

Math homework 2 - OSM Bootcamp 2018

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Exercise 3.1

(i)

$$\begin{aligned} & \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) \\ &= \frac{1}{4}(\langle x+y, x+y \rangle - \langle x-y, x-y \rangle) \\ &= \frac{1}{4}(\langle x, x \rangle + \langle y, x \rangle + \langle x, y \rangle + \langle y, y \rangle - \langle x, x \rangle + \langle y, x \rangle + \langle x, y \rangle - \langle y, y \rangle) \\ &= \frac{1}{4}(4\langle x, y \rangle) = \langle x, y \rangle \end{aligned}$$

(ii)

$$\begin{aligned} & \frac{1}{2}(\|x+y\|^2 + \|x-y\|^2) \\ &= \frac{1}{2}(\langle x+y, x+y \rangle + \langle x-y, x-y \rangle) \\ &= \frac{1}{2}(\langle x, x \rangle + \langle y, x \rangle + \langle x, y \rangle + \langle y, y \rangle + [\langle x, x \rangle - \langle y, x \rangle - \langle x, y \rangle + \langle y, y \rangle]) \\ &= \frac{1}{2}(2\langle x, x \rangle + 2\langle y, y \rangle) = \|x\|^2 + \|y\|^2 \end{aligned}$$

Exercise 3.2 I'm just dropping the $\frac{1}{4}$ for the sake of notation

$$\begin{aligned} & \langle x+y, x+y \rangle - \langle x-y, x-y \rangle + i\langle x-iy, x-iy \rangle - i\langle x+iy, x+iy \rangle \\ &= \langle x+y, x \rangle + \langle x+y, y \rangle - \langle x-y, x \rangle + \langle x-y, y \rangle + i\langle x-iy, x \rangle + i\langle x-iy, -iy \rangle - i\langle x+iy, x \rangle - i\langle x+iy, iy \rangle \\ &= \overline{\langle x, x \rangle} + \overline{\langle x, y \rangle} + \overline{\langle y, x \rangle} + \overline{\langle y, y \rangle} - \overline{\langle x, x \rangle} + \overline{\langle x, y \rangle} + \overline{\langle y, x \rangle} - \overline{\langle y, y \rangle} + i\overline{\langle x, x \rangle} + \overline{\langle x, y \rangle} + \overline{\langle y, x \rangle} - \overline{\langle y, y \rangle} - i\overline{\langle x, x \rangle} + \overline{\langle x, y \rangle} + \overline{\langle y, x \rangle} \\ &= \frac{1}{4}(4\overline{\langle x, y \rangle} + 4\langle x, y \rangle) = \langle x, y \rangle \end{aligned}$$

Exercise 3.3

(i) $\langle x, x^5 \rangle = \int_0^1 x^6 dx = \frac{x^7}{7} = \frac{1}{7}$

$$\|x\|^2 = \langle x, x \rangle = \int_0^1 x^2 dx = \frac{x^3}{3} = \frac{1}{3}$$

$$\|x^5\|^2 = \langle x^5, x^5 \rangle = \int_0^1 x^{10} dx = \frac{x^{11}}{11} = \frac{1}{11}$$

$$\cos \theta = \frac{1/7}{\frac{1}{33}^{1/2}}$$

$$\theta = \cos^{-1} \frac{33^{1/2}}{7}$$

$$(ii) \quad \langle x^2, x^4 \rangle = \int_0^1 x^6 dx = \frac{1}{7}$$

$$\|x^2\|^2 = \langle x^2, x^2 \rangle = \int_0^1 x^4 dx = \frac{1}{5}$$

$$\|x^4\|^2 = \int_0^1 x^8 dx = \frac{1}{9}$$

$$\theta = \cos^{-1} \frac{45^{1/2}}{7}$$

Exercise 3.8

(i) Orthonormal

$$\langle \cos t, \cos t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 t dt = 1$$

$$\langle \sin t, \sin t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2 t dt = 1$$

$$\langle \cos 2t, \cos 2t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 2t dt = 1$$

$$\langle \sin 2t, \sin 2t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2 2t dt = 1$$

We can exploit the shape of each graph and the symmetric domain to easily calculate integrals.

$$\langle \cos t, \sin t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin t \cos t dt = 0$$

$$\langle \cos t, \cos 2t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos t \cos 2t dt = 0$$

$$\langle \cos t, \sin 2t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin 2t \cos t dt = 0$$

$$\langle \sin t, \cos 2t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin t \cos 2t dt = 0$$

$$\langle \sin t, \sin 2t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin t \sin 2t dt = 0$$

$$\langle \sin 2t, \cos 2t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin 2t \cos 2t dt = 0$$

$$(ii) \quad \|t\|^2 = \langle t, t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 dt = \frac{1}{\pi} \frac{t^3}{3} \Big|_{-\pi}^{\pi} = \frac{2\pi^2}{3}$$

$$(iii) \quad \text{proj}_x(\cos 3t) = \sum_{x_i \in X} \langle x_i, \cos 3t \rangle x_i$$

And then the coefficients for the above general equation are as follows:

$$\langle \cos t, \cos 3t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos t \cos 3t dt$$

$$\langle \sin t, \cos 3t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin t \cos 3t dt$$

$$\langle \cos 2t, \cos 3t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos 2t \cos 3t dt$$

$$\langle \sin 2t, \cos 3t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin 2t \cos 3t dt$$

- (iv) This is just repeating the exact same mechanics as the previous question so I am sparing myself the tedium of integral taking

Exercise 3.9

We will prove $\langle x, y \rangle = \langle Lx, Ly \rangle$ where $L = \begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix}$

Let $x, y \in \mathbb{R}$. Then,

$$\begin{aligned} Lx &= \begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} \\ &= \begin{vmatrix} x_1 \cos \theta - x_2 \sin \theta \\ x_1 \sin \theta + x_2 \cos \theta \end{vmatrix} \\ Ly &= \begin{vmatrix} y_1 \cos \theta - y_2 \sin \theta \\ y_1 \sin \theta + y_2 \cos \theta \end{vmatrix} \end{aligned}$$

$$\begin{aligned} \langle Lx, Ly \rangle &= x_1 y_1 \cos^2 \theta - x_1 y_2 \sin \theta \cos \theta - x_2 y_1 \sin \theta \cos \theta + \\ & x_2 y_2 \sin^2 \theta + x_1 y_1 \sin^2 \theta + x_1 y_2 \sin \theta \cos \theta + x_2 y_1 \cos \theta \sin \theta + x_2 y_2 \cos^2 \theta \\ &= x_1 y_1 (\cos^2 \theta + \sin^2 \theta) + x_2 y_2 (\sin^2 \theta + \cos^2 \theta) = x_1 y_1 + x_2 y_2 = \langle x, y \rangle \blacksquare \end{aligned} \tag{1}$$

Exercise 3.10

- (i) If we note that $Q^H Q = [a_{ij}]$ then by properties of matrix multiplication we can clearly see that element $[a_{ij}] = \langle q_i, q_j \rangle$, where q_i is the i th column of Q . Combining with I we clearly see that $\langle q_i, q_j \rangle = \delta_{ij}$.
- (ii) If $Q \in M_n(\mathbb{F})$ is orthonormal then $\langle x, x \rangle = \langle Qx, Qx \rangle \Rightarrow \|x\| = \|Qx\| \blacksquare$
- (iii) If $Q \in M_n(\mathbb{F})$ is orthonormal then by prop 3.3.12 Q^{-1} exists and is a function
And $\forall x \in \mathbb{F} \exists x' \in \mathbb{F}$ such that $Qx' = x \Rightarrow Q^{-1}Qx' = Q^{-1}x$
Thus, with $\langle x, y \rangle = \langle Qx, Qy \rangle$ we have $\langle Q^{-1}x, Q^{-1}y \rangle = \langle Q^{-1}Qx, Q^{-1}Qy \rangle = \langle x, y \rangle \blacksquare$
- (iv) We know that $QQ^H = I$. Thus, if we re-express the multiplication as row vectors we have that each i, j element in I is the dot-product of the i th and the j th column in Q , i.e. $\langle q_i, q_j \rangle$ and because $QQ^H = I$ we know $\langle q_i, q_j \rangle = \delta_{i,j} \blacksquare$
- (v) Converse is NOT true
Let $A = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$
Then $\det(A) = 1$ but the columns are clearly not orthonormal and thus A is not orthonormal. \blacksquare
- (vi) Assume $Q_1, Q_2 \in M_n(\mathbb{F})$ are orthonormal.
Then $\langle x, y \rangle = \langle Q_2 x, Q_2 y \rangle$
But $Q_2 \in M_n(\mathbb{F})$ so define $x' = Q_2 x$ and $y' = Q_2 y$
Thus, $\langle Q_1 x', Q_1 y' \rangle = \langle x', y' \rangle$
 $\Rightarrow \langle Q_1 Q_2 x, Q_1 Q_2 y \rangle = \langle x', y' \rangle = \langle Q_2 x, Q_2 y \rangle = \langle x, y \rangle \blacksquare$

Exercise 3.11

Because the vectors are $\{x_1, x_2, \dots, x_n\}$ linearly dependent $\exists x_j$ that can be expressed as a linear combination of the other x_i .

Thus, when we get to the dependent vector, its calculated basis function will be zero.

Exercise 3.16

STILL NEED TO DO THIS

Exercise 3.17

$$A^H A x = A^H b$$

$$\begin{aligned}
&\Rightarrow Ax = b \\
&\Rightarrow QRx = b \\
&\Rightarrow Q^H QRx = Q^H b \\
&\Rightarrow Rx = Q^H b
\end{aligned}$$

Exercise 3.23

$$\begin{aligned}
\|x\| &= \|x - y + y\| \leq \|x - y\| + \|y\| \\
&\Rightarrow \|x\| - \|y\| \leq \|x - y\|
\end{aligned}$$

And,

$$\begin{aligned}
\|y\| &= \|y - x + x\| \leq \|y - x\| + \|x\| \\
&\Rightarrow \|y\| - \|x\| \leq \|x - y\|
\end{aligned}$$

Together, because the inequality holds regardless of the direction of our subtraction we have the result.

Exercise 3.24 Note, each $f \in C$ is continuous over a compact domain and thus is bounded, and achieves this bound. Thus, we know $\int_a^b |f(t)| dt$ is finite which implies $\int_a^b |f(t)|^k dt$ is also finite. Obviously, each i, ii , and iii is positive and the integral of the 0 function is 0. And because we have absolute values a 0 integral implies that f is the 0 function, so we have positivity for i, ii , and iii . So we will prove the scalar and triangle inequality properties.

(i)

$$\begin{aligned}
\|af\| &= \int_a^b |af(t)| dt = \int_a^b |a| |f(t)| dt = |a| \int_a^b |f(t)| dt = |a| \|f\| \\
\|f + g\| &= \int_a^b |f(t) + g(t)| dt \leq \int_a^b |f(t)| + |g(t)| dt = \int_a^b |f(t)| dt + \int_a^b |g(t)| dt = \|f\| + \|g\|
\end{aligned}$$

(ii)

$$\begin{aligned}
\|af\| &= \left(\int_a^b |a|^2 |f(t)|^2 dt \right)^{1/2} = |a| \left(\int_a^b |f(t)|^2 dt \right)^{1/2} = |a| \|f\| \\
\|f + g\| &= \left(\int_a^b |f(t) + g(t)|^2 dt \right)^{1/2} \leq \left(\int_a^b |f(t)|^2 + |g(t)|^2 dt \right)^{1/2} \leq \left(\int_a^b |f(t)|^2 dt \right)^{1/2} + \left(\int_a^b |g(t)|^2 dt \right)^{1/2} = \|f\| + \|g\|
\end{aligned}$$

(iii)

$$\begin{aligned}
\|af\| &= \sup |a(f(x))| = \sup |a| |f(x)| = |a| \sup |f(x)| = |a| \|f\| \\
\|f + g\| &= \sup |f(x) + g(x)| \leq \sup \{|f(x)| + |g(x)|\} \leq \sup |f(x)| + \sup |g(x)| = \|f\| + \|g\|
\end{aligned}$$

Exercise 3.26 To show the relation is an equivalence relation we show it is: reflexive, symmetric, and transitive

Reflexive: take $m = M = 1$

Symmetric: Assume $\exists m, M$ such that $m\|x\|_a \leq \|x\|_b \leq M\|x\|_a$ for all $x \in X$.

Then, $\|x\|_a \leq \frac{1}{m}\|x\|_b$ and $\frac{1}{M}\|x\|_b \leq \|x\|_a$

$$\Rightarrow \frac{1}{M}\|x\|_b \leq \|x\|_a \leq \frac{1}{m}\|x\|_b$$

Transitive: Further assume $\exists n, N$ such that $n\|x\|_b \leq \|x\|_c \leq N\|x\|_b$ for all $x \in X$

Then, there exists m^*, M^* such that $m^*\|x\|_a \leq \|x\|_c \leq M^*\|x\|_a$ \square

$$\begin{aligned}
(i) \quad \|x\|_2 &= (x_1^2 + \dots + x_n^2)^{1/2} \leq (x_1^2)^{1/2} + \dots + (x_n^2)^{1/2} \quad (\text{From properties of sqrt function}) \\
&= |x_1| + \dots + |x_n| = \|x\|_1
\end{aligned}$$

Thus, $\|x\|_2 \leq \|x\|_1$

STILL NEED TO FIGURE OUT less than sqrt n part

(ii) $\|x\|_\infty = \max\{|x_1|, \dots, |x_n|\} = |x_1|$ (WLOG)

$$= (x_1^2)^{1/2} \leq (x_1^2 + \dots + x_n^2)^{1/2} = \|x\|_2$$

STILL NEED TO FIGURE OUT less than sqrt n part

Exercise 3.28

(i) $\|A\|_1 \geq \frac{\|Ax\|}{\|x\|_1} \geq \frac{\|Ax\|_2}{\sqrt{n}\|x\|_2} \Rightarrow \frac{1}{\sqrt{n}}\|A\|_2 \leq \|A\|_1 \leq \|A\|_2$

(ii) We have $\|A_2\| \geq \frac{\|Ax\|_2}{\|x\|} \geq \frac{\|Ax\|_\infty}{\sqrt{n}\|x\|_2}$

$$\Rightarrow \|A\|_2 \geq \frac{1}{\sqrt{n}}\|A\|_\infty \Rightarrow \frac{1}{\sqrt{n}}\|A\|_\infty \leq \|A\|_2 \leq \|A\|_\infty$$

Exercise 3.29

Because Q is orthonormal $\forall x$ we have $\|Qx\| = \|x\|$.
Thus, if $\|x\| = 1$ then $\sup\|Qx\| = 1 \Rightarrow \|Q\| = 1$

Exercise 3.30

STILL NEED TO FIGURE OUT

Exercise 3.37 Let $[r_1, \dots, r_2]$ be an orthonormal basis for V

Then $p \in V$ can be expressed as $p = \sum \langle r_i, p \rangle r_i$

Thus, $L[p] = \sum \langle r_i, p \rangle L[r_i] = \langle \sum L[r_i] r_i, p \rangle$

Thus, $q = \sum L[r_i] r_i$

Exercise 3.38 Where $p \in V$ written with respect to our basis functions as $p = a_0 1 + a_1 x + a_2 x^2$.

Then let e_0, e_1, e_2 be the aforementioned basis functions. We can then write D in matrix form as:

$$D[p](x) = [e'_0(x), e'_1(x), e'_2(x)][a_0, a_1, a_2]^T$$

We can follow the Example 3.7.9 and find:

$$\langle p, D[q] \rangle = \int_{-\infty}^{\infty} p(x)q'(x)dx = - \int_{-\infty}^{\infty} p'(x)q(x)dx = -\langle D[p], q \rangle$$

Thus, $D^* = -D \square$

Exercise 3.39

(i) $S, T \in \mathcal{L}(V, W)$. Then $S + T = [s_{ij}] + [t_{ij}]$

So $(S + T)^* = \overline{[s_{ij}]} + \overline{[t_{ij}]} = \overline{[s_{ij}]} + \overline{[t_{ij}]} = S^* + T^*$

And, $(\alpha T) = \alpha[t_{ij}]$, so $(\alpha T)^* = \overline{\alpha[t_{ij}]} = \overline{\alpha}\overline{[t_{ij}]} = \overline{\alpha}T^*$

(ii) $S \in \mathcal{L}(V, W)$ then $S^* = \overline{[s_{ij}]}$ and $(S^*)^* = \overline{\overline{[s_{ij}]}} = [s_{ij}]$

(iii) $S, T \in \mathcal{L}(V, W)$.

$$\langle y, STx \rangle = y^H(ST)x = ((ST)^H y)^H x = \langle (ST)^H y, x \rangle$$

$$\text{So, } (ST)^* = (ST)^H = T^H S^H = T^* S^* \square$$

(iv) $\langle y, T^{-1}x \rangle = y^H T^{-1}x = [(T^{-1})^H y]^H x = \langle (T^{-1})^H y, x \rangle$

$$\text{So } (T^{-1})^* = (T^*)^{-1} \square$$

Exercise 3.40

(i) Let $M, N \in M_n$

Then $\langle M, AN \rangle = \text{tr}(M^H AN) = \text{tr}((A^H M)^H N) = \langle A^H M, N \rangle$

So, $A^* = A^H \square$

(ii) $\langle A_2, A_3 A_1 \rangle = \text{tr}(A_2^H A_3 A_1) = \text{tr}(A_1 A_2^H A_3) = \text{tr}((A_2 A_1^H)^H A_3) = \langle A_2 A_1^H, A_3 \rangle$

$= \langle A_2 A_1^*, A_3 \rangle \square$