Math 1 - Econ Bootcamp

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Exercise 1.3

- Let $A_1 := \{A : A \subset \mathbb{R}, open\}$. Then A^c is closed so $A \not\subset A_1$ and thus A_1 is not an algebra
- Let $A_2 := \{A : A$ is a finite union of the form $(a, b], (-\infty, b]$, and $(a, \infty)\}$.
  First, note that $(a, a]$ is the empty set and the real line is contained in $(-\infty, b] \cup (b, \infty)$
  If $A = (a, b]$ then $A^c = (-\infty, a] \cup (b, \infty) \in A_2$
  If $A = (-\infty, b]$ then $A^c = (b, \infty) \in A_2$
  If $A = (a, \infty)$ then $A^c = (-\infty, a] \in A_2$
  So any 'primitive' set type is closed under complement. Further, because the left end-point is always either negative infinity or open and the right end-point is always either infinity or closed any possible union of 'primitive' types does not change the nature of the end points and thus cannot result in a new set interval type. Thus, any finite union remains in A_2 and thus A_2 is an algebra.
- Let $A_3 := \{A : A$ is a countable union of the form $(a, b], (-\infty, b]$, and $(a, \infty)\}$.
  Then by the last example A_2 it is clearly an algebra and further, because an infinite union of these elements is still in A_3 it is also closed under countable infinite unions and thus it is a σ-algebra.

Exercise 1.7

An algebra must contain the empty set. Further, it must be closed under complements so X must be in the algebra. Further, the empty set and the union of X is just X so it is closed under unions, and we have the smallest possible algebra.
Further, if A is an algebra then it contains subsets of X, and $P(X)$ is by definition the largest set of subsets of X and thus $A \subset P(X)$

Exercise 1.10

Each S_α is a σ-algebra so $\emptyset \in S_\alpha \forall \alpha$. Thus, $\emptyset \in \bigcap_\alpha S_\alpha$
Further, let $A_i \in \bigcap_\alpha S_\alpha$ so $A_i \in S_\alpha \forall \alpha$. Then because each S_α is a σ-algebra $A_i^c \in S_\alpha \forall \alpha$. Thus, $A_i^c \in \bigcap_\alpha S_\alpha$ and it is closed under complements.
Finally, let $A_1, A_2, ... \in \bigcap_\alpha S_\alpha$ then $A_1, A_2, ... \in S_\alpha \forall \alpha$ and because each of these is a σ-algebra $\bigcup_\alpha^\infty A_i \in S_\alpha \forall \alpha$. Thus, σ-algebra $\bigcup_\alpha^\infty A_i \in \bigcap_\alpha S_\alpha$ and it is closed under a countable union, and is thus a sigma algebra.

Exercise 1.17

- Because $A \subset B$, define $C = (A^c \cap B)$.
  Then, $C \cap B = \emptyset$ and $A \cup C = B$.
  And because μ is a measure, $\mu(B) = \mu(A) + \mu(C)$ and $\mu(C) \geq 0$ so $\mu(B) \geq \mu(A)$.
- Because μ is a measure, if $A_i \cap A_j = \emptyset \ \forall i \neq j$, then we have pairwise disjoints set and we have equality and we are done.
  Therefore, assume $\exists \ i \neq j$ s.t. $A_i \cap A_j = C \neq \emptyset$. So $\mu(C) \geq 0$.
  Thus, $\mu(A_i) + \mu(A_j) = \mu(A_i \cup A_j) + \mu(C)$ and we are done.

Exercise 1.18
First, $\lambda(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$.
Second, let $\{A_i\}_{i=1}^{\infty}$ be pairwise disjoint collection of sets. Then $(A_i \cap B) \cap (A_j \cap B) = \emptyset \; \forall i \neq j$.
So,

$$\begin{aligned}
\lambda(\cup^{\infty} A_i) &= \mu((\cup^{\infty} A_i) \cap B) \\
&= \mu(\cup^{\infty}(A_i \cap B)) \\
&= \sum(\mu(A_i \cap B)) \\
&= \sum(\mu(A_i \cap B)) \\
&= \sum(\lambda(A_i))
\end{aligned}$$

Exercise 2.10
μ^* is an outer measure so it is countably subadditive and $(B \cap E)$ and $(B \cap E^c)$ are disjoint sets. Thus, $\mu^* \leq \mu^*(B \cap E) + \mu^*(B \cap E^c)$ and thus we must have equality.

Exercise 2.14 We can invoke Caratheodory once we show that $B(R) := \sigma(O) = \sigma(A)$, and then we have our result.

Exercise 3.1
Proof by contradiction. Let $a_i \in \mathbb{R}$, assume $\mu(a_i) > 0$, then let $(a, b) \subset \mathbb{R}$ such that $a_i \in (a, b)$
Then $(a, b) = (a, a_i) \cup \{a_i\} \cup (a_i, b)$
But $\mu((a, a_i) \cup (a_i, b)) = (a_i - a) + (b - a_i) = b - a = \mu((a, b))$
And $\mu(a_i) > 0$ so it must be that

$$\mu((a, a_i) \cup \{a_i\} \cup (a_i, b)) > \mu((a, a_i)) + \mu(a_i) + \mu((a_i, b))$$

Thus, $\mu(a_i) = 0 \; \forall$ single points in $\mathbb{R}$.
By extension $\mu(\cup a_i) = 0$

Exercise 3.4
$\mathcal{M}$ must be closed under complements so $f^{-1}((-\infty, a)) \in \mathcal{M} \iff f^{-1}([a, \infty)) \in \mathcal{M}$
Further, we know $\mu(\{a\}) = 0$ so we can freely add/remove this single point as we see fit without altering the meaning of the statement.

Exercise 3.7
We can define functions like:

$$\begin{aligned}
F(f(x), g(x)) &:= f(x) + g(x) \\
F(f(x), g(x)) &:= f(x)g(x) \\
F(f(x), g(x)) &:= max(f(x), g(x)) \\
F(f(x), g(x)) &:= min(f(x), g(x))
\end{aligned}$$

And we can use sup and inf result from 2. for $|f|$.

Exercise 3.14 If f is bounded $\exists B \in \mathbb{R}$ s.t. $f(x) \leq B \; \forall x \in X$
Further, $\{s_n\}$ each function is simple and thus has a finite range.
So, given $\epsilon > 0$ we know $s_n \to f$ pointwise, and we have a finite range so just take the max for uniform convergence.

Exercise 4.13
$f \in \mathcal{L}^1(E, \mu) \iff \int_E ||f|| d\mu < \infty \Rightarrow \int_E f^+ - \int_E f^-$
And also, $\mu(E) < \infty \Rightarrow \int_E f d\mu < \infty \Rightarrow \int_E ||f|| d\mu < \infty$

Exercise 4.14

Because $f \in \mathcal{L}^1(E, \mu)$ it is measureable. Then, define $E_n = \{x \in E, f(x) \geq n\}$ and thus $\int_E f d\mu < \int_E n d\mu = n\mu(E_n) < \infty$

Exercise 4.15

First, $f, g \in \mathcal{L}^1$ thus it is measureable. Then, by Prop 4.7, $\int_E f d\mu \leq \int_E g$

Exercise 4.16

Because $A \subset E$ and $A \in \mathcal{M}$ and f is measureable we have $\int_A f = \int_A f^+ - \int_A f^-$
Also, because f is absolutely integrable the integral of the subset is also finite so $||f|| = f^+ + f^- \Rightarrow f \in \mathcal{L}^1$

Exercise 4.21

$A = (A - B) \cup B \Rightarrow \mu(A) = \mu(A - B) + \mu(B) \Rightarrow \mu(A) - \mu(B) = 0 \Rightarrow \int_A f = \mu(A) - \mu(B) = \int_B f \Rightarrow \int_A f - \mu(A) = \int_B f + \mu(B)$

$$\Rightarrow \int_A f \leq \int_B f$$