Math homework 2 - OSM Bootcamp 2018

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Exercise 3.1

(i)

$$\frac{1}{4}(||x+y||^2 - ||x-y||^2)$$
$$= \frac{1}{4}(\langle x+y, x+y\rangle - \langle x-y, x-y\rangle)$$
$$= \frac{1}{4}(\langle x,x\rangle + \langle y,x\rangle + \langle x,y\rangle + \langle y,y\rangle - \langle x,x\rangle + \langle y,x\rangle + \langle x,y\rangle - \langle y,y\rangle)$$
$$= \frac{1}{4}(4\langle x,y\rangle) = \langle x,y\rangle$$

(ii)

$$\frac{1}{2}(||x+y||^2 + ||x-y||^2)$$
$$= \frac{1}{2}(\langle x+y, x+y\rangle + \langle x-y, x-y\rangle)$$
$$= \frac{1}{2}(\langle x,x\rangle + \langle y,x\rangle + \langle x,y\rangle + \langle y,y\rangle + [\langle x,x\rangle - \langle y,x\rangle - \langle x,y\rangle + \langle y,y\rangle$$
$$= \frac{1}{2}(2\langle x,x\rangle + 2\langle y,y\rangle) = ||x||^2 + ||y||^2$$

Exercise 3.2 I'm just dropping the $\frac{1}{4}$ for the sake of notation

$$\langle x+y, x+y\rangle - \langle x-y, x-y\rangle + i\langle x-iy, x-iy\rangle - i\langle x+iy, x+iy\rangle$$
$$= \langle x+y, x\rangle + \langle x+y, y\rangle - \langle x-y, x\rangle + \langle x-y, y\rangle + i\langle x-iy, x\rangle + i\langle x-iy, -iy\rangle - i\langle x+iy, x\rangle - i\langle x+iy, iy\rangle$$
$$= \overline{\langle x,x\rangle} + \overline{\langle x,y\rangle} + \overline{\langle y,x\rangle} + \overline{\langle y,y\rangle} - \overline{\langle x,x\rangle} + \overline{\langle x,y\rangle} + \overline{\langle y,x\rangle} - \overline{\langle y,y\rangle} + i\overline{\langle x,x\rangle} + \overline{\langle x,y\rangle} + \overline{\langle y,x\rangle} - \overline{\langle y,y\rangle} - i\overline{\langle x,x\rangle} + \overline{\langle x,y\rangle} + \overline{\langle y,x\rangle}$$
$$= \frac{1}{4}(4\overline{\langle x,y\rangle} + 4\langle x,y\rangle) = \langle x,y\rangle$$

Exercise 3.3

(i) $\langle x, x^5\rangle = \int_0^1 x^6 dx = \frac{x^7}{7} = \frac{1}{7}$

$$||x||^2 = \langle x,x\rangle = \int_0^1 x^2 dx = \frac{x^3}{3} = \frac{1}{3}$$
$$||x^5||^2 = \langle x^5, x^5\rangle = \int_0^1 x^1 0 dx = \frac{x^1 1}{11} = \frac{1}{11}$$
$$\cos\theta = \frac{1/7}{\frac{1}{33}^{1/2}}$$
$$\theta = \cos^{-1}\frac{33^{1/2}}{7}$$

(ii) $\langle x^2, x^4 \rangle = \int_0^1 x^6 dx = \frac{1}{7}$

$$||x^2||^2 = \langle x^2, x^2 \rangle = \int_0^1 x^4 dx = \frac{1}{5}$$

$$||x^4||^2 = \int_0^1 x^8 dx = \frac{1}{9}$$

$$\theta = \cos^{-1} \frac{45^{1/2}}{7}$$

Exercise 3.8

(i) Orthonormal

$$\langle \cos t, \cos t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 t dt = 1$$

$$\langle \sin t, \sin t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2 t dt = 1$$

$$\langle \cos 2t, \cos 2t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 2t dt = 1$$

$$\langle \sin 2t, \sin 2t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2 2t dt = 1$$

We can exploit the shape of each graph and the symmetric domain to easily calculate integrals.

$$\langle \cos t, \sin t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin t \cos t dt = 0$$

$$\langle \cos t, \cos 2t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos t \cos 2t dt = 0$$

$$\langle \cos t, \sin 2t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin 2t \cos t dt = 0$$

$$\langle \sin t, \cos 2t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin t \cos 2t dt = 0$$

$$\langle \sin t, \sin 2t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin t \sin 2t dt = 0$$

$$\langle \sin 2t, \cos 2t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin 2t \cos 2t dt = 0$$

(ii) $||t||^2 = \langle t, t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 = \frac{1}{\pi} \frac{t^3}{3} |_{-\pi}^{\pi} = \frac{2\pi^2}{3}$

(iii) $proj_x(\cos 3t) = \sum_{x_i \in X} \langle x_i, \cos 3t \rangle x_i$
And then the coefficients for the above general equation are as follows:

$$\langle \cos t, \cos 3t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos t \cos 3t dt$$

$$\langle \sin t, \cos 3t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin t \cos 3t dt$$

$$\langle \cos 2t, \cos 3t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos 2t \cos 3t dt$$

$$\langle \sin 2t, \cos 3t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin 2t \cos 3t dt$$

(iv) This is just repeating the exact same mechanics as the previous question so I am sparing myself the tedium of integral taking

Exercise 3.9

We will prove $\langle x, y\rangle = \langle Lx, Ly\rangle$ where $L = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

Let $x, y \in \mathbb{R}$. Then,

$$Lx = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix}$$
$$= \begin{vmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{vmatrix}$$
$$Ly = \begin{vmatrix} y_1\cos\theta - y_2\sin\theta \\ y_1\sin\theta + y_2\cos\theta \end{vmatrix}$$

$$\langle Lx, Ly\rangle = x_1y_1\cos^2\theta - x_1y_2\sin\theta\cos\theta - x_2y_1\sin\theta\cos\theta + \\ x_2y_2\sin^2\theta + x_1y_1\sin^2\theta + x_1y_2\sin\theta\cos\theta + x_2y_1\cos\theta\sin\theta + x_2y_2\cos^2\theta \tag{1}$$

$$= x_1y_1(\cos^2\theta + \sin^2\theta) + x_2y_2(\sin^2\theta + \cos^2\theta) = x_1y_1 + x_2y_2 = \langle x, y\rangle\blacksquare$$

Exercise 3.10

(i) If we note that $Q^HQ = [a_{ij}]$ then by properties of matrix multiplication we can clearly see that element $[a_{ij}] = \langle q_i, q_j\rangle$, where q_i is the ith column of Q. Combining with I we clearly see that $\langle q_i, q_j\rangle = \delta_{ij}$.

(ii) If $Q \in M_n(\mathbb{F})$ is orthonormal then $\langle x, x\rangle = \langle Qx, Qx\rangle \Rightarrow ||x|| = ||Qx||\blacksquare$

(iii) If $Q \in M_n(\mathbb{F})$ is orthonormal then by prop 3.3.12 Q^{-1} exists and is a function
And $\forall x \in \mathbb{F}\ \exists x' \in \mathbb{F}$ such that $Qx' = x \Rightarrow Q^{-1}Qx' = Q^{-1}x$
Thus, with $\langle x, y\rangle = \langle Qx, Qy\rangle$ we have $\langle Q^{-1}x, Q^{-1}y\rangle = \langle Q^{-1}Qx, Q^{-1}Qy\rangle = \langle x, y\rangle\blacksquare$

(iv) We know that $QQ^H = I$. Thus, if we re-express the multiplication as row vectors we have that each i, j element in I is the dot-product of the ith and the jth column in Q, i.e. $\langle q_i, q_j\rangle$ and because $QQ^H = I$ we know $\langle q_i, q_j\rangle = \delta_{i,j}\blacksquare$

(v) Converse is NOT true
Let $A = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$
Then $det(A) = 1$ but the columns are clearly not orthonormal and thus A is not orthonormal. $\blacksquare$

(vi) Assume $Q_1, Q_2 \in M_n(\mathbb{F})$ are orthonormal.
Then $\langle x, y\rangle = \langle Q_2x, Q_2y\rangle$
But $Q_2 \in M_n(\mathbb{F})$ so define $x' = Q_2x$ and $y' = Q_2y$
Thus, $\langle Q_1x', Q_1y'\rangle = \langle x', y'\rangle$
$\Rightarrow \langle Q_1Q_2x, Q_1Q_2y\rangle = \langle x', y'\rangle = \langle Q_2x, Q_2y\rangle = \langle x, y\rangle\blacksquare$

Exercise 3.11

Because the vectors are $\{x_1, x_2, ..., x_n\}$ linearly dependent $\exists x_j$ that can be expressed as a linear combination of the other x_i.
Thus, when we get to the dependent vector, its calculated basis function will be zero.

Exercise 3.16

STILL NEED TO DO THIS

Exericse 3.17

$$A^HAx = A^Hb$$

$$\Rightarrow Ax = b$$
$$\Rightarrow QRx = b$$
$$\Rightarrow Q^H QRx = Q^H b$$
$$\Rightarrow Rx = Q^H b$$

Exercise 3.23

$$||x|| = ||x - y + y|| \leq ||x - y|| + ||y||$$
$$\Rightarrow ||x|| - ||y|| \leq ||x - y||$$

And,

$$||y|| = ||y - x + x|| \leq ||y - x|| + ||x||$$
$$\Rightarrow ||y|| - ||x|| \leq ||x - y||$$

Together, because the inequality holds regardless of the direction of our subtraction we have the result.

Exercise 3.24 Note, each $f \in C$ is continuous over a compact domain and thus is bounded, and achieves this bound. Thus, we know $\int_a^b |f(t)|dt$ is finite which implies $\int_a^b |f(t)|^k dt$ is also finite. Obviously, each i, ii, and iii is positive and the integral of the 0 function is 0. And because we ahve absolute values a 0 integral implies that f is the 0 function, so we have positivity for i, ii, and iii. So we will prove the scalar and triangle inequality properties.

(i)

$$||af|| = \int_a^b |af(t)|dt = \int_a^b |a||f(t)|dt = |a| \int_a^b |f(t)|dt = |a|||f||$$
$$||f + g|| = \int_a^b |f(t) + g(t)|dt \leq \int_a^b |f(t)| + |g(t)|dt = \int_a^b |f(t)|dt + \int_a^b |g(t)|dt = ||f|| + ||g||$$

(ii)

$$||af|| = (\int_a^b |a|^2|f(t)|dt)^{1/2} = |a|(\int_a^b |f(t)|dt)^{1/2} = |a|||f||$$
$$||f+g|| = (\int_a^b |f(t)+g(t)|^2 dt)^{1/2} \leq (\int_a^b |f(t)|^2+|g(t)|^2 dt)^{1/2} \leq \int_a^b |f(t)|dt+\int_a^b |g(t)|dt = ||f||+||g||$$

(iii)

$$||af|| = sup|a(f(x)| = sup|a||f(x)| = |a|sup|f(x)| = |a|||f||$$
$$||f + g|| = sup|f(x) + g(x)|| \leq sup\{|f(x)| + |g(x)|\} \leq sup|f(x)| + sup|g(x)| = ||f|| + ||g||$$

Exercise 3.26 To show the relation is an equivalence relation we show it is: reflexive, symmetric, and transitive
Reflexive: take $m = M = 1$
Symmetric: Assume $\exists m, M$ such that $m||x||_a \leq ||x||_b \leq M||x||_a$ for all $x \in X$.
Then, $||x||_a \leq \frac{1}{m}||x||_b$ and $\frac{1}{M}||x||_b \leq ||x||_a$

$$\Rightarrow \frac{1}{M}||x||_b \leq ||x||_a \leq \frac{1}{m}||x||_b$$

Transitive: Further assume $\exists n, N$ such that $n||x||_b \leq ||x||_c \leq N||x||_b$ for all $x \in X$
Then, there exists $m*, M*$ such that $m * ||x||_a \leq ||x||_c \leq M * ||x||_a \square$

(i) $||x||_2 = (x_1^2 + ... + x_n^2)^{1/2} \leq (x_1^2)^{1/2} + ... + (x_n^2)^{1/2}$ (From properties of sqrt function)

$$= |x_1| + ... + |x_n| = ||x||_1$$

Thus, $||x||_2 \leq ||x||_1$
STILL NEED TO FIGURE OUT less than sqrt n part

(ii) $||x||_\infty = max\{|x_1|, ...|x_n|\} = |x_1|$ (WLOG)

$$= (x_1^2)^{1/2} \leq (x_1^2 + ... + x_n^2)^{1/2} = ||x||_2$$

STILL NEED TO FIGURE OUT less than sqrt n part

Exercise 3.28

(i) $||A||_1 \geq \frac{||Ax||}{||x||_1} \geq \frac{||Ax||_2}{\sqrt{n}||x||_2} \Rightarrow \frac{1}{\sqrt{n}}||A||_2 \leq ||A||_1 \leq ||A||_2$

(ii) We have $||A_2|| \geq \frac{||Ax||_2}{||x||} \geq \frac{||Ax||_\infty}{\sqrt{n}||x||_2}$

$$\Rightarrow ||A||_2 \geq \frac{1}{\sqrt{n}}||A||_\infty \Rightarrow \frac{1}{\sqrt{n}}||A||_\infty \leq ||A||_2 \leq ||A||_\infty$$

Exercise 3.29
Because Q is orthonormal $\forall x$ we have $||Qx|| = ||x||$.
Thus, if $||x|| = 1$ then $sup||Qx|| = 1 \Rightarrow ||Q|| = 1$

Exercise 3.30
STILL NEED TO FIGURE OUT

Exercise 3.37 Let $[r_1, ..., r_2]$ be an orthonormal basis for V
Then $p \in V$ can be expressed as $p = \sum \langle r_i, p \rangle r_i$
Thus, $L[p] = \sum \langle r_i, p \rangle L[r_i] = \langle \sum \overline{L[r_i]} r_i, p \rangle$
Thus, $q = \sum \overline{L[r_i]} r_i$

Exercise 3.38 Where $p \in V$ written with respect to our basis functions as $p = a_0 1 + a_1 x + a_2 x^2$. Then let e_0, e_1, e_2 be the aforementioned basis functions. We can then write D in matrix form as:

$$D[p](x) = [e_0'(x), e_1'(x), e_2'(x)][a_0, a_1, a_2]^T$$

We can follow the Example 3.7.9 and find:

$$\langle p, D[q] \rangle = \int_{-\infty}^{\infty} p(x)q'(x)dx = -\int_{-\infty}^{\infty} p'(x)q(x)dx = -\langle D[p], q \rangle$$

Thus, $D* = -D$ $\square$

Exercise 3.39

(i) $S, T \in \mathcal{L}(V, W)$. Then $S + T = [s_{ij}] + [t_{ij}]$
So $(S + T)^* = \overline{[s_{ij}] + [t_{ij}]} = \overline{[s_{ij}]} + \overline{[t_{ij}]} = S^* + T^*$
And, $(\alpha T) = \alpha[t_{ij}]$, so $(\alpha T)^* = \overline{\alpha[t_{ij}]} = \overline{\alpha}\overline{[t_{ij}]} = \overline{\alpha} T^*$

(ii) $S \in \mathcal{L}(V, W)$ then $S^* = \overline{[s_{ij}]}$ and $(S^*)^* = \overline{\overline{[s_{ij}]}} = [s_{ij}]$

(iii) $S, T \in \mathcal{L}(V, W)$.

$$\langle y, STx \rangle = y^H(ST)x = ((ST)^H y)^H x = \langle (ST)^H y, x \rangle$$

So, $(ST)^* = (ST)^H = T^H S^H = T^* S^* \square$

(iv) $\langle y, T^{-1}x \rangle = y^H T^{-1} x = [(T^{-1})^H y]^H x = \langle (T^{-1})^H y, x \rangle$
So $(T^{-1})^* = (T^*)^{-1} \square$

Exercise 3.40

(i) Let $M, N \in M_n$
Then $\langle M, AN\rangle = tr(M^H AN) = tr((A^H M)^H N) = \langle A^H M, N\rangle$
So, $A^* = A^H \square$

(ii) $\langle A_2, A_3A_1\rangle = tr(A_2^H A_3 A_1) = tr(A_1 A_2^H A_3) = tr((A_2 A_1^H)^H A_3) = \langle A_2 A_1^H, A_3\rangle$

$$= \langle A_2 A_1^*, A_3\rangle \square$$

Exercise 3.46

(i) $A^H Ax = 0 \Rightarrow A^H(Ax) = 0 \Rightarrow Ax \in N(A^H)$
And by definition $Ax \in R(A)$

(ii) Let $x \in N(A^H A)$ so $A^H Ax = 0 \Rightarrow x \in N(A)$
Let $x \in N(A)$ so $Ax = 0$ and $A^H 0 = 0$ so $A^H Ax = A^H 0 = 0$ so $x \in N(A^H A)$
Thus, each is a subset of each other implying we must have equality.

(iii) Because their null spaces are equal we can envoke rank-nullity theorem and thus their ranks are equal as well

(iv) If A has linearly independent columns then it has rank n so it's injective adn thus $A^H A$ is invertible, i.e. non-singular.

Exercise 3.47

(i) $P^2 = P$

$$P^2 = A(A^H A)^{-1} A^H A (A^H A)^{-1} A^H$$
$$= A(A^H A^{-1}) A^H = P$$

(ii) $P^H = P$

$$((A(A^H A)^{-1}) A^H)^H = A(A(A^A)^{-1})^H$$
$$= A(A^H A)^{-1} A^{-1}$$

(iii) $A^H P = A^H A (A^H A)^{-1} A^H$

$$\Rightarrow A^H P = A^H$$
$$\Rightarrow rankA = rankA^H = rankP = n$$

Exercise 3.48

(i) $P(\alpha A + B) = \frac{\alpha A + B + \alpha A^T + B^T}{2} =$

$$= \frac{\alpha A + \alpha A^T}{2} + \frac{B + B^T}{2} = \alpha P(A) + P(B)$$

(ii) $P(P(A)) = \frac{(.5(A+A^T)) + (.5(A+A^T))^T}{2} = \frac{(.5(A+A^T)) + (.5(A+A^T))}{2} = \frac{A+A^T}{2} = P(A)$

(iii) $\langle A, P(B)\rangle = tr(A^T \frac{B+B^T}{2}) = tr(A^T B + A^T B^T) = tr(A^T B) + tr(A^T B^T)$

$$= tr(A^T B) + tr(AB) = tr(A^T B + AB) = tr((A + A^T)^T B) = \langle P(A), B\rangle$$