Ch 12 Ex - MC

§ 12.1 – Sequences

Ex. 1: N/A

Ex. 2: $\left\{ \frac{3}{5}, \frac{-4}{25}, \frac{5}{125}, \frac{-6}{625}, \frac{7}{3125} \right\}$

$$a_n = \frac{(n+2)(-1)^{n+1}}{5^n}$$

Ex. 3: N/A

Ex. 4: $\lim_{n\to\infty} \frac{n}{n+1}$ → Same degree on num & denom.

as n becomes large, the diff. b/w num & denom. becomes less & less; therefore $\lim_{n\to\infty} \frac{n}{n+1} = 1$

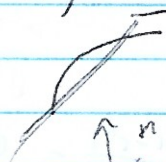
$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

Ex. 5: $a_n = \frac{n}{\sqrt{10+n}}$

n^{th} term test → num. degree > denom. degree;

limit approaches ∞; diverges.

Ex. 6: $\lim_{n\to\infty} \frac{\ln n}{n}$



$\lim_{n\to\infty} \frac{\ln n}{n} = 0$ ← n becomes much greater than ln n

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L'Hopital's rule: $\lim_{n\to\infty} \frac{\ln n}{n}$

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$$

Ex. 7: $a_n = (-1)^n$

alt. series test → $\lim_{n\to\infty}(1) \neq 0$; diverges

Ex. 8: $\lim_{n\to\infty} \frac{(-1)^n}{n}$

If $\lim_{n\to\infty} |a_n| = 0$, $\lim_{n\to\infty} a_n = 0$

$$\lim_{n\to\infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n\to\infty} \frac{1}{n} = 0$$

$$\therefore \lim_{n\to\infty} \left(\frac{(-1)^n}{n} \right) = 0$$

Ex. 9: $\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right)$

If $\lim_{n\to\infty} a_n = L$ and f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$

$$= \sin\left(\lim_{n\to\infty} \frac{1}{n}\right) = \sin(0) = 0$$

Ex. 10: $a_n = \frac{n!}{n^n}$

$$\cancel{\frac{n(n-1)(n-2)(n-3)\cdots}{n \cdot n \cdot n \cdot n \cdot n \cdots}}$$

bottom is > than top; converges to 0

Ex. 11 $a_n = r^n$

~~geometric~~

if r is a fraction $\left(\frac{1}{r}\right)$, converges to 0; if $r > 1$, diverges

$r > 1 \quad \{r^n\} \to \infty$

$0 < r < 1 \quad \{r^n\} \to 0$

but also, $\left(-\frac{1}{r}\right)^n \to 0$

11 cont'd; $r=1$ $(1^n)=1$

$\{r^n\}$ converges if $-1<r\leq 1$.

Ex. 12 = N/A

Ex. 13: $a_n = \frac{n}{n^2+1}$

the degree of the numerator < degree of denominator. therefore the next term will be smaller, and it is decreasing.

Ex. 14: $\{a_n\}$: $a_1=2$ $a_{n+1}=\frac{1}{2}(a_n+6)$

$a_1=2$, $a_2=4$, $a_3=5$, $a_4=\frac{11}{2}$

$a_5=\frac{23}{4}$ $a_6=\frac{47}{8}$

looks as if it is converging towards $\boxed{6}$.

§ 11.2 – Series

Ex-1 $S_n = a_1+a_2+a_3+\dots+a_n = \frac{2n}{3n+5}$

$\sum a_n = \lim_{n\to\infty} S_n = \boxed{\frac{2}{3}}$

Ex-2 geo. series $\to a+ar+ar^2+\dots ar^{n-1}$

diverges if $|r|\geq 1$

Converges if $|r|<1$; $S_n = \frac{a}{1-r}$

Ex-3 series:

$5-\frac{10}{3}+\frac{20}{9}-\frac{40}{27}$

$a_1=5$ $r=\left(-\frac{2}{3}\right)$ (as observed)

$|r| < 1 \to$ converges

$S_n = \frac{a}{1-r} = \frac{5}{1-(-\frac{2}{3})} = \frac{5}{\frac{5}{3}} = \boxed{3}$

Ex-4 $\to$ series: $\sum_{n=1}^{\infty} 2^{2n}\cdot 3^{1-n}$

rewrite in the form ar^n

$\sum_{n=1}^{\infty} (2^2)^n 3^{(1-n)} = 4^n\cdot 3^{(n-1)(-1)}$

$= \frac{4^n}{3^{n-1}} = \frac{4^{(n-1)}\cdot 4^1}{3^{n-1}}$

$= 4\left(\frac{4}{3}\right)^{n-1}$

diverges, $\frac{4}{3}>1$

Ex-5

a) $C_n = 0.2n + \frac{3}{10}C_{n-1} + \frac{3}{10}C_{n-2} + \dots$

$C_n = \sum_{n=1}^{\infty} (0.2)\left(\frac{3}{10}\right)^{n-1}$

where $a=0.2$ & $r=\frac{3}{10}$

if $n=3$, $C_3 = 0.2\left(\frac{3}{10}\right)^0 + 0.2\left(\frac{3}{10}\right)^1 + 0.2\left(\frac{3}{10}\right)^2$

$= \boxed{0.278 \text{ mg/mL}}$ ✓

b) $S_C = \frac{a}{1-r} = \frac{0.2}{(1-\frac{3}{10})} = \boxed{0.286 \text{ mg/mL}}$

c) limiting conc. $= \lim_{n\to\infty} (0.2)\left(\frac{3}{10}\right)^{n-1}$

$= 0.2$

b) $C_n = \frac{a(1-r^n)}{1-r} = \frac{0.2(1-(0.3)^n)}{0.7}$

$= \frac{2}{7}(1-(0.3)^n)$

c) $\lim_{n\to\infty} (0.3)^{n-1} = 0$

so the limiting conc. is $\boxed{\frac{2}{7} \text{ mg/mL}}$

Ex-6 $2.3\overline{17} = 2.3\underbrace{17}\underbrace{17}\underbrace{17}\ldots$

$= 2.3 + \frac{17}{1000} + \frac{17}{100{,}000} + \ldots$

~~$a = 2.3$~~

$= 2.3 + \frac{17}{1000} + \ldots$

$a = \frac{17}{1000}$ $r = \frac{1}{100}$

$2.3\overline{17} = 2.3 + S_n = 2.3 + \frac{a}{1-r}$

$= 2.3 + \frac{17}{1000} \cdot \frac{1}{1-\frac{1}{100}}$

$= 2.3 + \frac{17}{1000} \cdot \frac{100}{99}$

$= \frac{23}{10} + \frac{17}{990}$

$= \frac{23 \cdot 99 + 17}{990} = \frac{2294}{990}$

$= \boxed{\frac{1147}{495}}$

Ex. 7 $\sum_{n=0}^{\infty} x^n$

$a = 1$ $r = x$ $|x| < 1$

$S_n = \frac{a}{1-r} = \boxed{\frac{1}{1-x}}$

Ex-8 $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n^2+n}$

comparison test $\rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} \rightarrow$ p series, converges

$\frac{1}{n^2+n} < \frac{1}{n^2}$ ∴ it converges. ✓

partial fraction of $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$S_n = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right)$

$+ \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$

$\lim_{n\to\infty} \frac{1}{n+1} = 0$ $\boxed{S_n = 1}$

Ex. 9 $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$

p-series, $p = 1$ is divergent

The sum keeps getting larger & doesn't converge to a single #.

Ex. 10 $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$

n^{th} term test $\lim_{n\to\infty} \frac{n^2}{5n^2+4} = \frac{1}{5} \neq 0$

∴ the series diverges.

Ex. 11 $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$

$= \sum_{n=1}^{\infty} \frac{3}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{2^n} = 3\sum_{n=1}^{\infty} \frac{1}{(n+1)n} + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$

$S_n = 3 + 2 = \boxed{5}$

$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ $a = \frac{1}{2}$ $r = \frac{1}{2}$ ∴ $S_n = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

$S = 3 + 1 = \boxed{4}$

§ 12.3 – Integral Test + Estimates of Sums

Ex-1 $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Continuous, (+), decreasing

$\int_1^{\infty} \frac{1}{x^2+1} dx$

$= \lim_{b\to\infty} \int_1^b \frac{1}{x^2+1} dx$

$= \lim_{b\to\infty} (\arctan x)\Big|_1^b$

$= \lim_{b\to\infty} (\arctan b - \arctan 1)$

$= \frac{\pi}{2} - \frac{\pi}{4} = \boxed{\frac{\pi}{4}}$

Ex-2 $\sum \frac{1}{n^p}$ → p series

cont, (+), dec.

$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p} dx = \lim_{b\to\infty} \int_1^b x^{-p} dx$

$= \lim_{b\to\infty} \left(\frac{1}{-p+1} x^{-p+1}\right)\Big|_1^b$

$= \lim_{b\to\infty} \left(\frac{1}{-p+1} \cdot \frac{x}{x^p}\right)\Big|_1^b$?

Converges if $p<1$

diverges if $p \geq 1$

Ex-3 N/A

Ex-4

$\sum_{n=1}^{\infty} \frac{\ln n}{n}$ cont, (+), dec

$\int_1^{\infty} \frac{\ln x}{x} dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x} dx$

$\frac{du}{dx}(\ln x) = \frac{1}{x}$

$dx = du \cdot x$

$u = \ln x$

$\lim_{b\to\infty} \int_1^b u\,du = \lim_{b\to\infty} \left(\frac{1}{2}(\ln x)^2\right)\Big|_1^b$

$= \lim_{b\to\infty} \left(\frac{1}{2}(\ln b)^2 - \frac{1}{2}(\ln(1))^2\right)$

$= \lim_{b\to\infty} \left(\frac{1}{2}(\ln b)^2\right)$ → diverges

Ex. 5 $\sum \frac{1}{n^3}$

a) $S - S_n \leq \int_{10}^{\infty} \frac{1}{x^3} dx$

$S \approx S_{10} = S_1 + S_2 + \cdots + S_{10}$

$S_{10} \approx 1.1975$ → how?

$R_{10} \leq \int_{10}^{\infty} \frac{1}{x^3} dx = \lim_{b\to\infty} \int_{10}^b \frac{1}{x^3} dx$

$= \lim_{b\to\infty} \left(-\frac{1}{2x^2} dx\right)\Big|_{10}^b$

$= \lim_{b\to\infty} \left(-\frac{1}{2b^2} + \frac{1}{2(10)^2}\right)$

$= \frac{1}{200} = \boxed{0.005}$

$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_n^{\infty} f(x)\,dx$

$\int_n^{\infty} \frac{1}{x^3} dx = \frac{1}{2x^2}$ $0.0005 = \frac{1}{2000}$

$\frac{1}{2n^2} = \frac{1}{2000}$ $n^2 = 1000$

$n = \boxed{32}$

Ex-6

$S_{10} + \int_{11}^{\infty} \frac{1}{x^3} dx \leq S \leq S_{10} + \int_{10}^{\infty} \frac{1}{x^3} dx$

$\frac{1}{2n^2}$ $S_{10} + \frac{1}{2(11)^2} \leq S \leq S_{10} + \frac{1}{2(10)^2}$

$S_{10} = 1.197532$ did they just use a calculator?

$1.197532 + \frac{1}{242} \leq S \leq 1.197532 + \frac{1}{200}$

$1.20166 \leq S \leq 1.202532$

$S \approx 1.201$

§12.4 Comparison tests

Ex-1 $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$

compare to $\sum \frac{5}{2n^2} > \frac{5}{2n^2+4n+3}$

$\frac{5}{2n^2} = \frac{5}{2}\sum_{n=1}^{\infty} \frac{1}{n^2} \to$ p-series $\to$ converges

therefore $\sum \frac{5}{2n^2+4n+3}$ converges.

Ex-2 $\sum_{k=1}^{\infty} \frac{\ln k}{k}$

$\frac{\ln k}{k} > \frac{1}{k}$ $\quad \frac{1}{k} \to$ p-series, $p=1$, diverges

$\therefore \frac{\ln k}{k}$ diverges

Ex-3 $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

compare to $\frac{1}{2^n} \to$ geo. series, $a=\frac{1}{2}$, $r=\frac{1}{2}$

$\left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n-1}$

converges

$\frac{1}{2^n-1} : \frac{1}{2^n}$ inconclusive

limit test

$\lim_{n\to\infty} \frac{2^n}{2^{n-1}} = \lim_{n\to\infty} 2 = 2$

since $\frac{1}{2^n}$ converges, then $\frac{1}{2^n-1}$ converges.

$\lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{1}{1-\left(\frac{1}{2}\right)^n} = 1 > 0$

Ex-4 $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$

$\hookrightarrow \frac{n^2}{n^{\frac{5}{2}}} = \frac{1}{n^{\frac{1}{2}}}$

comp. $\frac{2}{n^{\frac{1}{2}}}$

$\lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{5+n^5}} \cdot \frac{n^{\frac{1}{2}}}{2} = \frac{2n^{\frac{5}{2}}+3n^{\frac{3}{2}}}{2\sqrt{5+n^5}} = \frac{2}{2} = 1$

$\frac{2}{n^{\frac{1}{2}}} \to$ p-series, $\frac{1}{2} < 1$, diverges

$\therefore$ the series diverges.

Ex-5 $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$

compare w/ $\frac{1}{n^3}$ $\quad \frac{1}{n^3+1} < \frac{1}{n^3}$

$T_n \le \int_n^{\infty} \frac{1}{x^3}\,dx = \frac{1}{2n^2}$ $\quad R_n \le T_n \le \frac{1}{2n^2}$

$R_{100} = 0.00005$

$S_{100} = 0.686$ $\quad$ err < 0.00005

§12.5 - Alt. series

Ex-1 N/A

Ex-2 N/A

Ex-3 $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$

$\frac{n^2}{n^3+1} \to$ is decreasing

$\lim_{n\to\infty} \frac{n^2}{n^3+1} \to = 0$

$\therefore$ the series converges

Test that it is decreasing, though.

$f'(x) = \frac{x(2-x^3)}{(x^3+1)^2}$, which is negative

$\hookrightarrow$ decreasing

Ex-4 $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

$|R_n| = |S - S_n| \le b_{n+1}$

$R_n \le b_{n+1} \le \frac{1}{1000} \to 0.0001$

$\frac{1}{(n+1)!} \le \frac{1}{1000}$

$S_6 \approx 0.368056$

$b_7 = \frac{1}{5040} < \frac{1}{5000} = 0.0002 \to$ three decimal places

$\boxed{S = 0.368}$

§11.6 Absolute convergence, ratio + root tests

Ex-1 N/A

Ex-2 N/A

Ex-3

$\sum_{n=1} \frac{\cos n}{n^2} \to$ cos n is the alt. series part

$\left|\frac{\cos n}{n^2}\right| = \frac{|\cos n|}{n^2}$

$|\cos n| \le 1 \therefore \frac{|\cos n|}{n^2} \le \frac{1}{n^2}$

$\frac{1}{n^2} \to$ converges, therefore $\frac{|\cos n|}{n^2}$ converges, and the original series also converges absolutely.

Ex-4

$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$

ratio test

$\lim_{n\to\infty} \left| \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{(n^3)} \right| = \frac{1}{3} < 1$

$\therefore$ the series converges absolutely.

Ex-5

$\sum_{n=1}^{\infty} \frac{n^n}{n!}$ ratio test

$\lim_{n\to\infty} \left| \frac{(n+1)!}{(n+1)^{n+1}} \cdot n! \right|$

$\lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right| = \lim_{n\to\infty} \left| \frac{\cancel{(n+1)}(n+1)^n \cancel{n!}}{\cancel{n!(n+1)} \cdot n^n} \right|$

$= 1 \Rightarrow$ inconclusive

$= \lim_{n\to\infty} \left| \frac{(n+1)^n}{n^n} \right| = \lim_{n\to\infty} \left| \left(1 + \frac{1}{n}\right)^n \right|$

$= e > 1 \therefore$ diverges

Ex-6 $\sum_{n=1} \left(\frac{2n+3}{3n+2}\right)^n$

root test

$\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n} = \lim_{n\to\infty} \left(\frac{2n+3}{3n+2}\right) = \frac{2}{3} < 1$

converges absolutely

§12.7 · Strategy for testing series

Ex-1 $\sum \frac{n-1}{2n+1}$

n^{th} term test $\to \lim = \frac{1}{2} \ne 0 \therefore$ diverges

Ex-2 $\sum \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$

Comparison test, $\frac{n^{3/2}}{3n^3} = \frac{1}{3n^{3/2}}$

$\frac{\sqrt{n^3+1}}{3n^3+4n^2+2} < \frac{1}{3n^{3/2}}$ $\quad \frac{1}{n^{3/2}}$ converges b/c of p-series

$\therefore$ converges

Ex-3 $\sum n e^{-n^2}$?

integral test or ratio test

Ex-4 $\sum (-1)^n \frac{n^3}{n^4+1} \to$ alt. series test

EX-5 $\sum \frac{2^k}{k!}$ → ratio test, factorial

EX-6 $\sum \frac{1}{2+3^n}$ comparison test + geo series $\frac{1}{3^n}$

§12-8 Power Series

EX-1 $\sum_{n=0}^{\infty} n! x^n$

ratio test → $\lim_{n\to\infty} \left|\frac{(n+1)! x^{n+1}}{n! x^n}\right|$

$= \lim_{n\to\infty} |(n+1)(x)| = 0$

$\boxed{x=0}$ → conv

EX-2 $\sum \frac{(x-3)^n}{n}$

ratio test → $\lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{(n+1)} \cdot \frac{n}{(x-3)^n}\right| < 1$

$= \lim_{n\to\infty} \left|\frac{(x-3)n}{n+1}\right| < 1$

$|x-3| < 1$

$-1 < x-3 < 1$

$2 < x < 4$

@ $x=2$, ~~$\lim_{n\to\infty} \frac{}{\sqrt{n+1}}$~~

$\sum \frac{(-1)^n}{n}$ → alt series, converges

@ $x=4$, $\sum \frac{1^n}{n} = \sum \frac{1}{n}$ → p-sum, diverges.

Converges on $[2, 4)$

EX-3 Find domain of

$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

ratio test - $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right|$

$= \frac{x^2}{4(n+1)^2} = 0 < 1$

converges on all x

EX-4 $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = \frac{(-1)^n \cdot 3^n x^n}{\sqrt{n+1}}$

ratio test ~~$\lim_{n\to\infty} \left|(-3)^{n+1}\right.$~~

$\lim_{n\to\infty} \left|\frac{3^{n+1} \cdot x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{3^n \cdot x^n}\right| < 1$

$= \lim_{n\to\infty} \left|\frac{3x\sqrt{n+1}}{\sqrt{n+2}}\right| < 1$

$|3x| < 1$ $R=3$

$-3 < x < 3$

@ $x=-3$, $\sum \frac{(-3)^n \cdot (-3)^n}{\sqrt{n+1}}$

$= \sum \frac{3^{2n}}{\sqrt{n+1}} > \frac{1}{\sqrt{n+1}}$

$\frac{1}{\sqrt{n+1}}$ → p-sum, diverges → div

@ $x=3$ $\frac{(-3)^n (3)^n}{\sqrt{n+1}} = \frac{(-1)^n \cdot 3^{2n}}{\sqrt{n+1}}$ → alt. ser. diverges

$\lim_{n\to\infty} \left(\frac{3^{2n}}{\sqrt{n+1}}\right)$ → diverges

Converges on $x: (-3, 3)$

$|3x|<1 \to -\frac{1}{3} < x < \frac{1}{3}$

$R = \frac{1}{3}$ when $x = -\frac{1}{3}$, $\frac{(-3)^n \cdot (-\frac{1}{3})^n}{\sqrt{n+1}}$

$= \frac{1}{\sqrt{n+1}} \to$ p-series, diverges

$x = \frac{1}{3} \to \frac{(-3)^n (\frac{1}{3})^n}{\sqrt{n+1}} \to \frac{(-1)^n}{\sqrt{n+1}}$

alt. series, converges.

$\therefore$ converges on $(-\frac{1}{3}, \frac{1}{3}]$

Ex. 5 $\sum \frac{n(x+2)^n}{3^{n+1}}$

ratio test $\lim_{n\to\infty} \left| \frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n} \right|$

$= \lim_{n\to\infty} \left| \frac{(x+2)(n+1)}{3n} \right| < 1$

$\left| \frac{x+2}{3} \right| < 1$

$-3 < x+2 < 3$ $\boxed{R=3}$

$-5 < x < 1$

when $x = -5$

$\sum \frac{n(-3)^n}{3^{n+1}} = \sum \frac{n(-1)^n \cancel{3^n}}{3 \cdot \cancel{3^n}}$

alt. series, diverges.

when $x = 1$, $\sum \frac{n(3)^n}{3 \cdot 3^n} = \sum \frac{n}{3}$

n^{th} term test, diverges.

$\therefore$ IOC: $(-5, 1)$

§12.9 Representations of Funcs. as Power series

Ex. 1 $f(x) = \frac{1}{1+x^2}$

Make it take the form $\frac{1}{1-x}$

$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$

geo series $- |-x^2| < 1 \to |x| < 1 \to (-1, 1)$

Ex. 2 $\frac{1}{x+2} = \frac{1}{2+x} = \frac{1}{1+(x+1)} = \frac{1}{1-(-1-x)}$

$= \sum_{n=0}^{\infty} (-x-1)^n$ geo series

$|-x-1| < 1$

$-1 < -x-1 < 1$

$1 > x+1 > -1$

$-2 < x < 0$ $(-2, 0)$

$\frac{1}{x+2} = \frac{1}{2+x} = \frac{1}{2} \cdot \frac{1}{(1+\frac{x}{2})}$

$= \frac{1}{2} \sum_{n=0}^{\infty} (-\frac{x}{2})^n = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^n}{2^n}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}}$

converges when $|-\frac{x}{2}| < 1 \to$ geo series

$\therefore$ ioc $(-2, 2)$

Ex. 3 $\frac{x^3}{x+2}$

$= x^3 \cdot \frac{1}{x+2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+3}}{2^{n+1}}$

IOC $(-2, 2)$

Ex. 4 N/A

Ex-5 $\frac{1}{(1-x)^2}$

$\frac{d}{dx}\frac{1}{1-x} = \frac{-1\cdot(-1)}{(1-x)^2}$ $\quad \frac{1}{u} \quad u^{-1} \quad -u^{-2}$

$u = 1-x \qquad = \frac{1}{(1-x)^2}$

$\frac{du}{dx} = -1$

$\frac{1}{(1-x)^2} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n$

$= \sum_{n=0}^{\infty} n x^{n-1}$ $\quad R=1$, from $\frac{1}{1-x}$

Ex. 6 $\ln(1+x)$

$\frac{d}{dx}\ln(1+x) = \frac{1}{1+x}$

$\ln(1+x) = \int \sum_{n=0}^{\infty} x^n \, dx$

$= \sum_{n=0}^{\infty} \frac{1}{n+1} \cdot x^{n+1}$

$= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$ $\quad R=1$, from $\frac{1}{1-x}$

$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int (1 - x + x^2 - x^3 + \cdots)\,dx$

$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + C$

$= \sum_{n=0}^{\infty} \frac{x^n \cdot (-1)^{n+1}}{n} + C$

C: $x=0$, $\ln(1+0) = 0 = C$

$\ln(1+x) = \sum_{n=0} \frac{(-1)^{n+1} x^n}{n}$; $R=1$

Ex-7 $f(x) = \arctan x$

$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$

$= \int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty} (x^2)^n \,dx$

$= \int (1 - x^2 + x^4 - x^6 + \cdots)\,dx$

$= C + x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \cdots$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + C$

when $x=0$, $\arctan 0 = 0 = C$

$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ $\quad R=1$

Ex-8

(a) $\int \frac{1}{1+x^7}\,dx$

$\frac{1}{1+x^7} = \frac{1}{1-(-x^7)} = \sum_{n=0}^{\infty} (-x^7)^n$

$\int \frac{1}{1+x^7} = \int (1 - x^7 + x^{14} - x^{21} + \cdots)\,dx$

$= x - \frac{1}{8}x^8 + \frac{1}{15}x^{15} - \frac{1}{22}x^{22} + \cdots + C$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{7n+1}}{7n+1} + C$

(b) $\int_0^{0.5} \frac{1}{1+x^7}\,dx$ $\quad 0.5^8 \to 0.004$

$0.5^{15} \to 0.00003$

$0.5^{22} \to 0.0000002 \to 10^{-7}$

$\int_0^{0.5} \frac{1}{1+x^7}\,dx \approx (0.5) - \frac{1}{8}(0.5)^8 + \frac{1}{15}(0.5)^{15} - \frac{1}{22}(0.5)^{22}$

$=$ ~~0.500490305~~ typed in wrong

$= 0.49951374$

§ 12.10 Taylor & Maclaurin series

Ex.1 - Maclaurin e^x $\quad f'(x)=e^x \ldots$

$f(0)=1 \quad f'(0)=1 \quad f''(0)=1 \ldots$

$$e^x = \sum_{n=0}^{\infty} \frac{(1)\cdot x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

ROC - ratio test $\Rightarrow \dfrac{x^{n+1}}{n!(n+1)} \cdot \dfrac{n!}{x^n} = \dfrac{x}{n+1}$

$\lim = 0$

converges for all x

Ex.2 prove that e^x is the sum of the Maclaurin series

$f(x)=e^x \quad f^{(n+1)}(x)=e^x$ for all n

if d is a (+) # and $|x| \le d$,

then $f^{(n+1)}(x) = e^x \le e^d$

$|R_n(x)| \le \dfrac{e^d}{(n+1)!}|x|^{n+1}$ for $|x|\le d$

$$\lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = e^d \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

$$\therefore\ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Ex.3 Taylor $f(x)=e^x \quad a=2$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)\,(x-a)^n}{n!}$$

$f^n(x)=e^x \qquad f^n(a)=e^2$

$$f(x) = \sum_{n=0}^{\infty} \frac{e^2 (x-2)^n}{n!}$$

Ex.4 Maclaurin $\sin x$

$f(x)=\sin x \quad f'(x)=\cos x \quad f''(x)=-\sin x$

$f'''(x)=-\cos x$

$f^{(4)}(x)=\sin x$

$$\sum \frac{f^n(0)\, x^n}{n!}$$

$f(0)=\sin 0=0$

$f'(0)=\cos 0=1$

$f''=0$

$f'''=-1$

$f^{(4)}=0$

$$\sin x = \frac{0\cdot x^0}{0!} + \frac{(1)\cdot x}{1!} + \frac{0\cdot x^2}{2!} + \frac{(-1)\cdot x^3}{3!} + \frac{0\cdot x^4}{4!} + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{x^{(2n+1)}(-1)^n}{(2n+1)!}$$

& cos?

Ex.5 Maclaurin $\cos x$

$$\cos x = \frac{d}{dx}\sin x = \frac{d}{dx}\sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)}}{(2n+1)!}$$

$$= \frac{(2n+1)\cdot(-1)^n\cdot x^{2n}}{(2n+1)!} = \frac{(-1)^n x^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Ex.6 Maclaurin $x\cos x$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}\cdot x}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!}$$

Ex.7 Taylor $\sin x \quad a=\frac{\pi}{3}$

$f(x)=\sin x \qquad f(a)=\frac{\sqrt{3}}{2}$

$f'(x)=\cos x \qquad f'(a)=\frac{1}{2}$

$f''(x)=-\sin x \qquad f''(a)=-\frac{\sqrt{3}}{2}$

$f'''(x)=-\cos x \qquad f'''(a)=-\frac{1}{2}$

$f^{(4)}(x)=\sin x \qquad f^{4}(a)=\frac{\sqrt{3}}{2}$

$\sin x = \cancel{\sum f^n(a)}$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)(x-a)^n}{n!}$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{1}{0!} + \frac{1}{2}\cdot\frac{(x-\frac{\pi}{3})}{1!} - \frac{\sqrt{3}}{2}\cdot\frac{(x-\frac{\pi}{3})^2}{2!}$$

$$-\frac{1}{2}\cdot\frac{(x-\frac{\pi}{3})^3}{3!} + \frac{\sqrt{3}}{2}\cdot\frac{(x-\frac{\pi}{3})^4}{4!} + \dots \checkmark$$

In Sigma notation: Separate terms containing $\sqrt{3}$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n\sqrt{3}}{2(2n!)}\left(x-\frac{\pi}{3}\right)^{2n} + \sum_{n=0}^{\infty}\frac{(-1)^n}{2(2n+1)!}\left(x-\frac{\pi}{3}\right)^{2n+1}$$

EX-8 Maclaurin $(1+x)^k$

$f(x) = (1+x)^k$
$f'(x) = k(1+x)^{k-1}$
$f''(x) = k(k-1)(1+x)^{k-2}$
$f'''(x) = k(k-1)(k-2)(1+x)^{k-3}$
$\vdots$

$f(0) = 1^k = 1 \quad f'(0) = k \quad f''(0) = k(k-1)$
$f'''(0) = k(k-1)(k-2)\dots$

$$f(x) = \sum_{n=0} \frac{f^n(0)\cdot x^n}{n!}$$

$$= 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}(k \cdot ?)$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}(k)(k-1)(k-2)\cdots(k-n+1)$$

EX-9 Maclaurin $f(x) = \frac{1}{\sqrt{4-x}}$

binomial $k = -\frac{1}{2} \quad x \Rightarrow -\frac{x}{4}$

$$\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4(1-\frac{x}{4})}} = \frac{1}{2}\cdot\frac{1}{\sqrt{1-\frac{x}{4}}} = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{x}{4}\right)^n$$

$$= \frac{1}{2}\left(1 + \left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(-\frac{x}{4}\right)^2 + \dots\right)$$

Converges when $\left|-\frac{x}{4}\right| < 1$

$|x| < 4 \quad R = 4$

EX-10 $\frac{1}{1\cdot 2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \dots$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\cdot 2^n} = \cancel{\sum_{n=0}\frac{(-1)^n}{(n-1)2^{(n-1)}}}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,2^{-n}}{n} \qquad \ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{x^n}{n}$$

$2^{-n} = \left(\frac{1}{2}\right)^n \quad x = \frac{1}{2} \qquad \ln\left(1+\frac{1}{2}\right) = \boxed{\ln\left(\frac{3}{2}\right)}$

EX-11 a) $\int e^{-x^2}dx$ b) $\int_0^1 e^{-x^2}dx$ w/in 0.001

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \qquad e^{-x^2} = \sum_{n=0}^{\infty}\frac{-x^{(2n)}}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{x^{2n}}{n!}(-1)^{2n}$$

$$\int e^{-x^2}dx = \int\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}dx = \int\left(\frac{1}{0!} + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots\right)dx$$

$$= C + \frac{1}{3}\cdot x^3 + \frac{x^5}{5\cdot 2!} + \frac{x^7}{7\cdot 3!} + \dots)$$

$$= \boxed{C + \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)\cdot n!}}$$

b) $x=1$, $n=2$: $\frac{1^5}{5\cdot 2!} = \frac{1}{10}$ $\quad$ $x=1$, $n=3$: $\frac{1^7}{7\cdot 3!} = \frac{1}{7\cdot 6} = 0.02$

$n=4$: $\frac{1^9}{9\cdot 4!} = \frac{1}{9\cdot 24} = 0.005$

$(n=4)$ $\int_0^1 e^{-x^2}\,dx = C + \frac{1}{3} + \frac{1}{5\cdot 2!} + \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!} - C$

$= \boxed{0.462}$

a) Keep $(-1)^n \to (-x^2)^n = (-1)^n \cdot x^{2n}$

$\int e^{-x^2}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)\cdot n!}$

b) $\int_0^1 e^{-x^2}\,dx = \left(x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!}\right)\Big|_0^1$

$= 1 - \frac{1}{3} + \frac{1}{5\cdot 2} - \frac{1}{7\cdot 6} + \frac{1}{9\cdot 24}$

next term $= \frac{1}{11\cdot 24\cdot 5} = 0.00075 < 0.001$

$= 0.7475$

Ex. 12 $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$

L'hospital's rule $\lim_{x\to 0} = \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$

Using Maclaurin series:

$\lim_{x\to 0} \frac{\left(\cancel{1} + \cancel{\frac{x}{1!}} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) \cancel{-1} \cancel{-x}}{x^2}$

$= \frac{\frac{x^2}{2} + \frac{x^3}{6} + \cdots}{x^2} = \boxed{\frac{1}{2}}$

Ex. 13 a) first 3 Maclaurin $e^x \sin x$, $\tan x$

a) $e^x \sin x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \cdot \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!}$

$= \frac{1(1)(x^1)}{(0!)(1!)} + \frac{x(-1)(x^3)}{(1!)(3!)} + \frac{x^2(1)(x^5)}{(2!)(5!)}$

$= x - \frac{x^4}{6} + \frac{x^7}{240} + \cdots$

multiply & divide polynomials

$e^x \sin x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \cdots\right)$

$1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$

$x \quad - \frac{1}{6}x^3$

$x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$

$-\frac{1}{6}x^3 - \frac{1}{6}x^4$

$x + x^2 + \frac{1}{3}x^3 + \cdots = e^x \sin x$

b) $\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!}}{1 - \frac{x^2}{2!} + \frac{x^4}{4!}}$

$x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$

$1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots \;\big)\; x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

$x - \frac{1}{2}x^3 + \frac{1}{24}x^5$

$\frac{1}{3}x^3 - \frac{1}{30}x^5$

$\frac{1}{3}x^3 - \frac{1}{6}x^5$

$\frac{2}{15}x^5 + \cdots$

$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5$

§12.11 Applications of Taylor Polynomials

Ex. 1 Approx $f(x) = \sqrt[3]{x} \to$ Taylor $a=8$, $n=2$

$f(x) = x^{1/3}$ $f'(x) = \frac{1}{3}x^{-2/3}$ $f''(x) = -\frac{2}{9}x^{-5/3}$

$f(8) = 2$ $f'(8) = \frac{1}{3}\cdot\frac{1}{2^2} = \frac{1}{12}$ $f''(8) = -\frac{2}{9}\cdot\frac{1}{2^5} = -\frac{1}{144}$

$$\sqrt[3]{x} = \frac{f^n(a)\cdot(x-a)^n}{n!} = \frac{(1)(2)}{0!} + \frac{(x-8)}{1\cdot 12} - \frac{(x-8)^2}{144\cdot 2}$$

$$= 2 + \frac{x-8}{12} - \frac{(x-8)^2}{288}$$

b) accuracy $7 \le x \le 9$

Taylor's inequality

$$|R_2(x)| \le \frac{M}{3!}|x-8|^3$$

$|f'''(x)| \le M$

$f'''(x) = \frac{10}{27}x^{-8/3}$

if $7 \le x \le 9$ ...?

$x \ge 7 \to x^{8/3} \ge 7^{8/3}$

$f'''(x) = \frac{10}{27}\left(\frac{1}{x^{8/3}}\right) \le \frac{10}{27}\cdot\frac{1}{7^{8/3}} < 0.0021$

$M = 0.0021$

$|R_2(x)| \le \frac{0.0021}{3!}\cdot 1^3 = \frac{0.0021}{6} < 0.0004$

Ex. 2 max err $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$-0.3 \le x \le 0.3$

alt. ~~series~~ error Taylor error?

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

$a=0$ $n=2$ $f^{n+1}(x) \le M$

$f'''(\sin x) = -\cos x$

$x \ge -0.3$ $-\cos x \ge -\cos(-0.3)$

$|-\cos(-0.3)| = 0.9553 = M$

$-0.3 \le x \le 0.3$

$$|R_2(x)| \le \frac{0.9553}{3!}(0.3)^3 = 0.0043$$

Use alt. series err, w/ the next term $\to -\frac{x^7}{7!}$

$$\left|\frac{x^7}{7!}\right| = \frac{|x|^7}{5040} = \frac{(0.3)^7}{5040} \approx 4.3\times10^{-8}$$

$\sin 12^\circ = \sin(0.209)$

$$= (0.209) - \frac{(0.209)^3}{3!} + \frac{(0.209)^5}{5!}$$

$= 0.2074818$

Use $12^\circ = \frac{\pi}{15}$ exact val

$\sin 12 \approx 0.20791169$

b) $\frac{|x|^7}{5040} \le 0.00005$

$|x|^7 \le 0.252$

$|x| \le 0.821$

Ex. 13

a) $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$

when $v \ll c$, $\sqrt{1-\frac{v^2}{c^2}} \to 1$; $m = m_0$

$K = mc^2 - m_0c^2$

as $v\uparrow$, $m\uparrow$ if m is very close to m_0, v is small, and $K \to 0$, but it agrees w/ $\frac{1}{2}m_0v^2$ b/c c is very large compared w/ v.

$$K = mc^2 - m_0c^2 = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0c^2$$

$$= m_0c^2\left(\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} - 1\right)$$

$x = -\frac{v^2}{c^2} \rightarrow$ Maclaurin, binomial $\binom{-\frac{1}{2}}{n}$

$$(1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}x^2 + \cdots$$

$$= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots$$

$$K = m_0c^2\left(\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots\right) - 1\right)$$

$$= m_0c^2\left(\frac{1}{2}\frac{v^2}{c^2} + \cdots\right)$$

$K \approx \frac{1}{2}m_0v^2$ if $v \ll c$

b) Taylor's inequality

$$|R_1(x)| \le \frac{M}{2!}x^2$$

$$f''(x) = \frac{3}{4}m_0c^2(1+x)^{-\frac{5}{2}} \qquad |v| \le 100\ m/s$$

$$= \frac{3m_0c^2}{4\left(1-\frac{100^2}{c^2}\right)^{\frac{5}{2}}} = M$$

$$|R_1(x)| \le \frac{1}{2}\cdot\frac{3m_0c^2}{4\left(1-\frac{100^2}{c^2}\right)^{\frac{5}{2}}}\cdot\frac{100^4}{c^4} < (4.17\times10^{-10})m_0$$