Chapter 11 Notes - MC

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11 Parametric Equations and Polar Coordinates

11.1 Curves Defined by Parametric Equations

- Parameter - 3rd variable that x and y are both a function of:

$$x = f(t) \text{ and } y = g(t)$$

- Points along the curve $(x, y) = (f(t), g(t))$
- Graphing calculators can be used to produce parametric curves that you wouldn't be able to make by hand.
- Equation 1: parametric equations for a cycloid:

$$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta) \qquad \theta \in \mathbb{R}$$

11.2 Calculus with Parametric Curves

- Equation 1: first derivative of a parametric equation:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{if} \quad \frac{dx}{dt} \neq 0$$

- Second derivative of a parametric equation:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \neq \frac{\frac{d^2}{dt^2}}{\frac{d^2x}{dt^2}}$$

- Equation 2: arc length of a curve:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

- Equation 3/Theorem 5: arc length of a parametric curve:

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

- Equation 6: surface area of a rotated parametric curve about the x axis:

$$S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

11.3 Polar Coordinates

- polar coordinates - (r, θ)
- Theta is always ccw
- Equations 1 and 2: polar coordinates:

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

- Derivative of a polar curve:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

11.4 Areas and Lengths in Polar Coordinates

- Equation 1: area of a sector of a circle: $A = \frac{1}{2}r^2\theta$
- Equations 3 and 4: polar area:

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta = \int_a^b \frac{1}{2} r^2 d\theta$$

- Equation 5: polar arc length:

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

11.5 Conic Sections

- Equation 1: vertical parabola with focus $(0, p)$ and directrix $y = -p$:

$$x^2 = 4py$$

- Equation 2: horizontal parabola with focus $(p, 0)$ and directrix $x = -p$:

$$y^2 = 4px$$

- Equation 3: general form of an ellipse:
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
- Equation 4: horizontal ellipse with foci $(\pm c, 0)$, verticies $(\pm a, 0)$, where $c^2 = a^2 - b^2$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a \geq b > 0$$
- Equation 5: vertical ellipse with foci $(0, \pm c)$, verticies $(0, \pm a)$, where $c^2 = a^2 - b^2$
$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad a \geq b > 0$$
- Equation 6: general form of a hyperbola:
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
- Equation 7: hyperbola with horizontal transverse axis, with foci $(\pm c, 0)$, verticies $(\pm a, 0)$, asymptotes $y = \pm \frac{b}{a} x$, where $c^2 = a^2 + b^2$:
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
- Equation 8: hyperbola with vertical transverse axis, foci $(0, \pm c)$, verticies $(0, \pm a)$, asymptotes $y = \pm \frac{a}{b} x$, where $c^2 = a^2 + b^2$:
$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

11.6 Conic Sections in Polar Coordinates

- Theorem 1: Let F be a fixed point (called the focus) and l be a fixed line (called the directrix) in a plane. Let e be a fixed positive number (called the eccentricity). The set of all points P in the plane such that
$$\frac{|PF|}{|Pl|} = e$$
is a conic section. (That is, the ratio of the distance from F to the distance from l is the constant e). The conic is:
  - (a) an ellipse if $e < 1$
  - a parabola if $e = 1$
  - a hyperbola if $e > 1$
- Theorem 6: A polar equation of the form
$$r = \frac{ed}{1 \pm e \cos\theta} \qquad \text{or} \qquad r = \frac{ed}{1 \pm e \sin\theta}$$
represents a conic section with eccentricity e. The conic is an ellipse if $e < 1$, parabola if $e = 1$, or a hyperbola if $e > 1$
  - d is the distance from focus to directrix
- $e = \frac{c}{a} \qquad \text{where} \qquad c^2 = a^2 + b^2$
- Kepler's laws:

- 1 - A planet revolves around the sun in an elliptical orbit with the sun at one focus.
- 2 - The line joining the sun to a planet sweeps out equal areas in equal times.
- 3 - The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

- Equation 7: The polar equation of an ellpise with focus at the origin, semimajor axis a, eccentricity e, and directirx $x = d$ can be written in the form:

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

- Equation 8: The perihelion distance from a planet to the sun is $a(1-e)$ and the aphelion distance is $a(1+e)$