Math 590 Homework 3 Solutions Fall 2018

Due: Thursday, September 27

1. **Powers of 2**

   Describe an algorithm for a Turing Machine that decides the language consisting of all strings of 0s whose length is a power of 2:

   $$\{0^{2^n} \mid n \geq 0\}.$$

   You may assume that the input alphabet in this case is $\Sigma = \{0\}$.

   Solution #1:

   (a) If the input is empty, reject.

   (b) If there is only a single 0, accept.

   (c) Go through the input and cross off every other 0.

   (d) If there were an odd number of 0s, reject. Else, return to step (b).

   Solution #2:

   (a) If the input is empty, reject.

   (b) If there is only a single 0, accept.

   (c) Mark the first unmarked 0 and cross off the last 0.
      - If there was no last 0 to cross off, reject.
      - If there are no unmarked 0s, unmark all marked 0s and return to step (b).

2. **Elementary Multiplication**

   Describe an algorithm for a Turing Machine that can decide some basic multiplication:

   $$\{a^n b^m c^k \mid n \times m = k \text{ and } n, m, k \geq 1\}.$$

   You may assume that the input alphabet is $\Sigma = \{a, b, c\}$. (Hint: multiplication is repeated addition.)

   (a) Scan the input and reject if not in the form $a^n b^m c^k$ with at least one a, b, and c.

   (b) While there is an unmarked a:

      (i) While there is an unmarked b:
         - Mark the first unmarked b.
         - Cross off the leftmost remaining c.
         - If there was no c to cross off, reject.

      (ii) Unmark all of the bs.

   (c) If there are any cs remaining, reject. Else, accept.

3. **3-tuples**

   Prove that the set of all 3-tuples of $\mathbb{N}$ is countably infinite, i.e., prove that the set

   $$\{(i, j, k) \mid i, j, k \in \mathbb{N}\}$$

   is a countably infinite set. (Hint: note that positive rational numbers m/n can be rewritten as 2-tuples (m, n) of natural numbers. We showed in class that the positive rational numbers were countably infinite. Can you somehow generalize the idea of that proof?)

   Given a 3-tuple (i, j, k), consider its sum $i+j+k$. Note that when we limit $i, j, k \in \mathbb{N}$, then the smallest possible sum is $i+j+k = 3$, and all possible sums are themselves natural numbers (i.e., $i+j+k \in \mathbb{N}$).

   Next we note that there are a finite number of 3-tuples that all have the same sum. For example, if we say $i+j+k = 5$, then the possible 3-tuples are:

   $$(1,1,3), \quad (1,3,1), \quad (3,1,1), \quad (1,2,2), \quad (2,1,2), \quad (2,2,1).$$

   Now, note that if we can describe a way to list our 3-tuples, then we have successfully described a way to count them (since there is a 1st thing in the list, a 2nd thing in the list, etc).

   To list the 3-tuples, consider all 3-tuples that sum to 3 and list those, then list all 3-tuples that sum to 4, then list all 3-tuples that sum to 5, etc. This would generate a list like:

   $$\{(1,1,1)\}, \quad \{(1,1,2), \quad (1,2,1), \quad (2,1,1)\},$$

   $$\{(1,1,3), \quad (1,2,2), \quad (1,3,1), \quad (2,1,2), \quad (2,2,1), \quad (3,1,1)\}, \ldots$$

4. **Finite Subsets of** $\mathbb{N}$

   Here you will prove that the set of all **finite** subsets of $\mathbb{N}$ is countably infinite, i.e.,

   $$\{\, \{n_1, n_2, \ldots n_k\} \mid n_1, n_2, \ldots, n_k \in \mathbb{N} \textbf{ and } k \in \mathbb{N} \,\}.$$

   (a) A first attempt at this problem might be to say "list all subsets of size 1, then all subsets of size 2, then all subsets of size 3, etc." Explain why this approach doesn't work.

   There are an infinite number of subsets of size 1:

   $$\{1\}, \quad \{2\}, \quad \{3\}, \quad \{4\}, \quad \{5\}, \quad \{6\}, \quad \{7\}, \quad \{8\}, \ldots$$

   So if we tried listing the subsets in this order, we would never get to the subsets of size 2.

(b) Provide the correct proof that the set of all finite subsets of $\mathbb{N}$ is countably infinite.

Proof #1: The idea to list the subsets based on the size of the subset was a good start. But, we somehow have to limit the number of each subset of a given size that we list. The idea now is to also limit the maximum number that we can see. There are a finite number of subsets that have at most size k whose elements are at most k. So we propose the following method to list all of the finite subsets of $\mathbb{N}$:

- List all subsets of **at most** size 1 whose elements are **at most** 1, leaving out any repeats from before.
- List all subsets of **at most** size 2 whose elements are **at most** 2, leaving out any repeats from before.
- List all subsets of **at most** size 3 whose elements are **at most** 3, leaving out any repeats from before.
- List all subsets of **at most** size 4 whose elements are **at most** 4, leaving out any repeats from before.
- etc

This generates a list like:

$$\begin{aligned}
&\{1\},\\
&\{2\},\ \{1,2\},\\
&\{3\},\ \{1,3\},\ \{2,3\},\ \{1,2,3\},\\
&\{4\},\ \{1,4\},\ \{2,4\},\ \{3,4\},\ \{1,2,4\},\ \{1,3,4\},\ \{2,3,4\},\ \{1,2,3,4\},\\
&\ldots
\end{aligned}$$

To make this ordering unique, we would:

- Require that each set have its elements written in ascending order so we have a unique representation for each set, i.e., the set $\{4,2,1\}$ must instead be always written as $\{1,2,4\}$.
- When listing all subsets of at most size k whose elements are at most k, sort them by size, then list them in ascending order: $\{3\}$, $\{1,3\}$, $\{2,3\}$, $\{1,2,3\}$.

Proof #2: We could have used the sum of the elements in the set to also provide a correct listing, i.e., list all subsets that have a sum of 1, then those that sum to 2, then those that sum to 3, etc.

(c) Use this fact to trivially prove problem 3.

For now, this was a free problem.

5. **Infinite Subsets of $\mathbb{N}$**

Use a diagonalization argument to prove that the set of all (including **infinite**) subsets of $\mathbb{N}$ are uncountable.

Proof:
Assume by way of contradiction that the set of all subsets of $\mathbb{N}$ is countable. That would mean that there exists some listing of these subsets. Suppose we are given such a list. We will construct a subset of $\mathbb{N}$, which we will call S, that is not on the list as follows:

- Look at the 1st subset on the list s_1. If $1 \in s_1$, then $1 \notin S$. If $1 \notin s_1$, then $1 \in S$.
- Look at the 2nd subset on the list s_2. If $2 \in s_2$, then $2 \notin S$. If $2 \notin s_2$, then $2 \in S$.
- Look at the 3rd subset on the list s_3. If $3 \in s_3$, then $3 \notin S$. If $3 \notin s_3$, then $3 \in S$.
- etc.

Clearly our new subset S is an subset of $\mathbb{N}$, but was different from every subset in the given list. Therefore the list was incomplete. Since we can do this for any list, all attempts at listing the subsets of $\mathbb{N}$ must be incomplete. Therefore the set of all subsets of $\mathbb{N}$ must be uncountable.

6. **Computer Programs**

(a) Given a **finite** alphabet Σ, prove that the set of all **finite** strings that can be generated from the alphabet is a countably infinite set, i.e., $\{w \mid w \in \Sigma^*,\ |w| \in \mathbb{N}\}$.

Proof: Note that there are a finite number of strings of length k. So, list all strings of length 0, then all strings of length 1, then all strings of length 2, then all strings of length 3, etc. We have found a listing, so the set is countable.

(b) Assuming that computers programs are finite in length, use part (a) to prove that there are a countably infinite number of possible computer programs.

The alphabet that can be used to generate a computer program (i.e., the alphabet that is all possible symbols that can be typed into a computer program) is a finite alphabet. Furthermore, all computer programs can be (trivially) represented as strings. So, any finite computer program can be considered a finite string generated from a finite alphabet. Therefore the set of all computer programs is a subset of the set of all finite strings generated from a finite alphabet. In part (a), we proved that the set of all finite strings generated from a finite alphabet was countable, and therefore there are a countable number of computer programs.