Math homework 5 - OSM Bootcamp 2018

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Exercise 8.1
See python code and associated png files for graphs
From the image we see the intersection corresponding to the point $(\frac{37}{7}, \frac{5}{7})$ which yields a function value of $\frac{165}{7}$ is the optimal point.

Exercise 8.2
See python code and associated png files for graphs
From the image we see the intersection corresponding to the point $(15, 12)$ which yields a function value of 132 is the optimal point.

Exercise 8.3
We can formulate this problem as:
$\max_{q_1,q_2} 7q_1 + 7q_2 - 3q_1 - 4q_2$

$$\text{subject to} \begin{cases} q_2 \leq 200 \\ .25q_1 + \frac{1}{6}q_2 \leq 30 \\ \frac{2}{60}q_1 + \frac{2}{60}q_2 \leq 5 \\ q_1, q_2 \geq 0 \end{cases}$$

Exercise 8.4
$\max 2x_{AB} + 5x_{BC} + 2x_{CF} + x_{AD} + 5x_{AD} + 2x_{BD} + 7x_{BE} + 9x_{BF} + 4x_{DE} + 3x_{EF}$

$$\text{subject to} \begin{cases} x_{AD} + x_{AB} = 10 \\ x_{BC} + x_{BE} - x_{AB} = 1 \\ x_{CF} - x_{BC} = -2 \\ x_{DE} - x_{AD} - x_{BD} = -3 \\ x_{EF} - x_{BE} - x_{DE} = 4 \\ -x_{CF} - x_{EF} - x_{BF} = -10 \\ 0 \leq x_i \leq 6 \end{cases}$$

Exercise 8.5

(i) Introducing slack variables to the problem we have:
$\max 3x_1 + x_2$

$$\text{subject to} \begin{cases} x_1 + 3x_2 + w_1 = 15 \\ 2x_1 + 3x_2 + w_2 = 18 \\ x_1 - x_2 + w_3 = 4 \end{cases}$$

$$\underline{\psi = 3x_1 + x_2}$$

$$w_1 = 15 - x_1 - 3x_2$$

$$w_2 = 18 - 2x_1 - 3x_2$$

$$w_3 = 4 - x_1 + x_2$$

Pivot: IN(x_1); OUT(w_3)
Sub: $x_1 = 4 - w_3 + x_2$

$$\underline{\psi = 12 - 3w_3 + 4x_2}$$
$$w_1 = 11 + w_3 - 4x_2$$
$$w_2 = 10 + 2w_3 - 2x_2$$
$$x_1 = 4 - w_3 + x_2$$

Pivot: IN(x_2); OUT(w_1)
Sub: $x_2 = \frac{11}{4} + \frac{w_3}{4} - \frac{w_1}{4}$

$$\underline{\psi = 23 - 2w_1 - w_2} = 23$$
$$x_2^* = 11/4$$
$$w_2 = 9/2$$
$$x_1^* = 27/4$$

(ii) Introducing slack variables to the problem we have:
$\max 4x + 6y$
subject to $\begin{cases} -x + y + w_1 = 11 \\ x + y + w_2 = 27 \\ 2x + 5y + w_3 = 90 \end{cases}$

$$\underline{\psi = 4x + 6y}$$
$$w_1 = 11 + x - y$$
$$w_2 = 27 - x - y$$
$$w_3 = 90 - 2x - 5y$$

Pivot: IN(x); OUT(w_2)
Sub: $x = 27 - w_2 - y$

$$\underline{\psi = 108 - 4w_2 + 2y}$$
$$w_1 = 38 - w_2 - 2y$$
$$x = 27 - w_2 - y$$
$$w_3 = 36 + 2w_2 - 3y$$

Pivot: IN(y); OUT(w_3)
Sub: $y = 12 + \frac{2}{3}w_2 - \frac{w_3}{3}$

$$\underline{\psi = 132 - (4 - 4/3)w_2 -]frac23w_3 = 132}$$
$$w_1 = 14$$
$$x = 15$$
$$y = 12$$

Exercise 8.6
Introducing slack variables and reducing some of the constraints we have:

$$\underline{\psi = 4x_1 + 3x_2}$$

$$w_1 = 360 - 3x_1 - 2x_2$$

$$w_2 = 150 - x_1 - x_2$$

$$w_3 = 200 - x_2$$

Pivot: IN(x_2); OUT(w_2)
Sub: $x_2 = 150 - x_1 - w_2$

$$\underline{\psi = x_1 + 450 - 3w_2}$$

$$w_1 = 60 - x_1 + 2w_2$$

$$x_2 = 150 - x_1 - w_2$$

$$w_3 = 50 + x_1 + w_2$$

Pivot: IN(x_1); OUT(w_3)
Sub: $x_2 = 150 - x_1 - w_2$
And thus the optimal is at point $(50, 150)$

Exercise 8.8
$\max x_1 + x_2 - x_3$

subject to $\begin{cases} x_1 \leq 1 \\ x_2 \leq 1 \\ x_1, x_2, x_3 \geq 0 \end{cases}$

Exercise 8.9
$\max x_1 + x_2 + x_3$

subject to $\begin{cases} x_1 \leq 1 \\ x_2 \leq 1 \\ x_1, x_2, x_3 \geq 0 \end{cases}$

Exercise 8.10
$\max x_1 + x_2 + x_3$

subject to $\begin{cases} x_1 \leq 1 \\ x_1 \leq -2 \\ x_1, x_2, x_3 \geq 0 \end{cases}$

Exercise 8.11
$\max x_1 + x_2 + x_3$

subject to $\begin{cases} x_1 \leq 1 \\ x_2 \leq 1 \\ -x_3 \leq -1 \\ x_3 \leq 2 \\ x_1, x_2, x_3 \geq 0 \end{cases}$

Exercise 8.15
Show $C^T x \leq b^T y$
$Ax \leq b$ and $A^T y \geq c$
So $y^T Ax \leq y^T b$ and $x^T A^T y \geq x^T c$ because of the non-negativity constraint

And, $y^T(Ax) = (Ax)^T y = x^T A^T y$
And by transitivity we have $y^T b \leq x^T c$ so $b^T y \geq c^T x$

Exercise 8.17
Let our primal be:
$\max c^T x$
subject to
$\begin{cases} Ax \leq b \\ x \geq 0 \end{cases}$

Then the associated dual problem is
$\min b^T y$
subject to
$\begin{cases} A^T y \geq c \\ y \geq 0 \end{cases}$

Which we rewrite in standard form as:
$\max -b^T y$
subject to
$\begin{cases} -A^T y \leq -c \\ y \geq 0 \end{cases}$

And we write this dual as:
Which we rewrite in standard form as:
$\min -c^T x'$
subject to
$\begin{cases} -(A^T)^T x' \geq -b \\ x' \geq 0 \end{cases}$
And when we rewrite this in standard form we have our original problem $\square$

Exercise 8.18
Solve the primal first.

$$\begin{array}{c} \underline{\psi = x_1 + x_2} \\ w_1 = 3 - 2x_1 - x_2 \\ w_2 = 5 - x_1 - 3x_2 \\ w_3 = 4 - 2x_1 - 3x_2 \end{array}$$

Pivot: IN(x_1); OUT(w_1)
Sub: $x_1 = 3/2 - w_1/2 - x_2/2$

$$\begin{array}{c} \underline{\psi = 3/2 - w_1/2 + x_2/2} \\ x_1 = 3/2 - w_1/2 - x_2/2 \\ w_2 = 7/2 + w_1/2 - 5/2x_2 \\ w_3 = 1 + w_1 - 2x_2 \end{array}$$

Pivot: IN(x_2); OUT(w_3)
Sub: $x_2 = 1/2 + w_1/2 - w_3/3$
$\Rightarrow x_1^* = 5/4$ and $x_2^* = 1/2$ for max value of $7/4$