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## 16 Multiple Integrals

### 16.1 Double Integrals over Rectangles

- The double integral of  $f$  over the rectangle  $R$  is

$$\iint_R f(x, y) dA = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$$

if this limit exists.

- If  $f(x, y) \geq 0$ , then the volume  $V$  of the solid that lies above the rectangle  $R$  and below the surface  $z = f(x, y)$  is

$$V = \iint_R f(x, y) dA$$

- Midpoint rule for double integrals:

$$\iint_R f(x, y) dA \approx \sum_{i=1}^m \sum_{j=1}^n f(\bar{x}_i, \bar{y}_j) \Delta A$$

where  $\bar{x}_i$  is the midpoint of  $[x_{i-1}, x_i]$  and  $\bar{y}_j$  is the midpoint of  $[y_{j-1}, y_j]$ .

- Fubini's Theorem: If  $f$  is continuous on the rectangle  $R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$ , then

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

More generally, this is true if we assume that  $f$  is bounded on  $R$ ,  $f$  is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

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$$\iint_R g(x)h(y) dA = \int_a^b g(x) dx \int_c^d h(y) dy \quad \text{where } R = [a, b] \times [c, d]$$

## 16.2 Double Integrals over General Regions

- If  $F$  is integrable over  $R$ , then we define the double integral of  $f$  over  $D$  by

$$\iint_D f(x, y) dA = \iint_R F(x, y) dA \quad \text{where } F \text{ is given by Equation 1}$$

- If  $f$  is continuous on a type I region  $D$  such that

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

then

$$\iint_D f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

- Type II plane regions:

$$D = \{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

- If  $D$  is a type II region,

$$\iint_D f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

- Properties of double integrals

—

$$\iint_D [f(x, y) + g(x, y)] dA = \iint_D f(x, y) dA + \iint_D g(x, y) dA$$

—

$$\iint_D c f(x, y) dA = c \iint_D f(x, y) dA$$

where  $c$  is a constant

- If  $f(x, y) \geq g(x, y)$  for all  $(x, y)$  in  $D$ , then

$$\iint_D f(x, y) dA \geq \iint_D g(x, y) dA$$

- If  $D = D_1 \cup D_2$ , where  $D_1$  and  $D_2$  don't overlap except perhaps on their boundaries, then

$$\iint_D f(x, y) dA = \iint_{D_1} f(x, y) dA + \iint_{D_2} f(x, y) dA$$

- 

$$\iint_D 1 dA = A(D)$$

- If  $m \leq f(x, y) \leq M$  for all  $(x, y)$  in  $D$ , then

$$mA(D) \leq \iint_D f(x, y) dA \leq MA(D)$$

### 16.3 Double Integrals in Polar Coordinates

- Recall:

$$r^2 = x^2 + y^2 \quad x = r \cos \theta \quad y = r \sin \theta$$

- Change to polar coordinates in a double integral: If  $f$  is continuous on a polar rectangle  $R$  given by  $0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta$ , where  $0 \leq \beta - \alpha \leq 2\pi$ , then

$$\iint_R f(x, y) dA = \int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) r dr d\theta$$

- If  $f$  is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$

then

$$\iint_D f(x, y) dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta$$

### 16.4 Applications of Double Integrals

- mass of a lamina:

$$m = \lim_{k, l \rightarrow \infty} \sum_{i=1}^k \sum_{j=1}^l \rho(x_{ij}^*, y_{ij}^*) \Delta A = \iint_D \rho(x, y) dA$$

- Total charge in a given area:

$$Q = \iint_D \sigma(x, y) dA$$

- Moment of a lamina about the  $x$  axis:

$$M_x = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n y_{ij}^* \rho(x_{ij}^*, y_{ij}^*) \Delta A = \iint_D y \rho(x, y) dA$$

- About the  $y$  axis:

$$M_y = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n x_{ij}^* \rho(x_{ij}^*, y_{ij}^*) \Delta A = \iint_D x \rho(x, y) dA$$

- The coordinates  $(\bar{x}, \bar{y})$  of the center of mass of a lamina occupying the region  $D$  and having density function  $\rho(x, y)$  are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x \rho(x, y) dA \quad \bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_D y \rho(x, y) dA$$

where the mass  $m$  is given by

$$m = \iint_D \rho(x, y) dA$$

- Moment of inertia about  $x$  axis:

$$I_x = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) \Delta A = \iint_D y^2 \rho(x, y) dA$$

- About the  $y$  axis:

$$I_y = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) \Delta A = \iint_D x^2 \rho(x, y) dA$$

- Moment of inertia about the origin, or polar moment of inertia:

$$I_0 = \lim_{m,n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n [(x_{ij}^*)^2 + (y_{ij}^*)^2] \rho(x_{ij}^*, y_{ij}^*) \Delta A = \iint_D (x^2 + y^2) \rho(x, y) dA$$

where  $I_0 = I_x + I_y$

- Radius of gyration of a lamina about an axis is the number  $R$  such that

$$mR^2 = I$$

- Radius of gyration  $\bar{\bar{y}}$  with respect to  $x$  axis and radius of gyration  $\bar{\bar{x}}$  with respect to the  $y$  axis are given by

$$m\bar{\bar{y}}^2 = I_x \quad m\bar{\bar{x}}^2 = I_y$$

## 16.5 Surface Area

- The surface area of a surface  $S$  is

$$A(S) = \lim_{m,n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n \Delta T_{ij}$$

- The area of the surface with equation  $z = f(x, y)$ ,  $(x, y) \in D$ , where  $f_x$  and  $f_y$  are continuous, is

$$A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} dA$$

which is also

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA$$

## 16.6 Triple Integrals

- The triple integral of  $f$  over the box  $B$  is

$$\iiint_B f(x, y, z) dV = \lim_{l,m,n \rightarrow \infty} \sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V$$

if this limit exists.

- If we choose the sample point to be  $(x_i, y_j, z_k)$ , we get

$$\iiint_B f(x, y, z) dV = \lim_{l,m,n \rightarrow \infty} \sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n f(x_i, y_j, z_k) \Delta V$$

- Fubini's theorem for triple integrals: If  $f$  is continuous on the rectangular box  $B = [a, b] \times [c, d] \times [r, s]$ , then

$$\iiint_B f(x, y, z) dV = \int_r^s \int_c^d \int_a^b f(x, y, z) dx dy dz$$

- A solid region  $E$  is said to be of type 1 if it lies between the graphs of two continuous functions of  $x$  and  $y$ , that is,

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

- If  $E$  is a type 1 region:

$$\iiint_E f(x, y, z) dV = \iint_D \left[ \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right] dA$$

- If the projection of  $D$  of  $E$  onto the  $xy$  plane is a type I plane region, then

$$E = \{(x, y, z) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x), u_1(x, y) \leq z \leq u_2(x, y)\}$$

, and

$$\iiint_E f(x, y, z) dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz dy dx$$

- If  $D$  is a type II plane region, then

$$E = \{(x, y, z) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y), u_1(x, y) \leq z \leq u_2(x, y)\}$$

, and

$$\iiint_E f(x, y, z) dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz dx dy$$

- A solid region  $E$  is of type 2 if:

$$E = \{(x, y, z) \mid (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$$

where  $D$  is the projection of  $E$  onto the  $yz$  plane. The back surface is  $x = u_1(y, z)$  and the front surface is  $x = u_2(y, z)$ , and

$$\iiint_E f(x, y, z) dV = \iint_D \left[ \int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z) dx \right] dA$$

- A type 3 region is of the form:

$$E = \{(x, y, z) \mid (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$$

where  $D$  is the projection of  $E$  onto the  $xz$  plane,  $y = u_1(x, z)$  is the left surface, and  $y = u_2(x, z)$  is the right surface. Thus,

$$\iiint_E f(x, y, z) dV = \iint_D \left[ \int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z) dy \right] dA$$

- If  $f(x, y, z) = 1$  for all points in  $E$ , then:

$$V(E) = \iiint_E dV$$

## 16.7 Triple Integrals in Cylindrical Coordinates

- Recall:

$$r^2 = x^2 + y^2 \quad x = r \cos \theta \quad y = r \sin \theta \quad z = z \quad \tan \theta = \frac{y}{x}$$

- Triple integration in cylindrical coordinates:

$$\iiint_E f(x, y, z) dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r \cos \theta, r \sin \theta)}^{u_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) r dz dr d\theta$$

## 16.8 Triple Integrals in Spherical Coordinates

- Recall:

$$x = \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi \quad \rho^2 = x^2 + y^2 + z^2$$

- Triple integral in spherical coordinates:

$$\iiint_E f(x, y, z) dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi d\rho d\theta d\phi$$

where  $E$  is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$$

## 16.9 Change of Variables in Multiple Integrals

- We can write the substitution rule as:

$$\int_a^b f(x) dx = \int_c^d f(g(u)) g'(u) du$$

where  $x = g(u)$  and  $a = g(c), b = g(d)$  which is also

$$\int_a^b f(x) dx = \int_c^d f(x(u)) \frac{dx}{du} du$$

- The Jacobian of the transformation  $T$  given by  $x = g(u, v)$  and  $y = h(u, v)$  is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$

- Approximation to the area  $\Delta A$  of  $R$ :

$$\Delta A \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \Delta v$$

where the Jacobian is evaluated at  $(u_0, v_0)$

- Change of variables in a double integral: Suppose that  $T$  is a  $C^1$  transformation whose Jacobian is nonzero and that  $T$  maps a region  $S$  in the  $uv$  plane onto a region  $R$  in the  $xy$  plane. Suppose that  $f$  is continuous on  $R$  and that  $R$  and  $S$  are type I or type II plane regions. Suppose also that  $T$  is one-to-one, except perhaps on the boundary of  $S$ . Then:

$$\iint_R f(x, y) dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

- If:

$$x = g(u, v, w) \quad y = h(u, v, w) \quad z = k(u, v, w)$$

then the Jacobian of  $T$  is given by:

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

- Change of variables for triple integrals:

$$\iiint_R f(x, y, z) dV = \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw$$