EXAMPLE: (CONTINUOUS, PARTIAL DERIVATIVES EXIST BUT NOT DIFFERENTIABLE)

$$f(x,y) = \begin{cases} \dfrac{x^3+2y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

i) CONTINUITY: $\displaystyle\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^3+2y^3}{x^2+y^2}$ (Necessary for differentiability)

Changing to polar coordinates:

$$\lim_{r \to 0} \frac{r^3\cos^3\theta + 2r^3\sin^3\theta}{r^2}$$

$$= \lim_{r \to 0} r\left(\cos^3\theta + 2\sin^3\theta\right) = 0 = f(0,0)$$

ALTERNATIVE:

$$|f(x,y) - 0| = \left|\frac{r^3\cos^3\theta + 2r^3\sin^3\theta}{r^2}\right|$$

(subst. $x = r\cos\theta$, $y = r\sin\theta$)

$$\leq r\,|\cos^3\theta| + 2r\,|\sin^3\theta|$$

$$< 3r \overset{!}{<} \varepsilon$$

Choose $\delta < \frac{\varepsilon}{3}$ then

$|f(x,y) - 0| < \varepsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$

$\Rightarrow$ $f(x,y)$ is continuous at $(0,0)$.

ii) Existence of partial derivatives: (Necessary for differentiability)

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x^3}{\Delta x^3} = 1.$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{2\Delta y^3}{\Delta y^3} = 2$$

iii) Differentiability:

$$\lim_{\Delta\rho \to 0} \frac{\Delta z - dz}{\Delta \rho} \neq 0$$

as.

$$\Delta z = f(0+\Delta x, 0+\Delta y) - f(0,0)$$

$$= \frac{\Delta x^3 + 2\Delta y^3}{\Delta x^2 + \Delta y^2}$$

$$dz = \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y$$

$$= \Delta x + 2\Delta y$$

Now :

$$\lim_{\Delta\rho \to 0} \frac{\Delta z - dz}{\Delta \rho}$$

$$= \lim_{\Delta\rho \to 0} \left[\frac{\Delta x^3 + 2\Delta y^3}{\Delta x^2 + \Delta y^2} - (\Delta x + 2\Delta y) \right] \frac{1}{\sqrt{\Delta x^2 + \Delta y^2}}$$

$$= \lim_{\Delta\rho \to 0} \frac{-\Delta x \Delta y^2 - 2\Delta x^2 \Delta y}{(\Delta x^2 + \Delta y^2)^{3/2}}$$

Along the path $\Delta y = m \Delta x$

$$\Rightarrow \lim_{\Delta x \to 0} - \frac{m^2 - 2m}{(1+m^2)^{3/2}}$$

limit depends on the path.

$\Rightarrow$ The given function is not differentiable.

EXAMPLE: (FUNCTION IS DIFFERENTIAL BUT f_x & f_y are not continuous)

$$f(x,y) = \begin{cases} (x^2+y^2)\cos\left[\dfrac{1}{\sqrt{x^2+y^2}}\right] & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

i) Continuity: $\displaystyle\lim_{\substack{x\to 0\\ y\to 0}} (x^2+y^2)\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right) = 0 = f(0,0)$

ii) Existence of partial derivatives:

$$f_x(0,0) = \lim_{\Delta x\to 0} \frac{f(\Delta x,0)-f(0,0)}{\Delta x} = \lim_{\Delta x\to 0} \Delta x \cos\left(\frac{1}{|\Delta x|}\right) = 0$$

$$f_y(0,0) = \lim_{\Delta y\to 0} \frac{f(0,\Delta y)-f(0,0)}{\Delta y} = \lim_{\Delta y\to 0} \Delta y \cos\left(\frac{1}{|\Delta y|}\right) = 0$$

iii) Differentiability:

$$dz = z_x \Delta x + z_y \Delta y = 0$$

$$\lim_{\Delta\rho\to 0}\left(\frac{\Delta z - dz}{\Delta \rho}\right) = \lim_{\Delta\rho\to 0} \frac{(\Delta x^2+\Delta y^2)}{\sqrt{\Delta x^2+\Delta y^2}}\cos\left(\frac{1}{\sqrt{\Delta x^2+\Delta y^2}}\right)$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \sqrt{\Delta x^2+\Delta y^2}\,\cos\left(\frac{1}{\Delta x^2+\Delta y^2}\right) = 0$$

Hence the function is differentiable.

iv) Continuity of f_x & f_y.

At $(x,y)\neq(0,0)$: $f_x(x,y) = -(x^2+y^2)\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right)\cdot\left(-\dfrac{1}{2}\,\dfrac{1\cdot 2x}{(x^2+y^2)^{3/2}}\right)$

$$+\ 2x\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

$$= \frac{x}{\sqrt{x^2+y^2}}\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) + 2x\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

Along x-axis: $\displaystyle\lim_{x\to 0} f_x(x,y) = \lim_{x\to 0}\left[\frac{x}{|x|}\sin\left(\frac{1}{|x|}\right) + 2x\cos\left(\frac{1}{|x|}\right)\right] \neq 0$

Hence f_x is not continuous at $(0,0)$. Similarly one can show that f_y is not continuous.

This example shows that continuity of partial 1st order derivatives is not a necessary condition for differentiabity. A function can be differentiable without having first order partial derivatives continuous.

Ex. For the function

$$f(x,y) = \begin{cases} \dfrac{x^2 y(x-y)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Find $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ at $(0,0)$.

Sol:

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0,0)}{\Delta x}$$

where

$$f_y(\Delta x, 0) = \lim_{\Delta y \to 0} \frac{f(\Delta x, \Delta y) - f(\Delta x, 0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\Delta x^2 \cancel{\Delta y}(\Delta x - \Delta y)}{(\Delta x^2 + \Delta y^2)\cancel{\Delta y}} = \Delta x$$

Hence:

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} = \lim_{\Delta x \to 0} \frac{\Delta x - 0}{\Delta x} = 1$$

because

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} = 0$$

Now:

$$\left.\frac{\partial^2 f}{\partial y\, \partial x}\right|_{(0,0)} = \lim_{\Delta y \to 0} \frac{f_x(0, \Delta y) - f_x(0,0)}{\Delta y}$$

where. $$f_x(0, \Delta y) = \lim_{\Delta x \to 0} \frac{f(\Delta x, \Delta y) - f(0, \Delta y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x^2 \Delta y (\Delta x - \Delta y)}{\Delta x (\Delta x^2 + \Delta y^2)} = 0$$

& $$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = 0$$

Hence

$$\left.\frac{\partial^2 f}{\partial y\, \partial x}\right|_{(0,0)} = \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} = 0$$

Ex: Test the continuity and existence of f_x & f_y at the origin of the followin function:

$$f(x,y) = \begin{cases} 0 & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0 \end{cases}$$

Sol. limit along $x = y$: $\lim_{x \to 0} f(x,y) = 0$

Since $f(0,0) = 1$, f is not continuous at $(0,0)$.

$$f_x|_{(0,0)} = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{1-1}{\Delta x} = 0$$

$$f_y|_{(0,0)} = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{1-1}{\Delta y} = 0.$$

$\Rightarrow$ First order partial derivatives exist at $(0,0)$

Ex. Test the differentiability of the following function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

at the origin.

Sol: Clearly the function is continuous at the origin as

$$\lim_{x\to 0, y\to 0} f(x,y) = \lim_{r\to 0} r\cos\theta \sin\theta = 0$$

Existence of partial derivatives:

$$f_x(0,0) = \lim_{\Delta x\to 0} \frac{f(\Delta x,0) - f(0,0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{0}{\Delta x} = 0$$

Similarly: $$f_y(0,0) = \lim_{\Delta y\to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} = 0$$

So, $$df = \Delta x . f_x + \Delta y f_y = 0$$

$$\lim_{\substack{\Delta x\to 0 \\ \Delta y \to 0}} \frac{\Delta f - df}{\sqrt{\Delta x^2 + \Delta y^2}} = \lim_{\substack{\Delta x\to 0 \\ \Delta y \to 0}} \frac{\Delta x \Delta y}{\Delta x^2 + \Delta y^2}$$

Along the path $\Delta y = m \Delta x$:

$$= \lim_{\Delta x\to 0} \frac{m}{1+m^2} = \frac{m}{1+m^2}$$ limit does not exist.

$\Rightarrow$ f is not differentiable.

Ex. Find the total differential and the total increment of the function $z = xy$ at the point $(2,3)$ for $\Delta x = 0.1$, $\Delta y = 0.2$.

Sol.

$$\Delta z = (x+\Delta x)(y+\Delta y) - xy$$

$$= x\,\Delta y + y\,\Delta x + \Delta x\,\Delta y$$

& $$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = y\,dx + x\,dy = y\,\Delta x + x\,\Delta y$$

Consequently: $$dz = 3.(0.1) + 2(0.2)$$

$$= 0.3 + 0.4 = 0.7$$

$$\Delta z = 0.7 + 0.1 \times 0.2$$

$$= 0.7 + 0.02 = 0.72$$

Q. Let $$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Discuss the continuity of f_{yx} at $(0,0)$.

Sol:
$$f_x = \frac{(x^2+y^2)\,2xy^2 - x^2y^2(2x)}{(x^2+y^2)^2} = \frac{2xy^4}{(x^2+y^2)^2}$$

$$f_{yx} = \frac{8x^3y^3}{(x^2+y^2)^3}$$

Along the path $y = mx$:

$$\lim_{x\to 0} f_{yx} = \lim_{x\to 0} \frac{8x^3m^3x^3}{(x^2+m^2x^2)^3} = \frac{8m^3}{(1+m^2)^3}$$

$\Rightarrow$ f_{yx} is not continuous at $(0,0)$.