Quantum Optics – M1 ICFP

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TD1: Properties and applications of the Wigner function

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1 Definition and properties of the density matrix: a reminder

1.1 Pure state density matrix

1. Recall the expression of the density matrix $\hat{\rho}$ for a pure state $|\psi\rangle$. Show that it is a positive semi-definite hemitian operator of trace 1.

2. Show that the evolution of $\hat{\rho}$ is given by the von Neumann equation:

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = \frac{1}{i\hbar}\left[\hat{H}, \hat{\rho}\right] \tag{1}$$

3. Show that the average value of an operator $\hat{A}$ is given by:

$$\left\langle \hat{A} \right\rangle = \mathrm{Tr}\left(\hat{\rho}\hat{A}\right) \tag{2}$$

1.2 Mixed state density matrix

We now consider the density matrix of a statistical mixture $\hat{\rho} = \sum_\mu P_\mu \, |\mu\rangle\langle\mu|$

4. Show that all the above-mentionned properties are still valid.

5. Find an inequality on $\mathrm{Tr}\left(\hat{\rho}^2\right)$, and give a characterization of pure states based on their density matrix.

1.3 Formalism of the partial trace

In order to explore the simplest possible example, we consider a system made of two spin-$\frac{1}{2}$s.

6. Write the form of the most general state of the universe made of the two two-level-systems in matrix form. Write its density matrix in terms of its coefficients, again in matrix form.

7. Given two operators $\hat{A}$ and $\hat{B}$, each acting of the Hilbert space of one of the spin-$\frac{1}{2}$'s, give the matrix representation of the tensor product operator $\hat{A} \otimes \hat{B}$.

8. Let $\hat{O}$ be a composite operator $\hat{O} = \hat{A} \otimes \hat{B}$ only acting on the first subsystem. How do you translate this property mathematically? Give the matrix expression for $\hat{O}$.

We introduce the reduced density matrix by performing a partial trace on subsystem B:

$$\hat{\rho}_A = \mathrm{Tr}_B\left(\hat{\rho}\right) = \sum_{b \in \mathcal{H}_B} \langle b|\hat{\rho}|b\rangle \tag{3}$$

9. Show that the expectation value of $\hat{O}$ is given by:

$$\left\langle \hat{O} \right\rangle = \mathrm{Tr}\left(\hat{\rho}\hat{O}\right) = \mathrm{Tr}\left(\hat{\rho}_A \hat{A}\right) \tag{4}$$

2 The Weyl transformation and the Wigner function

We define the Weyl transformation $\tilde{A}$ of an operator $\hat{A}$ as the function of the two variables x and p:

$$\tilde{A}(x,p) = \int e^{-ipy/\hbar} \langle x + y/2|\hat{A}|x - y/2\rangle \,\mathrm{d}y \tag{5}$$

2.1 Properties of the Weyl transform

1. Show that the Weyl transform can be expressed in terms of the matrix elements of the operator $\hat{A}$ in the p basis:

$$\tilde{A}(x,p) = \int e^{ixu/\hbar} \langle p + u/2|\hat{A}|p - u/2\rangle \,\mathrm{d}u \tag{6}$$

2. Show that we have:

$$\mathrm{Tr}\left(\hat{A}\hat{B}\right) = \frac{1}{h} \iint \tilde{A}(x,p)\tilde{B}(x,p) \,\mathrm{d}x \,\mathrm{d}p \tag{7}$$

2.2 Wigner function of a pure state

We now define the Wigner function W of a state as the Weyl transform of its density matrix $\hat{\rho}$, up to a normalization constant. From now on we will consider only pure states, and we will generalize to statistical mixtures later on.

$$W(x,p) = \frac{1}{h}\tilde{\rho} \tag{8}$$

3. Express the expectation value of an operator $\hat{A}$ in terms of its Weyl transform and the Wigner function of the state.

4. Show that integrating the Wigner function along p gives the probability density distribution in the x representation, and vice-versa:

$$\int W(x,p)\,\mathrm{d}p = |\psi(x)|^2 \tag{9}$$

$$\int W(x,p)\,\mathrm{d}x = |\varphi(p)|^2 \tag{10}$$

5. Show that the Wigner function is real.

6. Show that the Wigner function is normalized in the following sense:

$$\iint W(x,p)\,\mathrm{d}x\,\mathrm{d}p = 1 \tag{11}$$

7. Show that shifting the wavefunction in position ($\psi(x) \to \psi(x-b)$) representation leads to a reasonable result on the Wigner function. Do the same for a momentum shift ($\psi(x) \to \psi(x)e^{ixb_p/\hbar}$).

The Wigner function seems to be a good candidate for a "classical" probability density distribution in phase space. We will see why it cannot be interpreted as such.

8. We consider two pure states $|\psi_a\rangle$ and $|\psi_b\rangle$ with density matrices $\hat{\rho}_a$ and $\hat{\rho}_b$. Express the square of their inner product $|\langle\psi_a|\psi_b\rangle|^2$ in terms of their Wigner functions W_a and W_b. Conclude about the sign of the Wigner function.

9. Show that the Wigner function is bounded:

$$|W(x,p)| \leq \frac{2}{h} \tag{12}$$

10. Show that the Weyl transform of an operator solely dependant on position $\hat{A} = A(\hat{x})$ is $\tilde{A}(x,p) = A(x)$. Do the same thing for a solely p-dependant operator $\hat{B} = B(\hat{p})$. Conclude that the expectation value of the Hamiltonian is given by:

$$\left\langle \hat{H} \right\rangle = \iint W(x,p) H(x,p) \, \mathrm{d}x \, \mathrm{d}p \tag{13}$$

2.3 Temporal evolution of a harmonic oscillator

11. Show that the general equation of motion for the Wigner function of a pure state, for a quantum system evolving in a potential $U(x)$ is:

$$\frac{\partial W}{\partial t} = \frac{\partial W_T}{\partial t} + \frac{\partial W_U}{\partial t} \tag{14}$$

where

$$\frac{\partial W_T}{\partial t} = -\frac{p}{m}\frac{\partial W(x,p)}{\partial x} \tag{15}$$

$$\frac{\partial W_U}{\partial t} = \sum_{s=0}^{\infty}(-\hbar^2)^s \frac{1}{(2s+1)!}\left(\frac{1}{2}\right)^{2s} \frac{\partial^{2s+1}U(x)}{\partial x^{2s+1}} \left(\frac{\partial}{\partial p}\right)^{2s+1} W(x,p) \tag{16}$$

12. Show that for a harmonic potential, the equation of motion of the Wigner function reduces to the classical Liouville equation:

$$\frac{\partial W(x,p)}{\partial t} = -\frac{p}{m}\frac{\partial W}{\partial x} + \frac{\partial U}{\partial x}\frac{\partial W}{\partial p} \tag{17}$$

Predict the behavior of the Wigner function of a coherent state of light, and that of a squeezed state.

2.4 Generalization to statistical mixtures

We now consider the Wigner function of a mixed state with density matrix $\hat{\rho} = \sum_\mu P_\mu |\mu\rangle\langle\mu|$.

13. Show that all the properties demonstrated above are still valid.