B 24 R5
B 8
BB 32
BBR 37 — T 37

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1. Consider $y(t)$ and $x(t)$. Obtain
   A. $\dfrac{dy}{dx}$, B. $\dfrac{d^2y}{dx^2}$
   in terms of $\dfrac{dy}{dt}$, $\dfrac{dx}{dt}$, $\dfrac{d^2y}{dt^2}$, or $\dfrac{d^2x}{dt^2}$.

2. Obtain x and y in terms of r and θ in polar coordinates. Demonstrate this in a figure.

3. For the cardioid $r = 1 + \sin\theta$,
   A. find the slope of the tangent line when $\theta = \dfrac{\pi}{3}$
   B. find the points on the cardioid where the tangent line is horizontal or vertical.
   Hint: You may need to use l'Hospital's Rule.

4. Prove that the area of an object is given by $A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$ where r is a function of θ.

5. Find the area enclosed by the four-leaved rose $r = \cos 2\theta$

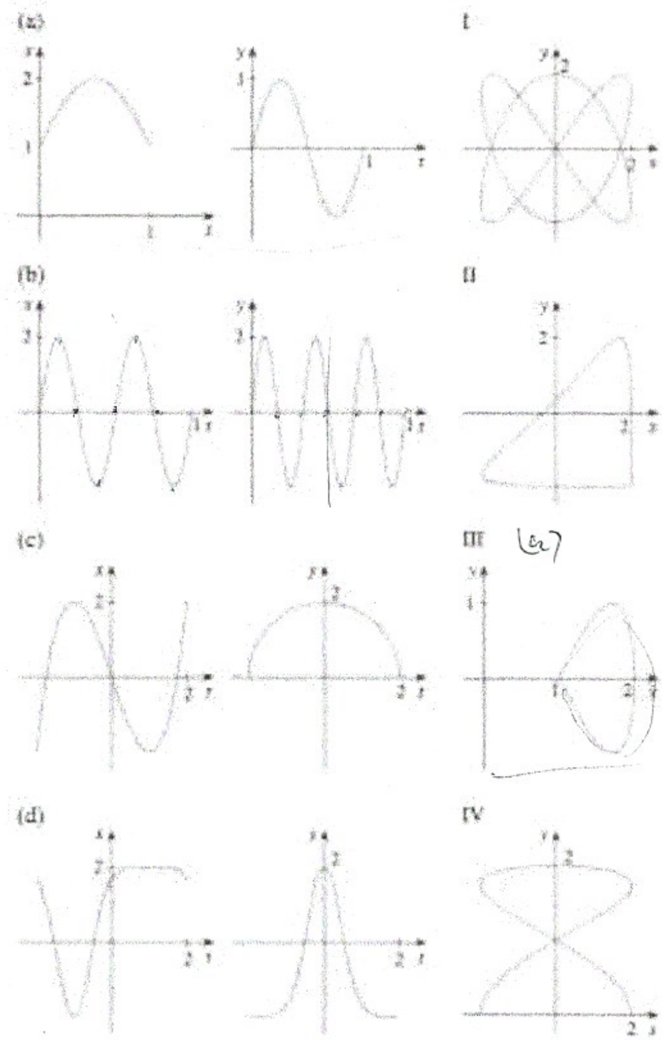
6. Find the foci and asymptotes of the hyperbola $ax^2 - by^2 = r$ in terms of a, b, r.

7.
   A. Show that the length formula of a curve C defined by the parametric equations *x=x(t), y=y(t)* is given by
   $$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$
   B. Use the result in part A to calculate the length of a curve defined by the a parametric equations $x=r\sin t$ and $y=r\cos t$ for $0 \le t < 2\pi$.

8. Obtain the expression for the surface area of revolution created by a line defined by the parametric *x=x(t), y=y(t)*
   A. when the line is revolved around the x-axis
   B. when the line is revolved around the y-axis.

9. If a projectile is fired with an initial velocity v_o at an angle α above the horizontal and air resistance is assumed to be negligible, then its position after t seconds is given by the parametric equations
   $x = v_o t\cos\alpha,\ \ y = v_o t\sin\alpha - \frac{1}{2}gt^2$ where g is the acceleration due to gravity.
   A. Obtain the maximum height the projectile reaches in terms of v_o, g, α.
   B. Show that the path is a parabola.

10. Match the graphs of the parametric equations *x=f(t), y=g(t)* in (a)-(d) with the parametric curves labeled I-IV. Give reasons for your choices.



11.

Mesut B. Çakır

J. Yang

$B + B = 32$

Bk	24	T_p	37
Bu	8	T_r	37
R	5		

24
13
37

Ch 11 Test

1) a) $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$ (chain rule)

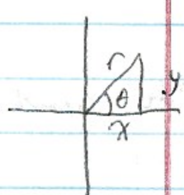
$= \boxed{\frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$ when $\frac{dx}{dt} \neq 0$ +1

b) $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt} \cdot \frac{dt}{dx}\left(\frac{dy}{dx}\right)$ (chain rule)

$= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$ $\frac{d}{dt}\left(\frac{dy}{dx}\right)$ ✓

Numerator: $\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$ ✓ ✓

$\frac{d^2y}{dx^2} = \boxed{\frac{\frac{d^2y}{dt^2}}{\left(\frac{dx}{dt}\right)\left(\frac{d^2x}{dt^2}\right)}}$ when $\frac{dx}{dt} \neq 0$ and $\frac{d^2x}{dt^2} \neq 0$ ✓

2) 

$x^2 + y^2 = r^2$

$x = r\cos\theta$ +1

$y = r\sin\theta$

3) when $\theta = \frac{\pi}{3}$, $r = 1 + \sin\frac{\pi}{3} = 1 + \frac{\sqrt{3}}{2} = \frac{2+\sqrt{3}}{2}$

$\left(\frac{2+\sqrt{3}}{2}, \frac{\pi}{3}\right)$ $x = r\cos\theta = \left(1 + \frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{\sqrt{3}}{4}$

$= \frac{2+\sqrt{3}}{4}$

$y = r\sin\theta = \left(\frac{2+\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2} + \frac{3}{4} = \frac{3+2\sqrt{3}}{4}$

$\left(\frac{2+\sqrt{3}}{4}, \frac{3+2\sqrt{3}}{4}\right)$

☆

$\frac{dy}{dx} = \frac{\frac{dy}{dr}}{\frac{dx}{dr}} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}\left((1+\sin\theta)\sin\theta\right)}{\frac{d}{d\theta}\left((1+\sin\theta)(\cos\theta)\right)}$ +1

$= \frac{\frac{d}{d\theta}(\sin\theta + \sin^2\theta)}{\frac{d}{d\theta}(\cos\theta + \sin\theta\cos\theta)}$

$= \frac{\cos\theta + 2\sin\theta\cos\theta}{-\sin\theta + \cos^2\theta - \sin^2\theta}$

$= \frac{\cos\theta + 2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta - \sin\theta}$ +1

when $\theta = \frac{\pi}{3}$, $\frac{dy}{dx} = \frac{\frac{1}{2} + 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2}}{\frac{1}{4} - \frac{3}{4} - \frac{\sqrt{3}}{2}}$

$= \frac{\sqrt{3}+1}{2} \Big/ -\frac{(\sqrt{3}+1)}{2} = \boxed{-1}$ +1

+3

b) 

horizontal: $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$

$\frac{dy}{d\theta} = \cos\theta + 2\sin\theta\cos\theta = 0$

$\cos\theta(1 + 2\sin\theta) = 0$

$\cos\theta = 0$ $\sin\theta = -\frac{1}{2}$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, -\frac{\pi}{6}, \frac{7\pi}{6}$ $\frac{11\pi}{6} = -\frac{\pi}{6}$ +1

at $\frac{3\pi}{2}$, $\frac{dx}{d\theta} = 0 - 1 + 1 = 0$

use l'hopital's rule: $\frac{-\sin\theta + 2\cos^2\theta - 2\sin^2\theta}{-2\sin\theta\cos\theta - 2\sin\theta\cos\theta - \cos\theta}$

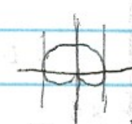
$\frac{1 + 0 - 2}{0 - 0 - 0} = -\frac{1}{0}$ ∴ it is vertical @ $\theta = \frac{3\pi}{2}$

horizontal: $\left(2, \frac{\pi}{2}\right), \left(\frac{1}{2}, -\frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{7\pi}{6}\right)$

Vertical: $\frac{dx}{d\theta} = 0$, $\frac{dy}{d\theta} \neq 0$

$\cos^2\theta - \sin^2\theta - \sin\theta = 0$

$\cos^2\theta =$ ~~[illegible]~~ $\sin\theta + \sin^2\theta$



from l'hopital's rule:

$-4\sin\theta\cos\theta + \cos\theta = 0$ +3

$\cos\theta(\cos\theta + 1) = 0$

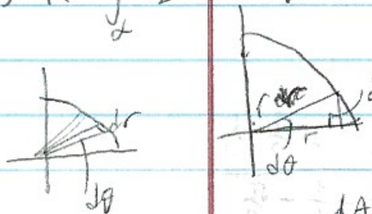
~~$\cos\theta = 0$~~ $\sin\theta = -\frac{1}{4}$

$\sin\theta = -\frac{1}{4}$

$\theta = -0.253,\ 3.395,\ \frac{3\pi}{2}$ ~

$(\frac{3}{4}, -0.253), (\frac{3}{4}, 3.395), (0, \frac{3\pi}{2})$ +1

4) $A = \int_\alpha^\beta \frac{1}{2} r^2\, d\theta$ when r is a function of θ



ds +1

$ds = r\,d\theta$

$dA = \frac{1}{2} bh = \frac{1}{2} r\,ds = \frac{1}{2} r^2 d\theta$

$\boxed{A = \int_\alpha^\beta \frac{1}{2} r^2 d\theta}$ +1

5) find one leaf

$\cos 2\theta = 0$

$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \ldots$

$A = 4 \cdot \frac{1}{2} \int_{\pi/4}^{3\pi/4} r^2 d\theta = 2 \int_{\pi/4}^{3\pi/4} \cos^2 2\theta\, d\theta$ ~ integrate by hand?

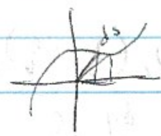
+1

$= \frac{\pi}{2}$ +1

6) $ax^2 - by^2 = r$ ~ ~

$\frac{ax^2}{r} - \frac{by^2}{r} = 1$? ~ ~

7) a) $L = \int_a^b ds$



~ ~ ~

$ds = r\,d\theta$

$r = \sqrt{x^2+y^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$ds = \sqrt{x^2+y^2}\, d\theta$

5

$L = \int dr$?

b) $L = \int_0^{2\pi} \sqrt{(r\cos t)^2 + (-r\sin t)^2}\, dt$ +1

$= \int_0^{2\pi} \sqrt{r^2(\sin^2 t + \cos^2 t)}\, dt$ +1

$= \int_0^{2\pi} r\,dt = rt\Big|_0^{2\pi} = \boxed{2\pi r}$ +1

circumference of a circle

when parameter equations are $x = r\sin t$ & $y = r\cos t$

8)

$dS = 2\pi r\,dl$

$r = y$

$dl = \sqrt{x'(t)^2 + y'(t)^2}\, dt$

$S = 2\pi \int_a^b y(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt$ Around x-axis

+1

$dS = 2\pi r\,dl$

$S = 2\pi \int_a^b x(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt$ around y

+1

9) $x = v_0 t\cos\alpha$ $y = v_0 t\sin\alpha - \frac{1}{2} g t^2$

a) find when $\frac{dy}{dx} = 0$, sub into y

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{v_0\sin\alpha - gt}{v_0\cos\alpha} = 0$

$\tan\alpha = \frac{gt}{v_0\cos\alpha}$

$t = \frac{v_0\cos\alpha\tan\alpha}{g} = \frac{v_0\sin\alpha}{g}$ +1

$y(t) = \frac{v_0\sin\alpha\, v_0\sin\alpha}{g} - \frac{1}{2}\frac{v_0^2\sin\alpha^2 \cdot g}{g^2}$

$= \boxed{\frac{v_0^2\sin\alpha^2}{2g}}$ +1

7

a) b) eliminate the parameter

$t = \frac{x}{v_0 \cos\alpha}$ +1

$\frac{1}{2} g t^2 - v_0 t \sin\alpha + y = 0$

quadratic → $t = \frac{v_0 \sin\alpha \pm \sqrt{v_0^2 \sin^2\alpha - 2gy}}{g}$

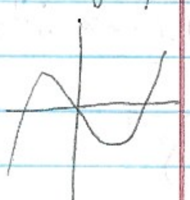
$\frac{gx}{v_0 \cos\alpha} = v_0 \sin\alpha \pm \sqrt{v_0^2 \sin^2\alpha - 2gy}$

takes on the form $x = \sqrt{y}$, or
$y = x^2$, which
is a parabola. +1

18) a) when $t = 0$, $x = 1$, $y = 0$

(x, y, t) $(1, 0, 0)$
$(2, 0, 0.5)$
$(1, 0, 1)$
$(1.5, 1, 0.25)$
↳ looks like graph III +1

b) by extension of part (a), they both start at (0,0) and vary sinusoidally
↳ graph I +1

c)   $(0, 2)$
$(-2, 0)$
↳ graph IV +1

d) must be graph II +1
$(2, 2)$, $(0, \text{almost } 0)$, etc.

6) $\frac{ax^2}{r} - \frac{by^2}{r} = 1 \rightarrow \frac{x^2}{\left(\frac{r}{a}\right)} - \frac{y^2}{\left(\frac{r}{b}\right)} = 1$

$c^2 = \frac{r^2}{a^2} + \frac{r^2}{b^2}$

foci: $\left(\sqrt{\frac{r^2}{a^2} + \frac{r^2}{b^2}}, 0\right), \left(-\sqrt{\frac{r^2}{a^2} + \frac{r^2}{b^2}}, 0\right)$

$= \left(r\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}, 0\right), \left(-r\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}, 0\right)$

asymptotes: $y = \pm \frac{b}{a} x$

$y = \frac{r}{b} \cdot \frac{a}{r} x = \frac{a}{b} x$

$y = \frac{a}{b} x$, $y = -\frac{a}{b} x$

7) a) $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ chain rule

$= \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2} \cdot \frac{dx}{dt} \cdot dt$

$= \int_\alpha^\beta \sqrt{1 + \frac{\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}} \cdot \frac{dx}{dt}\, dt$

$= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 \left(1 + \frac{\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}\right)}\, dt$

$= \int_\alpha^\beta \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$

alt. solution

6

3) $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$

$\frac{dr}{d\theta} = \cos\theta$ $\quad \frac{dy}{dx} = \frac{\sin\theta\cos\theta + \cos\theta + \sin\theta\cos\theta}{\cos^2\theta - \sin\theta - \sin^2\theta}$

(which is the same result from before) N/A

1) b) chain rule

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\cdot\frac{dt}{dx}\left(\frac{dy}{dx}\right)$ +1

$\frac{d^2y}{dx^2}\cdot\frac{dx}{dt} = \frac{d}{dt}\left(\frac{dy}{dx}\right)$

$= \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$

$\cancel{\frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}}$

$\frac{f'g - fg'}{g^2}$ +1

$\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{\frac{d^2y}{dt^2}\cdot\frac{dx}{dt} - \frac{dy}{dt}\cdot\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2}$ +1

$= \frac{d^2y}{dx^2}\cdot\frac{dx}{dt}$

$\frac{d^2y}{dx^2} = \frac{\frac{d^2y}{dt^2}\cdot\frac{dx}{dt} - \frac{dy}{dt}\cdot\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$ +1

+4

3) b) vertical: $\frac{dx}{d\theta} = 0$

$\cos^2\theta - \sin\theta - \sin^2\theta = 0$?

how to solve? ~

5) $2\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos^2 2\theta \, d\theta$ ~

6) 7/r $\quad ax^2 - by^2 = r \quad c = \sqrt{a^2+b^2}$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \frac{x^2}{(\sqrt{\frac{r}{a}})^2} - \frac{y^2}{(\sqrt{\frac{r}{b}})^2} = 1$ +1

foci: $\left(\pm\sqrt{\frac{r}{a} + \frac{r}{b}}, 0\right)$ +1 wow

asymptotes: $y = \pm\frac{b}{a}x$ +1

$y = \pm\frac{\sqrt{\frac{r}{b}}}{\sqrt{\frac{r}{a}}}x = \pm\sqrt{\frac{a}{b}}x$ +1 +4

3) b vert. $\frac{dx}{d\theta} = (1+\sin\theta)(1-2\sin\theta) = 0$

$\theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$ +1

5) $A = \int_0^{2\pi} \frac{1}{2}\cdot\frac{1}{2}(1+\cos 4\theta)\,d\theta$

$= \frac{1}{4}\left[1 + \frac{1}{4}\sin 4\theta\right]_0^{2\pi} = \frac{\pi}{2}$ +1

7) $ds^2 = ds\cdot ds = (i\,dx + j\,dy)\cdot(i\,dx + j\,dy)$

$= dx^2 + dy^2$

$ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$L = \int_d^f ds = \int_d^f \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ +3

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