Problem Set - 3 — Autumn 2018

MATHEMATICS-I(MA10001) — 25/07/2018

1. Determine the limits of the following functons, if they exist:

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$,

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \log\frac{y}{x}$,

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{|x|}{y^2}\exp(-\frac{|x|}{y^2})$,

(d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\tan(xy)}$,

(e) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$,

(f) $\displaystyle\lim_{(x,y)\to(0,0)} \log(\frac{\sqrt{x^2+y^2}+x}{\sqrt{x^2+y^2}-x})$,

(g) $\displaystyle\lim_{(x,y)\to(0,0)} (\sin\frac{x}{y}+\sin\frac{y}{x})$,

(h) $\displaystyle\lim_{(x,y)\to(0,0)} \cos^3(\sqrt{x^2+y^2})$,

(i) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2y+xy^2)}{xy}$

(j) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$,

(k) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy^2z^2}{x^4+y^4+z^8}$,

(l) $\displaystyle\lim_{(x,y)\to(0,0)} \tan^{-1}(\frac{|x|+|y|}{x^2+y^2})$,

(m) $\displaystyle\lim_{(x,y)\to(2,1)} \frac{\sin^{-1}(xy-2)}{\tan^{-1}(3xy-6)}$,

(n) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y}$,

(o) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2}$,

(p) $\displaystyle\lim_{(x,y)\to(0,0)} \cos\frac{x^3-y^3}{x^2+y^2}$.

2. Using $\epsilon-\delta$ method, prove the followings:

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{2x^2+y^2}} = 0$,

(b) $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)\sin(\frac{1}{xy}) = 0$,

(c) $\displaystyle\lim_{(x,y)\to(2,1)} (x^2-2y+y^2) = 3$,

(d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy^2}{y^2+x^2} = 0$,

(e) $\displaystyle\lim_{(x,y)\to(-1,-1)} (xy-2x^2) = -1$,

(f) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{(x-1)^2\ln x}{y^2+(x-1)^2} = 0$,

(g) $\displaystyle\lim_{(x,y)\to(-2,2)} \frac{x^2-y^2}{y+x} = -4$,

(h) $\displaystyle\lim_{(x,y)\to(0,0)} xy\frac{x^2-y^2}{y^2+x^2} = 0$,

(i) $\displaystyle\lim_{(x,y)\to(0,0)} x\sin x\cos y = 0$,

(j) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{y^2+x^2}} = 0$,

(k) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{y^2+x^2} = 0$,

(l) $\displaystyle\lim_{(x,y)\to(1,1)} (x^2+y^2-1) = 1$,

(m) $\lim_{(x,y)\to(0,0)} \dfrac{x^4y - 3x^2y^3 + y^5}{(x^2+y^2)^2} = 0,$

(n) $\lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^2} = 0.$

(o) $\lim_{(x,y)\to(0,0)} \left[y\sin\left(\dfrac{x}{y}\right) + x\sin\left(\dfrac{y}{x}\right)\right] = 0.$

(p) $\lim_{(x,y,z)\to(0,0,0)} \dfrac{xyz}{\sqrt{x^2+y^2+z^2}} \sin(\dfrac{1}{xyz}) = 0$

3. Using $\epsilon - \delta$ method, show that the following functions are continuous:

(a) $f(x,y) = \begin{cases} xy, & (x,y) \neq (2,3); \\ 6, & (x,y) = (2,3). \end{cases}$

(b) $f(x,y) = \begin{cases} \dfrac{5x^2y^2}{x^2+y^2}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$

(c) $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$

(d) $f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$

(e) $f(x,y) = \begin{cases} \dfrac{y(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$

(f) $f(x,y) = \begin{cases} \dfrac{x^2+y^2}{|x|+|y|}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$

4. Discuss the continuity of the following functions at $(0,0)$:

(a) $f(x,y) = \begin{cases} \dfrac{1}{x^2+y^2}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$

(b) $f(x,y) = \begin{cases} \dfrac{x^3y^3}{x^2+y^2}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$

(c) $f(x,y) = \begin{cases} \dfrac{|xy|}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$

(d) $f(x,y) = \begin{cases} \dfrac{|xy|}{xy}, & xy \neq 0; \\ 0, & xy = 0. \end{cases}$

(e) $f(x,y) = \begin{cases} \dfrac{e^{xy}}{x^2+y^2}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$

(f) $f(x,y) = \begin{cases} \dfrac{3x^2y-y^3}{x^2+y^2}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$

(g) $f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$

(h) $f(x,y) = \begin{cases} \dfrac{\sin xy}{xy}, & xy \neq 0; \\ 1, & xy = 0. \end{cases}$

(i) $f(x,y) = \begin{cases} x\sin\dfrac{1}{y} + y\sin\dfrac{1}{x}, & xy \neq 0; \\ 0, & xy = 0. \end{cases}$

(j) $f(x,y) = \begin{cases} \dfrac{(x-y)^2}{x^2+y^2}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$

(k) $f(x,y) = \begin{cases} \dfrac{2x^2+y^2}{3+\sin x}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$

(l) $f(x,y) = \begin{cases} \dfrac{x^2+\sin^2 y}{2x^2+y^2}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$

5. For what values of n, the following function f is continuous at $(0,0)$:

$$f(x,y) = \begin{cases} \dfrac{2xy}{(x^2+y^2)^n}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$$

6. Find the values of c for which the following functions f are continuous at $(0,0)$:

(a) $f(x,y) = \begin{cases} \dfrac{x^4-y^4}{x^2+y^2}, & (x,y) \neq (0,0); \\ c, & (x,y) = (0,0). \end{cases}$

(b) $f(x,y) = \begin{cases} x^2\log(x^2+y^2), & (x,y) \neq (0,0); \\ c, & (x,y) = (0,0). \end{cases}$

(c) $f(x,y) = \begin{cases} \dfrac{\sin(x^2+y^2)}{x^2+y^2}, & (x,y) \neq (0,0); \\ c, & (x,y) = (0,0). \end{cases}$

(d) $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2}, & (x,y) \neq (0,0); \\ c, & (x,y) = (0,0). \end{cases}$

(e) $f(x,y) = \begin{cases} \dfrac{(x-1)^2\log x}{(x-1)^2+y^2}, & (x,y) \neq (1,0); \\ c, & (x,y) = (1,0). \end{cases}$

(f) $f(x,y) = \begin{cases} \dfrac{e^{-(x^2+y^2)}-1}{x^2+y^2}, & (x,y) \neq (0,0); \\ c, & (x,y) = (0,0). \end{cases}$

(g) $f(x,y) = \begin{cases} \exp\left(-\dfrac{1}{|x-y|}\right), & x \neq y; \\ c, & x = y. \end{cases}$

(h) $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{1+x^2+y^2}, & (x,y) \neq (0,0); \\ c, & (x,y) \neq (0,0). \end{cases}$

7. Do the following functions have any point of discontinuities? Explain.

   (a) $f(x,y) = \dfrac{x-y}{1+x+y}$,

   (b) $f(x,y) = \dfrac{x-y}{1+x^2+y^2}$.

   (c) $f(x,y) = \dfrac{xy}{1+e^{x-y}}$.

8. Find the point of discontinuities of the following functions.

   (a) $f(x,y) = \dfrac{1}{\sin^2 \pi x + \sin^2 \pi y}$,

   (b) $f(x,y) = \dfrac{e^x + e^y}{e^{xy} - 1}$.

9. Is it possible to define the function $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ at $(0,0)$ such that the function is continuous?

10. Let $f(x,y) = \begin{cases} 0, & xy \neq 0; \\ 1, & xy = 0. \end{cases}$

   (i) Find the limit of $f(x,y)$ as $(x,y) \to (0,0)$ along the line $y = x$.

   (ii) Is $f(x,y)$ continuous at origin?