Chapter 3

VECTOR CALCULUS

No man really becomes a fool until he stops asking questions.

—CHARLES P. STEINMETZ

3.1 INTRODUCTION

Chapter 1 is mainly on vector addition, subtraction, and multiplication in Cartesian coordinates, and Chapter 2 extends all these to other coordinate systems. This chapter deals with vector calculus—integration and differentiation of vectors.

The concepts introduced in this chapter provide a convenient language for expressing certain fundamental ideas in electromagnetics or mathematics in general. A student may feel uneasy about these concepts at first—not seeing "what good" they are. Such a student is advised to concentrate simply on learning the mathematical techniques and to wait for their applications in subsequent chapters.

3.2 DIFFERENTIAL LENGTH, AREA, AND VOLUME

Differential elements in length, area, and volume are useful in vector calculus. They are defined in the Cartesian, cylindrical, and spherical coordinate systems.

A. Cartesian Coordinates

From Figure 3.1, we notice that

(1) Differential displacement is given by

$$\boxed{d\mathbf{l} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z} \tag{3.1}$$

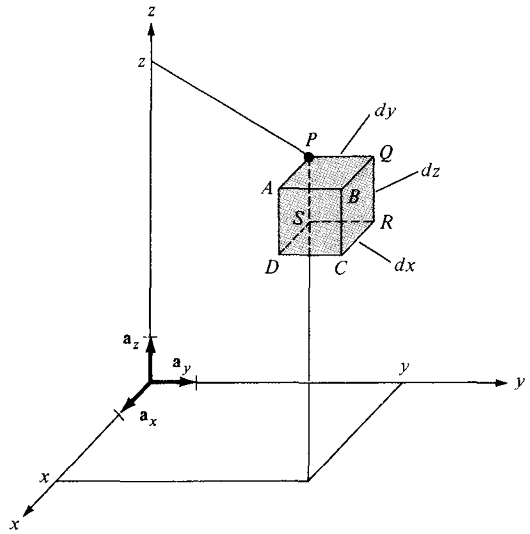

Figure 3.1 Differential elements in the right-handed Cartesian coordinate system.

(2) Differential normal area is given by

$$\boxed{\begin{aligned} d\mathbf{S} = dy\,dz\,\mathbf{a}_x \\ dx\,dz\,\mathbf{a}_y \\ dz\,dy\,\mathbf{a}_z \end{aligned}} \tag{3.2}$$

and illustrated in Figure 3.2.

(3) Differential volume is given by

$$\boxed{dv = dx\,dy\,dz} \tag{3.3}$$

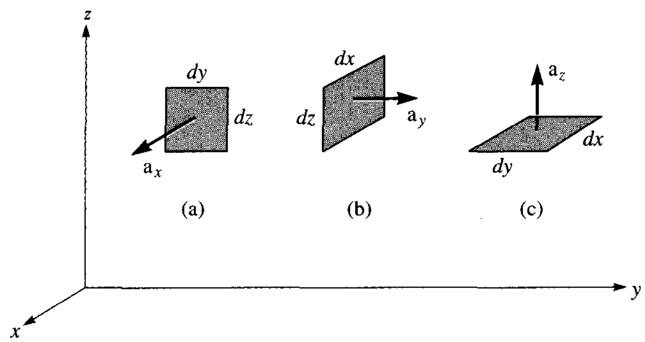

Figure 3.2 Differential normal areas in Cartesian coordinates: **(a)** $d\mathbf{S} = dy\,dz\,\mathbf{a}_x$, **(b)** $d\mathbf{S} = dx\,dz\,\mathbf{a}_y$, **(c)** $d\mathbf{S} = dx\,dy\,\mathbf{a}_z$

These differential elements are very important as they will be referred to again and again throughout the book. The student is encouraged not to memorize them, however, but to learn to derive them from Figure 3.1. Notice from eqs. (3.1) to (3.3) that $d\mathbf{l}$ and $d\mathbf{S}$ are vectors whereas dv is a scalar. Observe from Figure 3.1 that if we move from point P to Q (or Q to P), for example, $d\mathbf{l} = dy\,\mathbf{a}_y$ because we are moving in the y-direction and if we move from Q to S (or S to Q), $d\mathbf{l} = dy\,\mathbf{a}_y + dz\,\mathbf{a}_z$ because we have to move dy along y, dz along z, and $dx = 0$ (no movement along x). Similarly, to move from D to Q would mean that $d\mathbf{l} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z$.

The way $d\mathbf{S}$ is defined is important. The differential surface (or area) element $d\mathbf{S}$ may generally be defined as

$$d\mathbf{S} = dS\,\mathbf{a}_n \tag{3.4}$$

where dS is the area of the surface element and $\mathbf{a}_n$ is a unit vector normal to the surface dS (and directed away from the volume if dS is part of the surface describing a volume). If we consider surface $ABCD$ in Figure 3.1, for example, $d\mathbf{S} = dy\,dz\,\mathbf{a}_x$ whereas for surface $PQRS$, $d\mathbf{S} = -dy\,dz\,\mathbf{a}_x$ because $\mathbf{a}_n = -\mathbf{a}_x$ is normal to $PQRS$.

What we have to remember at all times about differential elements is $d\mathbf{l}$ and how to get $d\mathbf{S}$ and dv from it. Once $d\mathbf{l}$ is remembered, $d\mathbf{S}$ and dv can easily be found. For example, $d\mathbf{S}$ along $\mathbf{a}_x$ can be obtained from $d\mathbf{l}$ in eq. (3.1) by multiplying the components of $d\mathbf{l}$ along $\mathbf{a}_y$ and $\mathbf{a}_z$; that is, $dy\,dz\,\mathbf{a}_x$. Similarly, $d\mathbf{S}$ along $\mathbf{a}_z$ is the product of the components of $d\mathbf{l}$ along $\mathbf{a}_x$ and $\mathbf{a}_y$; that is $dx\,dy\,\mathbf{a}_z$. Also, dv can be obtained from $d\mathbf{l}$ as the product of the three components of $d\mathbf{l}$; that is, $dx\,dy\,dz$. The idea developed here for Cartesian coordinates will now be extended to other coordinate systems.

B. Cylindrical Coordinates

Notice from Figure 3.3 that in cylindrical coordinates, differential elements can be found as follows:

(1) Differential displacement is given by

$$\boxed{d\mathbf{l} = d\rho\,\mathbf{a}_\rho + \rho\,d\phi\,\mathbf{a}_\phi + dz\,\mathbf{a}_z} \tag{3.5}$$

(2) Differential normal area is given by

$$\boxed{\begin{aligned} d\mathbf{S} = {} & \rho\,d\phi\,dz\,\mathbf{a}_\rho \\ & d\rho\,dz\,\mathbf{a}_\phi \\ & \rho\,d\phi\,d\rho\,\mathbf{a}_z \end{aligned}} \tag{3.6}$$

and illustrated in Figure 3.4.

(3) Differential volume is given by

$$\boxed{dv = \rho\,d\rho\,d\phi\,dz} \tag{3.7}$$

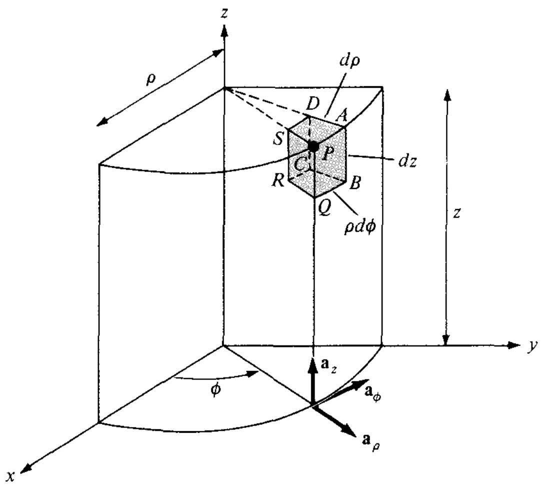

Figure 3.3 Differential elements in cylindrical coordinates.

As mentioned in the previous section on Cartesian coordinates, we only need to remember $d\mathbf{l}$; $d\mathbf{S}$ and dv can easily be obtained from $d\mathbf{l}$. For example, $d\mathbf{S}$ along $\mathbf{a}_z$ is the product of the components of $d\mathbf{l}$ along $\mathbf{a}_\rho$ and $\mathbf{a}_\phi$; that is, $d\rho\ \rho\ d\phi\ \mathbf{a}_z$. Also dv is the product of the three components of $d\mathbf{l}$; that is, $d\rho\ \rho\ d\phi\ dz$.

C. Spherical Coordinates

From Figure 3.5, we notice that in spherical coordinates,

(1) The differential displacement is

$$\boxed{d\mathbf{l} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi} \tag{3.8}$$

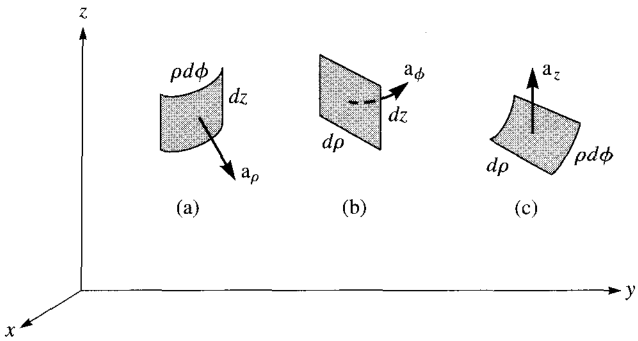

Figure 3.4 Differential normal areas in cylindrical coordinates: **(a)** $d\mathbf{S} = \rho\ d\phi\ dz\ \mathbf{a}_\rho$, **(b)** $d\mathbf{S} = d\rho\ dz\ \mathbf{a}_\phi$, **(c)** $d\mathbf{S} = \rho\ d\phi\ d\rho\ \mathbf{a}_z$

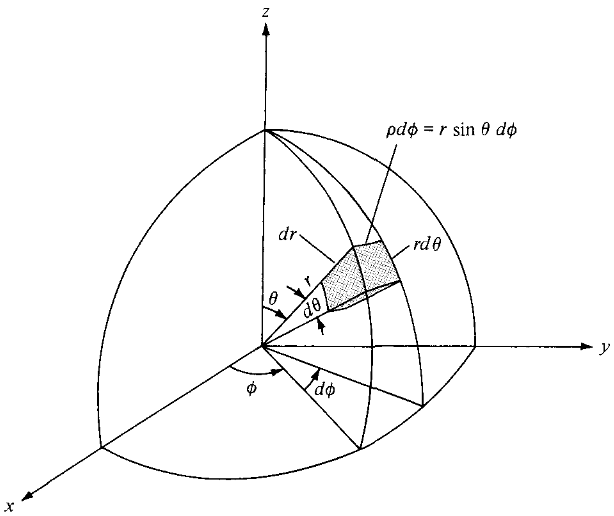

Figure 3.5 Differential elements in the spherical coordinate system.

(2) The differential normal area is

$$\boxed{\begin{aligned} d\mathbf{S} = {} & r^2 \sin \theta \, d\theta \, d\phi \, \mathbf{a}_r \\ & r \sin \theta \, dr \, d\phi \, \mathbf{a}_\theta \\ & r \, dr \, d\theta \, \mathbf{a}_\phi \end{aligned}} \tag{3.9}$$

and illustrated in Figure 3.6.

(3) The differential volume is

$$\boxed{dv = r^2 \sin \theta \, dr \, d\theta \, d\phi} \tag{3.10}$$

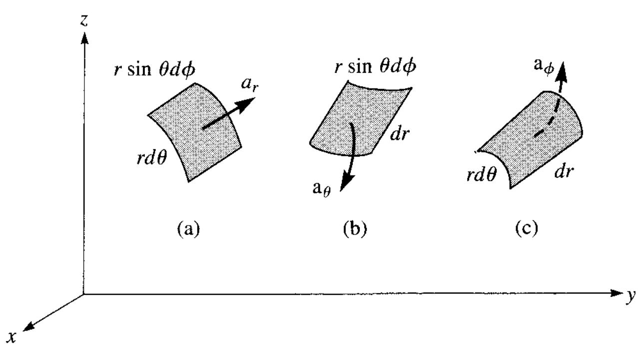

Figure 3.6 Differential normal areas in spherical coordinates: **(a)** $d\mathbf{S} = r^2 \sin \theta \, d\theta \, d\phi \, \mathbf{a}_r$, **(b)** $d\mathbf{S} = r \sin \theta \, dr \, d\phi \, \mathbf{a}_\phi$, **(c)** $d\mathbf{S} = r \, dr \, d\theta \, \mathbf{a}_\phi$

Again, we need to remember only $d\mathbf{l}$ from which $d\mathbf{S}$ and dv are easily obtained. For example, $d\mathbf{S}$ along $\mathbf{a}_\theta$ is obtained as the product of the components of $d\mathbf{l}$ along $\mathbf{a}_r$ and $\mathbf{a}_\phi$; that is, $dr \cdot r \sin\theta \, d\phi$; dv is the product of the three components of $d\mathbf{l}$; that is, $dr \cdot r \, d\theta \cdot r \sin\theta \, d\phi$.

EXAMPLE 3.1

Consider the object shown in Figure 3.7. Calculate

(a) The distance BC

(b) The distance CD

(c) The surface area $ABCD$

(d) The surface area ABO

(e) The surface area $AOFD$

(f) The volume $ABDCFO$

Solution:

Although points A, B, C, and D are given in Cartesian coordinates, it is obvious that the object has cylindrical symmetry. Hence, we solve the problem in cylindrical coordinates. The points are transformed from Cartesian to cylindrical coordinates as follows:

$$A(5, 0, 0) \rightarrow A(5, 0°, 0)$$

$$B(0, 5, 0) \rightarrow B\left(5, \frac{\pi}{2}, 0\right)$$

$$C(0, 5, 10) \rightarrow C\left(5, \frac{\pi}{2}, 10\right)$$

$$D(5, 0, 10) \rightarrow D(5, 0°, 10)$$

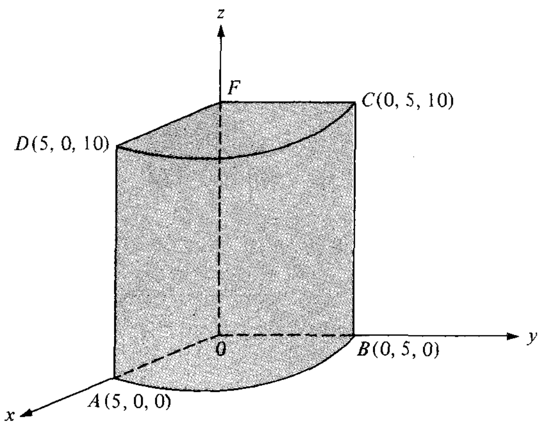

Figure 3.7 For Example 3.1.

(a) Along BC, $dl = dz$; hence,

$$BC = \int dl = \int_0^{10} dz = 10$$

(b) Along CD, $dl = \rho\, d\phi$ and $\rho = 5$, so

$$CD = \int_0^{\pi/2} \rho\, d\phi = 5\, \phi \Big|_0^{\pi/2} = 2.5\pi$$

(c) For $ABCD$, $dS = \rho\, d\phi\, dz$, $\rho = 5$. Hence,

$$\text{area } ABCD = \int dS = \int_{\phi=0}^{\pi/2} \int_{z=0}^{10} \rho\, d\phi\, dz = 5 \int_0^{\pi/2} d\phi \int_0^{10} dz \Big|_{\rho=5} = 25\pi$$

(d) For ABO, $dS = \rho\, d\phi\, d\rho$ and $z = 0$, so

$$\text{area } ABO = \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{5} \rho\, d\phi\, d\rho = \int_0^{\pi/2} d\phi \int_0^5 \rho\, d\rho = 6.25\pi$$

(e) For $AOFD$, $dS = d\rho\, dz$ and $\phi = 0°$, so

$$\text{area } AOFD = \int_{\rho=0}^{5} \int_{z=0}^{10} d\rho\, dz = 50$$

(f) For volume $ABDCFO$, $dv = \rho\, d\phi\, dz\, d\rho$. Hence,

$$v = \int dv = \int_{\rho=0}^{5} \int_{\phi=0}^{\pi/2} \int_{z=0}^{10} \rho\, d\phi\, dz\, d\rho = \int_0^{10} dz \int_0^{\pi/2} d\phi \int_0^5 \rho\, d\rho = 62.5\pi$$

PRACTICE EXERCISE 3.1

Refer to Figure 3.26; disregard the differential lengths and imagine that the object is part of a spherical shell. It may be described as $3 \le r \le 5$, $60° \le \theta \le 90°$, $45° \le \phi \le 60°$ where surface $r = 3$ is the same as $AEHD$, surface $\theta = 60°$ is $AEFB$, and surface $\phi = 45°$ is $ABCD$. Calculate

(a) The distance DH

(b) The distance FG

(c) The surface area $AEHD$

(d) The surface area $ABDC$

(e) The volume of the object

Answer: (a) 0.7854, (b) 2.618, (c) 1.179, (d) 4.189, (e) 4.276.

3.3 LINE, SURFACE, AND VOLUME INTEGRALS

The familiar concept of integration will now be extended to cases when the integrand involves a vector. By a line we mean the path along a curve in space. We shall use terms such as *line, curve,* and *contour* interchangeably.

The line integral $\int_L \mathbf{A} \cdot d\mathbf{l}$ is the integral of the tangential component of $\mathbf{A}$ along curve L.

Given a vector field $\mathbf{A}$ and a curve L, we define the integral

$$\int_L \mathbf{A} \cdot d\mathbf{l} = \int_a^b |\mathbf{A}| \cos \theta \, dl \tag{3.11}$$

as the *line integral* of $\mathbf{A}$ around L (see Figure 3.8). If the path of integration is a closed curve such as *abca* in Figure 3.8, eq. (3.11) becomes a closed contour integral

$$\oint_L \mathbf{A} \cdot d\mathbf{l} \tag{3.12}$$

which is called the *circulation* of $\mathbf{A}$ around L.

Given a vector field $\mathbf{A}$, continuous in a region containing the smooth surface S, we define the *surface integral* or the *flux* of $\mathbf{A}$ through S (see Figure 3.9) as

$$\Psi = \int_S |\mathbf{A}| \cos \theta \, dS = \int_S \mathbf{A} \cdot \mathbf{a}_n \, dS$$

or simply

$$\boxed{\Psi = \int_S \mathbf{A} \cdot d\mathbf{S}} \tag{3.13}$$

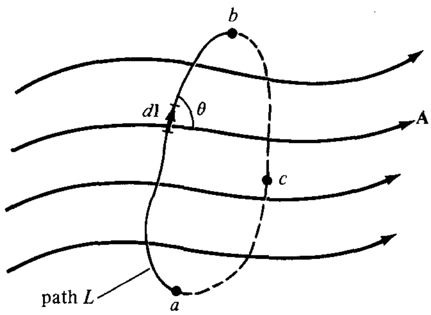

Figure 3.8 Path of integration of vector field **A**.

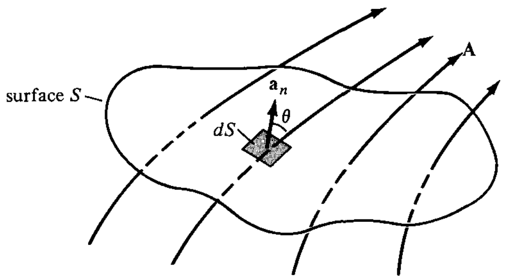

Figure 3.9 The flux of a vector field **A** through surface S.

where, at any point on S, $\mathbf{a}_n$ is the unit normal to S. For a closed surface (defining a volume), eq. (3.13) becomes

$$\Psi = \oint_S \mathbf{A} \cdot d\mathbf{S} \tag{3.14}$$

which is referred to as the *net outward flux of* **A** from S. Notice that a closed path defines an open surface whereas a closed surface defines a volume (see Figures 3.11 and 3.16).

We define the integral

$$\int_v \rho_v \, dv \tag{3.15}$$

as the *volume integral* of the scalar ρ_v over the volume v. The physical meaning of a line, surface, or volume integral depends on the nature of the physical quantity represented by **A** or ρ_v. Note that $d\mathbf{l}$, $d\mathbf{S}$, and dv are all as defined in Section 3.2.

EXAMPLE 3.2

Given that $\mathbf{F} = x^2\mathbf{a}_x - xz\mathbf{a}_y - y^2\mathbf{a}_z$, calculate the circulation of **F** around the (closed) path shown in Figure 3.10.

Solution:

The circulation of **F** around path L is given by

$$\oint_L \mathbf{F} \cdot d\mathbf{l} = \left(\int_1 + \int_2 + \int_3 + \int_4 \right) \mathbf{F} \cdot d\mathbf{l}$$

where the path is broken into segments numbered 1 to 4 as shown in Figure 3.10.

For segment 1, $y = 0 = z$

$$\mathbf{F} = x^2\mathbf{a}_x, \qquad d\mathbf{l} = dx\,\mathbf{a}_x$$

Notice that $d\mathbf{l}$ is always taken as along $+\mathbf{a}_x$ so that the direction on segment 1 is taken care of by the limits of integration. Thus,

$$\int_1 \mathbf{F} \cdot d\mathbf{l} = \int_1^0 x^2 dx = \frac{x^3}{3} \bigg|_1^0 = -\frac{1}{3}$$

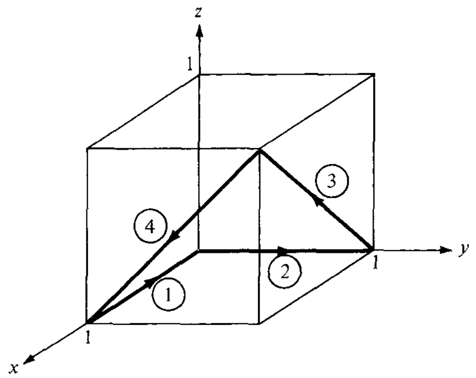

Figure 3.10 For Example 3.2.

For segment 2, $x = 0 = z$, $\mathbf{F} = -y^2\mathbf{a}_z$, $d\mathbf{l} = dy\,\mathbf{a}_y$, $\mathbf{F} \cdot d\mathbf{l} = 0$. Hence,

$$\int_2 \mathbf{F} \cdot d\mathbf{l} = 0$$

For segment 3, $y = 1$, $\mathbf{F} = x^2\mathbf{a}_x - xz\mathbf{a}_y - \mathbf{a}_z$, and $d\mathbf{l} = dx\,\mathbf{a}_x + dz\,\mathbf{a}_z$, so

$$\int_3 \mathbf{F} \cdot d\mathbf{l} = \int (x^2 dx - dz)$$

But on 3, $z = x$; that is, $dx = dz$. Hence,

$$\int_3 \mathbf{F} \cdot d\mathbf{l} = \int_0^1 (x^2 - 1)\, dx = \frac{x^3}{3} - x \Big|_0^1 = -\frac{2}{3}$$

For segment 4, $x = 1$, so $\mathbf{F} = \mathbf{a}_x - z\mathbf{a}_y - y^2\mathbf{a}_z$, and $d\mathbf{l} = dy\,\mathbf{a}_y + dz\,\mathbf{a}_z$. Hence,

$$\int_4 \mathbf{F} \cdot d\mathbf{l} = \int (-z\, dy - y^2 dz)$$

But on 4, $z = y$; that is, $dz = dy$, so

$$\int_4 \mathbf{F} \cdot d\mathbf{l} = \int_1^0 (-y - y^2)\, dy = -\frac{y^2}{2} - \frac{y^3}{3} \Big|_1^0 = \frac{5}{6}$$

By putting all these together, we obtain

$$\oint_L \mathbf{F} \cdot d\mathbf{l} = -\frac{1}{3} + 0 - \frac{2}{3} + \frac{5}{6} = -\frac{1}{6}$$

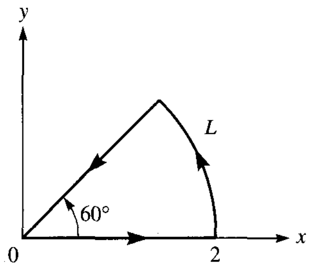

Figure 3.11 For Practice Exercise 3.2.

PRACTICE EXERCISE 3.2

Calculate the circulation of

$$\mathbf{A} = \rho \cos \phi \, \mathbf{a}_\rho + z \sin \phi \, \mathbf{a}_z$$

around the edge L of the wedge defined by $0 \leq \rho \leq 2$, $0 \leq \phi \leq 60°$, $z = 0$ and shown in Figure 3.11.

Answer: 1.

3.4 DEL OPERATOR

The del operator, written ∇, is the vector differential operator. In Cartesian coordinates,

$$\boxed{\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z} \tag{3.16}$$

This vector differential operator, otherwise known as the *gradient operator*, is not a vector in itself, but when it operates on a scalar function, for example, a vector ensues. The operator is useful in defining

1. The gradient of a scalar V, written as ∇V
2. The divergence of a vector $\mathbf{A}$, written as $\nabla \cdot \mathbf{A}$
3. The curl of a vector $\mathbf{A}$, written as $\nabla \times \mathbf{A}$
4. The Laplacian of a scalar V, written as $\nabla^2 V$

Each of these will be defined in detail in the subsequent sections. Before we do that, it is appropriate to obtain expressions for the del operator ∇ in cylindrical and spherical coordinates. This is easily done by using the transformation formulas of Section 2.3 and 2.4.

To obtain ∇ in terms of ρ, ϕ, and z, we recall from eq. (2.7) that[1]

$$\rho = \sqrt{x^2 + y^2}, \qquad \tan\phi = \frac{y}{x}$$

Hence

$$\frac{\partial}{\partial x} = \cos\phi \frac{\partial}{\partial \rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial \phi} \tag{3.17}$$

$$\frac{\partial}{\partial y} = \sin\phi \frac{\partial}{\partial \rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial \phi} \tag{3.18}$$

Substituting eqs. (3.17) and (3.18) into eq. (3.16) and making use of eq. (2.9), we obtain ∇ in cylindrical coordinates as

$$\boxed{\nabla = \mathbf{a}_\rho \frac{\partial}{\partial \rho} + \mathbf{a}_\phi \frac{1}{\rho}\frac{\partial}{\partial \phi} + \mathbf{a}_z \frac{\partial}{\partial z}} \tag{3.19}$$

Similarly, to obtain ∇ in terms of r, θ, and ϕ, we use

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \tan\theta = \frac{\sqrt{x^2 + y^2}}{z}, \qquad \tan\phi = \frac{y}{x}$$

to obtain

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi \frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial \theta} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial \phi} \tag{3.20}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\phi \frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial \theta} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial \phi} \tag{3.21}$$

$$\frac{\partial}{\partial z} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \tag{3.22}$$

Substituting eqs. (3.20) to (3.22) into eq. (3.16) and using eq. (2.23) results in ∇ in spherical coordinates:

$$\boxed{\nabla = \mathbf{a}_r \frac{\partial}{\partial r} + \mathbf{a}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{a}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}} \tag{3.23}$$

Notice that in eqs. (3.19) and (3.23), the unit vectors are placed to the right of the differential operators because the unit vectors depend on the angles.

[1] A more general way of deriving ∇, $\nabla \cdot \mathbf{A}$, $\nabla \times \mathbf{A}$, ∇V, and $\nabla^2 V$ is using the curvilinear coordinates. See, for example, M. R. Spiegel, *Vector Analysis and an Introduction to Tensor Analysis.* New York: McGraw-Hill, 1959, pp. 135–165.

3.5 GRADIENT OF A SCALAR

The **gradient** of a scalar field V is a vector that represents both the magnitude and the direction of the maximum space rate of increase of V.

A mathematical expression for the gradient can be obtained by evaluating the difference in the field dV between points P_1 and P_2 of Figure 3.12 where V_1, V_2, and V_3 are contours on which V is constant. From calculus,

$$\begin{aligned} dV &= \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \\ &= \left(\frac{\partial V}{\partial x} \mathbf{a}_x + \frac{\partial V}{\partial y} \mathbf{a}_y + \frac{\partial V}{\partial z} \mathbf{a}_z\right) \cdot (dx\, \mathbf{a}_x + dy\, \mathbf{a}_y + dz\, \mathbf{a}_z) \end{aligned} \tag{3.24}$$

For convenience, let

$$\mathbf{G} = \frac{\partial V}{\partial x} \mathbf{a}_x + \frac{\partial V}{\partial y} \mathbf{a}_y + \frac{\partial V}{\partial z} \mathbf{a}_z \tag{3.25}$$

Then

$$dV = \mathbf{G} \cdot d\mathbf{l} = G \cos\theta \, dl$$

or

$$\frac{dV}{dl} = G \cos\theta \tag{3.26}$$

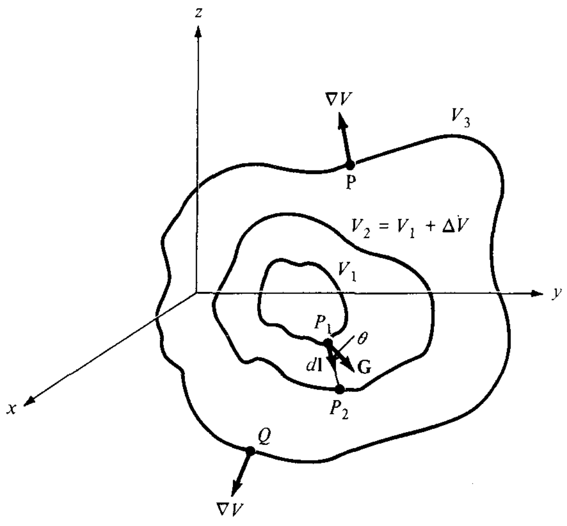

Figure 3.12 Gradient of a scalar.

where $d\mathbf{l}$ is the differential displacement from P_1 to P_2 and θ is the angle between $\mathbf{G}$ and $d\mathbf{l}$. From eq. (3.26), we notice that dV/dl is a maximum when $\theta = 0$, that is, when $d\mathbf{l}$ is in the direction of $\mathbf{G}$. Hence,

$$\left.\frac{dV}{dl}\right|_{\max} = \frac{dV}{dn} = G \tag{3.27}$$

where dV/dn is the normal derivative. Thus G has its magnitude and direction as those of the maximum rate of change of V. By definition, $\mathbf{G}$ is the gradient of V. Therefore:

$$\text{grad } V = \nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z \tag{3.28}$$

By using eq. (3.28) in conjunction with eqs. (3.16), (3.19), and (3.23), the gradient of V can be expressed in Cartesian, cylindrical, and spherical coordinates. For Cartesian coordinates

$$\boxed{\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z}$$

for cylindrical coordinates,

$$\boxed{\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z} \tag{3.29}$$

and for spherical coordinates,

$$\boxed{\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi} \tag{3.30}$$

The following computation formulas on gradient, which are easily proved, should be noted:

(a) $\nabla(V + U) = \nabla V + \nabla U$ (3.31a)

(b) $\nabla(VU) = V\nabla U + U\nabla V$ (3.31b)

(c) $\nabla\left[\frac{V}{U}\right] = \frac{U\nabla V - V\nabla U}{U^2}$ (3.31c)

(d) $\nabla V^n = nV^{n-1}\nabla V$ (3.31d)

where U and V are scalars and n is an integer.

Also take note of the following fundamental properties of the gradient of a scalar field V:

1. The magnitude of ∇V equals the maximum rate of change in V per unit distance.
2. ∇V points in the direction of the maximum rate of change in V.
3. ∇V at any point is perpendicular to the constant V surface that passes through that point (see points P and Q in Figure 3.12).

4. The projection (or component) of ∇V in the direction of a unit vector $\mathbf{a}$ is $\nabla V \cdot \mathbf{a}$ and is called the *directional derivative* of V along $\mathbf{a}$. This is the rate of change of V in the direction of $\mathbf{a}$. For example, dV/dl in eq. (3.26) is the directional derivative of V along P_1P_2 in Figure 3.12. Thus the gradient of a scalar function V provides us with both the direction in which V changes most rapidly and the magnitude of the maximum directional derivative of V.
5. If $\mathbf{A} = \nabla V$, V is said to be the scalar potential of $\mathbf{A}$.

EXAMPLE 3.3

Find the gradient of the following scalar fields:

(a) $V = e^{-z} \sin 2x \cosh y$

(b) $U = \rho^2 z \cos 2\phi$

(c) $W = 10r \sin^2 \theta \cos \phi$

Solution:

(a)
$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$
$$= 2e^{-z}\cos 2x \cosh y\, \mathbf{a}_x + e^{-z}\sin 2x \sinh y\, \mathbf{a}_y - e^{-z}\sin 2x \cosh y\, \mathbf{a}_z$$

(b)
$$\nabla U = \frac{\partial U}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial U}{\partial \phi}\mathbf{a}_\phi + \frac{\partial U}{\partial z}\mathbf{a}_z$$
$$= 2\rho z \cos 2\phi\, \mathbf{a}_\rho - 2\rho z \sin 2\phi\, \mathbf{a}_\phi + \rho^2 \cos 2\phi\, \mathbf{a}_z$$

(c)
$$\nabla W = \frac{\partial W}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial W}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial W}{\partial \phi}\mathbf{a}_\phi$$
$$= 10 \sin^2 \theta \cos \phi\, \mathbf{a}_r + 10 \sin 2\theta \cos \phi\, \mathbf{a}_\theta - 10 \sin \theta \sin \phi\, \mathbf{a}_\phi$$

PRACTICE EXERCISE 3.3

Determine the gradient of the following scalar fields:

(a) $U = x^2 y + xyz$

(b) $V = \rho z \sin \phi + z^2 \cos^2 \phi + \rho^2$

(c) $f = \cos \theta \sin \phi \ln r + r^2 \phi$

Answer: (a) $y(2x + z)\mathbf{a}_x + x(x + z)\mathbf{a}_y + xy\mathbf{a}_z$

(b) $(z \sin \phi + 2\rho)\mathbf{a}_\rho + \left(z \cos \phi - \frac{z^2}{\rho} \sin 2\phi\right)\mathbf{a}_\phi + (\rho \sin \phi + 2z \cos^2 \phi)\mathbf{a}_z$

(c) $\left(\frac{\cos \theta \sin \phi}{r} + 2r\phi\right)\mathbf{a}_r - \frac{\sin \theta \sin \phi}{r} \ln r\, \mathbf{a}_\theta + \left(\frac{\cot \theta}{r} \cos \phi \ln r + r \operatorname{cosec} \theta\right)\mathbf{a}_\phi$

EXAMPLE 3.4

Given $W = x^2y^2 + xyz$, compute ∇W and the direction derivative dW/dl in the direction $3\mathbf{a}_x + 4\mathbf{a}_y + 12\mathbf{a}_z$ at $(2, -1, 0)$.

Solution:

$$\begin{aligned} \nabla W &= \frac{\partial W}{\partial x}\mathbf{a}_x + \frac{\partial W}{\partial y}\mathbf{a}_y + \frac{\partial W}{\partial z}\mathbf{a}_z \\ &= (2xy^2 + yz)\mathbf{a}_x + (2x^2y + xz)\mathbf{a}_y + (xy)\mathbf{a}_z \end{aligned}$$

At $(2, -1, 0)$: $\nabla W = 4\mathbf{a}_x - 8\mathbf{a}_y - 2\mathbf{a}_z$
Hence,

$$\frac{dW}{dl} = \nabla W \cdot \mathbf{a}_l = (4, -8, -2) \cdot \frac{(3, 4, 12)}{13} = -\frac{44}{13}$$

PRACTICE EXERCISE 3.4

Given $\Phi = xy + yz + xz$, find gradient Φ at point $(1, 2, 3)$ and the directional derivative of Φ at the same point in the direction toward point $(3, 4, 4)$.

Answer: $5\mathbf{a}_x + 4\mathbf{a}_y + 3\mathbf{a}_z$, 7.

EXAMPLE 3.5

Find the angle at which line $x = y = 2z$ intersects the ellipsoid $x^2 + y^2 + 2z^2 = 10$.

Solution:

Let the line and the ellipsoid meet at angle ψ as shown in Figure 3.13. The line $x = y = 2z$ can be represented by

$$\mathbf{r}(\lambda) = 2\lambda\mathbf{a}_x + 2\lambda\mathbf{a}_y + \lambda\mathbf{a}_z$$

where λ is a parameter. Where the line and the ellipsoid meet,

$$(2\lambda)^2 + (2\lambda)^2 + 2\lambda^2 = 10 \rightarrow \lambda = \pm 1$$

Taking $\lambda = 1$ (for the moment), the point of intersection is $(x, y, z) = (2, 2, 1)$. At this point, $\mathbf{r} = 2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z$.

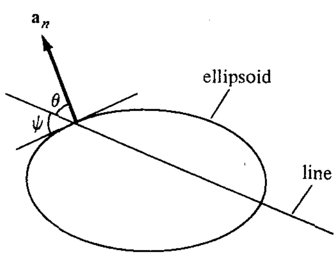

Figure 3.13 For Example 3.5; plane of intersection of a line with an ellipsoid.

The surface of the ellipsoid is defined by

$$f(x, y, z) = x^2 + y^2 + 2z^2 - 10$$

The gradient of f is

$$\nabla f = 2x\,\mathbf{a}_x + 2y\,\mathbf{a}_y + 4z\,\mathbf{a}_z$$

At (2, 2, 1), $\nabla f = 4\mathbf{a}_x + 4\mathbf{a}_y + 4\mathbf{a}_z$. Hence, a unit vector normal to the ellipsoid at the point of intersection is

$$\mathbf{a}_n = \pm\frac{\nabla f}{|\nabla f|} = \pm\frac{\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z}{\sqrt{3}}$$

Taking the positive sign (for the moment), the angle between $\mathbf{a}_n$ and $\mathbf{r}$ is given by

$$\cos\theta = \frac{\mathbf{a}_n \cdot \mathbf{r}}{|\mathbf{a}_n \cdot \mathbf{r}|} = \frac{2 + 2 + 1}{\sqrt{3}\sqrt{9}} = \frac{5}{3\sqrt{3}} = \sin\psi$$

Hence, $\psi = 74.21°$. Because we had choices of + or − for λ and $\mathbf{a}_n$, there are actually four possible angles, given by $\sin\psi = \pm 5/(3\sqrt{3})$.

PRACTICE EXERCISE 3.5

Calculate the angle between the normals to the surfaces $x^2y + z = 3$ and $x \log z - y^2 = -4$ at the point of intersection $(-1, 2, 1)$.

Answer: 73.4°.

3.6 DIVERGENCE OF A VECTOR AND DIVERGENCE THEOREM

From Section 3.3, we have noticed that the net outflow of the flux of a vector field **A** from a closed surface S is obtained from the integral $\oint \mathbf{A} \cdot d\mathbf{S}$. We now define the divergence of **A** as the net outward flow of flux per unit volume over a closed incremental surface.

The **divergence** of **A** at a given point P is the *outward* flux per unit volume as the volume shrinks about P.

Hence,

$$\operatorname{div}\mathbf{A} = \nabla \cdot \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} \tag{3.32}$$

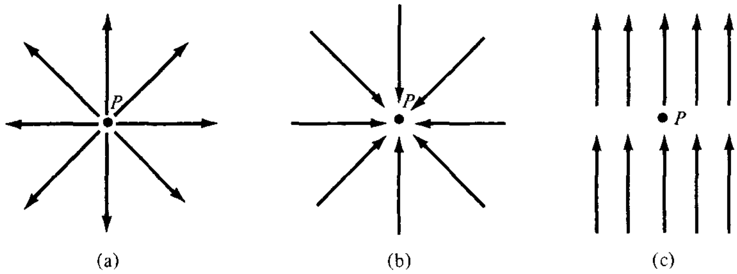

Figure 3.14 Illustration of the divergence of a vector field at P; **(a)** positive divergence, **(b)** negative divergence, **(c)** zero divergence.

where Δv is the volume enclosed by the closed surface S in which P is located. Physically, we may regard the divergence of the vector field $\mathbf{A}$ at a given point as a measure of how much the field diverges or emanates from that point. Figure 3.14(a) shows that the divergence of a vector field at point P is positive because the vector diverges (or spreads out) at P. In Figure 3.14(b) a vector field has negative divergence (or convergence) at P, and in Figure 3.14(c) a vector field has zero divergence at P. The divergence of a vector field can also be viewed as simply the limit of the field's source strength per unit volume (or source density); it is positive at a *source* point in the field, and negative at a *sink* point, or zero where there is neither sink nor source.

We can obtain an expression for $\nabla \cdot \mathbf{A}$ in Cartesian coordinates from the definition in eq. (3.32). Suppose we wish to evaluate the divergence of a vector field $\mathbf{A}$ at point $P(x_o, y_o, z_o)$; we let the point be enclosed by a differential volume as in Figure 3.15. The surface integral in eq. (3.32) is obtained from

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \left(\int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}} \right) \mathbf{A} \cdot d\mathbf{S} \tag{3.33}$$

A three-dimensional Taylor series expansion of A_x about P is

$$\begin{aligned} A_x(x, y, z) = A_x(x_o, y_o, z_o) &+ (x - x_o) \left.\frac{\partial A_x}{\partial x}\right|_P + (y - y_o) \left.\frac{\partial A_x}{\partial y}\right|_P \\ &+ (z - z_o) \left.\frac{\partial A_x}{\partial z}\right|_P + \text{higher-order terms} \end{aligned} \tag{3.34}$$

For the front side, $x = x_o + dx/2$ and $d\mathbf{S} = dy\, dz\, \mathbf{a}_x$. Then,

$$\int_{\text{front}} \mathbf{A} \cdot d\mathbf{S} = dy\, dz \left[A_x(x_o, y_o, z_o) + \frac{dx}{2} \left.\frac{\partial A_x}{\partial x}\right|_P \right] + \text{higher-order terms}$$

For the back side, $x = x_o - dx/2$, $d\mathbf{S} = dy\, dz(-\mathbf{a}_x)$. Then,

$$\int_{\text{back}} \mathbf{A} \cdot d\mathbf{S} = -dy\, dz \left[A_x(x_o, y_o, z_o) - \frac{dx}{2} \left.\frac{\partial A_x}{\partial x}\right|_P \right] + \text{higher-order terms}$$

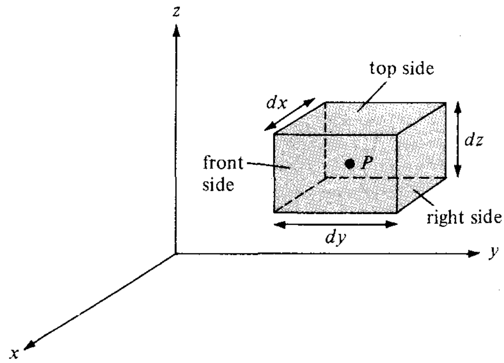

Figure 3.15 Evaluation of $\nabla \cdot \mathbf{A}$ at point $P(x_0, y_0, z_0)$.

Hence,

$$\int_{\text{front}} \mathbf{A} \cdot d\mathbf{S} + \int_{\text{back}} \mathbf{A} \cdot d\mathbf{S} = dx\, dy\, dz \left.\frac{\partial A_x}{\partial x}\right|_P + \text{higher-order terms} \tag{3.35}$$

By taking similar steps, we obtain

$$\int_{\text{left}} \mathbf{A} \cdot d\mathbf{S} + \int_{\text{right}} \mathbf{A} \cdot d\mathbf{S} = dx\, dy\, dz \left.\frac{\partial A_y}{\partial y}\right|_P + \text{higher-order terms} \tag{3.36}$$

and

$$\int_{\text{top}} \mathbf{A} \cdot d\mathbf{S} + \int_{\text{bottom}} \mathbf{A} \cdot d\mathbf{S} = dx\, dy\, dz \left.\frac{\partial A_z}{\partial z}\right|_P + \text{higher-order terms} \tag{3.37}$$

Substituting eqs. (3.35) to (3.37) into eq. (3.33) and noting that $\Delta v = dx\, dy\, dz$, we get

$$\lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} = \left.\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\right|_{\text{at } P} \tag{3.38}$$

because the higher-order terms will vanish as $\Delta v \to 0$. Thus, the divergence of **A** at point $P(x_o, y_o, z_o)$ in a Cartesian system is given by

$$\boxed{\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}} \tag{3.39}$$

Similar expressions for $\nabla \cdot \mathbf{A}$ in other coordinate systems can be obtained directly from eq. (3.32) or by transforming eq. (3.39) into the appropriate coordinate system. In cylindrical coordinates, substituting eqs. (2.15), (3.17), and (3.18) into eq. (3.39) yields

$$\boxed{\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}} \tag{3.40}$$

Substituting eqs. (2.28) and (3.20) to (3.22) into eq. (3.39), we obtain the divergence of **A** in spherical coordinates as

$$\boxed{\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}} \tag{3.41}$$

Note the following properties of the divergence of a vector field:

1. It produces a scalar field (because scalar product is involved).
2. The divergence of a scalar V, div V, makes no sense.
3. $\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$
4. $\nabla \cdot (V\mathbf{A}) = V\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla V$

From the definition of the divergence of **A** in eq. (3.32), it is not difficult to expect that

$$\boxed{\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{A}\, dv} \tag{3.42}$$

This is called the *divergence theorem,* otherwise known as the *Gauss–Ostrogradsky theorem.*

The **divergence theorem** states that the total outward flux of a vector field **A** through the *closed* surface S is the same as the volume integral of the divergence of **A**.

To prove the divergence theorem, subdivide volume v into a large number of small cells. If the kth cell has volume Δv_k and is bounded by surface S_k

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \sum_k \oint_{S_k} \mathbf{A} \cdot d\mathbf{S} = \sum_k \frac{\oint_{S_k} \mathbf{A} \cdot d\mathbf{S}}{\Delta v_k} \Delta v_k \tag{3.43}$$

Since the outward flux to one cell is inward to some neighboring cells, there is cancellation on every interior surface, so the sum of the surface integrals over S_k's is the same as the surface integral over the surface S. Taking the limit of the right-hand side of eq. (3.43) and incorporating eq. (3.32) gives

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{A}\, dv \tag{3.44}$$

which is the divergence theorem. The theorem applies to any volume v bounded by the closed surface S such as that shown in Figure 3.16 provided that **A** and $\nabla \cdot \mathbf{A}$ are continu-

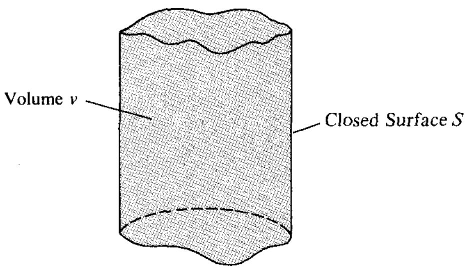

Figure 3.16 Volume v enclosed by surface S.

ous in the region. With a little experience, it will soon become apparent that volume integrals are easier to evaluate than surface integrals. For this reason, to determine the flux of **A** through a closed surface we simply find the right-hand side of eq. (3.42) instead of the left-hand side of the equation.

EXAMPLE 3.6

Determine the divergence of these vector fields:

(a) $\mathbf{P} = x^2yz\,\mathbf{a}_x + xz\,\mathbf{a}_z$

(b) $\mathbf{Q} = \rho \sin\phi\,\mathbf{a}_\rho + \rho^2 z\,\mathbf{a}_\phi + z\cos\phi\,\mathbf{a}_z$

(c) $\mathbf{T} = \dfrac{1}{r^2}\cos\theta\,\mathbf{a}_r + r\sin\theta\cos\phi\,\mathbf{a}_\theta + \cos\theta\,\mathbf{a}_\phi$

Solution:

(a)
$$\begin{aligned}\nabla \cdot \mathbf{P} &= \frac{\partial}{\partial x}P_x + \frac{\partial}{\partial y}P_y + \frac{\partial}{\partial z}P_z \\ &= \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(xz) \\ &= 2xyz + x\end{aligned}$$

(b)
$$\begin{aligned}\nabla \cdot \mathbf{Q} &= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho Q_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \phi}Q_\phi + \frac{\partial}{\partial z}Q_z \\ &= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho^2\sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(\rho^2 z) + \frac{\partial}{\partial z}(z\cos\phi) \\ &= 2\sin\phi + \cos\phi\end{aligned}$$

(c)
$$\begin{aligned}\nabla \cdot \mathbf{T} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(T_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(T_\phi) \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}(\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(r\sin^2\theta\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\cos\theta) \\ &= 0 + \frac{1}{r\sin\theta}2r\sin\theta\cos\theta\cos\phi + 0 \\ &= 2\cos\theta\cos\phi\end{aligned}$$

PRACTICE EXERCISE 3.6

Determine the divergence of the following vector fields and evaluate them at the specified points.

(a) $\mathbf{A} = yz\mathbf{a}_x + 4xy\mathbf{a}_y + y\mathbf{a}_z$ at $(1, -2, 3)$
(b) $\mathbf{B} = \rho z \sin\phi\, \mathbf{a}_\rho + 3\rho z^2 \cos\phi\, \mathbf{a}_\phi$ at $(5, \pi/2, 1)$
(c) $\mathbf{C} = 2r\cos\theta\cos\phi\, \mathbf{a}_r + r^{1/2}\mathbf{a}_\phi$ at $(1, \pi/6, \pi/3)$

Answer: (a) $4x$, 4, (b) $(2 - 3z)z\sin\phi$, -1, (c) $6\cos\theta\cos\phi$, 2.598.

EXAMPLE 3.7

If $\mathbf{G}(r) = 10e^{-2z}(\rho\mathbf{a}_\rho + \mathbf{a}_z)$, determine the flux of $\mathbf{G}$ out of the entire surface of the cylinder $\rho = 1, 0 \leq z \leq 1$. Confirm the result using the divergence theorem.

Solution:
If Ψ is the flux of $\mathbf{G}$ through the given surface, shown in Figure 3.17, then

$$\Psi = \oint \mathbf{G} \cdot d\mathbf{S} = \Psi_t + \Psi_b + \Psi_s$$

where Ψ_t, Ψ_b, and Ψ_s are the fluxes through the top, bottom, and sides (curved surface) of the cylinder as in Figure 3.17.

For Ψ_t, $z = 1$, $d\mathbf{S} = \rho\, d\rho\, d\phi\, \mathbf{a}_z$. Hence,

$$\Psi_t = \int \mathbf{G} \cdot d\mathbf{S} = \int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} 10e^{-2}\rho\, d\rho\, d\phi = 10e^{-2}(2\pi)\left.\frac{\rho^2}{2}\right|_0^1$$
$$= 10\pi e^{-2}$$

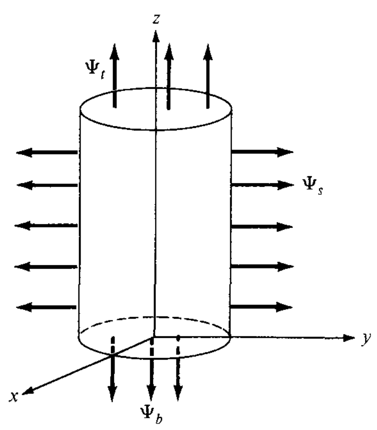

Figure 3.17 For Example 3.7.

For Ψ_b, $z = 0$ and $d\mathbf{S} = \rho\, d\rho\, d\phi(-\mathbf{a}_z)$. Hence,

$$\Psi_b = \int_b \mathbf{G} \cdot d\mathbf{S} = \int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} 10e^0 \rho\, d\rho\, d\phi = -10(2\pi) \frac{\rho^2}{2} \Big|_0^1$$
$$= -10\pi$$

For Ψ_s, $\rho = 1$, $d\mathbf{S} = \rho\, dz\, d\phi\, \mathbf{a}_\rho$. Hence,

$$\Psi_s = \int_s \mathbf{G} \cdot d\mathbf{S} = \int_{z=0}^{1} \int_{\phi=0}^{2\pi} 10e^{-2z} \rho^2\, dz\, d\phi = 10(1)^2(2\pi) \frac{e^{-2z}}{-2} \Big|_0^1$$
$$= 10\pi(1 - e^{-2})$$

Thus,

$$\Psi = \Psi_t + \Psi_b + \Psi_s = 10\pi e^{-2} - 10\pi + 10\pi(1 - e^{-2}) = 0$$

Alternatively, since S is a closed surface, we can apply the divergence theorem:

$$\Psi = \oint_S G \cdot d\mathbf{S} = \int_v (\nabla \cdot \mathbf{G})\, dv$$

But

$$\nabla \cdot \mathbf{G} = \frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho G_\rho) + \frac{1}{\rho} \frac{\partial}{\partial \phi} G_\phi + \frac{\partial}{\partial z} G_z$$
$$= \frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho^2 10e^{-2z}) - 20e^{-2z} = 0$$

showing that **G** has no source. Hence,

$$\Psi = \int_v (\nabla \cdot G)\, dv = 0$$

PRACTICE EXERCISE 3.7

Determine the flux of $\mathbf{D} = \rho^2 \cos^2 \phi\, \mathbf{a}_\rho + z \sin \phi\, \mathbf{a}_\phi$ over the closed surface of the cylinder $0 \leq z \leq 1$, $\rho = 4$. Verify the divergence theorem for this case.

Answer: 64π.

3.7 CURL OF A VECTOR AND STOKES'S THEOREM

In Section 3.3, we defined the circulation of a vector field **A** around a closed path L as the integral $\oint_L \mathbf{A} \cdot d\mathbf{l}$.

The curl of A is an axial (or rotational) vector whose magnitude is the maximum circulation of A per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.[2]

That is,

$$\text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \left(\lim_{\Delta S \to 0} \frac{\oint_L \mathbf{A} \cdot d\mathbf{l}}{\Delta S} \right)_{\max} \mathbf{a}_n \tag{3.45}$$

where the area ΔS is bounded by the curve L and $\mathbf{a}_n$ is the unit vector normal to the surface ΔS and is determined using the right-hand rule.

To obtain an expression for $\nabla \times \mathbf{A}$ from the definition in eq. (3.45), consider the differential area in the yz-plane as in Figure 3.18. The line integral in eq. (3.45) is obtained as

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = \left(\int_{ab} + \int_{bc} + \int_{cd} + \int_{da} \right) \mathbf{A} \cdot d\mathbf{l} \tag{3.46}$$

We expand the field components in a Taylor series expansion about the center point $P(x_o, y_o, z_o)$ as in eq. (3.34) and evaluate eq. (3.46). On side ab, $d\mathbf{l} = dy\,\mathbf{a}_y$ and $z = z_o - dz/2$, so

$$\int_{ab} \mathbf{A} \cdot d\mathbf{l} = dy \left[A_y(x_o, y_o, z_o) - \frac{dz}{2} \frac{\partial A_y}{\partial z} \bigg|_P \right] \tag{3.47}$$

On side bc, $d\mathbf{l} = dz\,\mathbf{a}_z$ and $y = y_o + dy/2$, so

$$\int_{bc} \mathbf{A} \cdot d\mathbf{l} = dz \left[A_z(x_o, y_o, z_o) + \frac{dy}{2} \frac{\partial A_z}{\partial y} \bigg|_P \right] \tag{3.48}$$

On side cd, $d\mathbf{l} = dy\,\mathbf{a}_y$ and $z = z_o + dz/2$, so

$$\int_{cd} \mathbf{A} \cdot d\mathbf{l} = -dy \left[A_y(x_o, y_o, z_o) + \frac{dz}{2} \frac{\partial A_y}{\partial z} \bigg|_P \right] \tag{3.49}$$

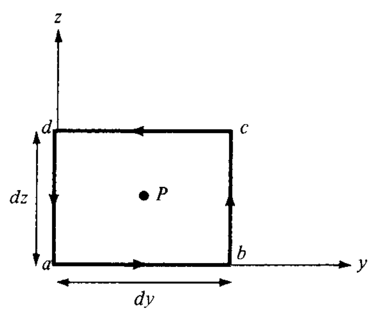

Figure 3.18 Contour used in evaluating the x-component of $\nabla \times \mathbf{A}$ at point $P(x_o, y_o, z_o)$.

[2]Because of its rotational nature, some authors use rot **A** instead of curl **A**.

On side da, $d\mathbf{l} = dz\,\mathbf{a}_z$ and $y = y_o - dy/2$, so

$$\int_{da} \mathbf{A} \cdot d\mathbf{l} = -dz \left[A_z(x_o, y_o, z_o) - \frac{dy}{2} \frac{\partial A_z}{\partial y} \bigg|_P \right] \tag{3.50}$$

Substituting eqs. (3.47) to (3.50) into eq. (3.46) and noting that $\Delta S = dy\,dz$, we have

$$\lim_{\Delta S \to 0} \oint_L \frac{\mathbf{A} \cdot d\mathbf{l}}{\Delta S} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

or

$$(\text{curl } \mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \tag{3.51}$$

The y- and x-components of the curl of $\mathbf{A}$ can be found in the same way. We obtain

$$(\text{curl } \mathbf{A})_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \tag{3.52a}$$

$$(\text{curl } \mathbf{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \tag{3.52b}$$

The definition of $\nabla \times \mathbf{A}$ in eq. (3.45) is independent of the coordinate system. In Cartesian coordinates the curl of $\mathbf{A}$ is easily found using

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \tag{3.53}$$

or

$$\boxed{\begin{aligned} \nabla \times \mathbf{A} = {} & \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right] \mathbf{a}_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right] \mathbf{a}_y \\ & + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right] \mathbf{a}_z \end{aligned}} \tag{3.54}$$

By transforming eq. (3.54) using point and vector transformation techniques used in Chapter 2, we obtain the curl of $\mathbf{A}$ in cylindrical coordinates as

$$\nabla \times \mathbf{A} = \frac{1}{\rho} \begin{vmatrix} \mathbf{a}_\rho & \rho\,\mathbf{a}_\phi & \mathbf{a}_z \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

or

$$\boxed{\begin{aligned}\nabla \times \mathbf{A} = {} & \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\mathbf{a}_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right]\mathbf{a}_\phi \\ & + \frac{1}{\rho}\left[\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right]\mathbf{a}_z\end{aligned}} \tag{3.55}$$

and in spherical coordinates as

$$\nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \mathbf{a}_r & r\,\mathbf{a}_\theta & r\sin\theta\,\mathbf{a}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta\,A_\phi \end{vmatrix}$$

or

$$\boxed{\begin{aligned}\nabla \times \mathbf{A} = {} & \frac{1}{r\sin\theta}\left[\frac{\partial(A_\phi \sin\theta)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right]\mathbf{a}_r \\ & + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial(rA_\phi)}{\partial r}\right]\mathbf{a}_\theta + \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right]\mathbf{a}_\phi\end{aligned}} \tag{3.56}$$

Note the following properties of the curl:

1. The curl of a vector field is another vector field.
2. The curl of a scalar field V, $\nabla \times V$, makes no sense.
3. $\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$
4. $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$
5. $\nabla \times (V\mathbf{A}) = V\nabla \times \mathbf{A} + \nabla V \times \mathbf{A}$
6. The divergence of the curl of a vector field vanishes, that is, $\nabla \cdot (\nabla \times \mathbf{A}) = 0$.
7. The curl of the gradient of a scalar field vanishes, that is, $\nabla \times \nabla V = 0$.

Other properties of the curl are in Appendix A.

The physical significance of the curl of a vector field is evident in eq. (3.45); the curl provides the maximum value of the circulation of the field per unit area (or circulation density) and indicates the direction along which this maximum value occurs. The curl of a vector field $\mathbf{A}$ at a point P may be regarded as a measure of the circulation or how much the field curls around P. For example, Figure 3.19(a) shows that the curl of a vector field around P is directed out of the page. Figure 3.19(b) shows a vector field with zero curl.

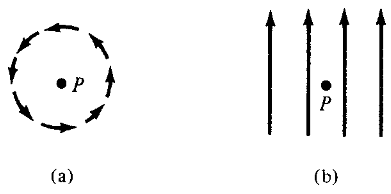

Figure 3.19 Illustration of a curl: **(a)** curl at P points out of the page; **(b)** curl at P is zero.

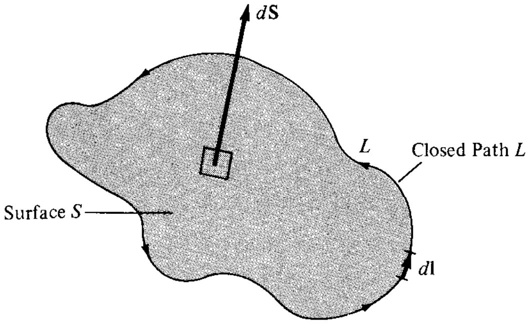

Figure 3.20 Determining the sense of $d\mathbf{l}$ and $d\mathbf{S}$ involved in Stokes's theorem.

Also, from the definition of the curl of **A** in eq. (3.45), we may expect that

$$\boxed{\oint_L \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}} \tag{3.57}$$

This is called *Stokes's theorem.*

Stokes's theorem states that the circulation of a vector field **A** around a (closed) path L is equal to the surface integral of the curl of **A** over the open surface S bounded by L (see Figure 3.20) provided that **A** and $\nabla \times \mathbf{A}$ are continuous on S.

The proof of Stokes's theorem is similar to that of the divergence theorem. The surface S is subdivided into a large number of cells as in Figure 3.21. If the kth cell has surface area ΔS_k and is bounded by path L_k.

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = \sum_k \oint_{L_k} \mathbf{A} \cdot d\mathbf{l} = \sum_k \frac{\oint_{L_k} \mathbf{A} \cdot d\mathbf{l}}{\Delta S_k} \Delta S_k \tag{3.58}$$

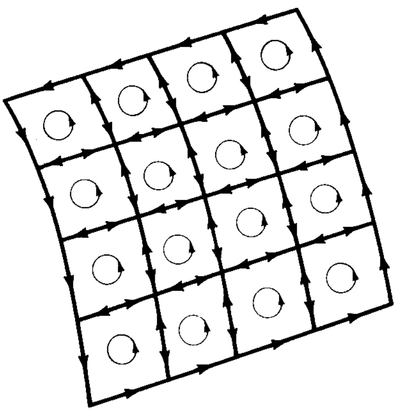

Figure 3.21 Illustration of Stokes's theorem.

As shown in Figure 3.21, there is cancellation on every interior path, so the sum of the line integrals around L_k's is the same as the line integral around the bounding curve L. Therefore, taking the limit of the right-hand side of eq. (3.58) as $\Delta S_k \to 0$ and incorporating eq. (3.45) leads to

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

which is Stokes's theorem.

The direction of $d\mathbf{l}$ and $d\mathbf{S}$ in eq. (3.57) must be chosen using the right-hand rule or right-handed screw rule. Using the right-hand rule, if we let the fingers point in the direction of $d\mathbf{l}$, the thumb will indicate the direction of $d\mathbf{S}$ (see Fig. 3.20). Note that whereas the divergence theorem relates a surface integral to a volume integral, Stokes's theorem relates a line integral (circulation) to a surface integral.

EXAMPLE 3.8

Determine the curl of the vector fields in Example 3.6.

Solution:

(a)
$$\begin{aligned}\nabla \times \mathbf{P} &= \left(\frac{\partial P_z}{\partial y} - \frac{\partial P_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial P_x}{\partial z} - \frac{\partial P_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial P_y}{\partial x} - \frac{\partial P_x}{\partial y}\right)\mathbf{a}_z \\ &= (0 - 0)\,\mathbf{a}_x + (x^2y - z)\,\mathbf{a}_y + (0 - x^2z)\,\mathbf{a}_z \\ &= (x^2y - z)\mathbf{a}_y - x^2z\mathbf{a}_z\end{aligned}$$

(b)
$$\begin{aligned}\nabla \times \mathbf{Q} &= \left[\frac{1}{\rho}\frac{\partial Q_z}{\partial \phi} - \frac{\partial Q_\phi}{\partial z}\right]\mathbf{a}_\rho + \left[\frac{\partial Q_\rho}{\partial z} - \frac{\partial Q_z}{\partial \rho}\right]\mathbf{a}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho Q_\phi) - \frac{\partial Q_\rho}{\partial \phi}\right]\mathbf{a}_z \\ &= \left(\frac{-z}{\rho}\sin\phi - \rho^2\right)\mathbf{a}_\rho + (0 - 0)\mathbf{a}_\phi + \frac{1}{\rho}(3\rho^2 z - \rho\cos\phi)\mathbf{a}_z \\ &= -\frac{1}{\rho}(z\sin\phi + \rho^3)\mathbf{a}_\rho + (3\rho z - \cos\phi)\mathbf{a}_z\end{aligned}$$

(c)
$$\begin{aligned}\nabla \times \mathbf{T} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(T_\phi \sin\theta) - \frac{\partial}{\partial \phi}T_\theta\right]\mathbf{a}_r \\ &\quad + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}T_r - \frac{\partial}{\partial r}(rT_\phi)\right]\mathbf{a}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(rT_\theta) - \frac{\partial}{\partial \theta}T_r\right]\mathbf{a}_\phi \\ &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\cos\theta\sin\theta) - \frac{\partial}{\partial \phi}(r\sin\theta\cos\phi)\right]\mathbf{a}_r \\ &\quad + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\frac{(\cos\theta)}{r^2} - \frac{\partial}{\partial r}(r\cos\theta)\right]\mathbf{a}_\theta \\ &\quad + \frac{1}{r}\left[\frac{\partial}{\partial r}(r^2\sin\theta\cos\phi) - \frac{\partial}{\partial \theta}\frac{(\cos\theta)}{r^2}\right]\mathbf{a}_\phi \\ &= \frac{1}{r\sin\theta}(\cos 2\theta + r\sin\theta\sin\phi)\mathbf{a}_r + \frac{1}{r}(0 - \cos\theta)\mathbf{a}_\theta \\ &\quad + \frac{1}{r}\left(2r\sin\theta\cos\phi + \frac{\sin\theta}{r^2}\right)\mathbf{a}_\phi \\ &= \left(\frac{\cos 2\theta}{r\sin\theta} + \sin\phi\right)\mathbf{a}_r - \frac{\cos\theta}{r}\mathbf{a}_\theta + \left(2\cos\phi + \frac{1}{r^3}\right)\sin\theta\,\mathbf{a}_\phi\end{aligned}$$

PRACTICE EXERCISE 3.8

Determine the curl of the vector fields in Practice Exercise 3.6 and evaluate them at the specified points.

Answer: (a) $\mathbf{a}_x + y\mathbf{a}_y + (4y - z)\mathbf{a}_z$, $\mathbf{a}_x - 2\mathbf{a}_y - 11\mathbf{a}_z$

(b) $-6\rho z \cos\phi\, \mathbf{a}_\rho + \rho \sin\phi\, \mathbf{a}_\phi + (6z - 1)z\cos\phi\, \mathbf{a}_z$, $5\mathbf{a}_\phi$

(c) $\dfrac{\cot\theta}{r^{1/2}}\mathbf{a}_r - \left(2\cot\theta\sin\phi + \dfrac{3}{2r^{1/2}}\right)\mathbf{a}_\theta + 2\sin\theta\cos\phi\,\mathbf{a}_\phi$,
$1.732\,\mathbf{a}_r - 4.5\,\mathbf{a}_\theta + 0.5\,\mathbf{a}_\phi$.

EXAMPLE 3.9

If $\mathbf{A} = \rho\cos\phi\,\mathbf{a}_\rho + \sin\phi\,\mathbf{a}_\phi$, evaluate $\oint \mathbf{A}\cdot d\mathbf{l}$ around the path shown in Figure 3.22. Confirm this using Stokes's theorem.

Solution:

Let

$$\oint_L \mathbf{A}\cdot d\mathbf{l} = \left[\int_a^b + \int_b^c + \int_c^d + \int_d^a\right]\mathbf{A}\cdot d\mathbf{l}$$

where path L has been divided into segments ab, bc, cd, and da as in Figure 3.22.

Along ab, $\rho = 2$ and $d\mathbf{l} = \rho\, d\phi\, \mathbf{a}_\phi$. Hence,

$$\int_a^b \mathbf{A}\cdot d\mathbf{l} = \int_{\phi=60^\circ}^{30^\circ} \rho\sin\phi\, d\phi = 2(-\cos\phi)\Big|_{60^\circ}^{30^\circ} = -(\sqrt{3} - 1)$$

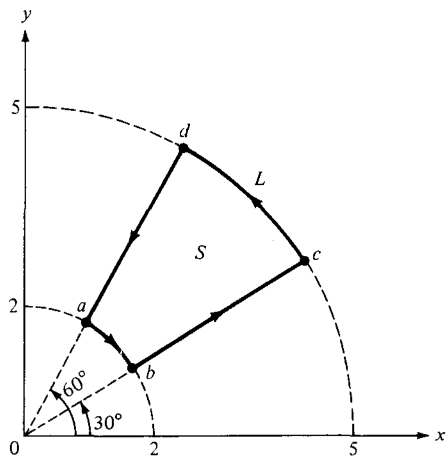

Figure 3.22 For Example 3.9.

Along bc, $\phi = 30°$ and $d\mathbf{l} = d\rho\, \mathbf{a}_\rho$. Hence,

$$\int_b^c \mathbf{A} \cdot d\mathbf{l} = \int_{\rho=2}^{5} \rho \cos\phi\, d\rho = \cos 30° \frac{\rho^2}{2}\bigg|_2^5 = \frac{21\sqrt{3}}{4}$$

Along cd, $\rho = 5$ and $d\mathbf{l} = \rho\, d\phi\, \mathbf{a}_\phi$. Hence,

$$\int_c^d \mathbf{A} \cdot d\mathbf{l} = \int_{\phi=30°}^{60°} \rho \sin\phi\, d\phi = 5(-\cos\phi)\bigg|_{30°}^{60°} = \frac{5}{2}(\sqrt{3} - 1)$$

Along da, $\phi = 60°$ and $d\mathbf{l} = d\rho\, \mathbf{a}_\rho$. Hence,

$$\int_d^a \mathbf{A} \cdot d\mathbf{l} = \int_{\rho=5}^{2} \rho \cos\phi\, d\rho = \cos 60° \frac{\rho^2}{2}\bigg|_5^2 = -\frac{21}{4}$$

Putting all these together results in

$$\begin{aligned} \oint_L \mathbf{A} \cdot d\mathbf{l} &= -\sqrt{3} + 1 + \frac{21\sqrt{3}}{4} + \frac{5\sqrt{3}}{2} - \frac{5}{2} - \frac{21}{4} \\ &= \frac{27}{4}(\sqrt{3} - 1) = 4.941 \end{aligned}$$

Using Stokes's theorem (because L is a closed path)

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

But $d\mathbf{S} = \rho\, d\phi\, d\rho\, \mathbf{a}_z$ and

$$\begin{aligned} \nabla \times \mathbf{A} &= \mathbf{a}_\rho\left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right] + \mathbf{a}_\phi\left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right] + \mathbf{a}_z \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi}\right] \\ &= (0 - 0)\mathbf{a}_\rho + (0 - 0)\mathbf{a}_\phi + \frac{1}{\rho}(1 + \rho)\sin\phi\, \mathbf{a}_z \end{aligned}$$

Hence:

$$\begin{aligned} \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} &= \int_{\phi=30°}^{60°} \int_{\rho=2}^{5} \frac{1}{\rho}(1 + \rho)\sin\phi\, \rho\, d\rho\, d\phi \\ &= \int_{30°}^{60°} \sin\phi\, d\phi \int_2^5 (1 + \rho)d\rho \\ &= -\cos\phi\bigg|_{30°}^{60°} \left(\rho + \frac{\rho^2}{2}\right)\bigg|_2^5 \\ &= \frac{27}{4}(\sqrt{3} - 1) = 4.941 \end{aligned}$$

PRACTICE EXERCISE 3.9

Use Stokes's theorem to confirm your result in Practice Exercise 3.2.

Answer: 1.

EXAMPLE 3.10

For a vector field $\mathbf{A}$, show explicitly that $\nabla \cdot \nabla \times \mathbf{A} = 0$; that is, the divergence of the curl of any vector field is zero.

Solution:

This vector identity along with the one in Practice Exercise 3.10 is very useful in EM. For simplicity, assume that $\mathbf{A}$ is in Cartesian coordinates.

$$\begin{aligned}
\nabla \cdot \nabla \times \mathbf{A} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\
&= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right), -\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right), \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right] \\
&= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \frac{\partial}{\partial x}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\
&= \frac{\partial^2 A_z}{\partial x\, \partial y} - \frac{\partial^2 A_y}{\partial x\, \partial z} - \frac{\partial^2 A_z}{\partial y\, \partial x} + \frac{\partial^2 A_x}{\partial y\, \partial z} + \frac{\partial^2 A_y}{\partial z\, \partial x} - \frac{\partial^2 A_x}{\partial z\, \partial y} \\
&= 0
\end{aligned}$$

because $\dfrac{\partial^2 A_z}{\partial x\, \partial y} = \dfrac{\partial^2 A_z}{\partial y\, \partial x}$ and so on.

PRACTICE EXERCISE 3.10

For a scalar field V, show that $\nabla \times \nabla V = 0$; that is, the curl of the gradient of any scalar field vanishes.

Answer: Proof.

3.8 LAPLACIAN OF A SCALAR

For practical reasons, it is expedient to introduce a single operator which is the composite of gradient and divergence operators. This operator is known as the *Laplacian.*

The **Laplacian** of a scalar field V, written as $\nabla^2 V$, is the divergence of the gradient of V.

Thus, in Cartesian coordinates,

Laplacian $V = \nabla \cdot \nabla V = \nabla^2 V$

$$= \left[\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right] \cdot \left[\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right] \tag{3.59}$$

that is,

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}} \tag{3.60}$$

Notice that the Laplacian of a scalar field is another scalar field.

The Laplacian of V in other coordinate systems can be obtained from eq. (3.60) by transformation. In cylindrical coordinates,

$$\boxed{\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}} \tag{3.61}$$

and in spherical coordinates,

$$\boxed{\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}} \tag{3.62}$$

A scalar field V is said to be *harmonic* in a given region if its Laplacian vanishes in that region. In other words, if

$$\nabla^2 V = 0 \tag{3.63}$$

is satisfied in the region, the solution for V in eq. (3.63) is harmonic (it is of the form of sine or cosine). Equation (3.63) is called *Laplace's equation.* Solving this equation will be our major task in Chapter 6.

We have only considered the Laplacian of a scalar. Since the Laplacian operator ∇^2 is a scalar operator, it is also possible to define the Laplacian of a vector $\mathbf{A}$. In this context, $\nabla^2 \mathbf{A}$ should not be viewed as the divergence of the gradient of $\mathbf{A}$, which makes no sense. Rather, $\nabla^2 \mathbf{A}$ is defined as the gradient of the divergence of $\mathbf{A}$ minus the curl of the curl of $\mathbf{A}$. That is,

$$\boxed{\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}} \tag{3.64}$$

This equation can be applied in finding $\nabla^2 \mathbf{A}$ in any coordinate system. In the Cartesian system (and only in that system), eq. (3.64) becomes

$$\nabla^2 \mathbf{A} = \nabla^2 A_x \mathbf{a}_x + \nabla^2 A_y \mathbf{a}_y + \nabla^2 A_z \mathbf{a}_z \tag{3.65}$$

EXAMPLE 3.11

Find the Laplacian of the scalar fields of Example 3.3; that is,

(a) $V = e^{-z} \sin 2x \cosh y$

(b) $U = \rho^2 z \cos 2\phi$

(c) $W = 10r \sin^2 \theta \cos \phi$

Solution:

The Laplacian in the Cartesian system can be found by taking the first derivative and later the second derivative.

$$\begin{aligned}
\text{(a)}\quad \nabla^2 V &= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \\
&= \frac{\partial}{\partial x}(2e^{-z} \cos 2x \cosh y) + \frac{\partial}{\partial y}(e^{-z} \cos 2x \sinh y) \\
&\quad + \frac{\partial}{\partial z}(-e^{-z} \sin 2x \cosh y) \\
&= -4e^{-z} \sin 2x \cosh y + e^{-z} \sin 2x \cosh y + e^{-z} \sin 2x \cosh y \\
&= -2e^{-z} \sin 2x \cosh y
\end{aligned}$$

$$\begin{aligned}
\text{(b)}\quad \nabla^2 U &= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} \\
&= \frac{1}{\rho}\frac{\partial}{\partial \rho}(2\rho^2 z \cos 2\phi) - \frac{1}{\rho^2} 4\rho^2 z \cos 2\phi + 0 \\
&= 4z \cos 2\phi - 4z \cos 2\phi \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\text{(c)}\quad \nabla^2 W &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial W}{\partial r}\right) + \frac{1}{r^2 \sin \theta}\frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial W}{\partial \theta}\right) + \frac{1}{r^2 \sin^2 \theta}\frac{\partial^2 W}{\partial \phi^2} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}(10r^2 \sin^2 \theta \cos \phi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(10r \sin 2\theta \sin \theta \cos \phi) \\
&\quad - \frac{10r \sin^2 \theta \cos \phi}{r^2 \sin^2 \theta} \\
&= \frac{20 \sin^2 \theta \cos \phi}{r} + \frac{20r \cos 2\theta \sin \theta \cos \phi}{r^2 \sin \theta} \\
&\quad + \frac{10r \sin 2\theta \cos \theta \cos \phi}{r^2 \sin \theta} - \frac{10 \cos \phi}{r} \\
&= \frac{10 \cos \phi}{r}(2 \sin^2 \theta + 2 \cos 2\theta + 2 \cos^2 \theta - 1) \\
&= \frac{10 \cos \phi}{r}(1 + 2 \cos 2\theta)
\end{aligned}$$

PRACTICE EXERCISE 3.11

Determine the Laplacian of the scalar fields of Practice Exercise 3.3, that is,

(a) $U = x^2y + xyz$
(b) $V = \rho z \sin\phi + z^2 \cos^2\phi + \rho^2$
(c) $f = \cos\theta \sin\phi \ln r + r^2 \phi$

Answer: (a) $2y$, (b) $4 + 2\cos^2\phi - \dfrac{2z^2}{\rho^2}\cos 2\phi$, (c) $\dfrac{1}{r^2}\cos\theta \sin\phi\,(1 - 2\ln r$ $\operatorname{cosec}^2\theta \ln r) + 6\phi$.

3.9 CLASSIFICATION OF VECTOR FIELDS

A vector field is uniquely characterized by its divergence and curl. Neither the divergence nor curl of a vector field is sufficient to completely describe the field. All vector fields can be classified in terms of their vanishing or nonvanishing divergence or curl as follows:

(a) $\nabla \cdot \mathbf{A} = 0, \nabla \times \mathbf{A} = 0$
(b) $\nabla \cdot \mathbf{A} \neq 0, \nabla \times \mathbf{A} = 0$
(c) $\nabla \cdot \mathbf{A} = 0, \nabla \times \mathbf{A} \neq 0$
(d) $\nabla \cdot \mathbf{A} \neq 0, \nabla \times \mathbf{A} \neq 0$

Figure 3.23 illustrates typical fields in these four categories.

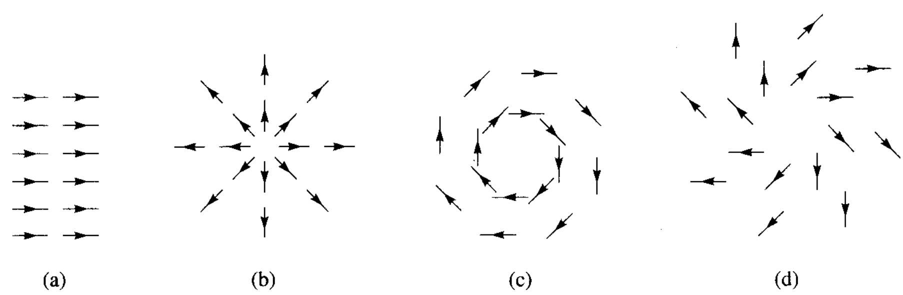

Figure 3.23 Typical fields with vanishing and nonvanishing divergence or curl.
(a) $\mathbf{A} = k\mathbf{a}_x$, $\nabla \cdot \mathbf{A} = 0$, $\nabla \times \mathbf{A} = 0$,
(b) $\mathbf{A} = k\mathbf{r}$, $\nabla \cdot \mathbf{A} = 3k$, $\nabla \times \mathbf{A} = 0$,
(c) $\mathbf{A} = \mathbf{k} \times \mathbf{r}$, $\nabla \cdot \mathbf{A} = 0$, $\nabla \times \mathbf{A} = 2\mathbf{k}$,
(d) $\mathbf{A} = \mathbf{k} \times \mathbf{r} + c\mathbf{r}$, $\nabla \cdot \mathbf{A} = 3c$, $\nabla \times \mathbf{A} = 2\mathbf{k}$.

A vector field **A** is said to be **solenoidal** (or **divergenceless**) if $\nabla \cdot \mathbf{A} = 0$.

Such a field has neither source nor sink of flux. From the divergence theorem,

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{A} \, dv = 0 \tag{3.66}$$

Hence, flux lines of **A** entering any closed surface must also leave it. Examples of solenoidal fields are incompressible fluids, magnetic fields, and conduction current density under steady state conditions. In general, the field of curl **F** (for any **F**) is purely solenoidal because $\nabla \cdot (\nabla \times \mathbf{F}) = 0$, as shown in Example 3.10. Thus, a solenoidal field **A** can always be expressed in terms of another vector **F**; that is,

if $\quad \nabla \cdot \mathbf{A} = 0$

then $\quad \oint_S \mathbf{A} \cdot d\mathbf{S} = 0 \quad$ and $\quad \mathbf{F} = \nabla \times \mathbf{A} \qquad$ (3.67)

A vector field **A** is said to be **irrotational** (or **potential**) if $\nabla \times \mathbf{A} = 0$.

That is, a *curl-free* vector is irrotational.[3] From Stokes's theorem

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_L \mathbf{A} \cdot d\mathbf{l} = 0 \tag{3.68}$$

Thus in an irrotational field **A**, the circulation of **A** around a closed path is identically zero. This implies that the line integral of **A** is independent of the chosen path. Therefore, an irrotational field is also known as a *conservative field.* Examples of irrotational fields include the electrostatic field and the gravitational field. In general, the field of gradient V (for any scalar V) is purely irrotational since (see Practice Exercise 3.10)

$$\nabla \times (\nabla V) = 0 \tag{3.69}$$

Thus, an irrotational field **A** can always be expressed in terms of a scalar field V; that is

if $\quad \nabla \times \mathbf{A} = 0$

then $\quad \oint_L \mathbf{A} \cdot d\mathbf{l} = 0 \quad$ and $\quad \mathbf{A} = -\nabla V \qquad$ (3.70)

For this reason, **A** may be called a *potential* field and V the scalar potential of **A**. The negative sign in eq. (3.70) has been inserted for physical reasons that will become evident in Chapter 4.

[3] In fact, curl was once known as *rotation,* and curl **A** is written as rot **A** in some textbooks. This is one reason to use the term *irrotational.*

A vector **A** is uniquely prescribed within a region by its divergence and its curl. If we let

$$\nabla \cdot \mathbf{A} = \rho_v \tag{3.71a}$$

and

$$\nabla \times \mathbf{A} = \boldsymbol{\rho}_S \tag{3.71b}$$

ρ_v can be regarded as the source density of **A** and $\boldsymbol{\rho}_S$ its circulation density. Any vector **A** satisfying eq. (3.71) with both $\boldsymbol{\rho}_v$ and $\boldsymbol{\rho}_S$ vanishing at infinity can be written as the sum of two vectors: one irrotational (zero curl), the other solenoidal (zero divergence). This is called *Helmholtz's theorem.* Thus we may write

$$\mathbf{A} = -\nabla V + \nabla \times \mathbf{B} \tag{3.72}$$

If we let $\mathbf{A}_i = -\nabla V$ and $\mathbf{A}_s = \nabla \times \mathbf{B}$, it is evident from Example 3.10 and Practice Exercise 3.10 that $\nabla \times \mathbf{A}_i = 0$ and $\nabla \times \mathbf{A}_s = 0$, showing that $\mathbf{A}_i$ is irrotational and $\mathbf{A}_s$ is solenoidal. Finally, it is evident from eqs. (3.64) and (3.71) that any vector field has a Laplacian that satisfies

$$\nabla^2 \mathbf{A} = \nabla \rho_v - \nabla \times \boldsymbol{\rho}_S \tag{3.73}$$

EXAMPLE 3.12

Show that the vector field **A** is conservative if **A** possesses one of these two properties:

(a) The line integral of the tangential component of **A** along a path extending from a point P to a point Q is independent of the path.

(b) The line integral of the tangential component of **A** around any closed path is zero.

Solution:

(a) If **A** is conservative, $\nabla \times \mathbf{A} = 0$, so there exists a potential V such that

$$\mathbf{A} = -\nabla V = -\left[\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right]$$

Hence,

$$\begin{aligned}\int_P^Q \mathbf{A} \cdot d\mathbf{l} &= -\int_P^Q \left[\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right] \\ &= -\int_P^Q \left[\frac{\partial V}{\partial x}\frac{dx}{ds} + \frac{\partial V}{\partial y}\frac{dy}{ds} + \frac{\partial V}{\partial z}\frac{dz}{ds}\right] ds \\ &= -\int_P^Q \frac{dV}{ds}\, ds = -\int_P^Q dV\end{aligned}$$

or

$$\int_P^Q \mathbf{A} \cdot d\mathbf{l} = V(P) - V(Q)$$

showing that the line integral depends only on the end points of the curve. Thus, for a conservative field, $\int_P^Q \mathbf{A} \cdot d\mathbf{l}$ is simply the difference in potential at the end points.

(b) If the path is closed, that is, if P and Q coincide, then

$$\oint \mathbf{A}\, d\mathbf{l} = V(P) - V(P) = 0$$

PRACTICE EXERCISE 3.12

Show that $\mathbf{B} = (y + z \cos xz)\mathbf{a}_x + x\mathbf{a}_y + x \cos xz\, \mathbf{a}_z$ is conservative, without computing any integrals.

Answer: Proof.

SUMMARY

1. The differential displacements in the Cartesian, cylindrical, and spherical systems are respectively

$$d\mathbf{l} = dx\, \mathbf{a}_x + dy\, \mathbf{a}_y + dz\, \mathbf{a}_z$$

$$d\mathbf{l} = d\rho\, \mathbf{a}_\rho + \rho\, d\phi\, \mathbf{a}_\phi + dz\, \mathbf{a}_z$$

$$d\mathbf{l} = dr\, \mathbf{a}_r + r\, d\theta\, \mathbf{a}_\theta + r \sin\theta\, d\phi\, \mathbf{a}_\phi$$

Note that $d\mathbf{l}$ is always taken to be in the positive direction; the direction of the displacement is taken care of by the limits of integration.

2. The differential normal areas in the three systems are respectively

$$\begin{aligned} d\mathbf{S} &= dy\, dz\, \mathbf{a}_x \\ & dx\, dz\, \mathbf{a}_y \\ & dx\, dy\, \mathbf{a}_z \end{aligned}$$

$$\begin{aligned} d\mathbf{S} &= \rho\, d\phi\, dz\, \mathbf{a}_\rho \\ & d\rho\, dz\, \mathbf{a}_\phi \\ & \rho\, d\rho\, d\phi\, \mathbf{a}_z \end{aligned}$$

$$\begin{aligned} d\mathbf{S} &= r^2 \sin\theta\, d\theta\, d\phi\, \mathbf{a}_r \\ & r \sin\theta\, dr\, d\phi\, \mathbf{a}_\theta \\ & r\, dr\, d\theta\, \mathbf{a}_\phi \end{aligned}$$

Note that $d\mathbf{S}$ can be in the positive or negative direction depending on the surface under consideration.

3. The differential volumes in the three systems are

$$dv = dx\, dy\, dz$$

$$dv = \rho\, d\rho\, d\phi\, dz$$

$$dv = r^2 \sin\theta\, dr\, d\theta\, d\phi$$