Econ homework 1 - OSM Bootcamp 2018

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Exercise 1

1. State variables are $p :=$ current price of oil and $B :=$ total oil endowment
2. Control variable is $q :=$ the oil extracted in the current period
3. The transition equation is:
$$q = B - B'$$
4. The sequence problem is defined:
$$max_{\{q\}_0^t} \sum_{t=1}^{\infty} (\frac{1}{1+r})^t q_t p_t$$
$$s.t. \sum_{t=1}^{\infty} q_t = B$$
And the corresponding Bellman equation is defined:
$$V(B) = max_{B'} (B - B')p + \frac{1}{1+r} V(B')$$
5. We can derive the Euler equation beginning with either the sequence problem or the Bellman equation. Regardless, we will arrive at the same FOC/Euler equations, hence showing that our two formulations of the problem are equivalent. First, we derive the Bellman equation-based Euler equation:
$$V(B) = max_{B'} (B - B')p + \frac{1}{1+r} V(B')$$
$$\frac{\partial V}{\partial B'} = -p + \frac{1}{1+r} V'(B') = 0$$
$$\frac{\partial V}{\partial B} = p - p\frac{\partial B'}{\partial B} + \frac{1}{1+r}\frac{\partial V}{\partial B'}\frac{\partial B'}{\partial B}$$
$$= p + [\frac{1}{1+r}\frac{\partial V}{\partial B'} - p]\frac{\partial B'}{\partial B}$$
$$= p$$
We can advance this relationship forward one period and substitute which then yields the Euler equation:
$$\frac{1}{1+r} p' = p$$
This makes intuitive since, as this represents the marginal utility across the two time periods, which should be equal at a maximum

Now we can derive the Euler equation from the sequence problem by forming the Lagrangian as:

$$\mathcal{L} = \sum_{t=1}^{\infty}(\frac{1}{1+r})^t q_t p_t + \lambda(B - \sum_{t=1}^{\infty} q_t)$$

$$\frac{\partial \mathcal{L}}{\partial q_t}(\frac{1}{1+r})^t p_t - \lambda = 0$$

$$\Rightarrow (\frac{1}{1+r})^t p_t = \lambda, \forall t$$

Then let $p = p_t$ and $p' = p_{t+1}$, then we have:

$$(\frac{1}{1+r})^t p = (\frac{1}{1+r})^{t+1} p'$$

$$p = (\frac{1}{1+r})p'$$

6. If $p_{t+1} = p_t$ for all t, then because of our discounting we will always sell all possible oil in the current period t.
   In contrast, if $p_{t+1} > (1+r)p_t$ for all t then the marginal utility of selling oil tomorrow is always greater than today. Thus, we will always sell 'tomorrow' which will result in us always holding the oil and never selling it.
   It is important to note that the marginal utility in a given period is *not* a function of the current amount sold. Thus, the utility maximizing strategy will always be to sell all oil in the period with the highest price - the same period maximizes utility be if for the first or the last barrel of oil sold. Thus, the oil time we will split the oil extraction between periods is if we are indifferent, i.e. when $p_{t+k} = (1+r)^k p_t$ for some k

Exercise 2

1. State variables are: y_t, k_t, z_t
2. Control variables are: c_t which then implicitly induces corresponding values of i_t, k_{t+1}
3. The Bellman equation, after substituting in the the transition equation and the constraints is:

$$V(k_t, z_t) = max_{k_{t+1}} u(c_t) + \beta E[V(k_{t+1}, z_{t+1})]$$

$$V(k_t, z_t) = max_{k_{t+1}} u(z_t k_t^{\alpha} - k_{t+1} + (1-\delta)k_t) + \beta E[V(k_{t+1}, z_{t+1})]$$

4. See Jupyter notebook for coding results

Exercise 3

1. The Bellman equation is the same, but the expectation is now conditional on the current z_t:

$$V(k_t, z_t) = max_{k_{t+1}} u(z_t k_t^{\alpha} - k_{t+1} + (1-\delta)k_t) + \beta E_{z_{t+1}|z_t}[V(k_{t+1}, z_{t+1})]$$

2. See Jupyter notebook for coding results

Exercise 4

1. The Bellman equation is now:

$$V(w) = max\{V^0(w), V^1(w)\}$$

   where:

$$V^0(w) = \sum_{i=0}^{\infty} \beta^t w = w\frac{1}{1-\beta}$$

$$V^1(w) = \sum_{i=1}^{\infty} \beta^t E[w] + b = b + E[w]\frac{\beta}{1-\beta}$$

We can actually solve for the "reservation wage" analytically and use the results to inform the corresponding computation approach.
The reservation wage is w $s.t.$ $V^0(w) = V^1(w)$. Thus, we have:

$$w\frac{1}{1-\beta} = b + E[w]\frac{\beta}{1-\beta}$$

And re-arranging we have the reservation wage as:

$$w^*(b, \beta) = b(1-\beta) + \beta E[w]$$

This makes intuitive sense. As $\beta \to 0$ we see $w^* \to b$. As we become extremely high discounting, any current wage over our gauranteed unemployment benefit will be the preferred strategy.
And as $\beta \to 1$ we see $w^* \to E[w]$. As we discount less and less, any draw less than the expected value of our wage is rejected - we are fine just waiting for another draw so anything *above* that threshold will cause a change in behavior and we lock in the higher than expected wage draw.

Finally, because we have an unconditional expectation for our random variable component, the expected future earnings is always the same. Thus, we essentially have a utility floor defined at the reservation wage. Therefore, we expect to see a flat value curve before the reservation wage and a monotonically increasing function after the wage - and this is exactly what we see from our Jupyter notebook results.

2. See Jupyter notebook for coding results