Constraint Satisfaction Problem

LESSON 5

Reading

Chapter 5

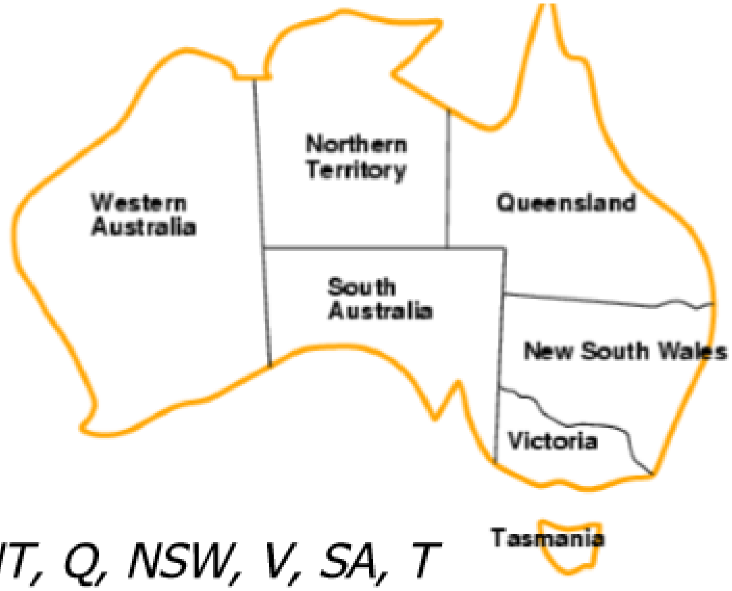
Outline

- Constraint satisfaction problems?
- Backtracking
- Variable ordering and value selection
- Forward checking
- Constraint propagation
- Iterative min-conflicts

Constraint satisfaction problems (CSPs)

- Standard search problem:
  - state is a "black box" – any data structure that supports successor function, heuristic function, and goal test
- CSP:
  - state is defined by variables X_i with values from domain D_i
  - goal test is a set of constraints specifying allowable combinations of values for subsets of variables
- Simple example of a formal representation language
- Allows useful general-purpose algorithms with more power than standard search algorithms

Example: Map-Coloring



- Variables: *WA, NT, Q, NSW, V, SA, T*
- Domains: D_i = {red,green,blue}
- Constraints: adjacent regions must have different colors

- e.g., WA ≠ NT, or (WA,NT) in {(red,green),(red,blue),(green,red), (green,blue),(blue,red),(blue,green)}

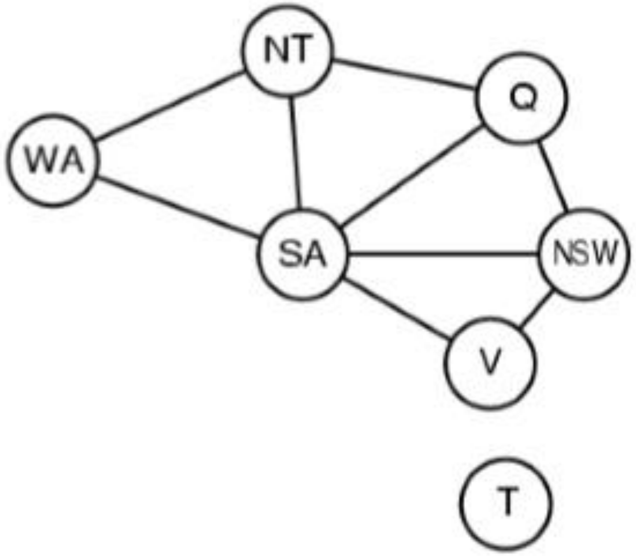
Example: Map-Coloring

- Solutions are complete and consistent assignments, e.g., WA = red, NT = green,Q = red, NSW = green, V = red, SA = blue, T = green

Constraint graph

Binary CSP: each constraint relates two variables

Constraint graph: nodes are variables, arcs are constraints



Varieties on the CSP formalism

- Discrete variables
  - finite domains:
    - n variables, domain size d -> O(d^n) complete assignments
    - e.g., Boolean CSPs, incl.~Boolean satisfiability (NP-complete)
  - infinite domains:
    - integers, strings, etc.
    - e.g., job scheduling, variables are start/end days for each job
    - need a constraint language, e.g., StartJob 1 + 5 ≤ StartJob 3
- Continuous variables
    - e.g., start/end times for Hubble Space Telescope observations
    - linear constraints solvable in polynomial time by linear programming

Varieties of constraints

- Unary constraints involve a single variable,
  - e.g., SA ≠ green
- Binary constraints involve pairs of variables,
  - e.g., SA ≠ WA
- Higher-order constraints involve 3 or more variables,
  - e.g., cryptarithmetic column constraints

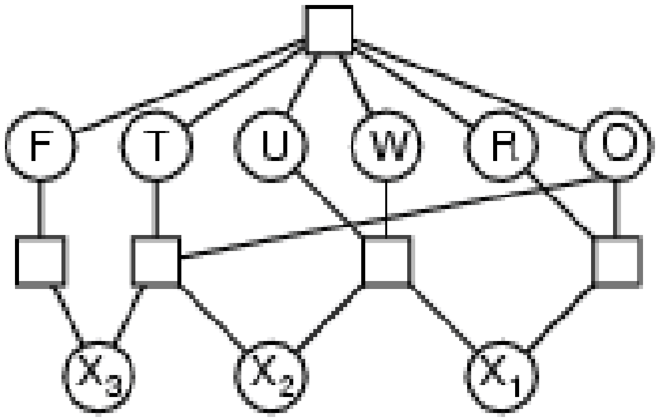
Example: Sudoku

		3		2		6		
9			3		5			1
		1	8		6	4		
		8	1		2	9		
								8
		6	7		8	2		
		2	6		9	5		
8			2		3			9
		5		1		3		

- Variables: *up to 81 variables*
- Domains: *{0,1,2,3,4,5,6,7,8,9}*
- Constraints: *Alldiff (…) * 27 (columns, rows, boxes)*

Example: Cryptarithmetic

```
    T W O
  + T W O
  -------
  F O U R
```



Variables: $F\ T\ U\ W$
$R\ O\ X_1\ X_2\ X_3$

Domains: $\{0,1,2,3,4,5,6,7,8,9\}$

Constraints: *Alldiff (F,T,U,W,R,O)*

- $O + O = R + 10 \cdot X_1$
- $X_1 + W + W = U + 10 \cdot X_2$
- $X_2 + T + T = O + 10 \cdot X_3$
- $X_3 = F,\ T \neq 0,\ F \neq 0$

Real-world CSPs

- Assignment problems
  - e.g., who teaches what class
- Timetabling problems
  - e.g., which class is offered when and where?
- Transportation scheduling
- Factory scheduling

- Many real-world problems involve real-valued variables
- Many problems also feature preferences (I don’t want to on Monday morning)

Standard search formulation (incremental)

Let's start with the straightforward approach, then fix it

States are defined by the values assigned so far

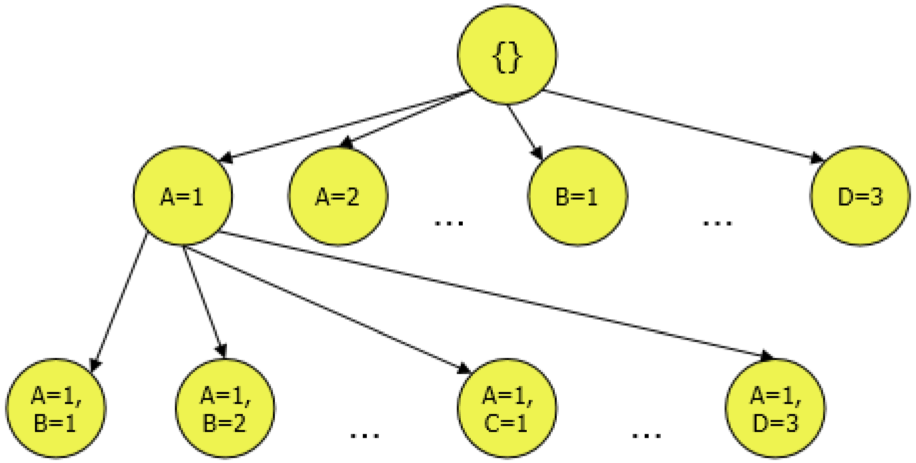
- Initial state: the empty assignment { }
- Successor function: assign a value to an unassigned variable that does not conflict with current assignment->fail if no legal assignments
- Goal test: the current assignment is complete

This is the same for all CSPs 2. Every solution appears at depth n with n variables -> use depth-first search

Path is irrelevant, so can also use complete-state formulation

CSP Search tree size

$b = (n - \ell)d$ at depth ℓ, hence $n! \cdot d^n$ leaves



Variables: A,B,C,D
Domains: 1,2,3

Depth 1: 4 variables x 3 domains = 12 states

Depth 2: 3 variables x 3 domains = 9 states

Depth 3: 2 variables x 3 domains = 6 states

Depth 4: 1 variable x 3 domains = 3 states (leaf level)

Backtracking search

- Variable assignments are commutative, i.e.,

[WA = red then NT = green] same as [NT = green then WA = red]

- Only need to consider assignments to a single variable at each node
  - Fix an order in which we'll examine the variables

-> $b = d$ and there are d^n leaves

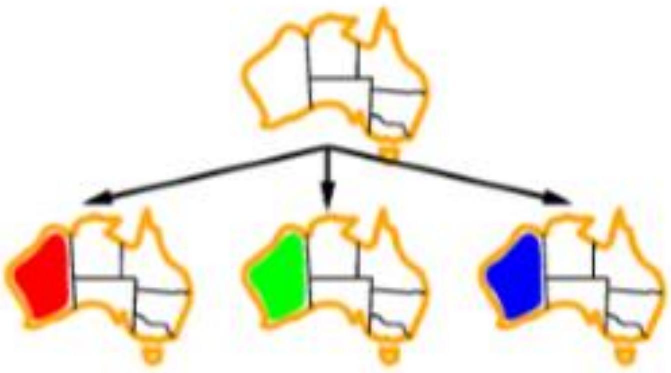
- Depth-first search for CSPs with single-variable assignments is called backtracking search
  - Is the basic uninformed algorithm for CSPs
  - Can solve n-quens for n ~ 25

Backtracking search

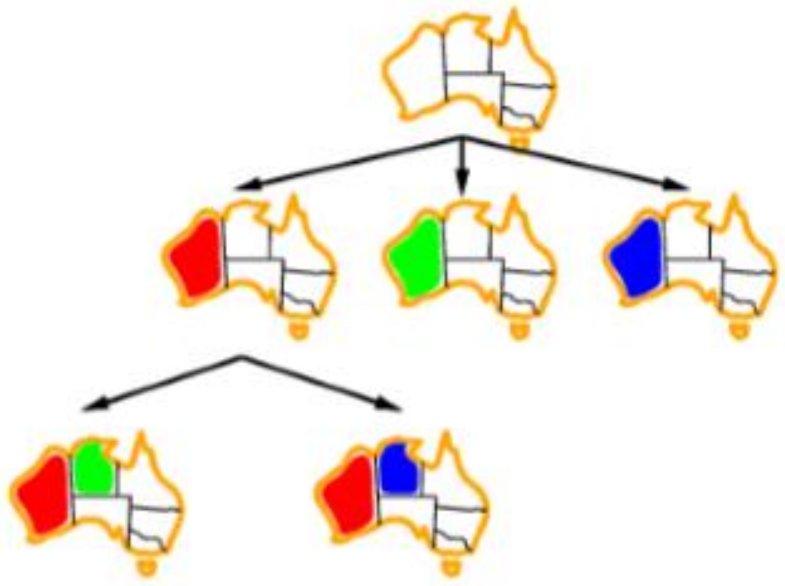
```
function BACKTRACKING-SEARCH( csp) returns a solution, or failure
    return RECURSIVE-BACKTRACKING({}, csp)

function RECURSIVE-BACKTRACKING( assignment,csp) returns a solution, or
failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE( Variables[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment according to Constraints[csp] then
            add { var = value } to assignment
            result ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failue then return result
            remove { var = value } from assignment
    return failure
```

Backtracking example

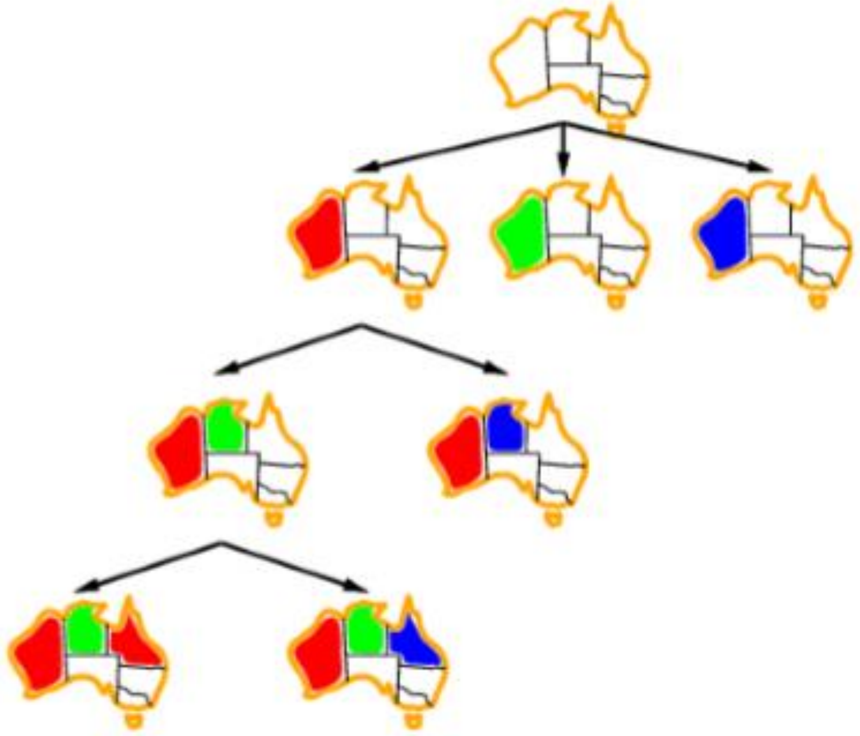
Backtracking example



Backtracking example



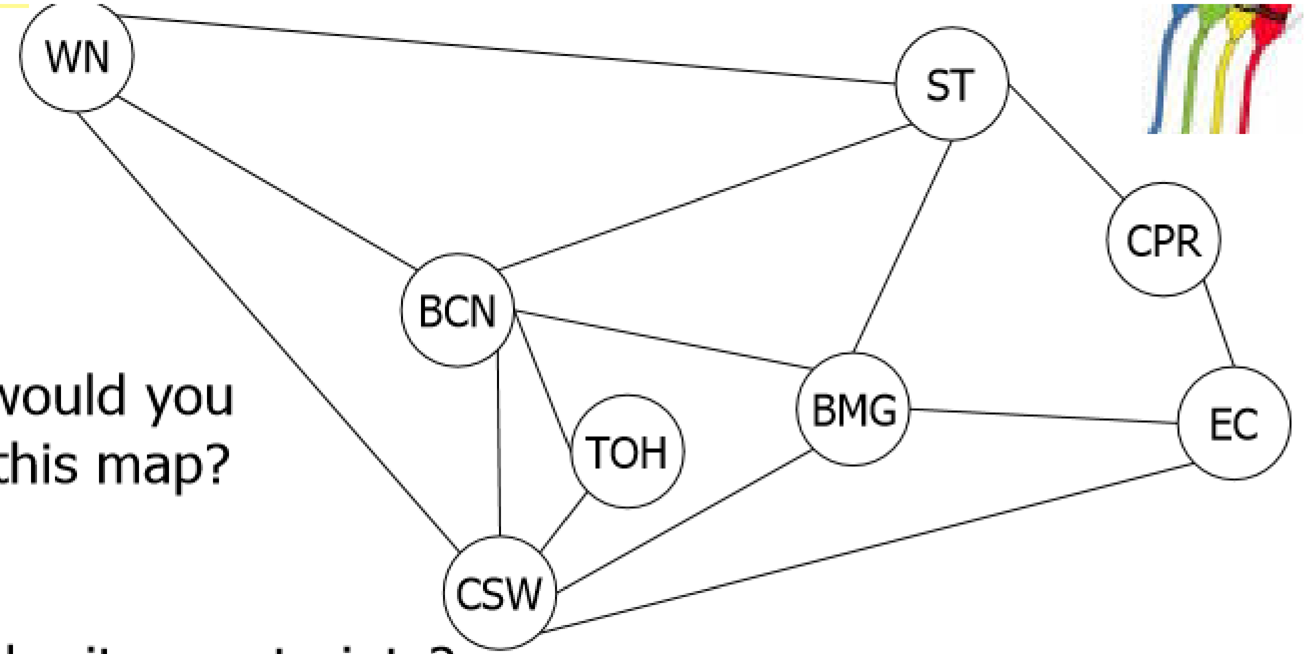
Backtracking example



Exercise - paint the town

- Districts across corners can be colored using the same color.

Constraint Graph



How would you
color this map?

Consider its constraints?
Can you do better than blind search?

Improving backtracking efficiency

- General-purpose methods can yield significant gains in speed:
  - Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?

Most constrained variable

Most constrained variable:

choose the variable with the fewest legal values

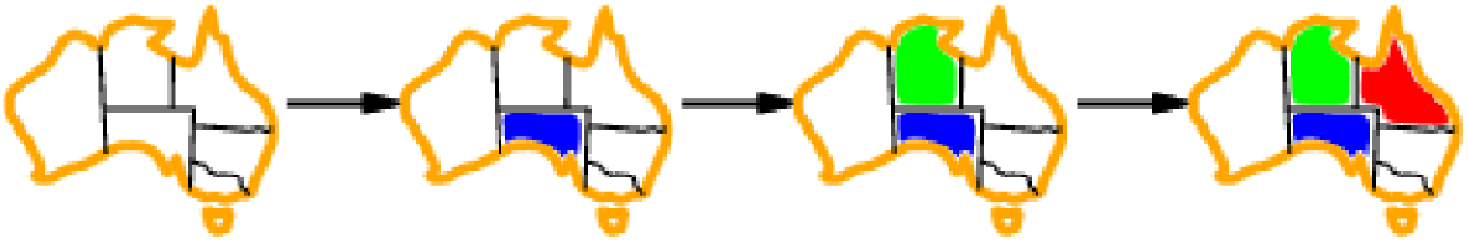
a.k.a. minimum remaining values (MRV) heuristic

Most constraining variable

Tie-breaker among most constrained variables

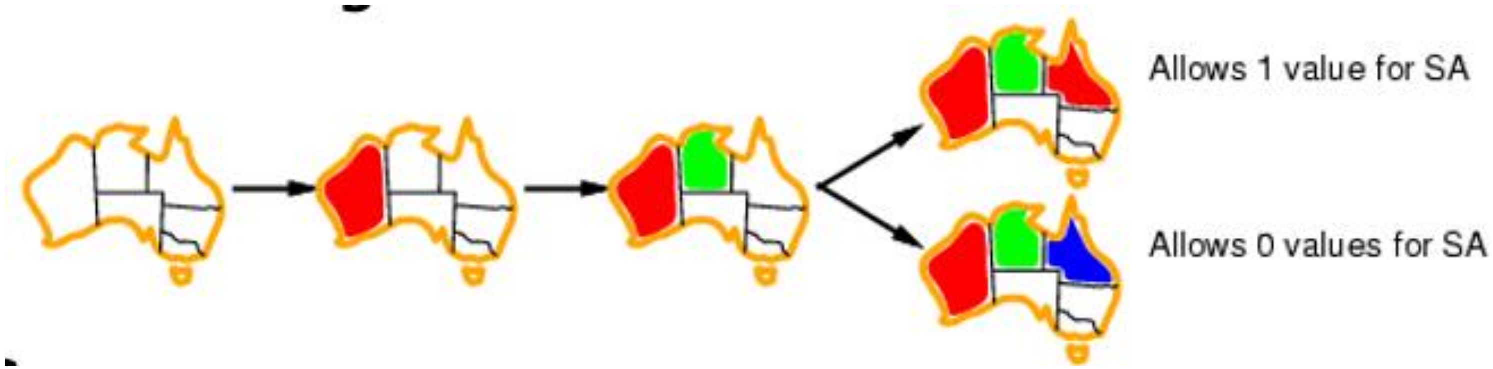
Most constraining variable:

- choose the variable with the most constraints on remaining variables



Least constraining value

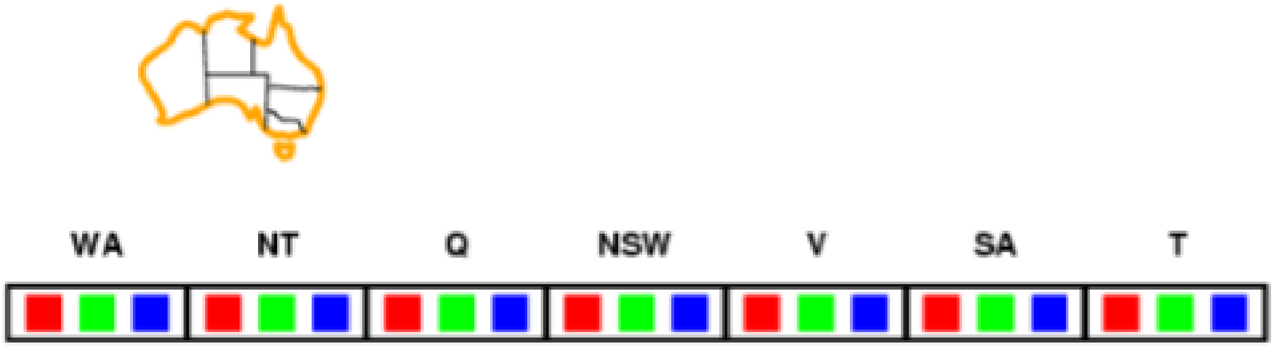
- Given a variable, choose the least constraining value:
  - the one that rules out the fewest values in the remaining variables



- Combining these heuristics makes 1000 queens feasible

Forward checking

- Idea:
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any variable has no legal values

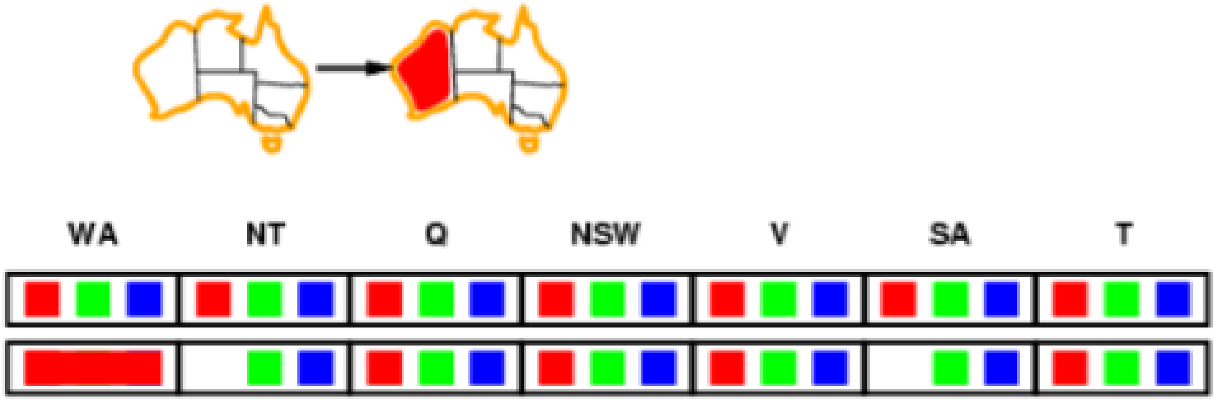


Difference: backtracking and forward checking

- Forward Checking is an improved version of simple backtracking. In forward checking, again initially a variable is instantiated to a value from its domain. Then repeatedly at each step, next variable is instantiated to a value that is consistent with the previous assignments.
- Different than backtracking, while assigning a value to the current variable, arc consistency between the current variable and the uninstantiated variables are maintained. By this way, current variable cannot take a value that causes an empty domain for one of the uninstantiated variables.
- If there is not such a value, then the algorithm backtracks to the point where it can start a new branch.
- In backtracking, the inconsistencies are detected when they occur, however in forward checking it is possible to detect inconsistencies much earlier. On the other hand, forward checking does more computations compared to backtracking although it has a smaller search tree.

Forward checking

- Idea:
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any variable has no legal values



Forward checking

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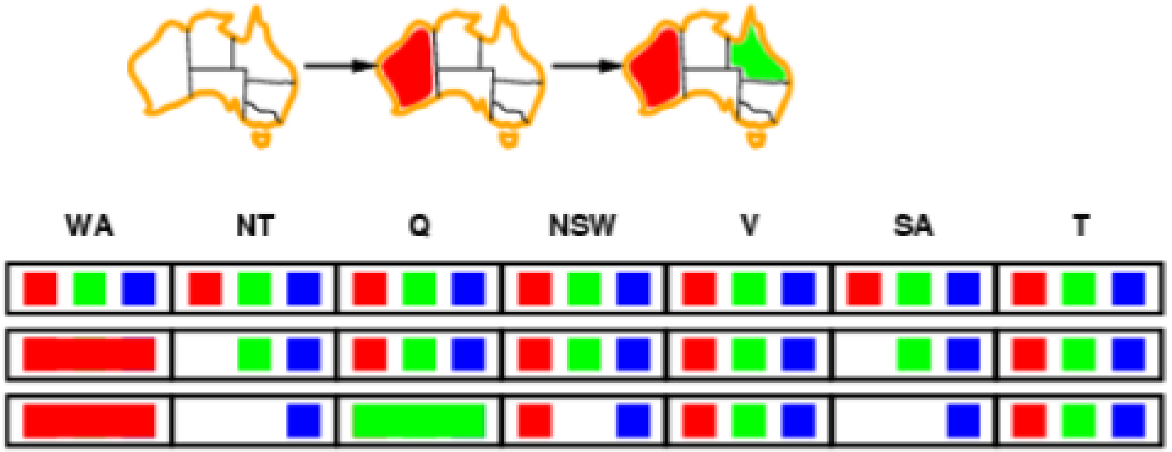

Forward checking

- Idea:
  - Keep track of remaining legal values for unassigned variables
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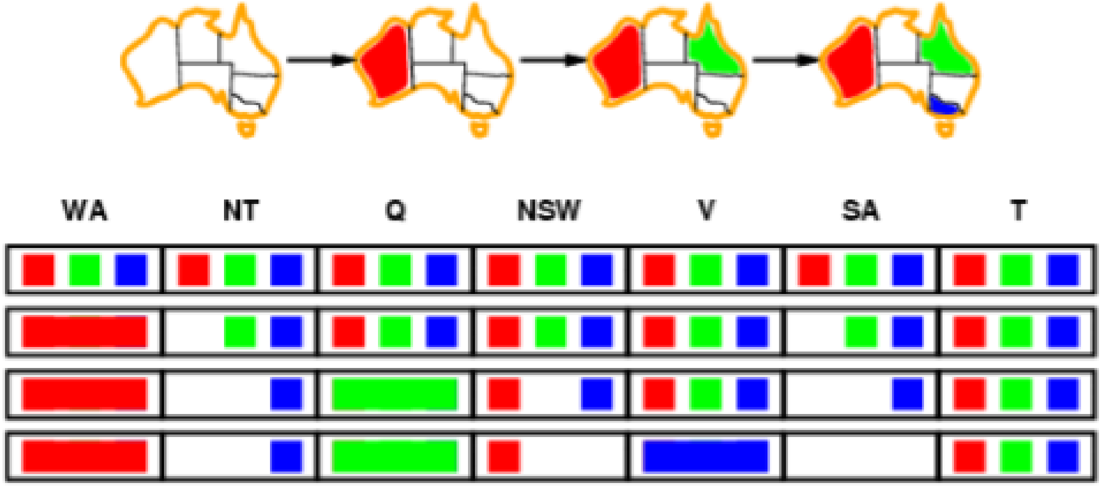
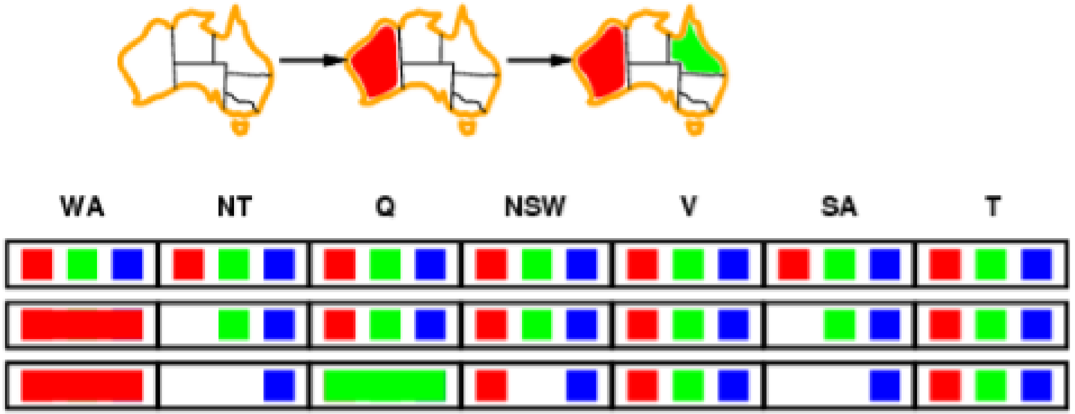


Constraint propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:

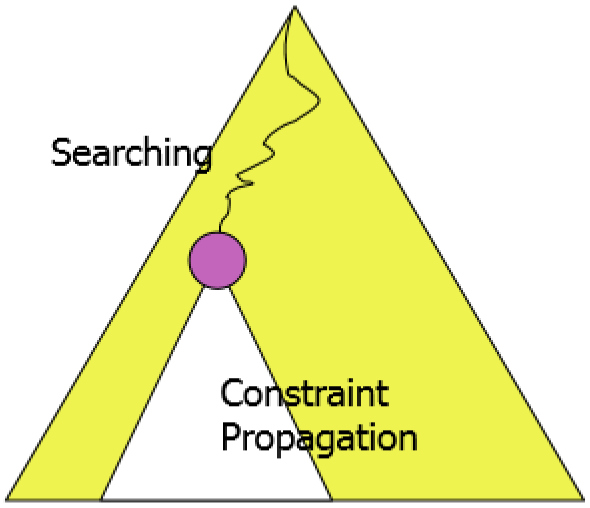


- NT and SA cannot both be blue!
- Constraint propagation repeatedly enforces constraints locally

Inference in CSPs

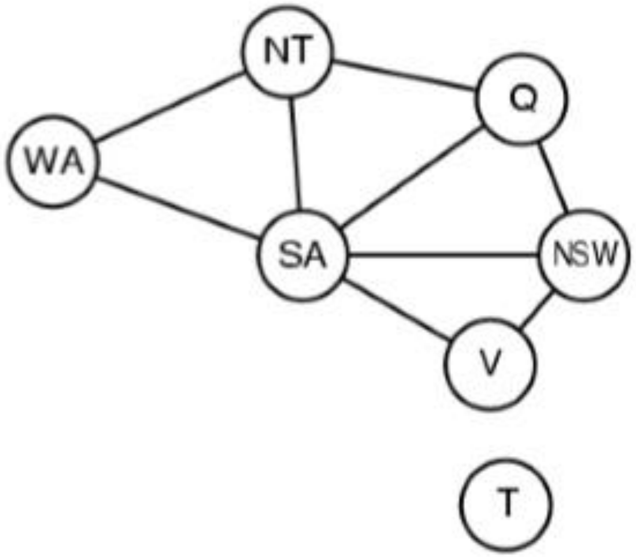
- Besides searching, in CSPs we can try to infer illegal values for variables by performing constraint propagation
  - Node consistency for unary constraints
  - Arc consistency for binary constraints
  - ...

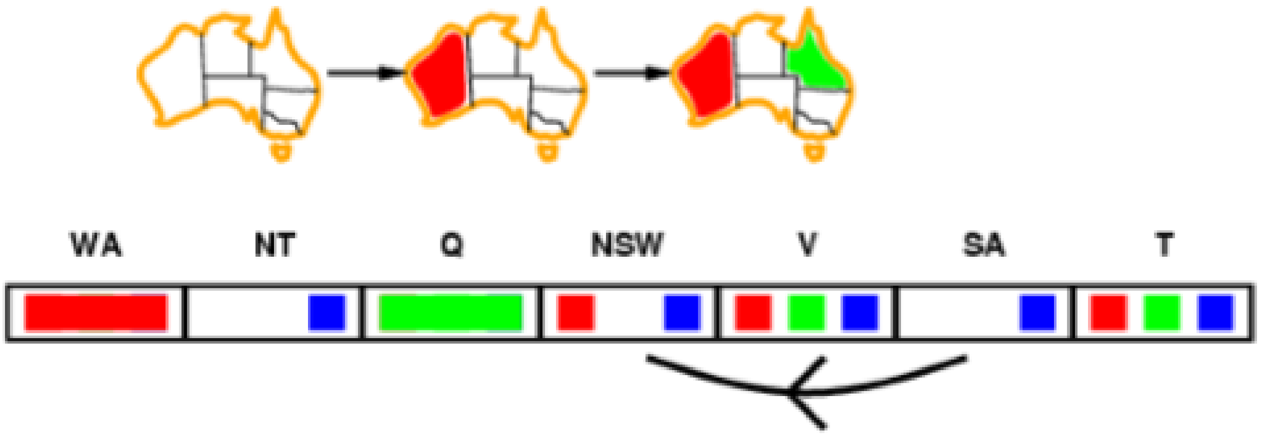
- Can interleave with searching or do as preprocessing



Arc consistency

- Simplest form of propagation makes each arc consisten
- X -> Y is consistent iff for every value x of X there is som



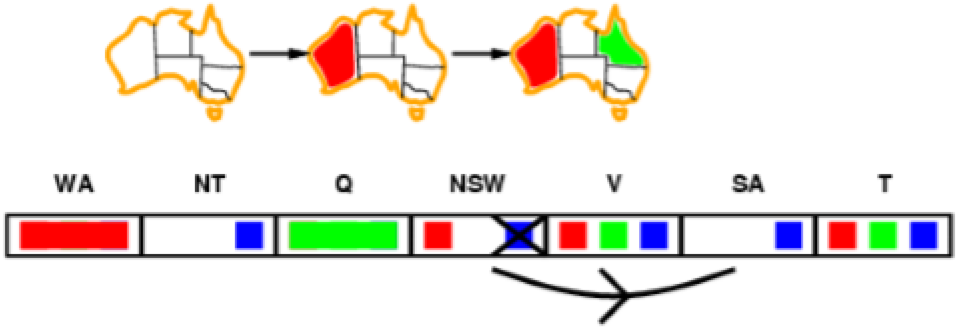


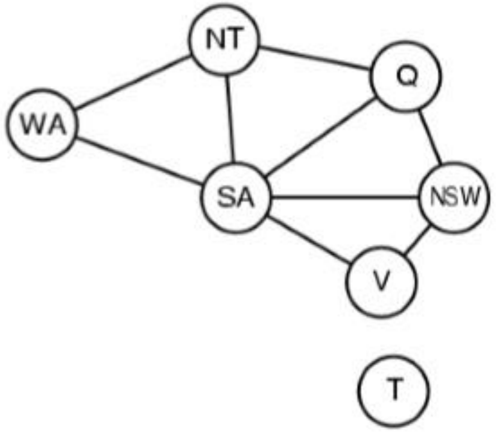
More on arc consistency

- Arc consistency is based on a very simple concept
  - if we can look at just one constraint and see that x=v is impossible …
  - obviously we can remove the value x=v from consideration
- How do we know a value is impossible?
- If the constraint provides no support for the value
- e.g. if Dx = {1,4,5} and Dy = {1, 2, 3}
  - then the constraint x > y provides no support for x=1
  - we can remove x=1 from Dx

Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \rightarrow Y$ is consistent iff
  for every value x of X there is some allowed y



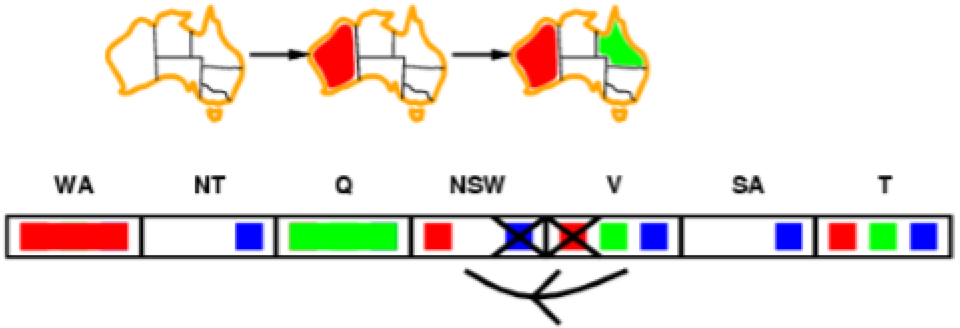


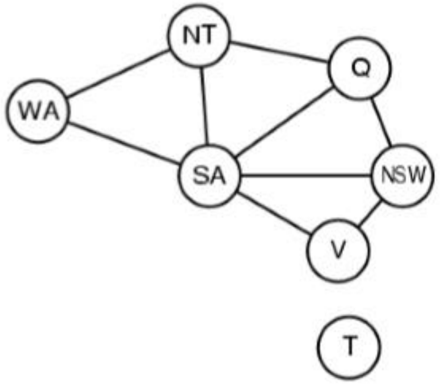
- Arcs are directed, a binary constraint becomes two arcs
- NSW $\Rightarrow$ SA arc originally not consistent, is consistent after deleting blue

Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \rightarrow Y$ is consistent iff

  for every value x of X there is some allowed y



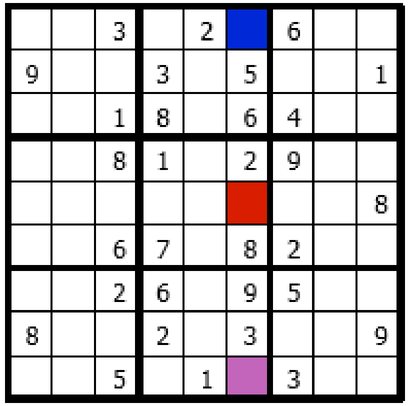


- If X loses a value, neighbors of X need to be (re)checked

Arc consistency propagation

- When we remove a value from Dx, we may get new removals because of it
- E.g. Dx = {1,4,5}, Dy = {1, 2, 3}, Dz= {2, 3, 4, 5}
  - x > y, z > x
  - As before we can remove 1 from Dx, so Dx = {4,5}
  - But now there is no support for Dz = 2,3,4
  - So we can remove those values, Dz = {5}, so z=5
  - Before AC applied to y-x, we could not change Dz
- This can cause a chain reaction

Sudoku Chain Reaction

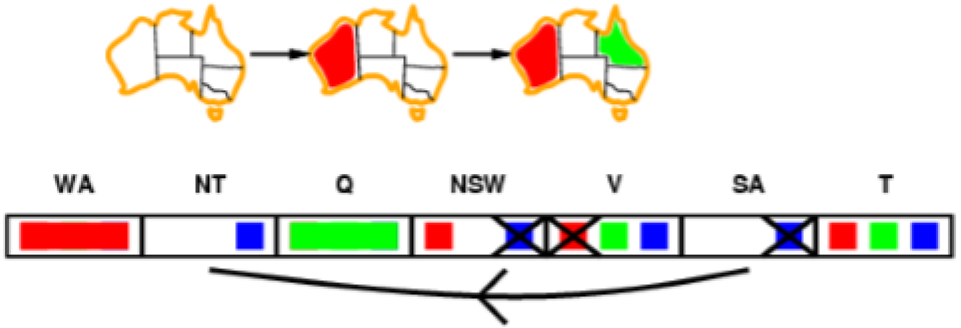


- Alldiff from box makes domain of red square {3,4,5,6,9} Column constraints reduces domain to {4}
- Then consider purple square. Original column and box constraints yield domain of {1,4}. Red square forces {1}
- Then final blue box must by {7} as column already has eight values.

Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \rightarrow Y$ is consistent iff

  for every value x of X there is some allowed y



- If X loses a value, neighbors of X need to be (re)checked
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment

Arc consistency algorithm AC-3

```
function AC-3( csp) returns the CSP, possibly with reduced domains
    inputs: csp, a binary CSP with variables {X_1, X_2, ..., X_n}
    local variables: queue, a queue of arcs, initially all the arcs in csp

    while queue is not empty do
        (X_i, X_j) ← REMOVE-FIRST(queue)
        if RM-INCONSISTENT-VALUES(X_i, X_j) then
            for each X_k in NEIGHBORS[X_i] do
                add (X_k, X_i) to queue
```

```
function RM-INCONSISTENT-VALUES( X_i, X_j) returns true iff remove a value
    removed ← false
    for each x in DOMAIN[X_i] do
        if no value y in DOMAIN[X_j] allows (x,y) to satisfy constraint(X_i, X_j)
            then delete x from DOMAIN[X_i];  removed ← true
    return removed
```

Time complexity: $O(n^2d^3)$

Time complexity of AC-3

- CSP has n^2 directed arcs
- Each arc X_i,X_j has d possible values.
- For each value we can reinsert the neighboring arc X_k,X_i at most d times because X_i has d values
- Checking an arc requires at most d^2 time
- ⍰ $O(n^2 * d * d^2) = O(n^2 d^3)$

```
function AC-3(csp) returns the CSP, possibly with reduced domains
   inputs: csp, a binary CSP with variables {X1, X2, ..., Xn}
   local variables: queue, a queue of arcs, initially all the arcs in csp

   while queue is not empty do
      (Xi, Xj) ← REMOVE-FIRST(queue)
      if RM-INCONSISTENT-VALUES(Xi, Xj) then
         for each Xk in NEIGHBORS[Xi] do
            add (Xk, Xi) to queue
```

```
function RM-INCONSISTENT-VALUES(Xi, Xj) returns true iff remove a value
   removed ← false
   for each x in DOMAIN[Xi] do
      if no value y in DOMAIN[Xj] allows (x,y) to satisfy constraint(Xi, Xj)
         then delete x from DOMAIN[Xi];  removed ← true
   return removed
```

Maintaining AC (MAC)

- We can use AC in search
- i.e. search proceeds as follows:
  - establish AC at the root
  - when AC3 terminates, choose a new variable/value
  - re-establish AC given the new variable choice (i.e. maintain AC) repeat;
  - backtrack if AC gives domain wipe out
- The hard part of implementation is undoing effects of AC

Special kinds of Consistency

- Some kinds of constraint lend themselves to special kinds of arc-consistency
- Consider the all-different constraint
  - the named variables must all take different values
  - not a binary constraint
  - can be expressed as n(n-1)/2 not-equals constraints
- We can apply (e.g.) AC3 as usual
- But there is a much better option

All Different

- Suppose Dx = {2,3} = Dy, Dz = {1,2,3}
- All the constraints $x \neq y$, $y \neq z$, $z \neq x$ are all arc consistent
  - e.g. x=2 supports the value z = 3
- The single ternary constraint AllDifferent(x,y,z) is not!
  - We must set z = 1
- A special purpose algorithm exists for All-Different to establish GAC in efficient time
  - Special purpose propagation algorithms are vital

K-consistency

Arc Consistency (2-consistency) can be extended to kconsistency

3-consistency (path consistency): any pair of adjacent variables can always be extended to a third neighbor.

- Catches problem with Dx, Dy and Dz, as assignment of Dz = 2 and Dx = 3 will lead to domain wipe out.
- But is expensive, exponential time

n -consistency means the problem is solvable in linear time

- As any selection of variables would lead to a solution

In general, need to strike a balance between consistency and search.

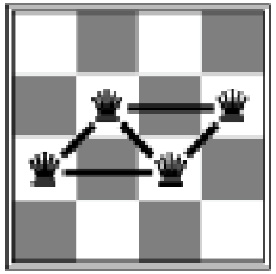
- This is usually done by experimentation.

Local search for CSPs

- Hill-climbing, simulated annealing typically work with "complete" states, i.e., all variables assigned
- To apply to CSPs:
  - allow states with unsatisfied constraints
  - operators re-assign variable values
- Variable selection: randomly select any conflicted variable
- Value selection by min-conflicts heuristic:
  - choose value that violates the fewest constraints
  - i.e., hill-climb with h(n) = total number of violated constraints

Example: 4-Queens

- States: 4 queens in 4 columns (4^4 = 256 states)
- Actions: move queen in column
- Goal test: no attacks
- Evaluation: $h(n)$ = number of attacks



h = 5

- Given random initial state, can solve *n*-queens in almost constant time for arbitrary *n* with high probability (e.g., *n* = 10,000,000)

Min-conflicts

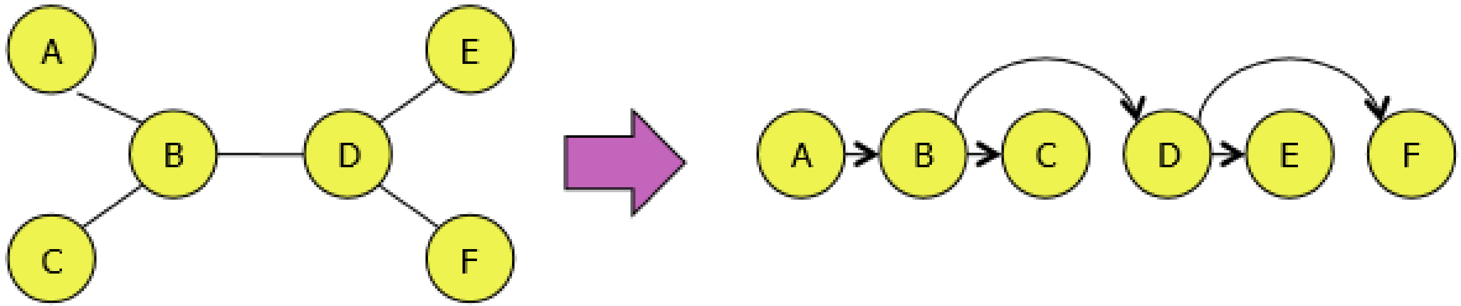
function MIN-CONFLICTS(*csp*, *max_steps*) **returns** a solution or failure
  inputs: *csp*, a constraint satisfaction problem
    max_steps, the number of steps allowed before giving up

  current ← an initial complete assignment for *csp*
  for *i* = 1 to *max_steps* **do**
    if *current* is a solution for *csp* **then return** *current*
    var ← a randomly chosen, conflicted variable from VARIABLES[*csp*]
    value ← the value *v* for *var* that minimizes CONFLICTS(*var*, *v*, *current*, *csp*)
    set *var* = *value* in *current*
  return *failure*

Figure 5.8 The MIN-CONFLICTS algorithm for solving CSPs by local search. The initial state may be chosen randomly or by a greedy assignment process that chooses a minimal-conflict value for each variable in turn. The CONFLICTS function counts the number of constraints violated by a particular value, given the rest of the current assignment.

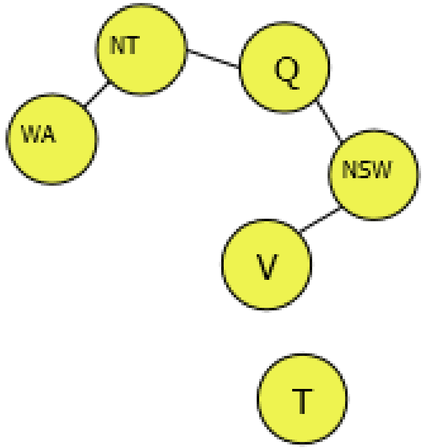
The structure of problems

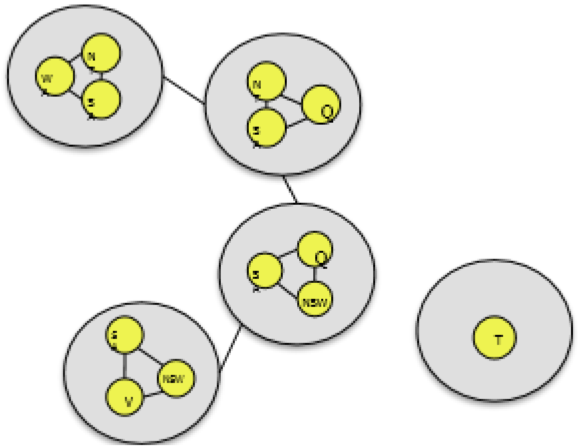
- Independent subproblems = unconnected components
- (Return to this point after midterm)
- Tree based CSPs can be solved by topological sort
  - Pick a root and “dangle” other nodes by it
  - Will have n-1 arcs, can make arc consistent in O(n)
  - O(nd^2)



Reducing CSP Trees

- Reduce other problems to trees, to use Tree-CSP-Solver, which yields solutions without backtracking. Aim to reduce to many small subproblems.
- Two approaches:
  - Remove nodes from CSP graph to make a tree
    - Assign values to removed nodes and remove used domains from tree nodes
  - Tree decomposition: make tree CSP with nodes as subproblems





Summary

- CSPs are a special kind of problem:
  - states defined by values of a fixed set of variables
  - goal test defined by constraints on variable values
- Backtracking = depth-first search with one variable assigned per node
- Variable ordering and value selection heuristics help significantly
- Forward checking prevents assignments that guarantee later failure
- Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies
- Iterative min-conflicts is usually effective in practice