Problem Set - 4 $\hfill$ Autumn 2018

MATHEMATICS-I(MA10001) $\hfill$ 25/07/2018

1. Show that the function

$$f(x,y) = \begin{cases} \dfrac{x^2+y^2}{|x|+|y|}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

is continuous at $(0,0)$, but $f_x(0,0)$ and $f_y(0,0)$ do not exist.

2. Show that the following functions

(a) $$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

(b) $$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y}, & \text{if } x \neq y; \\ 0, & \text{if } x = y. \end{cases}$$

possess partial derivatives at $(0,0)$, though it is not continuous at $(0,0)$.

3. Find $f_x(x,y)$ and $f_y(x,y)$ using definition for the followings :

(a) $f(x,y) = x^2 + y^2$,

(b) $f(x,y) = \sin(3x+4y)$,

(c) $f(x,y) = ye^{-x} + xy$.

(d) $f(x,y) = x^2 + xy + y^3$,

(e) $f(x,y) = x\sin y + x^2$,

(f) $f(x,y) = e^{xy} + \dfrac{x}{y}$.

4. Find $f_x(0,0)$, $f_y(0,0)$, $f_x(0,y)$ and $f_y(x,0)$ for the followings :

(a) $f(x,y) = \begin{cases} \dfrac{xy}{x+y}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

(b) $f(x,y) = \log(1+xy)$,

(c) $f(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 0 \text{ or both } x = 0 \text{ and } y = 0; \\ 0, & \text{Otherwise} \end{cases}$

(d) $f(x,y) = e^{x-y} - e^{y-x}$,

(e) $f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x+y}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

5. Show that the following functions

(a) $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & \text{if } (x^2+y^2) \neq 0; \\ 0, & \text{if } (x^2+y^2) = 0. \end{cases}$

(b) $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2}, & \text{if } (x^2+y^2) \neq 0; \\ 0, & \text{if } (x^2+y^2) = 0. \end{cases}$

have first order partial derivatives at $(0,0)$, and discuss the differentiability at $(0,0)$.

6. Show that following function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0) \end{cases}$$

is continuous, possess first order partial derivatives but it is not differentiable at the origin.

7. Prove that the function $f(x,y) = \sqrt{|xy|}$ is not differentiable at $(0,0)$, but that f_x and f_y both exists at origin and have the value 0. Show that f_x and f_y are continuous everywhere except at the origin.

8. Test the differentiability of the following functions at $(0,0)$

   (a) $f(x,y) = \begin{cases} (x^2+y^2)\sin\dfrac{1}{x^2+y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

   (b) $f(x,y) = \begin{cases} \dfrac{x^3-2y^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

9. Let $f(x,y) = \begin{cases} y\dfrac{x^2-y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

   Find $f_{xx}(x,y)$, $f_{xy}(x,y)$, $f_{yx}(x,y)$ and $f_{yy}(x,y)$ at $(0,0)$. Also check the differentiability of the function $f(x,y)$ at the origin.

10. For the function $f(x,y) = \begin{cases} \dfrac{x^2y(x-y)}{x^2+y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$
    check that $f_{xy}(0,0) \neq f_{yx}(0,0)$. Also check the differentiability of $f(x,y)$ at the origin

11. Find $f_{yxx}(x,y)$ and $f_{xyx}(x,y)$ for the following functions:

    (a) $f(x,y) = x^4\sin 3y + 5x - 6y$.
    (b) $f(x,y) = x^5y^3 + \log(xy) + 10x$.
    (c) $f(x,y) = e^{xy}\tan x + x^3y^2$.
    (d) $f(x,y) = x^3\sin y + y^3\cos x$
    (e) $f(x,y) = e^x\ln y + \cos y\ln x$
    (f) $f(x,y) = x^3y^2 + 2xy^3 + \cos(xy^2)$

12. Find the total differential of the following functions

(a) $w = x^2 + xy^2 + xy^2z^3$

(b) $z = \tan^{-1}(x/y)$,

(c) $u = e^{(x^2 + y^2 + z^2)}$,

(d) $w = \sin(3x + 4y) + 5e^z$

(e) $w = z \ln y + y \ln z + xyz$,

(f) $u = \sqrt{x^2 + y^2 + z^2}$,

(g) $w = e^x \sin(y + 2z) - x^2y^2$,

(h) $w = e^{\frac{x}{y}} + e^{\frac{z}{y}}$.