B38EM Introduction to Electricity and Magnetism
Lecture 10

Transmission Lines

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Outline & Outcome

- Lumped or distributed?
- Transmission line theory
- Terminated transmission line
- Smith Chart

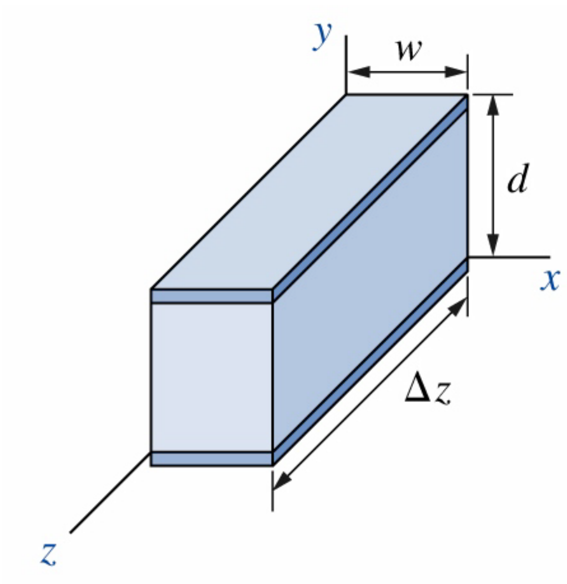
References & Resources

- Elements of Electromagnetics (7th Edition), by Sadiku, Oxford University Press
- Fundamentals of Applied Electromagnetics (7th Edition), by Ulaby and Ravaioli
- Field and Wave Electromagnetics (2nd Edition), by David Cheng

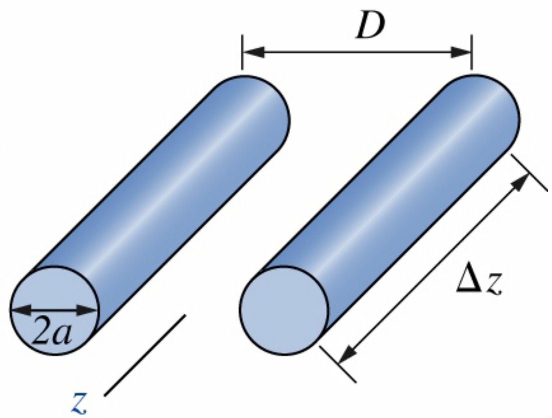
Transmission Lines

Common Types

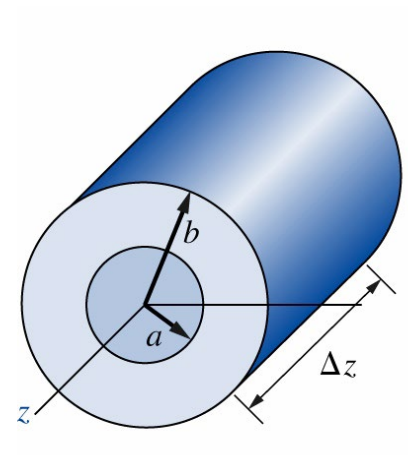
(TEM waves)



Parallel-plate transmission line



Two-wire (twin-lead) transmission line



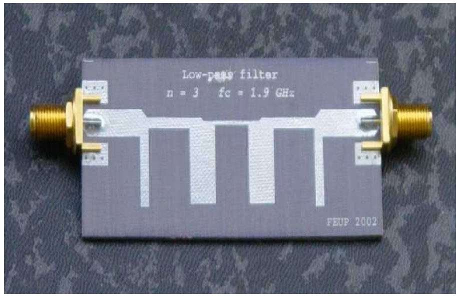
Coaxial transmission line

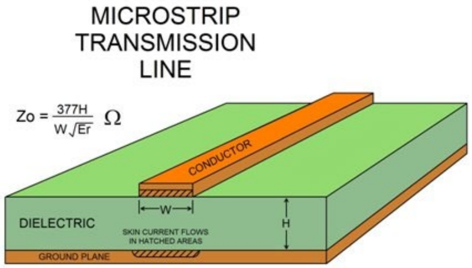
Each structure (including the twin lead) may have a dielectric between two conductors used to keep the separation between the metallic elements constant, so that the electrical properties would be constant.

Transmission Lines

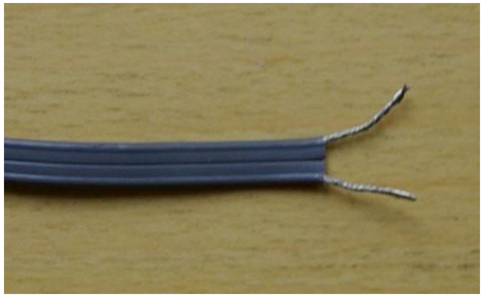
Common Types (TEM waves)

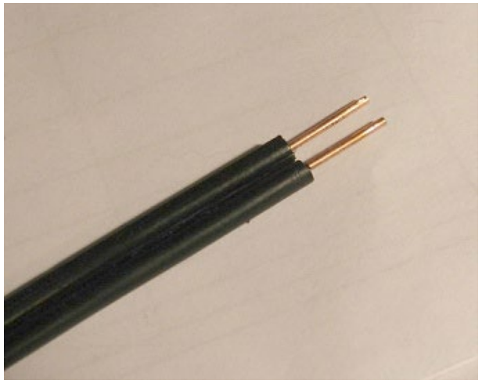
Microstrip line



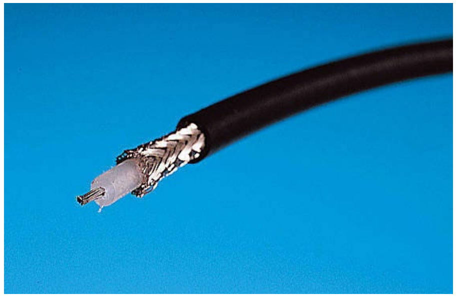


Twin lead





Coaxial cable



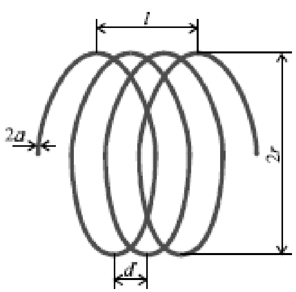
Transmission Lines

Lumped or Distributed?

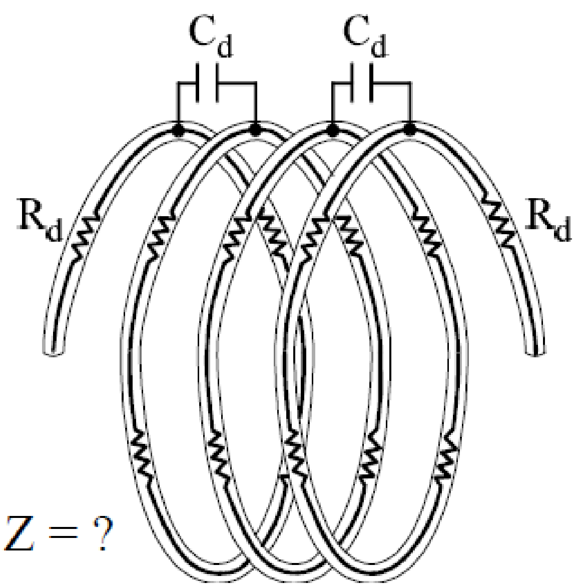
Example: Inductor

Low-frequency

(lumped)



High-frequency



$Z = R + j\omega L$

$Z = ?$

Transmission Lines

Lumped or Distributed?

At low frequencies:

Can simply use a wire to connect two components

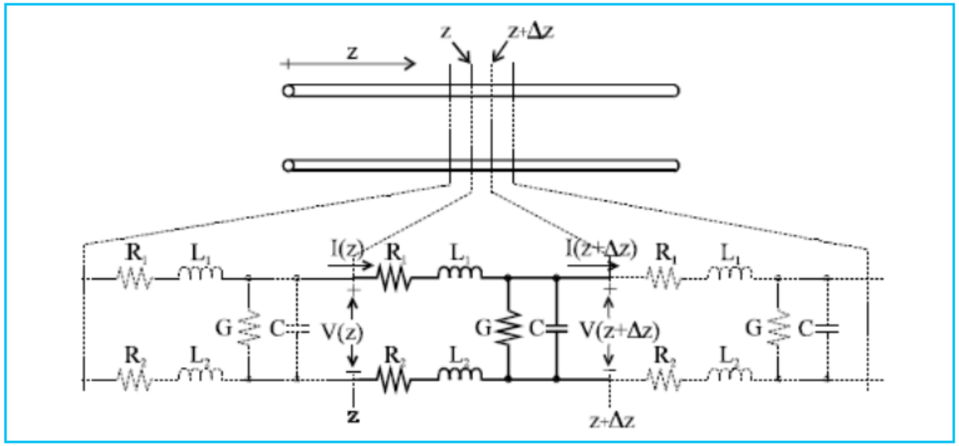
f =50 HZ, wavelength = 6 x 10^6m;

At high frequencies:

Cannot simply use a wire to connect two components

f =500 MHZ, wavelength = 0.6 m;

Transmission Lines



- R = series resistance per unit length, for both conductors, in Ω/m;
- L = series inductance per unit length, for both conductors, in H/m;
- G = parallel conductance per unit length, in S/m;
- C = parallel capacitance per unit length, in F/m.

- Loss: R (due to the finite conductivity) + G (due to the dielectric loss)

Transmission Lines

Transmission line theory

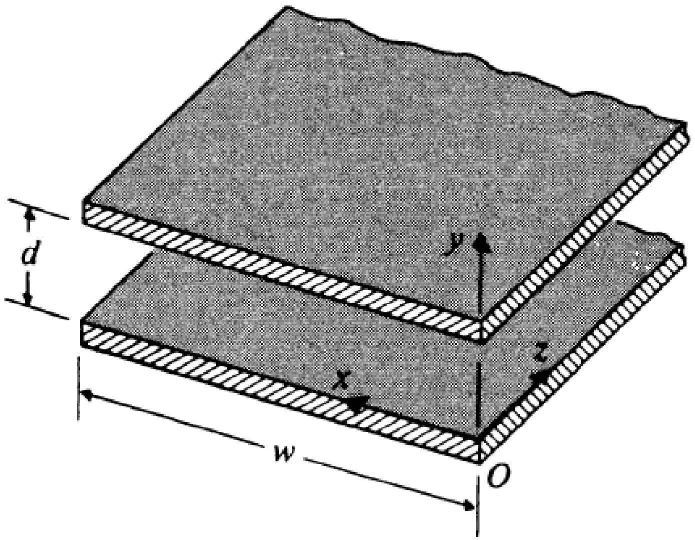
- **Bridges the gap between field analysis and basic circuit theory**
- **Extension from lumped to distributed theory**
- **A specialization of Maxwell's equations**
- **Significant importance in microwave network analysis**

The key difference between circuit theory and transmission line theory is electrical size. Circuit analysis assumes that the physical dimensions of a network are much smaller than the electrical wavelength, while transmission lines may be a considerable fraction of a wavelength, or many wavelengths, in size. Thus a transmission line is a distributed-parameter network, where voltages and currents can vary in magnitude and phase over its length.

Transmission Lines

Parallel-plate transmission line

From Maxwell's equations to Transmission line equation



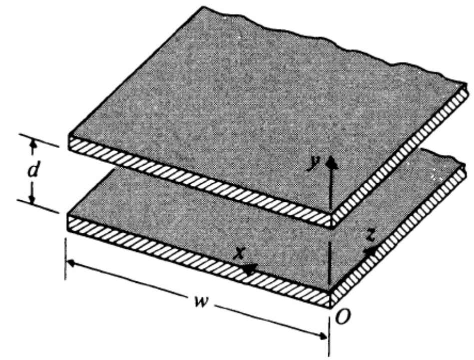
$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 0 \quad \text{since} \quad \mathbf{E} \propto e^{i\omega t} \;\rightarrow\; \nabla^2\mathbf{E} + \frac{\omega^2}{c^2}\mathbf{E} = 0 \; (\nabla^2\mathbf{E} + k_0^2\mathbf{E} = 0)$$

Transmission Lines

Parallel-plate transmission line

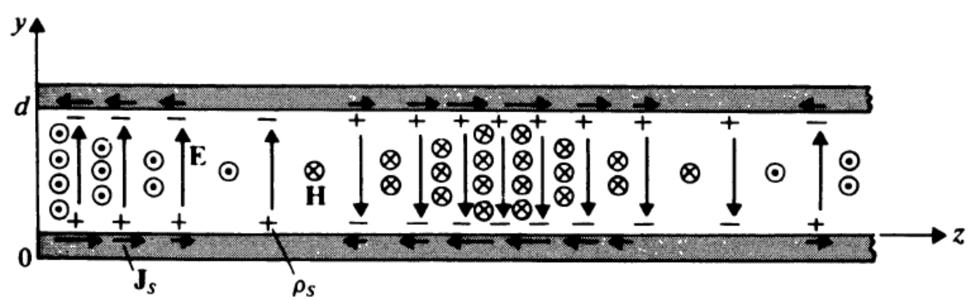
$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad \text{since} \quad \mathbf{E} \propto e^{i\omega t} \;\rightarrow\; \nabla^2 \mathbf{E} + \frac{\omega^2}{c^2}\mathbf{E} = 0 \, (\nabla^2 \mathbf{E} + k_0^2 \mathbf{E} = 0)$$

Assume it's a plane wave propagate in the *z*-axis with polarization in the y-axis direction.



$$\frac{d^2}{dz^2}E_y + k_0^2 E_y = 0 \;\rightarrow\; \mathbf{E} = \hat{y}\tilde{E}_0 e^{-ik_0 z + i\omega t}$$

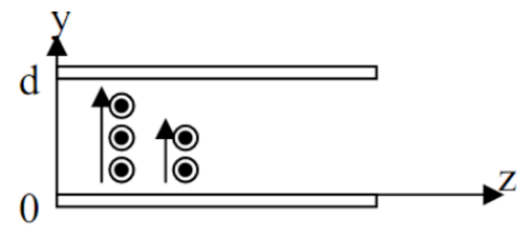
$$\mathbf{H} = -\hat{x}\frac{1}{\sqrt{\frac{\mu}{\varepsilon}}}\tilde{E}_0 e^{-ikz+i\omega t} = H_x \hat{x}$$



Transmission Lines

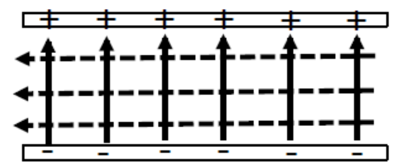
Parallel-plate transmission line

Crossing the boundary from dielectric medium to the perfect conduction plates:



$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \& \quad \nabla \times \vec{B} = \mu\varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\rightarrow \quad \nabla \times \vec{E} = -i\omega\mu\vec{H} \quad \& \quad \nabla \times \vec{H} = i\omega\varepsilon\vec{E}$$



$$\mathbf{E} = \hat{y}\tilde{E}_y(z,t), \quad \mathbf{H} = \hat{x}\tilde{H}_x(z,t) \quad (\vec{E} \rightarrow V, \vec{H} \rightarrow \sigma \rightarrow I)$$

Basic differential equations

$$\begin{vmatrix} \hat{i} & \hat{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} = -i\omega\mu H_x \quad \rightarrow \quad \frac{dE_y}{dz} = i\omega\mu H_x \quad \& \quad \frac{dH_x}{dz} = i\omega\varepsilon E_y$$

Transmission Lines

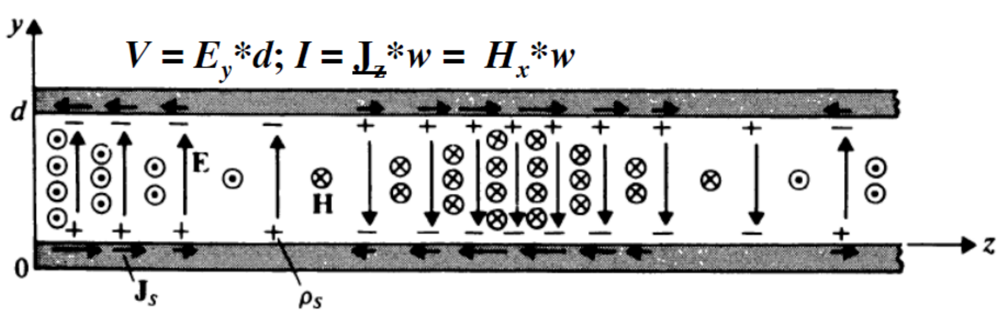
Parallel-plate transmission line

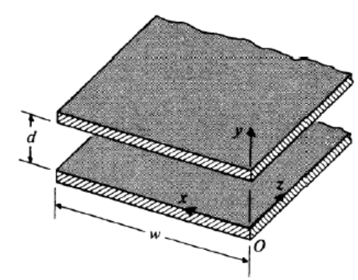
$$\int_0^d \frac{dE_y}{dz}\,dy = i\omega\mu\int_0^d H_x\,dy \quad \rightarrow \quad -\frac{dV(z)}{dz} = i\omega L I(z)$$

$L = \mu\dfrac{d}{w}$ is the inductance per unit length

$$\int_0^w \frac{dH_x}{dz}\,dx = i\omega\varepsilon\int_0^w E_y\,dx \quad \rightarrow \quad -\frac{dI(z)}{dz} = i\omega C V(z)$$

$C = \varepsilon\dfrac{w}{d}$ is the capacitance per unit length





Transmission Lines

Time-harmonic transmission line equations

$$\left.\begin{aligned} -\frac{dV(z)}{dz} &= i\omega L I(z) \\ -\frac{dI(z)}{dz} &= i\omega C V(z) \end{aligned}\right\}$$

$$\frac{d^2V(z)}{dz^2} = -\omega^2 LCV(z) \quad \Longrightarrow \quad V(z) = V_0 e^{-ikz}$$

$$(k = \omega\sqrt{LC})$$

$$\frac{d^2I(z)}{dz^2} = -\omega^2 LCI(z) \quad \Longrightarrow \quad I(z) = I_0 e^{-ikz}$$

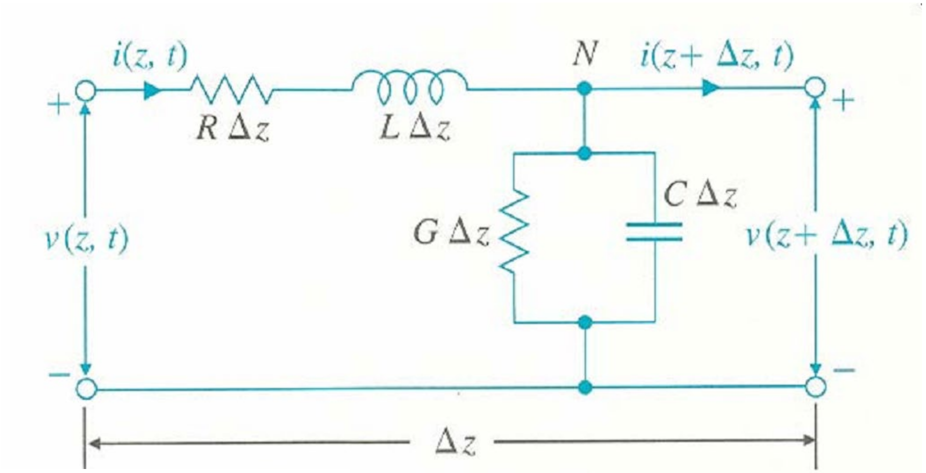
Characteristic impedance: $Z_0 = \frac{V(z)}{I(z)} = \frac{V_0}{I_0} = \frac{\omega L I_0}{k I_0} = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu d/w}{\varepsilon w/d}} = \frac{d}{w}\sqrt{\frac{\mu}{\varepsilon}}$

Phase velocity : $v = \frac{\omega}{k} = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(\mu d/w)(\varepsilon w/d)}} = \frac{1}{\sqrt{\mu\varepsilon}}$

$L = \mu\frac{d}{w}$ $\quad$ $C = \varepsilon\frac{w}{d}$

Transmission Lines

Time-harmonic transmission line equations



$$-\frac{dV(z)}{dz} = (R + i\omega L)I(z)$$

$$-\frac{dI(z)}{dz} = (G + i\omega C)V(z)$$

$$\frac{d^2V(z)}{dz^2} = \gamma^2 V(z)$$

$$\frac{d^2I(z)}{dz^2} = \gamma^2 I(z),$$

Where $\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \qquad (m^{-1})$

α : attenuation constant $\qquad \beta$: phase constant

Transmission Lines

Time-harmonic transmission line equations

$$\frac{d^2V(z)}{dz^2} = \gamma^2 V(z) \qquad \frac{d^2I(z)}{dz^2} = \gamma^2 I(z),$$

$$\begin{aligned} V(z) &= V^+(z) + V^-(z) \\ &= V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}, \end{aligned} \qquad \begin{aligned} I(z) &= I^+(z) + I^-(z) \\ &= I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}, \end{aligned}$$

Wave propagation in the positive z direction

Wave propagation in the negative -z direction

$$\frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \frac{R + j\omega L}{\gamma}$$

Characteristic impedance: $\boxed{Z_0 = \frac{R + j\omega L}{\gamma} = \frac{\gamma}{G + j\omega C} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \quad (\Omega)}$

Transmission Lines

Time-harmonic transmission line equations

Lossless Line ($R = 0$, $G = 0$).

a) Propagation constant:

$$\gamma = \alpha + j\beta = j\omega\sqrt{LC};$$
$$\alpha = 0,$$
$$\beta = \omega\sqrt{LC} \qquad \text{(a linear function of } \omega\text{)}.$$

b) Phase velocity:

$$u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \qquad \text{(constant)}.$$

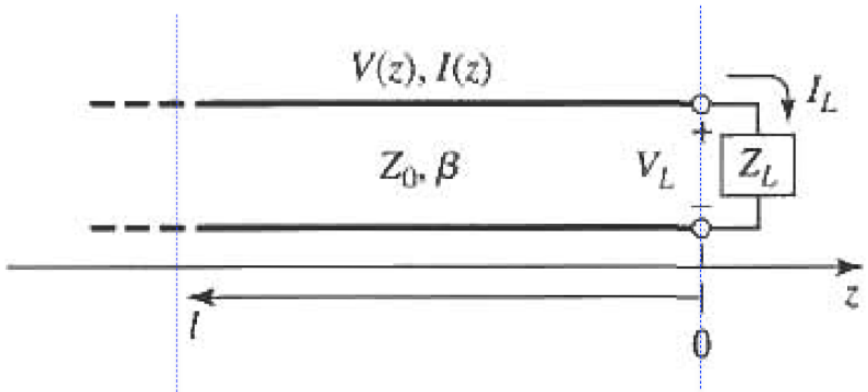
c) Characteristic impedance:

$$Z_0 = R_0 + jX_0 = \sqrt{\frac{L}{C}};$$
$$R_0 = \sqrt{\frac{L}{C}} \qquad \text{(constant)},$$
$$X_0 = 0.$$

Terminated transmission line

Lossless: $\alpha=0$; $\gamma=j\beta$

What is a voltage reflection coefficient?



$$V(z) = V^+(z) + V^-(z)$$
$$I(z) = I^+(z) + I^-(z)$$

- Assume an incident wave ($V_0^+ e^{-j\beta z}$) generated from a source at $z < 0$. We have seen that the ratio of voltage to current for such a traveling wave is Z_o, the characteristic impedance. But when the line is terminated in an arbitrary load $Z_L \neq Z_o$, the ratio of voltage to current at the load must be Z_L. Thus, a reflected wave must be excited with the appropriate amplitude to satisfy this condition.

Transmission Lines

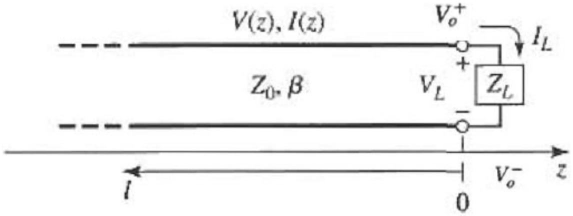
Terminated transmission line

Lossless: α=0; γ=$j\beta$

Total voltage and current on the line (superposition of incident and reflected waves):

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{j\beta z}$$



(V_0^+: incident voltage at z =0; V_0^-: reflected voltage at z = 0)

The total voltage and current at the load are related by the load impedance, so at z = 0, we must have

$$Z_L = \frac{V(0)}{I(0)} = \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} Z_0$$

Voltage reflection coefficient Γ_0:

$$\boxed{\Gamma_0 = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0}}$$

$\Gamma_0 = 0$ ($Z_l = Z_0$)

$\Gamma_0 = 1$ ($Z_L \to \infty$)

$\Gamma_0 = -1$ ($Z_L \to 0$)

(Phase difference: π)

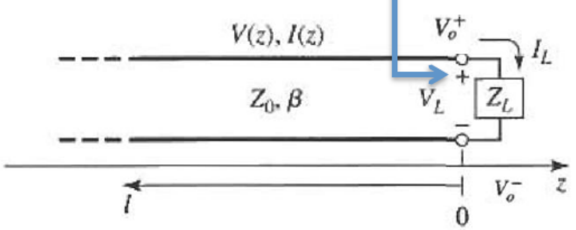
Transmission Lines

Terminated transmission line

Lossless: $\alpha=0$; $\gamma=j\beta$

$$\boxed{\Gamma_0 = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0}}$$

The total voltage and current waves on the line :



$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} = V_0^+[e^{-j\beta z} + \Gamma_0 e^{j\beta z}]$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{j\beta z} = \frac{V_0^+}{Z_0}[e^{-j\beta z} - \Gamma_0 e^{j\beta z}]$$

Consider the time-average power flow along the line at the point z:

$$P_{av} = \frac{1}{2}\mathrm{Re}[V(z)I(z)^*] = \frac{1}{2}\frac{|V_0^+|^2}{Z_0}\mathrm{Re}\{1 - \Gamma_0^* e^{-2j\beta z} + \Gamma_0 e^{2j\beta z} - |\Gamma_0|^2\}$$

which can be simplified:

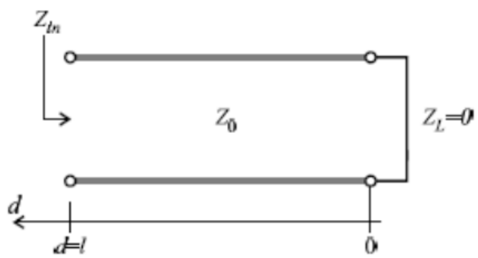
$$P_{av} = \frac{1}{2}\frac{|V_0^+|^2}{Z_0}(1 - |\Gamma_0|^2)$$

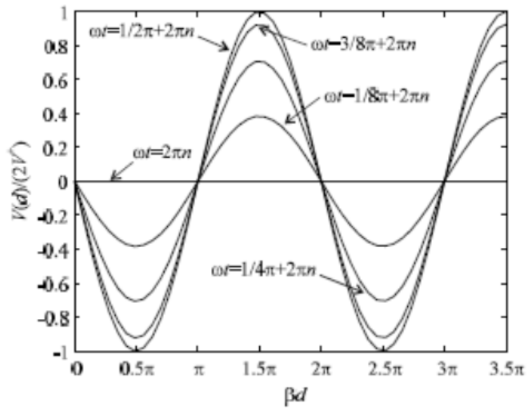
- **Constant average power flow at any point on the line;**
- **Total power delivered to the load = incident power – reflected power**

Transmission Lines

Terminated transmission line

(a). Standing wave (Γ_0 = -1) $V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} = V_0^+ [e^{-j\beta z} + \Gamma_0 e^{j\beta z}]$





$$v(d,t) = 2V^+ \sin(\beta d)\cos(\omega t + \pi/2)$$

(b). Voltage standing wave ratio ($|\Gamma_0| < 1$)

$$|V(z)| = |V_0^+||1 + \Gamma_0 e^{2j\beta z}| = |V_0^+||1 + \Gamma_0 e^{-2j\beta l}| \qquad (z = -l)$$

$$SWR = \frac{|V|_{max}}{|V|_{min}} = \frac{1+|\Gamma_0|}{1-|\Gamma_0|}$$ (1 ≤ SWR < ∞， where SWR=1 implied a match load.)

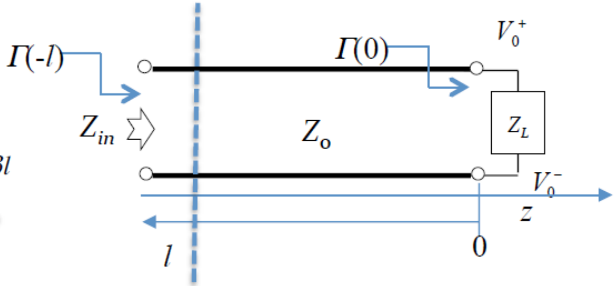
$$RL = -20\log|\Gamma_0| \qquad (dB) \quad \text{(return loss)}$$

Transmission Lines

Terminated transmission line

Reflection coefficient at z = −*l* and input impedance Z_{in}

Reflection coefficient Γ_l at $z = -l$:



$$\Gamma_l = \frac{V^-(z)}{V^+(z)}\bigg|_{z=-l} = \frac{V_0^- e^{j\beta^*(-l)}}{V_0^+ e^{-j\beta^*(-l)}} = \frac{V_0^- e^{-j\beta l}}{V_0^+ e^{j\beta l}} = \Gamma_0 e^{-2j\beta l}$$

$$[V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}, \quad \Gamma_0 = \frac{V_0^-}{V_0^+}]$$

At a distance *l* from the load, **the input impedance Z_{in}** seen looking toward the load is

$$Z_{in} = \frac{V(-l)}{I(-l)} = \frac{V_0^+(e^{j\beta l} + \Gamma_0 e^{-j\beta l})}{V_0^+(e^{j\beta l} - \Gamma_0 e^{-j\beta l})} Z_0$$

$$\boxed{\begin{aligned} V(z) &= V_0^+[e^{-j\beta z} + \Gamma_0 e^{j\beta z}] \\ I(z) &= \frac{V_0^+}{Z_0}[e^{-j\beta z} - \Gamma_0 e^{j\beta z}] \end{aligned}}$$

$$= \frac{1 + \Gamma_0 e^{-2j\beta l}}{1 - \Gamma_0 e^{-2j\beta l}} Z_0$$

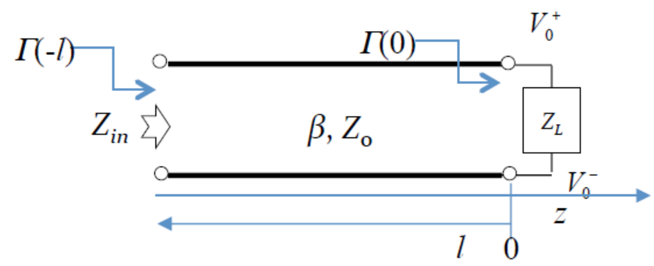
$$= \frac{1 + \Gamma_l}{1 - \Gamma_l} Z_0$$

$$\boxed{\Gamma_l = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}}$$

Transmission Lines

Terminated transmission line

Reflection coefficient at z = −*l* and input impedance Z_{in}



A more usable form of input impedance:

$$Z_{in} = \frac{1+\Gamma_0 e^{-2j\beta l}}{1-\Gamma_0 e^{-2j\beta l}} Z_0 \qquad \boxed{\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0}}$$

$$= Z_0 \frac{(Z_L + Z_0)e^{j\beta l} + (Z_L - Z_0)e^{-j\beta l})}{(Z_L + Z_0)e^{j\beta l} - (Z_L - Z_0)e^{-j\beta l})}$$

$$= Z_0 \frac{Z_L \cos\beta l + jZ_0 \sin\beta l}{Z_0 \cos\beta l + jZ_L \sin\beta l}$$

$$= \boxed{Z_0 \frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}}$$

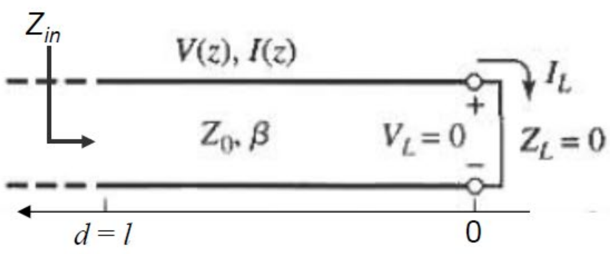
- Input impedance of a portion of transmission line with an arbitrary load impedance.

$$\boxed{\Gamma_l = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}}$$

Transmission Lines

(1). Short-circuit transmission line (Z_L=0, Γ_0=-1)

$$V(z) = V_0^+[e^{-j\beta z} + \Gamma_0 e^{j\beta z}]$$
$$I(z) = \frac{V_0^+}{Z_0}[e^{-j\beta z} - \Gamma_0 e^{j\beta z}]$$



Voltage:

$$V(d) = 2jV^+ \sin(\beta d)$$

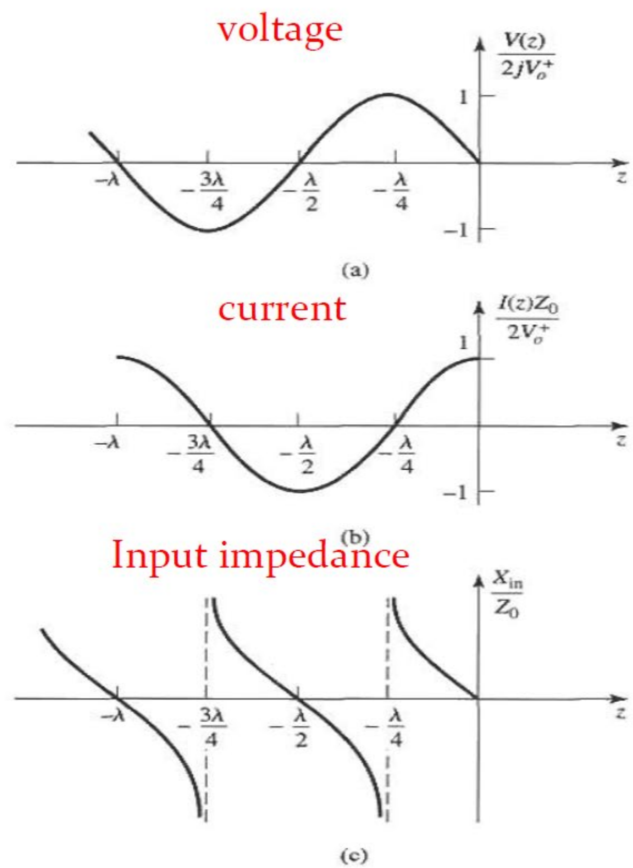
Current:

$$I(d) = \frac{2V^+}{Z_0}\cos(\beta d)$$

Impedance

$$\boxed{Z_{in}(d) = jZ_0 \tan(\beta d)}$$

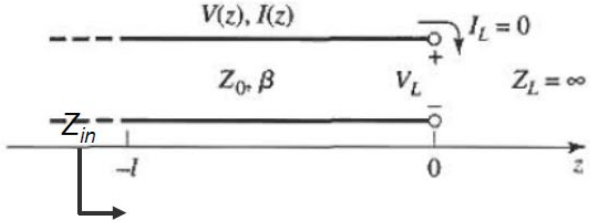
Imaginary number



Transmission Lines

(2). Open-circuit transmission line ($Z_L=\infty$, $\Gamma_0=1$)

$$V(z) = V_0^+[e^{-j\beta z} + \Gamma_0 e^{j\beta z}]$$
$$I(z) = \frac{V_0^+}{Z_0}[e^{-j\beta z} - \Gamma_0 e^{j\beta z}]$$



Voltage:

$$V(d) = 2V^+\cos(\beta d)$$

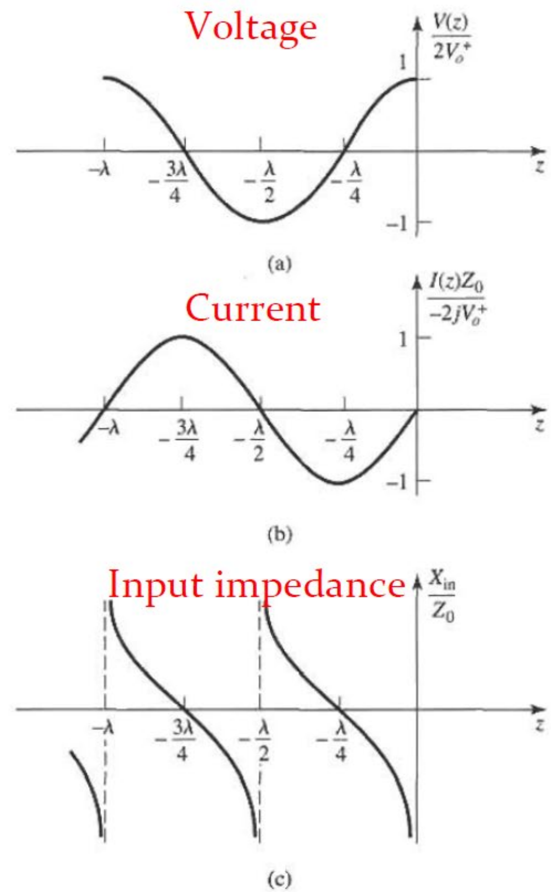
Current:

$$I(d) = \frac{2jV^+}{Z_0}\sin(\beta d)$$

Impedance

$$\boxed{Z_{in}(d) = -jZ_0\cot(\beta d)}$$

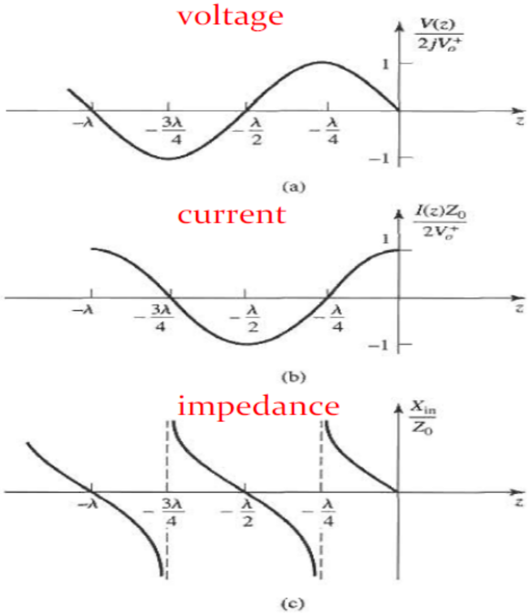
Imaginary number

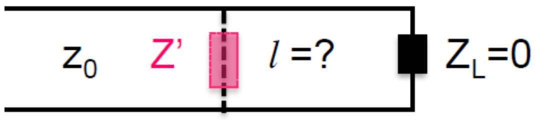


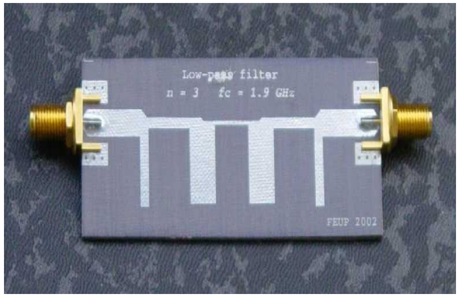
Transmission Lines

Q: Find the position l to place an impedance element Z' (Z'=Z_0) so that Γ is zero in the case of $Z_L = 0$ or infinite.

(1) $Z_L = 0$







$$\boxed{\Gamma_l = \frac{Z_{in}(l) - Z_0}{Z_{in}(l) + Z_0} = 0}$$

$$\boxed{Z_{in}(l) = Z_0}$$

$$\boxed{Z'_{in} = Z_0 \frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}}$$

$$\boxed{\frac{1}{Z_{in}(l)} = \frac{1}{Z'_{in}} + \frac{1}{Z'}}$$

$Z'_{in} \rightarrow \infty$ &

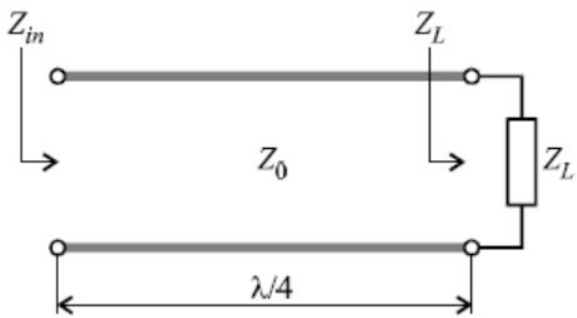
$l = (2m+1)*\lambda/4$

m=0, 1, 2, 3,…

Transmission Lines

(3). Quarter-wave transmission line

Quarter-wave transmission line



$$(l = \lambda/4 + n\lambda/2, \qquad n = 1,2,3,\ldots)$$

$$Z_{in}(\lambda/4) = Z_0 \frac{Z_L + jZ_0 \tan(\beta\lambda/4)}{Z_0 + jZ_L \tan(\beta\lambda/4)} = \frac{Z_0^{\ 2}}{Z_L}$$

Quarter-wave transformer model:

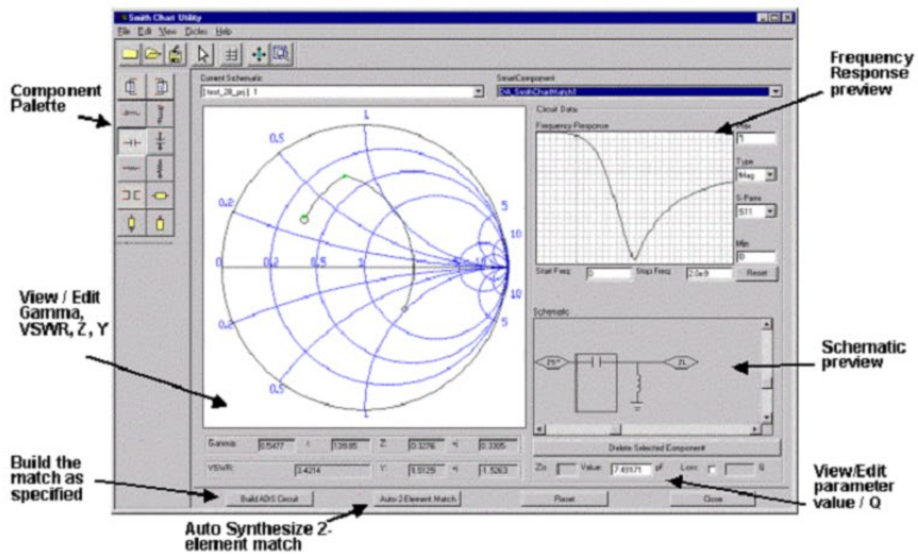
given input and output impedances $\xrightarrow{\text{Predict line impedance}}$ $\boxed{Z_0 = \sqrt{Z_L Z_{in}}}$

Smith Chart: Introduction

Introduction

A graphical tool used to solve transmission line problems.

Today, a presentation medium in computer-aided design (CAD) software and measuring equipment for displaying the performance of microwave circuits.



Smith Chart: Introduction

Reflection coefficient:

$$\Gamma = \frac{Z_L - R_0}{Z_L + R_0} = |\Gamma| e^{j\theta_\Gamma} = \Gamma_r + j\Gamma_i$$

Normalized load impedance:

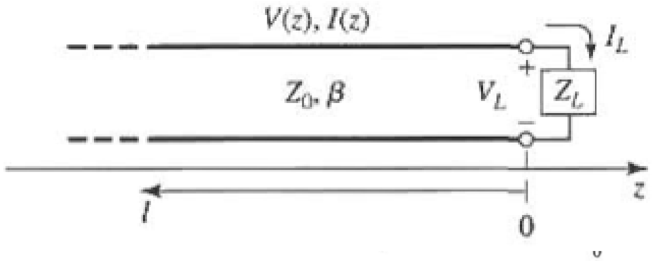
$$z_L = \frac{Z_L}{R_0} = \frac{R_L}{R_0} + j\frac{X_L}{R_0} = r + jx$$

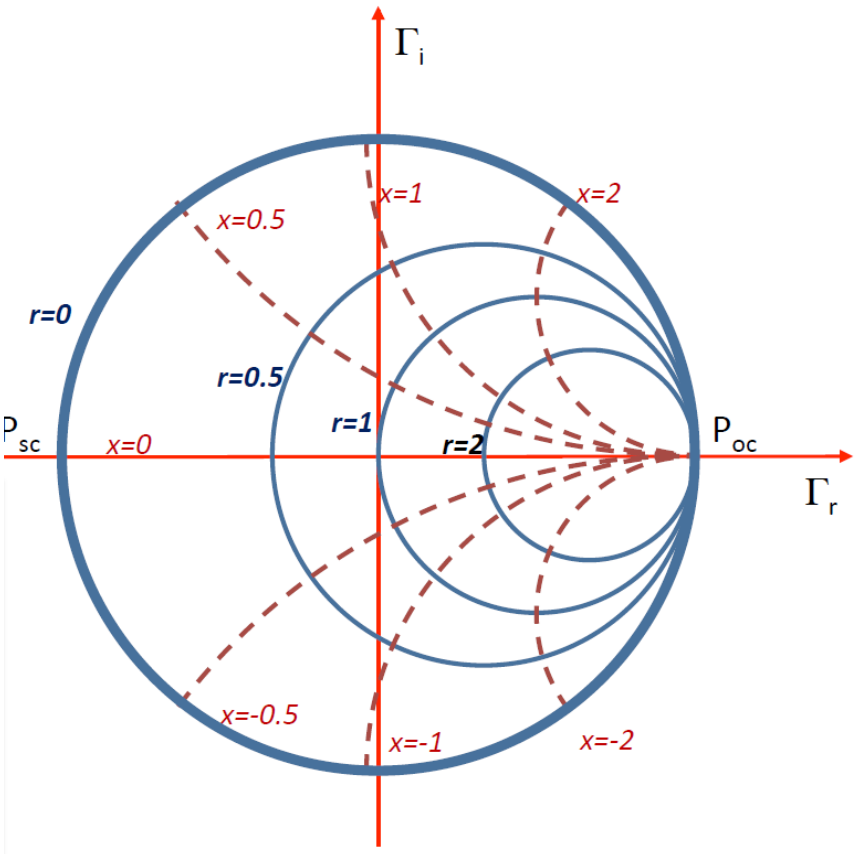
***r*-circles:**

$$\left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r}\right)^2.$$

***x*-circles:**

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2.$$





Smith Chart: Introduction

$$\left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r}\right)^2.$$

For the constant r circles:

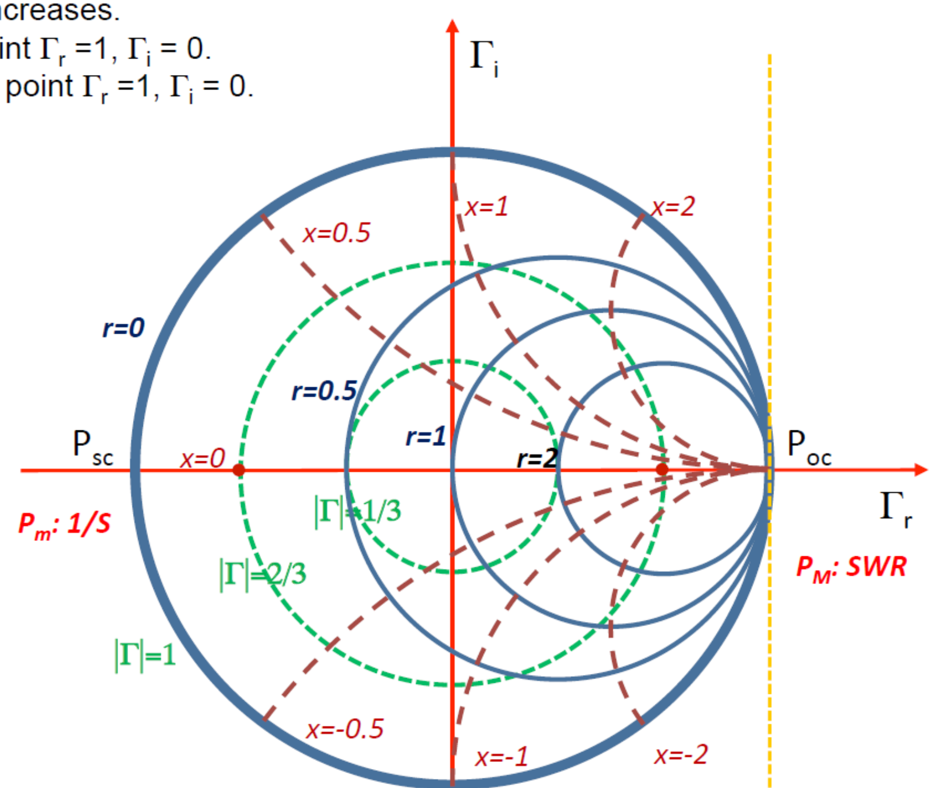
1. The centers of all the constant r circles are on the horizontal axis – real part of the reflection coefficient.
2. The radius of circles decreases when r increases.
3. All constant r circles pass through the point $\Gamma_r = 1$, $\Gamma_i = 0$.
4. The normalized resistance $r = \infty$ is at the point $\Gamma_r = 1$, $\Gamma_i = 0$.

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2.$$

For the constant x (partial) circles:

1. The centers of all the constant x circles are on the $\Gamma_r = 1$ line. The circles with $x > 0$ (inductive reactance) are above the Γ_r axis; the circles with $x < 0$ (capacitive) are below the Γ_r axis.
2. The radius of circles decreases when absolute value of x increases.
3. The normalized reactances $x = \pm\infty$ are at the point $\Gamma_r = 1$, $\Gamma_i = 0$

The constant r circles are orthogonal to the constant x circles at every intersection.

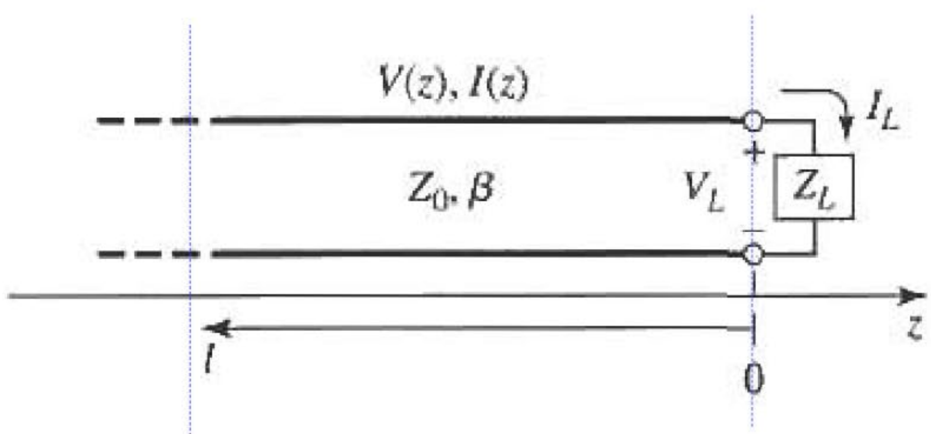


1. All $|\Gamma|$-circles are centered at the origin, and their radii vary uniformly from 0 to 1.
2. The angle, measured from the positive real axis, of the line drawn from the origin through the point representing z_L equals θ_Γ.
3. The value of the r-circle passing through the intersection of the $|\Gamma|$-circle and the positive real axis equals the standing-wave ratio S

Smith Chart: Introduction

Examples

EXAMPLE 9–14 A lossless transmission line of length 0.434λ and characteristic impedance 100 (Ω) is terminated in an impedance $260 + j180$ (Ω). Find (a) the voltage reflection coefficient, (b) the standing-wave ratio, (c) the input impedance, and (d) the location of a voltage maximum on the line.



Smith Chart: Introduction

Example 9-14

$l = 0.434\lambda$

$Z_0 = 100\Omega$

$Z_L = 260 + i180\Omega$

Find :

Γ_0

SWR

Z_{in}

$l_{\max}$