Chapter 1

VECTOR ALGEBRA

One thing I have learned in a long life: that all our science, measured against reality, is primitive and childlike—and yet is the most precious thing we have.

—ALBERT EINSTEIN

1.1 INTRODUCTION

Electromagnetics (EM) may be regarded as the study of the interactions between electric charges at rest and in motion. It entails the analysis, synthesis, physical interpretation, and application of electric and magnetic fields.

Electromagnetics (EM) is a branch of physics or electrical engineering in which electric and magnetic phenomena are studied.

EM principles find applications in various allied disciplines such as microwaves, antennas, electric machines, satellite communications, bioelectromagnetics, plasmas, nuclear research, fiber optics, electromagnetic interference and compatibility, electromechanical energy conversion, radar meteorology, and remote sensing.[1,2] In physical medicine, for example, EM power, either in the form of shortwaves or microwaves, is used to heat deep tissues and to stimulate certain physiological responses in order to relieve certain pathological conditions. EM fields are used in induction heaters for melting, forging, annealing, surface hardening, and soldering operations. Dielectric heating equipment uses shortwaves to join or seal thin sheets of plastic materials. EM energy offers many new and exciting possibilities in agriculture. It is used, for example, to change vegetable taste by reducing acidity.

EM devices include transformers, electric relays, radio/TV, telephone, electric motors, transmission lines, waveguides, antennas, optical fibers, radars, and lasers. The design of these devices requires thorough knowledge of the laws and principles of EM.

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[1] For numerous applications of electrostatics, see J. M. Crowley, *Fundamentals of Applied Electrostatics.* New York: John Wiley & Sons, 1986.

[2] For other areas of applications of EM, see, for example, D. Teplitz, ed., *Electromagnetism: Paths to Research.* New York: Plenum Press, 1982.

1.3 SCALARS AND VECTORS

Vector analysis is a mathematical tool with which electromagnetic (EM) concepts are most conveniently expressed and best comprehended. We must first learn its rules and techniques before we can confidently apply it. Since most students taking this course have little exposure to vector analysis, considerable attention is given to it in this and the next two chapters.[3] This chapter introduces the basic concepts of vector algebra in Cartesian coordinates only. The next chapter builds on this and extends to other coordinate systems.

A quantity can be either a scalar or a vector.

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†Indicates sections that may be skipped, explained briefly, or assigned as homework if the text is covered in one semester.

[3]The reader who feels no need for review of vector algebra can skip to the next chapter.

A scalar is a quantity that has only magnitude.

Quantities such as time, mass, distance, temperature, entropy, electric potential, and population are scalars.

A vector is a quantity that has both magnitude and direction.

Vector quantities include velocity, force, displacement, and electric field intensity. Another class of physical quantities is called *tensors*, of which scalars and vectors are special cases. For most of the time, we shall be concerned with scalars and vectors.[4]

To distinguish between a scalar and a vector it is customary to represent a vector by a letter with an arrow on top of it, such as $\vec{A}$ and $\vec{B}$, or by a letter in boldface type such as **A** and **B.** A scalar is represented simply by a letter—e.g., A, B, U, and V.

EM theory is essentially a study of some particular fields.

A field is a function that specifies a particular quantity everywhere in a region.

If the quantity is scalar (or vector), the field is said to be a scalar (or vector) field. Examples of scalar fields are temperature distribution in a building, sound intensity in a theater, electric potential in a region, and refractive index of a stratified medium. The gravitational force on a body in space and the velocity of raindrops in the atmosphere are examples of vector fields.

1.4 UNIT VECTOR

A vector **A** has both magnitude and direction. The *magnitude* of **A** is a scalar written as A or $|\mathbf{A}|$. A *unit vector* $\mathbf{a}_A$ along **A** is defined as a vector whose magnitude is unity (i.e., 1) and its direction is along **A**, that is,

$$\boxed{\mathbf{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{A}}{A}} \tag{1.5}$$

Note that $|\mathbf{a}_A| = 1$. Thus we may write **A** as

$$\mathbf{A} = A\mathbf{a}_A \tag{1.6}$$

which completely specifies **A** in terms of its magnitude A and its direction $\mathbf{a}_A$.

A vector **A** in Cartesian (or rectangular) coordinates may be represented as

$$(A_x, A_y, A_z) \qquad \text{or} \qquad A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z \tag{1.7}$$

[4]For an elementary treatment of tensors, see, for example, A. I. Borisenko and I. E. Tarapor, *Vector and Tensor Analysis with Application.* Englewood Cliffs, NJ: Prentice-Hall, 1968.

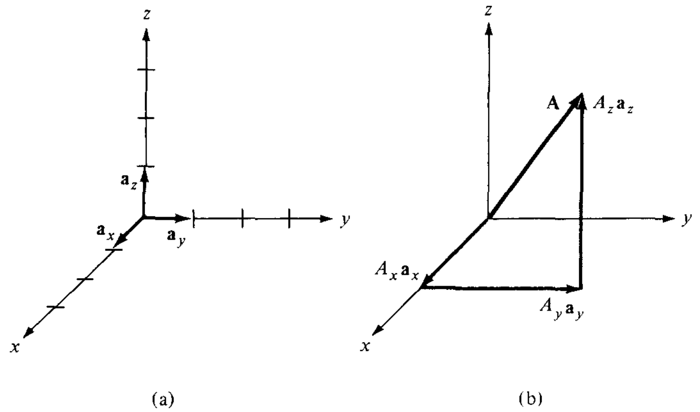


Figure 1.1 **(a)** Unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$, **(b)** components of **A** along $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$.

where A_x, A_y, and A_z are called the *components of* **A** in the x, y, and z directions respectively; $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ are unit vectors in the x, y, and z directions, respectively. For example, $\mathbf{a}_x$ is a dimensionless vector of magnitude one in the direction of the increase of the x-axis. The unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ are illustrated in Figure 1.1(a), and the components of **A** along the coordinate axes are shown in Figure 1.1(b). The magnitude of vector **A** is given by

$$\boxed{A = \sqrt{A_x^2 + A_y^2 + A_z^2}}$$ in rectangular coordinates

and the unit vector along **A** is given by

$$\mathbf{a}_A = \frac{A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \tag{1.9}$$

1.5 VECTOR ADDITION AND SUBTRACTION

Two vectors **A** and **B** can be added together to give another vector **C**; that is,

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \tag{1.10}$$

The vector addition is carried out component by component. Thus, if $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$.

$$\mathbf{C} = (A_x + B_x)\mathbf{a}_x + (A_y + B_y)\mathbf{a}_y + (A_z + B_z)\mathbf{a}_z \tag{1.11}$$

Vector subtraction is similarly carried out as

$$\begin{aligned}\mathbf{D} &= \mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) \\ &= (A_x - B_x)\mathbf{a}_x + (A_y - B_y)\mathbf{a}_y + (A_z - B_z)\mathbf{a}_z\end{aligned} \tag{1.12}$$

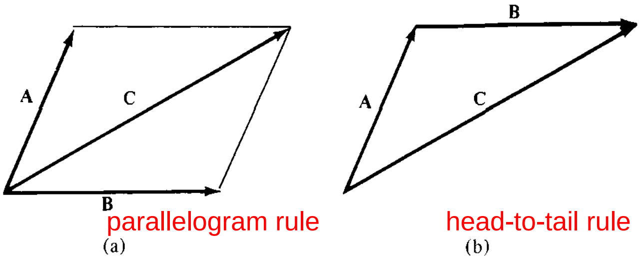


Figure 1.2 Vector addition $\mathbf{C} = \mathbf{A} + \mathbf{B}$: **(a)** parallelogram rule, **(b)** head-to-tail rule.

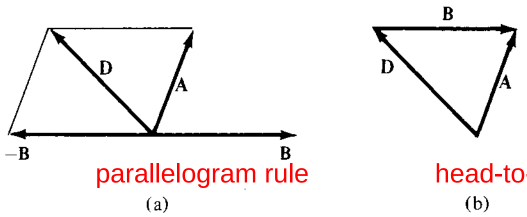


Figure 1.3 Vector subtraction $\mathbf{D} = \mathbf{A} - \mathbf{B}$: **(a)** parallelogram rule, **(b)** head-to-tail rule.

Graphically, vector addition and subtraction are obtained by either the parallelogram rule or the head-to-tail rule as portrayed in Figures 1.2 and 1.3, respectively.

The three basic laws of algebra obeyed by any given~~y~~ vectors **A**, **B**, and **C**, are summarized as follows:

Law	Addition	Multiplication
Commutative	$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$	$k\mathbf{A} = \mathbf{A}k$
Associative	$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$	$k(\ell\mathbf{A}) = (k\ell)\mathbf{A}$
Distributive	$k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}$	

where k and ℓ are scalars. Multiplication of a vector with another vector will be discussed in Section 1.7.

1.6 POSITION AND DISTANCE VECTORS

A point P in Cartesian coordinates may be represented by (x, y, z).

The **position vector** $\mathbf{r}_P$ (or **radius vector**) of point P is as the directed sitance from the origin O to P; i.e.,

$$\mathbf{r}_P = OP = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z \tag{1.13}$$

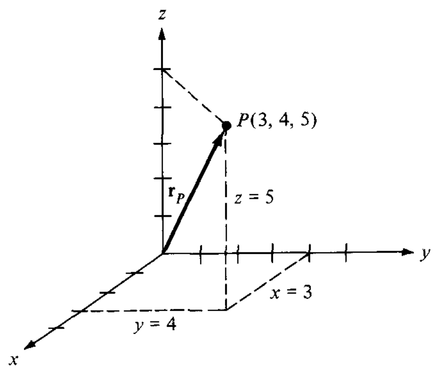


Figure 1.4 Illustration of position vector $\mathbf{r}_P = 3\mathbf{a}_x + 4\mathbf{a}_y + 5\mathbf{a}_z$.

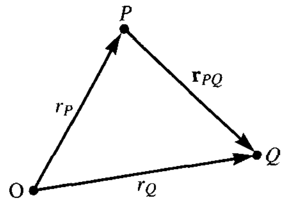


Figure 1.5 Distance vector $\mathbf{r}_{PQ}$.

The position vector of point P is useful in defining its position in space. Point (3, 4, 5), for example, and its position vector $3\mathbf{a}_x + 4\mathbf{a}_y + 5\mathbf{a}_z$ are shown in Figure 1.4.

The **distance vector** is the displacement from one point to another.

If two points P and Q are given by (x_P, y_P, z_P) and (x_Q, y_Q, z_Q), the *distance vector* (or *separation vector*) is the displacement from P to Q as shown in Figure 1.5; that is,

$$\begin{aligned}\mathbf{r}_{PQ} &= r_Q - r_P \\ &= (x_Q - x_P)\mathbf{a}_x + (y_Q - y_P)\mathbf{a}_y + (z_Q - z_P)\mathbf{a}_z\end{aligned} \tag{1.14}$$

The difference between a point P and a vector $\mathbf{A}$ should be noted. Though both P and $\mathbf{A}$ may be represented in the same manner as (x, y, z) and (A_x, A_y, A_z), respectively, the point P is not a vector; only its position vector $\mathbf{r}_P$ is a vector. Vector $\mathbf{A}$ may depend on point P, however. For example, if $\mathbf{A} = 2xy\mathbf{a}_x + y^2\mathbf{a}_y - xz^2\mathbf{a}_z$ and P is $(2, -1, 4)$, then $\mathbf{A}$ at P would be $-4\mathbf{a}_x + \mathbf{a}_y - 32\mathbf{a}_z$. A vector field is said to be *constant* or *uniform* if it does not depend on space variables x, y, and z. For example, vector $\mathbf{B} = 3\mathbf{a}_x - 2\mathbf{a}_y + 10\mathbf{a}_z$ is a uniform vector while vector $\mathbf{A} = 2xy\mathbf{a}_x + y^2\mathbf{a}_y - xz^2\mathbf{a}_z$ is not uniform because $\mathbf{B}$ is the same everywhere whereas $\mathbf{A}$ varies from point to point.

EXAMPLE 1.1

If $\mathbf{A} = 10\mathbf{a}_x - 4\mathbf{a}_y + 6\mathbf{a}_z$ and $\mathbf{B} = 2\mathbf{a}_x + \mathbf{a}_y$, find: (a) the component of $\mathbf{A}$ along $\mathbf{a}_y$, (b) the magnitude of $3\mathbf{A} - \mathbf{B}$, (c) a unit vector along $\mathbf{A} + 2\mathbf{B}$.

Solution:

(a) The component of **A** along $\mathbf{a}_y$ is $A_y = -4$.

(b) $$\begin{aligned} 3\mathbf{A} - \mathbf{B} &= 3(10, -4, 6) - (2, 1, 0) \\ &= (30, -12, 18) - (2, 1, 0) \\ &= (28, -13, 18) \end{aligned}$$

Hence

$$\begin{aligned} |3\mathbf{A} - \mathbf{B}| &= \sqrt{28^2 + (-13)^2 + (18)^2} = \sqrt{1277} \\ &= 35.74 \end{aligned}$$

(c) Let $\mathbf{C} = \mathbf{A} + 2\mathbf{B} = (10, -4, 6) + (4, 2, 0) = (14, -2, 6)$.

A unit vector along **C** is

$$\mathbf{a}_c = \frac{\mathbf{C}}{|\mathbf{C}|} = \frac{(14, -2, 6)}{\sqrt{14^2 + (-2)^2 + 6^2}}$$

or

$$\mathbf{a}_c = 0.9113\mathbf{a}_x - 0.1302\mathbf{a}_y + 0.3906\mathbf{a}_z$$

Note that $|\mathbf{a}_c| = 1$ as expected.

PRACTICE EXERCISE 1.1

Given vectors $\mathbf{A} = \mathbf{a}_x + 3\mathbf{a}_z$ and $\mathbf{B} = 5\mathbf{a}_x + 2\mathbf{a}_y - 6\mathbf{a}_z$, determine

(a) $|\mathbf{A} + \mathbf{B}|$

(b) $5\mathbf{A} - \mathbf{B}$

(c) The component of **A** along $\mathbf{a}_y$

(d) A unit vector parallel to $3\mathbf{A} + \mathbf{B}$

Answer: (a) 7, (b) (0, −2, 21), (c) 0, (d) ±(0.9117, 0.2279, 0.3419).

EXAMPLE 1.2

Points P and Q are located at (0, 2, 4) and (−3, 1, 5). Calculate

(a) The position vector P

(b) The distance vector from P to Q

(c) The distance between P and Q

(d) A vector parallel to PQ with magntude of 10

Solution:

(a) $\mathbf{r}_P = 0\mathbf{a}_x + 2\mathbf{a}_y + 4\mathbf{a}_z = 2\mathbf{a}_y + 4\mathbf{a}_z$

(b) $\mathbf{r}_{PQ} = \mathbf{r}_Q - \mathbf{r}_P = (-3, 1, 5) - (0, 2, 4) = (-3, -1, 1)$
or $\mathbf{r}_{PQ} = -3\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z$

(c) Since $\mathbf{r}_{PQ}$ is the distance vector from P to Q, the distance between P and Q is the magnitude of this vector; that is,

$$d = |\mathbf{r}_{PQ}| = \sqrt{9 + 1 + 1} = 3.317$$

Alternatively:

$$\begin{aligned} d &= \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2} \\ &= \sqrt{9 + 1 + 1} = 3.317 \end{aligned}$$

(d) Let the required vector be $\mathbf{A}$, then

$$\mathbf{A} = A\mathbf{a}_A$$

where $A = 10$ is the magnitude of $\mathbf{A}$. Since $\mathbf{A}$ is parallel to PQ, it must have the same unit vector as $\mathbf{r}_{PQ}$ or $\mathbf{r}_{QP}$. Hence,

$$\mathbf{a}_A = \pm\frac{\mathbf{r}_{PQ}}{|\mathbf{r}_{PQ}|} = \pm\frac{(-3, -1, 1)}{3.317}$$

and

$$\mathbf{A} = \pm\frac{10(-3, -1, -1)}{3.317} = \pm(-9.045\mathbf{a}_x - 3.015\mathbf{a}_y + 3.015\mathbf{a}_z)$$

1.7 VECTOR MULTIPLICATION

When two vectors **A** and **B** are multiplied, the result is either a scalar or a vector depending on how they are multiplied. Thus there are two types of vector multiplication:

1. Scalar (or dot) product: $\mathbf{A} \cdot \mathbf{B}$
2. Vector (or cross) product: $\mathbf{A} \times \mathbf{B}$

Multiplication of three vectors **A**, **B**, and **C** can result in either:

3. Scalar triple product: $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$

or

4. Vector triple product: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$

A. Dot Product

The **dot product** of two vectors **A** and **B**, written as $\mathbf{A} \cdot \mathbf{B}$, is defined geometrically as the product of the magnitudes of **A** and **B** and the cosine of the angle between them.

Thus:

$$\boxed{\mathbf{A} \cdot \mathbf{B} = AB \cos \theta_{AB}} \tag{1.15}$$

where θ_{AB} is the *smaller* angle between **A** and **B**. The result of $\mathbf{A} \cdot \mathbf{B}$ is called either the *scalar product* because it is scalar, or the *dot product* due to the dot sign. If $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$, then

$$\boxed{\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z} \tag{1.16}$$

which is obtained by multiplying **A** and **B** component by component. Two vectors **A** and **B** are said to be *orthogonal* (or perpendicular) with each other if $\mathbf{A} \cdot \mathbf{B} = 0$.

Note that dot product obeys the following:

(i) *Commutative law:*

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \tag{1.17}$$

(ii) *Distributive law:*

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \tag{1.18}$$

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2 = A^2 \tag{1.19}$$

(iii)

Also note that

$$\mathbf{a}_x \cdot \mathbf{a}_y = \mathbf{a}_y \cdot \mathbf{a}_z = \mathbf{a}_z \cdot \mathbf{a}_x = 0 \tag{1.20a}$$

$$\mathbf{a}_x \cdot \mathbf{a}_x = \mathbf{a}_y \cdot \mathbf{a}_y = \mathbf{a}_z \cdot \mathbf{a}_z = 1 \tag{1.20b}$$

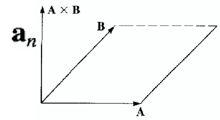
It is easy to prove the identities in eqs. (1.17) to (1.20) by applying eq. (1.15) or (1.16).

B. Cross Product

The **cross product** of two vectors **A** and **B**, written as **A** × **B**, is a vector quantity whose magnitude is the area of the parallelopiped formed by **A** and **B** (see Figure 1.7) and is in the direction of advance of a right-handed screw as **A** is turned into **B**.

Thus

$$\boxed{\mathbf{A} \times \mathbf{B} = AB \sin \theta_{AB} \mathbf{a}_n} \tag{1.21}$$



where $\mathbf{a}_n$ is a unit vector normal to the plane containing **A** and **B**. The direction of $\mathbf{a}_n$ is taken as the direction of the right thumb when the fingers of the right hand rotate from **A** to **B** as shown in Figure 1.8(a). Alternatively, the direction of $\mathbf{a}_n$ is taken as that of the advance of a right-handed screw as **A** is turned into **B** as shown in Figure 1.8(b).

The vector multiplication of eq. (1.21) is called *cross product* due to the cross sign; it is also called *vector product* because the result is a vector. If $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$ then

$$\boxed{\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}} \tag{1.22a}$$

$$= (A_yB_z - A_zB_y)\mathbf{a}_x + (A_zB_x - A_xB_z)\mathbf{a}_y + (A_xB_y - A_yB_x)\mathbf{a}_z \tag{1.22b}$$

which is obtained by "crossing" terms in cyclic permutation, hence the name cross product.

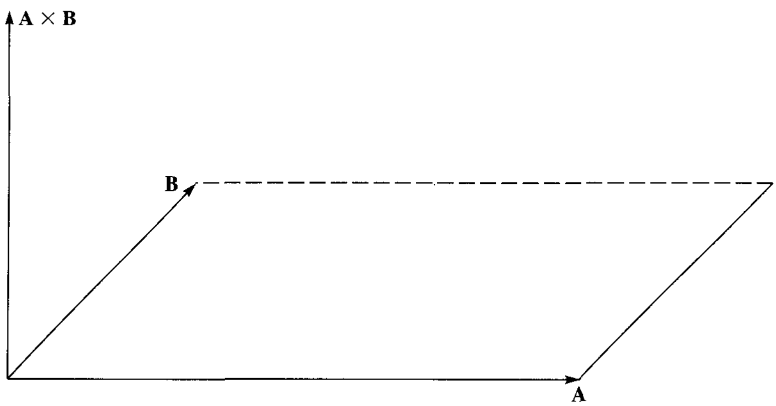


Figure 1.7 The cross product of **A** and **B** is a vector with magnitude equal to the area of the parallelogram and direction as indicated.

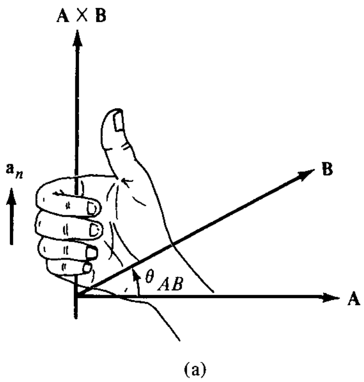


Figure 1.8 Direction of $\mathbf{A} \times \mathbf{B}$ and $\mathbf{a}_n$ using **(a)** right-hand rule, **(b)** right-handed screw rule.

1.8 COMPONENTS OF A VECTOR

A direct application of vector product is its use in determining the projection (or component) of a vector in a given direction. The projection can be scalar or vector. Given a vector **A**, we define the *scalar component* A_B of **A** along vector **B** as [see Figure 1.10(a)]

$$A_B = A \cos \theta_{AB} = |\mathbf{A}||\mathbf{a}_B| \cos \theta_{AB}$$

or

$$\boxed{A_B = \mathbf{A} \cdot \mathbf{a}_B} \tag{1.33}$$

The *vector component* $\mathbf{A}_B$ of **A** along **B** is simply the scalar component in eq. (1.33) multiplied by a unit vector along **B**; that is,

$$\boxed{\mathbf{A}_B = A_B\mathbf{a}_B = (\mathbf{A} \cdot \mathbf{a}_B)\mathbf{a}_B} \tag{1.34}$$

Both the scalar and vector components of **A** are illustrated in Figure 1.10. Notice from Figure 1.10(b) that the vector can be resolved into two orthogonal components: one component $\mathbf{A}_B$ parallel to **B**, another $(\mathbf{A} - \mathbf{A}_B)$ perpendicular to **B**. In fact, our Cartesian representation of a vector is essentially resolving the vector into three mutually orthogonal components as in Figure 1.1(b).

We have considered addition, subtraction, and multiplication of vectors. However, division of vectors **A/B** has not been considered because it is undefined except when **A** and **B** are parallel so that $\mathbf{A} = k\mathbf{B}$, where k is a constant. Differentiation and integration of vectors will be considered in Chapter 3.

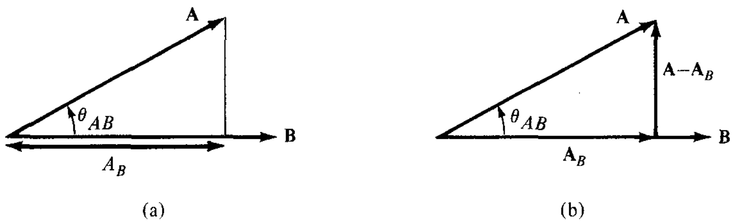


Figure 1.10 Components of **A** along **B**: **(a)** scalar component A_B, **(b)** vector component $\mathbf{A}_B$.

EXAMPLE 1.4

Given vectors $\mathbf{A} = 3\mathbf{a}_x + 4\mathbf{a}_y + \mathbf{a}_z$ and $\mathbf{B} = 2\mathbf{a}_y - 5\mathbf{a}_z$, find the angle between **A** and **B**.

Solution:

The angle θ_{AB} can be found by using either dot product or cross product.

$$\mathbf{A} \cdot \mathbf{B} = (3, 4, 1) \cdot (0, 2, -5)$$
$$= 0 + 8 - 5 = 3$$
$$|\mathbf{A}| = \sqrt{3^2 + 4^2 + 1^2} = \sqrt{26}$$
$$|\mathbf{B}| = \sqrt{0^2 + 2^2 + (-5)^2} = \sqrt{29}$$
$$\cos \theta_{AB} = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{3}{\sqrt{(26)(29)}} = 0.1092$$
$$\theta_{AB} = \cos^{-1} 0.1092 = 83.73°$$

Alternatively:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 3 & 4 & 1 \\ 0 & 2 & -5 \end{vmatrix}$$
$$= (-20 - 2)\mathbf{a}_x + (0 + 15)\mathbf{a}_y + (6 - 0)\mathbf{a}_z$$
$$= (-22, 15, 6)$$
$$|\mathbf{A} \times \mathbf{B}| = \sqrt{(-22)^2 + 15^2 + 6^2} = \sqrt{745}$$
$$\sin \theta_{AB} = \frac{|\mathbf{A} \times \mathbf{B}|}{|\mathbf{A}||\mathbf{B}|} = \frac{\sqrt{745}}{\sqrt{(26)(29)}} = 0.994$$
$$\theta_{AB} = \cos^{-1} 0.994 = 83.73°$$

PRACTICE EXERCISE 1.4

If $\mathbf{A} = \mathbf{a}_x + 3\mathbf{a}_z$ and $\mathbf{B} = 5\mathbf{a}_x + 2\mathbf{a}_y - 6\mathbf{a}_z$, find θ_{AB}.

Answer: 120.6°.

EXAMPLE 1.5

Three field quantities are given by

$$\mathbf{P} = 2\mathbf{a}_x - \mathbf{a}_z$$
$$\mathbf{Q} = 2\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z$$
$$\mathbf{R} = 2\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z$$

Determine

(a) $(\mathbf{P} + \mathbf{Q}) \times (\mathbf{P} - \mathbf{Q})$
(b) $\mathbf{Q} \cdot \mathbf{R} \times \mathbf{P}$