NAME: Qn: Ql: E: U:

14

RETURN THE TEST WITH YOUR ANSWERS! MCLA CHAPTER 14 TEST

1. The displacement vector of a particle is given by $\vec{r} = < a\sin t, b\cos t, ct >$.
   A. Obtain its velocity.
   B. Obtain its acceleration
   C. Draw
      1. displacement as a function of time
      2. velocity as a function of time
      3. acceleration as a function of time
      4. Explain the physical meaning of a,b,c.

2. Find a vector function that represents the curve of intersection of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the plane $y+z=b$.

3. An object moving with constant velocity $\vec{v}(t=0)=< v_O,0,0 >$ enters a region of space at (0,0,0) location where there is an acceleration in the z-direction given by $\vec{a}=\left\langle 0,b,-ct^2 \right\rangle$ where b andc are constants. Obtain the object's
   A. velocity
   B. displacement as a function of time.

4.
   A. Show that $\frac{d}{dt}|\vec{r}(t)| = \frac{\vec{r}(t)\bullet\vec{r}'(t)}{|\vec{r}(t)|}$.

      Hint $|\vec{r}(t)|^2 = \vec{r}(t)\bullet\vec{r}(t)$
   B. If $\vec{b}(t)=\vec{r}(t)\bullet[\vec{r}'(t)\times\vec{r}''(t)]$, show that $\vec{b}'(t)=\vec{r}(t)\bullet[\vec{r}'(t)\times\vec{r}'''(t)]$.
   C. Obtain $\vec{b}''(t)$

5. Use your knowledge of vectors
   A. to derive the displacement formula in 2D and 3D for a space curve parametrized in terms of t.
   B. Show that the arc length formula can be written in the compact form $L=\int_a^b v(t).dt$

      where $v(t)$ is the "speed" (the magnitude of the derivative of the displacement vector.

6. Use a graph to explain the geometric meaning of
   A. the unit tangent vector $\vec{T}(t)=\frac{\vec{r}'(t)}{|\vec{r}'(t)|}=\frac{\vec{v}(t)}{v(t)}$
   B. The curvature $\kappa=\left|\frac{d\vec{T}}{ds}\right|=\frac{|\vec{T}'|}{v}$
   C. the principal unit normal vector $\vec{N}(t)=\frac{\vec{T}'(t)}{|\vec{T}'(t)|}$
   D. the binormal vector $\vec{B}(t)=\vec{T}(t)\times\vec{N}(t)$

      Physics students, what do these remind you? Think rotational dynamics.

7. Use the definition in 6D to prove that $\frac{d}{ds}\vec{B}=\vec{T}\times\frac{d}{ds}\vec{N}$.

   FYI: The torsion function of a smooth curve is defined by $\tau=-\frac{d\vec{B}}{ds}\bullet\vec{N}$.

8. Obtain the curvature of the vector function given by $\vec{r}(t)=< b\cos\omega t, b\sin\omega t >$.

9. Prove that the curvature of the curve given by the vector function $\vec{r}(t)$ is $\kappa(t)=\frac{|\vec{r}'(t)\times\vec{r}''(t)|}{|\vec{r}'(t)|^3}$. Follow the steps below for partial credit
   A. $\vec{r}'(t)=v(t)=\frac{ds}{dt}$. Use question 6B, 6A, and write $\vec{r}'(t)$ in terms of $\vec{T}(t)$
   B. Take the derivative of $\vec{r}'(t)$ in 9A with respect to *t* to obtain $\vec{r}''(t)$ in terms of $\vec{T}$, $\vec{T}'$.
   C. Use your answers from 9A and 9B to calculate $\vec{r}'\times\vec{r}''$
   D. Do the necessary cancellations in 9C and use the fact that the magnitudes of $\vec{T}$, $\vec{T}'$ are 1, to obtain the final answer.

10. Use the definition of $\vec{T}(t)$ given in 6A in terms of the speed and velocity, the definition of κ in terms of v and $|\vec{T}'|$, and the definition of $\vec{N}(t)$ to prove that the acceleration of an object may be given by $\vec{a}(t)=v'\vec{T}+\kappa v^2\vec{N}$.

11. Obtain the tangential and normal components of acceleration for a particle whose displacement vector is $\vec{r}=< b\sin t, b\cos t, ct >$.

12. At time t=0, a projectile is fired from the origin with an initial velocity v_O at an angle θ with respect to the horizontal under the effect of gravity $g\downarrow$. In terms of the given quantities, obtain, as a function of time t, its
   A. acceleration
   B. velocity
   C. displacement
   D. maximum height
   E. x-range (where it hits the ground for the first time).

13. Using the "easy" methods introduced in class, obtain $d\vec{s}$, then ds^2 in
   A. cartesian
   B. cylindrical
   C. spherical coordinates

B 20 B 23 BB 43 R 47

BBR 90 T ~~B8~~ 95

S2 25

Mesut B. Çakır

14. Use the expressions

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z$$

$$x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$$

to show that the arc length $ds^2 = dx^2 + dy^2 + dz^2$ of a line is given by

A. $ds^2 = dr^2 + r^2 d\theta^2 + dz^2$ in cylindrical coordinates

B. $ds^2 = d\rho^2 + \rho^2 d\phi^2 + \rho^2 \sin^2\phi\ d\theta^2$ in spherical coordinates

Hint: write the displacement vector in each case and take the derivative. I will put up the picture you need on the board. You do **not** need to show me how you derived the expressions for x, y, z in each coordinate frame.

15. Use the expressions

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z$$

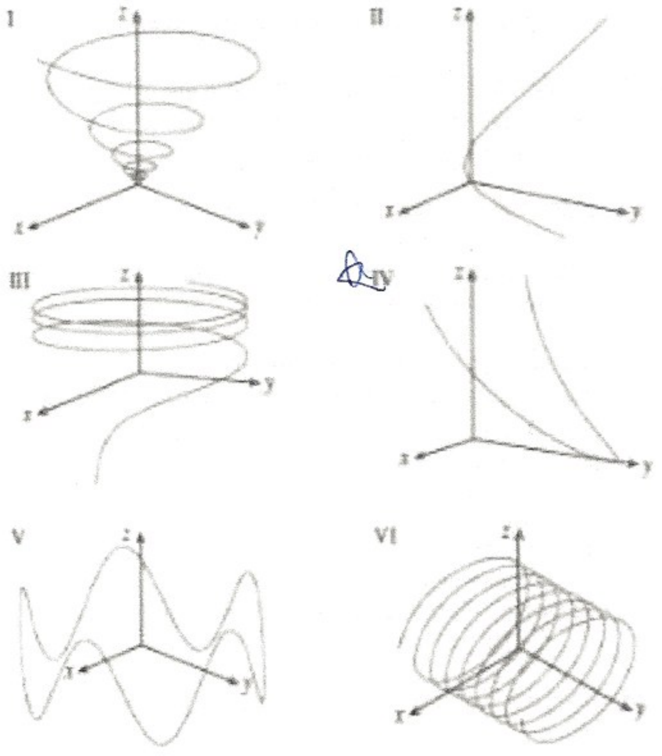
$$x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$$

and $\vec{ds}$ to obta(in terms of $\vec{i}$, $\vec{j}$, $\vec{k}$) the unit vectors

A. $\vec{e}_r$, $\vec{e}_\theta$, $\vec{k}$ of the cylindrical coordinates

B. $\vec{e}_\rho$, $\vec{e}_\theta$, $\vec{e}_\phi$ of the spherical coordinates

16. Deduce the functional behavior of x, y, and z with respect to a parameter t for the figures shown. Hint: You may want to identify the behavior on a plane first (xy-, yz-, or zx-plane) or along one of the axis (i.e. x, y, or z)

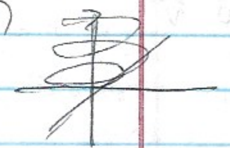
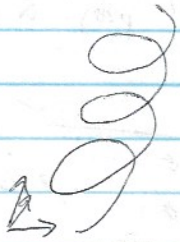


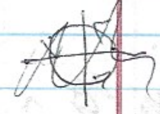
Ch 14 Test

1) $\vec{r} = \langle a\sin t, b\cos t, ct\rangle$

a) $v = r'(t) = \langle a\cos t, -b\sin t, c\rangle$ +1

b) $a = r''(t) = \langle -a\sin t, -b\cos t, 0\rangle$ +1

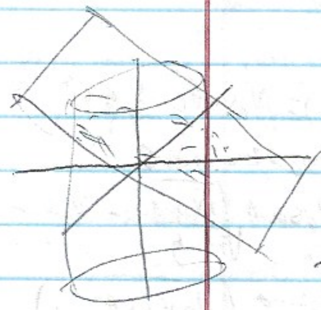
c) 1)  

2) 

3) 

4) Coefficients determine the width of the curve, and c determines the separation of the rings.

2) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; $y + z = h$



$\frac{y}{h} + \frac{z}{h} = 1$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{y}{h} + \frac{z}{h}$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{y}{h} - \frac{z}{h} = 0$

Let $y = t$

$\frac{x^2}{a^2} + \frac{t^2}{b^2} = 1$ ~ ^

$x^2 = a^2 - \frac{a^2}{b^2}t^2$

$x = a\sqrt{1 - \frac{t^2}{b^2}}$ $z = h - t$

$y = t$

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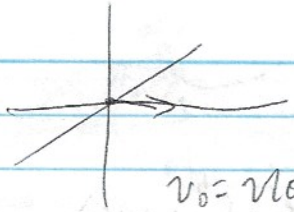
3) 

$v_0 = v(0) = \langle v_0, 0, 0\rangle$

$a = \langle 0, b, -ct^2\rangle$

a) $v_f = v_0 + \int a\,dt$ +1 $\left(\langle \int x\,dt, \int y\,dt, \int z\,dt\rangle\right)$

$v_f = \langle v_0, 0, 0\rangle + \langle 0, bt, -\frac{c}{3}t^3\rangle$ +1

$= \langle v_0, bt, -\frac{c}{3}t^3\rangle$ +1

b) $x(t) = x_0 + \int v\,dt$ +1

$= \langle 0, 0, 0\rangle + \langle v_0 t, \frac{1}{2}bt^2, -\frac{c}{12}t^4\rangle$ +1

$= \langle v_0 t, \frac{1}{2}bt^2, -\frac{c}{12}t^4\rangle$ +1

4) a) $\frac{d}{dt}|\vec{r}(t)| = \frac{\vec{r}(t)\cdot r'(t)}{|\vec{r}(t)|}$

$|r(t)|^2 = r(t)\cdot r(t)$ (take sqrt)

$\frac{d}{dt}|\vec{r}(t)| = \frac{d}{dt}\left|\frac{|r(t)|^2}{|r(t)|}\right| = \frac{d}{dt}\left|\frac{r(t)\cdot r(t)}{|r(t)|}\right|$

$= \cancel{\frac{r'(t)\cdot r(t) + r(t)\cdot r'(t)}{|r(t)|}}$

$\frac{f'g + fg'}{g^2}$ $= \cancel{\frac{(r(t)\cdot r'(t) + r'(t)\cdot r(t))|r(t)| + r(t)\cdot r(t)\cdot r'(t)}{|r(t)|^2}}$ ~

$= \cancel{\frac{2(r(t))^2 r'(t) + (r(t))^2 r'(t)}{(r(t))^2}}$

$= \cancel{3r'(t)}$

b) ~ ~ ~

c) $h'''(t) = r(t)\cdot[r''(t)\times r'''(t)]$ ~ ~

5) $\Delta x = \int_a^b r'(t)\,dt$ (?)

$v(t) = r'(t)$

6

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A

5 cont'd

$r'(t) = v(t)$

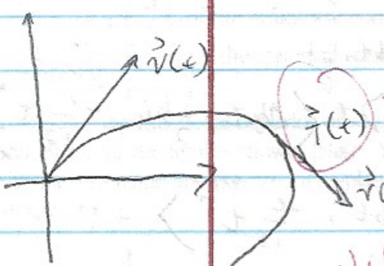
$\Delta x = \int_a^b r'(t)\,dt = \left\langle \int_a^b x'(t)\,dt, \int_a^b y'(t)\,dt \right\rangle$

position → velocity

velocity → displacement

5) neater

6) $\dfrac{\vec{v}(t)}{v(t)} = \left\langle \dfrac{v_x(t)}{v(t)}, \dfrac{v_y(t)}{v(t)} \right\rangle$

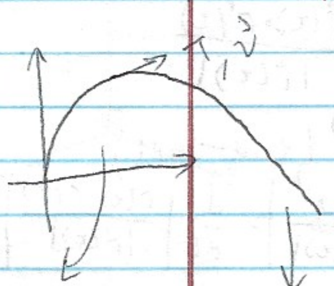


+1

unit vector tangent to direction

N, T, K, r

b)



Greater K　　lower K

$K = \left|\dfrac{dT}{ds}\right| = \dfrac{|T'|}{v}$

$T = \dfrac{\vec{v}(t)}{v(t)}$

$|T'| = \cancel{\ }$

7) $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

8) $K(t) = \dfrac{|r'(t) \times r''(t)|}{|r'(t)|^3}$

$= \dfrac{|\langle -b\omega \sin\omega t, b\omega \cos\omega t \rangle \times \langle -b\omega^2 \cos\omega t, -b\omega^2 \cos\omega t \rangle|}{|\langle -b\omega \sin\omega t, b\omega \cos\omega t \rangle|^3}$

9, 10

11) $r = \langle b\sin t, b\cos t, ct \rangle$

$r'(t) = v(t) = \langle b\cos t, -b\sin t, c \rangle$ +1

$r''(t) = a(t) = \langle -b\sin t, -b\cos t, 0 \rangle$

Tangential. proj. $\vec{a}$ onto $\hat{v}$

12)

$v_{0x} = v_0 \cos\theta$

$v_{0y} = v_0 \sin\theta$

v_0 is (+)

a) $a(t) = g \downarrow = \langle 0, -g \rangle$ -1　↑+

b) $v(t) = v_0 + at$

$= \langle v_0 \cos\theta, v_0 \sin\theta \rangle + \langle 0, g t \rangle$

$= \langle v_0 \cos\theta, v_0 \sin\theta + g t \rangle$ +1

c) $r(t) = v_0 t + \frac{1}{2} a t^2$

$= \langle v_0 \cos\theta\, t, v_0 \sin\theta\, t \rangle + \frac{1}{2}\langle 0, g t^2 \rangle$

$= \langle v_0 \cos\theta\, t, v_0 \sin\theta\, t + \frac{1}{2} g t^2 \rangle$ +1

d) y_{max} occurs when $a_y = 0$

$v_0 \sin\theta - \frac{1}{2} g t^2 = 0$

$\frac{1}{2} g t^2 = v_0 \sin\theta$

$t = \sqrt{\dfrac{2v_0 \sin\theta}{g}}$　　$y(t) = v_0 \sin\theta\, t - \frac{1}{2} g t^2$

~~$= v_0 \sin\theta\, t - v_0 \sin\theta$~~

$y(t)_{max} = v_0 \sin\theta \sqrt{\dfrac{2v_0 \sin\theta}{g}} - v_0 \sin\theta$

$= v_0 \sin\theta \left(\sqrt{\dfrac{2v_0 \sin\theta}{g}} - 1 \right)$

e) $t_{down} = 2 t_{max}$

$r = v_0 \cos\theta\, t = 2 v_0 \cos\theta \sqrt{\dfrac{2v_0 \sin\theta}{g}}$

(6) II - twisted cubic $x = t$ +1
$y = t^2$ +1
$z = t^3$

I) $x = t\cos t$ t^n
$y = t\sin t$ t^n
$z = t$ t^n

II) $x = a\cos t$ $a = b$ +1
$y = b\sin t$ +1
$z = \frac{1}{t}$?

IV ?

V) $x = a\cos t$ +1
$y = b\sin t$ +1
$z = \sin t$ 5 curves

VI $y = t$ +1
$x = a\cos t$ βt
$z = b\sin t$

7

2) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $y + z = h$

proj. of $\mathcal{C}$ into x-y plane is an ellipse; $x = a\cos t$ +1 $y = b\sin t$ +1

$z = h - b\sin t$ +1

$r(t) = a\cos t\,\hat{\imath} + b\sin t\,\hat{\jmath} + (h - b\sin t)\hat{k}$

9) b) $L = \int_a^b v(t)\,dt = \int_a^b |r'(t)|\,dt$

Since $v(t) = |r'(t)|$

$|r'(t)| = |x'(t)\hat{\imath} + y'(t)\hat{\jmath} + z'(t)\hat{k}|$

$= \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$

$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$

$= \int_a^b v(t)\,dt$,

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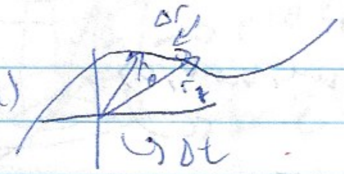
9) a)



add up all infinitesimals of displacements

$L = \int_a^b \frac{dr}{dt}\,dt = \int_a^b dr$

$L = \int_a^b dr = \int_a^b \frac{dr}{dt}\cdot dt$

$= \int_a^b |r'(t)|\,dt$

4) b) $K = \left|\frac{dT}{ds}\right|$

8) $K(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$

$r(t) = \langle b\cos\omega t, b\sin\omega t\rangle$

$r'(t) = \langle -b\omega\sin\omega t, b\omega\cos\omega t\rangle$

$r''(t) = \langle -b\omega^2\cos\omega t, -b\omega^2\sin\omega t\rangle$

$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ -b\omega\sin\omega t & b\omega\cos\omega t & 0 \\ -b\omega^2\cos\omega t & -b\omega^2\sin\omega t & 0 \end{vmatrix}$

$= b\omega^2\hat{k}$

$K(t) = \frac{\sqrt{b^2\cdot\omega^6}}{(\sqrt{b^2\omega^2\sin^2\omega t + b^2\omega^2\cos^2\omega t})^3}$

$= \frac{b\omega^3}{b^3\omega^3} = \boxed{\frac{1}{b^2}}$

11) $r(t) = \langle b\sin t, b\cos t, ct\rangle$

$r'(t) = v(t) = \langle b\cos t, -b\sin t, c\rangle$

$r''(t) = a(t) = \langle -b\sin t, -b\cos t, 0\rangle$

$T(a(t)) = \frac{a'(t)}{|a'(t)|} = \frac{1}{\sqrt{b^2(\sin^2 t + \cos^2 t)}}\,a'(t)$

$= \frac{1}{b}\langle -b\cos t, b\sin t, 0\rangle$ 0

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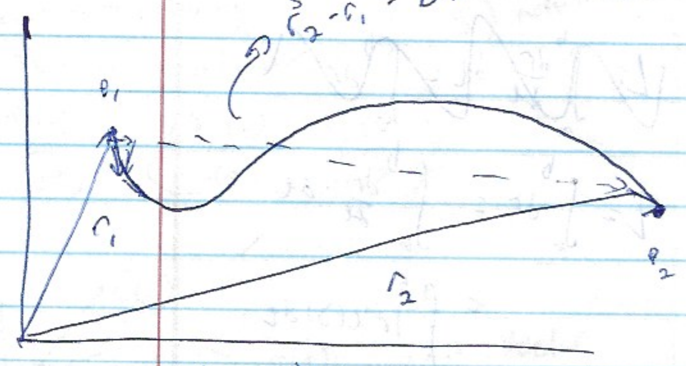
$N(t) = \frac{T'(t)}{|T'(t)|} = \frac{1}{b^2}\langle b\sin t, b\cos t, 0\rangle$

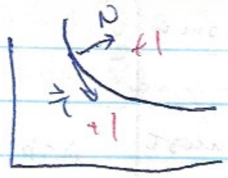
greater $T'(t) \Rightarrow$ steeper curves

9) [pg. 905, stewart]

(e)



c) $N(t) = \frac{T'(t)}{|T'(t)|}$



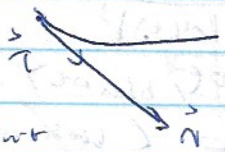
rate of change of speed is acceleration

rate of change of direction of velocity

magnitude $\Rightarrow$ gives a vector perpendicular to the velocity; the direction of the acceleration, centripetally.

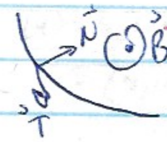
a) $T(t) = \frac{r'(t)}{|r'(t)|} = \frac{\vec{v}(t)}{v(t)}$

$\frac{d\vec{r}}{dt} = \vec{v}$



T is the unit vector of $\vec{v}$

d)



$\vec{N} \odot \vec{B}$ +1

$\vec{B} = \vec{T} \times \vec{N}$

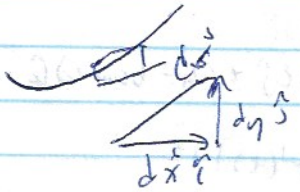
is perpendicular to both; using RHR

b) $\kappa = \left|\frac{dT}{ds}\right| = \frac{|\vec{T}'|}{v}$

$= \left|\frac{dT}{ds}\right| = \left|\frac{dT}{dt}\cdot\frac{dt}{ds}\right|$ (chain rule)

$= \frac{\left|\frac{dT}{dt}\right|}{\left|\frac{ds}{dt}\right|} = \frac{|T'|}{v}$

4) $\frac{d}{dt}|r(t)| = \frac{\vec{r}(t)\cdot\vec{r}'(t)}{|r(t)|}$



$T(t) = \frac{\vec{v}(t)}{v(t)}$

$T'(t) = \frac{\vec{v}'(t)v(t) + \vec{v}(t)\cdot v'(t)}{(v(t))^2}$

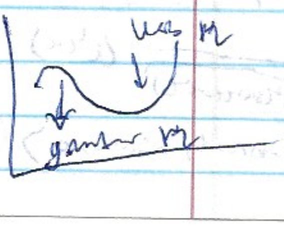
$ds \cdot ds = ds^2 = (\hat{i}dx + dy\hat{j})\cdot(\hat{i}dx + dy\hat{j})$

$\hat{i}\cdot\hat{i} = 1$

$\hat{j}\cdot\hat{j} = 1$

$\hat{i}\cdot\hat{j} = 0$

$\hookrightarrow ds^2 = dx^2 + dy^2$



$\kappa = \left|\frac{T'(t)}{v}\right| = \frac{\text{rate of change of direction}}{\text{speed}}$

$\to 0$

3

$$\frac{d}{dt}|r(t)| = \frac{\vec{r}(t)\cdot r'(t)}{|\vec{r}(t)|}$$

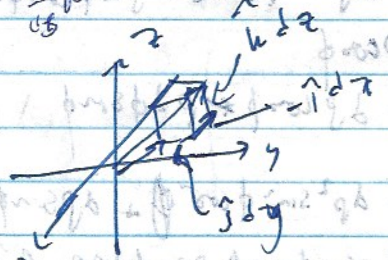
$\frac{d}{dt}|r(t)| \cdots =$

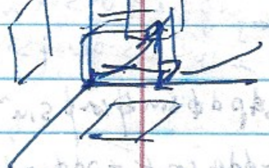
7) $B = T\times N$

$$\frac{d}{ds}B = \frac{d}{ds}(T\times N)$$

$$= \frac{d}{ds}T \times \frac{d}{ds}N$$

13) a) ds



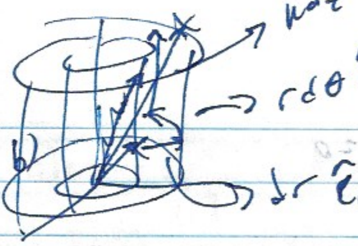


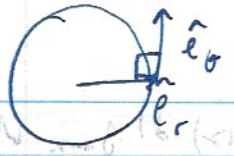
$$\vec{ds} = \hat{i}dx + \hat{j}dy + \hat{k}dz$$

$$ds^2 = \vec{ds}\cdot\vec{ds} = (\hat{i}dx + \hat{j}dy + \hat{k}dz)\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$\hat{i}\cdot\hat{i}=1$	$\hat{i}\cdot\hat{i}\,dx\,dx + \cancel{\hat{i}\cdot\hat{j}\,dx\,dy} + \cancel{\hat{i}\cdot\hat{k}\,dx\,dz}$
$\hat{j}\cdot\hat{j}=1$	$+ \cancel{\hat{j}\,dy\cdot\hat{i}\,dx} + \hat{j}\cdot\hat{j}\,dy\,dy + \cancel{\hat{j}\cdot\hat{k}\,dy\,dz}$
$\hat{k}\cdot\hat{k}=1$	$+ \cancel{\hat{k}\cdot\hat{i}\,dz\,dx} + \cancel{\hat{k}\cdot\hat{j}\,dz\,dy} + \hat{k}\cdot\hat{k}\,dz\,dz$
$\hat{i}\cdot\hat{j}=0$	$= \hat{i}\cdot\hat{i}\,dx^2 + \hat{j}\cdot\hat{j}\,dy^2 + \hat{k}\cdot\hat{k}\,dz^2$
$\hat{i}\cdot\hat{k}=0$	$= dx^2 + dy^2 + dz^2 = ds^2$
$\hat{j}\cdot\hat{k}=0$	

b)



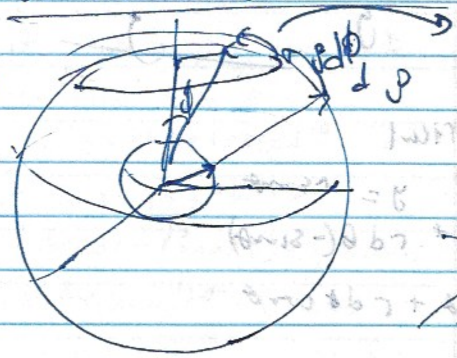


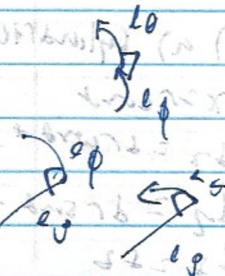
$$\vec{ds} = dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta + \hat{k}\,dz$$

$\hat{e}_r\cdot\hat{e}_r = 1$	$ds^2 = \vec{ds}\cdot\vec{ds} = (dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta + \hat{k}dz)\cdot(dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta + \hat{k}dz)$
$\hat{e}_\theta\cdot\hat{e}_\theta = 1$	$= dr\,dr\,\hat{e}_r\cdot\hat{e}_r + \cancel{dr\,r\,d\theta\,\hat{e}_r\hat{e}_\theta}$
$\hat{k}\cdot\hat{k}=1$	$+ \cancel{dr\,\hat{e}_r\,\hat{k}dz} + \cancel{r\,d\theta\,\hat{e}_\theta\,dr\,\hat{e}_r}$
$\hat{e}_r\cdot\hat{e}_\theta = 0$	$+ r^2\,d\theta\,d\theta\,\hat{e}_\theta\cdot\hat{e}_\theta + \cancel{r\,d\theta\,dz\,\hat{e}_\theta\hat{k}}$
$\hat{e}_r\cdot\hat{k} = 0$	$+ \cancel{\hat{k}dz\,dr\,\hat{e}_r} + \cancel{r\,d\theta\,dz\,\hat{k}\hat{e}_\theta}$
$\hat{e}_k\cdot\hat{e}_\theta = 0$	$+ \hat{k}\,dz\cdot dz\cdot\hat{k}$

$$= dr^2 + r^2d\theta^2 + dz^2 = ds^2$$

c)





$$\vec{ds} = d\rho\,\hat{e}_\rho + \rho\,d\phi\,\hat{e}_\phi + \rho\sin\phi\,d\theta\,\hat{e}_\theta$$

$\hat{e}_\theta\cdot\hat{e}_\theta = 1$	$ds^2 = \vec{ds}\cdot\vec{ds} = (d\rho\,\hat{e}_\rho + \rho\,d\phi\,\hat{e}_\phi + \rho\sin\phi\,d\theta\,\hat{e}_\theta)$
$\hat{e}_\rho\cdot\hat{e}_\rho = 1$	$\cdot(d\rho\,\hat{e}_\rho + \rho\,d\phi\,\hat{e}_\phi + \rho\sin\phi\,d\theta\,\hat{e}_\theta)$
$\hat{e}_\phi\cdot\hat{e}_\phi = 1$	$= d\rho\,d\rho\,\hat{e}_\rho\cdot\hat{e}_\rho + \cancel{d\rho\,\rho\,d\phi\,\hat{e}_\rho\hat{e}_\phi}$
	$+ \cancel{d\rho\,\rho\sin\phi\,d\theta\,\hat{e}_\theta\hat{e}_\rho} + \cancel{\rho\,d\phi\,\hat{e}_\phi\cdot d\rho\,\hat{e}_\rho}$
$\hat{e}_\theta\cdot\hat{e}_\phi = 0$	$+ \rho\,d\phi\,\hat{e}_\phi\cdot\rho\,d\phi\,\hat{e}_\phi + \cancel{\rho\,d\phi\,\rho\sin\phi\,d\theta\,\hat{e}_\phi\hat{e}_\theta}$
$\hat{e}_\rho\cdot\hat{e}_\phi = 0$	$+ \cancel{\rho\sin\phi\,d\theta\,\hat{e}_\theta\cdot d\rho\,\hat{e}_\rho} + \cancel{\rho\sin\phi\,d\theta\,\hat{e}_\theta\,\rho\,d\phi\,\hat{e}_\phi}$
$\hat{e}_\theta\cdot\hat{e}_\rho = 0$	$+ \rho\sin\phi\,d\theta\cdot\rho\sin\phi\,d\theta\cdot\hat{e}_\theta\cdot\hat{e}_\theta$

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + \rho^2\sin^2\phi\,d\theta^2$$

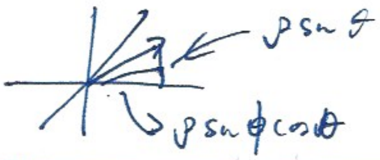
0

0

17

19

8



12) d) y_{max} when $v_y = 0$

$v_0 \sin\theta - g t = 0$

or $t = \dfrac{v_0 \sin\theta}{g}$ +1

$y(t) = v_0 \sin\theta t - \frac{1}{2} g t^2$

$= \dfrac{v_0^2 \sin^2\theta}{g} - \dfrac{1}{2} g \cdot \dfrac{v_0^2 \sin^2\theta}{g^2}$

$= \boxed{\dfrac{v_0^2 \sin^2\theta}{2g}}$ +1

e) $2t$ = range (x) +1

$x = v_0 \cos\theta \cdot \dfrac{v_0 \sin\theta}{g} = \boxed{\dfrac{v_0^2 \sin\theta \cos\theta}{g}}$ +1

14) a) cylindrical

$x = r\cos\theta \qquad y = r\sin\theta$

$dx = dr\cos\theta + r\,d\theta(-\sin\theta)$ +1

$dy = dr\sin\theta + r\,d\theta\cos\theta$ +1

$dz = dz$

$dx^2 = (dr\cos\theta - r\,d\theta\sin\theta)^2$

$= dr^2\cos^2\theta - 2dr\cos\theta \cdot r\,d\theta\sin\theta$

$+ r^2 d\theta^2 \sin^2\theta$ +1

$dy^2 = (dr\sin\theta + r\,d\theta\cos\theta)^2$

$= dr^2\sin^2\theta + 2dr\sin\theta \cdot r\,d\theta\cos\theta$

$+ r^2 d\theta^2 \cos^2\theta$ +1

$dz^2 = dz^2$

$ds^2 = dx^2 + dy^2 + dz^2$

$= dr^2\cos^2\theta - 2r\,dr\,\cancel{\cos\theta\sin\theta\, d\theta}$ +1

$+ r^2 d\theta^2 \sin^2\theta + dr^2\sin^2\theta + 2r\,dr\,\cancel{\cos\theta\sin\theta\, d\theta}$

$+ r^2 d\theta^2 \cos^2\theta + dz^2$

$= dr^2(\cos^2\theta + \sin^2\theta) + r^2 d\theta^2(\cos^2\theta + \sin^2\theta) + dz^2$

$= \underline{dr^2 + r^2 d\theta^2 + dz^2}$ +1

10

b) spherical

$x = (\rho\sin\phi)\cos\theta$

$dx = d\rho\sin\phi\cos\theta + \rho\,d\phi\cos\phi\cos\theta - \rho\sin\phi\,d\theta\sin\theta$ +1

$y = \rho\sin\phi\sin\theta$

$dy = d\rho\sin\phi\sin\theta + \rho\,d\phi\cos\phi\sin\theta + \rho\sin\phi\,d\theta\cos\theta$ +1

$z = \rho\cos\phi$

$dz = d\rho\cos\phi - \rho\,d\phi\sin\phi$ +1

$dx^2 = d\rho^2\sin^2\phi\cos^2\theta + d\rho\sin\phi\cos\theta \cdot \rho\,d\phi\cos\phi\cos\theta$

$- d\rho\sin\phi\cos\theta \cdot \rho\sin\phi\,d\theta\sin\theta + \rho^2 d\phi^2\cos^2\phi\cos^2\theta$

$+ \rho\,d\phi\cos\phi\cos\theta \cdot d\rho\sin\phi\cos\theta - \rho\,d\phi\cos\phi\cos\theta \cdot \rho\sin\phi\,d\theta\sin\theta$

$+ \rho^2\sin^2\phi\,d\theta^2\sin^2\theta - \rho\,d\rho\sin^2\phi\,d\theta\sin\theta\cos\theta - \rho^2 d\phi\sin\phi\cos\phi\sin\theta\cos\theta\,d\theta$

$dy^2 = d\rho^2\sin^2\phi\sin^2\theta + \rho\,d\rho\,d\phi\sin\phi\cos\phi\sin^2\theta$

$+ d\rho\sin\phi\sin\theta \cdot \rho\sin\phi\,d\theta\cos\theta + \rho\,d\phi\cos\phi\sin\theta \cdot d\rho\sin\phi\sin\theta$

$+ \rho^2 d\phi^2\cos^2\phi\sin^2\theta + \rho\,d\phi\cos\phi\sin\theta \cdot \rho\sin\phi\,d\theta\cos\theta$

$+ \rho\sin\phi\,d\theta\cos\theta \cdot d\rho\sin\phi\sin\theta + \rho\sin\phi\,d\theta\cos\theta \cdot \rho\,d\phi\cos\phi\sin\theta$

$+ \rho^2\sin^2\phi\,d\theta^2\cos^2\theta$

$dz^2 = d\rho^2\cos^2\phi - 2\rho\,d\rho\,d\phi\sin\phi\cos\phi + \rho^2 d\phi^2\sin^2\phi$

$ds^2 = dx^2 + dy^2 + dz^2$

$= \cancel{d\rho^2\sin^2\phi}\cos^2\theta + \rho\,d\rho\,d\phi\cancel{\sin\phi\cos\phi}\cos^2\theta$

$- \cancel{\rho\,d\rho\sin^2\phi\,d\theta\sin\theta\cos\theta} + \rho^2\cancel{d\phi^2}\cos^2\phi\cos^2\theta$

$+ \rho\,d\rho\,d\phi\sin\phi\cos\phi\cos^2\theta - \cancel{\rho^2 d\phi\sin\phi\cos\phi\,d\theta\sin\theta\cos\theta}$

$+ \rho^2\sin^2\phi\,d\theta^2\sin^2\theta + \cancel{d\rho^2\sin^2\phi}\sin^2\theta$

$+ \rho\,d\rho\,d\phi\sin\phi\cos\phi\sin^2\theta + \cancel{\rho\,d\rho\sin^2\phi\,d\theta\sin\theta\cos\theta}$

$+ \rho\,d\rho\,d\phi\sin\phi\cos\phi\sin^2\theta + \rho^2\cancel{d\phi^2\cos^2\phi}\sin^2\theta$

$+ \cancel{\rho^2 d\phi\sin\phi\cos\phi\,d\theta\sin\theta\cos\theta} + \cancel{\rho\,d\rho\sin^2\phi\,d\theta\sin\theta\cos\theta}$

$+ \cancel{\rho^2 d\phi\sin\phi\cos\phi\,d\theta\sin\theta\cos\theta} + \rho^2\sin^2\phi\,d\theta^2\cos^2\theta + d\rho^2\cos^2\phi$

$- 2\rho\,d\rho\,d\phi\sin\phi\cos\phi + \cancel{\rho^2 d\phi^2\sin^2\phi} - \cancel{\rho^2 d\phi\sin\phi\cos\phi\sin\theta\cos\theta\,d\theta}$

$- \cancel{\rho\,d\rho\sin^2\phi\,d\theta\sin\theta\cos\theta}$

3

14) b) cont'd

is $d\rho^2,\ \rho^2 d\phi^2,\ \rho^2\sin^2\phi\, d\theta^2$

$= d\rho^2\sin^2\phi\cos^2\theta + d\rho^2\sin^2\phi\sin^2\theta + d\rho^2\cos^2\phi$
$+ \rho^2 d\phi^2\cos^2\phi\cos^2\theta + \rho^2 d\phi^2\cos^2\phi\sin^2\theta$
$+ \rho^2 d\phi^2\sin^2\phi$
$+ \rho^2\sin^2\phi\, d\theta^2\sin^2\theta + \rho^2\sin^2\phi\, d\theta^2\cos^2\theta$
$\cancel{-\rho^2 d\phi\sin\phi\cos\phi\, d\theta\sin\theta\cos\theta\cdot 2}$
$\cancel{+\rho^2 d\phi\sin\phi\cos\phi\, d\theta\sin\theta\cos\theta\cdot 2}$
$+2\rho\, d\rho\, d\phi\sin\phi\cos\phi\cos^2\theta$
$+2\rho\, d\rho\, d\phi\sin\phi\cos\phi\sin^2\theta$
$-2\rho\, d\rho\, d\phi\sin\phi\cos\phi$ $= 0$ ($\cos^2\theta+\sin^2\theta=1$)

$= d\rho^2(\sin^2\phi\cos^2\theta+\sin^2\phi\sin^2\theta+\cos^2\phi)$
$+ \rho^2 d\phi^2(\cos^2\phi\cos^2\theta+\cos^2\phi\sin^2\theta+\sin^2\phi)$
$+ \rho^2\sin^2\phi\, d\theta^2(\sin^2\theta+\cos^2\theta)$
$= d\rho^2(\sin^2\phi(\sin^2\theta+\cos^2\theta)+\cos^2\phi)$
$+ \rho^2 d\phi^2(\cos^2\phi(\sin^2\theta+\cos^2\theta)+\sin^2\phi) + \rho^2\sin^2\phi\, d\theta^2$
$= \boxed{d\rho^2+\rho^2 d\phi^2+\rho^2\sin^2\phi\, d\theta^2 = ds^2}$

~~9~~, 15, 8, 7, 6, 4, 5a, 16 I, II-3, III-3, IV, V-3, VI-2,3
~~10~~

10) $\vec{v} = v\cdot\hat{v} = vT$, given that $T = \frac{r'(t)}{|r'(t)|} = \frac{\vec{r}'}{|\vec{r}'|} = \hat{v}$

$a(t) = \frac{dv}{dt} = \frac{d(vT)}{dt} = \frac{dv}{dt}\cdot T + v\frac{d\hat{T}}{dt}$

$= v'T + v\cdot\frac{dT}{dt}$

~~Since~~ $K = \frac{|T'|}{v}$ $= v'T + v|T'|\vec{N}$

because $\vec{N} = \frac{T'}{|T'|}$

$v'T + v\cdot|T'|\cdot\frac{v}{v}\vec{N}$ $\frac{|T'|}{v} = K$

$= \boxed{v'T + Kv^2\vec{N} = a(t)}$

4

9) a) $T = \frac{\vec{r}'}{|r'|} \rightarrow \vec{r}' = T|r'| \rightarrow \vec{v} = v\vec{T}$

b) $\frac{dv}{dt} = \frac{d(\vec{T}v)}{dt} = T'v + Tv'$

c) $r'\times r'' = v\times v' = \vec{v}\times(v\vec{T}'+\vec{T}v')$
$= v\vec{v}\times\vec{T}' + \vec{v}\times\vec{T}v'$

$\vec{v}\times\vec{T} = 0$ b/c $\vec{v} = v\vec{T}$

$\therefore \vec{r}'\times\vec{r}'' = v\vec{v}\times\vec{T}'$

d) $\frac{|r'\times r''|}{|r'|^3} = \frac{|\vec{v}\times\vec{T}'|}{v^2} = \frac{|v\vec{T}\times\vec{N}|\,|T'|}{v^2}$

$T' = N|T'|$; $B = T\times N$, $|B| = 1$

$\frac{|r'\times r''|}{|r'|^3} = \frac{v|T'|\,|T\times N|}{v^2} = \frac{|T'|}{v} = K$

7

15) a) $x = r\cos\theta$ $y = r\sin\theta$ $z = z$ Cylindrical

$dx = \cos\theta\, dr - r\sin\theta\, d\theta$

$dy = \sin\theta\, dr + \cos\theta\, r\, d\theta$, dz

$ds = i\,dx + j\,dy + k\,dz$

$= i(\cos\theta\, dr - r\sin\theta\, d\theta) + j(\sin\theta\, dr + r\cos\theta\, d\theta) + k\,dz$

$= (i\cos\theta + j\sin\theta)dr + (-i\sin\theta + j\cos\theta)r\,d\theta + k\,dz$

$\vec{e}_r = i\cos\theta + j\sin\theta$, $e_\theta = -i\sin\theta + j\cos\theta$, $\hat{k}$

b) spherical

$x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$

$dx = d\rho\sin\phi\cos\theta - \rho\sin\phi\sin\theta\, d\theta + \rho\cos\phi\, d\phi\cos\theta$

$dy = \sin\phi\sin\theta\, d\rho + \rho\sin\phi\cos\theta\, d\theta + \rho\cos\phi\sin\theta\, d\phi$

$dz = \cos\phi\, d\rho - \rho\sin\phi\, d\phi$

$ds = i\,dx + j\,dy + k\,dz$

$= \hat{i}(\sin\phi\cos\theta\, d\rho - \rho\sin\phi\sin\theta\, d\theta + \rho\cos\phi\cos\theta\, d\phi)$
$+ \hat{j}(\sin\phi\sin\theta\, d\rho + \rho\sin\phi\cos\theta\, d\theta + \rho\cos\phi\sin\theta\, d\phi)$
$+ \hat{k}(\cos\phi\, d\rho - \rho\sin\phi\, d\phi)$

$= d\rho(i\sin\phi\cos\theta + \cancel{j\sin\phi\sin\theta\, d\theta} + j\sin\phi\sin\theta + k\cos\phi)$
$+ \rho\, d\phi(\hat{i}\cos\phi\cos\theta + \hat{j}\cos\phi\sin\theta - k\sin\phi)$
$+ \rho\sin\phi\, d\theta(-i\sin\theta + \hat{j}\cos\theta)$ | $\hat{e}_\phi = \hat{i}\cos\phi\cos\theta + \hat{j}\cos\phi\sin\theta - k\sin\phi$

$\therefore e_\rho = i\sin\phi\cos\theta + j\sin\phi\sin\theta + k\cos\phi$ $\hat{e}_\theta = -i\sin\theta + \hat{j}\cos\theta$

14

21

17

19

8) $r(t) = \langle b\cos\omega t, b\sin\omega t\rangle$

$K = \left|\frac{dT}{ds}\right| = \frac{|T'|}{v}$; $T(t) = \frac{r'(t)}{|r'(t)|} = \frac{\vec{v}(t)}{v(t)}$

$T(t) = \frac{\vec{v}(t)}{v(t)} = \frac{b\omega\langle -\sin\omega t, \cos\omega t\rangle}{b\omega} = \langle -\sin\omega t, \cos\omega t\rangle$

$K = \frac{|T'|}{v} = \frac{|\omega\langle -\cos\omega t, -\sin\omega t\rangle|}{b\omega} = \boxed{\frac{1}{b}}$ +3

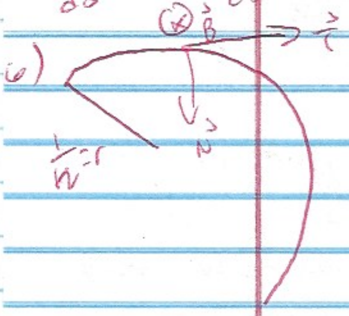
7) $\frac{dB}{ds} = \frac{d(T\times N)}{ds} = \frac{dT}{ds}\times N + T\times\frac{d}{ds}N$

$\frac{dT}{ds}\times N = \frac{dT}{dt}\frac{dt}{ds}\times N = \frac{dt}{ds}\cdot\frac{d\vec{T}}{dt}\times\vec{N}$

$= \frac{dt}{ds}\vec{N}\times\vec{N} = 0$

$\frac{dB}{ds} = \vec{T}\times\frac{d}{ds}\vec{N}$ +3

c)



+1

4) a) $\frac{d}{dt}|r(t)| = \frac{d}{dt}\sqrt{r(t)\cdot r(t)} = \frac{\frac{1}{2}\left(\frac{dr}{dt}\cdot r + r\cdot\frac{dr}{dt}\right)}{\sqrt{r(t)\cdot r(t)}}$

$\therefore \frac{d}{dt}|r(t)| = \frac{r(t)\cdot r'(t)}{|r(t)|}$

b) $\frac{d}{dt}h(t) = \frac{d}{dt}(r(t)\cdot(r'\times r''(t)))$

$h'(t) = r'\cdot(r'\times r'') + \vec{r}\cdot(r''\times r'') + \vec{r}\cdot(r'\times r''')$

$r'\cdot(r'\times r'') = 0$ since $r'\perp[r'\times r'']$

$r\cdot(r''\times r'') = 0$ as $r''\times r'' = 0$ since $r'' \parallel r''$

$h' = \vec{r}\cdot(r'\times r''')$

c) $\frac{d}{dt}h' = \frac{d}{dt}(r\cdot(r'\times r'''))$

$h'' = r'\cdot(r'\times r''') + \vec{r}\cdot(r''\times r''') + r\cdot(r'\times r'''')$

$= \vec{r}\cdot(r''\times r''') + \vec{r}\cdot(r'\times r'''')$

+6

13

5) a) $ds = i\,dx + j\,dy + k\,dz$

$ds = i\cdot\frac{dx}{dt}dt + j\frac{dy}{dt}dt + k\frac{dz}{dt}dt$

$S = \int_b^a \left(i\frac{dx}{dt} + j\frac{dy}{dt} + k\frac{dz}{dt}\right)dt$

6) I) $x = t^m\cos t$, $y = t^m\sin t$, $z = t^m\cos t$ +3

II) $z = \frac{1}{t^m}$, $\ln t$ +1

III) $z = t^m$, $\ln t$ +1

IV) $x = t$, $y = \frac{1}{1+t^2}$, $z = t^2$ +3

V) $z = b\sin 5t$ +1

VI) $y = t$, $z = b\sin\beta t$, $x = a\cos\beta t$ +2

5) b) $L = \int_a^b ds = \int_{t_a}^{t_b}\frac{ds}{dt}dt = \int_{t_a}^{t_b} v\,dt$ +2

$= v$

~~8) b) $L = \int_a^b ds = \int_{t_a}^{t_b}\frac{ds}{dt}dt$~~

~~$L = \int_{t_a}^{t_b} v\,dt$~~

~~$\frac{ds}{dt} = v$~~

+2

13

26

21

47

26

21

42

Sol 14

1. $\vec{r} = < a\sin t, b\cos t, ct >$

A. $\vec{v} = \frac{d\vec{r}}{dt} = < a\cos t, -b\sin t, c >$ 1pt

B. $\vec{a} = \frac{d\vec{v}}{dt} = < -a\sin t, -b\cos t, 0 >$ 1pt

C. Draw
   1. displacement as a function of time
   2. velocity as a function of time
   3. acceleration as a function of time
   4. Explain the physical meaning of a, b, c.

2. We need to have $x = a\cos(t+\varphi)$ and $y = b\sin(t+\varphi)$ which gives

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2\cos^2(t+\varphi)}{a^2} + \frac{b^2\sin^2(t+\varphi)}{b^2} = 1 \text{ and}$$

$z = b - y = b - b\sin(t+\varphi)$. As a result,

$\vec{r} = < a\cos(t+\varphi), b\sin(t+\varphi), b - b\sin(t+\varphi) >$ 3 pts

1pt per component with or without φ the phase angle.

5

3. $\vec{v}(t=0) = < v_o, 0, 0 >$, $\vec{r}(t=0) = < 0,0,0 >$, $\vec{a} = \left\langle 0, b, -ct^2 \right\rangle$

A. $\vec{v} = \int \vec{a}\,dt = \left\langle \int 0\,dt, \int b\,dt, -\int ct^2 dt \right\rangle$ 1pt

$= \left\langle v_x(0), bt + v_y(0), -\frac{ct^3}{3} + v_z(0) \right\rangle$ 1pt

$= \left\langle v_o, bt, -\frac{ct^3}{3} \right\rangle$, since $\vec{v}(t=0) = < v_o, 0, 0 >$ 1pt

B. $\vec{r} = \int \vec{v}\,dt = \left\langle \int v_o dt, \int bt\,dt, -\int \frac{ct^3}{3} dt \right\rangle$ 1pt

$= \left\langle v_o t + x(0), \frac{bt^2}{2} + y(0), -\frac{ct^4}{12} + z(0) \right\rangle$ 1pt

$= \left\langle v_o t, \frac{bt^2}{2}, -\frac{ct^4}{12} \right\rangle$, since $\vec{r}(t=0) = < 0,0,0 >$ 1pt

4.

A. $\frac{d}{dt}|\vec{r}(t)| = \frac{d}{dt}\sqrt{(\vec{r}(t)\bullet\vec{r}(t))} = \frac{\frac{1}{2}\left(\frac{d\vec{r}}{dt}\bullet\vec{r} + \vec{r}\bullet\frac{d\vec{r}}{dt}\right)}{\sqrt{(\vec{r}(t)\bullet\vec{r}(t))}}$ 1pt

$\frac{d}{dt}|\vec{r}(t)| = \frac{\vec{r}(t)\bullet\vec{r}'(t)}{|\vec{r}(t)|}$

B. $\frac{d}{dt}\vec{b}(t) = \frac{d}{dt}(\vec{r}(t)\bullet[\vec{r}'\times\vec{r}''(t)])$

$\vec{b}'(t) = \vec{r}'\bullet[\vec{r}'\times\vec{r}''] + \vec{r}\bullet[\vec{r}''\times\vec{r}''] + \vec{r}\bullet[\vec{r}'\times\vec{r}''']$ 1pt

$\vec{r}'\bullet[\vec{r}'\times\vec{r}''] = 0 \;\; b/c \;\; \vec{r}'\perp[\vec{r}'\times\vec{r}'']$ 1pt

$\vec{r}\bullet[\vec{r}''\times\vec{r}''] = 0 \;\; b/c \;\; \vec{r}''\times\vec{r}'' = 0$ since $\vec{r}'' // \vec{r}''$ 1pt

$\vec{b}' = \vec{r}\bullet[\vec{r}'\times\vec{r}''']$ 10

C. $\frac{d}{dt}\vec{b}' = \frac{d}{dt}(\vec{r}\bullet[\vec{r}'\times\vec{r}'''])$

$\vec{b}'' = \vec{r}'\bullet[\vec{r}'\times\vec{r}'''] + \vec{r}\bullet[\vec{r}''\times\vec{r}'''] + \vec{r}\bullet[\vec{r}'\times\vec{r}'''']$ 1pt

$\vec{b}'' = \vec{r}\bullet[\vec{r}''\times\vec{r}'''] + \vec{r}\bullet[\vec{r}'\times\vec{r}'''']$ 1pt

2

5.

A. $d\vec{s} = \hat{i}dx + \hat{j}dy + \hat{k}dz$

$d\vec{s} = \hat{i}\frac{dx}{dt}dt + \hat{j}\frac{dy}{dt}dt + \hat{k}\frac{dz}{dt}dt$

$\vec{s} = \int(\hat{i}\frac{dx}{dt} + \hat{j}\frac{dy}{dt} + \hat{k}\frac{dz}{dt})dt$

B. $L = \int_a^b ds = \int_{t_s}^{t_b}\frac{ds}{dt}dt$ 1pt $L = \int_{t_a}^{t_b} v\,dt$ since $\frac{ds}{dt} = v$ 1pt

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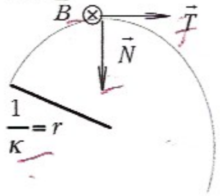
6. *1pt for each correct item. No points for incorrect directions.*



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7. $\frac{d\vec{B}}{ds} = \frac{d(\vec{T}\times\vec{N})}{ds} = \frac{d\vec{T}}{ds}\times\vec{N} + \vec{T}\times\frac{d}{ds}\vec{N}$ 1pt

$\frac{d\vec{T}}{ds}\times\vec{N} = \frac{d\vec{T}}{dt}\frac{dt}{ds}\times\vec{N}$ (1pt) $= \frac{dt}{ds}\frac{d\vec{T}}{dt}\times\vec{N} = \frac{dt}{ds}\vec{N}\times\vec{N} = 0$

1pt

3.

Therefore, $\frac{d\vec{B}}{ds} = \vec{T}\times\frac{d}{ds}\vec{N}$

8. $\vec{r}(t) = < b\cos\omega t, b\sin\omega t >$ looks like a circle of radius b; therefore, we expect the curvature to be $\kappa = b^{-1}$. Lets see.

$\kappa = \left|\frac{d\vec{T}}{ds}\right| = \frac{|\vec{T}'|}{v}$, with $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\vec{v}(t)}{v(t)}$ 1pt

$\vec{T}(t) = \frac{\vec{v}(t)}{v(t)} = \frac{b\omega < -\sin\omega t, \cos\omega t >}{b\omega} = < -\sin\omega t, \cos\omega t >$

1pt

$\kappa = \frac{|\vec{T}'|}{v} = \frac{|\omega < -\cos\omega t, -\sin\omega t >|}{b\omega} = \frac{1}{b}$ as expected. 1pt

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9.

A. $\vec{T} = \frac{\vec{r}'}{|\vec{r}'|}$, $\vec{r}' = \vec{T}|\vec{r}'|$ or $\vec{v} = \vec{T}v$ 1pt

B. $\frac{d\vec{v}}{dt} = \frac{d(\vec{T}v)}{dt} = \vec{T}'v + \vec{T}v'$ 1pt

C. $\vec{r}'\times\vec{r}'' = \vec{v}\times\vec{v}' = \vec{v}\times(v\vec{T}' + \vec{T}v') = v\vec{v}\times\vec{T}' + \vec{v}\times\vec{T}v'$ 1pt

but $\vec{v}\times\vec{T} = 0$ since $\vec{v} = v\vec{T}$ 1pt

$\vec{r}'\times\vec{r}'' = v\,\vec{v}\times\vec{T}'$

D. $\frac{|\vec{r}'\times\vec{r}''|}{|\vec{r}'|^3} = \frac{|\vec{v}\times\vec{T}'|}{v^2} = \frac{|v\vec{T}\times\vec{N}|\vec{T}'||}{v^2}$ 1pt

but $\vec{T}' = \vec{N}|\vec{T}'|$, $\vec{B} = \vec{T}\times\vec{N}$, $|\vec{B}| = 1$ 1pt

$\frac{|\vec{r}'\times\vec{r}''|}{|\vec{r}'|^3} = \frac{v|\vec{T}'||\vec{T}\times\vec{N}|}{v^2} = \frac{|\vec{T}'|}{v} = \kappa$ 1pt

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10. $\vec{v}=v\vec{T}\ ,\ \vec{a}=\dfrac{d\vec{v}}{dt}=\dfrac{d(v\vec{T})}{dt}=\dfrac{dv}{dt}\vec{T}+v\dfrac{d\vec{T}}{dt}$ 1pt

$\vec{a}=\dfrac{d\vec{v}}{dt}=\dfrac{d(v\vec{T})}{dt}=\dfrac{dv}{dt}\vec{T}+v\dfrac{d\vec{T}}{dt}=v'\vec{T}+v\vec{N}\,|\vec{T}'|$ 1pt

$\dfrac{dv}{dt}\vec{T}+v\dfrac{d\vec{T}}{dt}=v'\vec{T}+v\ \vec{N}\ |\ \vec{T}'|$ since

$\vec{N}(t)=\dfrac{\vec{T}\ '(t)}{|\vec{T}\ '(t)|}$ and $\kappa=\dfrac{|\vec{T}'|}{v}$ 1pt

$\vec{a}=v'\vec{T}+\kappa v^2\vec{N}$ 1pt which is $\vec{a}=a_t\vec{T}+\dfrac{v^2}{r}\vec{N}=a_t\vec{T}+a_c\vec{N}$

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11. $\vec{r}=<b\sin t, b\cos t, ct>$. We need to obtain $a_t=v'$ and κv^2.

$\dfrac{d\vec{r}}{dt}=\vec{v}=<b\cos t,-b\sin t,c>$ 1pt

$v=\sqrt{b^2\cos^2 t+b^2\sin^2 t+c^2}=\sqrt{b^2+c^2}$ 1pt

$a_t=\dfrac{dv}{dt}=\dfrac{d}{dt}\sqrt{b^2+c^2}=0$ 1pt

$\vec{T}=\dfrac{\vec{v}}{v}=\dfrac{<b\cos t,-b\sin t,c>}{\sqrt{b^2+c^2}}$ 1pt

$\vec{T}'=\dfrac{<-b\sin t,-b\cos t,0>}{\sqrt{b^2+c^2}},\ |\vec{T}'|=\dfrac{b}{\sqrt{b^2+c^2}}$ 1pt

$\kappa=\dfrac{|\vec{T}'|}{v}=\dfrac{b}{b^2+c^2}$ 1pt

$a_c=\kappa v^2=b$ 1pt

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12. At time t=0, a projectile is fired from the origin with an initial velocity v_0 at an angle θ with respect to the horizontal under the effect of gravity $g\downarrow$. In terms of the given quantities, obtain, as a function of time t, its

A. $\vec{a}=g\downarrow$ 1pt

B. $\vec{v}=\int\vec{a}dt=gt(-\hat{j})+\vec{v}_0=v_0\cos\theta\ \hat{i}+(v_0\sin\theta-gt)\hat{j}$ 1pt

C. $\vec{r}=\int\vec{v}dt=v_0\cos\theta\ t\ \hat{i}+(v_0\sin\theta\ t-\frac{1}{2}gt^2)\hat{j}$ 1pt

$\vec{r}_0=(0,0)$ since it starts at the origin.

D. It reaches its maximum height when $\vec{v}_y=0$,

$v_0\sin\theta-gt=0.\ t=\dfrac{v_0\sin\theta}{g}$ 1pt

$h=(v_0\sin\theta\ t-\frac{1}{2}gt^2)=\dfrac{v^2{}_0\sin^2\theta}{2g}$ 1pt

E. $y=0=v_0\sin\theta\ t-\frac{1}{2}gt^2,\ t=\dfrac{2v_0\sin\theta}{g}$ 1pt

$x=v_0\cos\theta\ t=v_0\cos\theta\dfrac{2v_0\sin\theta}{g}=\dfrac{2v^2{}_0\cos\theta\sin\theta}{g}$ 1pt

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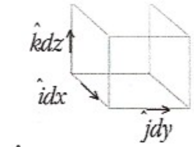
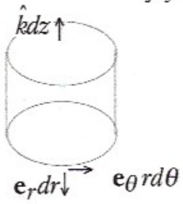
13. See class notes.



A. $\hat{i}dx\ \hat{j}dy\ \hat{k}dz$

$d\vec{s}=\mathbf{i}dx+\mathbf{j}dy+\mathbf{k}dz$

$ds^2=d\vec{s}\bullet d\vec{s}=dx^2+dy^2+dz^2$



B. $\mathbf{e}_r dr,\ \hat{\theta}\ \mathbf{e}_\theta rd\theta, \hat{k}dz$

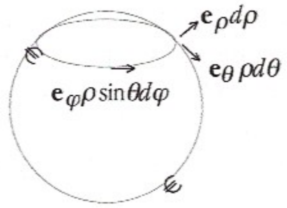
$d\vec{s}=\mathbf{e}_r dr+\mathbf{e}_\theta rd\theta+\mathbf{k}dz$

$ds^2=d\vec{s}\bullet d\vec{s}=dr^2+r^2d\theta^2+dz^2$

C. $\mathbf{e}_\rho d\rho,\ \mathbf{e}_\theta\rho d\theta, \mathbf{e}_\varphi\rho\sin\theta d\varphi$

$d\vec{s}=\mathbf{e}_\rho d\rho+\mathbf{e}_\theta\rho d\theta+\mathbf{e}_\varphi\rho\sin\theta d\varphi$

$ds^2=d\vec{s}\bullet d\vec{s}=d\rho^2+\rho^2d\theta^2+\rho^2\sin^2\theta d\varphi^2$



14.

A. $x=r\cos\theta,\ y=r\sin\theta,\ z=z$

$dx=\cos\theta dr-r\sin\theta d\theta$ 1pt

$dy=\sin\theta dr+\cos\theta d\theta\ ,\ dz$ 1pt

$dx^2=\cos^2\theta dr^2-2\cos\theta dr\ \ r\sin\theta d\theta+r^2\sin^2\theta d\theta^2$ 1pt

$dy^2=\sin^2\theta dr^2+2\sin\theta dr\ \ r\cos\theta d\theta+r^2\cos^2\theta d\theta^2$ 1pt

When we add these two, the cross terms cancel 1pt

Since $\sin^2\theta+\cos^2\theta=1$, we end up with 1pt

$ds^2=dx^2+dy^2+dz^2=dr^2+r^2d\theta^2+dz^2$

B. $x=\rho\sin\phi\cos\theta,\ y=\rho\sin\phi\sin\theta,\ z=\rho\cos\phi$

$dx=\sin\phi\cos\theta d\rho-\rho\sin\phi\sin\theta d\theta+\rho\cos\phi\cos\theta d\phi$ 1pt

$dy=\sin\phi\sin\theta d\rho+\rho\sin\phi\cos\theta d\theta+\rho\cos\phi\sin\theta d\phi$ 1pt

$dz=\cos\phi d\rho-\rho\sin\phi d\phi$ 1pt

$$dx^2=\cos^2\theta\sin^2\phi d\rho^2+\rho^2\sin^2\phi\sin^2\theta d\theta^2$$
$$+\rho^2\cos^2\theta\cos^2\phi d\phi^2$$
$$-2\rho os\theta\sin\theta\sin^2\phi d\rho d\theta$$
$$+2\rho\sin\phi\cos\phi\cos^2\theta d\rho d\phi$$
$$-2\rho^2\cos\phi\cos\theta\sin\phi\sin\theta d\theta d\phi$$

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$$dy^2=\sin^2\phi\sin^2\theta d\rho^2+\rho^2\sin^2\phi\cos^2\theta d\theta^2$$
$$+\rho^2\cos^2\phi\sin^2\theta d\phi^2$$
$$+2\rho\sin\theta\cos\theta\sin^2\phi d\rho d\theta$$
$$+2\rho\sin\phi\cos\phi\sin^2\theta d\rho d\phi$$
$$+2\rho^2\cos\phi\sin\theta\sin\phi\cos\theta d\theta d\phi$$

$$dz^2=\cos^2\phi d\rho^2+\rho^2\sin^2\phi d\phi^2-2\rho\cos\phi\sin\phi d\rho d\phi$$

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$$dx^2+dy^2+dz^2=$$
$$\cos^2\theta\sin^2\phi d\rho^2+\sin^2\theta\sin^2\phi d\rho^2+\cos^2\phi d\rho^2$$
$$+\rho^2(\sin^2\phi\sin^2\theta+\sin^2\phi\cos^2\theta)d\theta^2$$
$$+\rho^2(\cos^2\theta\cos^2\phi+\cos^2\phi\sin^2\theta+\sin^2\phi)d\phi^2$$
$$+2\rho(\sin\theta\cos\theta\sin^2\phi-os\theta\sin\theta\sin^2\phi)d\rho d\theta$$
$$+2\rho(\sin\phi\cos\phi\cos^2\theta+\sin\phi\cos\phi\sin^2\theta-\cos\phi\sin\phi)d\rho d\phi$$
$$+2\rho^2(\cos\phi\sin\theta\sin\phi\cos\theta-\cos\phi\cos\theta\sin\phi\sin\theta)d\theta d\phi$$
$$dx^2+dy^2+dz^2=$$
$$((\cos^2\theta+\sin^2\theta)\sin^2\phi+\cos^2\phi)d\rho^2\quad[=d\rho^2]$$
$$+\rho^2\sin^2\phi(\sin^2\theta+\cos^2\theta)d\theta^2\qquad[=\rho^2\sin^2\phi d\theta^2]$$
$$+\rho^2((\cos^2\theta+\sin^2\theta)\cos^2\phi+\sin^2\phi)d\phi^2\quad[=\rho^2 d\phi^2]$$
$$+2\rho(\sin\theta\cos\theta\sin^2\phi-\cos\theta\sin\theta\sin^2\phi)d\rho d\phi\quad[=0]$$
$$+2\rho(\sin\phi\cos\phi(\cos^2\theta+\sin^2\theta)-\cos\phi\sin\phi)d\rho d\phi\quad[=0]$$
$$+2\rho^2(\cos\theta\sin\phi\sin\theta\cos\phi-\cos\theta\cos\phi\sin\theta\sin\phi)d\theta d\phi\quad[=0]$$

$$ds^2=d\rho^2+\rho^2 d\phi^2+\rho^2\sin^2\phi\ d\theta^2$$

15.

A. $x=r\cos\theta,\ y=r\sin\theta,\ z=z$

$dx=\cos\theta dr-r\sin\theta d\theta$ 1pt

$dy=\sin\theta dr+\cos\theta r d\theta$, dz 1pt

$d\vec{s}=idx+jdy+kdz$

$=i(\cos\theta dr-r\sin\theta d\theta)+j(\sin\theta dr+r\cos\theta d\theta)+kdz$ 1pt

$=(\vec{i}\cos\theta+\vec{j}\sin\theta)dr+(-\vec{i}\sin\theta+\vec{j}\cos\theta)rd\theta+\vec{k}dz$ 1pt

$\vec{e}_r=\vec{i}\cos\theta+\vec{j}\sin\theta,\ \vec{e}_\theta=-\vec{i}\sin\theta+\vec{j}\cos\theta,\ \vec{k}$ 1pt

B. $x=\rho\sin\phi\cos\theta,\ y=\rho\sin\phi\sin\theta,\ z=\rho\cos\phi$

$dx=\sin\phi\cos\theta d\rho-\rho\sin\phi\sin\theta d\theta+\rho\cos\phi\cos\theta d\phi$ 1pt

$dy=\sin\phi\sin\theta d\rho+\rho\sin\phi\cos\theta d\theta+\rho\cos\phi\sin\theta d\phi$ 1pt

$dz=\cos\phi d\rho-\rho\sin\phi d\phi$ 1pt

$d\vec{s}=idx+jdy+kdz$ 8

$=\vec{i}(\sin\phi\cos\theta d\rho-\rho\sin\phi\sin\theta d\theta+\rho\cos\phi\cos\theta d\phi)$

$+\vec{j}(\sin\phi\sin\theta d\rho+\rho\sin\phi\cos\theta d\theta+\rho\cos\phi\sin\theta d\phi)$

$+\vec{k}(\cos\phi d\rho-\rho\sin\phi d\phi)$

$=(\vec{i}\sin\phi\cos\theta+\vec{j}\sin\phi\sin\theta+\vec{k}\cos\phi)d\rho$ 1pt

$+(\vec{i}\cos\phi\cos\theta+\vec{j}\cos\phi\sin\theta-\vec{k}\sin\phi)\rho d\phi$ 1pt

$+(-\vec{i}\sin\theta+\vec{j}\cos\theta)\rho\sin\phi d\theta$ 1pt

$\vec{e}_\rho=\vec{i}\sin\phi\cos\theta+\vec{j}\sin\phi\sin\theta+\vec{k}\cos\phi$ 1pt

$\vec{e}_\phi=\vec{i}\cos\phi\cos\theta+\vec{j}\cos\phi\sin\theta-\vec{k}\sin\phi$ 1pt

$\vec{e}_\theta=-\vec{i}\sin\theta+\vec{j}\cos\theta,\ \vec{e}_\phi$ 1pt

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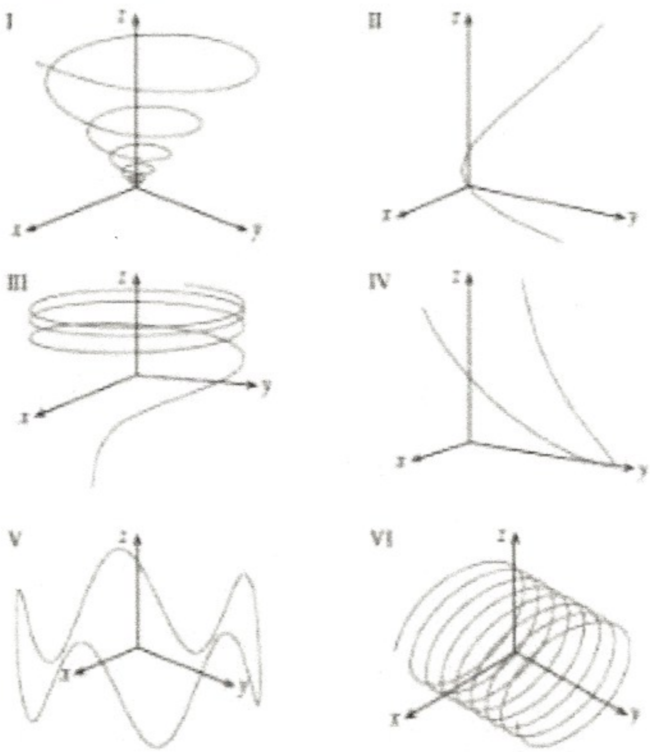
16. Deduce the functional behavior of x, y, and z with respect to a parameter t for the figures shown. Hint: You may want to identify the behavior on a plane first (xy-, yz-, or zx-plane) or along one of the axis (i.e. x, y, or z)



I. Circular motion in the x-y plane
Growing radius as a function of z
z is growing exponentially in t
$x=t^m\cos t,\ y=t^m\sin t,\ z=t^m$ or e^t 1pt+1pt+1pt

II. This one is difficulty visualize in 3D
$x=t,\ y=t^2,\ z=t^{-m}$ or e^{-t} 1pt+1pt+1pt

III. Similar to I but the radius is constant and the growth in z slowing down, proportionally to negtive power of t.
$x=a\cos t,\ y=a\sin t,\ z=t^{-m}$ or $\ln t$ 1pt+1pt+1pt

IV. In the x-z plane, this is a parabola; z is always positive.
$x=t,\ y=\dfrac{1}{1+t^2},\ z=t^2$ 1pt+1pt+1pt

V. Sinusoidal function in all axis.
Circle in the x-y plane
5 crests in z
$x=a\cos t,\ y=a\sin t,\ z=b\sin 5t$ 1pt+1pt+1pt

VI. Elliptical motion in the x-z plane
Constant growth in y
$x=a\cos\beta t,\ y=t,\ z=b\sin\beta t$ 1pt+1pt+1pt

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