

# CS 181 Spring 2017 Section 4

## Margin-Based Classification, SVMs

### 1 Motivation

The idea for Support Vector Machines is that, for all the linear hyperplanes that exist, we want one that will create the largest distance, or “margin”, with the training data. Larger margins tend to improve generalization error. To define the margin, we consider a hyperplane of the form

$$\mathbf{w}^\top \mathbf{x} + w_0 = 0$$

What is the perpendicular distance from an example  $\mathbf{x}_i$  to the decision boundary  $\mathbf{w}^\top \mathbf{x} + w_0 = 0$ ?

For two points  $\mathbf{x}_1$  and  $\mathbf{x}_2$  on the hyperplane, consider the projection with  $\mathbf{w}$ :

$$\mathbf{w}^\top (\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{w}^\top \mathbf{x}_1 - \mathbf{w}^\top \mathbf{x}_2 = -w_0 - (-w_0) = 0$$

Therefore,  $\mathbf{w}$  is orthogonal to the hyperplane. So to get the distance from a hyperplane and an arbitrary example  $\mathbf{x}$ , we just need the length in the direction of  $\mathbf{w}$  between the point and the hyperplane. We let  $r$  signify the distance between a point and the hyperplane. Then  $\mathbf{x}_\perp$  is the projection of the point onto the hyperplane, so that we can decompose a point  $\mathbf{x}$  as

$$\mathbf{x}_\perp + r \frac{\mathbf{w}}{\|\mathbf{w}\|} = \mathbf{x}$$

Left multiply by  $\mathbf{w}^\top$ :

$$\mathbf{w}^\top \mathbf{x}_\perp + r \frac{\mathbf{w}^\top \mathbf{w}}{\|\mathbf{w}\|} = \mathbf{w}^\top \mathbf{x} \Rightarrow r = \frac{\mathbf{w}^\top \mathbf{x} + w_0}{\|\mathbf{w}\|}$$

Scalar  $r$  then gives the signed, normalized distance between a point and the hyperplane.

For correctly classified data, we have  $y_i = +1$  when this distance is positive and  $y_i = -1$  when it is negative. Based on this, we can obtain a positive distance for both kinds of examples by multiplying by  $y_i$ .

We define the margin of the dataset as the minimum such distance over all examples:

$$\min_i \frac{y_i(\mathbf{w}^\top \mathbf{x}_i + w_0)}{\|\mathbf{w}\|}$$

What is the training problem for support vector machines?

We want the  $\mathbf{w}$  and  $w_0$  that maximize the margin:

$$\arg \max_{\mathbf{w}, w_0} \frac{1}{\|\mathbf{w}\|} \min_i y_i(\mathbf{w}^\top \mathbf{x}_i + w_0)$$

In the hard-margin training problem, we know that the data is linearly separable and therefore the optimal margin must be greater than 0:

$$\min_i \frac{y_i(\mathbf{w}^\top \mathbf{x}_i + w_0)}{\|\mathbf{w}\|} > 0$$

We can observe that  $\mathbf{w}$  and  $w_0$  are invariant to changes of scale. Because of this, it is without loss of generality to impose  $\min_i \frac{y_i(\mathbf{w}^\top \mathbf{x}_i + w_0)}{\|\mathbf{w}\|} > 1$ , and then because one or more of these constraints are binding, we can equivalently write the training problem as

$$\arg \max_{\mathbf{w}, w_0} \frac{1}{\|\mathbf{w}\|} \text{ s.t. } \forall i \ y_i(\mathbf{w}^\top \mathbf{x}_i + w_0) \geq 1$$

What is the corresponding minimization problem for hard-margin training for support vector machines?

Explain at a high level why the minimization problem for support vector machines is equivalent to the max-margin problem.

We can invert  $\mathbf{w}$  to change the max to a min:

$$\arg \min_{\mathbf{w}, w_0} \frac{1}{2} \|\mathbf{w}\|^2 \text{ s.t. } \forall i \ y_i(\mathbf{w}^\top \mathbf{x}_i + w_0) \geq 1$$

Informally, this is provide max-margin training because the constraint is binding for the examples closest to the decision boundary. For these examples we have  $y_i(\mathbf{w}^\top \mathbf{x}_i + w_0) = 1$ . The distance on these examples is  $1/\|\mathbf{w}\|$ , and is maximized by minimizing  $\|\mathbf{w}\|^2$ .

## 2 Soft Margin Formulation

For the hard margin formulation, we have been assuming that the data can be linearly separable. However, this is not always true, and even if the data can be linearly separable, it may not be best to find a separating hyperplane. In optimizing generalization error, there is a tradeoff between the size of the margin and the number of mistakes on the training data.

For the soft margin formulation, we introduce a slack variable  $\xi_i \geq 0$  for each  $i$  to relax the constraints on each example.

$$\xi_i \begin{cases} = 0 & \text{if correctly classified} \\ \in (0, 1] & \text{correctly classified but inside margin region} \\ > 1 & \text{if incorrectly classified} \end{cases}$$

We can now rewrite the training problem for a soft margin formulation to be

$$\begin{aligned} \arg \min_{\mathbf{w}, w_0} \quad & \frac{1}{2} \|\mathbf{w}\|^2 + C \sum_{i=1}^n \xi_i \\ \text{s.t. } \quad & \forall i \ y_i(\mathbf{w}^\top \mathbf{x}_i + w_0) \geq 1 - \xi_i \\ & \xi_i \geq 0 \end{aligned}$$

We add a regularization parameter  $C$ , that controls how much we penalize violating the hard margin constraints (and thus control the tradeoff.) A large  $C$  penalizes these violations and thus “respects” the data closely and has small regularization. A small  $C$  does not penalize the sum of slack variables as heavily, relaxing the constraint. This is increasing the regularization.

### 3 Dual Form of the Support Vector Machine Training Problem

Use Lagrange multipliers to convert the new minimization problem into dual form.

Our original hard-margin training problem, which looks for weights that maximize the margin on the training data, is

$$\mathbf{w}^*, w_0^* = \arg \min_{\mathbf{w}, w_0} \frac{1}{2} \|\mathbf{w}\|^2 \quad \text{s.t.} \quad y_i(\mathbf{w}^\top \mathbf{x}_i + w_0) \geq 1, \quad \forall i \in \{1, \dots, n\}. \quad (1)$$

We introduce Lagrange multipliers,  $\alpha_1, \dots, \alpha_n \geq 0$ , one for each inequality in Equation 1, i.e., one per example, to obtain the Lagrangian function (Bishop appendix E does a good job of explaining Lagrange multipliers):

$$L(\mathbf{w}, w_0, \boldsymbol{\alpha}) = \frac{1}{2} \|\mathbf{w}\|^2 - \sum_{i=1}^n \alpha_i (y_i(\mathbf{w}^\top \mathbf{x}_i + w_0) - 1) \quad (2)$$

Based on this, we now equivalently solve:

$$\mathbf{w}^*, w_0^* = \arg \min_{\mathbf{w}, w_0} \max_{\boldsymbol{\alpha} \geq 0} L(\mathbf{w}, w_0, \boldsymbol{\alpha})$$

By strong duality (out of scope!), we can equivalently write this as:

$$\max_{\boldsymbol{\alpha} \geq 0} \min_{\mathbf{w}, w_0} L(\mathbf{w}, w_0, \boldsymbol{\alpha})$$

This is useful because we can now solve analytically for the  $\min_{\mathbf{w}, w_0} [\cdot]$  part of this expression. Taking derivatives, we see

$$\frac{\partial L}{\partial w_j} = w_j - \sum_{i=1}^n \alpha_i y_i \mathbf{x}_i = 0 \Rightarrow \mathbf{w}^* = \sum_{i=1}^n \alpha_i y_i \mathbf{x}_i$$

$$\frac{\partial L}{\partial w_0} = - \sum_{i=1}^n \alpha_i y_i = 0 \Rightarrow \sum_{i=1}^n \alpha_i y_i = 0$$

Plugging these optimal values into the Lagrangian function, we get:

$$\begin{aligned} L(\mathbf{w}, w_0, \boldsymbol{\alpha}) &= \frac{1}{2} \left\| \sum_{i=1}^n \alpha_i y_i \mathbf{x}_i \right\|^2 - \sum_{i=1}^n \alpha_i y_i \left( \sum_{i'=1}^n \alpha_{i'} y_{i'} \mathbf{x}_{i'} \right)^\top \mathbf{x}_i - \sum_{i=1}^n \alpha_i y_i w_0 + \sum_{i=1}^n \alpha_i \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{i'=1}^n \alpha_i \alpha_{i'} y_i y_{i'} \mathbf{x}_i^\top \mathbf{x}_{i'} - \sum_{i=1}^n \sum_{i'=1}^n \alpha_i \alpha_{i'} y_i y_{i'} \mathbf{x}_i^\top \mathbf{x}_{i'} - w_0 \sum_{i=1}^n \alpha_i y_i + \sum_{i=1}^n \alpha_i \\ &= \sum_{i=1}^n \alpha_i - \frac{1}{2} \sum_{i=1}^n \sum_{i'=1}^n \alpha_i \alpha_{i'} y_i y_{i'} \mathbf{x}_i^\top \mathbf{x}_{i'} \end{aligned}$$

We then solve for  $\boldsymbol{\alpha}$ , maximizing  $L$ , subject to  $\boldsymbol{\alpha} \geq 0$ . The support vectors are defined to be  $Q = \{i : \alpha_i > 0\}$ . For the soft-margin variation, these include examples that lie in the margin region in addition to on the margin boundary.

## 4 Why bother with the dual form?

The dual form is useful because the number of variables to maximize is  $n$  (one for each training example), and does not depend on the dimension of the feature space. Moreover, there tends to be a small number of support vectors and thus the trained classifier can be interpretable. To classify a new example  $\mathbf{x}$ , we compute  $\sum_{i=1}^n \alpha_i^* y_i \mathbf{x}_i^\top \mathbf{x} + w_0^*$ , where  $\alpha^*$  and  $w_0^*$  solve the training problem. Based on this, we classify the example as  $+1$  if this discriminant value is  $> 0$ , and  $-1$  otherwise.

In addition, the dual has the very nice benefit that if we use a basis function to map  $\mathbf{x}$  to a higher dimensional space, this only comes in through the “kernel function.” The training problem is as explained earlier, except where  $\mathbf{x}_i^\top \mathbf{x}_{i'}$  appears, we use

$$K(\mathbf{x}_i, \mathbf{x}_{i'}) = \phi(\mathbf{x}_i)^\top \phi(\mathbf{x}_{i'}) \quad (3)$$

in its place.

Similarly, we classify a new example  $\mathbf{x}$  based on the value of discriminant  $\sum_{i=1}^n \alpha_i^* y_i K(\mathbf{x}_i, \mathbf{x}) + w_0^*$ .

The reason this is interesting is because we can directly compute the dot product  $\phi(\mathbf{x})^\top \phi(\mathbf{x}')$  without projecting to the higher-dimensional space. This is known as the “kernel trick.” As long as the kernel  $K$  corresponds to a valid inner product, the dual training problem can be solved without actually computing  $\phi$ .

## 5 Practice Problems

### 1. (Berkeley, Fall '11)

Suppose that we train two SVMs, the first containing all of the training data and the second trained on a data set constructed by removing some of the support vectors from the first training set. How does the size of the optimal margin change between the first and second training data?

### 2. Proof that margin is invariant to scalar multiplication

In reformulating our max margin

$$\frac{y_i(\mathbf{w}^\top \mathbf{x}_i + w_0)}{\|\mathbf{w}\|}$$

training problem, we use the fact that the margin is invariant to multiplying  $(\mathbf{w}, w_0)$  by any  $\beta > 0$ . Show that this property is true.

### 3. (Berkeley, Fall '11)

Consider  $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^2$ . Show that the polynomial kernel of degree 2,  $K(\mathbf{x}, \mathbf{x}') = (1 + \mathbf{x}^\top \mathbf{x}')^2$  is equivalent to a scalar product applied to basis function  $\phi(\mathbf{x}) = (x_1^2, x_2^2, x_1, x_2, x_1 x_2, 1)$ .

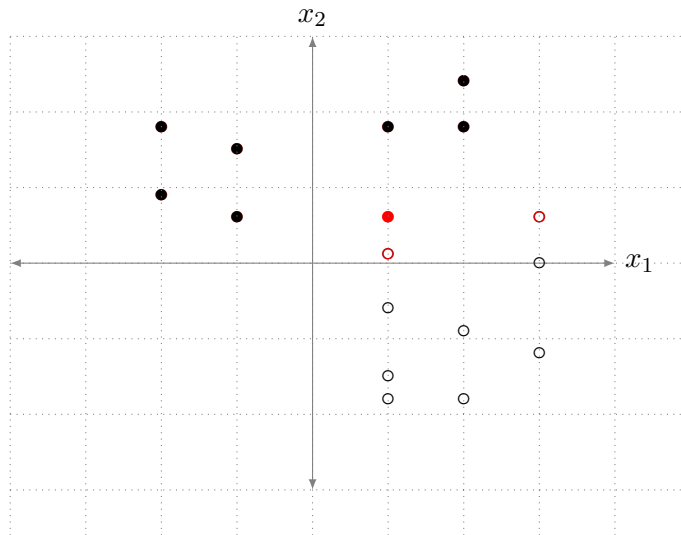
#### 4. Draw Margin Boundary

In the figures below, the red examples represent the support vectors. All other examples can be assumed to have  $\alpha_i = 0$ . Draw the margins for the boundary given this information.

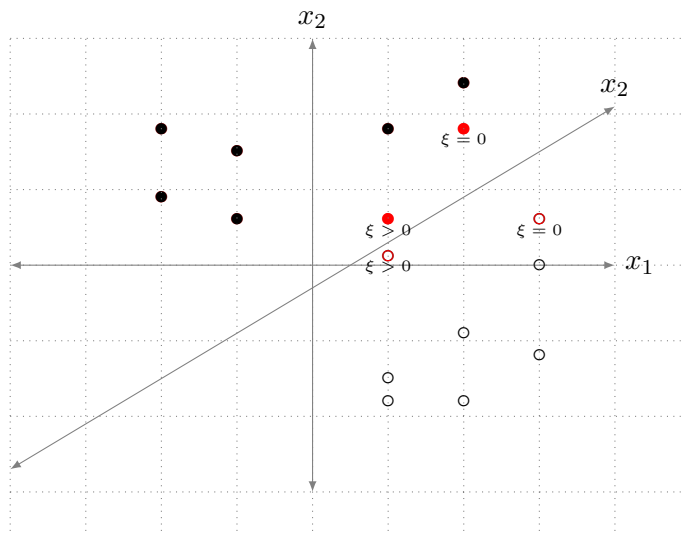
i) For the first example, you can assume the hard margin formulation. Draw the decision boundary as well as the two margin boundaries given the support vector.

ii) For the second example, you can assume the soft margin formulation and that all points are correctly classified with the optimal decision boundary. The decision boundary is already given. Draw the two margin boundaries given the support vector.

i)



ii)



## 5. String Kernel

Let  $s$  and  $s'$  be strings. To measure how similar  $s$  and  $s'$  are, consider the “string kernel”  $K(s, s')$ , which returns the total number of distinct substrings (of any length) that  $s$  and  $s'$  have in common. For example,  $K('aa', 'aab') = 3$  because the substrings  $' '$ ,  $'a'$ , and  $'aa'$  are in common.

(i) Compute  $K('aza', 'zaz')$ .

(ii) What is the number of possible substrings of length 1, 2, and 3 in strings that are composed from a 26-letter alphabet?

(iii) Suppose we wanted to project a string into a higher-dimensional space such that we could represent via a 0 or 1 each of all possible substrings of length  $\leq 3$ . How many dimensions would we need?