Math homework 4 - OSM Bootcamp 2018

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Exercise 6.6
The FOC are:

$$f_x = 6xy + 4y^2 = 0$$
$$f_y = 3x^2 + 8xy + x = 0$$

First, we need to go through some cases to check whether x or y can be 0.
If $x = 0$ then $4y^2 + y = 0 \Rightarrow y$ is imaginary
If $x \neq 0$ then $3x + 8y + 1 = 0$
Then if $y = 0, x \neq 0$ then $3x + 1 = 0 \Rightarrow x = \frac{-1}{3}$
If $x \neq 0, x \neq 0$ then:

$$f_x = 6x + 4y + 1 = 0$$
$$f_y = 3x + 8y + 1 = 0$$
$$(-16y - 1) + (4y + 1) = 0$$
$$-12y - 1 = 0$$
$$\Rightarrow y = \frac{-1}{12}$$
$$\Rightarrow 3x - \frac{8}{12} + \frac{12}{12} = 0$$
$$x = \frac{-4}{36}$$

And finally our critical points are $(0, 0)$, $(\frac{-1}{3}, 0)$, $(\frac{-4}{36}, \frac{-1}{12})$

Exercise 6.7

(i) Denote $A = [a_{i,j}]$
Then $A^T = [a_{j,i}]$ and $Q = [a_{j,i} + a_{i,j}] = [a_{i,j} + a_{j,i}] = Q^T$
And $x^T Q x = x^T(A^T + A)x = (x^T A^T + x^T A)x = x^T A^T x + x^T A x = 2x^T A x\square$

(ii) $f(x) = \frac{1}{2}x^T Q x - b^T x + c$

$$f'(x) = Q^T x - b = 0$$
$$\Rightarrow Q^T x^* = b$$

(iii) **NEED TO DO THIS QUESTION**

Exercise 6.11

$$x_1 = x_0 - f'(x_0)/f''(x_0)$$
$$x_1 = \frac{-b}{2ax_0} \Rightarrow f'(x_0) = 2ax_0 + b = -b + b = 0$$

And also, $f''(x_0) = 2a > 0$

Exercise 7.1
Let $x, y \in \text{conv}(S)$
Then x and y are of the form:

$$a_1x_1 + ... + a_nx_n = x \text{ and } b_1y_1 + ... + b_my_m = y$$

Where $\sum a_i = \sum b_i = 1$ and $x_i, y_i \in S$
Then if $\lambda \in [0,1]$ then $\lambda x + (1-\lambda)y = \sum_{i=1}^n \lambda a_i x_i + \sum_{i=1}^m (1-\lambda) b_i y_i \in \text{conv}(S)$
Note, the last part holds because $\sum \lambda a_i + \sum (1-\lambda) b_i = 1$

Exercise 7.2

(i) Let $P = \{x \in V | \langle a, x\rangle = b\}$
Then let $x, y \in P$ so $\langle a, x\rangle = \langle a, y\rangle = b$
Let $\lambda \in [0,1]$ then $\langle a, \lambda x + (1-\lambda)y\rangle =$

$$= \lambda\langle a, x\rangle + (1-\lambda)\langle a, y\rangle = b\square$$

(ii) Let $x, y \in H = \{x \in V | \langle a, x\rangle \leq b\}$ and $\lambda \in [0,1]$.
Then $\langle a, \lambda x + (1-\lambda)y\rangle =$

$$-\lambda\langle a, x\rangle + (1-\lambda)\langle a, y\rangle \leq b\square$$

Exercise 7.4

(i) $||x-y||^2 = ||x-p+p-y||^2 = \langle x-p+p-y, x-p+p-y\rangle$

$$= \langle x-p, x-p\rangle + \langle x-p, p-y\rangle + \langle p-y, x-p\rangle + \langle p-y, p-y\rangle$$

$$= ||x-p||^2 + ||p-y||^2 + 2\langle x-p, p-y\rangle$$

(ii) By (i) we have $||x-y||^2 = ||x-p||^2 + ||p-y||^2 + 2\langle x-p, p-y\rangle$
And by assumption we have $\langle x-p, p-y\rangle \geq 0$
And $||p-y||^2 > 0$ if $y \neq p$
Thus, together $||x-y||^2 > ||x-p||^2 \Rightarrow ||x-y|| > ||x-p||$, where the inequality is consistent across the square root due to the positivity of the norm.

(iii) By (i), $||x-y||^2 = ||x-p||^2 + \langle x-p, p-z\rangle + ||p-z||^2$ Then if $z = \lambda y + (1-\lambda)p$ we have

$$||x-z||^2 = ||x-p||^2 + 2\langle x-p, p-\lambda y - (1-\lambda)p\rangle + ||p-\lambda y - (1-\lambda)p||^2$$

$$= ||x-p||^2 + 2\lambda\langle x-p, p-y\rangle + \lambda^2||y-p||^2$$

(iv) **NEED TO DO THIS QUESTION**

Exercise 7.8
f is convex so $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$
I think there's a typo in the question because the matrix/vector operations in the argument are not compatible. If $A \in M_{m\times n}$ then $Ax\mathbb{R}^n$ but $b \in \mathbb{R}^m$ so the two can't be added?
So I'll assume $A \in M_{m\times m}$ for compatability.
Let $x, y \in \mathbb{R}^m$ then $x' = Ax + b$ and $y' = Ay + b$ are also in $\mathbb{R}^m$ Thus, $f(\lambda x' + (1-\lambda)y') \leq \lambda f(x') + (1-\lambda)f(y')$
and therefore $g(\lambda x + (1-\lambda)y) \leq g(x) + (1-\lambda)g(y)$

Exercise 7.12

(i) Let $A, B \in PD_n(\mathbb{R}$
Then $\lambda A + (1-\lambda)B = C$
And $v^TCv = \lambda v^TAv + (1-\lambda)v^TBv \geq 0$, so we have $C \in PDn(\mathbb{R})$.

(ii) **NEED TO DO THIS QUESTION**

Exercise 7.13

S'pose f is not constant. Then there exists some x, y such that $f(x) > f(y)$
By convexity we have $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$

$$\Rightarrow \frac{\lambda f(x) + (1-\lambda)f(y)}{\lambda} = \frac{f(x) - f(y)}{\lambda} \leq f(\frac{\lambda x + (1 - \lambda y)}{\lambda})$$

And $f(x) > f(y) \Rightarrow \frac{f(x)-f(y)}{\lambda}$ which goes to infinity as λ goes to zero, which would imply the function is not bounded, which is a contradiction.

Exercise 7.20

By the convexity of both f and $-f$ we have the following inequalities:

$$f(tx + (1-t)y) \leq tf(x) + (1-y)f(y)$$

$$-f(tx + (1-t)y) \leq -tf(x) - (1-y)f(y)$$

We can multiply the second equation by negative 1, and then see that for both to hold simulataneously we have equality. Thus,

$$f(tx + (1-t)y) = tf(x) + (1-y)f(y)$$

And we have our result.

Exercise 7.21

Let x^* be a local min. Then $f(x) \geq f(x^*)$ for all x in some epsilon neighborhood of x^*
And because ϕ is strictly increasing we have $\phi(f(x)) \geq \phi(f(x^*))$ for that same neighborhood so x^* is still a minimizer.
Similarly, going the other way, if $\phi(f(x)) \leq \phi(f(x^*))$ and because ϕ is monotone then $f(x^*) \leq f(x)$ for the neighborhood.