Math 590 Homework 3 Fall 2018

Due: Thursday, September 27

1. **Powers of 2**

   Describe an algorithm for a Turing Machine that decides the language consisting of all strings of 0s whose length is a power of 2:

   $$\{0^{2^n} \mid n \geq 0\}.$$

   You may assume that the input alphabet in this case is $\Sigma = \{0\}$.

2. **Elementary Multiplication**

   Describe an algorithm for a Turing Machine that can decide some basic multiplication:

   $$\{a^n b^m c^k \mid n \times m = k \text{ and } n, m, k \geq 1\}.$$

   You may assume that the input alphabet is $\Sigma = \{a, b, c\}$. (Hint: multiplication is repeated addition.)

3. **3-tuples**

   Prove that the set of all 3-tuples of $\mathbb{N}$ is countably infinite, i.e., prove that the set

   $$\{(i, j, k) \mid i, j, k \in \mathbb{N}\}$$

   is a countably infinite set. (Hint: note that positive rational numbers m/n can be rewritten as 2-tuples (m, n) of natural numbers. We showed in class that the positive rational numbers were countably infinite. Can you somehow generalize the idea of that proof?)

4. **Finite Subsets of $\mathbb{N}$**

   Here you will prove that the set of all **finite** subsets of $\mathbb{N}$ is countably infinite, i.e.,

   $$\{\, \{n_1, n_2, \ldots n_k\} \mid n_1, n_2, \ldots, n_k \in \mathbb{N} \textbf{ and } k \in \mathbb{N}\,\}.$$

   (a) A first attempt at this problem might be to say "list all subsets of size 1, then all subsets of size 2, then all subsets of size 3, etc." Explain why this approach doesn't work.

   (b) Provide the correct proof that the set of all finite subsets of $\mathbb{N}$ is countably infinite.

   (c) Use this fact to trivially prove problem 3. **Skip this, it is a free point.**

5. **Infinite Subsets of $\mathbb{N}$**

   Use a diagonalization argument to prove that the set of all (including **infinite**) subsets of $\mathbb{N}$ are uncountable.

6. **Computer Programs**

   (a) Given a **finite** alphabet Σ, prove that the set of all **finite** strings that can be generated from the alphabet is a countably infinite set, i.e., $\{w \mid w \in \Sigma^*,\ |w| \in \mathbb{N}\}$.

   (b) Assuming that computers programs are finite in length, use part (a) to prove that there are a countably infinite number of possible computer programs.