

Master ENS-ICFP M1  
Evolution and Measurement of Quantum States

Claude FABRE  
[claude.fabre@lkb.upmc.fr](mailto:claude.fabre@lkb.upmc.fr)

6 octobre 2016

The basic approach of Quantum Mechanics (recalled in the last chapters 9 and 10 of these notes) deals with state vectors, their collapse under measurement and their hamiltonian evolution. It describes the properties of the quantum world only when it is relatively simple : isolated system or system submitted to fixed forces, ideal non destructive measurements, perfectly determined quantum states. The aim of this course is to go one step further so as to characterize as precisely as possible quantum phenomena in more complex situations, often encountered in experiments : non-isolated physical system, imperfect preparation and non-ideal measurements. This will allow us to consider in the most general way the evolution of a quantum system under the effect of a generic interaction, even non-hamiltonian, such as in relaxation processes. For these extensions, we will use powerful tools such as the density matrix, Positive Operator Measures, Post-Measurement Operators, Kraus operators.

We will also investigate in which respect one can envision the measurement process as an actual physical evolution. We will then consider quantum correlations and entanglement, and the generation of quantum states by post-selection, i.e. conditioned by the result of a measurement. We will finally consider the quantum limits to physical measurements and estimations of physical parameters.

The reader will find more detailed accounts of these subjects in the references [1, 2, 3, 4, 5].

# Table des matières

<b>1</b>	<b>Introduction : experiments on single quantum objects</b>	<b>5</b>
1.1	Ensemble and single particle measurement on quantum systems .	5
1.2	Measurements on single quantum systems : observation of quantum jumps . . . . .	6
1.3	From quantum jumps to quantum fluctuations . . . . .	13
<b>2</b>	<b>The density matrix :</b>	
	<b>a general description of quantum systems</b>	<b>15</b>
2.1	Postulates of Basic Quantum Mechanics . . . . .	15
2.2	The density matrix as an alternative description of basic Quantum Mechanics . . . . .	16
2.3	First extension : imperfect state preparation . . . . .	19
2.4	Second extension : non-isolated systems . . . . .	23
2.5	General properties of the set $\mathcal{D}_{\mathcal{H}}$ of density matrices . . . . .	27
<b>3</b>	<b>Non ideal measurements on quantum systems</b>	<b>31</b>
3.1	Non ideal measurement devices . . . . .	31
3.2	New postulates for non ideal measurements . . . . .	32
3.3	Properties of the Probability Operator Measure . . . . .	35
3.4	Properties of the post-measurement operators $\hat{M}_n$ . . . . .	37
3.5	Examples . . . . .	40
<b>4</b>	<b>Evolution of Quantum systems</b>	<b>45</b>
4.1	Hamiltonian evolution of the density matrix . . . . .	45
4.2	Time evolution in the general case . . . . .	47
4.3	Two "colliding" systems . . . . .	53
4.4	Evolution of a small system coupled to a large one . . . . .	56
<b>5</b>	<b>The measurement as a physical evolution</b>	<b>65</b>
5.1	General ideas . . . . .	65
5.2	Zurek's model . . . . .	67
5.3	Back to the state collapse postulate M3 . . . . .	69
<b>6</b>	<b>Joint and conditional probabilities in quantum physics</b>	<b>71</b>
6.1	The Bayesian approach of statistics . . . . .	71
6.2	Statistics of correlations . . . . .	72
6.3	Quantum mechanics of successive measurements . . . . .	76

<b>7 Entanglement in bipartite systems</b>	<b>81</b>
7.1 Experimental implementations . . . . .	82
7.2 Case of a pure state $ \Psi_{AB}\rangle$ . . . . .	84
7.3 Case of a mixed state . . . . .	87
7.4 Conditional non destructive preparation of a quantum state . . . . .	89
7.5 The Einstein-Podolsky-Rosen argument . . . . .	91
<b>8 Use of entangled states</b>	<b>97</b>
8.1 Database search . . . . .	97
8.2 Quantum teleportation . . . . .	99
<b>9 Accuracy and uncertainty in Quantum measurements and estimations</b>	<b>102</b>
9.1 Quantum metrology . . . . .	102
9.2 The Heisenberg inequality . . . . .	103
9.3 Accuracy in the estimation of a physical parameter . . . . .	104
9.4 Measurement-induced perturbation of a quantum state . . . . .	113
<b>10 Reminder 1 : Basic Quantum Mechanics</b>	<b>119</b>
1 State vectors . . . . .	119
2 Observables . . . . .	122
3 Ideal measurements . . . . .	124
4 Temporal evolution . . . . .	126
<b>11 Reminder 2 : Description of composite systems</b>	<b>127</b>
1 Example . . . . .	127
2 Tensor product of Hilbert spaces . . . . .	128
3 Operators in tensor product of Hilbert spaces . . . . .	129
4 Examples . . . . .	129
5 No cloning theorem . . . . .	130

# Chapitre 1

## Introduction : experiments on single quantum objects

*We never experiment with just one electron or atom or (small) molecule. In thought experiments, we sometimes assume that we do ; this invariably entails ridiculous consequences . (E. Schrödinger 1952)*

### 1.1 Ensemble and single particle measurement on quantum systems

The sentence cited above reflects the level of sensitivity of the experiments performed at Schrödinger's time : with the sole exception of high energy physics experiments, for which the energy of the particles is so high that a single of them is enough to provide measurable effects in the detectors (Geiger counters, bubble chambers, wire chambers, ...), most measurements were, and are still, made on samples containing macroscopic quantities of quantum objects, of the order to the Avogadro number. For these experiments made on ensembles, the relevant quantities in the quantum mechanical description of the studied systems are *ensemble averaged quantities*, i.e. the mean values and sometimes the variances and the correlations. This has lead to great successes, in particular the ab initio determination from a quantum description of the microscopic world of the value of many macroscopic parameters such as thermal and electrical conductivity, light absorption and scattering parameters...

The situation has evolved with the development over the years of more and more sophisticated and sensitive experimental techniques : it is now possible to measure various properties of single quantum systems having an energy at the eV level or even at the meV le-

vel : single atoms, ions, molecules or photons are now the object of experimental studies : thought, or gedanken, experiments are now often real experiments. This allowed physicists to tackle new properties of the quantum world that often look strange, because they are far away from our intuition based on the macroscopic world : quantum fluctuations and correlations, entanglement, non-locality, conditional preparation and evolution are studied and even used in information processing, computation and precise measurements.

We give in the next section a few examples of measurements on single quantum systems. We will then study in the following chapters the so called *ridiculous consequences* of this new experimental situation.

## 1.2 Measurements on single quantum systems : observation of quantum jumps

There are nowadays throughout the world many experimental setups permitting to make measurements on single quantum systems. We will restrict ourselves in this section to a few selected examples.

### 1.2.1 Destructive detection of a single photon or particle

The photoelectric effect is the physical basis of photodetectors named photomultipliers (resp. avalanche photodiodes) : when a photon is incident on a metal (resp. on a semiconductor junction), it has some probability to expel an electron from the metal bulk (resp. to promote an electron to the conduction band). This free electron, by some kind of avalanche process, gives rise to a bunch of many electrons which constitutes a measurable pulsed electric signal, that we will call a "click" and witnesses the arrival of a single photon. When a strongly attenuated beam of laser light of constant intensity is sent on such detectors, one records clicks occurring at times that are unpredictable. If one makes the histogram  $H(n)$  of the number  $n$  of clicks recorded in a given time interval, one gets a *Poisson distribution* ( $H(n)$  proportional to  $\bar{n}^n/n!$ , where  $\bar{n}$  is the mean photon number), that is the sign that the photons are randomly distributed in the light beam.

Such devices are also sensitive to other kinds of incident particles, provided these particles bring an energy larger than the binding energy of the electron. This is for example the case of Helium atoms in its metastable state of very long lifetime and 20eV energy. Charged particles like electrons or ions of a few eV energy also are

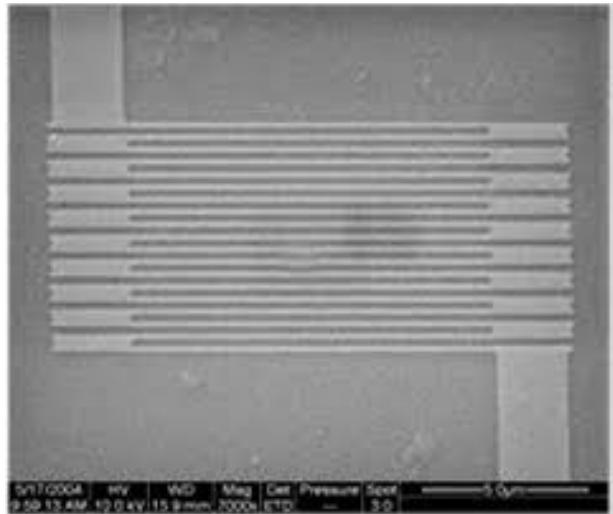


FIGURE 1.1 – niobium nitride Superconducting Single Photon Detector (J.-C. Villégier et al , CEA)

detected one by one by the same technique.

Bolometers constitute another kind of photodetectors. They record the temperature rise of an opaque body due to the absorption of light energy. Some of them can be extremely sensitive. The best ones consist of supraconducting thin (5nm) and narrow (100nm) nanowires cooled to very low temperatures (figure1.1) : they are able to monitor the transition from supraconductivity to normal conduction induced by the heating induced by the energy carried by a single photon ! Others are able to give distinguishable signals when 1, 2, 3 ... photons impinge on the detector : they are called Photon Number Resolving detectors.

### 1.2.2 Non destructive detection of the presence of a single ion

Appropriate configurations of electric and magnetic fields are able to create around some chosen point  $\mathbf{r}_0$  of the empty space a harmonic potential, so that a charged particle, like an electron or an ion, feels in its neighborhood a restoring force towards  $\mathbf{r}_0$  likely to trap it. With the help of some damping process, the ions can stop their oscillatory motion, and occupy the lowest vibrational level  $v = 0$  of this trap. Because of collisions with other particles present in the evacuated chamber in which the experiment is made, the ions can be expelled one by one from the trap, so that the number of ions

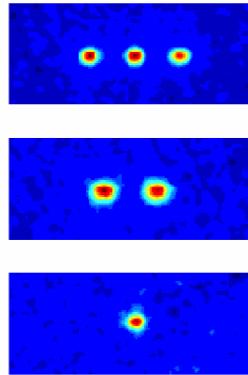


FIGURE 1.2 – fluorescence signal recorded on a CCD camera. Top to bottom : image of three, two ions and of a single ion trapped in a potential well

decreases with time, and can be equal to exactly 1 during some time, which ranges from second to days or even months.

To measure the presence of the single trapped ion, one shines on  $\mathbf{r}_0$  a laser beam which is resonant on the transition linking the ion ground state  $|g\rangle$  to an excited state  $|e\rangle$  of short lifetime  $T$  : the laser light excites the ion to level  $|e\rangle$ , which then decays back to  $|g\rangle$  by emitting a spontaneous emission photon in a random direction : this is the well-known phenomenon of fluorescence. A proportion of these photons is collected by a large aperture lens and monitored on a photodetector. Due to the continuous laser irradiation the ion is quickly excited back to  $|e\rangle$ , and the cycle can be repeated. The flux of detected photons is thus of the order of 1 photon per lifetime, i.e.  $10^8$  photons per second for a  $10\text{ns}$  lifetime. This is an easily detectable bunch of photons, even by the naked eye : it is possible to actually "see" and to take a picture of a single ion. Figure (1.2) shows the fluorescence of respectively three, two and finally one ion, recorded on a CCD camera. Note that the recorded fluorescence spot has a size which is much bigger than the ion size itself.

### 1.2.3 Non destructive detection of the energy level of a single trapped ion

Let us now describe the technique used to monitor not only the presence of the single ion, but its energy level, called the "shelving technique" (Fig1.3). Let us call  $|e'\rangle$  another excited state of the ion,

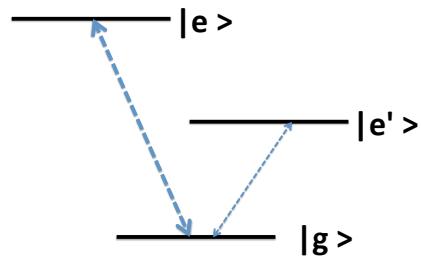


FIGURE 1.3 – Three-level atom with a strong transition  $|g\rangle \rightarrow |e\rangle$  and a weak transition  $|g\rangle \rightarrow |e'\rangle$

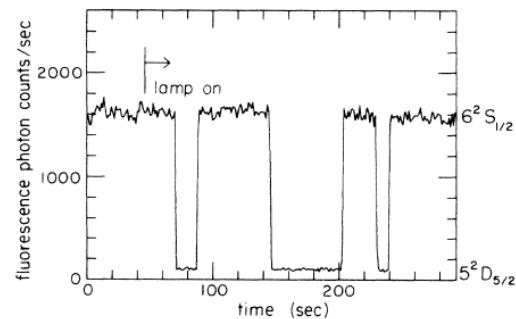


FIG. 2. A typical trace of the 493-nm fluorescence from the  $6^2P_{1/2}$  level showing the quantum jumps after the hollow cathode lamp is turned on. The atom is definitely known to be in the shelf level during the low fluorescence periods.

FIGURE 1.4 – fluorescence signal emitted by a single ion submitted to a strong resonant laser field and a weak lamp. The sudden changes in the level indicate the "quantum jumps" of the ion (Nagourney et al. Phys. Rev. Letters, **56**, 2797 (1986))

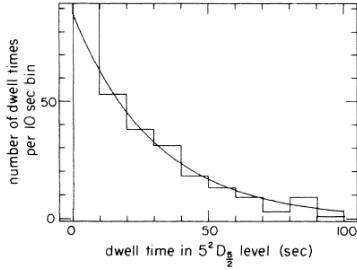


FIG. 3. Histogram of distribution of dwell times in the shelf level for 203 "off" times. A fitted theoretical (exponential) distribution for a metastable lifetime of 30 sec is superposed on the experimental histogram.

FIGURE 1.5 – histogram of times spent in the excited state (dwell times) corresponding to the periods of switched-off fluorescence in figure (1.2)

that is chosen to have a very long lifetime. If the ion is in the ground state  $|g\rangle$ , the laser, resonant on the transition  $|g\rangle \rightarrow |e\rangle$ , induces a strong fluorescence, as explained in paragraph 1.2.2, which is easily recorded. If some process excites the atom to the "shelf" level  $|e'\rangle$  (for example a weak lamp), the cycle  $|g\rangle - |e'\rangle$  stops immediately, and the fluorescence signal drops to zero. So the level of fluorescence is a direct measurement of the energy of the ion. Figure (1.4) shows one example of fluorescence signal : its switch-off indicates a passage from  $|g\rangle$  to  $|e'\rangle$ , whereas its switch-on indicates a passage from  $|e'\rangle$  to  $|g\rangle$ . Both are instantaneous, at least within the response time of the detection : they are witnesses of the famous *quantum jumps* of individual quantum systems, that need to be instantaneous because Quantum Mechanics forbid the ion to have an energy different from the eigenvalues of the Hamiltonian.

One observes that these quantum jumps occur at random times : *the quantum jump of a single ion is unpredictable*. Recording many successive jumps, one can build the histogram of the dwell times in the excited state  $|e'\rangle$  (Figure (1.5)). It can be fitted by an exponential function  $e^{-t/T'}$  which yields the lifetime  $T'$  of the excited state  $|e'\rangle$ . We see that, even though a single event on a single ion is random and cannot be predicted, the analysis of many successive events allows us to retrieve the relevant physical property of the ion.

#### 1.2.4 Destructive detection of the energy level of a single highly excited atom

Highly excited states of atoms, also called Rydberg states, (with principal numbers  $n$  of the order of 50) can be counted one by one, because their valence electron is weakly bound to the atomic core : an electrostatic field of  $100V/cm$  is enough to overcome the binding Coulomb field of the core and therefore to ionize it. The ejected electron can then be individually counted by an electron multiplier.

The technique can even be used to distinguish atoms in Rydberg states with different quantum numbers  $n$ , using the fact that the minimum electrostatic field needed to ionize such atoms varies with  $n$  (it is actually proportional to  $n^{-4}$ ). The set-up is sketched on figure (1.6) : the atom excited by laser beam to Rydberg state  $|e\rangle$  of principal number  $n$  travels in the vacuum of an evacuated chamber and, after various interactions described in section (1.2.5), crosses successively two capacitors. In the first one exists an electrostatic field which is just enough to ionize level  $|e\rangle$  of principal number  $n$  but not the more bound state  $|g\rangle$  of principal number  $|n - 1\rangle$ . Level  $|g\rangle$  is ionized in the second capacitor placed downstream, inside which exists a stronger electrostatic field that can ionize the atom when it is in state  $|g\rangle$ .

#### 1.2.5 Non destructive detection of a photon contained in a cavity

Two mirrors in front of each other constitute an optical cavity in which resonant light is trapped. In particular superconducting mirrors are able to "trap" photons of frequencies in the microwave domain (around  $50GHz$ ) during very long times, of the order of  $0.1s$ . S. Haroche and his group at the ENS (see also [1] page 321) have shown that it is possible to detect the presence of a single microwave photon in the cavity without destroying it. The set-up is sketched in figure (1.6) : an atom prepared in the Rydberg state  $|e\rangle$  by laser excitation enters cavity R1 in which it undergoes a  $\pi/2$  Rabi oscillation which brings it into the superposition  $(|e\rangle + |g\rangle)/\sqrt{2}$ . This state does not change if no photon is present in the cavity. Crossing cavity R2, the atom undergoes a second  $\pi/2$  Rabi oscillation which brings it into the state  $|e\rangle$ , detected in the first capacitor. If a single photon is present in the cavity, tuned to resonance with another transition  $|e\rangle \rightarrow |e'\rangle$  (where  $|e'\rangle$  is of quantum number  $n + 1$ ), the cavity finesse is so high that this single photon is enough to induce a Rabi oscillation of  $2\pi$  between  $|e\rangle$  and  $|e'\rangle$ , which changes the

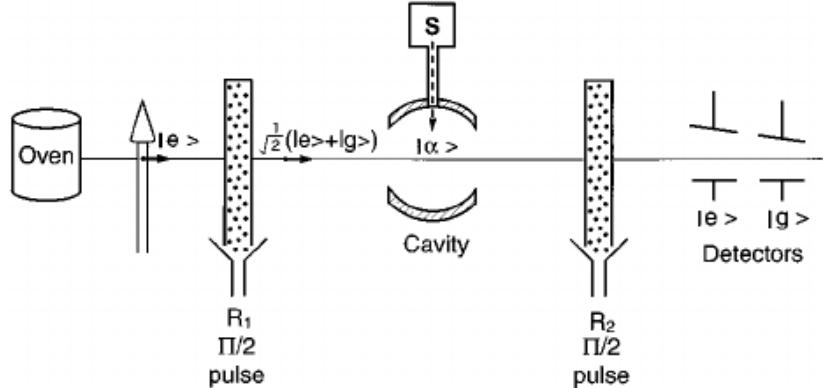


FIGURE 1.6 – Set-up for detecting a single microwave photon in cavity without destroying it by sending through the cavity Rydberg states.  $R_1$  and  $R_2$  are two auxiliary cavities where Rabi nutations between Rydberg states take place. Single atoms in Rydberg states  $|e\rangle$  and  $|g\rangle$  of principal quantum numbers  $n$  and  $n-1$  are detected by selective field ionization in the two capacitors on the right (see [1])

sign of the state  $|e\rangle$  while keeping the photon in the cavity, so that after interaction the atom is in state  $(-|e\rangle + |g\rangle)/\sqrt{2}$ . The  $\pi/2$  Rabi oscillation in cavity  $R_2$  brings it back into the state  $|g\rangle$ , which is detected by the second capacitor. The set-up is therefore able to detect the presence or absence of the single photon in the cavity without destroying it.

Figure (1.7) shows a temporal sequence of single Rydberg atom detections. The upper red bars are the "clicks" from the first capacitor, produced by ionization of atoms in state  $|e\rangle$ . The lower blue bars are the "clicks" from the second capacitor, produced by atoms in state  $|g\rangle$ . In the beginning, atoms are detected mostly in  $|g\rangle$ : the numerous blue clicks, very close to each other, witness the absence of photons in the cavity. A very weak microwave excitation of the cavity is produced by source S, which introduces at some random time by a quantum jump a single photon in the cavity, the presence of which is witnessed by the detection of atoms in level  $e$  (red clicks). This photon is monitored many times until it disappears suddenly, after fraction of a second, by a new quantum jumping occurring at an unpredictable time, so that atoms are then mostly detected afterwards in level  $g$  and give rise again to a great number of blue clicks. The statistical analysis of many traces like this one allows to determine the cavity lifetime. Note the presence of "stray clicks" (red in the blue sequence and blue in the red sequence) that correspond to

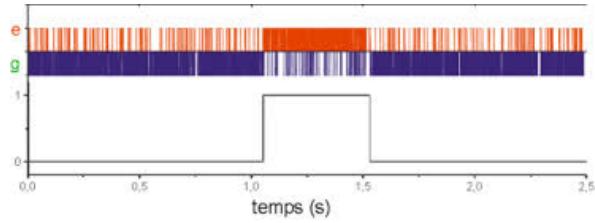


FIGURE 1.7 – Upper part : temporal sequence of recordings of Rydberg atoms in state  $|e\rangle$  (red "clicks") and  $|g\rangle$  (blue "clicks") by the set-up of figure (1.6). Lower part : time evolution of the number of photons in the cavity.

"false counts" due to imperfections in the selective field ionization technique.

### 1.3 From quantum jumps to quantum fluctuations

Let us consider the photon detector described in section (1.2.1) illuminated by a weak light source : it generates a photocurrent  $i(t)$  made of "clicks" occurring at random times (figure 1.8), that witness the presence of quantum jumps studied in the previous section. These clicks have a constant height  $i_0$  and a duration  $\tau$  due to the finite response time of the detector. If one increases the photon flux, the random clicks get closer to each other, and it may happen that they overlap : the photocurrent takes values 0,  $i_0$  and sometimes  $2i_0$ . Increasing further the light intensity, so that the clicks always overlap, the photocurrent takes now values that are multiples of  $i_0$ , and the randomness of the occurrence of quantum jumps translates into the randomness of the photocurrent values. At high light intensities (1mW of visible light corresponds to roughly  $10^{15}$  photons per second), it is not possible to see the discrete steps due to light energy quantization : the photocurrent intensity  $i(t)$  undergoes continuous temporal *quantum fluctuations*, also called *quantum noise*, around a non-zero mean value  $\langle i(t) \rangle$ , that is characterized like any noise by its spectral density  $S(\Omega)$  (see [17] p.38 ).

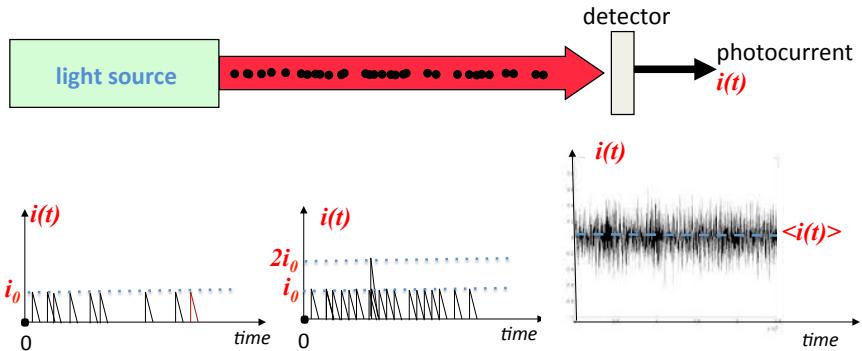


FIGURE 1.8 – photocurrent  $i(t)$  for different levels of the photon flux : at low flux (left), one observes at random times clicks of constant heights, that begin to overlap for intermediate fluxes (center); at high flux (right), one observes random temporal variations of the photocurrent (the vertical scale of the last recording on the right is strongly reduced with respect to the two left ones)

If the photons are randomly distributed, as it is the case for laser light, the clicks are distributed according to a Poisson law (variance equal to the mean). At high intensities, the photocurrent is affected by the corresponding quantum noise called "shot noise". Its spectral density is  $S(\Omega) = q < i(t) >$ . It is independent of the Fourier frequency  $\Omega$  (white noise). It is easily measurable using photodetectors having a quantum efficiency approaching 1.

Continuous quantum noise, in which the exact value of the photocurrent is unpredictable, is another avatar of the uncertainty inherent to the quantum world. Though the individual photons are no longer distinguishable, their random occurrence is at the origin of the quantum fluctuations. Note that quantum noise affects other quantities than the light intensity, like for example the phase of a monochromatic electromagnetic field, or the position and momentum of mechanical harmonic oscillators.

Quantum fluctuations can be mastered ([17] p. 387), within some constraints that are related to the Heisenberg inequality. For example, nonlinear optical processes, such as parametric down-conversion, generate a so-called "sub-Poissonian" light which have less quantum intensity fluctuations than shot noise. Such light can be modeled by photon distributions that are more regular than the Poisson distribution. The reduction of quantum noise on light intensity is accompanied by an increase of quantum noise in the phase of the field.

## Chapitre 2

# The density matrix : a general description of quantum systems

### 2.1 Postulates of Basic Quantum Mechanics

*If quantum mechanics has not profoundly shocked you, you have not understood it yet* (N. Bohr)

*Anyone who claims to understand quantum theory is either lying or crazy* (R. Feynman)

The reader will find in chapter 9 a detailed reminder of the basic principles of Quantum Mechanics. They concern state vectors and ideal measurements and are valid in the particular case of an isolated system (or of a system submitted to fixed forces) which is prepared in a totally controlled way. We will call a such isolated and controlled state a *pure state*.

We simply recall here the corresponding postulates :

**(S1) Superposition principle postulate** : an isolated physical system, prepared in a totally controlled way (pure state case), is described by a single mathematical object, called a *state vector*, or "ket",  $|\Psi\rangle$ , belonging to a well defined Hilbert space  $\mathcal{H}$ .

**(S2 ) Normalization postulate** : the state vector, or "ket", is normalized to 1 :

$$|\langle \Psi | \Psi \rangle|^2 = 1 \quad (2.1)$$

**(M1) Quantization postulate** : The measurement of the

physical quantity  $A$  can only give as a result one of the eigenvalues  $a_n$  of the associated observable  $\hat{A}$ .

**(M2) Born's rule postulate :** In general the result of the measurement cannot be predicted with certainty. The probability  $Proba(a_n, |\psi\rangle)$  of obtaining the result  $a_n$  when the system is in the quantum state described by the state vector  $|\psi\rangle$  is given by :

$$Proba(a_n, |\psi\rangle) = \langle\psi|\hat{P}_n|\psi\rangle \quad (2.2)$$

where  $\hat{P}_n$  is the projector on the eigen subspace associated with eigenvalue  $a_n$ .

**(M3) State collapse postulate :** The conditional state of the system just after the measurement and *and in the subset of cases where the measurement has given the result  $a_n$*  is :

$$|\psi^{after|a_n}\rangle = \frac{\hat{P}_n|\psi\rangle}{\sqrt{Proba(a_n, |\psi\rangle)}} \quad (2.3)$$

**(E1) Evolution postulate :** The time evolution of the state vector  $|\psi(t)\rangle$  is ruled by the Schrödinger equation :

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle \quad (2.4)$$

where  $\hat{H}(t)$  is the Hamiltonian of the system, which is the observable associated with its energy.

## 2.2 The density matrix as an alternative description of basic Quantum Mechanics

Let us now introduce a new tool, the density matrix or density operator  $\rho$ , which allows us to formulate the previous rules for pure states in a different way. We will then be able to extend these rules to more complex situations.

### 2.2.1 Definition

The density matrix  $\rho$  (also called density operator) of a pure state  $|\Psi\rangle$  is the operator acting on the Hilbert space  $\mathcal{H}$  given by :

$$\rho = |\Psi\rangle\langle\Psi| \quad (2.5)$$

### 2.2.2 Properties

From the definition (2.5), and using the following relations :

$$Tr|\phi_1\rangle\langle\phi_2| = \langle\phi_2|\phi_1\rangle \quad ; \quad Tr\hat{A}\hat{B} = Tr\hat{B}\hat{A} \quad (2.6)$$

one sees that the density matrix is an hermitian and positive operator of trace unity :

$$\rho = \rho^\dagger \quad ; \quad \forall |\phi\rangle, \langle\phi|\rho|\phi\rangle \geq 0 \quad ; \quad Tr\rho = 1 \quad (2.7)$$

It is easy to see that two state vectors differing by a phase factor  $e^{i\alpha}$  describe the same physical situation. This freedom of choice for the global phase does not exist for the density matrix. In addition, since  $\rho^2 = \rho$ , the density matrix is also a projector. Its eigenvalues are zeros and a single 1. The eigenvector associated with the eigenvalue 1 is precisely the state vector of the system.

So, state vectors and density matrices are two mathematical objects that one can use to describe a given physical system.

### 2.2.3 Expression of the previous postulates in terms of $\rho$

It is thus possible to re-write the postulates recalled in the previous section in the new framework :

**(S1)** Superposition principle postulate : an isolated physical system, prepared in a totally controlled way (pure state case), is described by a single mathematical object, the *density matrix*  $\rho$ , which is a projection operator acting in a well defined Hilbert space  $\mathcal{H}$ .

**(S2)** Normalization postulate : the density matrix is normalized to 1 :

$$Tr\rho = 1 \quad (2.8)$$

**(M1)** Quantization postulate : the measurement of the physical quantity  $A$  can only give as a result one of the eigenvalues  $a_n$  of the associated observable  $\hat{A}$ .

**(M2)** Born's rule postulate : in general the result of the measurement cannot be predicted with certainty. The probability  $Proba(a_n, \rho)$  of obtaining the result  $a_n$  when the system is in the quantum state described by the density matrix  $\rho$  is given by :

$$Proba(a_n, \rho) = Tr\hat{P}_n\rho \quad (2.9)$$

where  $\hat{P}_n$  is the projector on the eigen subspace associated with the eigenvalue  $a_n$ .

**(M3)** State collapse postulate : the conditional state of the system just after the measurement and *in the subset of cases where the measurement has given the result  $a_n$*  is described by the density matrix :

$$\rho^{after|a_n} = \frac{\hat{P}_n \rho \hat{P}_n}{Proba(a_n, \rho)} \quad (2.10)$$

**(E1)** Evolution postulate : the time evolution of the density matrix  $\rho(t)$  is given by

$$\rho(t) = \hat{U}(t)\rho(0)\hat{U}^\dagger(t) \quad (2.11)$$

where the unitary evolution operator  $\hat{U}(t)$  is ruled by the operatorial Schrödinger equation :

$$i\hbar \frac{d}{dt} \hat{U}(t) = \hat{H}(t) \hat{U}(t) \quad (2.12)$$

$\hat{H}$  being the Hamiltonian of the system, which is the observable associated with its energy.

## 2.2.4 Examples

-Qubit :

A qubit described by the state  $|\psi(t)\rangle = \alpha e^{-iE_- t/\hbar} |-\rangle + \beta e^{-iE_+ t/\hbar} |+\rangle$  is also described by the density matrix :

$$\rho = \begin{pmatrix} |\alpha|^2 & \alpha^* \beta e^{-i\omega t} \\ \alpha \beta^* e^{i\omega t} & |\beta|^2 \end{pmatrix} : \quad (2.13)$$

with  $\hbar\omega = E_+ - E_-$ . The diagonal terms are real, positive and time independent. They are called "*populations*" and give the probability to be in the corresponding state. The off-diagonal terms oscillate in time at the Bohr frequency  $\omega$  and are complex conjugate to each other. They are called "*coherences*" and are more subtle to understand. As for all oscillating quantities not only their amplitude but also their phase plays an important role. They contain most of the wave aspects of the quantum system.

-Particle :

A 1D particle described by the wavefunction  $\psi(x)$  is also descri-

bed by the two variable density matrix  $\rho(x_1, x_2)$  :

$$\rho(x_1, x_2) = \psi^*(x_2)\psi(x_1) \quad (2.14)$$

The diagonal part  $x_1 = x_2 = x$  give the probability for the particle to be at point  $x$ , whereas the off-diagonal part gives information about the correlations between position measurements made at two different points.

## 2.3 First extension : imperfect state preparation

The main interest of the density matrix is that the formalism can be extended to other situations than the simple one consisting of an isolated system in a perfectly controlled state on which perfect measurements are performed. In this section, we consider the case where one does not completely master the preparation of the quantum state, a frequent situation in real experiments.

### 2.3.1 Example

Let us consider a spin 1/2 particle. Its two basis states ,  $|+\rangle$  and  $|-\rangle$  , correspond to two opposite orientations of its magnetic moment. A spin pointing in the direction of polar angles  $\theta \phi$  is described by the state vector :

$$|\psi(\theta, \phi)\rangle = \cos(\theta/2)|-\rangle + e^{i\phi} \sin(\theta/2)|+\rangle \quad (2.15)$$

In the Stern et Gerlach experiment, the particle (more precisely the silver atom) is produced in a heated oven. As a result, the atom exiting the oven has a spin pointing in a direction which is not controlled. It is described by a state vector  $|\psi(\theta, \phi)\rangle$  with random  $\theta$  and  $\phi$  angles.

### 2.3.2 Definition of a statistical mixture

Let us consider in a general way the situation where the quantum system has a probability  $p_1$  to be prepared in state  $|\psi_1\rangle$ , ...,  $p_i$  to be prepared in state  $|\psi_i\rangle$ , with  $i = 1, \dots, N$  and  $\sum_i p_i = 1$  (these states are not necessarily orthogonal). One can then define an averaged density matrix, or *statistical mixture*,  $\rho_M$ , by :

$$\rho_M = \sum_{i=1}^N p_i |\psi_i\rangle \langle \psi_i| \quad (2.16)$$

One sees easily that  $\rho_M$  is of trace 1, and, because  $p_i \geq 0$ , is a positive operator.

### 2.3.3 Properties

The probability of obtaining the result  $a_n$  in the measurement of A is in the present situation the average of the probabilities that one obtains when the system is in one of the prepared states  $|\psi_i\rangle$ , i.e. :

$$Proba(a_n) = \sum_{i=1}^N p_i \text{Tr} \hat{P}_n |\psi_i\rangle \langle \psi_i| = \text{Tr} \hat{P}_n \rho_M \quad (2.17)$$

One sees that the usual Born's rule can be extended to the case of a statistical mixture in the density matrix formalism. This is not possible in the wave vector formalism. The reason is that the Born's expression for the probability is quadratic in  $|\psi\rangle$  but linear in  $\rho$ .

What about the state collapse postulate M3 ? The conditional state of the system when the measurement has given the result  $a_n$  is described by the statistical mixture of collapsed states  $\frac{\hat{P}_n |\psi_i\rangle \langle \psi_i| \hat{P}_n}{\langle \psi_i | \hat{P}_n | \psi_i \rangle}$ , weighted by the probability of getting such a state,  $\frac{p_i \langle \psi_i | \hat{P}_n | \psi_i \rangle}{\sum_i p_i \langle \psi_i | \hat{P}_n | \psi_i \rangle}$ , normalized product of the probability  $p_i$  of state preparation and the probability  $\langle \psi_i | \hat{P}_n | \psi_i \rangle$  of getting the result  $a_n$  :

$$\rho^{after|a_n} = \sum_i \frac{p_i}{\sum_i p_i \langle \psi_i | \hat{P}_n | \psi_i \rangle} \hat{P}_n |\psi_i\rangle \langle \psi_i| \hat{P}_n = \frac{\hat{P}_n \rho_M \hat{P}_n}{\text{Tr} \hat{P}_n \rho_M} \quad (2.18)$$

The state collapse postulate can therefore be extended to statistical mixtures.

### 2.3.4 Back to the example

In the spin 1/2 case, the density matrix describing a spin pointing in the direction of polar angles  $\theta \phi$  is, from (2.15) :

$$\rho(\theta, \phi) = \begin{pmatrix} \cos^2(\theta/2) & \cos(\theta/2) \sin(\theta/2) e^{i\phi} \\ \cos(\theta/2) \sin(\theta/2) e^{-i\phi} & \sin^2(\theta/2) \end{pmatrix} \quad (2.19)$$

When such a matrix is averaged over all the possible values of the angles, supposed equiprobable in equal solid angles  $d\Omega = \sin\theta d\theta d\phi$ , one gets the statistical mixture :

$$\rho_M = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{I}/2 \quad (2.20)$$

We see that the process of averaging has completely washed out the coherences, and equalized the population of the two levels. We note also that the same density matrix could have been obtained

in another set-up where one has a probability 1/2 of preparing the state  $|+\rangle$  and a probability 1/2 of preparing the state  $|-\rangle$  : different physical situations may lead to the same statistical mixture.

This example allows us to precise the role of the coherences, off-diagonal elements of the density matrix. Let us consider the spins pointing in the  $\theta = \pi/2, \phi = 0$  direction. It is described by the state vector  $(|+\rangle + |-\rangle)/\sqrt{2}$  or by the density matrix :

$$\rho(\theta = \pi/2, \phi = 0) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad (2.21)$$

Matrices (2.21) and (2.20) differ only by the coherences.  $\rho(\theta = \pi/2, \phi = 0)$  describes a pure state, coherent superposition of the two basis states with equal coefficients, representing a system which is at the same time in states  $|-\rangle$  AND  $|+\rangle$ . In contrast  $\rho_M$  describes a statistical mixture representing a system which is in a random way in states  $|-\rangle$  OR  $|+\rangle$ .

### 2.3.5 Purity

To distinguish the two situations described in the first and second sections of this chapter, we call *pure state* a quantum state described by a density matrix  $\rho$  which can be written as  $|\psi\rangle\langle\psi|$ , whatever  $|\psi\rangle$ . We call *mixed state* a quantum state for which this reduction is not possible.

Let us now calculate  $\rho_M^2$  :

$$\rho_M^2 = \sum_{i,j} p_i p_j |\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j| \quad (2.22)$$

This quantity is different from  $\rho_M$ , even when  $\langle\psi_i|\psi_j\rangle = \delta_{ij}$ , unless the sum reduces to a single term, i.e. when the density matrix is equal to  $|\psi_1\rangle\langle\psi_1|$  and corresponds to a pure state. Consequently *the density matrix of a mixed state is not a projector*.

One can then define the *purity*  $P_u$  of a state described by a density matrix  $\rho$  by :

$$P_u = Tr\rho^2 \quad (2.23)$$

One can easily show that the state is pure if and only if  $P_u$  is unity. In the general case  $P_u$  is smaller than 1, and it reaches its smaller value  $1/d$ , where  $d$  is the dimension of the Hilbert space, for the density matrix proportional to the identity  $\rho = \hat{1}/d$  (like in (2.20)), which corresponds to a situation of minimal control over the preparation : all states of any basis of the Hilbert space have an equal probability to have been prepared. The only information about the prepared state is that it belongs to a given Hilbert space.

### 2.3.6 Two kinds of randomness

Let us go back to the expression (2.17) of the probability of a measurement on the system in a mixed state :

$$Proba(a_n) = \sum_{i=1}^N p_i Tr \hat{P}_n |\psi_i\rangle\langle\psi_i| \quad (2.24)$$

We have in this formula a probability which is the result of two uncertainties of radically different natures : the uncertainty described by the probabilities  $p_i$  is due to the experimentalist's ignorance or laziness. It can be reduced and even suppressed by using more sophisticated experimental setups able to generate a unique quantum state. The uncertainty described by the Born's rule  $Tr \hat{P}_n |\psi_i\rangle\langle\psi_i|$  is of fundamental nature. It exists even when everything is perfectly controlled in the experiment and cannot be completely reduced, unless one restricts oneself to states which are eigenstates of the measured observable.

The existence of such a fundamental quantum uncertainty is not good news for physicists, the aim of which is to predict precisely every phenomenon in Nature. So, several of them felt uncomfortable with this situation. Einstein is the most famous of them. Let us quote him :

*(Quantum) theory brings us many things, but it hardly allows us to get closer to the secret of the Old Man. Anyway, I am convinced that at least He does not play dice* (A. Einstein, letter to M. Born 1926)

Physicists have wondered whether the quantum uncertainty is due to our partial ignorance of the quantum world. One can suppose that there exist supplementary degrees of freedom of the system under study that we do not know, and therefore do not master in the experiments : the uncertainty in the control of these "hidden variables" would induce a corresponding uncertainty in the measurement results, that we phenomenologically describe presently by the Born's rule. This was in particular the hope of Einstein, hence the title of his famous paper written with Podolski and Rosen : *is quantum Mechanics complete?*. We will examine further this issue in chapter 7. Let us just say here that that the existence of *local* supplementary degrees of freedom leads to conclusions (the famous Bell inequality) which are in contradiction with experimental results.

## 2.4 Second extension : non-isolated systems

We will now make a second step and consider systems which are no longer isolated, nor submitted to fixed forces.

### 2.4.1 Example and position of problem

Let us consider the single ion trapped in D. Wineland's experiment described in chapter 1. It cannot be considered as alone in the Universe. It had a long history before it was trapped and submitted to the measurements : it comes from a chemical reaction, and has been heated, evaporated and captured. In the trap it is submitted to collisions with the residual gas in the vacuum chamber and its motion induces currents in the electrodes which create the trapping potential.

So, in addition to the Hilbert space  $\mathcal{H}$  of the studied system S, one has to consider the Hilbert space of the systems with which it has interacted beforehand, and the Hilbert space of the systems with which it is presently interacting. We will call  $\mathcal{H}_R$  ("R" for "reservoir") the space describing all the systems with which the system S is or has been, interacting, so that one can consider that the total Hilbert space  $\mathcal{H} \otimes \mathcal{H}_R$ , tensor product of the Hilbert spaces  $\mathcal{H}$  and  $\mathcal{H}_R$ , is isolated. We will denote by  $|u_i\rangle$  ( $i = 1, \dots, d$ ) a basis of  $\mathcal{H}$ , by  $|v_j\rangle$  ( $j = 1, \dots, d_R$ ) a basis of  $\mathcal{H}_R$  and by  $|u_i\rangle \otimes |v_j\rangle = |u_i, v_j\rangle$  a basis of  $\mathcal{H} \otimes \mathcal{H}_R$  (for a precise introduction of tensor products of spaces and states, see [6] chapter 2F, and the reminder 2 in chapter 10 of these lecture notes).

### 2.4.2 Measurement on a non-isolated system : reduced density matrix

The whole system, including the reservoir, is described by a "total" density matrix  $\rho_{tot}$  acting on  $\mathcal{H} \otimes \mathcal{H}_R$ . We now want to measure a physical quantity  $A$ , that gives information about the system under study S and not about the reservoir. It is associated with an observable  $\hat{A}$  operating only in  $\mathcal{H}$ . The probability of measuring the eigenvalue  $a_n$  is

$$Proba(a_n, \rho_{tot}) = Tr \rho_{tot} \hat{P}_n = \sum_{i,j} \langle u_i, v_j | \rho_{tot} \hat{P}_n | u_i, v_j \rangle \quad (2.25)$$

Inserting the closure relation  $\sum_{i',j'} |u_{i'}, v_{j'}\rangle \langle u_{i'}, v_{j'}| = \hat{I}$  and taking into account the fact that the projector  $\hat{P}_n$  acts only in the Hilbert

space  $\mathcal{H}$ , one obtains :

$$Proba(a_n, \rho_{tot}) = \sum_{i,j,i',j'} \langle u_i, v_j | \rho_{tot} | u_{i'}, v_{j'} \rangle \langle u_{i'} | \hat{P}_n | u_i \rangle \delta_{j,j'} \quad (2.26)$$

$$= \sum_{i,j,i'} \langle u_i, v_j | \rho_{tot} | u_{i'}, v_j \rangle \langle u_{i'} | \hat{P}_n | u_i \rangle \quad (2.27)$$

Let us now introduce an operator  $\rho_{reduced}$  acting only in the "small" space  $\mathcal{H}$ , called the reduced density matrix, that is defined by its matrix elements :

$$\langle u_i | \rho_{reduced} | u_{i'} \rangle = \sum_j \langle u_i, v_j | \rho_{tot} | u_{i'}, v_j \rangle \quad (2.28)$$

Using the closure relation  $\sum_{i'} |u_{i'}\rangle\langle u_{i'}| = \hat{1}_{\mathcal{H}}$  we can write :

$$\begin{aligned} Proba(a_n, \rho_{tot}) &= \sum_{i,i'} \langle u_i | \rho_{reduced} | u_{i'} \rangle \langle u_{i'} | \hat{P}_n | u_i \rangle \quad (2.29) \\ &= \sum_i \langle u_i | \rho_{reduced} \hat{P}_n | u_i \rangle = Tr \rho_{reduced} \hat{P}_n \end{aligned}$$

One finally retrieves the usual Born's rule relative to the "small space" only, provided one uses the reduced density matrix defined in (2.28), which can be written in a more symbolic way as

$$\rho_{reduced} = \sum_j \langle v_j | \rho_{tot} | v_j \rangle = Tr_{\mathcal{H}_R} \rho_{tot} \quad (2.30)$$

This operation is called a "partial trace", as the trace is made by summing the diagonal elements only in the "big" Hilbert space  $\mathcal{H}_R$  : it does not yield a number, but an operator acting in the "small" space.

Let us note that we have for the partial trace the useful relation :

$$Tr_{\mathcal{H}_R} |u_{i'}, v_{j'}\rangle\langle u_i, v_j| = \langle v_j | v_{j'} \rangle |u_{i'}\rangle\langle u_i| \quad (2.31)$$

The Born's rule is therefore valid for a non isolated system S, whatever the strength of the interaction between S and the outer world, provided one uses in the formula the density matrix (2.28) or (2.30), obtained by making a partial trace on the total density matrix over the part of the system that one wants to ignore.

What about the state collapse postulate M3 ? The conditional state of the system just after the measurement on part  $\mathcal{H}$  when the measurement has given the result  $a_n$  is described by the total density matrix :

$$\rho_{tot}^{after|a_n} = \frac{\hat{P}_n \rho_{tot} \hat{P}_n}{Tr(\rho_{tot} \hat{P}_n)} \quad (2.32)$$

The reduced density matrix  $\rho_{reduced}^{after|a_n}$  is obtained by the partial trace operation over  $\mathcal{H}_R$ , which commutes with the protection operator  $\hat{P}_n$  of  $\mathcal{H}$ . Normalization to 1 of the reduced density matrix implies then that :

$$\rho_{reduced}^{after|a_n} = \frac{\hat{P}_n \rho_{reduced} \hat{P}_n}{Tr(\rho_{reduced} \hat{P}_n)} \quad (2.33)$$

The postulates  $M_2$  and  $M_3$  given at the beginning of this chapter in terms of the density matrix for ideal measurements apply thus to non isolated systems described in terms of reduced density matrices.

### 2.4.3 Non locality of Quantum Mechanics

It is often stated that when one wants to describe a non isolated quantum system S, one needs to consider it within the whole Hilbert space describing all the quantum systems with which it has interacted, even though these systems are gone far away from S, so that no interaction remains between them. When the whole system is in an entangled state, then non-local properties of the quantum world are likely to occur, as we will see in chapter 7.

It is interesting to note that a complete local quantum mechanical description of the system S is possible, using only reduced density matrices acting in the "small" Hilbert space  $\mathcal{H}$ , even if S is a part of a much larger ensemble. This is true provided one is only interested in "local properties", described by observables acting only inside  $\mathcal{H}$ . Of course, using  $\rho_{reduced}$  instead of  $\rho_{tot}$  one loses all the information about the reservoir and about the correlations between the reservoir and S. The puzzling non local properties of quantum systems appear only when one is considering such correlations.

### 2.4.4 Properties of the reduced density matrix

Using 2.30) one easily shows that  $\rho_{reduced}$  has the regular properties of a density matrix, i.e. unit trace :

$$Tr\rho_{reduced} = \sum_i \langle u_i | \rho_{reduced} | u_i \rangle = \sum_{i,j} \langle u_i, v_j | \rho_{tot} | u_i, v_j \rangle = Tr\rho_{tot} = 1 \quad (2.34)$$

and positivity :

$$\langle \phi | \rho_{reduced} | \phi \rangle = \langle \phi | (\sum_j \langle v_j | \rho_{tot} | v_j \rangle) | \phi \rangle = \sum_j \langle \phi, v_j | \rho_{tot} | v_j, \phi \rangle \geq 1 \quad (2.35)$$

as a sum of positive numbers whatever  $|\phi\rangle \in \mathcal{H}$ .

Obviously, the operation of partial trace partly destroys the purity  $Tr\rho^2$  of the total state, unless it can be factorized as the product of a pure state in S and a pure state in the reservoir R.

#### 2.4.5 Example 1 : one qubit coupled to another

Let us consider the very simple case where S is a qubit which is coupled to a reservoir consisting of another qubit, and assume that the total system of both qubits is described by the state vector which is a Bell state :

$$|\Psi_-\rangle = \frac{1}{\sqrt{2}}(|1:+,2:-\rangle - |1: -,2:+\rangle) \quad (2.36)$$

where 1 denotes the system S and 2 the reservoir R. Such an entangled state contains strong correlations between the two qubits. The corresponding total density matrix is :

$$\rho_{tot} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.37)$$

The reduced density matrix can be calculated using (2.28) :

$$\rho_{reduced} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \quad (2.38)$$

One finds the same density matrix as (2.20). It corresponds to a totally randomized qubit. We will come back to this example in chapter 7 about entanglement.

#### 2.4.6 Example 2 : spontaneous emission

Let us consider a two-level atom coupled to the electromagnetic field. Let us suppose that at time  $t = 0$  it is in the excited state  $|e\rangle$  without any photon present  $|\Psi(0)\rangle = |e\rangle \otimes |0\rangle$ , where  $|0\rangle$  is the vacuum state. From this state, the system spontaneously decays to the ground state by emitting a single photon in an electromagnetic field mode labeled  $|1\ell\rangle$ , these modes differing by the direction of

emission, polarization and frequency. The complete theory, found for example in reference [17], shows that the state of the total system  $|\Psi(t)\rangle$  is :

$$|\Psi(t)\rangle = \alpha(t)|e\rangle \otimes |0\rangle + \sum_{\ell} \beta_{\ell}(t)|g\rangle \otimes |1_{\ell}\rangle \quad (2.39)$$

with  $|\alpha(t)|^2 = e^{-\Gamma t}$ ,  $\Gamma$  being the decay constant and  $\sum_{\ell} |\beta_{\ell}(t)|^2 + |\alpha(t)|^2 = 1$ .

If one is only interested in the fate of the atom, one will use the atomic reduced density matrix obtained by tracing over the field degrees of freedom :

$$\rho_{at} = \langle 0|\Psi(t)\rangle\langle\Psi(t)|0\rangle + \sum_{\ell} \langle 1_{\ell}|\Psi(t)\rangle\langle\Psi(t)|1_{\ell}\rangle \quad (2.40)$$

$$= |\alpha(t)|^2|e\rangle\langle e| + (1 - |\alpha(t)|^2)|g\rangle\langle g| \quad (2.41)$$

The atomic density matrix is pure for  $t = 0$  and  $t \rightarrow \infty$ . The purity reaches the minimum value  $1/2$  of a fully randomized state for  $\Gamma t = \ln 2$ .

If one is only interested in the fluorescence light emitted by the decaying atom, one will use the photon reduced density matrix obtained by tracing over the atomic degree of freedom :

$$\rho_{photon} = \langle e|\Psi(t)\rangle\langle\Psi(t)|e\rangle + \langle g|\Psi(t)\rangle\langle\Psi(t)|g\rangle \quad (2.42)$$

$$= |\alpha(t)|^2|0\rangle\langle 0| + \sum_{\ell, \ell'} \beta_{\ell}(t)\beta_{\ell'}(t)|1_{\ell}\rangle\langle 1_{\ell'}| \quad (2.43)$$

The purity properties of the photon density matrix turn out to be the same as for the atomic density matrix. Note that at  $t \rightarrow \infty$ , the field is in the single photon pure state  $\sum_{\ell} \beta_{\ell}(\infty)|1_{\ell}\rangle$ .

These two reduced density matrices ignore of course the strong correlations existing between the atom and the field : if the atom is in the excited state, one is sure that no photon is present. If the atom is in the ground state, one is sure that a photon has been emitted.

## 2.5 General properties of the set $\mathcal{D}_{\mathcal{H}}$ of density matrices

We have shown that the density matrix formalism is able to describe all the possible situations encountered in quantum mechanics, and will now consider from a more general perspective the properties of these matrices.

### 2.5.1 Postulates S1 and S2 for general density matrices

We can now reformulate in their more general form the postulates dealing with the description of a physical system :

**(S1)** Superposition principle postulate : a physical system is described by a single mathematical object, the *density matrix*  $\rho$ , **which is a positive operator acting in a well defined Hilbert space  $\mathcal{H}$** .

**(S2 )** Normalization postulate : the density matrix is normalized to 1 :

$$Tr\rho = 1 \quad (2.44)$$

We will deal with the postulates (M1, M2, M3 and E1) about the measurement and the evolution in non-ideal situations in the next two chapters.

### 2.5.2 Mathematical properties of $\rho$

Let us call  $\mathcal{D}_{\mathcal{H}}$  the set of density matrices over  $\mathcal{H}$ , i.e. the set of linear operators acting on  $\mathcal{H}$  that are positive and of trace 1.

It can be shown that this implies that the density matrix is Hermitian. It can then be diagonalized, with real eigenvalues  $p_i$  comprised between 0 and 1 of sum 1, like probabilities, and eigenstates  $|\phi_i\rangle$ , so that one can always write  $\rho$  as the statistical mixture :

$$\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i| \quad (2.45)$$

A positive operator such as  $\rho$  can always be written as

$$\rho = \hat{A}^\dagger \hat{A} \quad (2.46)$$

where  $\hat{A}$  is any linear operator, hermitian or non hermitian, which is not unique : for example for a pure state  $\rho = |\psi\rangle\langle\psi|$ ,  $\hat{A} = |\phi\rangle\langle\psi|$  whatever  $|\phi\rangle$ .

The positivity implies for the matrix elements that  $|\rho_{ij}|^2 \leq \rho_{ii}\rho_{jj}$ , and therefore that non zero coherences exist only when the two states connected by the coherence are populated.

### 2.5.3 Properties of $\mathcal{D}_{\mathcal{H}}$

The set  $\mathcal{D}_{\mathcal{H}}$  of density matrices over  $\mathcal{H}$  is not a vector space, because a linear combination of positive operators is not in general a positive operator. This implies that there is no superposition principle for the density matrix : in a Young slit experiment for example, if  $\rho_1$  describes the state of the particle going through slit 1, and  $\rho_2$

the state of the particle going through slit 2, the whole system is not described by  $\rho = (\rho_1 + \rho_2)/2$ , which would rather describe the statistical mixture when the particle has 50% chances to go through slit 1 and 50% chances to go through slit 2. The interference pattern arises from the linear superposition of the corresponding wave vectors  $|\psi_1\rangle$  and  $|\psi_2\rangle$ , so that the density matrix describing the whole system is actually  $\rho_{tot} = (\rho_1 + \rho_2 + |\psi_1\rangle\langle\psi_2| + |\psi_2\rangle\langle\psi_1|)/2$  : the information about the interference is therefore more in the coherence  $|\psi_1\rangle\langle\psi_2|$  than in the populations.

The set  $\mathcal{D}_{\mathcal{H}}$  is actually a *convex space*, because of the following property :

$$\forall \rho_1, \rho_2 \in \mathcal{D}_{\mathcal{H}}, \forall p \in [0, 1] \quad p\rho_1 + (1 - p)\rho_2 \in \mathcal{D}_{\mathcal{H}} \quad (2.47)$$

meaning that one can superpose density matrices only in a probabilistic way. In the qubit case, the pure states belong to the surface of the Bloch sphere, whereas the mixed states, convex linear combinations of pure states, belong only to the interior of the Bloch sphere.

#### 2.5.4 Purification theorem

We now show that any density matrix acting in a Hilbert space  $\mathcal{H}$  can always be considered as the reduced density matrix of a pure state in a larger Hilbert space  $\mathcal{H}_{large}$

Let us consider a density matrix  $\rho \in \mathcal{D}_{\mathcal{H}}$ . As stated before, it can always be written as :

$$\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i| \quad (2.48)$$

Now let us introduce another Hilbert space, called the "ancilla" space,  $\mathcal{H}_a$ , of dimension equal to or larger than the number of non-zero eigenvalues of  $\rho$ , and its basis  $\{|v_j\rangle\}$ . Let us consider the state vector defined in  $\mathcal{H} \otimes \mathcal{H}_a$  :

$$|\psi_{tot}\rangle = \sum_i \sqrt{p_i} |\phi_i, v_i\rangle \quad (2.49)$$

The corresponding density matrix is :

$$\rho_{tot} = \sum_{i,j} \sqrt{p_i p_j} |\phi_i, v_i\rangle\langle\phi_j, v_j| \quad (2.50)$$

It is by construction the density matrix of a *pure state*. It is easy to check that the partial trace over  $\mathcal{H}_a$  of the pure case  $\rho_{tot}$  gives precisely the mixed state  $\rho$ .

The conclusion is that one can always "purify" a mixed state by going in a a Hilbert state of larger dimension. The purification can be actually performed in many different ways. We will use this property in the following.

## Chapitre 3

# Non ideal measurements on quantum systems

In chapter 2, we have considered ideal measurements (Von Neumann measurements) because they can be treated in a simple manner. They hardly describe realistic measurements, which very often destroy the quantum system under study and unavoidably present defects. In a way analogous to the previous chapter which has shown that it is possible to describe not perfectly prepared quantum states, we will present in this chapter a generalized approach of the measurement process which is able to incorporate in a simple way non ideal measurements in the formalism of Quantum Mechanics.

### 3.1 Non ideal measurement devices

We have quickly described in chapter 1 a few examples of real measurements that are currently used by experimentalists and are so sensitive that they give information about single quantum objects : single photon and single particle detectors, photon number resolving photon detectors, devices able to measure the energy level of a single trapped ion or the presence of a single photon inside an optical cavity. We have distinguished two kinds of measurement set-ups : the destructive ones, like the photomultiplier, the photodiode, the bolometer, the particle detectors, the selective ionization by an electrostatic field. In these measurement set-ups, the quantum system under study cannot be submitted to a second measurement. Others are non-destructive, like the "shelving" technique used to measure the energy level of an ion, or the detection of a photon in a cavity, which allow experimentalists to monitor in real time the whole evo-

lution of a single quantum object. These are called *Quantum Non Demolition* measurements.

In addition, all real detectors have more or less serious imperfections :

- They may "miss" events, i.e. be sensitive to individual quantum objects, but not to all of them. The ratio of actually recorded particles over the incident ones is called the *quantum efficiency* of the detector.
- They can give a wrong information, and indicate for example that an atom is in a given state whereas it is actually in another state. We have examples of wrong information about the Rydberg states  $|e\rangle$  and  $|g\rangle$  in the recording of figure(1.7), for example the rare presence of red clicks inside the series of blue clicks witnessing the absence of photon in the cavity. As the two levels can be seen as forming a qubit, this defect is characterized by what is called the *bit error rate*.
- A detector can give a non-zero signal, i.e. a "click", even when no particle is incident on it. This defect is characterized by the *dark count rate*.
- Most single particle detectors are able to detect the absence or presence of particles, but not to count exactly their number. Others have a *particle number resolving capacity*, up to a maximum number which is of the order of a few units.

## 3.2 New postulates for non ideal measurements

We will now modify the postulates M1, M2, M3 about the measurement so as to be able to describe non ideal measurements.

A generic measurement is sketched in figures (3.1) or (3.2) : a quantum system described by density matrix  $\rho$  acting in a Hilbert space  $\mathcal{H}$  interacts with a measurement device, or detector, designed to measure the physical parameter  $A$ . Many measurements are performed on successive copies of  $\rho$ . The measurement result is read and either

- stored in a computer memory (figure 3.1) which builds the histogram of the whole set of measured values. In the subset of recorded events where the measurement has given the result  $a_n$ , the conditional quantum state of the system exiting the detector is described by the density matrix  $\rho^{after|a_n}$ .

- used to open a shutter (figure 3.2) only when the measurement has given the result  $a_n$  and close it otherwise. The quantum state of the system exiting the detector is described by the density matrix

$$\rho^{after|a_n}.$$

The three generalized postulates ruling such a generic measurement are given below. They introduce new and important physical quantities that we will study in the following sections.

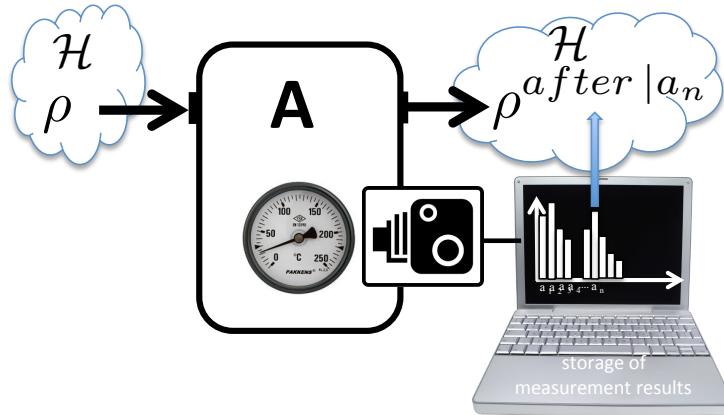


FIGURE 3.1 – Sketch of a measurement process (a) : results of the measurement are read then stored in a computer memory, which can display the histogram of results. Postulate M'3 gives the quantum state after the measurement if one selects the sub-set of events where the measurement has given the result  $a_n$ .

### 3.2.1 Quantization postulate

**(M'1) Quantization postulate :** the measurement of the physical property  $A$  can only give as a result a value  $a_n$  belonging to a list  $\{a_1, a_2, \dots, a_{N_A}\}$ . The number  $N_A$  of these possible values is not limited to the dimension of the Hilbert space  $\mathcal{H}$ .

For example, for a measurement determining the quantum state of the system as a result, and in the qubit case, the result list can be for example : 1 :  $|+\rangle$ , 2 :  $|-\rangle$ , and 3 :  $(|+\rangle + |-\rangle)/\sqrt{2}$ . In this case  $N_A = 3$ .

The list of measurement results is no longer the list of eigenvalues of an Hermitian operator, necessarily bounded by the dimension of the space : as a consequence, there is no observable operator  $\hat{A}$  associated with the measurement of the property  $A$  in the case of a non ideal measurement. The value  $a_n$  can just be the number  $n$  labeling the  $n^{th}$  possible result, like in the example given above

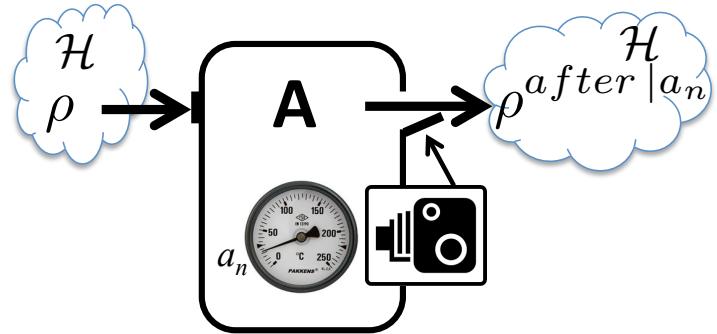


FIGURE 3.2 – Sketch of a generic measurement process (b) : results of the measurement are used to either open (when the result is  $a_n$ ) or close (when it is different from  $a_n$ ) a shutter. Postulate M'3 gives the quantum state actually exiting the device.

or any other number that one chooses to attach to this peculiar measurement result.

### 3.2.2 Born's rule postulate

**(M'2) Born's rule :** In general the result of the measurement cannot be predicted with certainty. The probability  $\text{Proba}(a_n, \rho)$  of obtaining the result  $a_n$  when the system is in the quantum state described by the density matrix  $\rho$  is given by :

$$\text{Proba}(a_n, \rho) = \text{Tr} \rho \hat{\Pi}_n \quad (3.1)$$

where  $\hat{\Pi}_n$  is a positive operator associated with the measurement device and the value  $a_n$ .

The set of operators  $\hat{\Pi}_n$  for all  $n$  values, that completely describes a given non ideal measurement device is usually called a "POVM" (Positive Operator-Valued Measure). Following S. Barnett[2], we will often call it a "POM" (Probability Operator Measure), each  $\hat{\Pi}_n$  being then a "PO" (Probability Operator).

### 3.2.3 State collapse postulate

(M'3) **State collapse postulate :** The conditional state of the system just after the measurement *and in the subset of measurements that have given the result  $a_n$* , called  $\rho^{after|a_n}$ , is :

$$\rho^{after|a_n} = \frac{1}{Proba(a_n, \rho)} \hat{M}_n \rho \hat{M}_n^\dagger \quad (3.2)$$

where the  $\hat{M}_n$  are linear operators acting on the Hilbert space  $\mathcal{H}$ .

The state described by  $\rho^{after|a_n}$  is called a *conditional state* because it is not the one which is always obtained after the measurement, but only in the special and uncontrolled opportunities when the measurement gives the precise result  $a_n$ . We will call "post-measurement operators" (PM) the operators  $\hat{M}_n$ .

### 3.2.4 Link with the previous postulates

It is simple to check that the new postulates include the old ones : in order to retrieve the usual formulation, one must simply take the projector  $\hat{P}_n$  (which is a positive operator) as both the probability operator  $\hat{\Pi}_n$  and the post-measurement operator  $\hat{M}_n$ .

## 3.3 Properties of the Probability Operator Measure

### 3.3.1 General properties

The positivity of operators  $\hat{\Pi}_n$  is a direct consequence of (3.1) and of the fact that a probability is a positive quantity.

A measurement performed on state  $\rho$  must necessarily give one of the results  $a_n$ , and therefore  $\sum_n Proba(a_n, \rho) = 1$  for any  $\rho$ . Therefore one has the following completeness relation in a POM :

$$\sum_{n=1}^{N_A} \hat{\Pi}_n = \hat{1} \quad (3.3)$$

The elements of a POM are not necessarily orthogonal i.e. the product  $\hat{\Pi}_n \hat{\Pi}_{n'}$  is not necessarily 0.

### 3.3.2 Gleason's theorem

The extended version (3.1) of Born's rule can be derived from very general arguments, as shown by Gleason's theorem[7], that we will state here without demonstration :

Let  $P(a_n, \rho)$  be a positive and smaller than 1 function of  $a_n$  and  $\rho$  such that

$$\sum_{n=1}^{N_A} P(a_n, \rho) = 1 \quad ; \quad P(a_n \text{ or } a_{n'}, \rho) = P(a_n, \rho) + P(a_{n'}, \rho) \quad (3.4)$$

and if the mathematical object describing the quantum state belongs to a Hilbert space  $\mathcal{H}$  of dimension larger than 2, then it exists a set of operators  $\hat{\Pi}_n$  independent of the state  $\rho$  of the system such that  $P(a_n, \rho) = \text{Tr}(\rho \hat{\Pi}_n)$ , i.e. relation (3.1). Conditions (3.4) are necessarily fulfilled by probabilities, so that this theorem applies obviously to the determination of the probability of measuring  $a_n$  on  $\rho$ .

The important point to remind here is that the generalized Born's rule is intrinsically linked to the linear character of the space in which the quantum states are defined, i.e. ultimately to the superposition principle.

### 3.3.3 Mean values, variances

Expression (3.1) allows us to calculate any statistical moment of the physical quantity  $A$  :

$$\langle A \rangle = \sum_{n=1}^{N_A} a_n \text{Proba}(a_n, \rho) = \text{Tr} \rho \hat{A}_1 \quad (3.5)$$

$$\langle A^k \rangle = \sum_{n=1}^{N_A} a_n^k \text{Proba}(a_n, \rho) = \text{Tr} \rho \hat{A}_k \quad (3.6)$$

where

$$\hat{A}_1 = \sum_{n=1}^{N_A} a_n \hat{\Pi}_n \quad ; \quad \hat{A}_k = \sum_{n=1}^{N_A} a_n^k \hat{\Pi}_n \quad (3.7)$$

$\hat{A}_1 = \sum_{n=1}^{N_A} a_n \hat{\Pi}_n$  can be considered as the extension to the case of non-ideal measurements of the observable  $\hat{A} = \sum_n a_n \hat{P}_n$  used in the case of a projective measurement. In this case  $\hat{A}_1 = \hat{A}$  and  $\hat{A}_k = \hat{A}^k$  because  $\hat{P}_n \hat{P}_{n'} = \delta_{n,n'}$ . This is no longer true for a non ideal measurement and there is no simple relation between  $(\hat{A}_1)^k$  and  $\hat{A}_k$ .

Therefore the observable  $\hat{A}_1 = \sum a_n \hat{\Pi}_n$  is not enough to determine all the statistical properties of the measurement results.

This is in particular the case for the variance of the measurement results. An interesting question then arises[8] : how to characterize the difference in terms of quantum fluctuations between a non-ideal and an ideal measurement, and how does it affect the usual Heisenberg inequality which is demonstrated for "usual" variances of hermitian operators ? One has, in the case of a non-ideal measurement :

$$Var(A) = \langle \hat{A}_2 \rangle - \langle \hat{A}_1 \rangle^2 = Var(A_1) + \Delta \quad (3.8)$$

where  $\Delta$  is the additional term which arises because one is performing a non-ideal measurement of quantity  $A$ , instead of an ideal measurement of the observable  $\hat{A}_1$ .  $\Delta$  can be written as

$$\Delta = \sum_n a_n^2 \hat{\Pi}_n - \hat{A}_1^2 = \sum_n (\hat{A}_1 - a_n) \hat{\Pi}_n (\hat{A}_1 - a_n) \quad (3.9)$$

remembering that  $\hat{A}_1 = \sum a_n \hat{\Pi}_n$  and  $\sum \hat{\Pi}_n = 1$ .  $\Delta$  is manifestly a non negative operator : as expected the non ideality of the measurement implies an additional uncertainty (equal to zero for a projective measurement) on top of the basic quantum one. As a consequence the Heisenberg inequality holds also for non-ideal measurements of non commuting operators.

### 3.4 Properties of the post-measurement operators $\hat{M}_n$

#### 3.4.1 General properties

The normalization of the density matrix  $Tr\rho = 1$  implies that :

$$\hat{\Pi}_n = \hat{M}_n^\dagger \hat{M}_n \quad (3.10)$$

A possible solution of (3.10) for  $\hat{M}_n$  is  $\hat{U} \sqrt{\hat{\Pi}_n}$  where  $\hat{U}$  is an arbitrary unitary operator (the operator  $\sqrt{\hat{\Pi}_n}$  exists because  $\hat{\Pi}_n$  is a positive operator). Post-measurement operators satisfying relation (3.10) for a given  $\hat{\Pi}_n$  are not at all unique : this is to be related to the fact that several experimental measurement setups can lead to the same probabilities of results, but to different post-measurement histories. To determine the operators  $\hat{M}_n$ , one needs to know the exact mechanism of the measurement apparatus.

### 3.4.2 Conditional state after a measurement made on a pure state

Let us now take a pure state  $\rho = |\psi\rangle\langle\psi|$  as the input state. The conditional state after a measurement giving result  $a_n$  is :

$$\rho^{after|a_n} = \frac{\hat{M}_n|\psi\rangle\langle\psi|\hat{M}_n^\dagger}{Proba(a_n, \psi)} = |\psi^{after|a_n}\rangle\langle\psi^{after|a_n}| \quad (3.11)$$

with :

$$|\psi^{after|a_n}\rangle = \frac{\hat{M}_n|\psi\rangle}{\sqrt{Proba(a_n, \psi)}} \quad (3.12)$$

*This means that the conditional state after a measurement, even non ideal, effected on a pure state is always a pure state.* Compared to the ideal case, one has just to replace the projection operator  $\hat{P}_n$  by the post-measurement operator  $\hat{M}_n$ .

### 3.4.3 Not post-selected measurement

Formula (3.2) gives the expression of the conditional state resulting from a post-selection process in the record of all performed measurements of those having given the result  $a_n$ , discarding all the other ones. If one does not perform this post-selection, one gets after the measurement a density matrix  $\rho^{after}$  that is the statistical mixture of all the post-selected states weighted by the corresponding probabilities, i.e. :

$$\rho^{after} = \sum_{n=1}^{N_A} \rho^{after|a_n} Proba(a_n, \rho) = \sum_{n=1}^{N_A} \hat{M}_n \rho \hat{M}_n^\dagger \quad (3.13)$$

This expression is also known as the result of an "unread measurement". Such a name can be misleading as it seems to imply a mysterious effect of the observer, which obtains different after-measurement states when he chooses to look or not at the dial of the measurement device. Actually the fact that the observer chooses to read or not to read the dial is not relevant : what is important is whether he chooses to consider the whole list of the measurement results, or only the smaller part of it corresponding to a given value  $a_n$ .

We note also that the equation (3.2) relating the conditional collapsed output state  $\rho^{after|a_n}$  to the input state  $\rho$  is *not linear in  $\rho$* , whereas the equation (3.13) relating the not post-selected output state  $\rho^{after}$  to the input state  $\rho$  is *linear in  $\rho$* . This statement is true also for an ideal Von Neumann measurement.

### 3.4.4 Naimark theorem

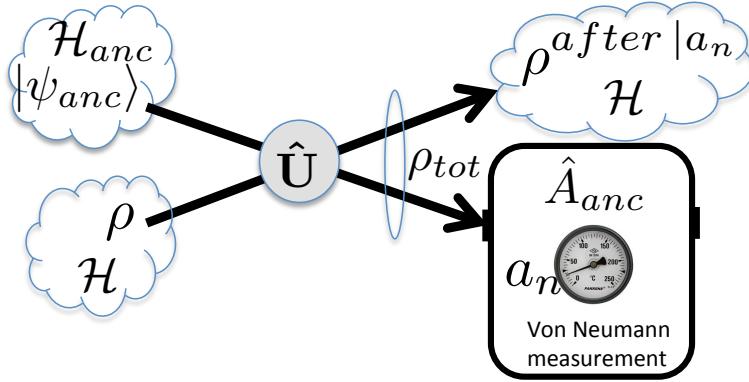


FIGURE 3.3 – Naimark theorem : a non ideal measurement in a given Hilbert space appears as a Von Neumann measurement in a larger space

The Naimark theorem[2], that we will not demonstrate here, is the equivalent for non-ideal measurements of the purification theorem for mixed states : any non ideal measurement in a given Hilbert space  $\mathcal{H}$  can be considered as a Von Neumann measurement in a larger Hilbert space.

More precisely, let us introduce an ancilla system  $S_{anc}$ , described by the Hilbert space  $\mathcal{H}_{anc}$ , that interacts during some time with the system S under study, described by states in Hilbert space  $\mathcal{H}$ . The initial state is the factorized state  $\rho \otimes |\psi_{anc}\rangle\langle\psi_{anc}|$ , where  $|\psi_{anc}\rangle$  is the initial pure state of the ancilla system. The evolution due to this coupling, ruled by the Schrödinger equation, is described by the unitary evolution operator  $\hat{U}$  on the whole space (see chapter 6). It creates the entangled state  $\rho_{tot} = \hat{U}\rho \otimes |\psi_{anc}\rangle\langle\psi_{anc}|\hat{U}^\dagger$ . After the interaction, one performs a Von Neumann ideal measurement within the ancilla space. It concerns an observable that we call  $\hat{A}_{anc}$  having the list of results  $\{a_n\}$  as eigenvalues and characterized by projectors  $\hat{P}_{anc,n}$  (which implies that the ancilla space has a dimension which can be larger than the dimension of  $\mathcal{H}$ ). The theorem shows that there is an initial ancilla state  $|\psi_{anc}\rangle$  and a unitary coupled evolution operator  $\hat{U}$  such that the probability  $Proba(a_n, \rho_{tot})$  of the ideal measurement calculated by (2.9) is equal to  $Proba(a_n, \rho)$  given by (3.1). In addition, when the measurement has given the result  $a_n$ , the reduced density matrix in space  $\mathcal{H}$  resulting from the global projective state collapse postulate (18) is precisely given by (3.2).

There are actually many different ways to perform this "idealization" of a non ideal measurement by considering a bigger space. Note also that the theorem implies that, if one considers only a part of a composite system on which a Von Neumann measurement is performed, the statistics of the measurement and the conditional state obey the postulates (3.1) and (3.2) of non ideal measurements.

## 3.5 Examples

### 3.5.1 Probability Operators

#### Bit error rate

Let us consider an imperfect measurement of the two qubit states : when the input state is  $|+\rangle$ , it has a probability  $p_+$  to give the answer "plus" (and a probability  $1 - p_+$  to give the answer "minus"). When the input state is  $|-\rangle$ , it has a probability  $p_-$  to give the answer "minus" (and a probability  $1 - p_-$  to give the answer "plus"). The POM is in this case :

$$\hat{\Pi}_{plus} = p_+|+\rangle\langle+| + (1-p_-)|-\rangle\langle-| \quad ; \quad \hat{\Pi}_{minus} = (1-p_+)|+\rangle\langle+| + p_-|-\rangle\langle-| \quad (3.14)$$

Note that, a priori, non-diagonal terms like  $\alpha|+\rangle\langle-| + \alpha^*|-\rangle\langle+|$  could be added in the expressions (3.14) of the POs without changing the value of the probabilities given above. But then these operators are no longer positive :  $Tr\rho\hat{\Pi}_{plus}$  and  $Tr\rho\hat{\Pi}_{min}$  take negative values for some states unless  $\alpha = 0$ .

One has obviously  $\hat{\Pi}_{plus} + \hat{\Pi}_{minus} = \hat{1}$ , but now  $\hat{\Pi}_{plus}^2 \neq \hat{\Pi}_{plus}$  : the POs are no longer projectors.

#### Quantum efficiency $\eta$ and dark count probability $d$

Let us consider an imperfect single photon detector : when the input state is a single photon state  $|1\rangle$ , it has a probability  $\eta$  to give a click (and a probability  $1 - \eta$  to give no click). When there is no input light, i.e. when the input state is the vacuum is  $|0\rangle$ , it has a probability  $d$  to give a click (and a probability  $1 - d$  to give no click). The POM is in this case :

$$\hat{\Pi}_{click} = d|0\rangle\langle 0| + \eta|1\rangle\langle 1| \quad (3.15)$$

$$\hat{\Pi}_{no\,click} = (1-d)|0\rangle\langle 0| + (1-\eta)|1\rangle\langle 1| \quad (3.16)$$

If one associates the measurement value  $n = 0$  to the no click situation, and  $n = 1$  to the click occurrence, then the effective number operator defined in (3.7),  $\hat{N}_1 = \sum n \hat{\Pi}_n$ , is equal to :

$$\hat{N}_1 = d|0\rangle\langle 0| + \eta|1\rangle\langle 1| \quad (3.17)$$

It has the same eigenstates as the ideal counter, but different eigenvalues. Let us recall that this number operator is only useful when one is interested in determining the mean value of the photon number.

### 3.5.2 Post-measurement operators

#### Case of an ideal but destructive measurement

Let us show that expression (3.2) includes the case of a measurement which fulfills the usual Born's rule (2.9), and is in this sense ideal, but which is destructive and does not fulfill the usual state collapse postulate (2.10).

This can be simply seen on the following simple case of an observable  $\hat{A}$  having non degenerate eigenvalues  $a_n$  and eigenstates  $|u_n\rangle$ . Then the POM is the set of projectors  $\hat{P}_n = |u_n\rangle\langle u_n|$ , and expression (3.1) reduces to the usual Born's rule (2.9). The measurement device destroys the input state and transforms it into the "destroyed state"  $|\phi_D\rangle$  whatever the result of the measurement (for example the quantum state describing the ion created by the field ionization process of figure (1.6) or from the photoelectric effect induced by the impinging photon in a photomultiplier). The corresponding post-measurement operator is :  $\hat{M}_n = |\phi_D\rangle\langle u_n|$ . In such a case :

$$\rho^{after|a_n} = \frac{|\phi_D\rangle\langle u_n|\rho|u_n\rangle\langle\phi_D|}{Tr\rho|u_n\rangle\langle u_n|} = |\phi_D\rangle\langle\phi_D| \quad (3.18)$$

whatever the result of the measurement.

#### Photodetection with a beamsplitter

A simple but imperfect non-destructive photodetector is sketched on figure (3.4) : the input beam to measure crosses a beamsplitter of amplitude transmission and reflexion coefficients  $t$  and  $r$  (with  $t^2 + r^2 = 1$ ). To simplify, we assume that the input state contains only 0 or 1 photon, so that we can restrict the Hilbert space to the one spanned by states  $|0\rangle$  and  $|1\rangle$ . The partially reflected light is measured by a photodetector, that we assume for simplicity to be perfect : no dark count and quantum efficiency 1. Note the similarity

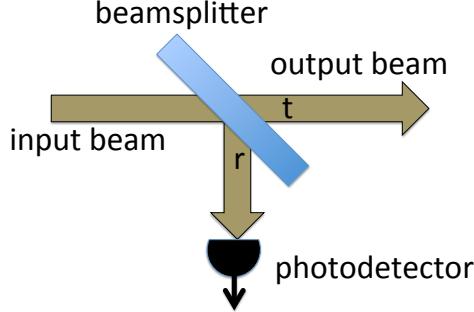


FIGURE 3.4 – Partially non destructive detection of a photon

of this scheme (figure 3.4 ) with the one considered by the Naimark theorem (figure 3.3 )

The system is simple enough to be the object of an exact calculation, with the help of the rules of quantum optics : if the input beam state is  $|0\rangle$ , then the output two-beam state is  $|0, 0\rangle$ , the two numbers giving the number of photons respectively in the transmitted and reflected beams ; if the input beam state is  $|1\rangle$ , then the output state is  $r|0, 1\rangle + t|1, 0\rangle$ . So the transformation giving the two-output beam density matrix for any input density matrix  $\rho$  is :

$$\rho^{out} = \begin{pmatrix} \rho_{00} & r\rho_{01} & t\rho_{10} & 0 \\ r\rho_{10} & r^2\rho_{11} & rt\rho_{11} & 0 \\ t\rho_{01} & rt\rho_{11} & t^2\rho_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.19)$$

the order of the basis elements being  $|0, 0\rangle, |0, 1\rangle, |1, 0\rangle, |1, 1\rangle$ .

The measurement being ideal, the click PO is the projector  $|1\rangle\langle 1|$  acting on the reflected beam, i.e. the operator  $\hat{P}_{click} = |0, 1\rangle\langle 0, 1| + |1, 1\rangle\langle 1, 1|$  in the total space. So the corresponding probability is  $Tr\rho\hat{P}_{click} = r^2\rho_{11}$ , and the total state after a click is :

$$\hat{\rho}_{tot}^{after|click} = \frac{\hat{P}_{click}\rho\hat{P}_{click}}{Tr\rho\hat{P}_{click}} = \frac{1}{r^2\rho_{11}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & r^2\rho_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = |01\rangle\langle 01| \quad (3.20)$$

The output beam state is obtained by a partial trace over the re-

flected beam space :

$$\hat{\rho}^{after|click} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 0| \quad (3.21)$$

which is rather obvious since the photon, measured on the reflected beam, cannot be at the same time on the transmitted beam. One notes that, considered as acting on an initial state belonging to the Hilbert space of the beam going straight through the beamsplitter, the measurement is not ideal and corresponds to POs and post-measurement operators given by :

$$\hat{\Pi}_{click} = r^2|1\rangle\langle 1| \quad ; \quad \hat{M}_{click} = r|0\rangle\langle 1| \quad (3.22)$$

One can do the same for the no-click case, with  $\hat{P}_{no\,click} = |0,0\rangle\langle 0,0| + |1,0\rangle\langle 1,0|$ . The corresponding probability is  $Tr\rho\hat{P}_{no\,click} = t^2\rho_{11} + \rho_{00}$ , and the total state after the measurement is :

$$\hat{\rho}_{tot}^{after|no\,click} = \frac{1}{t^2\rho_{11} + \rho_{00}} \begin{pmatrix} \rho_{00} & 0 & t\rho_{10} & 0 \\ 0 & 0 & 0 & 0 \\ t\rho_{01} & 0 & t^2\rho_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.23)$$

The partial trace gives the output beam state :

$$\hat{\rho}^{after|no\,click} = \frac{1}{t^2\rho_{11} + \rho_{00}} \begin{pmatrix} \rho_{00} & t\rho_{01} \\ t\rho_{10} & t^2\rho_{11} \end{pmatrix} \quad (3.24)$$

The corresponding PO and post-measurement operator in the output beam space are given by :

$$\hat{\Pi}_{no\,click} = |0\rangle\langle 0| + t^2|1\rangle\langle 1| \quad ; \quad \hat{M}_{no\,click} = |0\rangle\langle 0| + t|1\rangle\langle 1| \quad (3.25)$$

If the input state is the pure state  $\alpha|0\rangle + \beta|1\rangle$ , then the output state is also pure :  $|0\rangle$  in the click case, and  $(\alpha|0\rangle + \beta t|1\rangle)/\sqrt{|\alpha|^2 + |\beta|^2 t^2}$  in the no-click case.

The beamsplitter has the same POM as a detector with limited quantum efficiency  $\eta = t^2$ , but of course not the same post-measurement operator. One checks that  $\hat{M}_{click}^\dagger \hat{M}_{click} = \hat{\Pi}_{click}$ ,  $\hat{M}_{no\,click}^\dagger \hat{M}_{no\,click} = \hat{\Pi}_{no\,click}$  and  $\hat{\Pi}_{click} + \hat{\Pi}_{no\,click} = \hat{1}$ .

If the output state is not post-selected according to the measurement results, its expression after the crossing of the beamsplitter is :

$$\hat{\rho}^{after} = \begin{pmatrix} \rho_{00} + r^2\rho_{11} & t\rho_{01} \\ t\rho_{10} & t^2\rho_{11} \end{pmatrix} \quad (3.26)$$

This equation gives the effect of a beamsplitter on any input state. It could have been obtained more straightforwardly by taking the partial trace of  $\rho_{out}$  (Eq(3.4)). It is a linear transformation of the different components of the input density matrix which is trace preserving, i.e. a good example of a *quantum map* that we will study in a general way in chapter 4. Note that, as  $t < 1$ , *the coherence is reduced by the physical interaction with the measurement set-up*. We will come back to this in chapter 5.

## Chapitre 4

# Evolution of Quantum systems

Our aim in this chapter is to consider the evolution of quantum systems in the most general perspective.

### 4.1 Hamiltonian evolution of the density matrix

When the system is isolated or submitted to fixed forces and the quantum state of the system is perfectly controlled, we know from chapter 2 that the time evolution of the state vector  $|\psi(t)\rangle$  and of the associated density operator  $\rho(t)$  are given by

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle ; \quad \rho(t) = \hat{U}(t)\rho(0)\hat{U}^\dagger(t) \quad (4.1)$$

where the unitary evolution operator  $\hat{U}(t)$  obeys the differential equation

$$i\hbar \frac{d}{dt}\hat{U}(t) = \hat{H}(t)\hat{U}(t) \quad (4.2)$$

#### 4.1.1 Imperfect state preparation

Let us now consider an isolated quantum system the preparation of which is not perfectly controlled, as already considered in section (2.3). It is described by the statistical mixture :

$$\rho_M(t) = \sum_{i=1}^N p_i |\psi_i(t)\rangle\langle\psi_i(t)| \quad (4.3)$$

$p_i$  being the probability of preparing the system in state  $|\psi_i\rangle$ . One has then

$$\rho_M(t) = \hat{U}(t)\rho_M(0)\hat{U}^\dagger(t) \quad (4.4)$$

Note that equation (4.4) is the same for both pure and mixed states in the case of an isolated system.

#### 4.1.2 Non isolated system

We have seen in subsection (2.5) that a non-isolated quantum system can always be described by a density matrix  $\rho(t)$  which is the partial trace of a density matrix  $\rho_{tot}$  describing a pure state, therefore ruled by equation (4.4). The evolution of  $\rho_{tot}(t)$ , with the notations of (2.5.4), is given by :

$$\rho_{tot}(t) = \sum_{i,j} \sqrt{p_i p_j} \hat{U}_{tot}(t) |\phi_i, v_i\rangle \langle \phi_j, v_j| \hat{U}_{tot}^\dagger(t) \quad (4.5)$$

Because of the presence of interactions the evolution operator  $\hat{U}_{tot}(t)$  is not factorizable into a part acting on  $\mathcal{H}$  and a part acting on the reservoir space  $\mathcal{H}_R$  : as a result the evolution leads to an entangled state between these two spaces, the partial trace operation does not simply remove the reservoir part, and one cannot write the reduced density matrix  $\rho(t)$  as  $\hat{U}(t)\rho(0)\hat{U}^\dagger(t)$ .

Consequently the evolution of a non isolated system cannot be given by an equation like 4.4). We will call *Hamiltonian evolution* the one ruled by (4.4). It has some specific properties that we list now.

#### 4.1.3 Properties of Hamiltonian evolution

##### Conservation of purity

What about the Hamiltonian evolution of the purity of the system ? One has :

$$\rho^2(t) = \hat{U}(t)\rho^2(0)\hat{U}^\dagger(t) \quad \text{and} \quad Tr\rho^2(t) = Tr\rho^2(0) \quad (4.6)$$

The purity of a mixed state is therefore a constant of motion : a pure (mixed) state will stay pure (mixed).

##### Time independent Hamiltonian

If the system is isolated, it is characterized by a time-independent hamiltonian  $\hat{H}$  , and equation (4.2) leads to :

$$\hat{U}(t) = e^{-i\hat{H}t/\hbar} \quad (4.7)$$

## Reversibility

In addition, Hamiltonian evolution and *time reversibility* are often linked.

This is the case for a 1D particle submitted to a time-independent and *real* potential  $V(x)$ . The evolution of the wave function  $\psi(x, t)$  obeys the Schrödinger equation

$$i\hbar \frac{d}{dt} \psi(x, t) = \hat{H} \psi(x, t) \quad (4.8)$$

where  $\hat{H} = -(\hbar^2/2m)\partial^2/\partial x^2 + V(x)$ . Let us now take the complex conjugate of this equation :

$$-i\hbar \frac{d}{dt} \psi^*(x, t) = i\hbar \frac{d}{d(-t)} \psi^*(x, t) = \hat{H} \psi^*(x, t) \quad (4.9)$$

Therefore *the wave function  $\psi^*(x, t)$  is a solution of the same Schrödinger equation, but with a reversed time  $t \rightarrow -t$* .

If one starts from a given initial wave function  $\psi(x, 0)$ , the hamiltonian evolution will lead at time  $t_0$  to the wave function  $\psi(x, t_0) = \hat{U}(t_0)\psi(x, 0)$ . Let us then consider as a new initial condition at time  $t_0$  the wave function  $\psi^*(x, t_0)$ . The Hamiltonian evolution from  $t_0$  to  $2t_0$  due to  $\hat{H}$  will make the wave function evolve from  $\psi^*(x, t_0)$  to  $\hat{U}(t_0)\psi^*(x, t_0) = \psi^*(x, 0)$ , which has the same probability distribution of position than the initial one :  $|\psi(x, 2t_0)|^2 = |\psi(x, 0)|^2$ . Such an evolution is therefore *reversible*, like the corresponding one in classical mechanics.

More generally, one can show that any Hamiltonian evolution can be reversed. This means that, for any system characterized by a time independent Hamiltonian, there exists a *time reversal operator*,  $\hat{T}$ , such that, for any wave vector and any  $t_0$  :

$$\hat{U}(t_0)\hat{T}\hat{U}(t_0)|\psi(0)\rangle = \hat{T}|\psi(0)\rangle \quad (4.10)$$

For a particle  $\hat{T}$  is the complex conjugation of its wave function. For a spin 1/2 system, one can show that  $\hat{T}$  is proportional to the Pauli matrix  $\sigma_y$  :  $\hat{T} = -i\sigma_y$ .

## 4.2 Time evolution in the general case

### 4.2.1 Quantum map

We want now to determine in the most general case the quantum state  $\rho(t)$  of a system at time  $t$  knowing its state at time  $t = 0$ . We

can formally write it as

$$\rho(t) = \mathcal{L}(t)[\rho(0)] \quad (4.11)$$

This evolution, considered as connecting directly time  $t = 0$  to a later time  $t$  without determining the intermediate values is called a *Quantum map*. It does not necessarily consider the continuous evolution for all times in between. It can be also applied to the characterization of the *spatial propagation* of the quantum state, for example for states of light.

Because of the linearity of Quantum Mechanics and the convex nature of density matrices, one must have :

$$\mathcal{L}(t)[p\rho_1 + (1-p)\rho_2] = p\mathcal{L}(t)[\rho_1] + (1-p)\mathcal{L}(t)[\rho_2] \quad (4.12)$$

$\mathcal{L}(t)$  must therefore preserve linear superpositions of quantum states, and therefore be a linear operator acting on the Hilbert space of linear operators, sometimes called a "super-operator", or a "Liouvillean" .

Equation (4.11) implies another condition, namely that it transforms a density matrix into another density matrix, i.e. preserves during the evolution the positivity of the operator and the value 1 of the trace. This imposes strong constraints on the form of the quantum map that we consider in the next section.

#### 4.2.2 Kraus theorem

Its demonstration can be found for example in ref [1], p.173.

Actually, it turns out that the requirements of trace preservation and positivity are not enough for the demonstration. One must use a stronger, but quite natural, requirement, named *complete positivity*. We have seen in chapter 3 that a density matrix  $\rho$  acting on Hilbert space  $\mathcal{H}$  is often the result of a previous interaction with another physical system described in a Hilbert space  $\mathcal{H}_R$ . It is then the partial trace of a total density matrix  $\rho_{tot}$  in the whole space  $\mathcal{H} \otimes \mathcal{H}_R$ . The hypothesis of complete positivity is the requirement that the operator  $\mathcal{L}(t)[\rho_{tot}]$  is a positive operator of space  $\mathcal{H} \otimes \mathcal{H}_R$  for all the states  $\rho_{tot} \in \mathcal{H} \otimes \mathcal{H}_R$ .

Let us call  $d$  the dimension of the Hilbert space of state vectors. Under the hypothesis of complete positivity, the Kraus theorem states that there exists  $N \leq d^2$  linear operators  $\hat{K}_\ell(t)$  on  $\mathcal{H}$  such that :

$$\mathcal{L}(t)[\rho] = \sum_{\ell=1}^N \hat{K}_\ell(t)\rho(0)\hat{K}_\ell^\dagger(t) \quad (4.13)$$

where the operators  $\hat{K}_\ell(t)$  are called Kraus operators. Trace preservation imposes the extra-condition that :

$$\sum_{\ell=1}^N \hat{K}_\ell^\dagger(t) \hat{K}_\ell(t) = \hat{1} \quad (4.14)$$

Of course, for a Hamiltonian evolution  $\hat{K}_1(t) = \hat{U}(t)$  : there is a single, unitary, Kraus operator.

#### 4.2.3 Evolution Postulate

We have now a new evolution postulate which is universally valid :

**(E'1) Evolution postulate : a quantum system initially in state  $\rho(0)$  is at a later time  $t$  in state  $\mathcal{L}(t)[\rho(0)]$  given by :**

$$\mathcal{L}(t)[\rho] = \sum_{\ell=1}^N \hat{K}_\ell(t) \rho(0) \hat{K}_\ell^\dagger(t) \quad (4.15)$$

where  $\hat{K}_\ell(t)$  are linear operators satisfying the condition :

$$\sum_{\ell=1}^N \hat{K}_\ell^\dagger(t) \hat{K}_\ell(t) = \hat{1} \quad (4.16)$$

#### 4.2.4 Properties

One easily checks the positivity of the transformed density matrix. One has for any  $|\psi\rangle$  and any  $\rho$  density matrix written as in (2.45) :

$$\langle \psi | \mathcal{L}(t)[\rho] | \psi \rangle = \sum_{\ell=1}^N \sum_i p_i \left| \langle \psi | \hat{K}_\ell(t) | \phi_i \rangle \right|^2 > 0 \quad (4.17)$$

The set of  $\hat{K}_\ell(t)$  operators used in (4.15) is actually not unique. Terms like  $\hat{K}_\ell(t) \rho \hat{K}_\ell^\dagger(t)$  in the map "look like" the usual hamiltonian evolution  $\hat{U}(t) \rho \hat{U}^\dagger(t)$  but with non unitary evolution operators, i.e. which do not necessarily all tend to the identity when the time interval is very small. They will describe the instantaneous quantum jumps observed in experiments with single quantum objects (chapter 1).

Any linear operator like  $\hat{K}_\ell$  has a *polar decomposition*  $\hat{K}_\ell = \hat{P}_\ell \hat{U}_\ell$ , where  $\hat{P}_\ell$  is positive and  $\hat{U}_\ell$  is unitary, in a way analogous to the polar decomposition of a complex number  $|z|e^{i\text{Arg}(z)}$ . One has therefore :

$$\mathcal{L}[\rho] = \sum_{\ell=1}^N \hat{P}_\ell \hat{U}_\ell \rho \hat{U}_\ell^\dagger \hat{P}_\ell^\dagger \quad (4.18)$$

The evolution is thus decomposed into a Hamiltonian part  $\hat{U}_\ell$  and a non-hamiltonian part with a positive Kraus operator.

With formula (4.13), one does not have any longer  $\text{Tr}(\mathcal{L}[\rho])^2 = \text{Tr}\rho^2$  : the evolution does not conserve in general the purity of the state, so that a pure state may become mixed with time, but also the opposite is possible. In addition, the evolution given by such a map is in general not reversible.

Let us finally stress that the number of Kraus operators is limited to  $d^2$ , each being of dimension  $2d^2$  (2 because numbers are complex) : in a Hilbert space of finite dimension, there is a finite number of possible evolutions equal to  $2d^4$ , for example 32 for the qubits.

#### 4.2.5 Quantum map in the Heisenberg representation

We have so far adopted the so-called Schrödinger representation approach, where the quantum state of the system evolves in time, and not the observable  $\hat{A}$ . There is another approach, namely the Heisenberg representation, in which it is just the opposite.

We assume here that one performs on the system an ideal measurement of the physical quantity  $A$ , so that the usual observable  $\hat{A} = \sum a_n \hat{P}_n$  can be used to determine the mean value :

$$\langle A \rangle(t) = \text{Tr}(\rho(t)\hat{A}) = \sum_{\ell=1}^N \text{Tr}\hat{K}_\ell(t)\rho(0)\hat{K}_\ell^\dagger(t)\hat{A} \quad (4.19)$$

$$= \text{Tr}(\rho(0)\hat{A}(t)) \quad (4.20)$$

with

$$\hat{A}(t) = \sum_{\ell=1}^N \hat{K}_\ell^\dagger(t)\hat{A}\hat{K}_\ell(t) \quad (4.21)$$

Unfortunately, this time-dependent operator can be used only the evolution of the mean, but not that of the variance or the correlations, because

$$\langle A^2 \rangle(t) = \text{Tr}(\rho(t)\hat{A}^2) = \sum_{\ell=1}^N \text{Tr}\hat{K}_\ell(t)\rho(0)\hat{K}_\ell^\dagger(t)\hat{A}^2 \quad (4.22)$$

$$= \text{Tr}(\rho(0)\hat{A}_2(t)) \quad (4.23)$$

with

$$\hat{A}_2(t) = \sum_{\ell=1}^N \hat{K}_{\ell}^{\dagger}(t) \hat{A}^2 \hat{K}_{\ell}(t) \neq \left( \sum_{\ell=1}^N \hat{K}_{\ell}^{\dagger}(t) \hat{A} \hat{K}_{\ell}(t) \right)^2 \quad (4.24)$$

The difference between  $\langle A^2 \rangle(t)$  and  $\langle \hat{A}^2 \rangle(t)$  can be shown to be positive. This implies that quantum maps introduce an *extra fluctuating term* of zero mean value. It can be for example accounted for by introducing in the evolution equation of  $\hat{A}(t)$  a random force, often called Langevin force, if one wants to use the equation useful not only for calculating means, but also second moments

#### 4.2.6 "Unitarization" theorem

It is the analog for quantum maps of the purification theorem for statistical mixtures, or of the Naimark theorem for imperfect measurements : any non unitary quantum map can be seen as a Hamiltonian evolution when the system under study is embedded in a larger Hilbert space with the help of an appropriately chosen "ancilla" space  $\mathcal{H}_{anc}$ . Let assume that its dimension is  $N$  and call  $|v_j\rangle$  its basis elements.

One defines a linear transformation  $\hat{V}$  mapping  $\mathcal{H} \otimes \mathcal{H}_{anc}$  onto  $\mathcal{H} \otimes \mathcal{H}_{anc}$  satisfying the relation (omitting  $t$  for simplicity) :

$$\hat{V}|\psi, v_1\rangle = \sum_{\ell=1}^N \hat{K}_{\ell}|\psi, v_{\ell}\rangle \quad (4.25)$$

for any  $|\psi\rangle$  belonging to  $\mathcal{H}$ . It is easy to show that, because of (4.16),  $(\hat{V}|\psi, v_1\rangle)^{\dagger} \hat{V}|\psi', v_1\rangle = \langle\psi|\psi'\rangle$  : the transformation, restricted to input states  $|\psi, v_1\rangle$  is an isometry. Mathematicians show us that it is always possible to extend  $\hat{V}$  to all input states  $|\psi, v_j\rangle$  ( $j > 1$ ) of  $\mathcal{H} \otimes \mathcal{H}_{anc}$  in such a way that the transformation is unitary in the whole tensor product space.  $\hat{V}$  can thus be considered as a Hamiltonian evolution in the whole space, transforming an input density matrix  $\rho_{tot} = |\psi, v_1\rangle\langle\psi, v_1|$  into  $\mathcal{L}[\rho_{tot}] = \hat{V}|\psi, v_1\rangle\langle\psi, v_1|\hat{V}^{\dagger}$ . The reduced density matrix in space  $\mathcal{H}$  is obtained by a partial trace over the ancilla :

$$\mathcal{L}[|\psi\rangle\langle\psi|] = Tr_{\mathcal{H}_{anc}}(\hat{V}|\psi, v_1\rangle\langle\psi, v_1|\hat{V}^{\dagger}) \quad (4.26)$$

$$= \sum_{\ell, \ell'} Tr_{\mathcal{H}_{anc}}(\hat{K}_{\ell}|\psi, v_{\ell}\rangle\langle\psi, v'_{\ell'}|\hat{K}_{\ell'}^{\dagger}) \quad (4.27)$$

$$= \sum_{\ell, \ell'} \langle v_{\ell'} | v_{\ell} \rangle \hat{K}_{\ell} |\psi\rangle\langle\psi| \hat{K}_{\ell'}^{\dagger} = \sum_{\ell} \hat{K}_{\ell} |\psi\rangle\langle\psi| \hat{K}_{\ell}^{\dagger} \quad (4.28)$$

the extension to mixed input states is then straightforward, which ends the demonstration.

We see here that, knowing the Kraus operators, one can deduce (at least partially) the expression of the unitary evolution in the whole space. Reciprocally, from a known unitary evolution  $\hat{V}$  one can derive the expression of the Kraus operators by making explicit the partial trace in (4.26)

$$\mathcal{L}[|\psi\rangle\langle\psi|] = \sum_{\ell} \langle v_{\ell} | \hat{V} | \psi, v_1 \rangle \langle \psi, v_1 | \hat{V}^{\dagger} | v_{\ell} \rangle = \sum_{\ell} \hat{K}_{\ell} |\psi\rangle\langle\psi| \hat{K}_{\ell}^{\dagger} \quad (4.29)$$

with  $\hat{K}_{\ell} = \langle v_{\ell} | \hat{V} | v_1 \rangle$ . Actually, one could have chosen another basis  $\{|v'_{\ell}\rangle\}$  of the ancilla space, and ended up with different Kraus operators  $\hat{K}'_{\ell} = \langle v'_{\ell} | \hat{V} | v_1 \rangle$ . The two bases being linked by a unitary transformation  $U$  of matrix elements  $U_{\ell\ell'}$ , the different sets of Kraus operators are linked by :

$$\hat{K}_{\ell} = \sum_{\ell'} U_{\ell\ell'} \hat{K}_{\ell'} \quad (4.30)$$

#### 4.2.7 Examples of quantum maps

##### Propagation of single photons through a beamsplitter

We have already considered this case in the previous chapter (Figure(3.4)) : we send at most one photon onto a beamsplitter of amplitude transmission and reflection coefficients  $t$  and  $r$ , and consider the state that is transmitted through it, discarding the reflected part. This indeed corresponds to a case where the considered system is far from being isolated. We have shown that the transmitted state  $\rho_T$  is given as a function of the state  $\rho$  at the input by :

$$\rho_T = \begin{pmatrix} \rho_{00} + r^2 \rho_{11} & t \rho_{01} \\ t \rho_{10} & t^2 \rho_{11} \end{pmatrix} \quad (4.31)$$

It is easy to see that this map is generated by two Kraus operators :  $\hat{K}_1 = |0\rangle\langle 0| + t|1\rangle\langle 1|$  and  $\hat{K}_2 = r|0\rangle\langle 1|$ . One checks that  $\hat{K}_1^{\dagger}\hat{K}_1 + \hat{K}_2^{\dagger}\hat{K}_2 = \hat{1}$

##### Evolution of qubits

Let us now consider the evolution of qubits, where the Kraus operators are  $2 \times 2$  matrices. Two extreme cases of evolution can occur, that we characterize here without reference to the detailed physics of the interaction leading to such an evolution.

— *Coherence damping*

It corresponds to the map :

$$\mathcal{L} \left[ \begin{pmatrix} \rho_{++} & \rho_{+-} \\ \rho_{-+} & \rho_{--} \end{pmatrix} \right] = \begin{pmatrix} \rho_{++} & D\rho_{+-} \\ D\rho_{-+} & \rho_{--} \end{pmatrix} \quad (4.32)$$

with  $D < 1$ , so that the populations are unchanged and the coherence reduced. Such an evolution is ruled by three Kraus operators :  $\hat{K}_1 = \sqrt{D}\hat{1}$ ,  $\hat{K}_2 = \sqrt{1-D}|+\rangle\langle+|$ ,  $\hat{K}_3 = \sqrt{1-D}|-\rangle\langle-|$ .

— *Population damping*

Because of the trace preservation, the population of the two levels cannot be damped at the same time, but for example,  $\rho_{--}$  can evolve into  $D\rho_{--}$  with  $D < 1$ . This can be described by the map :

$$\mathcal{L} \left[ \begin{pmatrix} \rho_{++} & \rho_{+-} \\ \rho_{-+} & \rho_{--} \end{pmatrix} \right] = \begin{pmatrix} 1 - D\rho_{--} & \sqrt{D}\rho_{+-} \\ \sqrt{D}\rho_{-+} & D\rho_{--} \end{pmatrix} \quad (4.33)$$

corresponding to the two Kraus operators  $\hat{K}_1 = |+\rangle\langle+| + \sqrt{D}|-\rangle\langle-|$ ,  $\hat{K}_2 = \sqrt{1-D}|+\rangle\langle-|$ . The beamsplitter is a particular case of this kind of map, with  $D = t^2$ .

When the same interaction is repeated  $n$  times, in the first case the coherence evolves into  $T^n\rho_{+-}$  and in the second case the population in  $|-\rangle$  state evolves into  $D^n\rho_{--}$ . Both vanish when  $n$  goes to infinity.

### 4.3 Two "colliding" systems

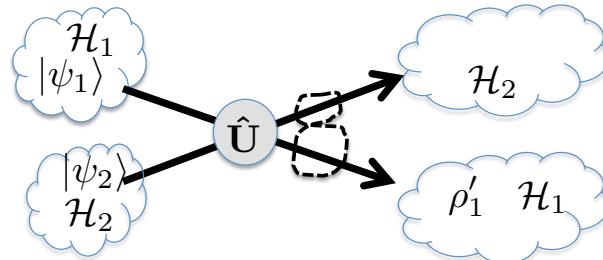


FIGURE 4.1 – "collisional" interaction between two systems

### 4.3.1 Evolution of a system under the effect of a "collision"

We consider now two quantum systems of any dimension, described in Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$  that are initially non interacting. We call respectively  $\{|u_n\rangle\}$  and  $\{|v_j\rangle\}$  bases of spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$ . The two systems "collide" from time  $t = 0$  for a short duration and then are again separated (see figure(4.1)). Let  $\hat{U}$  be the evolution operator of the total system after the end of the interaction, i.e. for a time  $t$  longer than the collision time. We assume that the whole system in  $\mathcal{H}_1 \otimes \mathcal{H}_2$  is initially a factorized state. Under the effect of the interaction, it will evolve into an entangled state described by a global density matrix  $\rho$ . We are interested only in the evolution of the system in  $\mathcal{H}_1$ , i.e. in the reduced density matrix  $\rho'_1$  obtained by a partial trace over space  $\mathcal{H}_2$  of  $\rho'_1$ .

Let us first assume that the initial state is a tensor product of two pure states  $\rho = |\psi_1\rangle \otimes |\psi_2\rangle$ . The reduced density matrix after the collision will be :

$$\rho'_1 = \sum_j \langle v_j | \hat{U} | \psi_1 \rangle \otimes |\psi_2\rangle \langle \psi_1| \otimes \langle \psi_2 | \hat{U}^\dagger | v_j \rangle = \sum_j \hat{K}_j |\psi_1\rangle \langle \psi_1| \hat{K}_j^\dagger \quad (4.34)$$

$\hat{K}_j$  being an operator acting on space  $\mathcal{H}_1$  and such that :

$$\langle u_n | \hat{K}_j | u_{n'} \rangle = \langle u_n | \otimes \langle v_j | \hat{U} | u_{n'} \rangle \otimes |\psi_2\rangle \quad (4.35)$$

The demonstration can be easily extended to the case of non pure initial states  $\rho_1$  in space  $\mathcal{H}_1$ , so that one retrieves in this particular case the Kraus sum :

$$\rho'_1 = \sum_j \hat{K}_j \rho_1 \hat{K}_j^\dagger \quad (4.36)$$

### 4.3.2 Particular case

We take now as an example a simple interaction Hamiltonian :

$$\hat{H}_{int} = \hbar g(t) \hat{A}_1 \hat{B}_2 \quad (4.37)$$

in which  $\hat{A}_1$  is an observable on  $\mathcal{H}_1$ , and  $\hat{B}_2$  is an observable on  $\mathcal{H}_2$  and  $g(t)$  the strength of the interaction which is zero before and after the collision.

$$= e^{-i \int_0^t g(t') dt' \hat{A}_1 \hat{B}_2} \quad (4.38)$$

so that the evolution for the whole collision is :

$$\hat{U} = e^{-ig\hat{A}_1\hat{B}_2} \quad (4.39)$$

with  $g = \int_0^\infty g(t')dt'$ .

We will choose for the basis  $\{|u_n\rangle\}$  of  $\mathcal{H}_1$  the eigenstates of  $\hat{A}_1$ , with corresponding eigenvalues  $a_n$ .  $|\psi_1\rangle$  can be written as  $\sum_n c_n |u_n\rangle$ . The global state after the collision will be :

$$|\psi'\rangle = \sum_n c_n |u_n\rangle \otimes |\psi_{2,n}\rangle \quad (4.40)$$

with  $|\psi_{2,n}\rangle = e^{-ig a_n \hat{B}_2} |\psi_2\rangle$ . We see that the  $\mathcal{H}_2$  components  $|\psi_{2,n}\rangle$  of the entangled state have received the "footprint" of their associated  $\mathcal{H}_1$  component  $|u_n\rangle$ .

The Kraus operators are diagonal in the basis  $\{|u_n\rangle\}$ , with :

$$\hat{K}_j |u_n\rangle = \langle v_j | \psi_{2,n} \rangle |u_n\rangle \quad (4.41)$$

so that the matrix elements of the reduced density matrix are<sup>1</sup> :

$$\rho'_{1,n,n'} = \sum_j \langle v_j | \psi_{2,n} \rangle \rho_{1,n,n'} \langle \psi_{2,n'} | v_j \rangle = \langle \psi_{2,n'} | \psi_{2,n} \rangle \rho_{1,n,n'} \quad (4.43)$$

Populations  $\rho'_{1,n,n}$  are unchanged, whereas coherences are multiplied by a number  $\alpha_{n,n'} = \langle \psi_2 | e^{ig(a_n - a_{n'}) \hat{B}_2} | \psi_2 \rangle$  which has, like any inner product of two normalized states, a modulus smaller than 1 : we are here in the *coherence damping* situation. When the two eigenvalues are significantly different,  $\alpha_{n,n'}$ , as the average of a rapidly oscillating term, is in general very small and the corresponding coherence will be reduced more efficiently than the one between levels of almost equal eigenvalues. The coherences between two levels of equal eigenvalues are not affected.

The conclusion of this analysis is that when two systems interact with a Hamiltonian of the form (4.37), they become as expected entangled and strongly correlated : *in the basis which diagonalizes the interaction*, the effect of this interaction (which does not affect the state vectors  $|u_n\rangle$ ) on one of the two systems is a strong decoherence between states of different eigenvalues, and the conservation of all the populations.

---

1. Note that this conclusion can be extended to the more general case where the basis states  $|u_n\rangle$  are eigenstates of the evolution operator, whatever its precise expression, more precisely when we can write :

$$\hat{U}(|u_n\rangle \otimes |\psi_2\rangle) = |u_n\rangle \otimes |\psi_{2n}\rangle \quad (4.42)$$

## 4.4 Evolution of a small system coupled to a large one

Let us now consider a quantum system, characterized by a Hilbert space  $\mathcal{H}_1$  that interacts with another system, called "bath" or "reservoir", characterized by a Hilbert space  $\mathcal{H}_R$  of very large size. The situation is analogous to the canonical ensemble in statistical physics, in which one defines the thermostat, or bath, as a large system which interacts with the small system of interest, but which is not affected much by this interaction and always stays in the same state.

### 4.4.1 Case of repeated collisions

As an example, let us consider a case for which we can directly use our previous results : the system 1 of interest is experiencing collisions with many systems 2, the totality of which constitute a large Hilbert space  $\mathcal{H}_R$  of the "reservoir".

In such a situation, it is reasonable to assume that system 1 will interact with a series of systems 2 that are all in a "fresh" initial state  $|\psi_2\rangle$ , i.e. not entangled with system 1 due to a previous collision with it. If one uses repeatedly the results of the previous section, one finds that the decoherence effect accumulates, and the reduced density matrix of system 1 gets soon completely diagonal, except between levels of degenerate eigenvalues. The final density matrix  $\rho_\infty$  is therefore the statistical mixture :

$$\rho_\infty = \sum_n \hat{P}_n \rho \hat{P}_n \quad (4.44)$$

The projectors  $\hat{P}_n$  on the eigen-spaces of  $\hat{A}_1$  are therefore the Kraus operators of the evolution under the effect of the bath.

### 4.4.2 Born-Markov approximation

Let us go back to the general case :the evolution from  $t$  to a later time  $t + \Delta t$  of the whole system, which is an isolated system, is governed by the unitary evolution operator  $\hat{U}_{1R}(t, t + \Delta t)$ . we make the important assumption that the initial state of the whole system at time  $t$  is factorized and that the bath is in a pure state :  $\rho_{tot} = \rho_1(t) \otimes |\Psi_R\rangle\langle\Psi_R|$ , so that the reduced density matrix  $\rho_1(t + \Delta t)$  of the small system is given by :

$$\rho_1(\Delta t) = Tr_R \hat{U}_{1R}(t, t + \Delta t) \rho_1(0) |\Psi_R\rangle\langle\Psi_R| \hat{U}_{1R}^\dagger(t, t + \Delta t) \quad (4.45)$$

Introducing a basis  $\{|v_j\rangle\}$  of the bath Hilbert space  $\mathcal{H}_R$ , one gets, like in section (4.3.1) :

$$\begin{aligned}\rho_1(t + \Delta t) &= \sum_j \langle v_j | \hat{U}_{1R}(t, t + \Delta t) \rho_1(t) | \Psi_R \rangle \langle \Psi_R | \hat{U}_{1R}^\dagger(t, t + \Delta t) | v_j \rangle \\ &= \sum_j \hat{K}_j(t, \Delta t) \rho_1(t) \hat{K}_j^\dagger(t, \Delta t)\end{aligned}\quad (4.47)$$

$\hat{K}_j$  is a linear operator acting only in the Hilbert state of the small system  $\mathcal{H}_1$ , equal to  $\hat{K}_j = \langle v_j | \hat{U}_{1R}(t, t + \Delta t) | \Psi_R \rangle$ .

Let us now consider the state of the system at time  $t + 2\Delta t$ . Generally speaking, though the reservoir is stationary, we cannot calculate  $\rho_1(t + 2\Delta t)$  from  $\rho_1(t + \Delta t)$  using formula 4.46 giving  $\rho_1(t + \Delta t)$  from  $\rho_1(t)$  because the new initial state is not factorized : the different "time slices" of duration  $\Delta t$  are not equivalent. The expression of the Kraus operators depend on  $t$  and are in general difficult to determine.

The problem is drastically simplified if one makes the approximation described in this section, called the "short memory" or Born-Markov approximation. Our aim is to describe only the "coarse grained" temporal evolution, i.e. the evolution by time steps  $\Delta t$  which are not infinitely small. We assume that these time steps are long enough compared to the "collision time" so that we can use (4.47). In addition, the reservoir is so big and the considered system so small that the reservoir state is insensitive to changes occurring in the system 1. It remains always in its initial state  $|\Psi_R\rangle$ , and therefore the total state of the system is in the factorized state  $\rho_1(t)|\Psi_R\rangle\langle\Psi_R|$  at all times. The bath in some way "forgets" that it has interacted with the small system<sup>2</sup>. As the global state  $\rho(\Delta t)$  is factorized, we can use the previous derivation, and we have for any  $t$  :

$$\rho_1(t + \Delta t) = \sum_j \hat{K}_j(\Delta t) \rho_1(t) \hat{K}_j^\dagger(\Delta t) \quad (4.48)$$

Note that the Kraus operators  $\hat{K}_j(\Delta t)$  do not depend now on  $t$  : at the Markov approximation the "time slices" of large enough duration  $\Delta t$  are equivalent.

#### 4.4.3 Differential expression of the quantum map : master equation

We assume now that  $\Delta t$  is much shorter than the characteristic evolution time of the density matrix  $\rho_1(t)$ , so that the state at time

---

2. This behavior must of course be checked for each particular form of the interaction between the system and the bath

$t + \Delta t$  can be written in form of a Taylor expansion, as it does not change much during  $\Delta t$  :

$$\rho_1(t + \Delta t) \simeq \rho_1(t) + \Delta t \frac{d\rho_1}{dt} \quad (4.49)$$

$\rho_1(t + \Delta t)$  is also given by the Kraus sum :

$$\rho_1(t + \Delta t) = \sum_{j=0}^N \hat{K}_j(\Delta t) \rho_1(t) \hat{K}_j^\dagger(\Delta t) \quad (4.50)$$

A comparison of equations (4.49) and (4.50) shows that the Taylor development of the Kraus operators  $\hat{K}_j(\Delta t)$  must contain terms of the form  $\hat{1} + \hat{A}_j \sqrt{\Delta t} + \hat{B}_j \Delta t$  that are more precisely of the form :

$$\hat{K}_0(\Delta t) = \hat{1} + \Delta t \hat{B}_0 \quad \hat{K}_j(\Delta t) = \sqrt{\Delta t} \hat{A}_j \quad (4.51)$$

so that :

$$\sum_{j=0}^N \hat{K}_j(\Delta t) \rho_1(t) \hat{K}_j^\dagger(\Delta t) \simeq \rho(t) + \Delta t \left( \hat{B}_0 \rho_1 + \rho_1 \hat{B}_0^\dagger + \sum_{j=1}^N \hat{A}_j \rho_1 \hat{A}_j^\dagger \right) \quad (4.52)$$

Comparing with (4.49) we get a simple differential equation for the evolution of the reduced density matrix at the Born-Markov approximation

$$\frac{d\rho_1}{dt} = \hat{B}_0 \rho_1 + \rho_1 \hat{B}_0^\dagger + \sum_{j=1}^N \hat{A}_j \rho_1 \hat{A}_j^\dagger \quad (4.53)$$

The operators  $\hat{A}_n$  being independent of time. The constraint  $\sum_{j=0}^N \hat{K}_j^\dagger \hat{K}_j = \hat{1}$  implies that :

$$\hat{B}_0 + \hat{B}_0^\dagger + \sum_{j=1}^N \hat{A}_j^\dagger \hat{A}_j = 0 \quad (4.54)$$

$\hat{B}_0$ , as any linear operator, is the sum of commuting anti-Hermitian and Hermitian operators, so that one can write :

$$\hat{B}_0 = -i\hat{H}/\hbar - \hat{\Gamma} \quad \hat{H}, \hat{\Gamma} \text{ Hermitian} \quad (4.55)$$

we obtain finally the equation governing the slow coarse grain evolution of the reduced density matrix, called *master equation*, in the form of a first order differential equation :

$$\frac{d\rho_1}{dt} = -\frac{i}{\hbar} [\hat{H}, \rho_1] - (\rho_1 \hat{\Gamma} + \hat{\Gamma} \rho_1) + \sum_{n=1}^N \hat{A}_n \rho_1 \hat{A}_n^\dagger \quad (4.56)$$

with :

$$\hat{\Gamma} = \frac{1}{2} \sum_{j=1}^N \hat{A}_j^\dagger \hat{A}_j \quad (4.57)$$

equivalent to :

$$\frac{d\rho_1}{dt} = -\frac{i}{\hbar} [\hat{H}, \rho_1] - \frac{1}{2} \sum_{n=1}^N (\rho_1 \hat{A}_j^\dagger \hat{A}_j + \hat{A}_j^\dagger \hat{A}_j \rho_1 - 2 \hat{A}_j \rho_1 \hat{A}_j^\dagger) \quad (4.58)$$

The master equation in the form (4.56) or (4.58) is known as the Lindblad equation.

#### 4.4.4 Discussion

Let us consider successively the different contributions to the evolution of the density matrix in equation(4.56) :

- The first term in the master equation (4.56) obviously corresponds to the well-known Hamiltonian evolution of the density matrix ;
- The second and third term are in the form of an anti-commutator. They can be grouped with the first one (4.56) so that this part of the evolution can also be written as

$$\frac{d\rho}{dt} = -\frac{i}{\hbar} (\hat{H}_{eff} \rho - \rho \hat{H}_{eff}^\dagger) \quad (4.59)$$

with  $\hat{H}_{eff} = \hat{H} - i\hbar\hat{\Gamma} = i\hbar\hat{A}_0$ . One has a kind of pseudo Hamiltonian evolution with a non-hermitian hamiltonian  $\hat{H}_{eff}$ . Let us assume for simplicity that the operators  $\hat{H}$  and  $\hat{\Gamma}$  commute. They diagonalize in the same basis, and the diagonal elements in this basis are of the form  $E_n - i\hbar\gamma_n$  : this first part of time evolution acting on an eigenstate of the Hamiltonian  $|\psi_n\rangle$  yields the state  $e^{-iE_nt} e^{-\gamma_nt} |\psi_n\rangle$ . It contains a decaying exponential, characterizing a non-unitary, irreversible, damping process.

- Let us consider one term of the last sum,  $\hat{A}_j \rho_1 \hat{A}_j^\dagger$ . Acting alone and on the pure state  $|\psi_n\rangle$ , it yields the state  $\hat{A}_j |\psi_n\rangle$  which can be very different from  $|\psi\rangle$  : the system experiences a quantum jump, exactly like the ones measured in experiments with single physical systems described in the first chapter.

Of course because the two parts of the evolution are present simultaneously in the equation, both contribute in a stochastic way. A very efficient numerical method, named "Quantum Monte-Carlo

method"[9] is based on these considerations : the calculated evolution of the wave vector alternates periods of smooth exponential decays and sudden quantum jumps occurring at random times. Note that the two effects come necessarily together because of relation (4.57) : exponential decay is always accompanied with quantum jumps. This is the quantum equivalent of the "fluctuation-dissipation" theorem of statistical physics.

#### 4.4.5 Example : the damped harmonic oscillator

Let us consider the trapped ion that was described in Chapter 1. It is a quantum harmonic oscillator, with annihilation operators  $\hat{a}$  and energies  $E_n = \hbar\omega(n + 1/2)$ . We suppose that it is not isolated, but coupled to a big set of other harmonic oscillators (phonons, photons ...) described by annihilation operators  $\hat{b}_n$ , called the reservoir. In addition we assume that the reservoir is not affected by the presence of the oscillator and stays always in the same state that we assume to be the vacuum state, to simplify the analysis. The Hamiltonian of the whole system is :

$$\begin{aligned}\hat{H} &= \hat{H}_0 + \hat{H}_{int} & (4.60) \\ \hat{H}_0 &= \hbar\omega\hat{a}^\dagger\hat{a} + \sum_n \hbar\omega_n\hat{b}_n^\dagger\hat{b}_n & ; \quad \hat{H}_{int} = \sum_n \hbar g_n(\hat{a}^\dagger\hat{b}_n + \hat{a}\hat{b}_n^\dagger)\end{aligned}$$

where the coupling term  $\hat{H}_{int}$  corresponds to a transfer of excitation from the ion harmonic oscillator to the others, and the reverse.  $g_n$  is assumed to be real.

The total system is isolated and obeys the usual Schrödinger equation, that we can write in the Heisenberg representation for the operators  $\hat{a}(t)$  and  $\hat{b}_n(t)$  :

$$\frac{d\hat{a}}{dt} = \frac{1}{i\hbar}[\hat{a}, \hat{H}] = -i\omega\hat{a} - i\sum_n g_n\hat{b}_n & (4.62)$$

$$\frac{d\hat{b}_n}{dt} = \frac{1}{i\hbar}[\hat{b}_n, \hat{H}] = -i\omega_n\hat{b}_n - ig_n\hat{a} & (4.63)$$

Solving formally the second equation and inserting the result into the first one, one gets :

$$\frac{d\hat{a}}{dt} = -i\omega\hat{a} - \int_0^t dt' \hat{a}(t') K(t-t') + \hat{f}(t) & (4.64)$$

where the Kernel K and additional noise term  $\hat{f}(t)$  are given by

$$K(t) = \sum_n g_n^2 e^{-i\omega_n t} ; \quad \hat{f}(t) = -i \sum_n g_n \hat{b}_n(0) e^{-i\omega_n t} & (4.65)$$

Being the sum of a great number of terms oscillating at different frequencies that are in phase only at  $t = 0$ ,  $K(t)$  is non negligible only for very small times, and we will approximate it by  $\kappa\delta(t)$  : this is an example of the Born-Markov approximation introduced in the previous section. We have then :

$$\frac{d\hat{a}}{dt} = -i\omega\hat{a} - \kappa\hat{a} + \hat{f}(t) \quad (4.66)$$

$\hat{f}(t)$  satisfies  $\langle \hat{f}(t) \rangle = 0$  and  $\langle \hat{f}^\dagger(t)\hat{f}(t') \rangle = \kappa\delta(t-t')$ . It is actually a noise term, similar to the one appearing in the classical equation of the Brownian motion of a particle submitted to the collision of a surrounding liquid :

$$\frac{dv}{dt} = -\kappa v + f(t) \quad (4.67)$$

where  $v$  is the velocity of the particle,  $\kappa$  the damping constant and  $f$  the fluctuating "Langevin" force responsible for the random path followed by the particle, and satisfying  $\overline{f(t)} = 0$ ,  $\overline{f(t)f(t')} \propto \delta(t-t')$ .

The two first terms of (4.66) correspond to the pseudo-hamiltonian evolution with a complex frequency leading to exponential decay. They are the consequence in the evolution of observable  $\hat{a}$  of the pseudo-hamiltonian part (4.59) of the master equation (4.56) governing the evolution of the density matrix. The effective non Hermitian operator is  $\hat{H}_{eff} = \hbar(\omega - i\kappa)\hat{a}^\dagger\hat{a}$ , so that the associated damping operator of equation (4.56) is  $\hat{\Gamma} = \kappa\hat{a}^\dagger\hat{a}$ . Relation (4.57) implies that the Kraus operator  $\hat{A}_1$  can be taken as  $\hat{A}_1 = \sqrt{2\kappa}\hat{a}$ . The Lindblad master equation (4.56) for  $\rho$  must necessarily have the form :

$$\frac{d\rho}{dt} = -i\omega(\hat{a}^\dagger\hat{a}\rho - \rho\hat{a}^\dagger\hat{a}) - \kappa(\hat{a}^\dagger\hat{a}\rho + \rho\hat{a}^\dagger\hat{a}) + 2\kappa\hat{a}\rho\hat{a}^\dagger \quad (4.68)$$

The quantum jump due to  $\hat{A}_1$  operator acting on wavevector  $|\psi\rangle$  leads to the state  $\sqrt{2\kappa}\hat{a}|\psi\rangle$  : it indeed annihilates one quantum excitation in the state.

The Lindblad master equation (4.68) for the density matrix evolution in the Schrödinger representation could have been derived more directly by calculating the Hamiltonian evolution of the total system and making the partial trace over the reservoir, but this calculation is far more complex than the one we present here concerning the evolution of observables in the Heisenberg representation. This shows that there are different ways of tackling the problem of the time evolution at the Born-Markov approximation : the Lindblad master equation for the density matrix and the quantum Langevin

evolution equation like (4.66) for observables that contain stochastic, fluctuating, terms called Langevin forces, in order to derive the evolution not only of mean values but also of second moments<sup>3</sup>.

The evolution of the mean values  $\langle \hat{a} \rangle = Tr\rho\hat{a}$  and of  $\langle \hat{a}^\dagger\hat{a} \rangle = Tr\rho\hat{a}^\dagger\hat{a}$  can be derived either from (4.68) or from (4.66). One gets :

$$\frac{d}{dt} \langle \hat{a} \rangle = (-i\omega - \kappa) \langle \hat{a} \rangle ; \quad \frac{d}{dt} \langle \hat{a}^\dagger\hat{a} \rangle = -2\kappa \langle \hat{a}^\dagger\hat{a} \rangle \quad (4.69)$$

The evolution of these mean values is in form of a smooth exponential decay, with or without oscillations, because the noise term  $\hat{f}(t)$  has a zero mean value. However, the evolution of a given realization of the system is governed by the random value of the noise term, like in the Brownian motion. Seen from the master equation, this randomness is related to the presence of the quantum jump operator which suddenly lowers the energy of the oscillator by one quantum.

---

3. Due to the Quantum Regression Theorem[10], the quantum Langevin equations are also valid to calculate the evolution of higher order moments.

## Conclusion of chapters 2 to 4 : General Postulates of Quantum Mechanics

We can now formulate postulates of Quantum Mechanics which are valid for any quantum system, whatever the way in which it is prepared, evolving and measured. They depend on a small number of linear operators acting on the Hilbert state : the density matrix  $\rho$ , the post-measurement operators  $\hat{M}_n$ , the Kraus evolution operators  $\hat{K}_\ell$ . All these quantities can be experimentally accessed by techniques called *quantum tomography*. The reader interested in state tomography, measurement tomography and process tomography will find more details in [12].

**(S'1) Superposition principle postulate :** a physical system is described by a single mathematical object, called *density matrix*,  $\rho$ , which is a positive operator acting in a well defined Hilbert space  $\mathcal{H}$ .

**(S'2 ) Normalization postulate :** the density matrix is normalized to 1 :

$$Tr\rho = 1 \quad (4.70)$$

**(M'1) Quantization postulate :** the measurement of the physical property  $A$  can only give as a result a value  $a_n$  belonging to a list  $\{a_1, a_2, \dots, a_{N_A}\}$ . The number  $N_A$  of these possible values is not limited to the dimension of the Hilbert space  $\mathcal{H}$ .

**(M'2) Born's rule :** In general the result of the measurement cannot be predicted with certainty. The probability  $Proba(a_n, \rho)$  of obtaining the result  $a_n$  when the system is in the quantum state described by the density matrix  $\rho$  is given by :

$$Proba(a_n, \rho) = Tr\rho\hat{\Pi}_n \quad (4.71)$$

where  $\hat{\Pi}_n$  is a positive operator associated with the measurement device and the value  $a_n$  satisfying the condition  $\sum_n \hat{\Pi}_n = \hat{1}$ .

**(M'3) State collapse postulate :** The conditional state of the system just after the measurement *and in the subset of measurements that have given the result  $a_n$* , called  $\rho^{after|a_n}$ , is :

$$\rho^{after|a_n} = \frac{1}{Tr\rho\hat{\Pi}_n} \hat{M}_n \rho \hat{M}_n^\dagger \quad (4.72)$$

where the  $\hat{M}_n$  are linear operators acting on the Hilbert space  $\mathcal{H}$  satisfying the condition  $\hat{M}_n^\dagger \hat{M}_n = \hat{\Pi}_n$ .

**(E'1) Evolution postulate :** a quantum system initially in state  $\rho$  is at a later time  $t$  in state  $\mathcal{L}(t)[\rho]$  given by :

$$\mathcal{L}(t)[\rho] = \sum_{\ell=1}^N \hat{K}_\ell(t) \rho \hat{K}_\ell^\dagger(t) \quad (4.73)$$

where  $\hat{K}_\ell(t)$  are linear operators satisfying the condition  $\sum_{\ell=1}^N \hat{K}_\ell^\dagger(t) \hat{K}_\ell(t) = \hat{1}$

# Chapitre 5

## The measurement as a physical evolution

### 5.1 General ideas

#### 5.1.1 Position of problem

As seen in the first chapters, the postulates of quantum mechanics pay a very special attention to the measurement process, which have specific rules distinct from the postulate of evolution. But a measurement set-up seems indeed to be a physical system, which, like any quantum physical system, must obey the postulates of "normal" evolution.

Physicists and philosophers have considered from the beginning of the quantum age this central problem of quantum mechanics and brought to it different answers, that can be found for example in [14].

An important contribution to the solution of this problem has been brought by Zurek. We will present his derivation in the following sections.

#### 5.1.2 Ideas for a physical model of a measurement device

One wants to measure the physical quantity  $A$  on a quantum system described by a density matrix  $\rho$  in Hilbert space  $\mathcal{H}$ . For this purpose we couple it to a measurement device, the "meter", which is another quantum physical system described by states in another Hilbert space  $\mathcal{M}$ . The fact that it is able to "measure"  $A$  means that experimentalists are able to "read" by one way or another specific states  $|\chi_n\rangle$  of the meter which form the "pointer basis" : each  $|\chi_n\rangle$  corresponds for example to a definite position

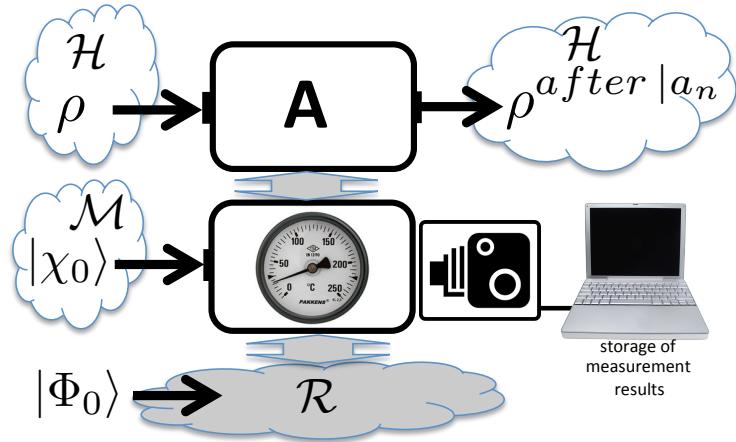


FIGURE 5.1 – A measurement device couples the system to measure to a physical "meter", the evolution of which is damped because of its interaction with a reservoir

of a needle which points to the value  $a_n$ , one of the eigenvalues of the associated observable  $\hat{A}$ . A good measurement device is a one in which the needle points to  $a_n$  when the input state in  $\mathcal{H}$  is the corresponding eigenstate  $|u_n\rangle$ . The result of the reading is then stored in a computer, and the whole process is repeated many times on the initial states all equal to  $\rho$ . From the collection of stored results one can infer the probability to get a specified pointer position  $a_n$ .

One immediately notes that the evolution resulting from the interaction between the system and the meter cannot be Hamiltonian, i.e. a combination of oscillatory terms : the needle in this case will oscillate for ever and never point to something, i.e. will never have a stationary final state. This is equally true for a classical measurement : imagine a weighing scale without damping : putting a weight on one platform of the scale will induce a never ending oscillation of the beam, and it will not be possible to infer the value of the weight. In contrast, if the motion of the beam is damped, the system will reach a final state associated with the value of the weight. This means that the measurement set-up is coupled to a big Hilbert space, the "reservoir"  $\mathcal{R}$ , which induces an irreversible evolution of the meter quantum state, as described in the previous chapter. The whole device is sketched in figure (5.1).

## 5.2 Zurek's model

### 5.2.1 Physical system

It is a toy model, with many assumptions that simplify the resolution of the problem. It is extensively described in [15].

- 1) One assumes that the coupling between the system and the meter is of the type described in section 4.1. The interaction Hamiltonian is

$$\hat{H}_{int} = \hbar g(t) \hat{A} \hat{M} \quad (5.1)$$

where  $\hat{M}$  is the "meter observable", which has the readable pointer states  $|\chi_n\rangle$  as eigenstates. For the sake of simplicity  $\hat{H}_{int}$  is supposed to act only during a limited amount of time.

-2) One assumes that the coupling of the meter with the reservoir, described by an interaction term  $\hat{H}_{MR}$ , takes place after the interaction between the meter and the system. This reservoir is not coupled to the reservoir, so that the system state does not change during the meter-reservoir interaction.

-3) One assumes that  $\hat{H}_{MR}$  is *diagonal in the pointer basis*  $\{|\chi_n\rangle\}$  :

$$\hat{H}_{MR} = \sum_n |\chi_n\rangle\langle\chi_n| \otimes \hat{V}_{MR}^n \quad (5.2)$$

where  $\hat{V}_{MR}^n$  is an operator acting only in the reservoir Hilbert space  $\mathcal{R}$ . This basic assumption implies that the pointer states  $|\chi_n\rangle$  are stationary and stay pointed to the appropriate value  $a_n$  even in presence of the reservoir (the beam of the scale stops its motion when it points to the right weight).

### 5.2.2 Evolution of the quantum system

The initial state in the system+meter+reservoir Hilbert space is the factorized state  $|\psi\rangle \otimes |\chi_0\rangle \otimes |\phi_0\rangle$ , with  $|\psi\rangle = \sum_n c_n |u_n\rangle$ . After the interaction (5.1) the system+meter is, according to the discussion of section 4.1, in the entangled state  $|\Psi_{SM}\rangle$  given by :

$$|\Psi_{SM}\rangle = \sum_n c_n |u_n\rangle \otimes |\chi_n\rangle \quad (5.3)$$

After a duration  $t$  of reservoir-meter interaction the system+meter+reservoir is in state :

$$|\Psi_{SMR}\rangle = \sum_n c_n |u_n\rangle \otimes |\chi_n\rangle \otimes |\phi_n(t)\rangle \quad (5.4)$$

with  $\phi_n(t)\rangle = \exp(-i\hat{V}_{MR}^n t/\hbar)|\Phi_0\rangle$ . One notices that the initial state of the system  $|u_n\rangle$  has created prints both in the meter and in the reservoir. But the reservoir is a large system that one does not measure. As a result we need to make a partial trace over it, which will give rise to the reduced system+meter density matrix :

$$\rho_{SM} = \sum_{n,n'} c_n c_{n'}^* \langle \phi_{n'}(t) | \phi_n(t) \rangle |u_n, \chi_n\rangle \langle u_{n'}, \chi_{n'}| \quad (5.5)$$

Because of the large size of the reservoir, the matrix elements  $\langle \phi_{n'}(t) | \phi_n(t) \rangle$  quickly decay to zero when  $n \neq n'$ . The final state is therefore described by the diagonal density matrix :

$$\rho_{SM}^{final} = \sum_n |c_n|^2 |u_n, \chi_n\rangle \langle u_n, \chi_n| \quad (5.6)$$

This state contains perfect correlations between the system and the meter that are required for the meter to properly measure the input system state. By performing partial traces, one also gets the reduced states of the meter alone and of the system alone :

$$\rho_M^{final} = \sum_n |c_n|^2 |\chi_n\rangle \langle \chi_n| \quad ; \quad \rho_S^{final} = \sum_n |c_n|^2 |u_n\rangle \langle u_n| \quad (5.7)$$

$\rho_M^{final}$  is a mixed state that gives the probability to find the needle pointing to the value  $a_n$  in agreement with Born's rule.  $\rho_S^{final}$  is the state of the system after its interaction with the measurement device, made of the meter and the reservoir. It can be also written in terms of the input density matrix  $\rho$  by :

$$\rho_S^{final} = \rho^{out} = \sum_n \hat{P}_n \rho \hat{P}_n \quad (5.8)$$

The Kraus operators of the quantum map associated with the interaction between the system and the measurement device are therefore the projectors on the eigenstates of the system-meter interaction.

### 5.2.3 Decoherence

The interaction with the measurement device has thus two consequences :

- it gives measurement results in agreement with the Born's rule for ideal Von Neumann measurements, which therefore does not appear in this simplified model as a postulate but as a consequence of the evolution of the whole system treated as a single quantum object ;

-it strongly modifies the state subject to the measurement and in an irreversible way, as expected from a damping process. The evolution leaves the diagonal blocks (in the basis of eigenstates of the observable  $\hat{A}$ ) unchanged and destroys all the off-diagonal blocks of the initial density matrix, i.e. all the coherent connections between the eigenstates. It is called a *decoherence process*.

Such a decoherence is not specific to the simple toy model used here to make simple calculations. It occurs in any more sophisticated physical model of the measurement, in particular when one tries to precisely model a specific physical device used to measure a specific observable (spin, photon number, energy...). Some realistic models take into account the defects of the detector (not perfect quantum efficiency for a photon detector, for example) and yield probabilities of measurement and final state of the system which obey the "postulates" of non ideal measurements given in chapter 4.

## 5.3 Back to the state collapse postulate M3

### 5.3.1 Conditional state after the measurement

We see that making a physical model of the measurement set-up has led us to the right value for the probability of measurement, and to an expression of the state after the measurement which is not the one given by the state collapse postulate M3. it rather corresponds to the non-postselected state, i.e. when one does not consider the measurement results and consider all the recorded values of the measurement on an equal footing. This is normal, as we never used in the derivation the fact that a specific value was obtained, nor the exact physics of the post-selection process.

### 5.3.2 The nature of the state collapse

Looking back at the expression of the conditional state  $\rho^{out|a_n}$  obtained after the measurement when this measurement has given the value  $a_n$  in the ideal and non ideal cases, we see that *the relations (2.33) and (??) are not quantum maps* (equation(4.15)). Consequently they cannot be obtained as the outcome of a physical evolution ; *they are even non-linear*, a rather unexpected property for quantum quantities. In contrast, expression (3.13) for the not post-selected state is indeed a quantum map, as expected for the result of the quantum calculation of the evolution of a specific system.

In conclusion, we have shown here that the expression of the conditional state after a measurement has a double origin :

- a physical evolution induced by the interaction between the system and the measurement device ;

- a change of point of view concerning the quantity of information that one can extract from the quantum state : instead of considering the physical system without restriction, and taking into account the data corresponding to all the possible measurements results, the observer decides to restrict his investigations to the subset of data corresponding to a given value  $a_n$ .

The state collapse is an information-related process. It takes place when the observer decides to make such a selection on its data.

## Chapitre 6

# Joint and conditional probabilities in quantum physics

*Un coup de dés jamais n'abolira le hasard* (S. Mallarmé)

### 6.1 The Bayesian approach of statistics

Let us consider a physical quantity  $A$ , and its possible measurement results  $\{a_n\}$ . The usual way of defining a probability is to consider a great number of copies of the same system on which  $A$  is measured : the probability  $P(a_n)$  is then the frequency of occurrence of  $a_n$  among all the recorded measurement results. This approach does not obviously apply to the prediction of the result of a single measurement, unless  $P(a_n) = 0$  or  $1$ .

Let us consider the following weather forecast report :"there is a 80% chance that it will rain to-morrow" : it is a statistical statement about a single event, and we do not consider it as meaningless, because we take our umbrella the next morning. What does this mean ? it means that given the information that meteorologists have gathered from pressure and temperature measurements, satellite photos, etc.. 80% of the runs of their computer simulations in which they slightly change the initial conditions have lead to rain, so that it is "very likely" that it rains in these conditions. In this case, probability is related to predictability, or "*likelihood*", (en français : "*vraisemblance*"), a not very precisely defined, but certainly meaningful, concept.

Ultimately likelihood is a measure of the quantity of *information* that we have on the system : if one knows nothing, all the possible

measurement outcomes have the same probability in this meaning. Each new information about the state of the system will increase or decrease the likelihood to get a given measurement result. The update of our information takes place when one knows the precise value of the measurement of another quantity, say  $B$ , performed on the same system. The relevant quantity in this point of view is the *conditional probability* of  $a_n$  given  $b_m$ , noted  $P(a_n|b_m)$ , of getting the result  $a_n$  for  $A$ , given that a previous measurement of another quantity  $B$  on the same system has yielded the result  $b_m$ .

The approach consisting in considering probabilities as quantifying information on the system acquired by successive measurements, and to manipulate conditional probabilities, has been introduced by Bayes in the 18th century and extended further by Fischer. It turns out to be very fruitful in many instances, and especially in Quantum Mechanics.

## 6.2 Statistics of correlations

*The only proper subjects of physics are correlations among different parts of the physical world. Correlations are fundamental, irreducible, and objective. They constitute the full content of physical reality* (N. Mermin 1996)

It is obvious that if there are no physical relations between the quantities  $A$  and  $B$ , information about  $B$  will not be useful to increase our information on  $A$ . So these two quantities must be correlated by one way or another. Let us now consider this correlation in more detail.

### 6.2.1 correlation functions, joint and conditional probabilities in classical physics

Let us assume that we make two successive measurements on a classical system : one measures first the physical quantity  $B$  (possible results in the list  $\{b_m\}$ ), which does not affect the system, since we are in the classical world. One then measures on the same system the physical quantity  $A$  (possible results in the list  $\{a_n\}$ ). One then gets a pair of results  $(a_n, b_m)$  that gives a point in the  $(A, B)$  representation plane. The order of the two measurements is not important in classical physics. If one repeats many times the double measurement, each time with a copy of the initial system, one will finally get a "cloud" of  $N$  points. (figure 6.1). If the points are randomly distributed in the plane (figure 6.1(a)), the knowledge of the

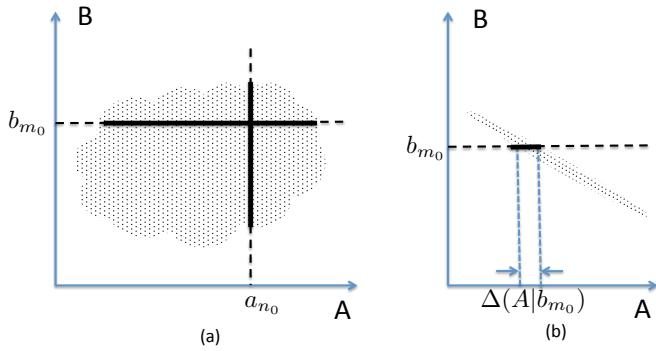


FIGURE 6.1 – Joint data in the  $(A, B)$  plane : each point corresponds to a given measured value of  $A$  and a given measured value of  $B$

specific value  $b_{m_0}$  for  $B$  does not modify the dispersion of results on the quantity  $A$  : the two quantities are statistically independent. If the points are well aligned (figure 6.1(b)), the knowledge of the specific value  $b_{m_0}$  for  $B$  provides an increase in the accuracy with which the value of  $A$  can be determined : there is a high degree of correlation between the quantities  $A$  and  $B$ .

The amount of correlation is characterized by the correlation function, given as usual by :  $\mathbf{C}(A, B) = \bar{AB} - \bar{A}\bar{B}$  (where the bar means the classical average value, or, better, by the *normalized correlation function*  $c(A, B)$ ) :

$$c(A, B) = \frac{\mathbf{C}(A, B)}{\Delta A \Delta B} \quad (6.1)$$

$c(A, B)$  is a number comprised between  $-1$  and  $1$ ,  $1$  corresponding to perfect correlation (exactly aligned points with positive slope),  $-1$  to perfect anticorrelation (exactly aligned points with negative slope). Note that the expectation values or means like  $\bar{A}$ , and the standard deviations like  $\Delta A = \sqrt{\bar{A}^2 - (\bar{A})^2}$ , are calculated over all the  $N$  recordings of measurements, i.e. over the total cloud of points in figure(6.1).

One can also define two quantities characterizing the correlation :

- the *joint probability*  $P(a_{n_0}, b_{m_0})$  as the frequency of occurrence of the pair  $(a_{n_0}, b_{m_0})$  within the total set of  $N$  recorded pairs

- $(a_n, b_m)$  ;
- the *conditional probability*  $P(a_n|b_{m_0})$  of measuring  $a_n$  on  $A$  given that the measurement of  $B$  on the same system has given beforehand the value  $b_{m_0}$ , as the frequency of occurrence of the value  $a_n$  within the subset of recorded pairs  $(a_n, b_{m_0})$ .

In figure (6.1), these selected pairs are on the horizontal line of ordinate  $b_{m_0}$ . All the other pairs are dropped, which represents a significant loss of data about the system. One defines more precisely the conditional probability as :

$$P(a_n|b_{m_0}) = \frac{P(a_n, b_{m_0})}{P(b_{m_0})} \quad (6.2)$$

where the *marginal probability*  $P(b_{m_0})$  of measuring  $b_{m_0}$  whatever the outcome of the measurement on  $A$ , defined as :

$$P(b_{m_0}) = \sum_n P(a_n, b_{m_0}) \quad (6.3)$$

allows to normalize to 1 the conditional probability. Note that this relation is analog to the operation of partial trace on a quantum system.

The conditional probability distribution  $P(a_n|b_{m_0})$  of  $A$  values allows us to define a *conditional mean* and a *conditional variance* or *scedastic function*  $\Delta^2(A|b_{m_0})$ , which gives the mean and the width of the horizontal cut in figure (6.1). If the statistical distribution of  $A$  and  $B$  values is Gaussian, the conditional variance  $\Delta^2(A|b_{m_0})$  is independent of  $b_{m_0}$  and given by

$$\Delta^2(A|b_{m_0}) = \Delta^2 A (1 - c(A, B)^2) \quad (6.4)$$

It is zero for perfectly correlated or anti-correlated quantities.

### 6.2.2 Bayes' rule

It is of course possible to measure first the quantity  $A$ , then the quantity  $B$ , and use now the correlation between these two quantities to determine the conditional probability  $P(b_m|a_{n_0})$  of measuring  $b_m$  on  $B$  knowing that the previous measurement of  $A$  on the same system has given the value  $a_{n_0}$ , as the probability  $b_m$  determined only from the data lying on the vertical line of abscissa  $a_{n_0}$  of figure (6.1).

The two conditional probabilities  $P(a_n|b_m)$  and  $P(b_m|a_n)$  are related to the same joint probability  $P(a_n, b_m)$  by eq(6.2). The following relation, known as Bayes' rule, relates one conditional probability to the other and to the two marginal probabilities  $P(b_m)$  and

$P(a_n) :$

$$P(a_n|b_m) = P(b_m|a_n) \frac{P(a_n)}{P(b_m)} \quad (6.5)$$

It can also be written as :

$$P(a_n|b_m) = P(a_n) \frac{P(b_m|a_n)}{P(b_m)} = P(a_n) \ell(a_n|b_m) \quad (6.6)$$

$\ell(a_n|b_m)$  is called the "Fischer likelihood". It characterizes the change in the probability distribution of  $A$  values when one acquires the additional information that the measurement on  $B$  has given the value  $b_m$ .

### 6.2.3 An example

Let us consider the classical example of a medical test that detects a deadly disease. It gives two answers, either "yes" or "no". But it is not perfect : statistical studies performed on a large set of contaminated persons, labeled (C), show that it gives the answer "yes" for 90% of them. Studies performed on non-contaminated persons, labeled (NC), show that it gives the answer "no" for 99% of them. A doctor receives a patient that has a test result "yes". What is the likelihood that the patient is contaminated ? This example is obviously a single event situation which may turn to drama in case of wrong decision. Can Bayes statistics be helpful in this case ?

Note that there are two kinds of conditional probabilities : probabilities of persons being contaminated or not knowing the result of the test, and probabilities of positive and negative answers of the test knowing the state of health of the person. They are related by Bayes' rule.

Let us write the situation in terms of conditional probabilities. We know from the statistical studies that  $P(\text{yes}|C) = 0.9$ ,  $P(\text{no}|C) = 0.1$ ,  $P(\text{no}|NC) = 0.99$ ,  $P(\text{yes}|NC) = 0.01$  and we want to determine  $P(C|\text{yes})$ . We need therefore to inverse the order of probabilities using Bayes' rule :

$$\begin{aligned} P(C|\text{yes}) &= P(\text{yes}|C) \frac{P(C)}{P(\text{yes})} \\ &= P(\text{yes}|C) \frac{P(C)}{P(\text{yes}|C)P(C) + P(\text{yes}|NC)(1 - P(C))} \end{aligned} \quad (6.7)$$

The conditional probability of a given person being contaminated is therefore a function of  $P(C)$ , the probability of being contaminated in the whole population. The fact that the test has given a

known result increases the amount of information about the possible contamination of the person.

The value of  $P(C)$  to put in equation (6.7) depends on some *a priori information* about the sanitary situation of the population :

- if there is an epidemic,  $P(C) = 0.1$  for example, then  $P(C|yes) = 0.91$  ; The best is to send immediately the patient to the hospital ;
- if one knows nothing about the frequency of this disease in the population,  $P(C) = P(NC) = 0.5$ , then  $P(C|yes) = 0.99$  and  $P(NC|no) = 0.91$  : the test is rather efficient in the screening ;
- if the disease is rare, say  $P(C) = 0.00001$ , then  $P(C|yes) = 0.0009$  ; The test is not sensitive enough, and therefore inconclusive. This seems counter-intuitive. The reason is that in that case, wrong answers dominate the statistics.

Note that the same reasoning applies to the more physical case of the beamsplitter of figure(3.4) but with a photon detector having some dark count, with the question : "when one click is recorded on the detector, what is the probability that it actually comes from an incident photon ?

### 6.3 Quantum mechanics of successive measurements

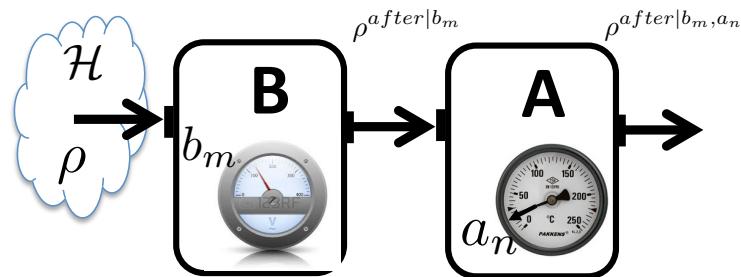


FIGURE 6.2 – Two successive measurements in quantum physics

So far, we have considered classical measurements that do not modify the measured system. We now consider the same problem in the quantum world, more precisely the situation sketched in figure

(6.2) : given an initial state  $\rho$ , one measures successively on it the quantities  $B$  and  $A$ , assuming no evolution between the two measurements. We want to know the joint and conditional probabilities associated with this situation. The quantum state of the system collapses because of the first measurement, so that the picture given by figure(6.1) is now not so meaningful. We can anyway use the quantum postulates of measurement to calculate the joint and conditional probabilities.

### 6.3.1 Case of two Von Neumann measurements

We have in this case two observables  $\hat{A}$ ,  $\hat{B}$ , and projection operators  $\hat{P}_n$ ,  $\hat{Q}_m$ , with  $\hat{A} = \sum_n a_n \hat{P}_n$ ,  $\hat{B} = \sum_m b_m \hat{Q}_m$ . After the first measurement giving a result  $b_m$  with probability  $\text{Tr} \rho \hat{Q}_m$ , the system enters the second measurement device in the state  $\rho^{\text{after}|b_m} = \hat{Q}_m \rho \hat{Q}_m / \text{Tr} \rho \hat{Q}_m$ .

#### Joint probability

Applying Von Neumann rules to this state, one finally gets an expression giving the *the joint probability for successive measurements*, as the product of the probabilities of measuring  $b_m$  on  $\rho$  by the probability of measuring  $a_n$  on  $\rho^{\text{after}|b_m}$  :

$$P(a_n, b_m) = \text{Tr}(\rho \hat{Q}_m \hat{P}_n \hat{Q}_m) \quad (6.8)$$

This formula is known as the Wigner formula.

We observe that two successive Von Neumann measurements do not constitute in general a Von Neumann measurement of the pair  $(b_m, a_n)$ , but instead a non ideal measurement described by the Probability Operator  $\hat{\Pi}_{m,n} = \hat{Q}_m \hat{P}_n \hat{Q}_m$ , which is not a projector. It is only when the operators  $\hat{A}$  and  $\hat{B}$  commute (and hence that the projectors  $\hat{P}_n$  and  $\hat{Q}_m$  commute) that the measurement of the pair is an ideal Von Neumann measurement.

Note that in general  $P(a_n, b_m) \neq P(b_m, a_n)$  : the order of the two measurements matters, unless the corresponding observables commute.

#### Conditional probability

Applying the definition of the conditional probability to the joint probability (6.8), we obtain, using the closure relation  $\sum_n \hat{P}_n = \hat{1}$  :

$$P(a_n|b_m) = \frac{\text{Tr}(\rho \hat{Q}_m \hat{P}_n \hat{Q}_m)}{\sum_n \text{Tr}(\rho \hat{Q}_m \hat{P}_n \hat{Q}_m)} = \frac{\text{Tr}(\rho \hat{Q}_m \hat{P}_n \hat{Q}_m)}{\text{Tr}(\rho \hat{Q}_m)} \quad (6.9)$$

### State of the system after the two measurements

The state after the two measurements is given by applying the state collapse postulate to the second measurement :

$$\rho^{\text{after}|b_m, a_n} = \frac{\hat{P}_n \hat{Q}_m \rho \hat{Q}_m \hat{P}_n}{\text{Tr}(\rho \hat{Q}_m \hat{P}_n \hat{Q}_m)} \quad (6.10)$$

We see that the post-measurement operator is, for each pair of results  $(b_m, a_n)$  :

$$\hat{K}_{m,n} = \hat{P}_n \hat{Q}_m \quad (6.11)$$

### 6.3.2 Case of two non ideal measurements

One can now make the same analysis, but with the general rules of non ideal quantum measurements of quantities  $B$  then  $A$ . We will call respectively  $\hat{\Theta}_m$  and  $\hat{\Pi}_n$  the corresponding POM,  $\hat{L}_m$  and  $\hat{K}_n$  the associated post-measurement operators.

#### Joint probability

For the joint probability, one gets :

$$\text{Proba}(a_n, b_m|\rho) = \text{Tr} \rho \hat{\Xi}_{m,n} \quad (6.12)$$

with a Probability Operator equal to  $\hat{\Xi}_{m,n} = \hat{L}_m^\dagger \hat{\Pi}_n \hat{L}_m = \hat{L}_m^\dagger \hat{K}_n^\dagger \hat{K}_n \hat{L}_m$ . Two successive non ideal measurements can thus be described as a joint non ideal measurement.

When the post-measurement operators commute  $[\hat{K}_n, \hat{L}_m] = 0$ , and hence the Probability Operators commute, one gets a simpler formula for the joint POM  $\hat{\Xi}_{m,n} = \hat{\Theta}_m \hat{\Pi}_n$  which obviously does not depend on the order of the two measurements.

#### Conditional probability

For the conditional probability one gets :

$$P(a_n|b_m) = \frac{\text{Tr}(\rho \hat{L}_m^\dagger \hat{\Pi}_n \hat{L}_m)}{\text{Tr}(\rho \hat{\Theta}_m)} \quad (6.13)$$

### State of the system after the two measurements

We have now :

$$\rho^{after|a_n, b_m} = \frac{M_{m,n}\rho M_{m,n}^\dagger}{Tr\rho\hat{\Xi}_{m,n}} \quad (6.14)$$

with the post-measurements operators :  $M_{m,n} = \hat{K}_n\hat{L}_m$ . We have obviously the relation  $\sum_{m,n} M_{m,n} = 1$ .

### Repeatability

What happens when one makes twice the same measurement ? We saw in chapter 2 that Quantum mechanics is repeatable with respect to Von Neumann measurements : one is sure to find in a second measurement the same result as in the first one.

For a non-ideal measurement, the joint probability of measuring  $a_{n'}$  after a first measurement giving the result  $a_n$  is :

$$P(a_{n'}|a_n) = \frac{Tr(\rho \hat{\Pi}_n \hat{\Pi}_{n'})}{Tr(\rho \hat{\Pi}_n)} \quad (6.15)$$

It is equal to  $\delta_{n,n'}$  only when  $\hat{\Pi}_n \hat{\Pi}_{n'} = \delta_{n,n'} \hat{\Pi}_n$ , i.e. when the measurement is projective, i.e. ideal.

One can therefore state that a measurement is actually a Von Neumann measurement if and only if it is repeatable.

### 6.3.3 The state collapse : updating the information on the system

When one knows the input state  $\rho$  of the system before any measurement has been effected on it, the conditional probability  $P(a_n|b_m)$  can be determined from the joint probability using relation (6.13) :

$$P(a_n|b_m) = \frac{Tr(\rho \hat{L}_m^\dagger \hat{\Pi}_n \hat{L}_m)}{Tr(\rho \hat{\Theta}_m)} \quad (6.16)$$

This conditional probability allows us to determine a conditional mean and conditional variance.

Note that it can also be written as in Born's rule :

$$P(a_n|b_m) = Tr(\rho_m \hat{\Pi}_n) = proba(a_n, \rho_m) \quad (6.17)$$

with :

$$\rho_m = \frac{\hat{L}_m \rho \hat{L}_m^\dagger}{Tr(\rho \hat{\Theta}_m)} \quad (6.18)$$

In other words *the conditional probability can be calculated as a regular, unconditional, probability, provided one uses the state  $\rho_m$  instead of the initial state  $\rho$ .*

Let's go back to the Bayes' conception that the relevant probabilities are the conditional probabilities, because they reflect our degree of information about a system, even for a single event, and let's apply it to quantum probabilities : each time one measures something on a quantum system, one acquires a piece of information, and as a result the conditional probabilities change. To account for this change, the tool used to calculate the probabilities, i.e. the density matrix  $\rho$ , must change accordingly,

Looking at (6.18), one sees that the "updated" state  $\rho_m$  is nothing else than the density matrix  $\rho^{after|b_m}$  resulting from the state collapse postulate applied after the first measurement.

More operationally, when two successive measurements (ideal or not ideal) are performed on a quantum system, one gets couples of values that are recorded in the memory of a computer. The recorded set of data gives us access to the statistical distribution for the joint probability of this couple of values. But in many instances, one is only interested in a subset of recorded events, those corresponding to a given value  $b_{m_0}$  of the  $B$  measurement, and discard all the other ones : this is called a "post-selection" process. We have just shown that one can calculate conditional probabilities, i.e. the statistical distribution of results of the  $A$  measurement in the post-selected set of data, by using the regular Born's rule for "normal" probabilities, but applied on a "conditional state", which is precisely the one given by the state collapse postulate.

The state collapse postulate appears in this point of view as a way to describe the effect of information updating on the quantum state. It is then quite natural that the updated state cannot be written as a quantum map, because it is not a physical evolution, but the result of a choice of relevant data.

## Chapitre 7

# Entanglement in bipartite systems

Let us consider a composite system, consisting of two parties (for example two light beams or two atoms) which have interacted in the past and are at the moment  $t$  of consideration separated by a distance which forbids any interaction between the two (from a relativistic point of view they are separated by a space-like interval, so that they cannot be connected by any interaction propagating at the velocity of light). We will call in the traditional way "Alice" and "Bob" the two parties,  $\mathcal{H}_A$  and  $\mathcal{H}_B$  the two corresponding Hilbert spaces (figure(7.1))

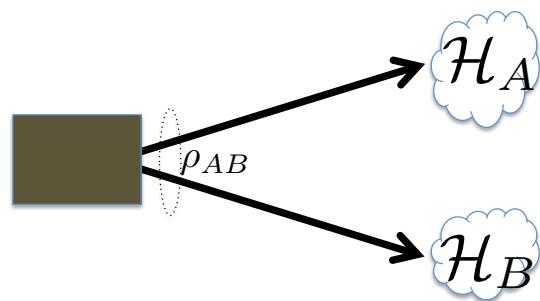


FIGURE 7.1 – Conditional non destructive generation of a specific "heralded" state from an entangled state

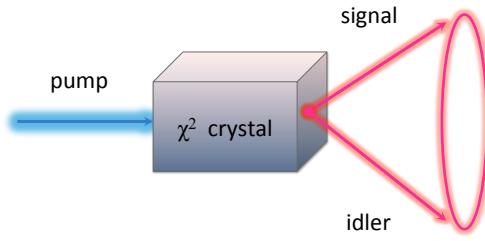


FIGURE 7.2 – Generation of an entangled state in the tensor product of two non-interacting Hilbert spaces

## 7.1 Experimental implementations

### 7.1.1 Photons

A first example is provided by the nonlinear process called *spontaneous parametric down-conversion*[17], which takes place in a nonlinear crystal pumped by laser light at frequency  $\omega_p$  and generates pairs of signal and idler photons at frequencies  $\omega_s$  and  $\omega_i$  (figure(7.2)). The energy conservation ensures that  $\omega_p = \omega_s + \omega_i$ . One shows that, when the effect is weak, the photon state  $|\psi_p\rangle$  produced by such a process can be written, within a normalization factor, as :

$$|\psi_p\rangle = |0\rangle + \int d\omega_s f(\omega_s) |1 : \omega_s\rangle \otimes |1 : \omega_i\rangle \quad (7.1)$$

where  $|0\rangle$  is the quantum state describing the absence of photons ("vacuum" state),  $f$  a coupling factor depending on the pumping light and on the characteristics of the nonlinear material : the signal and idler photons are usually emitted in symmetrical directions in a cone of down-converted light (figure(7.3)).

A second example is provided by the cascaded spontaneous emission of a three-level atom, with levels  $|f\rangle$ ,  $|e\rangle$  and  $|g\rangle$ , which is at time  $t = 0$  in its upper excited state  $|f\rangle$ . It decays to its ground state  $|g\rangle$  by the cascade  $|f\rangle \rightarrow |e\rangle$ ,  $|e\rangle \rightarrow |g\rangle$  by emitting two photons in two different random directions corresponding to wavevectors  $\mathbf{k}$  and  $\mathbf{k}'$ . It can be shown [16] that the photon state  $|\psi_c\rangle$  produced by such

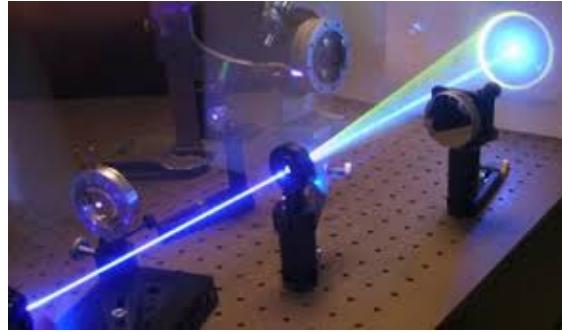


FIGURE 7.3 – Cone of down-converted light spontaneously emitted by the non-linear crystal when it is pumped by an intense laser

a process can be written as :

$$|\psi_c\rangle = |0\rangle + \int d\mathbf{k} d\mathbf{k}' \frac{g(\omega, \mathbf{k}, \omega', \mathbf{k}')}{(\omega + \omega' - \omega_\alpha - \omega_\beta + i\gamma_\alpha)(\omega - \omega_\beta + i\gamma_\beta)} |1 : \omega, \mathbf{k}\rangle \otimes |1 : \omega', \mathbf{k}'\rangle \quad (7.2)$$

where  $\omega_\alpha$  and  $\omega_\alpha$  are respectively the Bohr frequencies of transitions  $|f\rangle \rightarrow |e\rangle$  and  $|e\rangle \rightarrow |g\rangle$ , and  $g$  is a function depending on the properties of the atom.  $\gamma_\alpha$  and  $\gamma_\beta$  are the line widths of the two transitions.

One notes that these two states are not factorized as a tensor product of states belonging to Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_R$  : even though the generated photons have travelled over long distances and are very far apart, one cannot speak of specific quantum states for each of them. In order to fully account for their properties, one needs "non-local" states such as (7.1) and (7.2), which describe very extended physical systems as single global quantum objects.

### 7.1.2 Atoms

Suitable laser preparation is likely to produce non-factorized states of two two-level atoms or ions. One possibility consists in illuminating with resonant laser light two atoms (1) and (2) which are located at two points separated by a distance which is much smaller than the wavelength. Consequently, for very low excitation powers it is not possible to know which atom is excited, and the resulting two atom state  $|\psi_2\rangle$  is :

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|g_1, e_2\rangle + |g_2, e_1\rangle) \quad (7.3)$$

This state, which can be generalized to  $N$  atoms or ions, is not a tensor product either.

## 7.2 Case of a pure state $|\Psi_{AB}\rangle$

### 7.2.1 Definition

The states  $|\psi_c\rangle$ ,  $|\psi_p\rangle$  and  $|\psi_2\rangle$  that we have just mentioned are not factorized states, but may be so for some specific choice of the functions  $f$  or  $g$  if one performs a suitable "local" change of basis, which mixes only states inside  $\mathcal{H}_A$  or  $\mathcal{H}_B$  but does not mix states in  $\mathcal{H}_A$  to states in  $\mathcal{H}_B$  ).

If they cannot be factorized by any local basis change, they are called *entangled states*.

The two processes that we have just briefly outlined in the previous section are among the most widely used to produce photonic and atomic entangled states. In the two qubit case, the most famous entangled states are the so-called *Bell states*, given by :

$$|\Phi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|++\rangle \pm |--\rangle) \quad ; \quad |\Psi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|+-\rangle \pm |-+\rangle) \quad (7.4)$$

It is always straightforward but often cumbersome to check that un-factorized states remain unfactorized in any local basis. For example  $(|++\rangle - |--\rangle - |+-\rangle + |-+\rangle)/2$  is factorizable in the basis  $(|+\rangle \pm |-\rangle)/\sqrt{2}$ , whereas  $(|++\rangle - |--\rangle + |+-\rangle + |-+\rangle)/2$  is not.

Pure entangled states can also be described by the density matrix  $\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$ . Though the overall state is entangled, we have already noted that local properties measured on the Alice side for example using "local" measurement set-ups are fully predictable from the knowledge of the local reduced density matrix  $\rho_A = Tr_B \rho_{AB}$ , and the same is true for Bob using  $\rho_B = Tr_A \rho_{AB}$ .

Let us take the example of the Bell state  $|\Psi_-\rangle$  for example : - the joint probabilities of couples  $(+,+)$ ,  $(+,-)$ ,  $(-,+)$ ,  $(-,-)$  are respectively 0, 0.5, 0.5, 0. The conditional probabilities  $P(+|+)$ ,  $P(+|-)$ ,  $P(-|+)$ ,  $P(-|-)$  calculated by formula (6.2), are respectively 0, 1, 1, 0 : they correspond to deterministic predictions. Alice knows with certainty Bob's qubit value once she has measured her own qubit.

- in contrast the reduced density matrix on Alice side is equal to  $(|+\rangle\langle+||-\rangle\langle-|)/2$ . It is proportional to the identity and therefore describes a mixed state with a minimal information content. Entangled states featuring strong AB correlations that reduce to local

states obtained by partial trace that have no information content are called *maximally entangled states*.

It turns out that entangled states are among the most striking specific features of the quantum world. They are also the basic ingredient of the promising domain of *Quantum Computing*.

### 7.2.2 Schmidt decomposition

Erhard Schmidt has shown that any bipartite state can be written as :

$$|\Psi_{AB}\rangle = \sum_{n=1}^S \sqrt{p_n} |\phi_{nA}\rangle \otimes |\phi_{nB}\rangle \quad (7.5)$$

where  $0 \leq p_n \leq 1$ ,  $\sum p_n = 1$ ,  $\{\phi_{nA}\}$  and  $\{\phi_{nB}\}$  being two orthonormal sets of wavevectors respectively belonging to spaces  $\mathcal{H}_A$  and  $\mathcal{H}_B$ .

From 7.5, one gets

$$\rho_{AB} = \sum_{n,n'} \sqrt{p_n p_{n'}} |\phi_{nA}\rangle |\phi_{nB}\rangle \langle \phi_{n'A}| \langle \phi_{n'B}| \quad (7.6)$$

and the reduced density matrix is :

$$\rho_A = Tr_B \rho_{AB} = \sum_n p_n |\phi_{nA}\rangle \langle \phi_{nA}| \quad (7.7)$$

Therefore  $p_n$  is an eigenvalue, and  $|\phi_{nA}\rangle$  an eigenstate of the reduced density matrix  $\rho_A$ , whereas  $p_n$  is also an eigenvalue, and  $|\phi_{nB}\rangle$  an eigenstate of the reduced density matrix  $\rho_B$ . Being positive operators they have real positive eigenvalues with  $\sum p_n = 1$ .

The number  $S$  of terms in the Schmidt sum, smaller than or equal to the smaller dimension of the two Hilbert spaces, is called the *Schmidt number*. If  $S = 1$  the state is factorized. If  $S > 1$  the state is entangled.  $S$  is what is often called an *entanglement witness* : it gives a kind of "measure" of the strength of entanglement in the case of a pure two-partite state. Note that the Bell states (7.5) are directly written as a Schmidt sum with  $S = 2$  and  $p_n = 1/2$ .

Another entanglement witness is given by the local purity  $P_A = Tr \rho_A^2$  : if the two-partite pure state is factorized, then the local density matrix  $|\phi_{1A}\rangle \langle \phi_{1A}|$  is also pure and  $P_A = 1$ . Reciprocally, if  $P_A$  is one, then  $\sum p_n = 1$  and  $\sum p_n^2 = 1$ , which happens only when only one  $p_n$  is non zero : the state is factorized.

### 7.2.3 Alice-Bob Correlations

Alice and Bob can predict the probabilities of local measurement results on their own without caring about the other's result using their reduced density matrices. What happens now if they compare their results by sending them to a common place? They will be now able to discover possible correlations between what they have independently measured. To predict such correlations, they now need to use the full quantum state  $|\Psi_{AB}\rangle$ .

Let us consider local physical parameters,  $A$  on Alice's side and  $B$  on Bob's side, associated with observables  $\hat{A}$  acting on  $\mathcal{H}_A$  and  $\hat{B}$  acting on  $\mathcal{H}_B$ , with respective eigen-values and projectors  $\{a_\ell, \hat{P}_\ell\}$  and  $\{b_{\ell'}, \hat{Q}_{\ell'}\}$ . One has obviously  $[\hat{P}_\ell, \hat{Q}_{\ell'}] = 0$ , and therefore  $[\hat{A}, \hat{B}] = 0$ . For the sake of simplicity, we assume that both are ideal Von Neumann measurements.

We use the formalism introduced in chapter 6 about joint measurements. Then the joint probability of getting the couple of results  $(a_\ell, b_{\ell'})$  is

$$P(a_\ell, b_{\ell'}) = <\hat{P}_\ell \hat{Q}_{\ell'} \hat{P}_\ell> = <\hat{P}_\ell^2 \hat{Q}_{\ell'}> = <\hat{P}_\ell \hat{Q}_{\ell'}> \quad (7.8)$$

because the projection operators commute.

— if the state is factorized, then

$$P(a_\ell, b_{\ell'}) = \langle \phi_{1A} | \hat{P}_\ell | \phi_{1A} \rangle \langle \phi_{1B} | \hat{Q}_{\ell'} | \phi_{1B} \rangle = P(a_\ell) P(b_{\ell'}) \quad (7.9)$$

and therefore the two measurements are uncorrelated. One can also deduce it from the wave collapse postulate : when Alice obtains the result  $a_\ell$  of her prior measurement, the state collapse associated with the measurement affects only  $|\phi_{1A}\rangle$  and leaves Bob's part unchanged. So the measurement performed by Bob will not be affected by Alice's measurement. The reasoning is much simpler than the previous one, but seems to depend on the order of measurements. It is not the case in the reasoning involving joint probabilities.

— if the state is entangled, then the previous argument is no longer valid : the state collapse due to Alice's measurement will affect Bob's side, inducing a correlation between the measurement results.

More precisely, let us consider specific observables  $\hat{F}_A$  and  $\hat{F}_B$  that are diagonal in the Schmidt bases, with eigenvalues  $\alpha_\ell$  and  $\beta_{\ell'}$ . It is straightforward to show that the joint probability of getting the couple of results  $(\alpha_\ell, \beta_{\ell'})$  is :

$$P(\alpha_\ell, \beta_{\ell'}) = p_\ell \delta_{\ell, \ell'} \quad (7.10)$$

so that the conditional probability of measuring  $\hat{F}_B$  knowing the result on  $\hat{F}_A$  is :

$$P(\alpha_\ell|\beta_{\ell'}) = \frac{P(\alpha_\ell, \beta_{\ell'})}{P(\beta_{\ell'})} = \delta_{\ell,\ell'} \quad (7.11)$$

The two measurements results are therefore for this particular choice of observables *perfectly correlated* : after her measurement, Alice knows with certainty the result that Bob will obtain, even though they are separated by millions of kilometers !

It must be stressed here that *the existence of such a perfect correlation between remote measurement results is not at all a specific quantum effect* but exists also classically : let us consider for example a classical random process that draws integers ranging from 1 to S. For a given drawing lot yielding the value  $\ell$ , an operator sends an envelope to Alice with value  $\alpha_\ell$  written inside, and at the same time an envelope to Bob with value  $\beta_\ell$  written inside. When Alice, who knows the rules of the game but not the result of the draw, opens her envelope and "measures" the value  $\alpha_\ell$ , she knows for sure what Bob will measure when he opens his envelope, whatever the distance between them and the temporal order of their openings.

This correlation between remote locations does not violate the relativistic causality : the information that Alice has on Bob's results is "local" to Alice space. It cannot be transferred to Bob for comparison at a velocity larger than  $c$ . In addition it cannot be used to transfer information from Alice to Bob, because Alice does not master the outcome of her measurement.

## 7.3 Case of a mixed state

### 7.3.1 Definitions of entanglement and separability

- A *factorized state* is described by a density matrix of the form  $\rho_{AB} = \rho_A \rho_B$ , where  $\rho_A$  and  $\rho_B$  are density matrices in the two spaces  $\mathcal{H}_1$  and  $\mathcal{H}_R$ . The same reasoning as above shows that the measurements performed by Alice and Bob are not correlated.
- A *separable state* is described by a density matrix which is a statistical superposition of factorized states. It is therefore of the form

$$\rho_{AB} = \sum_n p_n \rho_{nA} \rho_{nB} \quad (7.12)$$

where  $\rho_{nA}$  and  $\rho_{nB}$  are density matrices in the two spaces  $\mathcal{H}_1$  and  $\mathcal{H}_R$ . It can be produced by making a classical draw with probability  $p_n$  of uncorrelated factorized states, like in the classical example of the envelopes given in the previous section. It is easy to show that one has  $P(a_\ell, b_{\ell'}) \neq P(a_\ell)P(b_{\ell'})$  and therefore that the Alice and Bob measurements have some degree of correlation, the classical or quantum origin of which is still an active debate between researchers.

- Finally, an *entangled state* is described by a density matrix which cannot be written as (7.12).

### 7.3.2 An entanglement witness : the Positive Partial Transpose criterion

We note that the definition of an entangled mixed state is more subtle than the definition of an entangled pure state. In addition, there is no Schmidt decomposition for a mixed bipartite state. So it is not a simple task to decide whether a given state is entangled or not. So far, only sufficient conditions can be found in the literature. We will describe here one of the most widely used entanglement criteria, called the "positive partial transpose" test[18] (PPT).

Let us introduce the basis  $|u_i, v_j\rangle$  of the bipartite Hilbert space  $\mathcal{H}_1 \otimes \mathcal{H}_R$ . It is easy to show that the transpose  $\rho^T$  of a positive operator  $\rho$ , defined as usual by  $\langle u_i, v_j | \rho^T | u_{i'} v_{j'} \rangle = \langle u_{i'}, v_{j'} | \rho | u_i v_j \rangle$  is also a positive operator : the transpose operation is a particular case of a quantum map.

Let us now define the *partial transpose operation PT* by :

$$\langle u_i, v_j | \rho^{PT} | u_{i'} v_{j'} \rangle = \langle u_{i'}, v_{j'} | \rho | u_i v_j \rangle \quad (7.13)$$

in which only the space  $\mathcal{H}_1$  is affected by the transposition. If the state is separable, one can write it as (7.12), and therefore :

$$\rho_{AB}^{PT} = \sum_n p_n \rho_{nA}^T \rho_{nB} \quad (7.14)$$

$\rho_{AB}^{PT}$  describes another separable state. It is a *bona fide* density matrix, and therefore is a positive operator.

Equivalently, we have shown that, if  $\rho_{AB}^{PT}$  is not a positive operator, then  $\rho_{AB}$  cannot be written like (7.12) and therefore describes an entangled state. The reciprocal is not true, so that the PPT criterion is only sufficient : some entangled states have a positive partial transpose operator, so that one cannot say anything about the entanglement of a quantum state described by a density matrix  $\rho_{AB}$

with a positive  $\rho_{AB}^{PT}$ , except for Hilbert spaces of dimensions  $2 \times 2$  (entangled qubits) and  $2 \times 3$ , for which the condition has been shown to be necessary and sufficient.

Let us give here a simple example : the Bell state  $|\Psi^-\rangle$  is described by the following density matrix in the basis  $(|++\rangle, |+-\rangle, |-+\rangle, |--\rangle)$  :

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (7.15)$$

which has positive eigenvalues  $(1,0,0,0)$ , like any pure state. Its partial transpose reads :

$$\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad (7.16)$$

It has eigenvalues  $(-1,1,1,1)$  and is therefore not positive :  $|\Psi^-\rangle$  is indeed an entangled state.

## 7.4 Conditional non destructive preparation of a quantum state

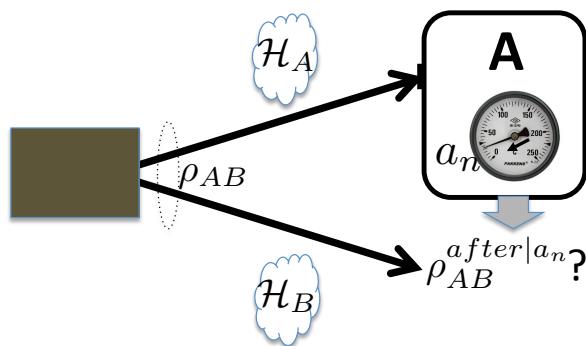


FIGURE 7.4 – Conditional non destructive generation of a specific "heralded" state from an entangled state

We consider two particles A and B, each described in Hilbert spaces respectively named  $\mathcal{H}_A$  and  $\mathcal{H}_B$ . A physical system, like the parametric crystal or the three-level atom of section 7.1, generates a two-particle quantum state, described by the density matrix  $\rho_{AB}$

operating in the tensor product  $\mathcal{H}_A \otimes \mathcal{H}_B$ , which may be entangled or not. After their production inside the physical system the two particles propagate in different directions. When they are, after some time, very far away from each other, they are still described by the common, non local, density matrix  $\rho_{AB}$ . One then makes a measurement on particle A that gives results in probabilistic way. We want to determine the state of particle B when one has measured on particle A the specific value  $a_n$ .

Let  $\hat{K}_n$  be the corresponding post-measurement operator, that acts only on  $\mathcal{H}_1$ , and  $\hat{\Pi}_n = \hat{K}_n^\dagger \hat{K}_n$  the associated POM. The two-particle state after the result  $a_n$  has been obtained is, according to the postulate  $M_3$ ,  $\hat{K}_n \rho_{AB} \hat{K}_n^\dagger / \text{Tr}(\hat{K}_n \rho_{AB} \hat{K}_n^\dagger)$ , and the state of particle B alone is obtained by a partial trace on space  $\mathcal{H}_1$ :

$$\rho_B^{after|a_n} = \frac{\text{Tr}_A(\hat{K}_n \rho_{AB} \hat{K}_n^\dagger)}{\text{Tr}(\hat{K}_n \rho_{AB} \hat{K}_n^\dagger)} \quad (7.17)$$

If the state is factorized  $\rho_{AB} = \rho_A \rho_B$ , then

$$\rho_B^{after|a_n} = \frac{\text{Tr}_A(\hat{K}_n \rho_A \hat{K}_n^\dagger) \rho_B}{\text{Tr}_A(\hat{K}_n \rho_A \hat{K}_n^\dagger) \text{Tr}_B(\rho_B)} = \rho_B \quad (7.18)$$

The conclusion is that, as intuitively expected, in the factorized and therefore uncorrelated case a measurement on A has no influence on particle B.

If the state is separable  $\rho_{AB} = \sum_i p_i \rho_{iA} \rho_{iB}$ , then

$$\rho_B^{after|a_n} = \frac{\sum_i p_i \text{Tr}_A(\hat{K}_n \rho_{iA} \hat{K}_n^\dagger) \rho_{iB}}{\sum_i p_i \text{Tr}_A(\hat{K}_n \rho_{iA} \hat{K}_n^\dagger) \text{Tr}_B(\rho_{iB})} = \frac{1}{\sum_i p_i q_{ni}} \sum_i p_i q_{ni} \rho_{iB} \quad (7.19)$$

where  $q_{ni} = \text{Tr}_A(\hat{K}_n \rho_{iA} \hat{K}_n^\dagger) = \text{Tr}_A(\rho_{iA} \hat{\Pi}_n)$  is the probability of getting the result  $a_n$  when the particle A is in state  $\rho_{iA}$ . The state of particle B, and therefore our knowledge of its properties, is changed in a conditioned way depending on the specific outcome of the measurement of particle A. This change takes place whatever the distance between them and without any physical action on particle B, hence the name "non-destructive" generation. In addition, if the measurement is chosen in such a way that there is a single value  $i_n$  for which  $q_{ni}$  is different from zero (and therefore equal to 1), then the generated state is simply

$$\rho_B^{after|a_n} = \rho_{i_n B} \quad (7.20)$$

This state is said to be "heralded" by the result of the measurement on the other particle. Note that it is generated when the initial state

$\rho_{AB}$  is separable : full entanglement is not necessary in this situation, only correlations. The heralded state is pure if  $\rho_{i_n B}$  is pure. Finally note that the heralded state is not generated in a deterministic way, but in a not-controlled, probabilistic, way. In this context, a very important parameter is the "success rate", i.e. the probability of obtaining the desired state, which is equal to the probability  $p_{i_n}$  of measuring  $a_n$ .

Let us now be more specific, and consider the case of twin photon generation by downconversion or atomic cascade, in which a pure, entangled state is generated, described by the Schmidt sum  $|\psi_{AB}\rangle = \sum_i \sqrt{p_i} |1:i, A\rangle |1:i, B\rangle$ . The corresponding density matrix is

$$\rho_{AB} = |0\rangle\langle 0| + \sum_{i,i'} \sqrt{p_i p_{i'}} |1:i, A\rangle\langle 1:i', A| \otimes |1:i, B\rangle\langle 1:i', B| \quad (7.21)$$

where  $|1:i, A\rangle$  for example describes a single photon in the optical mode labeled by  $i$ .

We suppose first that the detector is not selective and counts all photons of beam A with the same efficiency  $\eta$ , and has dark counts with probability  $d$ . The POM corresponding to a click, according to chapter 3, is  $\hat{\Pi}_{click} = \eta \sum_\ell |1:\ell, A\rangle\langle 1:\ell, A| + d|0\rangle\langle 0|$ . The heralded state is in this case :

$$\rho_B^{after|click} = \frac{1}{\eta + d} \left( \eta \sum_\ell p_\ell |1:\ell, B\rangle\langle 1:\ell, B| + d|0\rangle\langle 0| \right) \quad (7.22)$$

It is a mixed state, with a possibility of no-photons because of dark counts on A, and of a statistical mixture of single photon states. When there is no dark counts, the heralded state does not depend on the quantum efficiency of the detection, whereas the success rate to produce, equal to  $\eta$  depends on it.

Let us finally note that if the single photon detector is mode selective ( $\hat{\Pi}_{click} = \eta |1:\ell_0, A\rangle\langle 1:\ell_0, A|$ ), the heralded state, equal to  $|1:\ell_0, B\rangle\langle 1:\ell_0, B|$  is the pure single mode single photon state  $|1:\ell_0, B\rangle$ .

## 7.5 The Einstein-Podolsky-Rosen argument

The importance, uniqueness and strangeness of entanglement in Quantum Mechanics was recognized for the first time in 1935, in two famous papers by Einstein, Podolski and Rosen[19], and by Schrödinger[20] who introduced the word entanglement. We will expose here the argument given in the first paper, called EPR.

### 7.5.1 The EPR argument on position and momentum of two particles

Let us consider two one dimensional particles A and B of positions and momenta  $x_A, x_B, p_A, p_B$ .

Let us assume that the two particles A and B have interacted in the past so that their positions and momenta are somehow correlated. At the time of consideration, they are far apart and non-interacting. They are supposed to be in the quantum state described by the wavefunction :

$$\psi_{EPR}(x_A, x_B) = \delta(x_A - x_B - x_0) = \frac{1}{2\pi} \int dp e^{ipx_A} e^{-ipx_B} e^{-ipx_0} \quad (7.23)$$

where  $x_0$  is a fixed position. The second form of the state shows that it is not factorized in the basis  $(x_A, x_B)$ <sup>1</sup>. It is easy to show that in such a state,  $\Delta x_A, \Delta x_B, \Delta p_A$  and  $\Delta p_B$  are infinite, so that the two particles are completely delocalized and their velocities are completely random.

In addition  $\psi_{EPR}$  is an eigenstate of operator  $\hat{x}_A - \hat{x}_B$  with eigenvalue  $x_0$ , and of operator  $\hat{p}_A + \hat{p}_B$  with eigenvalue 0 :

$$(\hat{x}_A - \hat{x}_B)\psi_{EPR} = (x_A - x_B)\psi_{EPR} = x_0\psi_{EPR} \quad (7.24)$$

$$(\hat{p}_A + \hat{p}_B)\psi_{EPR} = \frac{\hbar}{i} \left( \frac{\partial}{\partial x_A} + \frac{\partial}{\partial x_B} \right) \delta(x_A - x_B - x_0) = 0 \quad (7.25)$$

This implies that  $\Delta(x_A - x_B) = 0$  and  $\Delta(p_A + p_B) = 0$ . There is a perfect position correlation and perfect momentum anticorrelation between the two particles. This is not forbidden by a Heisenberg inequality because  $[\hat{x}_A - \hat{x}_B, \hat{p}_A + \hat{p}_B] = 0$ . This correlation can be written in terms of the conditional variances of chapter 4 :

$$\Delta(x_A|x_B) = 0 \quad ; \quad \Delta(p_A|p_B) = 0 \quad (7.26)$$

When Bob measures a given value for the position of his particle, he knows for sure that Alice's particle has the same position ; When Bob measures later a given value for the momentum of his particle, he knows for sure that Alice's particle has just the opposite momentum. One has then  $\Delta(x_A|x_B)\Delta(p_A|p_B) = 0$ , so that it seems that a kind of Heisenberg-like inequality is violated in such a case, concerning conditional, post-selected variances instead of "true" variances calculated over all measurements of Alice and all measurements of Bob. This does not violate any rule of Quantum Mechanics, but is

---

1. This state is not normalizable, but can be considered as the limit of the well-behaved function  $\exp(-(x_A - x_B - x_0)^2/2\sigma^2)/\sqrt{2\pi\sigma}$  when  $\sigma \rightarrow 0$ .

puzzling, because it shows that Quantum Mechanics does not forbid us to know perfectly either the position or the momentum of a particle in a nondestructive way, and even without any interaction with this particle.

In their paper, EPR went further in the argument and claimed that this means that both position and momentum of a particle are objective elements of reality, so that usual Quantum Mechanics, which considers them as "complementary", antinomic, self-excluding quantities, is not complete, i.e. does not describe the whole physical reality of the particle.

### 7.5.2 The EPR argument on two qubits

Let us assume that two spin 1/2 particles have been created in the Bell state  $|\Psi^-\rangle$ . From an angular momentum point of view this state is actually the singlet state  $|S = 0, M_S = 0\rangle$  of the total spin  $\hat{\mathbf{S}} = \hat{\mathbf{S}}_A + \hat{\mathbf{S}}_B$ . Being a state of angular momentum 0, it is invariant with respect to rotations, so that :

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|+_x -_x\rangle - |-_x +_x\rangle) = \frac{1}{\sqrt{2}}(|+_y -_y\rangle - |-_y +_y\rangle) \quad (7.27)$$

where  $|\pm_x\rangle$ , for example is the eigenstate of  $\hat{S}_x$  with eigenvalue  $\pm 1/2$ .

On Alice's side, the reduced quantum state is  $\rho_A = (|+\rangle\langle+| + |-\rangle\langle-|)$ , which contains the minimal information about the state : a measurement of the angular momentum  $\hat{\mathbf{S}}_A$  projected over axes  $Ox$ ,  $Oy$  or  $Oz$  will give random results so that  $\langle \hat{S}_{xA} \rangle = \langle \hat{S}_{yA} \rangle = \langle \hat{S}_{zA} \rangle = 0$ , and the same is true on Bob's side.

Like the EPR state of the previous section,  $|\Psi^-\rangle$  features very strong correlations between the measurements performed by Alice and Bob : an argument based on state collapse postulate shows that when Bob measures  $\hat{S}_{xA}$  and finds  $1/2$ , he knows for sure that Alice will find the value  $-1/2$  for  $\hat{S}_{xB}$ , and when Bob finds  $-1/2$ , he knows for sure that Alice will find the value  $+1/2$ . The same is true for their measurements of the other components of the spin components of the two particles like  $\hat{S}_{yA}$  and  $\hat{S}_{yB}$ . This implies that  $\langle \hat{S}_{xA}\hat{S}_{xB} \rangle = \langle \hat{S}_{yA}\hat{S}_{yB} \rangle = \langle \hat{S}_{zA}\hat{S}_{zB} \rangle = -1/4$ , and for the normalized correlation functions :

$$c(S_{xA}, S_{xB}) = c(S_{yA}, S_{yB}) = c(S_{zA}, S_{zB}) = -1 \quad (7.28)$$

There is a perfect anticorrelation between the measurements. More generally, it is a simple exercice to show that

$$c(\mathbf{S}_A \cdot \mathbf{e}_A, \mathbf{S}_B \cdot \mathbf{e}_B) = -\mathbf{e}_A \cdot \mathbf{e}_B \quad (7.29)$$

where  $\mathbf{e}_A$  and  $\mathbf{e}_B$  are two unit vectors of arbitrary directions.

### 7.5.3 Bell inequality for classical correlations

As already noticed in section 7.2.3, classical correlations between remote objects exist also. Let us adapt the example given above to the two bit case : our operator is actually a shoemaker fabricating left and right shoes which are either red or blue. Side and color are two two-valued properties of the shoes that we call respectively  $S$  and  $S'$ , with the code :  $S = 1$  for red,  $S = -1$  for blue,  $S' = 1$  for left,  $S' = -1$  for right. The shoemaker has a supply of pairs containing the same number of blue and red pairs of shoes. He chooses randomly a pair among them and sends one shoe of the pair to Alice, and the other to Bob. When Alice opens her parcel and finds for example a right red shoe, she knows for sure what Bob will receive a parcel containing a left red shoe, whatever the distance between them and the order of their openings.

Let us note by an upper bar the average over many trials. One has obviously  $\bar{S}_A = \bar{S}_B = \bar{S}'_A = \bar{S}'_B = 0$  and  $\overline{S_A S_B} = 1$ ,  $\overline{S'_A S'_B} = -1$  and therefore a perfect correlation (or anticorrelation) between the measurements : the physical situation seems to be very similar with the spin case or EPR case.

Bell[21] made the striking discovery that classical, "obvious", correlations like the ones described in this section, are actually subject to a limit. He considered the following combination of measurement results :

$$B = S_A S_B + S_A S'_B + S'_A S_B - S'_A S'_B \quad (7.30)$$

Being a combination of four  $\pm 1$ , the mean value  $\overline{B}$  is a priori comprised between -4 and 4. But the bound is actually tighter. Let us write  $B$  as :

$$B = S_A(S_B + S'_B) + S'_A(S_B - S'_B) \quad (7.31)$$

If Bob finds in his draw identical values for  $S_B$  and  $S'_B$  then  $B = \pm 2S_A = \pm 2$ . If he finds opposite values then  $B = \pm 2S'_A = \pm 2$ . Then  $\overline{B}$  is an average of values  $\pm 2$  and therefore comprised between -2 and 2 :

$$-2 \leq \overline{B} = c(S_A, S_B) + c(S_A, S'_B) + c(S'_A, S_B) - c(S'_A, S'_B) \leq 2 \quad (7.32)$$

This is the famous *Bell inequality*[21, 22]. Note that  $\overline{B}$  is equal to 2 in the example with the shoes, because  $\overline{S_A S'_B} = \overline{S'_A S_B} = 0$ .

### 7.5.4 Quantum mechanics violates Bell inequality

Let us go back to the two spin 1/2 case, and consider that the two measurements made by Alice are spin component measurements  $\mathbf{S}_A \cdot \mathbf{e}_A$  and  $\mathbf{S}_A \cdot \mathbf{e}'_A$  with unit vectors  $\mathbf{e}_A$  and  $\mathbf{e}'_A$  taken along the  $Ox$  and  $Oy$  directions, and the two measurements made by Bob are spin component measurements  $\mathbf{S}_B \cdot \mathbf{e}_B$  with unit vectors  $\mathbf{e}_B$  and  $\mathbf{e}'_B$  taken along the diagonal and anti-diagonal directions of the  $xOy$  plane. Using relation (7.29), one has in this case :

$$\begin{aligned} & c(\mathbf{S}_A \cdot \mathbf{e}_A, \mathbf{S}_B \cdot \mathbf{e}_B) + c(\mathbf{S}_A \cdot \mathbf{e}'_A, \mathbf{S}_B \cdot \mathbf{e}_B) \\ & + c(\mathbf{S}_A \cdot \mathbf{e}_A, \mathbf{S}_B \cdot \mathbf{e}'_B) - c(\mathbf{S}_A \cdot \mathbf{e}'_A, \mathbf{S}_B \cdot \mathbf{e}'_B) \\ & = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) = 2\sqrt{2} \end{aligned} \quad (7.33)$$

This quantity, which is the quantum analogue of  $\bar{B}$  defined in the previous section, is larger than 2 for this specific choice of geometry. Bell has thus shown that there are situations where quantum mechanics predicts correlations between Alice and Bob measurements that cannot be attributed to correlations existing between classical remote objects having interacted in the past.

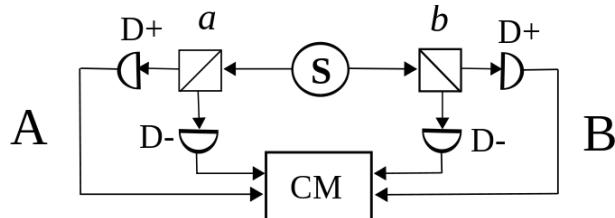


FIGURE 7.5 – sketch of an experiment testing Bell inequality on entangled photons

Several experiments, being closer and closer to the theoretical situation considered by Bell, have checked that the quantum prediction was indeed correct. For experimental reasons they used photons and polarization degrees of freedom, instead of massive spin 1/2 particles and Stern-Gerlach measurements. They are sketched on figure(7.5) : a source S, like the ones described at the beginning of this chapter (atomic cascade or spontaneous down conversion),

is designed in such a way that it produces pairs of photons with entangled polarization propagating in different directions. Their polarization are measured by Alice and Bob using polarizers a and b that can be rotated around the propagation axis. Alice and Bob measure successively the polarization of the photons propagating towards them using each two different directions of polarization, look at the correlations between their measurements events in CM and calculate the quantity  $B$  of equation (7.30). The first experiments were done by Freedman et al ([23]), Aspect et al ([24]), Weihs et al ([25]) respectively in 1972, 1982 and 1998. They measured  $\bar{B}$  values respectively equal to  $2.4 \pm 0.06$ ,  $2.101 \pm 0.020$  and  $2.73 \pm 0.02$ , violating the Bell inequality respectively by 3, 5 and 30 standard deviations.

### 7.5.5 Is quantum uncertainty due to a lack of knowledge ?

Quantum uncertainty is a puzzling feature of quantum mechanics, a limitation to our knowledge of Nature. A important question then arises : is this uncertainty intrinsic, or provisional and due to the fact that Quantum Mechanics does not describe completely the microscopic objects ? There are perhaps some physical supplementary parameters (the so-called *hidden variables*) characterizing the particles that are presently unknown and therefore not mastered. The randomness of their values could be responsible for the quantum uncertainty. In this case the quantum mean would be actually a classical mean over these quantities. Note that *the supplementary parameters are local*, i.e. have values which are fixed and carried by the particles after the end of their interaction, like the color and side of the shoes.

Bell inequality is a straightforward consequence of classical averaging over local parameters pre-existing to the measurement and carried by the particles. Its violation by Quantum mechanics shows that the hope to explain quantum uncertainty by uncontrolled supplementary parameters (shared by many physicists, in particular by Einstein) is wrong : it seems that Quantum uncertainty is intrinsic. In the quantum case the two measurements concern two non-commuting observables that cannot be measured simultaneously. One is led to the conclusion that the values found by Alice and Bob do not actually exist before the measurement, though the correlation between the values found by Alice and Bob are perfectly correlated.

## Chapitre 8

# Use of entangled states

Entangled states are characterized by correlations that are stronger than the classical ones. They can be used to perform new tasks, especially in the domain of quantum information and quantum computing. We give in this section two examples of such tasks. More details accounts of these applications can be found in [2, 3, 13].

### 8.1 Database search

This is the simplest example showing that the rules of quantum mechanics can lead to the resolution of a problem in a smaller number of operations than by using the classical rules. It is known as the *Grover algorithm*[26] : considering a given database, for example a list of  $N$  names  $X$  and corresponding telephone numbers  $N(X)$  in alphabetical order of names, the problem to solve is the "inverse problem", namely to find the name  $X_0$  corresponding to a given telephone number  $N_0 = N(x_0)$ . To simplify the following explanation, we suppose that  $N$  is a power of 2 :  $N = 2^p$ .

The classical way consists in browsing through the name list in a random or deterministic way. Since one can find  $N_0$  by chance at any place in the list with equal probability, the average number of trials leading to the searched  $X_0$  is  $N/2 = 2^{p-1}$ .

The quantum algorithm permitting to solve the search problem consists of :

- a Hilbert space spanned by a tensor product of  $p+1$  qubits (often called "qu-registers"), each qubit being either  $|0\rangle$  or  $|1\rangle$  instead of  $|+\rangle$  and  $|-\rangle$ . The list of the first  $p$  qubits gives a qu-register called  $|x\rangle$  which is associated with the name  $X$  in the list which is at a line given by the binary number  $x$ . For example  $|x\rangle = |1010\rangle$  is associated with the 10<sup>th</sup> name

in the telephone book. We will call  $|x_0\rangle$  the qu-register associated with the searched name  $X_0$ . The last qubit, called the "ancillary qubit"  $|\phi\rangle$  will be used in the computation. .

- A unitary quantum evolution operator  $\hat{U}_{N_0}$ , acting on this Hilbert space, that can be physically implemented on the qubits and depends on the "question"  $N_0$  and that will be used several times in the calculation. The number of iterations of  $\hat{U}_{N_0}$  corresponds to the number of trials in the classical search.
- An initial state  $|\psi_i\rangle$  such that  $(\hat{U}_{N_0})^m |\psi_i\rangle \simeq |x_0\rangle \otimes |\phi\rangle$  gives the solution  $x_0$  of the problem after  $m$  iterations.

More precisely  $\hat{U}_{N_0}$  is the product of two unitary evolutions  $\hat{U}_{N_0} = \hat{U}_2 \hat{U}_{1,N_0}$ .

-  $\hat{U}_{1,N_0}$  acting on a state  $|x\rangle \otimes |0 \text{ or } 1\rangle$  gives the same state if  $x \neq x_0$ , and gives  $|x\rangle \otimes |1 \text{ or } 0\rangle$  if  $x = x_0$ . This transformation can be easily implemented, as it uses the telephone book in the "right way", looking for the  $x^{th}$  name in the list, and checking whether it is the right one or not. One can find in ref [13] the details of the series of quantum gates needed to implement such a unitary transformation.

- The second evolution operator  $\hat{U}_2$  is equal to  $(2|s\rangle\langle s| - \hat{1}) \otimes \hat{1}$  where :

$$|s\rangle = \frac{1}{\sqrt{N}} \sum_{x=1}^N |x\rangle \quad (8.1)$$

One easily shows that  $\hat{U}_2$  is unitary. Acting on a given qu-register  $|x_0\rangle \otimes |\phi\rangle$  it gives

$$\hat{U}_2|x_0\rangle \otimes |\phi\rangle = \left( \frac{2}{N} \sum_{x=1}^N |x\rangle - |x_0\rangle \right) \otimes |\phi\rangle \quad (8.2)$$

So that, apart from being the identity for the last qubit  $|\phi\rangle$  its action on a general state  $|\psi\rangle = \sum c_x' |x'\rangle$  is such that :

$$\langle x_0 | \hat{U}_2 | \psi \rangle = \frac{2}{N} \sum_{x=1}^N c_x - c_{x_0} \quad (8.3)$$

$$\frac{1}{2} (\langle x_0 | \hat{U}_2 | \psi \rangle + c_{x_0}) = \frac{1}{N} \sum_{x=1}^N c_x \quad (8.4)$$

The transformation  $\hat{U}_2$  consists therefore in replacing each coefficient  $c_{x_0}$  of a state  $|\psi\rangle$  by its symmetrical value with respect to the mean  $(\sum c_x)/N$ .

- The input state is  $|\psi_i\rangle = |s\rangle \otimes |\phi'\rangle$ , where  $|s\rangle$  is the symmetrized state already mentioned and  $|\phi'\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ . The state  $|s\rangle$  is

obviously entangled with respect to the different qubits composing the quantum register. One notes that  $\hat{U}_{1,N_0}$  acting on  $|x\rangle \otimes |\phi'\rangle$  gives the same state if  $x \neq x_0$  and minus this state if  $x = x_0$ .

The calculation consists in applying  $m$  times the product of the two previous transformations to get the final state  $|\psi_f\rangle = (\hat{U}_2 \hat{U}_1)^m |\psi_i\rangle$

Let us consider the effect of these successive transformations when  $N = 8$  and  $p = 3$  in the following table. Each line contains the coefficients of the decomposition of the state over the 3 qubit-register basis starting from the initial state and after each step. We assume for example that the searched name is the fourth in the list :

$$\left| \begin{array}{ccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 4/8 & 4/8 & 4/8 & 20/8 & 4/8 & 4/8 & 4/8 \\ 4/8 & 4/8 & 4/8 & -20/8 & 4/8 & 4/8 & 4/8 \\ -2/8 & -2/8 & -2/8 & 22/8 & -2/8 & -2/8 & -2/8 \\ -2/8 & -2/8 & -2/8 & -22/8 & -2/8 & -2/8 & -2/8 \end{array} \right| \times 1/\sqrt{8} \quad (8.5)$$

One observes that the coefficient of the fourth column (the right "answer") grows step after step. After three steps the right answer has a probability weight of 95%. One can show that the coefficient "points" to the right answer after an average number of steps equal to  $\sqrt{N} = 2^{p/2}$  instead of  $N/2$ , which is quite a speed-up when the number  $N$  is very large.

Roughly speaking, the quantum advantage is due to the fact that one presents as an input the state  $|s\rangle$ , the linear superposition of all possible solutions. The algorithm processes then simultaneously, in a kind of parallel way, all the possibilities. What is exploited here are the two quantum resources of entanglement and superposition.

There are other algorithms for which the quantum advantage is even larger. This is for example the case of the *Shor algorithm*, a quantum algorithm able to factorize numbers in polynomial time instead of sub-exponential time for the corresponding classical algorithm.

## 8.2 Quantum teleportation

It should more properly named "quantum fax machine" as it deals with transportation of information and not of matter. A classical fax reads a page of unknown text at Alice's position, encodes it in a communication channel which transmits the information to another fax located at Bob's position. Bob's fax which decodes it and prints the recovered text on a blank page inserted in it. Whatever the input

page, there is no fundamental limitation to the accuracy with which the reading is made, and Bob's copy can be perfectly identical to the input text, leaving aside the technical imperfections of the machine which can be reduced without any limit, at least in principle.

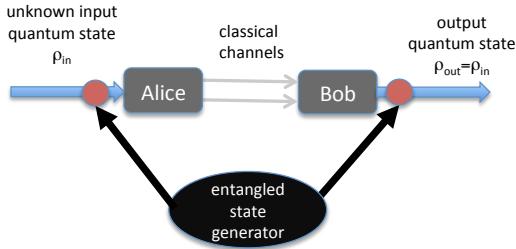


FIGURE 8.1 – Quantum teleportation : Alice makes measurements on the mixture between the state to teleport and the first part of an entangled state and sends the results to Bob through a classical communication channel. The teleported state results from the mixture between the second part of the entangled state and a quantum state produced by Bob on the basis of the results transmitted by Alice. The red circles correspond to linear couplers

We would like of course to extend this operation to the quantum domain, measure an unknown input quantum state  $|\psi_{Ain}\rangle$  of a quantum system, for example a particle A, transmit the information to Bob's place B and use it to modify the state of a particle B so that its final state  $|\psi_{Bout}\rangle$  is exactly identical to the input  $|\psi_{Ain}\rangle$ . This seems difficult, because any measurement made on particle A will give only a partial information on its quantum state, and furthermore will perturb it, so that it will be impossible to obtain another precise piece of information about the state in a second measurement. One can show that one can by this way obtain on particle B only a degraded copy of the initial state, described by a density matrix  $\rho_{Bout}$  such that :

$$\mathcal{F} = \langle \psi_{Ain} | \rho_{Bout} | \psi_{Ain} \rangle \leq 1/2 \quad (8.6)$$

$\mathcal{F}$  being called the *fidelity* of the process, which is one if the output state coincides with the input.

It is nevertheless possible to generate on particle B a perfect copy  $\rho_{Bout} = |\psi_{Ain}\rangle\langle\psi_{Ain}|$  with the help of entanglement if one uses the more complicated set-up of figure (8.1) : One produces in some other place a couple of particles A' and B' which are in an entangled state. Particle A' is sent to Alice and B' to Bob. Alice makes measurements

about the correlations between her own particle A in the unknown state  $|\psi_{Ain}\rangle$  and particle A'. This information is encoded in a classical channel, for example an optical telecommunication fiber and transmitted to Bob, who uses this information to modify the state of the entangled particle B'. One can show that the laws of quantum mechanics do not forbid that the final state of particle B' coincides exactly with the input state of particle A.

The details of the quantum teleportation operation are not given here. One can find them in refs [2, 13]. it has been experimentally implemented in different places in the world, with the help of different quantum systems, essentially with light, for example using polarization states of photons or field quadratures.

Note that the generation of a perfect copy of  $|\psi_{Ain}\rangle$  does not contradict the no-cloning theorem recalled in appendix 2, because the read-out process on Alice's side destroys the quantum state of particle A.

## Chapitre 9

# Accuracy and uncertainty in Quantum measurements and estimations

### 9.1 Quantum metrology

The probabilistic character of Quantum Mechanics, and the existence of the Heisenberg inequality, often called "indeterminacy principle" seems to imply that the Quantum world is blurred, undetermined, "veiled", and that the physical parameters of microscopic objects cannot be precisely measurable. It is easy to find many counter-examples of this statement. The most spectacular one is the  $1S_{1/2} \rightarrow 2S_{1/2}$  transition frequency in hydrogen, undoubtedly a quantum object, which has been measured at the following value :

$$f_{1S \rightarrow 2S} = 2\ 466\ 061\ 413\ 187\ 035 \pm 10 Hz \quad (9.1)$$

With a relative uncertainty of  $4 \cdot 10^{-15}$  it is the best experimentally known physical quantity in the whole physics !

Other physical parameters of quantum particles have been measured with a great accuracy, for example the magnetic moment of  $g\mu_B$  of the electron, where  $\mu_B$  is the Bohr magneton. The dimensionless  $g$  factor has been measured at the value :

$$g = 2.0023193043617(15), \quad (9.2)$$

The relative measurement uncertainty is here  $7 \cdot 10^{-13}$ .

So it is better to state that, even though some couples of parameters of microscopic objects cannot be measured with perfect accuracy, there are physical parameters which can be estimated with an outstanding precision. "Quantum metrology " is the domain of

quantum physics which studies these problems theoretically and experimentally, and looks for the possible quantum limits in measurements and the ways to reduce them. We will give here a brief introduction to the domain.

## 9.2 The Heisenberg inequality

### 9.2.1 Reminder

This inequality was actually rigorously proved by Kennard[27] and Robertson[28]. It states that

$$\forall |\psi\rangle, \forall \hat{A}, \hat{B} \text{ Hermitian} \quad \Delta A \Delta B \geq \frac{1}{2} |\langle \psi | [\hat{A}, \hat{B}] | \psi \rangle| \quad (9.3)$$

It has the following meaning : given N identically prepared copies of the state  $|\psi\rangle$ , one measures the physical quantity  $A$  on  $N/2$  copies, and the physical quantity  $B$  on  $N/2$  copies. One gets fluctuating results from which one calculates the standard deviations  $\Delta A$  and  $\Delta B$  about the means. These quantities fulfill inequality (9.3). This implies that, when  $[\hat{A}, \hat{B}] \neq 0$ , both  $\Delta A$  and  $\Delta B$  are non-zero : repeated measurements of these quantities on any quantum state are affected by unavoidable fluctuations around the mean, often called "quantum noise", and the smaller the quantum noise affecting one of these quantities, the bigger the other one.

Note that the inequality (9.3) has nothing to do with the perturbation of a quantum state by a measurement. We will study this latter problem in detail in section 8.4.

### 9.2.2 Generalized Heisenberg inequality

We give here the demonstration in the pure state case, which can be straightforwardly extended to mixed states. Let us consider a quantum state  $|\psi\rangle$  and two observables  $\hat{A}$  and  $\hat{B}$ . One introduces the "fluctuation operators" of zero mean :

$$\delta \hat{A} = \hat{A} - \langle \hat{A} \rangle \quad ; \quad \delta \hat{B} = \hat{B} - \langle \hat{B} \rangle \quad (9.4)$$

and write the Cauchy-Schwartz inequality  $\langle \alpha | \alpha \rangle \langle \beta | \beta \rangle \geq |\langle \alpha | \beta \rangle|^2$  when  $|\alpha\rangle = \delta \hat{A} |\psi\rangle$  and  $|\beta\rangle = \delta \hat{B} |\psi\rangle$ . One gets :

$$\Delta^2 A \Delta^2 B \geq |\langle \delta \hat{A} \delta \hat{B} \rangle|^2 \quad (9.5)$$

Let us write :

$$\delta \hat{A} \delta \hat{B} = \frac{1}{2} ([\delta \hat{A} \delta \hat{B}] + \{\delta \hat{A} \delta \hat{B}\}) \quad (9.6)$$

where  $\{\delta\hat{A}\delta\hat{B}\}$  is the anticommutator  $\delta\hat{A}\delta\hat{B} + \delta\hat{B}\delta\hat{A}$ . It is easy to show that the commutator is anti-hermitian and that the anti-commutator is hermitian, implying that the mean of the former is imaginary, and of the latter real. Finally, one gets an inequality which is "tighter" than the usual one :

$$\Delta^2 A \Delta^2 B \geq \frac{1}{4} |\langle \psi | [\hat{A}, \hat{B}] | \psi \rangle|^2 + \mathbf{C}_{A,B}^2 \quad (9.7)$$

where  $\mathbf{C}_{A,B}$  is the correlation function introduced in chapter 4.

### 9.2.3 Consequences

- One has  $\mathbf{C}_{A,B}^2 \leq \Delta^2 A \Delta^2 B$ , implying that the symmetrized version of the normalized correlation function  $c(A, B)$  defined in 7.29 is comprised between -1 and 1, like in the classical case.
- One has  $\Delta^2 A \Delta^2 B \geq \frac{1}{4} |\langle \psi | [\hat{A}, \hat{B}] | \psi \rangle|^2$ , the usual Heisenberg inequality. The saturation of the usual Heisenberg inequality implies that  $\mathbf{C}_{A,B} = 0$ , i.e. that there are no correlations between A and B in a minimum state.
- One has  $c(A, B)^2 \leq 1 - |\langle \psi | [\hat{A}, \hat{B}] | \psi \rangle|^2 / (4 \Delta^2 A \Delta^2 B)$  for the normalized correlation function : two quantities which do not commute cannot be perfectly correlated ;
- One has  $\Delta^2 A \Delta^2 B (1 - c(A, B)^2) \geq |\langle \psi | [\hat{A}, \hat{B}] | \psi \rangle|^2 / 4$ . One recognizes in the first term the conditional variance, at least when the statistics is Gaussian, so that one can write in this particular case an interesting Heisenberg inequality in the case of conditional measurements :

$$\Delta^2 A \Delta^2 (B|A) \geq \frac{1}{4} |\langle \psi | [\hat{A}, \hat{B}] | \psi \rangle|^2 \quad (9.8)$$

or equivalently :

$$\Delta^2 B \Delta^2 (A|B) \geq \frac{1}{4} |\langle \psi | [\hat{A}, \hat{B}] | \psi \rangle|^2 \quad (9.9)$$

## 9.3 Accuracy in the estimation of a physical parameter

### 9.3.1 Position of problem

One of the objectives of physics is to determine as precisely as possible the value of parameters of physical interest, like the fine structure constant, the mass or the magnetic moment of the electron, the energy levels of quantum objects, the position of propagating or

trapped particles. We will call  $p$  such a generic parameter. Let us give here three examples of parameter estimation that we will use in our discussion, chosen in the domain of optics, a widely used technique to perform very accurate measurements :

— *Measurement of concentration by intensity monitoring*

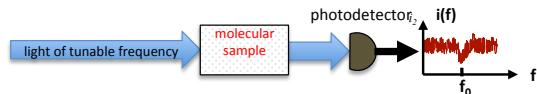


FIGURE 9.1 – sketch of an absorption measurement

One wants to determine the concentration  $n$  of molecules (for example a lethal gas) contained in a given sample of length  $L$  (figure(9.7)). For this purpose, a beam of light of tunable frequency  $f$  is sent through the sample, and the transmitted intensity  $I$  is measured by a photodetector. Let  $I_0$  be the intensity of the impinging beam in terms of photons per second. When the frequency of the light is resonant with an absorption frequency  $f_0$  of the atom, the transmitted intensity drops to the value  $I = I_0(1 - n\sigma L)$ , where  $\sigma$  is the absorption cross section of the atom. One then gets  $n$  from the formula  $n = (I_0 - I)/I_0\sigma L$ .

— *Measurement of length difference by phase monitoring*

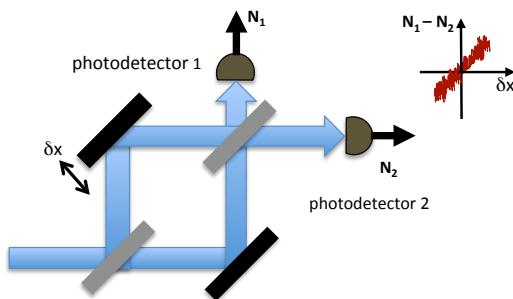


FIGURE 9.2 – sketch of an interferometric phase shift measurement

One wants to determine the length difference  $\delta x$  between two paths, for example to monitor the passage of a gravitational

wave. For this purpose, one uses an interferometric set-up, like a Mach-Zehnder interferometer (figure(9.2)), with a light of wavelength  $\lambda$  and intensity  $I_0$  (in photons per second), and records the variation with  $\delta x$  of the difference between the two output intensities  $I_1 - I_2 = I_0(\sin^2\pi\delta x/\lambda - \cos^2\pi\delta x/\lambda) = -I_0\cos2\pi\delta x/\lambda$ . Hence a small variation  $\delta x$  around a mean path difference of  $\lambda/4$  is related to the measured signal by  $\delta x = (\lambda/2\pi)(I_1 - I_2)/I_0$

— *Measurement of position by image monitoring*

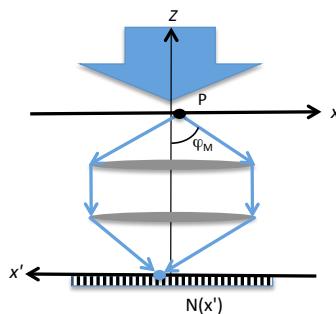


FIGURE 9.3 – measurement of the position of a particle P by imaging on a CCD detector the light scattered by the particle

One wants to determine the position  $x_0$  along  $Ox$  of a point-like object P (for example of biological interest). For this purpose, one uses a microscope along the  $z$  direction and records with the help of a CCD camera the intensity distribution  $I(x')$  in the image plane of the light scattered by the particle. Data processing, using the a priori knowledge of the theoretical shape  $PSF(x')$  of the diffraction-limited image through the microscope of a point-like object (the so-called "Point Spread Function") allows us to determine by deconvolution an estimation of the position  $x'_0 = Mx_0$  of the object P (figure(9.3)), where  $M$  is the magnification factor of the microscope.

In the three cases, from the measurement data and the knowledge of the physical properties of the measurement device, one gets a mathematical formula (or a data processing protocol) giving the parameter  $p$  of interest as a function of the measured quantities, that is called an *estimator*  $E(p)$  of  $p$ . The unavoidable imperfections of the device are responsible for a *technical noise* that limits the accuracy of the determination of  $p$ . This technical noise can be reduced without any priori limits by improving the measurement device. But

we also know that light is affected by a *quantum noise* of fundamental nature that cannot be completely eliminated. It gives rise to a *quantum limit* in the estimation of the parameter, that we would like to characterize and reduce as much as possible by optimizing the measurement protocol.

Note that measurements are very often performed with the help of electrical signals instead of optical ones. The quantum limit are a bit more difficult to determine and optimize in such a case.

### 9.3.2 Sketch of a general measurement

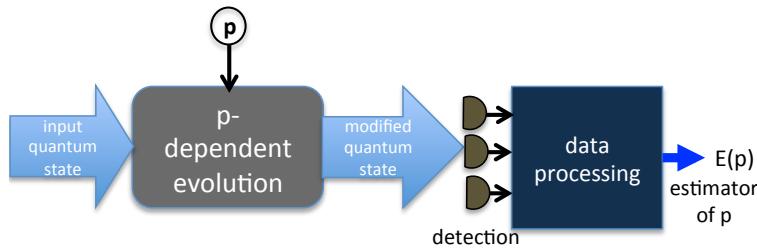


FIGURE 9.4 – Sketch of a generic measurement procedure of the parameter  $p$

From these examples, one can draw a general quantum scheme for the determination of a physical parameter  $p$  (figure(9.4) : a physical device (the measurement set-up) transforms the initial quantum state  $|\psi_0\rangle$  of the system into an output quantum state  $|\psi(p)\rangle$  that depends on the parameter of interest  $p$ . Let us define the unitary operator  $\hat{U}(p)$  and the generator  $\hat{G}(p)$  of translations in  $p$  by :

$$|\psi(p)\rangle = \hat{U}(p)|\psi_0\rangle \quad ; \quad \frac{d}{dp}|\psi(p)\rangle = i\hat{G}|\psi(p)\rangle \quad (9.10)$$

The state  $|\psi(p)\rangle$  is then submitted to a quantum measurement that records one or several quantities (for example a single light intensity or, for the CCD case, several intensities), and that can be repeated  $n$  times. The experimental data are then processed in order to generate the estimator  $E(p)$ , i.e. the experimental value of the parameter  $p$  is deduced. One assumes that all the technical

imperfections of the device have been eliminated, so that in particular, the state of the measurement device is a pure state. The only source of uncertainty is the quantum fluctuations due to the measurement process itself.  $E(p)$  is then a fluctuating quantity the mean value of which is assumed to be the searched value  $p$  (unbiased estimator :  $\text{Mean}(E(p)) = p$ ). Its variance  $\Delta E(p)$  gives a value characterizing the accuracy of the measurement. It gives in particular the minimum variation  $\delta p_{\min}$  of the parameter that gives rise to a measurable change in the measured data.

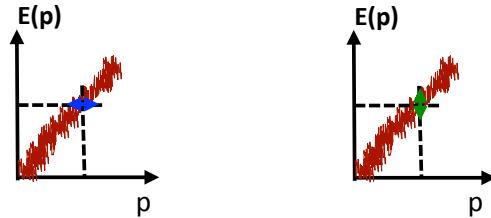


FIGURE 9.5 – Accuracy in the estimation of a parameter  $p$  : the blue arrow gives the uncertainty on the parameter  $p$  given a certain experimental result  $E(p)$  has been obtained ; the green arrow gives the dispersion of the measured values  $E(p)$  for a known value  $p$  of the parameter.

In this process, the experimentally determined data are not uncertain, because they are measured to a known value and recorded in the memory of the computer (fig(9.4)). What is not known is the value of the parameter  $p$  which leads to these recorded values of the data. One is back to the Bayesian approach of the probability : each new measurement adds information which changes the conditional probability. The measurement device is characterized by the conditional probabilities for the value of the parameter  $p$  knowing the result  $E(p)$  of the measurement :  $\text{proba}(p|E(p))$ . This probability is related through Bayes theorem to the reverse conditional probability  $\text{proba}(E(p)|p)$ , a quantity that one can determine provided one knows the details of the experimental set-up and the probability distribution of the quantum fluctuations of the measured quantities.

### 9.3.3 Minimization of the quantum fluctuations ; Quantum Cramer-Rao bound

There are different ways to improve the accuracy of a measurement :

- The simplest one is to repeat the measurement  $n$  times, so as to record more and more data about the system. This is not always possible, for example if one wants to measure a time-dependent parameter ;

- The second way is to improve the data processing technique that is used to deduce the estimator from the recorded measurement results. Information theory[29] tells us that there is a lower limit for  $\Delta p_{est}$  known as the *classical Cramer-Rao bound*, a quantity minimized over all possible data processing protocols, and equal to :

$$\Delta p_{est} \geq \Delta p_{classical\ Cramer-Rao} = \frac{1}{\sqrt{nF(p)}} \quad (9.11)$$

where  $F$  is the *Fisher information*, equal to :

$$F(p) = \int dp_{est} proba(p_{est}|p) \left| \frac{\partial \ln proba(p_{est}|p)}{\partial p} \right|^2 \quad (9.12)$$

This limit is very general and depends only on the conditional probability distribution  $proba(p_{est}|p)$ . It does not tell us the way to reach it. Note that the minimum measurable variation of  $p$  decreases with  $n$  as  $\frac{1}{\sqrt{n}}$ .

- The third way is to optimize the measurement procedure. One then reaches a new bound, optimized over all possible POVMs likely to be performed on the physical system described by the wavevector  $|\psi(p)\rangle$ , called the *Quantum Cramer-Rao bound*[30, 31, 32], equal for a pure state to :

$$\Delta p_{est} \geq \Delta p_{Quantum\ Cramer-Rao} = \frac{1}{\sqrt{nF_Q(p)}} \quad (9.13)$$

where  $F_Q$  is the *Quantum Fisher information*, equal to :

$$F_Q(p) = 4\Delta^2 \hat{G}(p) \quad (9.14)$$

$\Delta^2 \hat{G}$  being the variance in state  $|\psi_0\rangle$  of the operator  $\hat{G}(p)$  defined in (9.10).

For example if the parameter  $p$  that one wants to measure is the spatial displacement  $x$  of a quantum particle, then  $\hat{U}(x)$  is the spatial translation operator  $\hat{U}(x) = e^{i\hat{p}_x x/\hbar}$  and  $\hat{G}(x) = \hat{p}_x/\hbar$  is proportional to the conjugate momentum operator  $\hat{p}_x$ , so that the minimum measurable displacement  $\delta x_{min}$  is :

$$\delta x_{min} = \frac{\hbar}{2\sqrt{n}\Delta\hat{p}_x} \quad (9.15)$$

Actually the minimum measurable displacement  $\delta x_{min}$  can be much smaller than the dispersion  $\Delta \hat{x}$  of  $x$  values in the initial quantum state : suppose that the initial state fulfills  $\Delta \hat{p}_x \Delta \hat{x} = M\hbar/2$  where  $M$  is a large number. One has in this case :

$$\delta x_{min} = \frac{\Delta \hat{x}}{M \sqrt{n}} \quad (9.16)$$

A precise measurement is obtained by multiplying the measurements and by using a quantum state well above the Heisenberg inequality limit.

Note that, in the general case and for a single measurement ( $N = 1$ ), one can write (9.13) in the form of a Heisenberg-like inequality :

$$\delta p_{est} \hbar \Delta \hat{G} \geq \frac{\hbar}{2} \quad (9.17)$$

$\hbar \hat{G}(p)$  appears in this context as a kind of *complementary quantity* of the measured quantity  $p$ .

- The fourth and last way is to optimize the initial quantum state  $|\psi_0\rangle$  on which one imprints the information about  $p$ . So far, nobody has found a general bound optimized over all possible initial quantum states. So the determination of a state likely to increase the accuracy of a measurement is a question of intuition., and one is never sure that somebody else will come up with a better choice of quantum state.

Note also that if there is an observable  $\hat{P}$  such that  $\hat{P}|p\rangle = p|p\rangle$  with eigenvalues belonging to a continuous interval of values, then there are no quantum fluctuations when one measures repeatedly the parameter  $p$  on this state. Unfortunately these eigenstates do not always exist, like in the case of the phase of an electromagnetic field, or are not physical, like the eigenstates of position.

In short, the main interest of the Cramer-Rao bounds, classical or quantum, is that they gives an absolute limit to the accuracy that is independent of the precise device used in the measurement. But it does not give information about the ways allowing experimentalists to reach such a limit. In any given experimental situation, it is possible to determine the Signal to Noise ratio, and hence the minimum measurable parameter. This value must then be compared with the corresponding Quantum Cramer-Rao bound to determine whether there is or not room for improvement.

### 9.3.4 Examples

— *Measurement of concentration by intensity monitoring*

One can show[17] that laser light is described at the quantum level by a *coherent state*, that can be interpreted as describing a set of independently emitted photons, with Poissonian statistics  $\Delta N = \sqrt{< N >}$ , where  $< N > = IT$  is the mean number of recorded photons,  $I$  the intensity and  $T$  the measurement time. Therefore, if one uses a laser as the light source, one has :

$$n_{min} = \frac{1}{\sqrt{IT\sigma L}} \quad (9.18)$$

which can be reduced by increasing the laser intensity or the measurement time. This limit is often called the standard quantum noise limit, or the shot noise limit.

One can think that a perfect absorption measurement strategy would be to use a Fock state  $|N\rangle$  as the quantum state carrying the information, because it is an eigenstate of the photons number, and therefore has no intensity fluctuations. However, the spectrum of this operator is discrete and not continuous. Hence the minimum measurable intensity variation corresponds to the absorption of a single photon  $|N\rangle \rightarrow |N-1\rangle$ . The minimum detectable density of molecules is then given by :

$$n_{min} = \frac{1}{N\sigma L} = \frac{1}{IT\sigma L} \quad (9.19)$$

This is the so-called *Heisenberg limit*, which is inversely proportional to the number of photons in the Fock state. Up to now, Fock states have been generated experimentally only of  $n$  values up to tens, whereas coherent states may have mean photon numbers of the order of  $10^{15}$ , so, practically speaking in the present situation it is preferable to use lasers instead of Fock state generators.

As an absorption measurement implies losses and therefore is not a unitary, it is more difficult, and out of the scope of this textbook, to derive a quantum Cramer-Rao limit for this case.

#### — *Measurement of phase shift*

This example is important, as many ultra-sensitive measurements are performed by interferometric techniques. The phase-shift operator  $\hat{U}_\phi$  is in this case equal to  $\hat{U}_\phi = e^{-i\phi\hat{a}^\dagger\hat{a}}$ . It looks like the evolution operator  $\hat{U}(t) = e^{-i\omega t\hat{a}^\dagger\hat{a}}$  because phase shifts induced by spatial propagation or time evolution are closely related for monochromatic light. The  $\hat{G}$  operator introduced

in (9.14), equal to  $\hat{G} = \hat{a}^\dagger \hat{a}$ , is the photon number operator. One has therefore the quantum Cramer-Rao bound :

$$\delta\phi_{min} = \frac{1}{2\sqrt{n}\Delta N} \quad (9.20)$$

So the larger the photon number dispersion, the lower the quantum Cramer-Rao bound :

- If one uses lasers, i.e. coherent states in interferometric devices for which  $\Delta\hat{a}^\dagger \hat{a} = \langle N \rangle$ , the minimum detectable phase shift is :

$$\delta\phi_{min} = \frac{1}{2\sqrt{\langle N \rangle}} = \frac{1}{2\sqrt{IT}} \quad (9.21)$$

The  $1/\sqrt{N}$  dependence is the signature of a "standard quantum noise" or shot noise limit.

- Formula 9.20 shows that it is better to use a quantum state with a maximum spread of photon number, such as the state  $(|0\rangle + |N\rangle)/\sqrt{2}$ . The minimum detectable phase shift is then :

$$\delta\phi_{min} = \frac{2}{N} = \frac{2}{IT} \quad (9.22)$$

It corresponds to the Heisenberg limit for such a measurement. The operator  $\hat{U}_\phi$  given above corresponds to the overall transformation of a single mode input state. But one notices that in the Mach-Zehnder configuration of figure (9.2) the light takes two paths labeled 1 and 2, only the first one of which being submitted to the phase shift. For this configuration, a two-mode entangled quantum state has been proposed, the "NOON state"  $|\Psi_0\rangle = (|N, 0\rangle + |0, N\rangle)/\sqrt{2}$ . When it propagates through the interferometer, it evolves into

$$|\Psi_\phi\rangle = \frac{1}{\sqrt{2}} e^{-i\phi\hat{a}_1^\dagger \hat{a}_1} (|N, 0\rangle + |0, N\rangle) = \frac{1}{\sqrt{2}} (e^{-iN\phi} |N, 0\rangle + |0, N\rangle) \quad (9.23)$$

Note that the sensitivity of this scheme is increased because the phase coefficient  $e^{-iN\phi}$  evolves  $N$  times faster than in a single photon state. In addition  $\Delta N = \frac{N}{\sqrt{2}}$  is also on the order of  $N$ . The NOON state also allows us to reach the Heisenberg limit.

- *Measurement of a transverse displacement by optical means*  
In order to measure the displacement  $\delta x_P$  of particle P situated at initial position  $x_P$  (see figure(9.3)), one makes its optical

image through an optical microscope and monitors its intensity distribution on a CCD camera. Let us call  $E(x')$  the normalized amplitude distribution of the image ( $\int dx' |E(x')|^2 = 1$ ). One has  $I(x') = N|E(x')|^2$  where  $N$  is the total number of photons. It is out of the scope of this lecture notes to derive the expression of the quantum Fisher information in such a case. Details can be found in [33]. We just give here the result for a laser (coherent state) illumination. One finds

$$\delta x_{min} = \frac{x_0}{2M\sqrt{nN}} \quad (9.24)$$

where  $M$  is the magnification factor of the microscope, and

$$x_0 = \left( \int dx' \left( \frac{\partial E}{\partial x'} \right)^2 \right)^{-1/2} \quad (9.25)$$

is a scaling factor. For a  $TEM_{00}$  image of Gaussian shape of the particle of waist  $w_0$ ,  $x_0$  is equal to  $2^{1.5}\pi^{0.5}w_0$ . Expression 9.24 shows that, because of the  $\sqrt{N}$  factor, the minimum measurable displacement of  $P$  can be orders of magnitude smaller than the size  $w_0$  of the light spot on the CCD camera, provided one uses intense enough light and/or long exposure times. The minimum value of  $w_0$  for an optical microscope is of the order of the wavelength i.e.  $\simeq 1\mu m$ , and the minimum measurable displacement of such a spot can be as small as 10 femtometers ( $10^{-14}m$  in realistic situations : a focussed beam of total power 1mW and an exposure time of 1s. The use of non-classical states of light, such as squeezed light of optimized transverse shape, can reduce even further this limit.

## 9.4 Measurement-induced perturbation of a quantum state

### 9.4.1 The Heisenberg microscope

Any measurement performed on a quantum state either destroys it or perturbs it. When the state is not destroyed, the question is to minimize the perturbation while keeping a good accuracy in the measurement. This kind of issue was considered qualitatively by Heisenberg in his famous example of the "Heisenberg microscope" that we recall here in a modernized context. The experimental set-up, given in fig (9.6), is the same as sketched on figure (9.3).

In order to measure precisely the position of the particle along the  $x$  axis, supposed initially at rest, one uses light of wavelength  $\lambda$

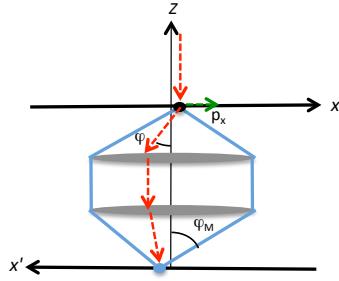


FIGURE 9.6 – The "Heisenberg microscope" : the red dotted line is the trajectory of a photon scattered by the particle; the green arrow shows the recoil effect induced by this scattering on the particle

propagating along  $Oz$ . The light that is scattered by the particle is collected by a lens that images the particle in the image plane. As we have seen in the previous section, the accuracy of the positioning is given by the size of the image of the particle divided by the square root of the photon number. The image size is more precisely limited by diffraction to the value  $\Delta x_{meas} \simeq \lambda/NA$ ,  $NA = \sin(\phi_M)$  being the numerical aperture of the lens, where  $\phi_M$  is the half-angle of the light cone focussed by the lens. It can be increased by using short wavelengths (UV, X-rays) or large numerical apertures like in microscopes.

This light obviously perturbs the particle, because there is a momentum transfer with light which pushes the particle. This perturbation can be reduced by lowering the light intensity, and the minimum perturbation will be obtained when a single photon is used. The momentum gained by the particle along  $x$  and due to the scattering of the photon in direction  $\phi$  with respect to  $Oz$  is  $p_x = \hbar k \sin \phi = h \sin \phi / \lambda$ . The photon is scattered by the particle in a random direction inside the cone of collected light, so that the uncertainty on the acquired momentum is  $\Delta p_{x,after} \simeq h \sin \phi_M / \lambda$ . One has finally :

$$\Delta x_{meas} \Delta p_{x,after} \simeq h \quad (9.26)$$

This relation looks like the Heisenberg inequality (9.3) and it is tempting to confuse the two relations. The Heisenberg microscope is often presented in textbooks as an illustration of the Heisenberg inequality. But actually it is very different from it as (9.26) deals with the effect of a measurement on the quantum state of the system after it has been submitted to a measurement process, whereas the

Heisenberg inequality is a general property of quantum states and does not depend on the measurement postulate M3 of Quantum Mechanics giving the quantum state after a measurement.

### 9.4.2 Noise-disturbance inequality

The rigorous noise-disturbance relation has been rigorously derived by Ozawa [34]. To demonstrate it, we must use the Heisenberg representation, in which the observables evolve in time instead of the quantum states.

We consider an observable  $\hat{A}$  and a measurement device of the quantity  $A$  which perturbs the input quantum state and that we treat here as a quantum device, like in chapter 6. We will also restrict ourselves to the case of an ideal Von Neumann measurement. We call  $\hat{M}$  the meter quantum observable, which has the pointer states as eigenstates, as described in section 5.2. In order to estimate the perturbing effect of the measurement of  $A$ , we consider its effect on another observable  $\hat{B}$ .

Let us note  $\hat{A}_{in}$ ,  $\hat{B}_{in}$ ,  $\hat{M}_{in}$ , and  $\hat{A}_{out}$ ,  $\hat{B}_{out}$ ,  $\hat{M}_{out}$  the values of the observables before and after the interaction with the measurement device. Ideally, the meter should give the exact information about the measured observable  $\hat{A}$ , and any other observable like  $\hat{B}$  should be left unperturbed by the interaction with the measurement set-up, so that we should have :

$$\hat{M}_{out} = \hat{A}_{in} \quad ; \quad \hat{B}_{out} = \hat{B}_{in} \quad (9.27)$$

This is not the case in Quantum Mechanics, as we will see. We must introduce additional "noise terms" in the previous relations :

$$\hat{M}_{out} = \hat{A}_{in} + \hat{N}_A \quad ; \quad \hat{B}_{out} = \hat{B}_{in} + \hat{D}_B \quad (9.28)$$

$\hat{N}_A$  represents the added quantum noise that prevents a perfect knowledge of the measured quantity  $A$ .  $\hat{D}_B$  represents the *disturbance* in the non measured quantity  $B$  arising from the interaction of the physical system with the measurement device. Let us first use three times the Robertson-Heisenberg inequality derived in section (8.2.1) :

$$\begin{aligned} \Delta N_A \Delta D_B + \Delta A_{in} \Delta D_B + \Delta N_A \Delta B_{in} &\geq \frac{1}{2} \left( |\langle [\hat{N}_A, \hat{D}_B] \rangle| + |\langle [\hat{A}_{in}, \hat{D}_B] \rangle| + |\langle [\hat{N}_A, \hat{B}_{in}] \rangle| \right) \\ &\geq \frac{1}{2} \left| \langle [\hat{N}_A, \hat{D}_B] \rangle + \langle [\hat{A}_{in}, \hat{D}_B] \rangle + \langle [\hat{N}_A, \hat{B}_{in}] \rangle \right| \end{aligned} \quad (9.29)$$

On the other hand,  $\hat{M}_{out}$  and  $\hat{B}_{out}$  are operators acting on different Hilbert spaces, so that  $[\hat{M}_{out}, \hat{B}_{out}] = 0$ , implying that :

$$[\hat{N}_A, \hat{D}_B] + [\hat{A}_{in}, \hat{D}_B] + [\hat{N}_A, \hat{B}_{in}] = -[\hat{A}_{in}, \hat{B}_{in}] \quad (9.30)$$

One has finally :

$$\Delta N_A \Delta D_B + \Delta A_{in} \Delta D_B + \Delta N_A \Delta B_{in} \geq \frac{1}{2} |\langle [\hat{A}_{in}, \hat{B}_{in}] \rangle| \quad (9.31)$$

This inequality, named the Heisenberg-Ozawa inequality, puts strong constraints on the different variances when the two input observables  $\hat{A}_{in}$  and  $\hat{B}_{in}$  do not commute.

To be more specific, we will focus on the long debated case of position and momentum measurements :

$$\Delta N_x \Delta D_p + \Delta x \Delta D_p + \Delta N_x \Delta p \geq \frac{\hbar}{2} \quad (9.32)$$

This inequality implies that :

- $\Delta N_x = \Delta D_p = 0$  is forbidden : there is always some added noise and/or some disturbance in a measurement process. The disturbance of the momentum  $p$  due to a measurement of  $x$  is called the "back-action" of the position measurement device.
- $\Delta N_x = 0$  is possible even for a finite momentum disturbance  $\Delta D_p$ , but then  $\Delta x \Delta D_p \geq \frac{\hbar}{2}$  : a perfect measurement of the position is possible, the associated back action on momentum being limited by the inverse of the dispersion of  $x$  values in the initial state.
- A non momentum-disturbing, called "back-action evading" or "Quantum Non Demolition" (QND) position measurement ( $\Delta D_p = 0$ ) is possible, but this measurement is not perfect : one has the constraint :  $\Delta N_x \Delta p \geq \frac{\hbar}{2}$  : the noise added in the measurement is limited by the inverse of the dispersion of  $p$  values in the initial state.
- The Heisenberg-Ozawa inequality does not forbid quantum states and measurement devices such that  $\Delta N_x \Delta D_p < \frac{\hbar}{2}$  : there exist physical situations where a (non-existing but sometimes invoked) Noise-Disturbance Heisenberg inequality is violated. This is possible when the initial dispersions  $\Delta x$  and  $\Delta p$  are large enough. Recent experiments have indeed put in evidence this apparent "violation" [35].

### 9.4.3 Example of a Quantum Non Demolition (QND) measurement

The intensity of a light beam, or equivalently the number of photons is usually measured by a photodetector which absorbs the beam : it is a totally destructive measurement. But one can envision devices that are able to measure  $\hat{n} = \hat{a}^\dagger \hat{a}$  without absorbing the light. Figure(9.7) gives one possible implementation, taken from [1] : it consists of an optical cavity containing the  $n$  photons to measure, one mirror of which, of mass  $M$ , is free to move from its initial position  $x_0$  from time  $t = 0$ . A weak auxiliary external light beam of frequency  $\omega$  is reflected by the moving mirror. One measures its frequency after reflection, which is shifted by Doppler effect, and therefore one knows the mirror velocity  $v$  and momentum  $p = Mv$  at a given time  $T$ .

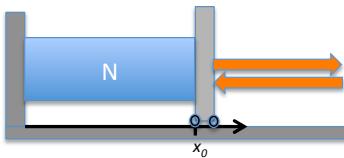


FIGURE 9.7 – Quantum Non Demolition measurement of the number of photons inside a resonant optical cavity

The system (i.e. the intracavity field) and the meter (the moving mirror) are coupled by the momentum exchange between the bouncing photons and the moving mirror, i.e. by the radiation pressure force  $F = n\hbar\omega/x_0$ . Classically the mirror momentum at time  $T$  is  $p(T) = n\hbar\omega T/x_0$ , and therefore the photon number is derived from  $p$  by  $n = px_0/(\hbar\omega T)$ . The quantity  $n_{est} = p(T)x_0/(\hbar\omega T)$  is therefore an *estimator* of the photon number.

From a quantum point of view, the set (system + meter) is described by the Hamiltonian

$$\hat{H} = \hbar\omega\hat{n} + \hat{p}^2/2M - \hbar\omega\hat{n}\hat{p}/x_0 \quad (9.33)$$

which commutes with  $\hat{n}$  : the photon number is thus a constant of motion even in presence of the meter. It is not modified by the measurement :

$$\hat{n}_{out} = \hat{n}_{in} \quad (9.34)$$

so that the device that we consider here allows for a QND measurement of the photon number. We will see now whether it is an accurate measurement of this quantity.

Working in the Heisenberg picture, one has at time  $T$  :

$$\hat{p}(T) = \hat{p}(0) + \frac{\hbar\omega T}{x_0} \hat{n} \quad (9.35)$$

and therefore :

$$\hat{n}_{est} = \hat{n} - \frac{x_0}{\hbar\omega T} \hat{p}(0) \quad (9.36)$$

where  $\hat{n}_{est} = \hat{p}(T)x_0/(\hbar\omega T)$  is the quantum estimator of the photon number. The second term in equation (9.36) is responsible for introducing unwanted noise in the photon number estimation. Its variance is given by

$$\Delta^2 n_{est} = \Delta^2 n + \frac{x_0^2}{\hbar^2 \omega^2 T^2} \Delta^2 p(0) \quad (9.37)$$

The excess noise, represented by the second term, that we call  $\Delta^2 N$ , is related to the initial unavoidable uncertainty of the mirror momentum. It decreases for long times.

Where is the "back-action" in such a device ? It turns out that it affects the phase  $\Phi$  of the intracavity field, which plays here the role of the conjugate quantity to the photon number. One has

$$\hat{\Phi}_{out} = \hat{\Phi}_{in} + \omega T \quad (9.38)$$

Actually  $\omega$  varies with time, because it undergoes a Doppler shift at each bounce of photons on the moving mirror. A simple calculation shows that variation of cavity length and variation of frequency are linked by  $\delta\omega/\omega = -\delta x_0/x_0$ . Let us note that in this device the intracavity field energy  $E = n\hbar\omega$  is not conserved (the kinetic energy of the mirror increases), but the "adiabatic invariant"  $E/\omega$  is indeed conserved. In addition, the frequency shift turns out to be just the one needed to keep the field resonant with the varying cavity.

Relation (9.38) implies for the noise acquired by the phase :

$$\Delta^2 \Phi_{out} = \Delta^2 \Phi_{in} + \Delta^2 \omega T^2 = \Delta^2 \Phi_{in} + (\omega^2 T^2 / x_0^2) \Delta^2 x_0 \quad (9.39)$$

The second term is precisely the variance of the measurement induced disturbance  $\Delta^2 D$ . The noise/disturbance relation writes in this specific case :

$$\Delta^2 N \Delta^2 D = \frac{1}{\hbar^2} \Delta^2 x_0 \Delta^2 p(0) \geq \frac{1}{4} \quad (9.40)$$

## Chapitre 10

# Reminder 1 : Basic Quantum Mechanics

Excerpt from Max Born's first paper about the interpretation of the wave function (in :*On the quantum theory of collisions*, Zeits. für Phys. **37**, 863 (1926)) :

...the wavefunction  $\Phi_{nm}(\alpha, \beta, \gamma)$  gives the probability(\*) for the electron to be thrown out in the direction designated by angles  $\alpha, \beta, \gamma$ ...  
(\*) note added in proof : More careful consideration shows that the probability is proportional to the square of the quantity  $\Phi_{nm}$ .

A Nobel prize-worth footnote !

## 1 State vectors

Let us begin by recalling the postulates concerning the system under study.

### 1.1 Postulates S1 and S2 about the system

Let us consider a physical system, atom, electron, spin, or light, that is isolated, or submitted to fixed forces, and prepared in a perfectly defined way.

(S1) Superposition principle postulate : a physical system, whatever its complexity, is described by a single mathematical object, called a *state vector*, or "ket",  $|\Psi\rangle$ , belonging to a well defined Hilbert space  $\mathcal{H}$ .

(S2) Normalization postulate : the state vector, or

"ket", is normalized to 1 :

$$|\langle \Psi | \Psi \rangle|^2 = 1 \quad (1)$$

## 1.2 The superposition principle

It is a consequence of the linear structure of the Hilbert space :

*If  $|\Psi_1\rangle$  and  $|\Psi_2\rangle$  describe two possible states of the physical system, then  $|\Psi_3\rangle = \lambda_1|\Psi_1\rangle + \lambda_2|\Psi_2\rangle$ , where  $\lambda_1$  and  $\lambda_2$  are any complex numbers (with  $|\lambda_1|^2 + |\lambda_2|^2 = 1$ ) is another possible state of the physical system.*

This property is not just a technical mathematical feature of the state vector. It is actually one of the deepest roots of the quantum world and comes from its wave-like character, since classical waves have the same property. This is why we call postulate S1 the superposition principle postulate. .

The superposition principle implies the possibility of *quantum interferences* : a classical example is the double slit experiment for photons or massive particles. If  $|\Psi_1\rangle$  describes an electron passing through slit 1, and  $|\Psi_2\rangle$  describes an electron passing through slit 2, then  $|\Psi_3\rangle = (|\Psi_1\rangle + |\Psi_2\rangle)/\sqrt{2}$  describes another possible state of the electron, namely the one in which the electron has the possibility of passing through both slits. The interference can be either constructive or destructive.

Another example is an optical system which has a state  $|\Psi_1\rangle$  which transmits light and a state  $|\Psi_2\rangle$  which reflects it. The superposition  $(|\Psi_1\rangle + |\Psi_2\rangle)/\sqrt{2}$  describes something which reflects and transmits light at the same time : in quantum optics and in contrast with literature, a door can be at the same time open **and** shut !

Let us call  $|a e^{i\phi}\rangle$  the quantum state describing a monochromatic light beam of amplitude  $a$  and phase  $\phi$ . Then  $|a e^{i(\phi+\pi)}\rangle$  is the quantum state describing a coherent light of amplitude  $a$  and phase  $\phi+\pi$ , i.e. having an electric field which is the opposite of the previous one at any time. Physicists have experimentally produced the states  $|a e^{i\phi}\rangle \pm |a e^{i(\phi+\pi)}\rangle$  (properly normalized to one), that describes a very strange object being the superposition of an oscillating electric field and its opposite !

An extreme case is provided by the famous "Schrödinger cat", in which the two states (cat alive, cat dead) are distinct states of a macroscopic, biological, object.

### 1.3 Examples of quantum systems

Let us briefly mention here different systems that will be used as examples in the notes :

#### Atoms

Though they have an infinite number of well defined energy eigenstates, in many instances, one can only consider states belonging to a sub-space of the total Hilbert space : the ground state  $|g\rangle$ , which is stationary, and one or two excited states  $|e\rangle$  and  $|e'\rangle$  which have finite lifetimes.

#### Qubits

They are two level quantum systems, and are the quantum analog of the classical bits which can be in two macroscopically different states. An important example is a spin 1/2 particle, which has two Zeeman substates that we will note  $|+\rangle$  and  $|-\rangle$ . It is called a "qubit", which can be in the most general state is  $|\Psi\rangle = \lambda_1|+\rangle + \lambda_2|-\rangle$ . It is the basic ingredient of quantum information, a part of quantum physics which is the object of numerous investigations over the world because of its fundamental implications, but also because of possible applications in quantum computing.

#### 1D particle

Let us consider a massive particle which can travel along the  $x$  axis only. The ket  $|x\rangle$  describes this particle when it is perfectly localized at point  $x$  (with  $\langle x|x'\rangle = \delta(x - x')$ ). The general quantum state of such a particle is therefore the ket  $|\Psi\rangle = \int dx \psi(x)|x\rangle$ , where the complex function  $\psi(x)$  is the wave function of the particle.

#### Light

Let us consider a light beam of given propagation direction, polarization, frequency and transverse shape that is enclosed in a cavity. One can show that it is described by a quantum state belonging to a Hilbert space that is formally equivalent to the one describing a harmonic oscillator. It has a basis of "number states"  $|n\rangle$  describing a state of light containing exactly  $n$  photons inside the cavity. In particular the number state  $|0\rangle$ , called "vacuum" describes the obscurity, i.e. the state of light when all light sources have been switched-off. The most general state of this light beam is  $|\Psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$ . One can for example consider the quantum state  $|\Psi_1\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$

, a superposition of a single photon state and of vacuum. The state  $|\Psi_1\rangle$  can be shown to have very different properties from the single photon state  $|1\rangle$ .

## 2 Observables

They are a basic tool of Quantum Mechanics, for which we recall some properties and give examples.

### 2.1 Definition

With any real physical quantity  $A$ , like the energy, the position, the electric field, the magnetic moment, ... is associated a hermitian operator  $\hat{A}$  operating in the Hilbert state of state vectors  $\mathcal{H}$ .

### 2.2 Properties

By definition  $\hat{A}^\dagger = \hat{A}$ , so that  $\hat{A}$  can be diagonalized :

$$\hat{A}|u_n\rangle = a_n|u_n\rangle \quad (2)$$

the eigenvalues  $a_n$  being real, and the eigenvectors  $|u_n\rangle$  forming an orthonormal basis of the Hilbert space  $\mathcal{H}$ . In some instances, in the so-called degenerate cases, several orthogonal eigenvectors  $|u_n^i\rangle$  are associated with the same eigenvalue  $a_n$ .

One defines *projectors* on eigenspaces  $\hat{P}_n$  by :

$$\hat{P}_n = \sum_i |u_n^i\rangle\langle u_n^i| \quad (3)$$

They are such that

$$\hat{P}_n \hat{P}_{n'} = \delta_{n,n'} \hat{P}_n \quad ; \quad \sum_n \hat{P}_n = \hat{1} \quad (4)$$

The observable  $\hat{A}$  can also be written as :

$$\hat{A} = \sum_n a_n \hat{P}_n \quad (5)$$

### 2.3 Examples

#### Atoms

With the atom energy  $E$  is associated the hermitian operator hamiltonian  $\hat{H}$  such that :

$$\hat{H}|\phi_n\rangle = E_n|\phi_n\rangle \quad (6)$$

## Spin 1/2, qubits

With the three components of the spin angular momentum  $\mathbf{S}$  of a spin 1/2 particle are associated three hermitian operators  $\hat{S}_i$  ( $i = x, y, z$ ) which are respectively equal to  $\sigma_i/2$  where  $\sigma_i$  ( $i = x, y, z$ ) are the Pauli matrices :

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (7)$$

the first and second columns corresponding respectively to qubit basis vectors  $|+\rangle$  and  $|-\rangle$ . These matrices have the following properties :

$$\sigma_i^2 = \hat{1} ; \quad \sigma_i \sigma_j = -\sigma_j \sigma_i = i \sigma_k \quad (8)$$

when  $(i, j, k)$  form a direct permutation of  $(x, y, z)$ .

The following non-hermitian operators, called spin flip operators, are also useful :

$$\sigma_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = |+\rangle\langle-| ; \quad \sigma_- = \sigma_+^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = |-\rangle\langle+| \quad (9)$$

This formalism is also useful for quantum states belonging to other kinds of two dimensional Hilbert spaces. It describes then "qubits", which are the quantum extension of the classical bits used in computers.

## 1D particle

With the position  $x$  of the particle is associated the hermitian operator  $\hat{X}$  such that :

$$\hat{X}|x\rangle = x|x\rangle \quad (10)$$

With the momentum  $p$  of the 1D particle is associated the hermitian operator  $\hat{P}$  such that :

$$\hat{P}|p\rangle = p|p\rangle \quad (11)$$

One can show that :

$$[\hat{X}, \hat{P}] = i\hbar\hat{1} ; \quad \langle x|p\rangle = \frac{1}{2\pi\hbar}e^{ipx/\hbar} \quad (12)$$

## Light

With the photon number  $N$  is associated the number operator  $\hat{N}$  such that :

$$\hat{N}|n\rangle = n|n\rangle \quad (13)$$

the electromagnetic energy being associated with the hamiltonian operator  $\hat{H} = (\hat{N} + 1/2)\hbar\omega$ .

In analogy with the harmonic oscillator one introduces the non hermitian photon annihilation and creation operators  $\hat{a}$  and  $\hat{a}^\dagger$  such that

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle; \quad \hat{a}|0\rangle = 0; \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle; \quad \hat{N} = \hat{a}^\dagger\hat{a} \quad (14)$$

With the electric field  $\mathbf{E}(\mathbf{r}, t)$  is associated the hermitian operator  $\hat{\mathbf{E}}(\mathbf{r}, t)$  which can be written as :

$$\hat{\mathbf{E}}(\mathbf{r}, t) = \hat{\mathbf{E}}^+(\mathbf{r}, t) + \hat{\mathbf{E}}^+(\mathbf{r}, t)^\dagger \quad (15)$$

with

$$\hat{\mathbf{E}}^+(\mathbf{r}, t) = \mathcal{E}\hat{a} \varepsilon e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \quad (16)$$

$\mathcal{E}$  being the electric field of a single photon and  $\varepsilon$  the polarization unit vector of the field.

In short, the wave properties of light (electric field) depend linearly on  $\hat{a}$  and  $\hat{a}^\dagger$ , whereas its particle properties (photon number) depend on  $\hat{a}^\dagger\hat{a}$ .

### 3 Ideal measurements

Let us now recall the usual postulates concerning the measurement. These postulates are valid only in the idealized case where the measurement process is as perfect as possible and does not destroy the physical object. We will call also "Von Neumann measurements" such ideal measurements.

#### 3.1 Postulates M1, M2 and M3 about the measurement

(M1) Quantization postulate : The measurement of the physical quantity  $A$  can only give as a result one of the eigenvalues  $a_n$  of the associated observable  $\hat{A}$ .

(M2) Born's rule postulate : In general the result of the measurement cannot be predicted with certainty. The probability  $Proba(a_n, |\psi\rangle)$  of obtaining the result  $a_n$  when the system is in the quantum state described by the state vector  $|\psi\rangle$  is given by :

$$Proba(a_n, |\psi\rangle) = \langle \psi | \hat{P}_n | \psi \rangle \quad (17)$$

where  $\hat{P}_n$  is the projector on the eigen subspace associated with eigenvalue  $a_n$ .

(M3) State collapse postulate : The conditional state of the system just after the measurement *and in the subset of cases where the measurement has given the result  $a_n$*  is :

$$|\psi^{after|a_n}\rangle = \frac{\hat{P}_n|\psi\rangle}{\sqrt{\langle\psi|\hat{P}_n|\psi\rangle}} \quad (18)$$

$|\psi^{after|a_n}\rangle$  is called a *conditional, or post-selected, state* because it is not the one which is always obtained after the measurement, but only in the special and uncontrolled opportunities when the measurement gives the precise result  $a_n$ .

Von Neumann measurements are also called "projective measurements", because they yield for the quantum state of the system after the measurement the normalized projection of the initial state onto the eigen-subspace corresponding to the measured value.

### 3.2 Consequences of M1 and M2

Let us first note that the number of possible measurement results  $a_n$  is necessarily limited to the dimension of the Hilbert space  $\mathcal{H}$ . In the case of degenerate eigenvalues, it can be even smaller.

Let us now assume that the preparation process is such that it delivers many identical copies of the system in state  $|\psi\rangle$ . One can then make many measurements of the same physical quantity  $A$ , and the averaged quantities can be predicted with almost certainty. In particular the mean value of  $A$ , that we will note  $\langle A \rangle$ , is :

$$\langle A \rangle = \sum_n a_n \text{Proba}(a_n||\psi\rangle) = \langle\psi|\hat{A}|\psi\rangle \quad (19)$$

as well as the mean of any function  $f(A)$  of  $A$  :

$$\langle f(A) \rangle = \langle\psi|f(\hat{A})|\psi\rangle \quad (20)$$

### 3.3 Consequences of M3

Let us now consider two successive measurements of quantity  $A$  on the same system (which must not be confused with two measurements of  $A$  on identically prepared quantum states). The probability

of getting successively results  $a_n$  and  $a_{n'}$  is :

$$Proba(a_n, a_{n'}, |\psi\rangle) = \frac{\langle\psi|\hat{P}_n\hat{P}_{n'}\hat{P}_n|\psi\rangle}{\langle\psi|\hat{P}_n|\psi\rangle} \quad (21)$$

Because  $\hat{P}_n\hat{P}_{n'} = \delta_{n,n'}\hat{P}_n$ , the probability of getting for the second measurement a result  $a_{n'}$  different from the first result  $a_n$  is zero, whereas the probability of getting for the second measurement the same result  $a_n$  as the first result  $a_n$  is one. Probabilities are in this special case certainties : after the first measurement we know with certainty the result of the measurement of  $A$  on the system. The first measurement acts as a *state preparation* : quantum physics has some intrinsic randomness, but is at least repeatable!

Let us also note that the state after the measurement, even ideal, is in general very different from the state before it. The measurement process results in a very strong perturbation of the system under study, during which most of the information about the initial state is lost. This is the reason why it is not possible to perfectly determine from successive measurements the quantum state of a system when its preparation is not known.

## 4 Temporal evolution

**(E1) Evolution postulate : The time evolution of the state vector  $|\psi(t)\rangle$  is ruled by the Schrödinger equation :**

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle \quad (22)$$

where  $\hat{H}(t)$  is the Hamiltonian of the system, which is the observable associated with its energy.

It is useful to introduce the unitary *evolution operator*, such that

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle \quad (23)$$

The evolution of  $\hat{U}(t)$  is ruled by the operatorial Schrödinger equation :

$$i\hbar \frac{d}{dt} \hat{U}(t) = \hat{H}(t) \hat{U}(t) \quad (24)$$

## Chapitre 11

# Reminder 2 : Description of composite systems

Very often, one is interested in the description not only of a single particle or physical object, but of several of them which are coupled by some interaction. Let us here briefly recall how such composite systems are described in Quantum Mechanics, with the help of tensor products of Hilbert spaces.

### 1 Example

Let us consider the system formed by two massive 1D particles traveling along the  $0x$  direction. It is characterized by a two-variable wave function  $\psi(x_1, x_2)$  such that  $|\psi(x_1, x_2)|^2$  gives the joint probability of detecting particle 1 at position  $x_1$  and particle 2 at position  $x_2$ . This way of characterizing the system is very different from the classical description of the system of two waves, for example two acoustic waves  $u_1(x) = u_0 e^{ikx}$  and  $u_2(x) = u_0 e^{-ikx}$  coming from in the opposite directions of the  $0x$  axis. The total system is then described by the standing wave formed by the *sum*  $u_{tot} = u_1(x) + u_2(x)$  of the two incoming waves. The total wave  $u_{tot}$  "lives" in the same space as the incoming waves, which is not the case for the total quantum wave function, which cannot be assimilated for this reason to a wave "living" in the geometrical usual space, but instead in a *configuration space*, which is larger than the geometrical space (much larger when one considers many particles). It yields therefore more information on the considered system, namely about the *quantum correlations* between position measurements performed on the two particles. These quantum correlations between two parts of a global system are one of the most striking, puzzling and even

"ridiculous" (according to Schrödinger's citation given in chapter 1) aspects of the quantum world.

let us stress that the situation considered in this section is very different from the one considered previously in section (1.2) which deals with the sum of two wavevectors and led to an interference phenomenon. The former describes a *single particle with two possible trajectories*, whereas the latter describes *two particles propagating along the same trajectory*.

## 2 Tensor product of Hilbert spaces

If a particle labeled 1 is described by a state vector  $|\psi_1\rangle$  belonging to a Hilbert space  $\mathcal{H}_1$  of dimension  $d_1$  spanned by a basis of vectors  $|u_i\rangle$  ( $i = 1, \dots, d_1$ ), and a particle 2 is described by a state vector  $|\psi_2\rangle$  belonging to a Hilbert space  $\mathcal{H}_2$  of dimension  $d_2$  spanned by a basis of vectors  $|v_j\rangle$  ( $j = 1, \dots, d_2$ ), then the system formed by the two particles is described by a state vector  $|\Psi\rangle$  belonging to the *tensor product*  $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$  of spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$ . This space has a dimension  $d_1 \times d_2$  (and not  $d_1 + d_2$ ), and is spanned by the basis of vectors  $|u_i\rangle \otimes |v_j\rangle$  ( $i = 1, \dots, d_1, j = 1, \dots, d_2$ ), that we will write  $|u_i, v_j\rangle$  for simplicity. It has a Hilbert inner product defined on its vector basis by :

$$\langle u_i, v_j | u_{i'}, v_{j'} \rangle = \delta_{i,i'} \delta_{j,j'} \quad (1)$$

A state belonging to  $\mathcal{H}$  has therefore the form

$$|\Psi\rangle = \sum_{i,j} \lambda_{i,j} |u_i, v_j\rangle \quad (2)$$

Note that it cannot be always written in a factorized way, as

$$|\Psi\rangle = \sum_{i,j} \lambda_i^1 \lambda_j^2 |u_i, v_j\rangle = \left( \sum_i \lambda_i^1 |u_i\rangle \right) \otimes \left( \sum_j \lambda_j^2 |v_j\rangle \right) \quad (3)$$

If such a factorization is not possible, the state is said *entangled*, or *intriqué* in french.

In the case of two 1D particles considered in the previous section, a factorized state has a wave function of the form  $\psi(x_1, x_2) = \psi_1(x_1)\psi_2(x_2)$ , whereas an entangled state is characterized by a wave function which cannot be written as a product.

### 3 Operators in tensor product of Hilbert spaces

Let us consider an operator  $\hat{A}_1$  operating on Hilbert space  $\mathcal{H}_1$ , and an operator  $\hat{B}_2$  operating on Hilbert space  $\mathcal{H}_2$ . The operator  $\hat{A}_1 \otimes \hat{B}_2$  operating on Hilbert space  $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$  is defined by :

$$\hat{A}_1 \otimes \hat{B}_2 |u_i, v_j\rangle = (\hat{A}_1 |u_i\rangle) \otimes (\hat{B}_2 |v_j\rangle) \quad (4)$$

In particular, one can extend an operator  $\hat{A}_1$  acting on space  $\mathcal{H}_1$  to an operator acting on tensor space  $\mathcal{H}$  by :

$$\hat{A}_1 |u_i, v_j\rangle = (\hat{A}_1 |u_i\rangle) \otimes |v_j\rangle \quad (5)$$

Note that one always has for the commutators between operators acting on different subspaces :

$$[\hat{A}_1, \hat{B}_2] = 0 \quad \forall \hat{A}_1 \forall \hat{B}_2 \quad (6)$$

For example in the case of two 1D particles  $[\hat{x}_1, \hat{p}_2] = 0$  whereas  $[\hat{x}_1, \hat{p}_1] = i\hbar$ .

### 4 Examples

Another example is provided by the system formed by two qubits, labeled 1 and 2, which is of dimension 4. A possible basis of the corresponding Hilbert space is the set  $\{|1+, 2+\rangle, |1-, 2+\rangle, |1-, 2+\rangle, |1-, 2-\rangle\}$ . Another useful basis is the set of entangled so-called "Bell states"  $\{|\Psi_{\pm}\rangle, |\Phi_{\pm}\rangle\}$  defined by

$$|\Psi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|1+, 2-\rangle \pm |1-, 2+\rangle) \quad (7)$$

$$|\Phi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|1+, 2+\rangle \pm |1-, 2-\rangle) \quad (8)$$

In many instances the two Hilbert spaces are of different natures. For example the system formed by a qubit traveling along the  $Ox$  axis is described by a state vector  $|\Psi\rangle$  belonging to the tensor product of a qubit and a 1D particle, spanned by the basis  $|\pm, x\rangle$  :

$$|\Psi\rangle = \sum_{\pm} \int dx \psi_{\pm}(x) |\pm, x\rangle \quad (9)$$

which requires the introduction of two wave functions  $\psi_+(x)$  and  $\psi_-(x)$ , forming what is called a "spinor", such that  $|\psi_+(x)|^2$  (resp.  $|\psi_-(x)|^2$ ) gives the probability of detecting the qubit in state  $|+\rangle$  (resp.  $|-\rangle$ ) at position  $x$ .

## 5 No cloning theorem

Is it possible to "clone" an unknown input quantum state  $|u_{in}\rangle$ , i.e. to make a perfect copy of it? W. Zurek[36] has shown that it is forbidden by the laws of quantum mechanics. The demonstration is actually very simple. A "cloning machine" would be a device in which two physical systems associated with Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$  of equal dimensions interact and then separate from each other. At the input, one sends the factorized state  $|\Psi_{in}\rangle = |u_{in}\rangle \otimes |v_0\rangle$ , where  $|v_0\rangle$  is a fixed quantum state. At the output of the device, a perfect cloning machine would generate the factorized state  $|\Psi_{out}\rangle = |u_{in}\rangle \otimes |u_{in}\rangle$  containing the desired identical copies.

Let us now consider two different unknown input states  $|u_{1,in}\rangle$  and  $|u_{2,in}\rangle$ . They lead to the final states  $|u_{1,in}\rangle \otimes |u_{1,in}\rangle$   $|u_{2,in}\rangle \otimes |u_{2,in}\rangle$ . What happens if we start now from the state  $(|u_{1,in}\rangle + |u_{2,in}\rangle)/\sqrt{2}$ ? The linearity of any quantum evolution in the cloning machine, linked to the superposition principle postulate, tells us that the output state must be  $(|u_{1,in}\rangle \otimes |u_{1,in}\rangle + |u_{2,in}\rangle \otimes |u_{2,in}\rangle)/\sqrt{2}$ . This state is different from the desired cloned state  $(|u_{1,in}\rangle + |u_{2,in}\rangle) \otimes (|u_{1,in}\rangle + |u_{2,in}\rangle)/2$ .

In addition, a cloning machine of particles would yield two copies on which one could perform simultaneously separate position and momentum measurements, which is also in strong contradiction with the very basis of quantum mechanics.

What is not forbidden is an imperfect cloning machine, leading to output states  $|u_{out}\rangle \otimes |u_{out}\rangle$  which are not perfectly identical to the input unknown state. It has been shown[37] that the overlap  $|\langle u_{out} | u_{in} \rangle|^2$ , also called "fidelity", cannot be larger than  $5/6$ .

# Bibliographie

- [1] S. Haroche, J.-M. Raimond *Exploring the quantum*, Oxford University Press (2006)
- [2] S. Barnett *Quantum Information*, Oxford University Press (2009)
- [3] M. Le Bellac, *Physique Quantique tomes 1 et 2*, EDP Sciences (2013)
- [4] J. Preskill, *Lecture Notes for Physics : Quantum Information and Computation* (1998)
- [5] M. Paris *The modern tools of quantum mechanics*, Eur. Phys. J Special Topics **203**, 61-86 (2012)
- [6] C. Cohen-Tannoudji, B. Diu, F. Laloe *Quantum Mechanics*
- [7] A. Gleason, M. Andrew *Measures on the closed sub-spaces of a Hilbert space* J. of Math. and Mech. **6**, 885 (1957)
- [8] S. Massar, *Uncertainty Relations for Positive Operator Valued Measures* Phys. Rev. A. **76**, 042114 (2007).
- [9] J.Dalibard, Y. Castin, K. Moelmer, *Wave function approach to dissipative processes in Quantum Mechanics*, Phys. Rev. Lett. **68**, 580 (1992).
- [10] C. Gardiner, P. Zoller, *Quantum Noise* Springer (2004)
- [11] G. Auletta, M. Fortunato, G. Parisi *Quantum Mechanics*, Cambridge University Press (2009).
- [12] M. D'Ariano, M. Paris, M. Sacchi *Quantum Tomography*, pp206-305 in "Advances in imaging and electron physics", vol 128, Elsevier (2003).
- [13] M. Nielsen, I. Chuang *Quantum Computation and Quantum Information*, Cambridge University Press (2010),
- [14] F. Laloë, *Do we really understand quantum mechanics ?* Cambridge University Press (2012)
- [15] C. Cohen-Tannoudji, Collège de France lectures 1989-1990
- [16] M. Scully, M. Zubairy, *Quantum Optics* Cambridge University Press (1997).
- [17] G. Grynberg, A. Aspect, C. Fabre *Introduction to Quantum Optics* Cambridge University Press (2010).
- [18] A. Peres, *Separability Criterion for Density Matrices* Phys. Rev. Lett. **77**, 1413 (1996).
- [19] A. Einstein, B. Podolsky, N. Rosen, *Can quantum-mechanical description of physical reality be considered complete ?* Phys. Rev A **47**, 777 (1935).
- [20] E. Schrödinger, *The present situation in quantum mechanics*, Naturwissenschaften **23**, pp.807-812 ; 823-828 ; 844-849 (1935).

- [21] J. S. Bell, *On the Einstein Podolsky Rosen Paradox*, Physics **1** 195 (1964).
- [22] J.F. Clauser, M.A. Horne, A. Shimony, R.A. Holt, *Proposed experiment to test local hidden-variable theories*, Phys. Rev. Lett. **23**, 15 (1969).
- [23] S.J. Freedman, J.F. Clauser, *Experimental test of local hidden-variable theories*, Phys. Rev. Lett. **28** 938 (1972).
- [24] A. Aspect, P. Grangier, G. Roger, *Experimental Tests of Realistic Local Theories via Bell's Theorem*, Phys. Rev. Lett. **47** 460 (1981).
- [25] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, A. Zeilinger, *Violation of Bell's inequality under strict Einstein locality conditions*, Phys. Rev. Lett. **81** 5039 (1998).
- [26] L. Grover, *Quantum Mechanics helps in searching for a needle in a haystack*, Phys. Rev Lett. **79**, 325 (1997).
- [27] E.H Kennard *Zur Quantenmechanik einfacher Bewegungstypen* Z. Physik **44**, 326 (1927).
- [28] H. P. Robertson *The uncertainty principle* Phys. Rev **34**, 163 (1929).
- [29] P. Réfrégier *Noise theory and applications to physics*, Springer (2003)
- [30] C.W. Helstrom *Quantum Detection and Estimation Theory* Academic Press (1973).
- [31] S. L. Braunstein and C. M. Caves, *Statistical distance and the geometry of quantum states*, Phys. Rev. Lett. **72**, 3439 (1994).
- [32] L. Davidovich *Intrication, Décohérence, et Métrologie Quantique* Cours (online) du collège de France 2011
- [33] O. Pinel, J. Fade, D. Braun, Pu Jian, N. Treps, C. Fabre, *Ultimate sensitivity of precision measurements with intense Gaussian quantum light : a multi-modal approach*, Phys. Rev A Rapid Com **85**, 010101 (2012)
- [34] M. Ozawa, *Universally valid reformulation of the Heisenberg uncertainty principle on noise and disturbance in measurement* Phys. Rev A **67**, 042105 (2003).
- [35] L. A. Rozema, A. Darabi, D. H. Mahler, A. Hayat, Y. Soudagar, and A. M. Steinberg, *Violation of Heisenberg's measurement-disturbance relationship by weak measurements* Phys. Rev. Lett. **109**, 100404 (2012).
- [36] W. Wootters, W. Zurek, *A Single Quantum Cannot be Cloned*. Nature **299** 802 (1982)
- [37] V. Buzek, M. Hillery, *Quantum Copying : Beyond the No-Cloning Theorem* Phys. Rev. A **54**, 1844 (1996)