Problem Set - 10

AUTUMN 2017

MATHEMATICS-I (MA10001)

1. Find the following limits (if exists)

   (a) $\lim_{z\to 0} \dfrac{(\mathrm{Re}(z)-\mathrm{Im}(z))^2}{|z|^2}$

   (b) $\lim_{z\to 0} \left[\dfrac{1}{1-e^{\frac{1}{x}}}+iy^2\right]$

   (c) $\lim_{z\to 0} \left(\dfrac{z}{\bar{z}}\right)^2$

   (d) $\lim_{z\to 0} \dfrac{z}{|z|}$

2. Test the continuity of the following functions:

$$(a)\quad f(z)=\begin{cases}\frac{\mathrm{Im}(z)}{|z|} & \text{if } z\neq 0\\ 0 & \text{if } z=0\end{cases} \qquad (b)\quad f(z)=\begin{cases}\frac{z^2}{|z|} & \text{if } z\neq 0\\ 0 & \text{if } z=0\end{cases} \qquad (c)\quad f(z)=\begin{cases}\frac{\mathrm{Re}(z^2)}{|z|^2} & \text{if } z\neq 0\\ 0 & \text{if } z=0\end{cases}$$

3. Examine the continuity of $f(z)$ at $z=0$, where $f(z)$ is

$$f(z)=\begin{cases}\frac{xy^2}{x^2+y^4} & \text{if } z\neq 0\\ 0 & \text{if } z=0\end{cases}$$

4. Show that

$$f(z)=\begin{cases}\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2} & \text{if } z\neq 0\\ 0 & \text{if } z=0\end{cases}$$

   satisfies Cauchy Reimann equations at $z=0$ but $f'(0)$ does not exist.

5. Show that for the function,

$$f(z)=\begin{cases}\frac{xy^2(x+iy)}{x^2+y^4} & \text{if } z\neq 0\\ 0 & \text{if } z=0\end{cases}$$

   $f'(0)$ does not exist but it satisfies Cauchy Reimann equations at $(0,0)$.

6. Using Cauchy Reimann equations show that

   (a) $f(z)=|z|^2$ is not analytic at any point.

   (b) $f(z)=\bar{z}$ is not analytic at any point.

   (c) $f(z)=\dfrac{1}{z}$, $z\neq 0$ is analytic at all points except at the point $z=0$.

7. Show that the function $Log z$ is analytic for all z except the point $\{z : \mathrm{Re}\, z \leq 0,\ \mathrm{Im}\, z = 0\}$.

8. Let $f(z)=u+iv$ be analytic in a domain D. Prove that f is constant in D if any one of the followings hold.

   (a) $f'(z)$ vanishes in D.

   (b) $\mathrm{Re}\, f(z)=u=$ constant.

   (c) $\mathrm{Im}\, f(z)=v=$ constant.

   (d) $|f(z)|=$ constant (non zero).

9. Show that the function $u=\cos x\cosh y$ is harmonic. Find its harmonic conjugate.

10. Show that following functions are harmonic:

   (a) $u(x,y)=2x+y^3-3x^2y$

(b) $v(x, y) = e^x \sin y$

and find their harmonic conjugates and the corresponding analytic functions $f(z)$.

11. $u(r, \theta) = r^2 \cos 2\theta$ is harmonic. Find its conjugate harmonic function and the corresponding analytic function $f(z)$.

12. If $f(z)$ is analytic function of z, then prove that

   (a) $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} = 4\dfrac{\partial^2}{\partial z \partial \bar{z}}$.

   (b) $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$.

13. If $\nabla = \left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right)$, then prove that (a) $\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial \bar{z}}$ (b) $\dfrac{\partial}{\partial y} = i\left(\dfrac{\partial}{\partial z} - \dfrac{\partial}{\partial \bar{z}}\right)$ (c) $\nabla = 2\dfrac{\partial}{\partial \bar{z}}$.

14. Find the values of constants a, b, c and d such that the function $f(z) = (x^2 + axy + by^2) + i(cx^2 + dxy + y^2)$ is analytic.

15. Suppose $f(z) = u + iv$ is analytic at $z_0 \neq 0$. Show that

$$f'(z_0) = -\frac{i}{z_0}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)$$

at $z = z_0$, where (r, θ) are the polar coordinates.