

Transmission Line Theory

Transmission line theory bridges the gap between field analysis and basic circuit theory and therefore is of significant importance in the analysis of microwave circuits and devices. As we will see, the phenomenon of wave propagation on transmission lines can be approached from an extension of circuit theory or from a specialization of Maxwell's equations; we shall present both viewpoints and show how this wave propagation is described by equations very similar to those used in Chapter 1 for plane wave propagation.

2.1

THE LUMPED-ELEMENT CIRCUIT MODEL FOR A TRANSMISSION LINE

The key difference between circuit theory and transmission line theory is electrical size. Circuit analysis assumes that the physical dimensions of the network are much smaller than the electrical wavelength, while transmission lines may be a considerable fraction of a wavelength, or many wavelengths, in size. Thus a transmission line is a *distributed-parameter* network, where voltages and currents can vary in magnitude and phase over its length, while ordinary circuit analysis deals with *lumped elements*, where voltage and current do not vary appreciably over the physical dimension of the elements.

As shown in Figure 2.1a, a transmission line is often schematically represented as a two-wire line since transmission lines (for transverse electromagnetic [TEM] wave propagation) always have at least two conductors. The piece of line of infinitesimal length Δz of Figure 2.1a can be modeled as a lumped-element circuit, as shown in Figure 2.1b, where R , L , G , and C are per-unit-length quantities defined as follows:

R = series resistance per unit length, for both conductors, in Ω/m .

L = series inductance per unit length, for both conductors, in H/m .

G = shunt conductance per unit length, in S/m .

C = shunt capacitance per unit length, in F/m .

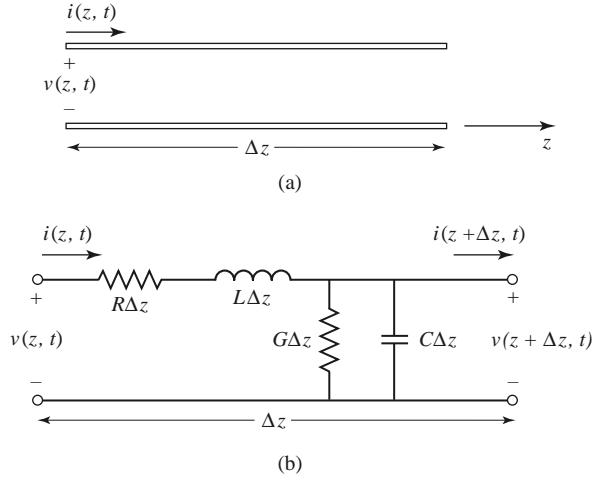


FIGURE 2.1 Voltage and current definitions and equivalent circuit for an incremental length of transmission line. (a) Voltage and current definitions. (b) Lumped-element equivalent circuit.

The series inductance L represents the total self-inductance of the two conductors, and the shunt capacitance C is due to the close proximity of the two conductors. The series resistance R represents the resistance due to the finite conductivity of the individual conductors, and the shunt conductance G is due to dielectric loss in the material between the conductors. R and G , therefore, represent loss. A finite length of transmission line can be viewed as a cascade of sections of the form shown in Figure 2.1b.

From the circuit of Figure 2.1b, Kirchhoff's voltage law can be applied to give

$$v(z, t) - R\Delta z i(z, t) - L\Delta z \frac{\partial i(z, t)}{\partial t} - v(z + \Delta z, t) = 0, \quad (2.1a)$$

and Kirchhoff's current law leads to

$$i(z, t) - G\Delta z v(z + \Delta z, t) - C\Delta z \frac{\partial v(z + \Delta z, t)}{\partial t} - i(z + \Delta z, t) = 0. \quad (2.1b)$$

Dividing (2.1a) and (2.1b) by Δz and taking the limit as $\Delta z \rightarrow 0$ gives the following differential equations:

$$\frac{\partial v(z, t)}{\partial z} = -Ri(z, t) - L \frac{\partial i(z, t)}{\partial t}, \quad (2.2a)$$

$$\frac{\partial i(z, t)}{\partial z} = -Gv(z, t) - C \frac{\partial v(z, t)}{\partial t}. \quad (2.2b)$$

These are the time domain form of the transmission line equations, also known as the *telegrapher equations*.

For the sinusoidal steady-state condition, with cosine-based phasors, (2.2a) and (2.2b) simplify to

$$\frac{dV(z)}{dz} = -(R + j\omega L)I(z), \quad (2.3a)$$

$$\frac{dI(z)}{dz} = -(G + j\omega C)V(z). \quad (2.3b)$$

Note the similarity in the form of (2.3a) and (2.3b) and Maxwell's curl equations of (1.41a) and (1.41b).

Wave Propagation on a Transmission Line

The two equations (2.3a) and (2.3b) can be solved simultaneously to give wave equations for $V(z)$ and $I(z)$:

$$\frac{d^2 V(z)}{dz^2} - \gamma^2 V(z) = 0, \quad (2.4a)$$

$$\frac{d^2 I(z)}{dz^2} - \gamma^2 I(z) = 0, \quad (2.4b)$$

where

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \quad (2.5)$$

is the complex propagation constant, which is a function of frequency. Traveling wave solutions to (2.4) can be found as

$$V(z) = V_o^+ e^{-\gamma z} + V_o^- e^{\gamma z}, \quad (2.6a)$$

$$I(z) = I_o^+ e^{-\gamma z} + I_o^- e^{\gamma z}, \quad (2.6b)$$

where the $e^{-\gamma z}$ term represents wave propagation in the $+z$ direction, and the $e^{\gamma z}$ term represents wave propagation in the $-z$ direction. Applying (2.3a) to the voltage of (2.6a) gives the current on the line:

$$I(z) = \frac{\gamma}{R + j\omega L} (V_o^+ e^{-\gamma z} - V_o^- e^{\gamma z}).$$

Comparison with (2.6b) shows that a *characteristic impedance*, Z_0 , can be defined as

$$Z_0 = \frac{R + j\omega L}{\gamma} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}, \quad (2.7)$$

to relate the voltage and current on the line as follows:

$$\frac{V_o^+}{I_o^+} = Z_0 = \frac{-V_o^-}{I_o^-}.$$

Then (2.6b) can be rewritten in the following form:

$$I(z) = \frac{V_o^+}{Z_0} e^{-\gamma z} - \frac{V_o^-}{Z_0} e^{\gamma z}. \quad (2.8)$$

Converting back to the time domain, we can express the voltage waveform as

$$\begin{aligned} v(z, t) = & |V_o^+| \cos(\omega t - \beta z + \phi^+) e^{-\alpha z} \\ & + |V_o^-| \cos(\omega t + \beta z + \phi^-) e^{\alpha z}, \end{aligned} \quad (2.9)$$

where ϕ^\pm is the phase angle of the complex voltage V_o^\pm . Using arguments similar to those in Section 1.4, we find that the wavelength on the line is

$$\lambda = \frac{2\pi}{\beta}, \quad (2.10)$$

and the phase velocity is

$$v_p = \frac{\omega}{\beta} = \lambda f. \quad (2.11)$$

The Lossless Line

The above solution is for a general transmission line, including loss effects, and it was seen that the propagation constant and characteristic impedance were complex. In many practical cases, however, the loss of the line is very small and so can be neglected, resulting in a simplification of the results. Setting $R = G = 0$ in (2.5) gives the propagation constant as

$$\gamma = \alpha + j\beta = j\omega\sqrt{LC},$$

or

$$\beta = \omega\sqrt{LC}, \quad (2.12a)$$

$$\alpha = 0. \quad (2.12b)$$

As expected for a lossless line, the attenuation constant α is zero. The characteristic impedance of (2.7) reduces to

$$Z_0 = \sqrt{\frac{L}{C}}, \quad (2.13)$$

which is now a real number. The general solutions for voltage and current on a lossless transmission line can then be written as

$$V(z) = V_o^+ e^{-j\beta z} + V_o^- e^{j\beta z}, \quad (2.14a)$$

$$I(z) = \frac{V_o^+}{Z_0} e^{-j\beta z} - \frac{V_o^-}{Z_0} e^{j\beta z}. \quad (2.14b)$$

The wavelength is

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{LC}}, \quad (2.15)$$

and the phase velocity is

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}. \quad (2.16)$$

2.2

FIELD ANALYSIS OF TRANSMISSION LINES

In this section we will rederive the time-harmonic form of the telegrapher's equations starting from Maxwell's equations. We will begin by deriving the transmission line parameters (R , L , G , C) in terms of the electric and magnetic fields of the transmission line and then derive the telegrapher equations using these parameters for the specific case of a coaxial line.

Transmission Line Parameters

Consider a 1 m length of a uniform transmission line with fields \vec{E} and \vec{H} , as shown in Figure 2.2, where S is the cross-sectional surface area of the line. Let the voltage between the conductors be $V_o e^{\pm j\beta z}$ and the current be $I_o e^{\pm j\beta z}$. The time-average stored magnetic

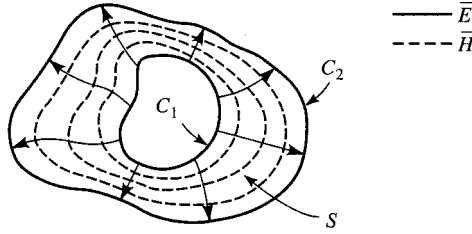


FIGURE 2.2 Field lines on an arbitrary TEM transmission line.

energy for this 1 m length of line can be written, from (1.86), as

$$W_m = \frac{\mu}{4} \int_S \bar{H} \cdot \bar{H}^* ds,$$

while circuit theory gives $W_m = L|I_o|^2/4$ in terms of the current on the line. We can thus identify the self-inductance per unit length as

$$L = \frac{\mu}{|I_o|^2} \int_S \bar{H} \cdot \bar{H}^* ds \text{ H/m.} \quad (2.17)$$

Similarly, the time-average stored electric energy per unit length can be found from (1.84) as

$$W_e = \frac{\epsilon}{4} \int_S \bar{E} \cdot \bar{E}^* ds,$$

while circuit theory gives $W_e = C|V_o|^2/4$, resulting in the following expression for the capacitance per unit length:

$$C = \frac{\epsilon}{|V_o|^2} \int_S \bar{E} \cdot \bar{E}^* ds \text{ F/m.} \quad (2.18)$$

From (1.131), the power loss per unit length due to the finite conductivity of the metallic conductors is

$$P_c = \frac{R_s}{2} \int_{C_1+C_2} \bar{H} \cdot \bar{H}^* d\ell$$

(assuming \bar{H} is tangential to S), while circuit theory gives $P_c = R|I_o|^2/2$, so the series resistance R per unit length of line is

$$R = \frac{R_s}{|I_o|^2} \int_{C_1+C_2} \bar{H} \cdot \bar{H}^* d\ell \text{ } \Omega/\text{m.} \quad (2.19)$$

In (2.19), $R_s = 1/\sigma\delta_s$ is the surface resistance of the conductors, and $C_1 + C_2$ represent integration paths over the conductor boundaries. From (1.92), the time-average power dissipated per unit length in a lossy dielectric is

$$P_d = \frac{\omega\epsilon''}{2} \int_S \bar{E} \cdot \bar{E}^* ds,$$

where ϵ'' is the imaginary part of the complex permittivity $\epsilon = \epsilon' - j\epsilon'' = \epsilon'(1 - j \tan \delta)$. Circuit theory gives $P_d = G|V_o|^2/2$, so the shunt conductance per unit length can be written as

$$G = \frac{\omega\epsilon''}{|V_o|^2} \int_S \bar{E} \cdot \bar{E}^* ds \text{ S/m.} \quad (2.20)$$

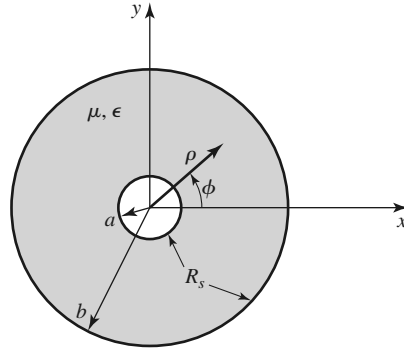
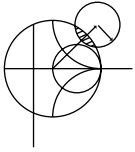


FIGURE 2.3 Geometry of a coaxial line with surface resistance R_s on the inner and outer conductors.



EXAMPLE 2.1 TRANSMISSION LINE PARAMETERS OF A COAXIAL LINE

The fields of a traveling TEM wave inside the coaxial line of Figure 2.3 can be expressed as

$$\begin{aligned}\bar{E} &= \frac{V_o \hat{\rho}}{\rho \ln b/a} e^{-\gamma z}, \\ \bar{H} &= \frac{I_o \hat{\phi}}{2\pi \rho} e^{-\gamma z},\end{aligned}$$

where γ is the propagation constant of the line. The conductors are assumed to have a surface resistivity R_s , and the material filling the space between the conductors is assumed to have a complex permittivity $\epsilon = \epsilon' - j\epsilon''$ and a permeability $\mu = \mu_0 \mu_r$. Determine the transmission line parameters.

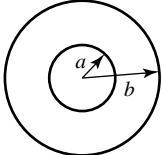
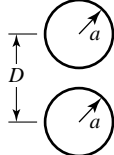
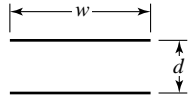
Solution

From (2.17)–(2.20) and the given fields the parameters of the coaxial line can be calculated as

$$\begin{aligned}L &= \frac{\mu}{(2\pi)^2} \int_{\phi=0}^{2\pi} \int_{\rho=a}^b \frac{1}{\rho^2} \rho d\rho d\phi = \frac{\mu}{2\pi} \ln b/a \text{ H/m}, \\ C &= \frac{\epsilon'}{(\ln b/a)^2} \int_{\phi=0}^{2\pi} \int_{\rho=a}^b \frac{1}{\rho^2} \rho d\rho d\phi = \frac{2\pi \epsilon'}{\ln b/a} \text{ F/m}, \\ R &= \frac{R_s}{(2\pi)^2} \left\{ \int_{\phi=0}^{2\pi} \frac{1}{a^2} a d\phi + \int_{\phi=0}^{2\pi} \frac{1}{b^2} b d\phi \right\} = \frac{R_s}{2\pi} \left(\frac{1}{a} + \frac{1}{b} \right) \Omega/\text{m}, \\ G &= \frac{\omega \epsilon''}{(\ln b/a)^2} \int_{\phi=0}^{2\pi} \int_{\rho=a}^b \frac{1}{\rho^2} \rho d\rho d\phi = \frac{2\pi \omega \epsilon''}{\ln b/a} \text{ S/m}. \quad \blacksquare\end{aligned}$$

Table 2.1 summarizes the parameters for coaxial, two-wire, and parallel plate lines. As we will see in the next chapter, the propagation constant, characteristic impedance, and attenuation of most transmission lines are usually derived directly from a field theory solution; the approach here of first finding the equivalent circuit parameters (L , C , R , G) is useful only for relatively simple lines. Nevertheless, it provides a helpful intuitive concept for understanding the properties of a transmission line and relates a transmission line to its equivalent circuit model.

TABLE 2.1 Transmission Line Parameters for Some Common Lines

	COAX	TWO-WIRE	PARALLEL PLATE
			
L	$\frac{\mu}{2\pi} \ln \frac{b}{a}$	$\frac{\mu}{\pi} \cosh^{-1} \left(\frac{D}{2a} \right)$	$\frac{\mu d}{w}$
C	$\frac{2\pi\epsilon'}{\ln b/a}$	$\frac{\pi\epsilon'}{\cosh^{-1}(D/2a)}$	$\frac{\epsilon' w}{d}$
R	$\frac{R_s}{2\pi} \left(\frac{1}{a} + \frac{1}{b} \right)$	$\frac{R_s}{\pi a}$	$\frac{2R_s}{w}$
G	$\frac{2\pi\omega\epsilon''}{\ln b/a}$	$\frac{\pi\omega\epsilon''}{\cosh^{-1}(D/2a)}$	$\frac{\omega\epsilon'' w}{d}$

The Telegrapher Equations Derived from Field Analysis of a Coaxial Line

We now show that the telegrapher equations of (2.3), derived using circuit theory, can also be obtained from Maxwell's equations. We will consider the specific geometry of the coaxial line of Figure 2.3. Although we will treat TEM wave propagation more generally in the next chapter, the present discussion should provide some insight into the relationship of circuit and field quantities.

A TEM wave on the coaxial line of Figure 2.3 will be characterized by $E_z = H_z = 0$; furthermore, due to azimuthal symmetry, the fields will have no ϕ variation, so $\partial/\partial\phi = 0$. The fields inside the coaxial line will satisfy Maxwell's curl equations,

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}, \quad (2.21a)$$

$$\nabla \times \vec{H} = j\omega\epsilon\vec{E}, \quad (2.21b)$$

where $\epsilon = \epsilon' - j\epsilon''$ may be complex to allow for a lossy dielectric filling. Conductor loss will be ignored here. A rigorous field analysis of conductor loss can be carried out but at this point would tend to obscure our purpose; the interested reader is referred to references [1] and [2].

Expanding (2.21a) and (2.21b) gives the following two vector equations:

$$-\hat{\rho}\frac{\partial E_\phi}{\partial z} + \hat{\phi}\frac{\partial E_\rho}{\partial z} + \hat{z}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\phi) = -j\omega\mu(\hat{\rho}H_\rho + \hat{\phi}H_\phi), \quad (2.22a)$$

$$-\hat{\rho}\frac{\partial H_\phi}{\partial z} + \hat{\phi}\frac{\partial H_\rho}{\partial z} + \hat{z}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\phi) = j\omega\epsilon(\hat{\rho}E_\rho + \hat{\phi}E_\phi). \quad (2.22b)$$

Since the \hat{z} components of these two equations must vanish, it is seen that E_ϕ and H_ϕ must have the forms

$$E_\phi = \frac{f(z)}{\rho}, \quad (2.23a)$$

$$H_\phi = \frac{g(z)}{\rho}. \quad (2.23b)$$

To satisfy the boundary condition that $E_\phi = 0$ at $\rho = a, b$, we must have $E_\phi = 0$ everywhere, due to the form of E_ϕ in (2.23a). Then from the $\hat{\rho}$ component of (2.22a), it is seen that $H_\rho = 0$. With these results, (2.22) can be reduced to

$$\frac{\partial E_\rho}{\partial z} = -j\omega\mu H_\phi, \quad (2.24a)$$

$$\frac{\partial H_\phi}{\partial z} = -j\omega\epsilon E_\rho. \quad (2.24b)$$

From the form of H_ϕ in (2.23b) and (2.24a), E_ρ must be of the form

$$E_\rho = \frac{h(z)}{\rho}. \quad (2.25)$$

Using (2.23b) and (2.25) in (2.24) gives

$$\frac{\partial h(z)}{\partial z} = -j\omega\mu g(z), \quad (2.26a)$$

$$\frac{\partial g(z)}{\partial z} = -j\omega\epsilon h(z). \quad (2.26b)$$

The voltage between the two conductors can be evaluated as

$$V(z) = \int_{\rho=a}^b E_\rho(\rho, z) d\rho = h(z) \int_{\rho=a}^b \frac{d\rho}{\rho} = h(z) \ln \frac{b}{a}, \quad (2.27a)$$

and the total current on the inner conductor at $\rho = a$ can be evaluated using (2.23b) as

$$I(z) = \int_{\phi=0}^{2\pi} H_\phi(a, z) a d\phi = 2\pi g(z). \quad (2.27b)$$

Then $h(z)$ and $g(z)$ can be eliminated from (2.26) by using (2.27) to give

$$\begin{aligned} \frac{\partial V(z)}{\partial z} &= -j \frac{\omega\mu \ln b/a}{2\pi} I(z), \\ \frac{\partial I(z)}{\partial z} &= -j\omega(\epsilon' - j\epsilon'') \frac{2\pi V(z)}{\ln b/a}. \end{aligned}$$

Finally, using the results for L , G , and C for a coaxial line as derived earlier, we obtain the telegrapher equations as

$$\frac{\partial V(z)}{\partial z} = -j\omega L I(z), \quad (2.28a)$$

$$\frac{\partial I(z)}{\partial z} = -(G + j\omega C) V(z). \quad (2.28b)$$

This result excludes R , the series resistance, since the conductors were assumed to have perfect conductivity. A similar analysis can be carried out for other simple transmission lines.

Propagation Constant, Impedance, and Power Flow for the Lossless Coaxial Line

Equations (2.24a) and (2.24b) for E_ρ and H_ϕ can be simultaneously solved to yield a wave equation for E_ρ (or H_ϕ):

$$\frac{\partial^2 E_\rho}{\partial z^2} + \omega^2 \mu \epsilon E_\rho = 0, \quad (2.29)$$

from which it is seen that the propagation constant is $\gamma^2 = -\omega^2 \mu \epsilon$, which, for lossless media, reduces to

$$\beta = \omega \sqrt{\mu \epsilon} = \omega \sqrt{LC}, \quad (2.30)$$

where the last result is from (2.12). Observe that this propagation constant is of the same form as that for plane waves in a lossless dielectric medium. This is a general result for TEM transmission lines.

The wave impedance for the coaxial line is defined as $Z_w = E_\rho / H_\phi$, which can be calculated from (2.24a), assuming an $e^{-j\beta z}$ dependence, to give

$$Z_w = \frac{E_\rho}{H_\phi} = \frac{\omega \mu}{\beta} = \sqrt{\mu / \epsilon} = \eta. \quad (2.31)$$

This wave impedance is seen to be identical to the intrinsic impedance of the medium, η , and is a general result for TEM transmission lines.

The characteristic impedance of the coaxial line is defined as

$$Z_0 = \frac{V_o}{I_o} = \frac{E_\rho \ln b/a}{2\pi H_\phi} = \frac{\eta \ln b/a}{2\pi} = \sqrt{\frac{\mu}{\epsilon}} \frac{\ln b/a}{2\pi}, \quad (2.32)$$

where the forms for E_ρ and H_ϕ from Example 2.1 have been used. The characteristic impedance is geometry dependent and will be different for other transmission line configurations.

Finally, the power flow (in the z direction) on the coaxial line may be computed from the Poynting vector as

$$P = \frac{1}{2} \int_s \bar{E} \times \bar{H}^* \cdot d\bar{s} = \frac{1}{2} \int_{\phi=0}^{2\pi} \int_{\rho=a}^b \frac{V_o I_o^*}{2\pi \rho^2 \ln b/a} \rho d\rho d\phi = \frac{1}{2} V_o I_o^*, \quad (2.33)$$

a result that is in clear agreement with circuit theory. This shows that the flow of power in a transmission line takes place entirely via the electric and magnetic fields between the two conductors; power is not transmitted through the conductors themselves. As we will see later, for the case of finite conductivity, power may enter the conductors, but this power is then lost as heat and is not delivered to the load.

2.3

THE TERMINATED LOSSLESS TRANSMISSION LINE

Figure 2.4 shows a lossless transmission line terminated in an arbitrary load impedance Z_L . This problem will illustrate wave reflection on transmission lines, a fundamental property of distributed systems.

Assume that an incident wave of the form $V_o^+ e^{-j\beta z}$ is generated from a source at $z < 0$. We have seen that the ratio of voltage to current for such a traveling wave is Z_0 , the characteristic impedance of the line. However, when the line is terminated in an arbitrary load $Z_L \neq Z_0$, the ratio of voltage to current at the load must be Z_L . Thus, a reflected wave

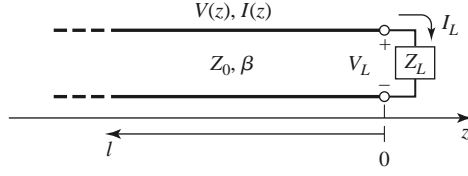


FIGURE 2.4 A transmission line terminated in a load impedance Z_L .

must be excited with the appropriate amplitude to satisfy this condition. The total voltage on the line can then be written as in (2.14a), as a sum of incident and reflected waves:

$$V(z) = V_o^+ e^{-j\beta z} + V_o^- e^{j\beta z}. \quad (2.34a)$$

Similarly, the total current on the line is described by (2.14b):

$$I(z) = \frac{V_o^+}{Z_0} e^{-j\beta z} - \frac{V_o^-}{Z_0} e^{j\beta z}. \quad (2.34b)$$

The total voltage and current at the load are related by the load impedance, so at $z = 0$ we must have

$$Z_L = \frac{V(0)}{I(0)} = \frac{V_o^+ + V_o^-}{V_o^+ - V_o^-} Z_0.$$

Solving for V_o^- gives

$$V_o^- = \frac{Z_L - Z_0}{Z_L + Z_0} V_o^+.$$

The amplitude of the reflected voltage wave normalized to the amplitude of the incident voltage wave is defined as the *voltage reflection coefficient*, Γ :

$$\Gamma = \frac{V_o^-}{V_o^+} = \frac{Z_L - Z_0}{Z_L + Z_0}. \quad (2.35)$$

The total voltage and current waves on the line can then be written as

$$V(z) = V_o^+ (e^{-j\beta z} + \Gamma e^{j\beta z}), \quad (2.36a)$$

$$I(z) = \frac{V_o^+}{Z_0} (e^{-j\beta z} - \Gamma e^{j\beta z}). \quad (2.36b)$$

From these equations it is seen that the voltage and current on the line consist of a superposition of an incident and a reflected wave; such waves are called *standing waves*. Only when $\Gamma = 0$ is there no reflected wave. To obtain $\Gamma = 0$, the load impedance Z_L must be equal to the characteristic impedance Z_0 of the transmission line, as seen from (2.35). Such a load is said to be *matched* to the line since there is no reflection of the incident wave.

Now consider the time-average power flow along the line at the point z :

$$P_{\text{avg}} = \frac{1}{2} \text{Re}\{V(z)I(z)^*\} = \frac{1}{2} \frac{|V_o^+|^2}{Z_0} \text{Re}\{1 - \Gamma^* e^{-2j\beta z} + \Gamma e^{2j\beta z} - |\Gamma|^2\},$$

where (2.36) has been used. The middle two terms in the brackets are of the form $A - A^* = 2j \text{Im}\{A\}$ and so are purely imaginary. This simplifies the result to

$$P_{\text{avg}} = \frac{1}{2} \frac{|V_o^+|^2}{Z_0} (1 - |\Gamma|^2), \quad (2.37)$$

which shows that the average power flow is constant at any point on the line and that the total power delivered to the load (P_{avg}) is equal to the incident power ($|V_o^+|^2/2Z_0$) minus the reflected power ($|V_o^-|^2/2Z_0$). If $\Gamma = 0$, maximum power is delivered to the load, while no power is delivered for $|\Gamma| = 1$. The above discussion assumes that the generator is matched, so that there is no re-reflection of the reflected wave from $z < 0$.

When the load is mismatched, not all of the available power from the generator is delivered to the load. This “loss” is called *return loss* (RL), and is defined (in dB) as

$$\text{RL} = -20 \log |\Gamma| \text{ dB}, \quad (2.38)$$

so that a matched load ($\Gamma = 0$) has a return loss of ∞ dB (no reflected power), while a total reflection ($|\Gamma| = 1$) has a return loss of 0 dB (all incident power is reflected). Note that return loss is a nonnegative number for reflection from a passive network.

If the load is matched to the line, $\Gamma = 0$ and the magnitude of the voltage on the line is $|V(z)| = |V_o^+|$, which is a constant. Such a line is sometimes said to be *flat*. When the load is mismatched, however, the presence of a reflected wave leads to standing waves, and the magnitude of the voltage on the line is not constant. Thus, from (2.36a),

$$\begin{aligned} |V(z)| &= |V_o^+| |1 + \Gamma e^{2j\beta z}| = |V_o^+| |1 + \Gamma e^{-2j\beta \ell}| \\ &= |V_o^+| |1 + |\Gamma| e^{j(\theta - 2\beta \ell)}|, \end{aligned} \quad (2.39)$$

where $\ell = -z$ is the positive distance measured from the load at $z = 0$, and θ is the phase of the reflection coefficient ($\Gamma = |\Gamma|e^{j\theta}$). This result shows that the voltage magnitude oscillates with position z along the line. The maximum value occurs when the phase term $e^{j(\theta - 2\beta \ell)} = 1$ and is given by

$$V_{\text{max}} = |V_o^+|(1 + |\Gamma|). \quad (2.40a)$$

The minimum value occurs when the phase term $e^{j(\theta - 2\beta \ell)} = -1$ and is given by

$$V_{\text{min}} = |V_o^+|(1 - |\Gamma|). \quad (2.40b)$$

As $|\Gamma|$ increases, the ratio of V_{max} to V_{min} increases, so a measure of the mismatch of a line, called the *standing wave ratio* (SWR), can be defined as

$$\text{SWR} = \frac{V_{\text{max}}}{V_{\text{min}}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}. \quad (2.41)$$

This quantity is also known as the *voltage standing wave ratio* and is sometimes identified as VSWR. From (2.41) it is seen that SWR is a real number such that $1 \leq \text{SWR} \leq \infty$, where $\text{SWR} = 1$ implies a matched load.

From (2.39), it is seen that the distance between two successive voltage maxima (or minima) is $\ell = 2\pi/2\beta = \pi\lambda/2\pi = \lambda/2$, while the distance between a maximum and a minimum is $\ell = \pi/2\beta = \lambda/4$, where λ is the wavelength on the transmission line.

The reflection coefficient of (2.35) was defined as the ratio of the reflected to the incident voltage wave amplitudes at the load ($\ell = 0$), but this quantity can be generalized to any point ℓ along the line as follows. From (2.34a), with $z = -\ell$, the ratio of the reflected component to the incident component is

$$\Gamma(\ell) = \frac{V_o^- e^{-j\beta \ell}}{V_o^+ e^{j\beta \ell}} = \Gamma(0) e^{-2j\beta \ell}, \quad (2.42)$$

where $\Gamma(0)$ is the reflection coefficient at $z = 0$, as given by (2.35). This result is useful when transforming the effect of a load mismatch down the line.

We have seen that the real power flow on the line is a constant (for a lossless line) but that the voltage amplitude, at least for a mismatched line, is oscillatory with position on the line. The perceptive reader may therefore have concluded that the impedance seen looking into the line must vary with position, and this is indeed the case. At a distance $\ell = -z$ from the load, the input impedance seen looking toward the load is

$$Z_{\text{in}} = \frac{V(-\ell)}{I(-\ell)} = \frac{V_o^+ (e^{j\beta\ell} + \Gamma e^{-j\beta\ell})}{V_o^+ (e^{j\beta\ell} - \Gamma e^{-j\beta\ell})} Z_0 = \frac{1 + \Gamma e^{-2j\beta\ell}}{1 - \Gamma e^{-2j\beta\ell}} Z_0, \quad (2.43)$$

where (2.36a,b) have been used for $V(z)$ and $I(z)$. A more usable form may be obtained by using (2.35) for Γ in (2.43):

$$\begin{aligned} Z_{\text{in}} &= Z_0 \frac{(Z_L + Z_0)e^{j\beta\ell} + (Z_L - Z_0)e^{-j\beta\ell}}{(Z_L + Z_0)e^{j\beta\ell} - (Z_L - Z_0)e^{-j\beta\ell}} \\ &= Z_0 \frac{Z_L \cos \beta\ell + j Z_0 \sin \beta\ell}{Z_0 \cos \beta\ell + j Z_L \sin \beta\ell} \\ &= Z_0 \frac{Z_L + j Z_0 \tan \beta\ell}{Z_0 + j Z_L \tan \beta\ell}. \end{aligned} \quad (2.44)$$

This is an important result giving the input impedance of a length of transmission line with an arbitrary load impedance. We will refer to this result as the *transmission line impedance equation*; some special cases will be considered next.

Special Cases of Lossless Terminated Lines

A number of special cases of lossless terminated transmission lines will frequently appear in our work, so it is appropriate to consider the properties of such cases here.

Consider first the transmission line circuit shown in Figure 2.5, where a line is terminated in a short circuit, $Z_L = 0$. From (2.35) it is seen that the reflection coefficient for a short circuit load is $\Gamma = -1$; it then follows from (2.41) that the standing wave ratio is infinite. From (2.36) the voltage and current on the line are

$$V(z) = V_o^+ (e^{-j\beta z} - e^{j\beta z}) = -2j V_o^+ \sin \beta z, \quad (2.45a)$$

$$I(z) = \frac{V_o^+}{Z_0} (e^{-j\beta z} + e^{j\beta z}) = \frac{2V_o^+}{Z_0} \cos \beta z, \quad (2.45b)$$

which shows that $V = 0$ at the load (as expected, for a short circuit), while the current is a maximum there. From (2.44), or the ratio $V(-\ell)/I(-\ell)$, the input impedance is

$$Z_{\text{in}} = j Z_0 \tan \beta\ell, \quad (2.45c)$$

which is seen to be purely imaginary for any length ℓ and to take on all values between $+j\infty$ and $-j\infty$. For example, when $\ell = 0$ we have $Z_{\text{in}} = 0$, but for $\ell = \lambda/4$ we have $Z_{\text{in}} = \infty$ (open circuit). Equation (2.45c) also shows that the impedance is periodic in ℓ ,

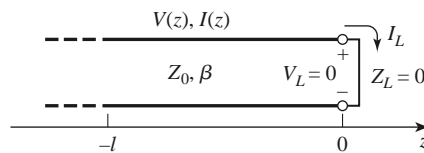


FIGURE 2.5 A transmission line terminated in a short circuit.

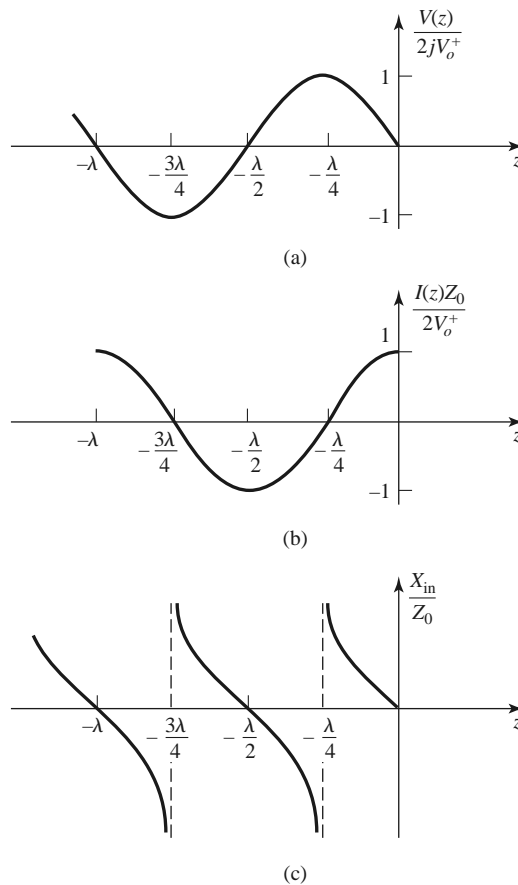


FIGURE 2.6 (a) Voltage, (b) current, and (c) impedance ($R_{in} = 0$ or ∞) variation along a short-circuited transmission line.

repeating for multiples of $\lambda/2$. The voltage, current, and input reactance for the short-circuited line are plotted in Figure 2.6.

Next consider the open-circuited line shown in Figure 2.7, where $Z_L = \infty$. Dividing the numerator and denominator of (2.35) by Z_L and allowing $Z_L \rightarrow \infty$ shows that the reflection coefficient for this case is $\Gamma = 1$, and the standing wave ratio is again infinite. From (2.36) the voltage and current on the line are

$$V(z) = V_o^+ (e^{-j\beta z} + e^{j\beta z}) = 2V_o^+ \cos \beta z, \quad (2.46a)$$

$$I(z) = \frac{V_o^+}{Z_0} (e^{-j\beta z} - e^{j\beta z}) = \frac{-2jV_o^+}{Z_0} \sin \beta z, \quad (2.46b)$$

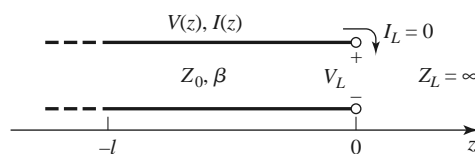


FIGURE 2.7 A transmission line terminated in an open circuit.

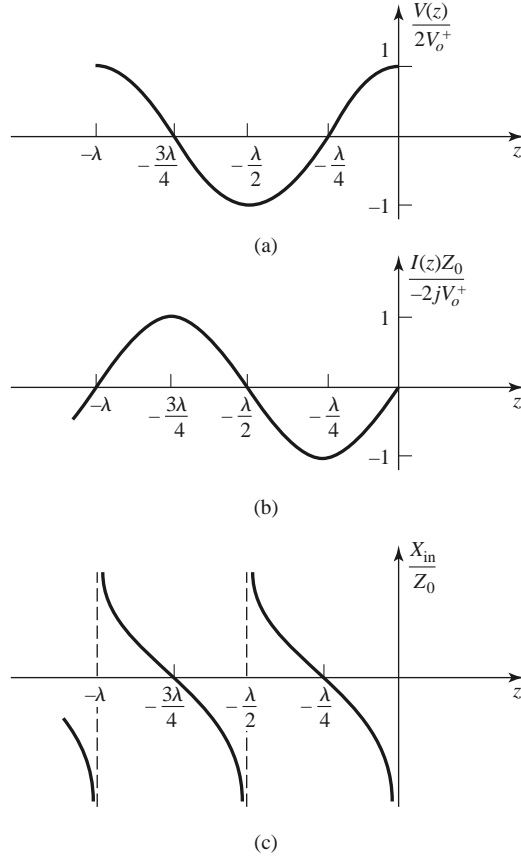


FIGURE 2.8 (a) Voltage, (b) current, and (c) impedance ($R_{\text{in}} = 0$ or ∞) variation along an open-circuited transmission line.

which shows that now $I = 0$ at the load, as expected for an open circuit, while the voltage is a maximum. The input impedance is

$$Z_{\text{in}} = -jZ_0 \cot \beta \ell, \quad (2.46c)$$

which is also purely imaginary for any length, ℓ . The voltage, current, and input reactance of the open-circuited line are plotted in Figure 2.8.

Now consider terminated transmission lines with some special lengths. If $\ell = \lambda/2$, (2.44) shows that

$$Z_{\text{in}} = Z_L, \quad (2.47)$$

meaning that a half-wavelength line (or any multiple of $\lambda/2$) does not alter or transform the load impedance, regardless of its characteristic impedance.

If the line is a quarter-wavelength long or, more generally, $\ell = \lambda/4 + n\lambda/2$, for $n = 1, 2, 3, \dots$, (2.44) shows that the input impedance is given by

$$Z_{\text{in}} = \frac{Z_0^2}{Z_L}. \quad (2.48)$$

Such a line is known as a *quarter-wave transformer* because it has the effect of transforming the load impedance in an inverse manner, depending on the characteristic impedance of the line. We will study this case more thoroughly in Section 2.5.

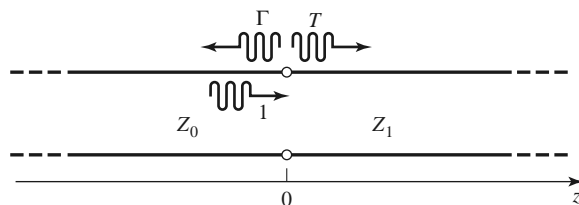


FIGURE 2.9 Reflection and transmission at the junction of two transmission lines with different characteristic impedances.

Next consider a transmission line of characteristic impedance Z_0 feeding a line of different characteristic impedance, Z_1 , as shown in Figure 2.9. If the load line is infinitely long, or if it is terminated in its own characteristic impedance, so that there are no reflections from its far end, then the input impedance seen by the feed line is Z_1 , so that the reflection coefficient Γ is

$$\Gamma = \frac{Z_1 - Z_0}{Z_1 + Z_0}. \quad (2.49)$$

Not all of the incident wave is reflected; some is transmitted onto the second line with a voltage amplitude given by a transmission coefficient.

From (2.36a) the voltage for $z < 0$ is

$$V(z) = V_o^+ (e^{-j\beta z} + \Gamma e^{j\beta z}), \quad z < 0, \quad (2.50a)$$

where V_o^+ is the amplitude of the incident voltage wave on the feed line. The voltage wave for $z > 0$, in the absence of reflections, is outgoing only and can be written as

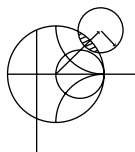
$$V(z) = V_o^+ T e^{-j\beta z} \quad \text{for } z > 0. \quad (2.50b)$$

Equating these voltages at $z = 0$ gives the *transmission coefficient*, T , as

$$T = 1 + \Gamma = 1 + \frac{Z_1 - Z_0}{Z_1 + Z_0} = \frac{2Z_1}{Z_1 + Z_0}. \quad (2.51)$$

The transmission coefficient between two points in a circuit is often expressed in dB as the *insertion loss*, IL,

$$\text{IL} = -20 \log |T| \text{ dB}. \quad (2.52)$$



POINT OF INTEREST: Decibels and Nepers

Often the ratio of two power levels P_1 and P_2 in a microwave system is expressed in decibels (dB) as

$$10 \log \frac{P_1}{P_2} \text{ dB}.$$

Thus, a power ratio of 2 is equivalent to 3 dB, while a power ratio of 0.1 is equivalent to -10 dB. Using power ratios in dB makes it easy to calculate power loss or gain through a series of components since multiplicative loss or gain factors can be accounted for by adding the loss or gain in dB for each stage. For example, a signal passing through a 6 dB attenuator followed by a 23 dB amplifier will have an overall gain of $23 - 6 = 17$ dB.

Decibels are used only to represent power ratios, but if $P_1 = V_1^2/R_1$ and $P_2 = V_2^2/R_2$, then the resulting power ratio in terms of voltage ratios is

$$10 \log \frac{V_1^2 R_2}{V_2^2 R_1} = 20 \log \frac{V_1}{V_2} \sqrt{\frac{R_2}{R_1}} \text{ dB},$$

where R_1 , R_2 are the load resistances and V_1 , V_2 are the voltages appearing across these loads. If the load resistances are equal, then this formula simplifies to

$$20 \log \frac{V_1}{V_2} \text{ dB}.$$

The ratio of voltages across equal load resistances can also be expressed in terms of nepers (Np) as

$$\ln \frac{V_1}{V_2} \text{ Np}.$$

The corresponding expression in terms of powers is

$$\frac{1}{2} \ln \frac{P_1}{P_2} \text{ Np},$$

since voltage is proportional to the square root of power. Transmission line attenuation is sometimes expressed in nepers. Since 1 Np corresponds to a power ratio of e^2 , the conversion between nepers and decibels is

$$1 \text{ Np} = 10 \log e^2 = 8.686 \text{ dB}.$$

Absolute power can also be expressed in decibel notation if a reference power level is assumed. If we let $P_2 = 1 \text{ mW}$, then the power P_1 can be expressed in dBm as

$$10 \log \frac{P_1}{1 \text{ mW}} \text{ dBm}$$

Thus a power of 1 mW is equivalent to 0 dBm, while a power of 1 W is equivalent to 30 dBm, and so on.