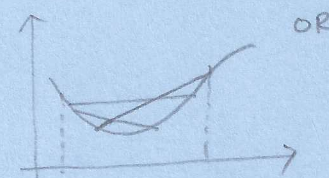
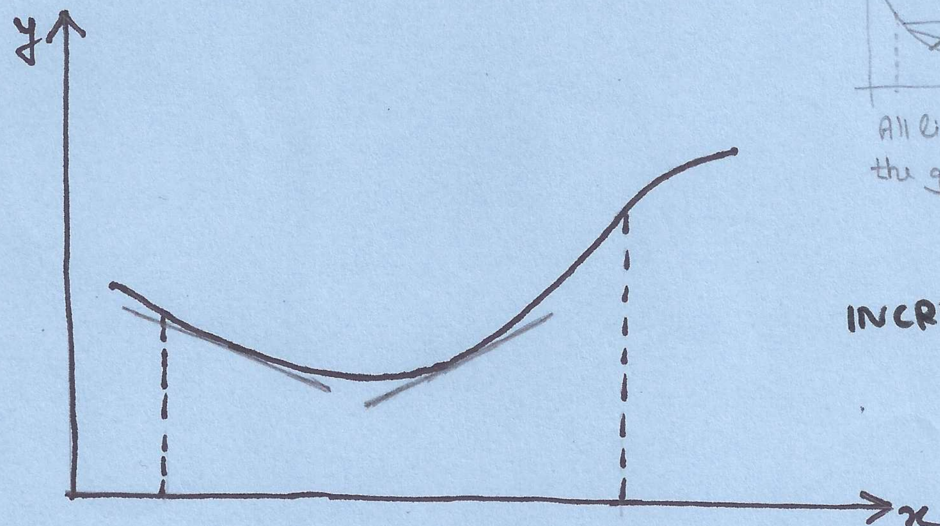
Convexity, concavity, point of inflection:

Def: A curve is convex (concave upwards, or convex downwards) on the interval (a, b) if all points of the curve lie above any tangent to it on the interval.

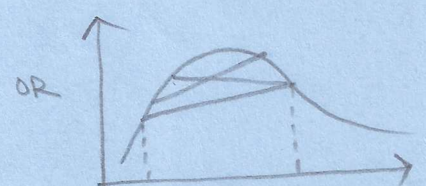


All line segments lie above the graph at any point



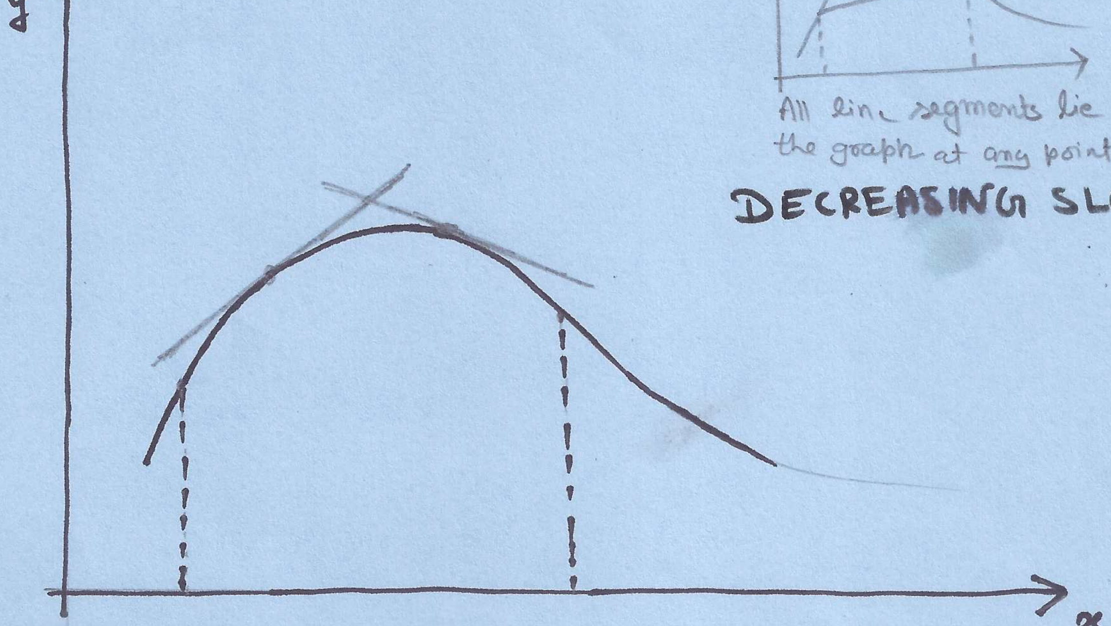
INCREASING SLOPE

Similarly, we say that a curve is concave (concave downwards or convex upwards) on the interval (a, b) if all points of the curve lie below any tangent to it on the interval.

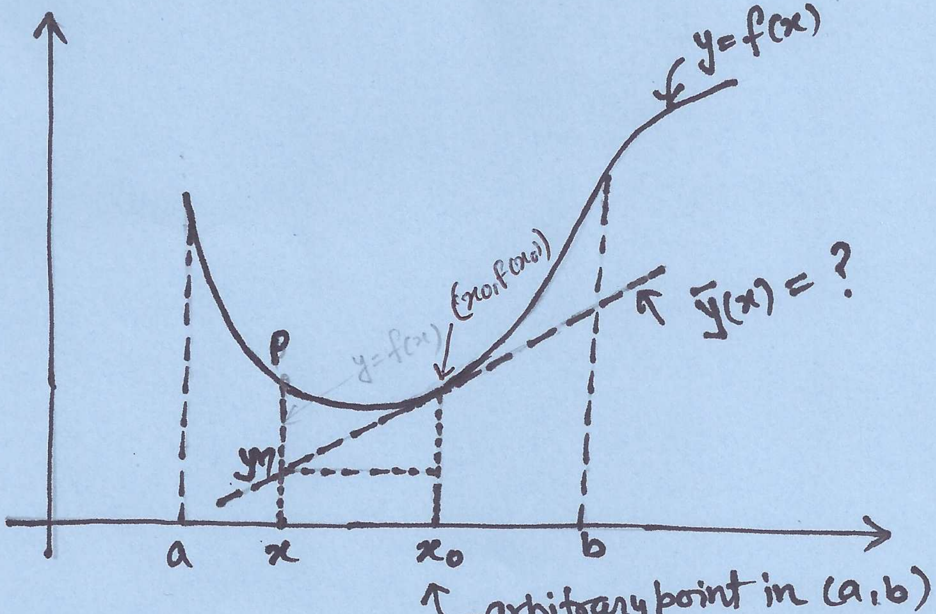


All line segments lie below the graph at any point.

DECREASING SLOPE.



<u>Th.</u> If at all points of an interval (a,b), the second derivative of the function $f(x)$ is positive, i.e., $f''(x)>0$, the curve $y=f(x)$ on this interval is convex downwards.



$$\text{slope of the tangent at } x_0 = f'(x_0) = \frac{f(x_0) - \bar{y}(x)}{x_0 - x}$$

$$\Rightarrow \bar{y}(x) = f(x_0) + f'(x_0)(x - x_0)$$

Aim: Curve lies above the tangent i.e., $(y-\bar{y}) > 0 \ \forall x \in (a,b)$

Consider

$$y - \bar{y} = f(x) - [f(x_0) + f'(x_0)(x - x_0)]$$

$$= \underbrace{f(x) - f(x_0)}_{f'(c_1)(x-x_0) = \text{ Lagrange's MVT}} - f'(x_0)(x - x_0)$$

$$= [f'(c_1) - f'(x_0)](x - x_0)$$

c_1 lies between x_0 and x.

Again applying Lagrange's MVT:

$$y - \bar{y} = f''(c_2)(c_1 - x_0)(x - x_0)$$

c_2 lies between x_0 & c_1

<u>Case I</u>: $x > x_0$

In this case $x_0 < c_2 < c_1 < x$

$\Rightarrow \quad (x-x_0)>0 \ \& \ (c_1-x_0)>0$

Also, it is given that $f''(c_1)>0$.

$\Rightarrow \quad y-\bar{y}>0$

Case II: $\quad x<x_0$:

In this case

$$x<c_1<c_2<x_0$$

$\Rightarrow (c_1-x_0)<0$ and $(x-x_0)<0$

Again $y-\bar{y}>0$

This proves that every point of the curve lies above the tangent to the curve.

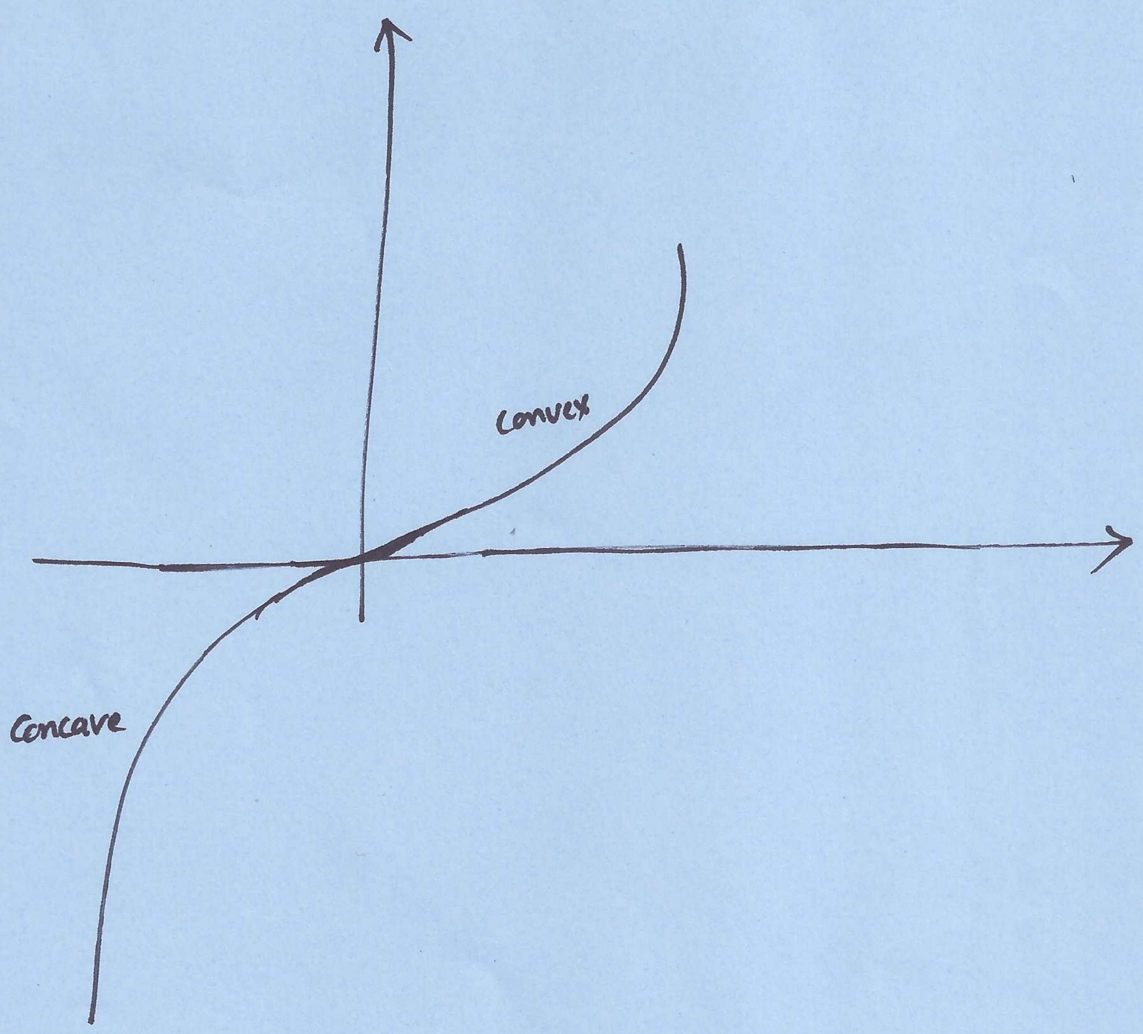
$\Rightarrow$ The curve is convex in this interval.

__Th.__: If at all points of the interval (a,b), the second derivative of the function $f(x)$ is negative, that is, $f''(x)<0$, then the curve $y=f(x)$ on this interval is concave (convex upwards)

__Proof__: Same as before.

Def. Point of inflection:

The point that separates the convex part of a continuous curve from the concave part is called the point of inflection of the curve



Th.: Let a curve be defined by the equation $y = f(x)$. If $f''(a) = 0$ or $f''(a)$ does not exist and if the derivative $f''(x)$ changes sign as x passes through a, then the point of the curve with abscissa $x = a$ is the point of inflection.

Examples:

1) $y = x^3$

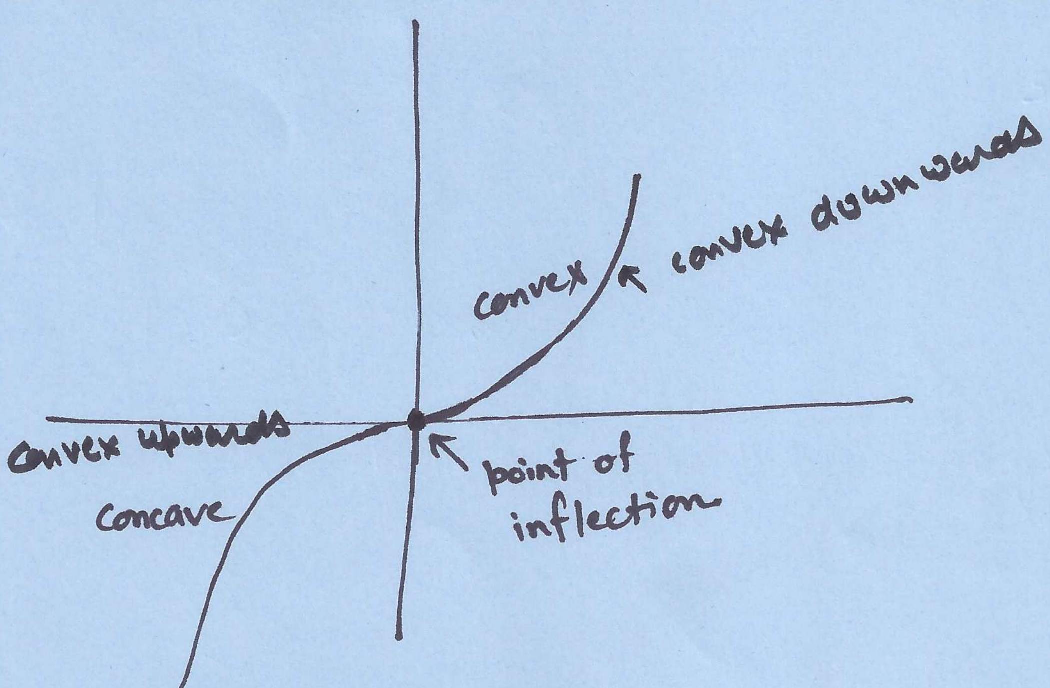
Since $y'' = 6x$

$y'' < 0$ for $x < 0$ & $y'' > 0$ for $x > 0$ & $y'' = 0$ for $x =$

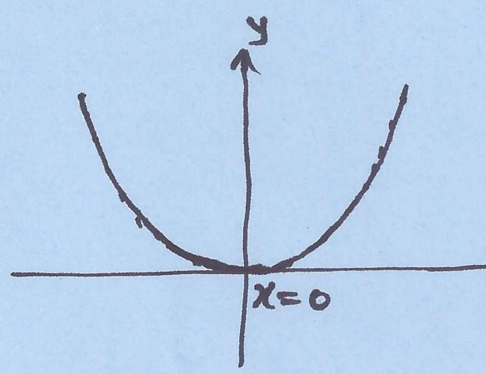
Hence for $x < 0$, the curve is concave (convex upwards)

$x > 0$, the curve is convex (convex downwards)

$(0, 0)$ ~~$x=0$~~, is a point of inflection



2) $y = x^4 \Rightarrow y'' = 4 \cdot 3 \cdot x^2$



The curve is convex (convex downward) for $x \in (-\infty, \infty)$

Also, note that at $x = 0$, $y'' = 0$ but y'' does not change sign passing through $x = 0$, so the curve has no point of inflection.

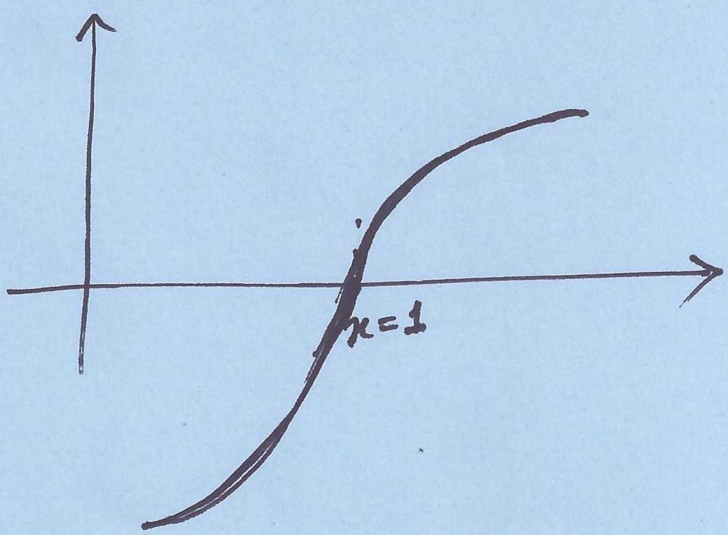
3) $y = (x-1)^{1/3}$ $\qquad y' = \frac{1}{3}(x-1)^{-2/3}, \quad x \neq 1$

$$y'' = -\frac{2}{9}(x-1)^{-5/3}, \qquad x \neq 1$$

For $x < 1$, $y'' > 0$ the curve is convex (convex downwards)

For $x > 1$, $y'' < 0$ the curve is concave (convex upwards)

y'' does not exist at $x = 1$. However, y'' changes its sign across $x = 1$, so there is a point of inflection at $x = 1$.



Homework:

Ex1: Find the interval in which the function

$$y = x + x^{5/3} + 5/3$$

is convex or concave (convex upwards)
(convex downwards)

Ex2: Investigate the point of inflection of the function

$$f(x) = \frac{x^2}{x-1}.$$