Chapter 1 Notes - LA

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Contents

1 Vectors

1.1 The Geometry and Algebra of Vectors

- A vector is a directed line segment that corresponds to a displacement from one point A to another point B.
- Column vectors and row vectors are different ways to express the same thing:

$$[3, 2] = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

- The point is that components of vectors are ordered.
- Two vectors are equal if they have the same magnitude and direction. Two vectors can still be equal if they have different initial and terminal points.
- Standard position of a vector - when the initial point is at the origin.
- Sum $\mathbf{u} + \mathbf{v} = [u_1 + v_1, u_2 + v_2]$
- Place vectors from head to tail.
- Scalar multiples: $c\mathbf{v} = [cv_1, cv_2]$ aka scaling a vector
- Subtraction is just adding the negative.
- Properties of vectors in $\mathbb{R}^n$: let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and let c and d be scalars. Then:
  - $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  - $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
  - $\mathbf{u} + \mathbf{0} = \mathbf{u}$
  - $\mathbf{u} + (-\mathbf{u}) = 0$
  - $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
  - $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

- $c(d\mathbf{u}) = (cd)\mathbf{u}$
- $1\mathbf{u} = \mathbf{u}$

- A vector $\mathbf{v}$ is a linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ if there are scalars $c_1, c_2, \cdots, c_k$ such that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$. Those scalars are called the coefficients of the linear combination.
- Binary vectors - the components are either 0 or 1.
- Modulus function - divide by a given number and you're left with the remainder.

1.2 Length and Angle: the Dot Product

- dot product: If

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

then the dot product of $\mathbf{u} \cdot \mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

- properties of dot product: let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and let c be a scalar. Then:
  - $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
  - $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
  - $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$
  - $\mathbf{u} \cdot \mathbf{u} \geq \mathbf{0}$ and $\mathbf{u} \cdot \mathbf{u} = 0$ IFF $\mathbf{u} = \mathbf{0}$
  - Length or norm of a vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$ is the nonnegative scalar $||\mathbf{v}||$ defined by

$$||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

- Normalizing a vector means finding the unit vector.
- Cauchy-Schwarz Inequality: For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,

$$|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}|| ||\mathbf{v}||$$

- Triangle inequality: for all vectors $\mathbf{u}$ and $\mathbf{v}$ and $\mathbb{R}^n$,

$$||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$$

- Distance between two vectors is defined by

$$d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$$

- Two vectors are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$
- For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, $||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$ IFF $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
- If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$ and $\mathbf{u} \neq \mathbf{0}$, then the projection of $\mathbf{v}$ onto $\mathbf{u}$ is the vector defined by

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\right) \mathbf{u}$$

1.3 Lines and Planes

- Normal form of the equation of a 2D line:

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = \mathbf{0} \qquad \text{or} \qquad \mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$$

  where $\mathbf{p}$ is a specific point on the line and $\mathbf{n} \neq \mathbf{0}$ is a normal vector for the line.

- The general form of the equation of the line is $ax + by = c$ where $\mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ is a normal vector for the line.

- The vector form of the equation of a 2D or 3D line is

$$\mathbf{x} = \mathbf{p} + t\mathbf{d}$$

  where $\mathbf{p}$ is a specific point on the line and $\mathbf{d} \neq \mathbf{0}$ is a direction vector for the line. The equations corresponding to the components of the vector form of the equations are called parametric equations of the line.

- Normal form of the equation of a plane $\mathscr{P}$ in $\mathbb{R}^3$ is

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0 \qquad \text{or} \qquad \mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$$

  where $\mathbf{p}$ is a specific point on $\mathscr{P}$ and $\mathbf{n} \neq \mathbf{0}$ is a normal vector for $\mathscr{P}$.

- The general form of the equation of $\mathscr{P}$ is $ax + by + cz = d$, where $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a normal vector for $\mathscr{P}$.

- The vector form of the equation of a plane $\mathscr{P}$ in $\mathbb{R}^3$ is

$$\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$$

  where $\mathbf{p}$ is a point on $\mathscr{P}$ and $\mathbf{u}$ and $\mathbf{v}$ are direction vectors for $\mathscr{P}$ ($\mathbf{u}$ and $\mathbf{v}$ are nonzero and parallel to $\mathscr{P}$, but not parallel to each other). The equations corresponding to the components of the vector form of the equation are called parametric equations of $\mathscr{P}$.

- Summary of equations of 2D lines:
  - Normal form: $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$
  - General form: $ax + by = c$
  - Vector form: $\mathbf{x} = \mathbf{p} + t\mathbf{d}$
  - Parametric form:

$$\begin{cases} x = p_1 + td_1 \\ y = p_2 + td_2 \end{cases}$$

- Summary of equations of 3D lines:
  - Normal form:

$$\begin{cases} \mathbf{n_1} \cdot \mathbf{x} = \mathbf{n_1} \cdot \mathbf{p_1} \\ \mathbf{n_2} \cdot \mathbf{x} = \mathbf{n_2} \cdot \mathbf{p_2} \end{cases}$$

  - General form:

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \end{cases}$$

  - Vector form: $\mathbf{x} = \mathbf{p} + t\mathbf{d}$

  - Parametric form:

$$\begin{cases} x = p_1 + td_1 \\ y = p_2 + td_2 \\ z = p_3 + td_3 \end{cases}$$

- Summary of equations of 3D planes:
  - Normal form: $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$
  - General form: $ax + by + cz = d$
  - Vector form: $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$
  - Parametric form:

$$\begin{cases} x = p_1 + su_1 + tv_1 \\ y = p_2 + su_2 + tv_2 \\ z = p_3 + su_3 + tv_3 \end{cases}$$

1.4 Applications

- Force vectors: if the resultant net force is zero, the system is in equilibrium.
- Resolve into components to work with the vectors.