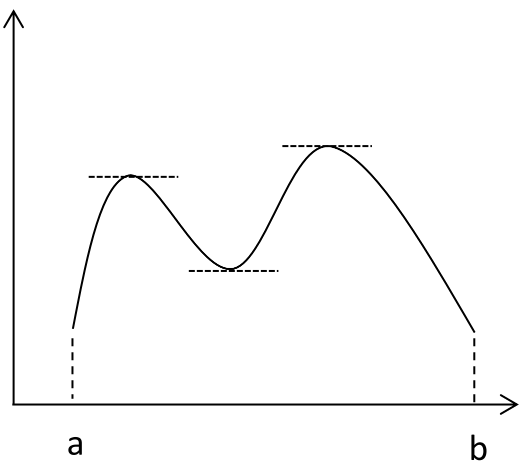
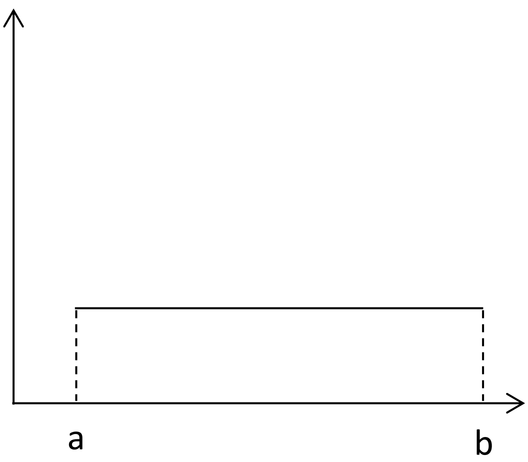
Rolle's Theorem:

If a function f is

a) continuous in [a, b]
b) differentiable in (a, b)
c) $f(a) = f(b)$

Then $\exists\ c \in (a, b)$ s.t. $f'(c) = 0$





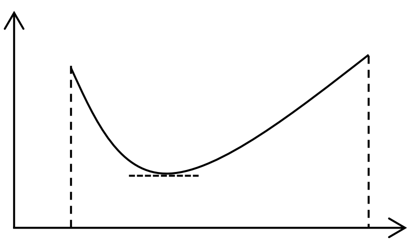
Proof: Suppose M & m are maximum and minimum of $f(x)$ in $[a, b]$.

(It will always exist because of Weierstrass extreme value theorem as f is continuous in [a, b])

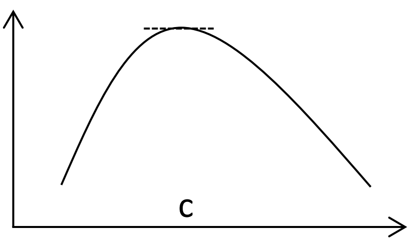
Case I: if $M = m$ i.e. $f(x) = M = m = constant$

This implies $f'(x) = 0 \quad \forall x \in (a, b)$

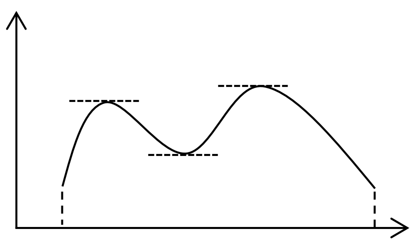
Case II: $M \neq m$. Then at least one of them must be different from equal values of $f(a)$ and $f(b)$.



Minimum is different from f(a) and f(b)



Maximum is different from f(a) and f(b)



Both are different from f(a) and f(b)

Let $M = f(c)$ be different. Since f is differentiable in (a, b), $f'(c)$ exists. Note that $f(c)$ is the maximum value, then

$f(c + \Delta x) - f(c) \leq 0$ for $\Delta x > 0$ or $\Delta x < 0$

This implies:

$$\frac{f(c+\Delta x) - f(c)}{\Delta x} \leq 0, \quad \text{for } \Delta x > 0$$

and

$$\frac{f(c+\Delta x) - f(c)}{\Delta x} \geq 0, \quad \text{for } \Delta x < 0$$

Since $f'(c)$ exists, passing limit as $\Delta x \to 0$, we get

$$\operatorname*{Lim}_{\substack{\Delta x \to 0 \\ (\Delta x > 0)}} \frac{f(c+\Delta x) - f(c)}{\Delta x} = f'(c) \leq 0 \tag{1}$$

and

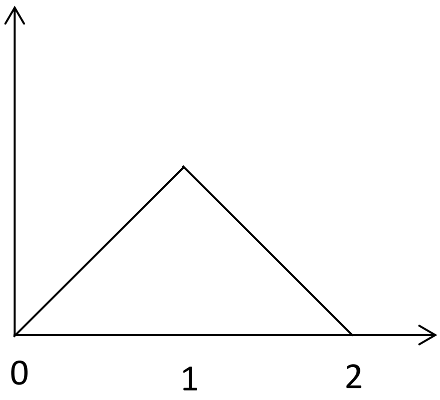
$$\operatorname*{Lim}_{\substack{\Delta x \to 0 \\ (\Delta x < 0)}} \frac{f(c+\Delta x) - f(c)}{\Delta x} = f'(c) \geq 0 \tag{2}$$

Inequality (1) and (2) implies $f'(c) = 0$.

Remark 1: The conclusion of Rolle's theorem may not hold for a function that does not satisfy any of its conditions.

Ex 1: Consider

$$f(x) = \begin{cases} x, & x \in [0, 1] \\ 2 - x, & x \in (1, 2] \end{cases}$$

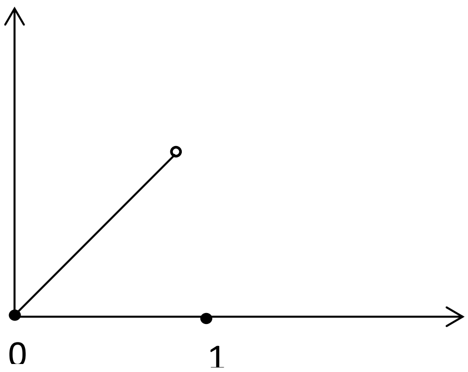


Note that $f'(x) \neq 0$ for any $x \in (1, 2)$. However, this does not contradict Rolle's Theorem, since $f'(1)$ does not exist.

Remark 2: The continuity condition for the function on the closed interval [a, b] is essential.

Ex: Consider

$$f(x) = \begin{cases} x, & x \in [0, 1) \\ 0, & x = 1 \end{cases}$$



Then, f is continuous and differentiable on (0, 1), and also $f(0) = f(1)$. But $f'(x) \neq 0$ for any $x \in (0, 1)$.

Remark 3: The hypotheses of Rolle's theorem are sufficient but not necessary for the conclusion. Meaning, if all three hypotheses are met then conclusion is guaranteed. Not necessary means if the hypotheses are not met then you may (or may not) reach the conclusion.

Example: Discuss the applicability of Rolle's theorem to the function

$$f(x) = \begin{cases} x^2 + 1\,, & x \in [0, 1] \\ 3 - x, & x \in (1, 2] \end{cases}$$

Solution:

1) Continuity

$$f(1+0) = \lim_{\substack{\Delta x \to 0 \\ \Delta x > 0}} 3 - (1 + \Delta x) = \lim_{\substack{\Delta x \to 0 \\ \Delta x > 0}} [2 - \Delta x] = 2 = f(1)$$

2) Differentiability

$$f'(1+0) = \lim_{\substack{\Delta x \to 0 \\ \Delta x > 0}} \frac{f(1+\Delta x) - f(1)}{\Delta x} = \lim_{\substack{\Delta x \to 0 \\ \Delta x > 0}} \frac{(2 - \Delta x) - 2}{\Delta x} = -1$$

$$f'(1-0) = \lim_{\substack{\Delta x \to 0 \\ \Delta x < 0}} \frac{f(1+\Delta x) - f(1)}{\Delta x} = \lim_{\substack{\Delta x \to 0 \\ \Delta x < 0}} \frac{(1+\Delta x)^2 + 1 - 2}{\Delta x}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta x < 0}} \frac{2\Delta x + \Delta x^2}{\Delta x} = 2$$

Thus $f'(1+0) \neq f'(1-0)$. This implies f is not differentiable.

Example: Using Rolle 's Theorem, show that the equation $x^{13} + 7x^3 - 5 = 0$ has exactly one real root in [0, 1].

Solution: Let $f(x) = x^{13} + 7x^3 - 5$ has two real roots, say α and β in [0, 1]. That is, we have $f(\alpha) = f(\beta) = 0$. All hypotheses of Rolle's theorem are satisfied in $[\alpha, \beta]$.

Rolle 's Theorem implies $f'(c) = 0$ for some $c \in (\alpha, \beta)$.

$\Rightarrow 13\, c^{12} + 21c^2 = 0$ for some $c \in (\alpha, \beta)$. Note that $c > 0$ as $\alpha \geq 0$. It contradicts our assumption of two real roots.

On the other hand $f(0) = -5$ and $f(1) = 3$. It confirms the existence of at least one root. Hence the function has exactly one root.

Lagrange's mean value theorem:

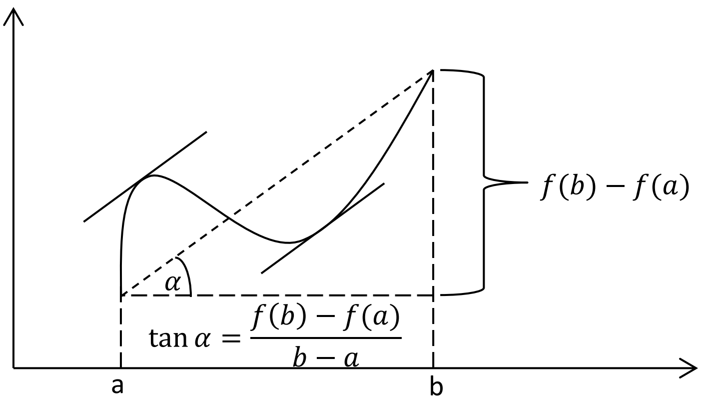
If a function f is

a) continuous in [a, b]
b) differentiable in (a, b)

then there exists at least one value $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

In other words, there is at least one tangent line in the interval that is parallel to the secant line that goes through the endpoints of the interval.



Proof: Define a function

$$\phi(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}\right] x$$

Note that the function $\phi(x)$ satisfies all the conditions of Rolle 's Theorem as $\phi(a) = \phi(b)$, and continuity and differentiability follows from the continuity and differentiability of $f(x)$. Rolle 's Theorem gives

$\phi'(c) = 0$ for some $c \in (a, b) \Rightarrow f'(c) - \frac{f(b)-f(a)}{b-a} = 0.$

Generalized mean value theorem (Cauchy mean value theorem):

If $f(x)$ and $g(x)$ are two functions continuous in [a, b] and differentiable in (a, b), and $g'(x)$ does not vanish anywhere inside the interval then ∃ a point c in (a, b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Proof: Define

$$\phi(x) = \big(f(x)-f(a)\big) - \left[\frac{f(b)-f(a)}{g(b)-g(a)}\right]\big(g(x)-g(a)\big)$$

Note that $g(b) \neq g(a)$ because g' does not vanish in (a, b). If $g(b) = g(a)$ then Rolle's Theorem implies $g'(c) = 0$, which contradicts the assumption that $g'(x) \neq 0$.

$\phi(x)$ satisfies all hypotheses of the Rolle's theorem on the interval [a, b]. Then there exists a point $c \in (a, b)$ such that $c \in (a, b)$ and $\phi'(c) = 0$.

$$\Rightarrow f'(c) - \left[\frac{f(b)-f(a)}{g(b)-g(a)}\right] g'(c) = 0$$

$$\Rightarrow \left[\frac{f(b)-f(a)}{g(b)-g(a)}\right] = \frac{f'(c)}{g'(c)} \ .$$

Notice that:

$$\text{Generalized MVT} \xRightarrow{g(x)=x} \text{Lagrange MVT} \xRightarrow{f(x)=f(a)} \text{Rolle's Theorem}$$

Ex: Using mean value theorem show that

$|\cos e^x - \cos e^y| \leq |x - y|$ for $x, y \leq 0$ (equality holds for x = y = 0)

Sol: Consider $f(t) = \cos e^t$ in the interval $[x, y]$. Using Lagrange mean value theorem

$$\frac{\cos e^x - \cos e^y}{x - y} = f'(c), \qquad c \in (x, y)$$

$$\Rightarrow |\cos e^x - \cos e^y| \leq |x - y| \max_{c \in (x,y)} f'(c) < |x - y|$$

as $f'(t) = -\mathrm{e}^{\mathrm{t}} \sin e^t \Rightarrow |f'(t)| = |e^t||\sin e^t| < 1 \text{ for } t < 0$

Ex: Using mean value theorem show that

$$\ln(1 + x) \leq \frac{x}{\sqrt{(1+x)}} \text{ for } x > 0.$$

Hint: Consider $f(t) = \ln(1 + t) - \frac{t}{\sqrt{(1+t)}}$ in the interval $[0, x]$.

Also use the inequality $1 + \frac{t}{2} \geq \sqrt{(1 + t)}$