Data Structures and Algorithms

Recursion

The mirrors

Acknowledgement

- The contents of these slides have origin from School of Computing, National University of Singapore.
- We greatly appreciate support from Mr. Aaron Tan Tuck Choy, and Dr. Low Kok Lim for kindly sharing these materials.

Policies for students

- These contents are only used for students PERSONALLY.
- Students are NOT allowed to modify or deliver these contents to anywhere or anyone for any purpose.

Recording of modifications

- Course website address is changed to http://sakai.it.tdt.edu.vn
- Slides “Practice Exercises” are eliminated.
- Course codes cs1010, cs1020, cs2010 are placed by 501042, 501043, 502043 respectively.

Objectives

1 • Strengthening the concept of recursion learned in 501042 (or equivalent)

2 • Demonstrating the application of recursion on some classic computer science problems

3 • Applying recursion on data structures

4 • Understanding recursion as a problem-solving technique known as divide-and-conquer paradigm

References

Book

- **Chapter 3:** Recursion: The Mirrors
- **Chapter 6:** Recursion as a Problem-Solving Technique, pages 337 to 345.

IT-TDT Sakai → 501043 website → Lessons

- http://sakai.it.tdt.edu.vn

Programs used in this lecture

- CountDown.java
- ConvertBase.java
- SortedLinkedList,java, TestSortedList.java
- Combination.java

Outline

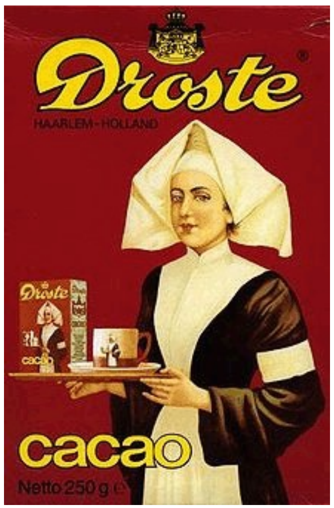
1. Basic Idea
2. How Recursion Works?
3. Examples
   - Count down
   - Display an integer in base b
   - Printing a Linked List
   - Printing a Linked List in reverse order
   - Inserting an element into a Sorted Linked List
   - Towers of Hanoi
   - Combinations: n choose k
   - Binary Search in a Sorted Array
   - K^{th} Smallest Number
   - Permutations of a string
   - The 8 Queens Problem

1 Basic Idea

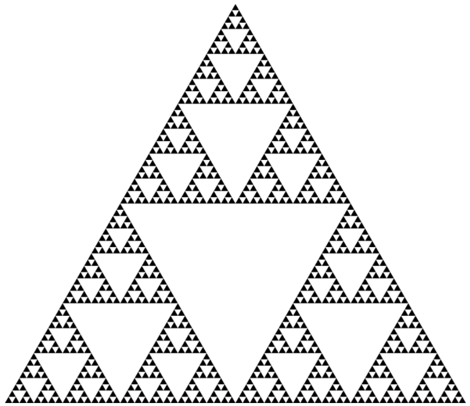
Also known as a central idea in CS

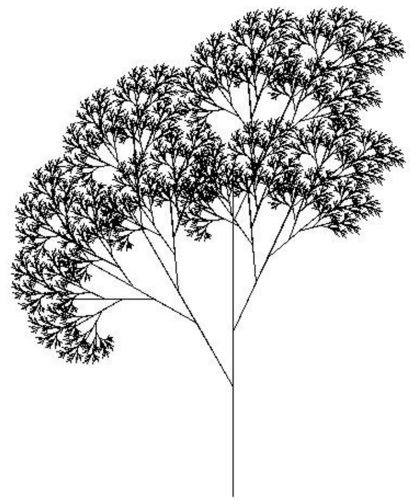
1.1 Pictorial examples

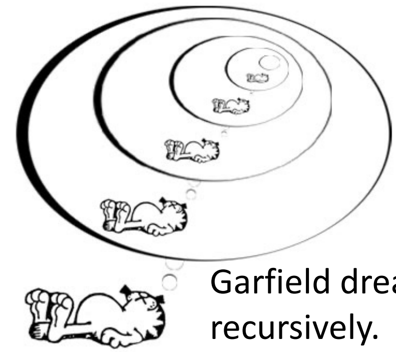
Some examples of recursion (inside and outside CS):



Droste effect


Sierpinksi triangle


Recursive tree


Garfield dreaming recursively.

Recursion is the process of repeating items in a self-similar way but with smaller size.

1.2 Textual examples

Definitions based on recursion:

Recursive definitions:

1. A person is a descendant of another if
   - the former is the latter's child, or
   - the former is one of the descendants of the latter's child.
2. A list of numbers is
   - a number, or
   - a number followed by a list of numbers.

Dictionary entry:

Recursion: See recursion.

To understand recursion, you must first understand recursion.

Recursive acronyms:

1. GNU = GNU's Not Unix
2. PHP = PHP: Hypertext Preprocessor

1.3 Divide-and-Conquer

- Divide: In top-down design (for program design or problem solving), break up a problem into sub-problems of the same type.
- Conquer: Solve the problem with the use of a function that calls itself to solve each sub-problem
  - one or more of these sub-problems are so simple that they can be solved directly without calling the function

A paradigm where the solution to a problem depends on solutions to smaller instances of the SAME problem.

1.4 Why recursion?

- Many algorithms can be expressed naturally in recursive form
- Problems that are complex or extremely difficult to solve using linear techniques may have simple recursive solutions
- It usually takes the following form:

```
SolveIt (problem) {
    if (problem is trivial) return result;
    else {
        simplify problem;
        return SolveIt (simplified problem) ;
    }
}
```

Calls itself!

2 How Recursion Works

Understanding Recursion

2.1 Recursion in 501042

- In 501042, you learned simple recursion
  - No recursion on data structures
  - Code consists of ‘if’ statement, no loop
  - How to trace recursive codes
- Examples covered in 501042
  - Factorial (classic example)
  - Fibonacci (classic example)
  - Greatest Common Divisor (classic example)
  - Other examples
  - Lecture slides and programs are available on 501043’s “501042 Stuffs” page: http://sakai.it.tdt.edu.vn

2.1 Recursion in 501042: Factorial (1/2)

$$n! = \begin{cases} 1, & n = 0 \\ n \times (n-1) \times \cdots \times 2 \times 1, & n > 0 \end{cases}$$

$$n! = \begin{cases} 1, & n = 0 \\ n \times (n-1)!, & n > 0 \end{cases}$$

Recurrence relation

Iterative solution

```
// Precond: n >= 0
int fact(int n) {
  int result = 1;
  for (int i=1;i<=n;i++)
    result *= i;
  return result;
}
```

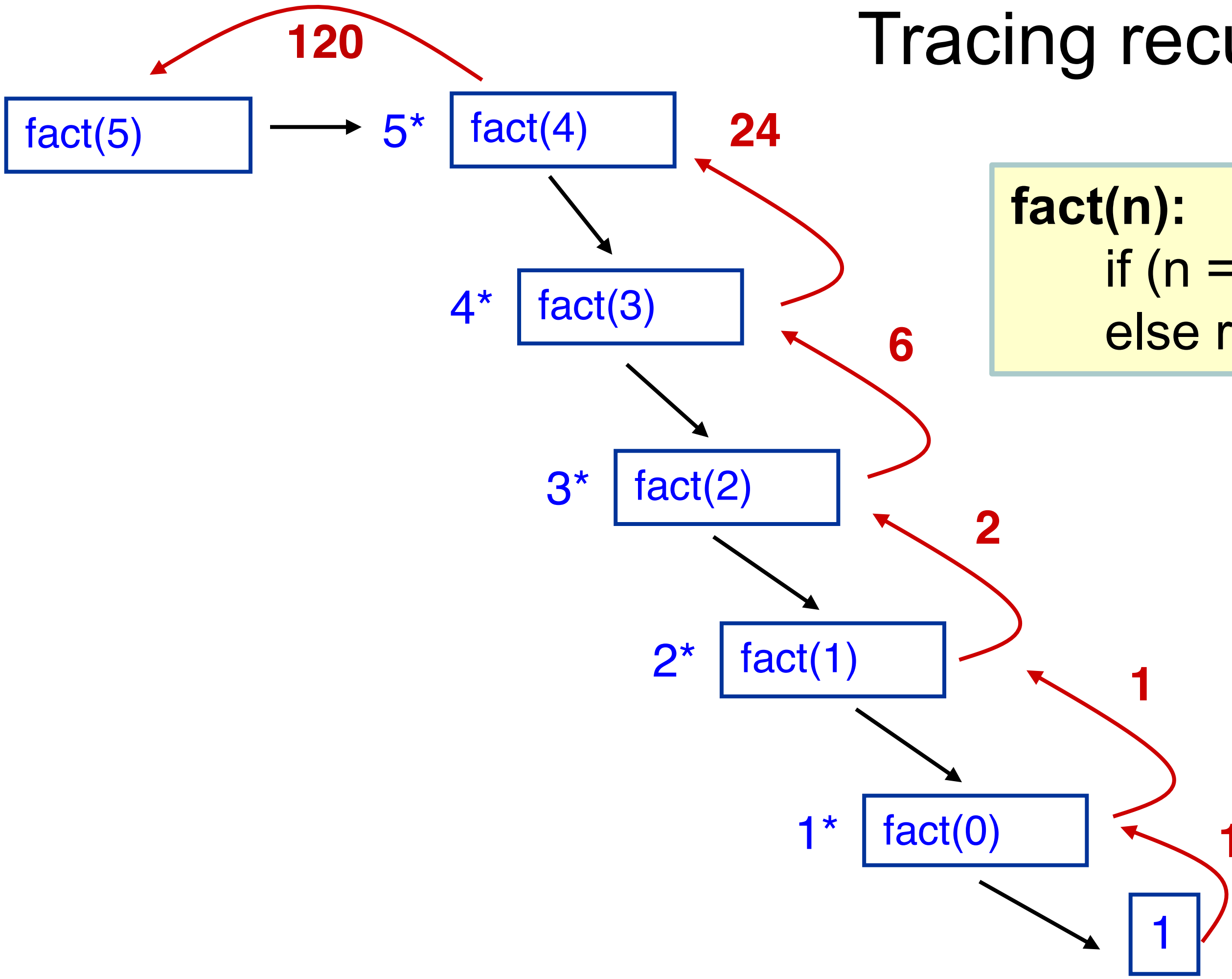
```
// Precond: n >= 0
int fact(int n) {
  if (n == 0)
    return 1;
  else
    return n * fact(n-1);
}
```

Remember to document pre-conditions, which are common for recursive codes.

Recursive call

Base case

2.1 Recursion in 501042: Factorial (2/2)



Tracing recursion

fact(n):
  if (n == 0) return 1;
  else return n * fact(n-1);

2.1 Recursion in 501042: Fibonacci (1/4)

- Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, …
  - The first two Fibonacci numbers are both 1 (arbitrary numbers)
  - The rest are obtained by adding the previous two together.
- Calculating the n^{th} Fibonacci number recursively:

Fib(n) = 1	for n=1, 2
= Fib(n-1) + Fib(n-2)	for n > 2

```
// Precond: n > 0
int fib(int n) {
  if (n <= 2)
    return 1;
  else
    return fib(n-1) + fib(n-2);
}
```

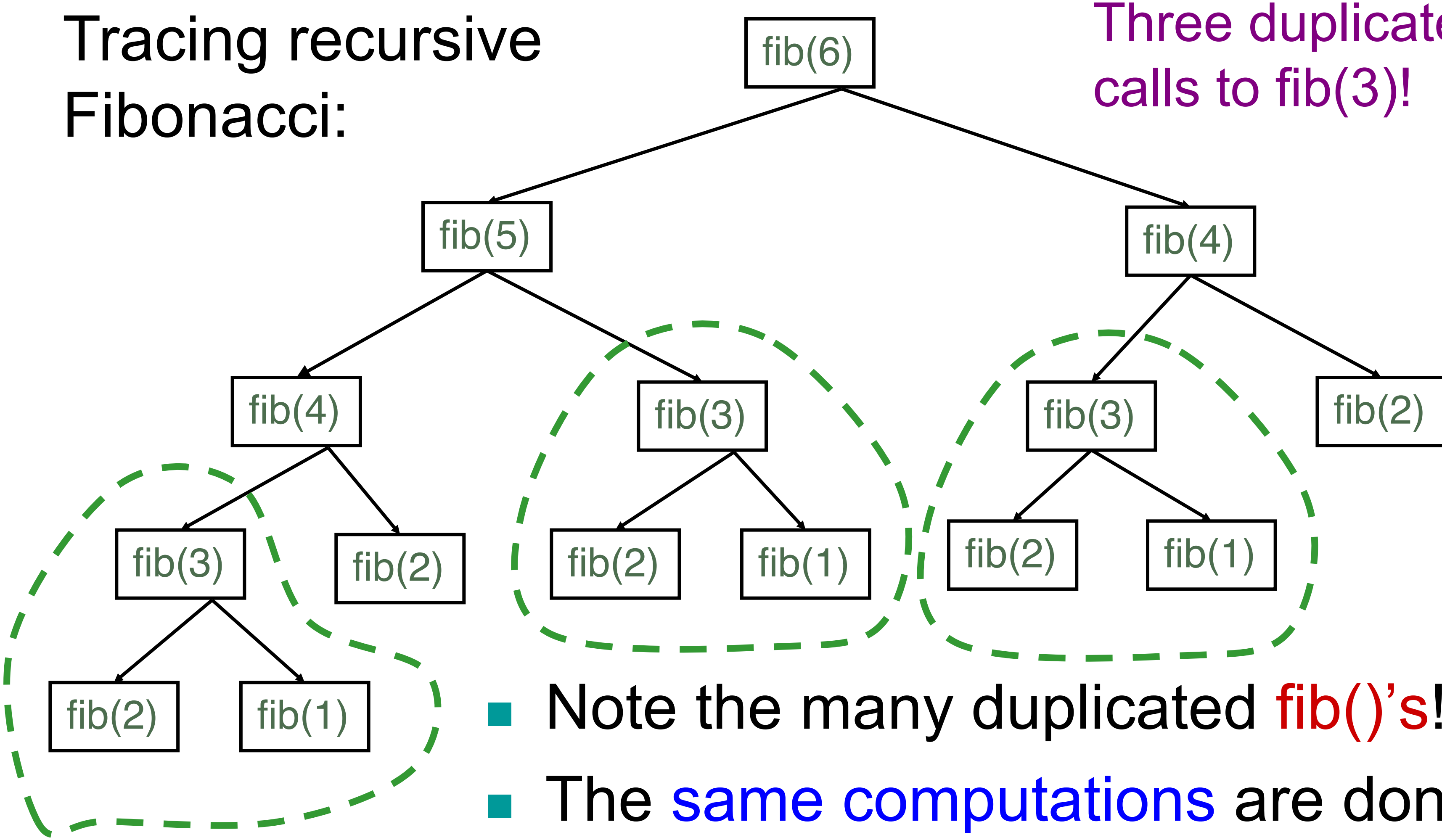
Elegant but extremely inefficient. Which is correct?

1. Recursion doesn't reach base case
2. A lot of repeated work
3. Should put recursive case above base case

2.1 Recursion in 501042: Fibonacci (2/4)

Tracing recursive Fibonacci:

Three duplicate calls to fib(3)!



- Note the many duplicated fib()'s!
- The same computations are done over and over again!

2.1 Recursion in 501042: Fibonacci (3/4)

Iterative Fibonacci

```
int fib(int n) {
  if (n <= 2)
    return 1;
  else {
    int prev1=1, prev2=1, curr;
    for (int i=3; i<=n; i++) {
      curr = prev1 + prev2;
      prev2 = prev1;
      prev1 = curr;
    }
    return curr;
  }
}
```

Q: Which part of the code is the key to the improved efficiency?

(1) Part A (red)
(2) Part B (blue)

2.1 Recursion in 501042: Fibonacci (4/4)

- Closed-form formula for Fibonacci numbers
- Take the ratio of 2 successive Fibonacci numbers (say A and B). The bigger the pair of numbers, the closer their ratio is to the Golden ratio φ which is $\approx$ 1.618034…

A	2	3	5	8	…	144	233
B	3	5	8	13	…	233	377
B/A	1.5	1.666…	1.6	1.625	…	1.61805…	1.61802…

- Using φ to compute the Fibonacci number x_n:

$$x_n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}$$

See
http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibFormula.html

2.1 Recursion in 501042: GCD (1/2)

- Greatest Common Divisor of two integers *a* and *b*, where *a* and *b* are non-negative and not both zeroes
- Iterative method given in Practice Exercise 11

```
// Precond: a, b non-negative,
//          not both zeroes
int gcd(int a, int b) {
  int rem;
  while (b > 0) {
     rem = a % b;
     a = b;
     b = rem;
  }
  return a;
}
```

2.1 Recursion in 501042: GCD (2/2)

- Recurrence relation:

$$\gcd(a, b) = \begin{cases} a, & \text{if } b = 0 \\ \gcd(b, a\%b), & \text{if } b > 0 \end{cases}$$

```
// Precond: a, b non-negative,
//          not both zeroes
int gcd(int a, int b) {
  if (b == 0)
     return a;
  else
     return gcd(b, a % b);
}
```

Artwork credit: ollie.olarte

2.2 Visualizing Recursion

- It’s easy to visualize the execution of non-recursive programs by stepping through the source code.
- However, this can be confusing for programs containing recursion.
  - Have to imagine each call of a method generating a copy of the method (including all local variables), so that if the same method is called several times, several copies are present.

2.2 Stacks for recursion visualization

int j = fact(5)

fact(0)	1
fact(1)	1 × 1
fact(2)	2 × 1
fact(3)	3 × 2
fact(4)	4 × 6
fact(5)	5 × 24

Use
push() for new recursive call
pop() to return a value from a call to the caller.

Example: fact (n)

```
if (n == 0) return 1;
else return n * fact (n-1);
```

j =120

2.3 Recipe for Recursion

Sometimes we call #1 the "**inductive step**"

To formulate a recursive solution:

1. General (recursive) case: Identify "simpler" instances of the same problem (so that we can make recursive calls to solve them)
2. Base case: Identify the "simplest" instance (so that we can solve it without recursion)
3. Be sure we are able to reach the "simplest" instance (so that we will not end up with infinite recursion)

2.4 Bad Recursion

$$\text{funct}(n) = \begin{cases} 1 & \text{if } (n==0) \\ \text{funct}(n-2)/n & \text{if } (n>0) \end{cases}$$

Q: What principle does the above code violate?

1. Doesn't have a simpler step.
2. No base case.
3. Can't reach the base case.
4. All's good. It's a ~trick~!

3 Examples

How recursion can be used

Example 1

3.1 Countdown

CountDown.java

```
public class CountDown {

   public static void countDown(int n) {
      if (n <= 0)    // don't use == (why?)
         System.out.println ("BLAST OFF!!!!");
      else {
         System.out.println("Count down at time " + n);
         countDown(n-1);
      }
   }

   public static void main(String[] args) {
      countDown(10);
   }
}
```

Example 2

3.2 Display an integer in base b

- See ConvertBase.java

E.g. One hundred twenty three is 123 in base 10; 173 in base 8

```
public static void displayInBase(int n, int base) {
  if (n > 0) {
    displayInBase(n / base, base);
    System.out.print(n % base);
  }
}
```

What is the precondition for parameter base?

Example 1:

n = 123, base = 10

123/10 =12 123 % 10 = 3
12/10 = 1 12 % 10 = 2
1/10 = 0 1 % 10 = 1

Answer: 123

Example 2:

n = 123, base = 8

123/8 = 15 123 % 8 = 3
15/8 = 1 15 % 8 = 7
1/8 = 0 1 % 8 = 1

Answer: 173

Example 3

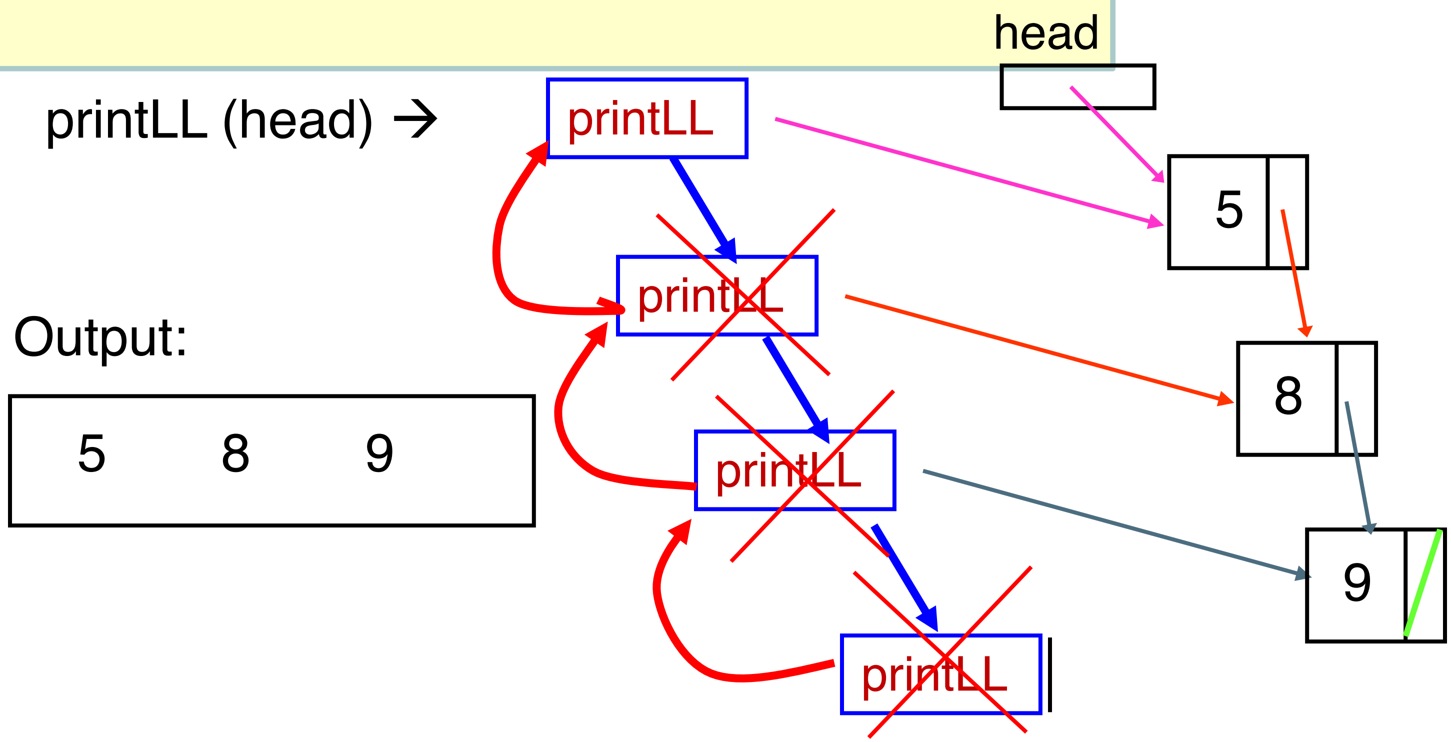
3.3 Printing a Linked List recursively

- See SortedLinkedList.java and TestSortedList.java

```
public static void printLL(ListNode n) {
  if (n != null) {
    System.out.print(n.value);
    printLL(n.next);
  }
}
```

Q: What is the base case?

Q: How about printing in reverse order?



Example 4

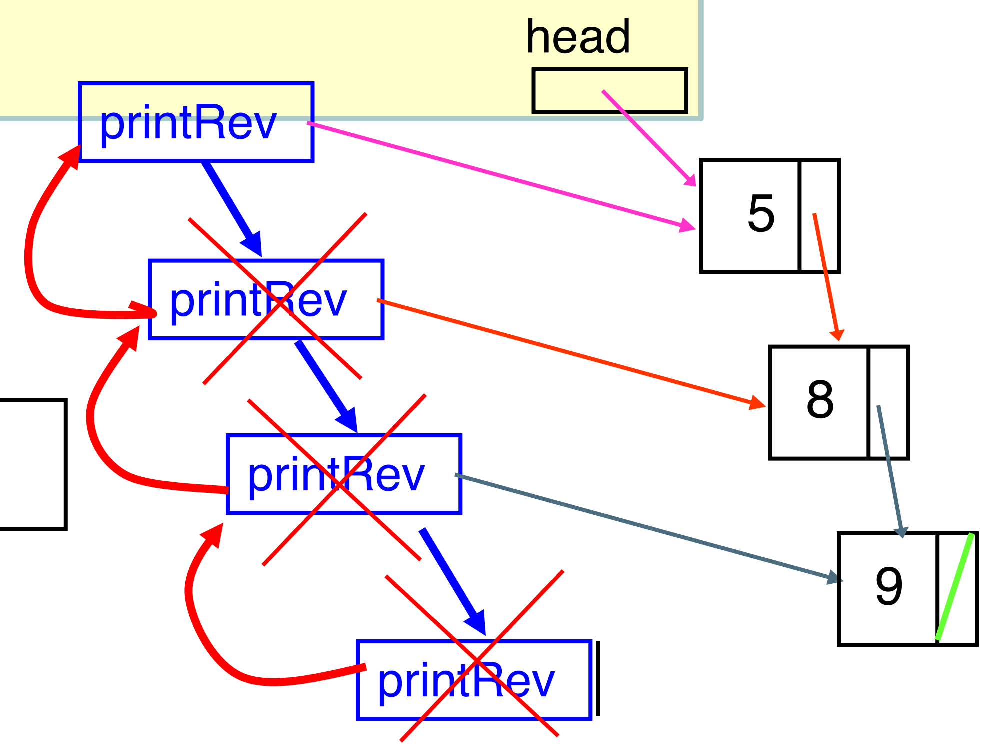
3.4 Printing a Linked List recursively in reverse order

- See SortedLinkedList.java and TestSortedList.java

```
public static void printRev(ListNode n) {
  if (n!=null) {
    printRev(n.next);
    System.out.print(n.value);
  }
}
```

Just change the name!
… Sure, right!

printRev(head) →



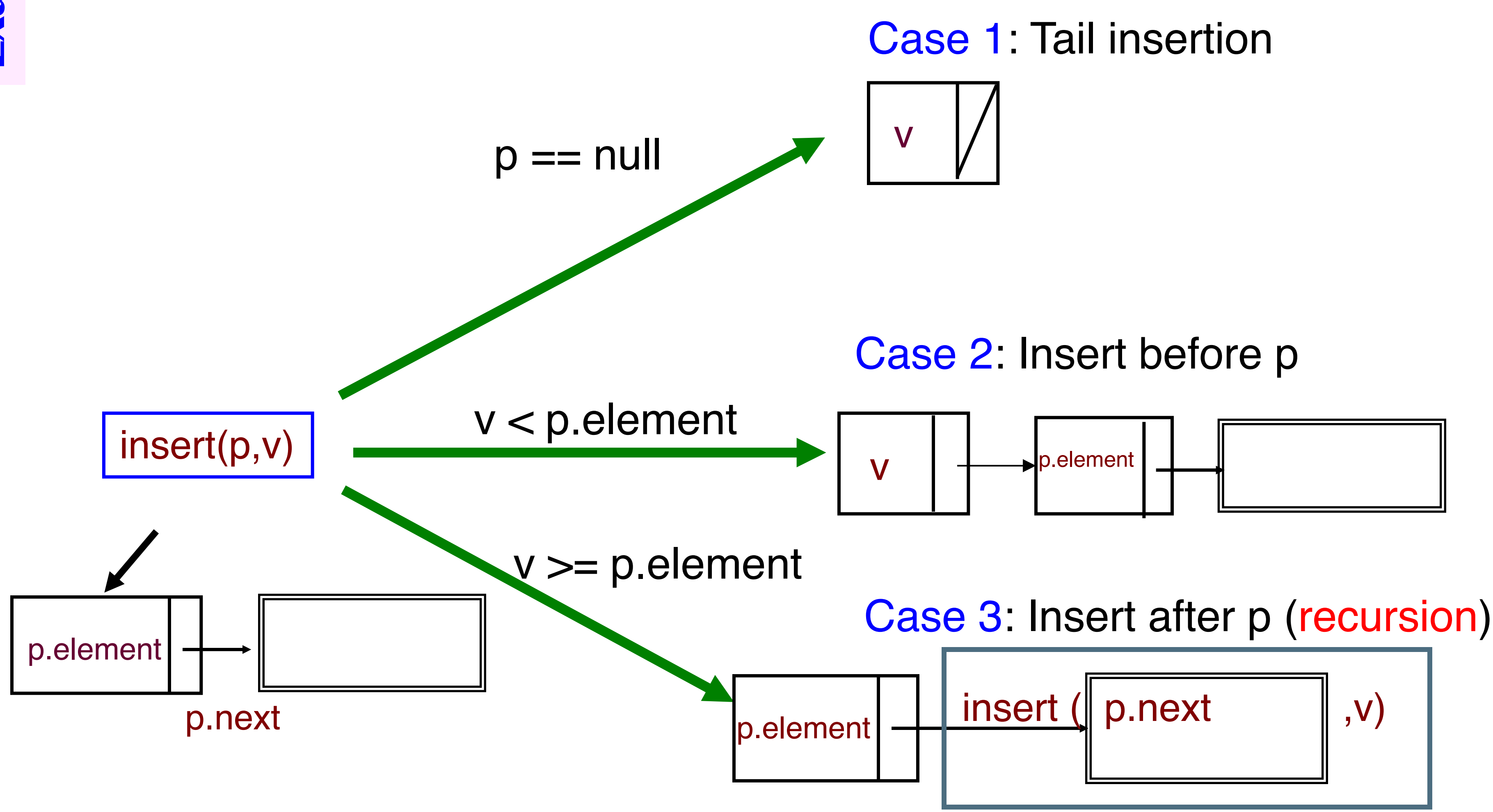
Output:

9	8	5

Example 5

3.5 Sorted Linked List Insertion (1/2)

- Insert an item v into the sorted linked list with head p



Example 5

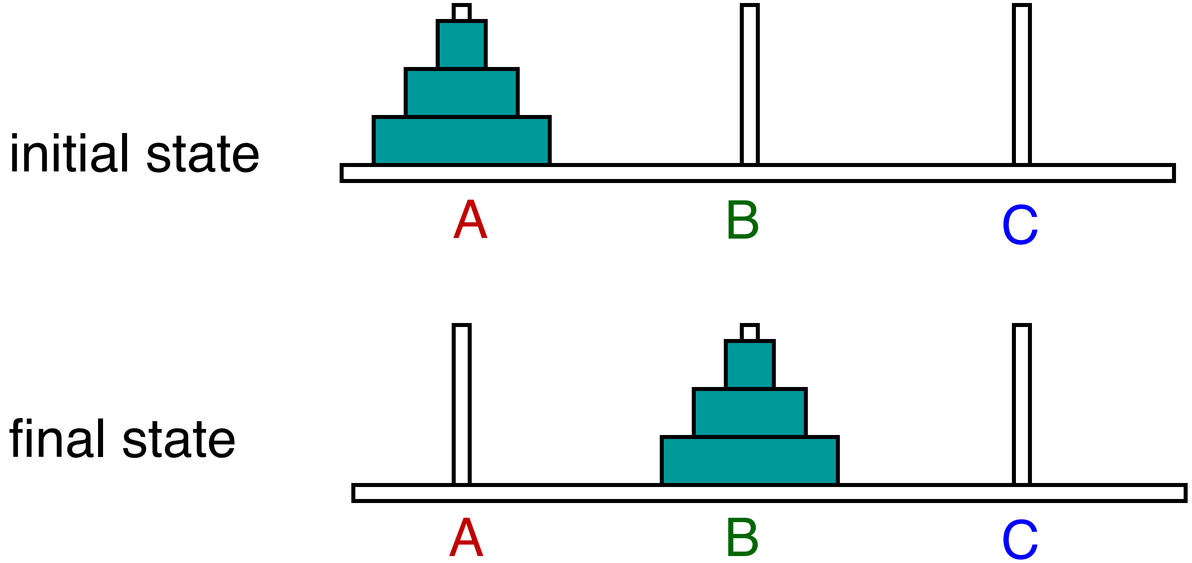
3.5 Sorted Linked List Insertion (2/2)

```
public static ListNode insert(ListNode p, int v) {

  // Find the first node whose value is bigger than v
  // and insert before it.
  // p is the “head” of the current recursion.
  // Returns the “head” after the current recursion.

  if (p == null || v < p.element)
    return new ListNode(v, p);
  else {
    p.next = insert(p.next, v);
    return p;
  }
}
```

To call this method:
head = insert(head, newItem);

3.6 Towers of Hanoi

- Given a stack of discs on peg A, move them to peg B, one disc at a time, with the help of peg C.
- A larger disc cannot be stacked onto a smaller one.



Only the first 3 steps shown.

Example 6

3.6 Towers of Hanoi – Quiz

From en.wikipedia.org

- What's the base case?
  - A: 1 disc
  - B: 0 disc
- What's the inductive step?
  - A: Move the top n-1 disks to another peg
  - B: Move the bottom n-1 disks to another peg
- How many times do I need to call the inductive step?
  - A: Once
  - B: Twice
  - C: Three times

Example 6

3.6 Tower of Hanoi solution

```
public static void Towers(int numDisks, char src, char dest, char temp) {

  if (numDisks == 1) {
     System.out.println("Move top disk from pole " + src + " to pole " + dest);
  } else {

     Towers(numDisks – 1, src, temp, dest);  // first recursive call
     Towers(1, src, dest, temp);
     Towers(numDisks – 1, temp, dest, src);  // second recursive call
  }
}
```

Example 6

3.6 Tower of Hanoi iterative solution (1/2)

```
public static void LinearTowers(int orig_numDisks, char orig_src,
                                char orig_dest, char orig_temp) {
  int numDisksStack[] = new int[100]; // Maintain the stacks manually!
  char srcStack[] = new char[100];
  char destStack[] = new char[100];
  char tempStack[] = new char[100];
  int stacktop = 0;
  numDisksStack[0] = orig_numDisks; // Init the stack with the 1st call
  srcStack[0] = orig_src;
  destStack[0] = orig_dest;
  tempStack[0] = orig_temp;
  stacktop++;
```

Complex!
This and the next slide are only for your reference.

Example 6

3.6 Tower of Hanoi iterative solution (2/2)

```
while (stacktop>0) {
   stacktop--;    // pop current off stack
   int numDisks = numDisksStack[stacktop];
   char src = srcStack[stacktop]; char dest  = destStack[stacktop];
   char temp = tempStack[stacktop];
   if (numDisks == 1) {
      System.out.println("Move top disk from pole "+src+" to pole "+dest);
   } else {
         /* Towers(numDisks-1,temp,dest,src); */ // second recursive call
      numDisksStack[stacktop] = numDisks -1;
      srcStack[stacktop] = temp;
      destStack[stacktop] = dest;
      tempStack[stacktop++] = src;
         /* Towers(1,src,dest,temp); */
      numDisksStack[stacktop] =1;
      srcStack[stacktop] = src; destStack[stacktop] = dest;
      tempStack[stacktop++] = temp;
         /* Towers(numDisks-1,src,temp,dest); */ // first recursive call
      numDisksStack[stacktop] = numDisks -1;
      srcStack[stacktop] = src; destStack[stacktop] = temp;
      tempStack[stacktop++] = dest;
   }
}
```

Q: Which version runs faster?

A: Recursive

B: Iterative (this version)

Example 6

3.6 Towers of Hanoi

Towers(4, src, dest, temp)

numDiskStack	srcStack	destStack	tempStack
3	src	temp	dest
1	src	dest	temp
3	temp	dest	src

Example 6

3.6 Time Efficiency of Towers()

Num of discs, n	Num of moves, f(n)	Time (1 sec per move)
1	1	1 sec
2	3	3 sec
3	3+1+3 = 7	7 sec
4	7+1+7 = 15	15 sec
5	15+1+15 = 31	31 sec
6	31+1+31 = 63	1 min
…	…	…
16	65,536	18 hours
32	4.295 billion	136 years
64	1.8×10^{10} billion	**584 billion years**
N	$\mathbf{2^N - 1}$	

Example 7

3.7 Being choosy…

"Photo" credits: Torley
(this pic is from 2nd life)

Suppose you visit an ice cream store with your parents.

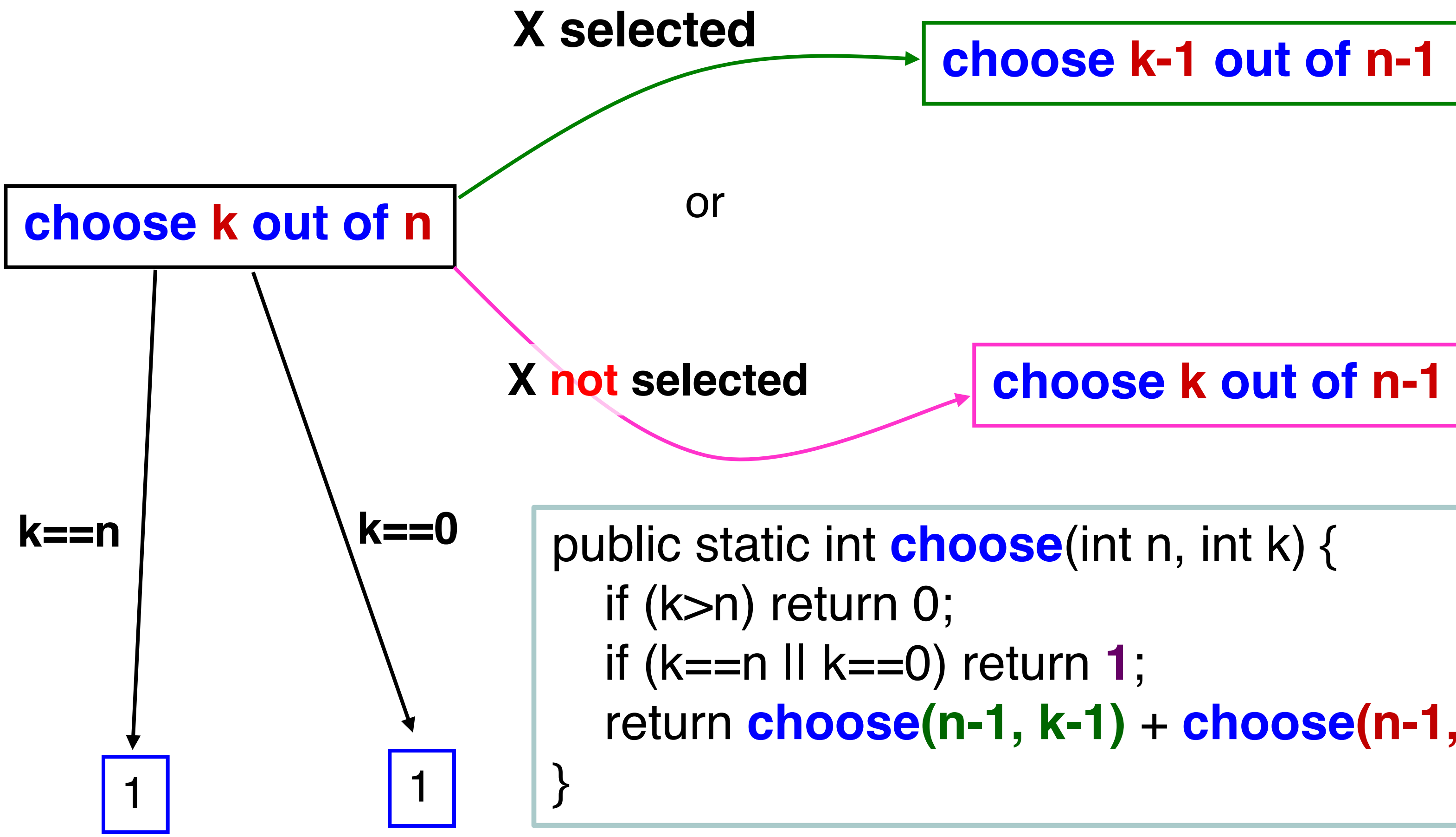
You've been good so they let you choose 2 flavors of ice cream.

The ice cream store stocks 10 flavors today. How many different ways can you choose your ice creams?

Example 7

3.7 n choose k

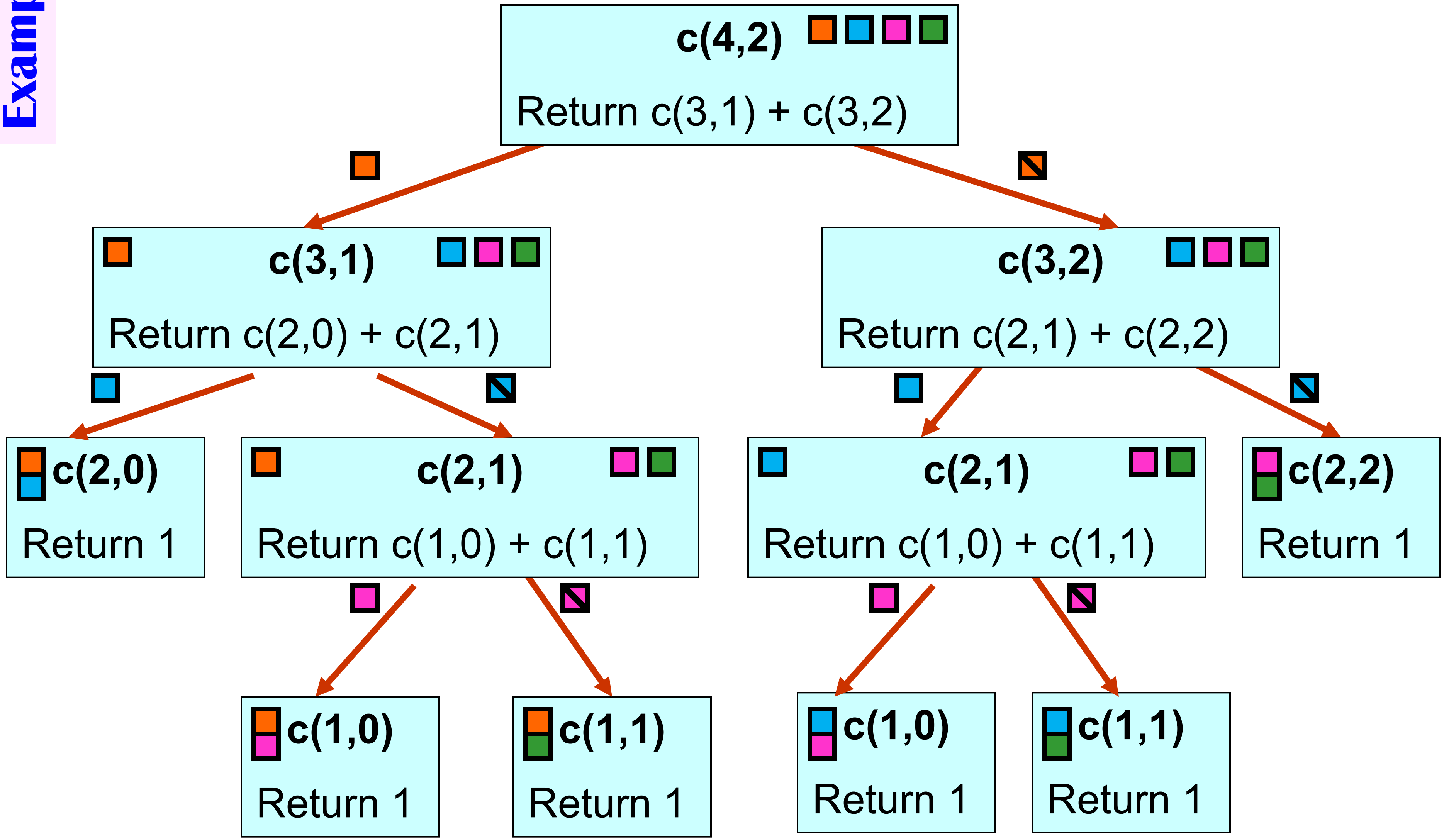
- See Combination.java




```java
public static int choose(int n, int k) {
    if (k>n) return 0;
    if (k==n || k==0) return 1;
    return choose(n-1, k-1) + choose(n-1, k);
}
```

Example 7

3.7 Compute c(4,2)



3.8 Searching within a sorted array

- Idea: narrow the search space by half at every iteration until a single element is reached.

Problem: Given a sorted int array a of *n* elements and int x, determine if x is in a.

a =

1	5	6	13	14	19	21	24	32

x = 15

Example 8

3.8 Binary Search by Recursion

```
public static int binarySearch(int [] a, int x, int low, int high)
                                                    throws ItemNotFound  {
   // low: index of the low value in the subarray
   // high: index of the highest value in the subarray
   if (low > high) // Base case 1: item not found
      throw new ItemNotFound("Not Found");

   int mid = (low + high) / 2;

   if (x > a[mid])
      return binarySearch(a, x, mid + 1, high);

   else if (x < a[mid])
      return binarySearch(a, x, low, mid - 1);

   else
      return mid; // Base case 2: item found
}
```

Q: Here, do we assume that the array is sorted in ascending or descending order?

A: Ascending

B: Descending

Example 8

3.8 Auxiliary functions for recursion

- Hard to use this function as it is.
- Users just want to find something in an array. They don't want to (or may not know how to) specify the low and high indices.
  - Write an auxiliary function to call the recursive function
  - Using overloading, the auxiliary function can have the same name as the actual recursive function it calls

Auxiliary function

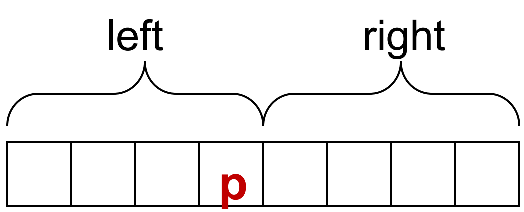
```
boolean binarySearch(int[] a, int x) {
   return binarySearch(a, x, 0, a.length-1);
}
```

Recursive function

Example 9

3.9 Find k^th smallest (unsorted array)

```
public static int kthSmallest(int k, int[] a) {   // k >= 1
    // Choose a pivot element p from a[]
    // and partition (how?) the array into 2 parts where
    //     left = elements that are <= p
    //     right = elements that are > p
     …
    int numLeft = sizeOf(left);

    if (1__________) 3___________;
    if (2__________) {
        return 4__________;
    else
        return 5__________;
}
```



Map the lines to the slots

A: 1i, 2ii, 3iii, 4iv, 5v

B: 1i, 2ii, 3v, 4iii, 5iv

C: 1ii, 2i, 3v, 4iii, 5iv

D: 1i, 2ii, 3v, 4iv, 5iii

where

i. k == numLeft

ii. k < numLeft

iii. return kthSmallest(k, left);

iv. return kthSmallest(k – numLeft, right);

v. return p;

Example 10

3.10 Find all Permutations of a String (1/3)

- For example, if the user types a word say *east*, the program should print all 24 permutations (anagrams), including *eats*, *etas*, *teas*, and non-words like *tsae*.
- Idea to generate all permutation:
  - Given *east*, we would place the first character i.e. *e* in front of all 6 permutations of the other 3 characters *ast* — *ast, ats, sat, sta, tas,* and *tsa* — to arrive at *east, eats, esat, esta, etas,* and *etsa,* then
  - we would place the second character, i.e. *a* in front of all 6 permutations of *est,* then
  - the third character i.e. *s* in front of all 6 permutations of *eat*, and
  - finally the last character i.e. *t* in front of all 6 permutations of *eas*.
  - Thus, there will be 4 (the size of the word) recursive calls to display all permutations of a four-letter word.
- Of course, when we're going through the permutations of the 3-character string e.g. *ast*, we would follow the same procedure.

3.10 Find all Permutations of a String (2/3)

- Recall overloaded substring() methods in String class

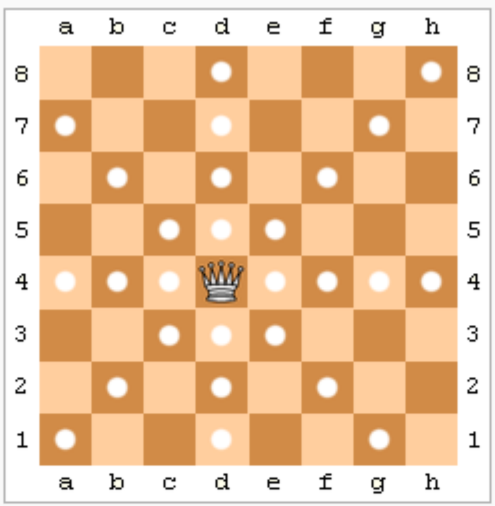
String	**substring**(int beginIndex) Returns a new string that is a substring of this string. The substring begins with the character at beginIndex and extends to the end of this string.
String	**substring**(int beginIndex, int endIndex) Returns a new string that is a substring of this string. The substring begins at beginIndex and extends to the character at index endIndex - 1. Thus the length of the substring is endIndex – beginIndex.

3.10 Find all Permutations of a String (3/3)

```
public class Permutations {
  public static void main(String args[]) {
    permuteString("", "String");
  }

  public static void permuteString(String beginningString, String endingString) {
    if (endingString.length() <= 1)
      System.out.println(beginningString + endingString);
    else
      for (int i = 0; i < endingString.length(); i++) {
        try {
          String newString = endingString.substring(0,i) + endingString.substring(i+1);
                    // newString is the endingString but without character at index i
          permuteString(beginningString + endingString.charAt(i), newString);
        } catch (StringIndexOutOfBoundsException exception) {
          exception.printStackTrace();
        }
      }
  }
}
```

Exercise: Eight Queens Problem

- Place eight Queens on the chess board so that they cannot attack one another



Possible moves of the queen are shown

- Q: How do you formulate this as a recursion problem?

Work with a partner on this.

http://en.wikipedia.org/wiki/Eight_queens_puzzle

Backtracking

- Recursion and stacks illustrate a key concept in search: backtracking
- We can show that the recursion technique can exhaustively search all possible results in a systematic manner
- Learn more about searching spaces in other CS classes.

More Recursion later

- You will see more examples of recursion later when we cover more advanced sorting algorithms
  - Examples: Quick Sort, Merge Sort

5 Summary

- Recursion – The Mirrors
- Base Case:
  - Simplest possible version of the problem which can be solved easily
- Inductive Step:
  - Must simplify
  - Must arrive at some base case
- Easily visualized by a Stack
- Operations before and after the recursive calls come in FIFO and LIFO order, respectively
- Elegant, but not always the best (most efficient) way to solve a problem

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