# SUMMARY

## DIFFERENTIAL CALCULUS : SINGLE VARIABLE

### MEAN VALUE THEOREM (GENERALIZED)

Consider two functions $f(x)$ & $g(x)$ to be

i) continuous in $[a,b]$

ii) differentiable in $(a,b)$

iii) $g'(x) \neq 0 \ \forall \ x \in (a,b)$

Then $\exists \ c \in (a,b)$ such that

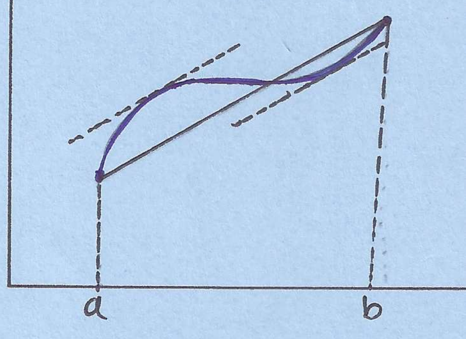
$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

PARTICULAR CASES:

a) $g(x) = x$

$$\Rightarrow \quad \frac{f(b)-f(a)}{b-a} = f'(c)$$

Tangent(s) is (are) parallel to the secant line.




b) $f(a) = f(b)$

$$\Rightarrow \quad f'(c) = 0$$

Tangent is parallel to the $x$-axis at some point $c \in (a,b)$.

## INDETERMINATE FORMS:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\cdot\infty,\ 0^{0},\ \infty^{0},\ 1^{\infty},\ \infty-\infty$$

### L'HOSPITAL RULE:

If $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$ OR $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\pm\infty}{\pm\infty}$

Then,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

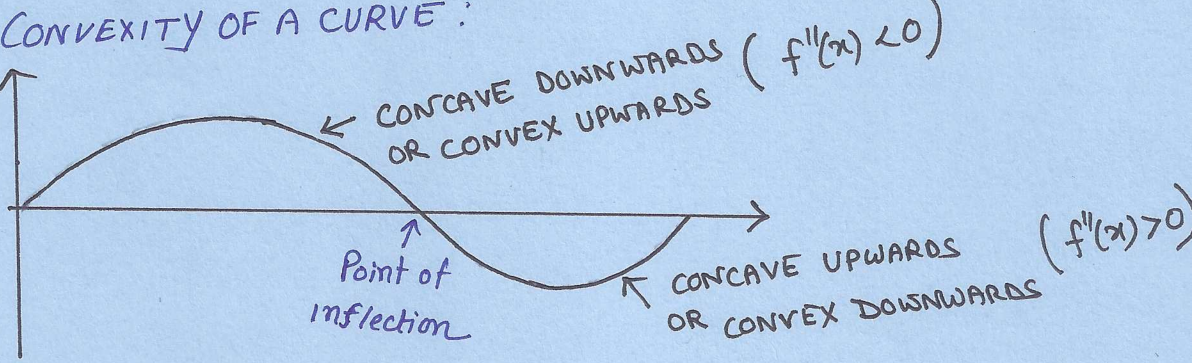
## TAYLOR'S and Maclaurin's THEOREMS:

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{\lfloor\!\underline{2}}f''(a)+\cdots+\frac{(x-a)^n}{\lfloor\!\underline{n}}f^{(n)}(a)$$

$$+\frac{(x-a)^{n+1}}{\lfloor\!\underline{n+1}}\cdot f^{(n+1)}(c)\ ;\quad a<c<x$$

For $a=0$, it is called *Maclaurin's formula*.

## CONCAVITY, CONVEXITY OF A CURVE:




If $f''$ changes sign as $x$ passes through $a$, then $x=a$ is the point of inflection.

Note that $f''(a)=0$ if it exists.

ASYMPTOTES!

VERTICAL: Find $x=a$ such that

$$\lim_{x \to a \pm 0} f(x) = \pm\infty \quad \text{OR} \quad \lim_{x \to a} f(x) = \pm\infty$$

INCLINED: $y = \underbrace{mx + c}_{f(x)}$ where

$$m = \lim_{x \to \infty} \frac{f(x)}{x} \quad (\text{if limit exists})$$

$$\& \quad c = \lim_{x \to \infty} (f(x) - mx)$$

CURVATURE:

CARTESIAN FORM: $y = f(x)$

$$K = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}} \quad ; \qquad R = \frac{1}{K}$$

PARAMETRIC FORM: $x = \varphi(t) \quad y = \psi(t)$

$$K = \frac{|\varphi'\psi'' - \psi'\varphi''|}{[\varphi'^2 + \psi'^2]^{3/2}}$$

POLAR COORDINATE: $\rho = f(\theta)$

$$K = \frac{|2\rho'^2 - \rho''\rho + \rho^2|}{[\rho'^2 + \rho^2]^{3/2}}$$

INCREASE AND DECREASE OF A FUNCTION (Application of Mean Value Theorem)

- A function $f$ is increasing (strictly) on on interval $I$ if

$$f(b) > f(a) \quad \forall\ b > a \text{ in } I.$$

Th.

If $f$ increases on an interval $[a,b]$, then $f'(x) \geq 0$ on this interval.

If $f'(x) > 0$ on $(a,b)$ then $f(x)$ increases on $[a,b]$.

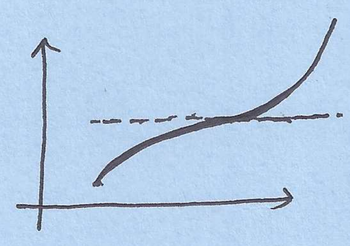
Proof: I. Since $f$ increases on $[a,b]$, then

$$f(x+\Delta x) > f(x) \quad \text{for } \Delta x > 0$$

$$\& \quad f(x+\Delta x) < f(x) \quad \text{for } \Delta x < 0$$

$$\Rightarrow \quad \frac{f(x+\Delta x) - f(x)}{\Delta x} > 0$$ in both cases for any sufficiently small $\Delta x$

$$\Rightarrow \quad \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \geqslant 0 \quad \Rightarrow \quad f'(x) \geq 0.$$



II. Consider two values $x_1$ & $x_2$ such that $x_1 < x_2$

By Lagrange mean value theorem:

$$f(x_2) - f(x_1) = f'(c)\,(x_2 - x_1), \qquad x_1 < c < x_2$$

It is given that $f'(c) > 0$

$$\Rightarrow \quad f(x_2) - f(x_1) > 0$$

$\Rightarrow$ $f(x)$ is an increasing function.

Similar result for decreasing function.

Ex. Determine the intervals of monotonicity for the function

$$f(x) = x^2 e^{-x}$$

Sol.

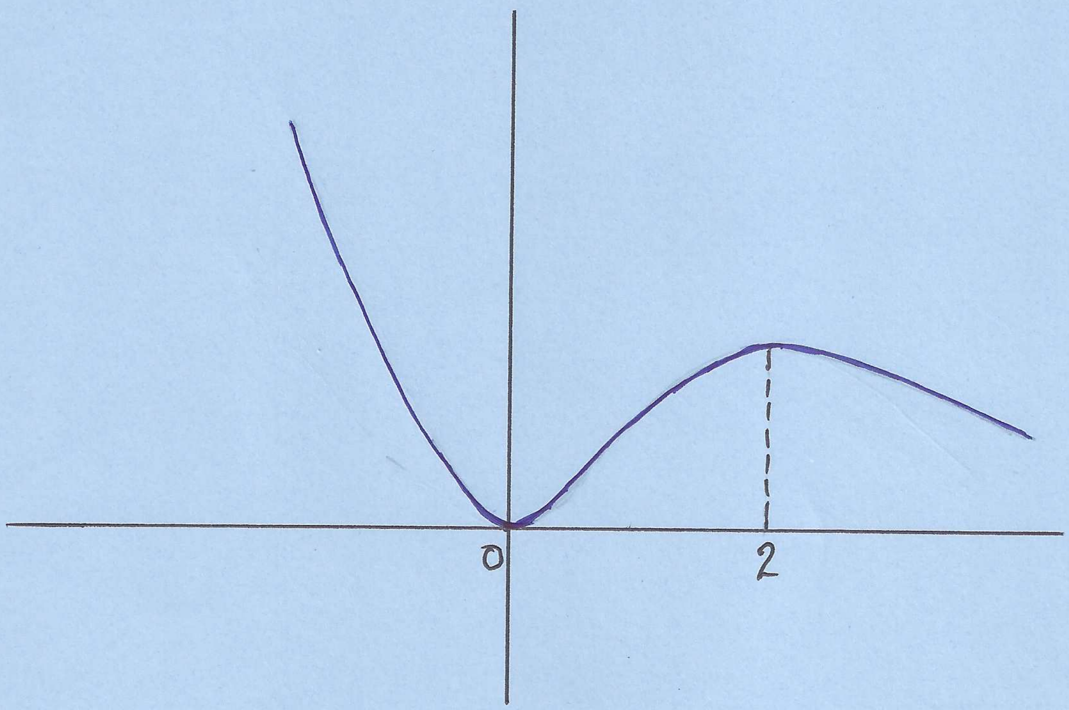
$$f'(x) = -x^2 e^{-x} + 2x e^{-x}$$

$$= x e^{-x}(2-x)$$

If $0<x<2$ ; $f'(x)>0$ $\Rightarrow$ function is increasing in $(0,2)$

If $x<0$ ; $f'(x)<0$ $\Rightarrow$ function is decreasing in $(-\infty,0)$

If $x>2$; $f'(x)<0$ $\Rightarrow$ function is decreasing in $(2,\infty)$



# LAGRANGE FORM OF REMAINDER (DERIVATION)

RECALL: TAYLOR'S POLYNOMIAL

$$P_n(x) = f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{\lfloor 2}f''(x_0) + \cdots + \frac{(x-x_0)^n}{\lfloor n} f^{(n)}(x_0).$$

Also,

$$R_n(x) = f(x) - P_n(x)$$

Question: $R_n(x)$ ?

## INTERMEDIATE RESULT:

Let $I$ be an open interval and let $n$ be a nonnegative integer and suppose that the function $f: I \to \mathbb{R}$ has $(n+1)$ derivatives. Suppose that at the point $x_0$ in $I$,

$$f^{(k)}(x_0) = 0 \quad \text{for} \quad 0 \le x \le n$$

Then for each point $x \neq x_0$ in $I$, there is a point $c$ between $x$ & $x_0$ at which

$$f(x) = \frac{f^{(n+1)}(c)}{\lfloor(n+1)}(x-x_0)^n$$

**PROOF:** Define $g(x) = (x-x_0)^{n+1} \quad \forall x \in I$

$$\Rightarrow g^{(k)}(x_0) = 0 \quad \& \quad g^{(n+1)}(x_0) = \lfloor(n+1)$$

Let $x$ be a point in $I$ and suppose $x > x_0$. By Cauchy mean value theorem to the functions $f(x)$ and $g(x)$ in $[x_0, x]$, we get

$$\frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f'(x_1)}{g'(x_1)} \Rightarrow \frac{f(x)}{g(x)} = \frac{f'(x_1)}{g'(x_1)} \qquad x_0 < x_1 < x. \quad \text{—(1)}$$

Applying CMVT for $f'$ & $g'$ in $[x_0, x_1]$

$$\frac{f'(x_1)}{g'(x_1)} = \frac{f''(x_2)}{g''(x_2)} \quad ; \quad x_0 < x_2 < x_1 \quad \text{—(2)}$$

By ① & ②:

$$\frac{f(x)}{g(x)} = \frac{f''(x_2)}{g''(x_2)}$$

Continuing, we get

$$\frac{f(x)}{g(x)} = \frac{f^{(n+1)}(x_{n+1})}{g^{(n+1)}(x_{n+1})}; \qquad x_0 < x_{n+1} < x_n < x$$

Setting $x_{n+1} = c$, we obtain

$$\frac{f(x)}{g(x)} = \frac{f^{(n+1)}(c)}{\lfloor(n+1)} \Rightarrow f(x) = \frac{f^{(n+1)}(c)}{\lfloor(n+1)} \cdot (x-x_0)^{n+1}$$

□

Coming back to $R_n(x) = f(x) - P_n(x)$.

Note that.

$$R_n(x_0) = R_n'(x_0) = \cdots = R_n^{(n)}(x_0) = 0$$

With the above intermediat result:

$$R_n(x) = \frac{R_n^{(n+1)}(c)}{\lfloor(n+1)} (x-x_0)^{n+1} \qquad x_0 < c < x.$$

Also not that

$$R_n^{(n+1)}(x) = f^{(n+1)}(x) - \underbrace{P_n^{(n+1)}(x)}_{=0} = f^{(n+1)}(x)$$

Therefore,

$$\boxed{R_n(x) = \frac{f^{(n+1)}(c)}{\lfloor(n+1)} (x-x_0)^{n+1}}$$