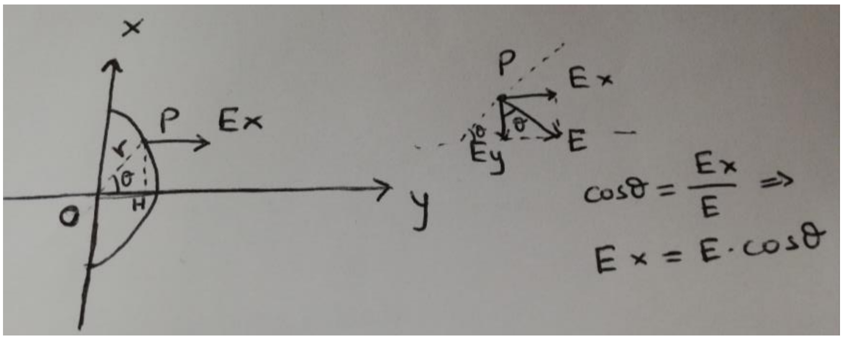
Introduction to Electric and Magnetic Fields B38EM

Tutorial Week #4 - Solutions

ε_0 = 8.85x10^{-12} Fm^{-1}, q_{e^-} = 1.6x10^{-19} C

1. Half a sphere of centre *O* is charged superficially with a constant surface charge density σ. Calculate the expression of the electric field at point *O*.

Sol:



Consider the half sphere with its flat surface along the *x*-direction. The field is directed along *Ox*. Consider the spherical annulus, defined as the portion of sphere limited by two planes parallel to the diameter plan limiting the half sphere.

If *P* is a point belonging to this annulus and *H* is its projection onto the ***x*-axis**

The radius of the annulus will be PH= $r\sin\theta$, where r is the radius of the annulus and θ is the angle between OP and OH.

Circle perimeter: 2πr
For the sphere: 2πrPHdθ
The surface of the annulus is: $2\pi r PH\, d\theta = 2\pi r^2 \sin\theta\, d\theta$

The electric filed is: $E = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2}$ and Ex=E cosθ (as seen from the diagram above)
The contribution in the **x-direction**, of the field created by an element dS of this annulus is:
$dE = \frac{dQ_{enc}}{4\pi\varepsilon_0 r^2} = \frac{\sigma dS}{4\pi\varepsilon_0 r^2} cos\theta$

The field created by the annulus is therefore: $dE = \frac{\sigma\cos\theta}{4\pi\varepsilon_0} . \frac{2\pi r^2 \sin\theta d\theta}{r^2} = \frac{\sigma}{4\varepsilon_0}\sin 2\theta d\theta$

Remember that $sinA \cdot cosB = \frac{1}{2}(\sin(A+B) + \sin(A-B))$
In this case A=B=θ, so $sin\theta \cdot cos\theta = \frac{1}{2}(\sin(\theta+\theta) + \sin(\theta-\theta)) = \frac{1}{2}sin2\theta$

The electric field is then: $E = \frac{\sigma}{4\varepsilon_0}\int_0^{\frac{\pi}{2}} sin2\theta\ d\theta$
NOTE: The limits of the integral are from **0 to π/2** because we have half a sphere.
To solve this integral: let's set a variable x=2θ, then θ=x/2 and dθ=dx/2.
The new limits will be: when θ=0, **x=0**. When θ=π/2, **x=π.**
The total field is therefore:

$$E = \frac{\sigma}{4\varepsilon_0}\int_0^{\pi/2} sin2\theta\ d\theta\ = \frac{\sigma}{4\varepsilon_0}\int_0^{\pi}\frac{1}{2}sinx\ dx\ = \frac{\sigma}{4\varepsilon_0}\frac{1}{2}\left|-cosx\right|_0^{\pi} = \frac{\sigma}{4\varepsilon_0}\frac{1}{2}\big(-(-1)-(-1)\big) = \frac{\sigma}{4\varepsilon_0}$$

2. Find the electric field inside a sphere which carries a charge density proportional to the distance from the origin. In other words, if ρ is the charge density, $\rho=kr$, where k is a constant and r is the distance from the origin.

Sol: Due to the symmetric nature of the charge distribution, the electric field vector **E** is radial and depends only on r. Taking a Gaussian sphere of radius r inside the physical sphere, we apply Gauss's theorem to that Gaussian surface (S) such that:

$$\oiint_S \mathbf{E.n}\,dA = E(r)\oiint_S dr' = E(r)4\pi r^2 = \frac{Q_{enc}}{\varepsilon_0}$$

The difficulty here is to calculate the charge enclosed within the Gaussian surface (S) since the distribution of charge varies with the distance from the origin.

$$Q_{enc} = \iiint_V \rho dV =$$
$$\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r'=0}^{r}(kr')(r'\sin\theta)dr'd\theta d\varphi = \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r'=0}^{r}(k(r')^3\sin\theta)dr'd\theta d\varphi =$$
$$k\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\left|\frac{(r')^4}{4}\right|_0^r \sin\theta\, d\theta d\varphi = k\frac{r^4}{4}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\sin\theta\, d\theta d\varphi = k\frac{r^4}{4}\int_{\varphi=0}^{2\pi}|-\cos\theta|_0^{\pi}d\varphi =$$
$$k\frac{r^4}{4}\int_{\varphi=0}^{2\pi}-(-1-1)d\varphi = k\frac{r^4}{4}\int_{\varphi=0}^{2\pi}2\,d\varphi = k\frac{r^4}{4}2\int_{\varphi=0}^{2\pi}d\varphi = k\frac{r^4}{2}2\pi = k\pi r^4$$

(V) is the volume defined by the closed surface (S). Therefore $\mathbf{E} = \frac{kr^2}{4\varepsilon_0}\mathbf{n}$

3. A hollow spherical sphere carries the charge density $\rho=k/r^2$ in the region a<r<b, where r is the distance from the origin of the sphere. In other words, there is void for r ranging from 0 to a, then matter between a and b, then void again for r greater than b.

Find the electric field intensity in the three regions (i) r<a, (ii) a<r<b and (iii) r>b. Plot the magnitude of the electric field **E** as a function of r.

Sol: Again due to the symmetric nature of the distribution of charge, we know that the electric field is radial and depends only on r, distance from the origin. To find the value of this field in the three regions, we need to apply a Gaussian surface of radius r centered at the same origin as the physical sphere and apply Gauss's law such that:

$$\oiint_S \mathbf{E.n}\,dA = E(r)\oiint_S dr' = E(r)4\pi r^2 = \frac{Q_{enc}}{\varepsilon_0} = \frac{\iiint_V \rho dV}{\varepsilon_0}$$

Where (V) is the volume defined by the Gaussian surface (S).

If $r<a$, the Gaussian surface does not enclose any charge as there is void ie ρ=0, therefore Q_{enc}=0 and E=0.

If $a<r<b$, the Gaussian surface encloses a volume which contains some distribution of charges. Therefore:

$$\iiint_V \rho dV = \int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\phi\int_a^r r'^2\frac{k}{r'^2}dr' = 4\pi k(r-a)$$

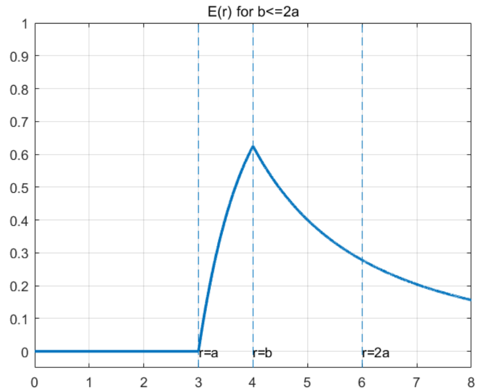
The last integral starts from a since it is only at the surface (r=a) that there is distribution of charge. The electric field is:

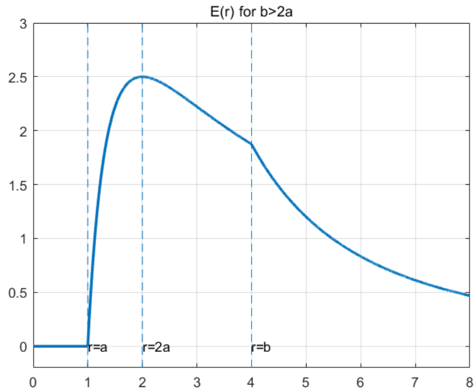
$$E = \frac{k(r-a)}{\varepsilon_0 r^2}$$

If $r>b$, the above last integral can be taken all the way from a to b. At b, there is again no matter and therefore no distribution of charges such that

$$\oiiint_V \rho dV = \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \int_a^b r'^2 \frac{k}{r'^2} dr' = 4\pi k(b-a)$$

and $$E = \frac{k(b-a)}{\varepsilon_0 r^2}$$ The graph is easy to plot. (1/r²)





4. Consider five point charges enclosed in a cylindrical surface (S). The charges are Q_1= 3nC, Q_2= -2nC, Q_3=2nC, Q_4=4nC and Q_5= -1nC. Calculate the flux through the closed surface.

Sol: The choice of a cylindrical surface (S) is a red herring. It does not matter what surface is being used here since from Gauss's law:

$$\phi_{\mathrm{E}} = \oiint_S \mathbf{E.n}\, dA = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sum_{i=1}^{5} Q_i}{\varepsilon_0} = \frac{6.10^{-9}}{8.85\,10^{-12}} \approx 678 Vm$$

5. A line charge with linear charge density $\lambda=10^{-12}$ C/m passes through the centre of a sphere. If the flux through the surface of the sphere is 1.13 10^{-3} Vm, calculate the radius *R* of the sphere.

Sol: The length of line enclosed by the sphere will be L=2R since the line passes through the centre of the sphere. By Gauss's law: $$\phi_{\mathrm{E}} = \frac{Q_{enc}}{\varepsilon_0} = \frac{\lambda 2R}{\varepsilon_0} = 1.13\,10^{-3}\,Vm$$

Therefore, the radius R=5 10^{-3} m.

6. We know that the E-field at a distance *r* from an infinitely charged line of linear charge density λ is given by: $\mathbf{E} = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r}\mathbf{n}$, where **n** is the vector radial to the line of charge. Calculate the electric flux passing through a cylinder of radius r and height H surrounding a portion of this infinite line. Verify that the enclosed charge is λH.

Sol: There is no contribution to the flux from the top and bottom surface of the cylinder since the normal to these surfaces are perpendicular to the electric field vector. Therefore, the surface integral for these portions of the surfaces is zero. The contribution to the flux comes solely from the side of the cylinder.

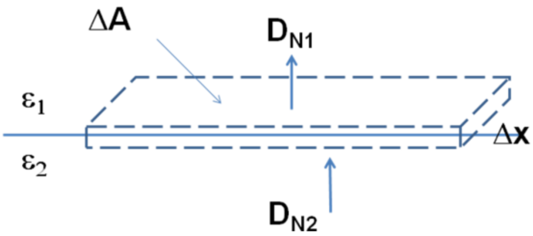
$$\phi_{\mathrm{E}} = \oiint_S \mathbf{E.n}\, dA = \iint_{side} \frac{\lambda}{2\pi\varepsilon_0 r} dA = \frac{\lambda}{2\pi\varepsilon_0 r}\iint_{side} dA = \frac{\lambda}{2\pi\varepsilon_0 r} 2\pi r H = \frac{\lambda H}{\varepsilon_0}$$

But according to Gauss's law, $\phi_E = \dfrac{Q_{enc}}{\varepsilon_0} = \dfrac{\lambda H}{\varepsilon_0}$

Therefore $Q_{enc} = \lambda H$.

7. Using Gauss's law, demonstrate that the normal component of the electric flux density, **D**, is continuous across the interface between two dielectric regions (i.e. have no charge) of permittivity ε_1 and ε_2.

Sol: According to Gauss's law, the flux of **D** across a Gaussian surface is equal to the charge enclosed within that surface. Consider a box of cross section ΔA and thickness ΔX surrounding the interface across the two regions as shown below.



For pure dielectric the flux of D is equal to the charge enclosed these dielectrics, which is zero. But If Δx tends to zero and ΔA is small but finite, the integral of the flux becomes:

$$D_{N1}\Delta A + 0(edges) - D_{N2}\Delta A = 0$$

Therefore D_{N1} is equal to D_{N2}. Note that the negative sign for DN2 is becomes the unit vector normal to the surface points outwards.