B38EM Introduction to Electricity and Magnetism
Lecture 6

Magnetostatics

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Outline & Outcome

- Revision on Electrostatics
- Lorentz force
- Biot-Savart law
- Ampere’s law
- Exercises

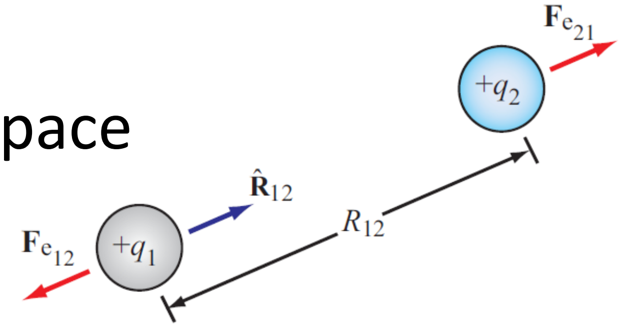
References & Resources

- Elements of Electromagnetics (7th Edition), by Sadiku, Oxford University Press
- Fundamentals of Applied Electromagnetics (7th Edition), by Ulaby and Ravaioli
- Field and Wave Electromagnetics (2nd Edition), by David K. Cheng
- ……

Electrostatics (1)

- Coulomb's Law

$$\boldsymbol{F}_{e21} = \hat{\boldsymbol{R}}_{12} \frac{q_1 q_2}{4\pi\varepsilon_0 R_{12}^2} \qquad (N) \qquad \text{in free space}$$



$1/4\pi\varepsilon_0$ = **electric constant** (AKA k_e) = $9 \cdot 10^9$ N·m^2 / C^2 in free-space.

ε_0 = **dielectric permitivity** of free space = $8.85 \cdot 10^{-12}$ F/m **(a constant)**

- Electric Field Intensity (***E***)

$$\boldsymbol{E} = \hat{\boldsymbol{R}} \frac{q}{4\pi\varepsilon_0 R^2} \qquad (V/m) \qquad \text{in free space}$$

force on a unit charge

Electrostatics (2)

- Electric flux density (***D***)

$$\boldsymbol{E} = \hat{\boldsymbol{R}} \frac{q}{4\pi\varepsilon_0 R^2} \qquad (V/m) \qquad \text{in free space}$$

We define *q*/*S* as ***D*** Electric displacement

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} \qquad (C/m^2)$$

Rules for electric field

- Field lines: point from (+) charge towards (-) charge.
- Field lines are perpendicular (⊥) to conducting surfaces.
- Field lines represent the **force vector** experienced by a test charge.
- Field lines never intersect. (Force has only one direction!)

Electrostatics (3)

- Gauss' law

The total electric flux is independent of the surface area and equal to the net charge enclosed.

$$\psi = DS = Q \qquad (C)$$

- Electric potential

$$V_{AB} = -\int_A^B \boldsymbol{E} \cdot d\boldsymbol{l} \qquad (volts)$$

$\boldsymbol{E} = -\nabla V$ The gradient of a scalar *V*

Magnetostatics

- Electrostatics

  Stationary or slow moving ELECTRIC CHARGES.

- Magnetostatics

  CURRENTS are steady.

Magnetostatics

- Steady electric current

Current consists of charges in motion.

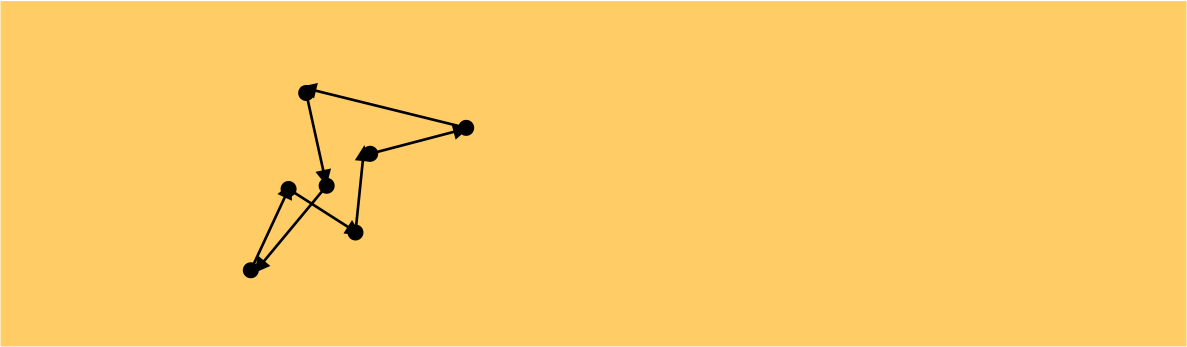
The unit of current is the ampere (A), corresponding to a flow of one coulomb per second (1 C/s).

Types of current:

- conduction
- polarization

Magnetostatics

- Charge motion – no electric field



conductor

There is no *net* motion of charge.

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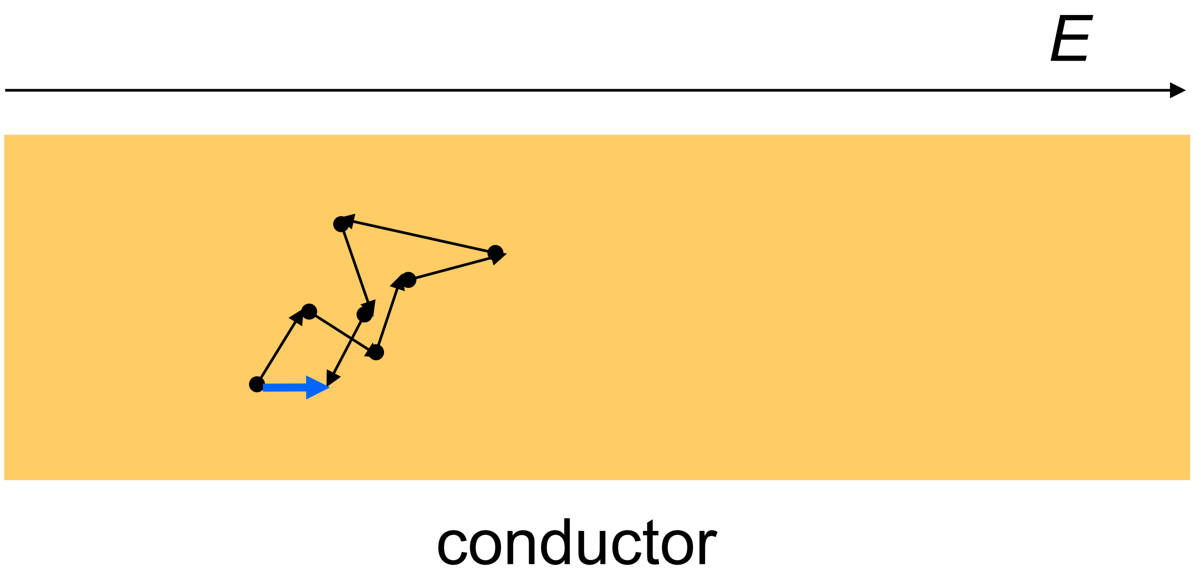
- Charge motion – electric field applied



There is a small *net* motion of charge in the direction of the field.

Magnetostatics

- Generalised Ohm's Law

The electric field imparts a small net *drift velocity* to free electrons. The resulting movement of positive charge in the direction of the field is described by Ohm's Law:

$$E = \rho J$$

ρ = resistivity ($\Omega \cdot$m)

J = current density (A / m^2)

Magnetostatics

- Properties of conductors

Temperature dependence:

$$R(T) = R_1\left[1 + \alpha(T - T_1)\right]$$

R_1 = resistance at temperature T_1

α = temperature coefficient of resistance

element	ρ **(** $\Omega\cdot m$ **)**	α **at** 0^0**C**
copper	1.76×10^{-8}	0.0043
iron	9.4×10^{-8}	0.0055
aluminium	2.83×10^{-8}	0.0043

Magnetostatics

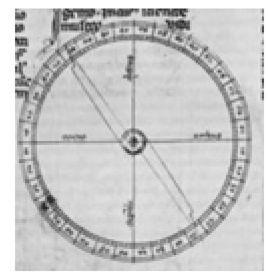
- History

Ancient Greeks (~ 500BC)

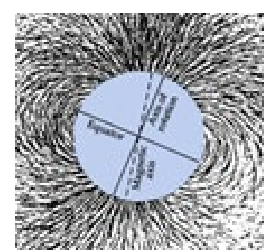
Invention of compass

China (~ 1000AD)

Europe (~1300AD)



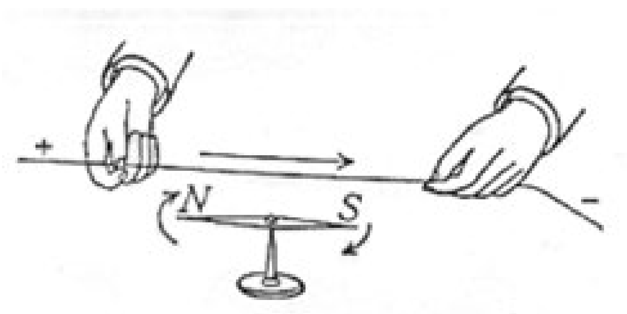
Willian Gilbert (16^{th} Century): Earth is a giant magnet

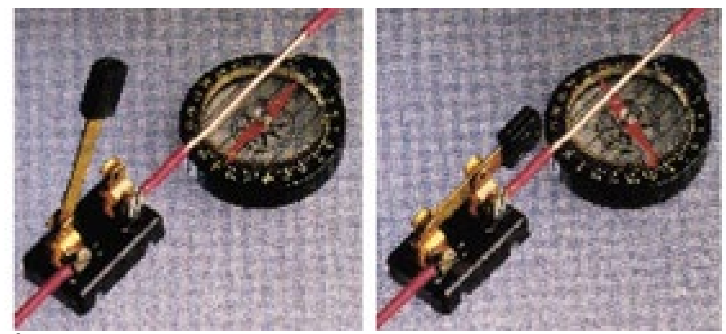


Magnetostatics

- History

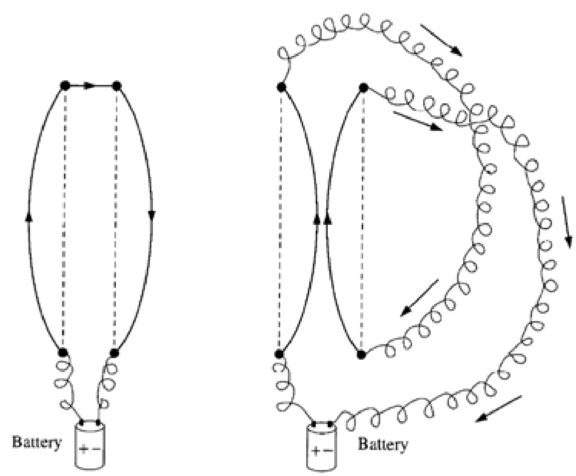
  1820: Hans Christian Orsted





This experiment united electricity and magnetism

One week later: Andre-Marie Ampere



Electrostatics cannot explain this effect
→ **magnetic fields**

Magnetostatics

- (Revision) What is Electric Field Vector ***E***?

  The electric force experienced at that point by a unit charge (1 Coulomb).

- What is Magnetic Field Vector ***H***? (***B*** = *μ**H***)

  B: The magnetic force experienced at that point by a unit charge (1 Coulomb) moving with *v* = 1m/s.

B: the number of field lines passing per unit area through a surface

$\mathbf{F}_{\mathrm{m}} = q(\mathbf{v}\times\mathbf{B}) = q\cdot v\cdot B\cdot\sin\theta \ (N)$

"***B*** = magnetic flux density"

θ = angle (***v***, ***B***)

Units of *B*: $(N\cdot s)/(C\cdot m) = 1\ N/(A\cdot m) = 1\ \mathrm{Wb/m^2} = 1$ (T)esla

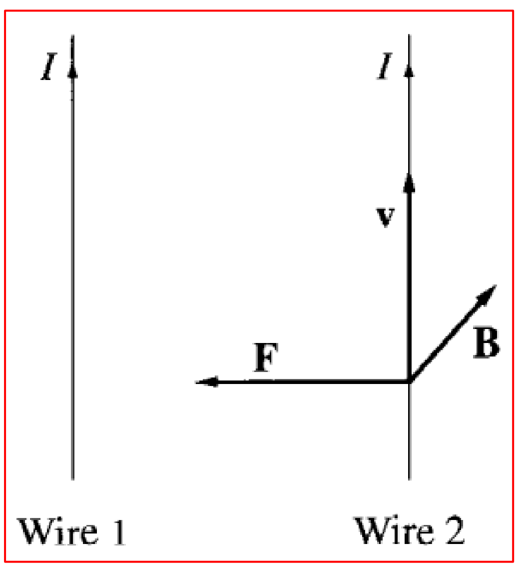
or **Gauss**: 1 Gauss = 10^{-4} T Earth ***B***-field ≈ 0.5Gauss

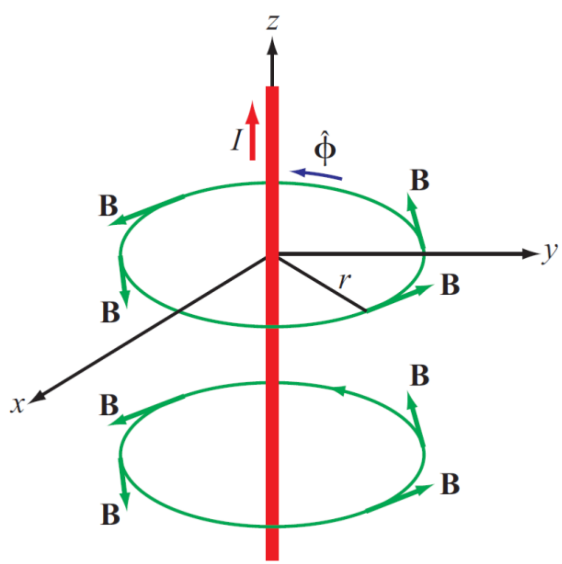
Magnetostatics

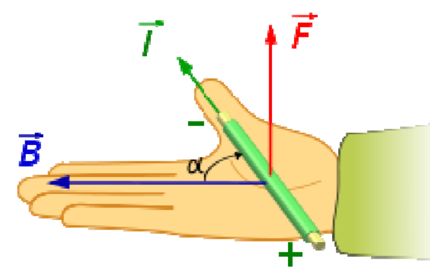
- Lorentz Force

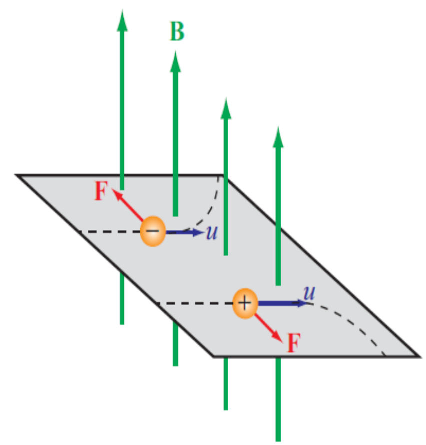
A particle of charge q moving with a velocity $\boldsymbol{v}$ in an electric field ***E*** and a magnetic field ***H*** experiences a force.

$$\boldsymbol{F} = \boldsymbol{F}_e + \boldsymbol{F}_m = q\boldsymbol{E} + q(\boldsymbol{v} \times \boldsymbol{B})$$









Magnetostatics

- Magnetic Flux Density

$\boldsymbol{B} = \mu \boldsymbol{H}$

$\mu = \mu_r \mu_0$

Permeability is the degree of magnetization of a material in response to a magnetic field.

$$\mu_o = 4\pi \times 10^{-7}\,\mathrm{H/m}$$

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \qquad (m/s)$$

Magnetostatics

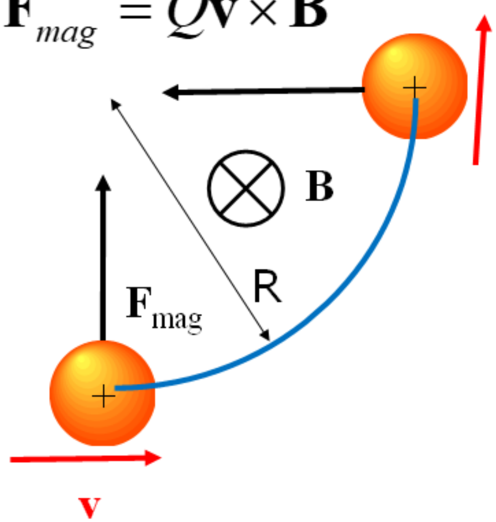
- Motion of charged particles in a magnetic field

Graphical convention:

vector goes into the screen: $\otimes$

vector comes out of the screen: $\odot$

$$\mathbf{F}_{mag} = Q\mathbf{v} \times \mathbf{B}$$



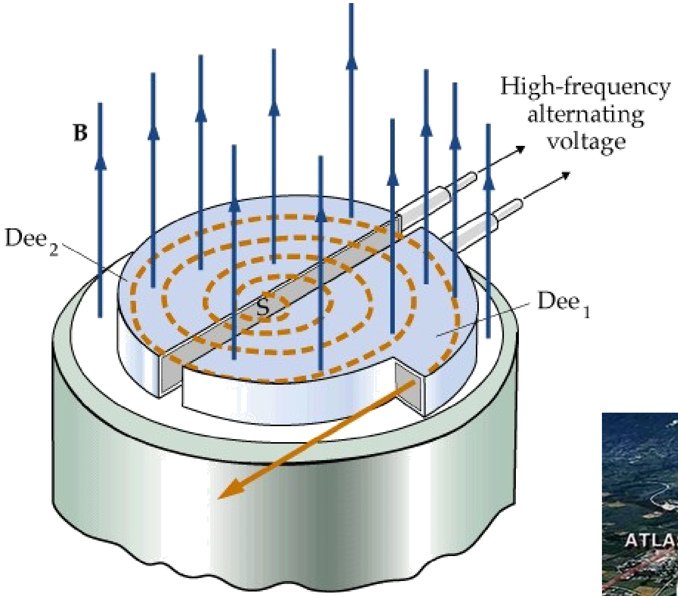
$$F_{mag} = QvB = \frac{mv^2}{R}$$

Lorentz force always radial

Magnetostatics

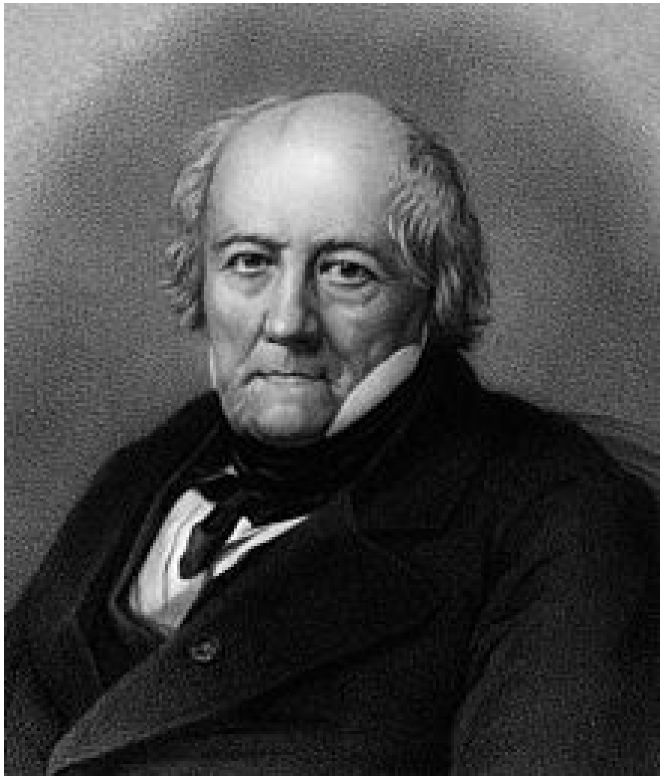
- Charged particle accelerator

$$F_{mag} = QvB = \frac{mv^2}{R}$$

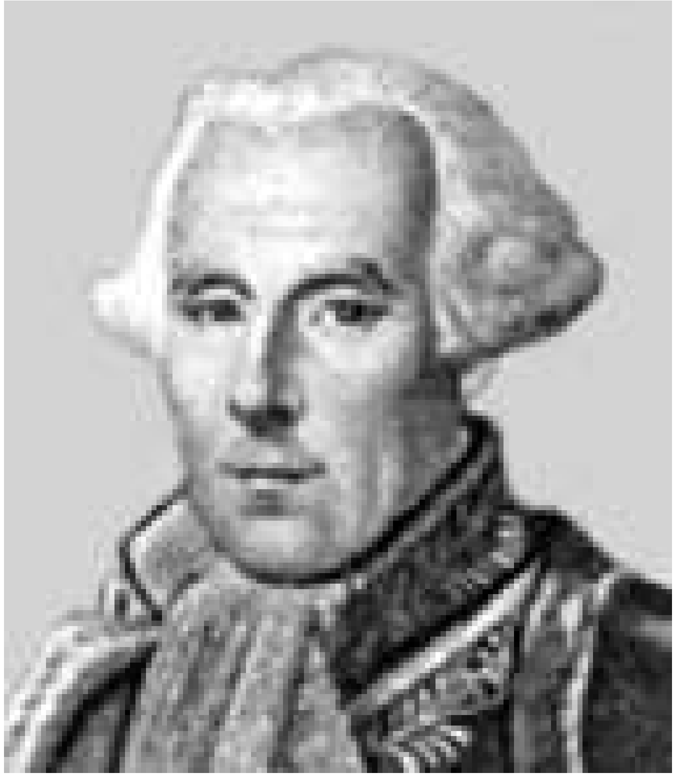


Magnetostatics

- Biot-Savart's Law



Jean-Baptiste Biot
1774 - 1862



Félix Savart
1791 - 1841

Magnetostatics

- Biot-Savart's Law

The differential magnetic field intensity dH produced at a point *P* by the differential current element, $I \cdot dl$, is proportional to the product $I \cdot dl$ and the sine of the angle θ between the element and the line joining *P* to the element and is inversely proportional to the square of the distance, R, between *P* and the element.

$$dH \propto \frac{I\, dl \sin\theta}{R^2}$$

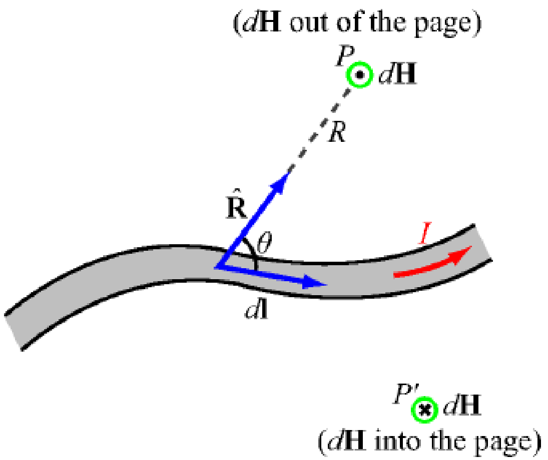
$$dH = \frac{kI\, dl \sin\theta}{R^2}$$

$$dH = \frac{I\, dl \sin\theta}{4\pi R^2}$$

$$d\vec{H} = \frac{I\, d\vec{l} \times \hat{a}_R}{4\pi R^2} = \frac{I\, d\vec{l}}{4\pi R^2} \times \hat{a}_R$$

$$d\vec{H} = \frac{I\, d\vec{l}}{4\pi R^2} \times \frac{\vec{R}}{|\vec{R}|} = \frac{I\, d\vec{l}}{4\pi R^2} \times \frac{\vec{R}}{R}$$

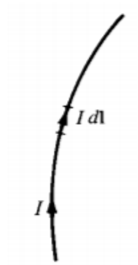
$$d\vec{H} = \frac{I\, d\vec{l} \times \vec{R}}{4\pi R^3}$$



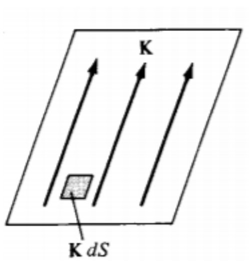
k is proportionality constant: $k = \dfrac{1}{4\pi}$

Magnetostatics

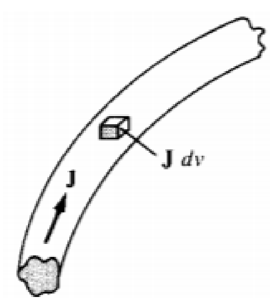
- Biot-Savart's Law



$$\vec{H} = \int_L \frac{I\, d\vec{l} \times \hat{a}_R}{4\pi R^2}$$ (line current)



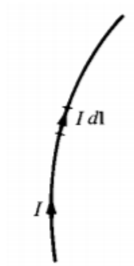
$$\vec{H} = \int_S \frac{\vec{K}\, d\vec{S} \times \hat{a}_R}{4\pi R^2}$$ (surface current)



$$\vec{H} = \int_v \frac{\vec{J}\, dv \times \hat{a}_R}{4\pi R^2}$$ (volume current)

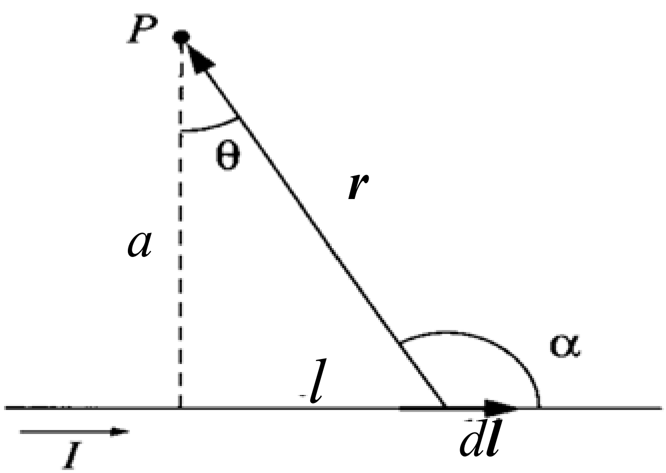
Magnetostatics

- Biot-Savart's Law (example)



$$H = \int_l \frac{I d\mathbf{l} \times \hat{\mathbf{r}}}{4\pi r^2}$$

Find the magnetic field at distance a from a long straight wire carrying a steady current I.



$d\boldsymbol{l} \times \hat{\boldsymbol{r}}$: points out of the screen, and has the magnitude of $dl\sin\alpha = dl\cos\theta$

$l = a\tan\theta,\qquad a = r\cos\theta$

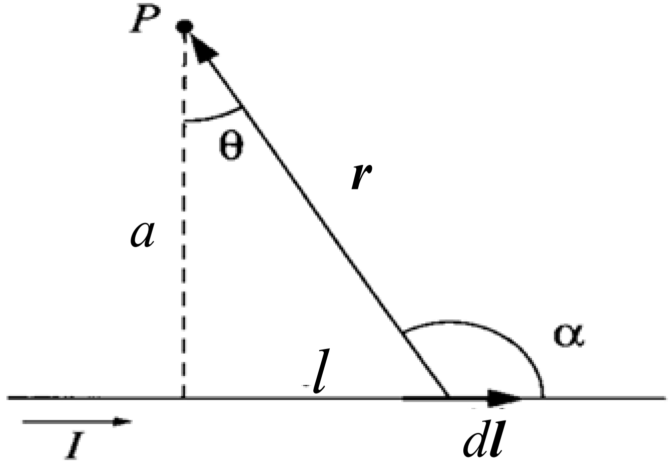
$$dl = \frac{a}{\cos^2\theta}d\theta \qquad \frac{1}{r^2} = \frac{\cos^2\theta}{a^2}$$

$$H = \frac{I}{4\pi}\int_{\theta_1}^{\theta_2} \frac{\cos^2\theta}{a^2}\frac{a}{\cos^2\theta}\cos\theta d\theta = \frac{I}{4\pi a}\int_{\theta_1}^{\theta_2}\cos\theta d\theta = \frac{I}{4\pi a}\left(\sin\theta_2 - \sin\theta_1\right)$$

Magnetostatics

- Biot-Savart's Law (example)

Find the magnetic field at distance a from a long straight wire carrying a steady current I.

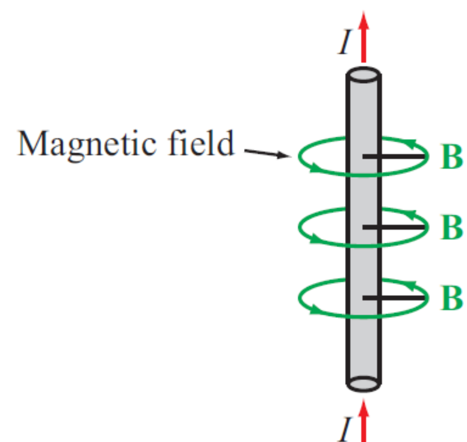


$$H = \frac{I}{4\pi a}\left(\sin\theta_2 - \sin\theta_1\right)$$

For **infinitely long wire**: $\theta_1 = -90°$ and $\theta_2 = +90°$, then

$$H = \frac{I}{2\pi a} \qquad B = \frac{\mu_0 I}{2\pi a}$$

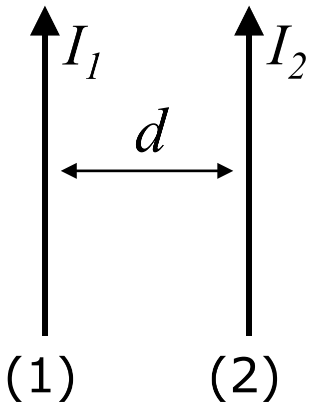
$$\boldsymbol{H} = \hat{\phi}\frac{I}{2\pi a} \qquad \boldsymbol{B} = \hat{\phi}\frac{\mu_0 I}{2\pi a}$$



Magnetostatics

- Two parallel wires

Determine the force of attraction between two long parallel wires, at distance d apart and carrying currents I_1 and I_2.



$$B = \frac{\mu_0 I_1}{2\pi d}$$

$$\sum \boldsymbol{F}_{mag} = \int \Delta Q_2 (\boldsymbol{v} \times \boldsymbol{B}) \qquad I_2 = \Delta Q_2 / \Delta t$$

$$\boldsymbol{F}_{mag} = \int I_2 (\boldsymbol{v} \times \boldsymbol{B}) dt \qquad v\Delta t = \Delta l$$

$$\boldsymbol{F}_{mag} = I_2 \int (d\boldsymbol{l} \times \boldsymbol{B})$$

$$F_{mag} = \frac{\mu_0 I_1 I_2}{2\pi d} \int dl$$

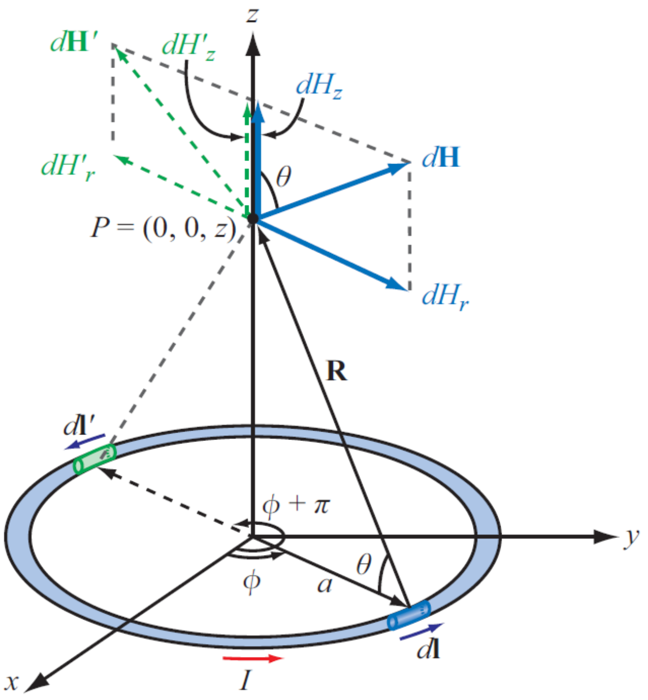
Force per unit length:

$$\frac{F_{mag}}{l} = \frac{\mu_0 I_1 I_2}{2\pi d}$$

Magnetostatics

- Magnetic field of a loop

$$dH = \frac{I}{4\pi R^2}\,|d\mathbf{l} \times \hat{\mathbf{R}}| = \frac{I\,dl}{4\pi(a^2+z^2)}$$



$d\mathbf{H}$ is in the r–z plane, therefore it has components dH_r and dH_z.

The dH_r-components due to **dl** and **dl′** cancel.

The dH_z-components due to **dl** and **dl′** add (same direction)

Hence for element dl:
$$d\mathbf{H} = \hat{\mathbf{z}}\,dH_z = \hat{\mathbf{z}}\,dH\cos\theta = \hat{\mathbf{z}}\,\frac{I\cos\theta}{4\pi(a^2+z^2)}\,dl$$

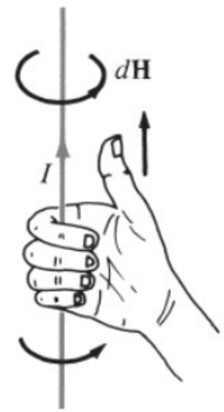
For entire loop, integrate =>
$$\mathbf{H} = \hat{\mathbf{z}}\,\frac{I\cos\theta}{4\pi(a^2+z^2)}\oint dl = \hat{\mathbf{z}}\,\frac{I\cos\theta}{4\pi(a^2+z^2)}(2\pi a).$$

at center of loop (z=0) =>
$$\mathbf{H} = \hat{\mathbf{z}}\,\frac{I}{2a} \qquad (\text{at } z = 0)$$

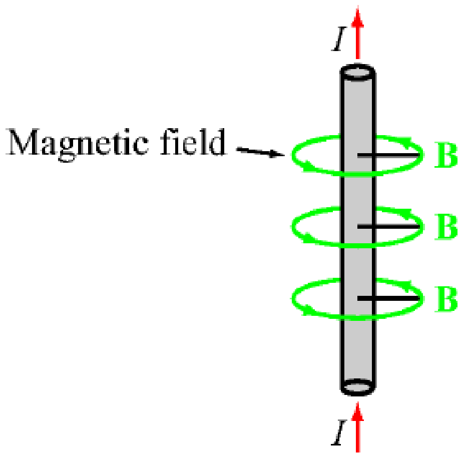
far from loop ($z^2 >> a^2$) =>
$$\mathbf{H} = \hat{\mathbf{z}}\,\frac{Ia^2}{2|z|^3} \qquad (\text{at } |z| \gg a).$$

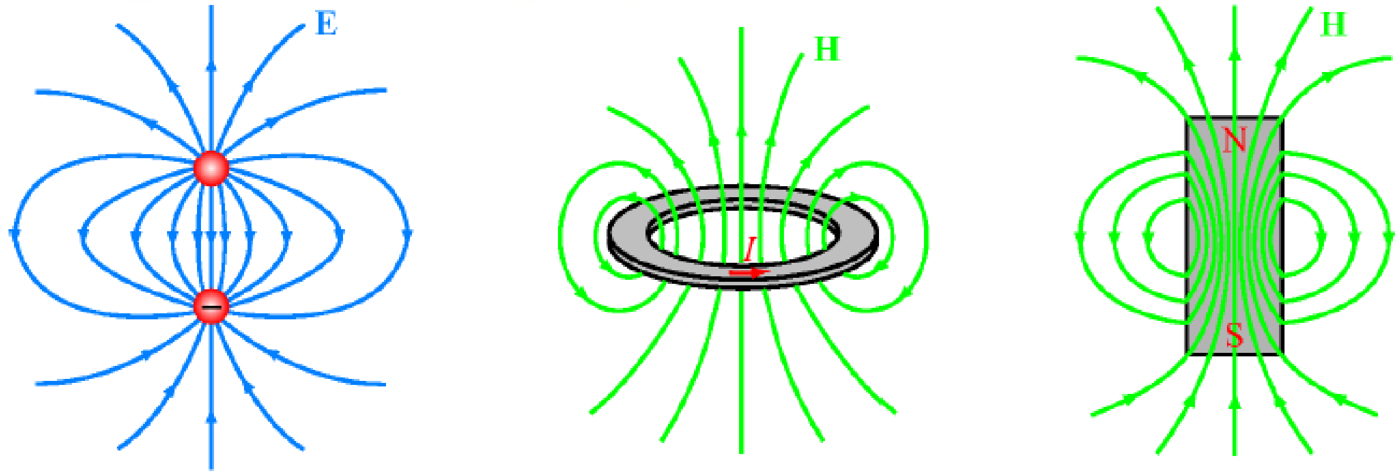
Magnetostatics

- Biot-Savart's Law (example)



Determining the direction of $d\mathbf{H}$ using the right-hand rule





(a) Electric dipole (b) Magnetic dipole (c) Bar magnet

Magnetostatics

- Gauss' Law of Magnetostatics

We know from Biot-Savart Law that: $$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{|\boldsymbol{\mathscr{r}}|^2} d\tau'$$

-- What is $\nabla\cdot\mathbf{B}$? The divergence of a vector

$$\nabla\cdot\mathbf{B} = \frac{\mu_0}{4\pi}\int \nabla\cdot\left(\mathbf{J}(\mathbf{r}')\times\frac{\hat{\boldsymbol{\mathscr{r}}}}{|\boldsymbol{\mathscr{r}}|^2}\right) d\tau'$$

with: $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$ (product rule)

follows..
$$\nabla\cdot\left(\mathbf{J}(\mathbf{r}')\times\frac{\hat{\boldsymbol{\mathscr{r}}}}{|\boldsymbol{\mathscr{r}}|^2}\right) = \frac{\hat{\boldsymbol{\mathscr{r}}}}{|\boldsymbol{\mathscr{r}}|^2}\cdot\left(\nabla\times\mathbf{J}(\mathbf{r}')\right) - \mathbf{J}\cdot\left(\nabla\times\frac{\hat{\boldsymbol{\mathscr{r}}}}{|\boldsymbol{\mathscr{r}}|^2}\right)$$

with $\nabla\times\mathbf{J}(\mathbf{r}') = 0$ ($\mathbf{J}$ depends on $\mathbf{r}'$ but not on $\mathbf{r}$)

$\nabla\times\frac{\hat{\boldsymbol{\mathscr{r}}}}{|\boldsymbol{\mathscr{r}}|^2} = 0$ (we know that from Electrostatics..)

follows

$$\boxed{\nabla\cdot\mathbf{B} = 0 \quad \Leftrightarrow \quad \oint_S \mathbf{B}\cdot d\mathbf{a} = 0}$$

Gauss' Law of Magnetostatics **no magnetic monopoles!**

Magnetostatics

- Ampere's law of magnetostatics

We know from Biot-Savart Law that: $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{|\boldsymbol{\mathscr{r}}|^2} d\tau'$

-- What is $\nabla \times \mathbf{B}$? The curl of a vector

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \times \left(\mathbf{J}(\mathbf{r}') \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{|\boldsymbol{\mathscr{r}}|^2} \right) d\tau'$$

with $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$ (another product rule)

follows $\nabla \times \left(\mathbf{J}(\mathbf{r}') \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{|\boldsymbol{\mathscr{r}}|^2} \right) = \mathbf{J} \left(\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{|\boldsymbol{\mathscr{r}}|^2} \right) - (\mathbf{J} \cdot \nabla) \frac{\hat{\boldsymbol{\mathscr{r}}}}{|\boldsymbol{\mathscr{r}}|^2}$ deriv. of $\mathbf{J}$ are zero

with $\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{|\boldsymbol{\mathscr{r}}|^2} = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$ and $\int -(\mathbf{J} \cdot \nabla) \frac{\hat{\boldsymbol{\mathscr{r}}}}{|\boldsymbol{\mathscr{r}}|^2} d\tau' = 0$ (using yet another product rule..)

follows

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}}$$

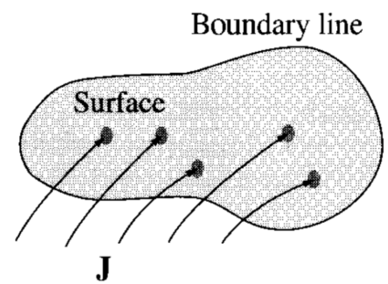
Ampère's Law of Magnetostatics integral form?

Magnetostatics

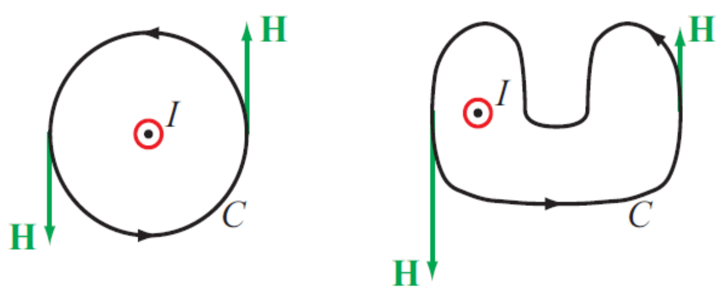
- Ampere's law of magnetostatics

Using Stokes' theorem:

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} \quad \longleftrightarrow \quad \oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I}$$



Ampere's Law states that the line integral of ***B*** (or ***H***) around a closed contour *C* is equal to the current traversing the surface bounded by the contour.



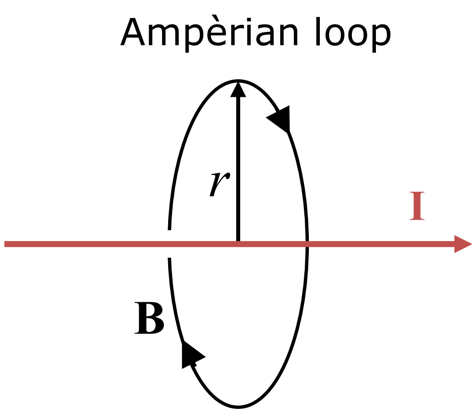
Ampere's Law equation offers an efficient way to calculate magnetic fields given appropriate current symmetries.

Magnetostatics

- Revisited: Magnetic Field of a Linear Conductor

$$\oint \boldsymbol{H} \cdot d\boldsymbol{l} = H\oint dl = H\int_0^{2\pi} r \cdot d\theta = H2\pi r = I$$

$$H = \frac{I}{2\pi r}$$



Much much easier!

Magnetostatics

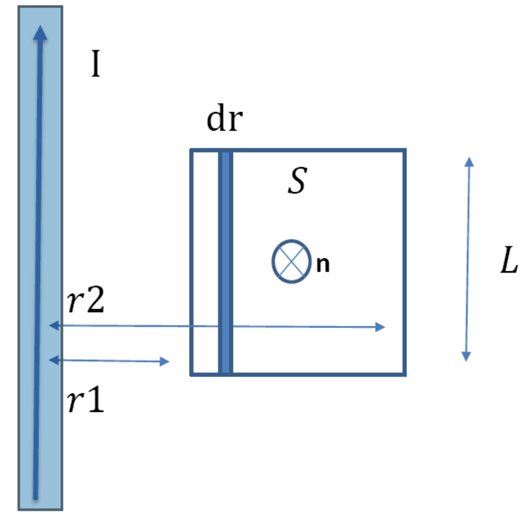
- Example: Ampere's law

Consider a straight non-magnetic conductor of circular cross-section and radius *a* carrying a current with uniform current density **J** (A/m^2) in the vertical direction. Using Ampere's law find the magnetic field inside and outside the conductor.

Magnetostatics

- Example: Magnetic flux

Find the magnetic flux Φ_B that passes through a wire frame placed next to a wire of current I.



Solution:

$$\Phi_B = \iint_s \boldsymbol{B} \cdot dS$$

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi r} \hat{\boldsymbol{n}}$$

$$d\boldsymbol{S} = L dr \cdot \hat{\boldsymbol{n}}$$

$$\Phi_B = \int_{r_1}^{r_2} \frac{\mu_0 I}{2\pi r} L \cdot dr \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}} = \frac{\mu_0 I}{2\pi} L \int_{r_1}^{r_2} \frac{1}{r} dr = \frac{\mu_0 I}{2\pi} L \ln\left(\frac{r_2}{r_1}\right)$$

Magnetostatics

- Example:

  In a Cartesian coordinate, z-axis carry currents of 20A along z-axis. Calculate $\vec{H}$ at point (6, 8, -6).

Magnetostatics

Solution: $I = 20\,A$ at z - axis $= (6,8,\text{-}6)$

$$\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$$

$$\hat{a}_\phi = \hat{a}_l \times \hat{a}_\rho$$

$$\hat{a}_\phi = \hat{a}_z \times \frac{(6-0,8-0,-6-(-6))}{|6-0,8-0,-6-(-6)|}$$

$$\hat{a}_\phi = \hat{a}_z \times \frac{(6,8,0)}{\sqrt{6^2+8^2}}$$

$$\hat{a}_\phi = \hat{a}_z \times \frac{(6,8,0)}{\sqrt{100}}$$

$$\hat{a}_\phi = \hat{a}_z \times \frac{(6,8,0)}{10}$$

Solution:

$$\hat{a}_\phi = \hat{a}_z \times \frac{(6,8,0)}{10}$$

$$\hat{a}_\phi = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0 & 0 & 1 \\ 6/10 & 8/10 & 0 \end{vmatrix}$$

$$\hat{a}_\phi = \hat{a}_x\left[(0)(0)-(1)(8/10)\right] - \hat{a}_y\left[(0)(0)-(1)(6/10)\right] + \hat{a}_z\left[0)(8/10)-(0)(6/10)\right]$$

$$\hat{a}_\phi = \frac{-8\hat{a}_x + 6\hat{a}_y}{10}$$

$$\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$$

$$\vec{H} = \frac{20}{2\pi\sqrt{(8)^2+6^2}}\left(\frac{-8\hat{a}_x + 6\hat{a}_y}{10}\right)$$

$$\vec{H} = \frac{20}{2\pi\sqrt{100}}\left(\frac{-8\hat{a}_x + 6\hat{a}_y}{10}\right)$$

$$\vec{H} = \frac{20}{2\pi 10}\left(\frac{-8\hat{a}_x + 6\hat{a}_y}{10}\right)$$

$$\vec{H} = -0.255\hat{a}_x + 0.191\hat{a}_y \text{ A/m}$$

Magnetostatics

- Example:

In a conducting medium,

$$\vec{H} = y^2 z \hat{a}_x + 2(x+1) yz \hat{a}_y - (x+1) z^2 \hat{a}_z \quad \text{A/m.}$$

Determine $\vec{J}$ at (1,0,-3) and the current, *I* passing through $y = 1, 0 \le x \le 1, 0 \le z \le 1.$

Magnetostatics

Solution:

$$\vec{H} = y^2 z\hat{a}_x + 2(x+1)yz\hat{a}_y - (x+1)z^2\hat{a}_z \quad \text{A/m.}$$

$$\nabla \times \vec{H} = \vec{J}$$

$$\vec{J} = \nabla \times \vec{H} = \hat{a}_x\left(\frac{\partial}{\partial y}H_z - \frac{\partial}{\partial z}H_y\right) + \hat{a}_y\left(\frac{\partial}{\partial z}H_x - \frac{\partial}{\partial x}H_z\right) + \hat{a}_z\left(\frac{\partial}{\partial x}H_y - \frac{\partial}{\partial y}H_x\right)$$

$$\vec{J} = \hat{a}_x\left(\frac{\partial}{\partial y}\left(-(x+1)z^2\right) - \frac{\partial}{\partial z}2(x+1)yz\right) + \hat{a}_y\left(\frac{\partial}{\partial z}\left(y^2 z\right) - \frac{\partial}{\partial x}\left(-(x+1)z^2\right)\right) + \hat{a}_z\left(\frac{\partial}{\partial x}2(x+1)yz - \frac{\partial}{\partial y}\left(y^2 z\right)\right)$$

$$\vec{J} = \hat{a}_x(0 - 2xy - 2y) + \hat{a}_y\left(y^2 + z^2\right) + \hat{a}_z(2yz - 2yz)$$

$$\vec{J} = \hat{a}_x(-2xy - 2y) + \hat{a}_y\left(y^2 + z^2\right)$$

$$\vec{J}(1,0,-3)$$

$$\vec{J} = \hat{a}_x(-2(1)(0) - 2(0)) + \hat{a}_y\left(0^2 + (-3)^2\right)$$

$$\vec{J} = 9\hat{a}_y \ \text{A/m}^2$$

Magnetostatics

Solution:

$$\vec{H} = y^2 z\hat{a}_x + 2(x+1)yz\hat{a}_y - (x+1)z^2\hat{a}_z \quad \text{A/m.}$$

$$\nabla \times \vec{H} = \vec{J}$$

$$\vec{J} = \nabla \times \vec{H} = \hat{a}_x(-2xy - 2y) + \hat{a}_y(y^2 + z^2)$$

$$\vec{J} = \hat{a}_x(-2xy - 2y) + \hat{a}_y(y^2 + z^2)$$

$$I = \int_S \vec{J} \cdot d\vec{S}$$

$$y = 1, 0 \le x \le 1, 0 \le z \le 1.$$

$$I = \int_S \vec{J} \cdot d\vec{S}$$

$$I = \int_0^1 \int_0^1 \vec{J} \cdot \hat{a}_y \Big|_{y=1} dxdz$$

$$I = \int_0^1 \int_0^1 \left[\hat{a}_x(-2xy - 2y) + \hat{a}_y(y^2 + z^2)\right] \cdot \hat{a}_y \Big|_{y=1} dxdz$$

$$I = \int_0^1 \int_0^1 \hat{a}_y(y^2 + z^2) \cdot \hat{a}_y \Big|_{y=1} dxdz; \quad \hat{a}_y \cdot \hat{a}_y = 1$$

$$I = \int_0^1 \int_0^1 (y^2 + z^2) \Big|_{y=1} dxdz$$

$$I = \int_0^1 \int_0^1 (1 + z^2) dxdz$$

$$I = \int_0^1 \left(z + \frac{z^3}{3} \right)_0^1 dx$$

$$I = \int_0^1 \left(1 + \frac{1}{3} \right) dx$$

$$I = \int_0^1 \frac{4}{3} dx$$

$$I = \left[\frac{4}{3} x \right]_0^1$$

$$I = \frac{4}{3}(1 - 0)$$

$$I = \frac{4}{3} A = 1.3333 A$$

Magnetostatics

Electric and magnetic forces

- **Electric Force**
  - acts in the direction of the electric field
  - acts on a charged particle regardless of whether the particle is moving
  - does work in displacing the particle
- **Magnetic Force**
  - acts perpendicular to the magnetic field
  - acts on a charged particle only when the particle is moving
  - does no work in displacing the particle

- **Electrical and magnetic fields are very different in electrostatics and magnetostatics!**
  - electrical charges can come individually with different charges (monopoles)
  - magnetic fields are always dipole fields (no magnetic monopoles!)

Magnetostatics

Table 5-1: Attributes of electrostatics and magnetostatics.

Attribute	Electrostatics	Magnetostatics
Sources	Stationary charges ρ_v	Steady currents $\mathbf{J}$
Fields and Fluxes	$\mathbf{E}$ and $\mathbf{D}$	$\mathbf{H}$ and $\mathbf{B}$
Constitutive parameter(s)	ε and σ	μ
Governing equations		
• **Differential form**	$\nabla \cdot \mathbf{D} = \rho_v$ $\nabla \times \mathbf{E} = 0$	$\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{H} = \mathbf{J}$
• **Integral form**	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$ $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$ $\oint_C \mathbf{H} \cdot d\mathbf{l} = I$
Potential	Scalar V, with $\mathbf{E} = -\nabla V$	Vector $\mathbf{A}$, with $\mathbf{B} = \nabla \times \mathbf{A}$
Energy density	$w_e = \frac{1}{2}\varepsilon E^2$	$w_m = \frac{1}{2}\mu H^2$
Force on charge q	$\mathbf{F}_e = q\mathbf{E}$	$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$
Circuit element(s)	C and R	L

Magnetostatics

Name	Integral equations	Differential equations
Gauss's law	$\oint\!\!\!\oint_{\partial\Omega} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0} \iiint_{\Omega} \rho \, \mathrm{d}V$	$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$
Gauss's law for magnetism	$\oint\!\!\!\oint_{\partial\Omega} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0$	$\nabla \cdot \mathbf{B} = 0$
Maxwell–Faraday equation (Faraday's law of induction)	$\oint_{\partial\Sigma} \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\Sigma} \mathbf{B} \cdot \mathrm{d}\mathbf{S}$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
Ampère's circuital law (with Maxwell's addition)	$\oint_{\partial\Sigma} \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 \iint_{\Sigma} \mathbf{J} \cdot \mathrm{d}\mathbf{S} + \mu_0 \varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \iint_{\Sigma} \mathbf{E} \cdot \mathrm{d}\mathbf{S}$	$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$