Math homework 3 - OSM Bootcamp 2018

Cooper Nederhood

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Exercise 4.2

Let A be the matrix form of L so we have:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Then the characteristic equation is: $p(z) = z^3 = 0 \Rightarrow \lambda = 0$ with algebraic multiplicity of 3

The corresponding eigenvector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ we geometric mulitiplicity of 1

Exercise 4.4

(i) If $A^H = A$ then it must be that:

$$A = \begin{bmatrix} a & b_0 + b_1 i \\ b_0 - b_1 i & d \end{bmatrix}$$

Thus, $p(\lambda) = \lambda^2 - ad\lambda + (ad - (b_0^2 + b_1^2))$
Which has real roots because $(-ad)^2 - 4(1)(ad - (b_0^2 + b_1^2)) > 0$

(ii) If $A^H = -A$ then it must be that:

$$A = \begin{bmatrix} ai & bi \\ bi & di \end{bmatrix}$$

Thus, $p(\lambda) = \lambda^2 + ad\lambda + (-ad + b^2) = 0$
Which has imaginary roots because $-4(1)(-ad + b^2) + (ad)^2 < 0$

Exercise 4.6

Because of the structure of a diagonal matrix its characteristic equation will always be of the form, which λ_i denotes a diagonal element:

$$p(z) = det(A - zI) = \prod_i^n (\lambda_i - z) = 0$$

And thus $z = \lambda_i$

Exercise 4.8

(i) To show S forms a basis we need only show the set to be linearly independent, namely that:

$$a\sin(x) + b\cos(x) + c\sin(2x) + d\cos(2x) = 0 \Rightarrow a = b = c = d = 0$$

We can chose clever x values and find:
Let $x = 0$ then $b + d = 0$
Let $x = \frac{\pi}{2}$ then $a - d = 0$

Let $x = \pi$ then $-b + d = 0$
Let $x = \frac{\pi}{3}$ then $a\sin(\frac{\pi}{3}) + b\cos(\frac{\pi}{3}) + c\sin(2\frac{\pi}{3}) + d\cos(2\frac{\pi}{3}) = 0$
Together, these equations show that $a = b = c = d = 0$ and thus they are linearly independent and span the space, forming a basis.

(ii)

$$D = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

(iii) Let $v_1 = \{\sin(2x), \cos(2x)\}$ and $v_2 = \{\sin(x), \cos(x)\}$

Exercise 4.13

To diagonalize A we need to find its eigenvectors and its eigenvalues. A has the characteristic equation:

$$p(z) = z^2 - 1.4z + .4 = 0 \Rightarrow \lambda_1 = 1 \text{ and } \lambda_2 = \frac{2}{5}$$

The eigenvectors are accordingly, $v_1 = [2, 1]^T$ and $v_2 = [1, -1]^T$
So we have $P = [v_1, v_2]$

Exercise 4.15

Because A is semi-simple, we can diagonalize and thus we denote this form as $A = PDP^{-1}$
Then, $f(A) = f(PDP^{-1}) = a_0I + a_1PDP^{-1} + ... + a_nPD^nP^{-1}$

$$= P[a_0I + a_1D + ... + a_nD^n]P^{-1}$$

$$= Pf(D)P^{-1}$$

Thus, $f(D)$ and $f(A)$ are similar matrices and thus have the same eigen-values. Further, because D is diagonal we can easily find the diagonal elements of $f(D)$ to be $f(D)_ii = a_0 + a_1dii + ... + a_nd_{ii}^n$ and thus the eigenvalues are just its diagonals which are $\{f(\lambda_1), ..., f(\lambda_n)\}$

Exercise 4.16

(i) $A = PDP^{-1}$ then $lim_{n\to\infty}A^n = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.4^n \end{bmatrix}\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$

$$= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

And clearly this is our limit

(ii) Our notions of convergence here will be invariant wrt to choice of metric/norm

(iii) Let $f(x) = 3 + 5x + x^3$ then the eigenvalues of $f(A)$ are just $f(\lambda)$ and we have $f(1) = 9, f(0.4) = 5.064$

Exercise 4.18

Let y be the eigenvector corresponding to the eigenvalue λ. Then we have:

$$Ay = \lambda y \Rightarrow x^TAy = x^T\lambda y \Rightarrow x^TAy = \lambda x^Ty$$

And thus, $x_TA = \lambda x^T$

Exercise 4.20

Note, $A^H = A$, then we have $B = U^HAU$ then $B^H = U^HA^HU = U^HAU = B$

Exercise 4.24
Note, that the denominator is just normalizing the vector x so we can just discard that and restrict to vectors of unit length. Thus, we just need to show that $\langle x, Ax\rangle \in \mathbb{R}$
$\langle x, Ax\rangle = \langle A^H x, x\rangle = -\langle Ax, x\rangle$
Then, $\langle x, Ax\rangle = \langle Ax, x\rangle$ and thus we have $\langle Ax, x\rangle = \langle A\overline{x}, x\rangle \in \mathbb{R}$ ■

Exercise 4.25

(i) Because $A \in M_n(\mathbb{C})$ is normal its eigenvectors span the ambient space.
Further, for all j we have that:

$$(x_1 x_a^H + ... + x_n x_n^h)x_j = x_1 x_a^H x_j + ... + x_n x_n^h x_j = x_j$$

And because the eigenvectors span the space for any $v \in \mathbb{C}^n$ we can write it as $v = \sum a_i x_i$
Then, let $B = x_1 x_1^H + ... + x_n x_n^H$
Thus, $Bv = \sum a_i B x_i = \sum a_i x_i = v$
So B behaves exactly like the identity, I, and we have our result.

(ii) Because A is normal we can diagonalize it wrt its eigenvectors and eigenvalues and thus we have $A = P\Lambda P^H$ where P has columns of eigenvectors and Λ is the diagonal matrix of eigenvalues. Basic manipulation of the linear algebra yields $A = \sum \lambda_i x_i x_i^H$

Exercise 4.27
Let $[a_{i,j}] = A$. Then, WLOG, define $x_1 = [1, 0, ..., 0]^T$. Then, $x_1^H A x_1 = a_{1,1}$ and we can similarly isolate any diagonal element of A. Given that this quantity must be positive for all x, then we see that the diagonals must be positive

Exercise 4.31

(i) S'pose A has rank r. Then $A^H A$ is pos def and has r distinct eigenvalues.
Let $s = \{v_1, ..., v_n\}$ be the orthonormal eigenvectors with corresponding eigenvalues $\{\sigma_1^2, ..., \sigma_n^2\}$ which are sorted decreasing in magnitude.
Since s spans $\mathbb{F}^n$ for arbitrary x we can write $x = \sum c_i v_i$
Thus, $||x||^2 = (\sum c_i v_i^H)(\sum c_i v_i) = \sum c_i^2$
It follows that $||Ax||^2 = (Ax)^H Ax = x^H A^H Ax = (\sum x_i v_i^H)(A^H A)(\sum c_i v_i)$

$$= (\sum c_i v_i^H)(\sum c_i \sigma_i^2 v_i^H) = \sum c_i \sigma_i^2$$

And by our ordering therefore we have $||Ax||^2 = \sigma_1^2$ and we have our result

(ii) The matrix A has SVD $A = U\Sigma V^H$ and thus $A^{-1} = (U\Sigma V^H)^{-1} = (V^H)^{-1}\Sigma^{-1}U^{-1}$
Then Σ^{-1} is another SVD with diagonals $\frac{1}{\sigma_i}$ and by result (i) we have that $||A|| = \frac{1}{\sigma_n}$

(iii) Let $A = U\Sigma V^H$ then $A^H = (U\Sigma V^H)^H = V\Sigma^H U = V\Sigma U$. Further, A^T is just a special case of A^H. Finally, the other results follow by applying the arguments of (i).

I had family in town this weekend and ran out of time to do these last few problems. My apologies.