Quantum Optics – M1

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Quantum limits in continuous displacements sensing

We will focus on the displacement sensing of a mechanical resonator with a Fabry-Perot cavity, with the mirror motion monitored through the phase of the reflected field.
The moving mirror is assumed to have a unity reflectivity, whereas the coupling mirror has a reflectivity $r = 1 - \gamma$, with $\gamma \ll 1$ (and a transmission coefficient t equal to $\sqrt{2\gamma}$). Optical losses are neglected.
The moving mirror is also characterized by its mechanical susceptibility $\chi(\Omega)$.

1 Field equations for a moving mirror Fabry-Perot cavity

The electric field amplitude $\mathcal{E}(t)$ at the coupling mirror location is described by a complex amplitude $a(t)$, slowly varying at the timescale $T = 2\pi/\omega_0 = 2\pi/k_0 c = \lambda_0/c$ of the wave, and normalized so that the mean field intensity $\overline{I} = |a|^2$ corresponds to the photon flux :

$$\mathcal{E}(t) = \left(\frac{\hbar\omega_0}{\mathcal{A}c}\right) a(t) \mathrm{e}^{-i\omega_0 t}. \tag{1}$$

$\mathcal{A}$ is the transverse area of the laser beam.

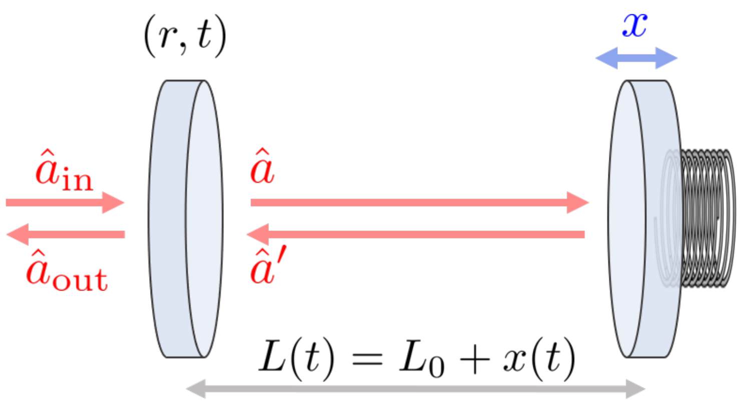

Figure 1 – Relations between the different fields in a moving mirror cavity.

We consider $L = L_0 + x(t)$ the cavity length, where $x(t)$ is the mirror displacement.

1. Show the following relations :

$$\begin{aligned} \hat{a}(t) &= t\hat{a}_{\text{in}}(t) + r\hat{a}'(t) \quad (2)\\ \hat{a}_{\text{out}}(t) &= t\hat{a}'(t) - r\hat{a}_{\text{in}}(t) \quad (3)\\ \hat{a}'(t) &= \hat{a}(t-\tau)\mathrm{e}^{i\Psi(t)}, \quad (4)\end{aligned}$$

   where τ is the round-trip time of light inside the cavity, and Ψ the field phase-shift :

$$\begin{aligned} \tau &= 2L_0/c \quad (5)\\ \Psi &= 2kL(t) \ [2\pi]. \quad (6)\end{aligned}$$

2. For a high-finesse ($\gamma \ll 1$) cavity close to resonance ($\Psi \ll 1$), the field operator is only slightly altered during one round-trip. Show then that :

$$\begin{aligned}
\tau \frac{d}{dt}\hat{a}(t) &= [-\gamma + i\Psi(t)]\ \hat{a}(t) + \sqrt{2\gamma}\,\hat{a}_{\rm in}(t) \qquad (7)\\
\hat{a}_{\rm out}(t) &= \sqrt{2\gamma}\,\hat{a}(t) - \hat{a}_{\rm in}(t) \qquad (8)\\
\Psi(t) &= \Psi_0 + 2k_0 x(t). \qquad (9)
\end{aligned}$$

2 Displacement measurement

We will now compute the sensitivity one can obtain on the displacement of the moving mirror, when taking into account the quantum phase fluctuations of the measurement beam.

Quadrature evolution by reflection upon the cavity

We first compute the input/output relations for the quadratures of the field. We will restrict the analysis to a resonant cavity. All mean fields therefore are in phase and we will assume they are real positive :

$$a_{\rm out} = a_{\rm in} = \sqrt{\frac{\gamma}{2}}\, a. \qquad (10)$$

3. By linearizing the field equations around the working point of the resonant cavity (with $\hat{a} \to a + \hat{a}$), establish :

$$\begin{aligned}
(\gamma - i\Omega\tau)\,\hat{a}[\Omega] &= \sqrt{2\gamma}\,\hat{a}_{\rm in}[\Omega] + 2iak_0 x\ [\Omega] \qquad (11)\\
\hat{a}_{\rm out}[\Omega] &= \sqrt{2\gamma}\,\hat{a}[\Omega] - \hat{a}_{\rm in}[\Omega]\,. \qquad (12)
\end{aligned}$$

4. For a real mean field, amplitude and phase quadratures are :

$$\begin{aligned}
\hat{X}[\Omega] &= \hat{a}[\Omega] + \hat{a}^\dagger[\Omega] \qquad (13)\\
\hat{Y}[\Omega] &= i(\hat{a}^\dagger[\Omega] - \hat{a}[\Omega]). \qquad (14)
\end{aligned}$$

   Establish and comment the following relations :

$$\begin{aligned}
\hat{X}[\Omega] &= \frac{\sqrt{2\gamma}}{\gamma - i\Omega\tau}\hat{X}_{\rm in}[\Omega] \qquad (15)\\
\hat{Y}[\Omega] &= \frac{\sqrt{2\gamma}}{\gamma - i\Omega\tau}\hat{Y}_{\rm in}[\Omega] + \frac{4ak_0}{\gamma - i\Omega\tau}\,x[\Omega] \qquad (16)\\
\hat{X}_{\rm out}[\Omega] &= \frac{\gamma + i\Omega\tau}{\gamma - i\Omega\tau}\hat{X}_{\rm in}[\Omega] \qquad (17)\\
\hat{Y}_{\rm out}[\Omega] &= \frac{\gamma + i\Omega\tau}{\gamma - i\Omega\tau}\hat{Y}_{\rm in}[\Omega] + \frac{4ak_0\sqrt{2\gamma}}{\gamma - i\Omega\tau}\,x[\Omega]\,. \qquad (18)
\end{aligned}$$

5. How are these equations transofmed if we assume $a \in i\mathbb{R}$?

Phase-noise-limited sensitivity

6. Establish that the phase noise spectrum of the reflected field is :

$$S_Y^{\mathrm{out}}\left[\Omega\right] = S_Y^{\mathrm{in}}\left[\Omega\right] + 256\frac{\mathcal{F}^2 \overline{I}_{\mathrm{in}}}{1+\left(\Omega/\Omega_{\mathrm{c}}\right)^2}\frac{S_x\left[\Omega\right]}{\lambda_0^2}, \tag{19}$$

where $\Omega_{\mathrm{c}} = \gamma/\tau$ is the cavity bandwidth and $\mathcal{F} = \pi/\gamma$ the cavity finesse.

7. Show that the measurement sensitivity can be written :

$$\tilde{x}_{\varphi} = \frac{\lambda_0}{16\mathcal{F}}\frac{1}{\sqrt{\overline{I}_{\mathrm{in}}}}\sqrt{1+\left(\Omega/\Omega_{\mathrm{c}}\right)^2}. \tag{20}$$

8. Comment on the dependence of $\tilde{x}_{\varphi}$ with the system parameters $\mathcal{F}$, $\overline{I}_{\mathrm{in}}$ and Ω_{c}.
9. Compute the corresponding sensitivity for the Virgo interferometer (1^{st} generation, 2010) : laser wavelength $\lambda_0 \simeq 1\mu$m, cavity finesse $\mathcal{F} = 50$ and optical input power $P_{\mathrm{in}} = 1\,$kW.

3 Radiation-pressure effects and the Standard Quantum Limit

Back-action of the meter beam

The displacement sensitivity $\tilde{x}_{\varphi}$ increases with the incident intensity (see Eq. 20), so that one might think that it could be arbitrarily increased just with the laser power. There is however a quantum limit related to the effect x_{rad} of the radiation-pressure fluctuations of the meter beam onto the mirror motion.
You may assume from now on $\Omega \ll \Omega_c$ (mechanical motion slow compared to the cavity timescale).

10. Rewrite equation (18) now taking into account x_{rad}.
11. Show that the noise spectrum S_Y^{out} now is the sum of two terms with an inverse dependence with the input optical power.

The Standard Quantum Limit

12. Deduce there is a fundamental sensitivity limit $\tilde{x}_{\mathrm{SQL}}$, called the Standard Quantum Limit.
13. What level is it ? For which intensity is it attained ?
14. At which frequency is it easier to demonstrate this SQL ?
15. Compute it numerically for the Virgo pendulum modes at 10 Hz.

Mechanical susceptibility for a harmonic oscillator :

$$\chi(\Omega) = \frac{1}{M}\frac{1}{\Omega_{\mathrm{m}}^2 - \Omega^2 - i\Omega\Omega_{\mathrm{m}}/Q}, \tag{21}$$

where M is the mass, $\Omega_{\mathrm{m}}/2\pi$ the resonance frequency and Q the mechanical quality factor.
For Virgo, $M \simeq 10$ kg, $\Omega_{\mathrm{m}}/2\pi \simeq 1$ Hz and $Q \gg 10^3$.