Introduction to Electricity and Magnetism B38EM

Tutorial #4

$\varepsilon_0 = 8.85\text{x}10^{-12}\ \text{Fm}^{-1}$, $e = 1.6\text{x}10^{-19}$ C, 1 nC = 10^{-9} C

1- Uniform surface charge densities of *6, 4,* and *2* nC/m^2 are present at $r = 2, 4,$ and *6* cm respectively. Assume potential $V=0$ at infinity. Find $V(r)$.

2- Calculate the divergence of the following vector functions:

(a) $\mathbf{V_a} = x^2\ \mathbf{i} + 3xz^2\ \mathbf{j} - 2xz\ \mathbf{k}$
(b) $\mathbf{V_b} = xy\ \mathbf{i} + 2yz\ \mathbf{j} + 3zx\ \mathbf{k}$
(c) $\mathbf{V_c} = y^2\ \mathbf{i} + (2xy+z^2)\mathbf{j} + 2yz\ \mathbf{k}$

3- By employing the appropriate line integral for the electric field, demonstrate that, at an interface between two dielectric regions the tangential electric field is continuous.
(Note: The normal components of the electric flux density are continuous across the interface).

4- Consider a straight non-magnetic conductor of circular cross-section and radius *a* carrying a current *I* in the vertical direction. Using Ampere's law find the magnetic field inside and outside the conductor.

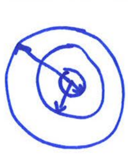
5- Using the same methodology as above, find the magnetic field everywhere in the cross-section of a coaxial cable. The radius of the inner conductor is *a*, the radius of the inner surface of the outer conductor is *b*, the radius of the outer surface of the outer conductor is *c*. The coaxial cable as a current in the inner conductor of I and a current in the outside conductor of *–I*.

6- Find the magnetic field due to an infinite sheet of current flowing in the *y*-direction.

1) Uniform surface charge densities of *6, 4,* and *2* nC/m² are present at $r = 2, 4,$ and 6 cm respectively. Assume potential $V=0$ at infinity. Find $V(r)$.

Solution:

① Consider set of concentric spheres



$r_1 = 2\,\text{cm} \quad \sigma_1 = 6\ \text{nC/m}^2 \qquad Q_1 = 4\pi\sigma_1 r_1^2$

$r_2 = 4\,\text{cm} \quad \sigma_2 = 4\ \text{nC/m}^2 \qquad Q_2 = 4\pi\sigma_2 r_2^2$

$r_3 = 6\,\text{cm} \quad \sigma_3 = 2\ \text{nC/m}^2 \qquad Q_3 = 4\pi\sigma_3 r_3^2$

Take observation point M at distance r from origin of the spheres.

We know that: $\vec{E}(M) = E(r)\,\vec{e_r}$

$r > 6$ cm Gauss's law $E(r)\,4\pi r^2 = \frac{4\pi}{\varepsilon_0}(Q_1+Q_2+Q_3)$

$$V(r) = -\int_\infty^r \frac{dr'}{\varepsilon_0 r'^2}(Q_1+Q_2+Q_3)$$

$$\boxed{V(r) = \frac{Q_1+Q_2+Q_3}{4\pi\varepsilon_0 r}} \quad \text{since } V(\infty)=0$$

$4 < r < 6$ cm Gauss's law $E(r)\,4\pi r^2 = \frac{4\pi}{\varepsilon_0}(Q_1+Q_2)$

$$V(r) = -\int_\infty^r dr'\,E(r) = -\int_\infty^{r_3} E(r')dr' - \int_{r_3}^{r} E(r')dr'$$

$$V(r) = \frac{Q_1+Q_2+Q_3}{4\pi\varepsilon_0 r_3} + \left[\frac{Q_1+Q_2}{4\pi\varepsilon_0 r} - \frac{Q_1+Q_2}{4\pi\varepsilon_0 r_3}\right]$$

$$\boxed{V(r) = \frac{Q_1+Q_2}{4\pi\varepsilon_0 r} + \frac{Q_3}{4\pi\varepsilon_0 r_3}}$$

$2 < r < 4$ In the same way $E(r)\,4\pi r^2 = \frac{4\pi}{\varepsilon_0}(Q_1)$

$$V(r) = -\int_\infty^{r_3} - \int_{r_3}^{r_2} - \int_{r_1}^{r}$$

$$V(r) = \frac{Q_1+Q_2+Q_3}{4\pi\varepsilon_0 r_3} + \frac{Q_1+Q_2}{4\pi\varepsilon_0 r_2} - \frac{Q_1+Q_2}{4\pi\varepsilon_0 r_3} + \frac{Q_1}{4\pi\varepsilon_0 r} - \frac{Q_1}{4\pi\varepsilon_0 r_2}$$

$$\boxed{V(r) = \frac{Q_1}{4\pi\varepsilon_0 r} + \frac{Q_3}{4\pi\varepsilon_0 r_3} + \frac{Q_2}{4\pi\varepsilon_0 r_2}}$$

$0 < r < 2$
$$V(r) = \frac{Q_1}{4\pi\varepsilon_0 r_1} + \frac{Q_2}{4\pi\varepsilon_0 r_2} + \frac{Q_3}{4\pi\varepsilon_0 r_3}$$

(since $E(r) = 0$ inside r_1)

Numerical calculations are easy to do then

2) Calculate the divergence of the following vector functions:

(d) $\mathbf{V_a} = x^2\,\mathbf{i} + 3xz^2\,\mathbf{j} - 2xz\,\mathbf{k}$

(e) $\mathbf{V_b} = xy\,\mathbf{i} + 2yz\,\mathbf{j} + 3zx\,\mathbf{k}$

(f) $\mathbf{V_c} = y^2\,\mathbf{i} + (2xy+z^2)\mathbf{j} + 2yz\,\mathbf{k}$

Solution:

Q2.

$$\vec{\nabla}\cdot\vec{V_A} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(3xz^2) + \frac{\partial}{\partial z}(-2xz) = 2x - 2x = 0$$

$$\vec{\nabla}\cdot\vec{V_B} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(3zx) = y + 2z + 3x$$

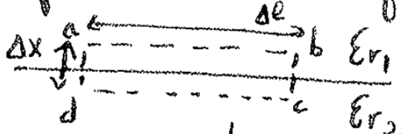
$$\vec{\nabla}\cdot\vec{V_C} = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(2xy + y^2) + \frac{\partial}{\partial z}(2yz) = 2x + 2y$$

3) By employing the appropriate line integral for the electric field, demonstrate that, at an interface between two dielectric regions the tangential electric field is continuous.

(Note: The normal components of the electric flux density are continuous across the interface).

3 In a static electric field $\quad Circ\,\vec{E} = \oint_{(c)} \vec{E}\cdot\vec{t}\,ds = 0$

Consider the application of the circulation to the closed path in the vicinity of the dielectric interface as shown below



$$Circ\,\vec{E} = 0 \implies \int_a^b \vec{E}\cdot\vec{ds} + \int_b^c \vec{E}\cdot\vec{ds} + \int_c^d \vec{E}\cdot\vec{ds} + \int_d^a \vec{E}\cdot\vec{ds} = 0$$

$$E_{T_1}\,\Delta l + 0 - E_{T_2}\,\Delta l + 0 = 0 \quad \text{as } \Delta x \to 0$$

If Δl small $\quad \boxed{E_{T_1} = E_{T_2}}$ at the interface

4) Consider a straight non-magnetic conductor of circular cross-section and radius a carrying a current I in the vertical direction. Using Ampere's law find the magnetic field inside and outside the conductor.

Solution:

(a) Outside the conductor:

Take an Amperian loop of radius larger than the radius of the conductor (r>a). The total enclosed within this loop will be I. We know that the magnetic field is orthoradial, i.e. the direction of **B** is circling around the wire. Ampere's law in its integral form yields:

$$\oint_{Amperian\,loop} \mathbf{B.dl} = B\oint dl = B\int_0^{2\pi} r d\phi = 2\pi r B = \mu_0 I$$

The magnetic field outside the conductor is therefore:

$$B(r) = \frac{\mu_0 I}{2\pi r}$$

(b) Inside the conductor:

For an Amperian loop of radius less than the radius of the conductor (r<a), we need to calculate the current that is enclosed by this loop. Assuming a uniform current distribution of current density **J**:

$$\mathbf{J} = \frac{I}{\pi a^2}\mathbf{k}$$

The current enclosed in the loop will therefore be:

$$I_{enc} = \iint_{surface\ of\ loop} \mathbf{J.k}\,dS = \mathbf{J.k}\iint dS = \frac{I}{\pi a^2}\pi r^2 = I\left(\frac{r}{a}\right)^2$$

and

$$\oint_{Amperian\,loop} \mathbf{B.dl} = B\oint dl = B\int_0^{2\pi} r d\phi = 2\pi r B = \mu_0 I\left(\frac{r}{a}\right)^2$$

Therefore

$$B(r) = \frac{\mu_0 I r}{2\pi a^2}$$

5) Using the same methodology as above, find the magnetic field everywhere in the cross-section of a coaxial cable. The radius of the inner conductor is *a*, the radius of the inner surface of the outer conductor is *b*, the radius of the outer surface of the outer conductor is *c*. The coaxial cable as a current in the inner conductor of I and a current in the outside conductor of *–I*.

Solution:

(a) Inside the wire of the inner conductor ($r<a$), we have the same value of the field as above i.e.

$$B(r) = \frac{\mu_0 I r}{2\pi a^2}$$

(b) Between the inner conductor (wire) and the outer conductor (the shield) ie for $a<r<b$, we have the same value of the field of above i.e.

$$B(r) = \frac{\mu_0 I}{2\pi r}$$

(c) Inside the outer conductor ($b<r<c$), the current enclosed is:

$$I_{enc} = \iint_{surface\,of\,the\,loop} \mathbf{J}.\mathbf{k}\,dS = I + \int_0^{2\pi} d\phi \int_b^r s\,ds(-\mathbf{J}_{outer}).\mathbf{k} = I - J_{outer}\pi(r^2 - b^2)$$

But we know that $J_{outer} = \dfrac{I}{\pi(c^2 - b^2)}$

Therefore $$\oint_{Amperian\,loop} \mathbf{B}.\mathbf{dl} = B\oint dl = B\int_0^{2\pi} r\,d\phi = 2\pi r B = \mu_0\left[I - I\frac{(r^2 - b^2)}{(c^2 - b^2)}\right] = \mu_0 I\frac{(c^2 - r^2)}{(c^2 - b^2)}$$

And $$B(r) = \frac{\mu_0 I}{2\pi r}\frac{(c^2 - r^2)}{(c^2 - b^2)}$$

(d) Outside the coaxial cable ($r>c$), the current enclosed is I_{enc}=I-I=0, therefore there is no magnetic field and B(r)=0.

6) Find the magnetic field due to an infinite sheet of current flowing in the *y*-direction.

Solution :

The magnetic field does not vary with x and y since the source does not vary with x and y. B_y is zero since the current is along y. Bz is zero due to the cancellation of contributions from two symmetrical elements along x. The resultant magnetic field has therefore only x-components.

Consider a countour along the x-z plane which cuts across the infinite sheet of current. There is no contribution from the vertical segments since B_z=0. The only contribution is from the horizontal segement of the contour. If the segments have for length L and J_{sy} is the current surface density flowing in the y-direction

$$\oint \mathbf{B.dl} = \mu_0 I_{enc} = B_{top} L - B_{bottom} L = J_{sy} L$$

From Biot-Savart law, the field is anti-symmetrical with respect of the current sheet plane therefore B_{top}=-$B_{bottom.}$

We have therefore B_{top}= J_{sy}/2 and B_{bottom}=-J_{sy}/2. This field does not depend on the distance from the infinite current sheet. This result is analogous to the **E**-field of an infinite charged sheet.