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A robust transportation signal control problem accounting for traffic dynamics

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This article is dedicated in honor of Professor Robert Paaswell on his 70th birthday for his outstanding contributions to transportation research and leadership

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ABSTRACT

Transportation system analysis must rely on predictions of the future that, by their very nature, contain substantial uncertainty. Future demand, demographics, and network capacities are only a few of the parameters that must be accounted for in both the planning and every day operations of transportation networks. While many repercussions of uncertainty exist, a primary concern in traffic operations is to develop efficient traffic signal designs that satisfy certain measures of short term future system performance while accounting for the different possible realizations of traffic state. As a result,uncertainty has to be incorporated in the design of traffic signal systems. Current dynamic traffic equilibrium models accounting for signal design, however, are not suitable for quantifying network performance over the range of possible scenarios and in analyzing the robust performance of the system. The purpose of this paper is to propose a new approach—robust system optimal signal control model; a supply-side within day operational transportation model where future transportation demand is assumed to be uncertain. A robust dynamic system optimal model with an embedded cell transmission model is formulated. Numerical analysis are performed on a test network to illustrate the benefits of accounting for uncertainty and robustness.

1. Introduction

Transportation system analysis has been traditionally concerned with supply and demand analysis in some nominally 'typical' conditions. However, since traffic volume and capacities continue to change both within day and day to day, dynamic models accounting for time varying conditions have recently been developed. In addition, the rapid advances in real-time sensors and information technology have provided us the ability to capture the inherent uncertainty and opportunities to apply robust optimization approaches to transportation problems. These opportunities have begun to challenge the idea of planning for 'nominal' conditions. While the initial impetus has been realized in the context of natural disasters—such as earthquakes [3,15]—affecting the 'connectivity' of a road network, recent thinking has focused on broadening this definition to both planning and operational decision making [29,30,5].

The main focus of this work is to develop a robust dynamic signal optimization formulation that integrates both dynamic traffic assignment and signal control. Because the transportation network performance depends on the control devices such as traffic lights, variable message signs, etc., and influences the 'optimal path' available for routing, an integrated model is highly beneficial to optimize the entire transportation network. In addition, there is also uncertainty in the total number of vehicles that will move from a given origin to a destination pair in a time interval. Estimates of how much of demand will move is forecasted for each O–D pair, but these estimates are at best educated guesses. Based on the realization of the demand, the signal control will vary at each intersection and the optimal routing pattern for vehicles would appropriately be different. In addition, the design of signal control should be *resilient* for all realizations of demand. Such resilient control mechanisms will improve the overall transportation network performance. Most of the past work account for this by using static user/stochastic user equilibrium models. These models are however not applicable because: (1) they are not applicable for real-time management; (2) they do not account for traffic dynamics and (3) they do not account for uncertainty and robustness.

The core question addressed in this analysis is: How should the traffic signal control settings be designed optimally over time to best meet the requirements of the transportation network performance, given uncertainties in point to point demand over time and the need to account for robustness, recognizing that there are constraints that limit the cycle times, green times at intersections and restrictions in capacities and jam densities on the road network?

To address this core question, we develop a robust optimization model of the dynamic system optimal signal control problem that focuses on developing optimal signal times at intersections. To

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correctly reflect the uncertainties in the O–D demand, and the resulting implications on network performance, the optimization must include uncertainty in the structure of some of its constraints. In addition to accounting for robustness, the objective function should measure the resiliency of the network. We have included this 'structural uncertainty' in the definition of discrete scenarios and develop a discrete two stage robust formulation. We are focused more on finding 'robust' solutions to the optimization problem, using the core concepts discussed in [30,23]. This robust optimization formulation allows us to study the tradeoff of different objectives with varying levels of system risk in the solutions that are accepted by the network manager.

Although, the main focus of this work is demonstrating the need to account for robustness in traffic signal control on simple transportation networks, the developed model can be extended to solve large scale networks by using efficient algorithms for the robust optimization model. The paper is organized as follows. Section 2 discusses the previous literature on robust optimization and applications in transportation problems. Section 3 describes the formulation of the robust signal control problem and discusses the intuition of the objective function and the constraint sets. Computational results on a test network are presented in Section 4. Section 5 focuses on discussing the insights obtained from the analysis and Section 6 concludes the paper with recommendations for future work.

2. Background

A common tool for modeling uncertainty is a two-stage stochastic program where the decision variables are partitioned into two sets. First stage variables are those that have to be decided before the uncertain parameters are realized. For transportation planning, these would be equivalent to any infrastructure added to the network before future demand is realized. The second stage (also referred to as recourse) variables represent decisions that are made after some uncertainty has been realized. The second stage problem can be viewed as an operational decision making problem following the first stage plan. It is important to realize that the second stage objective is to minimize the expected value of a function of the random second stage costs. This concept has been applied in linear, integer and non-linear programming problems. A compact formulation of a general stochastic linear programming is given in [13,7].

Waller and Ziliaskopoulos [31] proposed a dynamic network design model as a two stage stochastic linear programming problem where the cell transmission model (CTM) [10] is the embedded traffic flow model in the second stage and the demand was modeled as a random variable. Lo and Tung [21] discuss a chance-constrained reliability formulation of the traffic equilibrium problem under minor network disruptions. Their primary focus was on developing a probabilistic user equilibrium model under the assumption that users minimize the expected travel time and the flows would settle into equilibrium in the long term. In these formulations, minimizing expected costs often fails to appropriately account for extreme outcomes which are resilient to future variations. In other words, although the first stage variables will optimize the mean of the objective function, there may be scenarios for which the network performs poorly although on average it performs quite satisfactorily.

Long-term demand uncertainty can be accounted for using stochastic optimization methods with either a recourse or a chance-constrained formulation as demonstrated in [31]. In transportation, uncertainty has been primarily studied in terms of capacity reliability of a network [12]. 'Capacity reliability' has been defined in different ways by different researchers. A comprehensive review of these definitions is given in [2]. Chen et al. [8] defined capacity reliability as the probability that the network can accommodate a certain demand at a given service level. The studies were done on small networks, extending these studies for larger networks is a computationally intensive task. Du and Nicholson [11] proposed a conventional equilibrium approach with variable demand to describe flows in a network with degradable link capacities. Lo and Tung [22] define capacity reliability as a maximum flow that the network can carry, subject to link capacity and travel time reliability constraints.

The problem of robustness has been closely examined in the area of financial investment and is often addressed by including the variance of future cost as a measure for analysis (for example, refer to [23]). This approach minimizes variance as part of the objective function so that highly volatile solutions are discouraged. An efficient frontier is achieved by varying the weights on the expected value and volatility of network performance in the objective function. This approach is appropriate if input parameters are uncertain with known distributions, or if there exist multiple bounded random input parameters with unknown distributions. The model presented in [23] extends the volatility to higher norms of the random variables and has been extensively used in practice [24]. One drawback, however, is that it requires symmetrically distributed random variables. A second approach is based on the von Neumann–Morganstern expected utility models [14]. This presents a more general framework for handling risk aversion, the primary advantage being the ability to handle asymmetries in random variables. These models can also be extended to model multi-period planning problems. In our work we use this definition of robustness to model the minimization of network wide travel times.

A more recent definition of robustness is given in [6]. A robust solution is defined at an aggregate level as one that guarantees the feasibility of the solution if, for a given number i, less than i constraint coefficients change. Further, a probabilistic guarantee that the robust solution will be feasible is given if more than i coefficients change. Ben-Tal and Nemirovski [4] proposed a second-order cone programming approach to overcome the conservative solutions of Soyster [28]. The formulation is a non-linear program and has a difficult solution algorithm in the size of constraints and variables. This definition of robustness however converts our problem into the worst case problem [28].

3. Model formulation

In this section, we define the basic variables and the mathematical formulation for developing the robust signal control problem. The formulation utilizes an embedded cell transmission model [9,10], a mesoscopic traffic flow model to capture the vehicle movement in the network. CTM [9,10] provides a convergent numerical approximation to Lighthill and Whitham [16] and Richards' [25] (LWR) hydrodynamic model to simple difference constraints by assuming a piecewise linear relationship between traffic flow and density for each cell (or segment). The CTM approximates the fundamental diagram of flow-density shown in Fig. 1a by a piecewise linear model shown in Fig. 1b. The basic relationships of the cell transmission model are extensively discussed in [9,10,20,18,32]. To facilitate cross-reference, we adopt similar notation as in [32,29]. The CTM as proposed by Daganzo [9,10] does not explicitly model signalized intersections; however, the same basic building blocks can be extended to capture traffic realism. Beard and Ziliaskopoulos [1] develop an improvised CTM that can explicitly model intersection movements not accounting for the demand uncertainty. The intersection cell configuration adopted in this paper is similar to the one by Beard and Ziliaskopoulos [1] and is briefly described here. Each turning movement on each approach at the intersection is designated a separate single cell. Each cell uniquely handles a turning movement. The set of cells that represent all the movements at an intersection are together

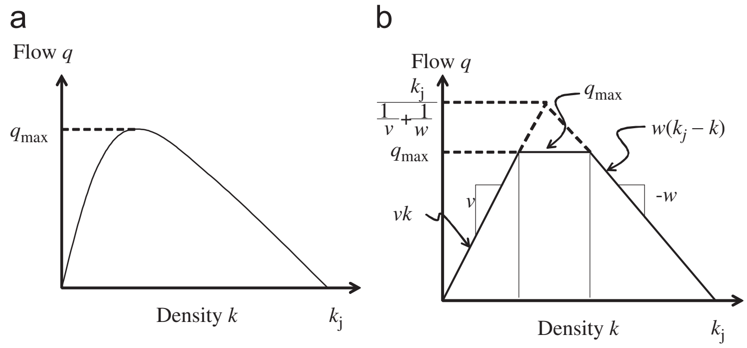

Fig. 1. (a) q–k diagram in LWR. (b) q–k diagram in CTM.

designated as intersection cell (C_I). A single diverging upstream cell ($\in C_D$—the set of diverging cells) transmits vehicles to the intersection cells. All turning movements feeding to a downstream direction of motion discharge the vehicles into a merge cell. Beard and Ziliaskopoulos [1] adopt a convenient numbering scheme for intersection cells that is consistent with the National Electrical Manufacturers Association (NEMA) numbering convention. The same numbering scheme is retained here for comparison. The first digit of intersection cell number represents the intersection number. The second and third digit denote NEMA signal phase numbers. For example, left-turn cells on the major street at intersection 1 are numbered 101 and 105 while the through movement cells on the major street arenumbered 102 and 106. Right turns are numbered 109–112. A complete numbering scheme for a test network is shown in Fig. 3.

The robust system optimal signal control (RO-SOSC) problem for a general traffic network is formulated as a linear program in this paper. Traffic flow is modeled using linear relationships while a linear positive deviation measure accounts for robustness of the solution.

The notations used in the RO-SOSC are listed below. The letters shown in bold represent the decision variables that are to be determined by the RO-SOSC model.

τ	discretization time interval (60 s)
T	set of discrete time intervals
C	set of cells—C_O: ordinary, C_D: diverging, C_M: merging, C_R: source, C_S: sink, C_I: intersection and C_{I_S} : $\{C_I \cup \Gamma^{-1}(i) \cup \Gamma^{-1}(\Gamma^{-1}(i)) : i \in C_I\}$ set of intersection cells and cells close to intersections which are used for computing number of stops and intersection delays.
$\mathbf{x_i^{odt}}$	number of vehicles in cell i during current time interval t oriented from source cell o to sink cell d
N_i^t	jam density (maximum number of vehicles that can be accommodated) of cell i at time t
$\mathbf{y_{ij}^{odt}}$	number of vehicles moving from cell i to cell j during current time interval t oriented from source cell o to sink cell d
Q_i	maximum number of vehicles that can flow into or out of cell i (saturation flow rate)
q_i^t	maximum number of vehicles that can flow into or out of cell i between time intervals t to $t+1$
δ_i^t	ratio of forward to backward propagation speed for cell i at current time interval t; $\delta_i^t :=1$
$\Gamma(i)$	set of successor cells to cell i
$\Gamma^{-1}(i)$	set of predecessor cells to cell i
I	set of intersections
$\tilde{d}_{odt}$	demandfrom cell o to cell d at current time interval t, $o \in C_R$ and $d \in C_S$
Ω	the set of demand realizations
$\mathbf{w_i^t}$	fraction of green time for cell i in time interval t
G_{min}	duration of minimum green phase
β	arbitrarily large constant
λ_1, λ_2, λ_3	weights for different performance measures in objective function—total system travel time, intersection delay and number of stops, respectively. $\lambda_1 = 1$, $\lambda_2 = 2$ or 0 and $\lambda_3 = 4\tau$ or 0.

In addition, the following letters will be used as indices: i, j, k for cells, od for origin–destination pairs, t for time steps and ω for demand scenario realizations. The model uses a set of discrete time intervals indexed by t $\{t = 0,1,2,3,\ldots,T\}$ and the network is represented by a set of discrete cells. The length of a time period is fixed at 60 s while the length of each cell is the distance traveled by a vehicle at free-flow speed.

Mathematical formulation of RO-SOSC problem:

Objective function:

$$\text{Min} \sum_{\omega \in \Omega} [\lambda p^{\omega} z^{\omega}(x,y) + (1-\lambda) p^{\omega} \Delta(z^{\omega}(x,y), \bar{z}(x,y)) + \beta z_4^{\omega}], \tag{1}$$

where $\bar{z}(x,y)$ is the expected value given by $\sum_{\omega \in \Omega}[p^{\omega} z^{\omega}(x,y)]$, $\Delta(z^{\omega}(x,y), \bar{z}(x,y))$ is a deviation measure defined thus: $\Delta(z^{\omega}(x,y), \bar{z}(x,y)) = \text{Max}(z^{\omega}(x,y) - \bar{z}(x,y), 0)$, $z^{\omega}(x,y)$ is a weighted performance measure, $z^{\omega}(x,y) = \lambda_1 z_1^{\omega}(x,y) + \lambda_2 z_2^{\omega}(x,y) + \lambda_3 z_3^{\omega}(x,y)$ where λ_i's are weights and z_i^{ω}'s represent the following: z_1^{ω} is the measure of total delay in network for a realization $\omega \in \Omega$ given by $\sum_{i \in C \setminus C_S} \sum_{\forall t} \tau(\sum_{\forall od} x_i^{odt}(\omega))$, z_2^{ω} the measure of intersection delays for a realization $\omega \in \Omega$ given by $\sum_{i \in C_{I_S}} \sum_{\forall t} \tau(\sum_{\forall od}(x_i^{odt}(\omega) - \sum_{j \in \Gamma(i)} y_{ij}^{odt}(\omega))$. The delay in each cell at each time interval in the network may be specified as [19] $\tau \sum_{\forall od}(x_i^{odt}(\omega) - \sum_{j \in \Gamma(i)} y_{ij}^{odt}(\omega))$. The sum of this delay over all cells that are 'close' to the intersection provides the intersection delay. The sum over all intersections over all time provides the total intersection delay in the network. z_3^{ω} the measure of number of stops for a realization $\omega \in \Omega$. The number of stops can be approximated as [17] $(\frac{1}{2}) * \sum_{j \in C_{I_S}} \sum_t \sum_{\forall od} |\sum_{k \in \Gamma(j)} y_{jk}^{odt}(\omega) - \sum_{i \in \Gamma^{-1}(j)} y_{ij}^{odt}(\omega)|$.

Constraints: The RO-SOSC should satisfy the following constraints. Each of the following constraints must hold for each $\omega \in \Omega$.

Flow conservation equations:

$$x_i^{odt}(\omega) - x_i^{odt-1}(\omega) - \sum_{k \in \Gamma^{-1}(i)} y_{ki}^{odt-1}(\omega) + \sum_{j \in \Gamma(i)} y_{ij}^{odt-1}(\omega) = \begin{cases} \tilde{d}^{odt-1}(\omega) & \text{if } i = o, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

$\forall o \in C_R, \forall d \in C_S, \forall i \in C, \forall t \in T$.

Demand satisfaction constraint:

$$\sum_t \sum_{i \in \Gamma^{-1}(d)} y_{id}^{odt}(\omega) = \sum_t \tilde{d}^{odt}(\omega) \quad \forall o \in C_R, \ \forall d \in C_S. \tag{3}$$

Flow propagation constraints:

$$\sum_{\forall j \in \Gamma(i)} y_{ij}^{odt}(\omega) - x_i^{odt}(\omega) \leqslant 0 \quad \forall o \in C_R,\ \forall d \in C_S,\ \forall i \in C,\ \forall t \in T, \tag{4}$$

$$\sum_{\forall od} \sum_{\forall j \in \Gamma(i)} y_{ij}^{odt}(\omega) \leqslant q_i^t \quad \forall i \in C,\ \forall t \in T, \tag{5}$$

$$\sum_{\forall od} \sum_{\forall i \in \Gamma^{-1}(j)} y_{ij}^{odt}(\omega) \leqslant q_j^t \quad \forall j \in C,\ \forall t \in T, \tag{6}$$

$$\sum_{\forall od} \sum_{\forall i \in \Gamma^{-1}(i)} y_{ij}^{odt}(\omega) + \sum_{\forall o \in C_R} \sum_{\forall d \in C_S} \delta_j^t x_j^{odt}(\omega) \leqslant \delta_j^t N_j^t \quad \forall j \in C,\ \forall t \in T. \tag{7}$$

Non-negativity and initialization constraints:

$$x_i^{odt}(\omega) \geqslant 0 \quad \forall i \in C,\ \forall t \in T,\ \forall o \in C_R,\ \forall d \in C_S, \tag{8}$$

$$y_{ij}^{od0}(\omega) = 0 \quad \forall (i,j) \in E,\ \forall o \in C_R,\ \forall d \in C_S, \tag{9}$$

$$y_{ij}^{odt}(\omega) \geqslant 0 \quad \forall (i,j) \in E,\ \forall t \in T,\ \forall o \in C_R,\ \forall d \in C_S. \tag{10}$$

Constraints restricting the maximum flow:

$$q_i^t = \begin{cases} w_i^t Q_i & \forall\ i \in C_I,\ \forall t \in T, \\ Q_i & \text{otherwise.} \end{cases} \tag{11}$$

Constraints to ensure correct coordination between different signal phases:

$$w_{i+p}^t = w_{i+p+4}^t \quad \forall i \in I,\ p \in \{1,2,3,4\},\ \forall t \in T, \tag{12}$$

$$\sum_{p=1}^{4} w_{i+p}^t \leqslant 1 \quad \forall i \in I,\ \forall t \in T, \tag{13}$$

$$w_{i+2p}^t = w_{i+p+8}^t \quad \forall i \in I,\ p \in \{1,2,3,4\},\ \forall t \in T. \tag{14}$$

The total green time for any phase should at least equal the minimum green time:

$$w_i^t \geqslant G_{min} \quad \forall i \in C_I,\ \forall t \in T. \tag{15}$$

The objective function (1) is composed of three terms. λ is the weight for term 1 and p^ω is the probability of realizing scenario $\omega \in \Omega$. The first term $\lambda p^\omega z^\omega(x,y)$ is a weighted measure of the expected value of performance measure function $z^\omega(x,y)$. The performance measure function $z^\omega(x,y)$ is a weighted combination of three different performance measures—the total system (network) travel time, delay at intersections and the number of stops. It is assumed that $\lambda_1 = 1, \lambda_2 = 2$ and $\lambda_3 = 4\tau$. That is, intersection delays contribute an additional disutility of twice the delay time, while the additional disutility of a stop is 4τ (that is 4τ units of travel time has the same disutility as a single stop). The reader would note that these weights are arbitrary and have been chosen for illustration purposes only—estimates based on real data should be used in real-life application of this model. The second term in the objective—$(1-\lambda)p^\omega \Delta(z^\omega(x,y), \bar{z}(x,y))$—is a measure of the positive deviation of the weighted performance measure function from its expected value; negative deviations do not affect the objective. The rationale for ignoring negative deviations is that travel times which are lesser than the expected values are always considered favorable; users view as unfavorable only those travel that takes longer than usual. Further, the adopted deviation measure retains the linearity of the objective function (the Max function may be removed by introducing an additional non-negative variable that is always at least as large as the difference between weighted performance measure function and the expected value). The final term in the objective function βz_4^ω ensures vehicles are not held back at the origin cells. This term may be computed as $z_4^\omega = \sum_t \sum_{\forall od} \sum_{j \in \Gamma(o)} t y_{oj}^{odt}$.

Constraints (2) represent the flow conservation relationships at a cell i. The number of vehicles from origin o to destination d in cell i at time $t(x_i^{odt})$ is equal to the number of vehicles at time $t-1$ plus the difference between inflow into and outflow from the cell. The same equation applies for the different types of cells. If the cell is an origin cell, there are no predecessor cells ($\Gamma^{-1}(o) = \emptyset$) therefore inflow from others cells is zero. However, origin cells get inflow from the demand generated at time $t-1$ at cell $o(\tilde{d}^{odt-1})$. On the other hand, destination cells have no successor cells ($\Gamma(d) = \emptyset$) and therefore zero outflow. Constraint (3) is the demand satisfaction constraint at the destination. Ignoring this constraint will result in all flow from an origin being routed to the nearest (based on travel time) destination. These constraints, however, are not required if the indices o, d for each cell are based on whether or not that cell lies on a path joining cells o and d.

Flow propagation is controlled by constraints (4)–(7). Constraint (4) restrict the outflow from a cell to its current occupancy; (5) and (6) restrict the outflow and inflow to the flow capacity of the sending and receiving cells, respectively. Constraint (7) restrict the inflow into the downstream cell to the available capacity. Constraints (8)–(10) are the non-negativity and initialization constraints. Initial occupancy and flow have been assigned to zero.

Flow restrictions at an intersection are modeled as reduced flow capacity. For example, if the fraction of green for a particular turning movement at time t is 0.3, the maximum flow out of the cell is $0.3Q_i$ for the time period t. Constraints (11) control these flow restrictions. The approximation for intersection flow restriction used here does not account for start-up delay and gap acceptance behavior (which were modeled in [1]). However accounting for these behaviors would require integer variables and therefore result in a significantly more complex problem. The present approximation was therefore adopted because of the significantly reduced computational burden.

Coordination between the different signal phases are controlled by constraints (12)–(14). Constraints (12) ensure that the left and through movements on the street share the same fraction of green times while constraint (14) ensures that there is synchronization of the right turn green times with the corresponding through green times (that is, 'No turn on Red'). Constraint (13) ensures that the green time fractions of independent phases add-up to 1. Finally, constraints (15) enforces a minimum green duration for each phase. The decision variables in this model are all constrained to be non-negative, but we do not explicitly impose integer restrictions. The green times are generally large enough for each phase that simply rounding a linear programming solution is quite acceptable in practice.

4. Test network and experimental setup

To illustrate the application of the model developed in Section 3, we implement the formulation on a test network (Fig. 2). In this section, the test network characteristics and the experimental setup are discussed and in the next section we provide key insights that are obtained from the RO-SOSC model for the test network. The test network and data are generally representative of the typical transportation network with signals, although the particular network or data used do not reflect actual values of any particular real transportation network.

The primary goals in conducting the analysis are to:

- measure the value of accounting for uncertainty in the signal control problem,
- understand the value of accounting for robustness (at different levels of risk) in the signal control problem and
- investigate the potential to develop signal timing plans for different levels of demand accounting for robustness.

The test network comprises of 45 cells with one intersection (numbered from 1 to 33 and 12 intersection cells numbered from

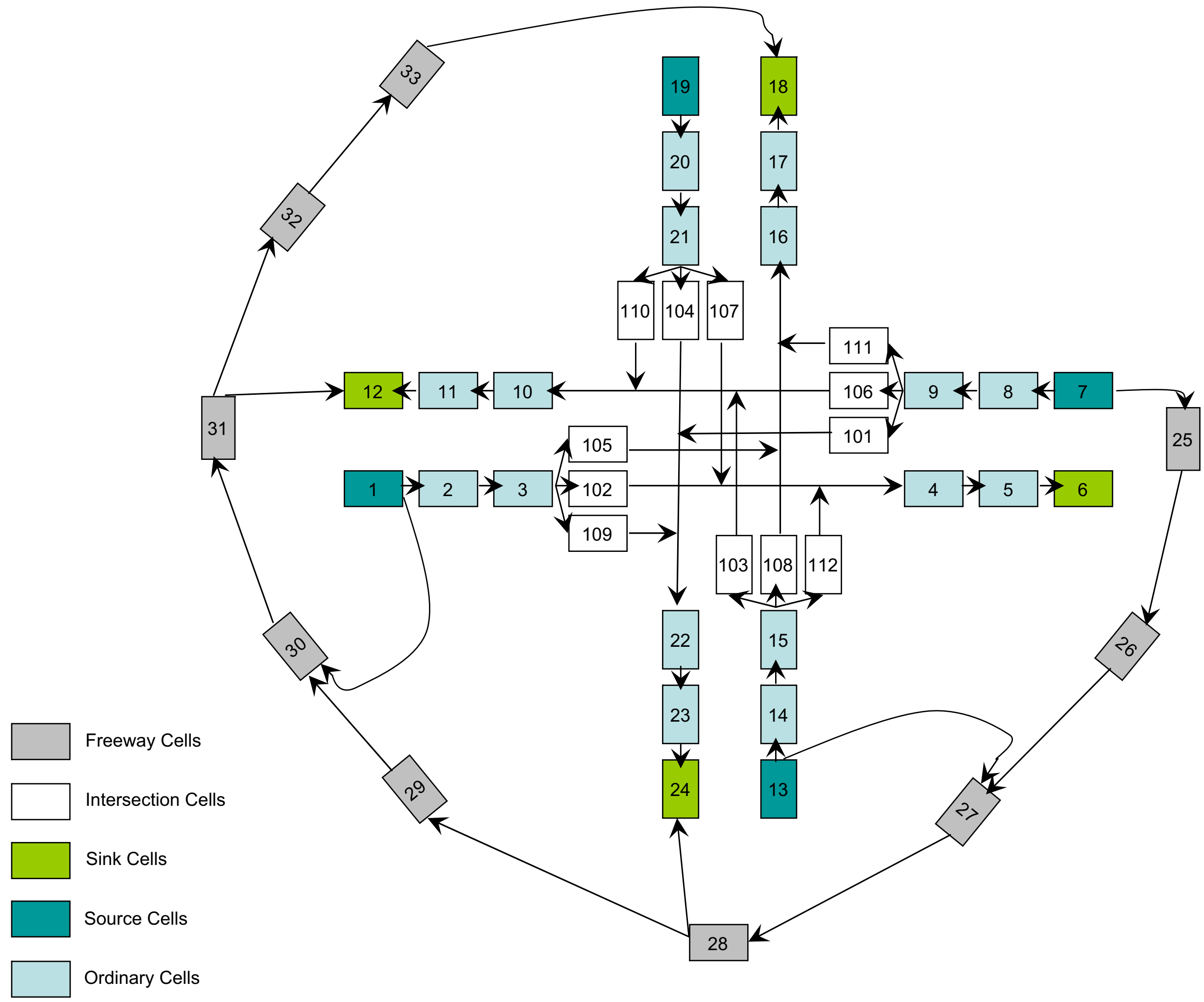

Fig. 2. Example network.

101 to 112) and eight O–D pairs. All eight O–D pairs have a path that passes through the intersection. Five of the O–D pairs also have an alternative route without intersections. Such a setup helps in understanding the value of robust signal timing optimization in the context when users also have route choice flexibility. The maximum flow capacity and jam density for each of the cells is shown in Table 1. The intersection operates under four mutually independent phases. Phase I (Phase III) corresponds to left turn from cells 101 and 105 (cells 103 and 107) and Phase II (Phase IV) corresponds to through flow from cells 103 and 107 (cells 104 and 108). Additional signal phases can be considered based on the intersection characteristics, however, for simplicity of illustration only four phases are considered in this work.

The primary source of uncertainty in the model is represented in the O–D demand. To reflect this uncertainty, we consider three realistic probability distributions from which a set of discrete scenarios are generated. Table 2 lists the distribution and the parameters for each origin–destination pair. The three distributions are Poisson, Uniform, and Beta distribution with shape parameters $\alpha=2$ and $\beta=4$. Four different cases for the Beta distribution were also tested. Case 1 corresponds to a high mean demand and a high variance case, case 2 has high mean, low variance, case 3 low mean, high variance and finally case 4 corresponds to a low mean and low variance distribution. The demands were generated from all origin–destination pairs from time steps 0 to 5. The total time steps modeled (T) is 15—this was chosen such that there was sufficient time for all demand generated to reach the destination in all cases.

Other factors that were varied in the experiments include the weight factor for the expected value and deviation measure in the objective function ($\lambda = \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.7, 1\}$), the number of demand realizations (1, 5, 10, 20, and 30 realizations), and different performance measure functions.

5. Results and discussion

Results from the experiments are presented in this section. Unless otherwise stated in the tables and charts, the reader may assume that the following default values for different parameters shall hold: demand distribution—Beta(2,4) with high demand and low variance (case 2 in Table 2), $\lambda = 0.2$, number of scenarios is 30, and the weighted combination of performance measure comprises all three performance measure functions.

Table 3 lists the performance measure values and the deviation measure for different distributions for the pure stochastic and pure robust cases for each distribution. The reader may note that the values of performance measure and deviation is the expected value over 30 scenario realizations. As expected the performance measure values are lesser for the stochastic case compared to the robust case

Table 1
Maximum flow capacity and jam density.

Cell number	Max. flow capacity	Jam density
1	120	9999
2	30	120
3	30	120
4	30	120
5	30	120
6	120	9999
7	120	9999
8	30	120
9	30	120
10	30	120
11	30	120
12	120	9999
13	120	9999
14	30	120
15	30	120
16	30	120
17	30	120
18	120	9999
19	120	9999
20	30	120
21	30	120
22	30	120
23	30	120
24	120	9999
25	30	480
26	30	480
27	60	960
28	60	960
29	60	720
30	60	960
31	60	960
32	60	720
33	60	720
101	30	120
102	30	120
103	30	120
104	30	120
105	30	120
106	30	120
107	30	120
108	30	120
109	30	120
110	30	120
111	30	120
112	30	120

where the objective is to minimize the deviation measure only. The deviations are minimized at the expense of recording very high travel times and intersections stops and delays. These results are consistent across the three different distributions. A possible reduction of the high values of performance measures may be achieved by assuming a value of $\lambda > 0$.

Table 4 presents the signal time values for the different distributions for both the pure stochastic and pure robust cases. The values presented for each phase (P1–P4) are the average percent green, the minimum value and the maximum value of percent green over the time intervals when there was non-zero flows from the intersection cells (typically this interval ranged from the 3rd time interval to the 12th/13th (10th/11th for low demand cases) time interval). The last column indicates the % of vehicles that choose the freeway route. The signal timing splits for the Poisson and Uniform distributions do not vary for the different time intervals; however, for the Beta distribution the splits show differences. The differences could be because of 'true' benefits that may be realized by adopting a dynamic signal time splits or may be 'false' bias introduced due to insufficient sample size. The effects could not be isolated and identified because of memory restrictions (problems were solved using CPLEX solver on a computer with 1 GB RAM) that did not allow us to run larger than 30 scenarios or 15 time intervals. The signal timing differences appear only minor from these results. Apart from the observation that the RO-SOSC provided solutions along expected lines for different distributional assumptions of demand, no conclusive results can be drawn on the impact of distribution on signal timing.

A related factor of interest is the performance of RO-SOSC for various levels of mean demand and variance. The Poisson distribution cannot be used in the experiment since its' coefficient of variance term is always 1. Between the Uniform and Beta distributions, the Beta distribution is chosen given the greater flexibility it affords in terms of varying shapes (for example, asymmetry). Four combinations arising from two different values (high and low) each for mean and variance were tested (see Table 2). The results are shown in Tables 5 and 6. The performance and deviation measures for the pure stochastic and pure robust cases are again along expected lines. In terms of the signal timings (Table 6) we again observe variation over time intervals (dynamic signal times). More pronounced is the variation between the stochastic and robust signal timing plans particularly for the low demand case. These variations are a result of greater flexibility in routing arising from excess unused capacity in the network. When the demand is high, there is less recourse in routing and therefore the signal timings of the stochastic and robust cases are close. This argument is further strengthened by the freeway route choice percentages—for the high demand case the difference between stochastic solution and robust solution is less than 5% while it is 20% for the low demand case. A greater proportion of the vehicles under low demand are routed through the freeway in the robust solution—perhaps, a result of the reliable travel times that freeways provide compared to the more unpredictable intersections. These results provide strong motivation to analyze the RO-SOSC problem, especially if the objective is to provide reliable service at low demands.

Given the importance of analyzing the RO-SOSC problem an important question facing the analyst is the degree or level of robustness in the objective function. To better understand the dependence of the solutions to varying levels of robustness, the RO-SOSC is solved by varying the value of λ from 0 to 1. This analysis was conducted at a high demand level, low variance, with # of scenarios equal to 30. As expected, there is a trade-off between the expected value of total system travel time (E(TSTT)) and the deviation measure (Fig. 3). As the weight on the risk factor increases it is observed that a conservative solution is obtained in terms of E(TSTT). However, there appears a value of $\lambda \approx 0.4$ beyond which any more increase in λ does not result in substantial changes in the performance measures. The particular value of λ obtained is expected to be dependent on several factors and may not be generalizable. The choice of the appropriate value of λ would depend on the level of risk that the analyst/planner is willing to accommodate. In terms of the signal timings, minor differences were observed for different values of lambda. However substantial differences or noticeable trends were not observed. An interesting future extension would be to observe the results under low demands since signal timings were found to be sensitive to the level of demand (Table 6).

The analysis reported till now were carried out with 30 scenarios for each case. This was the maximum number of scenarios that could be analyzed while solving the problem in CPLEX 9.0 on a personal computer running Windows XP operating system with 1 GB of RAM. Efficient decomposition and variable storage techniques could allow the analysis of larger networks with greater number of realizations. An interesting question in this context is whether 30 scenario realizations is enough to obtain reliable results. To test the sensitivity of solutions to number of scenarios different runs were carried out varying the number of replication. 1, 5, 10, 20 and 30 scenarios were analyzed. The value of the different performance measures and the deviation measure is plotted in Fig. 4. Given the multiplicity of solutions that are possible in system optimal problem formulations, the

Table 2
Origin–destination demands.

O–D pair	Poisson dist.	Uniform dist.	Beta(2,4) Range			
	Mean	Range	Case 1	Case 2	Case 3	Case 4
(1,6)	7	(0,14)	(0,14)	(3.5,10.5)	(0,7)	(1.75,5.25)
(1,18)	25	(0,50)	(0,50)	(12.5,37.5)	(0,25)	(6.25,18.75)
(7,12)	25	(0,50)	(0,50)	(12.5,37.5)	(0,25)	(6.25,18.75)
(7,24)	25	(0,50)	(0,50)	(12.5,37.5)	(0,25)	(6.25,18.75)
(13,12)	15	(0,30)	(0,30)	(7.5,22.5)	(0,15)	(3.75,11.25)
(13,18)	25	(0,50)	(0,50)	(12.5,37.5)	(0,25)	(6.25,18.75)
(19,6)	7	(0,14)	(0,14)	(3.5,10.5)	(0,7)	(1.75,5.25)
(19,24)	25	(0,30)	(0,30)	(7.5,22.5)	(0,15)	(3.75,11.25)

Table 3
Performance measure values for different distributions.

Distribution	Objective function	TSTT	No. of stops	Intersection delay	Deviation measure
Poisson	Stochastic ($\lambda=1$)	328888.00	74.21	18936.60	12120.20
	Robust ($\lambda=0$)	457512.00	306.08	30808.50	0.00
Uniform	Stochastic ($\lambda=1$)	352163.00	100.21	24624.80	15560.60
	Robust ($\lambda=0$)	464536.00	347.79	36010.90	0.00
Beta	Stochastic ($\lambda=1$)	201876.00	19.29	2279.00	6970.47
	Robust ($\lambda=0$)	316800.00	280.73	18047.00	0.00

Table 4
Signal times and route choice split for different distributions.

Dist.	Objective function	Avg. % green for phase				Avg. % of veh. freeway route
		P1	P2	P3	P4	
P	S	10.0 (10.0,10.0)	37.1 (37.1,37.1)	15.9 (15.9,15.9)	37.0 (37.0,37.0)	55.35
	R	14.7 (14.7,14.7)	32.4 (32.4,32.4)	15.9 (15.9,15.9)	37.0 (37.0,37.0)	54.47
U	S	11.3 (11.3,11.3)	23.3 (23.3,23.3)	20.3 (20.3,20.3)	45.0 (45.0,45.0)	56.50
	R	11.3 (11.3,11.3)	23.3 (23.3,23.3)	20.3 (20.3,20.3)	45.0 (45.0,45.0)	56.62
B	S	10.0 (10.0,10.0)	26.7 (26.7,26.7)	19.2 (17.5,31.7)	44.1 (31.7,45.8)	53.41
	R	10.0 (10.0,10.0)	27.3 (16.7,33.3)	18.8 (16.7,20.0)	0.44 (36.7,53.3)	58.70

Table 5
Performance measure values for different demand levels.

Demand level (mean, var.)	Objective function	TSTT	No. of stops	Intersection delay	Deviation measure
High, high	Stochastic ($\lambda=1$)	201876	19.29	2279	6970.47
	Robust ($\lambda=0$)	316800	280.73	18047	0
High, low	Stochastic ($\lambda=1$)	256921	20.45	3251	5107.87
	Robust ($\lambda=0$)	386262	338.70	18967.5	0
Low, high	Stochastic ($\lambda=1$)	94510	0.3	18	2668.93
	Robust ($\lambda=0$)	164909	187.71	9035	0
Low, low	Stochastic ($\lambda=1$)	119234	0.1	6	1461.33
	Robust ($\lambda=0$)	203578	192.92	10222.5	0

interpretations have to be cautious. Nevertheless the figures suggest that a minimum of 5 realizations are required to obtain reasonably stable performance measures while at least 20 realizations are required to obtain reasonable deviation measures. It is important to investigate whether the trends observed with 20 and 30 realizations continue for a greater number of realizations to conclusively answer the question on scenario sample size (Table 7).

Trading off different objectives—total delay, number of stops and intersection delay: because different objectives are possible for the robust signal control problem, it will be interesting to examine how the solution changes with the different parameters for λ_1, λ_2 and λ_3 are varied. Table 8 shows the performance measure values for different objective functions. It is observed that the network wide total travel time objective influences the greatest in the robust signal control problem as compared to the other objective functions. In other words, in the design of signal control systems, it is important to consider system wide impacts as opposed to local objectives such as intersection delays and number of stops.

Tables 9 and 10 present the value of robust system optimal signal design. This experiment is performed by evaluating the solutions obtained by the deterministic, stochastic and robust problems. The deterministic problem was solved as an expected value problem. To measure the value of robust design, the signal timings obtained from solving the deterministic, stochastic, and robust problems are taken as input to the system optimal dynamic assignment problem. Traffic assignment is then done assuming a purely stochastic objective

Table 6
Signal times and route choice split for demand levels.

Demand level (mean, var.)	Objective function	Avg. (min, max) % green for phase				Avg. % of veh. freeway route
		P1	P2	P3	P4	
High, high	S	10 (10,10)	26.7 (26.7,26.7)	19.3 (17.5,31.7)	44.1 (31.7,45.8)	53.4
	R	10 (10,10)	27.2 (16.7,33.3)	18.8 (16.7,20)	43.9 (36.7,53.3)	58.7
High, low	S	13.9 (10,45)	29.8 (28.3,35)	17.4 (10,18.3)	38.9 (10,43.3)	54.8
	R	10.5 (10,13.3)	29.9 (26.7,30.1)	17.1 (16.7,18.3)	42.4 (41.7,42.5)	57.4
Low, high	S	10 (10,10)	35.7 (33.3,36.7)	14.3 (13.3,16.7)	40 (40,40)	42.0
	R	14.5 (10,36.7)	23.6 (10,55)	35.5 (10,55)	26.4 (25,30)	62
Low, low	S	10 (10,10)	36.7 (36.7,36.7)	13.3 (13.3,13.3)	40 (40,40)	40.5
	R	13.3 (10,20)	27.7 (10.4,65)	27.6 (11.7,69.6)	31.8 (10,50)	59

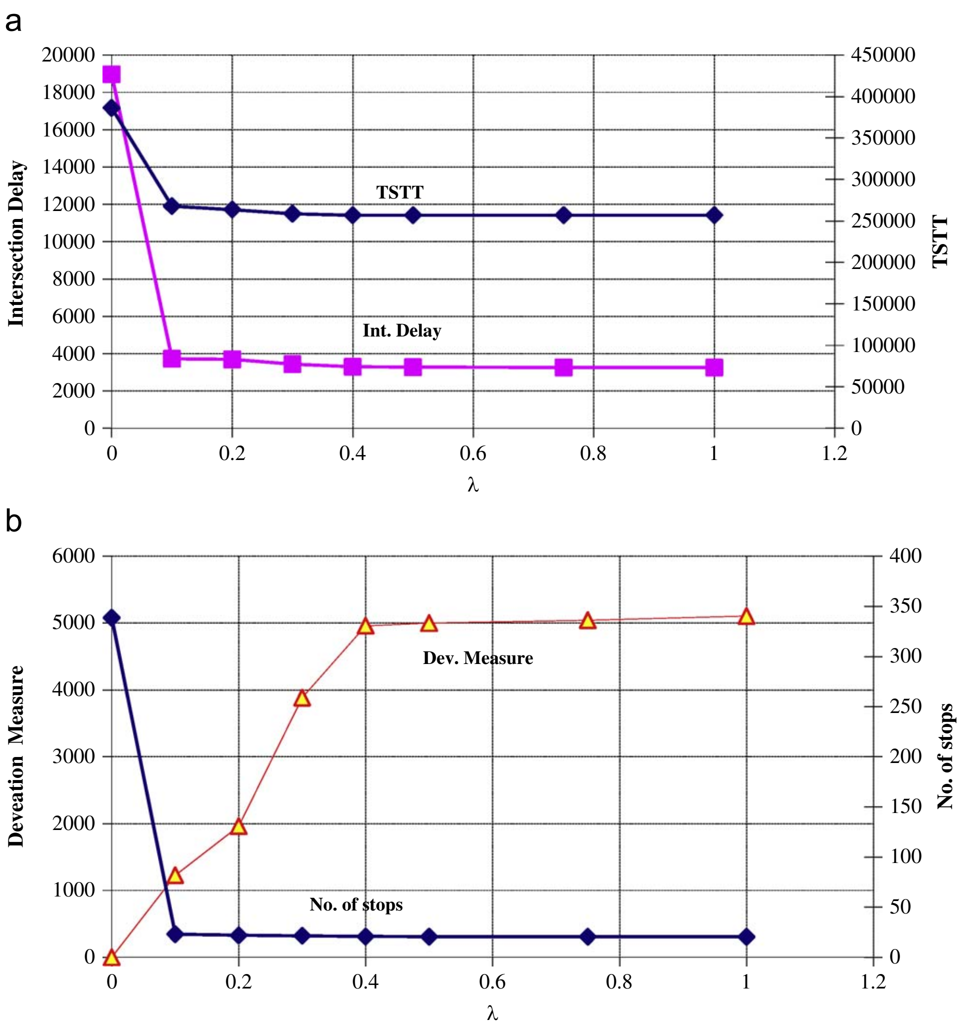

Fig. 3. Performance measure values for various robustness weights.

function or a weighted robust objective function. The value of total system travel time, number of stops, intersection delay, and the deviation measure are then compared. This comparison is repeated for two levels of demand: mean demand levels of high and low, with low variance in both cases. The assignment as well as the signal timing determination is based on the objective of minimizing the weighted measure of all three performance measures. The following insights are noteworthy:

- Robust assignment of traffic even when the signal times are set to deterministic or stochastic solutions provides deviation values (2106.01 and 2139 in high demand and 576 and 512 for low demand) that are within 10% of the deviation measure obtained in the pure robust case (1962.57 in high demand and 515.11 in low demand). This suggests that even if the signal timings are not optimized the deviation can be minimized substantially if at least the assignment of traffic is performed with a robust objective function.
- The deviation measures are high for robust signal times if the assignment objective is the pure stochastic problem. This suggests that it is not sufficient to determine signal times based on the robust problem to ensure risk minimization but that real time

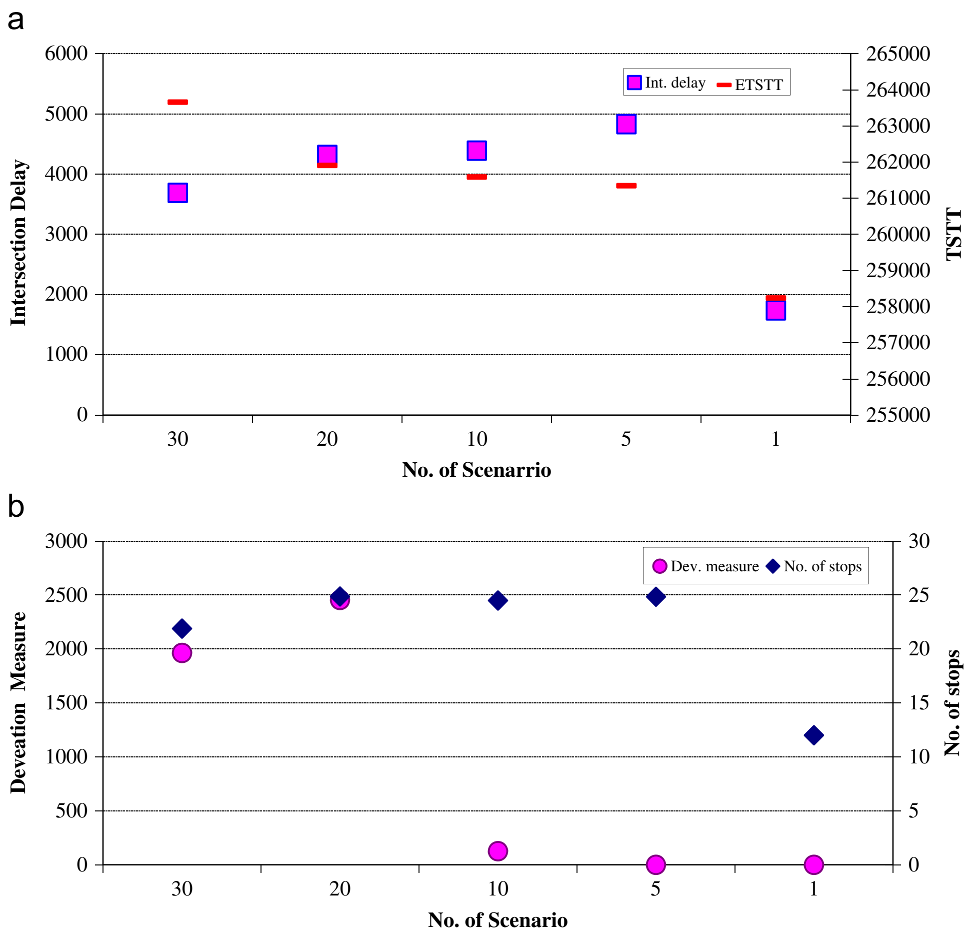

Fig. 4. Performance measure values for various number of scenarios.

Table 7
Signal times and route choice split for various robustness weights.

Value of λ	Avg. % green for phase				Avg. % veh. freeway
	P1	P2	P3	P4	
$\lambda = 0$	10.5 (10.0, 13.3)	29.9 (26.7, 30.8)	17.1 (16.7, 18.3)	42.4 (42.5, 41.7)	57.44
$\lambda = 0.1$	10.7 (10.0, 16.3)	30.3 (29.6, 32.7)	20.1 (16.7, 47.2)	38.9 (10.0, 43.7)	56.41
$\lambda = 0.2$	14.8 (10.0, 53.3)	30.2 (20.0, 31.4)	16.7 (16.7, 16.7)	38.3 (10.0, 41.9)	56.28
$\lambda = 0.3$	14.3 (10.0, 49.2)	30.0 (30.0, 30.0)	16.8 (10.8, 17.5)	38.9 (10.0, 42.5)	55.33
$\lambda = 0.4$	10.0 (10.0, 10.0)	29.6 (29.6, 29.6)	17.6 (17.6, 17.6)	42.8 (42.8, 42.8)	57.53
$\lambda = 0.5$	10.0 (10.0, 10.0)	30.2 (29.2, 33.9)	20.9 (17.8, 46.1)	38.8 (10.0, 43.1)	55.71
$\lambda = 0.75$	10.0 (10.0, 10.0)	29.6 (28.7, 35.4)	17.9 (17.9, 17.9)	42.5 (36.7, 43.3)	57.22
$\lambda = 1.0$	13.9 (10.0, 45.0)	29.8 (28.3, 35)	18.3 (18.3, 18.3)	42.5 (36.7, 43.3)	54.83

Table 8
Performance measure values for different objective functions.

Objective function includes	TSTT	No. of stops	Intersection delay	Deviation measure
TSTT only	262149	78.44	7039.27	1135.14
TSTT and intersection delay	261743	46.97	4073.84	1902.07
TSTT and # of stops	263075	21.86	3944.33	1326
All three terms	263660	21.88	3689.95	1962.57

control based on the robust objective function has to be enforced in order to minimize deviation.

- The expected values of performance measures are in general higher for the stochastic assignment of traffic when the signal timings are based on deterministic or robust problem compared to when signal timings are based on the stochastic problem (particularly for high demand case). In conclusion, traffic assignment provides some flexibility to work around when signal timings are not optimized for the same objective function as the assignment objective. The best results are obtained when both signal

Table 9
Value of robustness—high demand.

Signal timing objective function	Traffic assignment objective function	TSTT	No. of stops	Intersection delay	Deviation measure
Deterministic	Deterministic ($\lambda = 1$)	244800	0	0	0
Deterministic	Stochastic ($\lambda = 1$)	257146	21.57	3705.99	4874.73
	Robust ($\lambda = 0.2$)	263200	21.97	3793.99	2106.01
Stochastic ($\lambda = 1$)	Stochastic ($\lambda = 1$)	256921	20.45	3251	5107.87
	Robust ($\lambda = 0.2$)	264274	22.43	3472.16	2139
Robust ($\lambda = 0.2$)	Stochastic ($\lambda = 1$)	256944	21.08	3489.69	4845.48
	Robust ($\lambda = 0.2$)	263660	21.8813	3689.95	1962.57

Table 10
Value of robustness—low demand.

Signal timing objective function	Traffic assignment objective function	TSTT	No. of stops	Intersection delay	Deviation measure
Deterministic	Deterministic ($\lambda = 1$)	113760	0	0	0
Deterministic	Stochastic ($\lambda = 1$)	120028	0	0	1549.07
	Robust ($\lambda = 0.2$)	122500	0.03	6.0	576
Stochastic ($\lambda = 1$)	Stochastic ($\lambda = 1$)	119234	0.1	6	1461.33
	Robust ($\lambda = 0.2$)	121608	0.2	12	512
Robust ($\lambda = 0.2$)	Stochastic ($\lambda = 1$)	119230	0.16	10.89	1458.68
	Robust ($\lambda = 0.2$)	121588	0.21	14.25	515.11

optimization and assignment are done with the same objective function.

5.1. Summary

In summary, the main results of the experiments conducted on the test network are as follows:

- The introduction of uncertainty in the demand in the robust traffic control problem has a significant effect on the solution. The deterministic optimization using 'nominal' values for the input data yields conservative solutions and underestimate the network wide performance measures.
- The robust optimization solution is significantly different from the standard stochastic programming and the deterministic solution especially at low demand levels. It is observed that the signal timings are different for the robust and the stochastic cases.
- It was observed that the optimization problem is not overtly sensitive to the two objectives—total intersection delay and number of stops. The main important objective was the total system travel time. This states that signal setting design should consider network wide impacts and account for intersection delays and number of stops as secondary objectives.
- The number of stops and the intersection delay increases with increasing the value of robustness in the optimization problem.
- It is observed that the deviation can be minimized substantially if at least the assignment of traffic is performed with a robust objective function even when the signal controls are obtained by a different objective.
- High quality results are obtained when both the signal design objectives and the traffic assignment objectives are similar.
- The quality of solution changes with the increase in number of scenarios. In the computational results considered in this work, the maximum scenarios we could implement with the AMPL/CPLEX solver was 30 scenarios. It was observed that the solution is not stable. There is no conclusive evidence on the optimal number of scenarios required for robust signal design.

6. Conclusions

Accounting for uncertainty and robustness in the optimal signal control problem is an important illustration of the need to develop robust signal plans considering uncertainty. The signal plans have to be optimized for the intersection considering network wide impacts, traffic dynamics and the uncertainty in the O-D demand. One of the significant contributions of this paper is that we have developed a robust mathematical programming formulation of this problem that provides ample opportunity to explore the impact of uncertainty and robustness in optimal signal control.

Other contributions of the paper include: (i) embedding a mesoscopic traffic flow model (cell transmission) which captures shockwave and link spillover in traffic networks and (ii) accounting for traffic dynamics where the demand changes over time of day within the operation period of the traffic network. Further, the developed model can be extended to incorporate gap acceptance behavior of drivers, thereby developing more complicated signal phases at the intersection. This can be achieved by introducing a binary variable for each phase movement and restricting the flow of vehicle based on the turn movements. However, this transforms the problem from a linear program to an integer program for which efficient solution algorithms are unavailable. Our preliminary analysis with the integer version of the problem significantly reduces the problem size that can be solved with this formulation. Positive deviation measure of the travel time is used as a measure of risk in the robust optimization formulation. The risk term allows the examination of the tradeoffs of the risk against the expected value term. Extensive computational results on a test network illustrate that the introduction of the robust term is important in determining optimal signal timings that determine overall network performance. A tradeoff also exists between the signal timings obtained from the stochastic and robust problem. Evaluation of the deterministic, stochastic and robust problems are conducted with different weights' for the objective function.

In general, the multiobjective nature of the analysis facilitated by the model provides significantly greater insight into the tradeoffs that are available to traffic operations managers than would be the case if we focused on simply using the deterministic version or minimizing the expected costs. The robust optimization framework has substantial value in this regard.

Future research extending the current model should focus on solving large scale instances of the problem. Quite clearly, the developed formulation is computationally intensive and the addition of integer and binary variables will further add to the complexity of the problem. Enhancements related to variable reduction should be explored in future research. Further, more efficient solution techniques that exploit the structure of the formulation and utilize decomposition based approaches should be explored. Large scale instances of the problem can be solved by developing specialized sampling techniques.

Another direction of enhancement is to extend the model to incorporate different measures of risk. The current model used the positive deviation as a measure of risk, however the penalty parameters associated with this positive deviation were not calibrated. This extension can be explored by some of the recent advances in modeling risk measures, such as the conditional variance of risk [26,27].

Finally, further work in this area should explore the use of the models like this for informing policy makers to arrive at network wide control strategies. By providing policy makers with explicit need to account for robustness and the impact of uncertainty on traffic systems, it may be possible to identify technology based solutions that reduce the levels of uncertainty and also satisfy the concerns of other stakeholders.

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