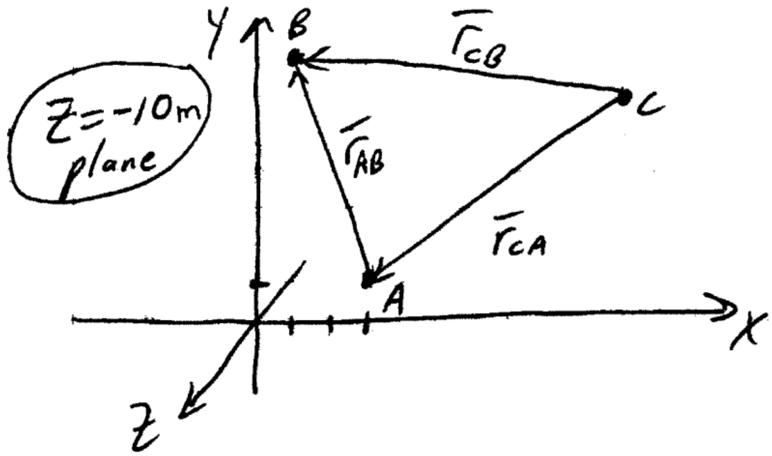
B38EM – Tutorial 1 (Week 2)

1) A triangle has vertices located at points *A*(3, 1, -10), *B*(1, 7, -10), and *C*(10, 6, -10) [units of meters]. Find the angle associated with vertex *C*, the area of the triangle, and the perimeter of the triangle.

Solution:



position vectors

$$\bar{r}_A = 3\hat{a}_x + \hat{a}_y - 10\hat{a}_z \quad m$$

$$\bar{r}_B = \hat{a}_x + 7\hat{a}_y - 10\hat{a}_z \quad m$$

$$\bar{r}_C = 10\hat{a}_x + 6\hat{a}_y - 10\hat{a}_z \quad m$$

distance vectors

→ from C to B

$$\bar{r}_{CB} = \bar{r}_B - \bar{r}_C = (1-10)\hat{a}_x + (7-6)\hat{a}_y + (-10-(-10))\hat{a}_z$$

$$\bar{r}_{CB} = -9\hat{a}_x + \hat{a}_y \quad m$$

→ from C to A

$$\bar{r}_{CA} = \bar{r}_A - \bar{r}_C = (3-10)\hat{a}_x + (1-6)\hat{a}_y + (-10+10)\hat{a}_z$$

$$\bar{r}_{CA} = -7\hat{a}_x - 5\hat{a}_y \quad m$$

→ from A to B

$$\bar{r}_{AB} = \bar{r}_B - \bar{r}_A = (1-3)\hat{a}_x + (7-1)\hat{a}_y + (-10+10)\hat{a}_z$$

$$\bar{r}_{AB} = -2\hat{a}_x + 6\hat{a}_y \quad m$$

Using dot product, find $\measuredangle \theta_c$

$$\Rightarrow \bar{r}_{CB} \cdot \bar{r}_{CA} = |\bar{r}_{CB}||\bar{r}_{CA}| \cos\theta_c$$

$$\bar{r}_{CB} \cdot \bar{r}_{CA} = (-9\hat{a}_x + \hat{a}_y) \cdot (-7\hat{a}_x - 5\hat{a}_y) = -9\cdot(-7) + 1(-5) = \underline{58}$$

$$|\bar{r}_{CB}| = \sqrt{\bar{r}_{CB} \cdot \bar{r}_{CB}} = \sqrt{(-9)^2 + 1^2} = \underline{\sqrt{82}}$$

$$|\bar{r}_{CA}| = \sqrt{\bar{r}_{CA} \cdot \bar{r}_{CA}} = \sqrt{(-7)^2 + (-5)^2} = \underline{\sqrt{74}}$$

$$\cos\theta_c = \frac{\bar{r}_{CB} \cdot \bar{r}_{CA}}{|\bar{r}_{CB}||\bar{r}_{CA}|} = \frac{58}{\sqrt{82}\sqrt{74}} = 0.74457$$

$$\underline{\underline{\theta_c = \cos^{-1}(0.74457) = 41.878^\circ}}$$

$$\bar{r}_{CA} \times \bar{r}_{AB} = \left|\begin{array}{ccc|cc} \hat{a}_x & \hat{a}_y & \hat{a}_z & \hat{a}_x & \hat{a}_y \\ -7 & -5 & 0 & -7 & -5 \\ -2 & 6 & 0 & -2 & 6 \end{array}\right|$$

$$= (\hat{a}_x 0 + \hat{a}_y 0 - \hat{a}_z 42) - (\hat{a}_x 0 + \hat{a}_y 0 + \hat{a}_z 10)$$

$$= \hat{a}_z(-52)$$

triangle area $= \frac{1}{2}(\text{base})(\text{height})$

$$= \frac{1}{2}\,|\text{side dist vector}_i \times \text{side dist. vector}_j|$$

$$= \frac{1}{2}\,|\bar{r}_{CA} \times \bar{r}_{AB}| = \frac{52}{2} = \underline{\underline{26\ m^2}}$$

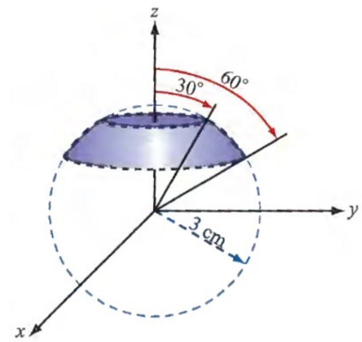
perimeter $= |\bar{r}_{AB}| + |\bar{r}_{CB}| + |\bar{r}_{CA}|$

$$= \sqrt{(-2)^2 + 6^2} + \sqrt{(-9)^2 + 1^2} + \sqrt{(-7)^2 + (-5)^2}$$

$$= \sqrt{40} + \sqrt{82} + \sqrt{74}$$

$$\underline{\underline{\text{perimeter} = 23.982\ m}}$$

2) **Surface Area in Spherical Coordinates:** The spherical strip shown in the Figure is a section of a sphere of radius 3 cm. Find the area of the strip.



Solution:

The area of an elemental spherical area with constant radius R is:

$$S = R^2 \int_{\theta=30^\circ}^{60^\circ} \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= 9(-\cos\theta)\Big|_{30^\circ}^{60^\circ} \; \phi\Big|_{0}^{2\pi} \qquad (\text{cm}^2)$$

$$= 18\pi(\cos 30^\circ - \cos 60^\circ) = 20.7 \text{ cm}^2.$$

Table 3-2: Coordinate transformation relations.

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2 + y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x\cos\phi + A_y\sin\phi$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r\cos\phi - A_\phi\sin\phi$ $A_y = A_r\sin\phi + A_\phi\cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt[+]{x^2 + y^2 + z^2}$ $\theta = \tan^{-1}[\sqrt[+]{x^2 + y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$	$A_R = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$ $A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$
Spherical to Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_x = A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$ $A_y = A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$
Cylindrical to spherical	$R = \sqrt[+]{r^2 + z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r\sin\theta + A_z\cos\theta$ $A_\theta = A_r\cos\theta - A_z\sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_r = A_R\sin\theta + A_\theta\cos\theta$ $A_\phi = A_\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$

3) **Cartesian to Cylindrical Transformations:** Given point $P_1 = (3, -4, 3)$ and vector $\mathbf{A} = \hat{x}2 - \hat{y}3 + \hat{z}4$, defined in Cartesian coordinates, express P_1 and $\mathbf{A}$ in cylindrical coordinates and evaluate $\mathbf{A}$ at P_1 .

Solution:

For point P_1, x = 3, y = - 4, and z = 3 we have:

$$r = \sqrt{x^2 + y^2} = 5, \quad \phi = \tan^{-1}\frac{y}{x} = -53.1^\circ = 306.9^\circ.$$

So, $P_1 = (5,306.9^\circ,3)$ in cylindrical coordinates.

The cylindrical components of vector $\mathbf{A} = \hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$ can be determined by applying Eqs. (3.58a) and (3.58b):

$$A_r = A_x \cos\phi + A_y \sin\phi = 2\cos\phi - 3\sin\phi,$$
$$A_\phi = -A_x \sin\phi + A_y \cos\phi = -2\sin\phi - 3\cos\phi,$$
$$A_z = 4.$$

Hence,

$$\mathbf{A} = \hat{\mathbf{r}}(2\cos\phi - 3\sin\phi) - \hat{\boldsymbol{\phi}}(2\sin\phi + 3\cos\phi) + \hat{\mathbf{z}}4.$$

At point P, $\phi = 306.9^\circ$, which gives

$$\mathbf{A} = \hat{\mathbf{r}}3.60 - \hat{\boldsymbol{\phi}}0.20 + \hat{\mathbf{z}}4.$$

4) Gradient of scalar function: Find the gradient of the following scalar function and then evaluate it at the given point.

$V_1 = 24x^2 - 3y + z$ at (1, 2, 3) in Cartesian coordinates.

Solution:

$$\nabla V_1 = \hat{\mathbf{x}}\frac{\partial V_1}{\partial x} + \hat{\mathbf{y}}\frac{\partial V_1}{\partial y} + \hat{\mathbf{z}}\frac{\partial V_1}{\partial z}$$
$$= \hat{\mathbf{x}}(2\cdot 24x) + \hat{\mathbf{y}}(-3) + \hat{\mathbf{z}}(1) =$$
$$= \hat{\mathbf{x}}\,48x - 3\hat{\mathbf{y}} + \hat{\mathbf{z}} =$$

At (1, 2, 3):

$$\nabla V_1|_{(1,2,3)} = \hat{\mathbf{x}}\,48 - 3\hat{\mathbf{y}} + \hat{\mathbf{z}}.$$

5) Calculating the Divergence: Determine the divergence of the vector field and then evaluate it at the indicated point:

$\mathbf{P} = x^2yz\ \mathbf{a}_x + xz\ \mathbf{a}_z$ at point: (3, 2, 4)

$$\text{(a)}\ \nabla \cdot \mathbf{P} = \frac{\partial}{\partial x}P_x + \frac{\partial}{\partial y}P_y + \frac{\partial}{\partial z}P_z$$

$$= \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(xz)$$

$$= 2xyz + x$$

At (3,2,4), $\nabla\cdot\mathbf{P}|_{(2,3,4)} = 2\cdot3\cdot2\cdot4 + 3 = 51$.

6) Calculating the Curl: Determine the curl of the vector field and then evaluate it at the indicated point:

$\mathbf{P} = x^2yz\ \mathbf{a}_x + xz\ \mathbf{a}_z$ at (5, 5, 1)

$$\nabla \times \mathbf{P} = \left(\frac{\partial P_z}{\partial y} - \frac{\partial P_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial P_x}{\partial z} - \frac{\partial P_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial P_y}{\partial x} - \frac{\partial P_x}{\partial y}\right)\mathbf{a}_z$$

$$= (0 - 0)\,\mathbf{a}_x + (x^2y - z)\,\mathbf{a}_y + (0 - x^2z)\,\mathbf{a}_z$$

$$= (x^2y - z)\mathbf{a}_y - x^2z\mathbf{a}_z$$

at (5, 5, 1): $\nabla\times\mathbf{P} = (5^2\cdot5 - 1)\ \mathbf{a}_x + 5^2\cdot1\mathbf{a}_z = 124\mathbf{a}_x + 25\mathbf{a}_z$ (vector)