Taylor theorem with Lagrange' form of Reminder

If f is such that $f^{(n-1)}$ is continuous on $[a, a+h]$, nth derivative exists on $(a, a+h)$, p is a positive integer there exists at least one number θ between 0 and 1 such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n(1-\theta)^{n-p}}{(n-1)!\,p} f^{(n)}(a+\theta h).$$

$f^{(n-1)}$ continuous on $[a, a+h] \Rightarrow$

$f, f', \ldots f^{n-2}$ $C[a, a+h]$.

Let
$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) \cdots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n-1)}(x) + A(a+h-x)^p$$

where A is to be determined such that $\phi(a) = \phi(a+h)$. $b = a+h$. (Rolle's thm)

by Rolle's thm $\exists\, \theta$. $0 < \theta < 1$ $\ni$ $\phi'(a+\theta h) = 0$

but
$$\phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n)}(x) - pA(a+h-x)^{p-1}$$

$$\therefore 0 = \phi'(a+\theta h) = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h)$$
$$- pA(1-\theta)^{p-1} h^{p-1}$$

$$\Rightarrow A = \frac{h^{n-p}(1-\theta)^{n-p}}{p(n-1)!} f^n(a+\theta h), \quad 1-\theta \neq 0, \ h \neq 0.$$

<u>Reminder</u> $\quad R_n = \frac{h^n(1-\theta)^{n-p}}{p(n-1)!} f^n(a+\theta h).$

<u>$p=1$</u> $\quad R_n = \frac{h^n(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h)$ Cauchy

$p=n$, $\quad R_n = \frac{h^n}{n!} f^n(a+\theta h).$ Lagrange

<u>Corollary</u> $a+h \to x$ i.e $h \to x-a.$

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a)$$
$$+ \cdot + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(x-a)^n}{p} \frac{(1-\theta)^{n-p}}{(n-1)!} f^n(a+\theta(x-a))$$
$$0 < \theta < 1.$$

<u>Cor 2</u> <u>Maclaurin</u> $a=0$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$$
$$+ \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + x^n \frac{(1-\theta)^{n-p}}{p(n-1)!} f^n(\theta x).$$