B38EM Introduction to Electricity and Magnetism
Lecture 7

Faraday’s Law & Displacement Current

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Outline & Outcome

- Revision
- Faraday's law
- Exercise
- Inductance & mutual inductance
- Displacement current

References & Resources

- Elements of Electromagnetics (7th Edition), by Sadiku, Oxford University Press

- Fundamentals of Applied Electromagnetics (7th Edition), by Ulaby and Ravaioli

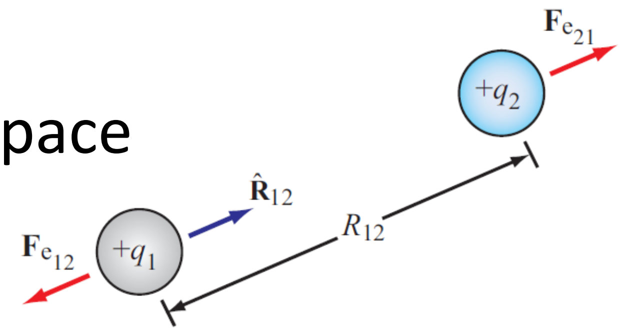
- Field and Wave Electromagnetics (2nd Edition), by David K. Cheng

- ……

Electrostatics

- Coulomb's Law

$$\boldsymbol{F}_{e21} = \hat{\boldsymbol{R}}_{12}\frac{q_1 q_2}{4\pi\varepsilon_0 R_{12}^2} \qquad (N) \qquad \text{in free space}$$



$1/4\pi\varepsilon_0$ = **electric constant** (AKA k_e) = $9 \cdot 10^9$ N·m^2 / C^2 in free-space.

ε_0 = **dielectric permitivity** of free space = $8.85 \cdot 10^{-12}$ F/m **(a constant)**

- Electric Field Intensity (***E***)

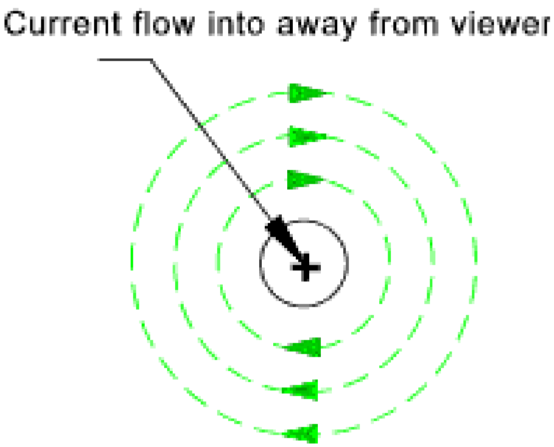
$$\boldsymbol{E} = \hat{\boldsymbol{R}}\frac{q}{4\pi\varepsilon_0 R^2} \qquad (V/m) \qquad \text{in free space}$$

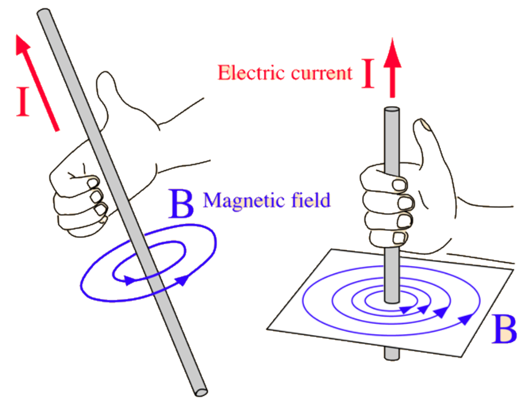
force on a unit charge

Magnetostatics

- Magnetic field (***H*** or ***B***) due to electric current

A **wire** with electric current I, produces an ***H***-field around that wire with direction given by the **right-hand thumb rule**. Thumb in direction of current → 4 fingers curl around in the direction of ***H***-field.





$$B = \frac{\mu_0 I}{2\pi r}$$

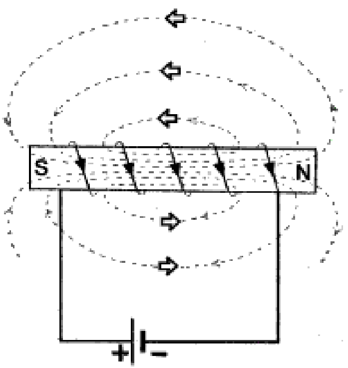
$$\mu_0 = 4\pi\times10^{-7}\ (\mathrm{H/m})$$

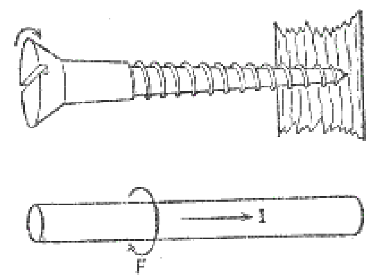
Magnetostatics

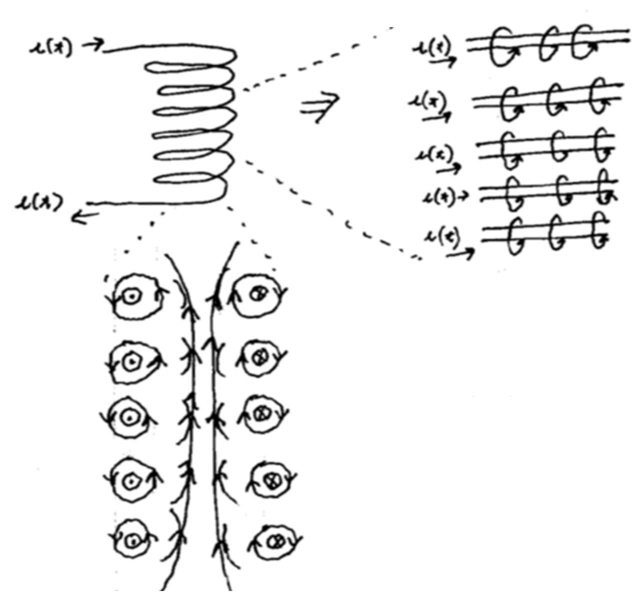
- Solenoid

Inside a **solenoid**, the resulting ***H***-field is uniform ($H=NI/l$)

*"The **right hand** grips the outside of a coil with the fingers in the direction of the current; the **thumb** points in the direction of the **H**-field **inside** the coil"*.







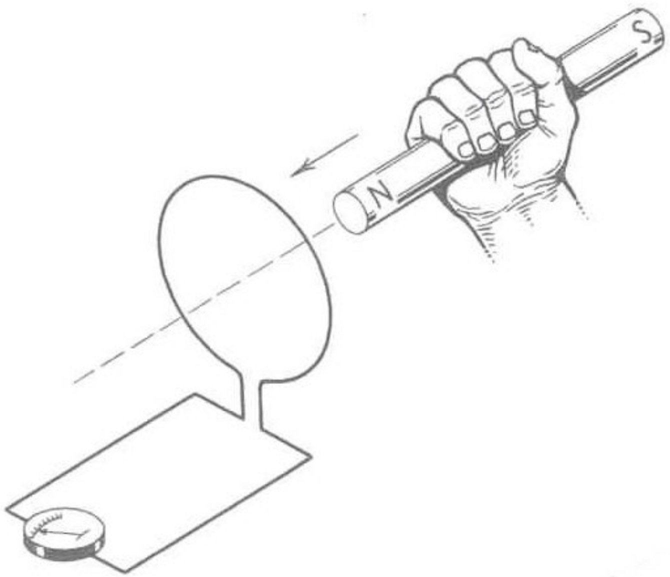
Faraday's law

- Observation

  - Stationary magnet in or out of a coil
    => voltmeter reads zero.

  - Magnet **moving into** the coil,
    => EMF is induced **in the coil** and
    appears as voltmeter reading.

  - Magnet **moving out of** the coil
    => reverse EMF is induced **in the coil**.



Changing ***H*** causes I.

Note: electromotive force, also called EMF, refers to voltage generated by a source

Faraday's Law

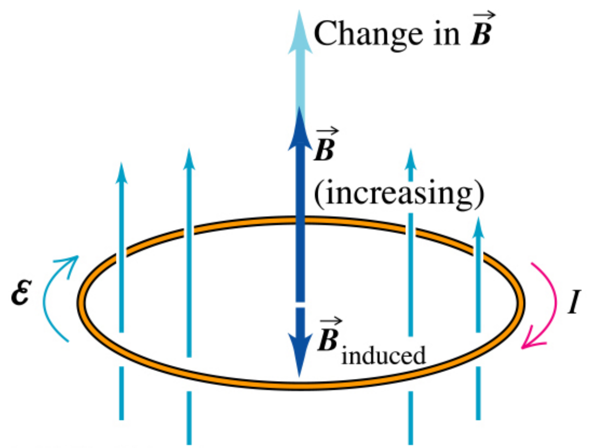
- Observation

"A changing magnetic flux in one coil (or ckt) produces (induces) a voltage across that coil (or ckt)" given by:

(Time rate of change of Magnetic Flux Φ linking the coil [Wb/sec])

$$v(t) = -N\frac{d\Phi}{dt} \qquad (volts)$$

" – " : *The direction of the induced EMF will always tend to setup a current that opposes the motion (or the change of flux) that induces that EMF.*



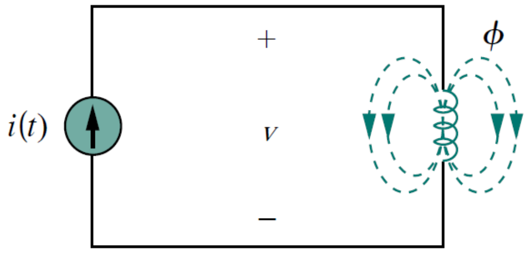


Figure 13.1 Magnetic flux produced by a single coil with N turns.

Since the coils 'oppose' to any change, a circuit with a coil will tend to oppose and thus 'delay' any change in its V or I.

https://phet.colorado.edu/en/simulation/faraday

https://phet.colorado.edu/sims/html/faradays-law/latest/faradays-law_en.html

Faraday's Law

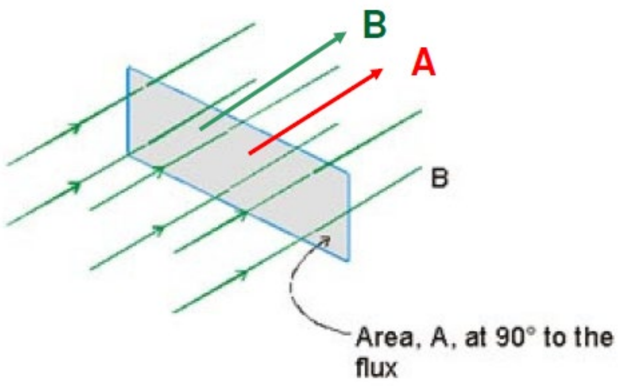
- Electromagnetic Induction

The process of using magnetic fields to produce voltage, and in a complete circuit, a current.

$$v(t) = -N\frac{d\Phi}{dt} \qquad (volts)$$

Magnetic Flux is a dot product:

$$\Phi = \vec{B} \cdot \vec{A} = B \cdot A \cdot cos(\theta)$$



How could we CHANGE the flux over a period of time?

- Change B; e.g. we could move the magnet away or towards (or the wire)
- Change A; e.g. we could increase or decrease the area
- Change ϑ; e.g. we could rotate either field or the area

Faraday’s Law

$$v(t) = -N\frac{d\Phi}{dt} \qquad (volts)$$

- Faraday’s Law

The induced EMF, V_{emf} (in volts), in any closed circuit is equal to the time rate of change of the magnetic flux through the circuit.

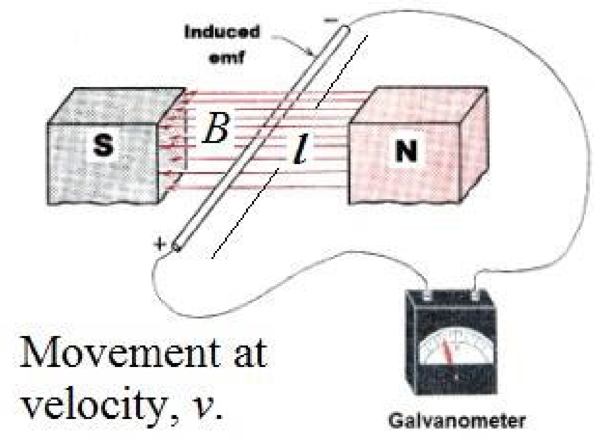
- Lenz’s Law

The direction of the induced EMF will always tend to setup a current that opposes the motion (or the change of flux) that induces the EMF".

Note: Lenz’s law gives the direction of the induced EMF and current resulting from electromagnetic induction.

Faraday's Law

Conductor moving in a B-field
(not necessarily carrying current),
will experience EMF (Voltage) given by:

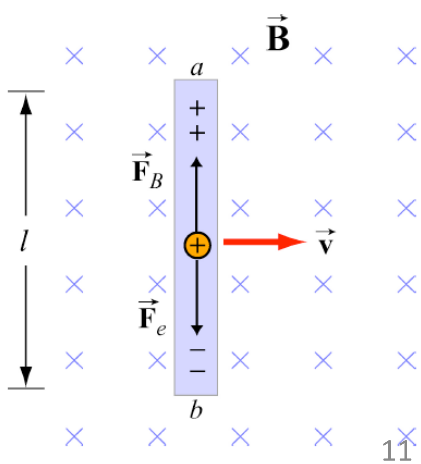


$$v(t) = -N\frac{d\Phi(t)}{dt} \qquad N = 1$$

emf = *Bvl* sinθ

B = flux density
v = velocity of conductor
l = length of conductor
θ = angle the conductor makes with the B-field

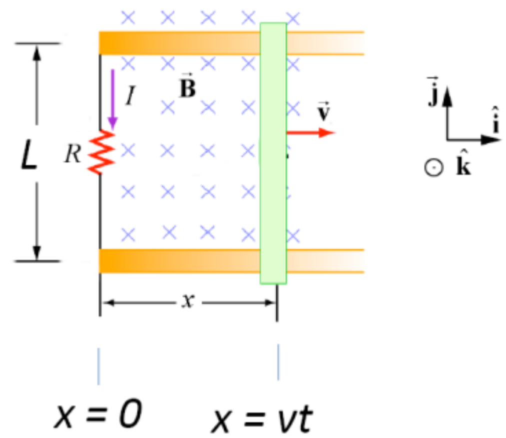
(e.g. below: $\theta = 90°$)



EMF **max** @ θ=90° (v $\perp$ B)
and **min** @ θ=0°(v // B)

Faraday's Law

If the conductor **slides** on 2 conducting rails connected with a resistor R:



$$V = -\frac{d\Phi}{dt} = \frac{d\left(BA\right)}{dt} = -\frac{d\left(Blvt\right)}{dt} = -Bvl$$

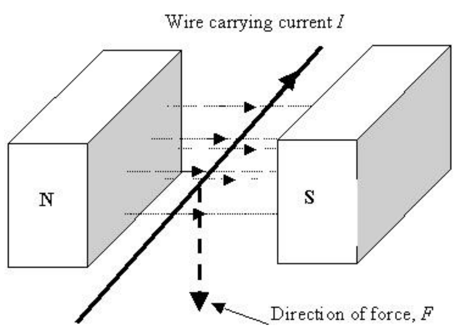
This will induce **current I** → Force:

(now we have a 'current-carrying conductor inside B)

$F = qvB\sin\theta$ This v is the speed of moving particles.

$I = \Delta q / \Delta t$ $v\Delta t = \Delta l$

$\Rightarrow F = BIL\sin\theta$

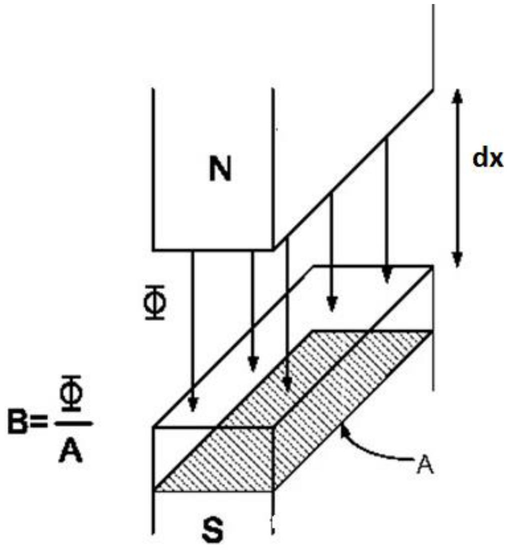


(again max at θ=90° (I ⊥ B) & **min** at θ=0°(I // B)

Faraday's Law

Lifting power of a magnet

Consider the magnetic circuit:



Due to the air-gap we can write:

$$B = \mu_0 H$$

Then:

Energy density: $$\frac{1}{2}BH = \frac{1}{2}\mu_0 H^2 = \frac{1}{2}\frac{B^2}{\mu_0}$$

If S is pulled apart by distance *dx*, then some work will have to be done against the force of attraction.

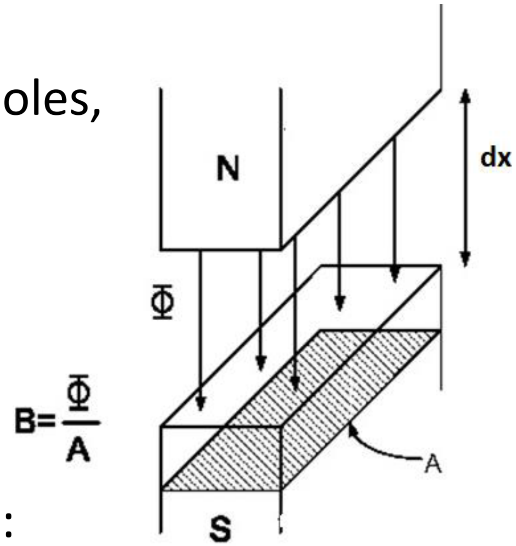
Faraday's Law

If we assume F is the force of attraction between the 2 poles,

the work W is:

W= – F·dx

This must equal the amount of energy stored in the field:



Energy = Energy density × volume = $\frac{1}{2}\frac{B^2 A dx}{\mu_0}$

$$F \cdot dx = \frac{1}{2}\frac{B^2 A dx}{\mu_0} \qquad \Rightarrow F \cdot = \frac{AB^2}{2\mu_0}$$

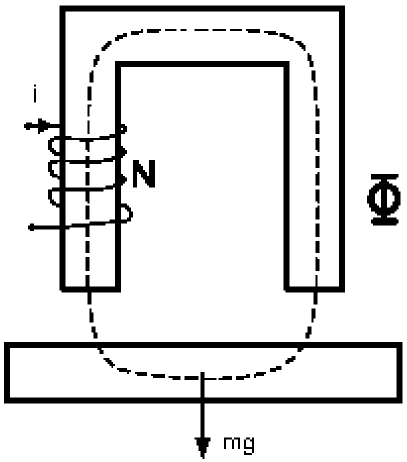
Newton's 2nd Law says: $F_{gravity} = mg$ (g=acceleration due to gravity, m=mass)

This means that: $mg = \frac{AB^2}{2\mu_0}$

Faraday's Law

It is now possible to design magnetic lifting machines.

The load m to be lifted "completes" the magnetic circuit with the weight divided equally between the 2 poles of the magnet



Example:
What is the flux density B needed to lift a load of 10 kg when the cross section of the magnetic material is 100 mm²?

Solution:
We know that: $$mg = \frac{AB^2}{2\mu_0} \quad \Rightarrow \quad B = \sqrt{\frac{2\mu_0 mg}{A}}$$

$$\mu_0 = 4\pi \cdot 10^{-7} \quad N/A^2$$

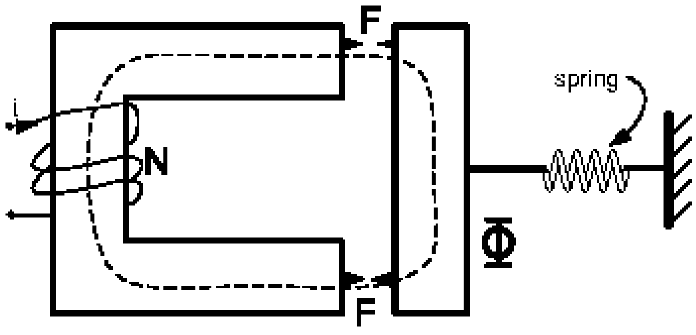
$$g = 9.8 \quad m/s^2$$

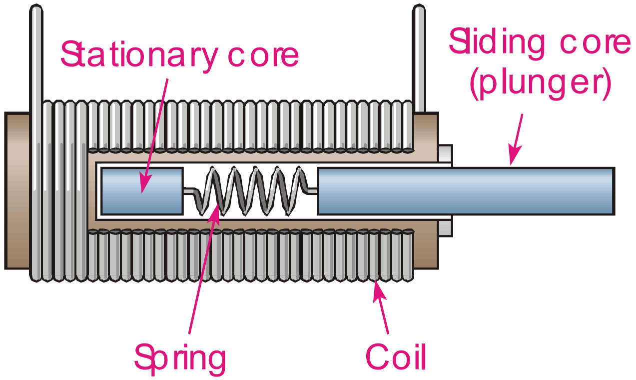
$$A = 100 \times 10^{-6} m^2 = 10^{-4} m^2$$

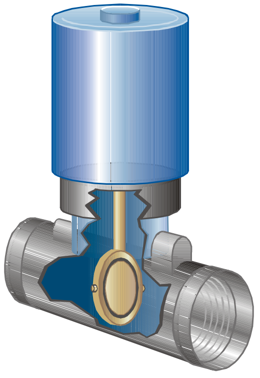
$$\Rightarrow \quad B = \sqrt{\frac{2\mu_0 mg}{A}} = 1.56T$$

Faraday’s Law

- Solenoids produce mechanical motion from electrical signals
- Often used to lift heavy objects and control fluids in pipes (e.g. sprinkler systems)
- A solenoid valve operates electro-mechanically:
  - Current passes through the coil and the right-hand part moves against the spring.

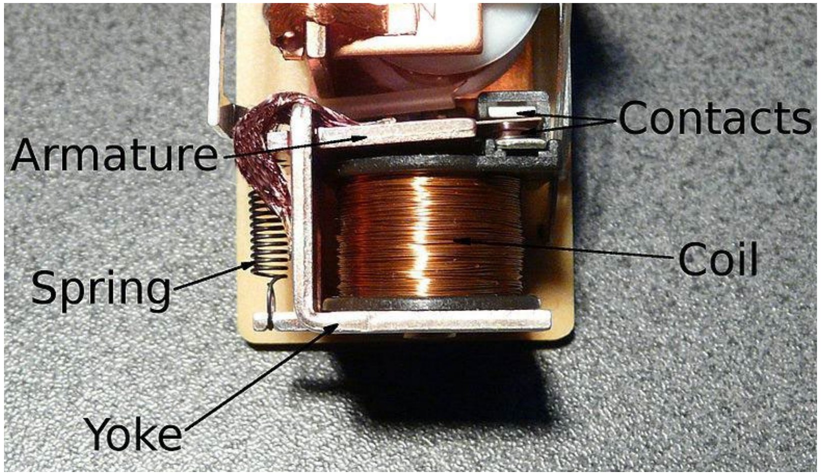


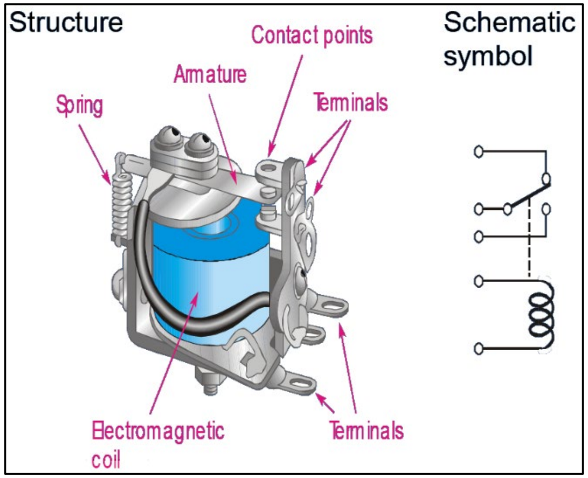




Faraday’s Law

Electromagnetic Relay



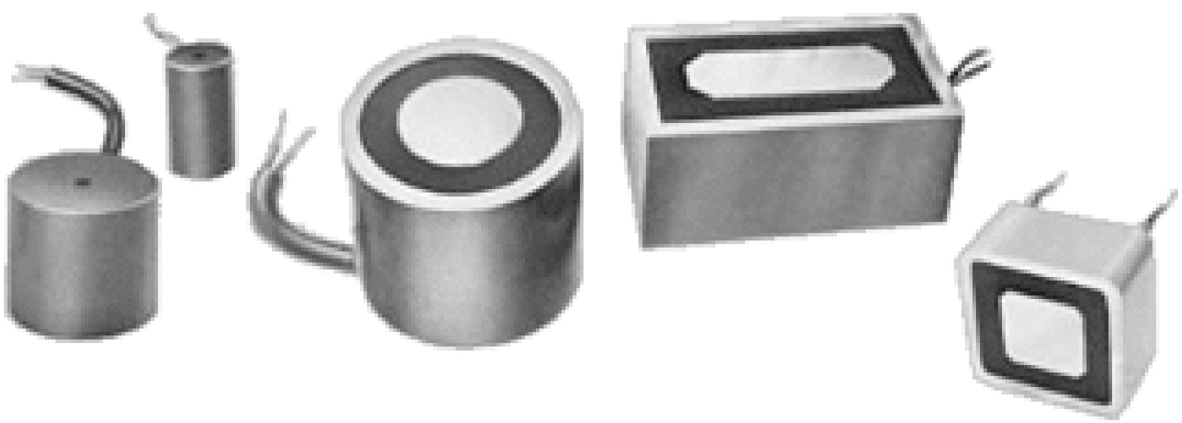


- A relay is an electrically controlled switch
- A small control voltage on the coil can control a large current through the contacts
- **Applications**: wherever we need to
  - control a circuit by a low-power signal, or
  - control several circuits by one signal

Faraday's Law

What is an electromagnet?

- Electromagnets use electric current to generate a magnetic field which can be turned ON or OFF as needed.
- Electromagnetic strength depends on N, I and μ_r.
- Types include Flat-faced, parallel pole magnets.
- Though all current-carrying conductors produce magnetic fields, an electromagnet is usually constructed in such a way as to maximise the strength of the magnetic field it produces for a special purpose.

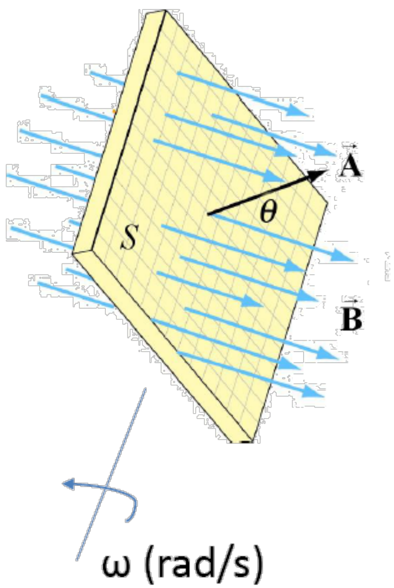


Faraday's Law

Example: Magnetic Flux through a rotating frame in homog**. B:**

How can an **AC E**-field be generated from a **B**-field?

Spinning frame inside a magnetic field:



$$\Phi = \iint \mathbf{B}d\mathbf{S} = BS\cos(\theta)$$

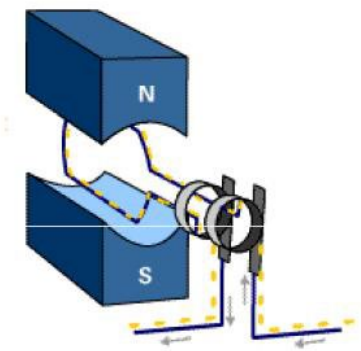
$$\theta = \omega t$$

$$\Phi = BS\cos(\omega t)$$

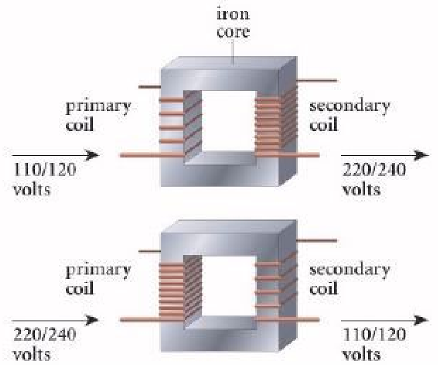
B interacts with the circuit and creates EMF
(**Basis of EM Generator/Motor**)

Faraday’s Law

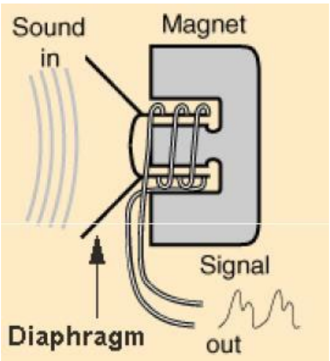
Applications



AC generators & motors



Transformers



Microphones

DC power supply

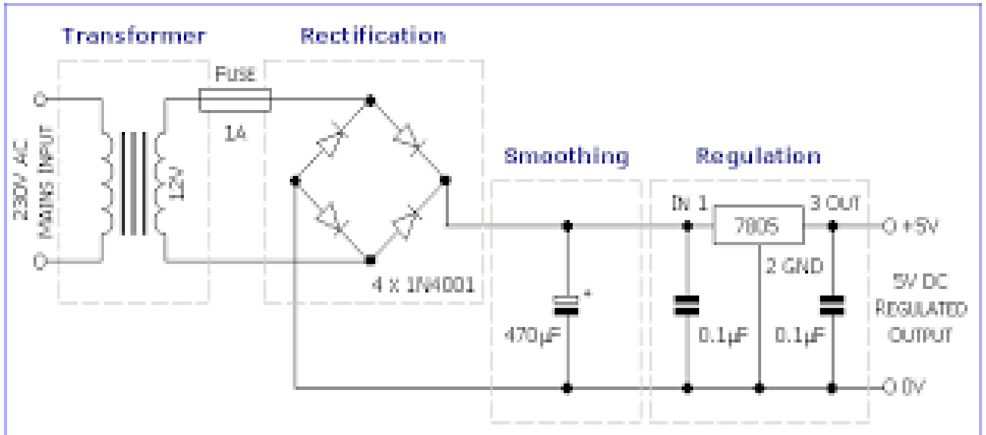
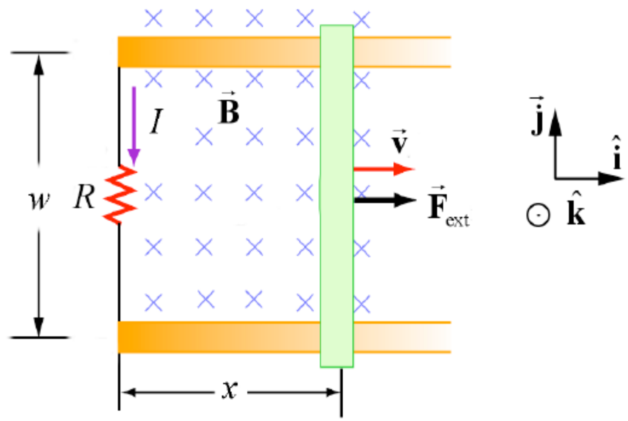


Figure 4: A simple 5V DC regulated power supply system

Self-study examples

1. Let the conductor move with a constant velocity $v = 10\ m/s$. The length of the conductor is $w = 10\ cm$ and the circuit resistance $R = 10\ Ohm$. What is the value of the generated current if the magnetic field is $B = 0.1\ T$?



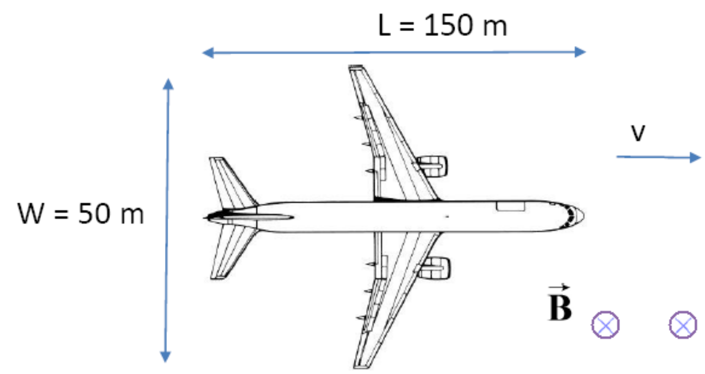
Lorentz force on the moving conductor?

Check: $I = Bvw/R = 0.1 \cdot 10 \cdot 0.1 / 10 = 0.01$ A

$F = 10^{-4}$ (N)

Self-study examples

2. Consider the B-field of the Earth B=50 μT, and an airplane flying with 1000 km/h as shown in the figure. What is the produced EMF?



Check: Emf = B·v·W= $50\cdot10^{-6}$ · 50 · 1000·1000/3600 => Emf=0.69 V

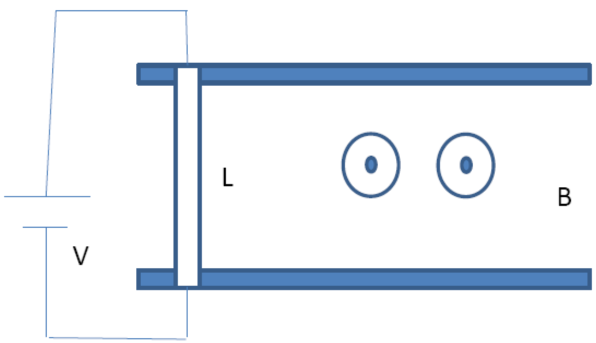
Self-study examples

3. Let $B = 1(T)$, $V = 2(V)$, $L = 1(m)$

a. Which direction will the conductor L move when we apply voltage V?

b. What is the constant velocity v_0 of the conductor in equilibrium?

c. What is the total current flowing in the conductor when moving at constant velocity v_0?



Check: $v_0 = V/BL = 2$ m/s

Self-study examples

Solution1:

According to Faraday's Law:

$$V = -\frac{d\Phi}{dt} = -\frac{d(\vec{B}\cdot\vec{A})}{dt} = -\frac{d[B\cdot(wx)(-\hat{k})\cdot(-\hat{k})]}{dt} = -\frac{d(Bwx)}{dt}$$

$$= -Bw\cdot\frac{dx}{dt} = -BwV = -0.1\times0.1\times10 = -0.1\ (V).$$

$$\therefore I = \frac{|V|}{R} = \frac{0.1}{10} = 0.01\,A.$$

Lorentz force

$$\cancel{\vec{F} = q\vec{v_e}\times\vec{B} = q v_e B\,\hat{j}\times(-\hat{k}) = q v_e B(-\hat{i})}$$

$$\Delta\vec{F} = \Delta q\,\vec{v_q}\times\vec{B} = \Delta q\,v_q B\,\hat{j}\times(-\hat{k}) = \Delta q\,v_q B(-\hat{i})$$

$$= -\Delta q\,\frac{\Delta w}{\Delta t}B\hat{i} = -I\Delta w B\hat{i}$$

$$\vec{F} = \int_0^W \Delta\vec{F} = -\int_0^W I B\hat{i}\,dw = -IBW\hat{i} = -0.01\times0.1\times0.1\,\hat{i}$$

$$= -1\times10^{-4}\,\hat{i}\ (N)$$

Self-study examples

Solution 2:

According to Faraday's Law:

$$V = -\frac{d\Phi}{dt} = -\frac{d(\vec{B}\cdot\vec{A})}{dt} = -\frac{d(BA\,\hat{n}\cdot\hat{n})}{dt} = -B\frac{dA}{dt} = -B\frac{d(W\cdot L)}{dt}$$

$$= -BW\frac{dL}{dt} = -BWV = -50\times10^{-6}\times50\times10^{6}/3600 = -0.69\ (V)$$

$\therefore$ EMF $= |V| = 0.69$ V pointing upward.

Solution 3:

a. $\longleftarrow$

b. in equilibrium $\longrightarrow$ no force on the conductor

$\longrightarrow$ no current flowing through the conductor

$\longrightarrow$ no voltage across the conductor

$\longrightarrow V_{emf} = 2(V)$.

$$|V_{emf}| = \frac{d\Phi}{dt} = \frac{d(\vec{B}\cdot\vec{A})}{dt} = \frac{d(BA\,\hat{n}\cdot\hat{n})}{dt} = \frac{d(BA)}{dt}$$

$$= B\frac{dA}{dt} = B\frac{d(L\cdot x)}{dt} = BLv_0$$

$$\Rightarrow v_0 = \frac{V_{emf}}{BL} = \frac{2}{1\times1} = 2\ (m/s)$$

Faraday's Law

- Inductance of solenoid

"*Inductance L* = the property of conductors (and solenoids) to oppose to the change of current", by inducing an EMF (voltage) in the conductor ".

$$v(t) = -N\frac{d\Phi}{dt} = -N\frac{d(kNi)}{dt} = -kN^2\frac{di}{dt} = -L\frac{di}{dt} \qquad H = NI/l$$

L determines how much voltage is generated per unit change of current:

E.g.: If a coil has *L* = 1H & its current changes @ *dI*/*dt* =1 Amp/sec**, => produces EMF = 1V that opposes this change in current.**

Faraday's Law

- Inductance of solenoid

$$v(t) = -N\frac{d\Phi}{dt} = -N\frac{d(kNi)}{dt} = -kN^2\frac{di}{dt} = -L\frac{di}{dt}$$

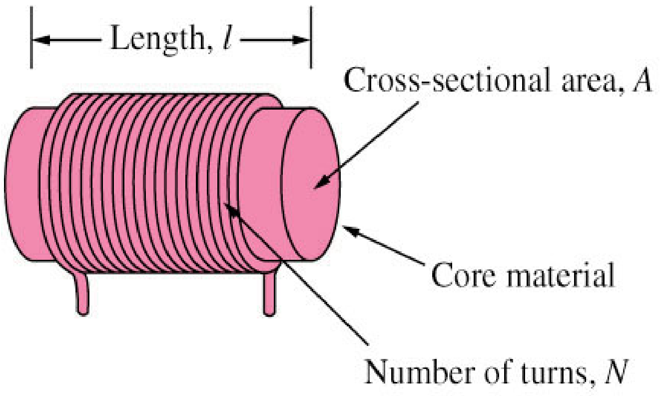
For averages of I, Φ or by $\int$ => $N\Phi = Li$ or $L = \frac{N\Phi}{i}$

$$B = \mu H = \mu\frac{Ni}{l}$$

$$\Phi = B \cdot A$$

$$L = \frac{N\Phi}{i}$$

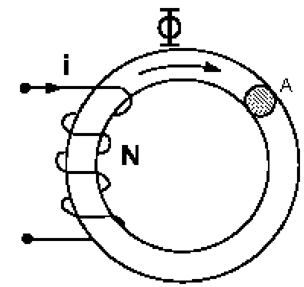
$$\Rightarrow L = \frac{\mu N^2 A}{l}$$ (H) or (Wb/A)



Faraday's Law

- Magnetic Field Energy and Energy Density

Definition:



Magnetic field energy *W*: $W = \frac{1}{2} L i^2$ (Joules)

$$L = \frac{\mu N H A}{i}$$

$$B = \frac{\mu N i}{l_c} \Rightarrow i = B \frac{l_c}{\mu N}$$

$$W = \frac{1}{2} B H \left(A l_c \right)$$

volume

Energy density = $\dfrac{W}{volume} = \dfrac{1}{2} B H$

Faraday's Law

- Mutual Inductance (M)

Mutual inductance (*M*) is the **ability** of one inductor to **induce a voltage** across a neighboring inductor.

M exists when:

→ 2 circuits are close to each other,

→ one at least has an AC source (I or V).
(remember: $L_{@\ \mathrm{DC}} \rightarrow \mathrm{SC}$)

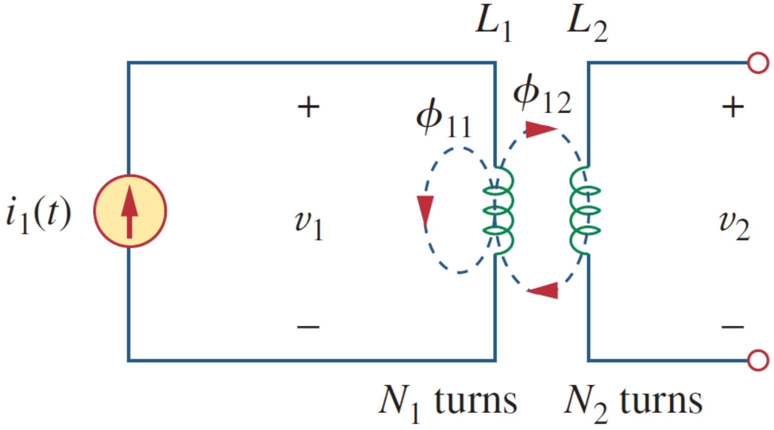
Faraday's Law

- Mutual Inductance (M)

Mutual inductance (M) is the **ability** of one inductor to **induce a voltage** across a neighboring inductor.

Assume 2 inductors close to each other (but NOT touching).

The mag. flux Φ_1 from coil-1 $\left(\begin{array}{c}\textbf{couples}\\\textbf{links}\\\textbf{affects}\end{array}\right)$ with coil-2 and induces voltage in it.



- $i_2 = 0$ (L_2 is O.C.)
- L_1 provides a total flux Φ_1 that has 2 components: Φ_{11} and Φ_{12}

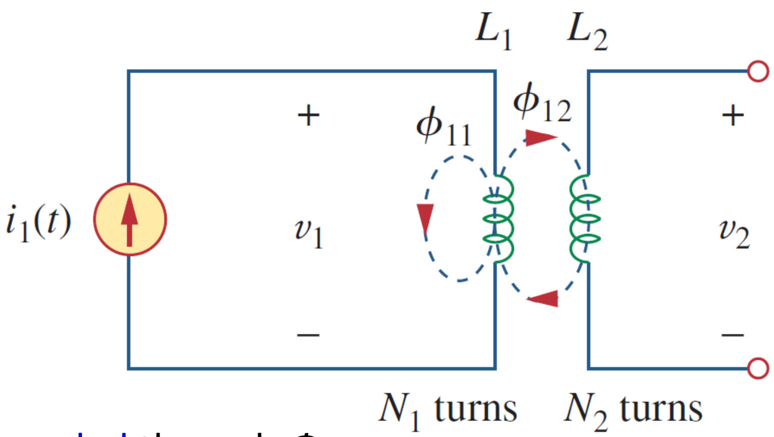
Assuming flux is uniform & omnidirectional, we can write

$$\Phi_1 = \Phi_{11} + \Phi_{12}$$

Coils are magnetically coupled through Φ_{12}

Faraday's Law

- Mutual Inductance (M)



$\Phi_1 = \Phi_{11} + \Phi_{12}$

Coils are magnetically coupled through Φ_{12}

The entire Φ_1 links L_1, so it induces in L_1 a voltage:

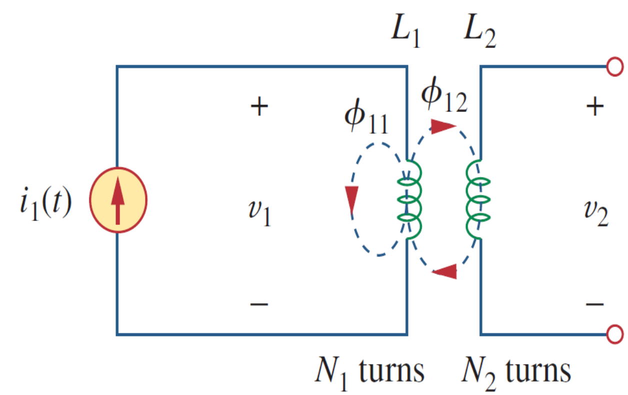
$$v_1(t) = -N_1 \frac{d\Phi_1(t)}{dt}$$

Only Φ_{12} links L_1 with L_2. Φ_{12} induces in L_2 a voltage:

$$v_2(t) = -N_2 \frac{d\Phi_{12}(t)}{dt}$$

Faraday's Law

- Mutual Inductance (M)



$$v_1(t) = -N_1 \frac{d\Phi_1(t)}{dt} = \underbrace{-N_1 \frac{d\Phi_1(t)}{di_1(t)}}_{L_1} \cdot \frac{di_1(t)}{dt}$$

L_1 'self'-inductance of coil 1

$$v_2(t) = -N_2 \frac{d\Phi_{12}(t)}{dt} = \underbrace{-N_2 \frac{d\Phi_{12}(t)}{di_1(t)}}_{M_{21}} \cdot \frac{di_1(t)}{dt}$$

M_{21} Mutual inductance of coil 2 due to current in coil 1

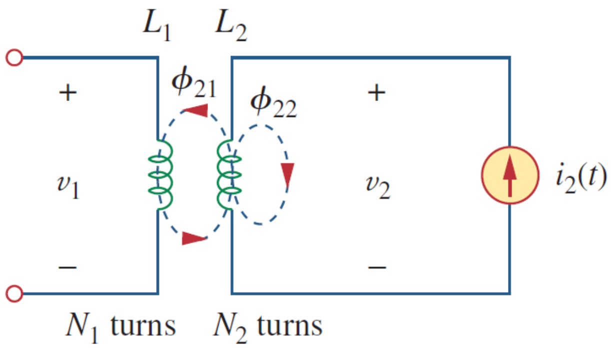
$$M_{21} = N_2 \frac{d\Phi_{12}(t)}{di_1(t)}$$

v_2 i_1 relates the v_2 with i_1

Faraday's Law

- Mutual Inductance (M)

Now let's move the source to the right, and OC coil L_1:



$$M_{12} = N_1 \frac{d\Phi_{21}(t)}{di_2(t)}$$

It can be approved that

$$M_{12} = M_{21} = M$$

To measure how much magnetic coupling exists between 2 coils, we use the coupling coefficient: $\boldsymbol{k = \frac{\Phi_{12}}{\Phi_1} = \frac{\Phi_{21}}{\Phi_2}}$.

k represents the fraction of the total flux emanating from one coil that links the other coil, so: $0 \le k \le 1$, so $\boldsymbol{M \le \sqrt{L_1 L_2}}$

M is always positive

Faraday's Law

- Mutual Inductance (M)

Mutual inductance (M) is the **ability** of one inductor to **induce a voltage** across a neighboring inductor.

M exists when:

- 2 circuits are close to each other
- and one at least has an AC source (I or V). (remember: $L_{@\,DC} \rightarrow$ SC)

M is **always** positive: $M > 0$

Obviously, $v(t)$ can be $\begin{cases} > 0 \\ < 0 \end{cases}$ because $v(t) = -M\dfrac{dI}{dt}$ (same as $-L\dfrac{di}{dt}$)

Maxwell’s Equations

Name	Integral equations	Differential equations
Gauss's law	$\oiint_{\partial\Omega} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0} \iiint_\Omega \rho \, \mathrm{d}V$	$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$
Gauss's law for magnetism	$\oiint_{\partial\Omega} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0$	$\nabla \cdot \mathbf{B} = 0$
Maxwell–Faraday equation (Faraday's law of induction)	$\oint_{\partial\Sigma} \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\frac{\mathrm{d}}{\mathrm{d}t} \iint_\Sigma \mathbf{B} \cdot \mathrm{d}\mathbf{S}$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
Ampère's circuital law (with Maxwell's addition)	$\oint_{\partial\Sigma} \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 \iint_\Sigma \mathbf{J} \cdot \mathrm{d}\mathbf{S} + \mu_0 \varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \iint_\Sigma \mathbf{E} \cdot \mathrm{d}\mathbf{S}$	$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$

$$\oint \mathbf{Edl} = -\frac{d}{dt} \iint \mathbf{BdS}$$

The circulation of the E-field over a closed path is proportional to the time-derivative of the magnetic flux through the surface enclosed by the path.

Maxwell's Equations

Name	Integral equations	Differential equations
Gauss's law	$\oiint_{\partial\Omega} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0} \iiint_{\Omega} \rho \, \mathrm{d}V$	$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$
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Maxwell's Equations

$$\oint \mathbf{B} \bullet \mathbf{dl} = \mu_0 \iint \mathbf{J} \bullet \mathbf{dS} + \mu_0 \varepsilon_0 \frac{d}{dt} \iint_S \mathbf{E} \bullet \mathbf{dS}$$

Ampere's law

The circulation of the **Magnetic Field B** over a closed path is proportional to the flux of the current density **J,** and the time derivative of the flux of the electric field through the surface enclosed by the path.

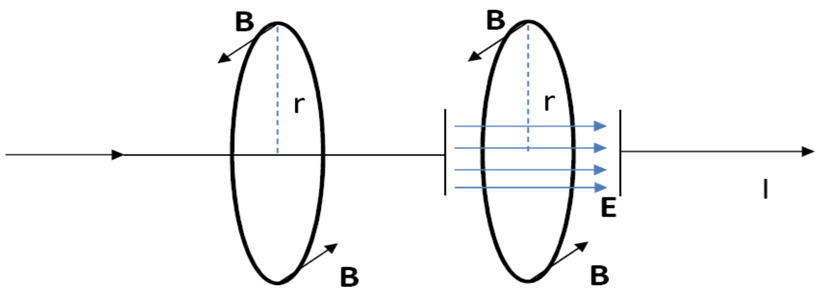
Maxwell's Equations

$$\oint \mathbf{B} \bullet \mathbf{dl} = \mu_0 \iint \mathbf{J} \bullet \mathbf{dS} + \mu_0 \varepsilon_0 \frac{d}{dt} \iint_S \mathbf{E} \bullet \mathbf{dS}$$

Maxwell's contribution to Ampere's law

In a capacitor there is no *J* between the plates, so how can we 'complete' the circuit?

$$\oint \boldsymbol{B} \cdot d\boldsymbol{l} = 2\pi r B = \mu_0 \varepsilon_0 \frac{dE}{dt} A$$



$$J_D = \frac{dE}{dt}$$ 'complete' the circuit.

Displacement current