B38DB: Digital Design and Programming
Combinational Logic Design – Karnaugh Maps

Mustafa Suphi Erden

Heriot-Watt University

School of Engineering & Physical Sciences

Electrical, Electronic and Computer Engineering

Room: EM 2.01
Phone: 0131-4514159
E-mail: m.s.erden@hw.ac.uk

Simplification of Boolean Functions with Karnaugh Maps

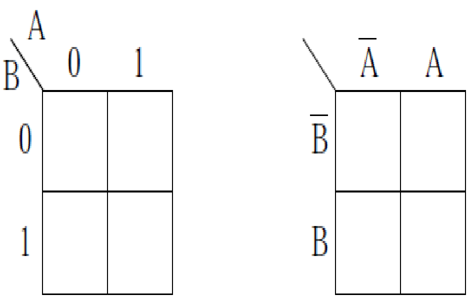
- Karnaugh maps → a powerful *graphical method* of logic simplification that **will always give the simplest sum-of-products form**.
- Transfer logic values from a Boolean statement or a truth table into a Karnaugh map
- The *arrangement of 0's and 1's* within the Karnaugh map leads directly to a simplified Boolean statement

Plotting Karnaugh Maps From Logic Expressions

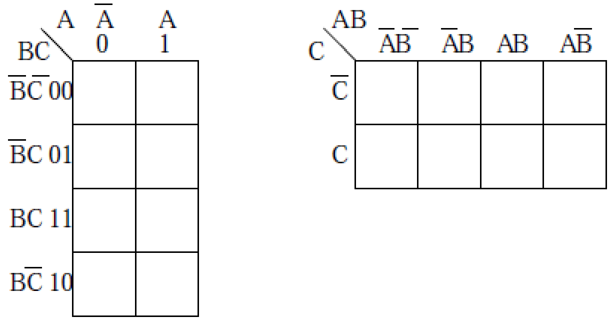
- **Cells** in the Karnaugh map are the equivalent of a truth table
- Each cell corresponds to a **minterm** (or a state from the function's truth table)
- A **minterm** is a product term that includes all the function's variables exactly once, in either true or complemented form

$$a = w'x'y'z' + w'x'yz' + w'x'yz + w'xy'z + w'xyz' + w'xyz + wx'y'z' + wx'y'z$$

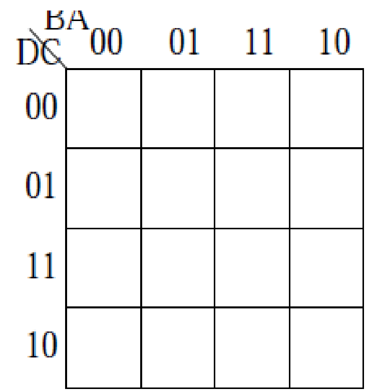
- <u>The adjacent areas of Karnaugh maps must always be neighbouring each other</u>
  - Each axis must be labelled using **Gray code** and cannot be extended to more than two variables.
- **2-D Kaunaugh maps** allow expressions with up to 4 variables; **3-D Karnaugh maps** have to be used for 5 or 6 variables (but cumbersome).



Two variable Karnaugh map



Three variable Karnaugh map



Four variable Karnaugh map

Example: Three Variables Karnaugh Map (2/7)

- Remember→ **A product term** in which all the variables appear is called a **minterm** of the function
- Every function can be written as **a sum of minterms**, which is a special kind of sum of products (SOP) form.

$$F(a,b,c)=\sum m(1,2,3,5,6)=\bar{a}\bar{b}c+\bar{a}b\bar{c}+\bar{a}bc+a\bar{b}c+ab\bar{c}$$

m(0) = $\bar{a}\bar{b}\bar{c}$	m(4) = $a\bar{b}\bar{c}$
m(1) = $\bar{a}\bar{b}c$	m(5) = $a\bar{b}c$
m(2) = $\bar{a}b\bar{c}$	m(6) = $ab\bar{c}$
m(3) = $\bar{a}bc$	m(7) = abc

c \ ab	00	01	11	10
0	0: $\bar{a}\bar{b}\bar{c}$	2: $\bar{a}b\bar{c}$ **1**	6: $ab\bar{c}$ **1**	4: $a\bar{b}\bar{c}$
1	1: $\bar{a}\bar{b}c$ **1**	3: $\bar{a}bc$ **1**	7: abc	5: $a\bar{b}c$ **1**

Plotting Karnaugh Maps From Truth Table

- Plotting a Karnaugh map from a truth table for a function
  - plotting **a '1' in the map of a particular minterm,**
  - a '1' is plotted for each map square where the output is 1 (G=1)
  - squares containing logic 0 terms are left blank

A	B	C	G
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

C \ AB	00	01	11	10
0		1	1	
1		1	1	

Simplification of Boolean Function Using Karnaugh Maps -Terminology-

- An **implicant** is a product term that may include fewer than all the function's variables, but is a term that evaluates to 1 only if the function should evaluate to 1.

- Graphically, it is any **legal-sized circle** including 1's in a Karnaugh map. Legal-sized circles in a Karnaugh map are one, two, four, eight, sixteen, or 2^k adjacent cells.

BC\A	0	1
00		
01	1	
11	1	1
10		1

→ 7 implicants

Prime Implicant of a Boolean Function

- A **prime implicant** of a function is an implicant with the property that if any variable were eliminated from the implicant, the result would be a term covering a minterm not in the function's on-set.
- Graphically, it is any **maximal circle** that covers 1's in a Karnaugh map.

BC\A	0	1
00		
01	1	
11	1	1
10		1

Essential Prime Implicant of a Boolean Function

- An **essential prime implicant** is a prime implicant that is the *only* prime implicant that covers a particular minterm in a function’s on-set.
- Graphically, it is the only circle (the largest possible, of course, since the circle must represent a prime implicant) that covers a particular 1.

BC\A	0	1
00		
01	1	
11	1	1
10		1

Minimal Cover of a Boolean Function

- A **minimal cover** of a function is **a** (there may be more than one) **smallest set** of prime implicants.
- Graphically, it is a **smallest set of circles** that cover all 1's in a Karnaugh map.

BC\A	0	1
00		
01	1	
11	1	1
10		1

Guidelines for Simplifying Functions

- Each cell on a K-map of *n* variables has *n* logically adjacent cells (i.e. differing in exactly one variable).
- When combining cells (with a circle), always group them in **powers of** $\mathbf{2}^{m}$ (m=0,1,2,….)!
- In general, grouping 2^m cells eliminates *m* variables.
- Group as many cells as possible.
- Make as few groups as possible. **Each group represents a separate product term**.
- You must **cover** each minterm at least once. However, it may be covered more than once.

K-Map Simplification Procedure

- **Step1**: Plot the K-map
- **Step2**: Circle **all** prime implicants on the K-map
- **Step3**: Identify and select all essential prime implicants for the cover
- **Step4**: Select a minimum subset of the remaining prime implicants to complete the cover
- **Step 5**: Read the K-map

Example 1: Three Variables Karnaugh Map (3/7)

- Every function can be written as **a sum of minterms**, which is a special kind of sum of products (SOP) form.

- Step 1: Plot the K-map

$$F(a,b,c)=\sum m(1,2,3,5,6)=\bar{a}\bar{b}c+\bar{a}b\bar{c}+\bar{a}bc+a\bar{b}c+ab\bar{c}$$

c \ ab	00	01	11	10
0		1	1	
1	1	1		1

Example 1: Three Variables Karnaugh Map (4/7)

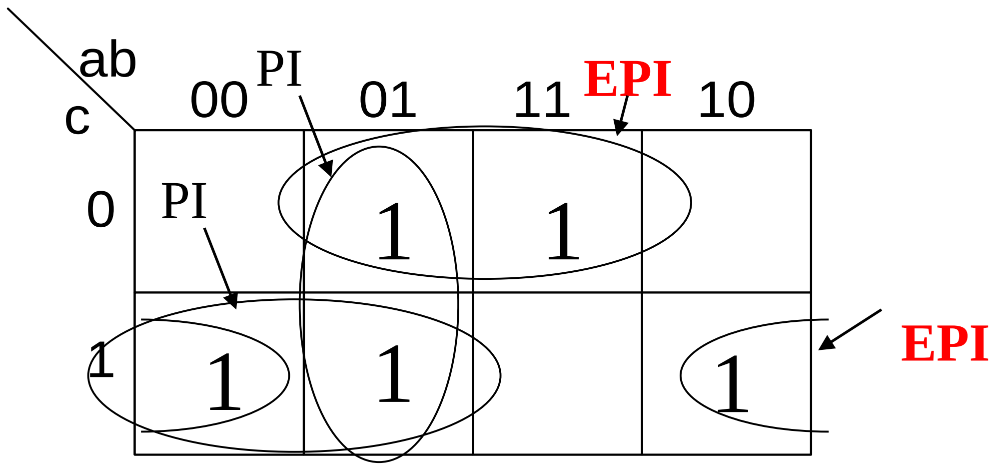
- Step 2: Circle **ALL** Prime Implicants

c \ ab	00	01	11	10
0		1	1	
1	1	1		1

$$F(a,b,c)=\sum m(1,2,3,5,6)$$

Example 1: Three Variables Karnaugh Map (5/7)

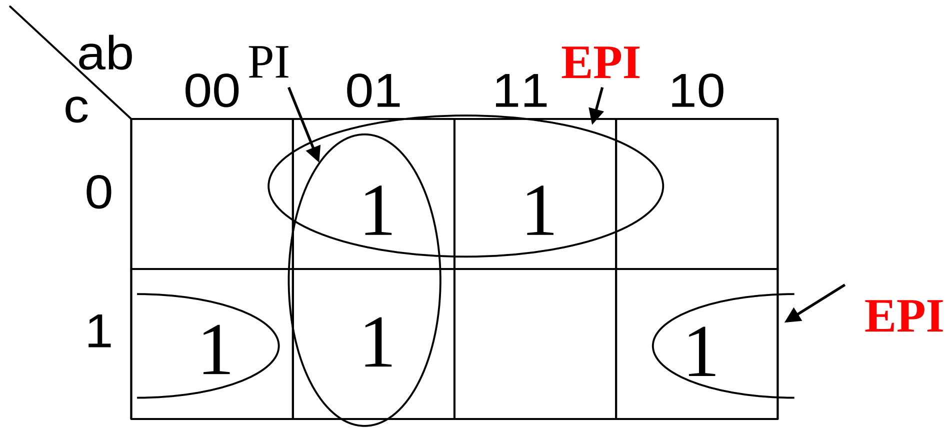
- Step 3: Identify Essential Prime Implicants



$$F(a,b,c) = \sum m(1,2,3,5,6)$$

Example 1: Three Variables Karnaugh Map (6/7)

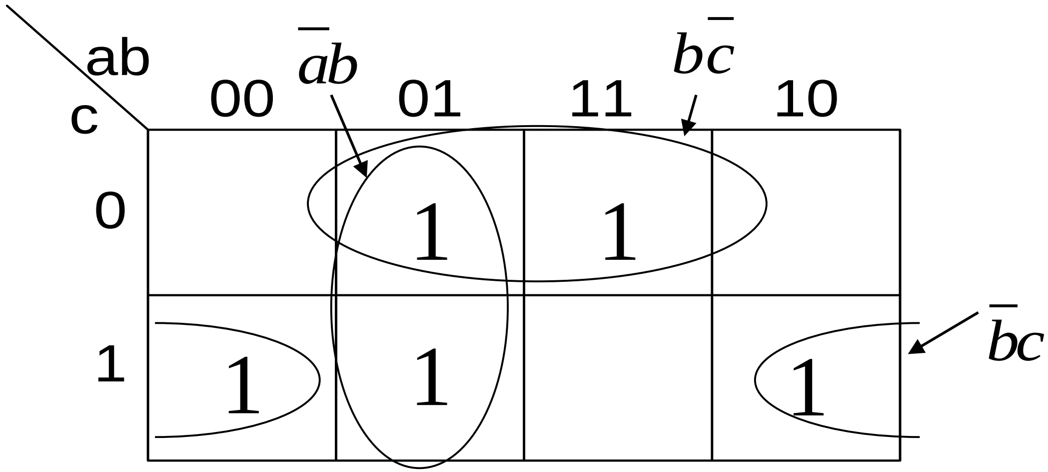
- Step 4: Select a minimum subset of remaining prime implicants to complete the cover



$$F(a,b,c)=\sum m(1,2,3,5,6)$$

Example 1: Three Variables Karnaugh Map (7/7)

- Step 5: Read the map



$$F(a,b,c)=\sum m(1,2,3,5,6)$$

$$F(a,b,c)=\bar{a}b+b\bar{c}+\bar{b}c$$

K-Map Simplification Procedure for POS Functions

- The above method is to simplify the Boolean expression into minimal **sum of products (SOP)**.
- If we want to get the minimal **product of sums (POS)**:
  - **Step1**: Plot the K-Map for the function $\overline{\mathbf{F}}$
  - **Step2**: Circle **all** prime implicants on the K-Map
  - **Step3**: Identify and select all essential prime implicants for the cover
  - **Step4**: Select a minimum subset of the remaining prime implicants to complete the cover
  - **Step5**: Read the K-Map
  - **Step6**: Use **DeMorgan's theorem** to convert $\overline{\mathbf{F}}$ to **F** in POS form

Maxterms

- A **maxterm is** a sum term that contains all the variables in complemented or un-complemented form.
- As before, if there are n variables, then there are 2^n maxterms.

$$F(a,b,c) = \prod M(1,2,3,5,6)$$

M(0) = $a+b+c$; M(4) = $\overline{a}+b+c$;

M(1) = $a+b+\overline{c}$; M(5) = $\overline{a}+b+\overline{c}$;

M(2) = $a+\overline{b}+c$; M(6) = $\overline{a}+\overline{b}+c$;

M(3) = $a+\overline{b}+\overline{c}$; M(7) = $\overline{a}+\overline{b}+\overline{c}$;

- According to the **DeMorgan's theorem**,

$$F(a,b,c) = \prod M(1,2,3,5,6)$$

$$= \sum m(0,4,7)$$

Example

- Use a K-Map to simplify the following Boolean expression into a **minimal product of sums (POS)**:

$$F(a,b,c)=\prod M(1,2,3,5,6)$$

- Please finish it by yourself.

- **Solution:**

$$\overline{F}=\overline{a}b+b\overline{c}+\overline{b}c$$

$$F=\overline{\overline{a}b+b\overline{c}+\overline{b}c}$$

$$=\left(a+\overline{b}\right)\left(\overline{b}+c\right)\left(b+\overline{c}\right)$$