B38EM Introduction to Electricity and Magnetism
Lecture 9

Electromagnetic Plane Waves

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Outline & Outcome

- Uniform plane EM waves
- Doppler effect
- Plane wave in lossless media
- Polarisation
- Poynting vector

References & Resources

- Elements of Electromagnetics (7th Edition), by Sadiku, Oxford University Press

- Fundamentals of Applied Electromagnetics (7th Edition), by Ulaby and Ravaioli

- Field and Wave Electromagnetics (2nd Edition), by David Cheng

- ……

Plane Wave

Time-harmonic electromagnetics (source-free)

$\rho = 0$, and $\boldsymbol{J} = 0$

Homogenous vector wave equations

$$\nabla^2 \mathbf{E} - \frac{1}{u^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0.$$

$$\nabla^2 \mathbf{H} - \frac{1}{u^2}\frac{\partial^2 \mathbf{H}}{\partial t^2} = 0.$$

Homogenous vector Helmholtz equations

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0$$

$$\nabla^2 \mathbf{H} + k^2 \mathbf{H} = 0,$$

Any twice differentiable function of $(t - R/u)$ or $(t + R/u)$ is a solution of the wave equation.

A simple solution [sine and cosine function $\sin(\omega t - kx)$, $\cos(\omega t - kx)$] can be immediately derived.

Plane Wave

Time-harmonic electromagnetics (source-free)

$\rho = 0$, and $\boldsymbol{J} = 0$

Homogenous vector Helmholtz equations

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = 0$$

$$\nabla^2\mathbf{H} + k^2\mathbf{H} = 0,$$

Time-harmonic plane wave: sine and cosine function $\sin(\omega t - kx)$, $\cos(\omega t - kx)$

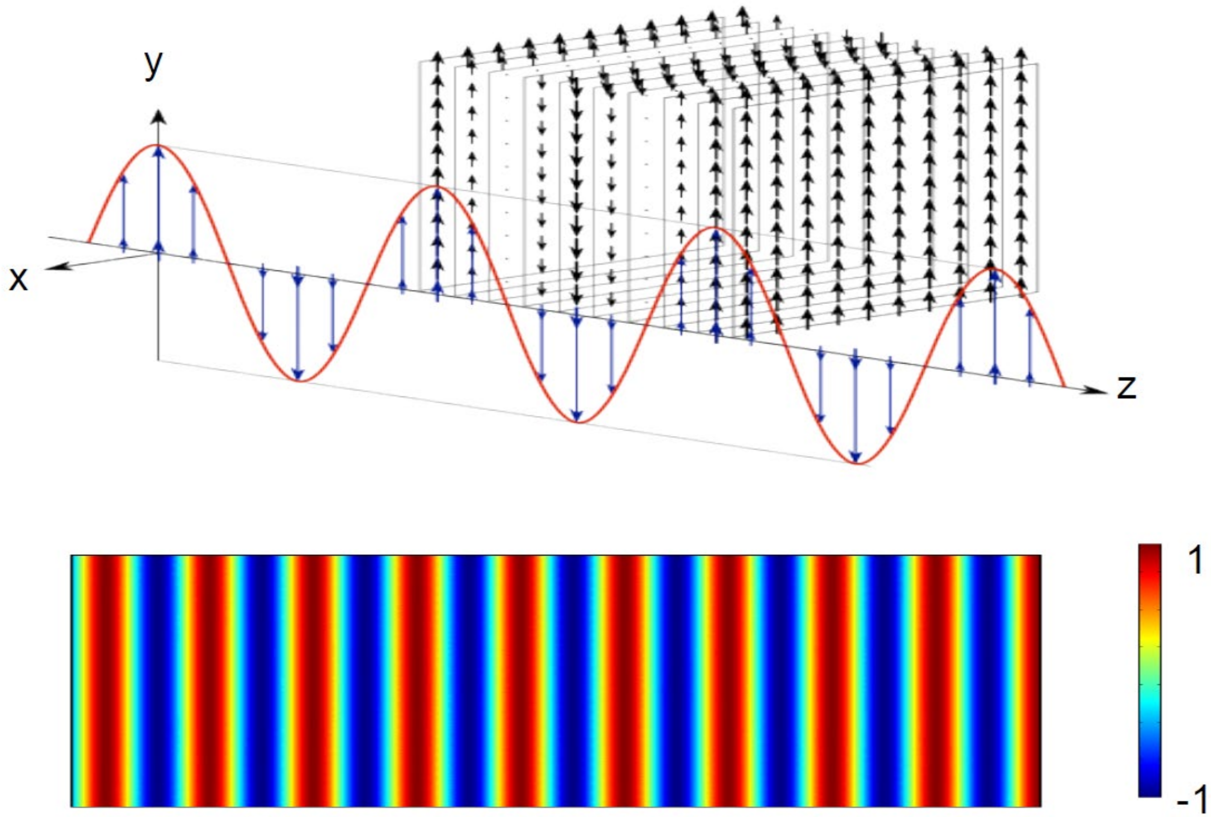
Uniform plane wave: the field with the same direction, same magnitude, and same phase in infinite planes (perpendicular to the direction of propagation)

Wavefront: the surface of constant phase

Note: The uniform plane wave does not exist in practice.

Plane Wave

Uniform plane wave



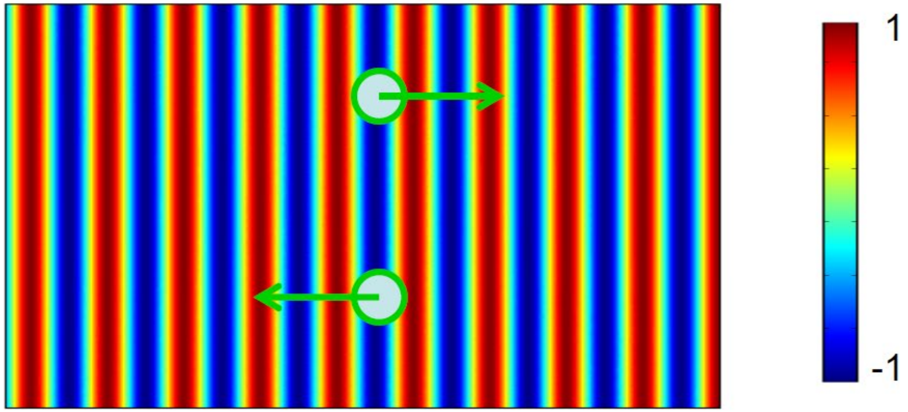
Plane Wave

Doppler effect

When there is relative motion between a time-harmonic source and a receiver, the frequency of the wave detected by the receiver tends to be different from that emitted by the source. This phenomenon is known as the ***Doppler effect***.† The Doppler effect manifests itself in acoustics as well as in electromagnetics.

Red shift:
Frequency decrease

Blue shift:
Frequency increase



Plane Wave

Plane waves in lossless media

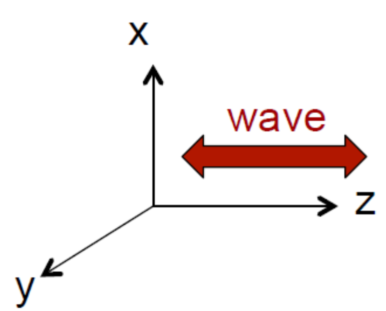
For free space, the source-free equation becomes a homogeneous vector **Helmholtz's equation**:

$$\nabla^2 \mathbf{E} + k_0^2 \mathbf{E} = 0,$$

Free-space wavenumber: $k_0 = \omega\sqrt{\mu_0 \epsilon_0} = \frac{\omega}{c} \qquad \text{(rad/m)}.$

In Cartesian coordinate, it can be expanded

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k_0^2\right) E_x = 0.$$



For uniform plane wave, E_x propagating along z axis, we have

$$\partial^2 E_x / \partial x^2 = 0 \qquad \partial^2 E_x / \partial y^2 = 0. \qquad \frac{d^2 E_x}{dz^2} + k_0^2 E_x = 0,$$

Plane Wave

Plane waves in lossless media

Solutions of Helmholtz equation of $\frac{d^2E_x}{dz^2} + k_0^2 E_x = 0$

$$E_x(z) = E_x^+(z) + E_x^-(z)$$
$$= \underbrace{E_0^+ e^{-jk_0z}}_{\text{Forward wave}} + \underbrace{E_0^- e^{jk_0z}}_{\text{Backward wave}}$$

Forward wave (propagating in the +z direction)

Backward wave (propagating in the -z direction)

Phasor: A quantity that contains amplitude and phase information but is independent of time t (D. K. Cheng, p. 337) .

Real electric field of a travelling wave (propagating in the +z direction):

$$E_x^+(z, t) = \mathscr{R}e[E_x^+(z)e^{j\omega t}]$$
$$= \mathscr{R}e[E_0^+ e^{j(\omega t - k_0 z)}]$$
$$= E_0^+ \cos(\omega t - k_0 z) \qquad \text{(V/m).}$$

Plane Wave

Plane waves in lossless media

If we fix our attention on a particular point (with a constant phase) on the wave

$$\omega t - k_0 z = \text{A constant phase}$$

Phase velocity (the velocity of propagation of an equiphase front):

$$u_p = \frac{dz}{dt} = \frac{\omega}{k_0} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c.$$

Wavenumber in vacuum:

$$k_0 = \frac{2\pi}{\lambda_0} \qquad \text{(rad/m)}$$

Plane Wave

Plane waves in lossless media

$$\nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H}$$

$$\nabla \times \mathbf{E} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ E_x^+(z) & 0 & 0 \end{vmatrix} = -j\omega\mu_0(\mathbf{a}_x H_x^+ + \mathbf{a}_y H_y^+ + \mathbf{a}_z H_z^+)$$

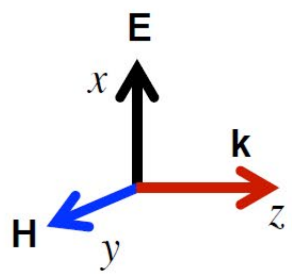
$$\boxed{E_0^+ e^{-jk_0 z}}$$

which leads to

$$H_x^+ = 0,$$

$$H_y^+ = \frac{1}{-j\omega\mu_0}\frac{\partial E_x^+(z)}{\partial z},$$

$$H_z^+ = 0.$$



Now you can see that **E**, **H**, **k** are in the x, y, z directions, respectively.

(D. K. Cheng, p. 357) **E, H, k form a right-handed system.**

Plane Wave

Plane waves in lossless media

Magnetic field calculated through electric field:

$$H_y^+ = \frac{1}{-j\omega\mu_0}\frac{\partial E_x^+(z)}{\partial z} \Leftarrow E_x^+(z) = E_0^+ e^{-jk_0 z}$$

$$= \frac{k_0}{\omega\mu_0} E_x^+(z) = \frac{1}{\eta_0} E_x^+(z) \quad \text{(A/m)}.$$

Intrinsic impedance of vacuum:

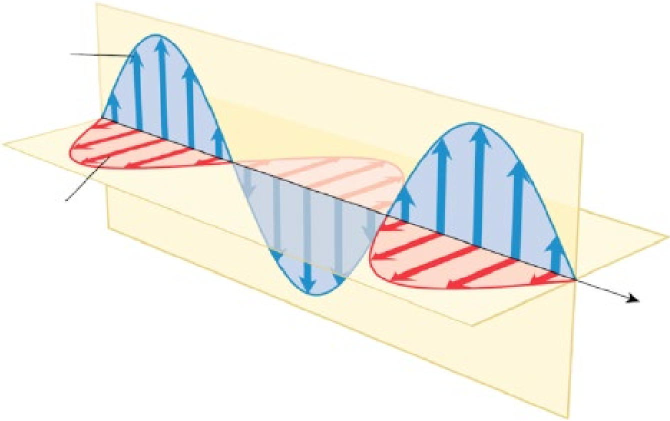
$$\boxed{\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \cong 120\pi \cong 377 \qquad (\Omega)}$$

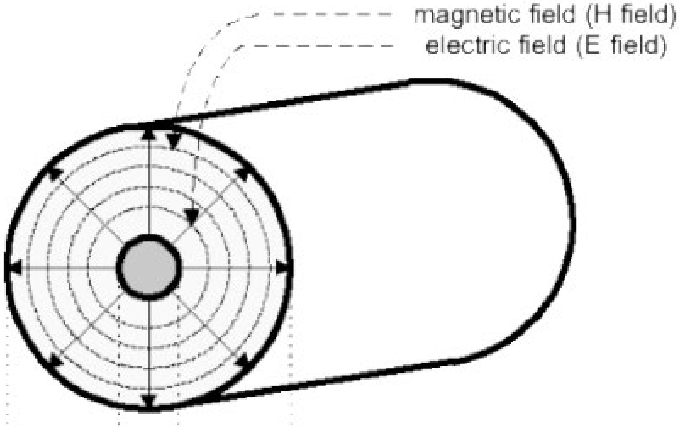
*For a uniform plane wave, the ratio of the magnitudes of **E** and **H** is the intrinsic impedance of the medium.*
(D. K. Cheng, p. 358)

Plane Wave

Plane waves in lossless media

Transverse electromagnetic (TEM) waves:
E **and** ***H*** are perpendicular to each other and both are transverse to the direction ***k*** of propagation.





Plane Wave

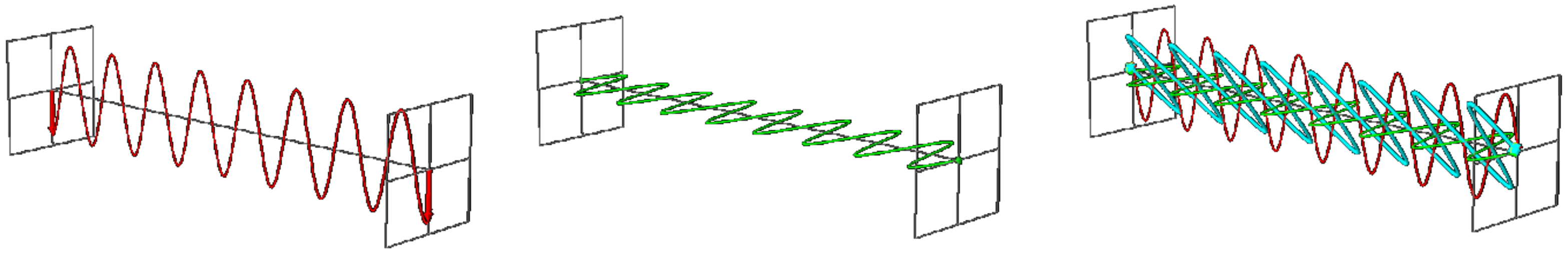
Polarisation

- E and H have components along the x,y,z directions (E_x, E_y, E_z and H_x, H_y, H_z)
- For a plane (single frequency) EM wave propagating along z
  - $E_z = H_z = 0$
  - And it is fully described by ***either* E *or* H** components (It is more usual to describe it in terms of its E components)

Polarization is a measurement of the electromagnetic field's alignment

Linear polarisation

The field oscillates in one plane only and is referred to as linear polarisation



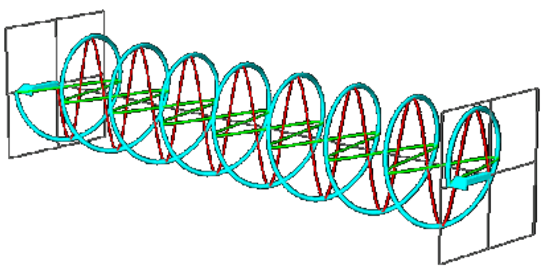
Plane Wave

Polarisation

B. Y. Toh, R. Cahill and V. F. Fusco, "Understanding and measuring circular polarization," in *IEEE Transactions on Education*, vol. 46, no. 3, pp. 313-318, Aug. 2003.

Circular polarisation

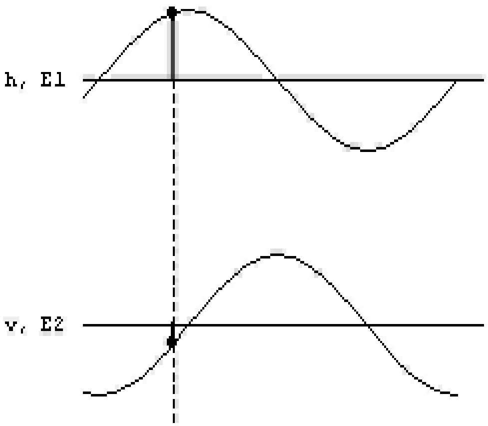
RHCP; LHCP



Two linear polarised plane waves of equal amplitude by differing 90°in phase.

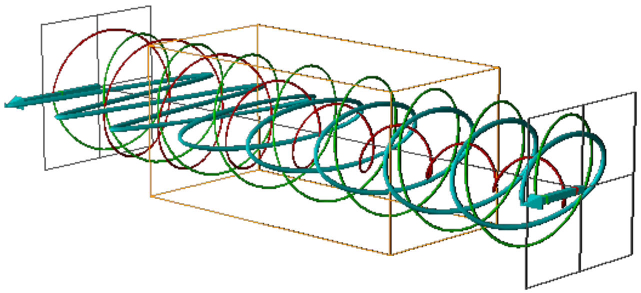
$$E_h = E_{oh} \sin \beta(z - vt)$$

$$E_v = E_{ov} \sin \beta\left(z - vt - \frac{\pi}{2}\right) = E_{ov} \cos \beta(z - vt)$$



Elliptical polarisation

Otherwise



Poynting Vector

Power flow density of an EM wave is given by the instantaneous **Poynting vector**

$$
\begin{aligned}
\mathbf{S}(t) &= \mathbf{E}(t) \times \mathbf{H}(t) = a_z \left(E_x H_y - E_y H_x \right) \\
&= a_z \eta \mathbf{H}^2 \\
&= a_z \frac{\mathbf{E}^2}{\eta}
\end{aligned}
$$

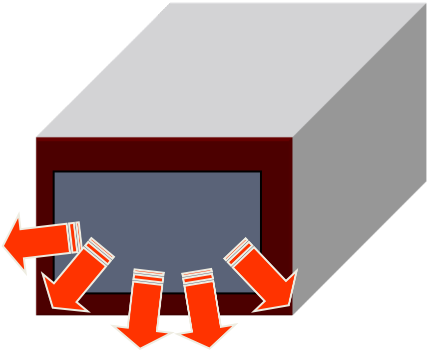
Time-average power flow density (for time harmonic fields):

$$
\begin{aligned}
\left\langle S(t) \right\rangle &= \frac{1}{T} \int_0^T S(t) \cdot dt \\
&= \frac{1}{2} \operatorname{Re}\left\{ \mathbf{E} \times \mathbf{H}^* \right\}
\end{aligned}
$$

S is the Poynting vector and indicates the direction and magnitude of power flow in the EM field.

Poynting Vector

The door of a microwave oven is left open



Estimate the peak *E* and *H* strengths in the aperture of the door.

DATA:

- Power-750 W
- Area of aperture - 0.3 m x 0.2 m
- impedance of free space - 377 Ω
- Poynting vector:

$$S = \frac{E^2}{\eta} = \eta H^2 \quad \mathrm{W/m^2}$$

Solution

E and *H* strengths in the aperture of the door

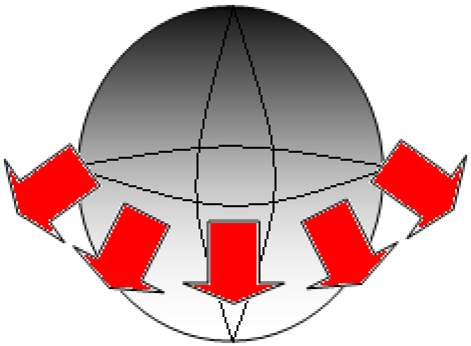
$$Power = SA = \frac{E^2}{\eta}A \quad = \eta H^2 \ \mathrm{A} \quad \mathrm{Watts}$$

$$E = \sqrt{\eta \frac{Power}{A}} = \sqrt{377 \frac{750}{0.3 \times 0.2}} = 2{,}171 k\mathrm{V/m}$$

$$\mathrm{H} = \frac{E}{\eta} = \frac{2170}{377} = 5.75 \mathrm{A/m}$$

Poynting Vector

What is the electric field strength due to an omnidirectional generator of radii 100Km radiating 1kW?



Power $P = 1\,kW$ $R = 100\,km$

Sphere surface area $S = 4\pi R^2$

Power flow density $= \frac{P}{S} = \frac{E^2}{\eta}$

$\therefore E = \sqrt{\eta \frac{P}{S}} = \sqrt{377 \frac{1\times10^3}{4\pi(1\times10^5)^2}} = 1.73\ mV/m$