B38EM Introduction to Electricity and Magnetism

Lecture 5

Electrostatics (part 3)

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Topics

- Coulomb Law & Gauss Law
- Parallel Plate Capacitor
- Energy Storage in a Capacitor
- Capacitance
- Dielectric
- Polarisation Density

References & Resources

- Elements of Electromagnetics (7th Edition), by Sadiku, Oxford University Press
- Fundamentals of Applied Electromagnetics (7th Edition), by Ulaby and Ravaioli
- Field and Wave Electromagnetics (2nd Edition), by David K. Cheng
- …

Coulomb Law & Gauss Law

Coulomb's Law:

$$F_e = \frac{Q_1 \times Q_2}{4\pi\varepsilon_0 r^2}$$

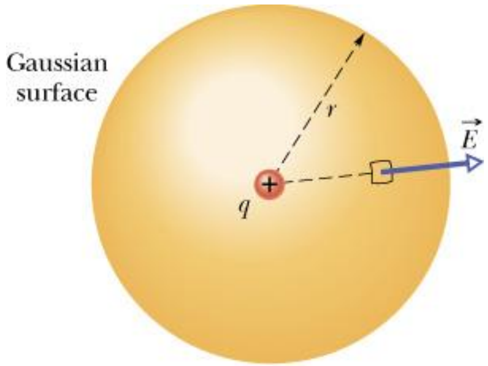
Gauss Law:

The total flux is independent of the surface area and equal to the net charge enclosed.

$$\Psi = DS = Q \qquad\qquad D = \varepsilon_0 E$$

Coulomb Law & Gauss Law

Assume for "Gaussian surface" a sphere at the centre of which lies the charge. Due to symmetry we can write



$$q = \varepsilon_o E \cdot \left(4\pi r^2\right) \Rightarrow E = \frac{q}{4\pi\varepsilon_o \cdot r^2}$$

And therefore the force experienced by a charge *Q* due to *q*

$$F = \frac{q \cdot Q}{4\pi\varepsilon_o \cdot r^2}$$

⇨ Coulomb's law can be derived by Gauss's law.

Capacitor

Capacitor: as a means to store energy

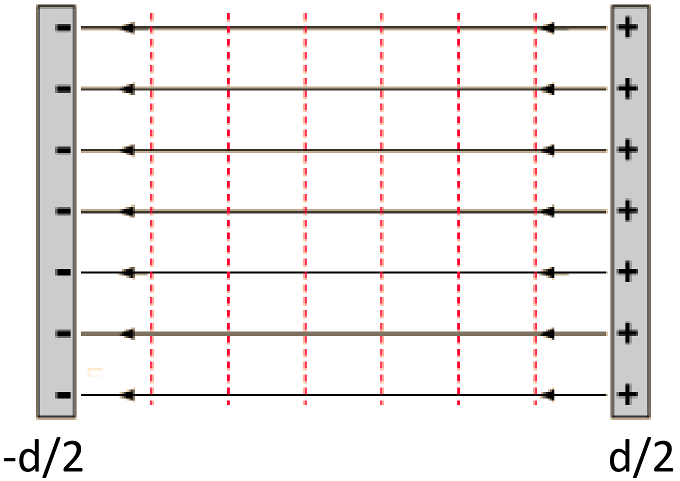
Advantages

- efficient
- low inertia, therefore high speed

Disadvantages

- low energy density – limited by flashover at e.g. 3 MV/m for air

Parallel Plate Capacitor

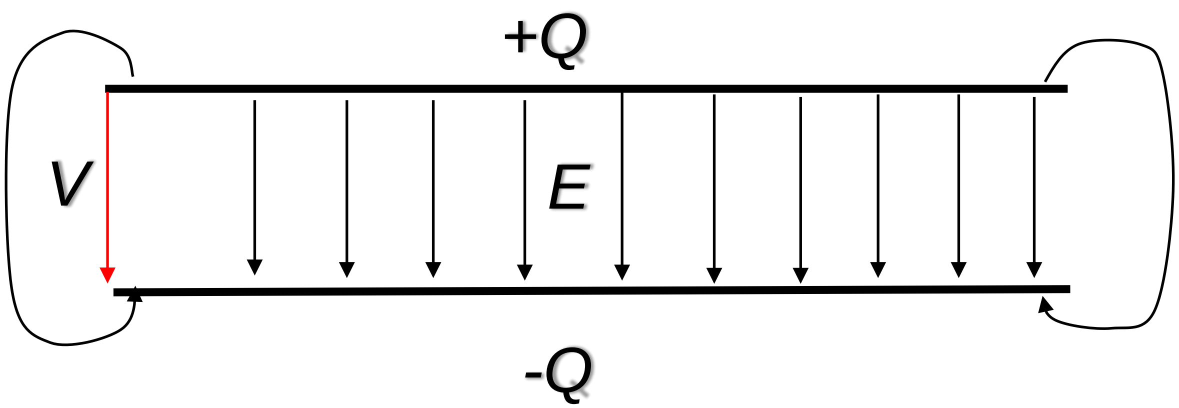


Electric field between two oppositely charged plates (assuming large surface and short separation) is uniform; therefore potential varies linearly

$$V = \int_{-d/2}^{d/2} \vec{E} \cdot \overrightarrow{dl} = \int_{-d/2}^{d/2} E \cdot dl = E \int_{-d/2}^{d/2} dl = E \cdot d$$

Parallel Plate Capacitor

Suppose that a potential difference *V* is applied across two parallel plates separated by air.



$$E = V / d$$

$$D = \varepsilon_0 E$$

$$\therefore \Psi \propto V$$

Parallel Plate Capacitor

$$E = V / d$$
$$D = \varepsilon_0 E$$
$$\therefore \Psi \propto V$$

From Gauss Theorem,

$$\Psi = Q$$
$$\therefore Q \propto V$$
$$\therefore Q = CV$$

where *C*, the constant of proportionality, is the ***capacitance***. Hence capacitance is

$$C = Q/V \, (Farads \quad or \quad F)$$

Energy Stored in a Capacitor

A charge of one coulomb (1 C) receives or delivers an energy of one joule (1 J) when it moves through a voltage of one volt (1 V).

$$\left.\begin{aligned} w &= qv \\ q &= Cv \end{aligned}\right\} \quad dw = q \cdot dv = Cv \cdot dv$$

$$W = \frac{1}{2} C V^2 = \frac{1}{2C} Q^2 = \frac{1}{2} QV$$

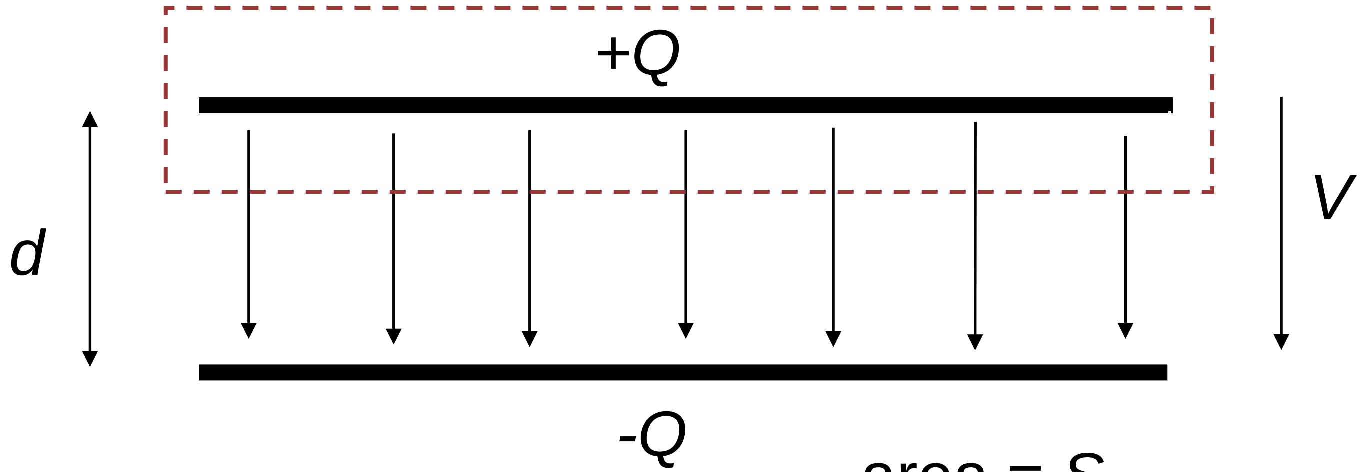
Determination of Capacitance

$$C = \frac{Q}{V}$$

- Start with arbitrary charges of Q and $-Q$ on the electrodes
- Use Gauss Theorem to obtain the electric flux Ψ
- Obtain the electric flux density D
- Obtain the electric field strength from $D = \varepsilon_0 E$
- Obtain the voltage between the electrodes from

$$V_{AB} = \int_A^B E\,dx$$

Parallel Plates

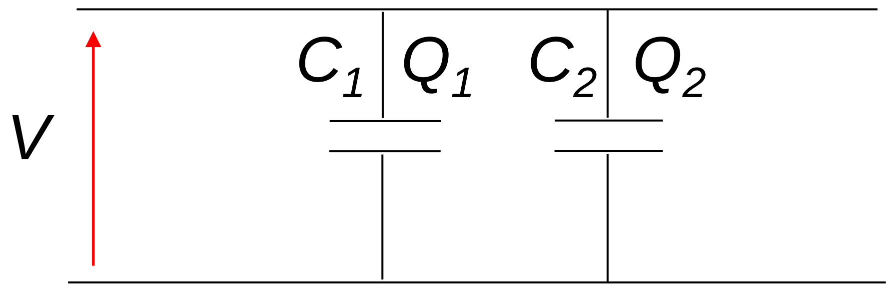


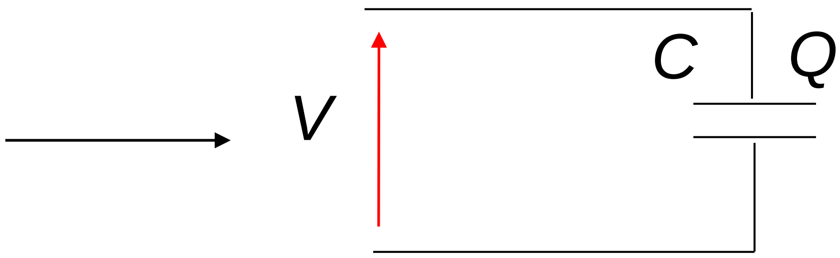
area = S

$$\Psi = D \cdot S = \varepsilon_o \cdot E \cdot S = Q \qquad V_{AB} = \int_A^B E dx = E \cdot d$$

$$\therefore C = \frac{\varepsilon_0 S}{d}$$

Capacitors in Parallel



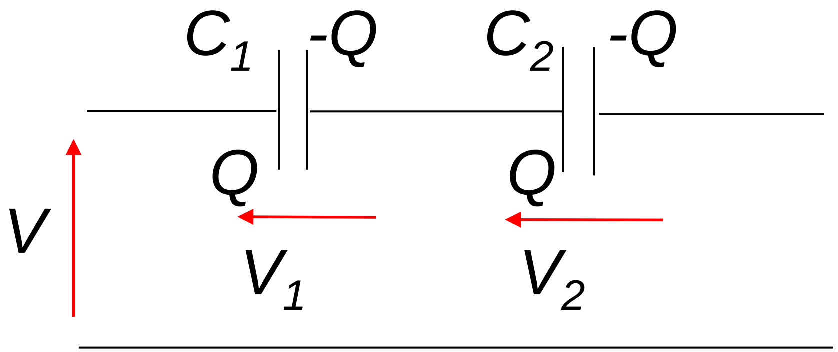


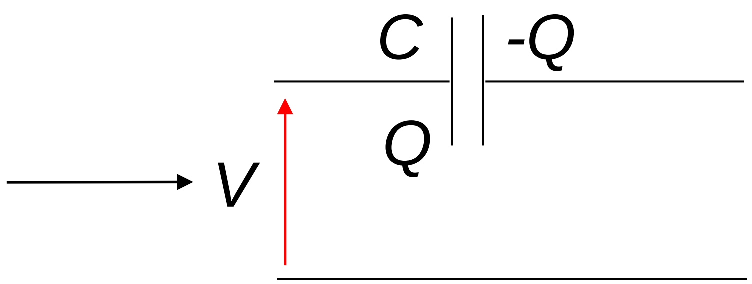
$Q = Q_1 + Q_2 + \ldots$

$$C_{eq} = \frac{Q}{V} = \frac{Q_1 + Q_2}{V} = \frac{Q_1}{V} + \frac{Q_2}{V}$$

$$C = C_1 + C_2 + \ldots$$

Capacitors in Series



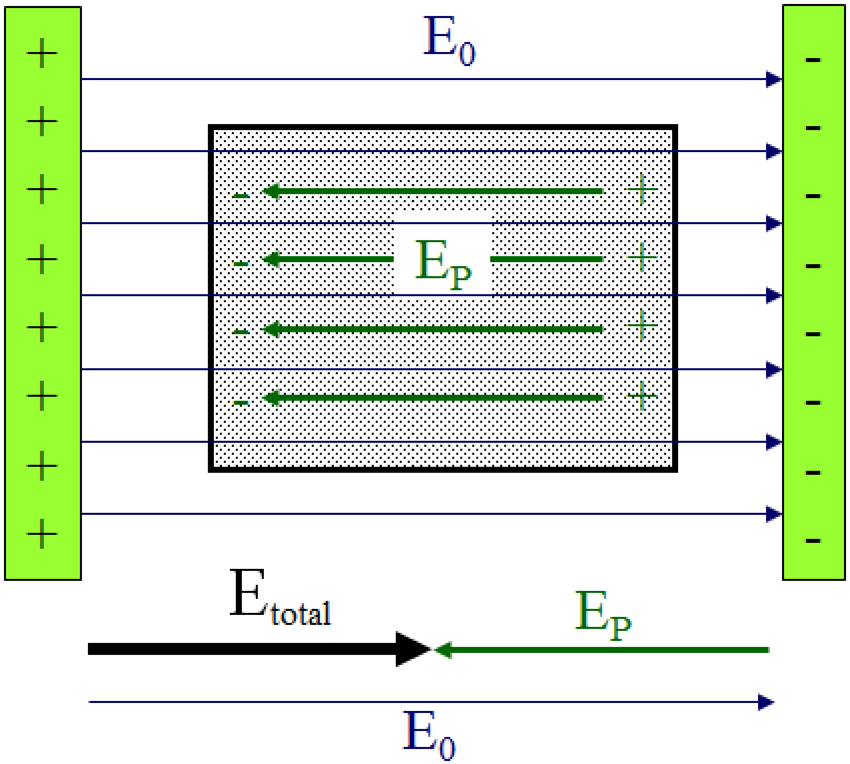


$$\frac{1}{C_{eq}} = \frac{V}{Q} = \frac{V_1 + V_2}{Q} = \frac{V_1}{Q} + \frac{V_2}{Q}$$

$$1/C = 1/C_1 + 1/C_2 + \ldots$$

Capacitors and Dielectrics

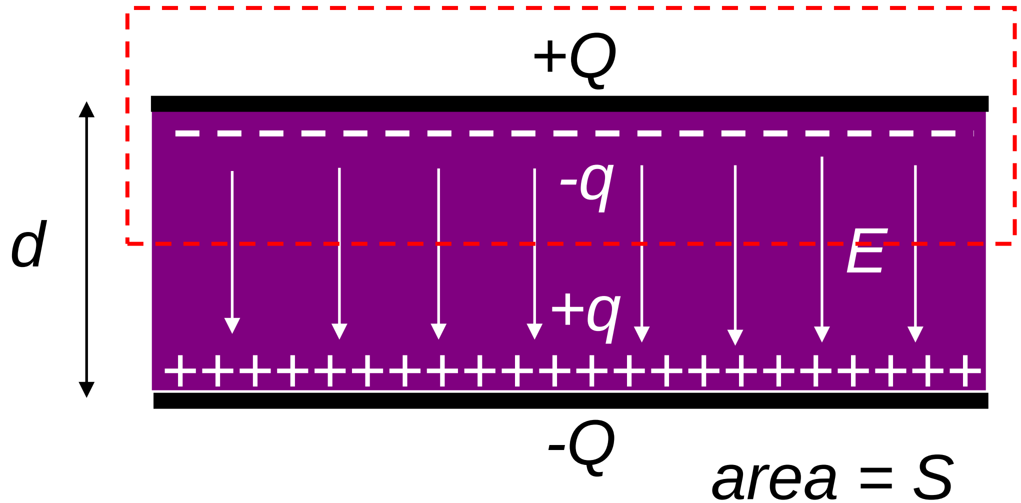
A block of insulating material becomes polarized as follows:



$E_{total} = E_0 - E_P$

Capacitors and Dielectrics

Capacitor with dielectric:



Gauss Law: $(Q - q) = \varepsilon_0 \cdot E \cdot S$, so that E is reduced once a dielectric is introduced.

Also we can rewrite Gauss Law:

$$Q = \frac{Q}{Q-q} \cdot \varepsilon_0 \cdot E \cdot S = \varepsilon_r \cdot \varepsilon_0 \cdot E \cdot S \quad \text{where} \quad \varepsilon_r = \frac{Q}{Q-q}$$

$$\therefore D = \varepsilon_r \cdot \varepsilon_0 \cdot E$$

Capacitors and Dielectrics

Since $V = E \cdot d$, a fixed charge Q will produce a lower voltage (since E is now reduced)

The capacitance is increased. Express this by a factor ε_r

$$C = \frac{\varepsilon_r \varepsilon_0 S}{d}$$

$$\varepsilon_r = \frac{Q}{Q - q}$$

ε_r is *relative permittivity* (1.0 for air).

Polarisation Density

Polarisation density is a vector field that expresses the density *electric dipole moments* in a dielectric material

The polarization density, $\vec{P}$, defines the electric flux density according to

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

(remember that so far we were using $\vec{D} = \varepsilon_r \varepsilon_0 \vec{E}$)

Energy Stored in a Capacitor

$$W = \frac{1}{2}CV^2 = \frac{1}{2}\frac{\varepsilon_r \varepsilon_0 S}{d}(Ed)^2$$

$$= \frac{1}{2}\varepsilon_r \varepsilon_0 E^2 \times volume$$

For example, ε_r for mica is 5 and E_{max} = 600 kV/cm.

$$\therefore W_{\max\_density} \approx 80,000 J / m^3$$