Data Structures and Algorithms

Finding Shortest Way

From Here to There, Part I

Acknowledgement

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- We greatly appreciate support from Dr. Steven Halim for kindly sharing these materials.

Policies for students

- These contents are only used for students PERSONALLY.
- Students are NOT allowed to modify or deliver these contents to anywhere or anyone for any purpose.

Recording of modifications

- Currently, there are no modification on these contents.

Outline

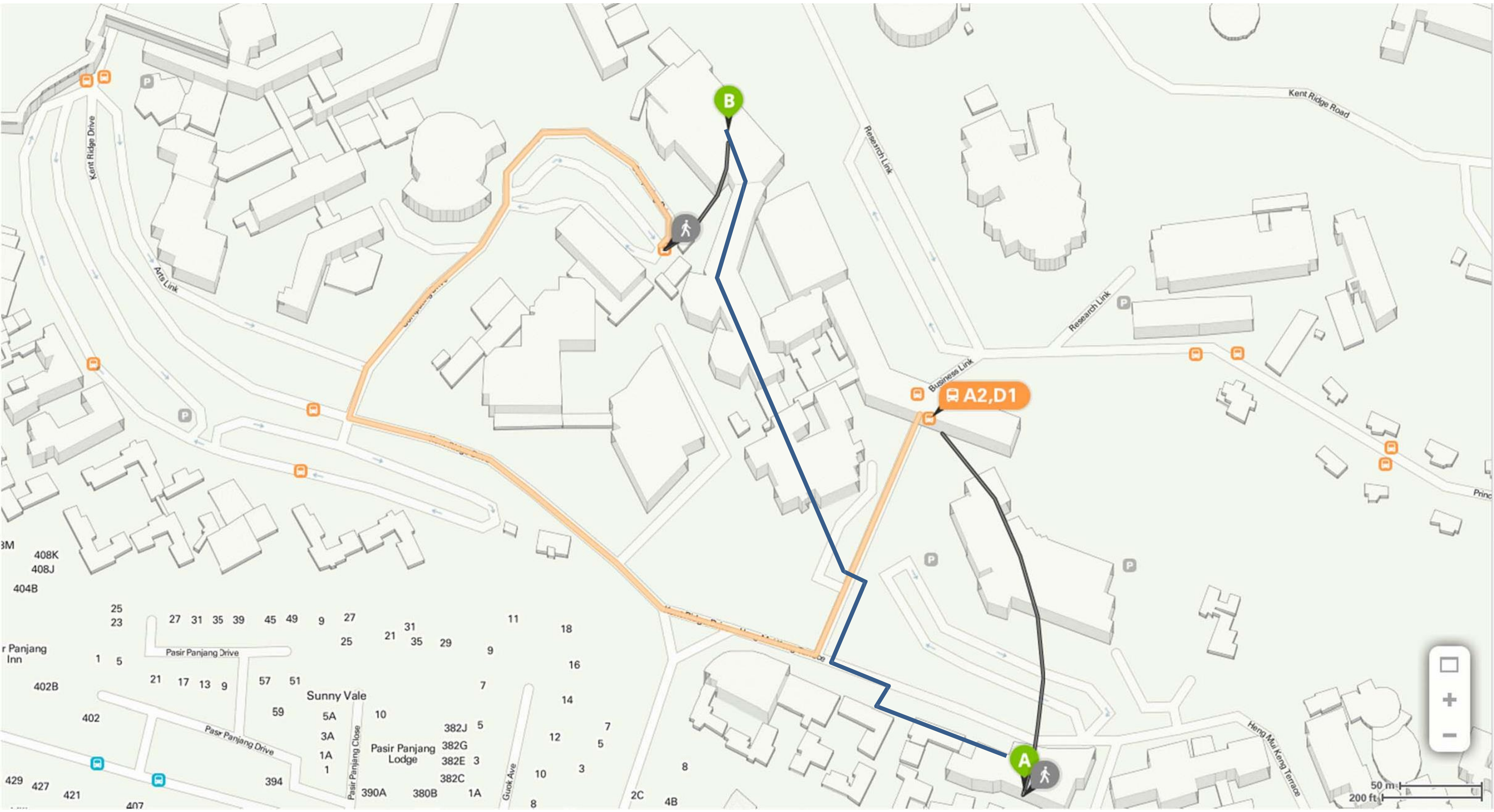
Single-Source Shortest Paths (SSSP) Problem

- Motivating example
- Some more definitions
- Discussion of negative weight edges and cycles

Algorithms to Solve SSSP Problem (CP3 Section 4.4)

- BFS algorithm cannot be used for the general SSSP problem
- Bellman Ford's algorithm
  - Pseudo code, example animation, and later: Java implementation
  - Theorem, proof, and corollary about Bellman Ford's algorithm

Motivating Example



Review: Definitions that you know (1)

- Vertex set **V** (e.g. street intersections, houses, etc)
- Edge set **E** (e.g. streets, roads, avenues, etc)
  - **Directed** (e.g. one way road, etc)
    - Note that we can simply use 2 edges (bi-directional) to model 1 undirected edge (e.g. two ways road, etc)
    - Recall that for the MST problem discussed in the previous lecture, we generally deal with **a connected** **undirected weighted graph**
  - **Weighted** (e.g. distance, time, toll, etc)
    - Weight function **w(a, b): E→R,** sets the weight of edge from **a** to **b**
- **Weighted** Graph: **G(V, E), w(a, b): E→R**

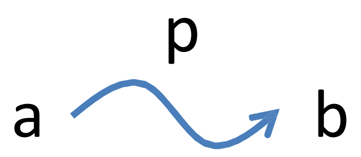
Review: Definitions that you know (2)

- **(Simple) Path** $p = \langle v_0, v_1, v_2, \square, v_k \rangle$
  - Where $(v_i, v_{i+1}) \in E, \forall_{0 \le i \le (k-1)}$
  - Simple = No repeated vertex!
- Shortcut notation: $v_0 \overset{p}{\rightsquigarrow} v_k$
  - Means that **p** is a path from v_0 to v_k
- **Path weight:** $PW(p) = \sum_{i=0}^{k-1} w(v_i, v_{i+1})$

More Definitions (1)

- **Shortest Path weight** from vertex **a** to **b**: **δ(a, b)**
  - δ is pronounced as ‘delta’

$$\delta(a,b) = \begin{cases} \min(PW(p)) & \text{If there exists such path} \\ \infty & \text{If } \mathbf{b} \text{ is unreachable from } \mathbf{a} \end{cases}$$

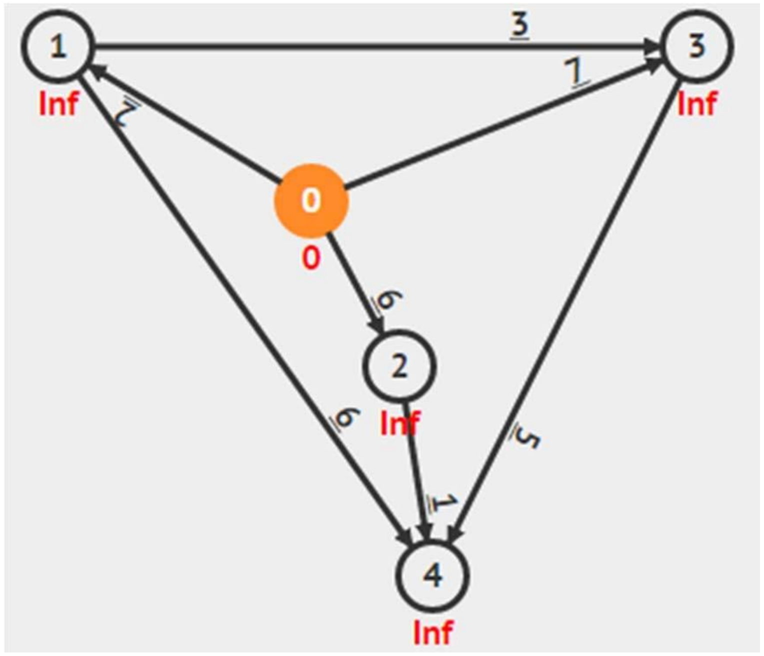


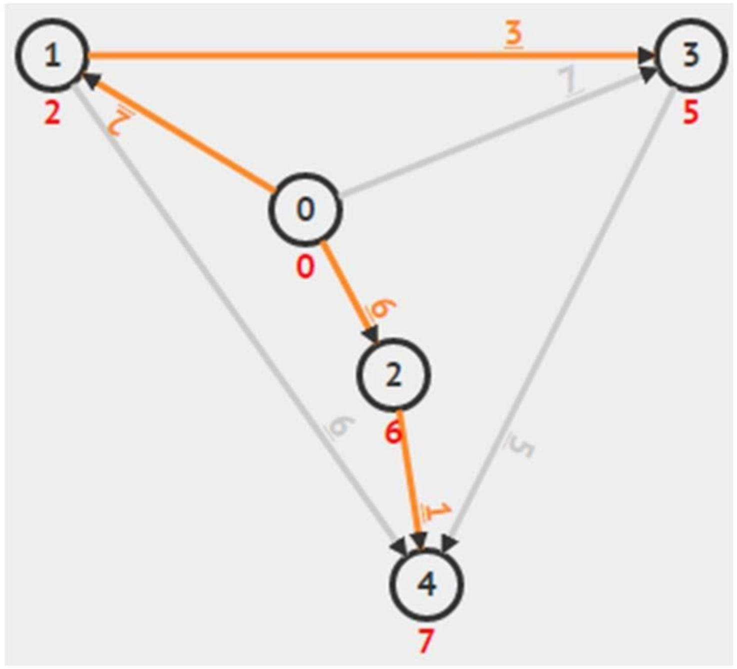
- **<u>Single-Source Shortest Paths</u>** (SSSP) Problem:
  - Given **G(V, E)**, **w(a, b): E->R**, and a **<u>source vertex s</u>**
  - Find **δ(s, b)** (+best paths) from vertex **s** to each vertex **b** ∈ **V**
    - i.e. From one source **to the rest**

More Definitions (2)

- **Additional Data Structures** to solve the SSSP Problem:
  - An array/Vector **D** of size **V** (**D** stands for ‘distance’)
    - Initially, **D[v] = 0** if **v** = **s**; otherwise **D[v] =** $\propto$ (a large number)
    - **D[v]** decreases as we find better paths
    - **D[v] ≥ δ(s, v)** throughout the execution of SSSP algorithm
    - **D[v] = δ(s, v)** at the end of SSSP algorithm
  - An array/Vector **p** of size **V**
    - **p[v]** = the predecessor on best path from source **s** to **v**
    - **p[s]** = NULL (not defined, we can use a value like -1 for this)
    - Recall: The usage of this array/Vector **p** is already discussed in BFS/DFS Spanning Tree (and also in PS4, Min Spanning Tree)

Example





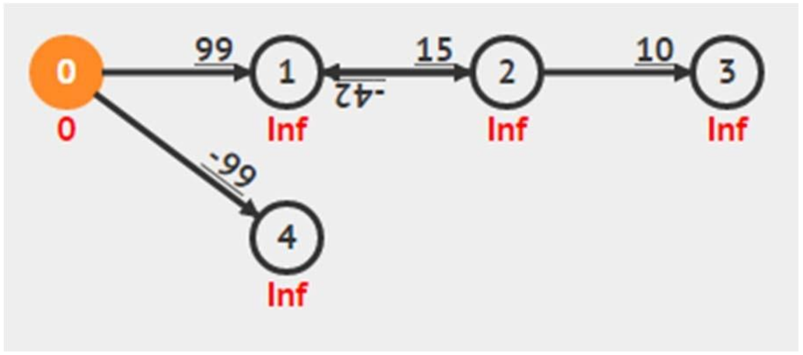
s = 0
Initially:
D[s] = D[0] = 0
D[v]= ∝ for the rest
Denoted as values in **red font/vertex**
p[s] = -1 (to say 'no predecessor')
p[v] = -1 for the rest
Denoted as **orange edges (none initially)**

s = 0
At the end of algorithm:
D[s] = D[0] = 0 (unchanged)
D[v] = δ(s, v) for the rest
e.g. D[2] = 6, D[4] = 7
p[s] = -1 (source has no predecessor)
p[v] = the origin of **orange edges** for the rest
e.g. p[0] = 2, p[4] = 0

Negative Weight Edges and Cycles

They exist in some applications

- Fictional application: Suppose you can travel back in time by passing through time tunnel (edges with negative weight)



- Shortest paths from 0 to {1, 2, 3} are **undefined**
  - 1→2→1 is a negative cycle as it has negative total path (cycle) weight
  - One can take 0→$\underline{1\rightarrow2\rightarrow1\rightarrow2\rightarrow1}$→… indefinitely to get -∞
- Shortest path from 0 to 4 is ok, with $\delta(0, 4) = -99$

SSSP Algorithms

This SSSP problem is a(nother) **well-known** CS problem

We will discuss three algorithms in this lecture:

1. O(**V**+**E**) BFS fails on *general case* of SSSP problem
   – Introducing the “initSSSP” and “Relax” operations
2. O(**VE**) Bellman Ford’s SSSP algorithm
   – General idea of SSSP algorithm
   – Trick to ensure termination of the algorithm
   – Bonus: Detecting negative weight cycle

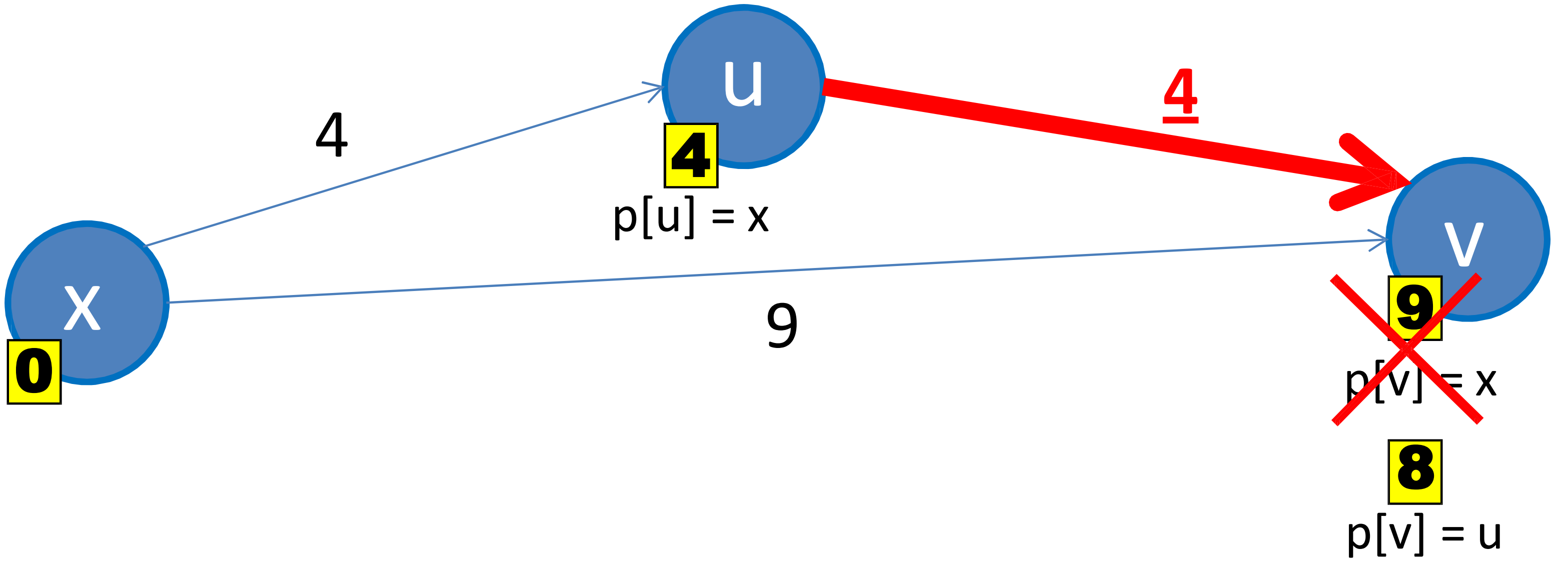
Initialization Step

We will use this initialization step
for all our SSSP algorithms

```
initSSSP(s)
  for each v  ∈V // initialization phase
    D[v] ← 1000000000 // use 1B to represent INF
    p[v] ← -1 // use -1 to represent NULL
  D[s] ← 0 // this is what we know so far
```

“Relax” Operation
(abbreviated name of these actions)

```
relax(u, v, w_u_v)
  if D[v] > D[u]+w_u_v // if SP can be shortened
    D[v] ← D[u]+w_u_v // relax this edge
    p[v] ← u // remember/update the predecessor
    // if necessary, update some data structure
```



Review: BFS

When the graph is **unweighted***, the SSSP can be viewed as a problem of finding the **least number of edges** traversed from source **s** to other vertices

* We can view each edge as having weight 1 or constant weight

The O(**V+E**) Breadth First Search (BFS) traversal algorithm precisely measures such thing

- BFS Spanning Tree = Shortest Paths Spanning Tree

Modified BFS

Do these <u>three </u>simple modifications:

1. Rename **`visited`** to **`D`** ☺
2. At the start of BFS, set **`D[v] = INF`** (say, 1B) for all **v** in **G**, except **`D[s] = 0`** ☺
3. Change this part (in the BFS loop) from:

```
if visited[v] = 0 // if v is not visited before
  visited[v] = 1; // set v as reachable from u
```

into:

```
if D[v] = INF // if v is not visited before
  D[v] = D[u]+1; // v is 1 step away from u ☺
```

Modified BFS Pseudo Code (1)

```
for all v in V
    D[v] ← INF
    p[v] ← -1
Q ← {s} // start from s
D[s] ← 0
```

Initialization phase

```
while Q is not empty
   u ←Q.dequeue()
   for all v adjacent to u // order of neighbor
      if D[v] = INF //  influences BFS
         D[v] ← D[u]+1 // visitation sequence
         p[v] ← u
         Q.enqueue(v)
```

Main loop

```
// we can then use information stored in D/p
```

Modified BFS Pseudo Code (2)

simpler form

```
initSSSP(s)
Q  ←{s} // start from s

while Q is not empty
  u  ←Q.dequeue()
  for all v adjacent to u // order of neighbor
    relax(u, v, 1); // the weight is 1

// we can then use information stored in D/p
```

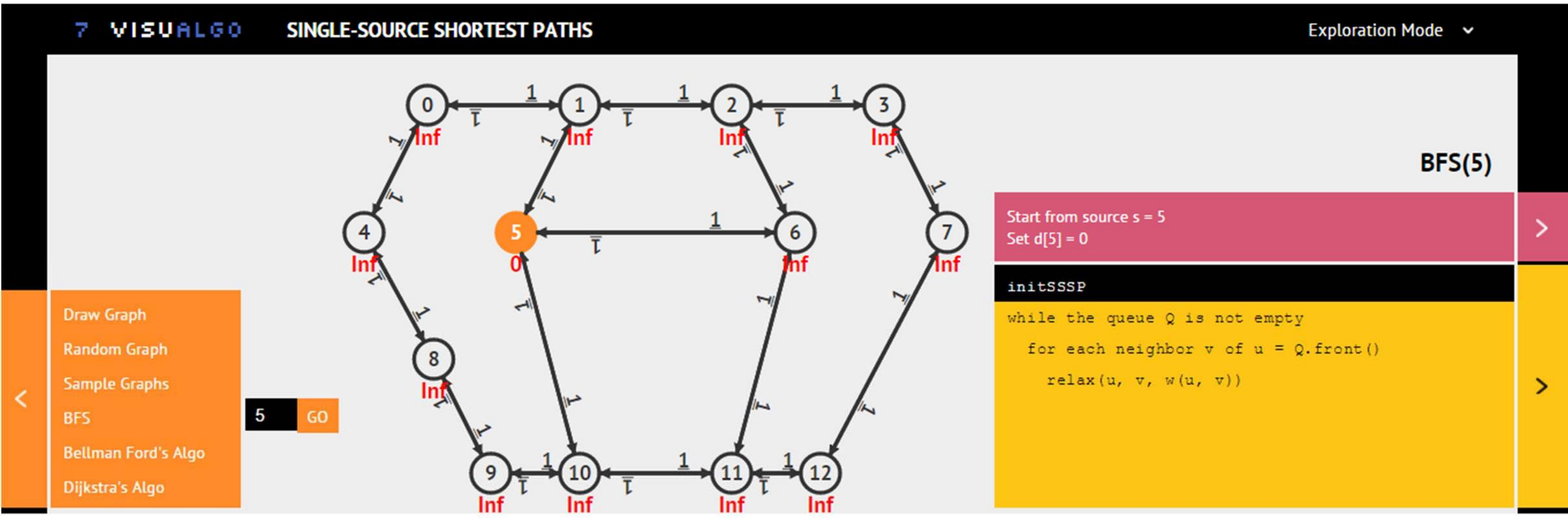
Initialization phase

Main loop

SSSP: BFS on Unweighted Graph

Ask VisuAlgo to perform BFS *from various sources* on the sample Graph (CP3 4.3)

In the screen shot below, we show the start of BFS from source vertex 5 (the same example as in Lecture 06, *it just looks messier due to bidirectional edges*)



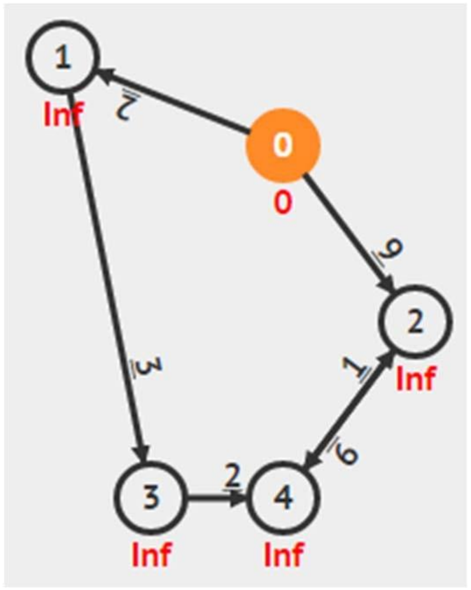
Start from source s = 5
Set d[5] = 0

```
initSSSP
while the queue Q is not empty
  for each neighbor v of u = Q.front()
    relax(u, v, w(u, v))
```

But BFS will not work on general cases

The shortest path from 0 to 2 is not path 0→2 with weight 9,
but a "detour" path 0→1→3→4→2 with weight 2+3+2+1= 8

- BFS cannot detect this and will only report path 0→2 (wrong answer)
- You can draw this graph @ VisuAlgo and try it for yourself



Rule of Thumb:
If you know for sure that your graph is unweighted (all edges have weight 1 or all edges have the same constant weight), then solve SSSP problem on it using the more efficient O(**V+E**) BFS algorithm

Reference: CP3 Section 4.4 (especially Section 4.4.4)

visualgo.net/sssp.html

BELLMAN FORD’S SSSP ALGORITHM

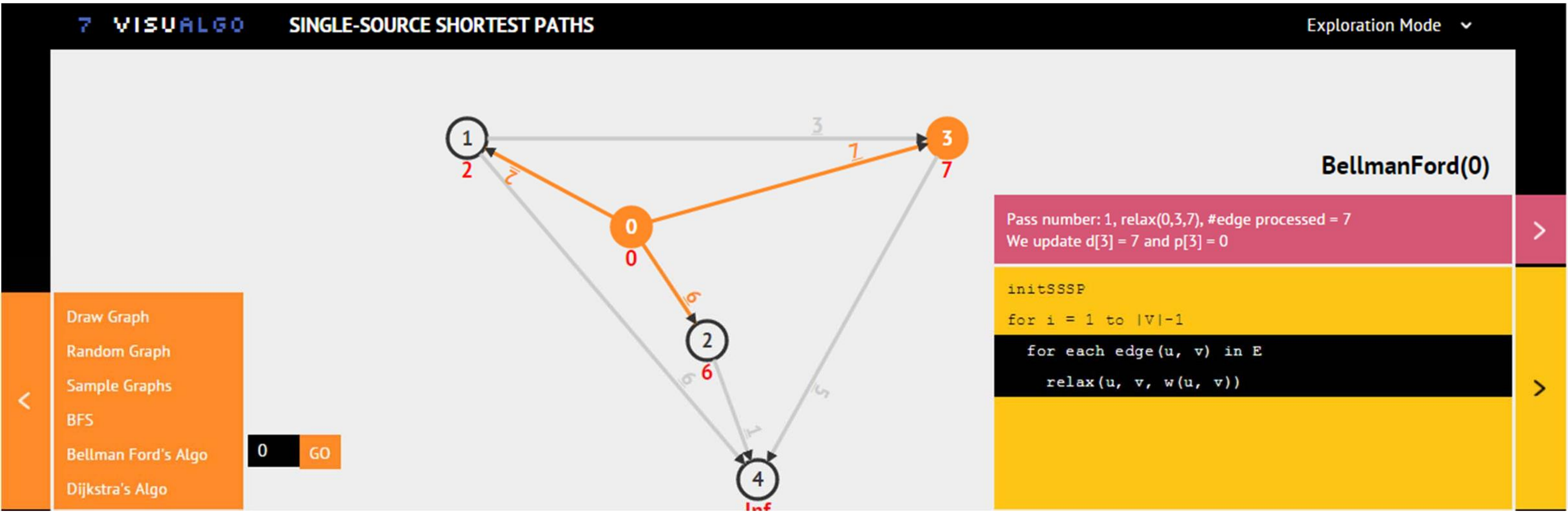
Bellman Ford's Algorithm

```
initSSSP(s)

// simple Bellman Ford's algorithm runs in O(VE)
for i = 1 to |V|-1 // O(V) here
  for each edge(u, v) ∊ E // O(E) here
    relax(u, v, w_u_v) // O(1) here

// At the end of Bellman Ford's algorithm,
// D[v] = δ(s, v) if no negative weight cycle exist

// Q: Why "relaxing all edges V-1 times" works?
```

SSSP: Bellman Ford’s

Ask VisuAlgo to perform Bellman Ford’s algorithm *<u>from various sources</u>* on the sample Graph (CP3 4.17)

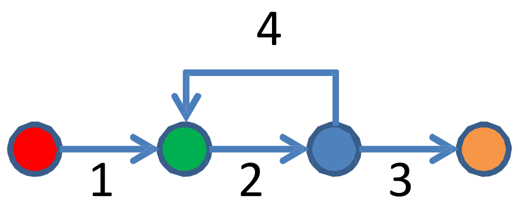
The screen shot below is *the first pass* of all **E** edges of **BellmanFord(0)**



Pass number: 1, relax(0,3,7), #edge processed = 7
We update d[3] = 7 and p[3] = 0

```
initSSSP
for i = 1 to |V|-1
  for each edge(u, v) in E
    relax(u, v, w(u, v))
```

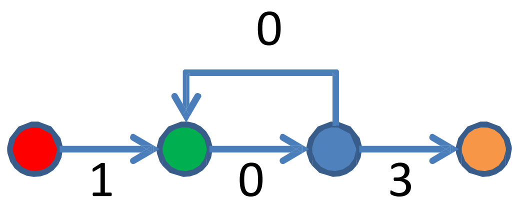
Theorem: If **G = (V, E)** contains no negative weight cycle, then the shortest path **p** from **s** to **v** is a **simple path**



Let's do a **Proof by Contradiction!**

1. Suppose the shortest path **p** is not a simple path
2. Then **p** contains one (or more) cycle(s)
3. Suppose there is a cycle **c** in **p** with positive weight
4. If we remove **c** from **p**,
   then we have a shorter 'shortest path' than **p**
5. This contradicts the fact that **p** is a shortest path

Theorem: If **G = (V, E)** contains no negative weight cycle, then the shortest path **p** from **s** to **v** is a **simple path**



6. Even if **c** is a cycle with zero total weight (it is possible!), we can still remove **c** from **p** without increasing the shortest path weight of **p**
7. So, **p** is a simple path (from point 5) or can always be made into a simple path (from point 6)

In another word, path **p** has at most **|V|-1** edges from the source **s** to the "furthest possible" vertex **v** in **G** (in terms of number of edges in the shortest path)

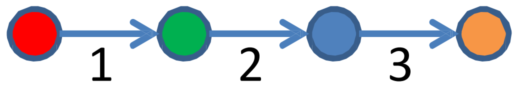
Theorem: If **G = (V, E)** contains no negative weight cycle, then after Bellman Ford's terminates **$D[v] = \delta(s, v), \forall v \in V$**

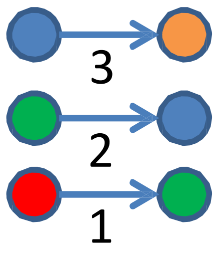
Let's do a **Proof by Induction**!

1. Consider the shortest path **p** from **s** to **v_i** (**p** will have minimum number of edges)
   - **v_i** is defined as a vertex which shortest path requires *i* hops (number of edges) from s
2. Initially **$D[v_0] = \delta(s, v_0) = 0$**, as **$v_0$** is just **s**
3. After **1** pass through **E**, we have **$D[v_1] = \delta(s, v_1)$**

Theorem: If $\mathbf{G = (V, E)}$ contains no negative weight cycle, then after Bellman Ford's terminates $\mathbf{D[v] = \delta(s, v), \forall v \in V}$

4. After 2 passes through E, we have $\mathbf{D[v_2] = \delta(s, v2)}$, ...
5. After **k** passes through **E**, we have $\mathbf{D[v_k] = \delta(s, v_k)}$
6. When there is no negative weight cycle, the shortest path **p** will be simple (see the previous proof)
7. Thus, after **|V|-1** iterations, the "furthest" vertex $\mathbf{v_{|V|-1}}$ from **s** has $\mathbf{D[v_{|V|-1}] = \delta(s, v_{|V|-1})}$
   – Even if edges in **E** are in *worst possible order*





“Side Effect” of Bellman Ford’s

Corollary: If a value **D[v]** *fails to converge* after **|V|-1** passes, then there exists a negative-weight cycle reachable from **s**

Additional check after running Bellman Ford’s:

```
for each edge(u, v) ∈ E
  if D[v] > D[u]+w(u, v)
    report negative weight cycle exists in G
```

Java Implementation (2)

See BellmanFordDemo.java

- Now implemented using **AdjacencyList** ☺
  - **AdjacencyList** or **EdgeList** can be used to have an O(**VE**) Bellman Ford's

Show performance on:

- Small graph without negative weight cycle → OK, in O(**VE**)
- Small graph with negative weight cycle → terminate in O(**VE**)
  - Plus we can report that negative weight cycle exists
- Small graph; some negative edges; no negative cycle → OK

Summary

Introducing the SSSP problem

Revisiting BFS algorithm for <u>unweighted</u> SSSP problem

- But it fails on general case

Introducing Bellman Ford's algorithm

- This one solves SSSP for general weighted graph in O(**VE**)
- Can also be used to detect the presence of -ve weight cycle

PS5* should now bedoable ☺

* The first Subtask of PS5…

(but I will only open it on Saturday, 17 Oct 2015, 8am)

Subtask B (easy), Subtask C (medium-hard), and Subtask E (R-option, also medium-hard) require something else ☺

Train first to check basic understanding of the past two lectures on graph algorithms:
http://visualgo.net/training.html?diff=Medium&n=5&tl=0&module=mst,sssp