# B38EM - Mathematical background

## Outline:

- Integral
    - Riemann integral
    - Work of a force
    - Surface integral
    - Volume integral
- Derivative
    - Tangential
    - Definition of derivative

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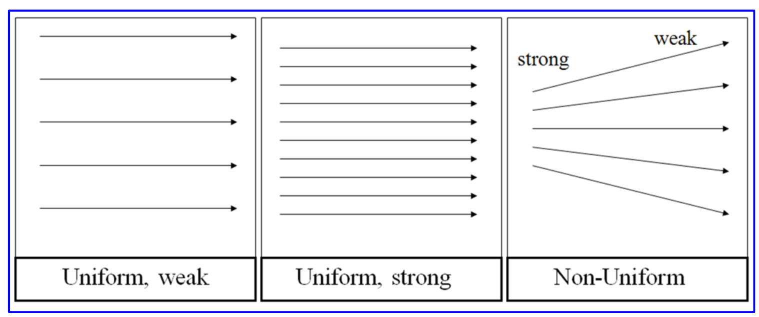
### Fields and Field Lines

- A plotted field contains information on field strength and uniformity.
- Describe the following field plots:

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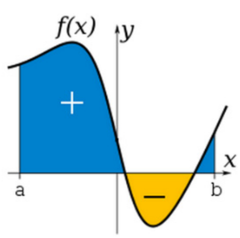
### Integral




$$\int_a^b f(x)\,dx$$

Describes the area under a **curve** (function $f(x)$ ).

That is the area of the region in the $x$-$y$ plane bounded by:

- the graph of $f(x)$,
- the $x$-axis, and
- the vertical lines $x = a$ and $x = b$,

The area **above** the $x$-axis ***adds*** to the total, while that **below** ***subtracts***.

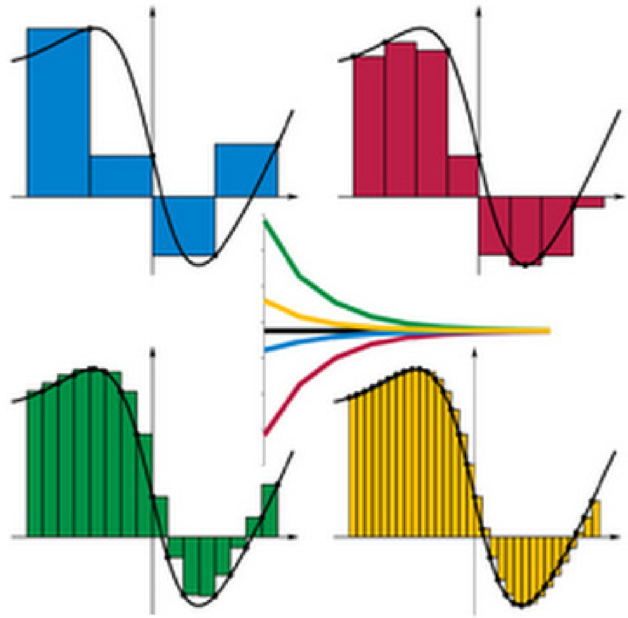
**Handy web page for evaluating integrals: http://integrals.wolfram.com**

## Calculation of Integral

Divide the area under the $f(x)$ into $i$ segments of length $\Delta x$.
Each segment will have: $Area = f(x_i)\cdot\Delta x_i$.

Total area under $n$ segments:

$$\sum_{i=1}^{n} f(x_i)\Delta_i$$



At the limit $\Delta \rightarrow 0$
"infinitesimal" length $dx$
('*differential*')

$$F = \int f(x)\,dx$$

$$\int f(x)\,dx$$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

*Antiderivatives*

### A. Cartesian Coordinates

From Figure 3.1, we notice that

(1) Differential displacement is given by

$$\boxed{d\mathbf{l} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z} \qquad (3.1)$$

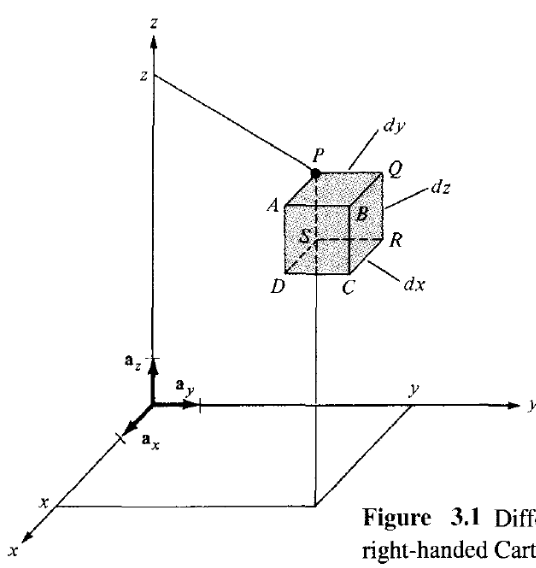



**Figure 3.1** Differential elements in the right-handed Cartesian coordinate system.

(2) Differential normal area is given by

$$\boxed{\begin{aligned} d\mathbf{S} = dy\, dz\, \mathbf{a}_x \\ dx\, dz\, \mathbf{a}_y \\ dz\, dy\, \mathbf{a}_z \end{aligned}} \qquad (3.2)$$

and illustrated in Figure 3.2.

(3) Differential volume is given by

$$\boxed{dv = dx\, dy\, dz} \qquad (3.3)$$

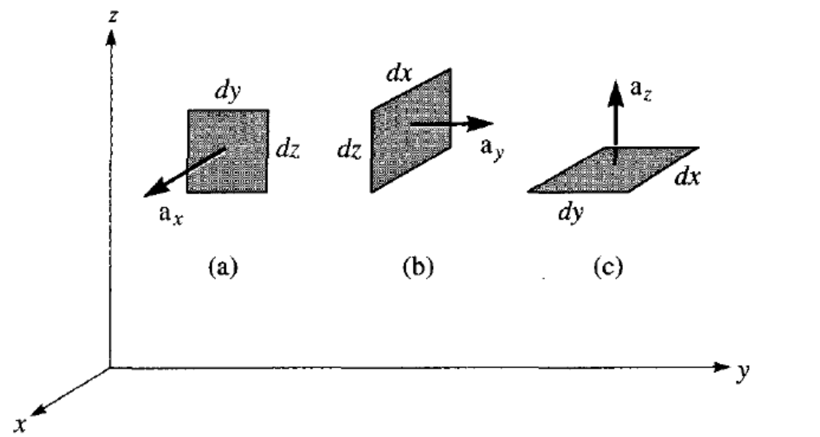



**Figure 3.2** Differential normal areas in Cartesian coordinates: **(a)** $d\mathbf{S} = dy\, dz\, \mathbf{a}_x$, **(b)** $d\mathbf{S} = dx\, dz\, \mathbf{a}_y$, **(c)** $d\mathbf{S} = dx\, dy\, \mathbf{a}_z$

These differential elements are important as they will be referred to often. Students are encouraged not to memorize them, but to learn to derive them, e.g. from Fig 3.1.

## B. Cylindrical Coordinates

Notice from Figure 3.3 that in cylindrical coordinates, differential elements can be found as follows:

(1) Differential displacement is given by

$$\boxed{d\mathbf{l} = d\rho\, \mathbf{a}_\rho + \rho\, d\phi\, \mathbf{a}_\phi + dz\, \mathbf{a}_z} \qquad (3.5)$$

(2) Differential normal area is given by

$$\boxed{\begin{aligned} d\mathbf{S} = \rho\, d\phi\, dz\, \mathbf{a}_\rho \\ d\rho\, dz\, \mathbf{a}_\phi \\ \rho\, d\phi\, d\rho\, \mathbf{a}_z \end{aligned}} \qquad (3.6)$$

and illustrated in Figure 3.4.

(3) Differential volume is given by

$$\boxed{dv = \rho\, d\rho\, d\phi\, dz} \qquad (3.7)$$

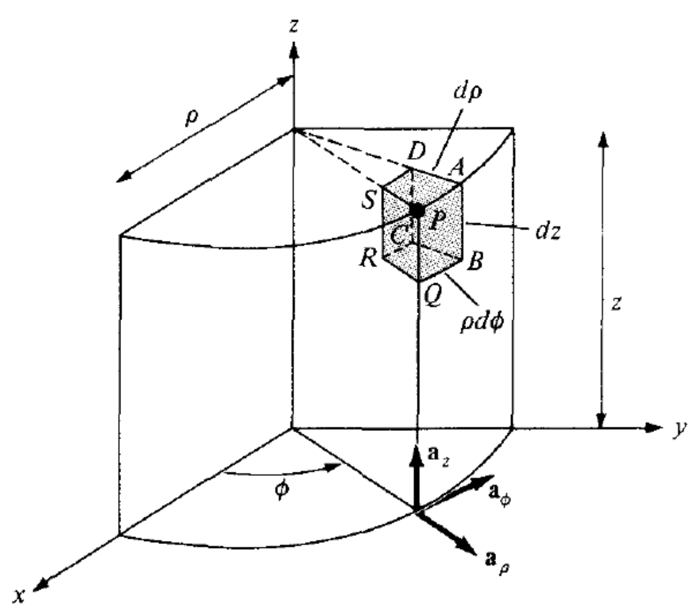



**Figure 3.3** Differential elements in cylindrical coordinates.

## C. Spherical Coordinates

From Figure 3.5, we notice that in spherical coordinates,

(1) The differential displacement is

$$\boxed{d\mathbf{l} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi} \qquad (3.8)$$

(2) The differential normal area is

$$\boxed{\begin{aligned} d\mathbf{S} &= r^2\sin\theta\,d\theta\,d\phi\,\mathbf{a}_r \\ &\quad r\sin\theta\,dr\,d\phi\,\mathbf{a}_\theta \\ &\quad r\,dr\,d\theta\,\mathbf{a}_\phi \end{aligned}} \qquad (3.9)$$

and illustrated in Figure 3.6.

(3) The differential volume is

$$\boxed{dv = r^2\sin\theta\,dr\,d\theta\,d\phi} \qquad (3.10)$$

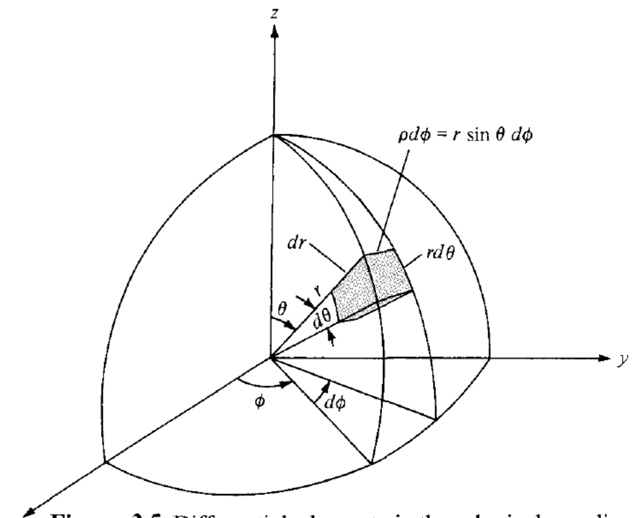



**Figure 3.5** Differential elements in the spherical coordinate system.

These differentials *dl*, ***dS*** and *dv* will be useful later in our calculations.

## 3.3 Line, Surface and Volume Integrals

## Line Integral - Circulation

The **line integral** $\int_L \mathbf{A} \cdot d\mathbf{l}$ is the integral of the tangential component of $\mathbf{A}$ along curve $L$.

Given a vector field **A** and a curve $L$, we define the

*line integral* of **A** around $L$: $$\int_L \mathbf{A} \cdot d\mathbf{l} = \int_a^b |\mathbf{A}| \cos\theta \, dl$$

If $L$ is a **closed path** → integral becomes **closed**

and expresses the **Circulation** of **A** around $L$ $$\oint_L \mathbf{A} \cdot d\mathbf{l}$$

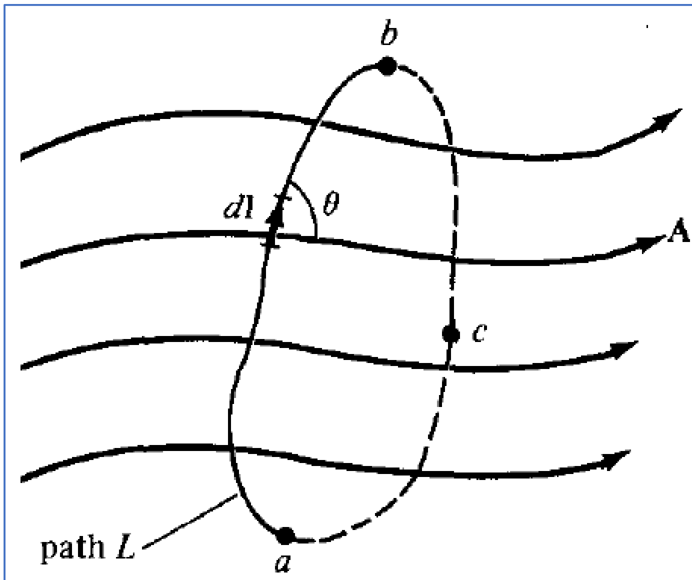



**Figure 3.8** Path of integration of vector field **A**.

## Surface Integral - Flux

Given a vector field **A** and a and a surface $S$, we define the

*surface integral* of **A through** $S$:

$$\boxed{\Psi = \int_S |\mathbf{A}| \cos\theta \, dS = \int_S \mathbf{A} \cdot \mathbf{a}_n \, dS}$$

or simply:

$$\boxed{\Psi = \int_S \mathbf{A} \cdot d\mathbf{S}}$$

If $S$ is a **closed surface** → ∫ becomes **closed (defines a volume)** and expresses the **outward Flux of A from** $S$

$$\boxed{\Psi = \oint_S \mathbf{A} \cdot d\mathbf{S}}$$

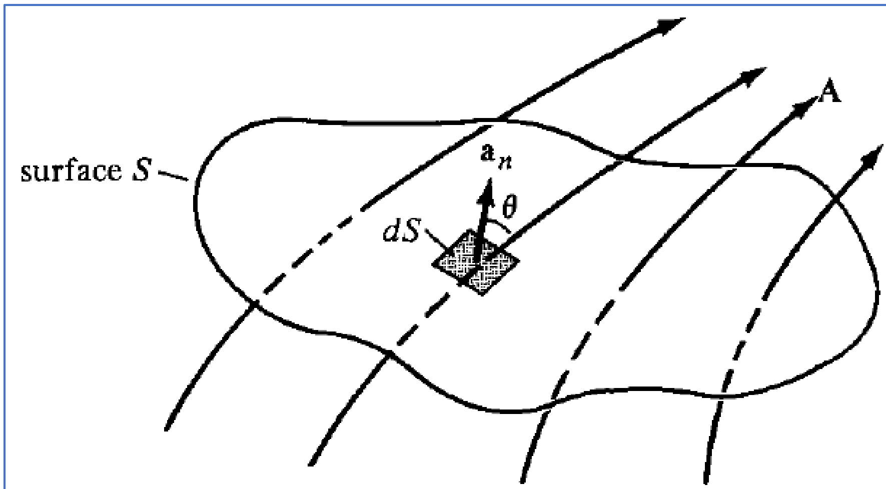



**Figure 3.9** The flux of a vector field **A** through surface $S$.

## Volume Integral

We define the integral

$$\int_v \rho_v \, dv$$

as the *volume integral* of the scalar $\rho_v$ over the volume $v$.

The **physical meaning** of a line, surface, or volume integral depends on the nature of the physical quantity represented by A or $\rho_v$.

# <u>Work of a Force</u>

When a constant force **F** is applied

on an object that moves along distance $\boldsymbol{l}$,

the work W done by the force is:

$$W = \boldsymbol{F} \cdot \boldsymbol{l} \;= F \cdot l \cdot \cos\theta$$

But if the force **F** is not constant, the above does not apply.

Then, we assume **F =** constant **only** for an infinitesimal distance $d\boldsymbol{l}$

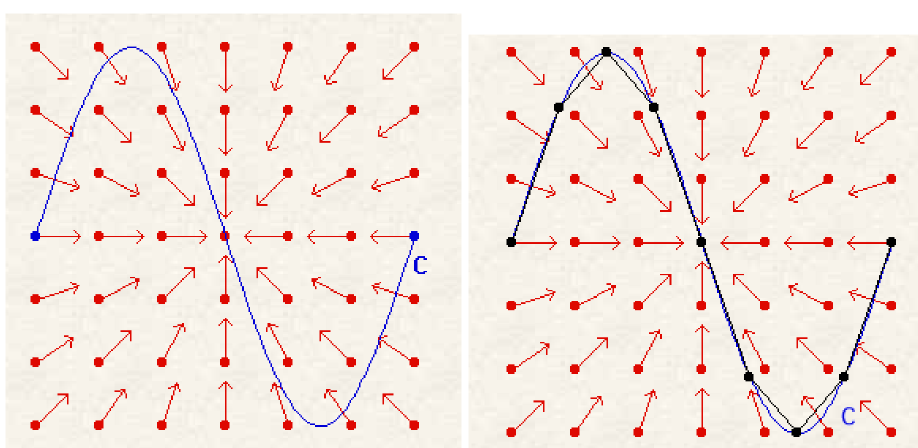
The associated work is then =>

$$dW = \boldsymbol{F} \cdot d\boldsymbol{l} \;= F \cdot dl \cdot \cos\theta$$

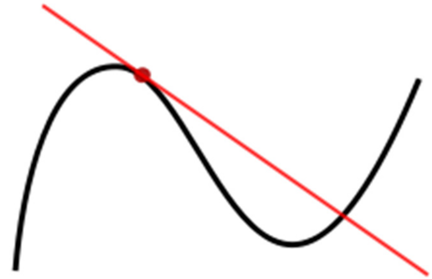
and the **total work done** over the entire distance $\boldsymbol{l}$ is:

$$\boxed{W \;=\; \lim_{\delta l \to 0} \sum f \cdot dl = \int f \cdot dl}$$

At the limit Δ→0, "infinitesimal" not along x-axis but along a line




This is the basis for a **<u>line integral</u>**: $\oint f \cdot dl$



**<u>Tangential</u> line** (tangent) to a curve at a given point is the straight line that "just touches" the curve at that point.

**<u>Derivative</u> is** a measure of **how that tangential function changes** as the input ($x$) changes. The simplest case is when $y$ is a linear function of $x$:

$$y = f(x) = m \cdot x + b$$

The **slope** $m$ of the tangent is: $m = \dfrac{\text{change in } y}{\text{change in } x} = \dfrac{\Delta y}{\Delta x}$

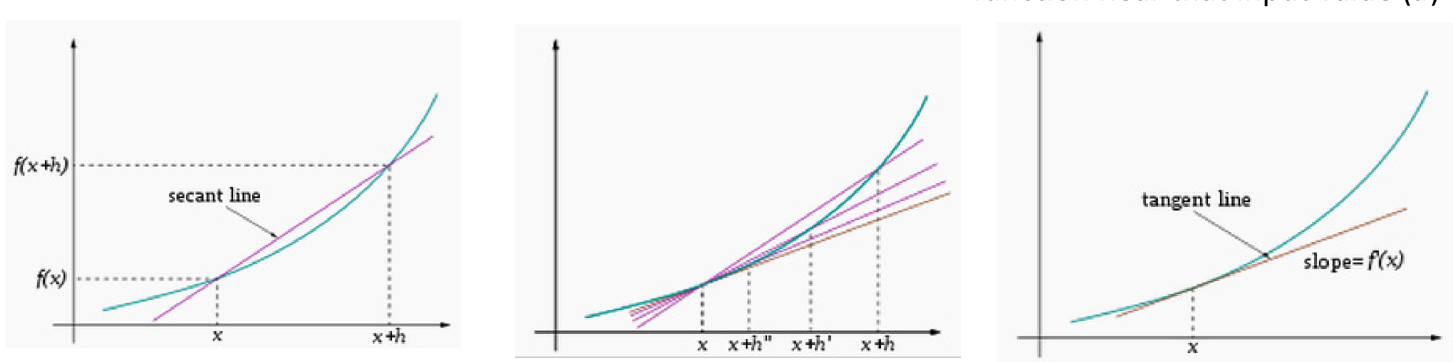
If the function $f$ is not a straight line => $\Delta y / \Delta x$ varies.

To find the **exact value** of $\Delta y / \Delta x$ at any given point $x$,

we use a method called **differentiation:**

Take the limit of $\Delta y / \Delta x$ as $\Delta x \rightarrow 0$ ($dx$): $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

best linear approximation of the function near that input value (*a*)




The value of the derivative function at a point equals to the tangent slope at that point.

# B38EM - Mathematical background

## Outline:

- Del Operator
  - Gradient of Scalar
  - Divergence and curl of a Vector
  - Laplacian of Scalar
- Vector Fields
  - Flux and circulation
  - Conservative fields

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### 3.4 Del Operator (∇)

The ∇ operator, is the **vector differential operator** (or gradient).

**Cartesian:** $$\boxed{\nabla = \frac{d}{dx}\boldsymbol{a_x} + \frac{d}{dy}\boldsymbol{a}_\mathbf{y} + \frac{d}{dz}\boldsymbol{a}_\mathbf{z} \ \textbf{(vector)}}$$

∇ is not a vector in itself.
When ∇ operates on a scalar => a **vector** ensues.

∇ is useful in defining:

1. The gradient of a scalar $V$: $\nabla V$
2. The divergence of a vector $\mathbf{A}$: $\nabla \bullet \mathbf{A}$
3. The curl of a vector $\mathbf{A}$: $\nabla \times \mathbf{A}$
4. The Laplacian of a scalar $V$: $\nabla^2 V$

In **cylindrical and spherical coordinates,** $\nabla$ can be obtained using the transformation formulas (see §2.3-2.4 / Appendix):

Cylindrical: $$\nabla = \mathbf{a}_\rho \frac{\partial}{\partial \rho} + \mathbf{a}_\phi \frac{1}{\rho}\frac{\partial}{\partial \phi} + \mathbf{a}_z \frac{\partial}{\partial z}$$

Spherical: $$\nabla = \mathbf{a}_r \frac{\partial}{\partial r} + \mathbf{a}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{a}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

## Gradient of Scalar: $\nabla V$

Gradient of a scalar field $V$ is a **vector** field whose **magnitude** is the rate of change of $V$ and **direction** the greatest rate of increase of $V$.

Cartesian: $$\boxed{\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z}$$

(for other coordinate systems please see textbook/notes)

## Divergence of Vector: $\nabla \cdot \mathbf{A}$

The **divergence** of **A** at a given point *P* is the ***outward*** **flux*** per unit volume as the volume ($\to$0) shrinks about *P.*

$$\text{div}\,\mathbf{A} = \nabla \cdot \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v}$$

Cartesian: $$\boxed{\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$ (scalar)

* For a better understanding, consider that "***flux***" expresses "***flow rate***"

From the definition of divergence, we can also show:

$$\boxed{\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{A} \, dv}$$

**Divergence theorem**:

"The total outward flux of a vector field **A** through the *closed* surface $S$ is the same as the vol. integral of the divergence of **A**."

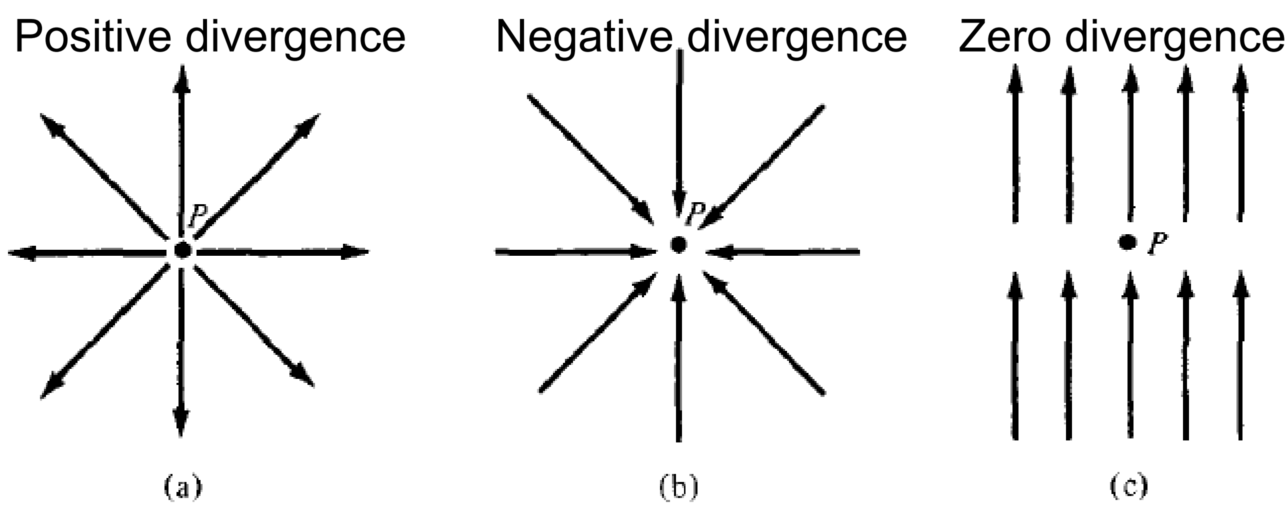
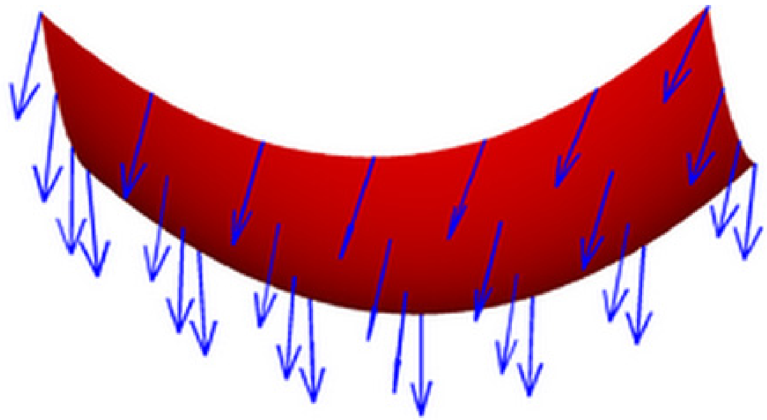



**Figure 3.14** Illustration of the divergence of a vector field at $P$; **(a)** positive divergence, **(b)** negative divergence, **(c)** zero divergence.

## Flow through a surface

Assume non-uniform liquid flow through a surface.



***Question***:

How can we calculate the total volume of water that crosses the surface?

**Answer:**

- Split surface into many small elements $d\boldsymbol{S}$.

- Consider each $d\mathbf{S}$ flat
- Apply the summation principle:
  - Total flux = sum of all fluxes through each $d\mathbf{S}$

$$\iint_S \vec{F} \cdot d\vec{A} = \lim_{m,n \to \infty} \sum_{i=j}^{n} \sum_{i=1}^{n} \vec{F}(P_{ij}) \cdot \Delta \vec{A}_{ij}$$

## Curl of Vector: $\nabla \times \mathbf{A}$

The curl describes the **rotation** of a vector field.

The curl of $\mathbf{A}$ is an axial (or rotational) **vector** whose **magnitude** is the maximum **circulation** of $\mathbf{A}$ per unit area (as the area tends to zero), and **direction** is normal to the area of maximum circulation.

$$\text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \left( \lim_{\Delta S \to 0} \frac{\oint_L \mathbf{A} \cdot d\mathbf{l}}{\Delta S} \right)_{\max} \mathbf{a}_n$$

Cartesian: $$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

or $$\nabla \times \mathbf{A} = \left[ \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right] \mathbf{a}_x + \left[ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right] \mathbf{a}_y + \left[ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right] \mathbf{a}_z$$

**Physical significance**:

The curl provides the maximum value of the circulation of the field per unit area (‘*circulation density*’) and indicates the direction along which this maximum value occurs.

The $\nabla \times \mathbf{A}$ at point $P$ shows the circulation or how much the field curls around $P$.

Figure 3.19(a) shows that the curl of a vector field around $P$ is directed out of the page.

Figure 3.19(b) shows a vector field with zero curl.

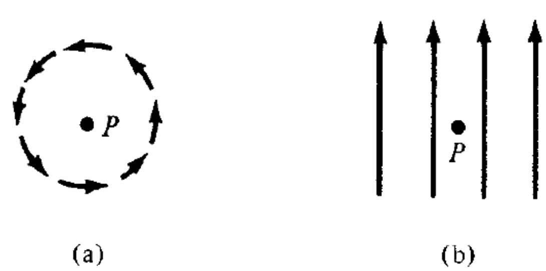


**Figure 3.19** Illustration of a curl:
(**a**) curl at $P$ points out of the page;
(**b**) curl at $P$ is zero.

From the definition of the curl we can show:

$$\boxed{\oint_L \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}}$$

**Stokes's theorem**:
"The circulation of a vector field **A** around a (closed) path *L* is equal to the surface integral of the curl of **A** over the open surface *S* bounded by *L*".

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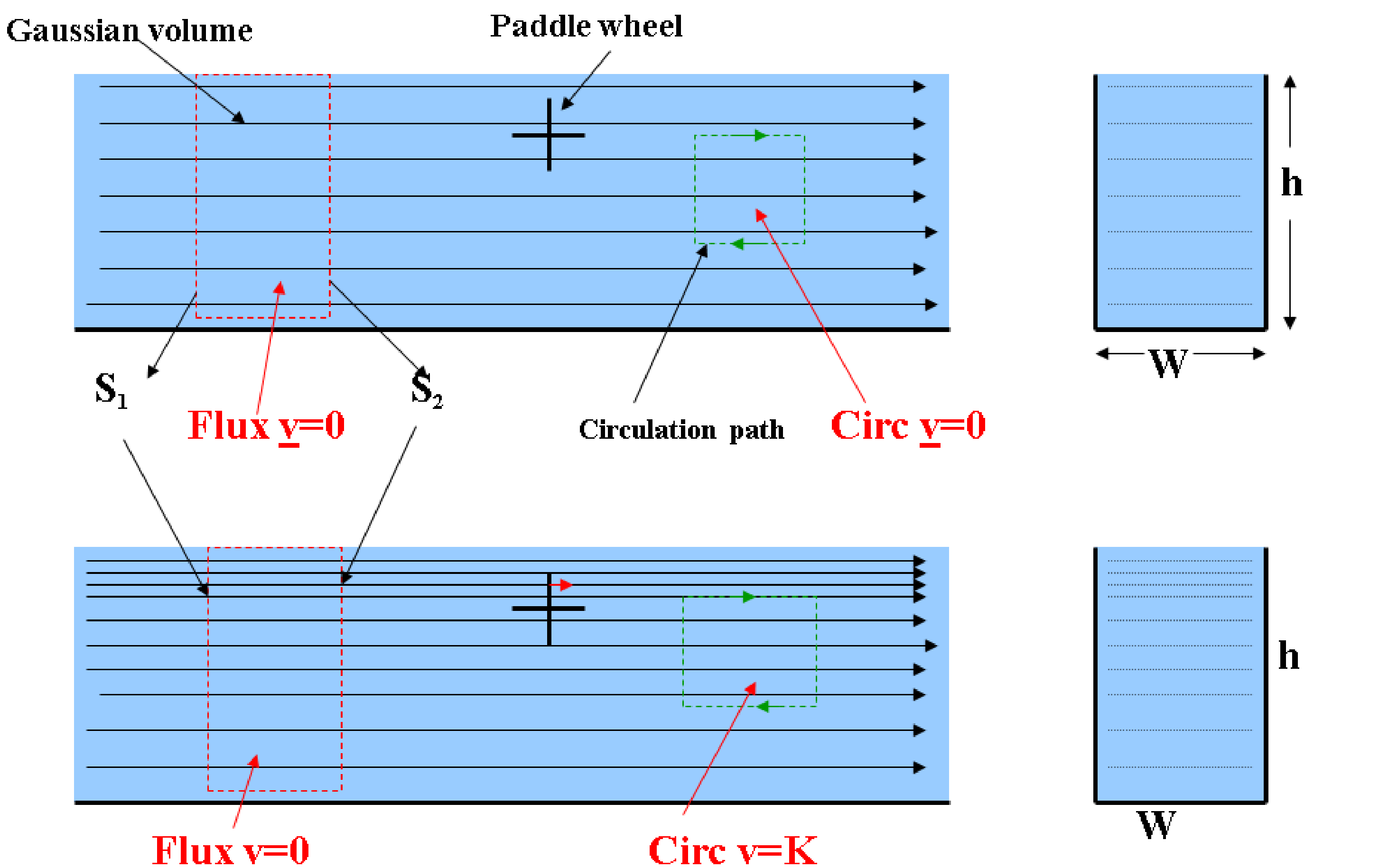
## FLUX and CIRCULATION



All vector fields can also be **uniquely and completely characterized** by their divergence and curl.

$\mathbf{A}$ is solenoidal (divergenceless) if $\nabla\cdot\mathbf{A}=0$

$\mathbf{A}$ is irrotational (or potential) if $\nabla\times\mathbf{A}=0$

All vector fields can be classified in terms of their vanishing or nonvanishing divergence or curl as follows:

(a) $\nabla \cdot \mathbf{A} = 0, \nabla \times \mathbf{A} = 0$ **Solenoidal, irrotational, and conservative**

(b) $\nabla \cdot \mathbf{A} \neq 0, \nabla \times \mathbf{A} = 0$ **Irrotational, and conservative**

(c) $\nabla \cdot \mathbf{A} = 0, \nabla \times \mathbf{A} \neq 0$ **Solenoidal**

(d) $\nabla \cdot \mathbf{A} \neq 0, \nabla \times \mathbf{A} \neq 0$ **Neither**

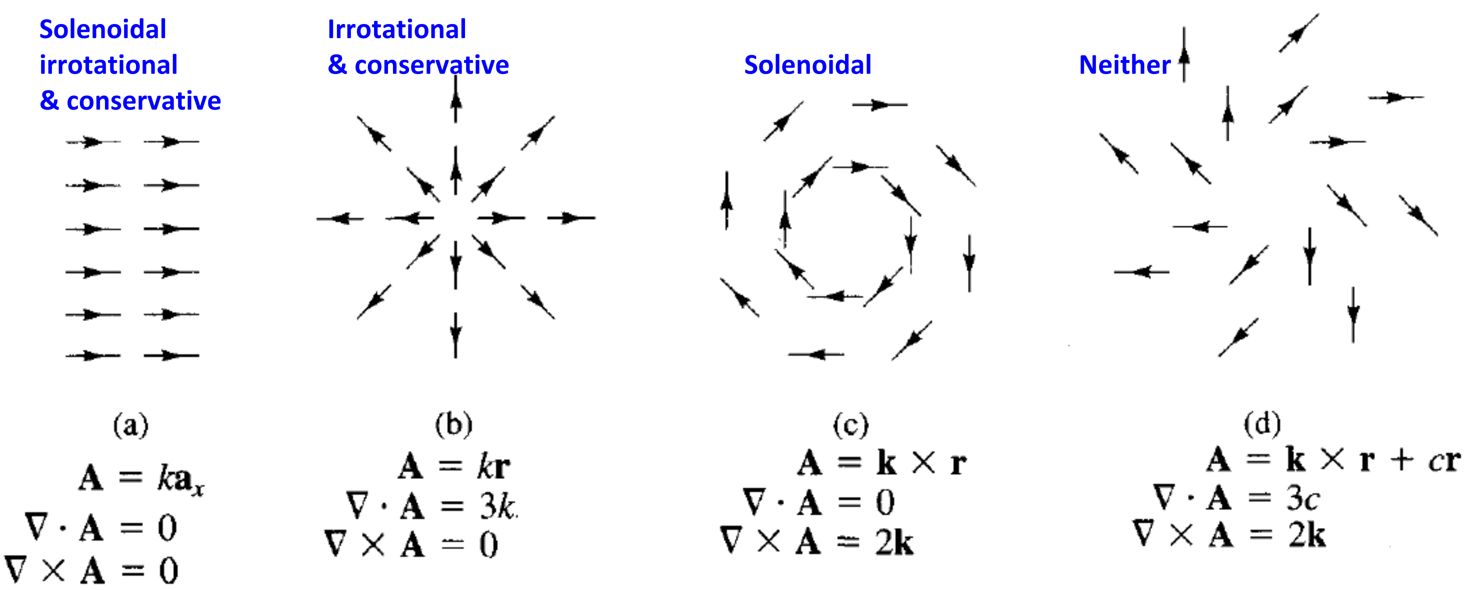
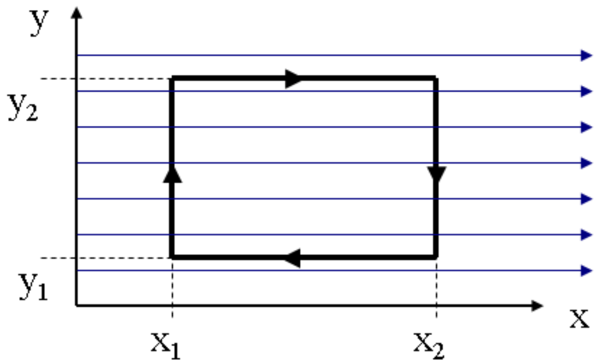



**Figure 3.23** Typical fields with vanishing and nonvanishing divergence or curl.

# Conservative Vector Fields

$$\oint_l \vec{A}.d\vec{l} = 0$$




**Vector field**

$$\vec{A} = A_1 \vec{a}_x$$

- A vector field with zero circulation is said to be conservative.
- Conservative refers to how energy is conserved around the integral path.
- A zero-curl field can also be described as irrotational.

All electrostatic fields are conservative as with gravitational fields.

$$\oint_l \vec{E}.d\vec{l} = 0$$

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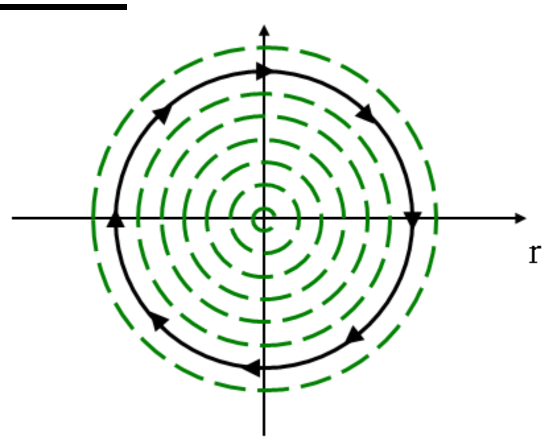
# Rotational Vector Fields

$$circA = \oint_l \vec{A}.d\vec{l} \neq 0$$




**Vector field**

$$\vec{A} = A_1 \vec{a}_\phi$$

A current carrying conductor will form closed loops of magnetic field around itself . Energy is not conserved as integration is carried out around a closed path.

Magnetostatic fields are not conservative.

$$circH = \oint_l \vec{H}.d\vec{l} = I$$

## Laplacian of Scalar: $\nabla^2 V$

Laplacian of a scalar field $V$ is the divergence of the gradient of $V$:

Cart: $\nabla \cdot \nabla V = \nabla^2 V = \left[\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right] \cdot \left[\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right]$

or $$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}}$$ (Scalar)

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## Laplacian of Vector: $\nabla^2 \mathbf{A}$

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$$

(equation valid for all coordinate systems)

Cartesian: $$\nabla^2 \mathbf{A} = \nabla^2 A_x \mathbf{a}_x + \nabla^2 A_y \mathbf{a}_y + \nabla^2 A_z \mathbf{a}_z$$