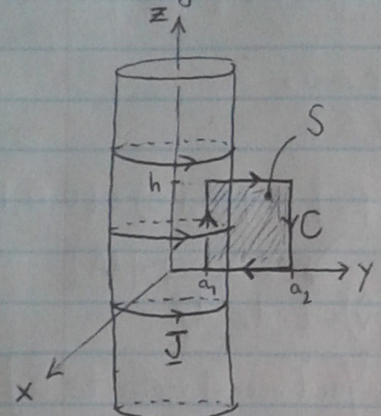
Stokes' theorem example

Consider an infinite cylinder of radius R, and with current $\underline{J}$ flowing around the surface:



$$\underline{J} = J_0\, \delta(r-R)\, \hat{v} \quad , \quad \hat{v} = \frac{(-y,x,0)}{\sqrt{x^2+y^2}}$$

cylindrical coordinates (r,θ,z)

unit vector field for rotation around the z-axis

Find a vector field $\underline{F}$ such that

$$\nabla \cdot \underline{F} = 0 \quad , \quad \nabla \times \underline{F} = \underline{J}$$

(and $\underline{F} \to 0$ as $r \to \infty$).

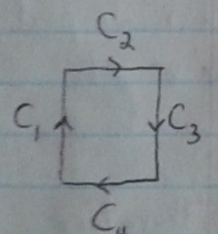
Solution: Symmetry considerations + what we know about curl / magnetic fields suggests we try to find a solution of the form

$$\underline{F}(r,\theta,z) = f(r)\,\hat{k} \quad , \quad \hat{k} = (0,0,1)$$

where $f(r)$ is some function with $f(r) \to 0$ as $r \to \infty$. Use Stokes' theorem to determine $f(r)$. We choose the surface S shown in the picture: it is a rectangle in the y,z-plane having height h and with its left and right sides ~~having~~ located at $y=a_1$ and $y=a_2$ respectively. By Stokes':

$$\oint_C \underline{F} \cdot d\underline{r} = \iint_S (\nabla \times \underline{F}) \cdot \hat{n}\, dS = \iint_S \underline{J} \cdot \hat{n}\, dS$$

LHS: C consists of four line segments C_1, C_2, C_3, C_4. The top and bottom line segments C_2, C_4 contribute 0 to the line integral because for these segments $\underline{F} = f(r)\hat{k}$ is orthogonal to the curve ($\underline{r}'(t)$ is parallel to $\hat{j} = (0,1,0)$ along these line segments).



$$\therefore \oint_C \underline{F} \cdot d\underline{r} = \int_{C_1} f(a_1)\,\hat{k} \cdot \hat{k}\, ds + \int_{C_3} f(a_2)\,\hat{k} \cdot (-\hat{k})\, ds = f(a_1)\,h - f(a_2)\,h$$

unit tangent vector to C_1

unit tangent vector to C_3

RHS: On the surface S, $x=0$ (S is contained in the y,z-plane), so

$$\hat{v}\big|_S = \frac{(-y,0,0)}{\sqrt{y^2}} = -\hat{\imath} \quad , \quad \hat{\imath} = (1,0,0)$$

☆ This agrees with the normal vector (unit) to the surface.

(We are using the right-hand-rule here: ~~x~~ $-\hat{\imath} = (-1,0,0)$ is the normal vector obtained using the right-hand-rule and the orientation of C in the picture.)

For ~~x~~ S, we can use r, z as coordinates, our parametrization is

$$\underline{p}(r,z) = (0, r, z) \qquad \begin{array}{l} a_1 \le r \le a_2 \\ 0 \le z \le h \end{array}$$

Note that the surface area factor: $|\underline{p}_r \times \underline{p}_z| = |(0,1,0)\times(0,0,1)| = 1$.

$$\therefore \iint_S \underline{J}\cdot\hat{n}\, dS = \int_{a_1}^{a_2}\int_0^h \underbrace{J_0\, \delta(r-R)(\hat{v}}_{\underline{J}} \cdot \underbrace{\hat{v}}_{\boxed{\text{unit normal vector to } S}}) \cdot 1\, \underbrace{dz\, dr}_{\boxed{dA}}$$

Now $\hat{v}\cdot\hat{v} = 1$, so this simplifies to

$$= J_0 h \int_{a_1}^{a_2} \delta(r-R)\, dr$$

Since we chose $a_1 < R < a_2$, the interval of integration contains the spike of the delta function, which gives 1. Therefore

$$\text{RHS} = J_0 h$$

Equating LHS = RHS gives

$$f(a_1) - f(a_2) = J_0$$

This equation holds for ANY a_1, a_2 such that $0 \le a_1 < R < a_2$. In particular, suppose we <u>fix</u> a_1 so $f(a_1)$ is just some number c. Then we get

$$f(a_2) = c - J_0$$

The RHS is <u>constant</u>! But we said $f(r) \to 0$ as $r \to \infty$. Therefore we must have

$$f(a_2) = 0 \quad \text{for all } a_2 > R.$$

$$\Rightarrow \quad f(a_1) = J_0 \quad \text{for} \quad 0 \le a_1 < R.$$

<u>Final answer</u>: $\underline{F} = \begin{cases} J_0 \hat{k}, & 0 \le r < R \\ 0 & r > R \end{cases}$