Quantum Optics – M1

Alexandre Journeaux & Pierre-François Cohadon

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TD6: Experimental measure of a cavity field mode's Wigner function

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We wish to find a practical way of measuring the Wigner function of a cavity field mode. This method has been proposed by L. G. Lutterbach and L. Davidovich in 1997 [1] and is now frequently used for practical readout of a cavity's Wigner function.

Since we are dealing with field quadratures X_0, P_0 instead of position and momentum x and p, we have to be a bit careful with the definition of the Wigner function: defining the eigenstate $|x\rangle$ of the field operator X_0 such that $X_0 |x\rangle = x |x\rangle$, we have:

$$W(x,p) = \frac{1}{\pi} \int e^{-2ipy} \langle x + y/2 | \hat{\rho} | x - y/2 \rangle \, \mathrm{d}y \tag{1}$$

1 A new expression for the Wigner function

1. Building on equations (9) and (10) of TD1, propose a (very impractical) way of reconstructing the cavity field mode's Wigner function.

2. Using Glauber's identity (given below), compute the action of the displacement operator $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$ on a an eigenstate $|x\rangle$ of the field operator X_0.

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \quad \text{if} \quad [A,[A,B]] = [B,[A,B]] = 0 \tag{2}$$

3. We introduce the parity operator $\mathcal{P} = e^{i\pi a^\dagger a}$. What are its eigenstates? What does its eigenvalues represent? Using your knowledge of the shape of the Fock states' wavefunctions, argue that we have $\mathcal{P}|x\rangle = |-x\rangle$ and $\mathcal{P}|p\rangle = |-p\rangle$.

4. Show that the Wigner function can be re-written in the simple form:

$$W(\alpha) = \frac{2}{\pi} \operatorname{Tr}(D(-\alpha) \hat{\rho} D(\alpha) \mathcal{P}) \tag{3}$$

2 Light shift of an atom in the cavity

We consider a microwave cavity possessing a mode of frequency ω_c, whose electric field can couple to an atom, modeled by a two-level system of frequency ω_a, *via* its electric dipole moment. We call $\Delta = \omega_a - \omega_c$ the atom-cavity detuning, and g the vacuum atom-cavity coupling.

1. Recall the Hamiltonians of the cavity, of the atom, as well as the coupling Hamiltonian between the two in the rotating wave approximation. The obtained Hamiltonian is known as the Jaynes Cummings Hamiltonian.

2. In the absence of atom-cavity coupling, propose a basis of the Hilbert space of the system that diagonalizes the Hamiltonian. What subspaces are left invariant by the total Hamiltonian, even in the non-zero coupling case?

3. Diagonalize the Hamiltonian in these subspaces, and derive its eigenenergies. In the dispersive limit ($\Delta \gg g$), show that the atom experiences a light shift proportional to the number of photons N present in the cavity:

$$\Delta E_{\mathrm{LS}} = \frac{2\hbar g(N+1)}{\Delta} \tag{4}$$

3 Measurement protocol of a cavity mode Wigner function

1. Recalling that the Ramsey sequence can be used for the precise frequency measurement of a two-level system, propose a simple way to measure the Wigner function of a the cavity mode, using the atom as a probe. How can it be implemented in a circuit QED experiment?

   Hint: Using a classical drive, i.e. a laser or a microwave source, we can perform any displacement of the field $D(\alpha)$, up to a phase.

Examples of these kind of measurement procedure can be found in the following articles:

- Marius Bild et al., Schrödinger cat states of a 16-microgram mechanical oscillator. Science 380, 274-278 (2023). `https://doi.org/10.1126/science.adf7553`
- von Lüpke, U., Yang, Y., Bild, M. et al. Parity measurement in the strong dispersive regime of circuit quantum acoustodynamics. Nat. Phys. 18, 794–799 (2022). `https://doi.org/10.1038/s41567-022-01591-2`

4 Homework: Wigner function of a mechanical resonator

The expression of a Shrödinger cat state of a harmonic oscillator is the following:

$$|\mathrm{cat}_\alpha\rangle = \frac{1}{\sqrt{2}}(|i\alpha\rangle + |-i\alpha\rangle) \tag{5}$$

where $|\alpha\rangle$ is the coherent state with amplitude α.

1. Reproduce the Wigner functions of figures 2C-D and 3A of the article `https://doi.org/10.1126/science.adf7553`. You can find additionnal help reading the Supplementary Materials of the article, providing all analytical formulas for the states along the evolution.