TD1

1.1

1. $\rho = |\psi\rangle\langle\psi|$

~~Proof~~ $\hat{A}$ Positive semi def $\Leftrightarrow \forall |\varphi\rangle \in \mathcal{H}$, $\langle\varphi|\hat{A}|\varphi\rangle \geq 0$

Here : $\langle\varphi|\rho|\varphi\rangle = \langle\varphi|\psi\rangle\langle\psi|\varphi\rangle$

$= |\langle\varphi|\psi\rangle|^2 \geq 0$

Hermitian : $\rho^+ = |\psi\rangle\langle\psi|^+ = |\psi\rangle\langle\psi| = \rho$

$Tr(\hat{\rho}) = \sum_{|\varphi\rangle \in \mathcal{B}(\mathcal{H})} \langle\varphi|\psi\rangle\langle\psi|\varphi\rangle$

$\mathcal{B}(\mathcal{H})$ basis : choose $\{|\psi\rangle, |\varphi_1\rangle, |\varphi_2\rangle, \dots\}$ by

Gram Schmidt orthonorm :

$Tr(\hat{\rho}) = \langle\psi|\psi\rangle\langle\psi|\psi\rangle + \sum_i \langle\underbrace{\varphi_i|\psi}_{=0}\rangle\langle\psi|\varphi_i\rangle$

$= 1$

2) $\partial_t|\psi\rangle = \frac{1}{i\hbar}\hat{H}|\psi\rangle$

$\partial_t\langle\psi| = -\frac{1}{i\hbar}\langle\psi|\hat{H}$

~~$\partial_t|\psi\rangle\langle\psi|$~~ $\partial_t\rho = (\partial_t|\psi\rangle)\langle\psi| + |\psi\rangle(\partial_t\langle\psi|)$

$= \frac{1}{i\hbar}\left[\hat{H}|\psi\rangle\langle\psi| - |\psi\rangle\langle\psi|\hat{H}\right]$

$= \frac{1}{i\hbar}[\hat{H}, \rho]$

3) $\cancel{\langle \hat{A} \rangle = \sum_{|\phi\rangle \in B(\mathcal{H})} \langle \phi | \hat{A} | \phi \rangle}$

$$\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle$$
$$= \mathrm{Tr}(\langle \psi | \hat{A} | \psi \rangle)$$
$$= \mathrm{Tr}(|\psi\rangle\langle\psi| \hat{A})$$
$$= \mathrm{Tr}(\hat{\rho} \hat{A})$$

<u>1.2</u> 4. LINEARITY

5. Pure: $\mathrm{Tr}(\hat{\rho}^2) = \mathrm{Tr}(|\psi\rangle\langle\psi|\psi\rangle\langle\psi|)$
$$= \mathrm{Tr}(|\psi\rangle\langle\psi|)$$
$$= 1$$

Mixt: $\mathrm{Tr}(\hat{\rho}^2) = \mathrm{Tr}\left(\left(\sum_\mu P_\mu |\mu\rangle\langle\mu| \right) \left(\sum_\nu P_\nu |\nu\rangle\langle\nu| \right) \right)$

$$= \mathrm{Tr} \sum_{\mu,\nu} P_\mu P_\nu \, |\mu\rangle\langle\mu|\nu\rangle\langle\nu|$$

$$\cancel{= \mathrm{Tr} \sum_{\mu,\nu} P_\mu P_\nu \, |\mu\rangle\langle\nu| \, \delta_{\mu\nu}}$$

$$= \sum_{\mu,\nu} \underset{\geqslant 0}{P_\mu} \; \underset{\geqslant 0}{P_\nu} \; \left| \begin{array}{l} \mathrm{Tr}(|\mu\rangle\langle\mu|\nu\rangle\langle\nu|) \\ \mathrm{Tr}(\langle\mu|\nu\rangle\langle\nu|\mu\rangle) \\ \mathrm{Tr}(|\langle\mu|\nu\rangle|^2) \\ |\langle\mu|\nu\rangle|^2 \\ 0 \leqslant .. \leqslant 1 \end{array} \right.$$

$$\mathrm{Tr}(\rho^2) \leq \sum_{\mu,\nu} p_\mu p_\nu$$

$$\leq \sum_\mu p_\mu \sum_\nu p_\nu$$

$$\leq 1$$

Egalité : CS ineq : equality if one $p_\mu = 1$ and others $= 0$

We saw pure $\Rightarrow \mathrm{Tr}(\rho^2) = 1$

Now $\mathrm{Tr}(\rho^2) = 1 \Rightarrow$?

$$\mathrm{Tr} \sum_{\mu\nu} p_\mu p_\nu |\mu\rangle\langle\mu|\nu\rangle\langle\nu| = 1$$

$$\Rightarrow \sum_{\mu\nu} p_\mu p_\nu \langle\mu|\nu\rangle\langle\nu|\mu\rangle = 1$$

$$\Rightarrow \sum_{\mu\nu} p_\mu p_\nu |\langle\mu|\nu\rangle|^2 = 1$$

populations diag

coherences hors-diag

(II)

1) $\mathcal{H} = \mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2}$

$\hookrightarrow$ basis : $\mathcal{B} = \{ |\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle \}$

$$|\psi\rangle = \alpha |\uparrow\uparrow\rangle + \beta |\uparrow\downarrow\rangle + \gamma |\downarrow\uparrow\rangle + \delta |\downarrow\downarrow\rangle .$$

We note : $|\psi\rangle = \sum_{a,b} \psi_{ab} \, |a\, b\rangle$, $a, b \in \uparrow, \downarrow$

$$|\psi\rangle = \begin{pmatrix} \psi_{\uparrow\uparrow} \\ \psi_{\uparrow\downarrow} \\ \psi_{\downarrow\uparrow} \\ \psi_{\downarrow\downarrow} \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \text{ in basis } \mathcal{B}$$

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$$\rho = |\psi\rangle\langle\psi| = \sum_{a_1 b_1, a_2 b_2} \psi_{a_1 b_1} \psi^*_{a_2 b_2} \, |a_1 b_1\rangle\langle a_2 b_2|$$

In basis $\mathcal{B}$:

$$\rho = \begin{pmatrix} \psi_{11}\psi_{11}^* & \psi_{11}\psi_{12}^* & \cdots \\ \psi_{12}\psi_{11}^* & \psi_{12}\psi_{12}^* & \\ \vdots & & \end{pmatrix}$$

2) $\hat{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\hat{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

$$\hat{A} \otimes \hat{B} = \begin{pmatrix} a_{11}\hat{B} & a_{12}\hat{B} \\ a_{21}\hat{B} & a_{22}\hat{B} \end{pmatrix} = \left(\begin{array}{cc|cc} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & \cdots \\ a_{11}b_{21} & a_{11}b_{22} & \vdots & \\ \hline a_{21}b_{11} & & a_{22}b_{11} & \cdots \\ & \ddots & \vdots & \ddots \end{array}\right)$$

3) $\hat{O} = \hat{A} \otimes \hat{B}$ Only acting on subsyst. A : $\hat{B} = \mathbb{1}$

$\hat{O} = \hat{A} \otimes \hat{\mathbb{1}}$

$$\hookrightarrow \hat{O} = \left(\begin{array}{cc|cc} a_{11} & 0 & a_{12} & 0 \\ 0 & a_{11} & 0 & a_{12} \\ \hline a_{21} & 0 & a_{22} & 0 \\ 0 & a_{21} & 0 & a_{22} \end{array}\right).$$

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$$\langle \hat{O} \rangle = \mathrm{Tr}(\hat{\rho}\hat{O})$$

4) We start with:

$$\mathrm{Tr}(\hat{\rho}\hat{O}) = \sum_{a_1 b_1} \langle a_1 b_1 | \hat{\rho}\hat{O} | a_1 b_1 \rangle$$

$$= \sum_{\substack{a_1 b_1 \\ a_2 b_2}} \langle a_1 b_1 | \hat{\rho} | a_2 b_2 \rangle \langle a_2 b_2 | \hat{O} | a_1 b_1 \rangle$$

$$= \sum_{\substack{a_1 b_1 \\ a_2 b_2}} \psi_{a_1 b_1} \psi^*_{a_2 b_2} \langle a_2 | A | a_1 \rangle \, \delta_{b_1 b_2}$$

$$= \sum_{\substack{a_1 b_1 \\ a_2}} \psi_{a_1 b_1} \psi^*_{a_2 b_1} \langle a_2 | A | a_1 \rangle .$$

And compare with :

$$\mathrm{Tr}(\rho_A A) = \sum_{a_3} \langle a_3 | \rho_A \, A | a_3 \rangle$$

$$= \sum_{a_3 a_4} \langle a_3 | \rho_A | a_4 \rangle \langle a_4 | A | a_3 \rangle .$$

$$= \sum_{\substack{a_1 b_1 \\ a_2 b_2 \\ a_3 a_4}} \psi_{a_1 b_1} \psi^*_{a_2 b_2} \underbrace{\langle a_3 | a_1 \rangle \langle a_2 | a_4 \rangle}_{\delta_{a_3 a_1} \, \delta_{a_2 a_4}} \langle a_4 | A | a_3 \rangle \underbrace{\mathrm{Tr}(|b_1 \rangle \langle b_2 |)}_{\delta_{b_1 b_2}}.$$

$$= \sum_{\substack{a_1 b_1 \\ a_2}} \psi_{a_1 b_1} \psi^*_{a_2 b_1} \langle a_2 | A | a_1 \rangle .$$

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$$= \mathrm{Tr}(\rho \hat{O}) = \langle \hat{O} \rangle .$$

TD Wigner

1) $\mathbb{1} = \int dp\, |p\rangle\langle p|$ $\quad \tilde{A}(x,p) = \int e^{-ipy/\hbar} \langle x + y/2 | \hat{A} | x - y/2 \rangle\, dy$

$$\tilde{A}(x,p) = \iiint e^{-ipy/\hbar} \langle x + y/2 | p_1 \rangle \langle p_1 | A | p_2 \rangle \langle p_2 | x - y/2 \rangle\, dy\, dp_1\, dp_2$$

$$\langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$$

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$$= \frac{1}{2\pi\hbar} \iiint dy\, dp_1\, dp_2\, e^{-ipy/\hbar}\, e^{ip_1(x+y/2)/\hbar}\, e^{-ip_2(x-y/2)/\hbar} \langle p_1 | \hat{A} | p_2 \rangle$$

$$= \frac{1}{2\pi\hbar} \iint dp_1\, dp_2\, \langle p_1 | \hat{A} | p_2 \rangle \underbrace{\int dy\, e^{\frac{i}{\hbar}\left(-py + p_1\left(x + \frac{y}{2}\right) - p_2\left(x - \frac{y}{2}\right)\right)}}_{\cancel{2\pi\hbar\, \delta\left(-py + p_1\left(x + \frac{y}{2}\right) - p_2\left(x - \frac{y}{2}\right)\right)} \;\; \cancel{= 2\pi\hbar\, \delta(}}$$

$$= \frac{1}{2\pi\hbar} \iint dp_1\, dp_2\, \langle p_1 | A | p_2 \rangle\, e^{i(p_1 - p_2)x/\hbar} \underbrace{\int dy\, e^{i\left(\frac{p_1 + p_2}{2} - p\right)y}}_{= 2\pi\hbar\, \delta\left(\frac{p_1 + p_2}{2} - p\right)}$$

$$= \iint dp_1\, dp_2\; \delta\!\left(\frac{p_1+p_2}{2} - p\right) \langle p_1 | A | p_2 \rangle\, e^{i(p_1-p_2)x/\hbar}$$

$$p_1 = v + \frac{u}{2} \qquad p_2 = v - \frac{u}{2}$$

$$u = p_1 - p_2 \;, \quad v = \frac{p_1+p_2}{2} \,. \qquad , \cancel{du\,dv = dp_1\,dp_2 = \tfrac{1}{2}(dp_1 - dp_2)(dp_1 + dp_2)}$$

$$\cancel{d} \qquad \cancel{p = 2v}$$

$$= \iint du\, dv \;\; \delta(v - p) \;\langle v + \tfrac{u}{2} | A | v - \tfrac{u}{2} \rangle \; e^{iux/\hbar}$$

$$= \int du \;\; e^{iux/\hbar} \;\langle p + \tfrac{u}{2} | A | p - \tfrac{u}{2} \rangle$$

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~~$\mathrm{Tr}[\hat A \hat B] =$~~

2) $$\frac{1}{h}\iint dx\,dp\;\tilde A(x,p)\,\tilde B(x,p) = \frac{1}{h}\iint dx\,dp \int dy_1\, e^{-ipy_1/\hbar}\langle x + y_1/2|A|x - y_1/2\rangle \int dy_2\, e^{-ipy_2/\hbar}\langle x + y_2/2|B|x - y_2/2\rangle$$

$$= \frac{1}{h}\iiiint dx\,dp\,dy_1\,dy_2\; e^{-ip(y_1+y_2)/\hbar}\langle x + y_1/2|A|x - y_1/2\rangle\langle x + y_2/2|B|x - y_2/2\rangle .$$

$$= \iiint dx\,dy_1\,dy_2\;\delta(y_1+y_2)\langle x + y_1/2|A|x - y_1/2\rangle\langle x + y_2/2|B|x - y_2/2\rangle$$

$$= \iint dx\,dy_1\;\langle x + y_1/2|A|\underbrace{x - y_1/2\rangle\langle x - y_1/2}|B|x + y_1/2\rangle$$

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$\cancel{= \int dy_1 \langle x}$ $\qquad u = x - \frac{y_1}{2}, \quad v = x + y_1/2$

$\Rightarrow dx\,dy_1 = du\,dv$

$$= \iint du\,dv\;\langle v|A|u\rangle\langle u|B|v\rangle$$

$$= \int dv\;\langle v|AB|v\rangle .$$

$$= \mathrm{Tr}(\hat A \hat B)$$

~~$\tilde A(x,p) = \int e^{-ipy/\hbar}\langle x + y/2|\hat A|x - y/2\rangle\,dy$~~

3) $\langle \hat{A} \rangle = \mathrm{Tr}(\hat{\rho}\hat{A}) = \frac{1}{h}\iint dx\,dp\ \tilde{\rho}(x,p)\tilde{A}(x,p) = \iint dx\,dp\ W(x,p)\ \tilde{A}(x,p)$

4) $\int dp\ W(x,p) = \frac{1}{h}\iint dp \int du\ e^{ixu/\hbar} \langle p+u/2|\hat{\rho}|p-u/2\rangle$

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$$= \frac{1}{h}\iint dp\,du\ e^{ixu/\hbar}\ \varphi(p+u/2)\,\varphi^*(p-u/2)$$

$\alpha = p + \frac{u}{2}, \quad \beta = p - \frac{u}{2}, \quad d\alpha\, d\beta = dp\, du$

$$= \frac{1}{h}\iint d\alpha\, d\beta\ e^{ix(\alpha-\beta)/\hbar}\ \varphi(\alpha)\,\varphi^*(\beta)$$

$$= \left(\frac{1}{\sqrt{h}}\int d\alpha\ e^{ix\alpha/\hbar}\,\varphi(\alpha)\right)\left(\frac{1}{\sqrt{h}}\int d\beta\ e^{-ix\beta/\hbar}\,\varphi^*(\beta)\right)$$

$$= \psi(x)\,\psi^*(x)$$

$$= |\psi(x)|^2.$$

Same for $\int dx\ W(x,p)$ w/ initial def of W.

5) $$W^*(x,p) = \frac{1}{h}\int e^{+ipy/\hbar} \left(\langle x + y/2 | \hat{\rho} | x - y/2 \rangle\right)^* dy$$

$$= \frac{1}{h}\int dy \; e^{+ipy/\hbar} \left(\psi(x + y/2)\,\psi^*(x - y/2)\right)^*.$$

$$= \frac{1}{h}\int dy \; e^{ipy/\hbar} \; \psi(x - y/2)\,\psi^*(x + y/2)$$

$y \to -y$

$$= \frac{1}{h}\int dy \; e^{-ipy/\hbar} \; \psi(x + y/2)\,\psi^*(x - y/2)$$

$$= W(x,p). \qquad \to \text{real}.$$

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6) $$\iint dx\,dp\; W(x,p) = \int dx \; |\psi(x)|^2 = 1 .$$

$\cancel{W^*(x,p) =}$

6) $\iint dx\,dp\; W(x,p) = \int dx\; |\psi(x)|^2 = 1$.

⑥

7) $x \to x - b$:

$$W'(x,p) = \frac{1}{h}\int dy\; e^{-ipy/\hbar}\; \psi\left(x - \frac{y}{2} - b\right)\psi^*\left(x + \frac{y}{2} - b\right) = W(x-b,p)$$

$p \to p - b_p$:

$$W''(x,p) = \frac{1}{h}\int du\; e^{iux/\hbar}\; \varphi\left(p - \frac{u}{2} - b_p\right)\varphi^*\left(p + \frac{u}{2} - b_p\right) = W(x, p - b_p) .$$

8) $|\langle\psi_a|\psi_b\rangle|^2 = \langle\psi_b|\psi_a\rangle\langle\psi_a|\psi_b\rangle$

$= \langle\rho_a\rangle_b$

$= \iint dx\,dp\; W_b(x,p)\;\tilde{\rho}_a$

$= h\iint dx\,dp\; W_b(x,p)\, W_a(x,p)$.

Take 2 orthogonal state $|\psi_a\rangle, |\psi_b\rangle$: $|\langle\psi_a|\psi_b\rangle|^2 = 0$

so $\iint dx\,dp\; W_a(x,p)\,W_b(x,p) = 0$: one of them has to be < 0 somewhere

9) Pose $\psi_1(y) = \frac{1}{\sqrt{2}} e^{-ipy/\hbar} \psi(x + y/2)$

$$\psi_2(y) = \frac{1}{\sqrt{2}} \psi(x - y/2)$$

Normalized ! Why? :

$$\int dy\, \psi_2^*(y)\, \psi_2(y) = \frac{1}{2} \int dy\, \psi^*(x - y/2)\, \psi(x - y/2)$$

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$$u = x - \frac{y}{2}, \quad dy = 2\,du$$

$$= \int dy\, \psi^*(u)\, \psi(u)$$

$$= 1 .$$

Then: $W(x,p) = \frac{2}{h} \int dy\, \psi_1(y)\, \psi_2^*(y)$

$$= \frac{2}{h} \langle \psi_1 | \psi_2 \rangle$$

$$|W(x,p)| \leqslant \frac{2}{h} |\langle \psi_1 | \psi_2 \rangle| \leqslant \frac{2}{h}$$

b) $\tilde{A} = \int dy\, e^{-ipy/\hbar} \langle x + y/2 | \tilde{A}(\hat{x}) | x - y/2 \rangle$

$= \cancel{\int dy\, e^{-ipy/\hbar} A \langle x + y/2 |}$

$= \int dy \quad e^{-ipy/\hbar} \quad A(x - y/2) \quad \langle x + y/2 | x - y/2 \rangle$

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$= \int dy \quad e^{-ipy/\hbar} \quad A(x - y/2) \quad \delta(y)$

$= \quad A(x).$

$\tilde{B} = B(\hat{p})$: iden w/ other formula ~~thorns~~. (4)

$\langle \hat{H} \rangle = \iint W(x,p)\, \tilde{H} \, dx\, dp \qquad \tilde{H} = \widetilde{\frac{p^2}{2m}} + \widetilde{U(x)}$

$= \frac{p^2}{2m} + U(x). = H(x,p).$

$= \iint W(x,p) \quad H(x,p) \; dx\, dp.$

~~Example of HO:~~ 2.4 : Cool temp HO.

$$\cancel{H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2}$$

~~Lowest energy es (solve Schrö eq):~~

~~Find wig fc° of 2 lowest eig states: (does they respect eq (10) (2021))~~

$$W(x,p) = \frac{1}{h} \int e^{-ipy/\hbar} \psi(x + y/2)\, \psi^*(x - y/2)\, dy$$

$$\frac{\partial W}{\partial t} = \frac{1}{h} \int e^{-ipy/\hbar} \left[\frac{\partial \psi^*(x - y/2)}{\partial t} \psi(x + y/2) + \frac{\partial \psi(x + y/2)}{\partial t} \psi^*(x - y/2) \right] dy$$

SE:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi + U(x) \psi \qquad (1)$$

So:

$$\begin{cases} \dfrac{\partial \psi}{\partial t} = -\dfrac{\hbar}{2im} \partial_x^2 \psi + \dfrac{1}{i\hbar} U(x) \psi \\[2ex] \dfrac{\partial \psi^*}{\partial t} = \dfrac{\hbar}{2im} \partial_x^2 \psi^* \cancel{+} - \dfrac{1}{i\hbar} U(x) \psi^* \end{cases}$$

$$\frac{\partial W}{\partial t} = \frac{\partial W_T}{\partial t} + \frac{\partial W_U}{\partial t} \quad w/$$

$$(*) \quad \frac{\partial W_T}{\partial t} = \frac{1}{4\pi i m} \int e^{-ipy/\hbar} \left[\frac{\partial^2 \psi^*(x - y/2)}{\partial x^2} \psi(x + y/2) + \frac{\partial^2 \psi(x + y/2)}{\partial x^2} \psi^*(x - y/2) \right] dy$$

$$\cancel{(**) \quad \frac{\partial W_U}{\partial t} = \frac{2\pi}{i\hbar^2} \int e^{-ipy/\hbar} \left[U(x + y/2) - U(x - y/2) \right] \psi^*(x - y/2) \psi(x + \frac{y}{2})\, dy}$$

$$\frac{\partial W_0}{\partial t} = \frac{2\pi}{i\hbar^2} \int e^{-ipy/\hbar} \left[U(x+y/2) - U(x-y/2) \right] \psi^*(x-y/2) \times \psi(x+y/2)\, dy \qquad (**)$$

Astuce :

$$(*) \quad \frac{\partial^2 \psi^*(x-y/2)}{\partial x^2} = -2 \frac{\partial^2 \psi^*(x-y/2)}{\partial x \, \partial y}$$

car : $$\frac{\partial \psi^*(x-y/2)}{\partial x} = -2 \frac{\partial \psi^*(f(x,y))}{\partial y}$$

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~~IPP~~

$$\frac{\partial W_T}{\partial t} = -2 \int e^{-ipy/\hbar} \frac{\partial^2 \psi^*(x-y/2)}{\partial y \, \partial x} \psi(x+y/2)\, dy$$

IPP

$$= -\frac{2ip}{\hbar} \int e^{-ipy/\hbar} \frac{\partial \psi^*(x-y/2)}{\partial x} \psi(x+y/2)\, dy$$

$$+ \int e^{-ipy/\hbar} \frac{\partial \psi^*(x-y/2)}{\partial x} \frac{\partial \psi(x+y/2)}{\partial x}\, dy$$

$$\int e^{-ipy/\hbar}\ \cancel{\hbar}\ \frac{\partial^2 \psi^*(x-y/2)}{\partial x^2}\ \psi(x+y/2)\, dy$$

$$= -2\int e^{-ipy/\hbar}\ \frac{\partial^2 \psi^*(x-y/2)}{\partial x\, \partial y}\ \psi(x+y/2)\, dy \qquad (3)$$

IPP

$$= -\frac{2ip}{\hbar}\int e^{-ipy/\hbar}\ \frac{\partial \psi^*(x-y/2)}{\partial x}\ \psi(x+y/2)\, dy$$

$$+2\int e^{-ipy/\hbar}\ \frac{\partial \psi^*(x-y/2)}{\partial x}\ \underbrace{\frac{\partial \psi(x+y/2)}{\partial y}}_{\text{transf again}}\, dy$$

$$= -\frac{2ip}{\hbar}\int e^{-ipy/\hbar}\ \frac{\partial \psi^*(x-y/2)}{\partial x}\ \psi(x+y/2)\, dy$$

$$+\int e^{-ipy/\hbar}\ \frac{\partial \psi^*(x-y/2)}{\partial x}\ \frac{\partial \psi(x+y/2)}{\partial x}\, dy$$

Same w/ 2^{nd} term of eq (4).

$$-\int e^{-ipy/\hbar}\ \psi^*(x-y/2)\ \frac{\partial^2 \psi(x+y/2)}{\partial x^2}\, dy =$$

$$= -2\int e^{-ipy/\hbar}\ \psi^*(x-y/2)\ \frac{\partial^2 \psi(x+y/2)}{\partial x\, \partial y}\, dy$$

$$= -\frac{2ip}{\hbar}\int e^{-ipy/\hbar}\ \psi^*(x-y/2)\ \frac{\partial \psi(x+y/2)}{\partial x}\, dy$$

$$-\ \cancel{\hbar}\int e^{-ipy/\hbar}\ \frac{\partial \psi^*(x-y/2)}{\partial x}\ \frac{\partial \psi(x+y/2)}{\partial x}\, dy$$

$(*) \Rightarrow$

(4)

$$\frac{\partial W_T}{\partial t} = \frac{1}{4\pi i m} \times -\frac{2ip}{\hbar} \int e^{-ipy/\hbar} \frac{\partial}{\partial x}\left(\psi^*(x-y/2)\,\psi(x+y/2)\right)$$

$$= -\frac{p}{\hbar m}\frac{\partial W}{\partial x}$$

$(**)$: $\cancel{U(x+y/2) = \sum_n \frac{1}{n!}\frac{\partial^n U(x)}{\partial x^n}}$

$$U(x+y/2) = \sum_n \frac{1}{n!}\frac{\partial^n U(x)}{\partial x^n}\Big|_x \left(\frac{y}{2}\right)^n$$

$$U(x+y/2) - U(x-y/2) = \sum_n \frac{1}{n!}\frac{\partial^n U(x)}{\partial x^n}\Big|_x \left[\left(\frac{y}{2}\right)^n - \left(-\frac{y}{2}\right)^n\right]$$

$$= \sum_s \frac{1}{(2s+1)!}\frac{\partial^{2s+1}U(x)}{\partial x^{2s+1}}\, 2\left(\frac{y}{2}\right)^{2s+1}$$

$$= \sum_s \frac{1}{(2s+1)!}\frac{\partial^{2s+1}U(x)}{\partial x^{2s+1}}\left(\frac{1}{2}\right)^{2s} y^{2s+1}$$

$$\frac{\partial W_U}{\partial t} = \frac{2\pi}{i\hbar^2}\sum_s \overbrace{\frac{1}{(2s+1)!}\frac{\partial^{2s+1}U(x)}{\partial x^{2s+1}}\left(\frac{1}{2}\right)^{2s}\frac{\partial^{2s+1}U(x)}{\partial x^{2s+1}}}^{A_s} \times$$

$$\times \int e^{-ipy/\hbar}\, y^{2s+1}\, \varphi^*(x-y/2)\,\psi(x+y/2)\, dy$$

$$= \cancel{\frac{2\pi}{i\hbar^2}\sum_s A_s \left(\frac{\hbar}{i}\right)^{2s+1}\left(\frac{\partial}{\partial p}\right)^{2s+1} W(x,p)}$$

$$= \sum_s (-\hbar^2)^s \frac{1}{(2s+1)!}\left(\frac{1}{2}\right)^{2s}\frac{\partial^{2s+1}U(x)}{\partial x^{2s+1}}\frac{\partial^{2s+1}}{\partial p^{2s+1}} W(x,p)$$

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HO or lin pot:

$$\frac{\partial^n U}{\partial x^n} = 0 \text{ for } n > 2.$$

$$\hookrightarrow \quad \frac{\partial W_c}{\partial t} = \frac{\partial U(x)}{\partial x} \frac{\partial W(x,p)}{\partial p}$$

So:

$$\frac{\partial W(x,p)}{\partial t} = -\frac{p}{m} \frac{W(x,p)}{\partial x} + \frac{\partial U(x)}{\partial x} \frac{\partial W(x,p)}{\partial p}$$

Classical point-like part:

$$x_0 = x \cos(\omega t) - \frac{p}{m\omega} \sin(\omega t)$$

$$p_0 = p \cos(\omega t) + m\omega x \sin(\omega t)$$

So:

$$W(x,p,t) = W\left(x \cos(\omega t) - \frac{p}{m\omega} \sin(\omega t),\right.$$

$$\left. p \cos(\omega t) + m\omega x \sin(\omega t), 0\right)$$