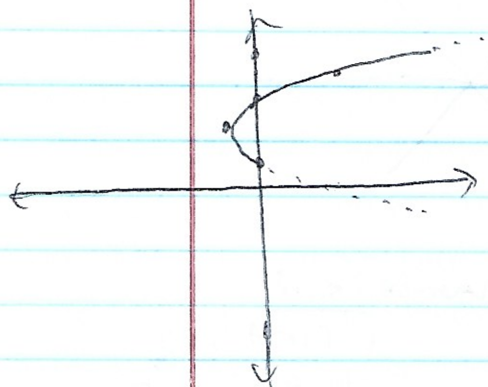
Ch 10 ex - MC

§10.1 - Parametric Curves

Ex. 1 $x = t^2 - 2t$ $y = t+1$



t	x	y
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5
5	.	.

horizontal parabola

eliminate parameter:

$t = y-1$

$x = y^2 - 2y + 1 - 2y + 2$

$x = y^2 - 4y + 3$

Ex. 2 $x = \cos t$ $y = \sin t$

$t = \sin^{-1}(y)$ $x = \cos(\sin^{-1} y)$

$= \sqrt{1-y^2}$

$x^2 = 1 - y^2$

$x^2 + y^2 = 1$ → circle, $r=1$

Ex. 3 $x = \sin 2t$ $y = \cos 2t$

$2t = \sin^{-1}(x)$

$y = \cos(\sin^{-1}(x)) = \sqrt{1-x^2}$

$x^2 + y^2 = 1$ → unit circle

Ex. 4 circle (h,k) $r=r$

$(x-h)^2 + (y-k)^2 = r^2$

$x = \cos t$ $y = \sin t$ → unit circle

$x = r\cos t$ $y = r\sin t$ → circ. 0, r

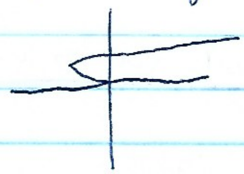
$x = h + r\cos t$ $y = k + r\sin t$ (h,k), r

Ex. 5 $x = \sin t$ $y = \sin^2 t$

$y = x^2$

Ex. 6 $x = y^4 - 3y^2$

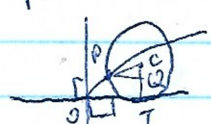
use param $y = t$ $x = t^4 - 3t^2$



Ex. 7 param. cycloid

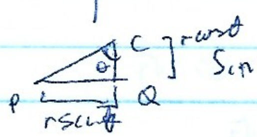
diam - $2r$ after $2\pi r$, one full rot.

param - θ



$OT = s = r\theta$ → distance travel in θ

height of $c = R$



$\sin\theta = \frac{PQ}{PC}$ $\cos\theta = \frac{CQ}{PC}$

$PC = r$ $PQ = r\sin\theta$ $CQ = r\cos\theta$

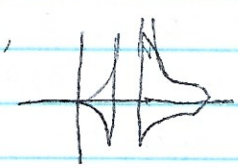
$x = r\theta - r\sin\theta = r(\theta - \sin\theta)$

$y = r - r\cos\theta = r(1-\cos\theta)$

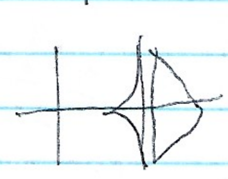
Ex. 8 $x = a + \cos t$ $y = a\tan t + \sin t$

if $a = 0$, circle

if $a = 1$,



$a = 2$



§10.2 Calculus w/ parameter Curves

Ex.1 $C \to x = t^2 \quad y = t^3 - 3t$

a) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-3}{2t}$

$t^2 = 3 \quad t^3 - 3t = 0$

$t(t^2-3) = 0$

$t = 0,\ t = \sqrt{3}$

~~One tangent: m = 3~~

One tangent is vertical (div by 0)

$t = 0$

~~[illegible]~~ when $t = 0$, $\boxed{x = 0}$

other tangent: $m = \frac{9-3}{2\sqrt{3}} = \frac{3}{\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}$

$y = \sqrt{3}x + b$

$0 = 3\sqrt{3} + b \quad b = -3\sqrt{3} \quad y = \sqrt{3}x - 3\sqrt{3}$

div by 0, take sqrt $\to \pm\sqrt{3}$

$m = \pm\sqrt{3} \quad y = -\sqrt{3}x + b$

$0 = -3\sqrt{3}x + b \quad b = 3\sqrt{3}$

$y = \sqrt{3}x - 3\sqrt{3} \quad y = -\sqrt{3}x + 3\sqrt{3}$

b) horiz when $y'(t) = 0$

$y'(t) = 3t^2 - 3 = 0$

$t = \pm 1$

when $t = 1$, $x = 1$, $y = -2$ $(1,-2)$

when $t = -1$ $x = 1$, $y = 2$ $(1,2)$

vert when $x'(t) = 0 = 2t \quad t = 0$

$x = 0,\ y = 0 \quad (0,0)$

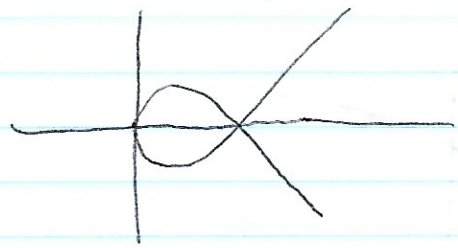
c) concavity - 2nd derivative

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{3}{2}\left(t - \frac{1}{t}\right)\right)}{2t} = \frac{\frac{3}{2} + \frac{3}{2t^2}}{2t}$

$= \frac{3}{4t} + \frac{3}{4t^3} = 0$ concave up $t > 0$

concave down $t < 0$

d) sketch using CAS



Ex.2

$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \left.\frac{r\sin\theta}{r - r\cos\theta}\right|_{\theta=\frac{\pi}{3}}$

$= \left.\frac{\sin\theta}{1-\cos\theta}\right|_{\frac{\pi}{3}} = \frac{\frac{\sqrt{3}}{2}}{1-\frac{1}{2}}$

$= \sqrt{3}$

$y = mx + b$

when $\theta = \frac{\pi}{3}$, $x = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$

$y = r\left(1 - \frac{1}{2}\right) = \frac{1}{2}r$

$\frac{1}{2}r = \sqrt{3}r\cdot\frac{\pi}{3} - \frac{\sqrt{3}r\cdot\sqrt{3}}{2} + b$

$\frac{1}{2}r + \frac{3}{2}r - \frac{\pi\sqrt{3}r}{3} = b$

$b = r\left(2 - \frac{\sqrt{3}}{3}\pi\right)$

$y = \sqrt{3}x + r\left(2 - \frac{\sqrt{3}}{3}\pi\right)$

b) horiz. tan when $y'(\theta) = 0 = r\sin\theta$

$\theta = 0, \pi$

vert. when $x'(\theta) = r - r\cos\theta = 0$

$\cos\theta = 1$

$\theta = 0$

horiz - $\sin\theta = 0$, $\cos\theta \neq 1 \to (2n-1)\pi$

vert. - expires as $\theta \to 2\pi n$

Ex.3

$A = \int_0^{2\pi} y(t)\, x'(t)\, dt$

$x(\theta) = r(\theta - \sin\theta)$

$y(\theta) = r(1-\cos\theta)$

$A = \int_0^{2\pi} (r(1-\cos\theta))(r(1-\cos\theta))\,d\theta$

$= r^2\int_0^{2\pi}(1-2\cos\theta+\cos^2\theta)\,d\theta$

$= r^2\left(\theta - 2\sin\theta + \frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2}\right)\Big|_0^{2\pi}$

$= r^2(2\pi+\pi-2(0)+0) = 3\pi r^2$

Ex.4 N/A

Ex.5 $x=r(\theta-\sin\theta)$ $y=r(1-\cos\theta)$

1 arch - $0 \to 2\pi$

$L=\int_0^{2\pi}\sqrt{x'(t)^2+y'(t)^2}\,dt$

$=\int_0^{2\pi}\sqrt{(r(1-\cos\theta))^2+(r\sin\theta)^2}\,d\theta$

$=\int_0^{2\pi}\sqrt{r^2(1-2\cos\theta+\cos^2\theta)+r^2(\sin^2\theta)}\,d\theta$

$= r\sqrt{2}\int_0^{2\pi}\sqrt{1-\cos x}\,dx$

$= \boxed{8r}$

Ex.6

circle $\to x = r\cos t$ $y = r\sin t$

use only 1 semicircle $\to 0 \to \pi$

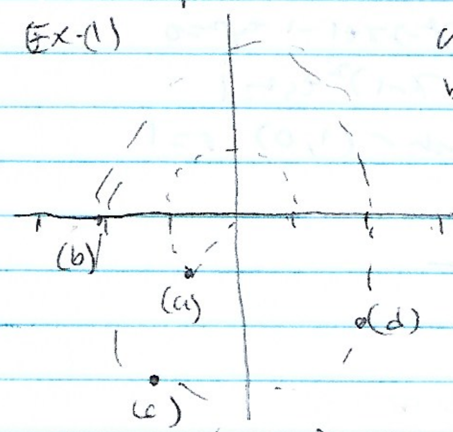
$S=\int_0^{\pi} 2\pi(r\sin t)\sqrt{(-r\sin t)^2+(r\cos t)^2}\,dt$

$=2\pi r^2\int_0^{\pi} \sin t\sqrt{\sin^2 t+\cos^2 t}\,dt$

$=2\pi r^2\int_0^{\pi}\sin t\,dt = \boxed{4\pi r^2}$

§10.3 polar coordinates

Ex.(1)



a - $(1, \frac{5\pi}{4})$

b - $(2, 3\pi)$

c - $(2, -\frac{2\pi}{3})$

d - $(-3, \frac{3\pi}{4})$

Ex.2 - $(2, \frac{\pi}{3}) \to$ cartesian

$x = r\cos\theta = 2\cos\frac{\pi}{3} = 1$

$y = r\sin\theta = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$

$(1, \sqrt{3})$

Ex.3 - $(1,-1) \to$ polar

$\theta = \tan^{-1}\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$

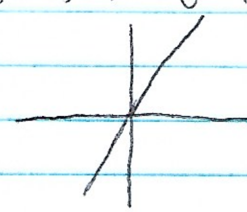
$r = \sqrt{x^2+y^2} = \sqrt{2}$

$(\sqrt{2}, -\frac{\pi}{4})$

Ex.4 $\to r=2 \to$ circle, $r=2$

Ex.5 $\to \theta=1 \to$ ~~all angles θ~~, all radii

$\theta=1$ angle



Ex-6 (a) $r = 2\cos\theta$ - sketch

(b) Cartesian

b → $y = 2\sin\theta\cos\theta$ $\quad x = 2\cos^2\theta$

$2\cos\theta = \frac{x}{\cos\theta}$

~~y = 2 sinθ~~ $y = \sin\theta \cdot \frac{x}{\cos\theta}$

$y = \cancel{x\tan\theta} = x \cdot \frac{y}{x}$

(b) → $\cos\theta = \frac{x}{r}$ $\quad r = 2\cos\theta = \frac{2x}{r}$

$r^2 = 2x = x^2 + y^2$

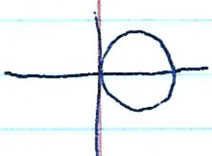
$x^2 - 2x + y^2 = 0$

$x^2 - 2x + 1 - 1 + y^2 = 0$

$(x-1)^2 + y^2 = 1$

circle w/ center $(1,0)$ $r = 1$

a)

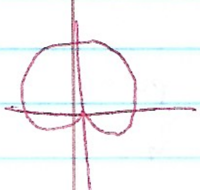


Ex. 7 → $r = 1 + \sin\theta$

$\sin\theta = \frac{y}{r}$ $\quad r = 1 + \frac{y}{r}$

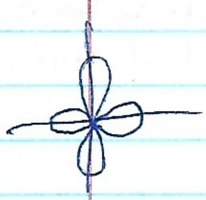
$r^2 = r + y$

cardioid



Ex. 8 $r = \cos 2\theta$

graphing calculator



Ex. 9 $r = 1 + \sin\theta$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

When $\theta = \frac{\pi}{3}$, $r = 1 + \frac{\sqrt{3}}{2} = \frac{\sqrt{3}+2}{2}$

$\frac{dr}{d\theta} = \cos\theta$

$$\frac{dy}{dx} = \frac{\sin\theta\cos\theta + r\cos\theta}{\cos^2\theta - r\sin\theta}$$

$$= \frac{\frac{\sqrt{3}}{2}\cdot\frac{1}{2} + \frac{\sqrt{3}+2}{2}\cdot\frac{1}{2}}{\frac{1}{4} - \frac{\sqrt{3}+2}{2}\cdot\frac{\sqrt{3}}{2}}$$

$$= \frac{\sqrt{3} + \sqrt{3} + 2}{4} \cdot \frac{4}{1 - 3 - 2\sqrt{3}}$$

$$= \frac{2\sqrt{3}+2}{-2\sqrt{3}-2} = \boxed{-1}$$

b) $\frac{dy}{dx} = 0 = \frac{\sin\theta\cos\theta + (1+\sin\theta)\cos\theta}{\cos^2\theta - (1+\sin\theta)(\sin\theta)}$

$$= \frac{2\sin\theta\cos\theta + \cos\theta}{\cos^2\theta - \sin^2\theta + \sin\theta} = 0$$

$\theta = \frac{(2n-1)\pi}{2}$ → horiz.

vert. $\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$

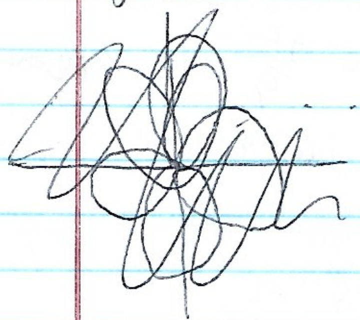
$= \cos^2\theta - \sin^2\theta + \sin\theta = 0$

$\theta =$

horiz. → $(2, \frac{\pi}{2}), (\frac{1}{2}, \frac{7\pi}{6}), (\frac{1}{2}, \frac{11\pi}{6})$

vert. → $(\frac{3}{2}, \frac{\pi}{6}), (\frac{3}{2}, \frac{5\pi}{6}), (0,0)$

EX.10 – CAS graph $r = \sin\left(\frac{8\theta}{5}\right)$



EX.11 $r = 1 + c\sin\theta$ – limaçon

$c = 0 \rightarrow$ circle, $r = 1$

$c = 1 \rightarrow$ cardioid

$c = 2 \rightarrow$ loop w/ cardioid

§11.4 Areas + Lengths in polar

EX.1 $r = \cos 2\theta \rightarrow$ 4 leaves $\rightarrow \theta \rightarrow \frac{\pi}{2}$

$$A = \frac{1}{2}\int_0^{\pi/2} (\cos 2\theta)^2 \, d\theta$$

$$= \frac{\pi}{8}$$

EX.2 $r = 3\sin\theta$ (inside) $r = 1 + \sin\theta$ (outside)

$3\sin\theta = 1 + \sin\theta \qquad 2\sin\theta = 1$

$\sin\theta = \frac{1}{2} \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$$r = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin\theta - (1+\sin\theta))^2 \, d\theta$$

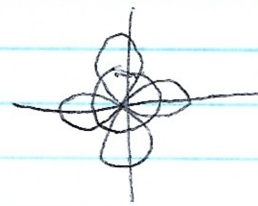
$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (2\sin\theta - 1)^2 \, d\theta = \boxed{\frac{-3\sqrt{3} + 2\pi}{2}}$$

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin\theta)^2 \, d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1+\sin\theta)^2 \, d\theta$$

$$= \pi$$

EX.3 $r = \cos 2\theta \quad r = \frac{1}{2} \rightarrow$ intersections

CAS



$\cos 2\theta = \frac{1}{2}$

$2\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

r also $= -\frac{1}{2}$

$\cos 2\theta = -\frac{1}{2} \qquad 2\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}$

$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

$\left(\frac{1}{2}, \frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{5\pi}{6}\right), \left(\frac{1}{2}, \frac{7\pi}{6}\right), \left(\frac{1}{2}, \frac{11\pi}{6}\right)$

$\left(\frac{1}{2}, \frac{\pi}{3}\right), \left(\frac{1}{2}, \frac{2\pi}{3}\right), \left(\frac{1}{2}, \frac{4\pi}{3}\right), \left(\frac{1}{2}, \frac{5\pi}{3}\right)$

EX.4 $r = 1 + \sin\theta$

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

$$= \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + (\cos\theta)^2} \, d\theta$$

$$= \int_0^{2\pi} \sqrt{1 + 2\sin\theta + \cancel{\sin^2\theta + \cos^2\theta}} \, d\theta$$

$$= \int_0^{2\pi} \sqrt{2 + 2\sin\theta} \, d\theta = \boxed{8}$$

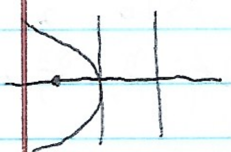
§11.5 Conic Sections

Ex. 1 $y^2+10x=0$

$y^2=-10x=4\left(-\frac{5}{2}\right)x$ $\quad p=-\frac{5}{2}$

directrix → $\boxed{x=\frac{5}{2}}$

focus → $\left(-\frac{5}{2}, 0\right)$



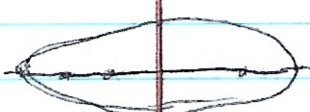
Ex. 2 $9x^2+16y^2=144$

$\cancel{\frac{x^2}{0.0625}+y^2}$ $\quad \frac{x^2}{16}+\frac{y^2}{9}=1$

$a>b$ → horiz ellipse

~~foci~~ $\quad c=\sqrt{a^2-b^2}=13.2$

foci $(\pm 13.2, 0)$



Ex. 3 → foci $(0,\pm 2)$

vertices $(0,\pm 3)$

vertical ellipse $\quad a=3$

$c=2$

$b^2=a^2-c^2$

$b=\sqrt{9-4}=\sqrt{5}$

$\frac{x^2}{5}+\frac{y^2}{9}=1$

$\cancel{4x^2+5y^2=20}$

$9x^2+5y^2=45$

Ex. 4 $9x^2-16y^2=144$

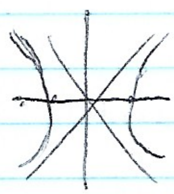
$\frac{x^2}{16}-\frac{y^2}{9}=1$ $\quad \cancel{a>b}$

vertical hyperbola

$c=\sqrt{a^2+b^2}=\sqrt{25}=5$

foci $(\pm 5, 0)$

asymptotes $y=\pm\frac{9}{16}x$



Ex. 5 vert. $(0,\pm 1)$ asym $y=2x$

horiz. $a=1$ $\quad \frac{a}{b}=2$

$b=\frac{1}{2}$ $\quad c=\sqrt{\frac{5}{4}}=\frac{\sqrt{5}}{2}$

$y^2-4x^2=1$

Ex. 6 foci $(2,-2)$, $(4,-2)$

$y\to -2$ $\quad c=1$

$x\to +3$

$(x-3)$, $(y+2)$

$a=2$ $\quad b=\sqrt{a^2-c^2}=\sqrt{4-1}=\sqrt{3}$

horiz.

$\frac{x^2}{4}+\frac{y^2}{3}=1$ (unshifted)

$\frac{(x-3)^2}{4}+\frac{(y+2)^2}{3}=1$

Ex. 7

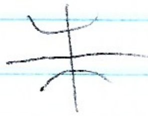
$9x^2 - 72x - 4y^2 + 8y + 176 = 0$

$\underbrace{9x^2 - 72x + 144} - 144 \underbrace{- 4y^2 + 8y - 4} + 4 + 176 = 0$

$9(x-4)^2 - 4(y-1)^2 = -36$

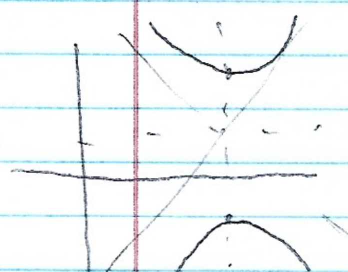
$\frac{(y-1)^2}{9} - \frac{(x-4)^2}{4} = 1$

vert. transvers axis



$x \to +4$ $c = \sqrt{13} = 3.6$

$y \to +1$ $a = 3$ $b = 2$



§10.6 Conic sections in polar

Ex. 1 dir. $y = -6$

$d = 6$ $e = 1 \to$ parabola

$r = \frac{6}{1 \pm \cos\theta}$

use picture $\to$ $r = \frac{6}{1 - \sin\theta}$

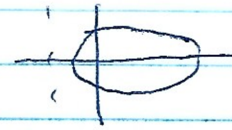
Ex. 2 $r = \frac{10}{3 - 2\cos\theta} = \frac{10/3}{1 - (\frac{2}{3})\cos\theta}$

$ed = \frac{10}{3}$ $e = \frac{2}{3}$ $d = 5$

$e = \frac{2}{3} \to$ ellipse

$d = 5$ $x = -5$ dir.

Sketch $\to$



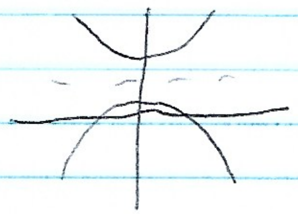
Ex. 3 $r = \frac{12}{2 + 4\sin\theta} = \frac{6}{1 + 2\sin\theta}$

$e = 2$ $d = 3$ $\to$ hyperbola

$\frac{ed}{1 + e\sin\theta}$ $\to$ vertical transverse axis

$\therefore$ directrix is horiz.

$y = 3$



find vertices

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$r = 2, -6$

Ex. 4 rotate an ellipse

$(\theta - \alpha) \to$ $r = \frac{10}{3 - 2\cos(\theta - \frac{\pi}{4})}$

Ex. 5

a) $r = \frac{a(1-e^2)}{1 + e\cos\theta}$ $a = \frac{m}{2} =$ 1.50×10^8 km

$\boxed{r = \frac{1.50 \times 10^8 \text{ km}}{1 + 0.017\cos\theta}}$

b) perihelion $\to a(1-e) = 1.47 \times 10^8$ km

aphelion $\to a(1+e) = 1.53 \times 10^8$ km