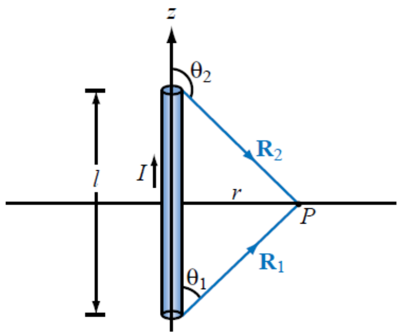
Introduction to Electricity and Magnetism B38EM

Tutorial #5 - Solutions

ε_0 = 8.85x10^{-12} Fm^{-1},   e = 1.6x10^{-19} C,   1 nC = 10^{-9} C

1) A semi-infinite linear conductor extends between z=0 and z=∞ along the z-axis. If the current I in the conductor flows along the positive z-direction, find **H** at a point in the x-y plane at a radial distance r from the conductor. (Ex. 5.6 Ulaby)



Solution: From (5.27),

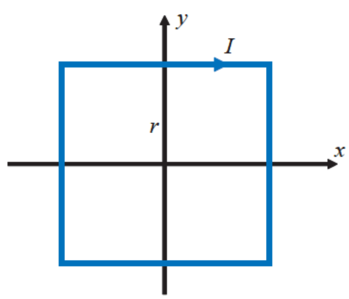
$$\mathbf{H} = \hat{\boldsymbol{\phi}} \frac{I}{4\pi r}(\cos\theta_1 - \cos\theta_2)$$

For a conductor extending from $z = 0$ to $z = \infty$, $\theta_1 = 0$ and $\theta_2 = \pi$. Hence,

$$\mathbf{H} = \hat{\boldsymbol{\phi}} \frac{I}{4\pi r}(1+1) = \hat{\boldsymbol{\phi}} \frac{I}{2\pi r} \quad \text{(A/m)}.$$

2) A wire is formed into a square loop and placed in the x-y plane with its centre at the origin and each of its sides parallel to either the x- or the y- axes. Each side is 40cm in length, and the wire carries a current of 5 A whose direction is clockwise when the loop is viewed from above. Calculate the magnetic field at the centre of the loop. (Ex. 5.8 Ulaby)

Solution:



The direction of the current will induce a magnetic field along $-\hat{\mathbf{z}}$ (according to the right-hand rule). At the center of the loop, each segment will contribute exactly the same amount. Each of the four contributions can be calculated using (5.29) with $\hat{\boldsymbol{\phi}}$ replaced with $-\hat{\mathbf{z}}$:

$$\mathbf{H}_1 = -\hat{\mathbf{z}} \frac{Il}{2\pi r\sqrt{4r^2+l^2}} .$$

In this case $r = l/2$. Hence,

$$\mathbf{H}_1 = -\hat{\mathbf{z}} \frac{Il}{2\pi(l/2)\sqrt{l^2+l^2}} = -\hat{\mathbf{z}} \frac{I}{\sqrt{2}\,\pi l} .$$

Finally,

$$\begin{aligned}\mathbf{H} = 4\mathbf{H}_1 &= -\hat{\mathbf{z}} \frac{4I}{\sqrt{2}\,\pi l} \\ &= -\hat{\mathbf{z}} \frac{4\times 5}{\sqrt{2}\,\pi \times 0.4} = -\hat{\mathbf{z}}\,11.25 \quad \text{(A/m)}.\end{aligned}$$

3) The metal niobium becomes a superconductor with zero electrical resistance when it is cooled to below 9 K, but its superconductive behavior ceases when the magnetic flux density at its surface exceeds 0.12 T. Determine the maximum current that a 0.1-mm-diameter niobium wire can carry and remain superconductive. (Ex. 5.10 Ulaby)

Solution: From (5.49), the magnetic field at $r \geq a$ from a wire is given by

$$H = \frac{I}{2\pi r}, \qquad r \geq a.$$

At the surface of the wire, $r = a$. Hence,

$$B = \mu_0 H = \frac{\mu_0 I}{2\pi a},$$

$$I = \frac{2\pi a B}{\mu_0}$$

$$= \frac{2\pi \times 0.05 \times 10^{-3} \times 0.12}{4\pi \times 10^{-7}} = 30 \text{ A}.$$

4) Find the internal and external magnetic field of long conductor

a) For $r < a$

$$\oint_{C_1} \mathbf{H}_1 \cdot d\mathbf{l}_1 = I_1,$$

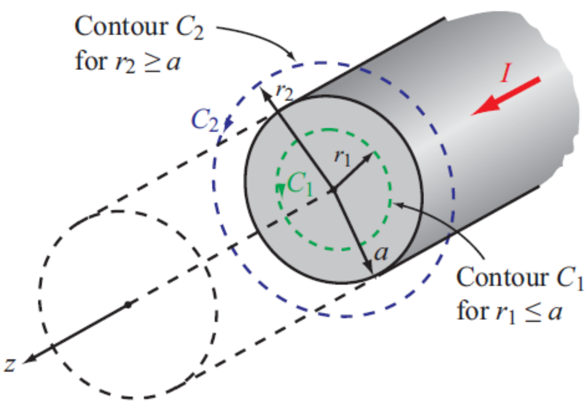
$$\oint_{C_1} \mathbf{H}_1 \cdot d\mathbf{l}_1 = \int_0^{2\pi} H_1(\hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}}) r_1 \, d\phi = 2\pi r_1 H_1.$$

The current I_1 flowing through the area enclosed by C_1 is equal to the total current I multiplied by the ratio of the area enclosed by C_1 to the total cross-sectional area of the wire:

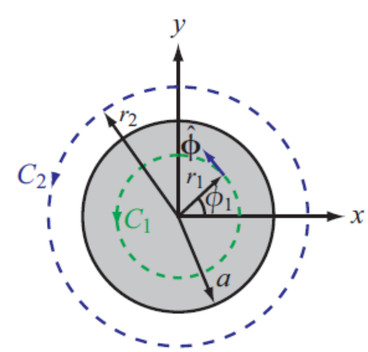
$$I_1 = \left(\frac{\pi r_1^2}{\pi a^2}\right) I = \left(\frac{r_1}{a}\right)^2 I.$$

Equating both sides of Eq. (5.48) and then solving for $\mathbf{H}_1$ yields

$$\boxed{\mathbf{H}_1 = \hat{\boldsymbol{\phi}} H_1 = \hat{\boldsymbol{\phi}} \frac{r_1}{2\pi a^2} I \qquad (\text{for } r_1 \leq a).} \qquad (5.49\text{a})$$



(a) Cylindrical wire



(b) Wire cross section

For $r > a$

(b) For $r = r_2 \geq a$, we choose path C_2, which encloses all the current I. Hence, $\mathbf{H}_2 = \hat{\boldsymbol{\phi}} H_2$, $d\boldsymbol{\ell}_2 = \hat{\boldsymbol{\phi}} r_2 \, d\phi$, and

$$\oint_{C_2} \mathbf{H}_2 \cdot d\mathbf{l}_2 = 2\pi r_2 H_2 = I,$$

which yields

$$\boxed{\mathbf{H}_2 = \hat{\boldsymbol{\phi}} H_2 = \hat{\boldsymbol{\phi}} \frac{I}{2\pi r_2} \qquad (\text{for } r_2 \geq a).} \qquad (5.49\text{b})$$

5) Find the magnetic field of a toroid.

Solution:
From symmetry, it is clear that H is uniform in the azimuthal direction.

r < a: *Construct a circular Amperian contour centered at origin:*

=> *no current flows through the surface of the contour (no current enclosed by the dashed line)*
=> **H** = 0 , for $r < a$.

a<r<b:
The current enclosed by the dashed line is just the number of loops times the current in each loop.

Applying Ampere's law over contour C:

$$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I$$

Ampere's law states that the line integral of **H** around a closed contour C *is equal to the current traversing the* surface bounded by the contour.

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_0^{2\pi} (-\hat{\boldsymbol{\phi}} H) \cdot \hat{\boldsymbol{\phi}} r \, d\phi = -2\pi r H = -NI.$$

Hence, $H = NI/(2\pi r)$ and

$$\mathbf{H} = -\hat{\boldsymbol{\phi}} H = -\hat{\boldsymbol{\phi}} \frac{NI}{2\pi r} \qquad (\text{for } a < r < b).$$

The magnetic field outside the toroid is zero. Why?

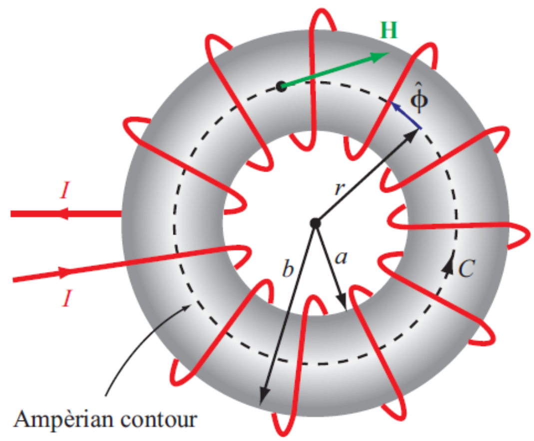


Figure 5-18: Toroidal coil with inner radius a and outer radius b. The wire loops usually are much more closely spaced than shown in the figure (Example 5-5).

r > b: An equal number of current coils cross the surface in both directions => Net current flowing through its surface is zero

=> **H = 0** , for $r > b$