  - $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
  - $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
  - $\mathbf{0} \cdot \mathbf{a} = 0$
- Theorem 3: If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then
$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$
- Corollary 6: If θ is the angle between the nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then
$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$
- Equation 7: Two vectors $\mathbf{a}$ and $\mathbf{b}$,are orthogonal IFF $\mathbf{a} \cdot \mathbf{b} = 0$
- Direction angles of a nonzero vector $\mathbf{a}$ are the angles α, β, and γ that $\mathbf{a}$ makes with the positive x-, y-, and z-axes respectively. The cosines of the direction angles are called direction cosines.
- Direction angles are given by:
$$\cos\alpha = \frac{a_1}{|\mathbf{a}|} \qquad \cos\beta = \frac{a_2}{|\mathbf{a}|} \qquad \cos\gamma = \frac{a_3}{|\mathbf{a}|}$$
  and,
$$\frac{1}{|\mathbf{a}|}\mathbf{a} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$
- Projections: think of it like a shadow.
- Scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ (aka component of $\mathbf{b}$ along $\mathbf{a}$):
$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$$
- Vector projection of $\mathbf{b}$ onto $\mathbf{a}$:
$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right) \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$$

13.4 The Cross Product

- Definition of the cross product: If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the cross product of $\mathbf{a}$ and $\mathbf{b}$ is the vector
$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$
- Cross product is also called vector product or external product.
- Cross product is only defined when both $\mathbf{a}$ and $\mathbf{b}$ are 3-D vectors.
- Determinant form of the cross product:
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
- Theorem 8: The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.
- Direction of the external product: use curling rhr - fingers curl from $\mathbf{a}$ to $\mathbf{b}$, thumb is the direction of the cross product.
- Theorem 9: magnitude of the cross product: If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leq \theta \leq \pi$), then
$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$
- Corollary 10: Two nonzero vector $\mathbf{a}$ and $\mathbf{b}$ are parallel IFF
$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$
- The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.
- Cross products of unit vectors:
$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$
$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$
- Cross product is neither commutative nor associative.
- Properties of the cross product: If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and c is a scalar, then:
  - $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
  - $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
  - $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
  - $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
  - $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
  - $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$
- Scalar triple product of vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$:
$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
- The scalar triple product is the volume of the parallelepiped determined by vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ and is given by:
$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

13.5 Equations of Lines and Planes

- Let the line L be any line in 3D space, which is determined when there is a point on L, $P_0(x_0, y_0, z_0)$, and we know the direction of L. Let $P(x, y, z)$ be any point on L and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P. Let $\mathbf{v}$ be a vector parallel to L. The vector equation of L is given by

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

  where t is a parameter.

- Parametric equations for a line L through the point (x_0, y_0, z_0) and parallel to the direction vector $\langle a, b, c\rangle$ are:

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

- Symmetric equations of L:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

- The line segment from $\mathbf{r_0}$ to $\mathbf{r_1}$ is given by the vector equation

$$\mathbf{r}(t) = (1 - t)\mathbf{r_0} + t\mathbf{r_1} \quad 0 \le t \le 1$$

- A plane is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and an orthogonal normal vector $\mathbf{n}$. The vector equation of the plane is given by

$$\mathbf{n}\cdot(\mathbf{r} - \mathbf{r_0}) = 0$$

  or

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}$$

- Scalar equation of the plane through point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c\rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

- The distance D from any point $P_1(x_1, y_1, z_1)$ to a plane $ax + by + cz + d = 0$ is given by

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

13.6 Cylinders and Quadric Surfaces

- A cylinder is a surface that consists of all lines (called rulings) that are parallel to a given line and pass through a given plane curve.
- A quadric surface is the graph of a second-degree equation in three variables x, y, and z. The most general form of a quadric surface is:

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

- It can also take one of two standard forms:

$$Ax^2 + By^2 + Cz^2 + J = 0 \qquad \text{or} \qquad Ax^2 + By^2 + Iz = 0$$

Common quadric surfaces (see page 877 for images):

- Ellipsoid - all traces are ellipses. Equation is given by:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

  If $a = b = c$, then the ellipsoid is a sphere.

- Elliptic paraboloid: horizontal traces are ellipses and vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

- Hyperbolic paraboloid: Horizontal traces are hyperbolas and vertical traces are parabolas.

$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

- Cone: Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

- Hyperboloid of one sheet: Horizontal traces are ellipses and vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

- Hyperboloid of two sheets: Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate the two sheets.

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$