Quantum Optics – M1

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October 9th, 2024

TD5: Introduction to decoherence: the quantum Brownian motion

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1 Introduction: Phase randomization in a Stern-Gerlach experiment

We consider spin-$\frac{1}{2}$ independant particles, e.g. silver atoms coming out of a furnace, with spin operator $\hat{\mathbf{S}}$.

1. Recall the expression of the components $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$ and their commutation relations.
2. Write the expression of the density matrix $\hat{\rho}$ of a spin-$\frac{1}{2}$ in terms of the coefficients of its decomposition in the z basis.
3. Show that $\hat{\rho}$ can be cast in the form $\hat{\rho} = \frac{1}{2}(\mathbb{1} + \mathbf{n} \cdot \hat{\boldsymbol{\sigma}})$ where $\|n\| = 1$. Express $\mathbf{n}$ in spherical coordinates.

The silver furnace is connected to a polarizing device that imposes to the outgoing atoms the spin orientation :

$$\left\{ \begin{array}{rcl} n_x & = & \cos(\phi) \\ n_y & = & \sin(\phi) \\ n_z & = & 0 \end{array} \right. \tag{1}$$

We place a Stern-Gerlach analyzer along the path of the silver beam to measure the projection of the spin along the y direction.

4. Write the operator $\hat{\mathcal{O}}$ of the quantity measured by this device. Compute its expectation value as a function of ϕ.

We now model very schematically the action of the environment by the addition of a random phase ϕ_R to the phase ϕ of the spin. The phase ϕ_R is evenly distributed in the interval

5. Compute the evolution of the ket $|\psi(t)\rangle$ in a uniform magnetic field along z starting from the state $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$. What does the random phase ϕ_R mimic?
6. Compute the new density matrix, taking into account the statistical mixture introduced by the random phase addition. Deduce the expectation value of the observable $\hat{\mathcal{O}}$.

We can see that the effect of the random phase shift introduced by the environment is the destruction of the non-diagonal elements of the density matrix, as well as the loss of the interference pattern, a purely quantum effect.

2 Partial tracing of the environment as a source of decoherence

In the previous example, we have very crudely modeled the environment by a classical source of random phase shift. The washout of the interference pattern is actually due to the fact that we haven't kept track of the phase ϕ_R introduced to each atom. We shall propose a refined model in this section, where the environment is treated as a quantum system, and decoherence is understood as the result of a partial knowledge of the total system.

2.1 Formalism of the partial trace

In order to explore the simplest possible example, we consider that our system is made of a single spin-$\frac{1}{2}$, and the environment is described by another single spin-$\frac{1}{2}$.

1. Write the form of the most general state of the universe made of the two two-level-systems in matrix form. Write its density matrix in terms of its coefficients, again in matrix form.
2. Given two operators $\hat{A}$ and $\hat{B}$, each acting of the Hilbert space of one of the spin-$\frac{1}{2}$'s, give the matrix representation of the tensor product operator $\hat{A} \otimes \hat{B}$.
3. Let $\hat{O}$ be a composite operator $\hat{O} = \hat{A} \otimes \hat{B}$ only acting on the first subsystem. How do you translate this property mathematically? Give the matrix expression for $\hat{O}$.

We introduce the reduced density matrix by performing a partial trace on subsystem B:

$$\hat{\rho}_A = \mathrm{Tr}_B\left(\hat{\rho}\right) = \sum_{b_1 b_2} \langle b_1 b_2 | \hat{\rho} | b_1 b_2 \rangle = \sum_{a_1 a_2 b_1 b_2} \psi_{a_1 b_1} \psi^*_{a_2 b_2} \left|a_1\right\rangle\!\left\langle a_2\right| \mathrm{Tr}(\left|b_1\right\rangle\!\left\langle b_2\right|) \tag{2}$$

4. Show that the expectation value of $\hat{O}$ is given by:

$$\left\langle \hat{O} \right\rangle = \mathrm{Tr}\left(\hat{\rho}\hat{O}\right) = \mathrm{Tr}\left(\hat{\rho}_A \hat{A}\right) \tag{3}$$

2.2 Application to an interferometric experiment

Our spin-$\frac{1}{2}$ (subsystem A) is used in an interferometric experiment. It is initially prepared in the state $\left|\psi_i\right\rangle = \frac{1}{\sqrt{2}}(\left|\uparrow\right\rangle + \left|\downarrow\right\rangle)$. After a given time evolution, the energy difference between the levels $\left|\uparrow\right\rangle$ and $\left|\downarrow\right\rangle$ introduce a phase shift ϕ: subsystem A is then in the state $\frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle + e^{i\phi}\left|\downarrow\right\rangle\right)$. We perform the unitary transformation $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ (also known as Hadamard gate) to subsystem A and measure the probability to be in $\left|\uparrow\right\rangle$.

5. Suppose the two subsystems did not interact. Compute the reduced density matrix of subsystem A before the application of the Hadamard gate. How is it transformed by the gate? Calculate the measured probability $P_\uparrow$.
6. Now suppose the two subsystems interact in a very simple way before the application of the gate: subsystem B aligns its spin projection with that of subsystem A. Tracing out the subsystem B, give the reduced density matrix of subsystem A before the gate. What happens to the probability of being in $\left|\uparrow\right\rangle$ after the gate?

3 Continuous decoherence: the quantum Brownian motion

We consider a one-dimensional free quantum mechanical system of mass m_s. We will show that coupling this system to a thermal bath of colliding particles leads to decoherence in the sense discovered above, i.e. a decay of the off-diagonal elements of the density matrix.

1. Give the differential equation governing the matrix elements $\rho_s(x, x') = \langle x|\hat{\rho}|x'\rangle$ of the density matrix in position representation in the absence of decoherence.

2. We consider a single collision between our system with initial impulsion p_s and a particle from the environment of mass m_e with initial impulsion p_e. In the case of an ideal elastic collision, give the impulsions $\tilde{p}_s$ and $\tilde{p}_e$ after the collision. Simplify these expressions assuming that the system is very massive and is moving much slower compared to the environment particle.

3. Writing down the plane wave decomposition of the wavefunction of the two particles in position representation $\tilde{\psi}(x_s, x_e)$ right after the collision, show that the effect of the collision is equivalent to the transformation :

$$\left\{ \begin{array}{lllll} x_e & \to & \tilde{x}_e & = & 2x_s - x_e \\ x_s & \to & \tilde{x}_s & = & x_s \end{array} \right. \tag{4}$$

We place the system in an environment containing an ideal gas at thermal equilibrium at temperature T. We assume that the system experiences Markovian collisions with the particles of the gas with a rate Γ. To compute the effect of the collisions on the density matrix of the system, we consider that the environment is composed of a single particle. We denote $\rho(x_s, x_e, x'_s, x'_e) = (\langle x_s| \otimes \langle x_e|)\hat{\rho}(|x'_s\rangle \otimes |x'_e\rangle)$ the matrix elements of the total density matrix.

4. Assuming the initial state is separable, i.e. $\hat{\rho} = \hat{\rho}_s \otimes \hat{\rho}_e$, write down the total density matrix elements $\tilde{\rho}(x_s, x_e, x'_s, x'_e)$ after a collision in terms of the initial density matrix elements and show that the reduced density matrix elements of subsystem A read:

$$\tilde{\rho}_s(x_s, x'_s) = \Lambda \rho_s(x_s, x'_s) \quad \text{where} \quad \Lambda = \mathrm{Tr}_e \left[\rho_e\big(2x_s - x_e, 2x'_s - x'_e\big)\right] \tag{5}$$

5. Expand Λ in powers of $2(x_s - x'_s)$ up to second order, and show that:

$$\Lambda \simeq 1 - \frac{2(x_s - x'_s)^2}{\hbar^2} \left\langle \hat{p}^2 \right\rangle \tag{6}$$

6. Under the Markovian approximation and assuming the ideal gas is at thermal equilibrium, show that the action of the environment acts as a new term in the differential equation governing the matrix elements of $\hat{\rho}_s$:

$$\left(\frac{\partial \rho_s}{\partial t}\right)_{\text{env}} = -2\big(x_s - x'_s\big)^2 \frac{\Gamma m_e k_B T}{\hbar^2} \rho_s \tag{7}$$

7. Interpret this term as an exponential decay of the off-diagonal elements of the density matrix.