B38EM Introduction to Electricity and Magnetism

Lecture 2

Mathematical Background

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Topics

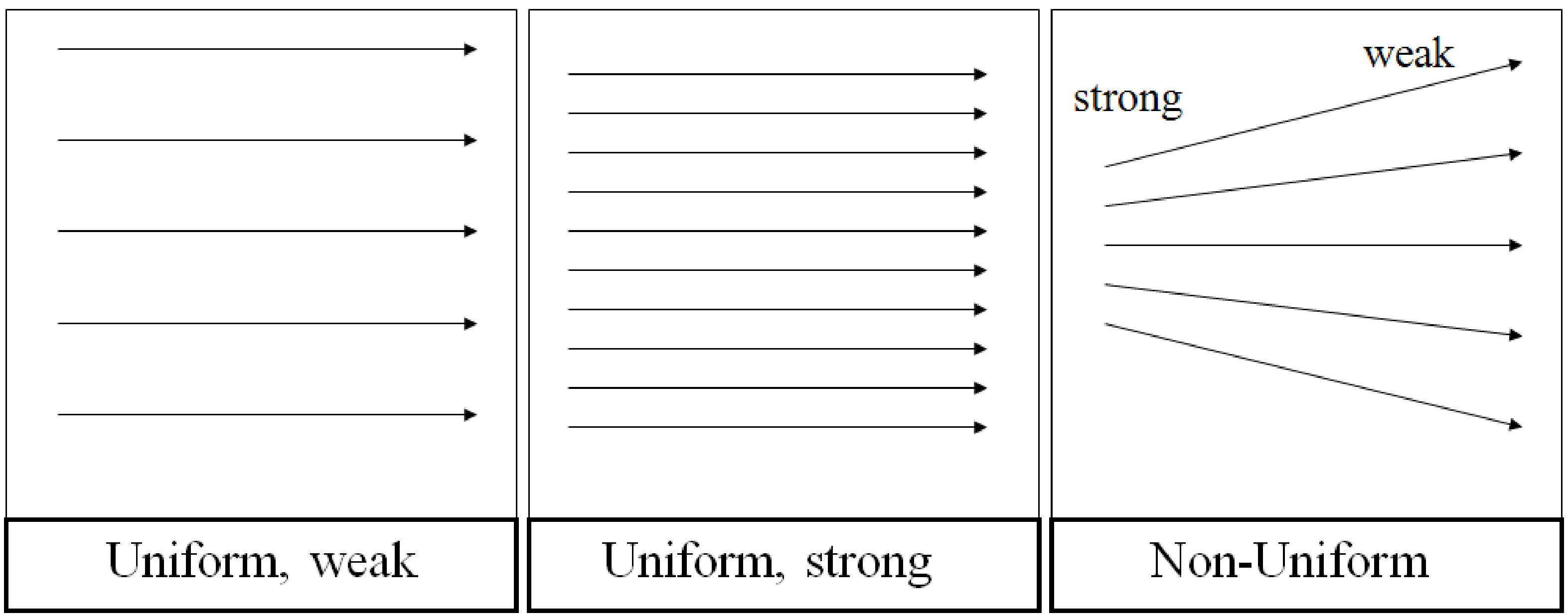
- Vectors
- Integral
- Derivative
- Coordinate Systems
- Gradient, Divergence & Curl

References & Resources

- Elements of Electromagnetics (7th Edition), by Sadiku, Oxford University Press
- Fundamentals of Applied Electromagnetics (7th Edition), by Ulaby and Ravaioli
- Field and Wave Electromagnetics (2nd Edition), by David K. Cheng
- Or many mathematical books, or YouTube

Fields and Field Lines

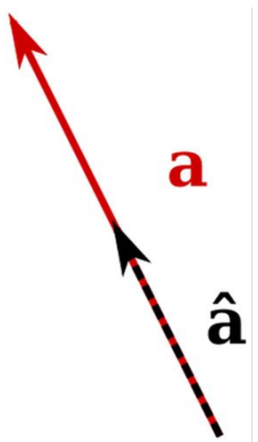
- A plotted field contains information on field strength and uniformity.
- Describe the following field plots:



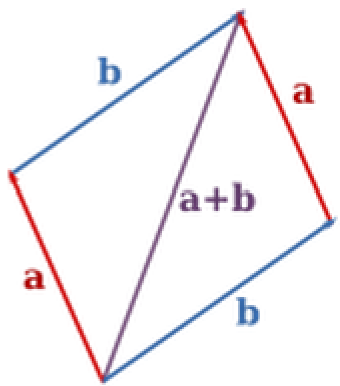
Vectors

Describes a quantity that has a magnitude (or length) and direction.

Vector algebra: calculations with vectors



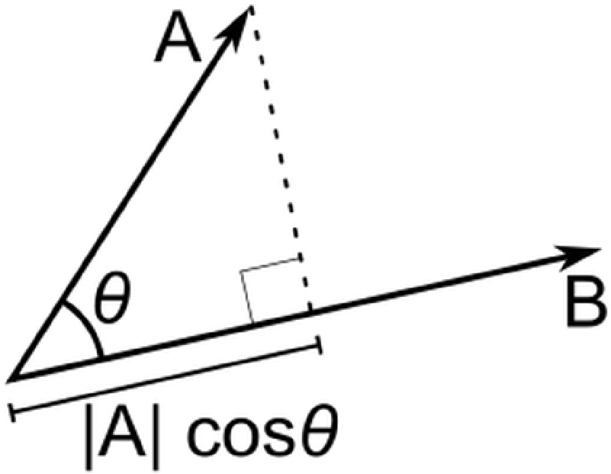
Unit vector



Vector addition

Vectors

Dot (or Scalar) product of two vectors



$$\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}| \cos\theta$$

It returns one scalar number.

Vectors

Dot product properties:

If $\boldsymbol{A}$ is orthogonal to $\boldsymbol{B}$ ($\vartheta = 90^o$): $\boldsymbol{A} \cdot \boldsymbol{B} = 0$

If $\boldsymbol{A}$ is parallel to $\boldsymbol{B}$ ($\vartheta = 0^o$): $\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}|$

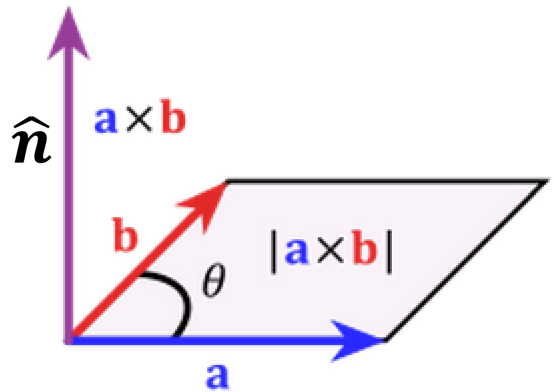
Commutative: $\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{A}$

Distributive: $\boldsymbol{A} \cdot (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{B} \cdot \boldsymbol{C}$

Scalar multiplication: $(c_1\boldsymbol{A}) \cdot (c_2\boldsymbol{B}) = c_1 c_2 \boldsymbol{A} \cdot \boldsymbol{B}$

Vectors

Cross product of two vectors



$$\boldsymbol{A} \times \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}| \sin\theta\, \hat{\boldsymbol{n}}$$

It returns another vector.

Vectors

Cross product properties:

If ***A*** is parallel to ***B*** ($\vartheta = 0^o$ or $\vartheta = 180^o$): $\boldsymbol{A} \times \boldsymbol{B} = \boldsymbol{0}$

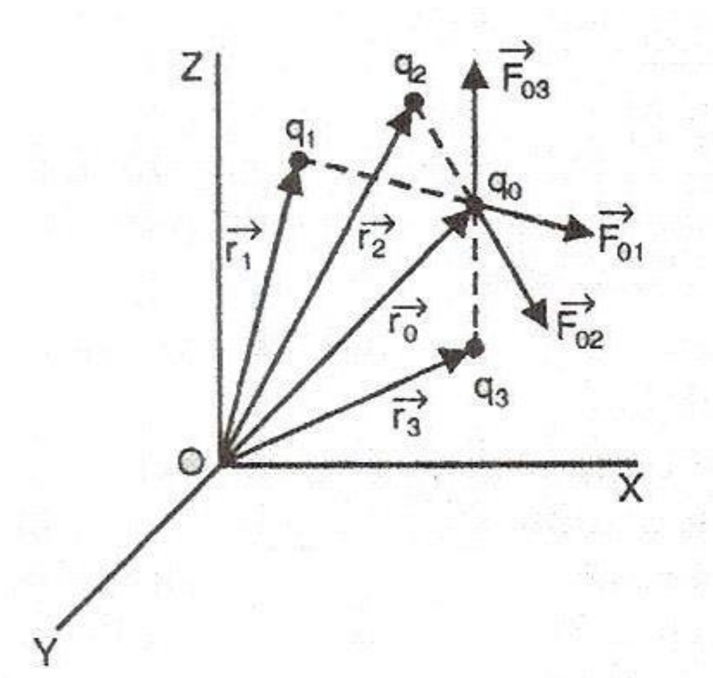
Anticommutative: $\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A}$

Distributive: $\boldsymbol{A} \times (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} \times \boldsymbol{B}) + (\boldsymbol{A} \times \boldsymbol{C})$

Scalar multiplication: $(c\boldsymbol{A}) \times \boldsymbol{B} = \boldsymbol{A} \times (c\boldsymbol{B}) = c(\boldsymbol{A} \times \boldsymbol{B})$

Superposition Principle

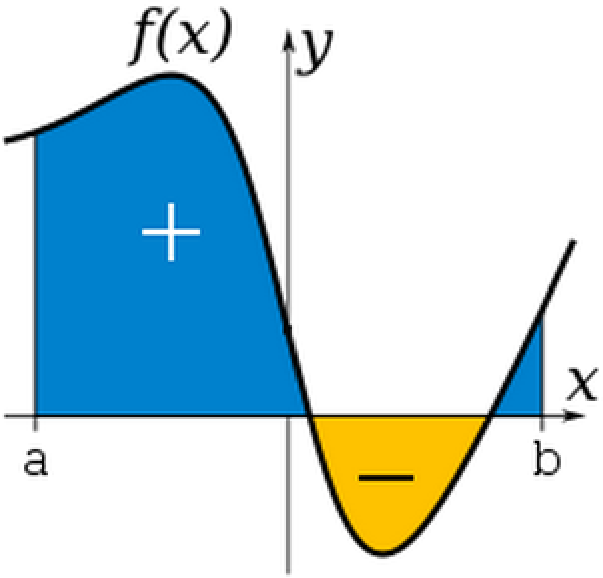
A field arising from a number of sources is determined by adding the individual fields from each source.



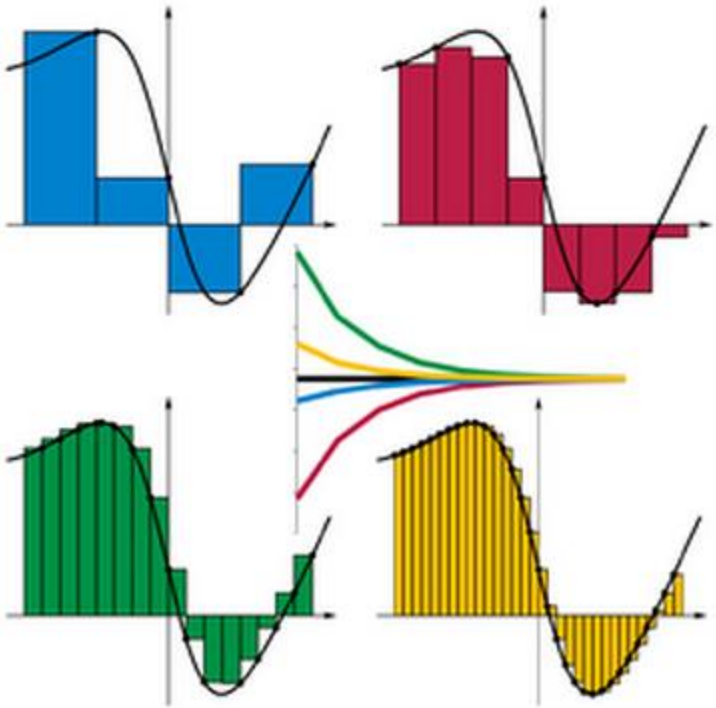
Integral

$$\int_a^b f(x)\, dx$$

The area of the region in the *xy*-plane bounded by the graph of *f*, the *x*-axis, and the vertical lines *x* = a and *x* = b, such that area above the *x*-axis adds to the total, and that below the *x*-axis subtracts from the total.



Calculation of Integral



$$\sum_{i=1}^{n} f(t_i)\Delta_i$$

At the limit Δ→0

"infinitesimal" dx

$$\int f(x)\,dx$$

$$F = \int f(x)\,dx$$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

'Indefinite integral' The derivative of the function is the given function *f*.

Indefinite Integral

$$\int cf(x)\,dx = c\int f(x)\,dx$$

$$\int \Big[f(x)+g(x)\Big]dx = \int f(x)dx + \int g(x)dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

Derivative

The derivative is a measure of how a function changes as its input changes.

The simplest case is when y is a linear function of *x*:

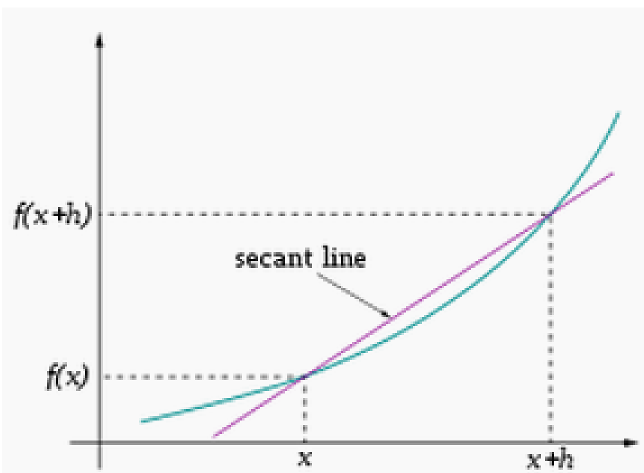
$y = f(x) = m \cdot x + b$,

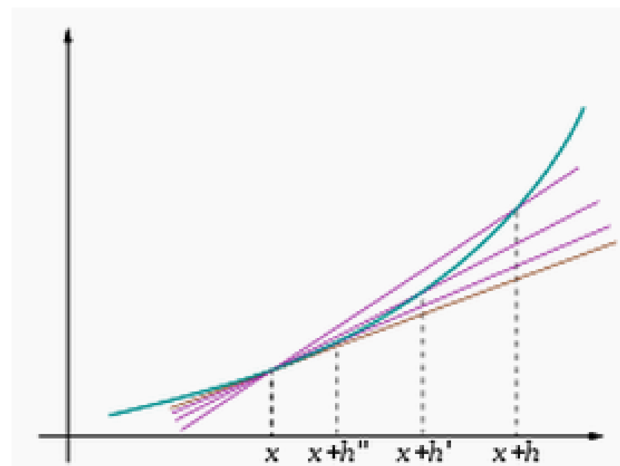
the slope *m* is given by

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}$$

Derivative

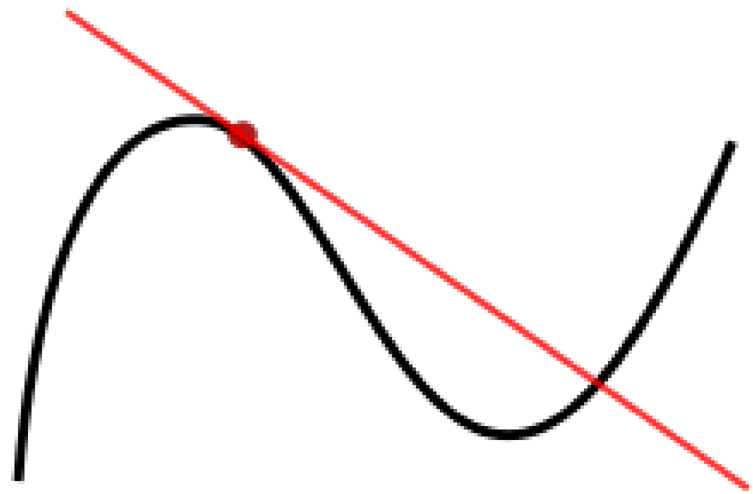
- If the function *f* is not linear (i.e. its graph is not a straight line), then the change in y divided by the change in *x* varies.
- Differentiation is a method to find an exact value for this rate of change at any given value of *x*.
- This can be done by taking the limit of $\Delta y/\Delta x$ for $\Delta x \to 0$ (d*x*)





Tangential

The tangent line (or simply the tangent) to a curve at a given point is the straight line that "just touches" the curve at that point.

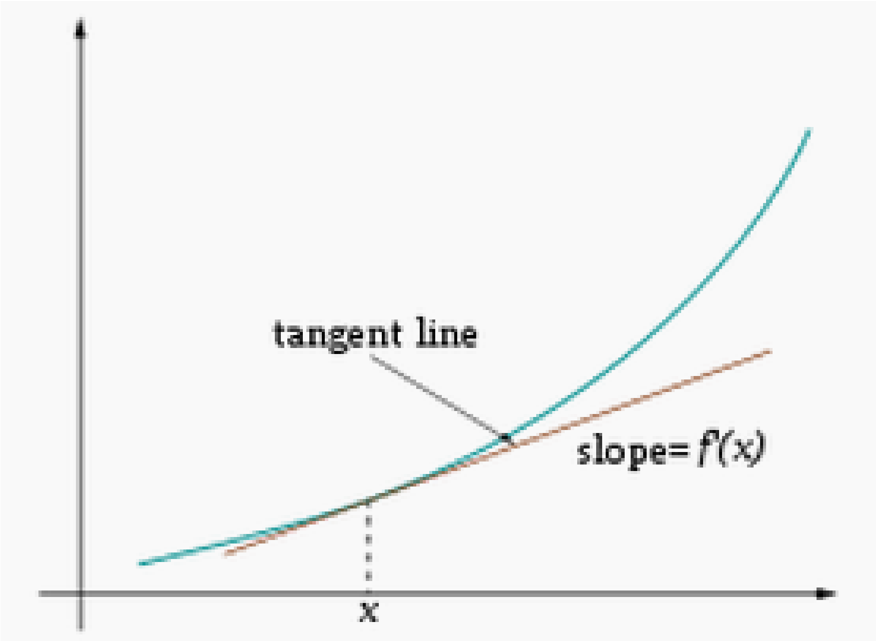


Derivative

The derivative at a point equals the slope of the tangent line to the graph of the function at that point.

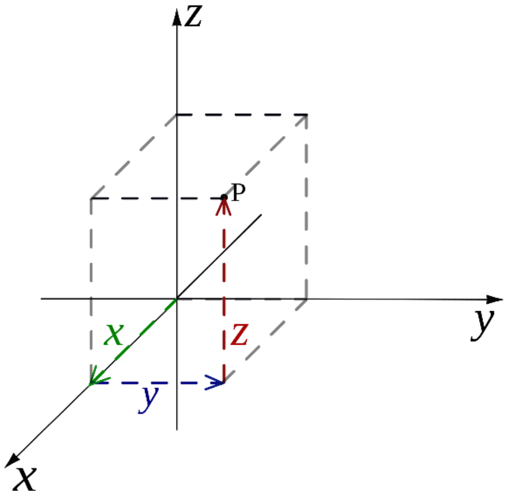
This is the best linear approximation of the function near that input value.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

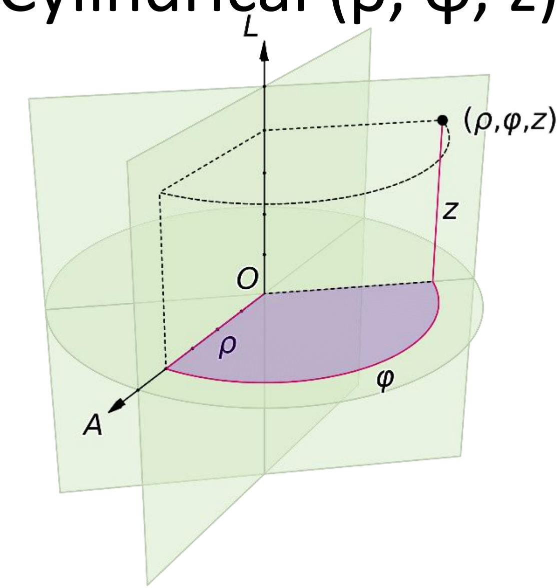


Coordinate Systems

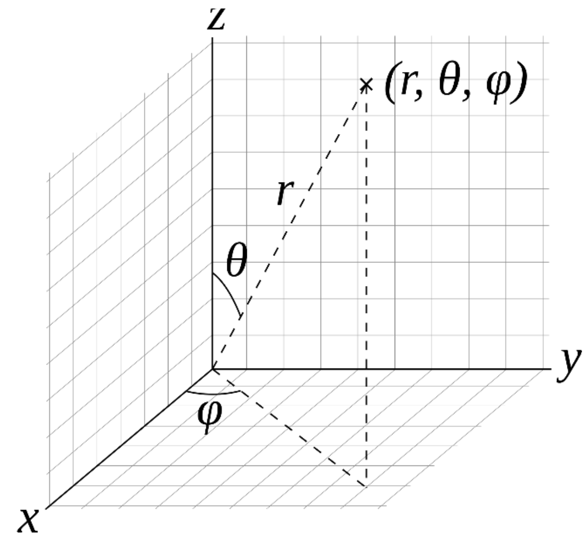
- Cartesian (x, y, z)



- Cylindrical (ρ, ϕ, z)



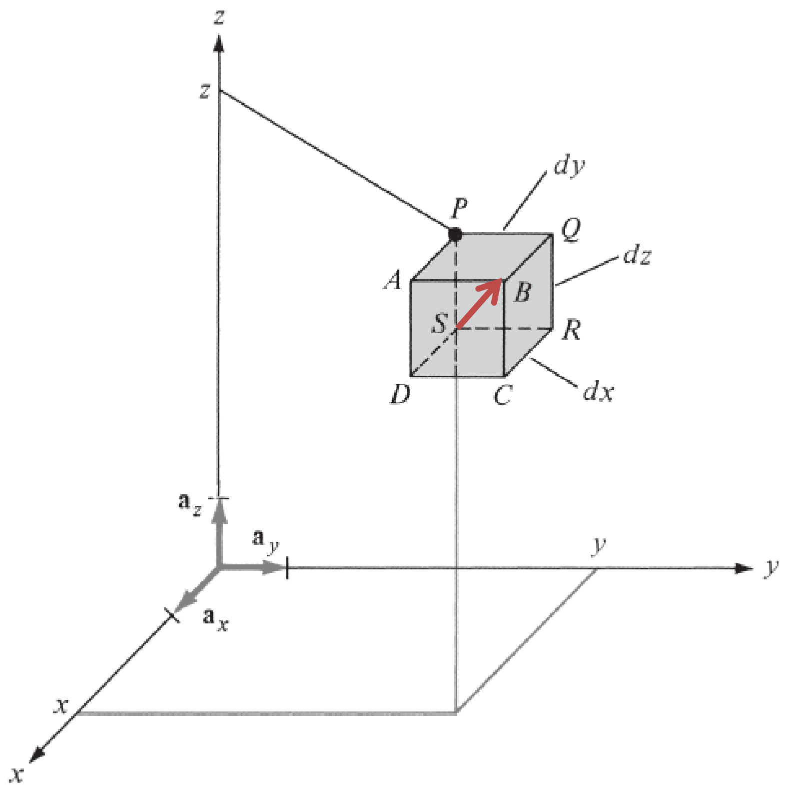
- Spherical (r, θ, ϕ)



https://en.wikipedia.org/wiki/Coordinate_system#Cylindrical_and_spherical_coordinate_systems

Cartesian Coordinate System

- Differential elements

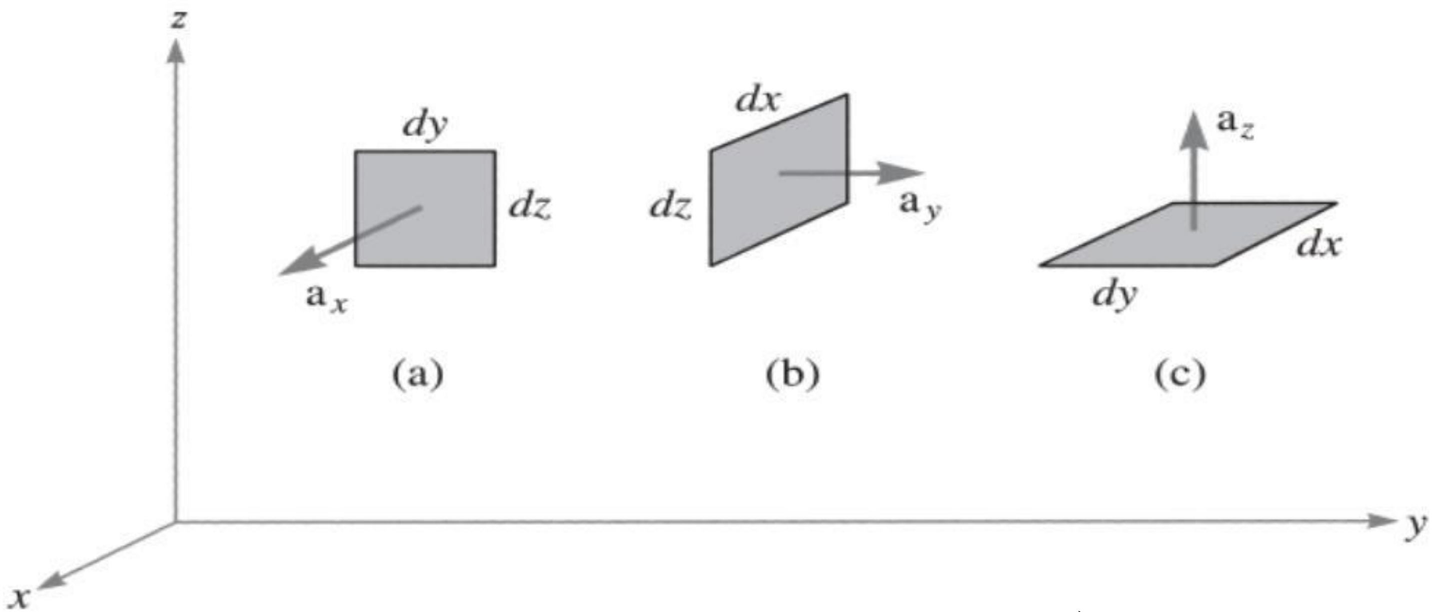


Differential displacement from point ***S***(*x*,*y*,*z*) to point ***B***(*x*+dx,*y*+dy,*z*+dz)
This is a vector, i.e. it has direction.

$$d\vec{l} = dx\ \hat{a}_x + dy\ \hat{a}_y + dz\ \hat{a}_z$$

Cartesian Coordinate System

- Differential normal surface areas



Differential normal surface area
This is a vector, i.e. it has direction.

$$(a) \quad d\vec{S} = dy\ dz\ \hat{a}_x$$

$$(b) \quad d\vec{S} = dx\ dz\ \hat{a}_y$$

$$(c) \quad d\vec{S} = dx\ dy\ \hat{a}_z$$

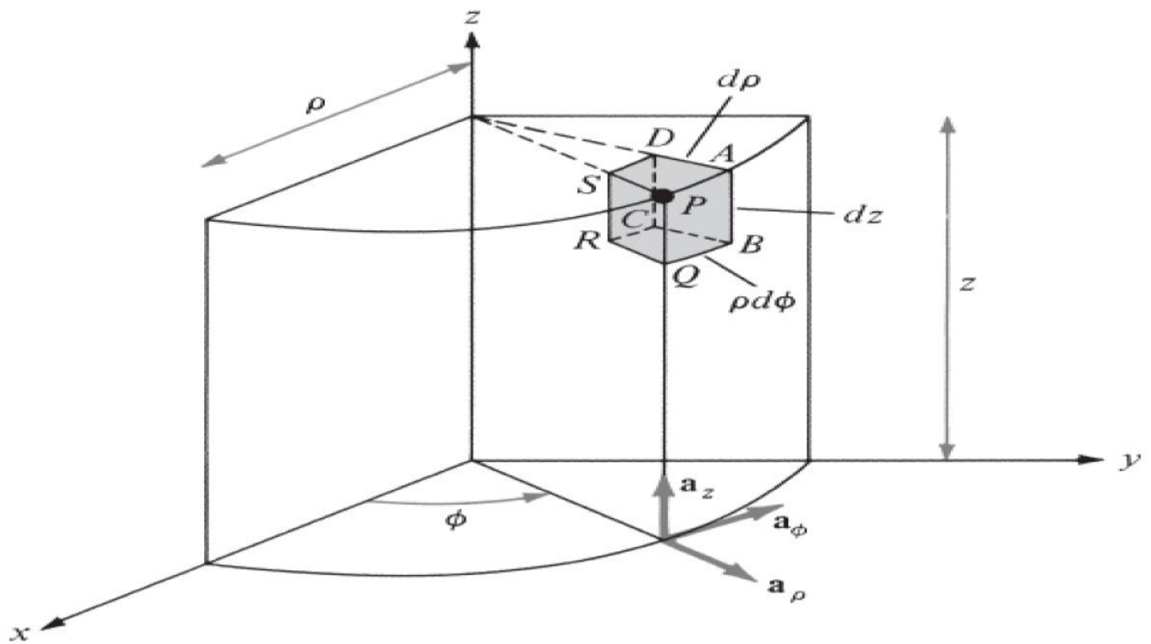
Differential volume
This is a scalar.

$$dv = dx\ dy\ dz$$

Cylindrical Coordinate System

- Differential elements

$0 \leq \rho < \infty$; radius of the cylinder
$0 \leq \phi < 2\pi$; azimuthal angle
$-\infty < z < \infty$; same as the Cartesian coordinate

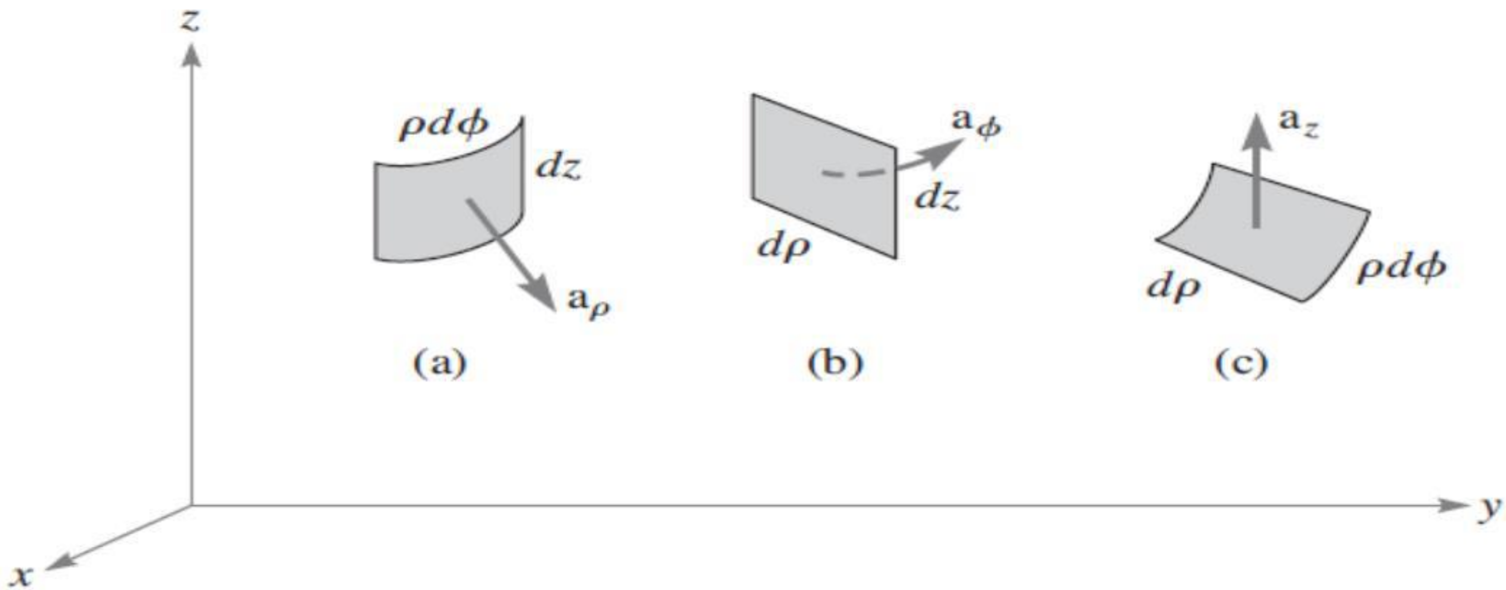


Differential displacement
This is a vector, i.e. it has direction.

$$d\vec{l} = d\rho\ \hat{a}_\rho + \rho d\phi\ \hat{a}_\phi + dz\ \hat{a}_z$$

Cylindrical Coordinate System

- Differential normal surface areas



Differential normal surface area
This is a vector, i.e. it has direction.

$$(a) \quad d\vec{S} = \rho d\phi \, dz \; \hat{a}_\rho$$

$$(b) \quad d\vec{S} = d\rho \; dz \; \hat{a}_\phi$$

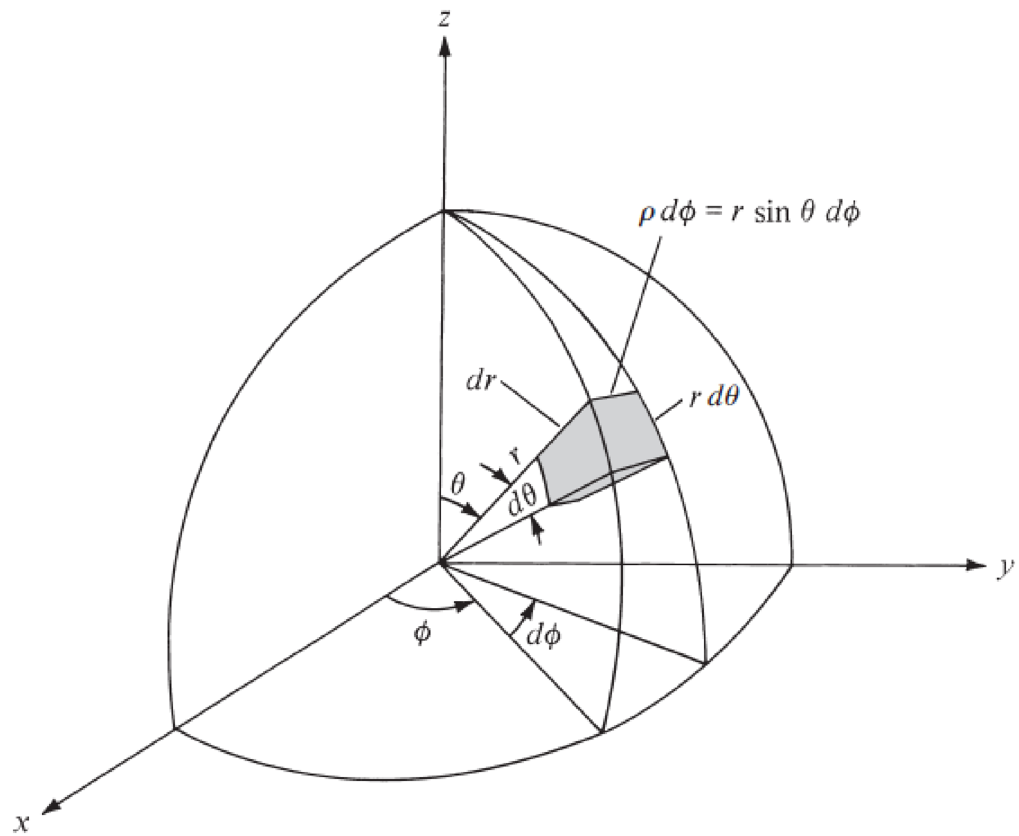
$$(c) \quad d\vec{S} = \rho d\rho \; d\phi \; \hat{a}_z$$

Differential volume
This is a scalar.

$$dv = \rho d\rho \; d\phi \; dz$$

Spherical Coordinate System

- Differential elements

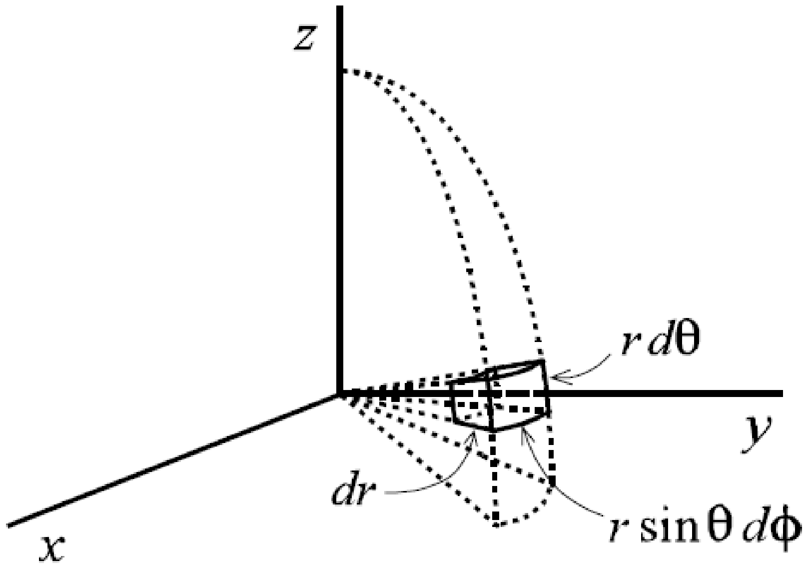


Differential displacement
This is a vector, i.e. it has direction.

$$d\vec{l} = dr\ \hat{a}_r + rd\vartheta\ \hat{a}_\vartheta + r\ sin\vartheta\ d\phi\ \hat{a}_\phi$$

Spherical Coordinate System

- Differential normal surface areas



Differential normal surface area
This is a vector, i.e. it has direction.

$$d\vec{S} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{a}_r$$

$$d\vec{S} = r \sin\theta \, dr \, d\phi \, \hat{a}_\theta$$

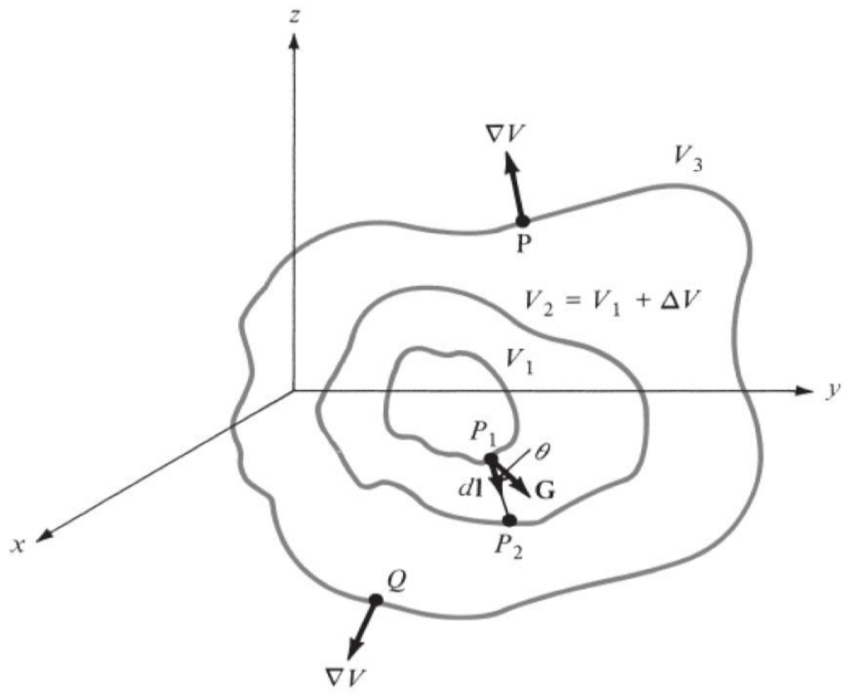
$$d\vec{S} = r \, dr \, d\theta \, \hat{a}_\phi$$

Differential volume
This is a scalar.

$$dv = r^2 sin\vartheta \; dr \; d\vartheta \; d\phi$$

Gradient of a Scalar Field

The **gradient** of a scalar field *V* is a vector that represents both the magnitude and the direction of the maximum space rate of increase of *V*.



https://www.youtube.com/watch?v=ynzRyIL2atU

Gradient of a Scalar Field

Cartesian coordinate

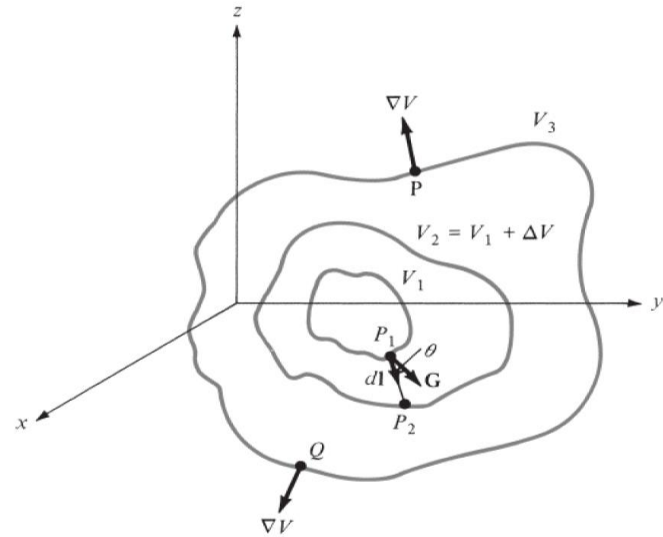
$$\text{grad } V = \nabla V = \left(\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z \right)$$

Cylindrical coordinate

$$\text{grad } V = \nabla V = \left(\frac{\partial V}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z \right)$$

Spherical coordinate

$$\text{grad } V = \nabla V = \frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi$$



- Del operator, ∇ is the ***vector differential operator***.

In Cartesian coordinate:

$$\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$$

In cylindrical coordinate:

$$\nabla = \frac{\partial}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\hat{a}_\phi + \frac{\partial}{\partial z}\hat{a}_z$$

In spherical coordinate:

$$\nabla = \frac{\partial}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{a}_\phi$$

Gradient of a Scalar Field

Let's Practice:

Find the gradient of:

$$(a)\quad T = x^2 + y^2 z$$

$$(b)\quad U = \rho^2 z \cos\left(2\phi\right)$$

$$(c)\quad W = 10r\sin^2\theta\cos\phi$$

Gradient of a Scalar Field

Solution:

Find the gradient of:

$$(a) \quad T = x^2 + y^2 z$$

$$\text{grad } T = \nabla T, \nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$$

$$\nabla T = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)\left(x^2 + y^2 z\right)$$

$$\nabla T = \frac{\partial\left(x^2 + y^2 z\right)}{\partial x}\hat{a}_x + \frac{\partial\left(x^2 + y^2 z\right)}{\partial y}\hat{a}_y + \frac{\partial\left(x^2 + y^2 z\right)}{\partial z}\hat{a}_z$$

$$\nabla T = (0 + 2x)\,\hat{a}_x + (0 + 2yz)\,\hat{a}_y + \left(0 + y^2\right)\hat{a}_z$$

$$\nabla T = 2x\,\hat{a}_x + 2yz\,\hat{a}_y + y^2\,\hat{a}_z$$

Gradient of a Scalar Field

Solution:

Find the gradient of:

$$(b) \quad U = \rho^2 z\cos(2\phi)$$

$$\nabla = \frac{\partial}{\partial\rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial\phi}\hat{a}_\phi + \frac{\partial}{\partial z}\hat{a}_z$$

$$U = \rho^2 z\cos 2\phi$$

$$\nabla U = \left(\frac{\partial}{\partial\rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial\phi}\hat{a}_\phi + \frac{\partial}{\partial z}\hat{a}_z\right)\left(\rho^2 z\cos 2\phi\right)$$

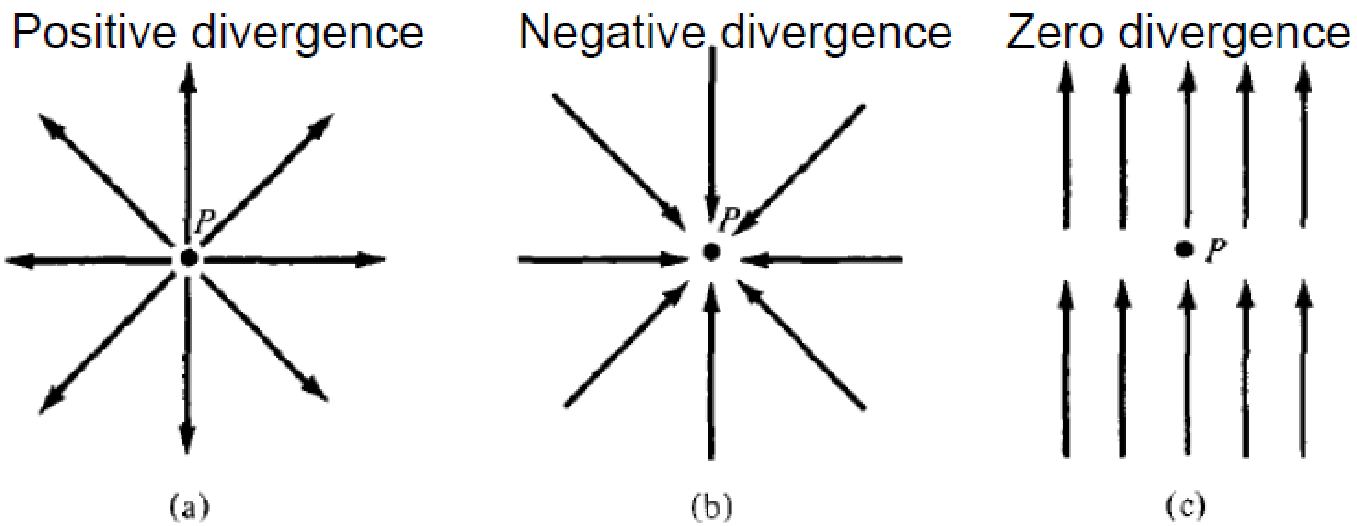
$$\nabla U = \frac{\partial U}{\partial\rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial U}{\partial\phi}\hat{a}_\phi + \frac{\partial U}{\partial z}\hat{a}_z$$

$$\nabla U = \frac{\partial\left(\rho^2 z\cos 2\phi\right)}{\partial\rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial\left(\rho^2 z\cos 2\phi\right)}{\partial\phi}\hat{a}_\phi + \frac{\partial\left(\rho^2 z\cos 2\phi\right)}{\partial z}\hat{a}_z$$

$$\nabla U = 2\rho z\cos 2\phi\,\hat{a}_\rho - 2\rho z\sin 2\phi\hat{a}_\phi + \rho^2\cos 2\phi\hat{a}_z$$

Divergence of a Vector Field

The **divergence** of $\vec{A}$ at a given point *P* is the outward flux per unit volume as the volume shrink about point *P*.



$$div\ \overrightarrow{A} = \nabla \cdot \vec{A} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{A} \cdot d\vec{S}}{\Delta v}$$

$$\int_v \nabla \cdot \vec{A}\, dv = \oint_S \vec{A} \cdot d\vec{S}$$

https://youtu.be/Cxc7ihZWq5o

https://youtu.be/c0MR-vWiUPU

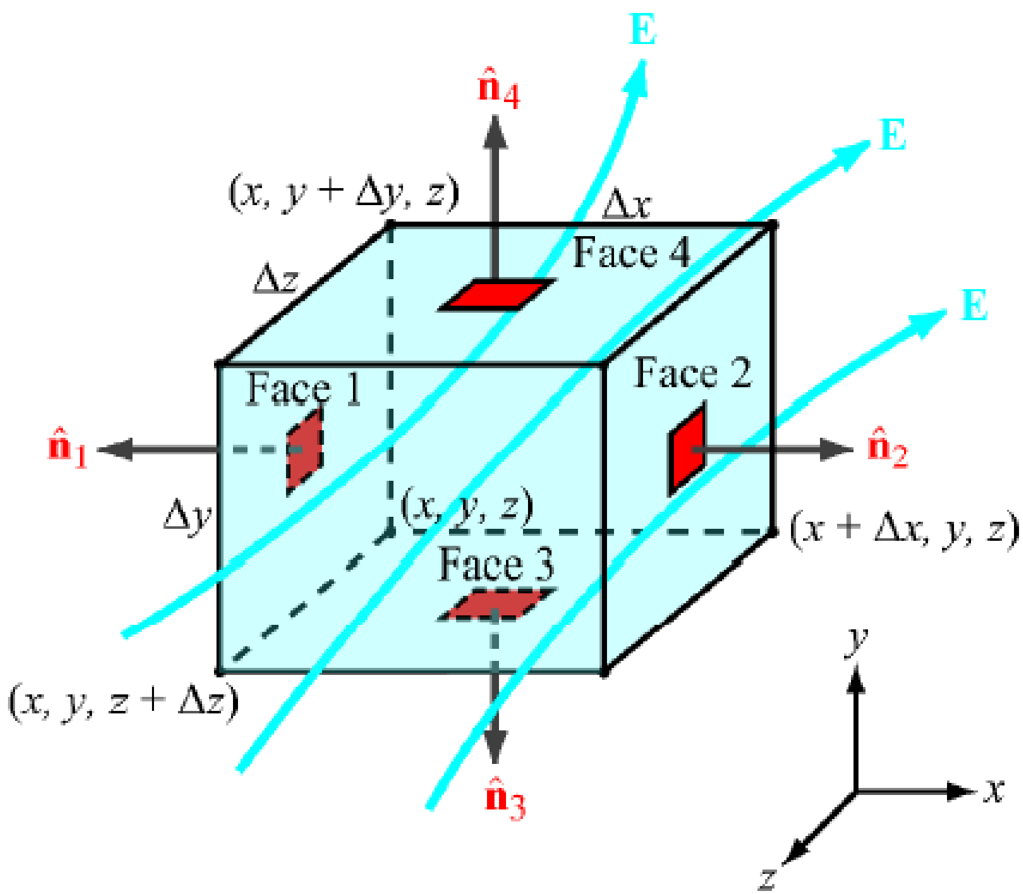
https://youtu.be/Yeie-aJT2eU

https://youtu.be/uOX7SijjH9w

https://youtu.be/TKlpZ0UUJTQ

Divergence of a Vector Field

Flux density is the amount of outward flux crossing a unit surface.



Divergence of a Vector Field

Divergence Theorem

The total outward flux of a vector $\vec{A}$ through the closed surface S is the same as the volume integral of the divergence of $\vec{A}$.

$$\int_v \nabla \cdot \vec{A} dv = \oint_S \vec{A} \cdot d\vec{S}$$

Divergence of a Vector Field

Divergence in Cartesian coordinate:

$$div\,\vec{A} = \nabla \cdot \vec{A} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z$$

Divergence in cylindrical coordinate:

$$div\,\vec{A} = \nabla \cdot \vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho A_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}A_\phi + \frac{\partial}{\partial z}A_z$$

Divergence in spherical coordinate:

$$div\,\vec{A} = \nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\,sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\,sin\theta}\frac{\partial}{\partial \phi}A_\phi$$

Divergence of a Vector Field

Let's Practice

Determine the divergence of these vector fields:

$$(a) \quad \vec{P} = x^2 yz\hat{a}_x + xz\hat{a}_z$$

$$(b) \quad \vec{Q} = \rho \sin\phi \hat{a}_\rho + \rho^2 z\hat{a}_\phi + z\cos\phi \hat{a}_z$$

$$(c) \quad \vec{T} = \frac{1}{r^2}\cos\theta \hat{a}_r + r\sin\theta\cos\phi \hat{a}_\theta + \cos\theta \hat{a}_\phi$$

Divergence of a Vector Field

Solutions

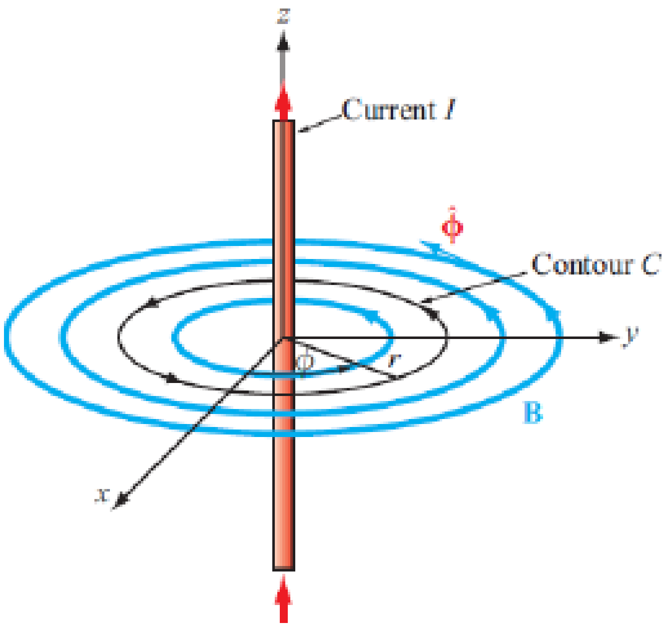
$$\text{(a)}\quad \nabla \cdot \mathbf{P} = \frac{\partial}{\partial x}P_x + \frac{\partial}{\partial y}P_y + \frac{\partial}{\partial z}P_z$$
$$= \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(xz)$$
$$= 2xyz + x$$

$$\text{(b)}\quad \nabla \cdot \mathbf{Q} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho Q_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \phi}Q_\phi + \frac{\partial}{\partial z}Q_z$$
$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho^2 \sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(\rho^2 z) + \frac{\partial}{\partial z}(z\cos\phi)$$
$$= 2\sin\phi + \cos\phi$$

$$\text{(c)}\quad \nabla \cdot \mathbf{T} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(T_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(T_\phi)$$
$$= \frac{1}{r^2}\frac{\partial}{\partial r}(\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(r\sin^2\theta\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\cos\theta)$$
$$= 0 + \frac{1}{r\sin\theta}2r\sin\theta\cos\theta\cos\phi + 0$$
$$= 2\cos\theta\cos\phi$$

Curl of a Vector Field

Curl: describes rotational property or circulation of a **vector** field.



The curl of $\vec{B}$ is a rotational vector whose magnitude is the maximum circulation of $\vec{B}$ per unit area (when it tends to zero) and whose direction is the normal direction of the area when the area is oriented to make the circulation maximum.

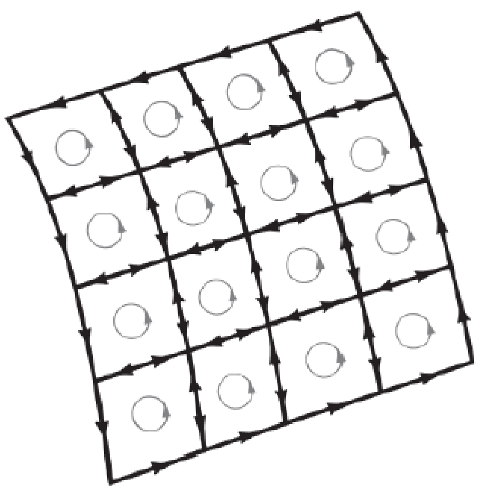
$$\operatorname{curl} \vec{B} = \nabla \times \vec{B} = \left(\lim_{\Delta S \to 0} \frac{\oint_L \vec{B} \cdot d\vec{l}}{\Delta S} \right)_{\max} \hat{a}_n$$

https://youtu.be/vvzTEbp9lrc

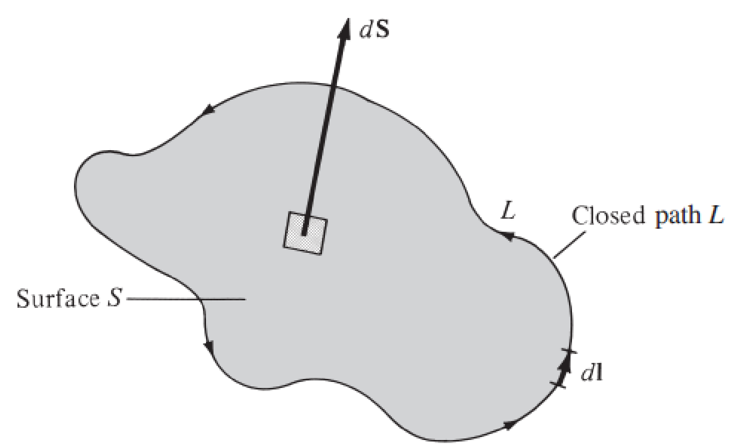
Stokes' Theorem

Stokes' theorem states that the circulation of a vector field $\vec{B}$ around a closed path L is equal to the surface integral of the curl of $\vec{B}$ over the open surface S bounded by L provided $\vec{B}$ and $\nabla \times \vec{B}$ are continuous on S.

$$\text{curl } \vec{B} = \nabla \times \vec{B} = \left(\lim_{\Delta S \to 0} \frac{\oint_L \vec{B} \cdot d\vec{l}}{\Delta S} \right)_{\max} \hat{a}_n$$

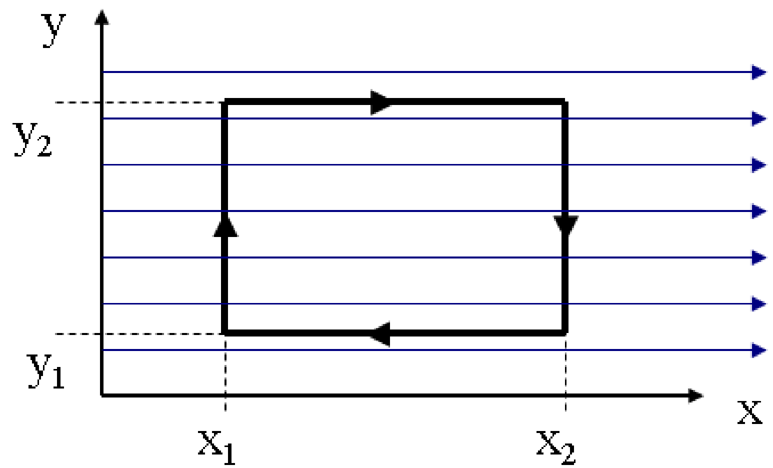


$$\oint_L \vec{B} \cdot d\vec{l} = \int_S (\nabla \times \vec{B}) \cdot d\vec{S}$$



Conservative Vector Field

$$\oint_l \vec{A}.d\vec{l} = 0$$



Vector field

$$\vec{A} = A_1 \vec{a}_x$$

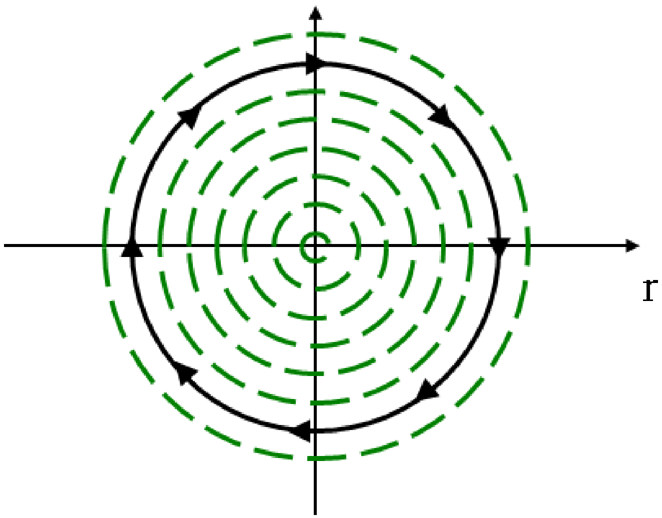
- A vector field with zero circulation is said to be conservative.
- Conservative refers to how energy is conserved around the integral path.
- A zero-curl field can also be described as irrotational.

All electrostatic fields are conservative as with gravitational fields.

$$\oint_l \vec{E}.d\vec{l} = 0$$

Rotational Vector Field

$$circA = \oint_l \vec{A}.d\vec{l} \neq 0$$



Vector field

$$\vec{A} = A_1 \vec{a}_\phi$$

A current carrying conductor will form closed loops of magnetic field around itself . Energy is not conserved as integration is carried out around a closed path.

Magnetostatic fields are not conservative.

$$circH = \oint_l \vec{H}.d\vec{l} = I$$

Curl of a Vector Field

$$\text{curl } \vec{B} = \nabla \times \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \hat{a}_x \left[\frac{\partial}{\partial y} B_z - \frac{\partial}{\partial z} B_y \right] - \hat{a}_y \left[\frac{\partial}{\partial x} B_z - \frac{\partial}{\partial z} B_x \right] + \hat{a}_z \left[\frac{\partial}{\partial x} B_y - \frac{\partial}{\partial y} B_x \right]$$

$$= \hat{a}_x \left[\frac{\partial}{\partial y} B_z - \frac{\partial}{\partial z} B_y \right] + \hat{a}_y \left[\frac{\partial}{\partial z} B_x - \frac{\partial}{\partial x} B_z \right] + \hat{a}_z \left[\frac{\partial}{\partial x} B_y - \frac{\partial}{\partial y} B_x \right]$$

$$\text{curl } \vec{B} = \nabla \times \vec{B} = \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho \hat{a}_\phi & \hat{a}_z \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ B_\rho & \rho B_\phi & B_z \end{vmatrix}$$

$$= \left[\frac{1}{\rho} \frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z} \right] \hat{a}_\rho + \left[\frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial \rho} \right] \hat{a}_\phi + \frac{1}{\rho} \left[\frac{\partial (\rho B_\phi)}{\partial \rho} - \frac{\partial B_\rho}{\partial \phi} \right] \hat{a}_z$$

Curl of a Vector Field

Let's Practice:

Determine the curl of these vector fields

$$(a) \quad \vec{P} = x^2 yz\hat{a}_x + xz\hat{a}_z$$

$$(b) \quad \vec{Q} = \rho \sin\phi\, \hat{a}_\rho + \rho^2 z\hat{a}_\phi + z\cos\phi\hat{a}_z$$

$$(c) \quad \vec{T} = \frac{1}{r^2}\cos\theta\hat{a}_r + r\sin\theta\cos\phi\hat{a}_\theta + \cos\theta\hat{a}_\phi$$

Curl of a Vector Field

Solutions:

(a) $$\nabla \times \mathbf{P} = \left(\frac{\partial P_z}{\partial y} - \frac{\partial P_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial P_x}{\partial z} - \frac{\partial P_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial P_y}{\partial x} - \frac{\partial P_x}{\partial y}\right)\mathbf{a}_z$$
$$= (0 - 0)\mathbf{a}_x + (x^2y - z)\mathbf{a}_y + (0 - x^2z)\mathbf{a}_z$$
$$= (x^2y - z)\mathbf{a}_y - x^2z\mathbf{a}_z$$

(b) $$\nabla \times \mathbf{Q} = \left[\frac{1}{\rho}\frac{\partial Q_z}{\partial \phi} - \frac{\partial Q_\phi}{\partial z}\right]\mathbf{a}_\rho + \left[\frac{\partial Q_\rho}{\partial z} - \frac{\partial Q_z}{\partial \rho}\right]\mathbf{a}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho Q_\phi) - \frac{\partial Q_\rho}{\partial \phi}\right]\mathbf{a}_z$$
$$= \left(\frac{-z}{\rho}\sin\phi - \rho^2\right)\mathbf{a}_\rho + (0 - 0)\mathbf{a}_\phi + \frac{1}{\rho}(3\rho^2 z - \rho\cos\phi)\mathbf{a}_z$$
$$= -\frac{1}{\rho}(z\sin\phi + \rho^3)\mathbf{a}_\rho + (3\rho z - \cos\phi)\mathbf{a}_z$$

Curl of a Vector Field

Solutions:

(c)
$$\nabla \times \mathbf{T} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(T_\phi \sin\theta) - \frac{\partial}{\partial\phi}T_\theta\right]\mathbf{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}T_r - \frac{\partial}{\partial r}(rT_\phi)\right]\mathbf{a}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(rT_\theta) - \frac{\partial}{\partial\theta}T_r\right]\mathbf{a}_\phi$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\cos\theta\sin\theta) - \frac{\partial}{\partial\phi}(r\sin\theta\cos\phi)\right]\mathbf{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\frac{(\cos\theta)}{r^2} - \frac{\partial}{\partial r}(r\cos\theta)\right]\mathbf{a}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(r^2\sin\theta\cos\phi) - \frac{\partial}{\partial\theta}\frac{(\cos\theta)}{r^2}\right]\mathbf{a}_\phi$$

$$= \frac{1}{r\sin\theta}(\cos 2\theta + r\sin\theta\sin\phi)\mathbf{a}_r + \frac{1}{r}(0 - \cos\theta)\mathbf{a}_\theta + \frac{1}{r}\left(2r\sin\theta\cos\phi + \frac{\sin\theta}{r^2}\right)\mathbf{a}_\phi$$

$$= \left(\frac{\cos 2\theta}{r\sin\theta} + \sin\phi\right)\mathbf{a}_r - \frac{\cos\theta}{r}\mathbf{a}_\theta + \left(2\cos\phi + \frac{1}{r^3}\right)\sin\theta\,\mathbf{a}_\phi$$

Summary

$$\text{Gradient of T} \rightarrow \nabla T$$

$$\text{Divergence of } \vec{A} \rightarrow \nabla \cdot \vec{A}$$

$$\text{Curl of } \vec{B} \rightarrow \nabla \times \vec{B}$$

Summary

Gradient

Cartesian Coordinate
$$\text{grad } V = \nabla V = \left(\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z \right)$$

Cylindrical Coordinate
$$\text{grad } V = \nabla V = \left(\frac{\partial V}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z \right)$$

Spherical Coordinate
$$\text{grad } V = \nabla V = \frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi$$

Summary

Divergence

Cartesian Coordinate
$div\,\vec{A} = \nabla \cdot \vec{A} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z$

Cylindrical Coordinate
$div\,\vec{A} = \nabla \cdot \vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho A_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}A_\phi + \frac{\partial}{\partial z}A_z$

Spherical Coordinate
$div\,\vec{A} = \nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\,sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\,sin\theta}\frac{\partial}{\partial \phi}A_\phi$

Summary

Curl

Cartesian Coordinate

$$\text{curl } \vec{B} = \nabla \times \vec{B}$$

$$= \hat{a}_x \left[\frac{\partial}{\partial y} B_z - \frac{\partial}{\partial z} B_y \right] + \hat{a}_y \left[\frac{\partial}{\partial z} B_x - \frac{\partial}{\partial x} B_z \right] + \hat{a}_z \left[\frac{\partial}{\partial x} B_y - \frac{\partial}{\partial y} B_x \right]$$

Cylindrical Coordinate

$$\text{curl } \vec{B} = \nabla \times \vec{B} =$$

$$= \left[\frac{1}{\rho} \frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z} \right] \hat{a}_\rho + \left[\frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial \rho} \right] \hat{a}_\phi + \frac{1}{\rho} \left[\frac{\partial (\rho B_\phi)}{\partial \rho} - \frac{\partial B_\rho}{\partial \phi} \right] \hat{a}_z$$

Spherical Coordinate

$$curl\ \vec{B} = \nabla \times \vec{B}$$

$$= \frac{1}{r \sin\theta} \left[\frac{\partial (A_\emptyset \sin\theta)}{\partial \theta} - \frac{\overset{\partial A_\vartheta}{\cancel{\partial A_\emptyset}}}{\partial \emptyset} \right] \hat{a}_r + \frac{1}{r} \left[\frac{1}{\sin\theta} \frac{\partial A_r}{\partial \emptyset} - \frac{\partial (r A_\emptyset)}{\partial r} \right] \hat{a}_\theta + \frac{1}{r} \left[\frac{\partial (r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right] \hat{a}_\emptyset$$

Summary

Divergence Theorem

$$\int_v \nabla \cdot \vec{A}\, dv = \oint_S \vec{A} \cdot d\vec{S}$$

Stokes' Theorem

$$\int_S \left(\nabla \times \vec{B}\right) \cdot d\vec{s} = \oint_L \vec{B} \cdot d\vec{l}$$

Trivia

- Is divergence of a vector a vector? $\nabla \cdot \vec{A}$
- Is gradient of a scalar a vector? ∇A
- Is curl of a vector a vector? $\nabla \times \vec{A}$
- What about gradient of a vector? $\nabla \vec{A}$
- What about curl of a scalar? $\nabla \times A$

Trivia

- Is divergence of a vector a vector? $\nabla \cdot \vec{A}$ (No)
- Is gradient of a scalar a vector? ∇A (Yes)
- Is curl of a vector a vector? $\nabla \times \vec{A}$ (Yes)
- What about gradient of a vector? $\nabla \vec{A}$

(The gradient operator is only applies to scalar field)

- What about curl of a scalar? $\nabla \times A$

(The curl operator is only applies to vector field)

Practice

The curl of the gradient of any scalar field is identically zero.

$$\nabla \times (\nabla V) = 0$$

The divergence of the curl of any vector field is identically zero.

$$\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$$