MAXIMA AND MINIMA OF A FUNCTION

DEF:

1. A function $z = f(x, y)$ has a maximum (or a minimum) at the point (x_0, y_0) if at every point in a neighbourhood of (x_0, y_0) the function assumes a smaller value (or a larger value) than at the point itself. Such a maximum or minimum is ofter called relative (or local) maximum or minimum respectively.

2. For a given closed and bounded domain, a function may also attain its greatest value (or least value) on the boundary of the domain.

   The smallest and the largest values attained by a function over the entire domain including the boundary are called the absolute (or global) minimum and absolute (or global) maximum, respectively.

3. The point (x_0, y_0) is called critical point (or stationary point) of $f(x, y)$ if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.

4. A critical point where the function has no minimum or maximum is called a saddle point.

5. Minimum and maximum values together are called extreme values.

Theorem (Necessary conditions for a function to have extremum)

Let $f(x,y)$ be continuous and have first order partial derivatives at a point $P(a,b)$. Then necessary conditions for the existence of an extreme value of it at the point P are

$$f_x(a,b) = 0 \quad \& \quad f_y(a,b) = 0.$$

OR

If the point (a,b) is a relative extrema of the function $f(x,y)$ then (a,b) is also a critical point of $f(x,y)$.

<u>Proof:</u> Let $(a+h, b+k)$ be a point in the neighbourhood of the point $P(a,b)$. Then P will be a point of maximum if

$$\Delta f = f(a+h, b+k) - f(a,b) \leq 0 \text{ for all sufficiently small } h \,\&\, k$$

and a point of minimum if

$$\Delta f = f(a+h, b+k) - f(a,b) \geq 0 \text{ for all sufficiently small } h \,\&$$

Taylor's series expansion about the point (a,b):

$$f(a+h, b+k) = f(a,b) + \left(hf_x + kf_y\right)_{(a,b)} + \frac{1}{2}\left(hf_x + kf_y\right)^2_{(a,b)} + \cdots$$

For sufficiently small h & k, we can neglect second and higher order terms, to set

$$\Delta f \simeq h f_x(a,b) + k f_y(a,b)$$

The sign of Δf depends on the sign of $h f_x(a,b) + k f_y(a,b)$.

Letting $h \to 0$ we find that Δf changes sign with k,

i.e., assuming $f_y(a,b) > 0$:

for $k > 0$; $\Delta f > 0$

for $k < 0$; $\Delta f < 0$

Therefore the function cannot have an extremum unless $f_y = 0$

Similarly, letting $k \to 0$, we find that the function f cannot have an extremum unless $f_x = 0$.

Therefore the necessary conditions for the existence of an extremum at the point (a,b) is that

$f_x(a,b) = 0$ & $f_y(a,b) = 0$.

$\square$.