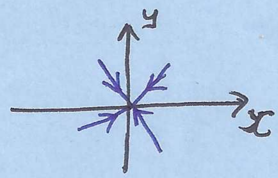
REMARK: Since $(x, y) \to (x_0, y_0)$ in the two dimensional plane, there are infinite number of paths joining (x, y) to (x_0, y_0). Since the limit, if exists, is unique, the limit should be the same along all the paths. Thus, the limit cannot be obtained by approaching the point P along a particular path and finding the limit of $f(x, y)$. If the limit is dependent on a path, then the limit does not exist.

<u>EXAMPLE</u> :

- $$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=mx}} \frac{x^2 y}{x^4+y^2}$$

- $$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=mx^2}} \frac{x^2 y}{x^4+y^2}$$

Along $y = mx$:

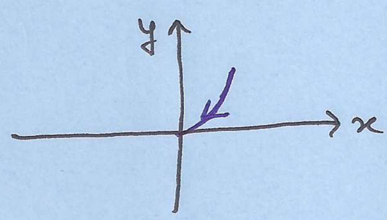


$$\lim_{x\to 0} \frac{x^2 \cdot mx}{x^4 + m^2x^2} = \lim_{x\to 0} \left(\frac{mx}{x^2+m^2} \right) = 0$$

Along $y = mx^2$:

$$\lim_{x\to 0} \frac{x^2 \cdot mx^2}{x^4 + m^2x^4} = \frac{m}{1+m^2}$$

In particular, $y = x^2$, we get $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \frac{1}{2}$



Hence limit does not exist in this case.

Ex. Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

does not exist.

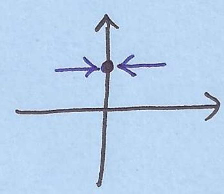
Sol. Consider $y = mx$:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{mx^2}{(1+m^2)x^2} = \frac{m}{1+m^2}$$

The limit depends on path. Hence the limit does not exist.

Ex. Show that the limit $\lim_{(x,y)\to(0,1)} \tan^{-1}\left(\frac{y}{x}\right)$ does not exist.

Sol. Let us fix $y = 1$ and calculate



$$\left.\begin{aligned} \lim_{x\to 0-0} \tan^{-1}\left(\frac{1}{x}\right) &= \tan^{-1}(-\infty) = -\frac{\pi}{2} \\ \lim_{x\to 0+0} \tan^{-1}\left(\frac{1}{x}\right) &= \tan^{-1}(+\infty) = \frac{\pi}{2} \end{aligned}\right\} \Rightarrow \text{limit does not exist.}$$

Working with limits: suppose, we have

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L_1 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = L_2$$

then

i) $\lim_{(x,y)\to(x_0,y_0)} [K f(x,y)] = K L_1$ for any real constant K.

ii) $\lim_{(x,y)\to(x_0,y_0)} [f(x,y) \pm g(x,y)] = L_1 \pm L_2$

iii) $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)\, g(x,y)] = L_1 \cdot L_2$

iv) $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)/g(x,y)] = \frac{L_1}{L_2}$ provided, $L_2 \neq 0$.

CONTINUITY:

A function $z = f(x,y)$ is said to be continuous at a point (x_0, y_0) if

i) $f(x,y)$ is defined at the point (x_0, y_0)

ii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists

iii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

OR

A function $f(x,y)$ is said to be continuous at (x_0,y_0) if for a given $\varepsilon>0$, there exists a real number $\delta>0$ such that

$$|f(x,y) - f(x_0,y_0)| < \varepsilon \quad \text{whenever} \quad \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$$

If $f(x_0,y_0)$ is defined and $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ exists but $f(x_0,y_0) \neq L$, then the point (x_0,y_0) is called a point of <u>removable discontinuity</u>.

If a function $f(x,y)$ is continuous at every point in a domain D, then it is said to be continuous in D.

EXAMPLE: Show that the following functions are continuous

i) $f(x,y) = \begin{cases} \dfrac{2x^4+3y^4}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

ii) $f(x,y) = \begin{cases} \dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)} & ; (x,y) \neq (0,0) \\ 1/2 & (x,y) = (0,0) \end{cases}$

Sol: i) Change of coordinate system:

$$x = r\cos\theta \qquad y = r\sin\theta$$

Note that $r = \sqrt{x^2+y^2} \neq 0$ since $(x,y) \neq (0,0)$.

Consider $|f(x,y) - f(0,0)| = \left| \dfrac{2r^4\cos^4\theta + 3r^4\sin^4\theta}{r^2(\cos^2\theta + \sin^2\theta)} \right| = |2r^2\cos^4\theta + 3r^2\sin^4\theta|$

$$\leq 2r^2\cos^4\theta + 3r^2\sin^4\theta < 5r^2 < 5\delta^2 < \varepsilon$$

$\Rightarrow$ Choose $\delta < \sqrt{\dfrac{\varepsilon}{5}}$

$\Rightarrow |f(x,y) - f(0,0)| < \varepsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$

Hence $f(x,y)$ is continuous at $(0,0)$.

Alternative Approach: Change of coordinate $x = r\cos\theta$, $y = r\sin\theta$

$$\lim_{(x,y)\to(0,0)} \frac{2x^4+3y^4}{x^2+y^2} = \lim_{\substack{(x,y)\to(0,0) \\ \text{or } r\to 0}} \frac{2\cdot r^4\cos^4\theta + 3r^4\sin^4\theta}{r^2} = 0$$

(No dependency on θ) If it depends on θ then limit does not exist.

ii) $$f(x,y) = \begin{cases} \dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)} & (x,y) \neq (0,0) \\ 1/2 & (x,y) = (0,0) \end{cases}$$

Choose $x+2y = t$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)} = \lim_{t\to 0} \frac{\sin^{-1}(t)}{\tan^{-1}(2t)} = \lim_{t\to 0} \frac{\left(\frac{1}{\sqrt{1-t^2}}\right)}{\left(\frac{2}{1+4t^2}\right)} = \frac{1}{2}$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = \frac{1}{2} = f(0,0)$$

$\Rightarrow$ The function is continuous.

Ex. Discuss the continuity of the functions

i) $$f(x,y) = \begin{cases} \dfrac{(x-y)^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Choosing the path $y = mx$.

$$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2} = \lim_{x\to 0} \frac{(x-mx)^2}{x^2+m^2x^2} = \lim_{x\to 0} \frac{(1-m)^2}{1+m^2} = \frac{(1-m)^2}{1+m^2}$$

The limit depends on the path. Therefore the limit does not exist and the function is not continuous at $(0,0)$.

ii) $$f(x,y) = \begin{cases} \dfrac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y)\to(0,0)} \frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \lim_{t\to 0} \frac{\sin\sqrt{t}}{\sqrt{t}} = 1 \neq f(0,0)$$

The function is discontinuous at $(0,0)$.

Note that the point $(0,0)$ is a point of removable discontinuity.

iii) $$f(x,y) = \begin{cases} \dfrac{e^{xy}}{x^2+1} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy}}{x^2+1} = \frac{1}{0+1} = 1 \neq f(0,0)$$

The function is not continuous at $(0,0)$

(iv) $$f(x,y) = \begin{cases} \dfrac{x^4 y^4}{(x^2+y^4)^3} ; & (x,y) \neq (0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$$

Choosing the path $y^2 = mx$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \left[\frac{x^4 y^4}{(x^2+y^4)^3}\right]_{y^2=mx} = \lim_{x\to 0} \frac{x^4 . m^2 x^2}{(x^2+m^2x^2)^3}$$

$$= \frac{m^2}{(1+m^2)^3}$$

The limit depends on the path and therefore does not exist.

The function is discontinuous at $(0,0)$.

v)

$$f(x,y)=\begin{cases}\dfrac{x^2+y^2}{\tan xy} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

Sol:

Take path $y=mx$

$$\lim_{(x,y)\to(0,0)}\frac{x^2+y^2}{\tan xy}=\lim_{x\to 0}\frac{x^2+m^2x^2}{\tan(mx^2)} \quad (m\neq 0)$$

$$=\lim_{x\to 0}\frac{(1+m^2)}{m\cdot\dfrac{\tan(mx^2)}{mx^2}}=\frac{1+m^2}{m}$$

as $\displaystyle\lim_{x\to 0}\frac{\tan(mx^2)}{mx^2}=1.$

The limit depends on the path, hence it does not exist.

The function is discontinuous at $(0,0)$.

Do not follow as:

$$\lim_{(x,y)\to(0,0)}\frac{x^2+y^2}{\tan xy}=\underbrace{\lim_{(x,y)\to(0,0)}\frac{x^2+y^2}{xy}}_{\text{does not exist }(y=mx)}\cdot\underbrace{\lim_{(x,y)\to(0,0)}\frac{1}{\dfrac{\tan(xy)}{xy}}}_{=1}\Rightarrow \text{limit does not exist.}$$

$\lim(fg)=\lim f\cdot\lim g$ is valid when both limits $\lim f$ & $\lim g$ exist!

Consider for example: $\displaystyle\lim_{(x,y)\to(0,0)}\left(\frac{x^2+y^2}{1+xy}\right)$, Clearly this limit exists and is equal to zero.

However, if we rewrite as $\displaystyle\underbrace{\lim_{(x,y)\to(0,0)}\frac{x^2+y^2}{xy}}_{\text{does not exist}}\cdot\underbrace{\lim_{(x,y)\to(0,0)}\left(\frac{xy}{1+xy}\right)}_{=0}$

$\Rightarrow$ limit does not exist.

REMARK: Changing to polar coordinate (subst. $x = r\cos\theta$, $y = r\sin\theta$) and investigating the limit of the resulting expression as $r \to 0$ is often very useful. for example:

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^3\theta}{r^2} = \lim_{r\to 0} r\cos^3\theta = 0.$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0 \quad \text{(Useful in finding limit)}$$

Also, Note that

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{(x^2+y^2)} = \lim_{r\to 0} \frac{r^2\cos^2\theta}{r^2} = \cos^2\theta \quad \text{(depends on } \theta)$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2}{(x^2+y^2)} \text{ does not exist.}$$

Shifting to polar coordinate does not always help, however, and may even tempt us to false conclusions.

For example:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{r\to 0} \frac{r^2\cos^2\theta * r\sin\theta}{r^4\cos^4\theta + r^2\sin^2\theta} = \lim_{r\to 0} \frac{r\cos^2\theta\sin\theta}{r^2\cos^4\theta + \sin^2\theta}$$

If we hold θ constant then

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = 0,$$ BUT this is not the limit, we should not fix θ. Taking the path $r\sin\theta = r^2\cos^2\theta$ ($y = x^2$):

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{r\to 0} \frac{\cancel{r}\cos^2\theta\,\cancel{\sin\theta}\,r^2\cos^2\theta}{r^2\cos^4\theta + r^2\cos^4\theta} = \frac{1}{2}$$

$\Rightarrow$ The limit does not exist.