(The elements of $F_q^{(q-1)kt+k+t}$ in this construction may be thought of as three-dimensional arrays where the elements of $\bar{x}_{ij}$ are z-lined, every underlined block is y-lined, and the tuple of blocks is x-lined. Naturally, the multary quasigroups V_i may be named "vertical" and H_i, "horizontal".)

The proof of the code distance is similar to that in [9], and the other properties of a $\bar{\mu}$-component are straightforward. The existence of admissible $(q-1)$-ary quasigroups v and h is the only restriction on the q (this concerns the next subsection as well). If F_q is a finite field, there are linear examples: $v(y_1,\ldots,y_{q-1}) = y_1+\ldots+y_{q-1}$, $v(y_1,\ldots,y_{q-1}) = \alpha_1 y_1+\ldots+\alpha_{q-1}y_{q-1}$ where α_1, ..., α_{q-1} are all the non-zero elements of F_q. If q is not a prime power, the existence of a q-ary perfect code of length $q+1$ is an open problem (with the only exception $q=6$, when the nonexistence follows from the nonexistence of two orthogonal 6×6 Latin squares [1, Th. 6]).

3.2. Generalized Phelps construction

Here we describe another way to construct $\bar{\mu}$-components, which generalizes the construction of binary perfect codes from [8].

Lemma 2. *Let $\bar{\mu}\in F_q^t$. Let for every i from 1 to $t+1$ the codes $C_{i,j}$, $j=0,1,\ldots,qk-k$ form a partition of F_q^k into perfect codes and $\gamma_i : F_q^k \to \{0,1,\ldots,qk-k\}$ be the corresponding partition function:*

$$\gamma_i(\bar{y}) = j \iff \bar{y}\in C_{i,j}.$$

Let v and h be $(q-1)$-ary quasigroups of order q such that the code $\{(\bar{y} \mid v(\bar{y}) \mid h(\bar{y})) : \bar{y}\in F_q^{q-1}\}$ is perfect. Let V_1, ..., V_t be $(k+1)$-ary quasigroups of order q and Q be a t-ary quasigroup of order $qk-k+1$.

$$\begin{aligned} K_{\bar{\mu}} \ = \ \Big\{ & (\underline{\bar{x}_{11} \mid ... \mid \bar{x}_{1k} \mid y_1} \mid \underline{\bar{x}_{21} \mid ... \mid \bar{x}_{2k} \mid y_2} \mid \ldots \mid \underline{\bar{x}_{t1} \mid ... \mid \bar{x}_{tk} \mid y_t} \mid \underline{z_1 \mid z_2 \mid ... \mid z_k}) : \\ & \bar{x}_{ij}\in F_q^{q-1}, \\ & (V_1(v(\bar{x}_{11}),...,v(\bar{x}_{1k}),y_1),\ldots,V_t(v(\bar{x}_{t1}),...,v(\bar{x}_{tk}),y_t)) = \bar{\mu}, \\ & Q(\gamma_1(h(\bar{x}_{11}),...,h(\bar{x}_{1k})),\ldots,\gamma_t(h(\bar{x}_{t1}),...,h(\bar{x}_{tk}))) = \gamma_{t+1}(z_1,...,z_k)\Big\} \end{aligned}$$

is a $\bar{\mu}$-component that satisfies the generalized parity-check law with

$$\sigma_i(\cdot,\ldots,\cdot,\cdot) = V_i(v(\cdot),...,v(\cdot),\cdot).$$

The proof consists of trivial verifications.

4. On the number of perfect codes

In this section we discuss some observations, which result in the best known lower bound on the number of q-ary perfect codes, $q\geq 3$. The basic facts are already contained in other known results: lower bounds on the number of multary quasigroups of order q, the