Chapter 2 Notes - LA

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2 Systems of Linear Equations

2.1 Introduction to Systems of Linear Equations

- A linear equation in the n variables $x_1, x_2, x_3, \cdots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

  where the coefficients $a_1, a_2, \cdots, a_n$ and the constant term b are constants.

- A solution of a linear equation $a_1x_1+a_2x_2+\cdots+a_nx_n = b$ is a vector $[s_1, s_2, \cdots, s_n]$ whose components satisfy the equation when we substitute $x_1 = s_1, x_2 = s_2, \cdots, x_n = s_n$.

- A system of linear equations is a finite set of linear equations, each with the same variables. A solution of a system of linear equations is a vector that is simultaneously a solution of each equation in the system. The solution set of a system of linear equations is the set of all solutions of the system.

- A system of linear equations is called consistent if it has at least one solution. A system with no solutions is inconsistent.

- Two linear systems are called equivalent if they have the same solution sets.

- Solving a matrix with a CAS may not always be the best choice.

2.2 Direct Methods for Solving Linear Systems

- The coefficient matrix contains the coefficients of the variables, and the augmented matrix is the coefficient matrix augmented by an extra column containing the constant terms.

- A matrix is in row echelon form if it satisfies the following properties:

  - Any rows consisting entirely of zeroes are at the bottom
  - In each nonzero row, the first nonzero entry (called the leading entry) is in a column to the left of any leading entries below it.

- Elementary row operations:

  - Interchange two rows
  - Multiply a row by a nonzero constant
  - Add a multiple of a row to another row

- Matrices A and B are row equivalent if there is a sequence of elementary row operations that converts A into B.

- Matrices A and B are row equivalent IFF they can be reduced to the same row echelon form.

- Gaussian elimination:
  - Write the augmented matrix of the system of linear equations.
  - Use elementary row operations to reduce the augmented matrix to row echelon form.
  - Using back substitution, solve the equivalent system that corresponds to the row-reduced matrix.

- The rank of a matrix is the number of nonzero rows in its row echelon form.

- The rank theorem: let A be the coefficient matrix of a system of linear equations with n variables. If the system is consistent, then

$$\text{number of free variables} = n - \text{rank}(A)$$

- A matrix is in reduced row echelon form if it satisfies the following:
  - It is in row echelon form.
  - The leading entry in each nonzero row is a 1 (called a leading 1)
  - Each column containing a leading 1 haas zeroes everywhere else.

- Steps for Gauss-Jordan Elimination:
  - Write the augmented matrix of the system of linear equations.
  - Use elementary row operations to reduce the augmented matrix to RREF
  - If the resulting system is consistent, solve for the leading variables in terms of any remaining free variables.

- A system of linear equations is called homogeneous if the constant term in each equation is zero.

- Theorem: If $[A|\mathbf{0}]$ is a homogeneous system of m linear equations with n variables, where $m < n$, then the system has infinitely many solutions.

2.3 Spanning Sets and Linear Independence

- Theorem: A system of linear equations with the augmented matrix $[A|\mathbf{b}]$ is consistent IFF $\mathbf{b}$ is a linear combination of the columns of A.

- Definition: If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ is a set of vectors in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ is called the span of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ and is denoted by $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k)$ or $\text{span}(S)$. If $\text{span}(S) = \mathbb{R}^n$, then S is called a spanning set for $\mathbb{R}^n$.

- Definition: A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ is linearly dependent if there are scalars $c_1, c_2, \cdots, c_k$, *at least one of which is not zero*, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

  A set of vectors that is not linearly dependent is called linearly independent.

- Theorem: Vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m$ in $\mathbb{R}^n$ are linearly dependent IFF at least one of the vectors can be expressed as a linear combination of the others.

- Theorem: Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m$ be (column) vectors in $\mathbb{R}^n$ and let A be the $n \times m$ matrix $[\, v_1 \mathbf{v}_2 \ \cdots \ \mathbf{v}_m]$ with these vectors as its columns. Then $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m$ are linearly dependent IFF the homogeneous linear system with augmented matrix $[A|\mathbf{o}]$ has a nontrivial solution.

- Theorem: Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m$ be (row) vectors in $\mathbb{R}^n$ and let $m \times n$ matrix $\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{bmatrix}$ with these vectors is its rows. Then $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m$ are linearly dependent IFF $\text{rank}(A) < m$.

- Theorem: Any set of m vectors in $\mathbb{R}^n$ is linearly dependent if $m > n$.

2.4 Applications

- Applications include:
  - Allocation of resources
  - Balancing chemical equations
  - Network analysis in transportation, economics, electricity and magnetism

2.5 Iterative Methods for Solving Linear Systems

- Two iterative methods: Jacobi's method and Gauss-Seidel method

- Theorem: If a set of n linear equations in n variables has a strictly diagonally dominant coefficient matrix, then it has a unique solution and both the Jacobi and Gauss-Seidel method converge to it.

- Theorem: If the Jacobi or the Gauss-Seidel method converges for a system of n linear equations in n varibales, then it must converge to the solution of the system.