Ch. 14 - Vector Functions

§14.1 Vector Functions & Space Curves

Ex.1 N/A

Ex.2 $\lim_{t\to 0} \vec{r}(t)$; $\vec{r}(t) = (1+t^3)\hat{i} + te^{-t}\hat{j} + \frac{\sin t}{t}\hat{k}$

$\lim_{t\to 0} r(t) = \langle 1, 0, \lim_{t\to 0} \frac{\cos t}{1} = 1 \rangle$

$= \langle 1, 0, 1 \rangle$

Ex.3 $r(t) = \langle 1+t, 2+5t, -1+6t \rangle$

$t=0$ $\langle 1, 2, -1 \rangle$
1 $\langle 2, 7, 5 \rangle$
2 $\langle 3, 12, 11 \rangle$

$x = 1+t$, $y = 2+5t$, $z = -1+6t$

parametric equations for line

Ex.4 $r(t) = \langle \cos t, \sin t, t \rangle$

$z \to$ increases steadily

$x, y \to x^2 + y^2 = \cos^2 t + \sin^2 t = 1$

↳ cylinder; since the particle traces out ~ path; it is a helix

Ex.5 $PQ \to (1, 3, -2)$ $(2, -1, 3)$

$\vec{v} = \langle 1, -4, 5 \rangle \to$ dir. vector

$\vec{r} = r_0 + tv$ r_0 is P

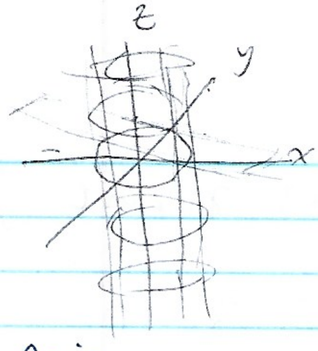
$$\left.\begin{aligned} x &= 1+t \\ y &= 3-4t \\ z &= -2+5t \end{aligned}\right\} \text{parametric}$$

$r(t) = (1+t)\hat{i} + (3-4t)\hat{j} + (-2+5t)\hat{k}$

(vector)

Ex.6 $x^2 + y^2 = 1$

$y + z = 2$



cylinder → circle → proj:

$x = \cos t$ $y = \sin t$ $0 \le t \le 2\pi$

$z = 2 - y = 2 - \sin t$

$r(t) = (\cos t)\hat{i} + \sin t\hat{j} + (2 - \sin t)\hat{k}$

Ex.7

$r(t) = \langle t, t^2, t^3 \rangle$

[computer graph]

~~Ex.8~~ §14.2 Derivatives + Integrals of vector functions

Ex.1

a) $\frac{dr}{dt} = (3t^2)\hat{i} - te^{-t}\hat{j} + 2\cos 2t\,\hat{k}$

b) $t = 0 \to r(0) = \langle 0, 0, 2 \rangle$

$u_0 = \frac{r(0)}{|r(0)|} = \frac{2\hat{k}}{2} = \hat{k}$

a) $r'(t) = (3t^2)\hat{i} + (1-t)e^{-t}\hat{j} + 2\cos 2t\,\hat{k}$

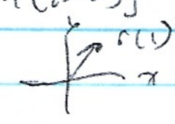
product rule

b) $r(0) = \langle 0, 1, 2 \rangle$

$u_0 = \frac{\hat{j} + 2\hat{k}}{\sqrt{1+4}} = \frac{1}{\sqrt{5}}\hat{j} + \frac{2}{\sqrt{5}}\hat{k}$

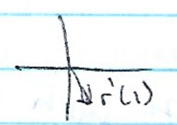
Ex.2 $r(t) = \sqrt{t}\,\hat{i} + (2-t)\hat{j}$

$r(1) = \hat{i} + \hat{j}$



$r'(t) = \frac{1}{2\sqrt{t}}\hat{i} - \hat{j}$

$r'(1) = \frac{1}{2}\hat{i} - \hat{j}$



Ex.3 $x = 2\cos t$ $y = \sin t$ $z = t$

$t = \frac{\pi}{2}$

$r'(t) = \langle -2\sin t, \cos t, 1 \rangle$

$\vec{v} = \langle -2, 0, 1 \rangle$
$\vec{r} = r_0 + t\vec{v}$

$x = -2t$ $y = 1$ $z = \frac{\pi}{2} + t$

Ex.4 if $|\vec{r}(t)| = c$, then $r'(t)$ is orthogonal to $r(t)$ for all t

$r(t) \cdot r(t) = |r(t)|^2 = c^2$

if $a \cdot b = 0$, they are orthogonal.

$\frac{d}{dt}(r(t) \cdot r(t)) = r'(t) \cdot r(t) + r(t) \cdot r'(t)$
$= 2r'(t) \cdot r(t) = 0$

$\therefore$ they are orthogonal

Ex.5 N/A

~~Ex.1~~

§13.3 Arc length & Curvature

Ex.1

$L = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1}\, dt$

$= \sqrt{2}\int_0^{2\pi} dt = 2\sqrt{2}\pi$

Ex.2 $r(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$

reparametrize with s

$\frac{ds}{dt} = |r'(t)| = \sqrt{2}$ $t = \frac{s}{\sqrt{2}}$

$s = s(t) = \int_0^t |r'(u)|\,du = \sqrt{2}\,t = s$

$r(t(s)) = \cos\left(\frac{s}{\sqrt{2}}\right)\hat{i} + \sin\left(\frac{s}{\sqrt{2}}\right)\hat{j} + \frac{s}{\sqrt{2}}\hat{k}$

Ex.3 curvature of a circle w/ radius a

$r(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j}$

$\to r'(t) = -a\sin t\,\hat{i} + a\cos t\,\hat{j}$

$|r'(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = \sqrt{a^2} = a$

$T(t) = \frac{r'(t)}{|r'(t)|} = -\sin t\,\hat{i} + \cos t\,\hat{j}$

$T'(t) = -\cos t\,\hat{i} - \sin t\,\hat{j}$

$|T'(t)| = 1$

$\kappa = \left|\frac{T'(t)}{r'(t)}\right| = \boxed{\frac{1}{a}}$ QED

Ex.4 $r(t) = \langle t, t^2, t^3 \rangle$

$r'(t) = \hat{i} + 2t\,\hat{j} + 3t^2\,\hat{k}$

~~$|r'(t)| = \sqrt{1 + 4t^2 + 6t^4}$~~

$r''(t) = 2\hat{j} + 6t\,\hat{k}$

$\kappa = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$

$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix}$

$= \langle 6t^2, -6t, 2 \rangle$

$\kappa = \frac{\sqrt{4 + 36t^2 + 36t^4}}{(1 + 4t^2 + 6t^4)^{3/2}} = \frac{2\sqrt{1 + 9t^2 + 9t^4}}{(1 + 4t^2 + 6t^4)^{3/2}}$

at $t = 0$,

$\kappa = \frac{2\sqrt{1}}{1^{3/2}} = 2$

EX 5 $y = x^2$

$$K(x) = \frac{|f''(x)|}{\left[1+(f'(x))^2\right]^{3/2}}$$

$f'(x) = 2x$

$f''(x) = 2$

$$K(x) = \frac{2}{(1+4x^2)^{3/2}}$$

$(0,0) \to K(0) = 2$

$(1,1) \to K(1) = \frac{2}{(5)^{3/2}} = \frac{2}{5\sqrt{5}}$

$(2,4) \to K(2) = \frac{2}{(17)^{3/2}} = \frac{2}{17\sqrt{17}}$

EX. 6 $r(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}$

$$T'(t) = \frac{r'(t)}{|r'(t)|} = \frac{-\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}}{\sqrt{2}}$$

$$|T'(t)| = \sqrt{\frac{\sin^2 t}{2} + \frac{1}{2}\cos^2 t + 1}$$

$$= \sqrt{\frac{3}{2}}$$

$$N(t) = \frac{-\sin t\,\hat{i} + \cos t\,\hat{j} + t\hat{k}}{\sqrt{3}}$$

$$T(t) = \frac{r(t)}{|r(t)|} = \frac{\cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}}{\sqrt{1+t^2}}$$

$B(t) = N(t) \times T(t)$

$$= \begin{vmatrix} i & j & k \\ \frac{-\sin t}{\sqrt{3}} & \frac{\cos t}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{\cos t}{\sqrt{1+t^2}} & \frac{\sin t}{\sqrt{1+t^2}} & \frac{t}{\sqrt{1+t^2}} \end{vmatrix}$$

$$= \frac{-\sqrt{3}}{3\sqrt{1+t^2}}\left(\hat{k}\cos^2 t - (t\hat{i} + \hat{j})\cos t + \sin t(\hat{k}\sin t - \hat{j}t + \hat{i})\right)$$

$$= \frac{-\sqrt{3}}{3\sqrt{1+t^2}}\left(\hat{k} - (t\cos t + \sin t)\hat{i} - (\cos t + t\sin t)\hat{j}\right)$$

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{\sqrt{2}}(-\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k})$$

$$T'(t) = \frac{1}{\sqrt{2}}(-\cos t\,\hat{i} - \sin t\,\hat{j})$$

$$N(t) = \frac{T'(t)}{|T'(t)|} = -\cos t\,\hat{i} - \sin t\,\hat{j}\ \left(\frac{1}{1}\right)$$

$$B(t) = T(t) \times N(t) = \frac{1}{\sqrt{2}}\begin{vmatrix} i & j & k \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}}(\sin t\,\hat{i} - \cos t\,\hat{j} + \hat{k})$$

EX. 7 $r(t) = \langle \cos t, \sin t, t \rangle$

~~osculating & binormal~~ vectors of $r(t)$

$T(t)$ & $N(t) \to$ osculating plane

$N(t)$ & $B(t) \to$ normal plane

normal plane $P(0, 1, \frac{\pi}{2})$

contains N & $B \to$ normal vector is $T(t)$?

normal plane $\to$ normal vector $r'(\frac{\pi}{2}) = \langle -1, 0, 1 \rangle$

plane $\Rightarrow -1(x-0) + 0(y-1) + 1(z - \frac{\pi}{2})$

$$z - x - \frac{\pi}{2} = 0$$

osculating plane $\to$ normal vector is $T \times N = B$

$$B(\tfrac{\pi}{2}) = \tfrac{1}{\sqrt{2}}\langle 1, 0, 1 \rangle = \left\langle \tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}} \right\rangle$$

plane $\Rightarrow \frac{1}{\sqrt{2}}(x-0) + 0(y-1) + \frac{1}{\sqrt{2}}(z - \frac{\pi}{2})$

$$\frac{x}{\sqrt{2}} + \frac{z}{\sqrt{2}} - \frac{\pi}{2\sqrt{2}} = 0$$

$$x + z - \frac{\pi}{2} = 0$$

Ex. 8 $y = x^2 \to$ osculating circle $(0,0)$

$\rho = \frac{1}{\kappa}$

$f(x) = x^2 \quad f'(x) = 2x \quad f''(x) = 2$

$$\kappa(0) = \frac{|2|}{(1)^{3/2}} = 2$$

$r = \frac{1}{2}$

tangent to origin $\to$ center is $\frac{1}{2}$ to y

$(0, \frac{1}{2})$

$$(x)^2 + (y - \tfrac{1}{2})^2 = \frac{1}{4}$$

§ 14.4 Motion in space: Velocity & Accel.

Ex. 1 $r(t) = t^3\hat{i} + t^2\hat{j}$

$r'(t) = 3t^2\hat{i} + 2t\hat{j} = v(t)$

$a(t) = r''(t) = 6t\hat{i} + 2\hat{j}$

$t = 1 \quad r(1) = \langle 1, 1 \rangle$

$v(1) = \langle 3, 2 \rangle$

$a(1) = \langle 6, 2 \rangle$

Speed $= |v(1)| = \sqrt{9+4} = \sqrt{13}$

Ex. 2 $r(t) = \langle t^2, e^t, te^t \rangle$

$v(t) = \langle 2t, e^t, (1+t)e^t \rangle$

$a(t) = \langle 2, e^t, e^t + (1+t)e^t \rangle$

$= \langle 2, e^t, (2+t)e^t \rangle$

$|v(t)| = \sqrt{4t^2 + e^{2t} + e^{2t} + t^2 e^{2t}}$

$= \sqrt{4t^2 + 2e^{2t} + t^2 e^{2t}}$

Ex. 3 $r(0) = \langle 1, 0, 0 \rangle$

$v(0) = \langle 1, -1, 1 \rangle$

$a(t) = \langle 4t, 6t, 1 \rangle$

$v(t) = \int a(t)\,dt = 2t^2\hat{i} + 3t^2\hat{j} + t\hat{k} + c$

$= (2t^2 + 1)\hat{i} + (3t^2 - 1)\hat{j} + (1+t)\hat{k}$

$r(t) = (\frac{2}{3}t^3 + t + 1)\hat{i} + (t^3 - t)\hat{j} + (\frac{1}{2}t^2 + t)\hat{k}$

Ex. 4 $r(t) = a\cos\omega t\,\hat{i} + a\sin\omega t\,\hat{j}$

$v(t) = -a\omega\sin\omega t\,\hat{i} + a\omega\cos\omega t\,\hat{j}$

$|v(t)| = \sqrt{a^2\omega^2\sin^2\omega t + a^2\omega^2\cos^2\omega t}$

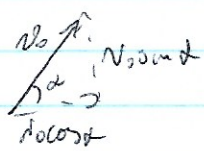
$= a\omega$

$$F_c = \frac{m|v(t)|^2}{r} = \frac{m \cdot a^2\omega^2}{a} = ma\omega^2$$

$a(t) = -a\omega^2\cos\omega t\,\hat{i} - a\omega^2\sin\omega t\,\hat{j}$

$F(t) = m\,a(t) = -ma\omega^2(\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j})$

Ex. 5



$v_0(t) = v_0\sin\alpha\,\hat{j} + v_0\cos\alpha\,\hat{i}$

$v_f = v_0 + at$

$v(t) = v_0\cos\alpha\,\hat{i} + v_0\sin\alpha\,\hat{j} - gt\,\hat{j}$

$= v_0\cos\alpha\,\hat{i} + (v_0\sin\alpha - gt)\hat{j}$

$r(t) = \int v(t)\,dt = v_0\cos\alpha\, t\,\hat{i} + (v_0\sin\alpha\, t - \frac{1}{2}gt^2)\hat{j}$

$d \to x$ when $y = 0$

$v_0\sin\alpha\, t - \frac{1}{2}gt^2 = 0$

$t(v_0\sin\alpha - \frac{1}{2}gt) = 0$

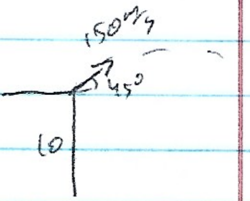
$$t = \frac{2v_0\sin\alpha}{g}$$

$$d_x = v_0\cos\alpha \cdot \frac{2v_0\sin\alpha}{g} = \frac{2v_0^2\sin\alpha\cos\alpha}{g}$$

$$= \frac{v_0^2\sin 2\alpha}{g}$$

max when $\sin 2\alpha = 1 \quad \alpha = 45$

Ex-6



$v_{0x} = v_0 \cos\theta = 150 \cos 45$

$v_{0y} = v_0 \sin\theta = 150 \sin 45$

$v_{fy}^2 = v_{0y}^2 + 2g\Delta y$

$= (150 \sin 45)^2 + 2(-9.8\ m/s^2)(-10\ m)$

$v_{fy} = 107\ m/s$

$v_f = v_0 + at$

$t = \frac{v_f - v_0}{a} = \frac{-107 - 150\sin 45}{-9.8\ m/s^2}$

$= 21.7\ s$

$v_{fx} = v_{0x} = 106.07\ m/s$

$d = v_{0x} t = (150 \cos 45)(21.7\ s)$

$= \boxed{2302\ m}$

$|v_f| = \sqrt{v_{fx}^2 + v_{fy}^2} = \sqrt{(150\cos 45)^2 + (107\ m/s)^2}$

$= \boxed{150.66\ m/s}$

Ex-7 $r(t) = \langle t, t^2, t^3 \rangle$

$r'(t) = \langle 1, 2t, 3t^2 \rangle$

$r''(t) = \langle 0, 2, 6t \rangle$

$a_T = \frac{r'(t) \cdot r''(t)}{|r'(t)|} = \frac{4t + 18t^3}{\sqrt{1 + 4t^2 + 9t^4}}$

$a_N = \frac{|r'(t) \times r''(t)|}{|r'(t)|}$

$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix}$

$= \langle 6t^2, -6t, 2 \rangle$

$a_N = \frac{\sqrt{36t^4 + 36t^2 + 4}}{\sqrt{1 + 4t^2 + 9t^4}} = \frac{2\sqrt{9t^4 + 9t^2 + 1}}{\sqrt{1 + 4t^2 + 9t^4}}$

$r(t) = \langle t^2, t^2, t^3 \rangle$ not $\langle t, t^2, t^3 \rangle$

$r'(t) = \langle 2t, 2t, 3t^2 \rangle$

$r''(t) = \langle 2, 2, 6t \rangle$

$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 2t & 2t & 3t^2 \\ 2 & 2 & 6t \end{vmatrix} = \langle 6t^2, -6t^2, 0 \rangle$

$a_T = \frac{4t + 4t + 18t^3}{|r'(t)|}$

$= \frac{8t + 18t^3}{\sqrt{8t^2 + 9t^4}} = \frac{8 + 18t^2}{\sqrt{8 + 9t^2}}$

$a_N = \frac{\sqrt{72t^4}}{\sqrt{8t^2 + 9t^4}} = \frac{6\sqrt{2}\, t^2}{t\sqrt{8 + 9t^2}}$

$= \frac{6\sqrt{2}\, t}{\sqrt{8 + 9t^2}}$