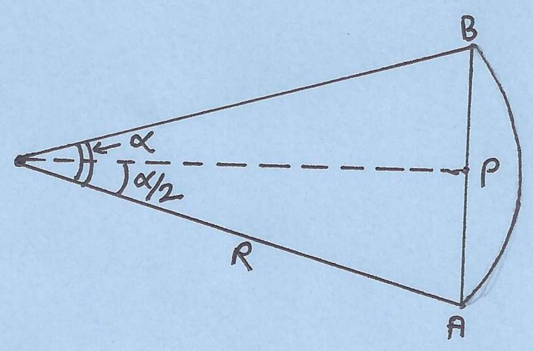
CURVATURE OF A CURVE

(RATE OF CHANGE OF BENTNESS)

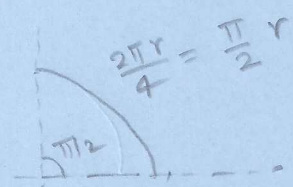
- RATIO OF THE LENGTH OF ARC OF A CIRCLE TO THE LENGTH OF ITS CORD



$$\lim_{\alpha \to 0} \frac{\text{arc } AB}{\text{Cord } AB} = ?$$

Note that $\quad AP = R \sin\frac{\alpha}{2} \Rightarrow AB = 2R\sin\frac{\alpha}{2}$

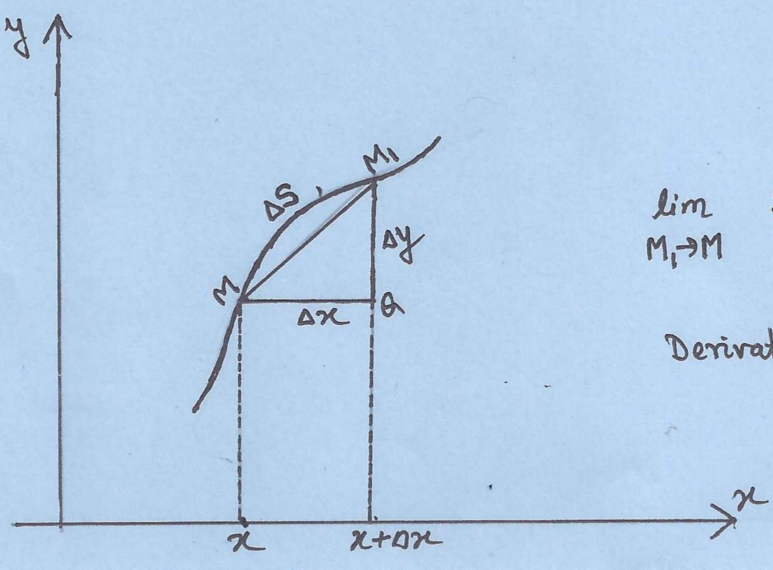
$$\frac{\text{arc } AB}{\text{Cord } AB} = \frac{\alpha R}{2R\sin\frac{\alpha}{2}} = \frac{(\alpha/2)}{\sin\left(\frac{\alpha}{2}\right)}$$



Now $\quad \lim_{\alpha \to 0} \frac{\text{arc } AB}{\text{Cord } AB} = \lim_{\alpha \to 0} \frac{(\alpha/2)}{\sin(\alpha/2)} = 1.$

NOTE: The above result holds for any curve (NOT ONLY FOR A CIRCLE)

- RATE OF CHANGE OF THE ARC WITH RESPECT TO ABSCISSA



$$\lim_{M_1 \to M} \frac{\Delta S}{\Delta x} = \frac{ds}{dx} = ?$$

Derivative of s with respect to x ?

From triangle ΔMM_1Q:

$$\overline{MM_1}^2 = \Delta x^2 + \Delta y^2 \qquad (\overline{MM_1} \text{ is the cord } MM_1)$$

$$\Rightarrow \frac{(\overline{MM_1})^2}{(\Delta s)^2} \cdot (\Delta s)^2 = \Delta x^2 + \Delta y^2$$

$$\Rightarrow \left(\frac{\overline{MM_1}}{\Delta s}\right)^2 \cdot \left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$$

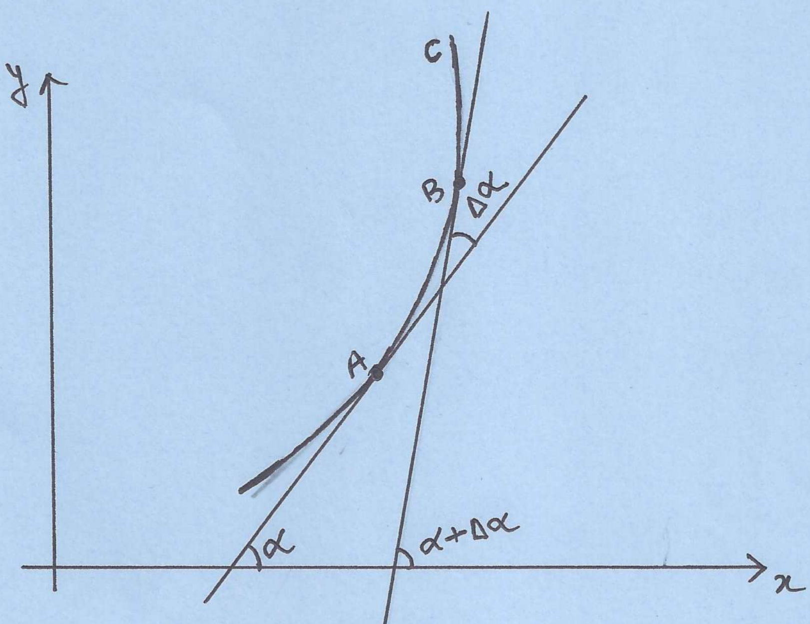
$$\Rightarrow \lim_{\overline{MM_1} \to 0} \left(\frac{\overline{MM_1}}{\Delta s}\right)^2 \cdot \lim_{\overline{MM_1} \to 0} \left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \lim_{\overline{MM_1} \to 0} \left(\frac{\Delta y}{\Delta x}\right)^2$$

$$\Rightarrow \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \boxed{\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

We make an convention that for the curve $y = f(x)$, 's' is measured positively in the direction of x increasing so that s increasi[ng] with x. Hence $\frac{ds}{dx}$ is positive.

CURVATURE:



Def.

1. Angle of Contingence ($\Delta\alpha$) of the arc AB of a curve C is the angle between the tangents A and B to the curve C.

2. The average curvature K_{av} of an arc $\widehat{AB}$ is the ratio of the corresponding angle of contingence $\Delta\alpha$ to the length of the arc:

$$K_{av} = \frac{\Delta\alpha}{\widehat{AB}}$$

3. The curvature K. of a curve at a given point A is the limit of the average curvature of the arc AB when the length of the arc approaches to zero.

$$K = \lim_{B \to A} K_{av} = \lim_{B \to A} \frac{\Delta\alpha}{\widehat{AB}} = \lim_{\Delta s \to 0} \frac{\Delta\alpha}{\Delta s} = \frac{d\alpha}{ds}$$

Curvature has to be positive as it measures how sharp does a curve bend, i.e.,

$$\boxed{K = \left|\frac{d\alpha}{ds}\right|}$$

CALCULATION OF CURVATURE: Let the curve is given by

$y = f(x)$ (CARTESIAN FORM)

We know:

$$\frac{d\alpha}{dx} = \frac{d\alpha}{ds}\frac{ds}{dx} \Rightarrow \frac{d\alpha}{ds} = \frac{d\alpha}{dx} \Big/ \frac{ds}{dx}$$

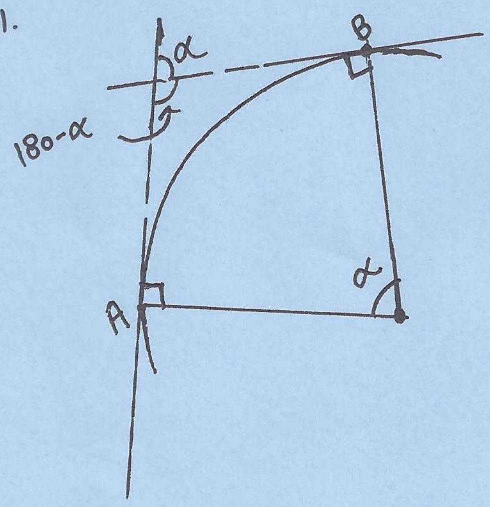
Note that $\tan\alpha = \frac{dy}{dx} \Rightarrow \alpha = \tan^{-1}\left(\frac{dy}{dx}\right)$

$$\Rightarrow \frac{d\alpha}{dx} = \frac{1}{1+\left(\frac{dy}{dx}\right)^2} \cdot \frac{d^2y}{dx^2}$$

Therefore: $\frac{d\alpha}{ds} = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}$, hence $\boxed{K = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}}$

Ex. For a given circle of radius r, determine the average curvature of the arc $\widehat{AB}$ and curvature at the point A.

Sol.



$$\widehat{AB} = \alpha r$$

$$K_{av} = \frac{\alpha}{\alpha r} = \frac{1}{r}$$

$$K = \lim_{\alpha \to 0} \frac{1}{r} = \frac{1}{r}$$

Smaller circles bend more sharply!

Ex. Curvature of a straight line $y = mx + c$.

$$\frac{dy}{dx} = m \qquad \frac{d^2y}{dx^2} = 0 \qquad \text{so} \qquad K = 0, \text{ i.e.,}$$

straight line has zero curvature.

- CARTESIAN FORM: $x = f(y)$

$$K = \frac{\left|\frac{d^2x}{dy^2}\right|}{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}} \qquad \left(\text{Useful when } \frac{dy}{dx} \to \infty\right)$$

- PARAMETRIC FORM: Let a curve be represented parametrically

$$x = \varphi(t) \qquad y = \psi(t)$$

then $$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} \Rightarrow \frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\cdot\frac{d^2y}{dt^2} - \frac{dy}{dt}\cdot\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2} \cdot \frac{dt}{dx}$$

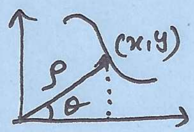
as $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)\cdot\frac{dt}{dx}$

Then the curvature

$$K = \frac{\left|\frac{\varphi'\psi'' - \psi'\varphi''}{\varphi'^3}\right|}{\left[1+\left(\frac{\psi'}{\varphi'}\right)^2\right]^{3/2}} = \frac{|\varphi'\psi'' - \psi'\varphi''|}{[\varphi'^2 + \psi'^2]^{3/2}}$$

so
$$\boxed{K = \frac{|\varphi'\psi'' - \psi'\varphi''|}{[\varphi'^2 + \psi'^2]^{3/2}}}$$

- POLAR FORM: $\rho = f(\theta)$



Transform polar coordinates to cartesian coordinates

$$\left.\begin{aligned} x &= \rho\cos\theta = f(\theta)\cos\theta \\ y &= \rho\sin\theta = f(\theta)\sin\theta \end{aligned}\right\}$$ same as parametric form with parameter θ.

Let $x = f(\theta)\cos\theta =: \varphi(\theta)$

$y = f(\theta)\sin\theta =: \psi(\theta)$

Then $\varphi'(\theta) = \frac{df}{d\theta}\cos\theta - f(\theta)\sin\theta$ $\quad\psi'(\theta) = \frac{df}{d\theta}\sin\theta + f\cos\theta$

$$\varphi''(\theta) = \frac{d^2f}{d\theta^2}\cos\theta - 2\frac{df}{d\theta}\sin\theta - f(\theta)\cos\theta$$

$$\psi'(\theta) = \frac{d^2f}{d\theta^2}\sin\theta + 2\frac{df}{d\theta}\cos\theta - f(\theta)\sin\theta$$

Therefore $\quad K = \frac{|\psi''\varphi' - \psi'\varphi''|}{|\varphi'^2 + \psi'^2]^{3/2}}$

After subst. we get: $\boxed{K = \frac{|2\rho'^2 - \rho''\rho + \rho^2|}{[\rho'^2 + \rho^2]^{3/2}}}$

THE RADIUS OF CURVATURE :

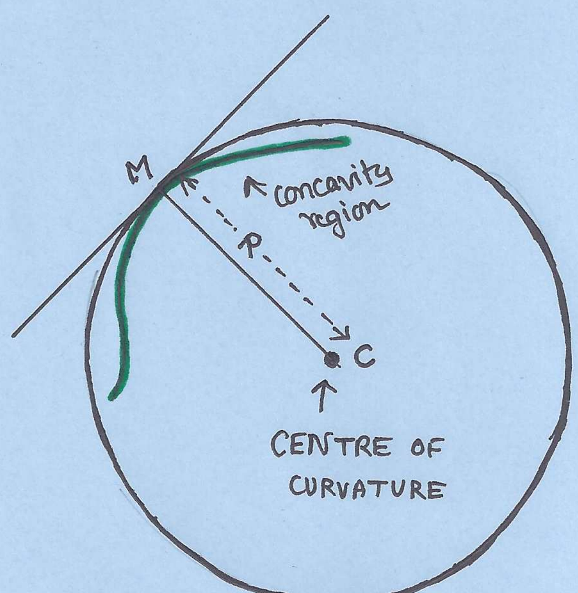
The reciprocal of the curvature of a curve at any point, in case it is not equal to zero, is called its radius of curvature at that point and is generally denoted by R,

$$\boxed{R = \frac{1}{K}}$$

Curve represented in a cartesian coordinate, R is given by

$$R = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$

Similarly we can write for curves represented parametrically or in polar coordinates.



Note that at a given point the curvature of a curve and the curvature of a circle of curvature are the same.

Centre of curvature for any point M is given at a distance R from M on the normal drawn in the direction of concavity of the curve.

The circle of radius R with centre at C is called the circle of curvature of the curve at the point M.

Ex. Find the curvature of $\rho = a\theta$ $(a>0)$ (SPIRAL)

Sol. $\rho' = a \qquad \rho'' = 0$

$$K = \frac{|2\rho'^2 - \rho''\rho + \rho^2|}{[\rho'^2 + \rho^2]^{3/2}} = \frac{|2a^2 - 0 + a^2\theta^2|}{[a^2 + a^2\theta^2]^{3/2}}$$

$$= \frac{1}{a}\left[\frac{2+\theta^2}{(\theta^2+1)^{3/2}}\right].$$

Ex. Curvature of the cycloid

$$x = a(t - \sin t) \qquad y = a(1-\cos t)$$

Sol:

$$\varphi' = \frac{dx}{dt} = a(1-\cos t) \qquad \psi' = a\sin t$$

$$\varphi'' = a\sin t \qquad \psi'' = a\cos t$$

$$K = \frac{|\varphi'\psi'' - \psi'\varphi''|}{[\varphi'^2 + \psi'^2]^{3/2}}$$

$$= \frac{|a(1-\cos t).a\cos t - a\sin t \cdot a\sin t|}{[a^2(1-\cos t)^2 + a^2\sin^2 t]^{3/2}}$$

$$= \frac{|a^2\cos t - a^2\cos^2 t - a^2\sin^2 t|}{a^3[1+\cos^2 t - 2\cos t + \sin^2 t]^{3/2}}$$

$$= \frac{a^2|\cos t - 1|}{a^3(2)^{3/2}[1-\cos t]^{3/2}} = \frac{1}{a\cdot 2^{3/2}}\cdot\frac{1}{(2\sin^2 t/2)^{1/2}}$$

$$= \frac{1}{4a}\cdot\frac{1}{|\sin t/2|}$$