Tutorial Sheet-1 **AUTUMN 2018**

MATHEMATICS-I (MA10001) *August 1, 2018*

1. Determine all the number(s) c which satisfy the conclusion of Rolle's Theorem for

   i. $f(x) = x^2 - 2x - 8 \qquad x \in [-1, 3]$

   ii. $g(x) = 2x - x^2 - x^3 \qquad x \in [-2, 1]$

2. Verify Rolle's theorem for $f(x) = x(x+3)e^{\frac{-x}{2}}$ in $[-3, 0]$.

3. If $f(x) = (x-a)^m(x-b)^n$, where $m, n \in \mathbb{N}$. Use Rolle's theorem to show that the point where $f'(x)$ vanishes divides the line segment $a \le x \le b$ in the ratio $m : n$.

4. Let $f(x) = (x-a)(x-b)(x-c)$, $\ a < b < c$, show that $f'(x) = 0$ has two roots one belonging to $]a, b[$ and other belonging to $]b, c[$.

5. Use Rolle's theorem to prove the following:

   i. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function on $[0, 1]$ satisfying the condition $\int_0^1 f(x)dx = 0$. Then there exists $c \in (0, 1)$ such that
   $$f(c) = \int_0^c f(x)dx.$$

   ii. Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$ and $f''(x)$ exists for all $x \in (a, b)$. Let $a < c < b$, then there exists a point ξ in (a, b) such that
   $$f(c) = \frac{b-c}{b-a}f(a) + \frac{c-a}{b-a}f(b) + \frac{1}{2}(c-a)(c-b)f''(\xi).$$

6. Determine all the number(s) c which satisfy the conclusion of Mean Value Theorem for $f(x) = 8t + e^{-3t}$ on $[-2, 3]$.

7. Suppose that $f(x)$ is continuous and differentiable everywhere and it has two roots. Then, show that $f'(x)$ must have atleast one root.

8. i. Suppose that $f(0) = -3$ and $f'(x) \le 5$ for all x. Use Lagrange's mean value theorem to find the largest possible value of $f(2)$.

   ii. Use Lagrange's mean value theorem to estimate $\sqrt[3]{28}$.

9. If $f(x)$ and $\phi(x)$ are continuous on [a,b] and differentiable on (a,b), then show that
   $$\begin{vmatrix} f(a) & f(b) \\ \phi(a) & \phi(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(b) & f'(c) \\ \phi(b) & \phi'(c) \end{vmatrix}, a < c < b.$$

10. Use Lagrange's mean value theorem to prove Bernoulli's inequality: for all $x > 0$ and for all $n \in \mathbb{N}$, $(1+x)^n > 1 + nx$.

11. Suppose $f(x)$ is continuous on $[-7,0]$ and diffentiable in $(-7,0)$ such that $f(-7)=-3$ and $f'(x)\leq 2$. Then, what is largest possible value of $f(0)$.

12. Using Cauchy's Mean value theorem, show that $1-\frac{x^2}{2}<\cos x \qquad$ for $x\neq 0$.

13. i. Let f be continuous on $[a,b]$, $a>0$ and differentiable on (a,b). Prove that there exists $c\in(a,b)$ such that
$$\frac{bf(a)-af(b)}{b-a}=f(c)-cf'(c).$$

    ii. If f is differentiable on $[0,1]$, show by Cauchy's mean value theorem that the equation $f(1)-f(0)=\frac{f'(x)}{2x}$ has at least one solution in $(0,1)$.

    iii. Let f be continuous on $[a,b]$ and differentiable on (a,b). Using Cauchy's Mean value theorem show that if $a\geq 0$ then there exist $x_1,x_2,x_3\in(a,b)$ such that
$$f'(x_1)=(b+a)\frac{f'(x_2)}{2x_2}=(b^2+ba+a^2)\frac{f'(x_3)}{3x_3^2}\,.$$

Answers and Hints-

1. i. Ans. $c = 1$

   ii. Ans. $c = -1.2153, \quad 0.5486$

2. Ans. $c = -2$

3. Hint- Apply Rolle's theorem on $f(x)$ and get the expression $\dfrac{n}{m} = \dfrac{b-c}{c-a}$.

4. Hint- Use Rolle's theorem twice.

5. i. Hint- Let $g(x) = e^{-x} \int_0^x f(t)dt$ and apply Rolle's theorem on $g(x)$.

   ii. Hint- Let $\phi(x) = f(x) - \dfrac{(x-b)(x-c)}{(a-b)(a-c)} f(a) - \dfrac{(x-c)(x-a)}{(b-c)(b-a)} f(b) - \dfrac{(x-a)(x-b)}{(c-a)(c-b)} f(c)$ and apply Rolle's theorem on $\phi(x)$.

6. Ans. $c = -1.0973$

7. Hint- Define $f(x)$ on $[a, b]$ such that $a,\ b$ are the roots of $f(x)$. Apply LMVT on $f(x)$.

8. i. Ans. $f(2) \leq 7$

   ii. Ans. $\sqrt[3]{28} \approx 3.037$ Hint- Let $f(x) = \sqrt[3]{28}$ and $x \in [27, 28]$. Use LMVT.

9. Hint- Let

$$g(x) = \begin{vmatrix} f(x) & f(b) \\ \phi(x) & \phi(b) \end{vmatrix}$$

   on $[a, b]$ and apply LMVT on $g(x)$.

10. Hint- Let $f(t) = (1+t)^n$ and $t \in [0, x]$.

11. Ans. $f(0) \leq 11$, Hint- Use LMVT.

12. Hint- Let $f(x) = 1 - \cos x$ and $g(x) = \dfrac{x^2}{2}$ on $x \in [0, x]$. Use CMVT.

13. i. Hint- Let $h(x) = \dfrac{f(x)}{x}$ and $g(x) = \dfrac{1}{x}$ on $[a, b]$. Use CMVT.

    ii. Hint- Let $h(x) = f(x)$ and $g(x) = x^2$ on $[0, 1]$. Use CMVT.

    iii. Hint- Apply CMVT thrice.