Linear Differential Equations of Higher order with constant Coefficients

The general form of linear equation with constant coeff.

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = X \quad \text{—①}$$

where $a_1, a_2, \cdots a_n$, are constants and X is a function of x.

General solution = Complementary function (C.F.) + Particular Integral (P.I.)

C.F.: Contain n arbitrary constants

P.I.: free from arbitrary constants

P.I.: If v is any particular solution, then

$$\frac{d^n v}{dx^n} + a_1 \frac{d^{n-1} v}{dx^{n-1}} + \cdots + a_n v = X$$

C.F.: It is the general solution of the homog. equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots a_n y = 0 \quad \text{—②}$$

Remarks:

1. Let y_1, y_2 be any two solutions of ②, then $c_1 y_1 + c_2 y_2$ is also a solution of ②, where c_1, c_2 are arbitrary constants:

$$\frac{d^n}{dt^n}(c_1 y_1 + c_2 y_2) + a_1 \frac{d^{n-1}}{dt^{n-1}}(c_1 y_1 + c_2 y_2) + \cdots + a_n (c_1 y_1 + c_2 y_2)$$
$$= c_1\left(\frac{d^n}{dt^n} y_1 + a_1 \frac{d^{n-1}}{dt^{n-1}} y_1 + \cdots + a_n y_1\right) + c_2\left(\frac{d^n}{dt^n} y_2 + a_1 \frac{d^{n-1}}{dt^{n-1}} y_2 + \cdots + a_n y_2\right)$$
$$= 0$$

Generalization: If $y_1, y_2 \cdots y_n$ be any n linearly independent * discussed later! solutions of ② then $c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ is the general solution of ②, where $c_1, c_2 \cdots c_n$ are arbitrary constants.

2. If u be the general solution of ② and v be any particular solution of ①, then $(u+v)$ is the general solution of ①.

(C.F. + P.I.)

↳ as it will contain n arbitrary constants.

$$\frac{d^n}{dt^n}(u+v) + a_1 \frac{d^{n-1}}{dt^{n-1}}(u+v) + \cdots + a_n(u+v)$$

$$= \left(\frac{d^n}{dt^n}u + a_1\frac{d^{n-1}}{dt^{n-1}}u + \cdots + a_n u\right) + \left(\frac{d^n}{dt^n}v + a_1\frac{d^{n-1}}{dt^{n-1}}v + \cdots + a_n v\right)$$

$$= 0 + X = X$$

OPERATOR: $\frac{d}{dx}, \frac{d^2}{dx^2} \cdots$

For the sake of convenience, the operators

$\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}$ are denoted by $D, D^2, \cdots, D^3, \cdots$

PRODUCT OF OPERATORS:

$$(D-\alpha)(D-\beta)y = (D-\beta)(D-\alpha)y, \quad \alpha, \beta \text{ being any constant.}$$

L.H.S: $(D-\alpha)(D-\beta)y = (D-\alpha)\left(\frac{dy}{dx} - \beta y\right)$

$$= \frac{d}{dx}\left(\frac{dy}{dx} - \beta y\right) - \alpha\left(\frac{dy}{dx} - \beta y\right)$$

$$= \frac{d^2y}{dx^2} - \beta\frac{dy}{dx} - \alpha\frac{dy}{dx} + \alpha\beta y$$

$$= \frac{d^2y}{dx^2} - (\alpha+\beta)\frac{dy}{dx} + \alpha\beta y$$

$$= \left[D^2 - (\alpha+\beta)D + \alpha\beta\right] y$$

Similarly, one can show that

$$(D-\beta)(D-\alpha)y = \left[D^2 - (\alpha+\beta)D + \alpha\beta\right] y$$

$$\therefore \quad (D-\alpha)(D-\beta) \equiv (D-\beta)(D-\alpha)$$

So the order of operational factors is immaterial.

Also note that

$$\underbrace{(D-\alpha)(D-\beta)}\, y = \underbrace{\left[D^2 - (\alpha+\beta)D + \alpha\beta\right]} y$$

same

Note: Treating D as a number, the ordinary laws of multiplication works.

In general:

$$\left[D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n\right] y = X$$

$$\Rightarrow \left[(D-\alpha_1)(D-\alpha_2)\cdots(D-\alpha_n)\right] y = X$$

Solution of $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0$

Write the equation in operator form

$$(D^2 + a_1 D + a_2)y = 0$$

- Case of non-repeated roots:

Suppose α_1 and α_2 are two non-repeated roots

$$(D-\alpha_1)(D-\alpha_2)y = 0$$

A solution of the equation:

$$(D-\alpha_2)y = 0$$

$$\Rightarrow \frac{dy}{dx} = \alpha_2 y \Rightarrow y = e^{\alpha_2 x}$$

Similarily the other solution:

$$y = e^{\alpha_1 x}$$

Thus the general solution:

$$c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x}$$

GENERALIZATION:

If $\alpha_1, \alpha_2, \ldots \alpha_n$ are distinct roots of

$(D^n + a_1 D^{n-1} + \cdots + a_n) = 0$ then

$e^{\alpha_1 x}, e^{\alpha_2 x}, \ldots, e^{\alpha_n x}$ will be n different independent solutions of the given equation and

$$c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x} + \cdots + c_n e^{\alpha_n x}$$

Is the general solution of the homogeneous equation

CASE OF REPEATED ROOTS:

$$(D-\alpha)(D-\alpha)y = 0$$

Let $(D-\alpha)y = Z$

then $(D-\alpha)Z = 0 \Rightarrow Z = c_1 e^{\alpha x}$

Now solving

$$(D-\alpha)y = c_1 e^{\alpha x}$$

$$\Rightarrow \frac{dy}{dx} - \alpha y = c_1 e^{\alpha x} \qquad (\text{linear in } y)$$

$$\text{I.F.} = e^{\int -\alpha\, dx} = e^{-\alpha x}$$

$$\Rightarrow y \cdot e^{-\alpha x} = \int c_1 e^{\alpha x} \cdot e^{-\alpha x}\, dx + c_2$$

$$\Rightarrow y = (c_1 x + c_2) e^{\alpha x}$$

GENERALIZATION: If a root α is repeated r times, the solution is

$$y = \left[c_1 x^{r-1} + c_2 x^{r-2} + \cdots + c_r\right] e^{\alpha x}$$

CASE OF IMAGINARY ROOTS:

Let $\alpha + i\beta$ and $\alpha - i\beta$ be two conjugate roots, the solution will be

$$y = \bar{c}_1 e^{(\alpha + i\beta)x} + \bar{c}_2 e^{(\alpha - i\beta)x}$$

$$\Rightarrow \quad y = \bar{c}_1 e^{\alpha x} e^{i\beta x} + \bar{c}_2 e^{\alpha x} e^{-i\beta x}$$

$$= e^{\alpha x}\left[\bar{c}_1 e^{i\beta x} + \bar{c}_2 e^{-i\beta x}\right]$$

$$= e^{\alpha x}\cdot\left[\bar{c}_1\{\cos\beta x + i\sin\beta x\} + \bar{c}_2\{\cos\beta x - i\sin\beta x\}\right]$$

$$= e^{\alpha x}\left[(\bar{c}_1+\bar{c}_2)\cos\beta x + i(\bar{c}_1-\bar{c}_2)\sin\beta x\right]$$

$$= e^{\alpha x}\left[c_1\cos\beta x + c_2\sin\beta x\right]$$

GENERALIZATION:

It can similarly be shown that if $(\alpha+i\beta)$ & $(\alpha-i\beta)$ are conjugate imaginary roots, each repeated r times, then the solution is

$$e^{\alpha x}\left[(p_1+p_2x+\cdots+p_r x^{r-1})\cos\beta x + (q_1+q_2x+\cdots+q_r x^{r-1})\sin\beta x\right]$$

Linear Dependence and Independence:

Two functions f and g are called linearly dependent on an open interval I if

$$f(x) = c\,g(x) \quad \forall x \text{ in } I \quad \text{for some constant } c.$$

$$\left(\text{OR } g(x) = c\,f(x)\right)$$

Otherwise they are called *linearly independent*.

Example: $\cos x$, $\sin x$

WRONSKIAN Test: To test whether two solutions of

$$y'' + p(x)\,y' + q(x)\,y = 0$$

are linearly independent.

Define the wronskian of solutions y_1 and y_2 to be

$$W(x) = y_1(x)\,y_2'(x) - y_1'(x)\,y_2(x)$$

$$= \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

Theorem: Let y_1 & y_2 be solutions of $y'' + p(x)y' + q(x)y = 0$ on an open interval I. Then

1. Either $W(x) = 0$ $\forall x$ in I, or $W(x) \neq 0$ $\forall x \in I$

2. y_1 and y_2 are linearly indep. on I iff $W(x) \neq 0$ on I.

<u>Example</u>: $y_1(x) = \cos x$ & $y_2(x) = \sin x$, solution of $y'' + y = 0$

$$W(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x$$

$$= 1 \neq 0$$

So y_1 & y_2 are linearly independent.

<u>Example</u>: Consider $y'' + xy = 0$ and its two solutions

$$y_1(x) = 1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 - \frac{1}{12960}x^9 + \cdots$$

$$y_2(x) = x - \frac{1}{12}x^4 + \frac{1}{504}x^7 - \frac{1}{45,360}x^{10} + \cdots$$

$\forall x \in \mathbb{R}$.

<u>Sol</u>: Note that calculating Wronskian at any nonzero x will be difficult, so we consider $x=0$ for Wronskian

$$W(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

Nonvanishing of the wronskian at this point alone is enough to conclude linear independence of these solution.