Q2 Calc 2

John Yang

pts 46

TA 9

1) $e^x \to e^{i\theta} = \cos\theta + i\sin\theta$ +1

$e^x = \sum_{n=0} \frac{x^n}{n!}$

$e^{-i\theta} = \cos\theta - i\sin\theta$

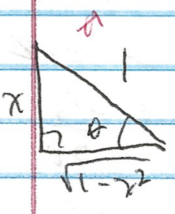
example: $\cos\theta_1 \sin\theta_2 = \frac{1}{2}(\sin(\theta_1+\theta_2) + \sin(\theta_1-\theta_2))$

4

b) given $\vec{a}\cdot\vec{b} = |a||b|\cos\theta$, $\cos\theta = \frac{a\cdot b}{|a||b|}$

and $|\vec{a}\times\vec{b}| = |a||b|\sin\theta$, you get law of cosines & law of sines.

c)



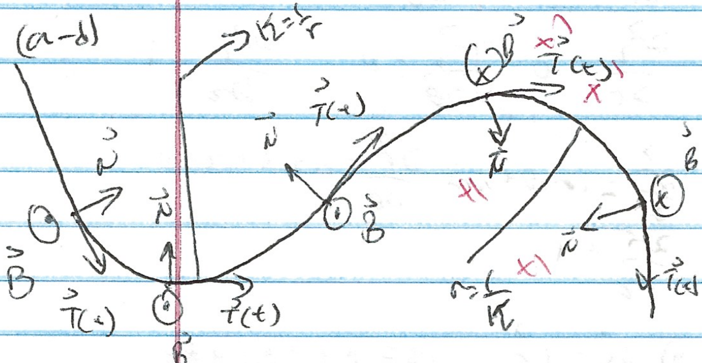
$a^2+b^2=c^2$

$\sin\theta = \frac{opp}{hyp} = x$ AH

$\cos\theta = \frac{adj}{hyp} = \sqrt{1-x^2}$ AH

$\tan\theta = \frac{x}{\sqrt{1-x^2}}$ +1 AH

2) (a-d)



d) $\frac{dB}{ds} = \vec{T}\times\frac{d\vec{N}}{ds}$

$\frac{dB}{ds} = \frac{d(\vec{T}\times\vec{N})}{ds} = \underbrace{\frac{d\vec{T}}{ds}\times\vec{N}} + \underbrace{\vec{T}\times\frac{d\vec{N}}{ds}}_{\text{remaining}}$ +1

$\frac{dT}{ds}\times\vec{N} = \underbrace{\frac{d\vec{T}}{dt}}_{=\vec{N}}\cdot\frac{dt}{ds}\times\vec{N} = \frac{dt}{ds}\,N\times N = 0$ +1

$\therefore \boxed{\frac{d\vec{B}}{ds} = \vec{T}\times\frac{d\vec{N}}{ds}}$ 9

3) $a(t) = \frac{d\vec{v}}{dt}$ $\quad \vec{v} = v\cdot\hat{v}$

$\frac{d\vec{v}}{dt} = \frac{d(v\cdot\hat{v})}{dt} = \frac{dv}{dt}\hat{v} + v\cdot\frac{d\hat{v}}{dt}$ +1

$\hat{v} = T(t) \Rightarrow \hat{v} = T = \frac{\vec{v}(t)}{v(t)}$

$\frac{d\vec{v}}{dt} = \vec{a} = \frac{dv}{dt}\hat{T} + v\cdot\frac{d\vec{T}}{dt}$
$= v'\vec{T} + v\vec{N}|T'|$ +1

$\frac{\frac{d\vec{T}}{dt}}{\left|\frac{dT}{dt}\right|} = \vec{N}$

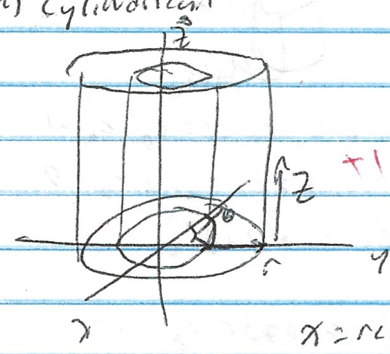
$\frac{d\vec{T}}{dt} = \vec{N}\left|\frac{dT}{dt}\right|$ +1

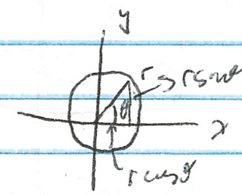
$\left|\frac{dT}{dt}\right| = \left|\frac{dT}{ds}\cdot\frac{ds}{dt}\right|$

$\left|\frac{dT}{ds}\right| = \kappa$ $\quad \frac{ds}{dt} = v$ $\quad \therefore |T'| = \kappa v$

$\therefore \frac{d\vec{v}}{dt} = v'T + v\vec{N}(\kappa v)$
$= v'T + \kappa v^2\vec{N}$ +1

4) a) cylindrical

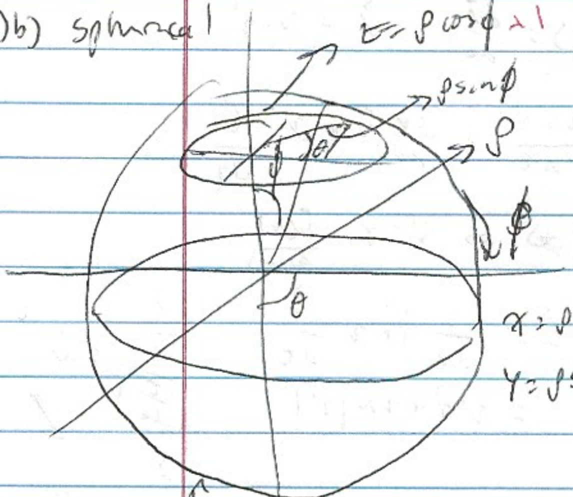




$x = r\cos\theta$ +1
$y = r\sin\theta$
$z = z$

6

u)b) spherical

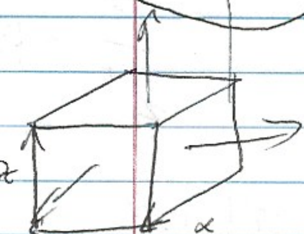


$z=\rho\cos\phi$ +1

$x=\rho\sin\phi\cos\theta$ +1

$y=\rho\sin\phi\sin\theta$ +1

c)



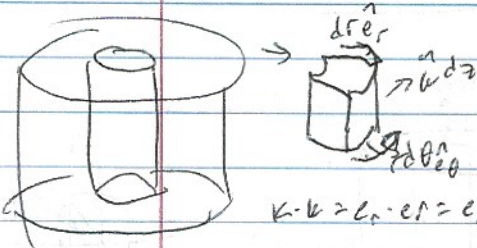
$dA_z = dxdy$ +1
$\hat{i}dx \times dy\hat{j}$
$= dxdy\,\hat{k}$

$dA_x = dy\hat{j} \times dz\hat{k}$ +1
$= dydz\,\hat{i}$

$dA_y = dx\hat{i} \times dz\hat{k}$ +1
$= dxdz\,\hat{j}$

$i\cdot i=1$ $i\cdot j=0$
$j\cdot j=1$ $i\cdot k=0$
$k\cdot k=1$ $j\cdot k=0$

d)



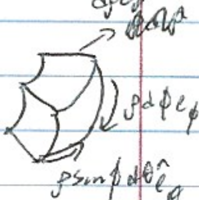
$k\cdot k = e_r\cdot e_r = e_\theta\cdot e_\theta = 1$
$e_r\cdot k = e_r\cdot e_\theta = k\cdot e_\theta = 0$

$dA_z = dr\hat{e}_r \times rd\theta\hat{e}_\theta = rdrd\theta\,\hat{k}$ +1

$dA_\theta = dz\hat{k} \times dr\hat{e}_r = dzdr\,\hat{e}_\theta$ +1

$dA_r = rd\theta\hat{e}_\theta \times dz\hat{k} = rd\theta dz\,\hat{e}_r$ +1

e)



$dA_\rho = \hat{e}_\phi\rho d\phi \times \rho\sin\phi d\theta\hat{e}_\theta$
$= \rho^2 d\phi\sin\phi d\theta e_\rho$ +1

$dA_\phi = \hat{e}_\theta\rho\sin\phi d\theta \times \hat{e}_\rho d\rho$
$= \rho d\rho\sin\phi d\theta\,\hat{e}_\phi$ +1

$dA_\theta = \hat{e}_\rho d\rho \times \hat{e}_\phi\rho d\phi =$
$\rho d\rho d\phi\,e_\theta$ +1

$e_\rho\cdot e_\rho = e_\theta\cdot e_\theta = e_\phi\cdot e_\phi = 1$
$e_\rho\cdot e_\theta = e_\rho\cdot e_\phi = e_\theta\cdot e_\phi = 0$

f) $dV = d\vec{A}_x\cdot dx\hat{i} = dxdydz$ +1

g) $dV = d\vec{A}_z\cdot dz\hat{k} = rdrd\theta dz$ +1

h) $dV = d\vec{A}_\rho\cdot d\rho\hat{e}_\rho = \rho^2\sin\phi\,d\rho d\phi d\theta$ +1

i)

$V = \iiint dV = \int_0^c\int_0^b\int_0^a dxdydz$ +1

[illegible]

j) $V = \int_0^h\int_{\theta_1}^{\theta_2}\int_a^b rdrd\theta dz$ +1

k) $V = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_a^b \rho^2\sin\phi\,d\rho d\phi d\theta$

TA1 Flipped θ & ϕ

5) $r(x,y,z) = x\hat{i} + y\hat{j} + z\hat{k}$

$x = r\cos\theta$ $y = r\sin\theta$ $z = z$

$\frac{\partial\vec{r}_x}{\partial r} = \cos\theta$ $\frac{\partial\vec{r}_x}{\partial\theta} = -r\sin\theta$ $\frac{\partial\vec{r}_x}{\partial z} = 0$

$\frac{\partial\vec{r}_y}{\partial r} = \sin\theta$ $\frac{\partial\vec{r}_y}{\partial\theta} = r\cos\theta$ $\frac{\partial\vec{r}_y}{\partial z} = 0$

$\frac{\partial\vec{r}_z}{\partial r} = 0$ $\frac{\partial\vec{r}_z}{\partial\theta} = 0$ $\frac{\partial\vec{r}_z}{\partial z} = 1$

+1 +1 +1

$\frac{\partial\vec{r}}{\partial r} = \cos\theta + \sin\theta$ $\frac{\partial\vec{r}}{\partial\theta} = -r\sin\theta + r\cos\theta$

$\frac{\partial\vec{r}}{\partial z} = 1$

b) $d\vec{s} = d\vec{r}$ $ds = dx + dy + dz$

$dx = (\cos\theta - r\sin\theta)\hat{i}$

$dy = (\sin\theta + r\cos\theta)\hat{j}$

$dz = (1)\hat{k}$

$ds = \hat{i}\sin\theta + \hat{j}\cos\theta + r(\cos\theta - \sin\theta)\hat{e}_r$

$d\vec{r} = d\vec{s}$
$= dr\hat{e}_r + rd\theta\hat{e}_\theta + \hat{k}dz$

c) ~~$ds = dr\hat{e}_r + rd\theta\hat{e}_\theta + \hat{k}dz$~~

~~$ds = dr(i\cos\theta + j\sin\theta) + rd\theta$~~

8 TA1

5) c) $\hat{r} = \hat{i}\sin\theta + \hat{j}\cos\theta$ ~

$\hat{\theta} = \cos\theta\,\hat{j} - \hat{i}\sin\theta$ +1

$\hat{h} = \hat{k}$

6) given $f(x,y)$, find the points where $f_x(a,b)=0$ & $f_y(a,b)=0$ +1 critical points.

Let $D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - (f_{xy}(a,b))^2$.

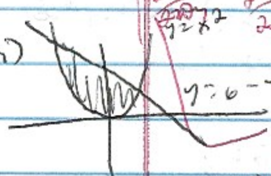
If $D>0$ & $f_{xx}(a,b)>0 \to$ local max (A)

If $D>0$ & $f_{xx}(a,b)<0 \to$ local min (A)

If $D<0 \to$ saddle points (A)

$D = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix} = f_{xx}(a,b)\, f_{yy}(a,b) - (f_{xy}(a,b))^2$.

8) a)



intersect @ $x=3,-2$ +1

$x=3 \to y=3$

$x=-2 \to y=8$

$y\,dx$ +1

$$\cancel{A = \int_{-2}^{3} (x^2+x-6)\,dx = \left[\tfrac{1}{3}x^3+\tfrac{1}{2}x^2-6x\right]_{-2}^{3} = 9+\tfrac{9}{2}-18+\tfrac{8}{3}-2-12}$$

$$= \int_{-2}^{3} (6+x-x^2)\,dx = \left[6x-\tfrac{1}{2}x^2-\tfrac{1}{3}x^3\right]_{-2}^{3}$$

$$= \frac{25}{6}$$

b) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\quad x=a\cos\theta$, $y=b\sin\theta$

$r^2 = x^2+y^2$

$r^2 = a^2\cos^2\theta + b^2\sin^2\theta$

$-a \le x \le a \quad -b \le y \le b$

+1

9) $f = xyz \quad \vec{F}(r) = y\hat{i} + z\hat{j} + x\hat{k}$

A) $\nabla f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$

B) $\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\cdot\left(y\hat{i}+z\hat{j}+x\hat{k}\right)$

$= \frac{\partial y}{\partial x} + \frac{\partial z}{\partial y} + \frac{\partial x}{\partial z}$

c) $\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$

$= \frac{\partial x}{\partial y}\hat{i} - \frac{\partial z}{\partial z}\hat{i} - \frac{\partial x}{\partial x}\hat{j} + \frac{\partial y}{\partial z}\hat{j} - \frac{\partial y}{\partial y}\hat{k} + \frac{\partial z}{\partial x}\hat{k}$

$= \frac{\partial x}{\partial y}\hat{i} - \hat{i} + \frac{\partial y}{\partial z}\hat{j} - \hat{j} + \frac{\partial z}{\partial x}\hat{k} - \hat{k}$

$= \left\langle \frac{\partial x}{\partial y} - 1, \frac{\partial y}{\partial z} - 1, \frac{\partial z}{\partial x} - 1 \right\rangle$

7) a) $\oint \vec{F}\cdot d\vec{r} = \iint (\nabla \times F)\cdot d\vec{A}$

$\oint \vec{F}\cdot d\vec{r} = \oint (F_x\,dx + F_y\,dy + F_z\,dz)$ +1

since $dr = dx + dy + dz$

$\vec{F} = F_x + F_y + F_z$

$= \iint \left(\frac{dF_x}{dx}dx + \frac{dF_y}{dx}dy + \frac{dF_z}{dx}dz\right)$ +1

$= \iint \Big[\left(\frac{\partial F_z}{\partial x}dz + \frac{\partial F_z}{\partial y}dy\right)dx + \left(\frac{\partial F_y}{\partial x}dx + \frac{\partial F_y}{\partial z}dz\right)dy + \left(\frac{\partial F_x}{\partial x}dx + \frac{\partial F_z}{\partial y}dy\right)dz\Big]$ +1

51A3

2(a) cont'd

$$= \iint \frac{\partial F_x}{\partial z} dz\,dx + \frac{\partial F_x}{\partial y} dy\,dx + \frac{\partial F_y}{\partial x} dx\,dy$$
$$+ \frac{\partial F_y}{\partial z} dz\,dy + \frac{\partial F_z}{\partial x} dx\,dz + \frac{\partial F_z}{\partial y} dy\,dz$$

$dz\,dx = -dx\,dz = \hat{j}\cdot d\vec{s}$

$dx\,dy = -dy\,dx = \hat{k}\cdot d\vec{s}$

$dy\,dz = -dz\,dy = \hat{i}\cdot d\vec{s}$

$$= \iint \frac{\partial F_x}{\partial z} dz\,dx - \frac{\partial F_z}{\partial x} dz\,dx + \frac{\partial F_y}{\partial x} dx\,dy - \frac{\partial F_x}{\partial y} dx\,dy$$
$$+ \frac{\partial F_z}{\partial y} dy\,dz - \frac{\partial F_y}{\partial z} dy\,dz$$

$$= \iint \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) dz\,dx + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dx\,dy$$
$$+ \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) dy\,dz \quad +1$$

$$= \iint \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) \hat{j}\cdot d\vec{s} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \hat{k}\cdot d\vec{s}$$
$$+ \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) \hat{i}\cdot d\vec{s}$$

$$\nabla \times F = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{j}$$
$$+ \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{k}$$

TA 1 found $\nabla\times F$ as a whole rather than the individual x, y, z

$$\therefore \iint (\nabla \times F)\cdot d\vec{s} = \oint F\cdot d\vec{r}$$

b) div. theorem

$$\oiint \vec{F}\cdot d\vec{s} = \iiint \nabla\cdot F\, dV$$

$$\oiint \vec{F}\cdot d\vec{s} = \oiint F_x\, dS_x + F_y\, dS_y + F_z\, dS_z \quad +1$$

$$= \iiint \partial F_x\, dy\,dz + \partial F_y\, dx\,dz + \partial F_z\, dx\,dy$$

TA 1 $\iint (F_x\, dy\,dz + F_y\, dz\,dx + F_z\, dx\,dy)$ is replaced in this step

since $dS_x = dy\,dz$

$dS_y = dz\,dx = -dx\,dz$

$dS_z = dx\,dy$

$$= \iiint \left(\frac{\partial F_x}{\partial x} + \cancel{\frac{\partial F_x}{\partial y}} + \cancel{\frac{\partial F_x}{\partial z}}\right) dy\,dz + \left(\cancel{\frac{\partial F_y}{\partial x}} + \frac{\partial F_y}{\partial y} + \cancel{\frac{\partial F_y}{\partial z}}\right) dx\,dz$$
$$+ \left(\cancel{\frac{\partial F_z}{\partial x}} + \cancel{\frac{\partial F_z}{\partial y}} + \frac{\partial F_z}{\partial z}\right) dx\,dy$$

+1 3 expanded. steps

$$= \iiint \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) dx\,dy\,dz \quad +1$$

$dV = dx\,dy\,dz$

$$\nabla\cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\therefore \oiint F\cdot dS = \iiint (\nabla\cdot \vec{F})\, dV$$

~~2)~~

(3) $\nabla\times B = \mu_0 J$

$$\iint (\nabla\times F)\, dS = \oint F\cdot dr$$

$$\iint (\nabla\times B)\cdot dS = \oint B\cdot dl$$

$$= \iint \mu_0 J\cdot dS =$$

7) a) $V(r,\theta,z) = r^2(\theta_2-\theta_1)z$

$= r^2\theta z$

b) $A(r,\theta,z) = 2r^2\theta + 2rz + r\theta z$

1 TA 1

3 TA 1

8) c) $r = a\sin\theta$, $r = b\sin\theta$ $2\pi, \pi$

$\frac{\pi}{4} \le \theta \le \pi$ $a \le r \le b$

+1

$$A = \frac{1}{2}\int_{\pi/4}^{\pi} (b\sin\theta - a\sin\theta)^2\, d\theta$$

(if $b > a$)

b) $z = 2z - 1$ $r = \sqrt{z}$, $r = \sqrt{2z-1}$

$z = 1$ elliptic paraboloids

$0 \le \theta \le 2\pi$

$x^2 + y^2 = z$

$x^2 + y^2 = 2z - 1$

$r^2 = x^2 + y^2$

$r^2 = z$

$r = \sqrt{2z-1}$

$dv = r\,dr\,d\theta\,dz$

$$V = \int_0^1\int_0^{2\pi}\int_{\sqrt{z}}^{\sqrt{2z-1}} r\,dr\,d\theta\,dz$$

(1a) $F(x) = \int_{ax}^{bx^2} (t^2 + x^2)\,dt$ $g(t) = t^2 + x^2$

$$\frac{dF(x)}{dx} = \frac{d}{dx}\int_{ax}^{bx^2} (t^2 + x^2)\,dt = g(x)$$

$u(x) = ax$ $v(x) = bx^2$

$$\frac{dF}{dx} = \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial u}\frac{du}{dx} + \frac{\partial \Phi}{\partial v}\frac{dv}{dx}$$

$\frac{du}{dx} = a$ $\frac{dv}{dx} = 2bx$

$$\frac{dF}{dx} = \int_{u(x)}^{v(x)} \frac{\partial g(x,t)}{\partial x}\,dt + \frac{dv}{dx} g(x, v(x)) - \frac{du}{dx} g(x, u(x))$$

$\frac{\partial g(x,t)}{\partial x} = 2x$ $\int_{ax}^{bx^2} 2x\,dt = 2x\left[t\right]_{ax}^{bx^2} = 2x(bx^2 - ax)$

+1 $\boxed{2x(bx^2 - ax)}$

$\frac{dv}{dx} g(x, v(x)) = 2bx(b^2x^4 + x^2)$

$t = bx^2$ +1 b/c they are

$-\frac{du}{dx} g(x, u(x)) = -a(a^2 + x^2)$ right except the 3rd term.

~~$\frac{dF}{dx} = 2x(bt^2 - at)$~~

$$\frac{dF}{dx} = 2x(bx^2 - ax) + 2bx(b^2x^4 + x^2) - a(a^2 + x^2)$$

10) $\vec{J} = \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ $\frac{\partial r}{\partial u}du$

$x, y, z \to u, v, w$ u, $\frac{\partial r}{\partial v}dz$, v

$d\vec{s} = \hat{e}_u du \times \hat{e}_v dv$

$= \underbrace{e_u \times e_v}\, du\,dv$

$= J \to \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial x} & \frac{\partial y}{\partial x} & \frac{\partial z}{\partial x} \end{vmatrix}$

(10) a) $\hat{e}_u = \frac{\partial r}{\partial u}$

$\hat{e}_v = \frac{\partial r}{\partial v}$

$$d\vec{s} = \left(\frac{\partial r}{\partial u}du\right) \times \left(\frac{\partial r}{\partial v}dv\right)$$

$$= \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right) du\,dv$$

$$= \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right) dA$$

+1

b) $dV = dx\,dy\,dz = |d\vec{s} \cdot d\vec{h}| = \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right) dA \cdot \frac{\partial r}{\partial w} dw$

$$= \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right) \cdot \left(\frac{\partial r}{\partial w}\right) du\,dv\,dw$$

+1

1

2 + 2

c) ~~$A(S) = \iint \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| dA$~~

$A(S) = \iint |dS| = \iint \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| du\,dv$

$= \iint_D \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| dA$

TA 1

d) $z = f(x,y)$ $\quad A(S) = \iint_D \left| \frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} \right| dA$

$$= \iint \left| \begin{matrix} i & j & k \\ \frac{\partial x}{\partial x} & \frac{\partial y}{\partial x} & \frac{\partial z}{\partial x} \\ \frac{\partial x}{\partial y} & \frac{\partial y}{\partial y} & \frac{\partial z}{\partial x} \end{matrix} \right| dA$$

$$= i\left(\frac{\partial y}{\partial x} \cdot \cancel{\frac{\partial z}{\partial y}} - \frac{\partial z}{\partial x} \cancel{\frac{\partial y}{\partial y}} \right) + j\left(\frac{\partial z}{\partial x} \frac{\partial x}{\partial y} - \cancel{\frac{\partial x}{\partial x}} \frac{\partial z}{\partial x} \right) + k\left(\cancel{\frac{\partial x}{\partial x}} \cancel{\frac{\partial y}{\partial y}} - \frac{\partial y}{\partial y} \frac{\partial y}{\partial x} \right)$$

$$= i\left(\frac{\partial y}{\partial x} - \frac{\partial z}{\partial x} \right) + j\left(\frac{\partial z}{\partial x} \frac{\partial x}{\partial y} \right) + k\left(1 - \frac{\partial x}{\partial y} \frac{\partial y}{\partial x} \right)$$

$$= \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

TA 1