Decision Theory

Lecture 10

Time to make a decision…

Exercise inspired by Mausam, University of Washington, CSE573

		State of Nature	
		Poor market performance	Good market performance
		Payoff	**Payoff**
Action	Buy Apple	-1,000	1,700
	Buy Google	-2,000	2,100
	Don’t invest	10	10

How to invest $10 in 2004?

Maximax

Optimism

Select the maximum of the maximum payoff

Action	**State of Nature** Poor market performance **Payoff**	 Good market performance **Payoff**	**Criterion** Maximum payoff for an action
Buy Apple	-1,000	1,700	1,700
Buy Google	-2,000	2,100	2,100
Don't invest	10	10	10

Maximax

Optimism

Select the maximum of the maximum payoff

		State of Nature		**Criterion**
		Poor market performance	Good market performance	Maximum payoff for an action
		Payoff	**Payoff**	
Action	Buy Apple	-1,000	1,700	1,700
	Buy Google	-2,000	2,100	2,100
	Don’t invest	10	10	10

← **Maximax**

Maximin

Pessimism

Select the maximum of the minimum payoffs

Action	**State of Nature**		**Criterion**
	Poor market performance **Payoff**	Good market performance **Payoff**	Minimum payoff for an action
Buy Apple	-1,000	1,700	-1,000
Buy Google	-2,000	2,100	-2,000
Don’t invest	10	10	10

Maximin

Pessimism

Select the maximum of the minimum payoffs

Action	**State of Nature** Poor market performance **Payoff**	Good market performance **Payoff**	**Criterion** Minimum payoff for an action
Buy Apple	-1,000	1,700	-1,000
Buy Google	-2,000	2,100	-2,000
Don’t invest	10	10	10

← **Maximin**

Minimax

Select the minimum maximum regret

	State of Nature				Criterion
	Poor market performance		Good market performance		Maximum regret for an action
Action	**Payoff**	**Regret**	**Payoff**	**Regret**	
Buy Apple	-1,000	1,010	1,700	400	1,010
Buy Google	-2,000	2,010	2,100	0	2,010
Don’t invest	10	0	10	2,090	2,090

If I knew the future, which decision would I regret least?

Minimax

Select the minimum maximum regret

	State of Nature				Criterion
	Poor market performance		Good market performance		Maximum regret for an action
Action	**Payoff**	**Regret**	**Payoff**	**Regret**	
Buy Apple	-1,000	1,010	1,700	400	1,010
Buy Google	-2,000	2,010	2,100	0	2,010
Don’t invest	10	0	10	2,090	2,090

← **Minimax**

If I knew the future, which decision would I regret least?

Equal likelihood

Select the highest average payoff ASSUMING all states are of equal probability

Action	**State of Nature** Poor market performance **Payoff**	 Good market performance **Payoff**	**Criterion** Average reward/ payoff
Buy Apple	-1,000	1,700	350
Buy Google	-2,000	2,100	50
Don't invest	10	10	10
State Probability:	0.5	0.5	

Equal likelihood

Select the highest average payoff ASSUMING all states are of equal probability

		State of Nature		**Criterion**
		Poor market performance **Payoff**	Good market performance **Payoff**	Average reward/ payoff
Action	Buy Apple	-1,000	1,700	350
	Buy Google	-2,000	2,100	50
	Don't invest	10	10	10
	State Probability:	0.5	0.5	

← **Equal likelihood**

Weighted average

Select the highest average payoff ASSUMING state probabilities from prior knowledge

		State of Nature		**Criterion**
		Poor market performance **Payoff**	Good market performance **Payoff**	Probability weighted average
Action	Buy Apple	-1,000	1,700	80
	Buy Google	-2,000	2,100	-360
	Don't invest	10	10	10
State Probability:		0.6	0.4	

Weighted average

Select the highest average payoff ASSUMING state probabilities from prior knowledge

	State of Nature		Criterion
	Poor market performance **Payoff**	Good market performance **Payoff**	Probability weighted average
Buy Apple	-1,000	1,700	80
Buy Google	-2,000	2,100	-360
Don't invest	10	10	10
State Probability:	0.6	0.4	

Action

← **Weighted average**

Decision making design pattern

1. Define a measure of risk or reward
2. Select the action that optimizes that metric

Investments: reward

		State of Nature (s)	
		Poor market performance	Excellent market performance
Action	Buy Apple	-1,000	1,700
	Buy Google	-2,000	2,100
	Don’t invest	10	10

Investments: reward

	State of Nature (s)	
Action	Poor market performance $s = s_0$	Excellent market performance $s = s_1$
Buy Apple	-1,000	1,700
Buy Google	-2,000	2,100
Don’t invest	10	10

Investments: reward

		State of Nature (s)	
		Poor market performance $s = s_0$	Excellent market performance $s = s_1$
Action	Buy Apple $a = a_0$	-1,000	1,700
	Buy Google $a = a_1$	-2,000	2,100
	Don't invest $a = a_2$	10	10

Investments: reward

State of Nature (s)

Action	Poor market performance $s = s_0$	Excellent market performance $s = s_1$
Buy Apple $a = a_0$	$V(a_0 \mid s_0)$ -1,000	$V(a_0 \mid s_1)$ 1,700
Buy Google $a = a_1$	$V(a_1 \mid s_0)$ -2,000	$V(a_1 \mid s_1)$ 2,100
Don’t invest $a = a_2$	$V(a_2 \mid s_0)$ 10	$V(a_2 \mid s_1)$ 10

Investments: reward

State of Nature (s)

Action	Poor market performance $s = s_0$	Excellent market performance $s = s_1$
Buy Apple $a = a_0$	$V(a_0 \mid s_0)$ -1,000	$V(a_0 \mid s_1)$ 1,700
Buy Google $a = a_1$	$V(a_1 \mid s_0)$ -2,000	$V(a_1 \mid s_1)$ 2,100
Don’t invest $a = a_2$	$V(a_2 \mid s_0)$ 10	$V(a_2 \mid s_1)$ 10

State Probability: $P(s_0) =$ 0.6 $P(s_1) =$ 0.4

Investments: reward

$EV(a_i) = V(a_i|s_0)P(s_0) + V(a_i|s_1)(s_1)$

Expected payoff

State of Nature (s)

Action

	Poor market performance $s = s_0$	Excellent market performance $s = s_1$
Buy Apple $a = a_0$	$V(a_0 \mid s_0)$ -1,000	$V(a_0 \mid s_1)$ 1,700
Buy Google $a = a_1$	$V(a_1 \mid s_0)$ -2,000	$V(a_1 \mid s_1)$ 2,100
Don't invest $a = a_2$	$V(a_2 \mid s_0)$ 10	$V(a_2 \mid s_1)$ 10
State Probability:	$P(s_0) =$ 0.6	$P(s_1) =$ 0.4

Investments: reward

$$EV(a_i) = V(a_i|s_0)P(s_0) + V(a_i|s_1)(s_1)$$

Expected payoff

State of Nature (s)

Action	Poor market performance $s = s_0$	Excellent market performance $s = s_1$	**Expected Reward** $EV(a_i)$
Buy Apple $a = a_0$	$V(a_0\|s_0)$ -1,000	$V(a_0\|s_1)$ 1,700	
Buy Google $a = a_1$	$V(a_1\|s_0)$ -2,000	$V(a_1\|s_1)$ 2,100	
Don’t invest $a = a_2$	$V(a_2\|s_0)$ 10	$V(a_2\|s_1)$ 10	

State Probability: $P(s_0) =$ 0.6 $P(s_1) =$ 0.4

Investments: reward

$$\underset{\uparrow}{EV(a_i)} = V(a_i|s_0)P(s_0) + V(a_i|s_1)(s_1)$$

Expected payoff

State of Nature (s)

Action

	Poor market performance $s = s_0$	Excellent market performance $s = s_1$	**Expected Reward** $EV(a_i)$
Buy Apple $a = a_0$	$V(a_0\|s_0)$ -1,000	$V(a_0\|s_1)$ 1,700	$(0.6)(-1000) + (0.4)(1700)$ $= \mathbf{80}$
Buy Google $a = a_1$	$V(a_1\|s_0)$ -2,000	$V(a_1\|s_1)$ 2,100	
Don’t invest $a = a_2$	$V(a_2\|s_0)$ 10	$V(a_2\|s_1)$ 10	
State Probability:	$P(s_0) =$ 0.6	$P(s_1) =$ 0.4	

Investments: reward

$$\underset{\text{Expected payoff}}{\underset{\uparrow}{EV(a_i)}} = V(a_i|s_0)P(s_0) + V(a_i|s_1)(s_1)$$

State of Nature (s)

		Poor market performance $s = s_0$	Excellent market performance $s = s_1$	**Expected Reward** $EV(a_i)$
Action	Buy Apple $a = a_0$	$V(a_0\|s_0)$ -1,000	$V(a_0\|s_1)$ 1,700	$(0.6)(-1000) + (0.4)(1700)$ $= \mathbf{80}$
	Buy Google $a = a_1$	$V(a_1\|s_0)$ -2,000	$V(a_1\|s_1)$ 2,100	$(0.6)(-2000) + (0.4)(2100)$ $= \mathbf{-360}$
	Don't invest $a = a_2$	$V(a_2\|s_0)$ 10	$V(a_2\|s_1)$ 10	

State Probability: $P(s_0) =$ 0.6 $P(s_1) =$ 0.4

Investments: reward

$$EV(a_i) = V(a_i|s_0)P(s_0) + V(a_i|s_1)(s_1)$$

Expected payoff

State of Nature (s)

Action

	Poor market performance $s = s_0$	Excellent market performance $s = s_1$	**Expected Reward** $EV(a_i)$
Buy Apple $a = a_0$	$V(a_0 \mid s_0)$ -1,000	$V(a_0 \mid s_1)$ 1,700	$(0.6)(-1000) + (0.4)(1700)$ $= \mathbf{80}$
Buy Google $a = a_1$	$V(a_1 \mid s_0)$ -2,000	$V(a_1 \mid s_1)$ 2,100	$(0.6)(-2000) + (0.4)(2100)$ $= \mathbf{-360}$
Don't invest $a = a_2$	$V(a_2 \mid s_0)$ 10	$V(a_2 \mid s_1)$ 10	$(0.6)(10) + (0.4)(10)$ $= \mathbf{10}$
State Probability:	$P(s_0) =$ 0.6	$P(s_1) =$ 0.4	

Risk = expected loss

(cost)

Loss: $\lambda(a_i|s_j) \triangleq$ Loss incurred by choosing action i and the state of nature being state j

Risk: $R(a_i) = \sum_{j=1}^{N_s} \lambda(a_i|s_j)P(s_j)$

Goal: Select action i for which $R(a_i)$ is minimum

Investments: loss

$\lambda(a_i) = \lambda(a_i|s_0)P(s_0) + V(a_i|s_1)(s_1)$

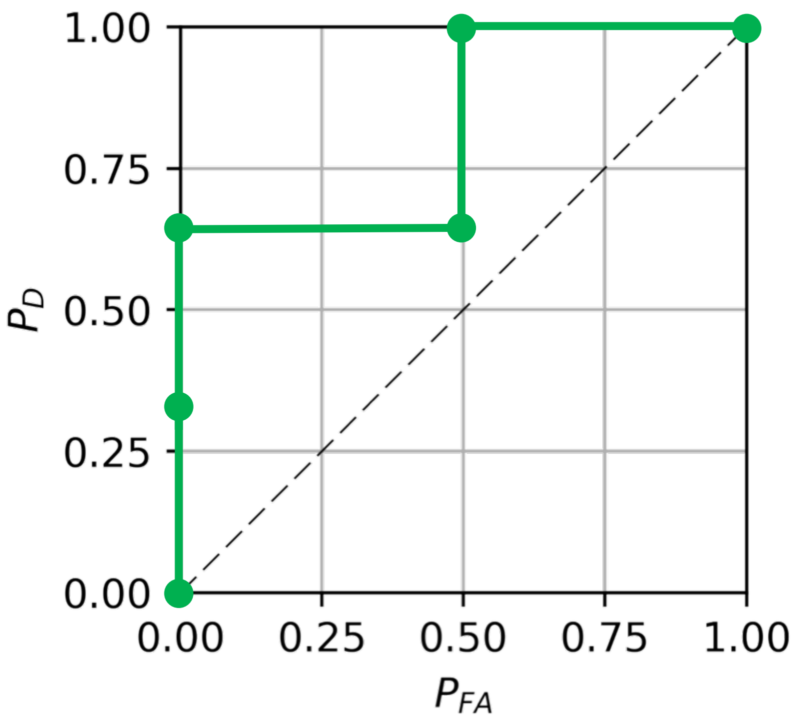
Expected loss

State of Nature (s)

Action	Poor market performance $s = s_0$	Excellent market performance $s = s_1$	**Expected Loss** $\lambda\,(a_i)$
Buy Apple $a = a_0$	$\lambda(a_0|s_0)$ 1,010	$\lambda\,(a_0|s_1)$ -1,690	$(0.6)(1010) + (0.4)(-1690)$ $= \mathbf{-70}$
Buy Google $a = a_1$	$\lambda\,(a_1|s_0)$ 2,010	$\lambda\,(a_1|s_1)$ -2,090	$(0.6)(2010) + (0.4)(-2090)$ $= \mathbf{370}$
Don't invest $a = a_2$	$\lambda\,(a_2|s_0)$ 0	$\lambda\,(a_2|s_1)$ 0	$(0.6)(0) + (0.4)(0)$ $= \mathbf{0}$
State Probability:	$P(s_0) =$ 0.6	$P(s_1) =$ 0.4	

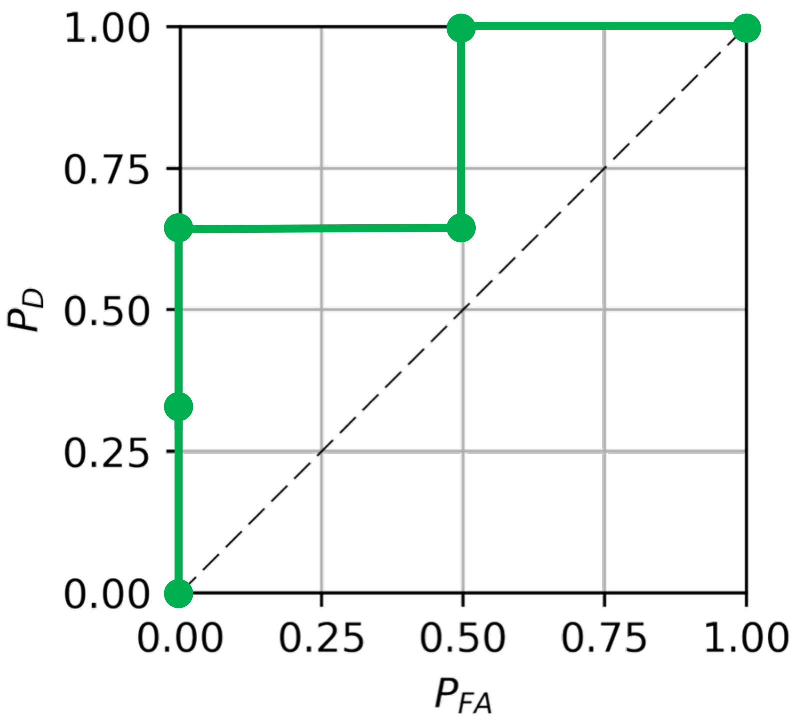
How does this relate to supervised learning?

Where to operate along ROC?



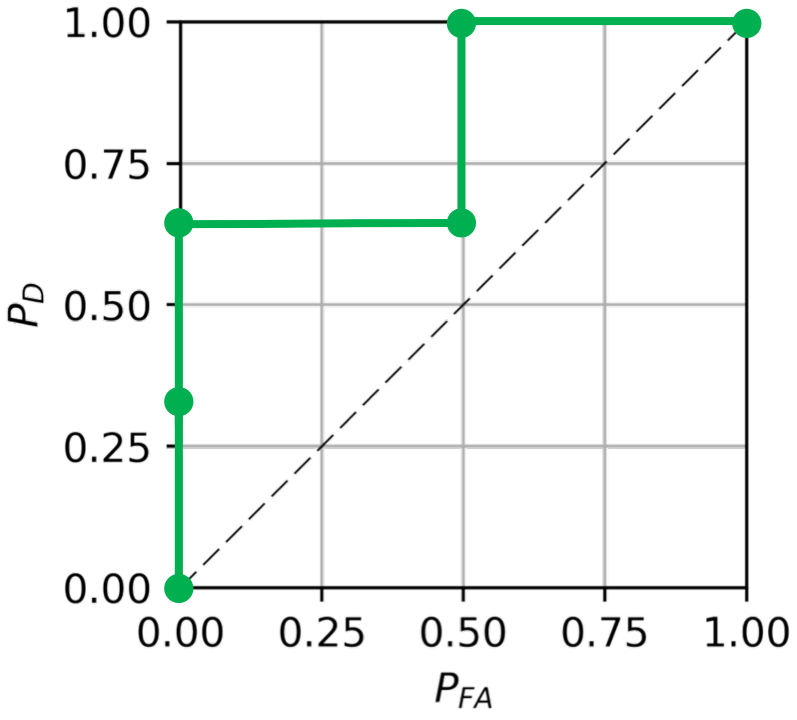
Where to operate along ROC?

		State of Nature	
		Class 0	Class 1
Estimate	Class 0	$\lambda_{00} = 0$	$\lambda_{01} = 100$
	Class 1	$\lambda_{10} = 1$	$\lambda_{11} = 0$



$\lambda_{ij} =$ Classify as class i when state of nature is class j

Where to operate along ROC?



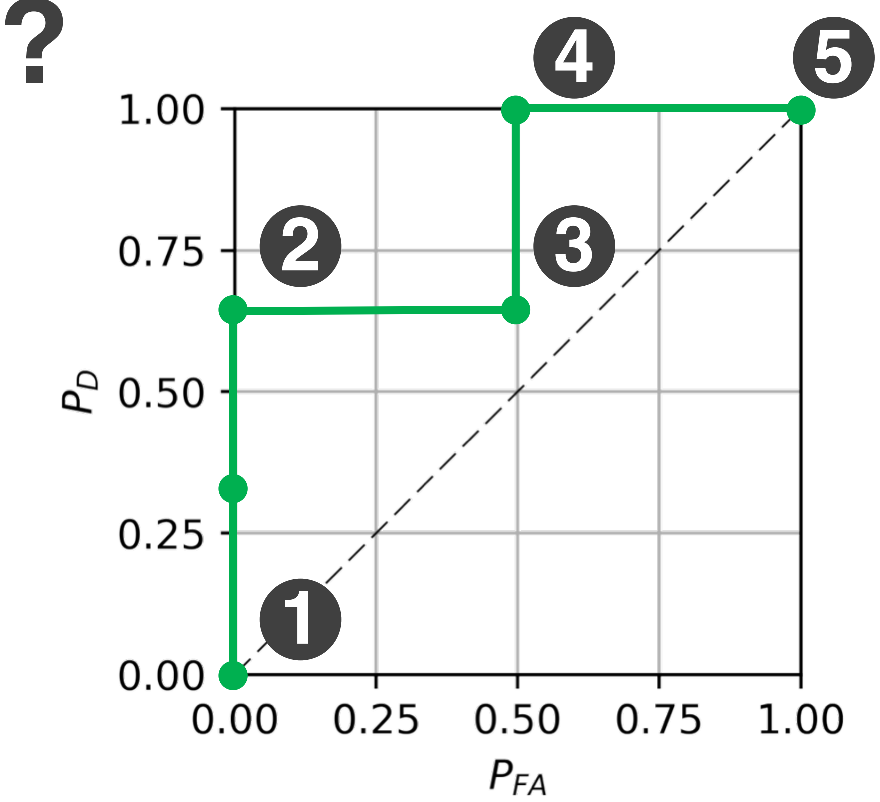
State of Nature

Estimate		Class 0	Class 1
	Class 0	$\lambda_{00} = 0$	$\lambda_{01} = 100$
	Class 1	$\lambda_{10} = 1$	$\lambda_{11} = 0$

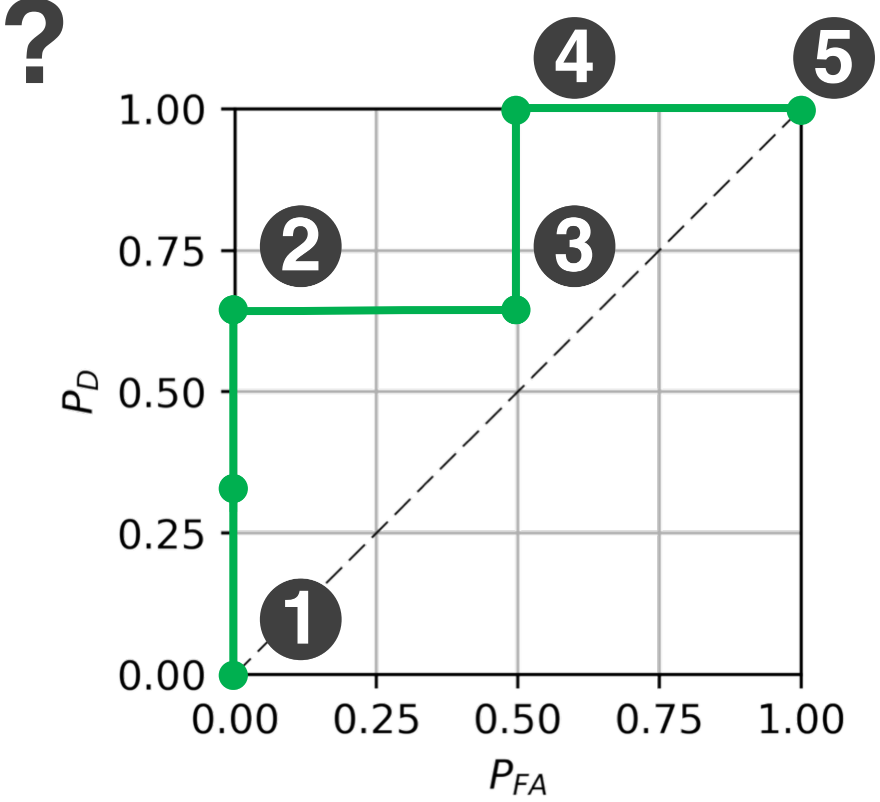
$\lambda_{ij} =$ Classify as class i when state of nature is class j

- Assume our classification problem is landmine detection
- A false alarm wastes some time and resources, but a missed detection may cost a life

Where to operate along ROC?



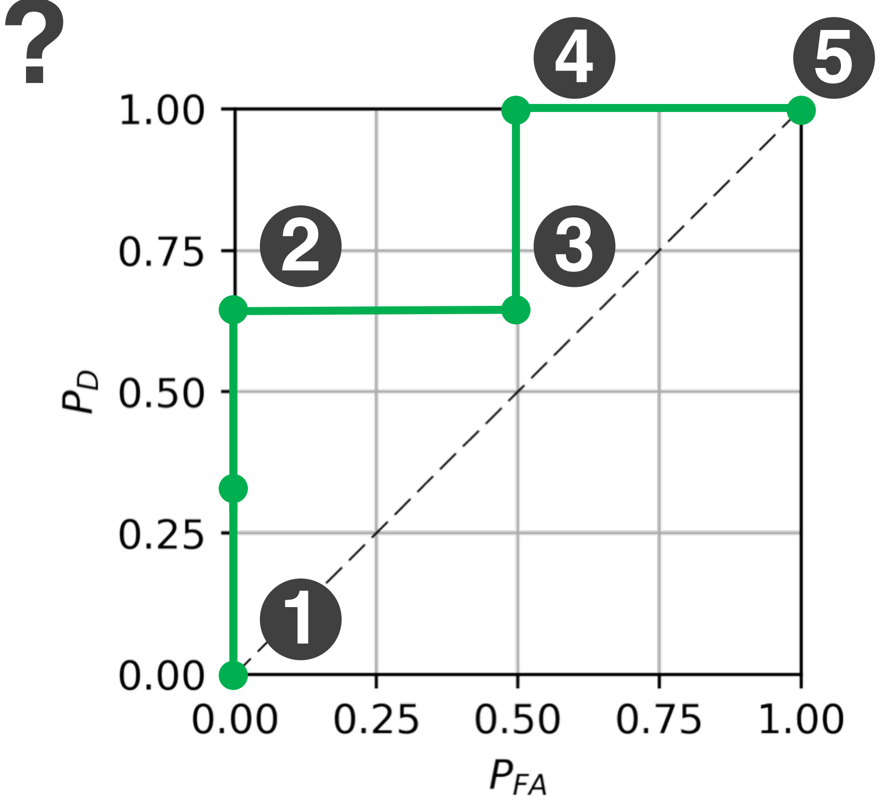
Where to operate along ROC?



State of Nature

		Class 0	Class 1
Estimate	Class 0	$\lambda_{00} = 0$	$\lambda_{01} = 100$
	Class 1	$\lambda_{10} = 1$	$\lambda_{11} = 0$

Where to operate along ROC?



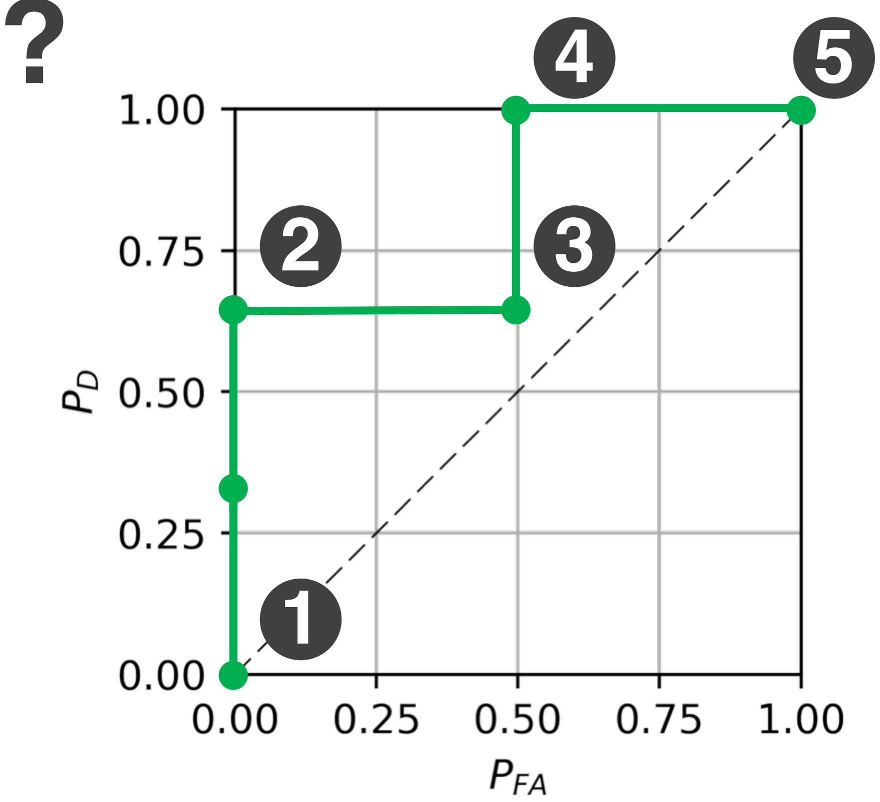
$$R(a_i) = \lambda_{01}(1 - P_D(i)) + \lambda_{10} P_{FA}(i)$$

State of Nature

Estimate		Class 0	Class 1
	Class 0	$\lambda_{00} = 0$	$\lambda_{01} = 100$
	Class 1	$\lambda_{10} = 1$	$\lambda_{11} = 0$

Where to operate along ROC?

Action: select operating point i	Probability of false alarm P_{FA}	Probability of missed detection $(1-Pd)$	Risk $R(a_i)$
1	0	1	100
2	0	0.33	33
3	0.5	0.33	33.5
4	0.5	0	0.5
5	1	0	1



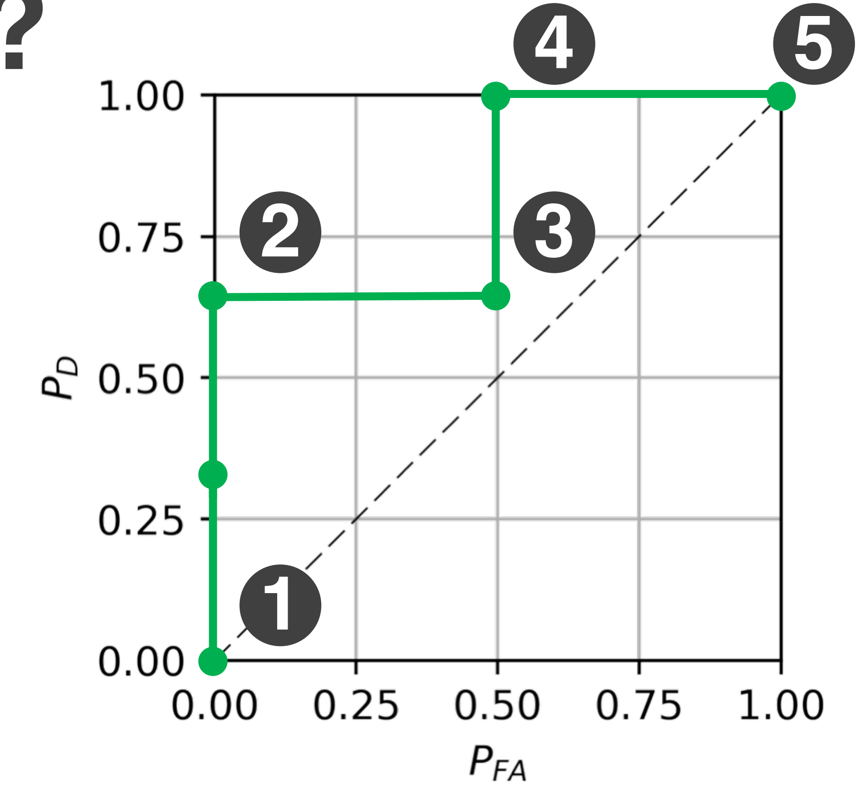
$$R(a_i) = \lambda_{01}(1 - P_D(i)) + \lambda_{10}P_{FA}(i)$$

State of Nature

Estimate	Class 0	Class 1
Class 0	$\lambda_{00} = 0$	$\lambda_{01} = 100$
Class 1	$\lambda_{10} = 1$	$\lambda_{11} = 0$

Where to operate along ROC?

Action: select operating point i	Probability of false alarm P_{FA}	Probability of missed detection $(1-Pd)$	Risk $R(a_i)$
1	0	1	100
2	0	0.33	33
3	0.5	0.33	33.5
4	0.5	0	0.5
5	1	0	1



$$R(a_i) = \lambda_{01}(1 - P_D(i)) + \lambda_{10}P_{FA}(i)$$

		State of Nature	
		Class 0	Class 1
Estimate	Class 0	$\lambda_{00} = 0$	$\lambda_{01} = 100$
	Class 1	$\lambda_{10} = 1$	$\lambda_{11} = 0$

Can we use this theory to develop a classifier?

Binary decision

State of Nature

		Class 0 $s = s_0$	Class 1 $s = s_1$
Estimate	Class 0 $a = a_0$	$\lambda(a_0 \vert s_0)$ λ_{00}	$\lambda\,(a_0 \vert s_1)$ λ_{01}
	Class 1 $a = a_1$	$\lambda\,(a_1 \vert s_0)$ λ_{10}	$\lambda\,(a_1 \vert s_1)$ λ_{11}

$\lambda_{ij} =$ Loss when you classify as class i when state of nature is class j

$$R(a_0|\boldsymbol{x}) = \lambda_{00}P(s_0|\boldsymbol{x}) + \lambda_{01}P(s_1|\boldsymbol{x})$$

$$R(a_1|\boldsymbol{x}) = \lambda_{10}P(s_0|\boldsymbol{x}) + \lambda_{11}P(s_1|\boldsymbol{x})$$

$$P(s_i|\boldsymbol{x}) = \frac{P(\boldsymbol{x}|s_i)P(s_i)}{P(\boldsymbol{x})}$$

❶

Define the risk associated with each of the two actions

$$R(a_0|\boldsymbol{x}) = \lambda_{00}P(s_0|\boldsymbol{x}) + \lambda_{01}P(s_1|\boldsymbol{x})$$

$$R(a_1|\boldsymbol{x}) = \lambda_{10}P(s_0|\boldsymbol{x}) + \lambda_{11}P(s_1|\boldsymbol{x})$$

❷

Create a decision rule based on the data

If $R(a_0|\boldsymbol{x}) < R(a_1|\boldsymbol{x})$ then a_0

Else $R(a_0|\boldsymbol{x}) > R(a_1|\boldsymbol{x})$ then a_1

❸

Interpret that rule as a function of loss

$$\lambda_{00}P(s_0|\boldsymbol{x}) + \lambda_{01}P(s_1|\boldsymbol{x}) > \lambda_{10}P(s_0|\boldsymbol{x}) + \lambda_{11}P(s_1|\boldsymbol{x})$$

$$\frac{P(s_1|\boldsymbol{x})}{P(s_0|\boldsymbol{x})} > \frac{\lambda_{10} - \lambda_{00}}{\lambda_{01} - \lambda_{11}} \quad \text{then} \quad a_1$$

Minimizing the misclassification rate

$$\frac{P(s_1|\boldsymbol{x})}{P(s_0|\boldsymbol{x})} > \frac{\lambda_{10} - \lambda_{00}}{\lambda_{01} - \lambda_{11}} \quad \text{then} \quad a_1$$

Assume that the loss is only for error, and it's the same for both types of error:

$$\lambda_{10} = \lambda_{01} \quad \text{and} \quad \lambda_{00} = \lambda_{11} = 0$$

Then the decision rule becomes:

$$\frac{P(s_1|\boldsymbol{x})}{P(s_0|\boldsymbol{x})} > 1 \quad \text{then} \quad a_1$$

Pick whichever class is more likely given the data

Recall Bayes Rule

Likelihood Prior

$$P(s_i|\boldsymbol{x}) = \frac{P(\boldsymbol{x}|s_i)P(s_i)}{P(\boldsymbol{x})}$$

Posterior

Evidence

④

Use Bayes rule to express this as a function of likelihoods

$$\frac{P(s_1|\boldsymbol{x})}{P(s_0|\boldsymbol{x})} > \frac{\lambda_{10} - \lambda_{00}}{\lambda_{01} - \lambda_{11}}$$

$$P(s_i|\boldsymbol{x}) = \frac{P(\boldsymbol{x}|s_i)P(s_i)}{P(\boldsymbol{x})}$$

$$\frac{P(\boldsymbol{x}|s_1)P(s_1)}{P(\boldsymbol{x}|s_0)P(s_0)} > \frac{\lambda_{10} - \lambda_{00}}{\lambda_{01} - \lambda_{11}}$$

then a_1

⑤

The decision rule can be expressed as a **likelihood ratio**

$$\frac{P(\boldsymbol{x}|s_1)}{P(\boldsymbol{x}|s_0)} > \frac{\lambda_{10} - \lambda_{00}}{\lambda_{01} - \lambda_{11}} \frac{P(s_0)}{P(s_1)}$$

then a_1

Class conditional likelihoods

False positive

$$\frac{P(\boldsymbol{x}|s_1)}{P(\boldsymbol{x}|s_0)} > \frac{\lambda_{10} - \lambda_{00}}{\lambda_{01} - \lambda_{11}} \frac{P(s_0)}{P(s_1)}$$

then a_1

Missed detection

		State of Nature	
		Class 0	Class 1
Estimate	Class 0	$\lambda_{00} = 0$	$\lambda_{01} = 100$
	Class 1	$\lambda_{10} = 1$	$\lambda_{11} = 0$

We can use this with generative models

$$\frac{P(\boldsymbol{x}|s_1)}{P(\boldsymbol{x}|s_0)} > \frac{\lambda_{10} - \lambda_{00}}{\lambda_{01} - \lambda_{11}} \frac{P(s_0)}{P(s_1)} \quad \text{then} \quad a_1$$

- If we have class conditional models for our data, we can classify them
- **Naïve Bayes**: assume all features are independent given their class
- **Linear discriminant analysis**: assume the class conditional distributions, e.g. $P(\boldsymbol{x}|s_1)$, are each Gaussian

Generative and discriminative models

Unobservable

Data Generating Process

$p(X,Y)$

Target Function for predicting y from x

$f(x) \rightarrow y$

Types of models. We can either model the full data generating process **OR** the target function, the mapping x to y

If we model this process, it's a **generative model**

- Models $P(x|y)$
- Can be used to generate synthetic data and impute missing values

If we model this function, it's a **discriminative model**

- May model $P(y|x)$ or directly map x to y without probabilities
- Often better performance for large sample sizes

Takeaways

To make a decision:

1. Define a measure of risk or reward
2. Select the action that optimizes that metric

Decision theory informs how to operate supervised learning algorithms in practice

Decision theory incorporates relative importance of errors

Generative models estimate $P(x|y)$, while discriminative models estimate $P(y|x)$