Effectful Type Theories

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- Finest types to describe your programs
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The Pinnacle of the Curry-Howard correspondence

Bird's Eye View of CIC

Recall from what you have seen

- A purely functional language
- ... plus Π generalizing arrows
- ... plus ADTs with generalized pattern-matching
- ... where types are terms
- Proof and computation living in harmony

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CIC is quite a complex beast!

“The earthly paradise of the proof-program correspondence”

¿ "The earthly paradise of the proof-program correspondence" ?

The CIC brothers

Actually not quite one single theory.

Several flags tweaking the Coq kernel:

- Impredicative Set
- Type-in-type
- Indices Matter
- Cumulative inductive types
- ...

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The Many Calculi of Inductive Constructions.

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A crazy amount of axioms used in the wild!

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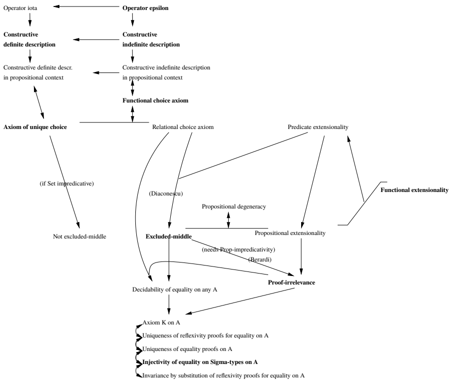
- Excluded middle, UIP, choice



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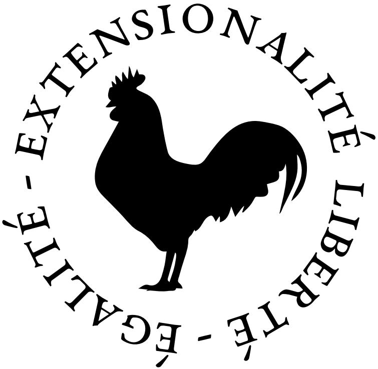

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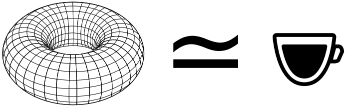

« A mathematician is a device for turning toruses into equalities (up to homotopy). »

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- Anti-classical axioms (???)

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Varying degrees of compatibility.

Reality Check

Theorem 0

Axioms Suck.

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- They break computation.
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□

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$\square$

Mathematicians may not care too much, but...

CIC suffers from an even more **fundamental** flaw.

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- You fire your favourite IDE
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COULD YOU WRITE A HELLO WORLD PROGRAM PLEASE?

Sad reality (a.k.a. Curry-Howard)

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Coq is even purer than Haskell:

- No mutable state (obviously)
- No exceptions (Haskell has them somehow)
- No arbitrary recursion
- and also no **HELLO WORLD** !

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We want a type theory with effects !

The Good News

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Intuitionistic Logic ⇔ **Functional** Programming

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Non-Intuitionistic Logic ⇔ **Impure** Programming

- `callcc` gives classical logic
- Delimited continuations prove Markov's principle
- Exceptions implement Markov's rule
- Well-behaved global cells provide univalence
- ...

We want a type theory with **effects** !

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There are some good properties we want to ensure...

We want a model of type theory with effects.

Good Properties

Consistency There is no proof of `False`.

Implementability Type-checking is decidable.

Canonicity Closed integers are indeed integers, i.e

$$\vdash M : \mathbb{N} \quad \text{implies} \quad M \equiv \mathtt{S} \ldots \mathtt{S}\,\mathtt{0}$$

Assuming we have a notion of reduction compatible with conversion:

Normalization Reduction is normalizing

Subject reduction Reduction is compatible with typing

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Some of these properties are interdependent

From Hell

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Roughly three standard families of models:

- **Set-theoretical** models
- **Realizability** models
- **Categorical** models

Let's review them quickly!

The Set-Theoretical Model

Because Sets are a (crappy) type theory.

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Interpret everything as sets and expect $\vdash_{\mathsf{CIC}} M : A \Rightarrow \vdash_{\mathsf{ZFC}} [\![M]\!] \in [\![A]\!]$.

Pro

- Well-known and trusted target
- Imports ZFC properties.

Con

- Forego syntax, computation and decidability
- No effects in sight.
- Imports ZFC properties.

The Realizability Models

Construct programs that respect properties.

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Construct programs that respect properties.

- Terms $M \rightsquigarrow$ programs $[M]$ (variable languages as a target)
- Types $A \rightsquigarrow$ meta-theoretical predicates $[\![A]\!]$
- $\vdash_{\mathsf{CIC}} M : A \quad \Rightarrow \quad [M] \in [\![A]\!]$

Pro

- Some preservation of syntax and computability

Con

- Usually crazily undecidable
- Meta-theory can be arbitrary crap, including ZFC

The Categorical Models

Abstract / obfuscated description of type theory.

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Abstract / obfuscated description of type theory.

Rephrase the rules of CIC in a categorical way.

Pro

- Very abstract and subsumes both previous examples
- Somewhat "easier" to show some structure is a model of TT

Con

- Same limitations as the previous examples
- Mostly useless to actually construct a model
- Yet another syntax, usually arcane and ill-fitted

Down With Semantics

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Syntactic Models

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« This is a *compiler*... »

... and you know it.

Curry-Howard Orthodoxy

Let's look at what Curry-Howard provides in simpler settings.

Program Translations $\Leftrightarrow$ Logical Interpretations

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On the **programming** side, enrich the language by program translation.

- Monadic style à la Haskell
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Program Translations $\Leftrightarrow$ Logical Interpretations

On the **programming** side, enrich the language by program translation.

- Monadic style à la Haskell
- Compilation of higher-level constructs down to assembly

On the **logic** side, extend expressivity through proof interpretation.

- Double-negation $\Rightarrow$ classical logic (`callcc`)
- Friedman's trick $\Rightarrow$ Markov's rule (exceptions)
- Forcing $\Rightarrow$ $\neg$CH (global monotonous cell)

Syntactic Models

Let us do the same thing with CIC: build **syntactic models**.

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We take the following act of faith for granted.

CIC is.

Not caring for its soundness, implementation, whatever. It just is.

Do everything by interpreting the new theories relatively to this foundation!

Suppress technical and cognitive burden by lowering impedance mismatch.

Syntactic Models II

Step 0: Fix a theory $\mathcal{T}$ as close as possible* to CIC, ideally CIC $\subseteq \mathcal{T}$.

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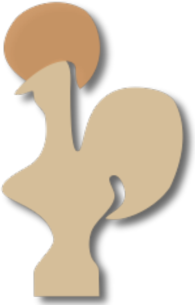
Step 3: Expand $\mathcal{T}$ by going down to the *CIC assembly language*, implementing new terms given by the $[\cdot]$ translation.

Anatomy of a syntactic model

COMPILATION



$\vdash_{\mathsf{CIC}^{++}} M : A \qquad \rightsquigarrow \qquad \vdash_{\mathsf{CIC}} [M] : [\![A]\!]$

« CIC, the LLVM of type theory »

Syntactic Models III

Obviously, that's subtle.

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Yet, a lot of nice consequences.

- Does not require non-type-theoretical foundations (*monism*)
- **Can be implemented in Coq** (*software monism*)
- Easy to show (relative) consistency, look at $[\![\texttt{False}]\!]$
- Inherit properties from CIC: computationality, decidability...

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We will show three simple models

- Warmup: Intentional types (not an effect)
- Part I: Read-only cell
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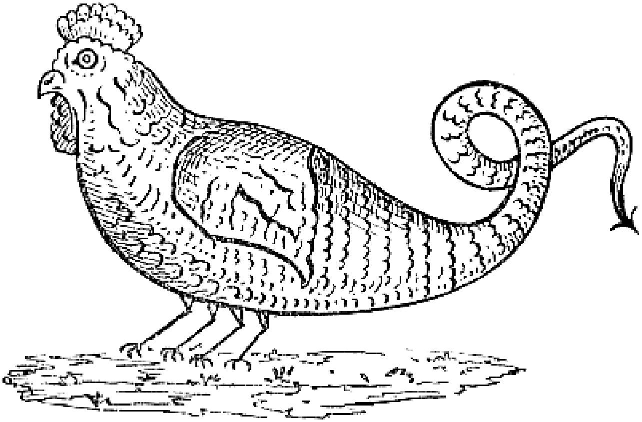
I and II feature fundamental interactions between effects and dependency.



Intensional Types, a.k.a. **Dynamically Typed CIC**

(effect-free introductory example)

Intensional Types

The intensional types translation extends type theory with

$$\begin{array}{lcl}
\texttt{flip} & : & \square \to \square \\
\texttt{flip_equiv} & : & \Pi(A : \square).\, \texttt{flip}\ A \cong A \\
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This breaks amongst other things univalence...

The Intensional Types Implementation

Intuitively:

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$$\begin{array}{lcl}
[\![A]\!] & \equiv & [A].\pi_1 \\
[\square] & \equiv & (\square \times \mathbb{B}, \mathtt{true}) \\
[\Pi x : A.\, B] & \equiv & (\Pi x : [\![A]\!].\, [\![B]\!], \mathtt{true}) \\
[x] & \equiv & x \\
[M\ N] & \equiv & [M]\ [N] \\
[\lambda x : A.\, M] & \equiv & \lambda x : [\![A]\!].\, [M]
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Types contain a boolean not used for their inhabitants!

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Soundness

If $\vec{x} : \Gamma \vdash M : A$ then $\vec{x} : [\![\Gamma]\!] \vdash [M] : [\![A]\!]$.

Extending the Intensional Types

Let's define the new operations obtained through the translation.

$$
\begin{array}{lcl}
[\texttt{flip}] & : & [\![\square \to \square]\!] \\
[\texttt{flip}] & : & \square \times \mathbb{B} \to \square \times \mathbb{B} \\
[\texttt{flip}] & \equiv & \lambda(A, b).\,(A, \texttt{negb}\ b) \\
\\
[\texttt{flip\_equiv}] & : & [\![\Pi A : \square.\, \texttt{flip}\ A \cong A]\!] \\
[\texttt{flip\_equiv}] & \equiv & \ldots \\
\\
[\texttt{flip\_neq}] & : & [\![\Pi A : \square.\, \texttt{flip}\ A \neq A]\!] \\
[\texttt{flip\_neq}] & : & \Pi A : \square \times \mathbb{B}.\, [\texttt{flip}]\ A \neq A \\
[\texttt{flip\_equiv}] & \equiv & \ldots
\end{array}
$$

- $[\![\texttt{flip}\ A]\!] \equiv [\![A]\!]$
- And isomorphism only depends on $[\![A]\!]$
- But (intensional) equality observes the boolean...

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- Assuming the target theory features induction-recursion
- Represent (source) types by their code
- This gives a real type-quote function in the source theory

$$
\begin{array}{rl}
\texttt{type\_rect} : & \Pi(P : \square \to \square). \\
& \quad P\ \square \to \\
& \quad (\Pi(A : \square)\,(B : A \to \square).\ P\ A \to (\Pi x : A.\ P\ (B\ x)) \to \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad P\ (\Pi x : A.\ B)) \to \\
& \quad P\ \mathbb{N} \to \\
& \quad \ldots \to \\
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& \quad P\ \mathbb{N} \to\\
& \quad \ldots \to\\
& \Pi(A : \square).\, P\ A
\end{aligned}
$$

Coq is compatible with dynamic types!!!

The reader translation, a.k.a. **Baby Forcing**

Overview

Essentially the same as Haskell's reader effect*.

- There is a global unnamed cell
- That can be read
- That can be updated **in a well-scoped way**

Not quite a state!

* main difference is that we're in call-by-name.

The Reader Translation

Assume some fixed cell type $\mathbb{R} : \square$.

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The reader translation extends CIC into $\mathsf{CIC}_{\mathbb{R}}$, with

$$\begin{array}{rrcl} & \texttt{read} & : & \mathbb{R} \\ & \texttt{into} & : & \square \to \mathbb{R} \to \square \\ & \texttt{enter} & : & \Pi(A : \square).\, A \to \Pi r : \mathbb{R}.\, \texttt{into}\ A\ r \\ (\text{morally} & \texttt{enter} & : & \Pi(A : \square).\, A \to \mathbb{R} \to A) \end{array}$$

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satisfying the expected definitional equations, e.g.

$$\begin{array}{rcl} \mathtt{enter}\ \mathbb{R}\ \mathtt{read}\ M & \equiv & M \\ \mathtt{enter}\ \mathbb{R}\ M\ \mathtt{read} & \equiv & M \end{array}$$

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Assuming a variable $r : \mathbb{R}$, intuitively:

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Soundness

We have $\Gamma \vdash M : A$ implies $[\![\Gamma]\!], r : \mathbb{R} \vdash [M]_r : [A]_r$.

The Reader Implementation: Inductive Types

PLT tells us we have to take $[A + B]_r \equiv [\![A]\!] + [\![B]\!]$.

$$\begin{array}{rcl} [A + B]_r & \equiv & [\![A]\!] + [\![B]\!] \\ [\mathtt{inl}\ M]_r & \equiv & \mathtt{inl}\ (\Pi s : \mathbb{R}.\, [M]_s) \\ [\mathtt{inr}\ M]_r & \equiv & \mathtt{inr}\ (\Pi s : \mathbb{R}.\, [M]_s) \end{array}$$

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It's possible to implement **non-dependent** pattern-matching as usual.

PLT tells us we have to take $[A + B]_r \equiv [\![A]\!] + [\![B]\!]$.

$$\begin{array}{lcl} [A + B]_r & \equiv & [\![A]\!] + [\![B]\!] \\ [\mathtt{inl}\ M]_r & \equiv & \mathtt{inl}\ (\Pi s : \mathbb{R}.\, [M]_s) \\ [\mathtt{inr}\ M]_r & \equiv & \mathtt{inr}\ (\Pi s : \mathbb{R}.\, [M]_s) \end{array}$$

It's possible to implement **non-dependent** pattern-matching as usual.

$$[\mathtt{elim}_+]_r : [\![\Pi P : \square.\, (A \to P) \to (B \to P) \to A + B \to P]\!] \equiv \Pi(P : \mathbb{R} \to \square).$$
$$(\Pi s : \mathbb{R}.\, [\![A]\!] \to P\ s) \to (\Pi s : \mathbb{R}.\, [\![B]\!] \to P\ s) \to (\mathbb{R} \to [\![A]\!] + [\![B]\!]) \to P\ r$$

$$\begin{array}{lcl} \mathtt{elim}_+\ P\ N_l\ N_r\ (\mathtt{inl}\ M) & \equiv & N_l\ M \\ \mathtt{elim}_+\ P\ N_l\ N_r\ (\mathtt{inr}\ M) & \equiv & N_r\ M \end{array}$$

Uh-oh

Unfortunately, It's **not possible** to implement **dependent** elimination!

$$[\![\Pi P.\,(\Pi(x:A).\,P\;(\mathtt{inl}\;x)) \to (\Pi(y:B).\,P\;(\mathtt{inr}\;y)) \to \Pi b:A+B.\,P\;b]\!]$$

$$\equiv$$

$$\begin{aligned}
&\Pi P : \mathbb{R} \to (\mathbb{R} \to [\![A]\!] + [\![B]\!]) \to \square.\\
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&\quad (\Pi(s:\mathbb{R})\,(y:[\![B]\!]).\,\boxed{P}\;s\;\boxed{(\lambda_:\mathbb{R}.\,\mathtt{inr}\;y)}) \to\\
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P only holds for two constant values but b can be anything!

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Horrible Realization

Dependent elimination is incompatible with effects.

Not All Predicates are Equal

In general through $[\cdot]_r$ predicates have the following type:

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In this case, induction principle becomes

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This **is** provable!

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Induction is still valid for predicates that evaluate **eagerly** their argument.

Linear Dependence is All You Need

We restrict dependent elimination in the following way:

$$\frac{\Gamma \vdash M : \mathbb{B} \qquad \ldots \qquad P \text{ eager in } b}{\Gamma \vdash \texttt{if } M \texttt{ then } N_1 \texttt{ else } N_2 : P\{b := M\}}$$

- Can be underapproximated by a generic, syntactic **guard condition**
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- The CBN doppelgänger of the dreaded **value restriction** in CBV!
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This restriction forms **Baclofen Type Theory**.

Outrageous claim

BTT is the generic theory to deal with dependent effects

The Exceptional Type Theory

(a.k.a. the Curry-Howard-Shadok correspondence)

An extension of CIC rooted in Shadok wisdom.

"THE MORE IT FAILS, THE MORE LIKELY IT WILL EVENTUALLY SUCCEED."

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The Exceptional Type Theory: Overview

The exceptional type theory $\mathcal{T}_{\mathbb{E}}$ extends CIC with

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Remark that in call-by-name, if $M : A \to B$, in general

$$M\ (\texttt{raise}\ A\ e) \quad \not\equiv \quad \texttt{raise}\ B\ e$$

Catch Me If You Can

Remember that on functions:

$$\texttt{raise}\ (\Pi x : A.\, B)\ e \quad \equiv \quad \lambda x : A.\, \texttt{raise}\ B\ e$$

It means catching exceptions is limited to positive datatypes!

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For inductive types, this is a **generalized induction principle**.

$$\begin{array}{ll} \texttt{catch}_{\mathbb{B}} : & \Pi P : \mathbb{B} \to \square. \\ & \quad P\ \texttt{true} \to \\ & \quad P\ \texttt{false} \to \\ & \quad (\Pi e : \mathbf{E}.\, P\ (\texttt{raise}\ \mathbb{B}\ e)) \to \\ & \Pi b : \mathbb{B}.\, P\ b \end{array} \qquad \begin{array}{ll} \mathbb{B}_{\texttt{rect}} : & \Pi P : \mathbb{B} \to \square. \\ & \quad P\ \texttt{true} \to \\ & \quad P\ \texttt{false} \to \\ & \\ & \Pi b : \mathbb{B}.\, P\ b \end{array}$$

where

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The Exceptional Implementation, Negative case

Intuition: $\vdash_{\mathcal{T}_{\mathbb{E}}} A : \square \quad \rightsquigarrow \quad \vdash_{\mathsf{CIC}} [A] : \Sigma A : \square.\, \mathbb{E} \to A.$

Every exceptional type comes with its own implementation of failure!

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$$[\![A]\!] : \square := \pi_1\ [A] \qquad \text{and} \qquad [A]_\varnothing : \mathbb{E} \to [\![A]\!] := \pi_2\ [A]$$

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If $\Gamma \vdash_{\mathsf{CIC}} M : A$ then $[\![\Gamma]\!] \vdash_{\mathsf{CIC}} [M] : [\![A]\!]$.

The Exceptional Implementation, Failure

It is straightforward to implement the failure operation.

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Computational rules trivially hold!

$$\begin{array}{ccc} [\texttt{raise}\ (\Pi x : A.\, B)\ e] & \equiv & [\lambda x : A.\, \texttt{raise}\ B\ e] \\ ||| & & ||| \\ \pi_2\ ((\Pi x : [\![A]\!].\, [\![B]\!]), (\lambda (e : \mathbb{E})\, (x : [\![A]\!]).\, \pi_2\ [\![B]\!]\ e))\ [e] & \equiv & \lambda x : [\![A]\!].\, \pi_2\ [B]\ [e] \end{array}$$

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The really interesting case is the inductive part of CIC.

How to implement $[\mathbb{B}]_\varnothing : \mathbb{E} \to [\![\mathbb{B}]\!]$?

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Could pose $[\![\mathbb{B}]\!] := \mathbb{B}$ and take an arbitrary boolean for $[\mathbb{B}]_{\varnothing}$...

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Could pose $[\![\mathbb{B}]\!] := \mathbb{B}$ and take an arbitrary boolean for $[\mathbb{B}]_\varnothing$...

... but that would not play well with computation, e.g. `catch`.

Worse, what about $[\bot]_\varnothing : \mathbb{E} \to [\![\bot]\!]$?

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Very elegant solution: add a default case to every inductive type!

$$\texttt{Inductive}\ [\![\mathbb{B}]\!]\ := [\texttt{true}] : [\![\mathbb{B}]\!]\ \mid [\texttt{false}] : [\![\mathbb{B}]\!]\ \mid \mathbb{B}_{\varnothing} : \mathbb{E} \to [\![\mathbb{B}]\!]$$

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Pattern-matching is translated pointwise, except for the new case.

$$[\![\Pi P : \mathbb{B} \to \square.\, P\ \texttt{true} \to P\ \texttt{false} \to \Pi b : \mathbb{B}.\, P\ b]\!]$$

$$\equiv \quad \Pi P : [\![\mathbb{B}]\!] \to [\![\square]\!].\, P\ [\texttt{true}] \to P\ [\texttt{false}] \to \Pi b : [\![\mathbb{B}]\!].\, P\ b$$

- If b is $[\texttt{true}]$, use first hypothesis
- If b is $[\texttt{false}]$, use second hypothesis
- If b is an error $\mathbb{B}_\varnothing\ e$, **reraise** e using $[P\ b]_\varnothing\ e$

Shadok Logic Strikes Back

Theorem

The exceptional translation interprets all of CIC.

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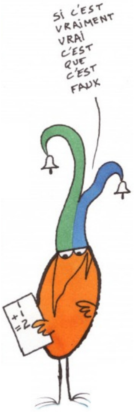

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The exceptional translation interprets all of CIC.

- ☺ A type theory with effects!
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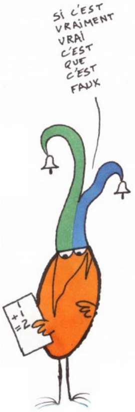
- ☹ Ah, yeah, and also, the theory is inconsistent.

It suffices to raise an exception to inhabit any type.



Consistency: A Social Construct

An Impure Dependently-typed Programming Language

Do you whine about the fact that OCaml is logically inconsistent?

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Theorem (Exceptional Canonicity a.k.a. Progress a.k.a. Meaningless explanations)

If $\vdash_{\mathcal{T}_{\mathbb{E}}} M : \bot$, *then* $M \equiv \mathtt{raise}\ \bot\ e$ *for some* $e : \mathbf{E}$.

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You can still use the CIC target to prove properties about $\mathcal{T}_{\mathbb{E}}$ programs!

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Cliffhanger

You can prove that a program does not raise uncaught exceptions.

Exception

Gotta catch 'em all!

If You Joined the Talk Recently

The exceptional type theory is logically inconsistent!

Cliffhanger (cont.)

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Let's call **valid** a program in $\mathcal{T}_{\mathbb{E}}$ that "does not raise exceptions".

For instance,

- there is no valid proof of $\perp$
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Validity is a type-directed notion!

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Let's locally write $M \Vdash A$ if M is valid at A.

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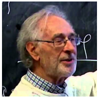

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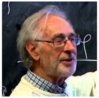

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Fools ! That's **parametricity**.

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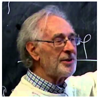
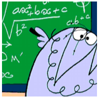


Come on. That's **intuitionistic realizability**.

Fools ! That's **parametricity**.



Zo!

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Idea:

From $\vdash M : A$ produce **two** sequents $\begin{cases} \vdash_{\text{CIC}} [M] : [\![A]\!] \\ \quad + \\ \vdash_{\text{CIC}} [M]_\varepsilon : [\![A]\!]_\varepsilon\ [M] \end{cases}$

where $[\![A]\!]_\varepsilon : [\![A]\!] \to \square$ is the validity predicate.

Parametric Exceptional Translation (Sketch)

Most notably,

$$\begin{array}{lcl}
[\![\Pi x : A.\, B]\!]_\varepsilon\ f & \equiv & \Pi(x : [\![A]\!])\,(x_\varepsilon : [\![A]\!]_\varepsilon\ x).\, [\![B]\!]_\varepsilon\ (f\, x) \\
[\![\mathbb{B}]\!]_\varepsilon\ b & \cong & b = [\mathtt{true}] + b = [\mathtt{false}] \\
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Every pure term is now automatically parametric.

If $\Gamma \vdash_{\mathsf{CIC}} M : A$ then $[\![\Gamma]\!]_\varepsilon \vdash_{\mathsf{CIC}} [M]_\varepsilon : [\![A]\!]_\varepsilon\ [M]$.

A Few Nice Results

Let's call $\mathcal{T}_{\mathbb{E}}^{p}$ the resulting theory. It inherits a lot from CIC!

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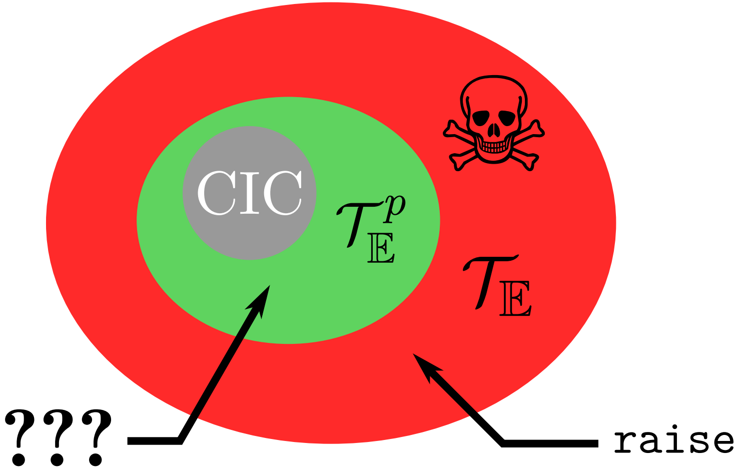
Theorem (Syntax)

$\mathcal{T}_{\mathbb{E}}^{p}$ *has decidable type-checking, strong normalization and whatnot.*

What If There Were No Cake?



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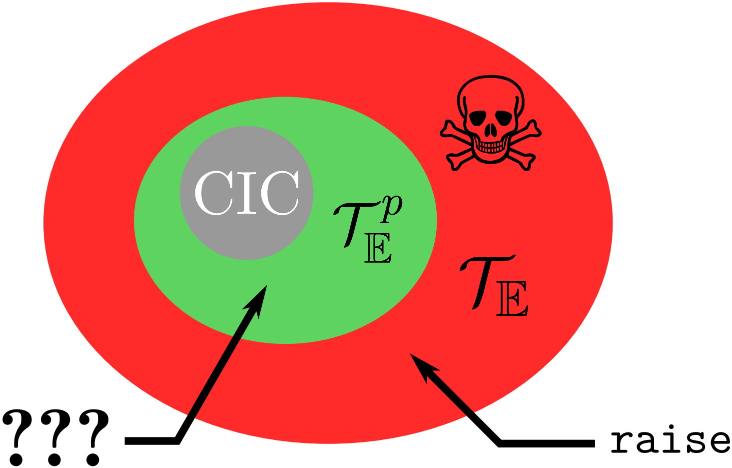
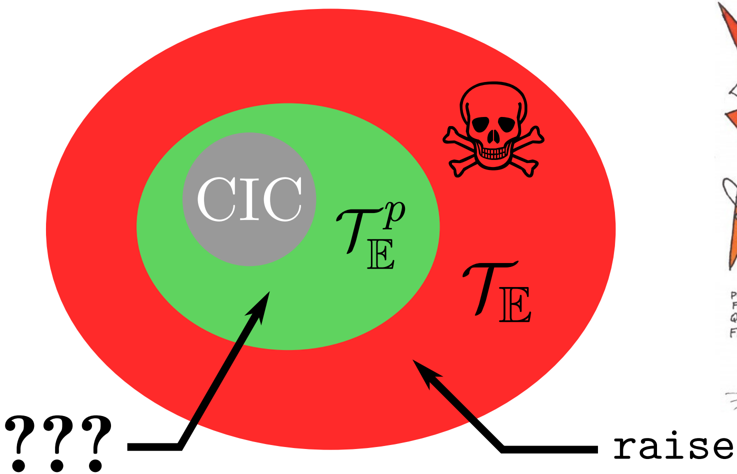
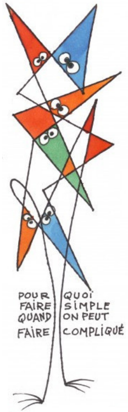


Bernardy-Lasson parametricity is a conservative extension of CIC...





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$\mathcal{T}_{\mathbb{E}}^{p}$ is **not** a conservative extension of CIC.

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Intuitively,

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- … but you can still raise them locally
- … as long as you prove they don't escape!

$\mathcal{T}_{\mathbb{E}}$ is the `unsafe` Coq fragment, and $\mathcal{T}_{\mathbb{E}}^{p}$ a semantical layer atop of it.

Independence of Premises

$$\mathsf{IP} : (\neg A \to \Sigma n : \mathbb{N}.\, P\ n) \to \Sigma n : \mathbb{N}.\, (\neg A \to P\ n)$$

Theorem (CIC + IP)

$\mathcal{T}_{\mathbb{E}}^{p}$ *validates* IP*, owing to the fact that in* $\mathcal{T}_{\mathbb{E}}$*, every type is inhabited.*

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Theorem (CIC + IP)

$\mathcal{T}^p_{\mathbb{E}}$ *validates* IP, *owing to the fact that in* $\mathcal{T}_{\mathbb{E}}$, *every type is inhabited.*

Proof (sketch).

In $\mathcal{T}_{\mathbb{E}}$, build a term `ip` : IP

- Given $f : \neg A \to \Sigma n : \mathbb{N}.\, P\ n$, apply it to `raise` $(\neg A)\ e$.
- If the returned integer is pure, return it with the associated proof.
- Otherwise, return a dummy integer and failing proof.

Easy to show that `ip` is actually valid in $\mathcal{T}^p_{\mathbb{E}}$. □

Stepping Back

Enter Effects

The end of the talk will focus on the big picture.

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Why did people have so much trouble mixing effects and dependency?

Because it's hard.

- Usual models are hard to grasp $\rightsquigarrow$ use syntactic models (done)
- Stuff breaks $\rightsquigarrow$ let's concentrate on that

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Typing rules embed the dynamics of programs!

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Recall conversion:

$$\frac{A \equiv_\beta B \qquad \Gamma \vdash M : B}{\Gamma \vdash M : A}$$

Bad news 1

Typing rules embed the dynamics of programs!

Combine that with this other observation and we're in trouble.

Bad news 2

Effects make reduction strategies relevant.

You Can't Have Your Cake and Eat It

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You Can't Have Your Cake and Eat It

Effects make reduction strategies relevant.

Call-by-value	**Call-by-name**
	
☹ Weaker conversion rule	☺ Full conversion rule
☺ Full dependent elimination	☹ Weaker dependent elimination
☺ Good old ML semantics	☹ Strange PL realm

What can go wronger?

- Call-by-name: **functions** well-behaved vs. **inductives** ill-behaved
- Call-by-value: **inductives** well-behaved vs. **functions** ill-behaved

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In **call-by-name** + effects:

$$(\lambda x.\, M)\ N \equiv M\{x := N\} \quad \rightsquigarrow \quad \textbf{arbitrary substitution}$$

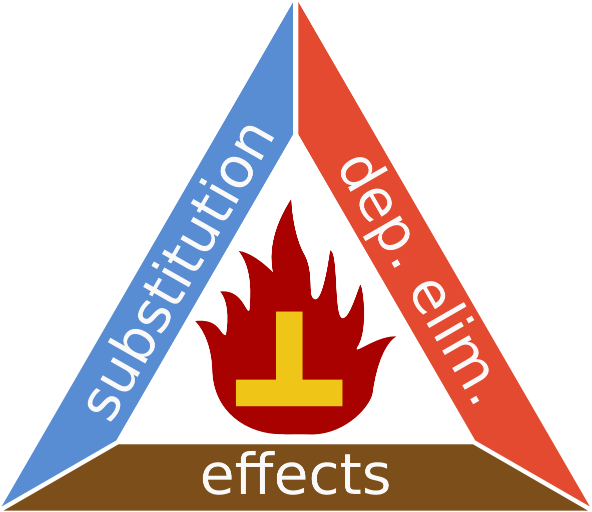
$$(\lambda b : \texttt{bool}.\, M)\ \textbf{fail} \quad \rightsquigarrow \quad \textbf{non-standard booleans}$$

In **call-by-value** + effects:

$$(\lambda x.\, M)\ V \equiv M\{x := V\} \quad \rightsquigarrow \quad \textbf{substitute only values}$$

$$(\lambda b : \texttt{unit}.\, \textbf{fail}\ b) \quad \rightsquigarrow \quad \textbf{invalid } \eta\textbf{-rule}$$

An Incompatibility



Conclusion

- Syntactic models bring semantics to the masses
- You can use them to add effects
- But you have to pick your poison
- Effects can be removed a posteriori for great profit

Scribitur ad narrandum, non ad probandum.

Thanks for your attention.