Quantum Optics – M1

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Quantum ground state and single-phonon control of a mechanical resonator

This execise sheet discusses the experimental results of this article: `https://doi.org/10.1038/nature08967`.

1 Description of the superconducting qubit

1.1 Current-biased Josephson qubit

Consider an electrical LC oscillator. We denote Φ the flux through the inductor and Q the charge on the capacitor's plates.

1. Writing the total energy of the system, show that the Hamiltonian of the oscillator can be cast in the form of a harmonic oscillator.

2. Promoting Φ and Q to quantum operators, recall the eigenenergies of this system. What temperature regime do we need to reach in order to perform quantum manipulations of this system. In view of performing quantum computations, justify the need of an anharmonicity in the oscillator.

3. We now replace the linear inductor by a Josephson junction: a superconducting element described by the two constitutive equations given below (1). Why is the junction called a "non-linear induction"? Show that the Hamiltonian of the oscillator now presents a strong anharmonicity.

$$\begin{cases} I(t) = I_0 \sin\left(2\pi \frac{\Phi(t)}{\Phi_0}\right), \quad \Phi_0 = \frac{h}{2e} \\ V(t) = \dfrac{\mathrm{d}\Phi}{\mathrm{d}t} \end{cases} \tag{1}$$

4. Show that biasing the Josephson junction with a constant current I results in a tilted washboard potential in the Hamiltonian. Explain how this can be used to read out the state of the qubit defined by the two lowest levels of the central well of the washboard.

1.2 Coherent control of the qubit

We restrict to the study of the two lowest energy states of the central well of our qubit, from now on respectively called $|0\rangle$ and $|1\rangle$, forming the qubit Hilbert space. The energy difference between these two states is denoted $\hbar\omega_q$.

5. In the article, the bias current is sent to the resonator through an inductive flux line. How does the Hamiltonian change if we send a microwave signal at frequency ω through the flux line? Write its restriction to the qubit Hilbert space and show it is of the form:

$$\frac{\hbar\omega_q}{2}\,\sigma_z + \hbar\Omega_R\cos(\omega t)\,\sigma_x \tag{2}$$

In order to deal with this time-dependant Hamiltonian, we perform the following unitary transformation to go to a rotating frame:

$$U(t) = e^{i\frac{\omega t}{2}\sigma_z} \tag{3}$$

6. Give the expression of the new Hamiltonian in the rotating frame, and justify that some terms may be neglected in the regime $\omega \gg \delta, \Omega_R$ where $\delta = \omega - \omega_q$.

7. Show that starting from $|0\rangle$ and applying a strong microwave pulse ($\Omega_R \gg \omega_q$), we can bring the qubit to the state $|1\rangle$. Give the duration of the pulse in terms of the Rabi frequency Ω_R.

2 Coupling to a mechanical resonator

We now consider the mechanical resonator nano-fabricated on the same chip. We write its motional eigenfrequency ω_m, and denote the "qubit-resonator detuning" by $\Delta = \omega_q - \omega_m$.

1. The resonator's mechanical eigenmodes are considered as harmonic. Give the Hamiltonian of the resonator in terms of a creation and annihilation operators a and $a^\dagger$.

2. We give an electrical equivalent model of our mechanical resonator as a LC oscillator with inductance L_0 and capacitance C_0. Give the values of L_0 and C_0 in terms of the resonator's properties and precise the analogies between the mechanical and electrical variables.

The resonator's bulk is made of a piezoelectric material, so that its motion generates an electrical signal, and vice versa. This piezoelectric device is coupled to the qubit by an interdigitated capacitor, modeled by a capacitance C_c.

3. Draw an electrical circuit equivalent of the whole system.

The coupling capacitor C_c adds a coupling term in the Hamiltonian of the form:

$$H_{\text{int}} = \beta Q_q Q_m \tag{4}$$

4. Show that the interaction Hamiltonian can be cast into the form:

$$H_{\text{int}} = \hbar g(a - a^\dagger)(\sigma_+ - \sigma_-) \tag{5}$$

5. Going to the Heisenberg picture, justify that some terms can be neglected in this Hamiltonian. We obtain the celebrated Jaynes-Cummings Hamiltonian, very often encountered in cavity quantum electrodynamics:

$$H_{JC} = \frac{\hbar\omega_q}{2}\sigma_z + \hbar\omega_m a^\dagger a + \hbar g(a\sigma_+ + a^\dagger\sigma_-) \tag{6}$$

6. What subspaces are left invariant by the Hamiltonian? Write its restriction to these subspaces and diagonalize it. Draw its eigenvalues as a function of Δ.

7. Justify that for some values of Δ, the qubit and the resonator will swap energy excitations, whereas for others they behave as independant systems. Propose a way to switch on and off the interaction between the qubit and the resonator at will, leading to controllable exchange of energy excitations between the two.

8. Explain the procedure used by the experimenters to prove that the mechanical resonator has been cryogenically cooled to its ground state.

9. Explain the procedure used by the experimenters to generate a single-phonon Fock state in the resonator.

2.1 Homework: Rabi oscillations with detuning

10. Reproduce figure 5b of the article `https://doi.org/10.1038/nature08967` by making a 2D plot of the population in the qubit excited state with respect to qubit-resonator detuning Δ and interaction time τ. As your model will be less precise than the one used in the article, you should find a symmetric result around $\Delta = 0$, looking like the white dot-dashed lines in the paper.