1. **Linear Regression** Consider a one-dimensional regression problem with training data $\{x_i, y_i\}$. We seek to fit a linear model with no bias term:

$$\hat{y} = wx$$

   (a) Assume a squared loss $\frac{1}{2}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$ and solve for the optimal value of $w^*$.

   (b) What is the prediction for some new observation $x$?

   (c) Suppose that we have a generative model of the form $y = wx + \epsilon$, where $\epsilon \sim \mathcal{N}(0, \sigma^2)$ and $w$ is known. Given a new $x$ what is the expression for the probability of $y$?

   Note: The univariate Gaussian distribution is $\mathcal{N}(x; \mu, \sigma^2)$:

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

   (d) Now assume that we have a known prior $s_0^2$ on $w \sim \mathcal{N}(0, s_0^2)$.

   Write down the form of the *posterior* in terms of the data and univariate Gaussian distribution. Take logs and drop terms that don't depend the data, but you do not need simplify further.

2. **Multiclass Classification**

   Suppose that we have a $c$-class classification scenario with training data $\{\mathbf{x}_i, \mathbf{y}_i\}_{i=1}^n$, where the $\mathbf{y}_i$ are 1-hot column vectors.

   We model this problem using a neural network with $d$ units in a single hidden layer, expressed as column vector $\boldsymbol{\phi}(\mathbf{x}; \mathbf{W}^1, \mathbf{w}_0^1) \in \mathbb{R}^d$, which we write as $\boldsymbol{\phi}$. We take a linear combination of these values and pass them to a softmax function to get a final set of $c$ outputs:

$$p(\mathbf{y} = C_k \,|\, \mathbf{x}; \{\mathbf{w}_\ell\}, \mathbf{W}^1, \mathbf{w}_0^1) = \frac{\exp(\mathbf{w}_k^\mathsf{T} \boldsymbol{\phi})}{\sum_{\ell=1}^c \exp(\mathbf{w}_\ell^\mathsf{T} \boldsymbol{\phi})}$$

   where $\mathbf{w} \in \mathbb{R}^d$ is a column vector of weights.

   (a) Suppose we add the same, global bias to each vector of weights in the final layer, ie replace $\mathbf{w}_k^\mathsf{T} \boldsymbol{\phi}$ with $\mathbf{w}_k^\mathsf{T} \boldsymbol{\phi} + w_0$ for some scalar $w_0$. Does that increase the expressivity of our model? Why or why not?

   (b) Write down and simplify the log likelihood of a particular observation $(\mathbf{x}_i, \mathbf{y}_i)$, including constants. Assume that we use a sigmoid activation function,

$$\boldsymbol{\phi}(\mathbf{x}; \mathbf{W}^1, \mathbf{w}_0^1) = \boldsymbol{\sigma}(\mathbf{W}^1 \mathbf{x} + \mathbf{w}_0^1)$$

(c) In information theory, cross-entropy $\mathbb{E}_p[-\ln(q)]$ is the number of bits required to explain a true distribution $p$ (our desired outputs) if given some other distribution $q$. Write down the expression for the cross-entropy of true multi-class prediction (samples from our data) compared to our predictions. Describe its relationship to the log loss in part (b).

(d) When running SGD we need to compute the gradient of the loss for a single example with respect to each of the parameters. Use the above definitions of $\boldsymbol{\phi}$ to compute this gradient for the bias vector for the neural network layer.

$$\frac{\partial}{\partial \mathbf{w}_0^1} - \ln p(\mathbf{y} = C_k | \mathbf{x}; \mathbf{W}^1, \mathbf{w}_0^1, \{\mathbf{w}_\ell\})$$

Note:

- You may use $\sigma'$ for the derivative of the sigmoid function.
- You may use this intermediate term:

$$\frac{\partial}{\partial \mathbf{w}_k^\top \boldsymbol{\phi}} - \ln p(\mathbf{y} = C_k | \mathbf{x}; \mathbf{W}^1, \mathbf{w}_0^1, \{\mathbf{w}_\ell\}) = p(\mathbf{y} = C_k | \mathbf{x}; \mathbf{W}^1, \mathbf{w}_0^1, \{\mathbf{w}_\ell\}) - y_k$$

3. **Overfitting and Underfitting**

Harvard Insta-Ice Unit (HI2U) has built a robot that can deliver 24-hour shaved ice to student houses. To prevent collisions, they train three different approaches to classify camera images as containing nearby tourists or open space; if the robot identifies a tourist in its path it is programmed to halt. The performances of the classifiers are

| | Training Accuracy | Test Accuracy |
|---|---|---|
| Classifier A | 75.3% | 74.8% |
| Classifier B | 80.3% | 77.8% |
| Classifier C | 90.2% | 60.0% |

where Classifier B has a more expressive model class than A, and classifier C has both a more expressive model class and more features than A. All the classifiers have closed-form solutions, so HI2U is pretty sure that the inference is not hindering performance.

(a) If you had to choose one: might Classifer A be overfitting or underfitting? Explain your reasoning.

(b) If you had to choose one: might Classifer C be overfitting or underfitting? Explain your reasoning.

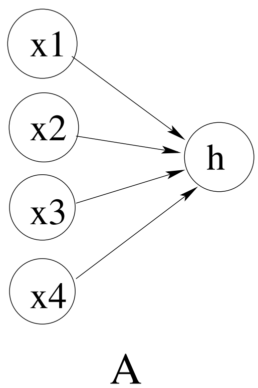
(c) If you had to guess yes or no: might more training examples significantly boost the test-time performance of Classifier A? Classifier C? Explain your reasoning.

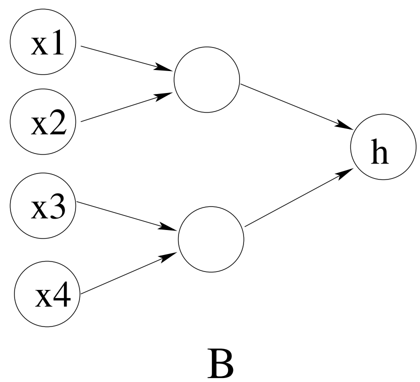
4. **Neural Networks.**

(a) Consider neural networks with a basis functions that uses a **0/1 activation** $f_{0/1}$, that switches at 0 from 0 to 1, with:

$$f_{0/1}(h) = \begin{cases} 1 & \text{if } h \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

for weighted sum of input for each $\phi_j(\mathbf{x}) = f_{0/1}(\mathbf{w}_j^{1\top}\mathbf{x} + w_0^1)$ . The input layer to the networks $x_1, x_2, x_3$ and $x_4$, **all take on values 0 or 1.** i.e. $\mathbf{x} \in \{0, 1\}^4$. Consider two networks, one which is linear in $\mathbf{x}$ and one that sparse connections in its first layer.




(i) Describe a logical formula on inputs that can be expressed by architecture (A) but not by (B), and provide weights that implement the formula in A.

(ii) Provide an argument for why B cannot express this formula (we don't expect a rigorous proof, but try to give a complete and convincing argument.)

(iii) How might you change the architecture of the second network to fix this issue? What downside might this have?

(b) What is the concern about training the networks as currently defined? What change would you make to the network to alleviate this concern?

(c) State **two** ways in which a validation set can be used when training neural networks. (One sentence for each is fine.)