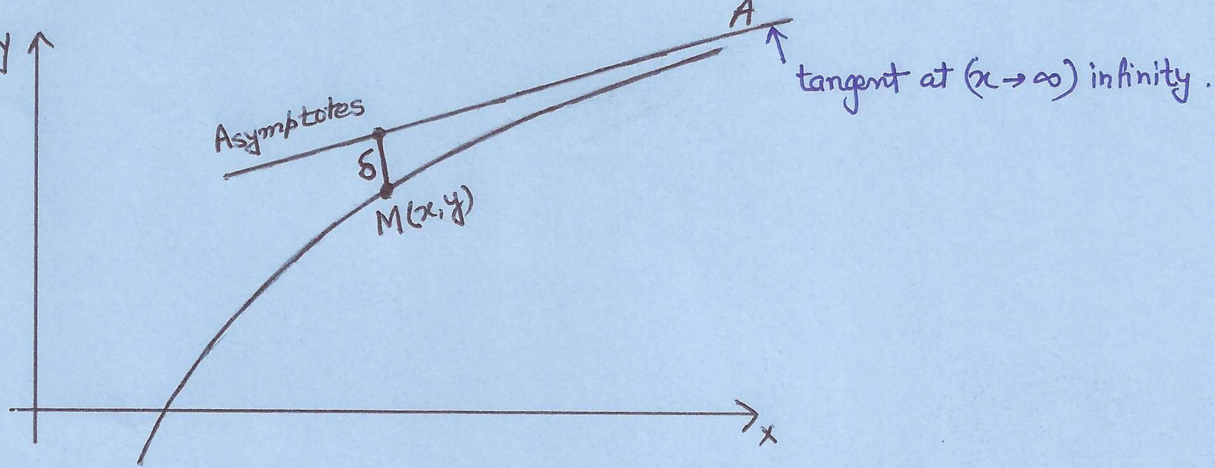
ASYMPTOTES:



Def: A straight line A is called an asymptote to a curve, if the parpendicular distance δ from the variable point M of the curve to this straight line approaches to zero as the point M recedes to infinity.

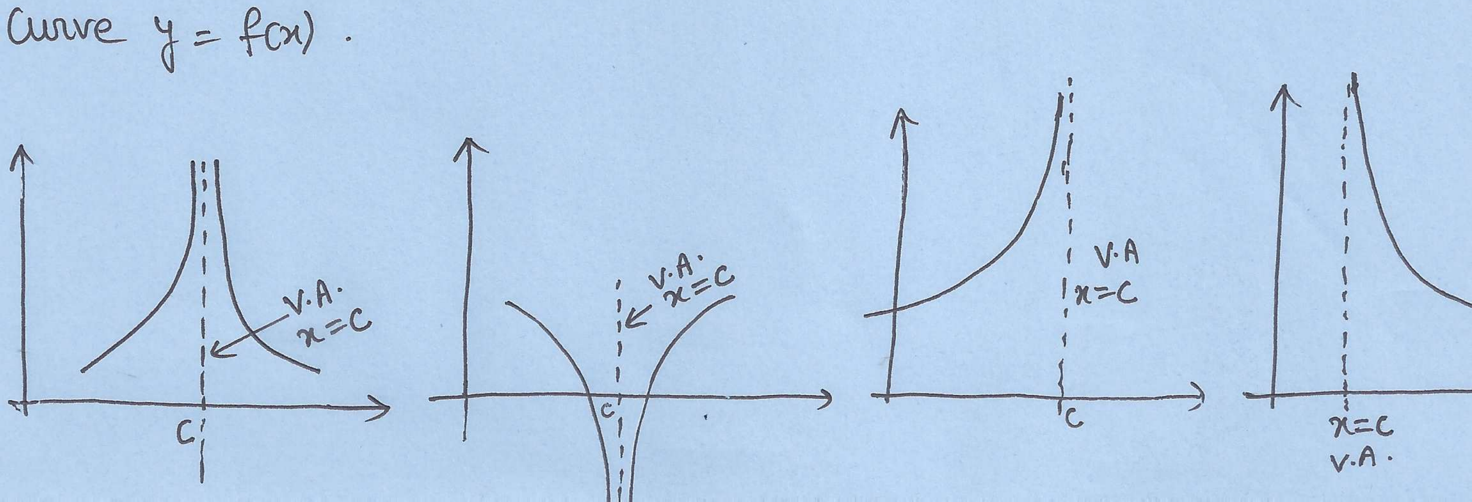
Asymptotes →
- Vertical asymptotes – **parallel to y-axis**
- Horizontal asymptotes – **parallel to x-axis**
- Inclined (oblique) asymptotes (Horizontal is also included)

1. VERTICAL ASYMPTOTES: (V.A.)

If $\lim_{x \to a+} f(x) = \pm\infty$ or $\lim_{x \to a-} f(x) = \pm\infty$ or

$$\lim_{x \to a} f(x) = \pm\infty,$$

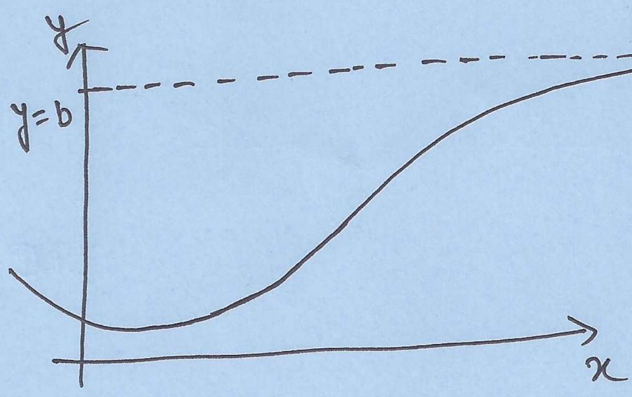
Then the straight line $x=a$ is an vertical asymptote to the curve $y = f(x)$.



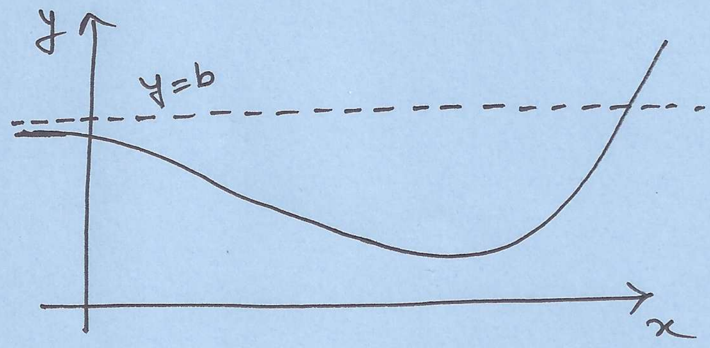
2. HORIZONTAL ASYMPTOTE:

The line $y=b$ is a horizontal asymptote for the function f if

$$\lim_{x \to \pm\infty} f(x) = b.$$



$f(x) \to b$ as $x \to +\infty$



$f(x) \to b$ as $x \to -\infty$

WORKING TIPS: ($f(x,y)=0$)

- VERTICAL ASYMPTOTES: These are obtained by equating to zero the coefficients of the highest powers of y in $f(x,y)=0$.

- HORIZONTAL ASYMPTOTES: These are obtained by equating to zero the coefficients of the highest powers of x in $f(x,y)=0$.

- NO VERTICAL OR HORIZONTAL ASYMPTOTES:

  If the coefficients of the highest powers of x and y in $f(x,y)=0$ are constants.

EXAMPLE: $x^2y - y - x = 0$.

VERTICAL: coeff. of highest power of y is x^2-1. V.A. are $x^2-1=0$
$\Rightarrow x = \pm 1$.

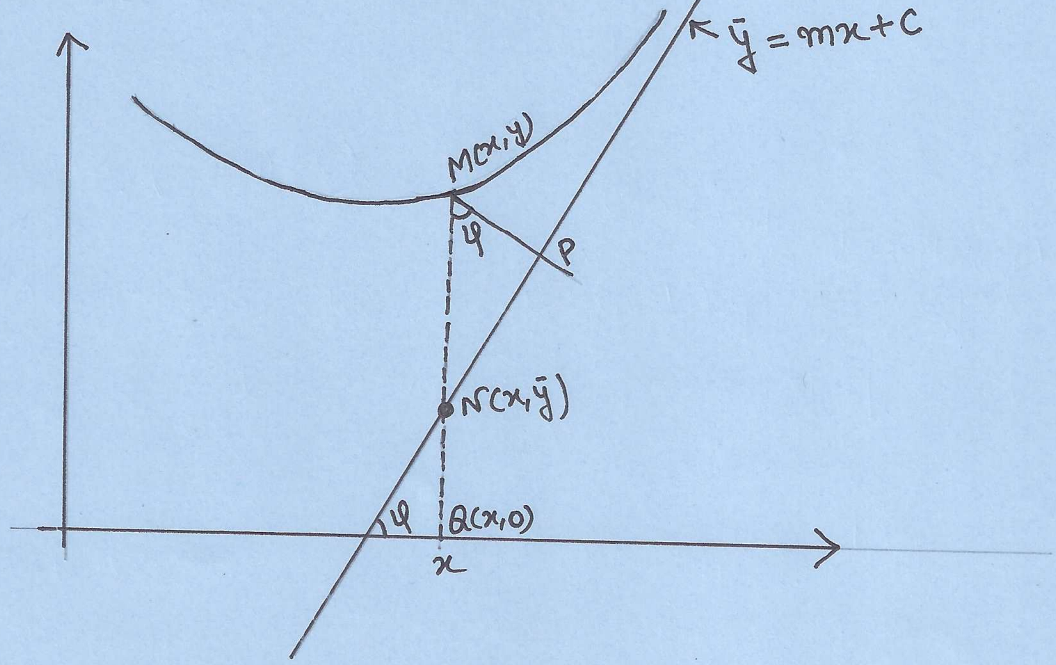
HORIZONTAL: coeff. of highest power of x is y. H.A is $y=0$.

EXAMPLE: The curve $x^3+y^3=3axy$ has no vertical and horizontal asymptotes because coefficients of x^3 and y^3 are constants.

3. **INCLINED ASYMPTOTES:** Let the curve $y=f(x)$ has an inclined asymptote whose equation is

$$\bar{y}=mx+c$$

Let $M(x,y)$ be a point on the curve and $N(x,\bar{y})$ a point on the asymptote.



Given: $\lim_{x\to\infty} MP=0$ (parpendicular distance from M to the asymp.)

From ΔNMP, we have

$$\cos\varphi=\frac{MP}{NM} \Rightarrow NM=\frac{MP}{\cos\varphi}$$

Note that φ is constant and $\varphi\neq\pi/2$ (Not a vertical asymptote)

Then, we have that

$$\lim_{x\to\infty} NM=0$$

Also, $NM = |QM - QN| = |y - \bar{y}|$

$$= |f(x) - (mx + c)|$$

So,

$$\lim_{x \to \infty} NM = 0 \Rightarrow \lim_{x \to \infty} f(x) - mx - c = 0 \quad \text{—} (1)$$

$$\Rightarrow \lim_{x \to \infty} x \left[\frac{f(x)}{x} - m - \frac{c}{x} \right] = 0$$

$$\Rightarrow \lim_{x \to \infty} \left(\frac{f(x)}{x} - m - \frac{c}{x} \right) = 0$$

$$\Rightarrow \lim_{x \to \infty} \frac{f(x)}{x} - m = 0 \Rightarrow \boxed{m = \lim_{x \to \infty} \frac{f(x)}{x}}$$

Again from (1):

$$\boxed{c = \lim_{x \to \infty} f(x) - mx}$$

WORKING STEPS:

1. Find $\lim_{x \to \pm\infty} \frac{f(x)}{x}$ and let $m = \lim_{x \to \pm\infty} \frac{f(x)}{x}$

2. Find $\lim_{x \to \pm\infty} (f(x) - mx)$ and let $c = \lim_{x \to \pm\infty} (f(x) - mx)$

3. Then $y = mx + c$ is an asymptote.

INCLINED ASYMPTOTE (ALTERNATIVE PROOF)

Let the equation of the curve be

$$y = f(x)$$

then the equation of the tangent to the curve at the point (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

$$\Rightarrow \quad Y = \frac{dy}{dx} X + \left(y - x\frac{dy}{dx}\right)$$

If $\lim_{x \to \infty} \frac{dy}{dx} = m$ and $\lim_{x \to \infty} \left(y - x\frac{dy}{dx}\right) = c$

Then the equation of the tangent:

$$\boxed{Y = mX + c}$$

This is called the asymptote of the curve.

EXAMPLE: Find the asymptotes of the curve

$$y = \frac{x^2+2x-1}{x}$$

Sol: VERTICAL ASYMPTOTES: Coeff. of highest power of y is x

$\Rightarrow$ $x=0$ is an vertical asymptote.

Also,
$$\lim_{x\to 0+} \frac{x^2+2x-1}{x} = -\infty$$

& $$\lim_{x\to 0-} \frac{x^2+2x-1}{x} = \infty$$

HORIZONTAL ASYMPTOTES: Coeff of highest power of x is constant

$\Rightarrow$ No horizontal asymptote.

INCLINED ASYMPTOTES:

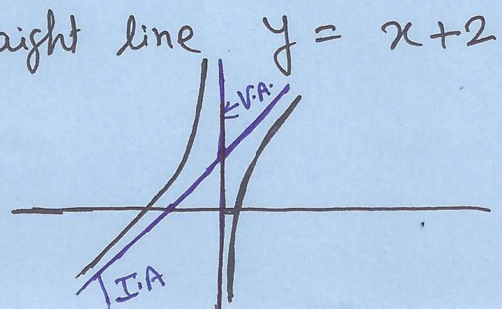
$$\lim_{x\to\pm\infty} \frac{y}{x} = \lim_{x\to\pm\infty} \left(\frac{x^2+2x-1}{x^2}\right) = \lim_{x\to\pm\infty}\left(1+\frac{2}{x}-\frac{1}{x^2}\right)$$
$$= 1$$

Hence $m = 1$.

Now, $$\lim_{x\to\pm\infty}(y-mx) = \lim_{x\to\pm\infty}(y-x) = \lim_{x\to\pm\infty}\left(2-\frac{1}{x}\right)$$
$$= 2$$

Hence $c = 2$.

$\Rightarrow$ The straight line $y = x+2$ is an inclined asymptote.



EXAMPLE: Find the asymptotes of the curve $x^3+y^3-3axy=0$.

We want to find $m=\lim_{x\to\infty}\frac{y}{x}$

& $c=\lim_{x\to\infty}(y-mx)$

then $y=mx+c$ is an asymptote.

- Clearly, there is no vertical or horizontal asymptote as the coeff. of highest power of x and y are constants.

Rewriting the given equation as:

$$1+\left(\frac{y}{x}\right)^3-3a\cdot\frac{1}{x}\cdot\left(\frac{y}{x}\right)=0$$

Taking limit as $x\to\infty$ and setting $m=\lim_{x\to\infty}\frac{y}{x}$, we get

$$1+m^3=0 \Rightarrow (m+1)(m^2-m+1)=0$$

$$\Rightarrow \boxed{m=-1} \text{ since } m^2-m+1 \text{ has no real root.}$$

Now $c=\lim_{x\to\infty}(y+x)$

Let us take $y+x=p$ then $c=\lim_{x\to\infty}p$.

Subst. $y=p-x$ in the equation:

$$x^3+(p-x)^3-3ax(p-x)=0$$

$$\Rightarrow \cancel{x^3}+p^3-\cancel{x^3}+3px^2-3p^2x-3axp+3ax^2=0$$

$$\Rightarrow 3x^2(p+a)-3x(p^2+ap)+p^3=0$$

Dividing by x^2:

$$3(p+a)-\frac{3(p^2+ap)}{x}+\frac{p^3}{x^2}=0$$

Let $x\to\infty \Rightarrow \boxed{c=-a} \Rightarrow y=-x-a$ is the only asymptote.