B_c 29   R 18
B_r 28   BBR 75
BB 57   T_p 75

13

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MCLA

1. Given $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

A. prove that

1. $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

2. $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

B. Use your answers in part A to prove that
$$\cos\theta_2 \sin\theta_1 = \frac{1}{2}[\sin(\theta_1 + \theta_2) + \sin(\theta_1 - \theta_2)]$$

C. Obtain the series expansions for $\cos\theta$ and $\sin\theta$

2. Consider two vectors **u** and **v**. The angle between them is θ. The difference vector is **w**=**u**-**v**.

A. Draw a figure representing the set up.
In terms of the given information, use vector algebra to prove

B. the law of cosines

C. the law of sines.

3. Given vectors $\vec{a} = [2,1,0]$ and $\vec{b} = [3,-2,1]$,

A. obtain the angle between the two vectors

B. find a **unit** vector $\vec{n}$ perpendicular to both $\vec{a}$ and $\vec{b}$

C. obtain the direction cosines of the unit vector $\vec{n}$

4. What region in $\Re^3$ is represented by
$1 \le x^2 + y^2 + z^2 \le 4,\ z \le 0$? Show the region in a figure.

5. Consider a triangle the corners of which are the three points P(1,4,6), Q(-2,5,-1), and R(1,-1,1). What is the area of the triangle?

6. A parallelepiped is defined by the three vectors $\vec{a} = \langle 1,4,-7 \rangle$, $\vec{b} = \langle 2,-1,4 \rangle$, $\vec{c} = \langle 0,-9,18 \rangle$. Obtain its volume.

7. Find the equation of the plane through the point (2, 4, -1) with normal vector $\vec{n} = \langle 2,\ 3,\ 4 \rangle$.

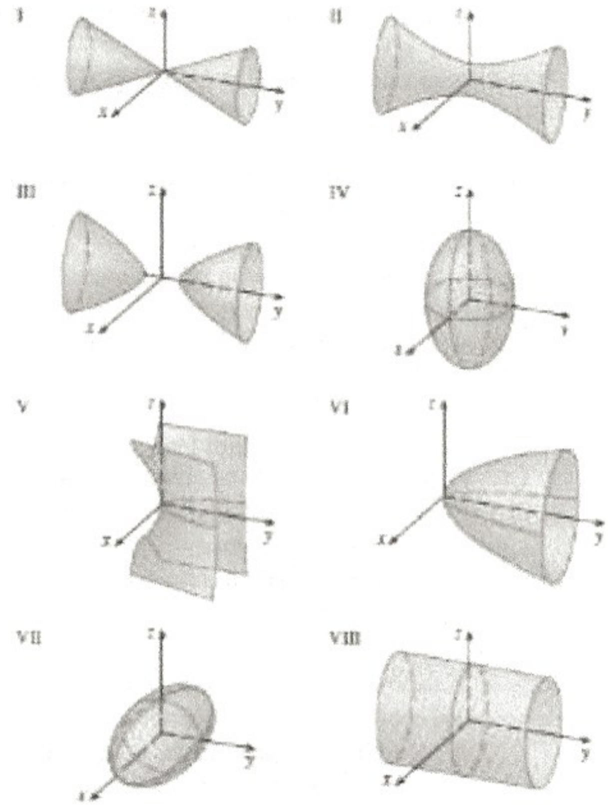
8. Write down the i) normal, ii) vector, and iii) parametric forms of the

A. line $ax + by = c$ and

B. plane $ax + by + cz = d$

9. Given the line $ax + by = c$ and the plane $ax + by + cz = d$, obtain a unit vectors representing each.

10. Match the equation with its graph



A. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1,\ a > b > c$

B. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1,\ a < b < c$

C. $x^2 - y^2 + z^2 = 1$

D. $-x^2 + y^2 - z^2 = 1$

E. $x^2 + cz^2 = 1,\ c > 1$

F. $y^2 = x^2 + cz^2,\ c > 1$

G. $y = ax^2 + z^2,\ a > 1$

H. $y = x^2 - z^2$

11. Given the plane $ax + by + cz = d$, obtain the distance $d(B,\wp)$ from the point $B = (x_o, y_o, z_o)$ to the plane.

A. Write down a vector perpendicular to the plane

B. Obtain the unit vector **n** normal to the plane

C. Use the vector form of the plane to obtain the distance.

Mesut B. Çakır

12. Given plane $\wp$, vector $\mathbf{v}$, the plane unit-normal vector $\mathbf{n}$,
   A. Write down the component of $\mathbf{v}$, parallel to $\mathbf{n}$ and perpendicular to $\wp$ in terms of a constant c and $\mathbf{n}$
   B. Write down the projection $\mathbf{p}$ of $\mathbf{v}$ onto the plane $\wp$ in terms of c, $\mathbf{v}$, and $\mathbf{n}$
   C. Use the fact that $\mathbf{n}$ is orthogonal to $\mathbf{p}$ to solve c in terms of $\mathbf{v}$ and $\mathbf{n}$.
   D. Obtain an expression for the projection $\mathbf{p}$ of $\mathbf{v}$ onto the plane $\wp$

13. Consider two vectors **a** and **b**. The vector **a** is at an angle α with the x-axis and the vector **b** is at an angle β with the x-axis.
   A. Draw a figure representing the set up.
   B. Use the vector operations to prove that the angle $\beta-\alpha$ between the two vectors is given by $\cos(\beta-\alpha)=\sin\alpha\sin\beta+\cos\alpha\cos\beta$ in terms of α and β.

14. A point P is represented by the ordered triple
   A. $(x,\ y,\ z)$ in the cartesian coordinates,
   B. $(r,\ \theta,\ z)$ in the cylindrical coordinates, $(r,\ \theta)$ are the polar coordinates in the xy-plane,
   C. $(\rho,\ \theta,\ \phi)$ in the spherical coordinate system.

   Given the information above, obtain expressions for $x,\ y,\ z$
   1. in the cylindrical coordinates, in terms of $(r,\ \theta,\ z)$ and obtain $r,\ \theta,\ z$ in terms of $x,\ y,\ z$.
   2. in the spherical coordinate system, in terms of $(\rho,\ \theta,\ \phi)$. and obtain $\rho,\ \theta,\ \phi$ in terms of $x,\ y,\ z$.

   Hint: Ask me to draw the figures showing the coordinates of the point in each system.

15. The generic equation for the ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Notice that, if you substitute $z=0$, you obtain the generic equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for the ellipse. The area of the ellipse is $A=\pi ab$. Notice that this area expression gives the area of a circle of radius r, $A=\pi r^2$, when $a=b=r$. Now, imagine you cut through the ellipsoid using the $z=h$ plane. This plane is perpendicular to the z-axis and parallel to the xy-plane.
   A. Rewrite the ellipsoid equation to obtain an ellipse equation in terms of z and c.
   B. Reduce the equation in part A in a way that it resembles the generic ellipse equation.
   C. Identify your new a and b in terms of a, b, c, z.
   D. Write the area for this ellipse as a function of a, b, c, z.
   E. The infinitesimal volume element for the ellipsoid is $dV=Adz$ where A is the area of the ellipse. Integrate the infinitesimal volume element from $-c$ to c to obtain the volume of the ellipsoid. You will know that your answer may be correct if you get the volume of a sphere of radius r, $V=\frac{4}{3}\pi r^3$, when you substitute $a=b=c=r$ in your answer.

Extra Curricular Information about proving **a.b**, **axb**, etc. You have to start with a set of axioms. For example, in Group Theory, a collection of elements G together with a binary operation is a group if it satisfies the axioms:

I. Associativity: If x, y, and z are in G, then $x\circ(y\circ z)=(x\circ y)\circ z$

II. Right identity : G contains an element e such that $x\circ e=x$
[There can be a left identity as well $e\circ x=x$]

III. Right Inverse: For every x in G, there is an element called , also in G, for which $x\circ x^{-1}=e$
[There can be a left identity and inverse as well]
A group is Abelian (commutative) if in addition

IV. $x\circ y=y\circ x$ *for all* $x,\ y$ *in G*

The ones you are used to are where $\circ$ is $+,\ -,\ \times,\ \bullet$. Unusual one for you is Grassman Algebra $x.y=-y.x$ which leads to $x.x=x^2=0$

I affirm that I did not compromize my integrity and that I did not violate the integrity of the course by using any questionable means to get a higher grade than *my studies and my knowledge of the material would justify* at the time of this test.

Name: Signature:

Ch 13 test

1) a) $e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \cos\theta + i\sin\theta$

$\cos\theta = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} - i\sin\theta$

$= \sum_{n=1}^{\infty} \frac{i^n \cdot \theta^n}{n!} - i\sin\theta$ ~ ~

taylor series for $f(\theta) = \sin\theta$?

b) $\cos\theta_2 \sin\theta_1 = \frac{1}{2}[\sin(\theta_1+\theta_2) + \sin(\theta_1-\theta_2)]$

$= \frac{e^{i\theta_2} + e^{-i\theta_2}}{2} \cdot \frac{e^{i\theta_1} - e^{-i\theta_1}}{2i}$ +1

$= \frac{1}{4i}\left(e^{(i\theta_2+i\theta_1)} + e^{(i\theta_2-i\theta_1)} + e^{(i\theta_1-i\theta_2)} - e^{(-i\theta_1-i\theta_2)}\right)$ +1

$= \frac{1}{4i}\left(e^{i(\theta_1+\theta_2)} - e^{-i(\theta_1+\theta_2)}\right) + \frac{1}{4i}\left(e^{i(\theta_1-\theta_2)} - e^{-i(\theta_1-\theta_2)}\right)$

$= \frac{1}{2}\sin(\theta_1+\theta_2) + \frac{1}{2}\sin(\theta_1-\theta_2)$

$= \frac{1}{2}[\sin(\theta_1+\theta_2) + \sin(\theta_1-\theta_2)]$ +1

c) ~~maclaurin~~ $\sin\theta$

$f^n(x) = \sin x, \cos x, -\sin x, -\cos x, \sin x \ldots$

$f^n(0) = 0, 1, 0, -1, 0, \ldots$

$\sin x = \sum_{n=0}^{\infty} \frac{x^{2n}(-1)^{n+1}}{n!}$ ~ ~ ~ ~

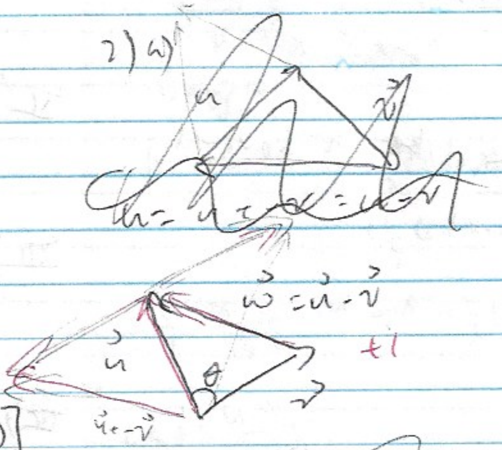
$\cos\theta$ $f^n(x) = \cos x, -\sin x, -\cos x, \sin x, \cos x, \ldots$

$f^n(0) = 1, 0, -1, 0, 1, \ldots$

3

$\cos x = \sum_{n=0}^{\infty} \frac{x^{2n}(-1)^{n+2}}{n!}$

2) a)



$\vec{w} = \vec{u} - \vec{v}$ +1

b) $(u+v)^2 = u \cdot u + v \cdot v + 2u \cdot v$
$= u^2 + v^2 + 2uv\cos\theta$

$w^2 = (u-v)^2 = u \cdot u + v \cdot v - 2u \cdot v$ +1
$= u^2 + v^2 - 2|u||v|\cos\theta$ +1

c) $\frac{\sin a}{a} = \frac{\sin b}{b} = \frac{\sin c}{c}$ ~ ~ ~ ~

5

3) $a = [2, 1, 0]$ $b = [3, -2, 1]$

a) $a \cdot b = |a||b|\cos\theta$ +1

$\theta = \cos^{-1}\left(\frac{a \cdot b}{|a||b|}\right)$ +1

$= \cos^{-1}\left(\frac{(6-2)}{\sqrt{4+1}\sqrt{9+4+1}}\right) = 1.072$ rad +1
$= 61.4$ deg. +1

b) $\hat{n} = \frac{a \times b}{|a \times b|}$ $|n| = 1$

$a \times b = \begin{vmatrix} i & j & k \\ 2 & 1 & 0 \\ 3 & -2 & 1 \end{vmatrix} = \langle 1, -2, -7 \rangle$ +1

0

$\hat{n} = \frac{1}{\sqrt{1+4+49}} \langle 1, -2, -7 \rangle = \langle 0.136, -0.272, -0.952 \rangle$ +1
$\sqrt{54}$

$= \left\langle \frac{1}{3\sqrt{6}}, \frac{-2}{3\sqrt{6}}, \frac{-7}{3\sqrt{6}} \right\rangle$ +1

$|n| = \sqrt{x^2+y^2+z^2} = 1$

c) ? ~

10

4) $x^2+y^2+z^2=1$ → sphere

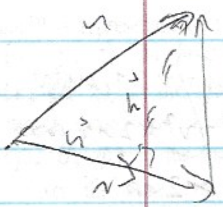
$x^2+y^2+z^2=4$ → sphere

$z \le 0$

like a bowl +1

+1

5)



$v = u - w$

~ ~ ~ ~ ~

$A = \frac{1}{2} bh$

$b = \text{proj}_v(u)$

6) $V = (a \times b) \cdot c = \begin{vmatrix} 0 & -9 & 18 \\ 1 & 4 & -7 \\ 2 & -1 & 4 \end{vmatrix}$ +1

$= 0$

$a \times b = \langle 9, -18, -9 \rangle \quad c = \langle 0, -9, 18 \rangle$

$a \times b \cdot c = 0 + 9 \cdot 18 - 9 \cdot 18 = 0$ +1

all in the same plane.

7) $P = P_0 + nt \cdot = 0$

$\langle 2, 4, -1 \rangle + \langle 2, 3, 4 \rangle t = 0$ ~ ~

8) eqn 9) eqn

~ ~ ~ ~ ~

4

10) I) $y=0$ → single point (F)

$x=0$ → two lines

+1

$z=0$ → two lines

II) $y=0$ → circle (C) +1

$x=0$ → hyp.

$z=0$ → hyp.

III) $y=0$ → n/a (D) +1

$z=0$ → hyp.

$x=0$ → hyp.

IV) $z=0$ - circle A or (B) +1

~~only $z \ge 0$~~

$x, y=0$ → ellipse

V) $x=0$ → parabola parabolic hyperboloid

$y=0$ → hyperbola (H) +1

$z=0$ → parabola

VI) $y=0$ → single point, traces in y are circles

$x, z=0$ → parabola (G) +1

VII $x=0$ → circle (A) or B

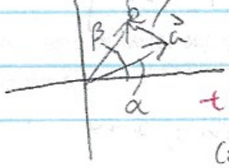
$y, z=0$ → ellipses +1

VIII $y \ne 0$ - circle - all y, same size.

$x, z=0$ → two lines → (E) $(x^2+cz^2=1)$ $c>1$ +1

11) eqn 12) eqn ~ ~ ~ ~ ~ ~

13)



$\vec{c} = \vec{b} + \vec{a}$

+1

$\vec{a} \cdot \vec{b} = ab \cos(\beta - \alpha)$

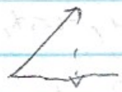
$\cos(\beta - \alpha) = \frac{\vec{a} \cdot \vec{b}}{|a||b|}$

$a = [x_1, y_1]$

$b = [x_2, y_2]$

$= \frac{x_1 x_2 + y_1 y_2}{\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}}$

$\sin\theta = \frac{\text{opp.}}{\text{hyp}}$



$\cos\theta = \frac{\text{adj.}}{\text{hyp.}}$

9

$$= \frac{x_1}{\sqrt{x_1^2+y_1^2}} \cdot \frac{x_2}{\sqrt{x_2^2+y_2^2}} + \frac{y_1}{\sqrt{x_1^2+y_1^2}} \cdot \frac{y_2}{\sqrt{x_2^2+y_2^2}}$$

+1 +1

$$\sin\alpha = \frac{y_1}{\sqrt{x_1^2+y_1^2}}$$

$$= \cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\beta-\alpha)$$ +1

(4) (5)

3

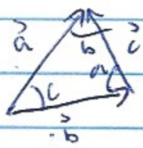
1) a) $e^{i\theta} = \cos\theta + i\sin\theta$

$$\cos\theta = e^{i\theta} - i\sin\theta$$

$$= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} - i\sum_{n=0}^{\infty} (-1)^n \cdot \frac{\theta^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{i^n\theta^n}{n!} - \frac{i(-1)^n \cdot \theta^{2n+1}}{(2n+1)!}$$

2) c) $\sin\theta$ arises from cross product: $|u \times v| = uv\sin\theta$



$\vec{c} = \vec{a} - \vec{b}$ +1

$\vec{c} \times \vec{a} = (\vec{a}-\vec{b}) \times \vec{a} = a \times a - b \times a$

$= 0 + a \times b$

+1 Since $a \times b = -b \times a$

$\vec{c} \times \vec{b} = (a-b) \times b = a \times b - b \times b$

$= a \times b$

$a \times b = c \times a = c \times b$ +1

$ab\sin C = ac\sin B = bc\sin A$

divide by abc $\frac{\sin C}{c} = \frac{\sin B}{b} = \frac{\sin A}{a}$ +1

3) c) $\vec{n} = \langle 0.136, -0.272, -0.952 \rangle$ $|\hat{n}| = 1$

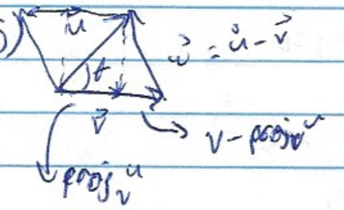
$\frac{\vec{n}}{|n|} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$

$\cos\alpha = 0.136$	$\frac{1}{3\sqrt{6}}$
$\cos\beta = -0.272$	$\frac{-2}{3\sqrt{6}}$
$\cos\gamma = -0.952$	$\frac{-7}{3\sqrt{6}}$

+1

5

5)



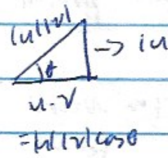
$h = u\sin\theta$ $A = \frac{1}{2}(\text{comp}_v u)|u|\sin\theta$

$+\frac{1}{2}(|v| - \text{comp}_v u)|u|\sin\theta$

5

$$= \frac{1}{2} \cdot \frac{u \cdot v}{|v|} \cdot |u|\sin\theta + \frac{1}{2}\left(|v| - \frac{u \cdot v}{|v|}\right)|u|\sin\theta$$

$$\cos\theta = \frac{u \cdot v}{|u||v|}$$



$\rightarrow |u|^2|v|^2 - (u \cdot v)^2$

$|u|^2|v|^2 - (|u||v|\cos\theta)^2$

$= |u|^2|v|^2(1-\cos^2\theta)$ $\sin^2\theta + \cos^2\theta = 1$

$= |u|^2|v|^2 \cdot \sin^2\theta$

$$\sin\theta = \frac{|u|^2|v|^2\sin^2\theta}{|u||v|}$$

$\frac{1}{\sin\theta} = |u||v|$

$\sin\theta = \frac{1}{|u||v|}$

$$= \frac{1}{2}\frac{u \cdot v}{|v|}\frac{|u|}{|u||v|} + \frac{1}{2}\frac{|v| \cdot |u|}{|u||v|} - \frac{1}{2}\frac{u \cdot v}{|v|}\frac{|u|}{|u||v|}$$

$$A = \frac{1}{2}\frac{|u||v|}{|u||v|} = \frac{1}{2} ??$$

0

7) $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

$2(x-2) + 3(y-4) - (z-4) = 0$

$2x - 4 + 3y - 12 - z + 4 = 0$

$2x + 3y - z - 12 = 0$

10

8) a) $r = r_0 + tv$ $ax + by = c$

$\frac{a}{c}x + \frac{b}{c}y = 1$

$\frac{ax}{c} + \frac{by}{c} - \frac{c}{c} = 0$

1

0

8)(a) general form: $ax + by = c$

normal: $n \cdot x = n \cdot p$

vector: $\vec{x} = \vec{p} + t\vec{d}$ +3

parametric: $\begin{cases} x = p_1 + td_1 \\ y = p_2 + td_2 \end{cases}$

normal: $n = \langle a, b \rangle$ $x = \langle x, y \rangle$

$\langle a, b \rangle \cdot \langle x_0, y_0 \rangle = c = n \cdot p$

vector: $\vec{x} = \vec{p} + t\vec{d}$

$\langle x, y \rangle = \langle x_0, y_0 \rangle + t \langle d_0, d_1 \rangle$

where $\vec{p}$ is a point on the line

and $\vec{d}$ is a direction vector parallel to the line.

parametric: $\begin{cases} x = p_1 + td_1 \\ y = p_2 + td_2 \end{cases}$ split up the vector form

b) general form: $ax + by + cz = d$

normal: $n \cdot x = n \cdot p$

vector: $\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$ +3

param: $\begin{cases} x = p_1 + su_1 + tv_1 \\ y = p_2 + su_2 + tv_2 \\ z = p_3 + su_3 + tv_3 \end{cases}$

9) a) $ax + by = c$

$\vec{x} = \vec{p} + t\vec{d}$

$\frac{\vec{d}}{|d|} = \vec{n}_e$

$a(x - x_0) + b(y - y_0) = 0$

$ax - ax_0 + by - by_0 = 0$

$c = ax_0 + by_0 = ax + by$

$n \cdot x = n \cdot p$

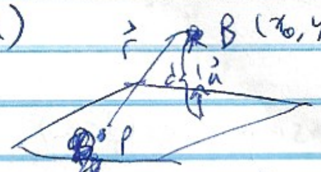
$\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ ✓

9(b) $ax + by + cz = d$

$\vec{n} = \langle a, b, c \rangle$

$\vec{u}_n = \frac{\vec{n}}{|n|} = \left\langle \frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}} \right\rangle$ +1

11)



$= P(x_1, y_1, z_1)$

$\vec{r} = \vec{PB} = \langle x_0 - x_1, y_0 - y_1, z_0 - z_1 \rangle$

$d = |\text{comp}_n \vec{r}| = \frac{|\vec{n} \cdot \vec{r}|}{|\vec{n}|}$ +1

where $\vec{n} = \langle a, b, c \rangle$ +1

$d = \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2+b^2+c^2}}$ +1

$= \frac{|ax_0 + by_0 + cz_0 - (ax_1 + by_1 + cz_1)|}{\sqrt{a^2+b^2+c^2}}$

but we know that P is in the plane,

so $ax_1 + by_1 + cz_1 = d$

$\therefore$ $d(B, P) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2+b^2+c^2}}$ +1

12, 14, 15 5

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1 a c 5 7 9a 12 14 15

1) a) $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \frac{1}{\cos\theta + i\sin\theta} = \frac{\cos\theta - i\sin\theta}{(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$

$= \frac{\cos\theta - i\sin\theta}{\cos^2\theta - (i^2)\sin^2\theta} = \cos\theta - i\sin\theta$

$\cos\theta = e^{i\theta} - i\sin\theta$

$= e^{i\theta} + e^{-i\theta} - \cos\theta$

$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ +1

1

$i\sin\theta = e^{i\theta} - \cos\theta$ $\quad$ $\cos\theta = i\sin\theta + e^{-i\theta}$

$= e^{i\theta} - e^{-i\theta} - i\sin\theta$

$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ +1

c) $\cos\theta = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} + \frac{(-i\theta)^n}{n!}\right)$

$= \sum_{n=0}^{\infty} \frac{1}{2n!} \cdot (i)^n (-i)^n (\theta)^{2n}$

$= \sum_{n=0}^{\infty} \frac{1}{2n!} (-1)^n \underbrace{(i)^{2n}}_{=(-1)^n} \cdot (\theta)^{2n}$

$= \sum_{n=0}^{\infty} \frac{\theta^{2n}}{2n!}$

$\sin\theta =$ ___

5) $A = \frac{1}{2}|a \times b|$ +1 PQ, PR

$= \frac{1}{2}\begin{vmatrix} i & j & k \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$ +1 $= \frac{1}{2}|\langle -40, -15, 15\rangle|$

$= \frac{1}{2}\sqrt{40^2 + 15^2 + 15^2}$ +1

$= 22.6$ +1

7) $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ +1

$2(x-2) + 3(y-4) + 4(z+1) = 0$

$2x + 3y + 4z - 4 - 12 + 4$

$2x + 3y + 4z - 12 = 0$ +1

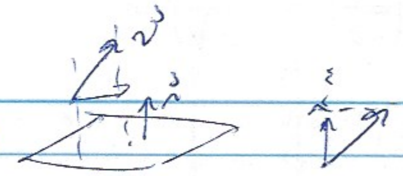
9) a) $ax + by = c$

direction vector $\langle a, b\rangle$

$\vec{u} = \frac{\vec{n}}{|n|} = \frac{\langle a, b\rangle}{\sqrt{a^2+b^2}}$ +1

9

12)

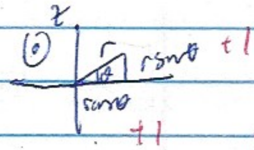


a)

$\text{comp}_n v = \frac{(n \cdot v)}{|n|} = \frac{q|n||v|}{|n|} = c|v|$

b) ?

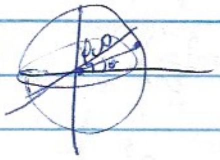
(4) b) cylindrical →



+1 $\quad$ 5

+1 $\quad$ 2

c)



1) c) $e^x = \sum_{n=0} \frac{x^n}{n!}$; $x = i\theta \rightarrow e^{i\theta} = \sum_{n=0} \frac{(i\theta)^n}{n!}$ +1

$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\sum_{n=0} \frac{(i\theta)^n + (-i\theta)^n}{n!}$ +1

odds cancel, evens add

$\cos\theta = \frac{1}{2}\sum_{n=0} \frac{(2i^{2n}\theta^{2n})}{(2n)!} = \sum_{n=0} \frac{(-1)^n \theta^{2n}}{(2n)!}$ +1

$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\sum_{n=0}^{\infty} \frac{((i\theta)^n - (-i\theta)^n)}{n!}$ +1

evens cancel, odds add

$\sin\theta = \frac{1}{2i}\sum_{n=0} \frac{(2i^{2n+1}\theta^{2n+1})}{(2n+1)!} = \sum_{n=0} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}$ +1

12) $\vec{v}_\perp = c\vec{n}$ +1

a)

b) $\vec{v} = v_\parallel + v_\perp = p + cn$, $p = v - cn$ +1

c) $n \cdot v = n \cdot p = 0 = n \cdot v - c(n \cdot n) = n \cdot v - c$ $\quad$ 10

$c = n \cdot v$ +1

d) $p = v - cn = v - (n \cdot v)n$, $p = v - (n \cdot v)n$ +1

$v = v_\parallel + v_\perp = v_p\hat{p} + v_n\hat{n}$

(4) c) $x = \rho\cos\theta\sin\phi$ +1

$y = \rho\sin\theta\sin\phi$ +1 $\quad$ 7

$z = \rho\cos\phi$ +1

12

15) a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{z^2}{c^2}$ +1

b) $\frac{x^2}{\left(a\sqrt{1-\frac{z^2}{c^2}}\right)^2}+\frac{y^2}{\left(b\sqrt{1-\frac{z^2}{c^2}}\right)^2}=1$ +1

c) $a_{new}=a\sqrt{1-\frac{z^2}{c^2}}$, $b_{new}=b\sqrt{1-\frac{z^2}{c^2}}$ +1

d) $A=\pi a_{new}b_{new}=\pi ab\left(1-\frac{z^2}{c^2}\right)$ +1

e) $V=\int_{-c}^{c}\pi ab\left(1-\frac{z^2}{c^2}\right)dz=\pi ab\left[z-\frac{z^3}{3c^2}\right]_{-c}^{c}$ +1

$=\frac{4}{3}\pi abc$ +1

$a=b=c=r \rightarrow V=\frac{4}{3}\pi r^3 \rightarrow$ sphere

6

1.

A. $e^{i\theta} = \cos\theta + i\sin\theta$, $e^{-i\theta} = \cos\theta - i\sin\theta$ 1 pt

$e^{i\theta} + e^{-i\theta} = 2\cos\theta$, $e^{i\theta} - e^{-i\theta} = -2i\sin\theta$ 1 pt

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2},\quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

B. $\cos\theta_2 \sin\theta_1 = \left(\dfrac{e^{i\theta_2}+e^{-i\theta_2}}{2}\right)\left(\dfrac{e^{i\theta_1}-e^{-i\theta_1}}{2i}\right)$ 1 pt

$$= \frac{1}{4i}\left[e^{i(\theta_1+\theta_2)} - e^{-i(\theta_1+\theta_2)} + e^{i(\theta_1-\theta_2)} - e^{-i(\theta_1-\theta_2)}\right]$$ 1 pt

$$= \frac{1}{2}\left[\sin(\theta_1+\theta_2) + \sin(\theta_1-\theta_2)\right]$$ 1 pt

C. $e^x = \sum_{n=0}^{\infty}\dfrac{x^n}{n!},\ x = i\theta,\ e^{i\theta} = \sum_{n=0}^{\infty}\dfrac{(i\theta)^n}{n!}$ 1 pt

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{[(i\theta)^n + (-i\theta)^n]}{n!}$$ 1 pt

All odd n's cancel and evens add.

$$\cos\theta = \frac{1}{2}\sum_{n=0}^{\infty}\frac{[2i^{2n}\theta^{2n}]}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n\theta^{2n}}{(2n)!}$$ 1 pt

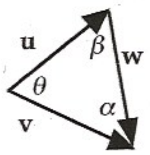
$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{2i}\sum_{n=0}^{\infty}\frac{[(i\theta)^n - (-i\theta)^n]}{n!}$$ 1 pt

All even n's cancel and odds add.

$$\sin\theta = \frac{1}{2i}\sum_{n=0}^{\infty}\frac{[2i^{2n+1}\theta^{2n+1}]}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{[(-1)^n\theta^{2n+1}]}{(2n+1)!}$$ 1 pt

10

2. Consider two vectors **u** and **v**. The angle between them is θ. The difference vector is **w**=**u**-**v**.



A. 1 pt

B. $w^2 = \vec{w}\cdot\vec{w} = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$ $u^2+v^2-2uv\cos\theta$ 1 pt

$= u^2 + w^2 - 2\vec{u}\cdot\vec{v} = \cancel{u^2+w^2} - 2uv\cos\theta$ 1 pt

C. $\vec{w}\times\vec{v} = (\vec{u}-\vec{v})\times\vec{v} = \vec{u}\times\vec{v}$, $\vec{w}\times\vec{u} = (\vec{u}-\vec{v})\times\vec{u} = \vec{v}\times\vec{u}$ 1 pt

Therefore, $\vec{u}\times\vec{v} = \vec{w}\times\vec{v} = \vec{w}\times\vec{u}$ 1 pt

and $uv\sin\theta = wv\sin\alpha = wu\sin\beta$ 1 pt

Divide by uvw to get the familiar form

$$\frac{\sin\theta}{w} = \frac{\sin\alpha}{u} = \frac{\sin\beta}{v}$$ 1 pt

3.

A. Use either $\vec{a}\cdot\vec{b} = ab\cos\theta$ or $|\vec{a}\times\vec{b}| = ab\sin\theta$ 1 pt

$ab = \sqrt{2^2+1+0}\sqrt{3^2+(-2)^2+1} = \sqrt{70}$ 1 pt

$\vec{a}\cdot\vec{b} = 6-2 = 4$ or $|\vec{a}\times\vec{b}| = \sqrt{1+4+49} = \sqrt{54}$ 1 pt

10

$\cos^{-1}\left(\dfrac{4}{\sqrt{70}}\right)$ or $\sin^{-1}\left(\dfrac{\sqrt{54}}{\sqrt{70}}\right)$, $\theta = 61°$ 1 pt

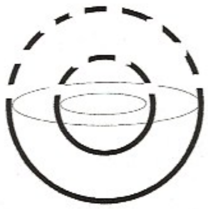
B. $\vec{a}\times\vec{b} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 1 & 0\\ 3 & -2 & 1\end{vmatrix} = \vec{i} - 2\vec{j} - 7\vec{k}$ 1 pt

$|\vec{a}\times\vec{b}| = \sqrt{1+4+49} = \sqrt{54}$ 1 pt

$\vec{n} = \dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \dfrac{1}{\sqrt{54}}(\vec{i} - 2\vec{j} - 7\vec{k})$ 1 pt

C. $\cos\alpha = n_x = \dfrac{1}{\sqrt{54}}$, $\cos\beta = n_x = \dfrac{-2}{\sqrt{54}}$, $\cos\gamma = n_x = \dfrac{-7}{\sqrt{54}}$ 1 pt 5

4. Since $\rho = \sqrt{x^2+y^2+z^2}$, it represents the lower hemispere between the radii 1 and 2. 1 pt



1 pt

5. $\vec{PQ} = \langle 1-(-2), 4-5, 6-(-1)\rangle = \langle 3,-1,7\rangle$ 1 pt

$\vec{PR} = \langle 0, 4-(-1), 6-1\rangle = \langle 0,5,5\rangle$ 1 pt

$\vec{A} = \dfrac{1}{2}\vec{PQ}\times\vec{PR} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & -1 & 7\\ 0 & 5 & 5\end{vmatrix} = \dfrac{1}{2}[(-5-35)\vec{i} - 15\vec{j} - 15\vec{k}]$ 1 pt

$\vec{A} = -20\vec{i} - \dfrac{15}{2}\vec{j} - \dfrac{15}{2}\vec{k}$, 1 pt

$Area = \dfrac{1}{2}\left|\vec{PQ}\times\vec{PR}\right| = \dfrac{1}{2}\sqrt{40^2+15^2+15^2} = 22.6$ *units* 1 pt

6.

$\vec{a}\times\vec{b} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 4 & -7\\ 2 & -1 & 4\end{vmatrix}$ 1 pt

$= (16-7)\vec{i} - (4+14)\vec{j} + (-1-8)\vec{k} = 9\vec{i} - 18\vec{j} - 9\vec{k}$

$V = (\vec{a}\times\vec{b})\cdot\vec{c} = (9\vec{i} - 18\vec{j} - 9\vec{k})\cdot(-9\vec{j} + 18\vec{k}) = 0$ 1 pt

7. $\vec{n}\cdot(\vec{r} - \vec{r}_o) = 0$ 1 pt 10

$2(x-2) + 3(y-4) + 4(z+1) = 2x + 3y + 4z - 12 = 0$ 1 pt

8. Table 1.3 in LA text

A. $ax + by = c$ *1 pt for each form [3 pts]*

B. $ax + by + cz = d$ *1 pt for each form [3 pts]*

9. The line $\dfrac{\langle a,b\rangle}{\sqrt{a^2+b^2}}$ and the plane $\dfrac{\langle a,b,c\rangle}{\sqrt{a^2+b^2+c^2}}$ 1+1 pts 9

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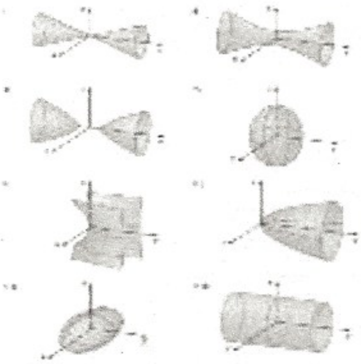
10.



A. VII $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ a>b>c$ This is an elipsoid with a larger radius in the x direction. *1 pt*

B. IV $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ a<b<c$ This is an elipsoid with a larger radius in the z direction. *1 pt*

C. II $x^2-y^2+z^2=1$ Circle in xz but in xy and yz *1 pt*

D. III $-x^2+y^2-z^2=1$ Parabola in xy and yz, circle in xz with $|y|>0$. *1 pt*

E. VIII $x^2+cz^2=1,\ c>1$ Ellipse in xz independent of y *1 pt*

F. I $y^2=x^2+cz^2,\ c>1$ Linear in xy, zy, ellipse in xz *1 pt*

G. VI $y=ax^2+z^2,\ a>1$ Parabola in y>0, ellipse in xz. *1 pt*

H. ~~VI~~ V $y=x^2-z^2$ Parabola in xy, hyporbola in yz *1 pt*

8

11.

A. $\vec{n}=\langle a,b,c\rangle$ is perpendicular to the plane $ax+by+cz=d$ and represents the plane's area vector. *1 pt*

B. $\hat{n}=\frac{\vec{n}}{|\vec{n}|}=\frac{\langle a,b,c\rangle}{\sqrt{a^2+b^2+c^2}}$ *1 pt*

C. $d(B,\wp)=\frac{|\vec{n}\cdot\vec{v}|}{|\vec{n}|}=\frac{|a(x-x_o)+b(y-y_o)+c(z-z_o)|}{\sqrt{a^2+b^2+c^2}}$ *1 pt*

since $ax+by+cz=d$,

$d(B,\wp)=\frac{|d-ax_o-by_o-cz_o|}{\sqrt{a^2+b^2+c^2}}=\frac{|ax_o+by_o+cz_o-d|}{\sqrt{a^2+b^2+c^2}}$ *1 pt*

12.

A. $\mathbf{v}_\perp=c\mathbf{n}$ *1 pt*

B. $\mathbf{v}=\mathbf{v}_=+\mathbf{v}_\perp=\mathbf{p}+c\mathbf{n},\ \mathbf{p}=\mathbf{v}-c\mathbf{n}$ *1 pt*

C. $\mathbf{n}\cdot\mathbf{v}_==\mathbf{n}\cdot\mathbf{p}=0=\mathbf{n}\cdot\mathbf{v}-c(\mathbf{n}\cdot\mathbf{n})=\mathbf{n}\cdot\mathbf{v}-c$

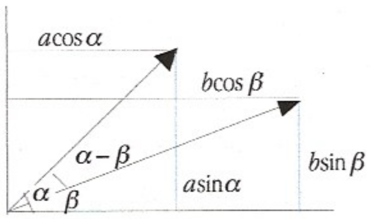
$c=\mathbf{n}\cdot\mathbf{v}$ *1 pt*

D. $\mathbf{p}=\mathbf{v}-c\mathbf{n}=\mathbf{v}-(\mathbf{n}\cdot\mathbf{v})\mathbf{n},\ \mathbf{p}=\mathbf{v}-(\mathbf{n}\cdot\mathbf{v})\mathbf{n}$ *1 pt*

8

This is obvious since $\mathbf{v}=\mathbf{v}_=+\mathbf{v}_\perp=v_p\hat{\mathbf{p}}+v_n\hat{\mathbf{n}}$

13.



A. *1 pt*

B. $\vec{a}\cdot\vec{b}=a_xb_x+a_yb_y=ab\cos\alpha\cos\beta+ab\sin\alpha\sin\beta$ *1 pt*

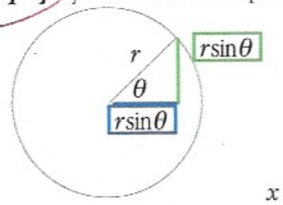
also $\vec{a}\cdot\vec{b}=ab\cos(\alpha-\beta)$ *1 pt*

Since both express the same scalar product, they ar equal.
Thus $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$ *1 pt*

4

14.

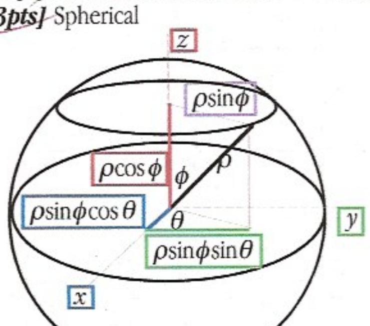
A. ***[2pts]*** Cylindrical: Same as polar with the z-axis.



$x=r\cos\theta,\ y=r\sin\theta,\ z=z$

1 pt for each correct expression in the figure

B. ***[3pts]*** Spherical



1 pt for each $x=\rho\cos\theta\sin\phi,\ y=\rho\sin\theta\sin\phi,\ z=\rho\cos\phi$ 5

15.

A. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{z^2}{c^2}$ *1 pt*

B. $\frac{x^2}{\left[a\sqrt{1-\frac{z^2}{c^2}}\right]^2}+\frac{y^2}{\left[b\sqrt{1-\frac{z^2}{c^2}}\right]^2}=1$ *1 pt*

C. $a_{New}=a\sqrt{1-\frac{z^2}{c^2}},\ b_{New}=b\sqrt{1-\frac{z^2}{c^2}}$ *1 pt*

D. $A=\pi a_{New}b_{New}=\pi ab\left(1-\frac{z^2}{c^2}\right)$ *1 pt*

E. $V=\int_{-c}^{c}\pi ab\left(1-\frac{z^2}{c^2}\right)dz=\pi ab\left[z-\frac{z^3}{3c^2}\right]_{-c}^{c}$ [1pt] $=\frac{4}{3}\pi abc$ [1pt]

$a=b=c=r$ gives $V=\frac{4}{3}\pi r^3$

6

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T, 75