TD decoherence :

(I)

1) $S_i = \frac{\hbar}{2}\sigma_i$, σ_i Pauli matrices .

1

$[S_i, S_j] = i\hbar\, \varepsilon_{ijk} S_k$.

2) $|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$:

$$\cancel{\rho = |\psi\rangle\langle\psi| = \begin{pmatrix}\alpha \\ \beta\end{pmatrix}(\alpha^* \ \beta^*) = \begin{pmatrix}\alpha\alpha^* & \alpha\beta^* \\ \alpha^*\beta & \beta\beta^*\end{pmatrix}}$$

~~or in kets :~~ $\cancel{\rho = \alpha\alpha^* |\uparrow\rangle\langle\uparrow| + \alpha\beta^* |}$

$$\rho = |\psi\rangle\langle\psi| = \alpha\alpha^* |\uparrow\rangle\langle\uparrow| + \alpha\beta^* |\uparrow\rangle\langle\downarrow| + \beta\alpha^* |\downarrow\rangle\langle\uparrow| + \beta\beta^* |\downarrow\rangle\langle\downarrow| .$$

$$= \begin{pmatrix}\alpha\alpha^* & \alpha\beta^* \\ \alpha^*\beta & \beta\beta^*\end{pmatrix}$$ in the $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis .

3) $\rho^\dagger = \rho$ so :

$$\rho = \begin{pmatrix} a & b-ic \\ b+ic & d\end{pmatrix}$$ a, b, c, d real

$$= \frac{a+d}{2}\mathbb{1} + \frac{a-d}{2}\sigma_z + b\sigma_x + c\,\sigma_z .$$

$\mathrm{Tr}(\rho) = 1$: $a + d = 1$. $\Rightarrow \rho = \frac{1}{2}(\mathbb{1} + \vec{n}\cdot\vec{\sigma})$

Pure state : $\cancel{\mathrm{Tr}(\rho^2) = 1 \ \ \rho^2 =}$ $\rho^2 = \rho$: so

$$\rho^2 = \frac{1}{4}(\mathbb{1} + \vec{n}\cdot\vec{\sigma})^2 = \frac{1}{4}(1 + n_x^2 + n_y^2 + n_z^2)\mathbb{1} + \dots \sigma_x + \dots$$

$\rho^2 = \rho \Rightarrow \frac{1}{4}\left(1 + \|\vec{n}\|^2\right) = \frac{1}{2}$

$\hookrightarrow \|\vec{n}\|^2 = 1$.

4) $\hat{O} = \frac{\hbar}{2}\hat{\sigma}_y$.

⚠ Here $\rho = \frac{1}{2}\begin{pmatrix} 1 & e^{-i\phi} \\ e^{i\phi} & 1 \end{pmatrix}$

$\langle O \rangle = \mathrm{Tr}(\rho O)$

$\rho O = \frac{\hbar}{4}\left(\mathbb{1} + \vec{n}\cdot\vec{\sigma}\right)\sigma_y$

2

Note: Pauli matrices are traceless:

$\rho O = \frac{\hbar}{4} n_y \mathbb{1} + \dots \sigma_x + \dots$

So: $\langle O \rangle = \mathrm{Tr}(\rho O) = \frac{\hbar}{2} n_y = \frac{\hbar}{2}\sin\phi$.

5) $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)$

Magnetic moment of the atom: $\mu_z = \gamma S_z$

gyromagnetic ratio.

Energy:

$H = \mu_z B_z = \gamma B_z S_z$.

So: $H|\uparrow\rangle = \gamma B_z \frac{\hbar}{2}|\uparrow\rangle$

$H|\downarrow\rangle = -\gamma B_z \frac{\hbar}{2}|\downarrow\rangle$.

So: $|\Psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\gamma B_z t/2}|\uparrow\rangle + e^{i\gamma B_z t/2}|\downarrow\rangle\right)$.

Hence:

$\rho = |\Psi(t)\rangle\langle\Psi(t)| = \frac{1}{2}\begin{pmatrix} 1 & e^{-i\beta} \\ e^{i\beta} & 1 \end{pmatrix}$, $\beta = \gamma B_z t$

Random phase Φ_R : fluctuating magnetic field along the Silver beam.

6) Φ_R uniform in $[0, 2\pi[$, so $\phi' = \phi + \Phi_R$ uniform in $[0, 2\pi[$.

So ρ in a stat mixture of all phases ϕ' w/equal weights:

$$\hat{\rho}_R = \int_0^{2\pi} \frac{d\phi'}{2\pi} \hat{\rho}(\phi')$$

$$= \frac{1}{2} \begin{pmatrix} \int_0^{2\pi} \frac{d\phi'}{2\pi} & \int_0^{2\pi} \frac{d\phi'}{2\pi} e^{-i\phi'} \\ \int_0^{2\pi} \frac{d\phi'}{2\pi} e^{i\phi'} & \int_0^{2\pi} \frac{d\phi'}{2\pi} \end{pmatrix}$$

3

$$= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\boxed{\rho_R = \frac{1}{2} \mathbb{1}}$. $\Rightarrow$ Decay of the coherences (off-diag elts)

So, $\langle O \rangle = \mathrm{Tr}(\rho_R O)$

$\boxed{\langle O \rangle = 0}$. $\Rightarrow$ Washout of the interferences

TD death.

TD Wigner.

II

1) $\mathcal{H} = \mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2}$

↳ basis : $\mathcal{B} = \{ |\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle \}$

$$|\psi\rangle = \alpha |\uparrow\uparrow\rangle + \beta |\uparrow\downarrow\rangle + \gamma |\downarrow\uparrow\rangle + \delta |\downarrow\downarrow\rangle.$$

We note : $|\psi\rangle = \sum_{a,b} \psi_{ab} \; |a\, b\rangle \quad , a, b \in \uparrow, \downarrow$

$$|\psi\rangle = \begin{pmatrix} \psi_{\uparrow\uparrow} \\ \psi_{\uparrow\downarrow} \\ \psi_{\downarrow\uparrow} \\ \psi_{\downarrow\downarrow} \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \text{ in basis } \mathcal{B}$$

4

$$\rho = |\psi\rangle\langle\psi| = \sum_{a_1 b_1, a_2 b_2} \psi_{a_1 b_1} \psi^*_{a_2 b_2} \; |a_1 b_1\rangle\langle a_2 b_2|$$

In basis $\mathcal{B}$:

$$\rho = \begin{pmatrix} \psi_{11}\psi_{11}^* & \psi_{11}\psi_{12}^* & \cdots \\ \psi_{12}\psi_{11}^* & \psi_{12}\psi_{12}^* & \\ \vdots & & \end{pmatrix}$$

2) $\hat{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad , \quad \hat{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

$$\hat{A} \otimes \hat{B} = \begin{pmatrix} a_{11}\hat{B} & a_{12}\hat{B} \\ a_{21}\hat{B} & a_{22}\hat{B} \end{pmatrix} = \left(\begin{array}{cc|cc} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & \cdots \\ a_{11}b_{21} & a_{11}b_{22} & \vdots & \\ \hline a_{21}b_{11} & & a_{22}b_{11} & \cdots \\ & \ddots & \vdots & \ddots \end{array}\right)$$

3) $\hat{O} = \hat{A} \otimes \hat{B}$ Only acting on subsyst. A $\therefore$ $\hat{B} = \mathbb{1}$

$\hat{O} = \hat{A} \otimes \hat{\mathbb{1}}$

$$\hookrightarrow \hat{O} = \left(\begin{array}{cc|cc} a_{11} & 0 & a_{12} & 0 \\ 0 & a_{11} & 0 & a_{12} \\ \hline a_{21} & 0 & a_{22} & 0 \\ 0 & a_{21} & 0 & a_{22} \end{array}\right).$$

5

$$\langle \hat{O} \rangle = \mathrm{Tr}(\hat{\rho}\hat{O})$$

4) We start with:

$$\mathrm{Tr}(\hat{\rho}\hat{O}) = \sum_{a_1 b_1} \langle a_1 b_1 | \hat{\rho}\hat{O} | a_1 b_1 \rangle$$

$$= \sum_{\substack{a_1 b_1 \\ a_2 b_2}} \langle a_1 b_1 | \hat{\rho} | a_2 b_2 \rangle \langle a_2 b_2 | \hat{O} | a_1 b_1 \rangle$$

$$= \sum_{\substack{a_1 b_1 \\ a_2 b_2}} \psi_{a_1 b_1} \psi^*_{a_2 b_2} \langle a_2 | A | a_1 \rangle \, \delta_{b_1 b_2}$$

$$= \sum_{\substack{a_1 b_1 \\ a_2}} \psi_{a_1 b_1} \psi^*_{a_2 b_1} \langle a_2 | A | a_1 \rangle .$$

And compare with :

$$\mathrm{Tr}(\rho_A A) = \sum_{a_3} \langle a_3 | \rho_A A | a_3 \rangle$$

$$= \sum_{a_3 a_4} \langle a_3 | \rho_A | a_4 \rangle \langle a_4 | A | a_3 \rangle .$$

$$= \sum_{\substack{a_1 b_1\\ a_2 b_2\\ a_3 a_4}} \psi_{a_1 b_1} \psi^*_{a_2 b_2} \underbrace{\langle a_3 | a_1 \rangle\langle a_2 | a_4 \rangle}_{\delta_{a_3 a_1}\, \delta_{a_2 a_4}} \langle a_4 | A | a_3 \rangle \underbrace{\mathrm{Tr}(|b_1\rangle\langle b_2|)}_{\delta_{b_1 b_2}.}$$

$$= \sum_{\substack{a_1 b_1\\ a_2}} \psi_{a_1 b_1} \psi^*_{a_2 b_1} \langle a_2 | A | a_1 \rangle .$$

6

$$= \mathrm{Tr}(\rho \hat{O}) = \langle \hat{O} \rangle .$$

TD Wigner

TD decoh.

5) $$\rho_A = \frac{1}{2}\left(|\uparrow\rangle + e^{i\phi}|\downarrow\rangle\right)\left(\langle\uparrow| + e^{-i\phi}\langle\downarrow|\right)$$

$$= \frac{1}{2}\Big(|\uparrow\rangle\langle\uparrow| + e^{-i\phi} |\uparrow\rangle\langle\downarrow| + e^{i\phi}|\downarrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\Big)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & e^{-i\phi} \\ e^{i\phi} & 1 \end{pmatrix}$$

Same as Stern-Gerlach exp!

$$\rho_A' = U \rho U^\dagger$$

$$= \cancel{\frac{1}{4}\begin{pmatrix} 1 & e^{-i\phi} \\ e^{i\phi} & 1 \end{pmatrix}} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & e^{-i\phi} \\ e^{i\phi} & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1+e^{-i\phi} & 1-e^{-i\phi} \\ 1+e^{i\phi} & e^{i\phi}-1 \end{pmatrix}$$

$$\rho_A' = \frac{1}{2}\begin{pmatrix} 1+\frac{1}{2}(e^{i\phi}+e^{-i\phi}) & \frac{1}{2}(e^{i\phi}-e^{-i\phi}) \\ \frac{1}{2}(e^{-i\phi}-e^{i\phi}) & 1 - \frac{1}{2}(e^{-i\phi}+e^{i\phi}) \end{pmatrix}$$

7

$$= \frac{1}{2}\begin{pmatrix} 1+\cos\phi & i\sin\phi \\ -i\sin\phi & 1-\cos\phi \end{pmatrix}.$$

$$P_\uparrow = \frac{1}{2}(1+\cos\phi)$$ interference pattern : we can measure the phase ϕ.

6) Now the state of the system is entangled :

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right).$$

So : before the gate .

$$\rho = \frac{1}{2}\left(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\uparrow\uparrow\rangle\langle\downarrow\downarrow| + |\downarrow\downarrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|\right)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

$$\rho_A = \mathrm{Tr}_B(\rho)$$

$$= \frac{1}{2}\begin{pmatrix} \mathrm{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \mathrm{Tr}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \mathrm{Tr}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \mathrm{Tr}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

⚠ We could use $\rho_A = \sum_{b_1 b_2} \langle b_1 b_2 | \rho | b_1 b_2 \rangle$. and play with the kets : easier if you're not sure

$$\rho_A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\mathbb{1}.$$

Off-diag elts disappeared : phase info is lost !

8

$$\rho_A' = U \rho_A U^+ = \rho_A .$$

$\hookrightarrow$ $P_\uparrow = \frac{1}{2}$ Interf washed out ! .

III

1) $\partial_t \rho = \partial_t \left(|\psi\rangle\langle\psi| \right)$

$$= \left(\partial_t |\psi\rangle\right) \langle\psi| + |\psi\rangle \left(\partial_t \langle\psi|\right)$$

Schrö. $\partial_t |\psi\rangle = \cancel{\frac{1}{i\hbar}\frac{\hat{p}^2}{2m}|\psi\rangle} = \frac{1}{i\hbar 2m} p^2 |\psi\rangle$

~~w/~~ $\cancel{\hat{p} = -i\hbar\partial_x}$ $\cancel{\partial_t \langle\psi| = \langle\psi| \frac{1}{i\hbar}\frac{\hat{p}^2}{2m}}$

$$\partial_t \langle\psi| = -\frac{1}{i\hbar 2m} \langle\psi| \hat{p}^2$$

$\hookrightarrow$ $\partial_t \rho = \frac{1}{2i\hbar m} \hat{p}^2 |\psi\rangle\langle\psi| - \frac{1}{2i\hbar m} |\psi\rangle\langle\psi| p^2$

$$= \frac{1}{2i\hbar m}\left(\hat{p}^2 |\psi\rangle\langle\psi| - |\psi\rangle\langle\psi| p^2 \right)$$

$$\partial_t \rho(x,x') = \langle x | \partial_t \rho | x' \rangle$$

$$= \frac{1}{2i\hbar m}\Big(\langle x | p^2 | \psi \rangle\langle \psi | x' \rangle - \langle x | \psi \rangle\langle \psi | p^2 | x' \rangle \Big)$$

9

$$= \frac{1}{2i\hbar m}\Big((-i\hbar\partial_x)^2 \langle x | \psi \rangle\langle \psi | x' \rangle - (i\hbar\partial_{x'})^2 \langle x | \psi \rangle\langle \psi | x' \rangle \Big)$$

$$= \frac{1}{2i\hbar m}\left(-\hbar^2 \partial_x^2 + \hbar^2 \partial_{x'}^2 \right) \rho(x,x')$$

$$\boxed{\partial_t \rho(x,x') = \frac{i\hbar}{2m}\left(\partial_x^2 - \partial_y^2 \right) \rho(x,x').}$$

2) Conservat° of energy and ~~impuls~~ momentum for elastic collis°:

$$\begin{cases} \dfrac{\tilde{p}_s^2}{2m_s} + \dfrac{\tilde{p}_e^2}{2m_e} = \dfrac{p_e^2}{2m_e} + \dfrac{p_s^2}{2m_s} & (1) \\ \tilde{p}_e + \tilde{p}_s = p_e + p_s & (2) \end{cases}$$

$$\text{So}\quad \frac{\tilde{p}_s^2}{2m_s} + \frac{(p_e + p_s - \tilde{p}_s)^2}{2m_e} - \frac{p_e^2}{2m_e} - \frac{p_s^2}{2m_s} = 0$$

$$\frac{1}{2}\left(\frac{m_e+m_s}{m_s m_e}\right)\tilde{p}_s^2 - \frac{1}{m_e}(p_e+p_s)\tilde{p}_s + \frac{1}{m_e}p_e p_s + \frac{1}{2}\left(\frac{1}{m_e} - \frac{1}{m_s}\right)p_s^2 = 0$$

$$\Delta = \frac{(p_e + p_s)^2}{m_e^2} - 2\,\frac{m_e + m_s}{m_e m_s}\left(\frac{1}{m_e} p_e p_s + \frac{1}{2}\left(\frac{m_s - m_e}{m_e m_s}\right) p_s^2\right)$$

$$= \frac{1}{m_e^2 m_s^2}\Big[m_s^2 p_e^2 + \cancel{2 m_s^2 p_e p_s} + \cancel{m_s^2 p_s^2} - 2 m_e m_s p_e p_s$$

$$\cancel{-2 m_s^2 p_e p_s} - \cancel{m_s m_e p_s^2} - \cancel{m_s^2 p_s^2} + m_e^2 p_s^2$$

$$\cancel{+ m_e m_s p_s^2}\Big]$$

$$\Delta = \frac{(m_s p_e - m_e p_s)^2}{m_e^2 m_s^2}$$

10

$$\tilde{p}_s = \frac{m_s m_e}{m_s + m_e}\left(\frac{1}{m_e}(p_e + p_s) \pm \frac{m_s p_e - m_e p_s}{m_e m_s}\right)$$

$$= \frac{m_s \mp m_e}{m_s + m_e} p_s + \frac{m_s \pm m_s}{m_s + m_e} p_e$$

Solut° ? $\ominus \Rightarrow \tilde{p}_s = p_s$ = pas de collis°.

$\oplus$ ✓

$$\boxed{\tilde{p}_s = \frac{m_s - m_e}{m_s + m_e} p_s + \frac{2 m_s}{m_s + m_e} p_e}$$

$$\tilde{p}_e = p_e + p_s - \tilde{p}_s$$

$$\boxed{\tilde{p}_e = \frac{2 m_e}{m_s + m_e} p_s - \frac{m_s - m_e}{m_s + m_e} p_e}$$

If $m_s \gg m_e$:

$$\begin{cases} \tilde{p}_s \simeq p_s + 2 p_e \\ \tilde{p}_e \simeq 2 \frac{m_e}{m_s} p_s - p_e \end{cases}$$

11

If $\frac{p_e}{m_e} \gg \frac{p_s}{m_s}$:

$$\begin{cases} \tilde{p}_s = p_s + 2 p_e \\ \tilde{p}_e = - p_e \end{cases}$$

3) After the collis°, plane wave decomposit° of the wavefunct° :

$$\tilde{\Psi}(x_e, x_s) = \int \frac{d\tilde{p}_e \, d\tilde{p}_s}{2\pi\hbar} \tilde{\varphi}(\tilde{p}_e, \tilde{p}_s) \, e^{i(\tilde{p}_e x_e + \tilde{p}_s x_s)/\hbar}$$

$\cancel{p_e, p_s}$ $(p_e, p_s) \rightarrow (\tilde{p}_e, \tilde{p}_s)$ one-to-one :

$$\tilde{\varphi}(\tilde{p}_e, \tilde{p}_s) = \varphi(p_e, p_s)$$

i.e. proba to be in $\tilde{p}_e, \tilde{p}_s$ after = proba to be in p_e, p_s before.

$$\begin{pmatrix} \tilde{p}_s \\ \tilde{p}_e \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}}_{= J} \begin{pmatrix} p_s \\ p_e \end{pmatrix}$$

$$\det J = -1$$

$$\tilde{\Psi}(x_e, x_s) = \int |\det J| \frac{dp_e \, dp_s}{2\pi\hbar} \varphi(p_e, p_s) \, e^{+i(-p_e x_e + (p_s + 2p_e) x_s)}$$

$$= \int \frac{dp_e \, dp_s}{2\pi\hbar} \varphi(p_e, p_s) \, e^{i(p_e \tilde{x}_e , p_s \tilde{x}_s)}$$

with $\begin{cases} \tilde{x}_e = 2x_s - x_e \\ \tilde{x}_s = x_s \end{cases}$

(12)

Colliso $\equiv$ $(x_e, x_s) \to (\tilde{x}_e, \tilde{x}_s)$.

4) $$\tilde{\rho}(x_e, x_s, x_e', x_s') = \rho(\tilde{x}_e, \tilde{x}_s, \tilde{x}_e', \tilde{x}_s').$$

$$= \left(\langle \tilde{x}_s | \otimes \langle \tilde{x}_e |\right) \hat{\rho} \left(|\tilde{x}_s'\rangle \otimes |\tilde{x}_e'\rangle \right)$$

$$= \langle \tilde{x}_s | \hat{\rho}_s | \tilde{x}_s' \rangle \langle \tilde{x}_e | \hat{\rho}_e | \tilde{x}_e' \rangle.$$

$$= \rho_s(\tilde{x}_s, \tilde{x}_s') \, \rho_e(\tilde{x}_e, \tilde{x}_e')$$

$$= \rho_s(x_s, x_s') \, \rho_e(2x_s - x_e, 2x_s' - x_e')$$

$\cancel{\tilde{\rho}_s(x_s, x_s')} =$

$$\tilde{\rho}_s(x_s, x_s') = \mathrm{Tr}_e\left(\tilde{\rho}(x_s, x_e, x_s', x_e')\right)$$

$$= \rho_s(x_s, x_s') \; \mathrm{Tr}_e\left(\rho_e(2x_s - x_e, 2x_s' - x_e')\right)$$

$$= \Lambda \rho_s(x_s, x_s')$$

$$\Lambda = \mathrm{Tr}_e\left(\rho_e(2x_s - x_e, 2x_s' - x_e')\right)$$

5) $\Lambda = \mathrm{Tr}_e \left(\rho_e \left(2x_s - x_e , 2x_s' - x_e' \right) \right)$

$$= \int_{-\infty}^{+\infty} dx_e \, \rho_e \left(2x_s - x_e , 2x_s' - x_e \right)$$

13

$\xi = 2x_s' - x_e$:

$$\Lambda = \int d\xi \, \rho_e \left(\xi + 2(x_s - x_s') , \xi \right)$$

DL in powers of $2(x_s - x_s')$:

$$\Lambda \simeq \int d\xi \, \rho_e(\xi, \xi) + 2(x_s - x_s') \int d\xi \, \frac{\partial}{\partial X} \rho_e(X, \xi) \Big|_{X=\xi}$$

$$+ 2 (x_s - x_s')^2 \int d\xi \, \frac{\partial^2}{\partial X^2} \rho_e (X, \xi) \Big|_{X = \xi} .$$

Let's Compute the terms :

$$\int d\xi \, \rho_e(\xi, \xi) = \mathrm{Tr}_e (\hat{\rho}_e) = 1 .$$

Use the def of momentum operator:

$$\int d\xi \, \partial_X \, \rho_e (X, \xi) \Big|_{X=\xi} = \int d\xi \left(-\frac{\hat{p}}{i\hbar} \hat{\rho}_e \right) (\xi, \xi)$$

$$= - \frac{1}{i\hbar} \mathrm{Tr} \left(\hat{\rho}_e \, \hat{p} \right)$$

$$= - \frac{1}{i\hbar} \langle \hat{p} \rangle_e$$

$$= 0$$

Random velocities for env. part : at least symm $p \to -p$.

$$\int d\xi \, \frac{\partial^2}{\partial x^2} \rho(x,\xi)\Big|_{x=\xi} = \int d\xi \left[\left(-\frac{\hat{p}}{i\hbar}\right)^2 \hat{\rho}\right](\xi,\xi)$$

$$= -\frac{1}{\hbar^2} \operatorname{Tr}(\hat{p}^2 \hat{\rho})$$

$$= \cancel{\frac{1}{\hbar}} - \frac{\langle \hat{p}^2 \rangle}{\hbar^2} \qquad 14$$

$$\hookrightarrow \boxed{\Lambda = 1 - \frac{2(x_s - x_s')^2}{\hbar^2} \langle \hat{p}^2 \rangle_e} .$$

6) Env = ideal 1D gas : $\frac{\langle p_e^2 \rangle}{2 m_e} = \frac{1}{2} k_B T$.

$$\hookrightarrow \Lambda = 1 - \frac{2 m_e k_B T}{\hbar^2} (x_s - x_s')^2$$

After 1 collision:

$$\tilde{\rho}_s(x_s, x_s') = \rho_s(x_s, x_s') - \frac{2 m_e k_B T}{\hbar^2} (x_s - x_s')^2 \rho_s(x_s, x_s')$$

Time evol w/ Markovian collision at rate Γ :

$$\rho_s(x_s, x_s', t+\Delta t) = \rho(x_s, x_s', t) - \underbrace{\Gamma \Delta t}_{\text{n° of coll in } \Delta t} \frac{2 m_e k_B T}{\hbar^2} (x_s - x_s')^2 \rho_s(x_s, x_s', t)$$

So:

$$\boxed{\frac{\partial \rho_s}{\partial t} = -\frac{2 \Gamma m_e k_B T}{\hbar^2}} (x_s - x_s')^2 \rho_s$$

7) So total evolut' eq for ρ :

$$\frac{\partial \rho}{\partial t} = \frac{i\hbar}{2m}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2}\right)\rho(x,x') - \frac{1}{\gamma(x,x')}\rho(x,x')$$

where : $\gamma(x,x') = \frac{2\Gamma \, m_e k_B T}{\hbar^2}(x_S - x_S')^2$

↳ exponential decay at rate $\gamma \sim (x-x')^2$

→ decay of off-diag coeffs!