

Chapter 6 Notes - LA

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6 Vector Spaces

6.1 Vector Spaces and Subspaces

- In the past, we studied vectors in a concrete situation, \mathbb{R}^n . Now, we generalize “vectors” by abstracting them into a general setting.
- Let V be a set on which two operations, called addition and scalar multiplication, have been defined. If \mathbf{u} and \mathbf{v} are in V , the sum of \mathbf{u} and \mathbf{v} is denoted by $\mathbf{u} + \mathbf{v}$, and if c is a scalar, the scalar multiple of \mathbf{u} by c is denoted by $c\mathbf{u}$. If the following axioms hold for all \mathbf{u} , \mathbf{v} , and \mathbf{w} in V and for all scalars c and d , then V is called a vector space and its elements are vectors.

1. $\mathbf{u} + \mathbf{v}$ is in V .
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There exists an element $\mathbf{0}$ in V , called a zero vector, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$
5. For each \mathbf{u} in V , there is an element $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
6. $c\mathbf{u}$ is in V .
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

- Let V be a vector space \mathbf{u} a vector in V , and c a scalar.
 - $0\mathbf{u} = \mathbf{0}$

- $c\mathbf{0} = \mathbf{0}$
- $(-1)\mathbf{u} = -\mathbf{u}$
- If $c\mathbf{u} = \mathbf{0}$, then $c = 0$ or $\mathbf{u} = \mathbf{0}$
- A subset W of a vector space V is called a subspace of V if W is itself a vector space with the same scalars, addition, and scalar multiplication as V .
- Let V be a vector space and let W be a nonempty subset of V . Then W is a subspace of V IFF the following conditions hold:
 - If \mathbf{u} and \mathbf{v} are in W , then $\mathbf{u} + \mathbf{v}$ is in W
 - If \mathbf{u} is in W and c is a scalar, then $c\mathbf{u}$ is in W .
- If W is a subspace of a vector space V , then W contains the zero vector $\mathbf{0}$ of V .
- If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a set of vectors in a vector space V , then the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ is called the span of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ and is denoted by $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$ or $\text{span}(S)$. If $V = \text{span}(S)$, then S is called a spanning set of V and V is said to be spanned by S .
- Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ be vectors in a vector space V .
 - $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$ is a subspace of V .
 - $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$ is the smallest subspace of V that contains $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$

6.2 Linear Independence, Basis, and Dimension

- A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ in a vector space V is linearly dependent if there are scalars c_1, c_2, \dots, c_k , at least one of which is not zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k = \mathbf{0}$$

A set of vectors that is not linearly dependent is said to be linearly independent.

- A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ in a vector space V is linearly dependent IFF at least one of the vectors can be expressed as a linear combination of the others.
- A set S of vectors in a vector space V is linearly dependent if it contains finitely many linearly dependent vectors. A set of vectors that is not linearly dependent is said to be linearly independent.
- A subset \mathcal{B} of a vector space V is a basis for V if
 - \mathcal{B} spans V and
 - \mathcal{B} is linearly independent.
- Let V be a vector space and let \mathcal{B} be a basis for V . For every vector \mathbf{v} in V , there is exactly one way to write \mathbf{v} as a linear combination of the basis vectors in \mathcal{B}
- Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis for the vector space V . Let \mathbf{v} be a vector in V , and write $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n$. Then c_1, c_2, \dots, c_n are called the coordinates of \mathbf{v} with respect to \mathcal{B} , and the column vector

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

is called the coordinate vector of \mathbf{v} with respect to \mathcal{B} .

- Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a basis for a vector space V . Let \mathbf{u} and \mathbf{v} be vectors in V and let c be a scalar. Then

- $[\mathbf{u} + \mathbf{v}]_{\mathcal{B}} = [\mathbf{u}]_{\mathcal{B}} + [\mathbf{v}]_{\mathcal{B}}$
- $[c\mathbf{u}]_{\mathcal{B}} = c[\mathbf{u}]_{\mathcal{B}}$
- $$[c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k]_{\mathcal{B}} = c_1[\mathbf{u}_1]_{\mathcal{B}} + \cdots + c_k[\mathbf{u}_k]_{\mathcal{B}}$$
- Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a basis for a vector space V and let $\mathbf{u}_1, \dots, \mathbf{u}_k$ be vectors in V . Then $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is linearly independent in V IFF $\{[\mathbf{u}_1]_{\mathcal{B}}, \dots, [\mathbf{u}_k]_{\mathcal{B}}\}$ is linearly independent in \mathbb{R}^n .
- Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a basis for a vector space V .
 - Any set of more than n vectors in V must be linearly dependent.
 - Any set of fewer than n vectors in V cannot span V .
- The Basis Theorem: If a vector space V has a basis with n vectors, then every basis for V has exactly n vectors.
- A vector space V is called finite-dimensional if it has a basis consisting of finitely many vectors. The dimension of V , denoted by $\dim V$, is the number of vectors in a basis for V . The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. A vector space that has no finite basis is called infinite-dimensional.
- Let V be a vector space with $\dim V = n$. Then:
 - Any linearly independent set in V contains at most n vectors.
 - Any spanning set for V contains at least n vectors.
 - Any linearly independent set of exactly n vectors in V is a basis for V .
 - Any spanning set for V consisting of exactly n vectors is a basis for V .
 - Any linearly independent set in V can be extended to a basis for V .
 - Any spanning set for V can be reduced to a basis for V .
- Let W be a subspace of a finite-dimensional vector space V . Then:
 - W is finite-dimensional and $\dim W \leq \dim V$.
 - $\dim W = \dim V$ IFF $W = V$

6.3 Change of Basis

- Let $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be bases for a vector space V . The $n \times n$ matrix whose columns are the coordinate vectors $[\mathbf{u}_1]_{\mathcal{C}}, \dots, [\mathbf{u}_n]_{\mathcal{C}}$ of the vectors in \mathcal{B} with respect to \mathcal{C} is denoted by $P_{\mathcal{C} \leftarrow \mathcal{B}}$ and is called the change-of-basis matrix from \mathcal{B} to \mathcal{C} . That is,

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = [[\mathbf{u}_1]_{\mathcal{C}} [\mathbf{u}_2]_{\mathcal{C}} \cdots [\mathbf{u}_n]_{\mathcal{C}}]$$

- Let $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be bases for a vector space V and let $P_{\mathcal{C} \leftarrow \mathcal{B}}$ be the change-of-basis matrix from \mathcal{B} to \mathcal{C} . Then
 - $P_{\mathcal{C} \leftarrow \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$ for all \mathbf{x} in V .
 - $P_{\mathcal{C} \leftarrow \mathcal{B}}$ is the unique matrix P with the property that $P[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$ for all \mathbf{x} in V .
 - $P_{\mathcal{C} \leftarrow \mathcal{B}}$ is invertible and $(P_{\mathcal{C} \leftarrow \mathcal{B}})^{-1} = P_{\mathcal{B} \leftarrow \mathcal{C}}$.
- Let $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be bases for a vector space V . Let $B = [[\mathbf{u}_1]_{\mathcal{E}} \cdots [\mathbf{u}_n]_{\mathcal{E}}]$ and $C = [[\mathbf{v}_1]_{\mathcal{E}} \cdots [\mathbf{v}_n]_{\mathcal{E}}]$, where \mathcal{E} is any basis for V . Then the row reduction applied to the $n \times 2n$ augmented matrix $[C|B]$ produces

$$[C|B] \rightarrow [I|P_{\mathcal{C} \leftarrow \mathcal{B}}]$$

6.4 Linear Transformations

- A linear transformation from a vector space V to a vector space W is a mapping $T : V \rightarrow W$ such that, for all \mathbf{u} and \mathbf{v} in V and for all scalars c ,

$$- T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

$$- T(c\mathbf{u}) = cT(\mathbf{u})$$

- $T : V \rightarrow W$ is a linear transformation IFF

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_kT(\mathbf{v}_k)$$

for all $\mathbf{v}_1, \dots, \mathbf{v}_k$ in V and scalars c_1, \dots, c_k .

- Let $T : V \rightarrow W$ be a linear transformation. Then:

$$- T(\mathbf{0}) = \mathbf{0}$$

$$- T(-\mathbf{v}) = -T(\mathbf{v}) \text{ for all } \mathbf{v} \text{ in } V.$$

$$- T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}) \text{ for all } \mathbf{u} \text{ and } \mathbf{v} \text{ in } V.$$

- Let $T : V \rightarrow W$ be a linear transformation and let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a spanning set for V . Then $T(\mathcal{B}) = \{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$ spans the range of T .

- If $T : U \rightarrow V$ and $S : V \rightarrow W$ are linear transformations, then the composition of S with T is the mapping $S \circ T$, defined by

$$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$$

where \mathbf{u} is in U .

- If $T : U \rightarrow V$ and $S : V \rightarrow W$ are linear transformations, then $S \circ T : U \rightarrow W$ is a linear transformation.

- $R \circ (S \circ T) = (R \circ S) \circ T$

- A linear transformation $T : V \rightarrow W$ is invertible if there is a linear transformation $T' : W \rightarrow V$ such that

$$T' \circ T = I_V \quad \text{and} \quad T \circ T' = I_W$$

In this case, T' is called an inverse for T .

- If T is an invertible linear transformation, then its inverse is unique.

6.5 The Kernel and Range of a Linear Transformation

- Let $T : V \rightarrow W$ be a linear transformation. The kernel of T , denoted $\ker(T)$, is the set of all vectors in V that are mapped by T to $\mathbf{0}$ in W . That is,

$$\ker(T) = \{\mathbf{v} \text{ in } V : T(\mathbf{v}) = \mathbf{0}\}$$

The range of T , denoted $\text{range}(T)$, is the set of all vectors in W that are images of vectors in V under T . That is,

$$\text{range}(T) = \{T(\mathbf{v}) : \mathbf{v} \text{ in } V\} = \{\mathbf{w} \text{ in } W : \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \text{ in } V\}$$

- Let $T : V \rightarrow W$ be a linear transformation. Then:

$$- \text{The kernel of } T \text{ is a subspace of } V.$$

$$- \text{The range of } T \text{ is a subspace of } W.$$

- Let $T : V \rightarrow W$ be a linear transformation. The rank of T is the dimension of the range of T and is denoted by $\text{rank}(T)$. The nullity of T is the dimension of the kernel of T and is denoted by $\text{nullity}(T)$.

- The rank theorem: Let $T : V \rightarrow W$ be a linear transformation from a finite-dimensional vector space V into a vector space W . Then

$$\text{rank}(T) + \text{nullity}(T) = \dim V$$

- A linear transformation $T : V \rightarrow W$ is called one-to-one if T maps distinct vectors in V to distinct vectors in W . If $\text{range}(T) = W$, then T is called onto.
- $T : V \rightarrow W$ is one-to-one if, for all \mathbf{u} and \mathbf{v} in V ,

$$\mathbf{u} \neq \mathbf{v} \text{ implies that } T(\mathbf{u}) \neq T(\mathbf{v})$$

- Which is to say, if $T : V \rightarrow W$ is one-to-one if, for all \mathbf{u} and \mathbf{v} in V ,

$$T(\mathbf{u}) = T(\mathbf{v}) \text{ implies that } \mathbf{u} = \mathbf{v}$$

- $T : V \rightarrow W$ is onto if, for all \mathbf{w} in W , there is at least one \mathbf{v} in V such that

$$\mathbf{w} = T(\mathbf{v})$$

- A linear transformation $T : V \rightarrow W$ is one-to-one IFF $\ker(T) = \{\mathbf{0}\}$.
- Let $\dim V = \dim W = n$. Then a linear transformation $T : V \rightarrow W$ is one-to-one IFF it is onto
- Let $T : V \rightarrow W$ be a one-to-one linear transformation. If $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a linearly independent set in V , then $T(S) = \{T(\mathbf{v}_1), \dots, T(\mathbf{v}_k)\}$ is a linearly independent set in W .
- Let $\dim V = \dim W = n$. Then a one-to-one linear transformation $T : V \rightarrow W$ maps a basis for V to a basis for W .
- A linear transformation $T : V \rightarrow W$ is invertible IFF it is one-to-one and onto.
- A linear transformation $T : V \rightarrow W$ is called an isomorphism if it is one-to-one and onto. If V and W are two vector spaces such that there is an isomorphism from V to W , then we say that V is isomorphic to W and write $V \cong W$.
- Let V and W be two finite-dimensional vector spaces (over the same field of scalars). Then V is isomorphic to W IFF $\dim V = \dim W$.

6.6 The Matrix of a Linear Transformation

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6.7 Applications

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