FunExt vs Weak FunExt

Nicolas Tabareau

January 4, 2020

For every $A : \mathsf{Type}$, $B : A \to \mathsf{Type}$ and $f\ g : \forall x : A.B\ x$, we define pointwise equality as

$$f == g := \forall x : A.f\ x = g\ x.$$

The axiom of weak extensionality is stated as the existence of a function

$$\text{funext}_{weak} := \forall A(B : A \to \mathsf{Type})(f\ g : \forall x : A, B\ x).f == g \to f = g.$$

Recall that the axiom of (strong) function extensionality is stated as the fact that the canonical map

$$\text{apD}_{10} : \forall A(B : A \to \mathsf{Type})(f\ g : \forall x : A, B\ x).f = g \to f == g.$$

is an equivalence

$$\text{funext}_{strong} := \forall A(B : A \to \mathsf{Type})(f\ g : \forall x : A, B\ x).\,\text{IsEquiv}(\text{apD}_{10}\ A\ B\ f\ g).$$

It is clear that strong function extensionality implies weak function extensionality, but the converse is not obvious and even looks wrong.

In this exercise, we show that weak function extensionality implies strong funextionality.

Lemma 1

Show that for every $A : \mathsf{Type}$ $B : A \to \mathsf{Type}$ and $f : \forall x : A.B\ x$, the space

$$\Sigma g : \forall x : A.B\ x\ \&\ f == g$$

is a retraction of the space

$$\forall x : A.\Sigma y : B\ x\ \&\ f\ x = y.$$

Recall that a retraction between two types A and B is given by

$$\text{retract}\,A\ B := \Sigma f : A \to B\ \&\ \Sigma g : B \to A\ \&\ \forall x : A.g(f\ x) = x.$$

As usual, $\Sigma x : A\ \&\ B$ denotes the dependent sum, with first and second projections noted .1 and .2.

Singleton are contractible

Show that singletons are contractible, that is for every type A and $a : A$, we have

$$\text{IsContr}(\Sigma a' : A\ \&\ a = a')$$

where $\text{IsContr}\,X := \Sigma x : X\ \&\ \forall x' : X.x = x'$.

Lemma 2

Show assuming funext_{weak} that for every type A, $B : A \to \mathsf{Type}$ and $f : \forall x : A.B\ x$, we have

$$\text{IsContr}(\forall x : A.\ \Sigma y : B\ x\ \&\ f\ x = y).$$

Iscontr $\Rightarrow$ retract

Show that if a type B is contractible and A is a retract of B, then A is contractible.

Lemma 3

Show for every type A, $B : A \to \mathsf{Type}$ $f : \forall x : A.B\ x$ and assuming $wf : \text{funext}_{weak}$, that if

$$wf\ \ A\ \ B\ \ f\ \ f(\lambda x.\ \mathsf{refl}\,(f\ \ x)) = \mathsf{refl}\,f$$

then

$$\forall (g : \forall x : A.B\ x)(h : f == g), \text{apD}_{10}(wf\ \ A\ \ B\ \ f\ \ g\ \ h) = h.$$

Lemma 4

Show that from any $wf : \text{funext}_{weak}$, we can define $wf' : \text{funext}_{weak}$ such that

$$wf'\ \ A\ \ B\ \ f\ \ f(\lambda x.\ \mathsf{refl}\,(f\ \ x)) = \mathsf{refl}\,f.$$

Conclude that

$$\text{funext}_{weak} \to \text{funext}_{strong}.$$